Manifolds

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The Sage Development Team

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CONTENTS

This is the Sage implementation of manifolds resulting from the SageManifolds project. This section describes only the "manifold" part of SageManifolds; the pure algebraic part is described in the section Tensors on free modules of finite rank.

More documentation (in particular example worksheets) can be found here.

CHAPTER

ONE

TOPOLOGICAL MANIFOLDS

1.1 Topological Manifolds

Given a topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$) and a nonnegative integer n, a *topological manifold of dimension* n *over K* is a topological space M such that

- M is a Hausdorff space,
- M is second countable,
- every point in M has a neighborhood homeomorphic to K^n.

Topological manifolds are implemented via the class *TopologicalManifold*. Open subsets of topological manifolds are also implemented via *TopologicalManifold*, since they are topological manifolds by themselves.

In the current setting, topological manifolds are mostly described by means of charts (see *Chart*).

TopologicalManifold serves as a base class for more specific manifold classes.

The user interface is provided by the generic function *Manifold()*, with with the argument `structure` set to `'topological'`.

Example 1: the 2-sphere as a topological manifold of dimension 2 over R

One starts by declaring S^2 as a 2-dimensional topological manifold:

```
sage: M = Manifold(2, 'S^2', structure='topological')
sage: M
2-dimensional topological manifold S^2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', structure='topological')
>>> M
2-dimensional topological manifold S^2
```

Since the base topological field has not been specified in the argument list of `Manifold`, $\mathbf{R}$ is assumed:

```
sage: M.base_field()
Real Field with 53 bits of precision
sage: dim(M)
2
```

```
>>> from sage.all import *
>>> M.base_field()
```

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```
Real Field with 53 bits of precision
>>> dim(M)
2
```

Let us consider the complement of a point, the "North pole" say; this is an open subset of S^2, which we call U:

```
sage: U = M.open_subset('U'); U
Open subset U of the 2-dimensional topological manifold S^2
```

```
>>> from sage.all import *
>>> U = M.open_subset('U'); U
Open subset U of the 2-dimensional topological manifold S^2
```

A standard chart on U is provided by the stereographic projection from the North pole to the equatorial plane:

```
sage: stereoN.<x,y> = U.chart(); stereoN
Chart (U, (x, y))
```

```
>>> from sage.all import *
>>> stereoN = U.chart(names=('x', 'y',)); (x, y,) = stereoN._first_ngens(2); stereoN
Chart (U, (x, y))
```

Thanks to the operator `<x,y>` on the left-hand side, the coordinates declared in a chart (here x and y), are accessible by their names; they are Sage's symbolic variables:

```
sage: y
y
sage: type(y)
<class 'sage.symbolic.expression.Expression'>
```

```
>>> from sage.all import *
>>> y
y
>>> type(y)
<class 'sage.symbolic.expression.Expression'>
```

The South pole is the point of coordinates $(x, y) = (0, 0)$ in the above chart:

```
sage: S = U.point((0,0), chart=stereoN, name='S'); S
Point S on the 2-dimensional topological manifold S^2
```

```
>>> from sage.all import *
>>> S = U.point((Integer(0),Integer(0)), chart=stereoN, name='S'); S
Point S on the 2-dimensional topological manifold S^2
```

Let us call V the open subset that is the complement of the South pole and let us introduce on it the chart induced by the stereographic projection from the South pole to the equatorial plane:

```
sage: V = M.open_subset('V'); V
Open subset V of the 2-dimensional topological manifold S^2
sage: stereoS.<u,v> = V.chart(); stereoS
Chart (V, (u, v))
```

```
>>> from sage.all import *
>>> V = M.open_subset('V'); V
Open subset V of the 2-dimensional topological manifold S^2
>>> stereoS = V.chart(names=('u', 'v',)); (u, v,) = stereoS._first_ngens(2); stereoS
Chart (V, (u, v))
```

The North pole is the point of coordinates $(u, v) = (0, 0)$ in this chart:

```
sage: N = V.point((0,0), chart=stereoS, name='N'); N
Point N on the 2-dimensional topological manifold S^2
```

```
>>> from sage.all import *
>>> N = V.point((Integer(0),Integer(0)), chart=stereoS, name='N'); N
Point N on the 2-dimensional topological manifold S^2
```

To fully construct the manifold, we declare that it is the union of U and V:

```
sage: M.declare_union(U,V)
```

```
>>> from sage.all import *
>>> M.declare_union(U,V)
```

and we provide the transition map between the charts `stereoN` = $(U, (x, y))$ and `stereoS` = $(V, (u, v))$, denoting by W the intersection of U and V (W is the subset of U defined by $x^2 + y^2 \neq 0$, as well as the subset of V defined by $u^2 + v^2 \neq 0$):

```
sage: stereoN_to_S = stereoN.transition_map(stereoS, [x/(x^2+y^2), y/(x^2+y^2)],
....:                       intersection_name='W', restrictions1= x^2+y^2!=0,
....:                                              restrictions2= u^2+v^2!=0)
sage: stereoN_to_S
Change of coordinates from Chart (W, (x, y)) to Chart (W, (u, v))
sage: stereoN_to_S.display()
u = x/(x^2 + y^2)
v = y/(x^2 + y^2)
```

```
>>> from sage.all import *
>>> stereoN_to_S = stereoN.transition_map(stereoS, [x/(x**Integer(2)+y**Integer(2)),␣
↪y/(x**Integer(2)+y**Integer(2))],
...                       intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                                              restrictions2=␣
↪u**Integer(2)+v**Integer(2)!=Integer(0))
>>> stereoN_to_S
Change of coordinates from Chart (W, (x, y)) to Chart (W, (u, v))
>>> stereoN_to_S.display()
u = x/(x^2 + y^2)
v = y/(x^2 + y^2)
```

We give the name `W` to the Python variable representing $W = U \cap V$:

```
sage: W = U.intersection(V)
```

```
>>> from sage.all import *
>>> W = U.intersection(V)
```

The inverse of the transition map is computed by the method *sage.manifolds.chart.CoordChange.inverse()*:

```
sage: stereoN_to_S.inverse()
Change of coordinates from Chart (W, (u, v)) to Chart (W, (x, y))
sage: stereoN_to_S.inverse().display()
x = u/(u^2 + v^2)
y = v/(u^2 + v^2)
```

```
>>> from sage.all import *
>>> stereoN_to_S.inverse()
Change of coordinates from Chart (W, (u, v)) to Chart (W, (x, y))
>>> stereoN_to_S.inverse().display()
x = u/(u^2 + v^2)
y = v/(u^2 + v^2)
```

At this stage, we have four open subsets on S^2:

```
sage: M.subset_family()
Set {S^2, U, V, W} of open subsets of the 2-dimensional topological manifold S^2
```

```
>>> from sage.all import *
>>> M.subset_family()
Set {S^2, U, V, W} of open subsets of the 2-dimensional topological manifold S^2
```

W is the open subset that is the complement of the two poles:

```
sage: N in W or S in W
False
```

```
>>> from sage.all import *
>>> N in W or S in W
False
```

The North pole lies in V and the South pole in U:

```
sage: N in V, N in U
(True, False)
sage: S in U, S in V
(True, False)
```

```
>>> from sage.all import *
>>> N in V, N in U
(True, False)
>>> S in U, S in V
(True, False)
```

The manifold's (user) atlas contains four charts, two of them being restrictions of charts to a smaller domain:

```
sage: M.atlas()
[Chart (U, (x, y)), Chart (V, (u, v)),
 Chart (W, (x, y)), Chart (W, (u, v))]
```

```
>>> from sage.all import *
>>> M.atlas()
[Chart (U, (x, y)), Chart (V, (u, v)),
 Chart (W, (x, y)), Chart (W, (u, v))]
```

Let us consider the point of coordinates $(1, 2)$ in the chart `stereoN`:

```
sage: p = M.point((1,2), chart=stereoN, name='p'); p
Point p on the 2-dimensional topological manifold S^2
sage: p.parent()
2-dimensional topological manifold S^2
sage: p in W
True
```

```
>>> from sage.all import *
>>> p = M.point((Integer(1),Integer(2)), chart=stereoN, name='p'); p
Point p on the 2-dimensional topological manifold S^2
>>> p.parent()
2-dimensional topological manifold S^2
>>> p in W
True
```

The coordinates of p in the chart `stereoS` are computed by letting the chart act on the point:

```
sage: stereoS(p)
(1/5, 2/5)
```

```
>>> from sage.all import *
>>> stereoS(p)
(1/5, 2/5)
```

Given the definition of p, we have of course:

```
sage: stereoN(p)
(1, 2)
```

```
>>> from sage.all import *
>>> stereoN(p)
(1, 2)
```

Similarly:

```
sage: stereoS(N)
(0, 0)
sage: stereoN(S)
(0, 0)
```

```
>>> from sage.all import *
>>> stereoS(N)
(0, 0)
>>> stereoN(S)
(0, 0)
```

A continuous map $S^2 \to \mathbf{R}$ (scalar field):

```
sage: f = M.scalar_field({stereoN: atan(x^2+y^2), stereoS: pi/2-atan(u^2+v^2)},
....:                    name='f')
sage: f
Scalar field f on the 2-dimensional topological manifold S^2
sage: f.display()
f: S^2 → R
on U: (x, y) ↦ arctan(x^2 + y^2)
on V: (u, v) ↦ 1/2*pi - arctan(u^2 + v^2)
sage: f(p)
arctan(5)
sage: f(N)
1/2*pi
sage: f(S)
0
sage: f.parent()
Algebra of scalar fields on the 2-dimensional topological manifold S^2
sage: f.parent().category()
Join of Category of commutative algebras over Symbolic Ring and Category of homsets␣
↪of topological spaces
```

```
>>> from sage.all import *
>>> f = M.scalar_field({stereoN: atan(x**Integer(2)+y**Integer(2)), stereoS: pi/
↪Integer(2)-atan(u**Integer(2)+v**Integer(2))},
...                    name='f')
>>> f
Scalar field f on the 2-dimensional topological manifold S^2
>>> f.display()
f: S^2 → R
on U: (x, y) ↦ arctan(x^2 + y^2)
on V: (u, v) ↦ 1/2*pi - arctan(u^2 + v^2)
>>> f(p)
arctan(5)
>>> f(N)
1/2*pi
>>> f(S)
0
>>> f.parent()
Algebra of scalar fields on the 2-dimensional topological manifold S^2
>>> f.parent().category()
Join of Category of commutative algebras over Symbolic Ring and Category of homsets␣
↪of topological spaces
```

Example 2: the Riemann sphere as a topological manifold of dimension 1 over $\mathbb{C}$

We declare the Riemann sphere $\mathbf{C}^*$ as a 1-dimensional topological manifold over $\mathbf{C}$:

```
sage: M = Manifold(1, 'ℂ*', structure='topological', field='complex'); M
Complex 1-dimensional topological manifold ℂ*
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(1), 'ℂ*', structure='topological', field='complex'); M
Complex 1-dimensional topological manifold ℂ*
```

We introduce a first open subset, which is actually $\mathbf{C} = \mathbf{C}^* \setminus \{\infty\}$ if we interpret $\mathbf{C}^*$ as the Alexandroff one-point compactification of $\mathbf{C}$:

```
sage: U = M.open_subset('U')
```

```
>>> from sage.all import *
>>> U = M.open_subset('U')
```

A natural chart on U is then nothing but the identity map of $\mathbf{C}$, hence we denote the associated coordinate by z:

```
sage: Z.<z> = U.chart()
```

```
>>> from sage.all import *
>>> Z = U.chart(names=('z',)); (z,) = Z._first_ngens(1)
```

The origin of the complex plane is the point of coordinate $z = 0$:

```
sage: O = U.point((0,), chart=Z, name='O'); O
Point O on the Complex 1-dimensional topological manifold ℂ*
```

```
>>> from sage.all import *
>>> O = U.point((Integer(0),), chart=Z, name='O'); O
Point O on the Complex 1-dimensional topological manifold ℂ*
```

Another open subset of $\mathbf{C}^*$ is $V = \mathbf{C}^* \setminus \{O\}$:

```
sage: V = M.open_subset('V')
```

```
>>> from sage.all import *
>>> V = M.open_subset('V')
```

We define a chart on V such that the point at infinity is the point of coordinate 0 in this chart:

```
sage: W.<w> = V.chart(); W
Chart (V, (w,))
sage: inf = M.point((0,), chart=W, name='inf', latex_name=r'\infty')
sage: inf
Point inf on the Complex 1-dimensional topological manifold ℂ*
```

```
>>> from sage.all import *
>>> W = V.chart(names=('w',)); (w,) = W._first_ngens(1); W
Chart (V, (w,))
```

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```
>>> inf = M.point((Integer(0),), chart=W, name='inf', latex_name=r'\infty')
>>> inf
Point inf on the Complex 1-dimensional topological manifold ℂ*
```

To fully construct the Riemann sphere, we declare that it is the union of U and V:

```
sage: M.declare_union(U,V)
```

```
>>> from sage.all import *
>>> M.declare_union(U,V)
```

and we provide the transition map between the two charts as $w = 1/z$ on $A = U \cap V$:

```
sage: Z_to_W = Z.transition_map(W, 1/z, intersection_name='A',
....:                           restrictions1= z!=0, restrictions2= w!=0)
sage: Z_to_W
Change of coordinates from Chart (A, (z,)) to Chart (A, (w,))
sage: Z_to_W.display()
w = 1/z
sage: Z_to_W.inverse()
Change of coordinates from Chart (A, (w,)) to Chart (A, (z,))
sage: Z_to_W.inverse().display()
z = 1/w
```

```
>>> from sage.all import *
>>> Z_to_W = Z.transition_map(W, Integer(1)/z, intersection_name='A',
...                           restrictions1= z!=Integer(0), restrictions2= w!
↪=Integer(0))
>>> Z_to_W
Change of coordinates from Chart (A, (z,)) to Chart (A, (w,))
>>> Z_to_W.display()
w = 1/z
>>> Z_to_W.inverse()
Change of coordinates from Chart (A, (w,)) to Chart (A, (z,))
>>> Z_to_W.inverse().display()
z = 1/w
```

Let consider the complex number i as a point of the Riemann sphere:

```
sage: i = M((I,), chart=Z, name='i'); i
Point i on the Complex 1-dimensional topological manifold ℂ*
```

```
>>> from sage.all import *
>>> i = M((I,), chart=Z, name='i'); i
Point i on the Complex 1-dimensional topological manifold ℂ*
```

Its coordinates w.r.t. the charts `Z` and `W` are:

```
sage: Z(i)
(I,)
sage: W(i)
(-I,)
```

```
>>> from sage.all import *
>>> Z(i)
(I,)
>>> W(i)
(-I,)
```

and we have:

```
sage: i in U
True
sage: i in V
True
```

```
>>> from sage.all import *
>>> i in U
True
>>> i in V
True
```

The following subsets and charts have been defined:

```
sage: M.subset_family()
Set {A, U, V, ℂ*} of open subsets of the Complex 1-dimensional topological manifold ℂ*
sage: M.atlas()
[Chart (U, (z,)), Chart (V, (w,)), Chart (A, (z,)), Chart (A, (w,))]
```

```
>>> from sage.all import *
>>> M.subset_family()
Set {A, U, V, ℂ*} of open subsets of the Complex 1-dimensional topological manifold ℂ*
>>> M.atlas()
[Chart (U, (z,)), Chart (V, (w,)), Chart (A, (z,)), Chart (A, (w,))]
```

A constant map $\mathbf{C}^* \to \mathbf{C}$:

```
sage: f = M.constant_scalar_field(3+2*I, name='f'); f
Scalar field f on the Complex 1-dimensional topological manifold ℂ*
sage: f.display()
f: ℂ* → ℂ
on U: z ↦ 2*I + 3
on V: w ↦ 2*I + 3
sage: f(O)
2*I + 3
sage: f(i)
2*I + 3
sage: f(inf)
2*I + 3
sage: f.parent()
Algebra of scalar fields on the Complex 1-dimensional topological
 manifold ℂ*
sage: f.parent().category()
Join of Category of commutative algebras over Symbolic Ring and Category of homsets␣
↪of topological spaces
```

```
>>> from sage.all import *
>>> f = M.constant_scalar_field(Integer(3)+Integer(2)*I, name='f'); f
Scalar field f on the Complex 1-dimensional topological manifold ℂ*
>>> f.display()
f: ℂ* → ℂ
on U: z ↦ 2*I + 3
on V: w ↦ 2*I + 3
>>> f(O)
2*I + 3
>>> f(i)
2*I + 3
>>> f(inf)
2*I + 3
>>> f.parent()
Algebra of scalar fields on the Complex 1-dimensional topological
 manifold ℂ*
>>> f.parent().category()
Join of Category of commutative algebras over Symbolic Ring and Category of homsets␣
↪of topological spaces
```

AUTHORS:

- Eric Gourgoulhon (2015): initial version
- Travis Scrimshaw (2015): structure described via `TopologicalStructure` or `RealTopologicalStructure`
- Michael Jung (2020): topological vector bundles and orientability

REFERENCES:

- [Lee2011]
- [Lee2013]
- [KN1963]
- [Huy2005]

`sage.manifolds.manifold.`**`Manifold`**`(`*dim*, *name*, *latex_name=None*, *field='real'*, *structure=None*, *start_index=0*, ***extra_kwds*`)`

Construct a manifold of a given type over a topological field.

Given a topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$) and a nonnegative integer n, a *topological manifold of dimension* n *over K* is a topological space M such that

- M is a Hausdorff space,
- M is second countable, and
- every point in M has a neighborhood homeomorphic to K^n.

A *real manifold* is a manifold over $\mathbf{R}$. A *differentiable* (resp. *smooth*, resp. *analytic*) *manifold* is a manifold such that all transition maps are *differentiable* (resp. *smooth*, resp. *analytic*). A *pseudo-Riemannian manifold* is a real differentiable manifold equipped with a metric tensor g (i.e. a field of non-degenerate symmetric bilinear forms), with the two subcases of *Riemannian manifold* (g positive-definite) and *Lorentzian manifold* (g has signature $n-2$ or $2-n$).

INPUT:

- `dim` – positive integer; dimension of the manifold
- `name` – string; name (symbol) given to the manifold

- `latex_name` – (default: `None`) string; LaTeX symbol to denote the manifold; if none are provided, it is set to `name`
- `field` – (default: `'real'`) field K on which the manifold is defined; allowed values are
  - `'real'` or an object of type `RealField` (e.g. `RR`) for a manifold over **R**
  - `'complex'` or an object of type `ComplexField` (e.g. `CC`) for a manifold over **C**
  - an object in the category of topological fields (see `Fields` and `TopologicalSpaces`) for other types of manifolds
- `structure` – (default: `'smooth'`) to specify the structure or type of manifold; allowed values are
  - `'topological'` or `'top'` for a topological manifold
  - `'differentiable'` or `'diff'` for a differentiable manifold
  - `'smooth'` for a smooth manifold
  - `'analytic'` for an analytic manifold
  - `'pseudo-Riemannian'` for a real differentiable manifold equipped with a pseudo-Riemannian metric; the signature is specified via the keyword argument `signature` (see below)
  - `'Riemannian'` for a real differentiable manifold equipped with a Riemannian (i.e. positive definite) metric
  - `'Lorentzian'` for a real differentiable manifold equipped with a Lorentzian metric; the signature convention is specified by the keyword argument `signature='positive'` (default) or `'negative'`
- `start_index` – (default: 0) integer; lower value of the range of indices used for "indexed objects" on the manifold, e.g. coordinates in a chart
- `extra_kwds` – keywords meaningful only for some specific types of manifolds:
  - `diff_degree` – (only for differentiable manifolds; default: `infinity`): the degree of differentiability
  - `ambient` – (only to construct a submanifold): the ambient manifold
  - `metric_name` – (only for pseudo-Riemannian manifolds; default: `'g'`) string; name (symbol) given to the metric
  - `metric_latex_name` – (only for pseudo-Riemannian manifolds; default: `None`) string; LaTeX symbol to denote the metric; if none is provided, the symbol is set to `metric_name`
  - `signature` – (only for pseudo-Riemannian manifolds; default: `None`) signature S of the metric as a single integer: $S = n_+ - n_-$, where n_+ (resp. n_-) is the number of positive terms (resp. negative terms) in any diagonal writing of the metric components; if `signature` is not provided, S is set to the manifold's dimension (Riemannian signature); for Lorentzian manifolds the values `signature='positive'` (default) or `signature='negative'` are allowed to indicate the chosen signature convention.

OUTPUT:

- a manifold of the specified type, as an instance of *`TopologicalManifold`* or one of its subclasses *`DifferentiableManifold`* or *`PseudoRiemannianManifold`*, or, if the keyword `ambient` is used, one of the subclasses *`TopologicalSubmanifold`*, *`DifferentiableSubmanifold`*, or *`PseudoRiemannianSubmanifold`*.

EXAMPLES:

A 3-dimensional real topological manifold:

```
sage: M = Manifold(3, 'M', structure='topological'); M
3-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological'); M
3-dimensional topological manifold M
```

Given the default value of the parameter `field`, the above is equivalent to:

```
sage: M = Manifold(3, 'M', structure='topological', field='real'); M
3-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological', field='real'); M
3-dimensional topological manifold M
```

A complex topological manifold:

```
sage: M = Manifold(3, 'M', structure='topological', field='complex'); M
Complex 3-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological', field='complex'); M
Complex 3-dimensional topological manifold M
```

A topological manifold over **Q**:

```
sage: M = Manifold(3, 'M', structure='topological', field=QQ); M
3-dimensional topological manifold M over the Rational Field
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological', field=QQ); M
3-dimensional topological manifold M over the Rational Field
```

A 3-dimensional real differentiable manifold of class C^4:

```
sage: M = Manifold(3, 'M', field='real', structure='differentiable',
....:              diff_degree=4); M
3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', field='real', structure='differentiable',
...              diff_degree=Integer(4)); M
3-dimensional differentiable manifold M
```

Since the default value of the parameter `field` is `'real'`, the above is equivalent to:

```
sage: M = Manifold(3, 'M', structure='differentiable', diff_degree=4)
sage: M
3-dimensional differentiable manifold M
sage: M.base_field_type()
'real'
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='differentiable', diff_
```

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```
↪degree=Integer(4))
>>> M
3-dimensional differentiable manifold M
>>> M.base_field_type()
'real'
```

A 3-dimensional real smooth manifold:

```
sage: M = Manifold(3, 'M', structure='differentiable', diff_degree=+oo)
sage: M
3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='differentiable', diff_degree=+oo)
>>> M
3-dimensional differentiable manifold M
```

Instead of `structure='differentiable', diff_degree=+oo`, it suffices to use `structure='smooth'` to get the same result:

```
sage: M = Manifold(3, 'M', structure='smooth'); M
3-dimensional differentiable manifold M
sage: M.diff_degree()
+Infinity
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='smooth'); M
3-dimensional differentiable manifold M
>>> M.diff_degree()
+Infinity
```

Actually, since `'smooth'` is the default value of the parameter `structure`, the creation of a real smooth manifold can be shortened to:

```
sage: M = Manifold(3, 'M'); M
3-dimensional differentiable manifold M
sage: M.diff_degree()
+Infinity
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M'); M
3-dimensional differentiable manifold M
>>> M.diff_degree()
+Infinity
```

Other parameters can change the default of the parameter `structure`:

```
sage: M = Manifold(3, 'M', diff_degree=0); M
3-dimensional topological manifold M
sage: M = Manifold(3, 'M', diff_degree=2); M
3-dimensional differentiable manifold M
```

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```
sage: M = Manifold(3, 'M', metric_name='g'); M
3-dimensional Riemannian manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', diff_degree=Integer(0)); M
3-dimensional topological manifold M
>>> M = Manifold(Integer(3), 'M', diff_degree=Integer(2)); M
3-dimensional differentiable manifold M
>>> M = Manifold(Integer(3), 'M', metric_name='g'); M
3-dimensional Riemannian manifold M
```

For a complex smooth manifold, we have to set the parameter `field`:

```
sage: M = Manifold(3, 'M', field='complex'); M
3-dimensional complex manifold M
sage: M.diff_degree()
+Infinity
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', field='complex'); M
3-dimensional complex manifold M
>>> M.diff_degree()
+Infinity
```

Submanifolds are constructed by means of the keyword `ambient`:

```
sage: N = Manifold(2, 'N', field='complex', ambient=M); N
2-dimensional differentiable submanifold N immersed in the
 3-dimensional complex manifold M
```

```
>>> from sage.all import *
>>> N = Manifold(Integer(2), 'N', field='complex', ambient=M); N
2-dimensional differentiable submanifold N immersed in the
 3-dimensional complex manifold M
```

The immersion $N \to M$ has to be specified in a second stage, via the method *set_immersion()* or *set_embedding()*.

For more detailed examples, see the documentation of *TopologicalManifold*, *DifferentiableManifold* and *PseudoRiemannianManifold*, or the documentation of *TopologicalSubmanifold*, *DifferentiableSubmanifold* and *PseudoRiemannianSubmanifold* for submanifolds.

Uniqueness of manifold objects

Suppose we construct a manifold named M:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
```

At some point, we change our mind and would like to restart with a new manifold, using the same name M and keeping the previous manifold for reference:

```
sage: M_old = M  # for reference
sage: M = Manifold(2, 'M', structure='topological')
```

```
>>> from sage.all import *
>>> M_old = M  # for reference
>>> M = Manifold(Integer(2), 'M', structure='topological')
```

This results in a brand new object:

```
sage: M.atlas()
[]
```

```
>>> from sage.all import *
>>> M.atlas()
[]
```

The object `M_old` is intact:

```
sage: M_old.atlas()
[Chart (M, (x, y))]
```

```
>>> from sage.all import *
>>> M_old.atlas()
[Chart (M, (x, y))]
```

Both objects have the same display:

```
sage: M
2-dimensional topological manifold M
sage: M_old
2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M
2-dimensional topological manifold M
>>> M_old
2-dimensional topological manifold M
```

but they are different:

```
sage: M != M_old
True
```

```
>>> from sage.all import *
>>> M != M_old
True
```

Let us introduce a chart on `M`, using the same coordinate symbols as for `M_old`:

```
sage: X.<x,y> = M.chart()
```

```
>>> from sage.all import *
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
```

The charts are displayed in the same way:

```
sage: M.atlas()
[Chart (M, (x, y))]
sage: M_old.atlas()
[Chart (M, (x, y))]
```

```
>>> from sage.all import *
>>> M.atlas()
[Chart (M, (x, y))]
>>> M_old.atlas()
[Chart (M, (x, y))]
```

but they are actually different:

```
sage: M.atlas()[0] != M_old.atlas()[0]
True
```

```
>>> from sage.all import *
>>> M.atlas()[Integer(0)] != M_old.atlas()[Integer(0)]
True
```

Moreover, the two manifolds `M` and `M_old` are still considered distinct:

```
sage: M != M_old
True
```

```
>>> from sage.all import *
>>> M != M_old
True
```

This reflects the fact that the equality of manifold objects holds only for identical objects, i.e. one has `M1 == M2` if, and only if, `M1 is M2`. Actually, the manifold classes inherit from `WithEqualityById`:

```
sage: isinstance(M, sage.misc.fast_methods.WithEqualityById)
True
```

```
>>> from sage.all import *
>>> isinstance(M, sage.misc.fast_methods.WithEqualityById)
True
```

class `sage.manifolds.manifold.`**`TopologicalManifold`**(*n*, *name*, *field*, *structure*, *base_manifold=None*, *latex_name=None*, *start_index=0*, *category=None*, *unique_tag=None*)

Bases: *`ManifoldSubset`*

Topological manifold over a topological field K.

Given a topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$) and a nonnegative integer n, a *topological manifold of dimension* n *over K* is a topological space M such that

- M is a Hausdorff space,
- M is second countable, and
- every point in M has a neighborhood homeomorphic to K^n.

This is a Sage *parent* class, the corresponding *element* class being `ManifoldPoint`.

INPUT:

- `n` – positive integer; dimension of the manifold
- `name` – string; name (symbol) given to the manifold
- `field` – field K on which the manifold is defined; allowed values are
  - `'real'` or an object of type `RealField` (e.g., `RR`) for a manifold over $\mathbf{R}$
  - `'complex'` or an object of type `ComplexField` (e.g., `CC`) for a manifold over $\mathbf{C}$
  - an object in the category of topological fields (see `Fields` and `TopologicalSpaces`) for other types of manifolds
- `structure` – manifold structure (see `TopologicalStructure` or `RealTopologicalStructure`)
- `base_manifold` – (default: `None`) if not `None`, must be a topological manifold; the created object is then an open subset of `base_manifold`
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the manifold; if none are provided, it is set to `name`
- `start_index` – (default: 0) integer; lower value of the range of indices used for "indexed objects" on the manifold, e.g., coordinates in a chart
- `category` – (default: `None`) to specify the category; if `None`, `Manifolds(field)` is assumed (see the category `Manifolds`)
- `unique_tag` – (default: `None`) tag used to force the construction of a new object when all the other arguments have been used previously (without `unique_tag`, the `UniqueRepresentation` behavior inherited from `ManifoldSubset` would return the previously constructed object corresponding to these arguments)

EXAMPLES:

A 4-dimensional topological manifold (over $\mathbf{R}$):

```
sage: M = Manifold(4, 'M', latex_name=r'\mathcal{M}', structure='topological')
sage: M
4-dimensional topological manifold M
sage: latex(M)
\mathcal{M}
sage: type(M)
<class 'sage.manifolds.manifold.TopologicalManifold_with_category'>
sage: M.base_field()
Real Field with 53 bits of precision
sage: dim(M)
4
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', latex_name=r'\mathcal{M}', structure=
↪'topological')
```

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```
>>> M
4-dimensional topological manifold M
>>> latex(M)
\mathcal{M}
>>> type(M)
<class 'sage.manifolds.manifold.TopologicalManifold_with_category'>
>>> M.base_field()
Real Field with 53 bits of precision
>>> dim(M)
4
```

The input parameter `start_index` defines the range of indices on the manifold:

```
sage: M = Manifold(4, 'M', structure='topological')
sage: list(M.irange())
[0, 1, 2, 3]
sage: M = Manifold(4, 'M', structure='topological', start_index=1)
sage: list(M.irange())
[1, 2, 3, 4]
sage: list(Manifold(4, 'M', structure='topological', start_index=-2).irange())
[-2, -1, 0, 1]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='topological')
>>> list(M.irange())
[0, 1, 2, 3]
>>> M = Manifold(Integer(4), 'M', structure='topological', start_index=Integer(1))
>>> list(M.irange())
[1, 2, 3, 4]
>>> list(Manifold(Integer(4), 'M', structure='topological', start_index=-
↪Integer(2)).irange())
[-2, -1, 0, 1]
```

A complex manifold:

```
sage: N = Manifold(3, 'N', structure='topological', field='complex'); N
Complex 3-dimensional topological manifold N
```

```
>>> from sage.all import *
>>> N = Manifold(Integer(3), 'N', structure='topological', field='complex'); N
Complex 3-dimensional topological manifold N
```

A manifold over $\mathbf{Q}$:

```
sage: N = Manifold(6, 'N', structure='topological', field=QQ); N
6-dimensional topological manifold N over the Rational Field
```

```
>>> from sage.all import *
>>> N = Manifold(Integer(6), 'N', structure='topological', field=QQ); N
6-dimensional topological manifold N over the Rational Field
```

A manifold over $\mathbf{Q}_5$, the field of 5-adic numbers:

```
sage: N = Manifold(2, 'N', structure='topological', field=Qp(5)); N           #␣
↪needs sage.rings.padics
2-dimensional topological manifold N over the 5-adic Field with capped
 relative precision 20
```

```
>>> from sage.all import *
>>> N = Manifold(Integer(2), 'N', structure='topological', field=Qp(Integer(5)));␣
↪N              # needs sage.rings.padics
2-dimensional topological manifold N over the 5-adic Field with capped
 relative precision 20
```

A manifold is a Sage *parent* object, in the category of topological manifolds over a given topological field (see `Manifolds`):

```
sage: isinstance(M, Parent)
True
sage: M.category()
Category of manifolds over Real Field with 53 bits of precision
sage: from sage.categories.manifolds import Manifolds
sage: M.category() is Manifolds(RR)
True
sage: M.category() is Manifolds(M.base_field())
True
sage: M in Manifolds(RR)
True
sage: N in Manifolds(Qp(5))                                                   #␣
↪needs sage.rings.padics
True
```

```
>>> from sage.all import *
>>> isinstance(M, Parent)
True
>>> M.category()
Category of manifolds over Real Field with 53 bits of precision
>>> from sage.categories.manifolds import Manifolds
>>> M.category() is Manifolds(RR)
True
>>> M.category() is Manifolds(M.base_field())
True
>>> M in Manifolds(RR)
True
>>> N in Manifolds(Qp(Integer(5)))                                            ␣
↪     # needs sage.rings.padics
True
```

The corresponding Sage *elements* are points:

```
sage: X.<t, x, y, z> = M.chart()
sage: p = M.an_element(); p
Point on the 4-dimensional topological manifold M
sage: p.parent()
4-dimensional topological manifold M
sage: M.is_parent_of(p)
```

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```
True
sage: p in M
True
```

```
>>> from sage.all import *
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_ngens(4)
>>> p = M.an_element(); p
Point on the 4-dimensional topological manifold M
>>> p.parent()
4-dimensional topological manifold M
>>> M.is_parent_of(p)
True
>>> p in M
True
```

The manifold's points are instances of class *ManifoldPoint*:

```
sage: isinstance(p, sage.manifolds.point.ManifoldPoint)
True
```

```
>>> from sage.all import *
>>> isinstance(p, sage.manifolds.point.ManifoldPoint)
True
```

Since an open subset of a topological manifold M is itself a topological manifold, open subsets of M are instances of the class *TopologicalManifold*:

```
sage: U = M.open_subset('U'); U
Open subset U of the 4-dimensional topological manifold M
sage: isinstance(U, sage.manifolds.manifold.TopologicalManifold)
True
sage: U.base_field() == M.base_field()
True
sage: dim(U) == dim(M)
True
sage: U.category()
Join of Category of subobjects of sets and Category of manifolds over
 Real Field with 53 bits of precision
```

```
>>> from sage.all import *
>>> U = M.open_subset('U'); U
Open subset U of the 4-dimensional topological manifold M
>>> isinstance(U, sage.manifolds.manifold.TopologicalManifold)
True
>>> U.base_field() == M.base_field()
True
>>> dim(U) == dim(M)
True
>>> U.category()
Join of Category of subobjects of sets and Category of manifolds over
 Real Field with 53 bits of precision
```

The manifold passes all the tests of the test suite relative to its category:

```
sage: TestSuite(M).run()
```

```
>>> from sage.all import *
>>> TestSuite(M).run()
```

See also

sage.manifolds.manifold

atlas()

Return the list of charts that have been defined on the manifold.

EXAMPLES:

Let us consider $\mathbf{R}^2$ as a 2-dimensional manifold:

```
sage: M = Manifold(2, 'R^2', structure='topological')
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', structure='topological')
```

Immediately after the manifold creation, the atlas is empty, since no chart has been defined yet:

```
sage: M.atlas()
[]
```

```
>>> from sage.all import *
>>> M.atlas()
[]
```

Let us introduce the chart of Cartesian coordinates:

```
sage: c_cart.<x,y> = M.chart()
sage: M.atlas()
[Chart (R^2, (x, y))]
```

```
>>> from sage.all import *
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)
>>> M.atlas()
[Chart (R^2, (x, y))]
```

The complement of the half line $\{y = 0, x \geq 0\}$:

```
sage: U = M.open_subset('U', coord_def={c_cart: (y!=0,x<0)})
sage: U.atlas()
[Chart (U, (x, y))]
sage: M.atlas()
[Chart (R^2, (x, y)), Chart (U, (x, y))]
```

```
>>> from sage.all import *
>>> U = M.open_subset('U', coord_def={c_cart: (y!=Integer(0),x<Integer(0))})
>>> U.atlas()
[Chart (U, (x, y))]
>>> M.atlas()
[Chart (R^2, (x, y)), Chart (U, (x, y))]
```

Spherical (polar) coordinates on `U`:

```
sage: c_spher.<r, ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi')
sage: U.atlas()
[Chart (U, (x, y)), Chart (U, (r, ph))]
sage: M.atlas()
[Chart (R^2, (x, y)), Chart (U, (x, y)), Chart (U, (r, ph))]
```

```
>>> from sage.all import *
>>> c_spher = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi', names=('r', 'ph',)); (r,␣
↪ph,) = c_spher._first_ngens(2)
>>> U.atlas()
[Chart (U, (x, y)), Chart (U, (r, ph))]
>>> M.atlas()
[Chart (R^2, (x, y)), Chart (U, (x, y)), Chart (U, (r, ph))]
```

See also

top_charts()

base_field()

Return the field on which the manifold is defined.

OUTPUT: a topological field

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: M.base_field()
Real Field with 53 bits of precision
sage: M = Manifold(3, 'M', structure='topological', field='complex')
sage: M.base_field()
Complex Field with 53 bits of precision
sage: M = Manifold(3, 'M', structure='topological', field=QQ)
sage: M.base_field()
Rational Field
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological')
>>> M.base_field()
Real Field with 53 bits of precision
>>> M = Manifold(Integer(3), 'M', structure='topological', field='complex')
>>> M.base_field()
Complex Field with 53 bits of precision
>>> M = Manifold(Integer(3), 'M', structure='topological', field=QQ)
```

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```
>>> M.base_field()
Rational Field
```

base_field_type()

Return the type of topological field on which the manifold is defined.

OUTPUT:

A string describing the field, with three possible values:

- `'real'` for the real field **R**
- `'complex'` for the complex field **C**
- `'neither_real_nor_complex'` for a field different from **R** and **C**

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: M.base_field_type()
'real'
sage: M = Manifold(3, 'M', structure='topological', field='complex')
sage: M.base_field_type()
'complex'
sage: M = Manifold(3, 'M', structure='topological', field=QQ)
sage: M.base_field_type()
'neither_real_nor_complex'
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological')
>>> M.base_field_type()
'real'
>>> M = Manifold(Integer(3), 'M', structure='topological', field='complex')
>>> M.base_field_type()
'complex'
>>> M = Manifold(Integer(3), 'M', structure='topological', field=QQ)
>>> M.base_field_type()
'neither_real_nor_complex'
```

chart(*coordinates=''*, *names=None*, *calc_method=None*, *coord_restrictions=None*)

Define a chart, the domain of which is the manifold.

A *chart* is a pair (U, φ), where U is the current manifold and $\varphi : U \to V \subset K^n$ is a homeomorphism from U to an open subset V of K^n, K being the field on which the manifold is defined.

The components $(x^1, \ldots, x^n)$ of φ, defined by $\varphi(p) = (x^1(p), \ldots, x^n(p)) \in K^n$ for any point $p \in U$, are called the *coordinates* of the chart (U, φ).

See `Chart` for a complete documentation.

INPUT:

- `coordinates` – (default: `''` (empty string)) string defining the coordinate symbols, ranges and possible periodicities, see below
- `names` – (default: `None`) unused argument, except if `coordinates` is not provided; it must then be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)

- `calc_method` – (default: `None`) string defining the calculus method to be used on this chart; must be one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy
  - `None`: the current calculus method defined on the manifold is used (cf. `set_calculus_method()`)
- `coord_restrictions` – additional restrictions on the coordinates. See below.

The coordinates declared in the string `coordinates` are separated by `' '` (whitespace) and each coordinate has at most four fields, separated by a colon (`':'`):

1. The coordinate symbol (a letter or a few letters).
2. (optional, only for manifolds over **R**) The interval I defining the coordinate range: if not provided, the coordinate is assumed to span all **R**; otherwise I must be provided in the form `(a,b)` (or equivalently `]a,b[`) The bounds `a` and `b` can be `+/-Infinity`, `Inf`, `infinity`, `inf` or `oo`. For *singular* coordinates, non-open intervals such as `[a,b]` and `(a,b]` (or equivalently `]a,b]`) are allowed. Note that the interval declaration must not contain any space character.
3. (optional) Indicator of the periodic character of the coordinate, either as `period=T`, where `T` is the period, or, for manifolds over **R** only, as the keyword `periodic` (the value of the period is then deduced from the interval I declared in field 2; see the example below)
4. (optional) The LaTeX spelling of the coordinate; if not provided the coordinate symbol given in the first field will be used.

The order of fields 2 to 4 does not matter and each of them can be omitted. If it contains any LaTeX expression, the string `coordinates` must be declared with the prefix 'r' (for "raw") to allow for a proper treatment of the backslash character (see examples below). If no interval range, no period and no LaTeX spelling is to be set for any coordinate, the argument `coordinates` can be omitted when the shortcut operator `<,>` is used to declare the chart (see examples below).

Additional restrictions on the coordinates can be set using the argument `coord_restrictions`.

A restriction can be any symbolic equality or inequality involving the coordinates, such as `x > y` or `x^2 + y^2 != 0`. The items of the list (or set or frozenset) `coord_restrictions` are combined with the `and` operator; if some restrictions are to be combined with the `or` operator instead, they have to be passed as a tuple in some single item of the list (or set or frozenset) `coord_restrictions`. For example:

```
coord_restrictions=[x > y, (x != 0, y != 0), z^2 < x]
```

means `(x > y) and ((x != 0) or (y != 0)) and (z^2 < x)`. If the list `coord_restrictions` contains only one item, this item can be passed as such, i.e. writing `x > y` instead of the single element list `[x > y]`. If the chart variables have not been declared as variables yet, `coord_restrictions` must be `lambda`-quoted.

OUTPUT:

- the created chart, as an instance of `Chart` or one of its subclasses, like `RealDiffChart` for differentiable manifolds over **R**.

EXAMPLES:

Chart on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X = M.chart('x y'); X
Chart (M, (x, y))
```

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```
sage: X[0]
x
sage: X[1]
y
sage: X[:]
(x, y)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart('x y'); X
Chart (M, (x, y))
>>> X[Integer(0)]
x
>>> X[Integer(1)]
y
>>> X[:]
(x, y)
```

The declared coordinates are not known at the global level:

```
sage: y
Traceback (most recent call last):
...
NameError: name 'y' is not defined
```

```
>>> from sage.all import *
>>> y
Traceback (most recent call last):
...
NameError: name 'y' is not defined
```

They can be recovered by the operator `[:]` applied to the chart:

```
sage: (x, y) = X[:]
sage: y
y
sage: type(y)
<class 'sage.symbolic.expression.Expression'>
```

```
>>> from sage.all import *
>>> (x, y) = X[:]
>>> y
y
>>> type(y)
<class 'sage.symbolic.expression.Expression'>
```

But a shorter way to proceed is to use the operator `<,>` in the left-hand side of the chart declaration (there is then no need to pass the string 'x y' to chart()):

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart(); X
Chart (M, (x, y))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2); X
Chart (M, (x, y))
```

Indeed, the declared coordinates are then known at the global level:

```
sage: y
y
sage: (x,y) == X[:]
True
```

```
>>> from sage.all import *
>>> y
y
>>> (x,y) == X[:]
True
```

Actually the instruction `X.<x,y> = M.chart()` is equivalent to the combination of the two instructions `X = M.chart('x y')` and `(x,y) = X[:]`.

As an example of coordinate ranges and LaTeX symbols passed via the string `coordinates` to `chart()`, let us introduce polar coordinates:

```
sage: U = M.open_subset('U', coord_def={X: x^2+y^2 != 0})
sage: P.<r,ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):periodic:\phi'); P
Chart (U, (r, ph))
sage: P.coord_range()
r: (0, +oo); ph: [0, 2*pi] (periodic)
sage: latex(P)
\left(U,(r, {\phi})\right)
```

```
>>> from sage.all import *
>>> U = M.open_subset('U', coord_def={X: x**Integer(2)+y**Integer(2) !=␣
↪Integer(0)})
>>> P = U.chart(r'r:(0,+oo) ph:(0,2*pi):periodic:\phi', names=('r', 'ph',));␣
↪(r, ph,) = P._first_ngens(2); P
Chart (U, (r, ph))
>>> P.coord_range()
r: (0, +oo); ph: [0, 2*pi] (periodic)
>>> latex(P)
\left(U,(r, {\phi})\right)
```

Using `coord_restrictions`:

```
sage: D = Manifold(2, 'D', structure='topological')
sage: X.<x,y> = D.chart(coord_restrictions=lambda x,y: [x^2+y^2<1, x>0]); X
Chart (D, (x, y))
sage: X.valid_coordinates(0, 0)
False
sage: X.valid_coordinates(1/2, 0)
True
```

```
>>> from sage.all import *
>>> D = Manifold(Integer(2), 'D', structure='topological')
>>> X = D.chart(coord_restrictions=lambda x,y: [x**Integer(2)+y**Integer(2)
↪<Integer(1), x>Integer(0)], names=('x', 'y',)); (x, y,) = X._first_ngens(2);
↪ X
Chart (D, (x, y))
>>> X.valid_coordinates(Integer(0), Integer(0))
False
>>> X.valid_coordinates(Integer(1)/Integer(2), Integer(0))
True
```

See the documentation of classes `Chart` and `RealChart` for more examples, especially regarding the coordinates ranges and restrictions.

constant_scalar_field(*value*, *name=None*, *latex_name=None*)

Define a constant scalar field on the manifold.

INPUT:

- `value` – constant value of the scalar field, either a numerical value or a symbolic expression not involving any chart coordinates
- `name` – (default: `None`) name given to the scalar field
- `latex_name` – (default: `None`) LaTeX symbol to denote the scalar field; if `None`, the LaTeX symbol is set to `name`

OUTPUT:

- instance of `ScalarField` representing the scalar field whose constant value is `value`

EXAMPLES:

A constant scalar field on the 2-sphere:

```
sage: M = Manifold(2, 'M', structure='topological') # the 2-dimensional␣
↪sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                                intersection_name='W',
....:                                restrictions1= x^2+y^2!=0,
....:                                restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: f = M.constant_scalar_field(-1) ; f
Scalar field on the 2-dimensional topological manifold M
sage: f.display()
M → ℝ
on U: (x, y) ↦ -1
on V: (u, v) ↦ -1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological') # the 2-
↪dimensional sphere S^2
```

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```
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                                 intersection_name='W',
...                                 restrictions1= x**Integer(2)+y**Integer(2)!
↪=Integer(0),
...                                 restrictions2= u**Integer(2)+v**Integer(2)!
↪=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> f = M.constant_scalar_field(-Integer(1)) ; f
Scalar field on the 2-dimensional topological manifold M
>>> f.display()
M → ℝ
on U: (x, y) ↦ -1
on V: (u, v) ↦ -1
```

We have:

```
sage: f.restrict(U) == U.constant_scalar_field(-1)
True
sage: M.constant_scalar_field(0) is M.zero_scalar_field()
True
```

```
>>> from sage.all import *
>>> f.restrict(U) == U.constant_scalar_field(-Integer(1))
True
>>> M.constant_scalar_field(Integer(0)) is M.zero_scalar_field()
True
```

See also

zero_scalar_field(), *one_scalar_field()*

continuous_map(*codomain*, *coord_functions=None*, *chart1=None*, *chart2=None*, *name=None*, *latex_name=None*)

Define a continuous map from `self` to `codomain`.

INPUT:

- `codomain` – *TopologicalManifold*; the map's codomain
- `coord_functions` – (default: `None`) if not `None`, must be either
  - (i) a dictionary of the coordinate expressions (as lists (or tuples) of the coordinates of the image expressed in terms of the coordinates of the considered point) with the pairs of charts `(chart1, chart2)` as keys (`chart1` being a chart on `self` and `chart2` a chart on `codomain`);

  – (ii) a single coordinate expression in a given pair of charts, the latter being provided by the arguments `chart1` and `chart2`;

  in both cases, if the dimension of the codomain is 1, a single coordinate expression can be passed instead of a tuple with a single element

- `chart1` – (default: `None`; used only in case (ii) above) chart on `self` defining the start coordinates involved in `coord_functions` for case (ii); if `None`, the coordinates are assumed to refer to the default chart of `self`
- `chart2` – (default: `None`; used only in case (ii) above) chart on `codomain` defining the target coordinates involved in `coord_functions` for case (ii); if `None`, the coordinates are assumed to refer to the default chart of `codomain`
- `name` – (default: `None`) name given to the continuous map
- `latex_name` – (default: `None`) LaTeX symbol to denote the continuous map; if `None`, the LaTeX symbol is set to `name`

OUTPUT:

- the continuous map as an instance of `ContinuousMap`

EXAMPLES:

A continuous map between an open subset of S^2 covered by regular spherical coordinates and $\mathbf{R}^3$:

```
sage: M = Manifold(2, 'S^2', structure='topological')
sage: U = M.open_subset('U')
sage: c_spher.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: N = Manifold(3, 'R^3', latex_name=r'\RR^3', structure='topological')
sage: c_cart.<x,y,z> = N.chart()   # Cartesian coord. on R^3
sage: Phi = U.continuous_map(N, (sin(th)*cos(ph), sin(th)*sin(ph), cos(th)),
....:                         name='Phi', latex_name=r'\Phi')
sage: Phi
Continuous map Phi from the Open subset U of the 2-dimensional topological␣
↪manifold S^2 to the 3-dimensional topological manifold R^3
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', structure='topological')
>>> U = M.open_subset('U')
>>> c_spher = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi', names=('th', 'ph',
↪)); (th, ph,) = c_spher._first_ngens(2)
>>> N = Manifold(Integer(3), 'R^3', latex_name=r'\RR^3', structure=
↪'topological')
>>> c_cart = N.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_cart._first_
↪ngens(3)# Cartesian coord. on R^3
>>> Phi = U.continuous_map(N, (sin(th)*cos(ph), sin(th)*sin(ph), cos(th)),
...                         name='Phi', latex_name=r'\Phi')
>>> Phi
Continuous map Phi from the Open subset U of the 2-dimensional topological␣
↪manifold S^2 to the 3-dimensional topological manifold R^3
```

The same definition, but with a dictionary with pairs of charts as keys (case (i) above):

```
sage: Phi1 = U.continuous_map(N,
....:        {(c_spher, c_cart): (sin(th)*cos(ph), sin(th)*sin(ph), cos(th))},
....:        name='Phi', latex_name=r'\Phi')
```

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```
sage: Phi1 == Phi
True
```

```
>>> from sage.all import *
>>> Phi1 = U.continuous_map(N,
...         {(c_spher, c_cart): (sin(th)*cos(ph), sin(th)*sin(ph), cos(th))},
...         name='Phi', latex_name=r'\Phi')
>>> Phi1 == Phi
True
```

The continuous map acting on a point:

```
sage: p = U.point((pi/2, pi)) ; p
Point on the 2-dimensional topological manifold S^2
sage: Phi(p)
Point on the 3-dimensional topological manifold R^3
sage: Phi(p).coord(c_cart)
(-1, 0, 0)
sage: Phi1(p) == Phi(p)
True
```

```
>>> from sage.all import *
>>> p = U.point((pi/Integer(2), pi)) ; p
Point on the 2-dimensional topological manifold S^2
>>> Phi(p)
Point on the 3-dimensional topological manifold R^3
>>> Phi(p).coord(c_cart)
(-1, 0, 0)
>>> Phi1(p) == Phi(p)
True
```

See also

See `ContinuousMap` for the complete documentation and more examples.

Todo

Allow the construction of continuous maps from `self` to the base field (considered as a trivial 1-dimensional manifold).

coord_change(*chart1*, *chart2*)

Return the change of coordinates (transition map) between two charts defined on the manifold.

The change of coordinates must have been defined previously, for instance by the method `transition_map()`.

INPUT:

- `chart1` – chart 1
- `chart2` – chart 2

OUTPUT:

- instance of `CoordChange` representing the transition map from chart 1 to chart 2

EXAMPLES:

Change of coordinates on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: c_uv.<u,v> = M.chart()
sage: c_xy.transition_map(c_uv, (x+y, x-y)) # defines the coord. change
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
sage: M.coord_change(c_xy, c_uv) # returns the coord. change defined above
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> c_uv = M.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> c_xy.transition_map(c_uv, (x+y, x-y)) # defines the coord. change
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
>>> M.coord_change(c_xy, c_uv) # returns the coord. change defined above
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
```

coord_changes()

Return the changes of coordinates (transition maps) defined on subsets of the manifold.

OUTPUT: dictionary of changes of coordinates, with pairs of charts as keys

EXAMPLES:

Various changes of coordinates on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: c_uv.<u,v> = M.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, [x+y, x-y])
sage: M.coord_changes()
{(Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y)) to Chart␣
↪(M, (u, v))}
sage: uv_to_xy = xy_to_uv.inverse()
sage: M.coord_changes()   # random (dictionary output)
{(Chart (M, (u, v)),
  Chart (M, (x, y))): Change of coordinates from Chart (M, (u, v)) to Chart␣
↪(M, (x, y)),
 (Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y)) to Chart␣
↪(M, (u, v))}
sage: c_rs.<r,s> = M.chart()
sage: uv_to_rs = c_uv.transition_map(c_rs, [-u+2*v, 3*u-v])
sage: M.coord_changes()   # random (dictionary output)
{(Chart (M, (u, v)),
  Chart (M, (r, s))): Change of coordinates from Chart (M, (u, v)) to Chart␣
↪(M, (r, s)),
```

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```
 (Chart (M, (u, v)),
  Chart (M, (x, y))): Change of coordinates from Chart (M, (u, v)) to Chart␣
↪(M, (x, y)),
 (Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y)) to Chart␣
↪(M, (u, v))}
sage: xy_to_rs = uv_to_rs * xy_to_uv
sage: M.coord_changes()  # random (dictionary output)
{(Chart (M, (u, v)),
  Chart (M, (r, s))): Change of coordinates from Chart (M, (u, v)) to Chart␣
↪(M, (r, s)),
 (Chart (M, (u, v)),
  Chart (M, (x, y))): Change of coordinates from Chart (M, (u, v)) to Chart␣
↪(M, (x, y)),
 (Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y)) to Chart␣
↪(M, (u, v)),
 (Chart (M, (x, y)),
  Chart (M, (r, s))): Change of coordinates from Chart (M, (x, y)) to Chart␣
↪(M, (r, s))}
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> c_uv = M.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> xy_to_uv = c_xy.transition_map(c_uv, [x+y, x-y])
>>> M.coord_changes()
{(Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y)) to Chart␣
↪(M, (u, v))}
>>> uv_to_xy = xy_to_uv.inverse()
>>> M.coord_changes()  # random (dictionary output)
{(Chart (M, (u, v)),
  Chart (M, (x, y))): Change of coordinates from Chart (M, (u, v)) to Chart␣
↪(M, (x, y)),
 (Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y)) to Chart␣
↪(M, (u, v))}
>>> c_rs = M.chart(names=('r', 's',)); (r, s,) = c_rs._first_ngens(2)
>>> uv_to_rs = c_uv.transition_map(c_rs, [-u+Integer(2)*v, Integer(3)*u-v])
>>> M.coord_changes()  # random (dictionary output)
{(Chart (M, (u, v)),
  Chart (M, (r, s))): Change of coordinates from Chart (M, (u, v)) to Chart␣
↪(M, (r, s)),
 (Chart (M, (u, v)),
  Chart (M, (x, y))): Change of coordinates from Chart (M, (u, v)) to Chart␣
↪(M, (x, y)),
 (Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y)) to Chart␣
↪(M, (u, v))}
>>> xy_to_rs = uv_to_rs * xy_to_uv
>>> M.coord_changes()  # random (dictionary output)
```

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```
{(Chart (M, (u, v)),
  Chart (M, (r, s))): Change of coordinates from Chart (M, (u, v)) to Chart␣
↪(M, (r, s)),
 (Chart (M, (u, v)),
  Chart (M, (x, y))): Change of coordinates from Chart (M, (u, v)) to Chart␣
↪(M, (x, y)),
 (Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y)) to Chart␣
↪(M, (u, v)),
 (Chart (M, (x, y)),
  Chart (M, (r, s))): Change of coordinates from Chart (M, (x, y)) to Chart␣
↪(M, (r, s))}
```

default_chart()

Return the default chart defined on the manifold.

Unless changed via *set_default_chart()*, the *default chart* is the first one defined on a subset of the manifold (possibly itself).

OUTPUT:

- instance of *Chart* representing the default chart

EXAMPLES:

Default chart on a 2-dimensional manifold and on some subsets:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: M.chart('x y')
Chart (M, (x, y))
sage: M.chart('u v')
Chart (M, (u, v))
sage: M.default_chart()
Chart (M, (x, y))
sage: A = M.open_subset('A')
sage: A.chart('t z')
Chart (A, (t, z))
sage: A.default_chart()
Chart (A, (t, z))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> M.chart('x y')
Chart (M, (x, y))
>>> M.chart('u v')
Chart (M, (u, v))
>>> M.default_chart()
Chart (M, (x, y))
>>> A = M.open_subset('A')
>>> A.chart('t z')
Chart (A, (t, z))
>>> A.default_chart()
Chart (A, (t, z))
```

dim()

Return the dimension of the manifold over its base field.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: M.dimension()
2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> M.dimension()
2
```

A shortcut is `dim()`:

```
sage: M.dim()
2
```

```
>>> from sage.all import *
>>> M.dim()
2
```

The Sage global function `dim` can also be used:

```
sage: dim(M)
2
```

```
>>> from sage.all import *
>>> dim(M)
2
```

dimension()

Return the dimension of the manifold over its base field.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: M.dimension()
2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> M.dimension()
2
```

A shortcut is `dim()`:

```
sage: M.dim()
2
```

```
>>> from sage.all import *
>>> M.dim()
2
```

The Sage global function `dim` can also be used:

```
sage: dim(M)
2
```

```
>>> from sage.all import *
>>> dim(M)
2
```

get_chart(*coordinates*, *domain=None*)

Get a chart from its coordinates.

The chart must have been previously created by the method `chart()`.

INPUT:

- `coordinates` – single string composed of the coordinate symbols separated by a space
- `domain` – (default: `None`) string containing the name of the chart's domain, which must be a subset of the current manifold; if `None`, the current manifold is assumed

OUTPUT:

- instance of `Chart` (or of the subclass `RealChart`) representing the chart corresponding to the above specifications

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: M.get_chart('x y')
Chart (M, (x, y))
sage: M.get_chart('x y') is X
True
sage: U = M.open_subset('U', coord_def={X: (y!=0,x<0)})
sage: Y.<r, ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi')
sage: M.atlas()
[Chart (M, (x, y)), Chart (U, (x, y)), Chart (U, (r, ph))]
sage: M.get_chart('x y', domain='U')
Chart (U, (x, y))
sage: M.get_chart('x y', domain='U') is X.restrict(U)
True
sage: U.get_chart('r ph')
Chart (U, (r, ph))
sage: M.get_chart('r ph', domain='U')
Chart (U, (r, ph))
sage: M.get_chart('r ph', domain='U') is Y
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> M.get_chart('x y')
Chart (M, (x, y))
>>> M.get_chart('x y') is X
True
```

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```
>>> U = M.open_subset('U', coord_def={X: (y!=Integer(0),x<Integer(0))})
>>> Y = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi', names=('r', 'ph',)); (r, ph,)
↪= Y._first_ngens(2)
>>> M.atlas()
[Chart (M, (x, y)), Chart (U, (x, y)), Chart (U, (r, ph))]
>>> M.get_chart('x y', domain='U')
Chart (U, (x, y))
>>> M.get_chart('x y', domain='U') is X.restrict(U)
True
>>> U.get_chart('r ph')
Chart (U, (r, ph))
>>> M.get_chart('r ph', domain='U')
Chart (U, (r, ph))
>>> M.get_chart('r ph', domain='U') is Y
True
```

has_orientation()

Check whether `self` admits an obvious or by user set orientation.

See also

Consult *orientation()* for details about orientations.

Note

Notice that if *has_orientation()* returns `False` this does not necessarily mean that the manifold admits no orientation. It just means that the user has to set an orientation manually in that case, see *set_orientation()*.

EXAMPLES:

The trivial case:

```
sage: M = Manifold(3, 'M', structure='top')
sage: c.<x,y,z> = M.chart()
sage: M.has_orientation()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> c = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c._first_ngens(3)
>>> M.has_orientation()
True
```

The non-trivial case:

```
sage: M = Manifold(2, 'M', structure='top')
sage: U = M.open_subset('U'); V = M.open_subset('V')
sage: M.declare_union(U, V)
sage: c_xy.<x,y> = U.chart(); c_uv.<u,v> = V.chart()
```

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```
sage: M.has_orientation()
False
sage: M.set_orientation([c_xy, c_uv])
sage: M.has_orientation()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> U = M.open_subset('U'); V = M.open_subset('V')
>>> M.declare_union(U, V)
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> M.has_orientation()
False
>>> M.set_orientation([c_xy, c_uv])
>>> M.has_orientation()
True
```

homeomorphism(*codomain*, *coord_functions=None*, *chart1=None*, *chart2=None*, *name=None*, *latex_name=None*)

Define a homeomorphism between the current manifold and another one.

See `ContinuousMap` for a complete documentation.

INPUT:

- `codomain` – `TopologicalManifold`; codomain of the homeomorphism
- `coord_functions` – (default: `None`) if not `None`, must be either
  - (i) a dictionary of the coordinate expressions (as lists (or tuples) of the coordinates of the image expressed in terms of the coordinates of the considered point) with the pairs of charts `(chart1, chart2)` as keys (`chart1` being a chart on `self` and `chart2` a chart on `codomain`);
  - (ii) a single coordinate expression in a given pair of charts, the latter being provided by the arguments `chart1` and `chart2`;

  in both cases, if the dimension of the codomain is 1, a single coordinate expression can be passed instead of a tuple with a single element
- `chart1` – (default: `None`; used only in case (ii) above) chart on `self` defining the start coordinates involved in `coord_functions` for case (ii); if `None`, the coordinates are assumed to refer to the default chart of `self`
- `chart2` – (default: `None`; used only in case (ii) above) chart on `codomain` defining the target coordinates involved in `coord_functions` for case (ii); if `None`, the coordinates are assumed to refer to the default chart of `codomain`
- `name` – (default: `None`) name given to the homeomorphism
- `latex_name` – (default: `None`) LaTeX symbol to denote the homeomorphism; if `None`, the LaTeX symbol is set to `name`

OUTPUT:

- the homeomorphism, as an instance of `ContinuousMap`

EXAMPLES:

Homeomorphism between the open unit disk in $\mathbf{R}^2$ and $\mathbf{R}^2$:

```
sage: forget()  # for doctests only
sage: M = Manifold(2, 'M', structure='topological')  # the open unit disk
sage: c_xy.<x,y> = M.chart('x:(-1,1) y:(-1,1)', coord_restrictions=lambda x,
↪y: x^2+y^2<1)
....:     # Cartesian coord on M
sage: N = Manifold(2, 'N', structure='topological')  # R^2
sage: c_XY.<X,Y> = N.chart()  # canonical coordinates on R^2
sage: Phi = M.homeomorphism(N, [x/sqrt(1-x^2-y^2), y/sqrt(1-x^2-y^2)],
....:                          name='Phi', latex_name=r'\Phi')
sage: Phi
Homeomorphism Phi from the 2-dimensional topological manifold M to
 the 2-dimensional topological manifold N
sage: Phi.display()
Phi: M → N
   (x, y) ↦ (X, Y) = (x/sqrt(-x^2 - y^2 + 1), y/sqrt(-x^2 - y^2 + 1))
```

```
>>> from sage.all import *
>>> forget()  # for doctests only
>>> M = Manifold(Integer(2), 'M', structure='topological')  # the open unit␣
↪disk
>>> c_xy = M.chart('x:(-1,1) y:(-1,1)', coord_restrictions=lambda x,y:␣
↪x**Integer(2)+y**Integer(2)<Integer(1), names=('x', 'y',)); (x, y,) = c_xy._
↪first_ngens(2)
...     # Cartesian coord on M
>>> N = Manifold(Integer(2), 'N', structure='topological')  # R^2
>>> c_XY = N.chart(names=('X', 'Y',)); (X, Y,) = c_XY._first_ngens(2)#␣
↪canonical coordinates on R^2
>>> Phi = M.homeomorphism(N, [x/sqrt(Integer(1)-x**Integer(2)-y**Integer(2)),␣
↪y/sqrt(Integer(1)-x**Integer(2)-y**Integer(2))],
...                          name='Phi', latex_name=r'\Phi')
>>> Phi
Homeomorphism Phi from the 2-dimensional topological manifold M to
 the 2-dimensional topological manifold N
>>> Phi.display()
Phi: M → N
   (x, y) ↦ (X, Y) = (x/sqrt(-x^2 - y^2 + 1), y/sqrt(-x^2 - y^2 + 1))
```

The inverse homeomorphism:

```
sage: Phi^(-1)
Homeomorphism Phi^(-1) from the 2-dimensional topological
 manifold N to the 2-dimensional topological manifold M
sage:  (Phi^(-1)).display()
Phi^(-1): N → M
   (X, Y) ↦ (x, y) = (X/sqrt(X^2 + Y^2 + 1), Y/sqrt(X^2 + Y^2 + 1))
```

```
>>> from sage.all import *
>>> Phi**(-Integer(1))
Homeomorphism Phi^(-1) from the 2-dimensional topological
 manifold N to the 2-dimensional topological manifold M
>>>  (Phi**(-Integer(1))).display()
Phi^(-1): N → M
   (X, Y) ↦ (x, y) = (X/sqrt(X^2 + Y^2 + 1), Y/sqrt(X^2 + Y^2 + 1))
```

See the documentation of `ContinuousMap` for more examples.

identity_map()

Identity map of `self`.

The identity map of a topological manifold M is the trivial homeomorphism:

$$\begin{array}{llll} \mathrm{Id}_M : & M & \longrightarrow & M \\ & p & \longmapsto & p \end{array}$$

OUTPUT:

- the identity map as an instance of `ContinuousMap`

EXAMPLES:

Identity map of a complex manifold:

```
sage: M = Manifold(2, 'M', structure='topological', field='complex')
sage: X.<x,y> = M.chart()
sage: id = M.identity_map(); id
Identity map Id_M of the Complex 2-dimensional topological manifold M
sage: id.parent()
Set of Morphisms from Complex 2-dimensional topological manifold M
 to Complex 2-dimensional topological manifold M in Category of
 manifolds over Complex Field with 53 bits of precision
sage: id.display()
Id_M: M → M
   (x, y) ↦ (x, y)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological', field='complex')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> id = M.identity_map(); id
Identity map Id_M of the Complex 2-dimensional topological manifold M
>>> id.parent()
Set of Morphisms from Complex 2-dimensional topological manifold M
 to Complex 2-dimensional topological manifold M in Category of
 manifolds over Complex Field with 53 bits of precision
>>> id.display()
Id_M: M → M
   (x, y) ↦ (x, y)
```

The identity map acting on a point:

```
sage: p = M((1+I, 3-I), name='p'); p
Point p on the Complex 2-dimensional topological manifold M
sage: id(p)
Point p on the Complex 2-dimensional topological manifold M
sage: id(p) == p
True
```

```
>>> from sage.all import *
>>> p = M((Integer(1)+I, Integer(3)-I), name='p'); p
Point p on the Complex 2-dimensional topological manifold M
>>> id(p)
```

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```
Point p on the Complex 2-dimensional topological manifold M
>>> id(p) == p
True
```

See also

See `ContinuousMap` for the complete documentation.

index_generator(*nb_indices*)

Generator of index series.

INPUT:

- `nb_indices` – number of indices in a series

OUTPUT:

- an iterable index series for a generic component with the specified number of indices

EXAMPLES:

Indices on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological', start_index=1)
sage: list(M.index_generator(2))
[(1, 1), (1, 2), (2, 1), (2, 2)]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological', start_
↪index=Integer(1))
>>> list(M.index_generator(Integer(2)))
[(1, 1), (1, 2), (2, 1), (2, 2)]
```

Loops can be nested:

```
sage: for ind1 in M.index_generator(2):
....:     print("{} : {}".format(ind1, list(M.index_generator(2))))
(1, 1) : [(1, 1), (1, 2), (2, 1), (2, 2)]
(1, 2) : [(1, 1), (1, 2), (2, 1), (2, 2)]
(2, 1) : [(1, 1), (1, 2), (2, 1), (2, 2)]
(2, 2) : [(1, 1), (1, 2), (2, 1), (2, 2)]
```

```
>>> from sage.all import *
>>> for ind1 in M.index_generator(Integer(2)):
...     print("{} : {}".format(ind1, list(M.index_generator(Integer(2)))))
(1, 1) : [(1, 1), (1, 2), (2, 1), (2, 2)]
(1, 2) : [(1, 1), (1, 2), (2, 1), (2, 2)]
(2, 1) : [(1, 1), (1, 2), (2, 1), (2, 2)]
(2, 2) : [(1, 1), (1, 2), (2, 1), (2, 2)]
```

irange(*start=None*, *end=None*)

Single index generator.

INPUT:

- `start` – (default: `None`) initial value i_0 of the index; if `None`, the value returned by `start_index()` is assumed
- `end` – (default: `None`) final value i_n of the index; if `None`, the value returned by `start_index()` plus $n-1$, where n is the manifold dimension, is assumed

OUTPUT: an iterable index, starting from i_0 and ending at $i_0 + i_n$

EXAMPLES:

Index range on a 4-dimensional manifold:

```
sage: M = Manifold(4, 'M', structure='topological')
sage: list(M.irange())
[0, 1, 2, 3]
sage: list(M.irange(start=2))
[2, 3]
sage: list(M.irange(end=2))
[0, 1, 2]
sage: list(M.irange(start=1, end=2))
[1, 2]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='topological')
>>> list(M.irange())
[0, 1, 2, 3]
>>> list(M.irange(start=Integer(2)))
[2, 3]
>>> list(M.irange(end=Integer(2)))
[0, 1, 2]
>>> list(M.irange(start=Integer(1), end=Integer(2)))
[1, 2]
```

Index range on a 4-dimensional manifold with starting index=1:

```
sage: M = Manifold(4, 'M', structure='topological', start_index=1)
sage: list(M.irange())
[1, 2, 3, 4]
sage: list(M.irange(start=2))
[2, 3, 4]
sage: list(M.irange(end=2))
[1, 2]
sage: list(M.irange(start=2, end=3))
[2, 3]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='topological', start_
↪index=Integer(1))
>>> list(M.irange())
[1, 2, 3, 4]
>>> list(M.irange(start=Integer(2)))
[2, 3, 4]
>>> list(M.irange(end=Integer(2)))
[1, 2]
```

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```
>>> list(M.irange(start=Integer(2), end=Integer(3)))
[2, 3]
```

In general, one has always:

```
sage: next(M.irange()) == M.start_index()
True
```

```
>>> from sage.all import *
>>> next(M.irange()) == M.start_index()
True
```

is_manifestly_coordinate_domain()

Return `True` if the manifold is known to be the domain of some coordinate chart and `False` otherwise.

If `False` is returned, either the manifold cannot be the domain of some coordinate chart or no such chart has been declared yet.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: U = M.open_subset('U')
sage: X.<x,y> = U.chart()
sage: U.is_manifestly_coordinate_domain()
True
sage: M.is_manifestly_coordinate_domain()
False
sage: Y.<u,v> = M.chart()
sage: M.is_manifestly_coordinate_domain()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> U = M.open_subset('U')
>>> X = U.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> U.is_manifestly_coordinate_domain()
True
>>> M.is_manifestly_coordinate_domain()
False
>>> Y = M.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)
>>> M.is_manifestly_coordinate_domain()
True
```

is_open()

Return if `self` is an open set.

In the present case (manifold or open subset of it), always return `True`.

one_scalar_field()

Return the constant scalar field with value the unit element of the base field of `self`.

OUTPUT:

- a *ScalarField* representing the constant scalar field with value the unit element of the base field of `self`

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.one_scalar_field(); f
Scalar field 1 on the 2-dimensional topological manifold M
sage: f.display()
1: M → ℝ
   (x, y) ↦ 1
sage: f.parent()
Algebra of scalar fields on the 2-dimensional topological manifold M
sage: f is M.scalar_field_algebra().one()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.one_scalar_field(); f
Scalar field 1 on the 2-dimensional topological manifold M
>>> f.display()
1: M → ℝ
   (x, y) ↦ 1
>>> f.parent()
Algebra of scalar fields on the 2-dimensional topological manifold M
>>> f is M.scalar_field_algebra().one()
True
```

open_subset(*name*, *latex_name=None*, *coord_def={}*, *supersets=None*)

Create an open subset of the manifold.

An open subset is a set that is (i) included in the manifold and (ii) open with respect to the manifold's topology. It is a topological manifold by itself. Hence the returned object is an instance of `TopologicalManifold`.

INPUT:

- `name` – name given to the open subset
- `latex_name` – (default: `None`) LaTeX symbol to denote the subset; if none are provided, it is set to `name`
- `coord_def` – (default: {}) definition of the subset in terms of coordinates; `coord_def` must a be dictionary with keys charts on the manifold and values the symbolic expressions formed by the coordinates to define the subset
- `supersets` – (default: only `self`) list of sets that the new open subset is a subset of

OUTPUT: the open subset, as an instance of `TopologicalManifold`

EXAMPLES:

Creating an open subset of a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: A = M.open_subset('A'); A
Open subset A of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> A = M.open_subset('A'); A
Open subset A of the 2-dimensional topological manifold M
```

As an open subset of a topological manifold, `A` is itself a topological manifold, on the same topological field and of the same dimension as `M`:

```
sage: isinstance(A, sage.manifolds.manifold.TopologicalManifold)
True
sage: A.base_field() == M.base_field()
True
sage: dim(A) == dim(M)
True
sage: A.category() is M.category().Subobjects()
True
```

```
>>> from sage.all import *
>>> isinstance(A, sage.manifolds.manifold.TopologicalManifold)
True
>>> A.base_field() == M.base_field()
True
>>> dim(A) == dim(M)
True
>>> A.category() is M.category().Subobjects()
True
```

Creating an open subset of `A`:

```
sage: B = A.open_subset('B'); B
Open subset B of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> B = A.open_subset('B'); B
Open subset B of the 2-dimensional topological manifold M
```

We have then:

```
sage: frozenset(A.subsets())  # random (set output)
{Open subset B of the 2-dimensional topological manifold M,
 Open subset A of the 2-dimensional topological manifold M}
sage: B.is_subset(A)
True
sage: B.is_subset(M)
True
```

```
>>> from sage.all import *
>>> frozenset(A.subsets())  # random (set output)
{Open subset B of the 2-dimensional topological manifold M,
 Open subset A of the 2-dimensional topological manifold M}
>>> B.is_subset(A)
True
```

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```
>>> B.is_subset(M)
True
```

Defining an open subset by some coordinate restrictions: the open unit disk in $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'R^2', structure='topological')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: U = M.open_subset('U', coord_def={c_cart: x^2+y^2<1}); U
Open subset U of the 2-dimensional topological manifold R^2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', structure='topological')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#
↪Cartesian coordinates on R^2
>>> U = M.open_subset('U', coord_def={c_cart: x**Integer(2)+y**Integer(2)
↪<Integer(1)}); U
Open subset U of the 2-dimensional topological manifold R^2
```

Since the argument `coord_def` has been set, `U` is automatically provided with a chart, which is the restriction of the Cartesian one to `U`:

```
sage: U.atlas()
[Chart (U, (x, y))]
```

```
>>> from sage.all import *
>>> U.atlas()
[Chart (U, (x, y))]
```

Therefore, one can immediately check whether a point belongs to `U`:

```
sage: M.point((0,0)) in U
True
sage: M.point((1/2,1/3)) in U
True
sage: M.point((1,2)) in U
False
```

```
>>> from sage.all import *
>>> M.point((Integer(0),Integer(0))) in U
True
>>> M.point((Integer(1)/Integer(2),Integer(1)/Integer(3))) in U
True
>>> M.point((Integer(1),Integer(2))) in U
False
```

options = Current options for manifolds - omit_function_arguments: False - textbook_output: True

orientation()

Get the preferred orientation of `self` if available.

An *orientation* of an n-dimensional topological manifold is an atlas of charts whose transition maps are orientation preserving. A homeomorphism $f: U \to V$ for open subsets $U, V \subset \mathbf{R}^n$ is called *orientation* *preserving* if for each $x \in U$ the following map between singular homologies is the identity:

$$H_n(\mathbf{R}^n, \mathbf{R}^n - 0; \mathbf{Z}) \cong H_n(U, U - x; \mathbf{Z}) \xrightarrow{f_*} H_n(V, V - f(x)) \cong H_n(\mathbf{R}^n, \mathbf{R}^n - 0; \mathbf{Z})$$

See this link for details.

Note

Notice that for differentiable manifolds, the notion of orientability does not need homology theory at all. See `orientation()` for details

The trivial case corresponds to the manifold being covered by one chart. In that case, if no preferred orientation has been manually set before, one of those charts (usually the default chart) is set to the preferred orientation and returned here.

EXAMPLES:

If the manifold is covered by only one chart, it certainly admits an orientation:

```
sage: M = Manifold(3, 'M', structure='top')
sage: c.<x,y,z> = M.chart()
sage: M.orientation()
[Chart (M, (x, y, z))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> c = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c._first_ngens(3)
>>> M.orientation()
[Chart (M, (x, y, z))]
```

Usually, an orientation cannot be obtained so easily:

```
sage: M = Manifold(2, 'M', structure='top')
sage: U = M.open_subset('U'); V = M.open_subset('V')
sage: M.declare_union(U, V)
sage: c_xy.<x,y> = U.chart(); c_uv.<u,v> = V.chart()
sage: M.orientation()
[]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> U = M.open_subset('U'); V = M.open_subset('V')
>>> M.declare_union(U, V)
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> M.orientation()
[]
```

In that case, the orientation can be set by the user manually:

```
sage: M.set_orientation([c_xy, c_uv])
sage: M.orientation()
[Chart (U, (x, y)), Chart (V, (u, v))]
```

```
>>> from sage.all import *
>>> M.set_orientation([c_xy, c_uv])
>>> M.orientation()
[Chart (U, (x, y)), Chart (V, (u, v))]
```

The orientation on submanifolds are inherited from the ambient manifold:

```
sage: W = U.intersection(V, name='W')
sage: W.orientation()
[Chart (W, (x, y))]
```

```
>>> from sage.all import *
>>> W = U.intersection(V, name='W')
>>> W.orientation()
[Chart (W, (x, y))]
```

scalar_field(*coord_expression=None*, *chart=None*, *name=None*, *latex_name=None*)

Define a scalar field on the manifold.

See *ScalarField* (or *DiffScalarField* if the manifold is differentiable) for a complete documentation.

INPUT:

- `coord_expression` – (default: `None`) coordinate expression(s) of the scalar field; this can be either
  - a single coordinate expression; if the argument `chart` is `'all'`, this expression is set to all the charts defined on the open set; otherwise, the expression is set in the specific chart provided by the argument `chart`
  - a dictionary of coordinate expressions, with the charts as keys
- `chart` – (default: `None`) chart defining the coordinates used in `coord_expression` when the latter is a single coordinate expression; if `None`, the default chart of the open set is assumed; if `chart=='all'`, `coord_expression` is assumed to be independent of the chart (constant scalar field)
- `name` – (default: `None`) name given to the scalar field
- `latex_name` – (default: `None`) LaTeX symbol to denote the scalar field; if `None`, the LaTeX symbol is set to `name`

If `coord_expression` is `None` or does not fully specified the scalar field, other coordinate expressions can be added subsequently by means of the methods *add_expr()*, *add_expr_by_continuation()*, or *set_expr()*

OUTPUT:

- instance of *ScalarField* (or of the subclass *DiffScalarField* if the manifold is differentiable) representing the defined scalar field

EXAMPLES:

A scalar field defined by its coordinate expression in the open set's default chart:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: U = M.open_subset('U')
sage: c_xyz.<x,y,z> = U.chart()
sage: f = U.scalar_field(sin(x)*cos(y) + z, name='F'); f
Scalar field F on the Open subset U of the 3-dimensional topological manifold
↪M
```

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```
sage: f.display()
F: U → ℝ
   (x, y, z) ↦ cos(y)*sin(x) + z
sage: f.parent()
Algebra of scalar fields on the Open subset U of the 3-dimensional␣
↪topological manifold M
sage: f in U.scalar_field_algebra()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological')
>>> U = M.open_subset('U')
>>> c_xyz = U.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> f = U.scalar_field(sin(x)*cos(y) + z, name='F'); f
Scalar field F on the Open subset U of the 3-dimensional topological manifold␣
↪M
>>> f.display()
F: U → ℝ
   (x, y, z) ↦ cos(y)*sin(x) + z
>>> f.parent()
Algebra of scalar fields on the Open subset U of the 3-dimensional␣
↪topological manifold M
>>> f in U.scalar_field_algebra()
True
```

Equivalent definition with the chart specified:

```
sage: f = U.scalar_field(sin(x)*cos(y) + z, chart=c_xyz, name='F')
sage: f.display()
F: U → ℝ
   (x, y, z) ↦ cos(y)*sin(x) + z
```

```
>>> from sage.all import *
>>> f = U.scalar_field(sin(x)*cos(y) + z, chart=c_xyz, name='F')
>>> f.display()
F: U → ℝ
   (x, y, z) ↦ cos(y)*sin(x) + z
```

Equivalent definition with a dictionary of coordinate expression(s):

```
sage: f = U.scalar_field({c_xyz: sin(x)*cos(y) + z}, name='F')
sage: f.display()
F: U → ℝ
   (x, y, z) ↦ cos(y)*sin(x) + z
```

```
>>> from sage.all import *
>>> f = U.scalar_field({c_xyz: sin(x)*cos(y) + z}, name='F')
>>> f.display()
F: U → ℝ
   (x, y, z) ↦ cos(y)*sin(x) + z
```

See the documentation of class `ScalarField` for more examples.

> **See also**
>
> `constant_scalar_field()`, `zero_scalar_field()`, `one_scalar_field()`

scalar_field_algebra()

Return the algebra of scalar fields defined the manifold.

See `ScalarFieldAlgebra` for a complete documentation.

OUTPUT:

- instance of `ScalarFieldAlgebra` representing the algebra $C^0(U)$ of all scalar fields defined on U = `self`

EXAMPLES:

Scalar algebra of a 3-dimensional open subset:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: U = M.open_subset('U')
sage: CU = U.scalar_field_algebra() ; CU
Algebra of scalar fields on the Open subset U of the 3-dimensional␣
↪topological manifold M
sage: CU.category()
Join of Category of commutative algebras over Symbolic Ring and Category of␣
↪homsets of topological spaces
sage: CU.zero()
Scalar field zero on the Open subset U of the 3-dimensional topological␣
↪manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological')
>>> U = M.open_subset('U')
>>> CU = U.scalar_field_algebra() ; CU
Algebra of scalar fields on the Open subset U of the 3-dimensional␣
↪topological manifold M
>>> CU.category()
Join of Category of commutative algebras over Symbolic Ring and Category of␣
↪homsets of topological spaces
>>> CU.zero()
Scalar field zero on the Open subset U of the 3-dimensional topological␣
↪manifold M
```

The output is cached:

```
sage: U.scalar_field_algebra() is CU
True
```

```
>>> from sage.all import *
>>> U.scalar_field_algebra() is CU
True
```

set_calculus_method(*method*)

Set the calculus method to be used for coordinate computations on this manifold.

The provided method is transmitted to all coordinate charts defined on the manifold.

INPUT:

- `method` – string specifying the method to be used for coordinate computations on this manifold; one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy

EXAMPLES:

Let us consider a scalar field `f` on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field(x^2 + cos(y)*sin(x), name='F')
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field(x**Integer(2) + cos(y)*sin(x), name='F')
```

By default, the coordinate expression of `f` returned by `expr()` is a Sage's symbolic expression:

```
sage: f.expr()
x^2 + cos(y)*sin(x)
sage: type(f.expr())
<class 'sage.symbolic.expression.Expression'>
sage: parent(f.expr())
Symbolic Ring
sage: f.display()
F: M → ℝ
   (x, y) ↦ x^2 + cos(y)*sin(x)
```

```
>>> from sage.all import *
>>> f.expr()
x^2 + cos(y)*sin(x)
>>> type(f.expr())
<class 'sage.symbolic.expression.Expression'>
>>> parent(f.expr())
Symbolic Ring
>>> f.display()
F: M → ℝ
   (x, y) ↦ x^2 + cos(y)*sin(x)
```

If we change the calculus method to SymPy, it becomes a SymPy object instead:

```
sage: M.set_calculus_method('sympy')
sage: f.expr()
x**2 + sin(x)*cos(y)
sage: type(f.expr())
<class 'sympy.core.add.Add'>
```

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```
sage: parent(f.expr())
<class 'sympy.core.add.Add'>
sage: f.display()
F: M → ℝ
   (x, y) ↦ x**2 + sin(x)*cos(y)
```

```
>>> from sage.all import *
>>> M.set_calculus_method('sympy')
>>> f.expr()
x**2 + sin(x)*cos(y)
>>> type(f.expr())
<class 'sympy.core.add.Add'>
>>> parent(f.expr())
<class 'sympy.core.add.Add'>
>>> f.display()
F: M → ℝ
   (x, y) ↦ x**2 + sin(x)*cos(y)
```

Back to the Symbolic Ring:

```
sage: M.set_calculus_method('SR')
sage: f.display()
F: M → ℝ
   (x, y) ↦ x^2 + cos(y)*sin(x)
```

```
>>> from sage.all import *
>>> M.set_calculus_method('SR')
>>> f.display()
F: M → ℝ
   (x, y) ↦ x^2 + cos(y)*sin(x)
```

The calculus method chosen via `set_calculus_method()` applies to any chart defined subsequently on the manifold:

```
sage: M.set_calculus_method('sympy')
sage: Y.<u,v> = M.chart()  # a new chart
sage: Y.calculus_method()
Available calculus methods (* = current):
 - SR (default)
 - sympy (*)
```

```
>>> from sage.all import *
>>> M.set_calculus_method('sympy')
>>> Y = M.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)# a new chart
>>> Y.calculus_method()
Available calculus methods (* = current):
 - SR (default)
 - sympy (*)
```

See also

`calculus_method()` for a control of the calculus method chart by chart

set_default_chart(*chart*)

Changing the default chart on `self`.

INPUT:

- `chart` – a chart (must be defined on some subset `self`)

EXAMPLES:

Charts on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: c_uv.<u,v> = M.chart()
sage: M.default_chart()
Chart (M, (x, y))
sage: M.set_default_chart(c_uv)
sage: M.default_chart()
Chart (M, (u, v))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> c_uv = M.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> M.default_chart()
Chart (M, (x, y))
>>> M.set_default_chart(c_uv)
>>> M.default_chart()
Chart (M, (u, v))
```

set_orientation(*orientation*)

Set the preferred orientation of `self`.

INPUT:

- `orientation` – a chart or a list of charts

Warning

It is the user's responsibility that the orientation set here is indeed an orientation. There is no check going on in the background. See `orientation()` for the definition of an orientation.

EXAMPLES:

Set an orientation on a manifold:

```
sage: M = Manifold(2, 'M', structure='top')
sage: c_xy.<x,y> = M.chart(); c_uv.<u,v> = M.chart()
sage: M.set_orientation(c_uv)
sage: M.orientation()
[Chart (M, (u, v))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪M.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> M.set_orientation(c_uv)
>>> M.orientation()
[Chart (M, (u, v))]
```

Set an orientation in the non-trivial case:

```
sage: M = Manifold(2, 'M', structure='top')
sage: U = M.open_subset('U'); V = M.open_subset('V')
sage: M.declare_union(U, V)
sage: c_xy.<x,y> = U.chart(); c_uv.<u,v> = V.chart()
sage: M.set_orientation([c_xy, c_uv])
sage: M.orientation()
[Chart (U, (x, y)), Chart (V, (u, v))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> U = M.open_subset('U'); V = M.open_subset('V')
>>> M.declare_union(U, V)
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> M.set_orientation([c_xy, c_uv])
>>> M.orientation()
[Chart (U, (x, y)), Chart (V, (u, v))]
```

set_simplify_function(*simplifying_func*, *method=None*)

Set the simplifying function associated to a given coordinate calculus method in all the charts defined on `self`.

INPUT:

- `simplifying_func` – either the string `'default'` for restoring the default simplifying function or a function `f` of a single argument `expr` such that `f(expr)` returns an object of the same type as `expr` (hopefully the simplified version of `expr`), this type being
  - `Expression` if `method` = `'SR'`
  - a SymPy type if `method` = `'sympy'`
- `method` – (default: `None`) string defining the calculus method for which `simplifying_func` is provided; must be one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy
  - `None`: the currently active calculus method on each chart is assumed

See also

calculus_method() and *sage.manifolds.calculus_method.CalculusMethod.simplify()* for a control of the calculus method chart by chart

EXAMPLES:

Les us add two scalar fields on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field((x+y)^2 + cos(x)^2)
sage: g = M.scalar_field(-x^2-2*x*y-y^2 + sin(x)^2)
sage: f.expr()
(x + y)^2 + cos(x)^2
sage: g.expr()
-x^2 - 2*x*y - y^2 + sin(x)^2
sage: s = f + g
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field((x+y)**Integer(2) + cos(x)**Integer(2))
>>> g = M.scalar_field(-x**Integer(2)-Integer(2)*x*y-y**Integer(2) +␣
↪sin(x)**Integer(2))
>>> f.expr()
(x + y)^2 + cos(x)^2
>>> g.expr()
-x^2 - 2*x*y - y^2 + sin(x)^2
>>> s = f + g
```

The outcome is automatically simplified:

```
sage: s.expr()
1
```

```
>>> from sage.all import *
>>> s.expr()
1
```

The simplification is performed thanks to the default simplifying function on chart `X`, which is *`simplify_chain_real()`* in the present case (real manifold and `SR` calculus):

```
sage: X.calculus_method().simplify_function() is \
....: sage.manifolds.utilities.simplify_chain_real
True
```

```
>>> from sage.all import *
>>> X.calculus_method().simplify_function() is sage.manifolds.utilities.
↪simplify_chain_real
True
```

Let us change it to the generic Sage function `simplify()`:

```
sage: M.set_simplify_function(simplify)
sage: X.calculus_method().simplify_function() is simplify
True
```

```
>>> from sage.all import *
>>> M.set_simplify_function(simplify)
>>> X.calculus_method().simplify_function() is simplify
True
```

`simplify()` is faster, but it does not do much:

```
sage: s = f + g
sage: s.expr()
(x + y)^2 - x^2 - 2*x*y - y^2 + cos(x)^2 + sin(x)^2
```

```
>>> from sage.all import *
>>> s = f + g
>>> s.expr()
(x + y)^2 - x^2 - 2*x*y - y^2 + cos(x)^2 + sin(x)^2
```

We can replaced it by any user defined function, for instance:

```
sage: def simpl_trig(a):
....:     return a.simplify_trig()
sage: M.set_simplify_function(simpl_trig)
sage: s = f + g
sage: s.expr()
1
```

```
>>> from sage.all import *
>>> def simpl_trig(a):
...     return a.simplify_trig()
>>> M.set_simplify_function(simpl_trig)
>>> s = f + g
>>> s.expr()
1
```

The default simplifying function is restored via:

```
sage: M.set_simplify_function('default')
```

```
>>> from sage.all import *
>>> M.set_simplify_function('default')
```

Then we are back to:

```
sage: X.calculus_method().simplify_function() is \
....: sage.manifolds.utilities.simplify_chain_real
True
```

```
>>> from sage.all import *
>>> X.calculus_method().simplify_function() is sage.manifolds.utilities.
↪simplify_chain_real
True
```

Thanks to the argument `method`, one can specify a simplifying function for a calculus method distinct from the current one. For instance, let us define a simplifying function for SymPy (note that `trigsimp()` is a

SymPy method only):

```
sage: def simpl_trig_sympy(a):
....:     return a.trigsimp()
sage: M.set_simplify_function(simpl_trig_sympy, method='sympy')
```

```
>>> from sage.all import *
>>> def simpl_trig_sympy(a):
...     return a.trigsimp()
>>> M.set_simplify_function(simpl_trig_sympy, method='sympy')
```

Then, it becomes active as soon as we change the calculus engine to SymPy:

```
sage: M.set_calculus_method('sympy')
sage: X.calculus_method().simplify_function() is simpl_trig_sympy
True
```

```
>>> from sage.all import *
>>> M.set_calculus_method('sympy')
>>> X.calculus_method().simplify_function() is simpl_trig_sympy
True
```

We have then:

```
sage: s = f + g
sage: s.expr()
1
sage: type(s.expr())
<class 'sympy.core.numbers.One'>
```

```
>>> from sage.all import *
>>> s = f + g
>>> s.expr()
1
>>> type(s.expr())
<class 'sympy.core.numbers.One'>
```

start_index()

Return the first value of the index range used on the manifold.

This is the parameter `start_index` passed at the construction of the manifold.

OUTPUT:

- the integer i_0 such that all indices of indexed objects on the manifold range from i_0 to $i_0 + n - 1$, where n is the manifold's dimension

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: M.start_index()
0
sage: M = Manifold(3, 'M', structure='topological', start_index=1)
sage: M.start_index()
1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological')
>>> M.start_index()
0
>>> M = Manifold(Integer(3), 'M', structure='topological', start_
↪index=Integer(1))
>>> M.start_index()
1
```

top_charts()

Return the list of charts defined on subsets of the current manifold that are not subcharts of charts on larger subsets.

OUTPUT:

- list of charts defined on open subsets of the manifold but not on larger subsets

EXAMPLES:

Charts on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: U = M.open_subset('U', coord_def={X: x>0})
sage: Y.<u,v> = U.chart()
sage: M.top_charts()
[Chart (M, (x, y)), Chart (U, (u, v))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> U = M.open_subset('U', coord_def={X: x>Integer(0)})
>>> Y = U.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)
>>> M.top_charts()
[Chart (M, (x, y)), Chart (U, (u, v))]
```

Note that the (user) atlas contains one more chart: `(U, (x,y))`, which is not a "top" chart:

```
sage: M.atlas()
[Chart (M, (x, y)), Chart (U, (x, y)), Chart (U, (u, v))]
```

```
>>> from sage.all import *
>>> M.atlas()
[Chart (M, (x, y)), Chart (U, (x, y)), Chart (U, (u, v))]
```

See also

atlas() for the complete list of charts defined on the manifold.

vector_bundle(*rank*, *name*, *field='real'*, *latex_name=None*)

Return a topological vector bundle over the given field with given rank over this topological manifold.

INPUT:

- `rank` – rank of the vector bundle
- `name` – name given to the total space
- `field` – (default: `'real'`) topological field giving the vector space structure to the fibers
- `latex_name` – (optional) LaTeX name for the total space

OUTPUT:

- a topological vector bundle as an instance of `TopologicalVectorBundle`

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top')
sage: M.vector_bundle(2, 'E')
Topological real vector bundle E -> M of rank 2 over the base space
 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> M.vector_bundle(Integer(2), 'E')
Topological real vector bundle E -> M of rank 2 over the base space
 2-dimensional topological manifold M
```

zero_scalar_field()

Return the zero scalar field defined on `self`.

OUTPUT:

- a `ScalarField` representing the constant scalar field with value 0

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.zero_scalar_field() ; f
Scalar field zero on the 2-dimensional topological manifold M
sage: f.display()
zero: M → ℝ
   (x, y) ↦ 0
sage: f.parent()
Algebra of scalar fields on the 2-dimensional topological manifold M
sage: f is M.scalar_field_algebra().zero()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.zero_scalar_field() ; f
Scalar field zero on the 2-dimensional topological manifold M
>>> f.display()
zero: M → ℝ
   (x, y) ↦ 0
>>> f.parent()
Algebra of scalar fields on the 2-dimensional topological manifold M
>>> f is M.scalar_field_algebra().zero()
True
```

1.2 Subsets of Topological Manifolds

The class `ManifoldSubset` implements generic subsets of a topological manifold. Open subsets are implemented by the class `TopologicalManifold` (since an open subset of a manifold is a manifold by itself), which inherits from `ManifoldSubset`. Besides, subsets that are images of a manifold subset under a continuous map are implemented by the subclass `ImageManifoldSubset`.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015): initial version
- Travis Scrimshaw (2015): review tweaks; removal of facade parents
- Matthias Koeppe (2021): Families and posets of subsets

REFERENCES:

- [Lee2011]

EXAMPLES:

Two subsets on a manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: a = M.subset('A'); a
Subset A of the 2-dimensional topological manifold M
sage: b = M.subset('B'); b
Subset B of the 2-dimensional topological manifold M
sage: M.subset_family()
Set {A, B, M} of subsets of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> a = M.subset('A'); a
Subset A of the 2-dimensional topological manifold M
>>> b = M.subset('B'); b
Subset B of the 2-dimensional topological manifold M
>>> M.subset_family()
Set {A, B, M} of subsets of the 2-dimensional topological manifold M
```

The intersection of the two subsets:

```
sage: c = a.intersection(b); c
Subset A_inter_B of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> c = a.intersection(b); c
Subset A_inter_B of the 2-dimensional topological manifold M
```

Their union:

```
sage: d = a.union(b); d
Subset A_union_B of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> d = a.union(b); d
Subset A_union_B of the 2-dimensional topological manifold M
```

Families of subsets after the above operations:

```
sage: M.subset_family()
Set {A, A_inter_B, A_union_B, B, M} of subsets of the 2-dimensional topological␣
↪manifold M
sage: a.subset_family()
Set {A, A_inter_B} of subsets of the 2-dimensional topological manifold M
sage: c.subset_family()
Set {A_inter_B} of subsets of the 2-dimensional topological manifold M
sage: d.subset_family()
Set {A, A_inter_B, A_union_B, B} of subsets of the 2-dimensional topological manifold␣
↪M
```

```
>>> from sage.all import *
>>> M.subset_family()
Set {A, A_inter_B, A_union_B, B, M} of subsets of the 2-dimensional topological␣
↪manifold M
>>> a.subset_family()
Set {A, A_inter_B} of subsets of the 2-dimensional topological manifold M
>>> c.subset_family()
Set {A_inter_B} of subsets of the 2-dimensional topological manifold M
>>> d.subset_family()
Set {A, A_inter_B, A_union_B, B} of subsets of the 2-dimensional topological manifold␣
↪M
```

class sage.manifolds.subset.**ManifoldSubset**(*manifold*, *name: str*, *latex_name=None*, *category=None*)

Bases: `UniqueRepresentation`, `Parent`

Subset of a topological manifold.

The class *ManifoldSubset* inherits from the generic class `Parent`. The corresponding element class is *ManifoldPoint*.

Note that open subsets are not implemented directly by this class, but by the derived class *TopologicalManifold* (an open subset of a topological manifold being itself a topological manifold).

INPUT:

- `manifold` – topological manifold on which the subset is defined
- `name` – string; name (symbol) given to the subset
- `latex_name` – string (default: `None`); LaTeX symbol to denote the subset; if none are provided, it is set to `name`
- `category` – (default: `None`) to specify the category; if `None`, the category for generic subsets is used

EXAMPLES:

A subset of a manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: from sage.manifolds.subset import ManifoldSubset
sage: A = ManifoldSubset(M, 'A', latex_name=r'\mathcal{A}')
sage: A
Subset A of the 2-dimensional topological manifold M
sage: latex(A)
\mathcal{A}
```

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```
sage: A.is_subset(M)
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> from sage.manifolds.subset import ManifoldSubset
>>> A = ManifoldSubset(M, 'A', latex_name=r'\mathcal{A}')
>>> A
Subset A of the 2-dimensional topological manifold M
>>> latex(A)
\mathcal{A}
>>> A.is_subset(M)
True
```

Instead of importing `ManifoldSubset` in the global namespace, it is recommended to use the method `subset()` to create a new subset:

```
sage: B = M.subset('B', latex_name=r'\mathcal{B}'); B
Subset B of the 2-dimensional topological manifold M
sage: M.subset_family()
Set {A, B, M} of subsets of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> B = M.subset('B', latex_name=r'\mathcal{B}'); B
Subset B of the 2-dimensional topological manifold M
>>> M.subset_family()
Set {A, B, M} of subsets of the 2-dimensional topological manifold M
```

The manifold is itself a subset:

```
sage: isinstance(M, ManifoldSubset)
True
sage: M in M.subsets()
True
```

```
>>> from sage.all import *
>>> isinstance(M, ManifoldSubset)
True
>>> M in M.subsets()
True
```

Instances of `ManifoldSubset` are parents:

```
sage: isinstance(A, Parent)
True
sage: A.category()
Category of subobjects of sets
sage: p = A.an_element(); p
Point on the 2-dimensional topological manifold M
sage: p.parent()
Subset A of the 2-dimensional topological manifold M
sage: p in A
```

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```
True
sage: p in M
True
```

```
>>> from sage.all import *
>>> isinstance(A, Parent)
True
>>> A.category()
Category of subobjects of sets
>>> p = A.an_element(); p
Point on the 2-dimensional topological manifold M
>>> p.parent()
Subset A of the 2-dimensional topological manifold M
>>> p in A
True
>>> p in M
True
```

Element

alias of `ManifoldPoint`

ambient()

Return the ambient manifold of `self`.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: A = M.subset('A')
sage: A.manifold()
2-dimensional topological manifold M
sage: A.manifold() is M
True
sage: B = A.subset('B')
sage: B.manifold() is M
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> A = M.subset('A')
>>> A.manifold()
2-dimensional topological manifold M
>>> A.manifold() is M
True
>>> B = A.subset('B')
>>> B.manifold() is M
True
```

An alias is `ambient`:

```
sage: A.ambient() is A.manifold()
True
```

```
>>> from sage.all import *
>>> A.ambient() is A.manifold()
True
```

closure(*name=None*, *latex_name=None*)

Return the topological closure of `self` as a subset of the manifold.

INPUT:

- `name` – (default: `None`) name given to the difference in the case the latter has to be created; the default prepends `cl_` to `self._name`
- `latex_name` – (default: `None`) LaTeX symbol to denote the difference in the case the latter has to be created; the default is built upon the operator cl

OUTPUT:

- if `self` is already known to be closed (see *is_closed()*), `self`; otherwise, an instance of *ManifoldSubsetClosure*

EXAMPLES:

```
sage: M = Manifold(2, 'R^2', structure='topological')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: M.closure() is M
True
sage: D2 = M.open_subset('D2', coord_def={c_cart: x^2+y^2<2}); D2
Open subset D2 of the 2-dimensional topological manifold R^2
sage: cl_D2 = D2.closure(); cl_D2
Topological closure cl_D2 of the
 Open subset D2 of the 2-dimensional topological manifold R^2
sage: cl_D2.is_closed()
True
sage: cl_D2 is cl_D2.closure()
True

sage: D1 = D2.open_subset('D1'); D1
Open subset D1 of the 2-dimensional topological manifold R^2
sage: D1.closure().is_subset(D2.closure())
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', structure='topological')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#
↪Cartesian coordinates on R^2
>>> M.closure() is M
True
>>> D2 = M.open_subset('D2', coord_def={c_cart: x**Integer(2)+y**Integer(2)
↪<Integer(2)}); D2
Open subset D2 of the 2-dimensional topological manifold R^2
>>> cl_D2 = D2.closure(); cl_D2
Topological closure cl_D2 of the
 Open subset D2 of the 2-dimensional topological manifold R^2
>>> cl_D2.is_closed()
True
```

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```
>>> cl_D2 is cl_D2.closure()
True

>>> D1 = D2.open_subset('D1'); D1
Open subset D1 of the 2-dimensional topological manifold R^2
>>> D1.closure().is_subset(D2.closure())
True
```

complement(*superset=None*, *name=None*, *latex_name=None*, *is_open=False*)

Return the complement of `self` in the manifold or in `superset`.

INPUT:

- `superset` – (default: `self.manifold()`) a superset of `self`
- `name` – (default: `None`) name given to the complement in the case the latter has to be created; the default is `superset._name` minus `self._name`
- `latex_name` – (default: `None`) LaTeX symbol to denote the complement in the case the latter has to be created; the default is built upon the symbol \
- `is_open` – boolean (default: `False`); if `True`, the created subset is assumed to be open with respect to the manifold's topology

OUTPUT:

- instance of *ManifoldSubset* representing the subset that is `superset` minus `self`

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: A = M.subset('A')
sage: B1 = A.subset('B1')
sage: B2 = A.subset('B2')
sage: B1.complement()
Subset M_minus_B1 of the 2-dimensional topological manifold M
sage: B1.complement(A)
Subset A_minus_B1 of the 2-dimensional topological manifold M
sage: B1.complement(B2)
Traceback (most recent call last):
...
TypeError: superset must be a superset of self
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> A = M.subset('A')
>>> B1 = A.subset('B1')
>>> B2 = A.subset('B2')
>>> B1.complement()
Subset M_minus_B1 of the 2-dimensional topological manifold M
>>> B1.complement(A)
Subset A_minus_B1 of the 2-dimensional topological manifold M
>>> B1.complement(B2)
Traceback (most recent call last):
...
TypeError: superset must be a superset of self
```

Demanding that the complement is open makes `self` a closed subset:

```
sage: A.is_closed()  # False a priori
False
sage: A.complement(is_open=True)
Open subset M_minus_A of the 2-dimensional topological manifold M
sage: A.is_closed()
True
```

```
>>> from sage.all import *
>>> A.is_closed()  # False a priori
False
>>> A.complement(is_open=True)
Open subset M_minus_A of the 2-dimensional topological manifold M
>>> A.is_closed()
True
```

declare_closed()

Declare `self` to be a closed subset of the manifold.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: A = M.subset('A')
sage: B1 = A.subset('B1')
sage: B1.is_closed()
False
sage: B1.declare_closed()
sage: B1.is_closed()
True

sage: B2 = A.subset('B2')
sage: cl_B2 = B2.closure()
sage: A.declare_closed()
sage: cl_B2.is_subset(A)
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> A = M.subset('A')
>>> B1 = A.subset('B1')
>>> B1.is_closed()
False
>>> B1.declare_closed()
>>> B1.is_closed()
True

>>> B2 = A.subset('B2')
>>> cl_B2 = B2.closure()
>>> A.declare_closed()
>>> cl_B2.is_subset(A)
True
```

declare_empty()

Declare that `self` is the empty set.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: A = M.subset('A', is_open=True)
sage: AA = A.subset('AA')
sage: A
Open subset A of the 2-dimensional topological manifold M
sage: A.declare_empty()
sage: A.is_empty()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> A = M.subset('A', is_open=True)
>>> AA = A.subset('AA')
>>> A
Open subset A of the 2-dimensional topological manifold M
>>> A.declare_empty()
>>> A.is_empty()
True
```

Empty sets do not allow to define points on them:

```
sage: A.point()
Traceback (most recent call last):
...
TypeError: cannot define a point on the
  Open subset A of the 2-dimensional topological manifold M
  because it has been declared empty
```

```
>>> from sage.all import *
>>> A.point()
Traceback (most recent call last):
...
TypeError: cannot define a point on the
  Open subset A of the 2-dimensional topological manifold M
  because it has been declared empty
```

Emptiness transfers to subsets:

```
sage: AA.is_empty()
True
sage: AA.point()
Traceback (most recent call last):
...
TypeError: cannot define a point on the
  Subset AA of the 2-dimensional topological manifold M
  because it has been declared empty
sage: AD = A.subset('AD')
sage: AD.is_empty()
True
```

```
>>> from sage.all import *
>>> AA.is_empty()
True
>>> AA.point()
Traceback (most recent call last):
...
TypeError: cannot define a point on the
  Subset AA of the 2-dimensional topological manifold M
  because it has been declared empty
>>> AD = A.subset('AD')
>>> AD.is_empty()
True
```

If points have already been defined on `self` (or its subsets), it is an error to declare it to be empty:

```
sage: B = M.subset('B')
sage: b = B.point(name='b'); b
Point b on the 2-dimensional topological manifold M
sage: B.declare_empty()
Traceback (most recent call last):
...
TypeError: cannot be empty because it has defined points
```

```
>>> from sage.all import *
>>> B = M.subset('B')
>>> b = B.point(name='b'); b
Point b on the 2-dimensional topological manifold M
>>> B.declare_empty()
Traceback (most recent call last):
...
TypeError: cannot be empty because it has defined points
```

Emptiness is recorded as empty open covers:

```
sage: P = M.subset_poset(open_covers=True, points=[b])                    #␣
↪needs sage.graphs
sage: def label(element):
....:     if isinstance(element, str):
....:         return element
....:     try:
....:         return element._name
....:     except AttributeError:
....:         return '[' + ', '.join(sorted(x._name for x in element)) + ']'
sage: P.plot(element_labels={element: label(element) for element in P})   #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 10 graphics primitives
```

```
>>> from sage.all import *
>>> P = M.subset_poset(open_covers=True, points=[b])                    #␣
↪needs sage.graphs
>>> def label(element):
...     if isinstance(element, str):
...         return element
```

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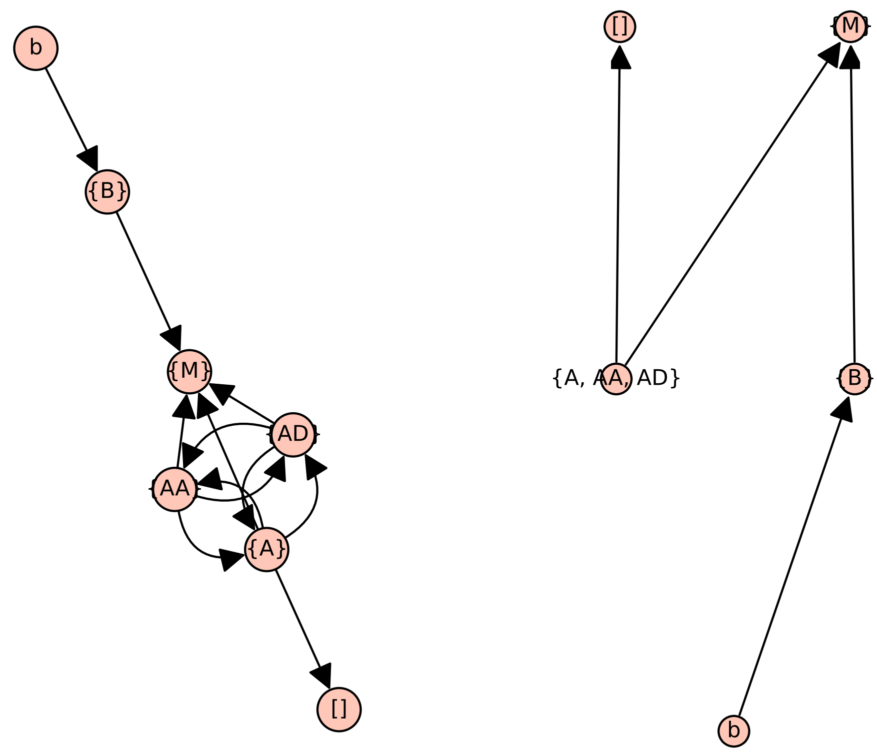
```
...     try:
...         return element._name
...     except AttributeError:
...         return '[' + ', '.join(sorted(x._name for x in element)) + ']'
>>> P.plot(element_labels={element: label(element) for element in P})      #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 10 graphics primitives
```



declare_equal(**others*)

Declare that `self` and `others` are the same sets.

INPUT:

- `others` – finitely many subsets or iterables of subsets of the same manifold as `self`

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U')
sage: V = M.open_subset('V')
sage: Vs = [M.open_subset(f'V{i}') for i in range(2)]
sage: UV = U.intersection(V)
sage: W = UV.open_subset('W')
sage: P = M.subset_poset()                                                  #␣
↪needs sage.graphs
sage: def label(element):
....:     return element._name
sage: P.plot(element_labels={element: label(element) for element in P})      #␣
```

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```
↪needs sage.graphs sage.plot
Graphics object consisting of 15 graphics primitives
sage: V.declare_equal(Vs)
sage: P = M.subset_poset()                                                    #␣
↪needs sage.graphs
sage: P.plot(element_labels={element: label(element) for element in P})      #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 11 graphics primitives
sage: W.declare_equal(U)
sage: P = M.subset_poset()                                                    #␣
↪needs sage.graphs
sage: P.plot(element_labels={element: label(element) for element in P})      #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 6 graphics primitives
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U')
>>> V = M.open_subset('V')
>>> Vs = [M.open_subset(f'V{i}') for i in range(Integer(2))]
>>> UV = U.intersection(V)
>>> W = UV.open_subset('W')
>>> P = M.subset_poset()                                                    #␣
↪needs sage.graphs
>>> def label(element):
...     return element._name
>>> P.plot(element_labels={element: label(element) for element in P})      #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 15 graphics primitives
>>> V.declare_equal(Vs)
>>> P = M.subset_poset()                                                    #␣
↪needs sage.graphs
>>> P.plot(element_labels={element: label(element) for element in P})      #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 11 graphics primitives
>>> W.declare_equal(U)
>>> P = M.subset_poset()                                                    #␣
↪needs sage.graphs
>>> P.plot(element_labels={element: label(element) for element in P})      #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 6 graphics primitives
```

declare_nonempty()

Declare that `self` is nonempty.

Once declared nonempty, `self` (or any of its supersets) cannot be declared empty.

This is equivalent to defining a point on `self` using *point()* but is cheaper than actually creating a *ManifoldPoint* instance.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
```

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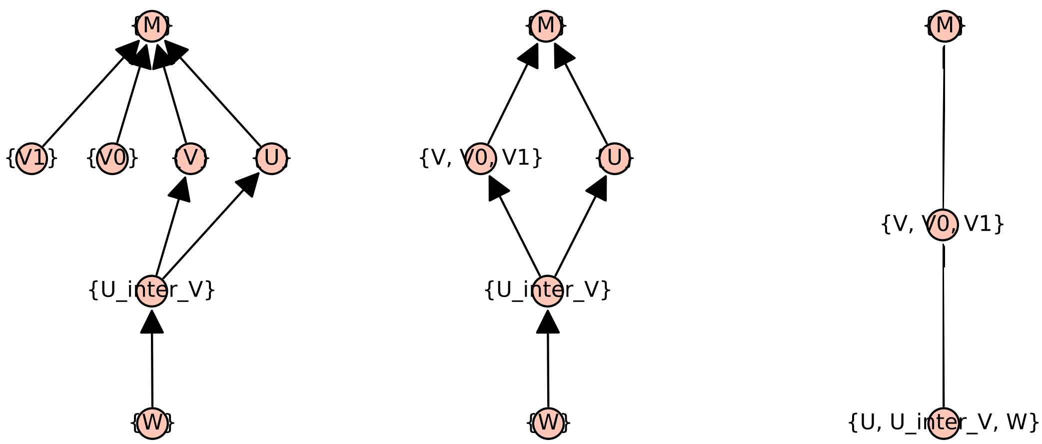

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```
sage: A = M.subset('A', is_open=True)
sage: AA = A.subset('AA')
sage: AA.declare_nonempty()
sage: A.has_defined_points()
True
sage: A.declare_empty()
Traceback (most recent call last):
...
TypeError: cannot be empty because it has defined points
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> A = M.subset('A', is_open=True)
>>> AA = A.subset('AA')
>>> AA.declare_nonempty()
>>> A.has_defined_points()
True
>>> A.declare_empty()
Traceback (most recent call last):
...
TypeError: cannot be empty because it has defined points
```

declare_subset(**supersets*)

Declare `self` to be a subset of each of the given supersets.

INPUT:

- `supersets` – other subsets of the same manifold

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: U1 = M.open_subset('U1')
sage: U2 = M.open_subset('U2')
sage: V = M.open_subset('V')
sage: V.superset_family()
Set {M, V} of open subsets of the 2-dimensional differentiable manifold M
```

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```
sage: U1.subset_family()
Set {U1} of open subsets of the 2-dimensional differentiable manifold M
sage: P = M.subset_poset()                                                    #␣
↪needs sage.graphs
sage: def label(element):
....:     return element._name
sage: P.plot(element_labels={element: label(element) for element in P})      #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 8 graphics primitives
sage: V.declare_subset(U1, U2)
sage: V.superset_family()
Set {M, U1, U2, V} of open subsets of the 2-dimensional differentiable␣
↪manifold M
sage: P = M.subset_poset()                                                    #␣
↪needs sage.graphs
sage: P.plot(element_labels={element: label(element) for element in P})      #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 9 graphics primitives
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U1 = M.open_subset('U1')
>>> U2 = M.open_subset('U2')
>>> V = M.open_subset('V')
>>> V.superset_family()
Set {M, V} of open subsets of the 2-dimensional differentiable manifold M
>>> U1.subset_family()
Set {U1} of open subsets of the 2-dimensional differentiable manifold M
>>> P = M.subset_poset()                                                    #␣
↪needs sage.graphs
>>> def label(element):
...     return element._name
>>> P.plot(element_labels={element: label(element) for element in P})      #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 8 graphics primitives
>>> V.declare_subset(U1, U2)
>>> V.superset_family()
Set {M, U1, U2, V} of open subsets of the 2-dimensional differentiable␣
↪manifold M
>>> P = M.subset_poset()                                                    #␣
↪needs sage.graphs
>>> P.plot(element_labels={element: label(element) for element in P})      #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 9 graphics primitives
```

Subsets in a directed cycle of inclusions are equal:

```
sage: M.declare_subset(V)
sage: M.superset_family()
Set {M, U1, U2, V} of open subsets of the 2-dimensional differentiable␣
↪manifold M
sage: M.equal_subset_family()
```

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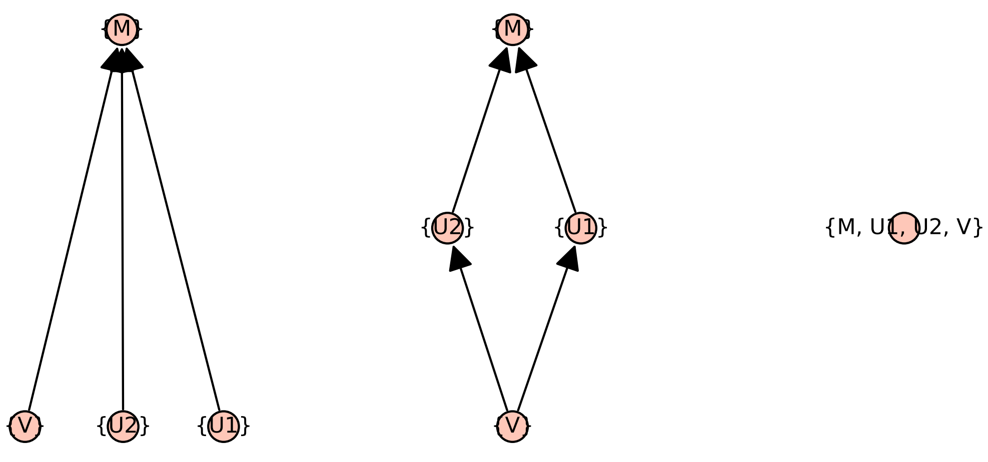
```
Set {M, U1, U2, V} of open subsets of the 2-dimensional differentiable manifold M
sage: P = M.subset_poset()                                                    # needs sage.graphs
sage: P.plot(element_labels={element: label(element) for element in P})      # needs sage.graphs sage.plot
Graphics object consisting of 2 graphics primitives
```

```
>>> from sage.all import *
>>> M.declare_subset(V)
>>> M.superset_family()
Set {M, U1, U2, V} of open subsets of the 2-dimensional differentiable manifold M
>>> M.equal_subset_family()
Set {M, U1, U2, V} of open subsets of the 2-dimensional differentiable manifold M
>>> P = M.subset_poset()                                                    # needs sage.graphs
>>> P.plot(element_labels={element: label(element) for element in P})      # needs sage.graphs sage.plot
Graphics object consisting of 2 graphics primitives
```



declare_superset(**subsets*)

Declare `self` to be a superset of each of the given subsets.

INPUT:

- `subsets` – other subsets of the same manifold

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U')
sage: V1 = M.open_subset('V1')
sage: V2 = M.open_subset('V2')
sage: W = V1.intersection(V2)
sage: U.subset_family()
```

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```
Set {U} of open subsets of the 2-dimensional differentiable manifold M
sage: P = M.subset_poset()                                              #␣
↪needs sage.graphs
sage: def label(element):
....:     return element._name
sage: P.plot(element_labels={element: label(element) for element in P})     #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 11 graphics primitives
sage: U.declare_superset(V1, V2)
sage: U.subset_family()
Set {U, V1, V1_inter_V2, V2} of open subsets of the 2-dimensional␣
↪differentiable manifold M
sage: P = M.subset_poset()                                              #␣
↪needs sage.graphs
sage: P.plot(element_labels={element: label(element) for element in P})     #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 11 graphics primitives
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U')
>>> V1 = M.open_subset('V1')
>>> V2 = M.open_subset('V2')
>>> W = V1.intersection(V2)
>>> U.subset_family()
Set {U} of open subsets of the 2-dimensional differentiable manifold M
>>> P = M.subset_poset()                                              #␣
↪needs sage.graphs
>>> def label(element):
...     return element._name
>>> P.plot(element_labels={element: label(element) for element in P})     #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 11 graphics primitives
>>> U.declare_superset(V1, V2)
>>> U.subset_family()
Set {U, V1, V1_inter_V2, V2} of open subsets of the 2-dimensional␣
↪differentiable manifold M
>>> P = M.subset_poset()                                              #␣
↪needs sage.graphs
>>> P.plot(element_labels={element: label(element) for element in P})     #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 11 graphics primitives
```

Subsets in a directed cycle of inclusions are equal:

```
sage: W.declare_superset(U)
sage: W.subset_family()
Set {U, V1, V1_inter_V2, V2} of open subsets of the 2-dimensional␣
↪differentiable manifold M
sage: W.equal_subset_family()
Set {U, V1, V1_inter_V2, V2} of open subsets of the 2-dimensional␣
↪differentiable manifold M
```

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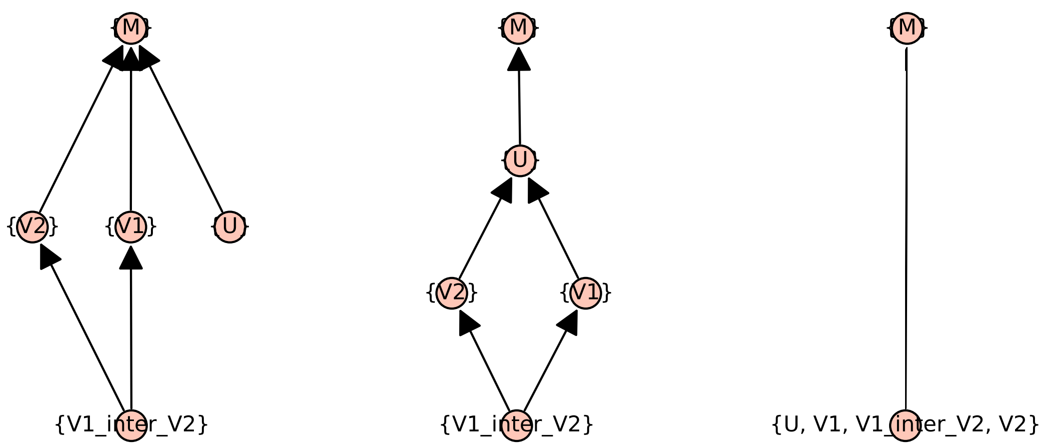
```
sage: P = M.subset_poset()                                                   #␣
↪needs sage.graphs
sage: P.plot(element_labels={element: label(element) for element in P})      #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 4 graphics primitives
```

```
>>> from sage.all import *
>>> W.declare_superset(U)
>>> W.subset_family()
Set {U, V1, V1_inter_V2, V2} of open subsets of the 2-dimensional␣
↪differentiable manifold M
>>> W.equal_subset_family()
Set {U, V1, V1_inter_V2, V2} of open subsets of the 2-dimensional␣
↪differentiable manifold M
>>> P = M.subset_poset()                                                   #␣
↪needs sage.graphs
>>> P.plot(element_labels={element: label(element) for element in P})      #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 4 graphics primitives
```



declare_union(*disjoint*, **subsets_or_families*)

Declare that the current subset is the union of two subsets.

Suppose U is the current subset, then this method declares that $U = \bigcup_{S \in F} S$.

INPUT:

- `subsets_or_families` – finitely many subsets or iterables of subsets
- `disjoint` – boolean (default: `False`); whether to declare the subsets pairwise disjoint

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: AB = M.subset('AB')
sage: A = AB.subset('A')
sage: B = AB.subset('B')
sage: def label(element):
```

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```
....:     try:
....:         return element._name
....:     except AttributeError:
....:         return '[' + ', '.join(sorted(x._name for x in element)) + ']'
sage: P = M.subset_poset(open_covers=True); P                                  #␣
↪needs sage.graphs
Finite poset containing 4 elements
sage: P.plot(element_labels={element: label(element) for element in P})      #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 8 graphics primitives

sage: AB.declare_union(A, B)
sage: A.union(B)
Subset AB of the 2-dimensional topological manifold M
sage: P = M.subset_poset(open_covers=True); P                                  #␣
↪needs sage.graphs
Finite poset containing 4 elements
sage: P.plot(element_labels={element: label(element) for element in P})      #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 8 graphics primitives

sage: B1 = B.subset('B1', is_open=True)
sage: B2 = B.subset('B2', is_open=True)
sage: B.declare_union(B1, B2, disjoint=True)
sage: P = M.subset_poset(open_covers=True); P                                  #␣
↪needs sage.graphs
Finite poset containing 9 elements
sage: P.plot(element_labels={element: label(element) for element in P})      #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 19 graphics primitives
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> AB = M.subset('AB')
>>> A = AB.subset('A')
>>> B = AB.subset('B')
>>> def label(element):
...     try:
...         return element._name
...     except AttributeError:
...         return '[' + ', '.join(sorted(x._name for x in element)) + ']'
>>> P = M.subset_poset(open_covers=True); P                                  #␣
↪needs sage.graphs
Finite poset containing 4 elements
>>> P.plot(element_labels={element: label(element) for element in P})      #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 8 graphics primitives

>>> AB.declare_union(A, B)
>>> A.union(B)
Subset AB of the 2-dimensional topological manifold M
>>> P = M.subset_poset(open_covers=True); P                                  #␣
```

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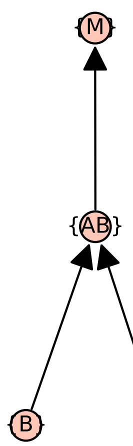
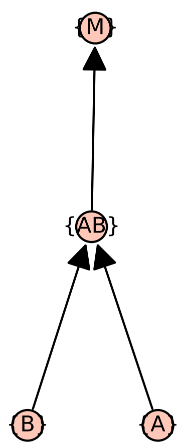
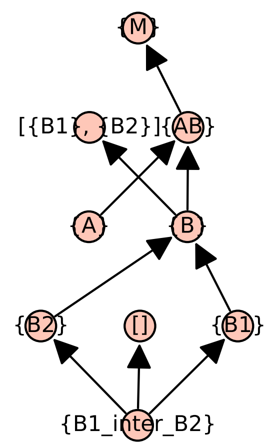
```
    needs sage.graphs
Finite poset containing 4 elements
>>> P.plot(element_labels={element: label(element) for element in P})      # needs sage.graphs sage.plot
Graphics object consisting of 8 graphics primitives

>>> B1 = B.subset('B1', is_open=True)
>>> B2 = B.subset('B2', is_open=True)
>>> B.declare_union(B1, B2, disjoint=True)
>>> P = M.subset_poset(open_covers=True); P                                  # needs sage.graphs
Finite poset containing 9 elements
>>> P.plot(element_labels={element: label(element) for element in P})      # needs sage.graphs sage.plot
Graphics object consisting of 19 graphics primitives
```







difference(*other*, *name=None*, *latex_name=None*, *is_open=False*)

Return the set difference of `self` minus `other`.

INPUT:

- `other` – another subset of the same manifold
- `name` – (default: `None`) name given to the difference in the case the latter has to be created; the default is `self._name` minus `other._name`
- `latex_name` – (default: `None`) LaTeX symbol to denote the difference in the case the latter has to be created; the default is built upon the symbol $\setminus$
- `is_open` – boolean (default: `False`); if `True`, the created subset is assumed to be open with respect to the manifold's topology

OUTPUT:

- instance of `ManifoldSubset` representing the subset that is `self` minus `other`

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: A = M.subset('A')
```

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```
sage: CA = M.difference(A); CA
Subset M_minus_A of the 2-dimensional topological manifold M
sage: latex(CA)
M\setminus A
sage: A.intersection(CA).is_empty()
True
sage: A.union(CA)
2-dimensional topological manifold M

sage: O = M.open_subset('O')
sage: CO = M.difference(O); CO
Subset M_minus_O of the 2-dimensional topological manifold M
sage: M.difference(O) is CO
True

sage: CO2 = M.difference(O, is_open=True, name='CO2'); CO2
Open subset CO2 of the 2-dimensional topological manifold M
sage: CO is CO2
False
sage: CO.is_subset(CO2) and CO2.is_subset(CO)
True
sage: M.difference(O, is_open=True)
Open subset CO2 of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> A = M.subset('A')
>>> CA = M.difference(A); CA
Subset M_minus_A of the 2-dimensional topological manifold M
>>> latex(CA)
M\setminus A
>>> A.intersection(CA).is_empty()
True
>>> A.union(CA)
2-dimensional topological manifold M

>>> O = M.open_subset('O')
>>> CO = M.difference(O); CO
Subset M_minus_O of the 2-dimensional topological manifold M
>>> M.difference(O) is CO
True

>>> CO2 = M.difference(O, is_open=True, name='CO2'); CO2
Open subset CO2 of the 2-dimensional topological manifold M
>>> CO is CO2
False
>>> CO.is_subset(CO2) and CO2.is_subset(CO)
True
>>> M.difference(O, is_open=True)
Open subset CO2 of the 2-dimensional topological manifold M
```

Since O is open and we have asked $M \setminus O$ to be open, O is a clopen set (if $O \neq M$ and $O \neq \emptyset$, this implies that M is not connected):

```
sage: O.is_closed() and O.is_open()
True
```

```
>>> from sage.all import *
>>> O.is_closed() and O.is_open()
True
```

equal_subset_family()

Generate the declared manifold subsets that are equal to `self`.

> **Note**
>
> If you only need to iterate over the equal sets in arbitrary order, you can use the generator method *equal_subsets()* instead.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: U = M.open_subset('U')
sage: V = U.subset('V')
sage: V.declare_equal(M)
sage: V.equal_subset_family()
Set {M, U, V} of subsets of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> U = M.open_subset('U')
>>> V = U.subset('V')
>>> V.declare_equal(M)
>>> V.equal_subset_family()
Set {M, U, V} of subsets of the 2-dimensional topological manifold M
```

equal_subsets()

Generate the declared manifold subsets that are equal to `self`.

> **Note**
>
> To get the equal subsets as a family, sorted by name, use the method *equal_subset_family()* instead.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: U = M.open_subset('U')
sage: V = U.subset('V')
sage: V.declare_equal(M)
sage: sorted(V.equal_subsets(), key=lambda v: v._name)
[2-dimensional topological manifold M,
 Open subset U of the 2-dimensional topological manifold M,
 Subset V of the 2-dimensional topological manifold M]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> U = M.open_subset('U')
>>> V = U.subset('V')
>>> V.declare_equal(M)
>>> sorted(V.equal_subsets(), key=lambda v: v._name)
[2-dimensional topological manifold M,
 Open subset U of the 2-dimensional topological manifold M,
 Subset V of the 2-dimensional topological manifold M]
```

get_subset(*name*)

Get a subset by its name.

The subset must have been previously created by the method `subset()` (or `open_subset()`)

INPUT:

- `name` – string; name of the subset

OUTPUT:

- instance of `ManifoldSubset` (or of the derived class `TopologicalManifold` for an open subset) representing the subset whose name is `name`

EXAMPLES:

```
sage: M = Manifold(4, 'M', structure='topological')
sage: A = M.subset('A')
sage: B = A.subset('B')
sage: U = M.open_subset('U')
sage: M.subset_family()
Set {A, B, M, U} of subsets of the 4-dimensional topological manifold M
sage: M.get_subset('A')
Subset A of the 4-dimensional topological manifold M
sage: M.get_subset('A') is A
True
sage: M.get_subset('B') is B
True
sage: A.get_subset('B') is B
True
sage: M.get_subset('U')
Open subset U of the 4-dimensional topological manifold M
sage: M.get_subset('U') is U
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='topological')
>>> A = M.subset('A')
>>> B = A.subset('B')
>>> U = M.open_subset('U')
>>> M.subset_family()
Set {A, B, M, U} of subsets of the 4-dimensional topological manifold M
>>> M.get_subset('A')
Subset A of the 4-dimensional topological manifold M
>>> M.get_subset('A') is A
```

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```
True
>>> M.get_subset('B') is B
True
>>> A.get_subset('B') is B
True
>>> M.get_subset('U')
Open subset U of the 4-dimensional topological manifold M
>>> M.get_subset('U') is U
True
```

has_defined_points(*subsets=True*)

Return whether any points have been defined on `self` or any of its subsets.

INPUT:

- `subsets` – boolean (default: `True`); if `False`, only consider points that have been defined directly on `self`. If `True`, also consider points on all subsets.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: A = M.subset('A', is_open=True)
sage: AA = A.subset('AA')
sage: AA.point()
Point on the 2-dimensional topological manifold M
sage: AA.has_defined_points()
True
sage: A.has_defined_points(subsets=False)
False
sage: A.has_defined_points()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> A = M.subset('A', is_open=True)
>>> AA = A.subset('AA')
>>> AA.point()
Point on the 2-dimensional topological manifold M
>>> AA.has_defined_points()
True
>>> A.has_defined_points(subsets=False)
False
>>> A.has_defined_points()
True
```

intersection(*name*, *latex_name*, **others*)

Return the intersection of the current subset with other subsets.

This method may return a previously constructed intersection instead of creating a new subset. In this case, `name` and `latex_name` are not used.

INPUT:

- `others` – other subsets of the same manifold

- `name` – (default: `None`) name given to the intersection in the case the latter has to be created; the default is `self._name` inter `other._name`
- `latex_name` – (default: `None`) LaTeX symbol to denote the intersection in the case the latter has to be created; the default is built upon the symbol $\cap$

OUTPUT:

- instance of `ManifoldSubset` representing the subset that is the intersection of the current subset with `others`

EXAMPLES:

Intersection of two subsets:

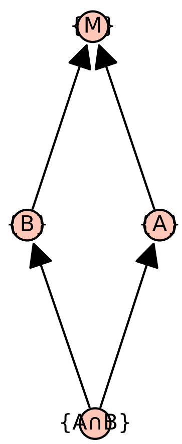
```
sage: M = Manifold(2, 'M', structure='topological')
sage: a = M.subset('A')
sage: b = M.subset('B')
sage: c = a.intersection(b); c
Subset A_inter_B of the 2-dimensional topological manifold M
sage: a.subset_family()
Set {A, A_inter_B} of subsets of the 2-dimensional topological manifold M
sage: b.subset_family()
Set {A_inter_B, B} of subsets of the 2-dimensional topological manifold M
sage: c.superset_family()
Set {A, A_inter_B, B, M} of subsets of the 2-dimensional topological manifold␣
↪M
```

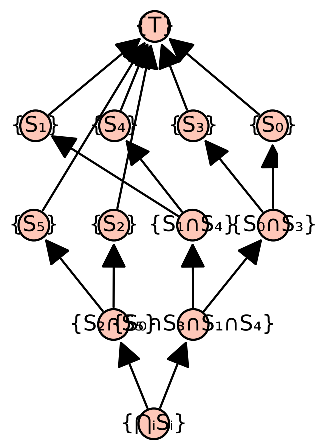
```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> a = M.subset('A')
>>> b = M.subset('B')
>>> c = a.intersection(b); c
Subset A_inter_B of the 2-dimensional topological manifold M
>>> a.subset_family()
Set {A, A_inter_B} of subsets of the 2-dimensional topological manifold M
>>> b.subset_family()
Set {A_inter_B, B} of subsets of the 2-dimensional topological manifold M
>>> c.superset_family()
Set {A, A_inter_B, B, M} of subsets of the 2-dimensional topological manifold␣
↪M
```

Intersection of six subsets:

```
sage: T = Manifold(2, 'T', structure='topological')
sage: S = [T.subset(f'S{i}') for i in range(6)]
sage: [S[i].intersection(S[i+3]) for i in range(3)]
[Subset S0_inter_S3 of the 2-dimensional topological manifold T,
 Subset S1_inter_S4 of the 2-dimensional topological manifold T,
 Subset S2_inter_S5 of the 2-dimensional topological manifold T]
sage: inter_S_i = T.intersection(*S, name='inter_S_i'); inter_S_i
Subset inter_S_i of the 2-dimensional topological manifold T
sage: inter_S_i.superset_family()
Set {S0, S0_inter_S3, S0_inter_S3_inter_S1_inter_S4, S1, S1_inter_S4,
     S2, S2_inter_S5, S3, S4, S5, T, inter_S_i} of
 subsets of the 2-dimensional topological manifold T
```

```
>>> from sage.all import *
>>> T = Manifold(Integer(2), 'T', structure='topological')
>>> S = [T.subset(f'S{i}') for i in range(Integer(6))]
>>> [S[i].intersection(S[i+Integer(3)]) for i in range(Integer(3))]
[Subset S0_inter_S3 of the 2-dimensional topological manifold T,
 Subset S1_inter_S4 of the 2-dimensional topological manifold T,
 Subset S2_inter_S5 of the 2-dimensional topological manifold T]
>>> inter_S_i = T.intersection(*S, name='inter_S_i'); inter_S_i
Subset inter_S_i of the 2-dimensional topological manifold T
>>> inter_S_i.superset_family()
Set {S0, S0_inter_S3, S0_inter_S3_inter_S1_inter_S4, S1, S1_inter_S4,
     S2, S2_inter_S5, S3, S4, S5, T, inter_S_i} of
 subsets of the 2-dimensional topological manifold T
```





is_closed()

Return if `self` is a closed set.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: M.is_closed()
True
sage: also_M = M.subset('also_M')
sage: M.declare_subset(also_M)
sage: also_M.is_closed()
True

sage: A = M.subset('A')
sage: A.is_closed()
False
sage: A.declare_empty()
sage: A.is_closed()
True

sage: N = M.open_subset('N')
sage: N.is_closed()
False
```

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```
sage: complement_N = M.subset('complement_N')
sage: M.declare_union(N, complement_N, disjoint=True)
sage: complement_N.is_closed()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> M.is_closed()
True
>>> also_M = M.subset('also_M')
>>> M.declare_subset(also_M)
>>> also_M.is_closed()
True

>>> A = M.subset('A')
>>> A.is_closed()
False
>>> A.declare_empty()
>>> A.is_closed()
True

>>> N = M.open_subset('N')
>>> N.is_closed()
False
>>> complement_N = M.subset('complement_N')
>>> M.declare_union(N, complement_N, disjoint=True)
>>> complement_N.is_closed()
True
```

is_empty()

Return whether the current subset is empty.

By default, manifold subsets are considered nonempty: The method *point()* can be used to define points on it, either with or without coordinates some chart.

However, using *declare_empty()*, a subset can be declared empty, and emptiness transfers to all of its subsets.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: A = M.subset('A', is_open=True)
sage: AA = A.subset('AA')
sage: A.is_empty()
False
sage: A.declare_empty()
sage: A.is_empty()
True
sage: AA.is_empty()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
```

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```
>>> A = M.subset('A', is_open=True)
>>> AA = A.subset('AA')
>>> A.is_empty()
False
>>> A.declare_empty()
>>> A.is_empty()
True
>>> AA.is_empty()
True
```

is_open()

Return if `self` is an open set.

This method always returns `False`, since open subsets must be constructed as instances of the subclass *TopologicalManifold* (which redefines `is_open`)

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: A = M.subset('A')
sage: A.is_open()
False
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> A = M.subset('A')
>>> A.is_open()
False
```

is_subset(*other*)

Return `True` if and only if `self` is included in `other`.

EXAMPLES:

Subsets on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: a = M.subset('A')
sage: b = a.subset('B')
sage: c = M.subset('C')
sage: a.is_subset(M)
True
sage: b.is_subset(a)
True
sage: b.is_subset(M)
True
sage: a.is_subset(b)
False
sage: c.is_subset(a)
False
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
```

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```
>>> a = M.subset('A')
>>> b = a.subset('B')
>>> c = M.subset('C')
>>> a.is_subset(M)
True
>>> b.is_subset(a)
True
>>> b.is_subset(M)
True
>>> a.is_subset(b)
False
>>> c.is_subset(a)
False
```

lift(*p*)

Return the lift of `p` to the ambient manifold of `self`.

INPUT:

- `p` – point of the subset

OUTPUT: the same point, considered as a point of the ambient manifold

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: A = M.open_subset('A', coord_def={X: x>0})
sage: p = A((1, -2)); p
Point on the 2-dimensional topological manifold M
sage: p.parent()
Open subset A of the 2-dimensional topological manifold M
sage: q = A.lift(p); q
Point on the 2-dimensional topological manifold M
sage: q.parent()
2-dimensional topological manifold M
sage: q.coord()
(1, -2)
sage: (p == q) and (q == p)
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> A = M.open_subset('A', coord_def={X: x>Integer(0)})
>>> p = A((Integer(1), -Integer(2))); p
Point on the 2-dimensional topological manifold M
>>> p.parent()
Open subset A of the 2-dimensional topological manifold M
>>> q = A.lift(p); q
Point on the 2-dimensional topological manifold M
>>> q.parent()
2-dimensional topological manifold M
>>> q.coord()
```

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```
(1, -2)
>>> (p == q) and (q == p)
True
```

list_of_subsets()

Return the list of subsets that have been defined on the current subset.

The list is sorted by the alphabetical names of the subsets.

OUTPUT:

- a list containing all the subsets that have been defined on the current subset

> **Note**
>
> This method is deprecated.
>
> To get the subsets as a *ManifoldSubsetFiniteFamily* instance (which sorts its elements alphabetically by name), use *subset_family()* instead.
>
> To loop over the subsets in an arbitrary order, use the generator method *subsets()* instead.

EXAMPLES:

List of subsets of a 2-dimensional manifold (deprecated):

```
sage: M = Manifold(2, 'M', structure='topological')
sage: U = M.open_subset('U')
sage: V = M.subset('V')
sage: M.list_of_subsets()
doctest:...: DeprecationWarning: the method list_of_subsets of ManifoldSubset
 is deprecated; use subset_family or subsets instead...
[2-dimensional topological manifold M,
 Open subset U of the 2-dimensional topological manifold M,
 Subset V of the 2-dimensional topological manifold M]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> U = M.open_subset('U')
>>> V = M.subset('V')
>>> M.list_of_subsets()
doctest:...: DeprecationWarning: the method list_of_subsets of ManifoldSubset
 is deprecated; use subset_family or subsets instead...
[2-dimensional topological manifold M,
 Open subset U of the 2-dimensional topological manifold M,
 Subset V of the 2-dimensional topological manifold M]
```

Using *subset_family()* instead (recommended when order matters):

```
sage: M.subset_family()
Set {M, U, V} of subsets of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M.subset_family()
Set {M, U, V} of subsets of the 2-dimensional topological manifold M
```

The method `subsets()` generates the subsets in an unspecified order. To create a set:

```
sage: frozenset(M.subsets())  # random (set output)
{Subset V of the 2-dimensional topological manifold M,
 2-dimensional topological manifold M,
 Open subset U of the 2-dimensional topological manifold M}
```

```
>>> from sage.all import *
>>> frozenset(M.subsets())  # random (set output)
{Subset V of the 2-dimensional topological manifold M,
 2-dimensional topological manifold M,
 Open subset U of the 2-dimensional topological manifold M}
```

manifold()

Return the ambient manifold of `self`.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: A = M.subset('A')
sage: A.manifold()
2-dimensional topological manifold M
sage: A.manifold() is M
True
sage: B = A.subset('B')
sage: B.manifold() is M
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> A = M.subset('A')
>>> A.manifold()
2-dimensional topological manifold M
>>> A.manifold() is M
True
>>> B = A.subset('B')
>>> B.manifold() is M
True
```

An alias is `ambient`:

```
sage: A.ambient() is A.manifold()
True
```

```
>>> from sage.all import *
>>> A.ambient() is A.manifold()
True
```

open_cover_family(*trivial=True*, *supersets=False*)

Return the family of open covers of the current subset.

If the current subset, A say, is a subset of the manifold M, an *open cover* of A is a `ManifoldSubsetFiniteFamily` F of open subsets $U \in F$ of M such that

$$A \subset \bigcup_{U \in F} U.$$

If A is open, we ask that the above inclusion is actually an identity:

$$A = \bigcup_{U \in F} U.$$

The family is sorted lexicographically by the names of the subsets forming the open covers.

Note

If you only need to iterate over the open covers in arbitrary order, you can use the generator method `open_covers()` instead.

INPUT:

- `trivial` – boolean (default: `True`); if `self` is open, include the trivial open cover of `self` by itself
- `supersets` – boolean (default: `False`); if `True`, include open covers of all the supersets. It can also be an iterable of supersets to include

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: M.open_cover_family()
Set {{M}} of objects of the 2-dimensional topological manifold M
sage: U = M.open_subset('U')
sage: U.open_cover_family()
Set {{U}} of objects of the 2-dimensional topological manifold M
sage: A = U.open_subset('A')
sage: B = U.open_subset('B')
sage: U.declare_union(A,B)
sage: U.open_cover_family()
Set {{A, B}, {U}} of objects of the 2-dimensional topological manifold M
sage: U.open_cover_family(trivial=False)
Set {{A, B}} of objects of the 2-dimensional topological manifold M
sage: V = M.open_subset('V')
sage: M.declare_union(U,V)
sage: M.open_cover_family()
Set {{A, B, V}, {M}, {U, V}} of objects of the 2-dimensional topological
↪manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> M.open_cover_family()
Set {{M}} of objects of the 2-dimensional topological manifold M
>>> U = M.open_subset('U')
>>> U.open_cover_family()
```

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```
Set {{U}} of objects of the 2-dimensional topological manifold M
>>> A = U.open_subset('A')
>>> B = U.open_subset('B')
>>> U.declare_union(A,B)
>>> U.open_cover_family()
Set {{A, B}, {U}} of objects of the 2-dimensional topological manifold M
>>> U.open_cover_family(trivial=False)
Set {{A, B}} of objects of the 2-dimensional topological manifold M
>>> V = M.open_subset('V')
>>> M.declare_union(U,V)
>>> M.open_cover_family()
Set {{A, B, V}, {M}, {U, V}} of objects of the 2-dimensional topological␣
↪manifold M
```

open_covers(*trivial=True*, *supersets=False*)

Generate the open covers of the current subset.

If the current subset, A say, is a subset of the manifold M, an *open cover* of A is a `ManifoldSubsetFiniteFamily` F of open subsets $U \in F$ of M such that

$$A \subset \bigcup_{U \in F} U.$$

If A is open, we ask that the above inclusion is actually an identity:

$$A = \bigcup_{U \in F} U.$$

Note

To get the open covers as a family, sorted lexicographically by the names of the subsets forming the open covers, use the method `open_cover_family()` instead.

INPUT:

- `trivial` – boolean (default: `True`); if `self` is open, include the trivial open cover of `self` by itself
- `supersets` – boolean (default: `False`); if `True`, include open covers of all the supersets. It can also be an iterable of supersets to include.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: M.open_covers()
<generator ...>
sage: list(M.open_covers())
[Set {M} of open subsets of the 2-dimensional topological manifold M]
sage: U = M.open_subset('U')
sage: list(U.open_covers())
[Set {U} of open subsets of the 2-dimensional topological manifold M]
sage: A = U.open_subset('A')
sage: B = U.open_subset('B')
sage: U.declare_union(A,B)
```

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```
sage: list(U.open_covers())
[Set {U} of open subsets of the 2-dimensional topological manifold M,
 Set {A, B} of open subsets of the 2-dimensional topological manifold M]
sage: list(U.open_covers(trivial=False))
[Set {A, B} of open subsets of the 2-dimensional topological manifold M]
sage: V = M.open_subset('V')
sage: M.declare_union(U,V)
sage: list(M.open_covers())
[Set {M} of open subsets of the 2-dimensional topological manifold M,
 Set {U, V} of open subsets of the 2-dimensional topological manifold M,
 Set {A, B, V} of open subsets of the 2-dimensional topological manifold M]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> M.open_covers()
<generator ...>
>>> list(M.open_covers())
[Set {M} of open subsets of the 2-dimensional topological manifold M]
>>> U = M.open_subset('U')
>>> list(U.open_covers())
[Set {U} of open subsets of the 2-dimensional topological manifold M]
>>> A = U.open_subset('A')
>>> B = U.open_subset('B')
>>> U.declare_union(A,B)
>>> list(U.open_covers())
[Set {U} of open subsets of the 2-dimensional topological manifold M,
 Set {A, B} of open subsets of the 2-dimensional topological manifold M]
>>> list(U.open_covers(trivial=False))
[Set {A, B} of open subsets of the 2-dimensional topological manifold M]
>>> V = M.open_subset('V')
>>> M.declare_union(U,V)
>>> list(M.open_covers())
[Set {M} of open subsets of the 2-dimensional topological manifold M,
 Set {U, V} of open subsets of the 2-dimensional topological manifold M,
 Set {A, B, V} of open subsets of the 2-dimensional topological manifold M]
```

open_subset(*name*, *latex_name=None*, *coord_def={}*, *supersets=None*)

Create an open subset of the manifold that is a subset of `self`.

An open subset is a set that is (i) included in the manifold and (ii) open with respect to the manifold's topology. It is a topological manifold by itself. Hence the returned object is an instance of *TopologicalManifold*.

INPUT:

- `name` – name given to the open subset
- `latex_name` – (default: `None`) LaTeX symbol to denote the subset; if none are provided, it is set to `name`
- `coord_def` – (default: {}) definition of the subset in terms of coordinates; `coord_def` must a be dictionary with keys charts on the manifold and values the symbolic expressions formed by the coordinates to define the subset
- `supersets` – (default: only `self`) list of sets that the new open subset is a subset of

OUTPUT:

- the open subset, as an instance of `TopologicalManifold` or one of its subclasses

EXAMPLES:

```
sage: M = Manifold(2, 'R^2', structure='topological')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: cl_D = M.subset('cl_D'); cl_D
Subset cl_D of the 2-dimensional topological manifold R^2
sage: D = cl_D.open_subset('D', coord_def={c_cart: x^2+y^2<1}); D
Open subset D of the 2-dimensional topological manifold R^2
sage: D.is_subset(cl_D)
True
sage: D.is_subset(M)
True

sage: M = Manifold(2, 'R^2', structure='differentiable')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: cl_D = M.subset('cl_D'); cl_D
Subset cl_D of the 2-dimensional differentiable manifold R^2
sage: D = cl_D.open_subset('D', coord_def={c_cart: x^2+y^2<1}); D
Open subset D of the 2-dimensional differentiable manifold R^2
sage: D.is_subset(cl_D)
True
sage: D.is_subset(M)
True

sage: M = Manifold(2, 'R^2', structure='Riemannian')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: cl_D = M.subset('cl_D'); cl_D
Subset cl_D of the 2-dimensional Riemannian manifold R^2
sage: D = cl_D.open_subset('D', coord_def={c_cart: x^2+y^2<1}); D
Open subset D of the 2-dimensional Riemannian manifold R^2
sage: D.is_subset(cl_D)
True
sage: D.is_subset(M)
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', structure='topological')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#
↪Cartesian coordinates on R^2
>>> cl_D = M.subset('cl_D'); cl_D
Subset cl_D of the 2-dimensional topological manifold R^2
>>> D = cl_D.open_subset('D', coord_def={c_cart: x**Integer(2)+y**Integer(2)
↪<Integer(1)}); D
Open subset D of the 2-dimensional topological manifold R^2
>>> D.is_subset(cl_D)
True
>>> D.is_subset(M)
True

>>> M = Manifold(Integer(2), 'R^2', structure='differentiable')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#
↪Cartesian coordinates on R^2
```

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```
>>> cl_D = M.subset('cl_D'); cl_D
Subset cl_D of the 2-dimensional differentiable manifold R^2
>>> D = cl_D.open_subset('D', coord_def={c_cart: x**Integer(2)+y**Integer(2)
↪<Integer(1)}); D
Open subset D of the 2-dimensional differentiable manifold R^2
>>> D.is_subset(cl_D)
True
>>> D.is_subset(M)
True

>>> M = Manifold(Integer(2), 'R^2', structure='Riemannian')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#␣
↪Cartesian coordinates on R^2
>>> cl_D = M.subset('cl_D'); cl_D
Subset cl_D of the 2-dimensional Riemannian manifold R^2
>>> D = cl_D.open_subset('D', coord_def={c_cart: x**Integer(2)+y**Integer(2)
↪<Integer(1)}); D
Open subset D of the 2-dimensional Riemannian manifold R^2
>>> D.is_subset(cl_D)
True
>>> D.is_subset(M)
True
```

open_superset_family()

Return the family of open supersets of `self`.

The family is sorted by the alphabetical names of the subsets.

OUTPUT:

- a *ManifoldSubsetFiniteFamily* instance containing all the open supersets that have been defined on the current subset

> **Note**
>
> If you only need to iterate over the open supersets in arbitrary order, you can use the generator method *open_supersets()* instead.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: U = M.open_subset('U')
sage: V = U.subset('V')
sage: W = V.subset('W')
sage: W.open_superset_family()
Set {M, U} of open subsets of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> U = M.open_subset('U')
>>> V = U.subset('V')
>>> W = V.subset('W')
```

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```
>>> W.open_superset_family()
Set {M, U} of open subsets of the 2-dimensional topological manifold M
```

open_supersets()

Generate the open supersets of `self`.

> **Note**
>
> To get the open supersets as a family, sorted by name, use the method *open_superset_family()* instead.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: U = M.open_subset('U')
sage: V = U.subset('V')
sage: W = V.subset('W')
sage: sorted(W.open_supersets(), key=lambda S: S._name)
[2-dimensional topological manifold M,
 Open subset U of the 2-dimensional topological manifold M]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> U = M.open_subset('U')
>>> V = U.subset('V')
>>> W = V.subset('W')
>>> sorted(W.open_supersets(), key=lambda S: S._name)
[2-dimensional topological manifold M,
 Open subset U of the 2-dimensional topological manifold M]
```

point(*coords=None*, *chart=None*, *name=None*, *latex_name=None*)

Define a point in `self`.

See *ManifoldPoint* for a complete documentation.

INPUT:

- `coords` – the point coordinates (as a tuple or a list) in the chart specified by `chart`
- `chart` – (default: `None`) chart in which the point coordinates are given; if `None`, the coordinates are assumed to refer to the default chart of the current subset
- `name` – (default: `None`) name given to the point
- `latex_name` – (default: `None`) LaTeX symbol to denote the point; if `None`, the LaTeX symbol is set to `name`

OUTPUT:

- the declared point, as an instance of *ManifoldPoint*

EXAMPLES:

Points on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: p = M.point((1,2), name='p'); p
Point p on the 2-dimensional topological manifold M
sage: p in M
True
sage: a = M.open_subset('A')
sage: c_uv.<u,v> = a.chart()
sage: q = a.point((-1,0), name='q'); q
Point q on the 2-dimensional topological manifold M
sage: q in a
True
sage: p._coordinates
{Chart (M, (x, y)): (1, 2)}
sage: q._coordinates
{Chart (A, (u, v)): (-1, 0)}
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> p = M.point((Integer(1),Integer(2)), name='p'); p
Point p on the 2-dimensional topological manifold M
>>> p in M
True
>>> a = M.open_subset('A')
>>> c_uv = a.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> q = a.point((-Integer(1),Integer(0)), name='q'); q
Point q on the 2-dimensional topological manifold M
>>> q in a
True
>>> p._coordinates
{Chart (M, (x, y)): (1, 2)}
>>> q._coordinates
{Chart (A, (u, v)): (-1, 0)}
```

retract (*p*)

Return the retract of `p` to `self`.

INPUT:

- `p` – point of the ambient manifold

OUTPUT: the same point, considered as a point of the subset

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: A = M.open_subset('A', coord_def={X: x>0})
sage: p = M((1, -2)); p
Point on the 2-dimensional topological manifold M
sage: p.parent()
2-dimensional topological manifold M
sage: q = A.retract(p); q
```

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```
Point on the 2-dimensional topological manifold M
sage: q.parent()
Open subset A of the 2-dimensional topological manifold M
sage: q.coord()
(1, -2)
sage: (q == p) and (p == q)
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> A = M.open_subset('A', coord_def={X: x>Integer(0)})
>>> p = M((Integer(1), -Integer(2))); p
Point on the 2-dimensional topological manifold M
>>> p.parent()
2-dimensional topological manifold M
>>> q = A.retract(p); q
Point on the 2-dimensional topological manifold M
>>> q.parent()
Open subset A of the 2-dimensional topological manifold M
>>> q.coord()
(1, -2)
>>> (q == p) and (p == q)
True
```

Of course, if the point does not belong to `A`, the `retract` method fails:

```
sage: p = M((-1, 3))  #  x < 0, so that p is not in A
sage: q = A.retract(p)
Traceback (most recent call last):
...
ValueError: the Point on the 2-dimensional topological manifold M
 is not in Open subset A of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> p = M((-Integer(1), Integer(3)))  #  x < 0, so that p is not in A
>>> q = A.retract(p)
Traceback (most recent call last):
...
ValueError: the Point on the 2-dimensional topological manifold M
 is not in Open subset A of the 2-dimensional topological manifold M
```

subset(*name*, *latex_name=None*, *is_open=False*)

Create a subset of the current subset.

INPUT:

- `name` – name given to the subset
- `latex_name` – (default: `None`) LaTeX symbol to denote the subset; if none are provided, it is set to `name`
- `is_open` – boolean (default: `False`); if `True`, the created subset is assumed to be open with respect to the manifold's topology

OUTPUT:

- the subset, as an instance of `ManifoldSubset`, or of the derived class `TopologicalManifold` if `is_open` is `True`

EXAMPLES:

Creating a subset of a manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: a = M.subset('A'); a
Subset A of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> a = M.subset('A'); a
Subset A of the 2-dimensional topological manifold M
```

Creating a subset of `A`:

```
sage: b = a.subset('B', latex_name=r'\mathcal{B}'); b
Subset B of the 2-dimensional topological manifold M
sage: latex(b)
\mathcal{B}
```

```
>>> from sage.all import *
>>> b = a.subset('B', latex_name=r'\mathcal{B}'); b
Subset B of the 2-dimensional topological manifold M
>>> latex(b)
\mathcal{B}
```

We have then:

```
sage: b.is_subset(a)
True
sage: b in a.subsets()
True
```

```
>>> from sage.all import *
>>> b.is_subset(a)
True
>>> b in a.subsets()
True
```

subset_digraph(*loops=False*, *quotient=False*, *open_covers=False*, *points=False*, *lower_bound=None*)

Return the digraph whose arcs represent subset relations among the subsets of `self`.

INPUT:

- `loops` – boolean (default: `False`); whether to include the trivial containment of each subset in itself as loops of the digraph
- `quotient` – boolean (default: `False`); whether to contract directed cycles in the graph, replacing equivalence classes of equal subsets by a single vertex. In this case, each vertex of the digraph is a set of `ManifoldSubset` instances.
- `open_covers` – boolean (default: `False`); whether to include vertices for open covers

- points – boolean (default: False); whether to include vertices for declared points; this can also be an iterable for the points to include
- lower_bound – (default: None) only include supersets of this

OUTPUT: a digraph; each vertex of the digraph is either:

- a *ManifoldSubsetFiniteFamily* containing one instance of *ManifoldSubset*.
- (if open_covers is True) a tuple of *ManifoldSubsetFiniteFamily* instances, representing an open cover.

EXAMPLES:

```
sage: # needs sage.graphs
sage: M = Manifold(3, 'M')
sage: U = M.open_subset('U'); V = M.open_subset('V'); W = M.open_subset('W')
sage: D = M.subset_digraph(); D
Digraph on 4 vertices
sage: D.edges(sort=True, key=lambda e: (e[0]._name, e[1]._name))           #␣
↪needs sage.graphs
[(Set {U} of open subsets of the 3-dimensional differentiable manifold M,
  Set {M} of open subsets of the 3-dimensional differentiable manifold M,
  None),
 (Set {V} of open subsets of the 3-dimensional differentiable manifold M,
  Set {M} of open subsets of the 3-dimensional differentiable manifold M,
  None),
 (Set {W} of open subsets of the 3-dimensional differentiable manifold M,
  Set {M} of open subsets of the 3-dimensional differentiable manifold M,
  None)]
sage: D.plot(layout='acyclic')                                              #␣
↪needs sage.plot
Graphics object consisting of 8 graphics primitives
sage: def label(element):
....:     try:
....:         return element._name
....:     except AttributeError:
....:         return '[' + ', '.join(sorted(x._name for x in element)) + ']'
sage: D.relabel(label, inplace=False).plot(layout='acyclic')                #␣
↪needs sage.plot
Graphics object consisting of 8 graphics primitives
sage: VW = V.union(W)
sage: D = M.subset_digraph(); D
Digraph on 5 vertices
sage: D.relabel(label, inplace=False).plot(layout='acyclic')                #␣
↪needs sage.plot
Graphics object consisting of 12 graphics primitives
```

```
>>> from sage.all import *
>>> # needs sage.graphs
>>> M = Manifold(Integer(3), 'M')
>>> U = M.open_subset('U'); V = M.open_subset('V'); W = M.open_subset('W')
>>> D = M.subset_digraph(); D
Digraph on 4 vertices
>>> D.edges(sort=True, key=lambda e: (e[Integer(0)]._name, e[Integer(1)]._
↪name))            # needs sage.graphs
```

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```
[(Set {U} of open subsets of the 3-dimensional differentiable manifold M,
  Set {M} of open subsets of the 3-dimensional differentiable manifold M,
  None),
 (Set {V} of open subsets of the 3-dimensional differentiable manifold M,
  Set {M} of open subsets of the 3-dimensional differentiable manifold M,
  None),
 (Set {W} of open subsets of the 3-dimensional differentiable manifold M,
  Set {M} of open subsets of the 3-dimensional differentiable manifold M,
  None)]
>>> D.plot(layout='acyclic')                                            #␣
↪needs sage.plot
Graphics object consisting of 8 graphics primitives
>>> def label(element):
...     try:
...         return element._name
...     except AttributeError:
...         return '[' + ', '.join(sorted(x._name for x in element)) + ']'
>>> D.relabel(label, inplace=False).plot(layout='acyclic')              #␣
↪needs sage.plot
Graphics object consisting of 8 graphics primitives
>>> VW = V.union(W)
>>> D = M.subset_digraph(); D
Digraph on 5 vertices
>>> D.relabel(label, inplace=False).plot(layout='acyclic')              #␣
↪needs sage.plot
Graphics object consisting of 12 graphics primitives
```

If `open_covers` is `True`, the digraph includes a special vertex for each nontrivial open cover of a subset:

```
sage: D = M.subset_digraph(open_covers=True)                             #␣
↪needs sage.graphs
sage: D.relabel(label, inplace=False).plot(layout='acyclic')             #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 14 graphics primitives
```

```
>>> from sage.all import *
>>> D = M.subset_digraph(open_covers=True)                              #␣
↪needs sage.graphs
>>> D.relabel(label, inplace=False).plot(layout='acyclic')              #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 14 graphics primitives
```

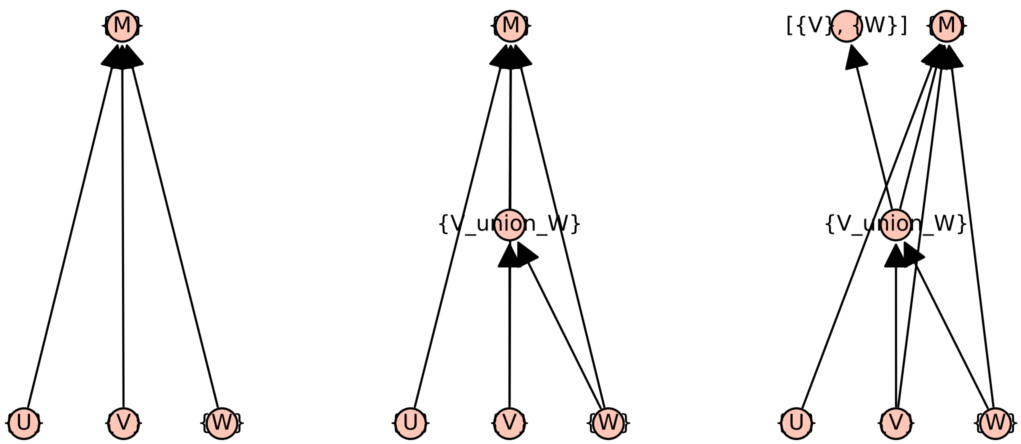
subset_family()

Return the family of subsets that have been defined on the current subset.

The family is sorted by the alphabetical names of the subsets.

OUTPUT:

- a *ManifoldSubsetFiniteFamily* instance containing all the subsets that have been defined on the current subset



Note: If you only need to iterate over the subsets in arbitrary order, you can use the generator method *subsets()* instead.

EXAMPLES:

Subsets of a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: U = M.open_subset('U')
sage: V = M.subset('V')
sage: M.subset_family()
Set {M, U, V} of subsets of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> U = M.open_subset('U')
>>> V = M.subset('V')
>>> M.subset_family()
Set {M, U, V} of subsets of the 2-dimensional topological manifold M
```

subset_poset(*open_covers=False*, *points=False*, *lower_bound=None*)

Return the poset of equivalence classes of the subsets of `self`.

Each element of the poset is a set of *ManifoldSubset* instances, which are known to be equal.

INPUT:

- `open_covers` – boolean (default: `False`); whether to include vertices for open covers
- `points` – boolean (default: `False`); whether to include vertices for declared points; this can also be an iterable for the points to include
- `lower_bound` – (default: `None`) only include supersets of this

EXAMPLES:

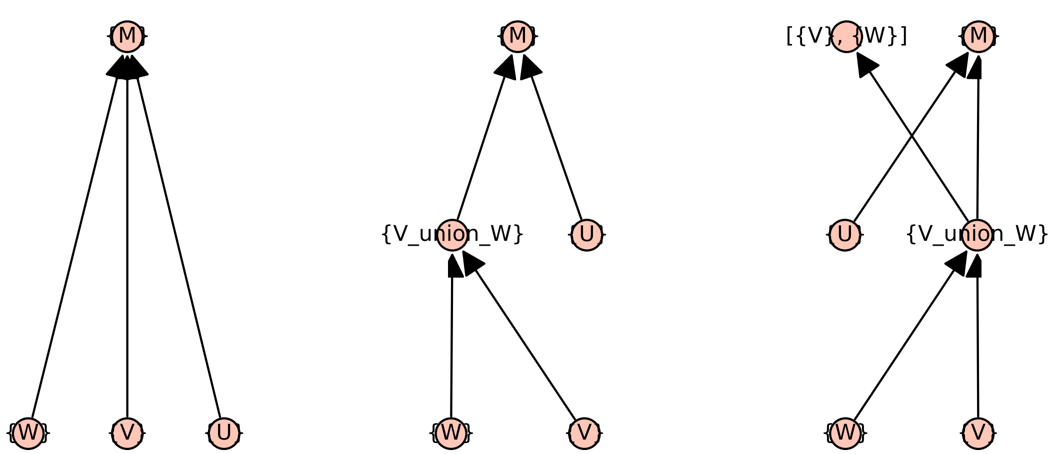
```
sage: # needs sage.graphs
sage: M = Manifold(3, 'M')
sage: U = M.open_subset('U'); V = M.open_subset('V'); W = M.open_subset('W')
sage: P = M.subset_poset(); P
Finite poset containing 4 elements
sage: P.plot(element_labels={element: element._name for element in P})      #␣
↪needs sage.plot
Graphics object consisting of 8 graphics primitives
sage: VW = V.union(W)
sage: P = M.subset_poset(); P
Finite poset containing 5 elements
sage: P.maximal_elements()
[Set {M} of open subsets of the 3-dimensional differentiable manifold M]
sage: sorted(P.minimal_elements(), key=lambda v: v._name)
 [Set {U} of open subsets of the 3-dimensional differentiable manifold M,
  Set {V} of open subsets of the 3-dimensional differentiable manifold M,
  Set {W} of open subsets of the 3-dimensional differentiable manifold M]
sage: from sage.manifolds.subset import ManifoldSubsetFiniteFamily
sage: sorted(P.lower_covers(ManifoldSubsetFiniteFamily([M])), key=str)
 [Set {U} of open subsets of the 3-dimensional differentiable manifold M,
  Set {V_union_W} of open subsets of the 3-dimensional differentiable␣
↪manifold M]
sage: P.plot(element_labels={element: element._name for element in P})      #␣
↪needs sage.plot
Graphics object consisting of 10 graphics primitives
```

```
>>> from sage.all import *
>>> # needs sage.graphs
>>> M = Manifold(Integer(3), 'M')
>>> U = M.open_subset('U'); V = M.open_subset('V'); W = M.open_subset('W')
>>> P = M.subset_poset(); P
Finite poset containing 4 elements
>>> P.plot(element_labels={element: element._name for element in P})      #␣
↪needs sage.plot
Graphics object consisting of 8 graphics primitives
>>> VW = V.union(W)
>>> P = M.subset_poset(); P
Finite poset containing 5 elements
>>> P.maximal_elements()
[Set {M} of open subsets of the 3-dimensional differentiable manifold M]
>>> sorted(P.minimal_elements(), key=lambda v: v._name)
 [Set {U} of open subsets of the 3-dimensional differentiable manifold M,
  Set {V} of open subsets of the 3-dimensional differentiable manifold M,
  Set {W} of open subsets of the 3-dimensional differentiable manifold M]
>>> from sage.manifolds.subset import ManifoldSubsetFiniteFamily
>>> sorted(P.lower_covers(ManifoldSubsetFiniteFamily([M])), key=str)
 [Set {U} of open subsets of the 3-dimensional differentiable manifold M,
  Set {V_union_W} of open subsets of the 3-dimensional differentiable␣
↪manifold M]
>>> P.plot(element_labels={element: element._name for element in P})      #␣
↪needs sage.plot
Graphics object consisting of 10 graphics primitives
```

If `open_covers` is `True`, the poset includes a special vertex for each nontrivial open cover of a subset:

```
sage: # needs sage.graphs
sage: P = M.subset_poset(open_covers=True); P
Finite poset containing 6 elements
sage: from sage.manifolds.subset import ManifoldSubsetFiniteFamily
sage: sorted(P.upper_covers(ManifoldSubsetFiniteFamily([VW])), key=str)
[(Set {V} of open subsets of the 3-dimensional differentiable manifold M,
  Set {W} of open subsets of the 3-dimensional differentiable manifold M),
 Set {M} of open subsets of the 3-dimensional differentiable manifold M]
sage: def label(element):
....:     try:
....:         return element._name
....:     except AttributeError:
....:         return '[' + ', '.join(sorted(x._name for x in element)) + ']'
sage: P.plot(element_labels={element: label(element) for element in P})      # needs sage.plot
Graphics object consisting of 12 graphics primitives
```

```
>>> from sage.all import *
>>> # needs sage.graphs
>>> P = M.subset_poset(open_covers=True); P
Finite poset containing 6 elements
>>> from sage.manifolds.subset import ManifoldSubsetFiniteFamily
>>> sorted(P.upper_covers(ManifoldSubsetFiniteFamily([VW])), key=str)
[(Set {V} of open subsets of the 3-dimensional differentiable manifold M,
  Set {W} of open subsets of the 3-dimensional differentiable manifold M),
 Set {M} of open subsets of the 3-dimensional differentiable manifold M]
>>> def label(element):
...     try:
...         return element._name
...     except AttributeError:
...         return '[' + ', '.join(sorted(x._name for x in element)) + ']'
>>> P.plot(element_labels={element: label(element) for element in P})      # needs sage.plot
Graphics object consisting of 12 graphics primitives
```



subsets()

Generate the subsets that have been defined on the current subset.

Note: To get the subsets as a family, sorted by name, use the method *subset_family()* instead.

EXAMPLES:

Subsets of a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: U = M.open_subset('U')
sage: V = M.subset('V')
sage: frozenset(M.subsets())  # random (set output)
{Subset V of the 2-dimensional topological manifold M,
 2-dimensional topological manifold M,
 Open subset U of the 2-dimensional topological manifold M}
sage: U in M.subsets()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> U = M.open_subset('U')
>>> V = M.subset('V')
>>> frozenset(M.subsets())  # random (set output)
{Subset V of the 2-dimensional topological manifold M,
 2-dimensional topological manifold M,
 Open subset U of the 2-dimensional topological manifold M}
>>> U in M.subsets()
True
```

The method *subset_family()* returns a family (sorted alphabetically by the subset names):

```
sage: M.subset_family()
Set {M, U, V} of subsets of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M.subset_family()
Set {M, U, V} of subsets of the 2-dimensional topological manifold M
```

superset(*name*, *latex_name=None*, *is_open=False*)

Create a superset of the current subset.

A *superset* is a manifold subset in which the current subset is included.

INPUT:

- `name` – name given to the superset
- `latex_name` – (default: `None`) LaTeX symbol to denote the superset; if none are provided, it is set to `name`
- `is_open` – boolean (default: `False`); if `True`, the created subset is assumed to be open with respect to the manifold's topology

OUTPUT:

- the superset, as an instance of *ManifoldSubset* or of the derived class *TopologicalManifold* if `is_open` is `True`

EXAMPLES:

Creating some superset of a given subset:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: a = M.subset('A')
sage: b = a.superset('B'); b
Subset B of the 2-dimensional topological manifold M
sage: b.subset_family()
Set {A, B} of subsets of the 2-dimensional topological manifold M
sage: a.superset_family()
Set {A, B, M} of subsets of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> a = M.subset('A')
>>> b = a.superset('B'); b
Subset B of the 2-dimensional topological manifold M
>>> b.subset_family()
Set {A, B} of subsets of the 2-dimensional topological manifold M
>>> a.superset_family()
Set {A, B, M} of subsets of the 2-dimensional topological manifold M
```

The superset of the whole manifold is itself:

```
sage: M.superset('SM') is M
True
```

```
>>> from sage.all import *
>>> M.superset('SM') is M
True
```

Two supersets of a given subset are a priori different:

```
sage: c = a.superset('C')
sage: c == b
False
```

```
>>> from sage.all import *
>>> c = a.superset('C')
>>> c == b
False
```

superset_digraph(*loops=False*, *quotient=False*, *open_covers=False*, *points=False*, *upper_bound=None*)

Return the digraph whose arcs represent subset relations among the supersets of `self`.

INPUT:

- `loops` – boolean (default: `False`); whether to include the trivial containment of each subset in itself as loops of the digraph

- `quotient` – boolean (default: `False`); whether to contract directed cycles in the graph, replacing equivalence classes of equal subsets by a single vertex. In this case, each vertex of the digraph is a set of *ManifoldSubset* instances.
- `open_covers` – boolean (default: `False`); whether to include vertices for open covers
- `points` – boolean (default: `False`); whether to include vertices for declared points; this can also be an iterable for the points to include
- `upper_bound` – (default: `None`) only include subsets of this

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: U = M.open_subset('U'); V = M.open_subset('V'); W = M.open_subset('W')
sage: VW = V.union(W)
sage: P = V.superset_digraph(loops=False, upper_bound=VW); P                  #␣
↪needs sage.graphs
Digraph on 2 vertices
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> U = M.open_subset('U'); V = M.open_subset('V'); W = M.open_subset('W')
>>> VW = V.union(W)
>>> P = V.superset_digraph(loops=False, upper_bound=VW); P                  #␣
↪needs sage.graphs
Digraph on 2 vertices
```

superset_family()

Return the family of declared supersets of the current subset.

The family is sorted by the alphabetical names of the supersets.

OUTPUT:

- a *ManifoldSubsetFiniteFamily* instance containing all the supersets

> **Note**
>
> If you only need to iterate over the supersets in arbitrary order, you can use the generator method *supersets()* instead.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: U = M.open_subset('U')
sage: V = M.subset('V')
sage: V.superset_family()
Set {M, V} of subsets of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> U = M.open_subset('U')
>>> V = M.subset('V')
>>> V.superset_family()
Set {M, V} of subsets of the 2-dimensional topological manifold M
```

superset_poset (*open_covers=False*, *points=False*, *upper_bound=None*)

Return the poset of the supersets of `self`.

INPUT:

- `open_covers` – boolean (default: `False`); whether to include vertices for open covers
- `points` – boolean (default: `False`); whether to include vertices for declared points; this can also be an iterable for the points to include
- `upper_bound` – (default: `None`) only include subsets of this

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: U = M.open_subset('U'); V = M.open_subset('V'); W = M.open_subset('W')
sage: VW = V.union(W)
sage: P = V.superset_poset(); P                                              #␣
↪needs sage.graphs
Finite poset containing 3 elements
sage: P.plot(element_labels={element: element._name for element in P})      #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 6 graphics primitives
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> U = M.open_subset('U'); V = M.open_subset('V'); W = M.open_subset('W')
>>> VW = V.union(W)
>>> P = V.superset_poset(); P                                               #␣
↪needs sage.graphs
Finite poset containing 3 elements
>>> P.plot(element_labels={element: element._name for element in P})       #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 6 graphics primitives
```

supersets ()

Generate the declared supersets of the current subset.

> **Note**
>
> To get the supersets as a family, sorted by name, use the method *superset_family()* instead.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: U = M.open_subset('U')
sage: V = M.subset('V')
sage: sorted(V.supersets(), key=lambda v: v._name)
[2-dimensional topological manifold M,
 Subset V of the 2-dimensional topological manifold M]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> U = M.open_subset('U')
```

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```
>>> V = M.subset('V')
>>> sorted(V.supersets(), key=lambda v: v._name)
[2-dimensional topological manifold M,
 Subset V of the 2-dimensional topological manifold M]
```

union (*name*, *latex_name*, **others*)

Return the union of the current subset with other subsets.

This method may return a previously constructed union instead of creating a new subset. In this case, `name` and `latex_name` are not used.

INPUT:

- `others` – other subsets of the same manifold
- `name` – (default: `None`) name given to the union in the case the latter has to be created; the default is `self._name` union `other._name`
- `latex_name` – (default: `None`) LaTeX symbol to denote the union in the case the latter has to be created; the default is built upon the symbol $\cup$

OUTPUT:

- instance of *ManifoldSubset* representing the subset that is the union of the current subset with `others`

EXAMPLES:

Union of two subsets:

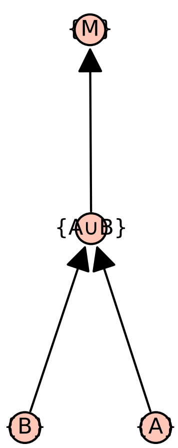
```
sage: M = Manifold(2, 'M', structure='topological')
sage: a = M.subset('A')
sage: b = M.subset('B')
sage: c = a.union(b); c
Subset A_union_B of the 2-dimensional topological manifold M
sage: a.superset_family()
Set {A, A_union_B, M} of subsets of the 2-dimensional topological manifold M
sage: b.superset_family()
Set {A_union_B, B, M} of subsets of the 2-dimensional topological manifold M
sage: c.superset_family()
Set {A_union_B, M} of subsets of the 2-dimensional topological manifold M
```

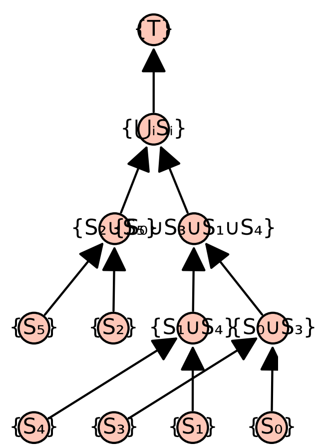
```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> a = M.subset('A')
>>> b = M.subset('B')
>>> c = a.union(b); c
Subset A_union_B of the 2-dimensional topological manifold M
>>> a.superset_family()
Set {A, A_union_B, M} of subsets of the 2-dimensional topological manifold M
>>> b.superset_family()
Set {A_union_B, B, M} of subsets of the 2-dimensional topological manifold M
>>> c.superset_family()
Set {A_union_B, M} of subsets of the 2-dimensional topological manifold M
```

Union of six subsets:

```
sage: T = Manifold(2, 'T', structure='topological')
sage: S = [T.subset(f'S{i}') for i in range(6)]
sage: [S[i].union(S[i+3]) for i in range(3)]
[Subset S0_union_S3 of the 2-dimensional topological manifold T,
 Subset S1_union_S4 of the 2-dimensional topological manifold T,
 Subset S2_union_S5 of the 2-dimensional topological manifold T]
sage: union_S_i = S[0].union(S[1:], name='union_S_i'); union_S_i
Subset union_S_i of the 2-dimensional topological manifold T
sage: T.subset_family()
Set {S0, S0_union_S3, S0_union_S3_union_S1_union_S4, S1,
     S1_union_S4, S2, S2_union_S5, S3, S4, S5, T, union_S_i}
 of subsets of the 2-dimensional topological manifold T
```

```
>>> from sage.all import *
>>> T = Manifold(Integer(2), 'T', structure='topological')
>>> S = [T.subset(f'S{i}') for i in range(Integer(6))]
>>> [S[i].union(S[i+Integer(3)]) for i in range(Integer(3))]
[Subset S0_union_S3 of the 2-dimensional topological manifold T,
 Subset S1_union_S4 of the 2-dimensional topological manifold T,
 Subset S2_union_S5 of the 2-dimensional topological manifold T]
>>> union_S_i = S[Integer(0)].union(S[Integer(1):], name='union_S_i'); union_
↪S_i
Subset union_S_i of the 2-dimensional topological manifold T
>>> T.subset_family()
Set {S0, S0_union_S3, S0_union_S3_union_S1_union_S4, S1,
     S1_union_S4, S2, S2_union_S5, S3, S4, S5, T, union_S_i}
 of subsets of the 2-dimensional topological manifold T
```





1.3 Manifold Structures

These classes encode the structure of a manifold.

AUTHORS:

- Travis Scrimshaw (2015-11-25): Initial version
- Eric Gourgoulhon (2015): add *DifferentialStructure* and *RealDifferentialStructure*

- Eric Gourgoulhon (2018): add *PseudoRiemannianStructure*, *RiemannianStructure* and *LorentzianStructure*

class sage.manifolds.structure.**DegenerateStructure**

Bases: Singleton

The structure of a degenerate manifold.

chart

alias of *RealDiffChart*

homset

alias of *DifferentiableManifoldHomset*

name = 'degenerate_metric'

scalar_field_algebra

alias of *DiffScalarFieldAlgebra*

subcategory(*cat*)

Return the subcategory of `cat` corresponding to the structure of `self`.

EXAMPLES:

```
sage: from sage.manifolds.structure import DegenerateStructure
sage: from sage.categories.manifolds import Manifolds
sage: DegenerateStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

```
>>> from sage.all import *
>>> from sage.manifolds.structure import DegenerateStructure
>>> from sage.categories.manifolds import Manifolds
>>> DegenerateStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

class sage.manifolds.structure.**DifferentialStructure**

Bases: Singleton

The structure of a differentiable manifold over a general topological field.

chart

alias of *DiffChart*

homset

alias of *DifferentiableManifoldHomset*

name = 'differentiable'

scalar_field_algebra

alias of *DiffScalarFieldAlgebra*

subcategory(*cat*)

Return the subcategory of `cat` corresponding to the structure of `self`.

EXAMPLES:

```
sage: from sage.manifolds.structure import DifferentialStructure
sage: from sage.categories.manifolds import Manifolds
sage: DifferentialStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

```
>>> from sage.all import *
>>> from sage.manifolds.structure import DifferentialStructure
>>> from sage.categories.manifolds import Manifolds
>>> DifferentialStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

class `sage.manifolds.structure.`**`LorentzianStructure`**

Bases: `Singleton`

The structure of a Lorentzian manifold.

`chart`

alias of *`RealDiffChart`*

`homset`

alias of *`DifferentiableManifoldHomset`*

`name = 'Lorentzian'`

`scalar_field_algebra`

alias of *`DiffScalarFieldAlgebra`*

`subcategory`(*cat*)

Return the subcategory of `cat` corresponding to the structure of `self`.

EXAMPLES:

```
sage: from sage.manifolds.structure import LorentzianStructure
sage: from sage.categories.manifolds import Manifolds
sage: LorentzianStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

```
>>> from sage.all import *
>>> from sage.manifolds.structure import LorentzianStructure
>>> from sage.categories.manifolds import Manifolds
>>> LorentzianStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

class `sage.manifolds.structure.`**`PseudoRiemannianStructure`**

Bases: `Singleton`

The structure of a pseudo-Riemannian manifold.

`chart`

alias of *`RealDiffChart`*

`homset`

alias of *`DifferentiableManifoldHomset`*

`name = 'pseudo-Riemannian'`

scalar_field_algebra

alias of *DiffScalarFieldAlgebra*

subcategory(*cat*)

Return the subcategory of `cat` corresponding to the structure of `self`.

EXAMPLES:

```
sage: from sage.manifolds.structure import PseudoRiemannianStructure
sage: from sage.categories.manifolds import Manifolds
sage: PseudoRiemannianStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

```
>>> from sage.all import *
>>> from sage.manifolds.structure import PseudoRiemannianStructure
>>> from sage.categories.manifolds import Manifolds
>>> PseudoRiemannianStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

class sage.manifolds.structure.**RealDifferentialStructure**

Bases: `Singleton`

The structure of a differentiable manifold over **R**.

chart

alias of *RealDiffChart*

homset

alias of *DifferentiableManifoldHomset*

name = 'differentiable'

scalar_field_algebra

alias of *DiffScalarFieldAlgebra*

subcategory(*cat*)

Return the subcategory of `cat` corresponding to the structure of `self`.

EXAMPLES:

```
sage: from sage.manifolds.structure import RealDifferentialStructure
sage: from sage.categories.manifolds import Manifolds
sage: RealDifferentialStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

```
>>> from sage.all import *
>>> from sage.manifolds.structure import RealDifferentialStructure
>>> from sage.categories.manifolds import Manifolds
>>> RealDifferentialStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

class sage.manifolds.structure.**RealTopologicalStructure**

Bases: `Singleton`

The structure of a topological manifold over **R**.

chart

alias of *RealChart*

homset

alias of *TopologicalManifoldHomset*

name = 'topological'

scalar_field_algebra

alias of *ScalarFieldAlgebra*

subcategory(*cat*)

Return the subcategory of `cat` corresponding to the structure of `self`.

EXAMPLES:

```
sage: from sage.manifolds.structure import RealTopologicalStructure
sage: from sage.categories.manifolds import Manifolds
sage: RealTopologicalStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

```
>>> from sage.all import *
>>> from sage.manifolds.structure import RealTopologicalStructure
>>> from sage.categories.manifolds import Manifolds
>>> RealTopologicalStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

class sage.manifolds.structure.**RiemannianStructure**

Bases: `Singleton`

The structure of a Riemannian manifold.

chart

alias of *RealDiffChart*

homset

alias of *DifferentiableManifoldHomset*

name = 'Riemannian'

scalar_field_algebra

alias of *DiffScalarFieldAlgebra*

subcategory(*cat*)

Return the subcategory of `cat` corresponding to the structure of `self`.

EXAMPLES:

```
sage: from sage.manifolds.structure import RiemannianStructure
sage: from sage.categories.manifolds import Manifolds
sage: RiemannianStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

```
>>> from sage.all import *
>>> from sage.manifolds.structure import RiemannianStructure
>>> from sage.categories.manifolds import Manifolds
```

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```
>>> RiemannianStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

class sage.manifolds.structure.**TopologicalStructure**

Bases: `Singleton`

The structure of a topological manifold over a general topological field.

chart

alias of *Chart*

homset

alias of *TopologicalManifoldHomset*

name = 'topological'

scalar_field_algebra

alias of *ScalarFieldAlgebra*

subcategory(*cat*)

Return the subcategory of `cat` corresponding to the structure of `self`.

EXAMPLES:

```
sage: from sage.manifolds.structure import TopologicalStructure
sage: from sage.categories.manifolds import Manifolds
sage: TopologicalStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

```
>>> from sage.all import *
>>> from sage.manifolds.structure import TopologicalStructure
>>> from sage.categories.manifolds import Manifolds
>>> TopologicalStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

1.4 Points of Topological Manifolds

The class *ManifoldPoint* implements points of a topological manifold.

A *ManifoldPoint* object can have coordinates in various charts defined on the manifold. Two points are declared equal if they have the same coordinates in the same chart.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015) : initial version

REFERENCES:

- [Lee2011]
- [Lee2013]

EXAMPLES:

Defining a point in $\mathbf{R}^3$ by its spherical coordinates:

```
sage: M = Manifold(3, 'R^3', structure='topological')
sage: U = M.open_subset('U')  # the domain of spherical coordinates
sage: c_spher.<r,th,ph> = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):periodic:\
↪phi')
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'R^3', structure='topological')
>>> U = M.open_subset('U')  # the domain of spherical coordinates
>>> c_spher = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):periodic:\phi', names=(
↪'r', 'th', 'ph',)); (r, th, ph,) = c_spher._first_ngens(3)
```

We construct the point in the coordinates in the default chart of `U` (`c_spher`):

```
sage: p = U((1, pi/2, pi), name='P')
sage: p
Point P on the 3-dimensional topological manifold R^3
sage: latex(p)
P
sage: p in U
True
sage: p.parent()
Open subset U of the 3-dimensional topological manifold R^3
sage: c_spher(p)
(1, 1/2*pi, pi)
sage: p.coordinates(c_spher) # equivalent to above
(1, 1/2*pi, pi)
```

```
>>> from sage.all import *
>>> p = U((Integer(1), pi/Integer(2), pi), name='P')
>>> p
Point P on the 3-dimensional topological manifold R^3
>>> latex(p)
P
>>> p in U
True
>>> p.parent()
Open subset U of the 3-dimensional topological manifold R^3
>>> c_spher(p)
(1, 1/2*pi, pi)
>>> p.coordinates(c_spher) # equivalent to above
(1, 1/2*pi, pi)
```

Computing the coordinates of `p` in a new chart:

```
sage: c_cart.<x,y,z> = U.chart() # Cartesian coordinates on U
sage: spher_to_cart = c_spher.transition_map(c_cart,
....:                     [r*sin(th)*cos(ph), r*sin(th)*sin(ph), r*cos(th)])
sage: c_cart(p)  # evaluate P's Cartesian coordinates
(-1, 0, 0)
```

```
>>> from sage.all import *
>>> c_cart = U.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_cart._first_ngens(3)#␣
```

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```
↪Cartesian coordinates on U
>>> spher_to_cart = c_spher.transition_map(c_cart,
...                     [r*sin(th)*cos(ph), r*sin(th)*sin(ph), r*cos(th)])
>>> c_cart(p)  # evaluate P's Cartesian coordinates
(-1, 0, 0)
```

Points can be compared:

```
sage: p1 = U((1, pi/2, pi))
sage: p1 == p
True
sage: q = U((2, pi/2, pi))
sage: q == p
False
```

```
>>> from sage.all import *
>>> p1 = U((Integer(1), pi/Integer(2), pi))
>>> p1 == p
True
>>> q = U((Integer(2), pi/Integer(2), pi))
>>> q == p
False
```

even if they were initially not defined within the same coordinate chart:

```
sage: p2 = U((-1,0,0), chart=c_cart)
sage: p2 == p
True
```

```
>>> from sage.all import *
>>> p2 = U((-Integer(1),Integer(0),Integer(0)), chart=c_cart)
>>> p2 == p
True
```

The 2π-periodicity of the ϕ coordinate is also taken into account for the comparison:

```
sage: p3 = U((1, pi/2, 5*pi))
sage: p3 == p
True
sage: p4 = U((1, pi/2, -pi))
sage: p4 == p
True
```

```
>>> from sage.all import *
>>> p3 = U((Integer(1), pi/Integer(2), Integer(5)*pi))
>>> p3 == p
True
>>> p4 = U((Integer(1), pi/Integer(2), -pi))
>>> p4 == p
True
```

class sage.manifolds.point.**ManifoldPoint**(*parent*, *coords=None*, *chart=None*, *name=None*, *latex_name=None*, *check_coords=True*)

Bases: `Element`

Point of a topological manifold.

This is a Sage *element* class, the corresponding *parent* class being `TopologicalManifold` or `ManifoldSubset`.

INPUT:

- `parent` – the manifold subset to which the point belongs
- `coords` – (default: `None`) the point coordinates (as a tuple or a list) in the chart `chart`
- `chart` – (default: `None`) chart in which the coordinates are given; if `None`, the coordinates are assumed to refer to the default chart of `parent`
- `name` – (default: `None`) name given to the point
- `latex_name` – (default: `None`) LaTeX symbol to denote the point; if `None`, the LaTeX symbol is set to `name`
- `check_coords` – boolean (default: `True`); determines whether `coords` are valid coordinates for the chart `chart`. For symbolic coordinates, it is recommended to set `check_coords` to `False`.

EXAMPLES:

A point on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: (a, b) = var('a b') # generic coordinates for the point
sage: p = M.point((a, b), name='P'); p
Point P on the 2-dimensional topological manifold M
sage: p.coordinates()  # coordinates of P in the subset's default chart
(a, b)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> (a, b) = var('a b') # generic coordinates for the point
>>> p = M.point((a, b), name='P'); p
Point P on the 2-dimensional topological manifold M
>>> p.coordinates()  # coordinates of P in the subset's default chart
(a, b)
```

Since points are Sage *elements*, the *parent* of which being the subset on which they are defined, it is equivalent to write:

```
sage: p = M((a, b), name='P'); p
Point P on the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> p = M((a, b), name='P'); p
Point P on the 2-dimensional topological manifold M
```

A point is an element of the manifold subset in which it has been defined:

```
sage: p in M
True
sage: p.parent()
```

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```
2-dimensional topological manifold M
sage: U = M.open_subset('U', coord_def={c_xy: x>0})
sage: q = U.point((2,1), name='q')
sage: q.parent()
Open subset U of the 2-dimensional topological manifold M
sage: q in U
True
sage: q in M
True
```

```
>>> from sage.all import *
>>> p in M
True
>>> p.parent()
2-dimensional topological manifold M
>>> U = M.open_subset('U', coord_def={c_xy: x>Integer(0)})
>>> q = U.point((Integer(2),Integer(1)), name='q')
>>> q.parent()
Open subset U of the 2-dimensional topological manifold M
>>> q in U
True
>>> q in M
True
```

By default, the LaTeX symbol of the point is deduced from its name:

```
sage: latex(p)
P
```

```
>>> from sage.all import *
>>> latex(p)
P
```

But it can be set to any value:

```
sage: p = M.point((a, b), name='P', latex_name=r'\mathcal{P}')
sage: latex(p)
\mathcal{P}
```

```
>>> from sage.all import *
>>> p = M.point((a, b), name='P', latex_name=r'\mathcal{P}')
>>> latex(p)
\mathcal{P}
```

Points can be drawn in 2D or 3D graphics thanks to the method `plot()`.

add_coord(*coords*, *chart=None*)

Add some coordinates in the specified chart.

The previous coordinates with respect to other charts are kept. To clear them, use `set_coord()` instead.

INPUT:

- `coords` – the point coordinates (as a tuple or a list)

- `chart` – (default: `None`) chart in which the coordinates are given; if none are provided, the coordinates are assumed to refer to the subset's default chart

> **Warning**
>
> If the point has already coordinates in other charts, it is the user's responsibility to make sure that the coordinates to be added are consistent with them.

EXAMPLES:

Setting coordinates to a point on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: p = M.point()
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> p = M.point()
```

We give the point some coordinates in the manifold's default chart:

```
sage: p.add_coordinates((2,-3))
sage: p.coordinates()
(2, -3)
sage: X(p)
(2, -3)
```

```
>>> from sage.all import *
>>> p.add_coordinates((Integer(2),-Integer(3)))
>>> p.coordinates()
(2, -3)
>>> X(p)
(2, -3)
```

A shortcut for `add_coordinates` is `add_coord`:

```
sage: p.add_coord((2,-3))
sage: p.coord()
(2, -3)
```

```
>>> from sage.all import *
>>> p.add_coord((Integer(2),-Integer(3)))
>>> p.coord()
(2, -3)
```

Let us introduce a second chart on the manifold:

```
sage: Y.<u,v> = M.chart()
sage: X_to_Y = X.transition_map(Y, [x+y, x-y])
```

```
>>> from sage.all import *
>>> Y = M.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)
>>> X_to_Y = X.transition_map(Y, [x+y, x-y])
```

If we add coordinates for `p` in chart `Y`, those in chart `X` are kept:

```
sage: p.add_coordinates((-1,5), chart=Y)
sage: p._coordinates  # random (dictionary output)
{Chart (M, (u, v)): (-1, 5), Chart (M, (x, y)): (2, -3)}
```

```
>>> from sage.all import *
>>> p.add_coordinates((-Integer(1),Integer(5)), chart=Y)
>>> p._coordinates  # random (dictionary output)
{Chart (M, (u, v)): (-1, 5), Chart (M, (x, y)): (2, -3)}
```

On the contrary, with the method `set_coordinates()`, the coordinates in charts different from `Y` would be lost:

```
sage: p.set_coordinates((-1,5), chart=Y)
sage: p._coordinates
{Chart (M, (u, v)): (-1, 5)}
```

```
>>> from sage.all import *
>>> p.set_coordinates((-Integer(1),Integer(5)), chart=Y)
>>> p._coordinates
{Chart (M, (u, v)): (-1, 5)}
```

add_coordinates(*coords*, *chart=None*)

Add some coordinates in the specified chart.

The previous coordinates with respect to other charts are kept. To clear them, use `set_coord()` instead.

INPUT:

- `coords` – the point coordinates (as a tuple or a list)
- `chart` – (default: `None`) chart in which the coordinates are given; if none are provided, the coordinates are assumed to refer to the subset's default chart

> **Warning**
>
> If the point has already coordinates in other charts, it is the user's responsibility to make sure that the coordinates to be added are consistent with them.

EXAMPLES:

Setting coordinates to a point on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: p = M.point()
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
```

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```
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> p = M.point()
```

We give the point some coordinates in the manifold's default chart:

```
sage: p.add_coordinates((2,-3))
sage: p.coordinates()
(2, -3)
sage: X(p)
(2, -3)
```

```
>>> from sage.all import *
>>> p.add_coordinates((Integer(2),-Integer(3)))
>>> p.coordinates()
(2, -3)
>>> X(p)
(2, -3)
```

A shortcut for `add_coordinates` is `add_coord`:

```
sage: p.add_coord((2,-3))
sage: p.coord()
(2, -3)
```

```
>>> from sage.all import *
>>> p.add_coord((Integer(2),-Integer(3)))
>>> p.coord()
(2, -3)
```

Let us introduce a second chart on the manifold:

```
sage: Y.<u,v> = M.chart()
sage: X_to_Y = X.transition_map(Y, [x+y, x-y])
```

```
>>> from sage.all import *
>>> Y = M.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)
>>> X_to_Y = X.transition_map(Y, [x+y, x-y])
```

If we add coordinates for `p` in chart `Y`, those in chart `X` are kept:

```
sage: p.add_coordinates((-1,5), chart=Y)
sage: p._coordinates  # random (dictionary output)
{Chart (M, (u, v)): (-1, 5), Chart (M, (x, y)): (2, -3)}
```

```
>>> from sage.all import *
>>> p.add_coordinates((-Integer(1),Integer(5)), chart=Y)
>>> p._coordinates  # random (dictionary output)
{Chart (M, (u, v)): (-1, 5), Chart (M, (x, y)): (2, -3)}
```

On the contrary, with the method *set_coordinates()*, the coordinates in charts different from `Y` would be lost:

```
sage: p.set_coordinates((-1,5), chart=Y)
sage: p._coordinates
{Chart (M, (u, v)): (-1, 5)}
```

```
>>> from sage.all import *
>>> p.set_coordinates((-Integer(1),Integer(5)), chart=Y)
>>> p._coordinates
{Chart (M, (u, v)): (-1, 5)}
```

coord(*chart=None*, *old_chart=None*)

Return the point coordinates in the specified chart.

If these coordinates are not already known, they are computed from known ones by means of change-of-chart formulas.

An equivalent way to get the coordinates of a point is to let the chart acting on the point, i.e. if `X` is a chart and `p` a point, one has `p.coordinates(chart=X) == X(p)`.

INPUT:

- `chart` – (default: `None`) chart in which the coordinates are given; if none are provided, the coordinates are assumed to refer to the subset's default chart
- `old_chart` – (default: `None`) chart from which the coordinates in `chart` are to be computed; if `None`, a chart in which the point's coordinates are already known will be picked, privileging the subset's default chart

EXAMPLES:

Spherical coordinates of a point on $\mathbf{R}^3$:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: c_spher.<r,th,ph> = M.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\
→phi') # spherical coordinates
sage: p = M.point((1, pi/2, pi))
sage: p.coordinates()  # coordinates in the manifold's default chart
(1, 1/2*pi, pi)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological')
>>> c_spher = M.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi', names=(
→'r', 'th', 'ph',)); (r, th, ph,) = c_spher._first_ngens(3)# spherical␣
→coordinates
>>> p = M.point((Integer(1), pi/Integer(2), pi))
>>> p.coordinates()  # coordinates in the manifold's default chart
(1, 1/2*pi, pi)
```

Since the default chart of `M` is `c_spher`, it is equivalent to write:

```
sage: p.coordinates(c_spher)
(1, 1/2*pi, pi)
```

```
>>> from sage.all import *
>>> p.coordinates(c_spher)
(1, 1/2*pi, pi)
```

An alternative way to get the coordinates is to let the chart act on the point (from the very definition of a chart):

```
sage: c_spher(p)
(1, 1/2*pi, pi)
```

```
>>> from sage.all import *
>>> c_spher(p)
(1, 1/2*pi, pi)
```

A shortcut for `coordinates` is `coord`:

```
sage: p.coord()
(1, 1/2*pi, pi)
```

```
>>> from sage.all import *
>>> p.coord()
(1, 1/2*pi, pi)
```

Computing the Cartesian coordinates from the spherical ones:

```
sage: c_cart.<x,y,z> = M.chart()  # Cartesian coordinates
sage: c_spher.transition_map(c_cart, [r*sin(th)*cos(ph),
....:                                 r*sin(th)*sin(ph), r*cos(th)])
Change of coordinates from Chart (M, (r, th, ph)) to Chart (M, (x, y, z))
```

```
>>> from sage.all import *
>>> c_cart = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_cart._first_
↪ngens(3)# Cartesian coordinates
>>> c_spher.transition_map(c_cart, [r*sin(th)*cos(ph),
...                                 r*sin(th)*sin(ph), r*cos(th)])
Change of coordinates from Chart (M, (r, th, ph)) to Chart (M, (x, y, z))
```

The computation is performed by means of the above change of coordinates:

```
sage: p.coord(c_cart)
(-1, 0, 0)
sage: p.coord(c_cart) == c_cart(p)
True
```

```
>>> from sage.all import *
>>> p.coord(c_cart)
(-1, 0, 0)
>>> p.coord(c_cart) == c_cart(p)
True
```

Coordinates of a point on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: (a, b) = var('a b') # generic coordinates for the point
sage: P = M.point((a, b), name='P')
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> (a, b) = var('a b') # generic coordinates for the point
>>> P = M.point((a, b), name='P')
```

Coordinates of P in the manifold's default chart:

```
sage: P.coord()
(a, b)
```

```
>>> from sage.all import *
>>> P.coord()
(a, b)
```

Coordinates of P in a new chart:

```
sage: c_uv.<u,v> = M.chart()
sage: ch_xy_uv = c_xy.transition_map(c_uv, [x-y, x+y])
sage: P.coord(c_uv)
(a - b, a + b)
```

```
>>> from sage.all import *
>>> c_uv = M.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> ch_xy_uv = c_xy.transition_map(c_uv, [x-y, x+y])
>>> P.coord(c_uv)
(a - b, a + b)
```

Coordinates of P in a third chart:

```
sage: c_wz.<w,z> = M.chart()
sage: ch_uv_wz = c_uv.transition_map(c_wz, [u^3, v^3])
sage: P.coord(c_wz, old_chart=c_uv)
(a^3 - 3*a^2*b + 3*a*b^2 - b^3, a^3 + 3*a^2*b + 3*a*b^2 + b^3)
```

```
>>> from sage.all import *
>>> c_wz = M.chart(names=('w', 'z',)); (w, z,) = c_wz._first_ngens(2)
>>> ch_uv_wz = c_uv.transition_map(c_wz, [u**Integer(3), v**Integer(3)])
>>> P.coord(c_wz, old_chart=c_uv)
(a^3 - 3*a^2*b + 3*a*b^2 - b^3, a^3 + 3*a^2*b + 3*a*b^2 + b^3)
```

Actually, in the present case, it is not necessary to specify `old_chart='uv'`. Note that the first command erases all the coordinates except those in the chart `c_uv`:

```
sage: P.set_coord((a-b, a+b), c_uv)
sage: P._coordinates
{Chart (M, (u, v)): (a - b, a + b)}
sage: P.coord(c_wz)
(a^3 - 3*a^2*b + 3*a*b^2 - b^3, a^3 + 3*a^2*b + 3*a*b^2 + b^3)
sage: P._coordinates  # random (dictionary output)
{Chart (M, (u, v)): (a - b, a + b),
 Chart (M, (w, z)): (a^3 - 3*a^2*b + 3*a*b^2 - b^3,
                     a^3 + 3*a^2*b + 3*a*b^2 + b^3)}
```

```
>>> from sage.all import *
>>> P.set_coord((a-b, a+b), c_uv)
>>> P._coordinates
{Chart (M, (u, v)): (a - b, a + b)}
>>> P.coord(c_wz)
(a^3 - 3*a^2*b + 3*a*b^2 - b^3, a^3 + 3*a^2*b + 3*a*b^2 + b^3)
>>> P._coordinates  # random (dictionary output)
{Chart (M, (u, v)): (a - b, a + b),
 Chart (M, (w, z)): (a^3 - 3*a^2*b + 3*a*b^2 - b^3,
                     a^3 + 3*a^2*b + 3*a*b^2 + b^3)}
```

coordinates(*chart=None*, *old_chart=None*)

Return the point coordinates in the specified chart.

If these coordinates are not already known, they are computed from known ones by means of change-of-chart formulas.

An equivalent way to get the coordinates of a point is to let the chart acting on the point, i.e. if `X` is a chart and `p` a point, one has `p.coordinates(chart=X) == X(p)`.

INPUT:

- `chart` – (default: `None`) chart in which the coordinates are given; if none are provided, the coordinates are assumed to refer to the subset's default chart
- `old_chart` – (default: `None`) chart from which the coordinates in `chart` are to be computed; if `None`, a chart in which the point's coordinates are already known will be picked, privileging the subset's default chart

EXAMPLES:

Spherical coordinates of a point on $\mathbf{R}^3$:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: c_spher.<r,th,ph> = M.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\
↪phi') # spherical coordinates
sage: p = M.point((1, pi/2, pi))
sage: p.coordinates()  # coordinates in the manifold's default chart
(1, 1/2*pi, pi)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological')
>>> c_spher = M.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi', names=(
↪'r', 'th', 'ph',)); (r, th, ph,) = c_spher._first_ngens(3)# spherical␣
↪coordinates
>>> p = M.point((Integer(1), pi/Integer(2), pi))
>>> p.coordinates()  # coordinates in the manifold's default chart
(1, 1/2*pi, pi)
```

Since the default chart of `M` is `c_spher`, it is equivalent to write:

```
sage: p.coordinates(c_spher)
(1, 1/2*pi, pi)
```

```
>>> from sage.all import *
>>> p.coordinates(c_spher)
(1, 1/2*pi, pi)
```

An alternative way to get the coordinates is to let the chart act on the point (from the very definition of a chart):

```
sage: c_spher(p)
(1, 1/2*pi, pi)
```

```
>>> from sage.all import *
>>> c_spher(p)
(1, 1/2*pi, pi)
```

A shortcut for `coordinates` is `coord`:

```
sage: p.coord()
(1, 1/2*pi, pi)
```

```
>>> from sage.all import *
>>> p.coord()
(1, 1/2*pi, pi)
```

Computing the Cartesian coordinates from the spherical ones:

```
sage: c_cart.<x,y,z> = M.chart()  # Cartesian coordinates
sage: c_spher.transition_map(c_cart, [r*sin(th)*cos(ph),
....:                                   r*sin(th)*sin(ph), r*cos(th)])
Change of coordinates from Chart (M, (r, th, ph)) to Chart (M, (x, y, z))
```

```
>>> from sage.all import *
>>> c_cart = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_cart._first_
↪ngens(3)# Cartesian coordinates
>>> c_spher.transition_map(c_cart, [r*sin(th)*cos(ph),
...                                   r*sin(th)*sin(ph), r*cos(th)])
Change of coordinates from Chart (M, (r, th, ph)) to Chart (M, (x, y, z))
```

The computation is performed by means of the above change of coordinates:

```
sage: p.coord(c_cart)
(-1, 0, 0)
sage: p.coord(c_cart) == c_cart(p)
True
```

```
>>> from sage.all import *
>>> p.coord(c_cart)
(-1, 0, 0)
>>> p.coord(c_cart) == c_cart(p)
True
```

Coordinates of a point on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: (a, b) = var('a b') # generic coordinates for the point
sage: P = M.point((a, b), name='P')
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> (a, b) = var('a b') # generic coordinates for the point
>>> P = M.point((a, b), name='P')
```

Coordinates of `P` in the manifold's default chart:

```
sage: P.coord()
(a, b)
```

```
>>> from sage.all import *
>>> P.coord()
(a, b)
```

Coordinates of `P` in a new chart:

```
sage: c_uv.<u,v> = M.chart()
sage: ch_xy_uv = c_xy.transition_map(c_uv, [x-y, x+y])
sage: P.coord(c_uv)
(a - b, a + b)
```

```
>>> from sage.all import *
>>> c_uv = M.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> ch_xy_uv = c_xy.transition_map(c_uv, [x-y, x+y])
>>> P.coord(c_uv)
(a - b, a + b)
```

Coordinates of `P` in a third chart:

```
sage: c_wz.<w,z> = M.chart()
sage: ch_uv_wz = c_uv.transition_map(c_wz, [u^3, v^3])
sage: P.coord(c_wz, old_chart=c_uv)
(a^3 - 3*a^2*b + 3*a*b^2 - b^3, a^3 + 3*a^2*b + 3*a*b^2 + b^3)
```

```
>>> from sage.all import *
>>> c_wz = M.chart(names=('w', 'z',)); (w, z,) = c_wz._first_ngens(2)
>>> ch_uv_wz = c_uv.transition_map(c_wz, [u**Integer(3), v**Integer(3)])
>>> P.coord(c_wz, old_chart=c_uv)
(a^3 - 3*a^2*b + 3*a*b^2 - b^3, a^3 + 3*a^2*b + 3*a*b^2 + b^3)
```

Actually, in the present case, it is not necessary to specify `old_chart='uv'`. Note that the first command erases all the coordinates except those in the chart `c_uv`:

```
sage: P.set_coord((a-b, a+b), c_uv)
sage: P._coordinates
{Chart (M, (u, v)): (a - b, a + b)}
```

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```
sage: P.coord(c_wz)
(a^3 - 3*a^2*b + 3*a*b^2 - b^3, a^3 + 3*a^2*b + 3*a*b^2 + b^3)
sage: P._coordinates  # random (dictionary output)
{Chart (M, (u, v)): (a - b, a + b),
 Chart (M, (w, z)): (a^3 - 3*a^2*b + 3*a*b^2 - b^3,
                     a^3 + 3*a^2*b + 3*a*b^2 + b^3)}
```

```
>>> from sage.all import *
>>> P.set_coord((a-b, a+b), c_uv)
>>> P._coordinates
{Chart (M, (u, v)): (a - b, a + b)}
>>> P.coord(c_wz)
(a^3 - 3*a^2*b + 3*a*b^2 - b^3, a^3 + 3*a^2*b + 3*a*b^2 + b^3)
>>> P._coordinates  # random (dictionary output)
{Chart (M, (u, v)): (a - b, a + b),
 Chart (M, (w, z)): (a^3 - 3*a^2*b + 3*a*b^2 - b^3,
                     a^3 + 3*a^2*b + 3*a*b^2 + b^3)}
```

plot(*chart=None*, *ambient_coords=None*, *mapping=None*, *label=None*, *parameters=None*, *size=10*, *color='black'*, *label_color=None*, *fontsize=10*, *label_offset=0.1*, ***kwds*)

For real manifolds, plot `self` in a Cartesian graph based on the coordinates of some ambient chart.

The point is drawn in terms of two (2D graphics) or three (3D graphics) coordinates of a given chart, called hereafter the *ambient chart*. The domain of the ambient chart must contain the point, or its image by a continuous manifold map Φ.

INPUT:

- `chart` – (default: `None`) the ambient chart (see above); if `None`, the ambient chart is set the default chart of `self.parent()`
- `ambient_coords` – (default: `None`) tuple containing the 2 or 3 coordinates of the ambient chart in terms of which the plot is performed; if `None`, all the coordinates of the ambient chart are considered
- `mapping` – (default: `None`) `ContinuousMap`; continuous manifold map Φ providing the link between the current point p and the ambient chart `chart`: the domain of `chart` must contain $\Phi(p)$; if `None`, the identity map is assumed
- `label` – (default: `None`) label printed next to the point; if `None`, the point's name is used
- `parameters` – (default: `None`) dictionary giving the numerical values of the parameters that may appear in the point coordinates
- `size` – (default: 10) size of the point once drawn as a small disk or sphere
- `color` – (default: `'black'`) color of the point
- `label_color` – (default: `None`) color to print the label; if `None`, the value of `color` is used
- `fontsize` – (default: 10) size of the font used to print the label
- `label_offset` – (default: 0.1) determines the separation between the point and its label

OUTPUT:

- a graphic object, either an instance of `Graphics` for a 2D plot (i.e. based on 2 coordinates of the ambient chart) or an instance of `Graphics3d` for a 3D plot (i.e. based on 3 coordinates of the ambient chart)

EXAMPLES:

Drawing a point on a 2-dimensional manifold:

```
sage: # needs sage.plot
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: p = M.point((1,3), name='p')
sage: g = p.plot(X)
sage: print(g)
Graphics object consisting of 2 graphics primitives
sage: gX = X.plot(max_range=4) # plot of the coordinate grid
sage: g + gX # display of the point atop the coordinate grid
Graphics object consisting of 20 graphics primitives
```

```
>>> from sage.all import *
>>> # needs sage.plot
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> p = M.point((Integer(1),Integer(3)), name='p')
>>> g = p.plot(X)
>>> print(g)
Graphics object consisting of 2 graphics primitives
>>> gX = X.plot(max_range=Integer(4)) # plot of the coordinate grid
>>> g + gX # display of the point atop the coordinate grid
Graphics object consisting of 20 graphics primitives
```

Actually, since `X` is the default chart of the open set in which `p` has been defined, it can be skipped in the arguments of `plot`:

```
sage: # needs sage.plot
sage: g = p.plot()
sage: g + gX
Graphics object consisting of 20 graphics primitives
```

```
>>> from sage.all import *
>>> # needs sage.plot
>>> g = p.plot()
>>> g + gX
Graphics object consisting of 20 graphics primitives
```

Call with some options:

```
sage: # needs sage.plot
sage: g = p.plot(chart=X, size=40, color='green', label='$P$',
....:            label_color='blue', fontsize=20, label_offset=0.3)
sage: g + gX
Graphics object consisting of 20 graphics primitives
```

```
>>> from sage.all import *
>>> # needs sage.plot
>>> g = p.plot(chart=X, size=Integer(40), color='green', label='$P$',
...            label_color='blue', fontsize=Integer(20), label_
↪offset=RealNumber('0.3'))
>>> g + gX
```

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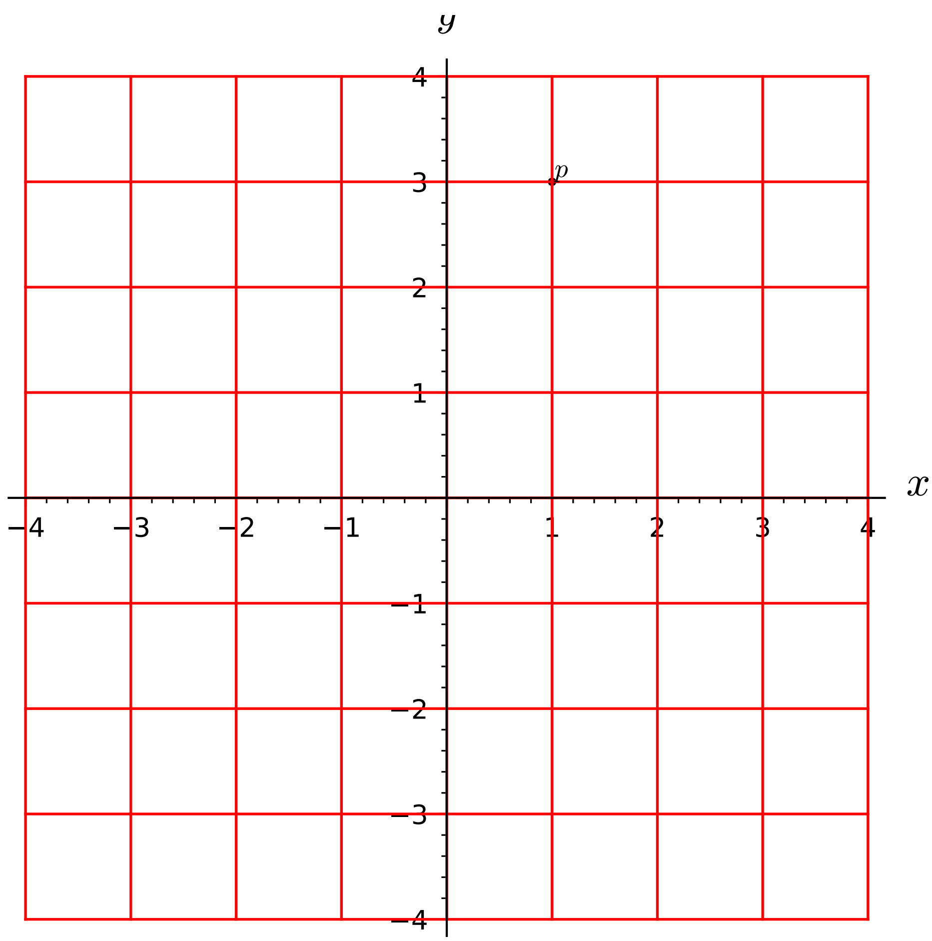
y
x
p
4
3
2
1
−1
−2
−3
−4
−4
−3
−2
−1
1
2
3
4

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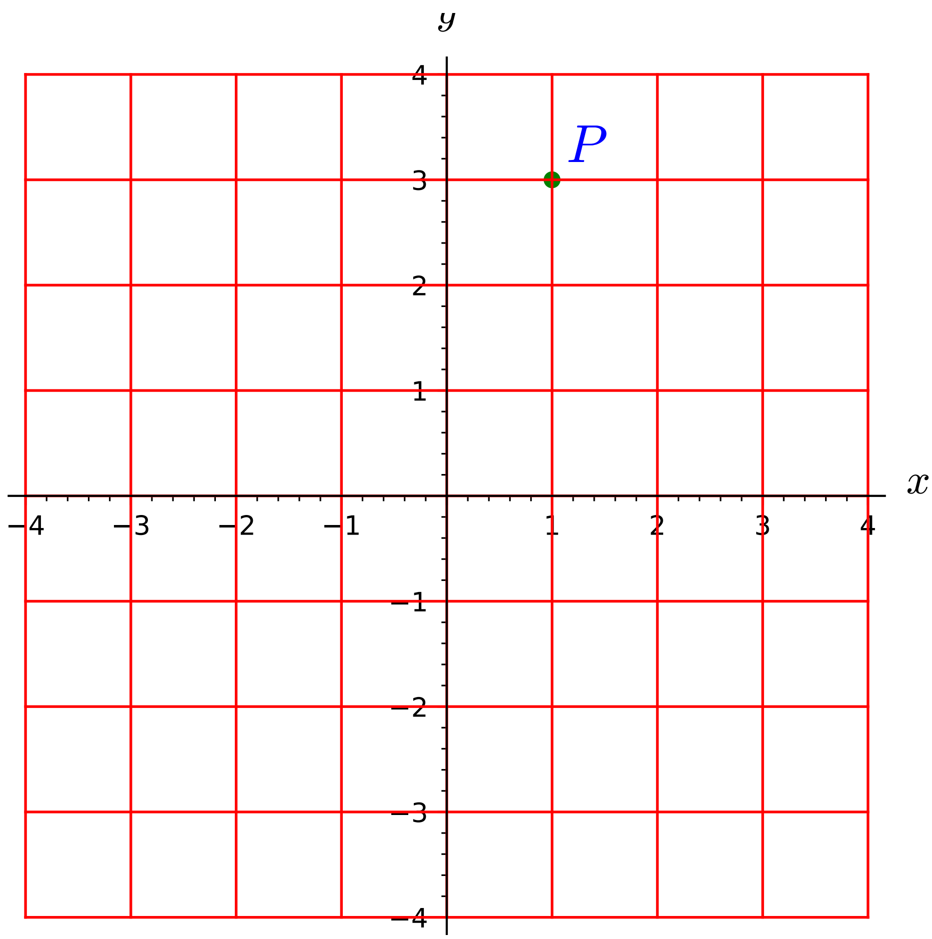
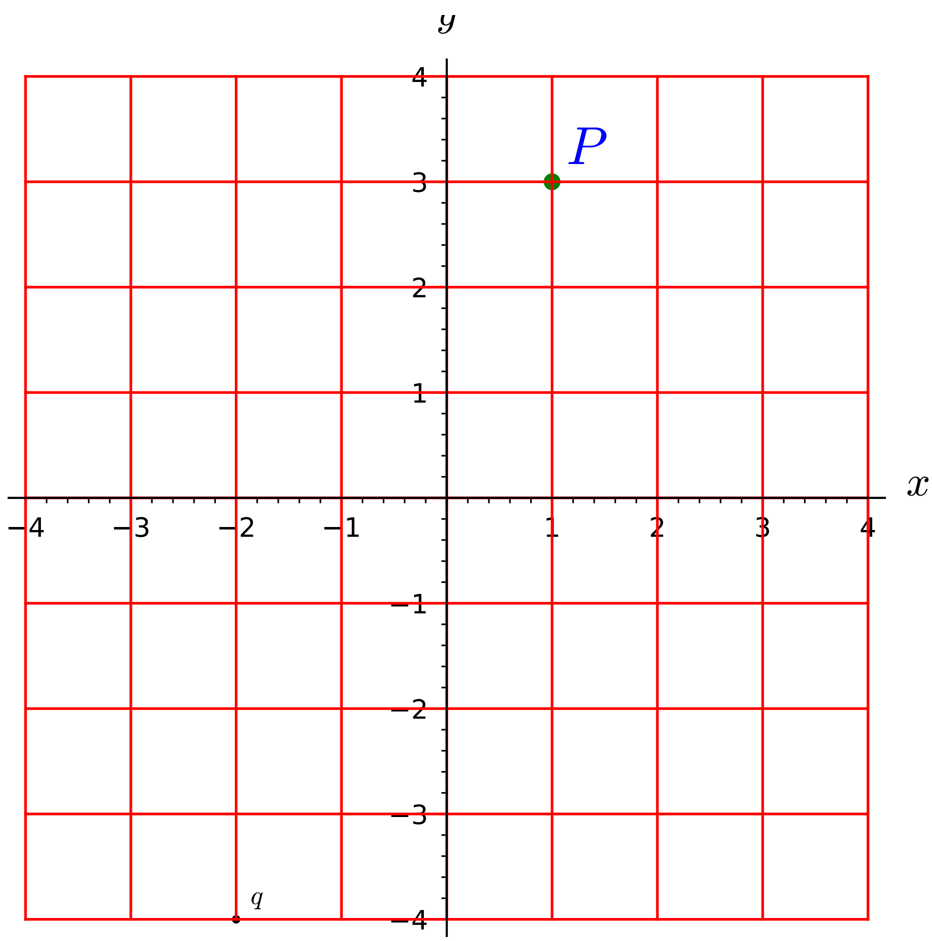
```
Graphics object consisting of 20 graphics primitives
```



Use of the `parameters` option to set a numerical value of some symbolic variable:

```
sage: a = var('a')
sage: q = M.point((a,2*a), name='q')                                   #␣
↪needs sage.plot
sage: gq = q.plot(parameters={a:-2}, label_offset=0.2)                 #␣
↪needs sage.plot
sage: g + gX + gq                                                      #␣
↪needs sage.plot
Graphics object consisting of 22 graphics primitives
```

```
>>> from sage.all import *
>>> a = var('a')
>>> q = M.point((a,Integer(2)*a), name='q')                            ␣
↪     # needs sage.plot
>>> gq = q.plot(parameters={a:-Integer(2)}, label_offset=RealNumber('0.2'))  ␣
↪                     # needs sage.plot
>>> g + gX + gq                                                      #␣
↪needs sage.plot
Graphics object consisting of 22 graphics primitives
```



y
x
P
q
4
3
2
1
−1
−2
−3
−4

The numerical value is used only for the plot:

```
sage: q.coord()                                                          #␣
↪needs sage.plot
(a, 2*a)
```

```
>>> from sage.all import *
>>> q.coord()                                                          #␣
↪needs sage.plot
(a, 2*a)
```

Drawing a point on a 3-dimensional manifold:

```
sage: # needs sage.plot
sage: M = Manifold(3, 'M', structure='topological')
sage: X.<x,y,z> = M.chart()
sage: p = M.point((2,1,3), name='p')
sage: g = p.plot()
sage: print(g)
Graphics3d Object
sage: gX = X.plot(number_values=5) # coordinate mesh cube
sage: g + gX # display of the point atop the coordinate mesh
Graphics3d Object
```

```
>>> from sage.all import *
>>> # needs sage.plot
>>> M = Manifold(Integer(3), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> p = M.point((Integer(2),Integer(1),Integer(3)), name='p')
>>> g = p.plot()
>>> print(g)
Graphics3d Object
>>> gX = X.plot(number_values=Integer(5)) # coordinate mesh cube
>>> g + gX # display of the point atop the coordinate mesh
Graphics3d Object
```

Call with some options:

```
sage: g = p.plot(chart=X, size=40, color='green', label='P_1',         #␣
↪needs sage.plot
....:             label_color='blue', fontsize=20, label_offset=0.3)
sage: g + gX                                                             #␣
↪needs sage.plot
Graphics3d Object
```

```
>>> from sage.all import *
>>> g = p.plot(chart=X, size=Integer(40), color='green', label='P_1',     ␣
↪     # needs sage.plot
...             label_color='blue', fontsize=Integer(20), label_
↪offset=RealNumber('0.3'))
>>> g + gX                                                             #␣
↪needs sage.plot
Graphics3d Object
```

An example of plot via a mapping: plot of a point on a 2-sphere viewed in the 3-dimensional space `M`:

```
sage: # needs sage.plot
sage: S2 = Manifold(2, 'S^2', structure='topological')
sage: U = S2.open_subset('U')  # the open set covered by spherical coord.
sage: XS.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: p = U.point((pi/4, pi/8), name='p')
sage: F = S2.continuous_map(M, {(XS, X): [sin(th)*cos(ph),
....:                          sin(th)*sin(ph), cos(th)]}, name='F')
sage: F.display()
F: S^2 → M
on U: (th, ph) ↦ (x, y, z) = (cos(ph)*sin(th), sin(ph)*sin(th), cos(th))
sage: g = p.plot(chart=X, mapping=F)
sage: gS2 = XS.plot(chart=X, mapping=F, number_values=9)
sage: g + gS2
Graphics3d Object
```

```
>>> from sage.all import *
>>> # needs sage.plot
>>> S2 = Manifold(Integer(2), 'S^2', structure='topological')
>>> U = S2.open_subset('U')  # the open set covered by spherical coord.
>>> XS = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi', names=('th', 'ph',));␣
↪(th, ph,) = XS._first_ngens(2)
>>> p = U.point((pi/Integer(4), pi/Integer(8)), name='p')
>>> F = S2.continuous_map(M, {(XS, X): [sin(th)*cos(ph),
...                          sin(th)*sin(ph), cos(th)]}, name='F')
>>> F.display()
F: S^2 → M
on U: (th, ph) ↦ (x, y, z) = (cos(ph)*sin(th), sin(ph)*sin(th), cos(th))
>>> g = p.plot(chart=X, mapping=F)
>>> gS2 = XS.plot(chart=X, mapping=F, number_values=Integer(9))
>>> g + gS2
Graphics3d Object
```

Use of the option `ambient_coords` for plots on a 4-dimensional manifold:

```
sage: # needs sage.plot
sage: M = Manifold(4, 'M', structure='topological')
sage: X.<t,x,y,z> = M.chart()
sage: p = M.point((1,2,3,4), name='p')
sage: g = p.plot(X, ambient_coords=(t,x,y), label_offset=0.4)  # the␣
↪coordinate z is skipped
sage: gX = X.plot(X, ambient_coords=(t,x,y), number_values=5)  # long time
sage: g + gX # 3D plot  # long time
Graphics3d Object
sage: g = p.plot(X, ambient_coords=(t,y,z), label_offset=0.4)  # the␣
↪coordinate x is skipped
sage: gX = X.plot(X, ambient_coords=(t,y,z), number_values=5)  # long time
sage: g + gX # 3D plot  # long time
Graphics3d Object
sage: g = p.plot(X, ambient_coords=(y,z), label_offset=0.4)  # the␣
↪coordinates t and x are skipped
sage: gX = X.plot(X, ambient_coords=(y,z))
```

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```
sage: g + gX # 2D plot
Graphics object consisting of 20 graphics primitives
```

```
>>> from sage.all import *
>>> # needs sage.plot
>>> M = Manifold(Integer(4), 'M', structure='topological')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> p = M.point((Integer(1),Integer(2),Integer(3),Integer(4)), name='p')
>>> g = p.plot(X, ambient_coords=(t,x,y), label_offset=RealNumber('0.4'))  #␣
↪the coordinate z is skipped
>>> gX = X.plot(X, ambient_coords=(t,x,y), number_values=Integer(5))  # long␣
↪time
>>> g + gX # 3D plot  # long time
Graphics3d Object
>>> g = p.plot(X, ambient_coords=(t,y,z), label_offset=RealNumber('0.4'))  #␣
↪the coordinate x is skipped
>>> gX = X.plot(X, ambient_coords=(t,y,z), number_values=Integer(5))  # long␣
↪time
>>> g + gX # 3D plot  # long time
Graphics3d Object
>>> g = p.plot(X, ambient_coords=(y,z), label_offset=RealNumber('0.4'))  #␣
↪the coordinates t and x are skipped
>>> gX = X.plot(X, ambient_coords=(y,z))
>>> g + gX # 2D plot
Graphics object consisting of 20 graphics primitives
```

set_coord(*coords*, *chart=None*)

Set the point coordinates in the specified chart.

Coordinates with respect to other charts are deleted, in order to avoid any inconsistency. To keep them, use the method *add_coord()* instead.

INPUT:

- coords – the point coordinates (as a tuple or a list)
- chart – (default: None) chart in which the coordinates are given; if none are provided, the coordinates are assumed to refer to the subset's default chart

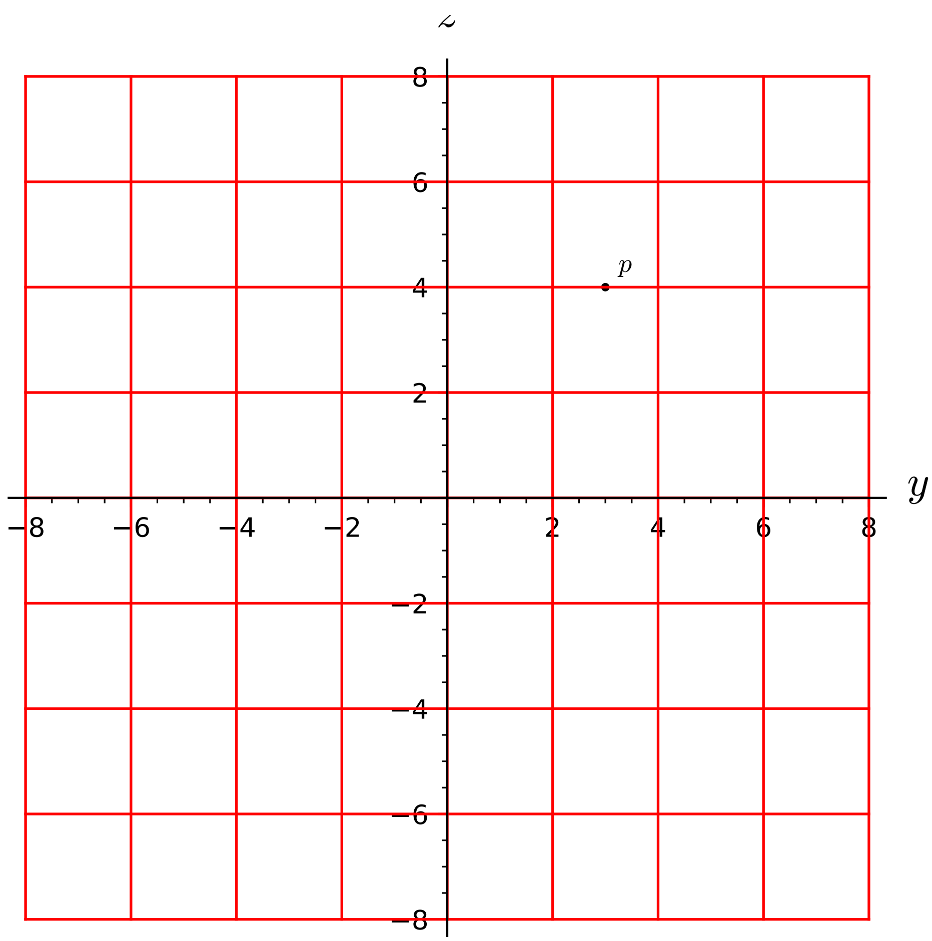
EXAMPLES:

Setting coordinates to a point on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: p = M.point()
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> p = M.point()
```

We set the coordinates in the manifold's default chart:


z
8
6
4
2
p
y
−8
−6
−4
−2
2
4
6
8
−2
−4
−6
−8

```
sage: p.set_coordinates((2,-3))
sage: p.coordinates()
(2, -3)
sage: X(p)
(2, -3)
```

```
>>> from sage.all import *
>>> p.set_coordinates((Integer(2),-Integer(3)))
>>> p.coordinates()
(2, -3)
>>> X(p)
(2, -3)
```

A shortcut for `set_coordinates` is `set_coord`:

```
sage: p.set_coord((2,-3))
sage: p.coord()
(2, -3)
```

```
>>> from sage.all import *
>>> p.set_coord((Integer(2),-Integer(3)))
>>> p.coord()
(2, -3)
```

Let us introduce a second chart on the manifold:

```
sage: Y.<u,v> = M.chart()
sage: X_to_Y = X.transition_map(Y, [x+y, x-y])
```

```
>>> from sage.all import *
>>> Y = M.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)
>>> X_to_Y = X.transition_map(Y, [x+y, x-y])
```

If we set the coordinates of `p` in chart `Y`, those in chart `X` are lost:

```
sage: Y(p)
(-1, 5)
sage: p.set_coord(Y(p), chart=Y)
sage: p._coordinates
{Chart (M, (u, v)): (-1, 5)}
```

```
>>> from sage.all import *
>>> Y(p)
(-1, 5)
>>> p.set_coord(Y(p), chart=Y)
>>> p._coordinates
{Chart (M, (u, v)): (-1, 5)}
```

set_coordinates(*coords*, *chart=None*)

Set the point coordinates in the specified chart.

Coordinates with respect to other charts are deleted, in order to avoid any inconsistency. To keep them, use the method *add_coord()* instead.

INPUT:

- `coords` – the point coordinates (as a tuple or a list)
- `chart` – (default: `None`) chart in which the coordinates are given; if none are provided, the coordinates are assumed to refer to the subset's default chart

EXAMPLES:

Setting coordinates to a point on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: p = M.point()
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> p = M.point()
```

We set the coordinates in the manifold's default chart:

```
sage: p.set_coordinates((2,-3))
sage: p.coordinates()
(2, -3)
sage: X(p)
(2, -3)
```

```
>>> from sage.all import *
>>> p.set_coordinates((Integer(2),-Integer(3)))
>>> p.coordinates()
(2, -3)
>>> X(p)
(2, -3)
```

A shortcut for `set_coordinates` is `set_coord`:

```
sage: p.set_coord((2,-3))
sage: p.coord()
(2, -3)
```

```
>>> from sage.all import *
>>> p.set_coord((Integer(2),-Integer(3)))
>>> p.coord()
(2, -3)
```

Let us introduce a second chart on the manifold:

```
sage: Y.<u,v> = M.chart()
sage: X_to_Y = X.transition_map(Y, [x+y, x-y])
```

```
>>> from sage.all import *
>>> Y = M.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)
>>> X_to_Y = X.transition_map(Y, [x+y, x-y])
```

If we set the coordinates of `p` in chart `Y`, those in chart `X` are lost:

```
sage: Y(p)
(-1, 5)
sage: p.set_coord(Y(p), chart=Y)
sage: p._coordinates
{Chart (M, (u, v)): (-1, 5)}
```

```
>>> from sage.all import *
>>> Y(p)
(-1, 5)
>>> p.set_coord(Y(p), chart=Y)
>>> p._coordinates
{Chart (M, (u, v)): (-1, 5)}
```

1.5 Coordinate Charts

1.5.1 Coordinate Charts

The class `Chart` implements coordinate charts on a topological manifold over a topological field K. The subclass `RealChart` is devoted to the case $K = \mathbf{R}$, for which the concept of coordinate range is meaningful. Moreover, `RealChart` is endowed with some plotting capabilities (cf. method `plot()`).

Transition maps between charts are implemented via the class `CoordChange`.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015) : initial version
- Travis Scrimshaw (2015): review tweaks
- Eric Gourgoulhon (2019): periodic coordinates, add `calculus_method()`

REFERENCES:

- Chap. 2 of [Lee2011]
- Chap. 1 of [Lee2013]

class `sage.manifolds.chart.`**`Chart`**(*domain*, *coordinates*, *calc_method=None*, *periods=None*, *coord_restrictions=None*)

Bases: `UniqueRepresentation`, `SageObject`

Chart on a topological manifold.

Given a topological manifold M of dimension n over a topological field K, a *chart* on M is a pair (U, φ), where U is an open subset of M and $\varphi : U \to V \subset K^n$ is a homeomorphism from U to an open subset V of K^n.

The components $(x^1, \ldots, x^n)$ of φ, defined by $\varphi(p) = (x^1(p), \ldots, x^n(p)) \in K^n$ for any point $p \in U$, are called the *coordinates* of the chart (U, φ).

INPUT:

- `domain` – open subset U on which the chart is defined (must be an instance of `TopologicalManifold`)
- `coordinates` – (default: '' (empty string)) single string defining the coordinate symbols, with `' '` (whitespace) as a separator; each item has at most three fields, separated by a colon (`:`):
  1. the coordinate symbol (a letter or a few letters)
  2. (optional) the period of the coordinate if the coordinate is periodic; the period field must be written as `period=T`, where `T` is the period (see examples below)

3. (optional) the LaTeX spelling of the coordinate; if not provided the coordinate symbol given in the first field will be used

The order of fields 2 and 3 does not matter and each of them can be omitted. If it contains any LaTeX expression, the string `coordinates` must be declared with the prefix 'r' (for "raw") to allow for a proper treatment of LaTeX's backslash character (see examples below). If no period and no LaTeX spelling are to be set for any coordinate, the argument `coordinates` can be omitted when the shortcut operator `<,>` is used to declare the chart (see examples below).

- `calc_method` – (default: `None`) string defining the calculus method for computations involving coordinates of the chart; must be one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy
  - `None`: the default of *CalculusMethod* will be used
- `names` – (default: `None`) unused argument, except if `coordinates` is not provided; it must then be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)
- `coord_restrictions` – additional restrictions on the coordinates. A restriction can be any symbolic equality or inequality involving the coordinates, such as `x > y` or `x^2 + y^2 != 0`. The items of the list (or set or frozenset) `coord_restrictions` are combined with the `and` operator; if some restrictions are to be combined with the `or` operator instead, they have to be passed as a tuple in some single item of the list (or set or frozenset) `coord_restrictions`. For example:

```
coord_restrictions=[x > y, (x != 0, y != 0), z^2 < x]
```

means `(x > y) and ((x != 0) or (y != 0)) and (z^2 < x)`. If the list `coord_restrictions` contains only one item, this item can be passed as such, i.e. writing `x > y` instead of the single element list `[x > y]`. If the chart variables have not been declared as variables yet, `coord_restrictions` must be `lambda`-quoted.

EXAMPLES:

A chart on a complex 2-dimensional topological manifold:

```
sage: M = Manifold(2, 'M', field='complex', structure='topological')
sage: X = M.chart('x y'); X
Chart (M, (x, y))
sage: latex(X)
\left(M,(x, y)\right)
sage: type(X)
<class 'sage.manifolds.chart.Chart'>
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', field='complex', structure='topological')
>>> X = M.chart('x y'); X
Chart (M, (x, y))
>>> latex(X)
\left(M,(x, y)\right)
>>> type(X)
<class 'sage.manifolds.chart.Chart'>
```

To manipulate the coordinates (x, y) as global variables, one has to set:

```
sage: x,y = X[:]
```

```
>>> from sage.all import *
>>> x,y = X[:]
```

However, a shortcut is to use the declarator `<x,y>` in the left-hand side of the chart declaration (there is then no need to pass the string `'x y'` to `chart()`):

```
sage: M = Manifold(2, 'M', field='complex', structure='topological')
sage: X.<x,y> = M.chart(); X
Chart (M, (x, y))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', field='complex', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2); X
Chart (M, (x, y))
```

The coordinates are then immediately accessible:

```
sage: y
y
sage: x is X[0] and y is X[1]
True
```

```
>>> from sage.all import *
>>> y
y
>>> x is X[Integer(0)] and y is X[Integer(1)]
True
```

Note that `x` and `y` declared in `<x,y>` are mere Python variable names and do not have to coincide with the coordinate symbols; for instance, one may write:

```
sage: M = Manifold(2, 'M', field='complex', structure='topological')
sage: X.<x1,y1> = M.chart('x y'); X
Chart (M, (x, y))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', field='complex', structure='topological')
>>> X = M.chart('x y', names=('x1', 'y1',)); (x1, y1,) = X._first_ngens(2); X
Chart (M, (x, y))
```

Then `y` is not known as a global Python variable and the coordinate y is accessible only through the global variable `y1`:

```
sage: y1
y
sage: latex(y1)
y
sage: y1 is X[1]
True
```

```
>>> from sage.all import *
>>> y1
```

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```
y
>>> latex(y1)
y
>>> y1 is X[Integer(1)]
True
```

However, having the name of the Python variable coincide with the coordinate symbol is quite convenient; so it is recommended to declare:

```
sage: M = Manifold(2, 'M', field='complex', structure='topological')
sage: X.<x,y> = M.chart()
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', field='complex', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
```

In the above example, the chart X covers entirely the manifold `M`:

```
sage: X.domain()
Complex 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> X.domain()
Complex 2-dimensional topological manifold M
```

Of course, one may declare a chart only on an open subset of `M`:

```
sage: U = M.open_subset('U')
sage: Y.<z1, z2> = U.chart(r'z1:\zeta_1 z2:\zeta_2'); Y
Chart (U, (z1, z2))
sage: Y.domain()
Open subset U of the Complex 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> U = M.open_subset('U')
>>> Y = U.chart(r'z1:\zeta_1 z2:\zeta_2', names=('z1', 'z2',)); (z1, z2,) = Y._
↪first_ngens(2); Y
Chart (U, (z1, z2))
>>> Y.domain()
Open subset U of the Complex 2-dimensional topological manifold M
```

In the above declaration, we have also specified some LaTeX writing of the coordinates different from the text one:

```
sage: latex(z1)
{\zeta_1}
```

```
>>> from sage.all import *
>>> latex(z1)
{\zeta_1}
```

Note the prefix `r` in front of the string `r'z1:\zeta_1 z2:\zeta_2'`; it makes sure that the backslash character is treated as an ordinary character, to be passed to the LaTeX interpreter.

Periodic coordinates are declared through the keyword `period=` in the coordinate field:

```
sage: N = Manifold(2, 'N', field='complex', structure='topological')
sage: XN.<Z1,Z2> = N.chart('Z1:period=1+2*I Z2')
sage: XN.periods()
(2*I + 1, None)
```

```
>>> from sage.all import *
>>> N = Manifold(Integer(2), 'N', field='complex', structure='topological')
>>> XN = N.chart('Z1:period=1+2*I Z2', names=('Z1', 'Z2',)); (Z1, Z2,) = XN._
↪first_ngens(2)
>>> XN.periods()
(2*I + 1, None)
```

Coordinates are Sage symbolic variables (see `sage.symbolic.expression`):

```
sage: type(z1)
<class 'sage.symbolic.expression.Expression'>
```

```
>>> from sage.all import *
>>> type(z1)
<class 'sage.symbolic.expression.Expression'>
```

In addition to the Python variable name provided in the operator `<.,.>`, the coordinates are accessible by their indices:

```
sage: Y[0], Y[1]
(z1, z2)
```

```
>>> from sage.all import *
>>> Y[Integer(0)], Y[Integer(1)]
(z1, z2)
```

The index range is that declared during the creation of the manifold. By default, it starts at 0, but this can be changed via the parameter `start_index`:

```
sage: M1 = Manifold(2, 'M_1', field='complex', structure='topological',
....:               start_index=1)
sage: Z.<u,v> = M1.chart()
sage: Z[1], Z[2]
(u, v)
```

```
>>> from sage.all import *
>>> M1 = Manifold(Integer(2), 'M_1', field='complex', structure='topological',
...               start_index=Integer(1))
>>> Z = M1.chart(names=('u', 'v',)); (u, v,) = Z._first_ngens(2)
>>> Z[Integer(1)], Z[Integer(2)]
(u, v)
```

The full set of coordinates is obtained by means of the slice operator `[:]`:

```
sage: Y[:]
(z1, z2)
```

```
>>> from sage.all import *
>>> Y[:]
(z1, z2)
```

Some partial sets of coordinates:

```
sage: Y[:1]
(z1,)
sage: Y[1:]
(z2,)
```

```
>>> from sage.all import *
>>> Y[:Integer(1)]
(z1,)
>>> Y[Integer(1):]
(z2,)
```

Each constructed chart is automatically added to the manifold's user atlas:

```
sage: M.atlas()
[Chart (M, (x, y)), Chart (U, (z1, z2))]
```

```
>>> from sage.all import *
>>> M.atlas()
[Chart (M, (x, y)), Chart (U, (z1, z2))]
```

and to the atlas of the chart's domain:

```
sage: U.atlas()
[Chart (U, (z1, z2))]
```

```
>>> from sage.all import *
>>> U.atlas()
[Chart (U, (z1, z2))]
```

Manifold subsets have a *default chart*, which, unless changed via the method `set_default_chart()`, is the first defined chart on the subset (or on a open subset of it):

```
sage: M.default_chart()
Chart (M, (x, y))
sage: U.default_chart()
Chart (U, (z1, z2))
```

```
>>> from sage.all import *
>>> M.default_chart()
Chart (M, (x, y))
>>> U.default_chart()
Chart (U, (z1, z2))
```

The default charts are not privileged charts on the manifold, but rather charts whose name can be skipped in the argument list of functions having an optional `chart=` argument.

The chart map φ acting on a point is obtained by passing it as an input to the map:

```
sage: p = M.point((1+i, 2), chart=X); p
Point on the Complex 2-dimensional topological manifold M
sage: X(p)
(I + 1, 2)
sage: X(p) == p.coord(X)
True
```

```
>>> from sage.all import *
>>> p = M.point((Integer(1)+i, Integer(2)), chart=X); p
Point on the Complex 2-dimensional topological manifold M
>>> X(p)
(I + 1, 2)
>>> X(p) == p.coord(X)
True
```

Setting additional coordinate restrictions:

```
sage: M = Manifold(2, 'M', field='complex', structure='topological')
sage: X.<x,y> = M.chart(coord_restrictions=lambda x,y: abs(x) > 1)
sage: X.valid_coordinates(2+i, 1)
True
sage: X.valid_coordinates(i, 1)
False
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', field='complex', structure='topological')
>>> X = M.chart(coord_restrictions=lambda x,y: abs(x) > Integer(1), names=('x', 'y
↪',)); (x, y,) = X._first_ngens(2)
>>> X.valid_coordinates(Integer(2)+i, Integer(1))
True
>>> X.valid_coordinates(i, Integer(1))
False
```

See also

sage.manifolds.chart.RealChart for charts on topological manifolds over **R**.

add_restrictions(*restrictions*)

Add some restrictions on the coordinates.

This is deprecated; provide the restrictions at the time of creating the chart.

INPUT:

- `restrictions` – list of restrictions on the coordinates, in addition to the ranges declared by the intervals specified in the chart constructor

A restriction can be any symbolic equality or inequality involving the coordinates, such as `x > y` or `x^2 + y^2 != 0`. The items of the list `restrictions` are combined with the `and` operator; if some restrictions are to be combined with the `or` operator instead, they have to be passed as a tuple in some single item of the list `restrictions`. For example:

```
restrictions = [x > y, (x != 0, y != 0), z^2 < x]
```

means `(x > y) and ((x != 0) or (y != 0)) and (z^2 < x)`. If the list `restrictions` contains only one item, this item can be passed as such, i.e. writing `x > y` instead of the single element list `[x > y]`.

EXAMPLES:

```
sage: M = Manifold(2, 'M', field='complex', structure='topological')
sage: X.<x,y> = M.chart()
sage: X.add_restrictions(abs(x) > 1)
doctest:warning...
DeprecationWarning: Chart.add_restrictions is deprecated; provide the
restrictions at the time of creating the chart
See https://github.com/sagemath/sage/issues/32102 for details.
sage: X.valid_coordinates(2+i, 1)
True
sage: X.valid_coordinates(i, 1)
False
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', field='complex', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> X.add_restrictions(abs(x) > Integer(1))
doctest:warning...
DeprecationWarning: Chart.add_restrictions is deprecated; provide the
restrictions at the time of creating the chart
See https://github.com/sagemath/sage/issues/32102 for details.
>>> X.valid_coordinates(Integer(2)+i, Integer(1))
True
>>> X.valid_coordinates(i, Integer(1))
False
```

calculus_method()

Return the interface governing the calculus engine for expressions involving coordinates of this chart.

The calculus engine can be one of the following:

- Sage's symbolic engine (Pynac + Maxima), implemented via the Symbolic Ring `SR`
- SymPy

See also

CalculusMethod for a complete documentation.

OUTPUT: an instance of *CalculusMethod*

EXAMPLES:

The default calculus method relies on Sage's Symbolic Ring:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: X.calculus_method()
```

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```
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> X.calculus_method()
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
```

Accordingly the method `expr()` of a function `f` defined on the chart `X` returns a Sage symbolic expression:

```
sage: f = X.function(x^2 + cos(y)*sin(x))
sage: f.expr()
x^2 + cos(y)*sin(x)
sage: type(f.expr())
<class 'sage.symbolic.expression.Expression'>
sage: parent(f.expr())
Symbolic Ring
sage: f.display()
(x, y) ↦ x^2 + cos(y)*sin(x)
```

```
>>> from sage.all import *
>>> f = X.function(x**Integer(2) + cos(y)*sin(x))
>>> f.expr()
x^2 + cos(y)*sin(x)
>>> type(f.expr())
<class 'sage.symbolic.expression.Expression'>
>>> parent(f.expr())
Symbolic Ring
>>> f.display()
(x, y) ↦ x^2 + cos(y)*sin(x)
```

Changing to SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f.expr()
x**2 + sin(x)*cos(y)
sage: type(f.expr())
<class 'sympy.core.add.Add'>
sage: parent(f.expr())
<class 'sympy.core.add.Add'>
sage: f.display()
(x, y) ↦ x**2 + sin(x)*cos(y)
```

```
>>> from sage.all import *
>>> X.calculus_method().set('sympy')
>>> f.expr()
x**2 + sin(x)*cos(y)
>>> type(f.expr())
```

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```
<class 'sympy.core.add.Add'>
>>> parent(f.expr())
<class 'sympy.core.add.Add'>
>>> f.display()
(x, y) ↦ x**2 + sin(x)*cos(y)
```

Back to the Symbolic Ring:

```
sage: X.calculus_method().set('SR')
sage: f.display()
(x, y) ↦ x^2 + cos(y)*sin(x)
```

```
>>> from sage.all import *
>>> X.calculus_method().set('SR')
>>> f.display()
(x, y) ↦ x^2 + cos(y)*sin(x)
```

codomain()

Return the codomain of `self` as a set.

EXAMPLES:

```
sage: M = Manifold(2, 'M', field='complex', structure='topological')
sage: X.<x,y> = M.chart()
sage: X.codomain()
Vector space of dimension 2 over Complex Field with 53 bits of precision
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', field='complex', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> X.codomain()
Vector space of dimension 2 over Complex Field with 53 bits of precision
```

domain()

Return the open subset on which the chart is defined.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: X.domain()
2-dimensional topological manifold M
sage: U = M.open_subset('U')
sage: Y.<u,v> = U.chart()
sage: Y.domain()
Open subset U of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> X.domain()
2-dimensional topological manifold M
```

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```
>>> U = M.open_subset('U')
>>> Y = U.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)
>>> Y.domain()
Open subset U of the 2-dimensional topological manifold M
```

function(*expression*, *calc_method=None*, *expansion_symbol=None*, *order=None*)

Define a coordinate function to the base field.

If the current chart belongs to the atlas of a n-dimensional manifold over a topological field K, a *coordinate function* is a map

$$\begin{array}{llcl} f: & V \subset K^n & \longrightarrow & K \\ & (x^1, \ldots, x^n) & \longmapsto & f(x^1, \ldots, x^n), \end{array}$$

where V is the chart codomain and $(x^1, \ldots, x^n)$ are the chart coordinates.

INPUT:

- `expression` – a symbolic expression involving the chart coordinates, to represent $f(x^1, \ldots, x^n)$
- `calc_method` – string (default: `None`); the calculus method with respect to which the internal expression of the function must be initialized from `expression`; one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy
  - `None`: the chart current calculus method is assumed
- `expansion_symbol` – (default: `None`) symbolic variable (the "small parameter") with respect to which the coordinate expression is expanded in power series (around the zero value of this variable)
- `order` – integer (default: `None`); the order of the expansion if `expansion_symbol` is not `None`; the *order* is defined as the degree of the polynomial representing the truncated power series in `expansion_symbol`.

> **Warning**
>
> The value of `order` is $n - 1$, where n is the order of the big O in the power series expansion

OUTPUT:

- instance of *ChartFunction* representing the coordinate function f

EXAMPLES:

A symbolic coordinate function:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(sin(x*y))
sage: f
sin(x*y)
sage: type(f)
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>
sage: f.display()
```

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```
(x, y) ↦ sin(x*y)
sage: f(2,3)
sin(6)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(sin(x*y))
>>> f
sin(x*y)
>>> type(f)
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>
>>> f.display()
(x, y) ↦ sin(x*y)
>>> f(Integer(2),Integer(3))
sin(6)
```

Using SymPy for the internal representation of the function (dictionary `_express`):

```
sage: g = X.function(x^2 + x*cos(y), calc_method='sympy')
sage: g._express
{'sympy': x**2 + x*cos(y)}
```

```
>>> from sage.all import *
>>> g = X.function(x**Integer(2) + x*cos(y), calc_method='sympy')
>>> g._express
{'sympy': x**2 + x*cos(y)}
```

On the contrary, for `f`, only the `SR` part has been initialized:

```
sage: f._express
{'SR': sin(x*y)}
```

```
>>> from sage.all import *
>>> f._express
{'SR': sin(x*y)}
```

See `ChartFunction` for more examples.

function_ring()

Return the ring of coordinate functions on `self`.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: X.function_ring()
Ring of chart functions on Chart (M, (x, y))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
```

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```
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> X.function_ring()
Ring of chart functions on Chart (M, (x, y))
```

manifold()

Return the manifold on which the chart is defined.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: U = M.open_subset('U')
sage: X.<x,y> = U.chart()
sage: X.manifold()
2-dimensional topological manifold M
sage: X.domain()
Open subset U of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> U = M.open_subset('U')
>>> X = U.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> X.manifold()
2-dimensional topological manifold M
>>> X.domain()
Open subset U of the 2-dimensional topological manifold M
```

multifunction(**expressions*)

Define a coordinate function to some Cartesian power of the base field.

If n and m are two positive integers and (U, φ) is a chart on a topological manifold M of dimension n over a topological field K, a *multi-coordinate function* associated to (U, φ) is a map

$$\begin{array}{llll} f: & V \subset K^n & \longrightarrow & K^m \\ & (x^1, \ldots, x^n) & \longmapsto & (f_1(x^1, \ldots, x^n), \ldots, f_m(x^1, \ldots, x^n)), \end{array}$$

where V is the codomain of φ. In other words, f is a K^m-valued function of the coordinates associated to the chart (U, φ).

See `MultiCoordFunction` for a complete documentation.

INPUT:

- `expressions` – list (or tuple) of m elements to construct the coordinate functions f_i ($1 \leq i \leq m$); for symbolic coordinate functions, this must be symbolic expressions involving the chart coordinates, while for numerical coordinate functions, this must be data file names

OUTPUT:

- a `MultiCoordFunction` representing f

EXAMPLES:

Function of two coordinates with values in $\mathbf{R}^3$:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.multifunction(x+y, sin(x*y), x^2 + 3*y); f
```

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```
Coordinate functions (x + y, sin(x*y), x^2 + 3*y) on the Chart (M, (x, y))
sage: f(2,3)
(5, sin(6), 13)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.multifunction(x+y, sin(x*y), x**Integer(2) + Integer(3)*y); f
Coordinate functions (x + y, sin(x*y), x^2 + 3*y) on the Chart (M, (x, y))
>>> f(Integer(2),Integer(3))
(5, sin(6), 13)
```

one_function()

Return the constant function of the coordinates equal to one.

If the current chart belongs to the atlas of a n-dimensional manifold over a topological field K, the "one" coordinate function is the map

$$\begin{array}{llcl} f: & V \subset K^n & \longrightarrow & K \\ & (x^1, \ldots, x^n) & \longmapsto & 1, \end{array}$$

where V is the chart codomain.

See class *ChartFunction* for a complete documentation.

OUTPUT:

- a *ChartFunction* representing the one coordinate function f

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: X.one_function()
1
sage: X.one_function().display()
(x, y) ↦ 1
sage: type(X.one_function())
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> X.one_function()
1
>>> X.one_function().display()
(x, y) ↦ 1
>>> type(X.one_function())
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>
```

The result is cached:

```
sage: X.one_function() is X.one_function()
True
```

```
>>> from sage.all import *
>>> X.one_function() is X.one_function()
True
```

One function on a p-adic manifold:

```
sage: # needs sage.rings.padics
sage: M = Manifold(2, 'M', structure='topological', field=Qp(5)); M
2-dimensional topological manifold M over the 5-adic Field with
 capped relative precision 20
sage: X.<x,y> = M.chart()
sage: X.one_function()
1 + O(5^20)
sage: X.one_function().display()
(x, y) ↦ 1 + O(5^20)
```

```
>>> from sage.all import *
>>> # needs sage.rings.padics
>>> M = Manifold(Integer(2), 'M', structure='topological',␣
↪field=Qp(Integer(5))); M
2-dimensional topological manifold M over the 5-adic Field with
 capped relative precision 20
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> X.one_function()
1 + O(5^20)
>>> X.one_function().display()
(x, y) ↦ 1 + O(5^20)
```

periods()

Return the coordinate periods.

OUTPUT:

- a tuple containing the period of each coordinate, with the value `None` if the coordinate is not periodic

EXAMPLES:

A chart without any periodic coordinate:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: X.periods()
(None, None)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> X.periods()
(None, None)
```

Charts with a periodic coordinate:

```
sage: Y.<u,v> = M.chart("u v:(0,2*pi):periodic")
sage: Y.periods()
(None, 2*pi)
sage: Z.<a,b> = M.chart(r"a:period=sqrt(2):\alpha b:\beta")
sage: Z.periods()
(sqrt(2), None)
```

```
>>> from sage.all import *
>>> Y = M.chart("u v:(0,2*pi):periodic", names=('u', 'v',)); (u, v,) = Y._
↪first_ngens(2)
>>> Y.periods()
(None, 2*pi)
>>> Z = M.chart(r"a:period=sqrt(2):\alpha b:\beta", names=('a', 'b',)); (a, b,
↪) = Z._first_ngens(2)
>>> Z.periods()
(sqrt(2), None)
```

Complex manifold with a periodic coordinate:

```
sage: M = Manifold(2, 'M', field='complex', structure='topological',
....:              start_index=1)
sage: X.<x,y> = M.chart("x y:period=1+I")
sage: X.periods()
(None, I + 1)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', field='complex', structure='topological',
...              start_index=Integer(1))
>>> X = M.chart("x y:period=1+I", names=('x', 'y',)); (x, y,) = X._first_
↪ngens(2)
>>> X.periods()
(None, I + 1)
```

preimage(*codomain_subset*, *name=None*, *latex_name=None*)

Return the preimage (pullback) of `codomain_subset` under `self`.

It is the subset of the domain of `self` formed by the points whose coordinate vectors lie in `codomain_subset`.

INPUT:

- `codomain_subset` – an instance of `ConvexSet_base` or another object with a `__contains__` method that accepts coordinate vectors
- `name` – string; name (symbol) given to the subset
- `latex_name` – string (default: `None`); LaTeX symbol to denote the subset; if none are provided, it is set to `name`

OUTPUT:

- either a *TopologicalManifold* or a *ManifoldSubsetPullback*

EXAMPLES:

```
sage: M = Manifold(2, 'R^2', structure='topological')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', structure='topological')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#␣
↪Cartesian coordinates on R^2
```

Pulling back a polytope under a chart:

```
sage: # needs sage.geometry.polyhedron
sage: P = Polyhedron(vertices=[[0, 0], [1, 2], [2, 1]]); P
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
sage: McP = c_cart.preimage(P); McP
Subset x_y_inv_P of the 2-dimensional topological manifold R^2
sage: M((1, 2)) in McP
True
sage: M((2, 0)) in McP
False
```

```
>>> from sage.all import *
>>> # needs sage.geometry.polyhedron
>>> P = Polyhedron(vertices=[[Integer(0), Integer(0)], [Integer(1),␣
↪Integer(2)], [Integer(2), Integer(1)]]); P
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
>>> McP = c_cart.preimage(P); McP
Subset x_y_inv_P of the 2-dimensional topological manifold R^2
>>> M((Integer(1), Integer(2))) in McP
True
>>> M((Integer(2), Integer(0))) in McP
False
```

Pulling back the interior of a polytope under a chart:

```
sage: # needs sage.geometry.polyhedron
sage: int_P = P.interior(); int_P
Relative interior of
 a 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
sage: McInt_P = c_cart.preimage(int_P, name='McInt_P'); McInt_P
Open subset McInt_P of the 2-dimensional topological manifold R^2
sage: M((0, 0)) in McInt_P
False
sage: M((1, 1)) in McInt_P
True
```

```
>>> from sage.all import *
>>> # needs sage.geometry.polyhedron
>>> int_P = P.interior(); int_P
Relative interior of
 a 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
>>> McInt_P = c_cart.preimage(int_P, name='McInt_P'); McInt_P
Open subset McInt_P of the 2-dimensional topological manifold R^2
>>> M((Integer(0), Integer(0))) in McInt_P
False
>>> M((Integer(1), Integer(1))) in McInt_P
True
```

Pulling back a point lattice:

```
sage: W = span([[1, 0], [3, 5]], ZZ); W
Free module of degree 2 and rank 2 over Integer Ring
Echelon basis matrix:
[1 0]
[0 5]
sage: McW = c_cart.pullback(W, name='McW'); McW
Subset McW of the 2-dimensional topological manifold R^2
sage: M((4, 5)) in McW
True
sage: M((4, 4)) in McW
False
```

```
>>> from sage.all import *
>>> W = span([[Integer(1), Integer(0)], [Integer(3), Integer(5)]], ZZ); W
Free module of degree 2 and rank 2 over Integer Ring
Echelon basis matrix:
[1 0]
[0 5]
>>> McW = c_cart.pullback(W, name='McW'); McW
Subset McW of the 2-dimensional topological manifold R^2
>>> M((Integer(4), Integer(5))) in McW
True
>>> M((Integer(4), Integer(4))) in McW
False
```

Pulling back a real vector subspaces:

```
sage: V = span([[1, 2]], RR); V
Vector space of degree 2 and dimension 1 over Real Field with 53 bits of␣
↪precision
Basis matrix:
[1.00000000000000 2.00000000000000]
sage: McV = c_cart.pullback(V, name='McV'); McV
Subset McV of the 2-dimensional topological manifold R^2
sage: M((2, 4)) in McV
True
sage: M((1, 0)) in McV
False
```

```
>>> from sage.all import *
>>> V = span([[Integer(1), Integer(2)]], RR); V
Vector space of degree 2 and dimension 1 over Real Field with 53 bits of␣
↪precision
Basis matrix:
[1.00000000000000 2.00000000000000]
>>> McV = c_cart.pullback(V, name='McV'); McV
Subset McV of the 2-dimensional topological manifold R^2
>>> M((Integer(2), Integer(4))) in McV
True
>>> M((Integer(1), Integer(0))) in McV
False
```

Pulling back a finite set of points:

```
sage: F = Family([vector(QQ, [1, 2], immutable=True),
....:             vector(QQ, [2, 3], immutable=True)])
sage: McF = c_cart.pullback(F, name='McF'); McF
Subset McF of the 2-dimensional topological manifold R^2
sage: M((2, 3)) in McF
True
sage: M((0, 0)) in McF
False
```

```
>>> from sage.all import *
>>> F = Family([vector(QQ, [Integer(1), Integer(2)], immutable=True),
...             vector(QQ, [Integer(2), Integer(3)], immutable=True)])
>>> McF = c_cart.pullback(F, name='McF'); McF
Subset McF of the 2-dimensional topological manifold R^2
>>> M((Integer(2), Integer(3))) in McF
True
>>> M((Integer(0), Integer(0))) in McF
False
```

Pulling back the integers:

```
sage: R = manifolds.RealLine(); R
Real number line ℝ
sage: McZ = R.canonical_chart().pullback(ZZ, name='ℤ'); McZ
Subset ℤ of the Real number line ℝ
sage: R((3/2,)) in McZ
False
sage: R((-2,)) in McZ
True
```

```
>>> from sage.all import *
>>> R = manifolds.RealLine(); R
Real number line ℝ
>>> McZ = R.canonical_chart().pullback(ZZ, name='ℤ'); McZ
Subset ℤ of the Real number line ℝ
>>> R((Integer(3)/Integer(2),)) in McZ
False
>>> R((-Integer(2),)) in McZ
True
```

pullback(*codomain_subset*, *name=None*, *latex_name=None*)

Return the preimage (pullback) of `codomain_subset` under `self`.

It is the subset of the domain of `self` formed by the points whose coordinate vectors lie in `codomain_subset`.

INPUT:

- `codomain_subset` – an instance of `ConvexSet_base` or another object with a `__contains__` method that accepts coordinate vectors
- `name` – string; name (symbol) given to the subset

- `latex_name` – string (default: `None`); LaTeX symbol to denote the subset; if none are provided, it is set to `name`

OUTPUT:

- either a `TopologicalManifold` or a `ManifoldSubsetPullback`

EXAMPLES:

```
sage: M = Manifold(2, 'R^2', structure='topological')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', structure='topological')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#␣
↪Cartesian coordinates on R^2
```

Pulling back a polytope under a chart:

```
sage: # needs sage.geometry.polyhedron
sage: P = Polyhedron(vertices=[[0, 0], [1, 2], [2, 1]]); P
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
sage: McP = c_cart.preimage(P); McP
Subset x_y_inv_P of the 2-dimensional topological manifold R^2
sage: M((1, 2)) in McP
True
sage: M((2, 0)) in McP
False
```

```
>>> from sage.all import *
>>> # needs sage.geometry.polyhedron
>>> P = Polyhedron(vertices=[[Integer(0), Integer(0)], [Integer(1),␣
↪Integer(2)], [Integer(2), Integer(1)]]); P
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
>>> McP = c_cart.preimage(P); McP
Subset x_y_inv_P of the 2-dimensional topological manifold R^2
>>> M((Integer(1), Integer(2))) in McP
True
>>> M((Integer(2), Integer(0))) in McP
False
```

Pulling back the interior of a polytope under a chart:

```
sage: # needs sage.geometry.polyhedron
sage: int_P = P.interior(); int_P
Relative interior of
 a 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
sage: McInt_P = c_cart.preimage(int_P, name='McInt_P'); McInt_P
Open subset McInt_P of the 2-dimensional topological manifold R^2
sage: M((0, 0)) in McInt_P
False
sage: M((1, 1)) in McInt_P
True
```

```
>>> from sage.all import *
>>> # needs sage.geometry.polyhedron
>>> int_P = P.interior(); int_P
Relative interior of
 a 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
>>> McInt_P = c_cart.preimage(int_P, name='McInt_P'); McInt_P
Open subset McInt_P of the 2-dimensional topological manifold R^2
>>> M((Integer(0), Integer(0))) in McInt_P
False
>>> M((Integer(1), Integer(1))) in McInt_P
True
```

Pulling back a point lattice:

```
sage: W = span([[1, 0], [3, 5]], ZZ); W
Free module of degree 2 and rank 2 over Integer Ring
Echelon basis matrix:
[1 0]
[0 5]
sage: McW = c_cart.pullback(W, name='McW'); McW
Subset McW of the 2-dimensional topological manifold R^2
sage: M((4, 5)) in McW
True
sage: M((4, 4)) in McW
False
```

```
>>> from sage.all import *
>>> W = span([[Integer(1), Integer(0)], [Integer(3), Integer(5)]], ZZ); W
Free module of degree 2 and rank 2 over Integer Ring
Echelon basis matrix:
[1 0]
[0 5]
>>> McW = c_cart.pullback(W, name='McW'); McW
Subset McW of the 2-dimensional topological manifold R^2
>>> M((Integer(4), Integer(5))) in McW
True
>>> M((Integer(4), Integer(4))) in McW
False
```

Pulling back a real vector subspaces:

```
sage: V = span([[1, 2]], RR); V
Vector space of degree 2 and dimension 1 over Real Field with 53 bits of␣
↪precision
Basis matrix:
[1.00000000000000 2.00000000000000]
sage: McV = c_cart.pullback(V, name='McV'); McV
Subset McV of the 2-dimensional topological manifold R^2
sage: M((2, 4)) in McV
True
sage: M((1, 0)) in McV
False
```

```
>>> from sage.all import *
>>> V = span([[Integer(1), Integer(2)]], RR); V
Vector space of degree 2 and dimension 1 over Real Field with 53 bits of␣
↪precision
Basis matrix:
[1.00000000000000 2.00000000000000]
>>> McV = c_cart.pullback(V, name='McV'); McV
Subset McV of the 2-dimensional topological manifold R^2
>>> M((Integer(2), Integer(4))) in McV
True
>>> M((Integer(1), Integer(0))) in McV
False
```

Pulling back a finite set of points:

```
sage: F = Family([vector(QQ, [1, 2], immutable=True),
....:             vector(QQ, [2, 3], immutable=True)])
sage: McF = c_cart.pullback(F, name='McF'); McF
Subset McF of the 2-dimensional topological manifold R^2
sage: M((2, 3)) in McF
True
sage: M((0, 0)) in McF
False
```

```
>>> from sage.all import *
>>> F = Family([vector(QQ, [Integer(1), Integer(2)], immutable=True),
...             vector(QQ, [Integer(2), Integer(3)], immutable=True)])
>>> McF = c_cart.pullback(F, name='McF'); McF
Subset McF of the 2-dimensional topological manifold R^2
>>> M((Integer(2), Integer(3))) in McF
True
>>> M((Integer(0), Integer(0))) in McF
False
```

Pulling back the integers:

```
sage: R = manifolds.RealLine(); R
Real number line ℝ
sage: McZ = R.canonical_chart().pullback(ZZ, name='ℤ'); McZ
Subset ℤ of the Real number line ℝ
sage: R((3/2,)) in McZ
False
sage: R((-2,)) in McZ
True
```

```
>>> from sage.all import *
>>> R = manifolds.RealLine(); R
Real number line ℝ
>>> McZ = R.canonical_chart().pullback(ZZ, name='ℤ'); McZ
Subset ℤ of the Real number line ℝ
>>> R((Integer(3)/Integer(2),)) in McZ
False
```

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```
>>> R((-Integer(2),)) in McZ
True
```

restrict(*subset*, *restrictions=None*)

Return the restriction of `self` to some open subset of its domain.

If the current chart is (U, φ), a *restriction* (or *subchart*) is a chart (V, ψ) such that $V \subset U$ and $\psi = \varphi|_V$.

If such subchart has not been defined yet, it is constructed here.

The coordinates of the subchart bare the same names as the coordinates of the current chart.

INPUT:

- `subset` – open subset V of the chart domain U (must be an instance of *TopologicalManifold*)
- `restrictions` – (default: `None`) list of coordinate restrictions defining the subset V

A restriction can be any symbolic equality or inequality involving the coordinates, such as `x > y` or `x^2 + y^2 != 0`. The items of the list `restrictions` are combined with the `and` operator; if some restrictions are to be combined with the `or` operator instead, they have to be passed as a tuple in some single item of the list `restrictions`. For example:

```
restrictions = [x > y, (x != 0, y != 0), z^2 < x]
```

means `(x > y) and ((x != 0) or (y != 0)) and (z^2 < x)`. If the list `restrictions` contains only one item, this item can be passed as such, i.e. writing `x > y` instead of the single element list `[x > y]`.

OUTPUT:

- chart (V, ψ) as a *Chart*

EXAMPLES:

Coordinates on the unit open ball of $\mathbf{C}^2$ as a subchart of the global coordinates of $\mathbf{C}^2$:

```
sage: M = Manifold(2, 'C^2', field='complex', structure='topological')
sage: X.<z1, z2> = M.chart()
sage: B = M.open_subset('B')
sage: X_B = X.restrict(B, abs(z1)^2 + abs(z2)^2 < 1); X_B
Chart (B, (z1, z2))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'C^2', field='complex', structure='topological')
>>> X = M.chart(names=('z1', 'z2',)); (z1, z2,) = X._first_ngens(2)
>>> B = M.open_subset('B')
>>> X_B = X.restrict(B, abs(z1)**Integer(2) + abs(z2)**Integer(2) <␣
↪Integer(1)); X_B
Chart (B, (z1, z2))
```

transition_map(*other*, *transformations*, *intersection_name=None*, *restrictions1=None*, *restrictions2=None*)

Construct the transition map between the current chart, (U, φ) say, and another one, (V, ψ) say.

If n is the manifold's dimension, the *transition map* is the map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \subset K^n \to \psi(U \cap V) \subset K^n,$$

where K is the manifold's base field. In other words, the transition map expresses the coordinates $(y^1, \ldots, y^n)$ of (V, ψ) in terms of the coordinates $(x^1, \ldots, x^n)$ of (U, φ) on the open subset where the two charts intersect, i.e. on $U \cap V$.

INPUT:

- `other` – the chart (V, ψ)
- `transformations` – tuple (or list) $(Y_1, \ldots, Y_n)$, where Y_i is the symbolic expression of the coordinate y^i in terms of the coordinates $(x^1, \ldots, x^n)$
- `intersection_name` – (default: `None`) name to be given to the subset $U \cap V$ if the latter differs from U or V
- `restrictions1` – (default: `None`) list of conditions on the coordinates of the current chart that define $U \cap V$ if the latter differs from U
- `restrictions2` – (default: `None`) list of conditions on the coordinates of the chart (V, ψ) that define $U \cap V$ if the latter differs from V

A restriction can be any symbolic equality or inequality involving the coordinates, such as `x > y` or `x^2 + y^2 != 0`. The items of the list `restrictions` are combined with the `and` operator; if some restrictions are to be combined with the `or` operator instead, they have to be passed as a tuple in some single item of the list `restrictions`. For example:

```
restrictions = [x > y, (x != 0, y != 0), z^2 < x]
```

means `(x > y) and ((x != 0) or (y != 0)) and (z^2 < x)`. If the list `restrictions` contains only one item, this item can be passed as such, i.e. writing `x > y` instead of the single element list `[x > y]`.

OUTPUT:

- the transition map $\psi \circ \varphi^{-1}$ defined on $U \cap V$ as a `CoordChange`

EXAMPLES:

Transition map between two stereographic charts on the circle S^1:

```
sage: M = Manifold(1, 'S^1', structure='topological')
sage: U = M.open_subset('U') # Complement of the North pole
sage: cU.<x> = U.chart() # Stereographic chart from the North pole
sage: V = M.open_subset('V') # Complement of the South pole
sage: cV.<y> = V.chart() # Stereographic chart from the South pole
sage: M.declare_union(U,V)   # S^1 is the union of U and V
sage: trans = cU.transition_map(cV, 1/x, intersection_name='W',
....:                           restrictions1= x!=0, restrictions2 = y!=0)
sage: trans
Change of coordinates from Chart (W, (x,)) to Chart (W, (y,))
sage: trans.display()
y = 1/x
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(1), 'S^1', structure='topological')
>>> U = M.open_subset('U') # Complement of the North pole
>>> cU = U.chart(names=('x',)); (x,) = cU._first_ngens(1)# Stereographic␣
↪chart from the North pole
>>> V = M.open_subset('V') # Complement of the South pole
>>> cV = V.chart(names=('y',)); (y,) = cV._first_ngens(1)# Stereographic␣
↪chart from the South pole
```

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```
>>> M.declare_union(U,V)    # S^1 is the union of U and V
>>> trans = cU.transition_map(cV, Integer(1)/x, intersection_name='W',
...                           restrictions1= x!=Integer(0), restrictions2 = y!
↪=Integer(0))
>>> trans
Change of coordinates from Chart (W, (x,)) to Chart (W, (y,))
>>> trans.display()
y = 1/x
```

The subset W, intersection of U and V, has been created by `transition_map()`:

```
sage: F = M.subset_family(); F
Set {S^1, U, V, W} of open subsets of the 1-dimensional topological manifold␣
↪S^1
sage: W = F['W']
sage: W is U.intersection(V)
True
sage: M.atlas()
[Chart (U, (x,)), Chart (V, (y,)), Chart (W, (x,)), Chart (W, (y,))]
```

```
>>> from sage.all import *
>>> F = M.subset_family(); F
Set {S^1, U, V, W} of open subsets of the 1-dimensional topological manifold␣
↪S^1
>>> W = F['W']
>>> W is U.intersection(V)
True
>>> M.atlas()
[Chart (U, (x,)), Chart (V, (y,)), Chart (W, (x,)), Chart (W, (y,))]
```

Transition map between the spherical chart and the Cartesian one on $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'R^2', structure='topological')
sage: c_cart.<x,y> = M.chart()
sage: U = M.open_subset('U') # the complement of the half line {y=0, x >= 0}
sage: c_spher.<r,phi> = U.chart(r'r:(0,+oo) phi:(0,2*pi):\phi')
sage: trans = c_spher.transition_map(c_cart, (r*cos(phi), r*sin(phi)),
....:                                restrictions2=(y!=0, x<0))
sage: trans
Change of coordinates from Chart (U, (r, phi)) to Chart (U, (x, y))
sage: trans.display()
x = r*cos(phi)
y = r*sin(phi)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', structure='topological')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)
>>> U = M.open_subset('U') # the complement of the half line {y=0, x >= 0}
>>> c_spher = U.chart(r'r:(0,+oo) phi:(0,2*pi):\phi', names=('r', 'phi',));␣
↪(r, phi,) = c_spher._first_ngens(2)
>>> trans = c_spher.transition_map(c_cart, (r*cos(phi), r*sin(phi)),
...                                restrictions2=(y!=Integer(0), x
```

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```
↪<Integer(0)))
>>> trans
Change of coordinates from Chart (U, (r, phi)) to Chart (U, (x, y))
>>> trans.display()
x = r*cos(phi)
y = r*sin(phi)
```

In this case, no new subset has been created since $U \cap M = U$:

```
sage: M.subset_family()
Set {R^2, U} of open subsets of the 2-dimensional topological manifold R^2
```

```
>>> from sage.all import *
>>> M.subset_family()
Set {R^2, U} of open subsets of the 2-dimensional topological manifold R^2
```

but a new chart has been created: $(U, (x, y))$:

```
sage: M.atlas()
[Chart (R^2, (x, y)), Chart (U, (r, phi)), Chart (U, (x, y))]
```

```
>>> from sage.all import *
>>> M.atlas()
[Chart (R^2, (x, y)), Chart (U, (r, phi)), Chart (U, (x, y))]
```

valid_coordinates(**coordinates*, ***kwds*)

Check whether a tuple of coordinates can be the coordinates of a point in the chart domain.

INPUT:

- `*coordinates` – coordinate values
- `**kwds` – options:
  - `parameters=None`, dictionary to set numerical values to some parameters (see example below)

OUTPUT:

- `True` if the coordinate values are admissible in the chart image, `False` otherwise

EXAMPLES:

```
sage: M = Manifold(2, 'M', field='complex', structure='topological')
sage: X.<x,y> = M.chart(coord_restrictions=lambda x,y: [abs(x)<1, y!=0])
sage: X.valid_coordinates(0, i)
True
sage: X.valid_coordinates(i, 1)
False
sage: X.valid_coordinates(i/2, 1)
True
sage: X.valid_coordinates(i/2, 0)
False
sage: X.valid_coordinates(2, 0)
False
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', field='complex', structure='topological')
>>> X = M.chart(coord_restrictions=lambda x,y: [abs(x)<Integer(1), y!
↪=Integer(0)], names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> X.valid_coordinates(Integer(0), i)
True
>>> X.valid_coordinates(i, Integer(1))
False
>>> X.valid_coordinates(i/Integer(2), Integer(1))
True
>>> X.valid_coordinates(i/Integer(2), Integer(0))
False
>>> X.valid_coordinates(Integer(2), Integer(0))
False
```

Example of use with the keyword `parameters` to set a specific value to a parameter appearing in the coordinate restrictions:

```
sage: var('a')  # the parameter is a symbolic variable
a
sage: Y.<u,v> = M.chart(coord_restrictions=lambda u,v: abs(v)<a)
sage: Y.valid_coordinates(1, i, parameters={a: 2})  # setting a=2
True
sage: Y.valid_coordinates(1, 2*i, parameters={a: 2})
False
```

```
>>> from sage.all import *
>>> var('a')  # the parameter is a symbolic variable
a
>>> Y = M.chart(coord_restrictions=lambda u,v: abs(v)<a, names=('u', 'v',));␣
↪(u, v,) = Y._first_ngens(2)
>>> Y.valid_coordinates(Integer(1), i, parameters={a: Integer(2)})  # setting␣
↪a=2
True
>>> Y.valid_coordinates(Integer(1), Integer(2)*i, parameters={a: Integer(2)})
False
```

zero_function()

Return the zero function of the coordinates.

If the current chart belongs to the atlas of a n-dimensional manifold over a topological field K, the zero coordinate function is the map

$$\begin{array}{llcl} f: & V \subset K^n & \longrightarrow & K \\ & (x^1, \ldots, x^n) & \longmapsto & 0, \end{array}$$

where V is the chart codomain.

See class `ChartFunction` for a complete documentation.

OUTPUT:

- a `ChartFunction` representing the zero coordinate function f

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: X.zero_function()
0
sage: X.zero_function().display()
(x, y) ↦ 0
sage: type(X.zero_function())
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> X.zero_function()
0
>>> X.zero_function().display()
(x, y) ↦ 0
>>> type(X.zero_function())
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>
```

The result is cached:

```
sage: X.zero_function() is X.zero_function()
True
```

```
>>> from sage.all import *
>>> X.zero_function() is X.zero_function()
True
```

Zero function on a p-adic manifold:

```
sage: # needs sage.rings.padics
sage: M = Manifold(2, 'M', structure='topological', field=Qp(5)); M
2-dimensional topological manifold M over the 5-adic Field with
 capped relative precision 20
sage: X.<x,y> = M.chart()
sage: X.zero_function()
0
sage: X.zero_function().display()
(x, y) ↦ 0
```

```
>>> from sage.all import *
>>> # needs sage.rings.padics
>>> M = Manifold(Integer(2), 'M', structure='topological',␣
↪field=Qp(Integer(5))); M
2-dimensional topological manifold M over the 5-adic Field with
 capped relative precision 20
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> X.zero_function()
0
>>> X.zero_function().display()
(x, y) ↦ 0
```

class sage.manifolds.chart.**CoordChange**(*chart1*, *chart2*, **transformations*)

Bases: SageObject

Transition map between two charts of a topological manifold.

Giving two coordinate charts (U, φ) and (V, ψ) on a topological manifold M of dimension n over a topological field K, the *transition map from* (U, φ) *to* (V, ψ) is the map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \subset K^n \to \psi(U \cap V) \subset K^n.$$

In other words, the transition map $\psi \circ \varphi^{-1}$ expresses the coordinates $(y^1, \ldots, y^n)$ of (V, ψ) in terms of the coordinates $(x^1, \ldots, x^n)$ of (U, φ) on the open subset where the two charts intersect, i.e. on $U \cap V$.

INPUT:

- chart1 – chart (U, φ)
- chart2 – chart (V, ψ)
- transformations – tuple (or list) $(Y_1, \ldots, Y_2)$, where Y_i is the symbolic expression of the coordinate y^i in terms of the coordinates $(x^1, \ldots, x^n)$

EXAMPLES:

Transition map on a 2-dimensional topological manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: Y.<u,v> = M.chart()
sage: X_to_Y = X.transition_map(Y, [x+y, x-y])
sage: X_to_Y
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
sage: type(X_to_Y)
<class 'sage.manifolds.chart.CoordChange'>
sage: X_to_Y.display()
u = x + y
v = x - y
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> Y = M.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)
>>> X_to_Y = X.transition_map(Y, [x+y, x-y])
>>> X_to_Y
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
>>> type(X_to_Y)
<class 'sage.manifolds.chart.CoordChange'>
>>> X_to_Y.display()
u = x + y
v = x - y
```

disp()

Display of the coordinate transformation.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

EXAMPLES:

From spherical coordinates to Cartesian ones in the plane:

```
sage: M = Manifold(2, 'R^2', structure='topological')
sage: U = M.open_subset('U') # the complement of the half line {y=0, x>= 0}
sage: c_cart.<x,y> = U.chart()
sage: c_spher.<r,ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi')
sage: spher_to_cart = c_spher.transition_map(c_cart, [r*cos(ph), r*sin(ph)])
sage: spher_to_cart.display()
x = r*cos(ph)
y = r*sin(ph)
sage: latex(spher_to_cart.display())
\left\{\begin{array}{lcl} x & = & r \cos\left({\phi}\right) \\
 y & = & r \sin\left({\phi}\right) \end{array}\right.
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', structure='topological')
>>> U = M.open_subset('U') # the complement of the half line {y=0, x>= 0}
>>> c_cart = U.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)
>>> c_spher = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi', names=('r', 'ph',)); (r,
↪ph,) = c_spher._first_ngens(2)
>>> spher_to_cart = c_spher.transition_map(c_cart, [r*cos(ph), r*sin(ph)])
>>> spher_to_cart.display()
x = r*cos(ph)
y = r*sin(ph)
>>> latex(spher_to_cart.display())
\left\{\begin{array}{lcl} x & = & r \cos\left({\phi}\right) \\
 y & = & r \sin\left({\phi}\right) \end{array}\right.
```

A shortcut is `disp()`:

```
sage: spher_to_cart.disp()
x = r*cos(ph)
y = r*sin(ph)
```

```
>>> from sage.all import *
>>> spher_to_cart.disp()
x = r*cos(ph)
y = r*sin(ph)
```

display()

Display of the coordinate transformation.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

EXAMPLES:

From spherical coordinates to Cartesian ones in the plane:

```
sage: M = Manifold(2, 'R^2', structure='topological')
sage: U = M.open_subset('U') # the complement of the half line {y=0, x>= 0}
sage: c_cart.<x,y> = U.chart()
sage: c_spher.<r,ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi')
sage: spher_to_cart = c_spher.transition_map(c_cart, [r*cos(ph), r*sin(ph)])
sage: spher_to_cart.display()
x = r*cos(ph)
y = r*sin(ph)
```

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```
sage: latex(spher_to_cart.display())
\left\{\begin{array}{lcl} x & = & r \cos\left({\phi}\right) \\
 y & = & r \sin\left({\phi}\right) \end{array}\right.
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', structure='topological')
>>> U = M.open_subset('U') # the complement of the half line {y=0, x>= 0}
>>> c_cart = U.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)
>>> c_spher = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi', names=('r', 'ph',)); (r,
↪ph,) = c_spher._first_ngens(2)
>>> spher_to_cart = c_spher.transition_map(c_cart, [r*cos(ph), r*sin(ph)])
>>> spher_to_cart.display()
x = r*cos(ph)
y = r*sin(ph)
>>> latex(spher_to_cart.display())
\left\{\begin{array}{lcl} x & = & r \cos\left({\phi}\right) \\
 y & = & r \sin\left({\phi}\right) \end{array}\right.
```

A shortcut is `disp()`:

```
sage: spher_to_cart.disp()
x = r*cos(ph)
y = r*sin(ph)
```

```
>>> from sage.all import *
>>> spher_to_cart.disp()
x = r*cos(ph)
y = r*sin(ph)
```

inverse()

Return the inverse coordinate transformation.

If the inverse is not already known, it is computed here. If the computation fails, the inverse can be set by hand via the method *set_inverse()*.

OUTPUT:

- an instance of *CoordChange* representing the inverse of the current coordinate transformation

EXAMPLES:

Inverse of a coordinate transformation corresponding to a rotation in the Cartesian plane:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: c_uv.<u,v> = M.chart()
sage: phi = var('phi', domain='real')
sage: xy_to_uv = c_xy.transition_map(c_uv,
....:                                 [cos(phi)*x + sin(phi)*y,
....:                                  -sin(phi)*x + cos(phi)*y])
sage: M.coord_changes()
{(Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y)) to Chart
↪(M, (u, v))}
```

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```
sage: uv_to_xy = xy_to_uv.inverse(); uv_to_xy
Change of coordinates from Chart (M, (u, v)) to Chart (M, (x, y))
sage: uv_to_xy.display()
x = u*cos(phi) - v*sin(phi)
y = v*cos(phi) + u*sin(phi)
sage: M.coord_changes()   # random (dictionary output)
{(Chart (M, (u, v)),
  Chart (M, (x, y))): Change of coordinates from Chart (M, (u, v)) to Chart␣
↪(M, (x, y)),
 (Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y)) to Chart␣
↪(M, (u, v))}
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> c_uv = M.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> phi = var('phi', domain='real')
>>> xy_to_uv = c_xy.transition_map(c_uv,
...                                 [cos(phi)*x + sin(phi)*y,
...                                  -sin(phi)*x + cos(phi)*y])
>>> M.coord_changes()
{(Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y)) to Chart␣
↪(M, (u, v))}
>>> uv_to_xy = xy_to_uv.inverse(); uv_to_xy
Change of coordinates from Chart (M, (u, v)) to Chart (M, (x, y))
>>> uv_to_xy.display()
x = u*cos(phi) - v*sin(phi)
y = v*cos(phi) + u*sin(phi)
>>> M.coord_changes()   # random (dictionary output)
{(Chart (M, (u, v)),
  Chart (M, (x, y))): Change of coordinates from Chart (M, (u, v)) to Chart␣
↪(M, (x, y)),
 (Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y)) to Chart␣
↪(M, (u, v))}
```

The result is cached:

```
sage: xy_to_uv.inverse() is uv_to_xy
True
```

```
>>> from sage.all import *
>>> xy_to_uv.inverse() is uv_to_xy
True
```

We have as well:

```
sage: uv_to_xy.inverse() is xy_to_uv
True
```

```
>>> from sage.all import *
>>> uv_to_xy.inverse() is xy_to_uv
True
```

restrict(*dom1*, *dom2=None*)

Restriction to subsets.

INPUT:

- `dom1` – open subset of the domain of `chart1`
- `dom2` – (default: `None`) open subset of the domain of `chart2`; if `None`, `dom1` is assumed

OUTPUT:

- the transition map between the charts restricted to the specified subsets

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: Y.<u,v> = M.chart()
sage: X_to_Y = X.transition_map(Y, [x+y, x-y])
sage: U = M.open_subset('U', coord_def={X: x>0, Y: u+v>0})
sage: X_to_Y_U = X_to_Y.restrict(U); X_to_Y_U
Change of coordinates from Chart (U, (x, y)) to Chart (U, (u, v))
sage: X_to_Y_U.display()
u = x + y
v = x - y
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> Y = M.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)
>>> X_to_Y = X.transition_map(Y, [x+y, x-y])
>>> U = M.open_subset('U', coord_def={X: x>Integer(0), Y: u+v>Integer(0)})
>>> X_to_Y_U = X_to_Y.restrict(U); X_to_Y_U
Change of coordinates from Chart (U, (x, y)) to Chart (U, (u, v))
>>> X_to_Y_U.display()
u = x + y
v = x - y
```

The result is cached:

```
sage: X_to_Y.restrict(U) is X_to_Y_U
True
```

```
>>> from sage.all import *
>>> X_to_Y.restrict(U) is X_to_Y_U
True
```

set_inverse(**transformations*, ***kwds*)

Set the inverse of the coordinate transformation.

This is useful when the automatic computation via `inverse()` fails.

INPUT:

- `transformations` – the inverse transformations expressed as a list of the expressions of the "old" coordinates in terms of the "new" ones
- `kwds` – optional arguments; valid keywords are
  - `check` – boolean (default: `True`); whether the provided transformations are checked to be indeed the inverse coordinate transformations
  - `verbose` – boolean (default: `False`); whether some details of the check are printed out; if `False`, no output is printed if the check is passed (see example below)

EXAMPLES:

From spherical coordinates to Cartesian ones in the plane:

```
sage: M = Manifold(2, 'R^2', structure='topological')
sage: U = M.open_subset('U') # complement of the half line {y=0, x>= 0}
sage: c_cart.<x,y> = U.chart()
sage: c_spher.<r,ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi')
sage: spher_to_cart = c_spher.transition_map(c_cart,
....:                                        [r*cos(ph), r*sin(ph)])
sage: spher_to_cart.set_inverse(sqrt(x^2+y^2), atan2(y,x))
Check of the inverse coordinate transformation:
  r == r  *passed*
  ph == arctan2(r*sin(ph), r*cos(ph))  **failed**
  x == x  *passed*
  y == y  *passed*
NB: a failed report can reflect a mere lack of simplification.
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', structure='topological')
>>> U = M.open_subset('U') # complement of the half line {y=0, x>= 0}
>>> c_cart = U.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)
>>> c_spher = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi', names=('r', 'ph',)); (r,
↪ph,) = c_spher._first_ngens(2)
>>> spher_to_cart = c_spher.transition_map(c_cart,
...                                        [r*cos(ph), r*sin(ph)])
>>> spher_to_cart.set_inverse(sqrt(x**Integer(2)+y**Integer(2)), atan2(y,x))
Check of the inverse coordinate transformation:
  r == r  *passed*
  ph == arctan2(r*sin(ph), r*cos(ph))  **failed**
  x == x  *passed*
  y == y  *passed*
NB: a failed report can reflect a mere lack of simplification.
```

As indicated, the failure for `ph` is due to a lack of simplification of the `arctan2` term, not to any error in the provided inverse formulas.

We have now:

```
sage: spher_to_cart.inverse()
Change of coordinates from Chart (U, (x, y)) to Chart (U, (r, ph))
sage: spher_to_cart.inverse().display()
r = sqrt(x^2 + y^2)
ph = arctan2(y, x)
sage: M.coord_changes()  # random (dictionary output)
```

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```
{(Chart (U, (r, ph)),
  Chart (U, (x, y))): Change of coordinates from Chart (U, (r, ph))
   to Chart (U, (x, y)),
 (Chart (U, (x, y)),
  Chart (U, (r, ph))): Change of coordinates from Chart (U, (x, y))
   to Chart (U, (r, ph))}
```

```
>>> from sage.all import *
>>> spher_to_cart.inverse()
Change of coordinates from Chart (U, (x, y)) to Chart (U, (r, ph))
>>> spher_to_cart.inverse().display()
r = sqrt(x^2 + y^2)
ph = arctan2(y, x)
>>> M.coord_changes()   # random (dictionary output)
{(Chart (U, (r, ph)),
  Chart (U, (x, y))): Change of coordinates from Chart (U, (r, ph))
   to Chart (U, (x, y)),
 (Chart (U, (x, y)),
  Chart (U, (r, ph))): Change of coordinates from Chart (U, (x, y))
   to Chart (U, (r, ph))}
```

One can suppress the check of the provided formulas by means of the optional argument `check=False`:

```
sage: spher_to_cart.set_inverse(sqrt(x^2+y^2), atan2(y,x),
....:                           check=False)
```

```
>>> from sage.all import *
>>> spher_to_cart.set_inverse(sqrt(x**Integer(2)+y**Integer(2)), atan2(y,x),
...                           check=False)
```

However, it is not recommended to do so, the check being (obviously) useful to avoid some mistake. For instance, if the term `sqrt(x^2+y^2)` contains a typo (`x^3` instead of `x^2`), we get:

```
sage: spher_to_cart.set_inverse(sqrt(x^3+y^2), atan2(y,x))
Check of the inverse coordinate transformation:
  r == sqrt(r*cos(ph)^3 + sin(ph)^2)*r  **failed**
  ph == arctan2(r*sin(ph), r*cos(ph))  **failed**
  x == sqrt(x^3 + y^2)*x/sqrt(x^2 + y^2)  **failed**
  y == sqrt(x^3 + y^2)*y/sqrt(x^2 + y^2)  **failed**
NB: a failed report can reflect a mere lack of simplification.
```

```
>>> from sage.all import *
>>> spher_to_cart.set_inverse(sqrt(x**Integer(3)+y**Integer(2)), atan2(y,x))
Check of the inverse coordinate transformation:
  r == sqrt(r*cos(ph)^3 + sin(ph)^2)*r  **failed**
  ph == arctan2(r*sin(ph), r*cos(ph))  **failed**
  x == sqrt(x^3 + y^2)*x/sqrt(x^2 + y^2)  **failed**
  y == sqrt(x^3 + y^2)*y/sqrt(x^2 + y^2)  **failed**
NB: a failed report can reflect a mere lack of simplification.
```

If the check is passed, no output is printed out:

```
sage: M = Manifold(2, 'M')
sage: X1.<x,y> = M.chart()
sage: X2.<u,v> = M.chart()
sage: X1_to_X2 = X1.transition_map(X2, [x+y, x-y])
sage: X1_to_X2.set_inverse((u+v)/2, (u-v)/2)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X1 = M.chart(names=('x', 'y',)); (x, y,) = X1._first_ngens(2)
>>> X2 = M.chart(names=('u', 'v',)); (u, v,) = X2._first_ngens(2)
>>> X1_to_X2 = X1.transition_map(X2, [x+y, x-y])
>>> X1_to_X2.set_inverse((u+v)/Integer(2), (u-v)/Integer(2))
```

unless the option `verbose` is set to `True`:

```
sage: X1_to_X2.set_inverse((u+v)/2, (u-v)/2, verbose=True)
Check of the inverse coordinate transformation:
  x == x  *passed*
  y == y  *passed*
  u == u  *passed*
  v == v  *passed*
```

```
>>> from sage.all import *
>>> X1_to_X2.set_inverse((u+v)/Integer(2), (u-v)/Integer(2), verbose=True)
Check of the inverse coordinate transformation:
  x == x  *passed*
  y == y  *passed*
  u == u  *passed*
  v == v  *passed*
```

class `sage.manifolds.chart.`**`RealChart`**(*domain*, *coordinates*, *calc_method=None*, *bounds=None*, *periods=None*, *coord_restrictions=None*)

Bases: `Chart`

Chart on a topological manifold over $\mathbf{R}$.

Given a topological manifold M of dimension n over $\mathbf{R}$, a *chart* on M is a pair (U, φ), where U is an open subset of M and $\varphi : U \to V \subset \mathbf{R}^n$ is a homeomorphism from U to an open subset V of $\mathbf{R}^n$.

The components $(x^1, \ldots, x^n)$ of φ, defined by $\varphi(p) = (x^1(p), \ldots, x^n(p)) \in \mathbf{R}^n$ for any point $p \in U$, are called the *coordinates* of the chart (U, φ).

INPUT:

- `domain` – open subset U on which the chart is defined
- `coordinates` – (default: '' (empty string)) single string defining the coordinate symbols, with `' '` (whitespace) as a separator; each item has at most four fields, separated by a colon (`:`):
  1. the coordinate symbol (a letter or a few letters)
  2. (optional) the interval I defining the coordinate range: if not provided, the coordinate is assumed to span all $\mathbf{R}$; otherwise I must be provided in the form `(a,b)` (or equivalently `]a,b[`); the bounds `a` and `b` can be `+/-Infinity`, `Inf`, `infinity`, `inf` or `oo`; for *singular* coordinates, non-open intervals such as `[a,b]` and `(a,b]` (or equivalently `]a,b]`) are allowed; note that the interval declaration must not contain any whitespace

3. (optional) indicator of the periodic character of the coordinate, either as `period=T`, where `T` is the period, or as the keyword `periodic` (the value of the period is then deduced from the interval I declared in field 2; see examples below)

4. (optional) the LaTeX spelling of the coordinate; if not provided the coordinate symbol given in the first field will be used

The order of fields 2 to 4 does not matter and each of them can be omitted. If it contains any LaTeX expression, the string `coordinates` must be declared with the prefix 'r' (for "raw") to allow for a proper treatment of LaTeX's backslash character (see examples below). If interval range, no period and no LaTeX spelling are to be set for any coordinate, the argument `coordinates` can be omitted when the shortcut operator `<,>` is used to declare the chart (see examples below).

- `calc_method` – (default: `None`) string defining the calculus method for computations involving coordinates of the chart; must be one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy
  - `None`: the default of *`CalculusMethod`* will be used
- `names` – (default: `None`) unused argument, except if `coordinates` is not provided; it must then be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)
- `coord_restrictions` – additional restrictions on the coordinates. A restriction can be any symbolic equality or inequality involving the coordinates, such as `x > y` or `x^2 + y^2 != 0`. The items of the list (or set or frozenset) `coord_restrictions` are combined with the `and` operator; if some restrictions are to be combined with the `or` operator instead, they have to be passed as a tuple in some single item of the list (or set or frozenset) `coord_restrictions`. For example:

```
coord_restrictions=[x > y, (x != 0, y != 0), z^2 < x]
```

  means `(x > y) and ((x != 0) or (y != 0)) and (z^2 < x)`. If the list `coord_restrictions` contains only one item, this item can be passed as such, i.e. writing `x > y` instead of the single element list `[x > y]`. If the chart variables have not been declared as variables yet, `coord_restrictions` must be `lambda`-quoted.

EXAMPLES:

Cartesian coordinates on $\mathbf{R}^3$:

```
sage: M = Manifold(3, 'R^3', r'\RR^3', structure='topological',
....:              start_index=1)
sage: c_cart = M.chart('x y z'); c_cart
Chart (R^3, (x, y, z))
sage: type(c_cart)
<class 'sage.manifolds.chart.RealChart'>
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'R^3', r'\RR^3', structure='topological',
...              start_index=Integer(1))
>>> c_cart = M.chart('x y z'); c_cart
Chart (R^3, (x, y, z))
>>> type(c_cart)
<class 'sage.manifolds.chart.RealChart'>
```

To have the coordinates accessible as global variables, one has to set:

```
sage: (x,y,z) = c_cart[:]
```

```
>>> from sage.all import *
>>> (x,y,z) = c_cart[:]
```

However, a shortcut is to use the declarator `<x,y,z>` in the left-hand side of the chart declaration (there is then no need to pass the string `'x y z'` to `chart()`):

```
sage: M = Manifold(3, 'R^3', r'\RR^3', structure='topological',
....:              start_index=1)
sage: c_cart.<x,y,z> = M.chart(); c_cart
Chart (R^3, (x, y, z))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'R^3', r'\RR^3', structure='topological',
...              start_index=Integer(1))
>>> c_cart = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_cart._first_ngens(3);
↪ c_cart
Chart (R^3, (x, y, z))
```

The coordinates are then immediately accessible:

```
sage: y
y
sage: y is c_cart[2]
True
```

```
>>> from sage.all import *
>>> y
y
>>> y is c_cart[Integer(2)]
True
```

Note that `x, y, z` declared in `<x,y,z>` are mere Python variable names and do not have to coincide with the coordinate symbols; for instance, one may write:

```
sage: M = Manifold(3, 'R^3', r'\RR^3', structure='topological',
....:              start_index=1)
sage: c_cart.<x1,y1,z1> = M.chart('x y z'); c_cart
Chart (R^3, (x, y, z))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'R^3', r'\RR^3', structure='topological',
...              start_index=Integer(1))
>>> c_cart = M.chart('x y z', names=('x1', 'y1', 'z1',)); (x1, y1, z1,) = c_cart._
↪first_ngens(3); c_cart
Chart (R^3, (x, y, z))
```

Then `y` is not known as a global variable and the coordinate y is accessible only through the global variable `y1`:

```
sage: y1
y
```

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```
sage: y1 is c_cart[2]
True
```

```
>>> from sage.all import *
>>> y1
y
>>> y1 is c_cart[Integer(2)]
True
```

However, having the name of the Python variable coincide with the coordinate symbol is quite convenient; so it is recommended to declare:

```
sage: forget()   # for doctests only
sage: M = Manifold(3, 'R^3', r'\RR^3', structure='topological', start_index=1)
sage: c_cart.<x,y,z> = M.chart()
```

```
>>> from sage.all import *
>>> forget()   # for doctests only
>>> M = Manifold(Integer(3), 'R^3', r'\RR^3', structure='topological', start_
↪index=Integer(1))
>>> c_cart = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_cart._first_ngens(3)
```

Spherical coordinates on the subset U of $\mathbf{R}^3$ that is the complement of the half-plane $\{y = 0, x \geq 0\}$:

```
sage: U = M.open_subset('U')
sage: c_spher.<r,th,ph> = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: c_spher
Chart (U, (r, th, ph))
```

```
>>> from sage.all import *
>>> U = M.open_subset('U')
>>> c_spher = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi', names=('r',
↪'th', 'ph',)); (r, th, ph,) = c_spher._first_ngens(3)
>>> c_spher
Chart (U, (r, th, ph))
```

Note the prefix 'r' for the string defining the coordinates in the arguments of `chart`.

Coordinates are Sage symbolic variables (see `sage.symbolic.expression`):

```
sage: type(th)
<class 'sage.symbolic.expression.Expression'>
sage: latex(th)
{\theta}
sage: assumptions(th)
[th is real, th > 0, th < pi]
```

```
>>> from sage.all import *
>>> type(th)
<class 'sage.symbolic.expression.Expression'>
>>> latex(th)
{\theta}
```

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```
>>> assumptions(th)
[th is real, th > 0, th < pi]
```

Coordinate are also accessible by their indices:

```
sage: x1 = c_spher[1]; x2 = c_spher[2]; x3 = c_spher[3]
sage: [x1, x2, x3]
[r, th, ph]
sage: (x1, x2, x3) == (r, th, ph)
True
```

```
>>> from sage.all import *
>>> x1 = c_spher[Integer(1)]; x2 = c_spher[Integer(2)]; x3 = c_spher[Integer(3)]
>>> [x1, x2, x3]
[r, th, ph]
>>> (x1, x2, x3) == (r, th, ph)
True
```

The full set of coordinates is obtained by means of the slice `[:]`:

```
sage: c_cart[:]
(x, y, z)
sage: c_spher[:]
(r, th, ph)
```

```
>>> from sage.all import *
>>> c_cart[:]
(x, y, z)
>>> c_spher[:]
(r, th, ph)
```

Let us check that the declared coordinate ranges have been taken into account:

```
sage: c_cart.coord_range()
x: (-oo, +oo); y: (-oo, +oo); z: (-oo, +oo)
sage: c_spher.coord_range()
r: (0, +oo); th: (0, pi); ph: (0, 2*pi)
sage: bool(th>0 and th<pi)
True
sage: assumptions()  # list all current symbolic assumptions
[x is real, y is real, z is real, r is real, r > 0, th is real,
 th > 0, th < pi, ph is real, ph > 0, ph < 2*pi]
```

```
>>> from sage.all import *
>>> c_cart.coord_range()
x: (-oo, +oo); y: (-oo, +oo); z: (-oo, +oo)
>>> c_spher.coord_range()
r: (0, +oo); th: (0, pi); ph: (0, 2*pi)
>>> bool(th>Integer(0) and th<pi)
True
>>> assumptions()  # list all current symbolic assumptions
```

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```
[x is real, y is real, z is real, r is real, r > 0, th is real,
 th > 0, th < pi, ph is real, ph > 0, ph < 2*pi]
```

The coordinate ranges are used for simplifications:

```
sage: simplify(abs(r)) # r has been declared to lie in the interval (0,+oo)
r
sage: simplify(abs(x)) # no positive range has been declared for x
abs(x)
```

```
>>> from sage.all import *
>>> simplify(abs(r)) # r has been declared to lie in the interval (0,+oo)
r
>>> simplify(abs(x)) # no positive range has been declared for x
abs(x)
```

A coordinate can be declared periodic by adding the keyword `periodic` to its range:

```
sage: V = M.open_subset('V')
sage: c_spher1.<r,th,ph1> = \
....: V.chart(r'r:(0,+oo) th:(0,pi):\theta ph1:(0,2*pi):periodic:\phi_1')
sage: c_spher1.periods()
(None, None, 2*pi)
sage: c_spher1.coord_range()
r: (0, +oo); th: (0, pi); ph1: [0, 2*pi] (periodic)
```

```
>>> from sage.all import *
>>> V = M.open_subset('V')
>>> c_spher1 = V.chart(r'r:(0,+oo) th:(0,pi):\theta ph1:(0,2*pi):periodic:\phi_1',
↪ names=('r', 'th', 'ph1',)); (r, th, ph1,) = c_spher1._first_ngens(3)
>>> c_spher1.periods()
(None, None, 2*pi)
>>> c_spher1.coord_range()
r: (0, +oo); th: (0, pi); ph1: [0, 2*pi] (periodic)
```

It is equivalent to give the period as `period=2*pi`, skipping the coordinate range:

```
sage: c_spher2.<r,th,ph2> = \
....: V.chart(r'r:(0,+oo) th:(0,pi):\theta ph2:period=2*pi:\phi_2')
sage: c_spher2.periods()
(None, None, 2*pi)
sage: c_spher2.coord_range()
r: (0, +oo); th: (0, pi); ph2: [0, 2*pi] (periodic)
```

```
>>> from sage.all import *
>>> c_spher2 = V.chart(r'r:(0,+oo) th:(0,pi):\theta ph2:period=2*pi:\phi_2',␣
↪names=('r', 'th', 'ph2',)); (r, th, ph2,) = c_spher2._first_ngens(3)
>>> c_spher2.periods()
(None, None, 2*pi)
>>> c_spher2.coord_range()
r: (0, +oo); th: (0, pi); ph2: [0, 2*pi] (periodic)
```

Each constructed chart is automatically added to the manifold's user atlas:

```
sage: M.atlas()
[Chart (R^3, (x, y, z)), Chart (U, (r, th, ph)),
 Chart (V, (r, th, ph1)), Chart (V, (r, th, ph2))]
```

```
>>> from sage.all import *
>>> M.atlas()
[Chart (R^3, (x, y, z)), Chart (U, (r, th, ph)),
 Chart (V, (r, th, ph1)), Chart (V, (r, th, ph2))]
```

and to the atlas of its domain:

```
sage: U.atlas()
[Chart (U, (r, th, ph))]
```

```
>>> from sage.all import *
>>> U.atlas()
[Chart (U, (r, th, ph))]
```

Manifold subsets have a *default chart*, which, unless changed via the method `set_default_chart()`, is the first defined chart on the subset (or on a open subset of it):

```
sage: M.default_chart()
Chart (R^3, (x, y, z))
sage: U.default_chart()
Chart (U, (r, th, ph))
```

```
>>> from sage.all import *
>>> M.default_chart()
Chart (R^3, (x, y, z))
>>> U.default_chart()
Chart (U, (r, th, ph))
```

The default charts are not privileged charts on the manifold, but rather charts whose name can be skipped in the argument list of functions having an optional `chart=` argument.

The chart map φ acting on a point is obtained by means of the call operator, i.e. the operator `()`:

```
sage: p = M.point((1,0,-2)); p
Point on the 3-dimensional topological manifold R^3
sage: c_cart(p)
(1, 0, -2)
sage: c_cart(p) == p.coord(c_cart)
True
sage: q = M.point((2,pi/2,pi/3), chart=c_spher) # point defined by its spherical␣
↪coordinates
sage: c_spher(q)
(2, 1/2*pi, 1/3*pi)
sage: c_spher(q) == q.coord(c_spher)
True
sage: a = U.point((1,pi/2,pi)) # the default coordinates on U are the spherical␣
↪ones
sage: c_spher(a)
(1, 1/2*pi, pi)
```

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```
sage: c_spher(a) == a.coord(c_spher)
True
```

```
>>> from sage.all import *
>>> p = M.point((Integer(1),Integer(0),-Integer(2))); p
Point on the 3-dimensional topological manifold R^3
>>> c_cart(p)
(1, 0, -2)
>>> c_cart(p) == p.coord(c_cart)
True
>>> q = M.point((Integer(2),pi/Integer(2),pi/Integer(3)), chart=c_spher) # point␣
↪defined by its spherical coordinates
>>> c_spher(q)
(2, 1/2*pi, 1/3*pi)
>>> c_spher(q) == q.coord(c_spher)
True
>>> a = U.point((Integer(1),pi/Integer(2),pi)) # the default coordinates on U are␣
↪the spherical ones
>>> c_spher(a)
(1, 1/2*pi, pi)
>>> c_spher(a) == a.coord(c_spher)
True
```

Cartesian coordinates on U as an example of chart construction with coordinate restrictions: since U is the complement of the half-plane $\{y = 0, x \geq 0\}$, we must have $y \neq 0$ or $x < 0$ on U. Accordingly, we set:

```
sage: c_cartU.<x,y,z> = U.chart(coord_restrictions=lambda x,y,z: (y!=0, x<0))
sage: U.atlas()
[Chart (U, (r, th, ph)), Chart (U, (x, y, z))]
sage: M.atlas()
[Chart (R^3, (x, y, z)), Chart (U, (r, th, ph)),
 Chart (V, (r, th, ph1)), Chart (V, (r, th, ph2)),
 Chart (U, (x, y, z))]
sage: c_cartU.valid_coordinates(-1,0,2)
True
sage: c_cartU.valid_coordinates(1,0,2)
False
sage: c_cart.valid_coordinates(1,0,2)
True
```

```
>>> from sage.all import *
>>> c_cartU = U.chart(coord_restrictions=lambda x,y,z: (y!=Integer(0), x
↪<Integer(0)), names=('x', 'y', 'z',)); (x, y, z,) = c_cartU._first_ngens(3)
>>> U.atlas()
[Chart (U, (r, th, ph)), Chart (U, (x, y, z))]
>>> M.atlas()
[Chart (R^3, (x, y, z)), Chart (U, (r, th, ph)),
 Chart (V, (r, th, ph1)), Chart (V, (r, th, ph2)),
 Chart (U, (x, y, z))]
>>> c_cartU.valid_coordinates(-Integer(1),Integer(0),Integer(2))
True
>>> c_cartU.valid_coordinates(Integer(1),Integer(0),Integer(2))
```

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```
False
>>> c_cart.valid_coordinates(Integer(1),Integer(0),Integer(2))
True
```

Note that, as an example, the following would have meant $y \neq 0$ *and* $x < 0$:

```
c_cartU.<x,y,z> = U.chart(coord_restrictions=lambda x,y,z: [y!=0, x<0])
```

Chart grids can be drawn in 2D or 3D graphics thanks to the method *plot()*.

add_restrictions(*restrictions*)

Add some restrictions on the coordinates.

This is deprecated; provide the restrictions at the time of creating the chart.

INPUT:

- `restrictions` – list of restrictions on the coordinates, in addition to the ranges declared by the intervals specified in the chart constructor

A restriction can be any symbolic equality or inequality involving the coordinates, such as `x > y` or `x^2 + y^2 != 0`. The items of the list `restrictions` are combined with the `and` operator; if some restrictions are to be combined with the `or` operator instead, they have to be passed as a tuple in some single item of the list `restrictions`. For example:

```
restrictions = [x > y, (x != 0, y != 0), z^2 < x]
```

means `(x > y) and ((x != 0) or (y != 0)) and (z^2 < x)`. If the list `restrictions` contains only one item, this item can be passed as such, i.e. writing `x > y` instead of the single element list `[x > y]`.

EXAMPLES:

Cartesian coordinates on the open unit disc in $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'M', structure='topological') # the open unit disc
sage: X.<x,y> = M.chart()
sage: X.add_restrictions(x^2+y^2<1)
doctest:warning...
DeprecationWarning: Chart.add_restrictions is deprecated; provide the
restrictions at the time of creating the chart
See https://github.com/sagemath/sage/issues/32102 for details.
sage: X.valid_coordinates(0,2)
False
sage: X.valid_coordinates(0,1/3)
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological') # the open unit␣
↪disc
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> X.add_restrictions(x**Integer(2)+y**Integer(2)<Integer(1))
doctest:warning...
DeprecationWarning: Chart.add_restrictions is deprecated; provide the
restrictions at the time of creating the chart
See https://github.com/sagemath/sage/issues/32102 for details.
>>> X.valid_coordinates(Integer(0),Integer(2))
```

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```
False
>>> X.valid_coordinates(Integer(0),Integer(1)/Integer(3))
True
```

The restrictions are transmitted to subcharts:

```
sage: A = M.open_subset('A') # annulus 1/2 < r < 1
sage: X_A = X.restrict(A, x^2+y^2 > 1/4)
sage: X_A._restrictions
[x^2 + y^2 < 1, x^2 + y^2 > (1/4)]
sage: X_A.valid_coordinates(0,1/3)
False
sage: X_A.valid_coordinates(2/3,1/3)
True
```

```
>>> from sage.all import *
>>> A = M.open_subset('A') # annulus 1/2 < r < 1
>>> X_A = X.restrict(A, x**Integer(2)+y**Integer(2) > Integer(1)/Integer(4))
>>> X_A._restrictions
[x^2 + y^2 < 1, x^2 + y^2 > (1/4)]
>>> X_A.valid_coordinates(Integer(0),Integer(1)/Integer(3))
False
>>> X_A.valid_coordinates(Integer(2)/Integer(3),Integer(1)/Integer(3))
True
```

If appropriate, the restrictions are transformed into bounds on the coordinate ranges:

```
sage: U = M.open_subset('U')
sage: X_U = X.restrict(U)
sage: X_U.coord_range()
x: (-oo, +oo); y: (-oo, +oo)
sage: X_U.add_restrictions([x<0, y>1/2])
sage: X_U.coord_range()
x: (-oo, 0); y: (1/2, +oo)
```

```
>>> from sage.all import *
>>> U = M.open_subset('U')
>>> X_U = X.restrict(U)
>>> X_U.coord_range()
x: (-oo, +oo); y: (-oo, +oo)
>>> X_U.add_restrictions([x<Integer(0), y>Integer(1)/Integer(2)])
>>> X_U.coord_range()
x: (-oo, 0); y: (1/2, +oo)
```

codomain()

Return the codomain of `self` as a set.

EXAMPLES:

```
sage: M = Manifold(2, 'R^2', structure='topological')
sage: U = M.open_subset('U') # the complement of the half line {y=0, x >= 0}
sage: c_spher.<r,phi> = U.chart(r'r:(0,+oo) phi:(0,2*pi):\phi')
```

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```
sage: c_spher.codomain()
The Cartesian product of ((0, +oo), (0, 2*pi))

sage: M = Manifold(3, 'R^3', r'\RR^3', structure='topological', start_index=1)
sage: c_cart.<x,y,z> = M.chart()
sage: c_cart.codomain()
Vector space of dimension 3 over Real Field with 53 bits of precision
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', structure='topological')
>>> U = M.open_subset('U') # the complement of the half line {y=0, x >= 0}
>>> c_spher = U.chart(r'r:(0,+oo) phi:(0,2*pi):\phi', names=('r', 'phi',));
↪(r, phi,) = c_spher._first_ngens(2)
>>> c_spher.codomain()
The Cartesian product of ((0, +oo), (0, 2*pi))

>>> M = Manifold(Integer(3), 'R^3', r'\RR^3', structure='topological', start_
↪index=Integer(1))
>>> c_cart = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_cart._first_
↪ngens(3)
>>> c_cart.codomain()
Vector space of dimension 3 over Real Field with 53 bits of precision
```

In the current implementation, the codomain of periodic coordinates are represented by a fundamental domain:

```
sage: V = M.open_subset('V')
sage: c_spher1.<r,th,ph1> = \
....: V.chart(r'r:(0,+oo) th:(0,pi):\theta ph1:(0,2*pi):periodic:\phi_1')
sage: c_spher1.codomain()
The Cartesian product of ((0, +oo), (0, pi), [0, 2*pi))
```

```
>>> from sage.all import *
>>> V = M.open_subset('V')
>>> c_spher1 = V.chart(r'r:(0,+oo) th:(0,pi):\theta ph1:(0,2*pi):periodic:\
↪phi_1', names=('r', 'th', 'ph1',)); (r, th, ph1,) = c_spher1._first_ngens(3)
>>> c_spher1.codomain()
The Cartesian product of ((0, +oo), (0, pi), [0, 2*pi))
```

coord_bounds(*i=None*)

Return the lower and upper bounds of the range of a coordinate.

For a nicely formatted output, use `coord_range()` instead.

INPUT:

- `i` – (default: `None`) index of the coordinate; if `None`, the bounds of all the coordinates are returned

OUTPUT:

- the coordinate bounds as the tuple `((xmin, min_included), (xmax, max_included))` where
  - `xmin` is the coordinate lower bound
  - `min_included` is a boolean, indicating whether the coordinate can take the value `xmin`, i.e. `xmin` is a strict lower bound iff `min_included` is `False`

- `xmin` is the coordinate upper bound
- `max_included` is a boolean, indicating whether the coordinate can take the value `xmax`, i.e. `xmax` is a strict upper bound iff `max_included` is `False`

EXAMPLES:

Some coordinate bounds on a 2-dimensional manifold:

```
sage: forget()  # for doctests only
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart('x y:[0,1)')
sage: c_xy.coord_bounds(0)  # x in (-oo,+oo) (the default)
((-Infinity, False), (+Infinity, False))
sage: c_xy.coord_bounds(1)  # y in [0,1)
((0, True), (1, False))
sage: c_xy.coord_bounds()
(((-Infinity, False), (+Infinity, False)), ((0, True), (1, False)))
sage: c_xy.coord_bounds() == (c_xy.coord_bounds(0), c_xy.coord_bounds(1))
True
```

```
>>> from sage.all import *
>>> forget()  # for doctests only
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> c_xy = M.chart('x y:[0,1)', names=('x', 'y',)); (x, y,) = c_xy._first_
↪ngens(2)
>>> c_xy.coord_bounds(Integer(0))  # x in (-oo,+oo) (the default)
((-Infinity, False), (+Infinity, False))
>>> c_xy.coord_bounds(Integer(1))  # y in [0,1)
((0, True), (1, False))
>>> c_xy.coord_bounds()
(((-Infinity, False), (+Infinity, False)), ((0, True), (1, False)))
>>> c_xy.coord_bounds() == (c_xy.coord_bounds(Integer(0)), c_xy.coord_
↪bounds(Integer(1)))
True
```

The coordinate bounds can also be recovered via the method *coord_range()*:

```
sage: c_xy.coord_range()
x: (-oo, +oo); y: [0, 1)
sage: c_xy.coord_range(y)
y: [0, 1)
```

```
>>> from sage.all import *
>>> c_xy.coord_range()
x: (-oo, +oo); y: [0, 1)
>>> c_xy.coord_range(y)
y: [0, 1)
```

or via Sage's function `sage.symbolic.assumptions.assumptions()`:

```
sage: assumptions(x)
[x is real]
sage: assumptions(y)
[y is real, y >= 0, y < 1]
```

```
>>> from sage.all import *
>>> assumptions(x)
[x is real]
>>> assumptions(y)
[y is real, y >= 0, y < 1]
```

coord_range(*xx=None*)

Display the range of a coordinate (or all coordinates), as an interval.

INPUT:

- `xx` – (default: `None`) symbolic expression corresponding to a coordinate of the current chart; if `None`, the ranges of all coordinates are displayed

EXAMPLES:

Ranges of coordinates on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: X.coord_range()
x: (-oo, +oo); y: (-oo, +oo)
sage: X.coord_range(x)
x: (-oo, +oo)
sage: U = M.open_subset('U', coord_def={X: [x>1, y<pi]})
sage: XU = X.restrict(U)  # restriction of chart X to U
sage: XU.coord_range()
x: (1, +oo); y: (-oo, pi)
sage: XU.coord_range(x)
x: (1, +oo)
sage: XU.coord_range(y)
y: (-oo, pi)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> X.coord_range()
x: (-oo, +oo); y: (-oo, +oo)
>>> X.coord_range(x)
x: (-oo, +oo)
>>> U = M.open_subset('U', coord_def={X: [x>Integer(1), y<pi]})
>>> XU = X.restrict(U)  # restriction of chart X to U
>>> XU.coord_range()
x: (1, +oo); y: (-oo, pi)
>>> XU.coord_range(x)
x: (1, +oo)
>>> XU.coord_range(y)
y: (-oo, pi)
```

The output is LaTeX-formatted for the notebook:

```
sage: latex(XU.coord_range(y))
y :\ \left( -\infty, \pi \right)
```

```
>>> from sage.all import *
>>> latex(XU.coord_range(y))
y :\ \left( -\infty, \pi \right)
```

plot(*chart=None*, *ambient_coords=None*, *mapping=None*, *fixed_coords=None*, *ranges=None*, *number_values=None*, *steps=None*, *parameters=None*, *max_range=8*, *color='red'*, *style='-'*, *thickness=1*, *plot_points=75*, *label_axes=True*, ***kwds*)

Plot `self` as a grid in a Cartesian graph based on the coordinates of some ambient chart.

The grid is formed by curves along which a chart coordinate varies, the other coordinates being kept fixed. It is drawn in terms of two (2D graphics) or three (3D graphics) coordinates of another chart, called hereafter the *ambient chart*.

The ambient chart is related to the current chart either by a transition map if both charts are defined on the same manifold, or by the coordinate expression of some continuous map (typically an immersion). In the latter case, the two charts may be defined on two different manifolds.

INPUT:

- `chart` – (default: `None`) the ambient chart (see above); if `None`, the ambient chart is set to the current chart
- `ambient_coords` – (default: `None`) tuple containing the 2 or 3 coordinates of the ambient chart in terms of which the plot is performed; if `None`, all the coordinates of the ambient chart are considered
- `mapping` – (default: `None`) `ContinuousMap`; continuous manifold map providing the link between the current chart and the ambient chart (cf. above); if `None`, both charts are supposed to be defined on the same manifold and related by some transition map (see `transition_map()`)
- `fixed_coords` – (default: `None`) dictionary with keys the chart coordinates that are not drawn and with values the fixed value of these coordinates; if `None`, all the coordinates of the current chart are drawn
- `ranges` – (default: `None`) dictionary with keys the coordinates to be drawn and values tuples `(x_min, x_max)` specifying the coordinate range for the plot; if `None`, the entire coordinate range declared during the chart construction is considered (with `-Infinity` replaced by `-max_range` and `+Infinity` by `max_range`)
- `number_values` – (default: `None`) either an integer or a dictionary with keys the coordinates to be drawn and values the number of constant values of the coordinate to be considered; if `number_values` is a single integer, it represents the number of constant values for all coordinates; if `number_values` is `None`, it is set to 9 for a 2D plot and to 5 for a 3D plot
- `steps` – (default: `None`) dictionary with keys the coordinates to be drawn and values the step between each constant value of the coordinate; if `None`, the step is computed from the coordinate range (specified in `ranges`) and `number_values`. On the contrary if the step is provided for some coordinate, the corresponding number of constant values is deduced from it and the coordinate range.
- `parameters` – (default: `None`) dictionary giving the numerical values of the parameters that may appear in the relation between the two coordinate systems
- `max_range` – (default: 8) numerical value substituted to +Infinity if the latter is the upper bound of the range of a coordinate for which the plot is performed over the entire coordinate range (i.e. for which no specific plot range has been set in `ranges`); similarly `-max_range` is the numerical valued substituted for `-Infinity`
- `color` – (default: `'red'`) either a single color or a dictionary of colors, with keys the coordinates to be drawn, representing the colors of the lines along which the coordinate varies, the other being kept constant; if `color` is a single color, it is used for all coordinate lines

- `style` – (default: `'-'`) either a single line style or a dictionary of line styles, with keys the coordinates to be drawn, representing the style of the lines along which the coordinate varies, the other being kept constant; if `style` is a single style, it is used for all coordinate lines; NB: `style` is effective only for 2D plots
- `thickness` – (default: 1) either a single line thickness or a dictionary of line thicknesses, with keys the coordinates to be drawn, representing the thickness of the lines along which the coordinate varies, the other being kept constant; if `thickness` is a single value, it is used for all coordinate lines
- `plot_points` – (default: 75) either a single number of points or a dictionary of integers, with keys the coordinates to be drawn, representing the number of points to plot the lines along which the coordinate varies, the other being kept constant; if `plot_points` is a single integer, it is used for all coordinate lines
- `label_axes` – boolean (default: `True`); determining whether the labels of the ambient coordinate axes shall be added to the graph; can be set to `False` if the graph is 3D and must be superposed with another graph

OUTPUT:

- a graphic object, either a `Graphics` for a 2D plot (i.e. based on 2 coordinates of the ambient chart) or a `Graphics3d` for a 3D plot (i.e. based on 3 coordinates of the ambient chart)

EXAMPLES:

A 2-dimensional chart plotted in terms of itself results in a rectangular grid:

```
sage: R2 = Manifold(2, 'R^2', structure='topological')  # the Euclidean plane
sage: c_cart.<x,y> = R2.chart()  # Cartesian coordinates
sage: g = c_cart.plot(); g  # equivalent to c_cart.plot(c_cart)            #␣
↪needs sage.plot
Graphics object consisting of 18 graphics primitives
```

```
>>> from sage.all import *
>>> R2 = Manifold(Integer(2), 'R^2', structure='topological')  # the␣
↪Euclidean plane
>>> c_cart = R2.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#␣
↪Cartesian coordinates
>>> g = c_cart.plot(); g  # equivalent to c_cart.plot(c_cart)            #␣
↪needs sage.plot
Graphics object consisting of 18 graphics primitives
```

Grid of polar coordinates in terms of Cartesian coordinates in the Euclidean plane:

```
sage: U = R2.open_subset('U', coord_def={c_cart: (y!=0, x<0)})  # the␣
↪complement of the segment y=0 and x>0
sage: c_pol.<r,ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi')  # polar␣
↪coordinates on U
sage: pol_to_cart = c_pol.transition_map(c_cart, [r*cos(ph), r*sin(ph)])
sage: g = c_pol.plot(c_cart); g                                           #␣
↪needs sage.plot
Graphics object consisting of 18 graphics primitives
```

```
>>> from sage.all import *
>>> U = R2.open_subset('U', coord_def={c_cart: (y!=Integer(0), x<Integer(0))}
↪)  # the complement of the segment y=0 and x>0
```

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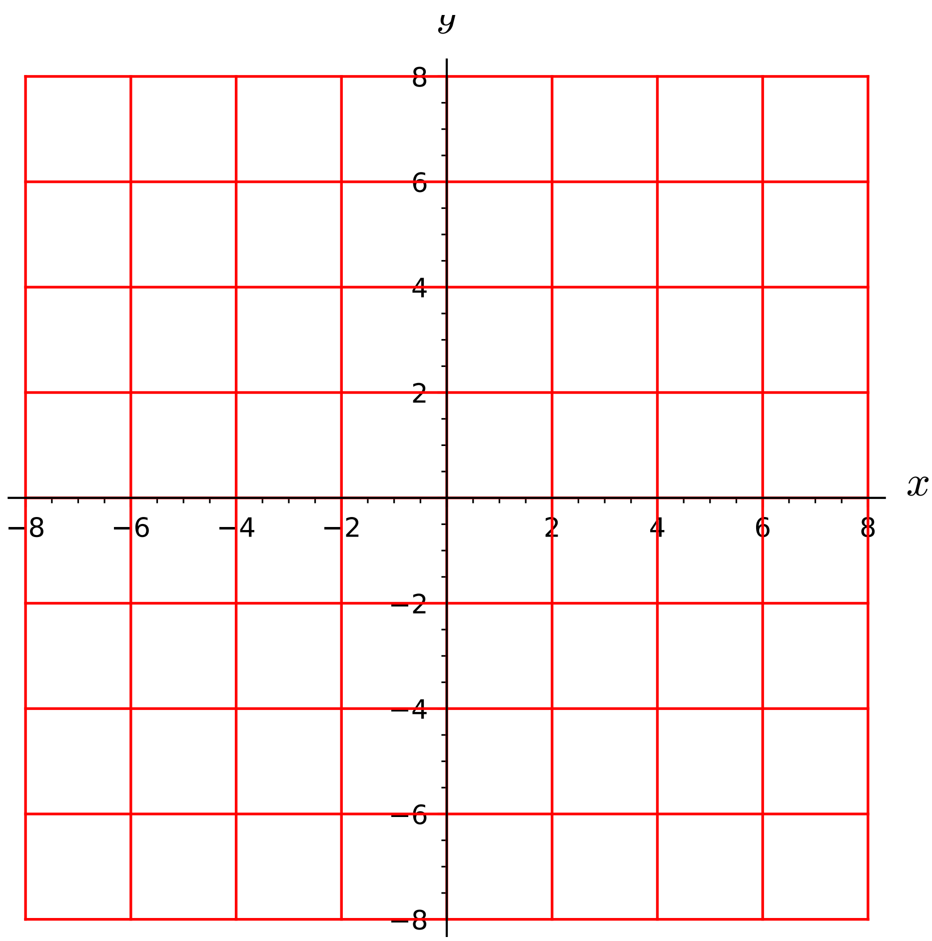

y
x
8
6
4
2
−2
−4
−6
−8
−8
−6
−4
−2
2
4
6
8

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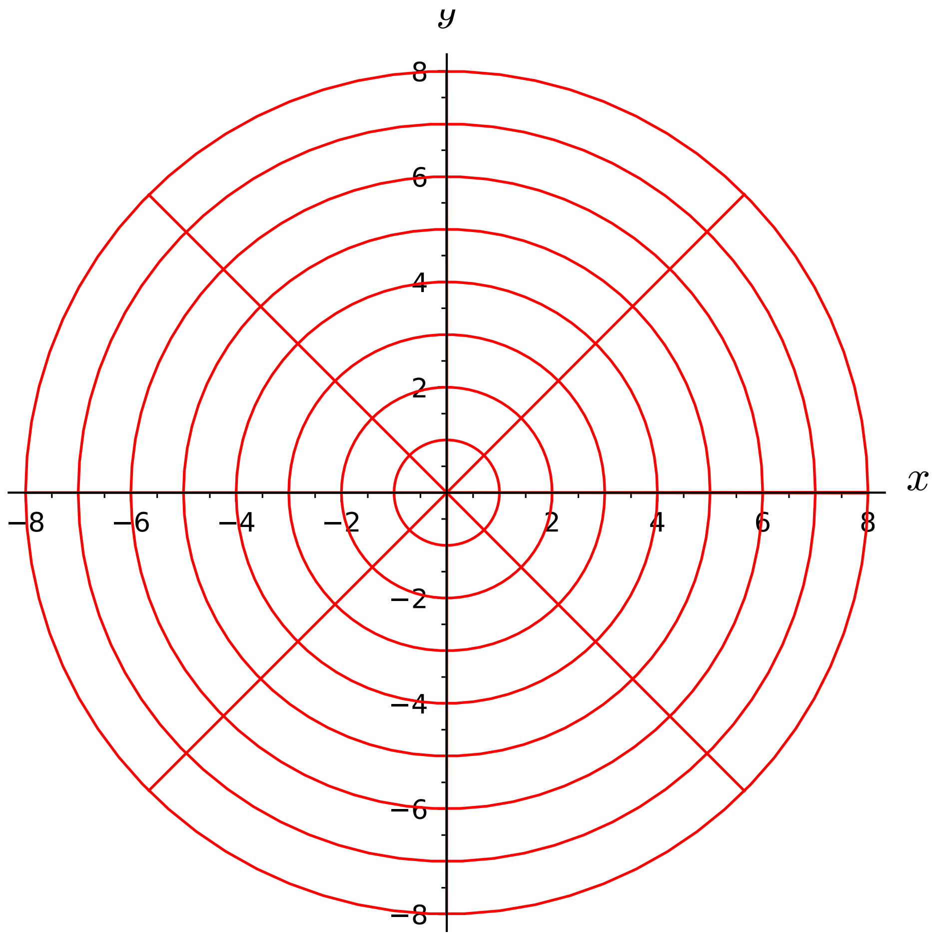
```
>>> c_pol = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi', names=('r', 'ph',)); (r,␣
↪ph,) = c_pol._first_ngens(2)# polar coordinates on U
>>> pol_to_cart = c_pol.transition_map(c_cart, [r*cos(ph), r*sin(ph)])
>>> g = c_pol.plot(c_cart); g                                          #␣
↪needs sage.plot
Graphics object consisting of 18 graphics primitives
```

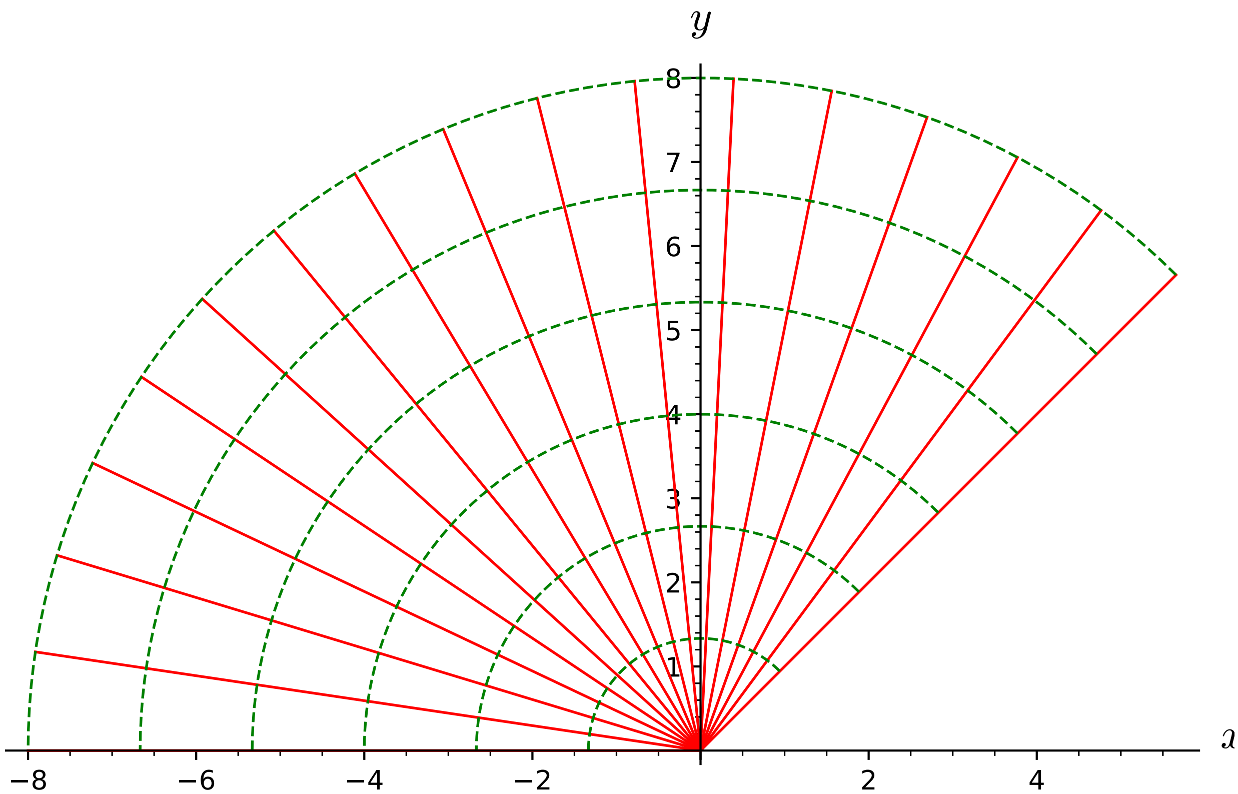


Call with non-default values:

```
sage: g = c_pol.plot(c_cart, ranges={ph:(pi/4,pi)},                    #␣
↪needs sage.plot
....:                  number_values={r:7, ph:17},
....:                  color={r:'red', ph:'green'},
....:                  style={r:'-', ph:'--'})
```

```
>>> from sage.all import *
>>> g = c_pol.plot(c_cart, ranges={ph:(pi/Integer(4),pi)},             ␣
↪     # needs sage.plot
...                  number_values={r:Integer(7), ph:Integer(17)},
...                  color={r:'red', ph:'green'},
...                  style={r:'-', ph:'--'})
```

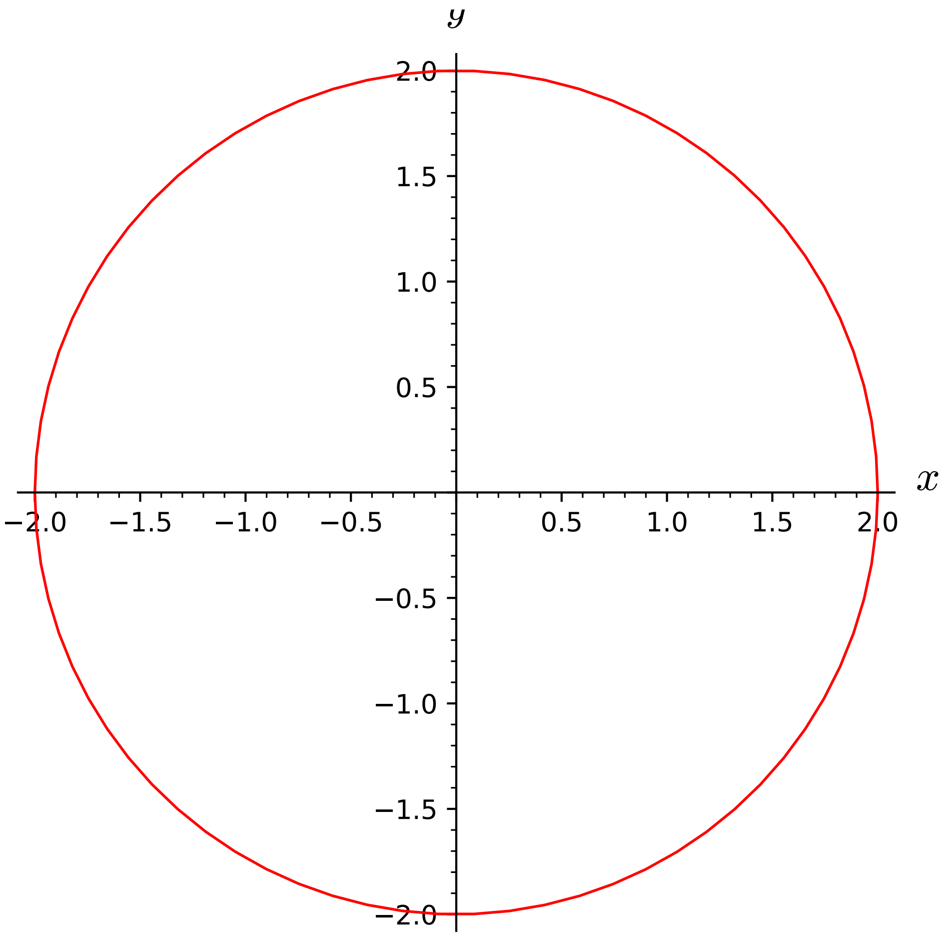
A single coordinate line can be drawn:



y
x
−8
−6
−4
−2
2
4
1
2
3
4
5
6
7
8

```
sage: g = c_pol.plot(c_cart,     # draw a circle of radius r=2                #␣
↪needs sage.plot
....:                  fixed_coords={r: 2})
```

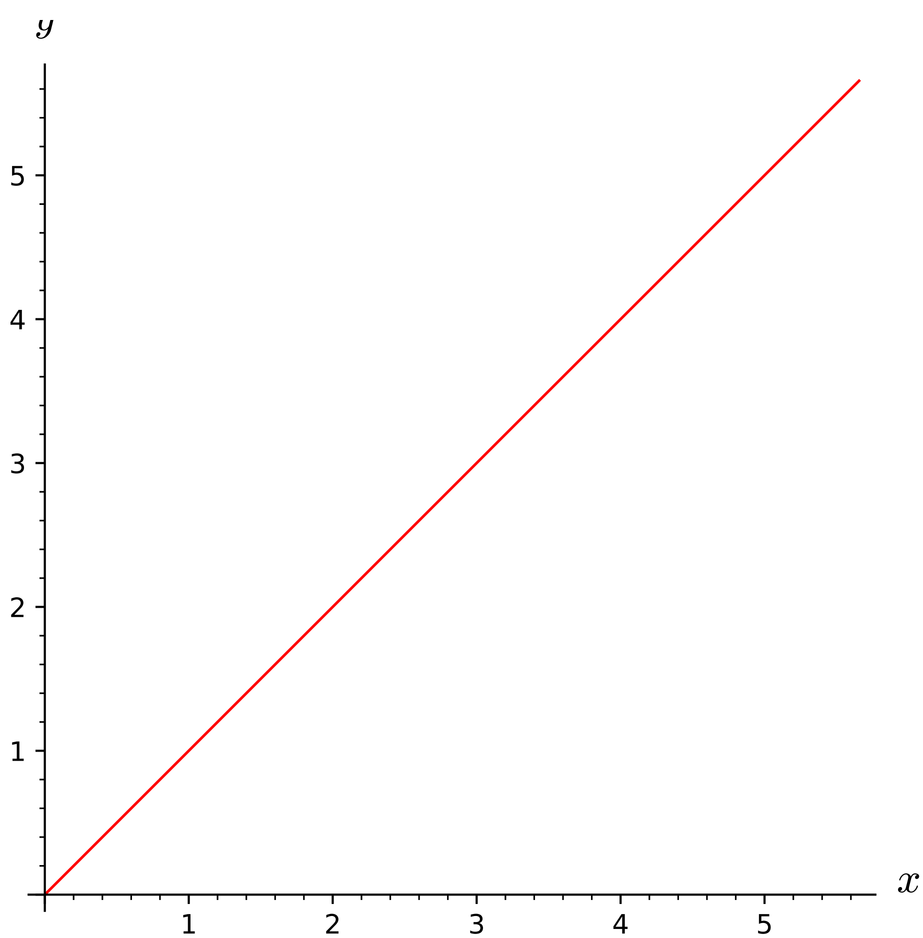
```
>>> from sage.all import *
>>> g = c_pol.plot(c_cart,     # draw a circle of radius r=2                #␣
↪needs sage.plot
...                  fixed_coords={r: Integer(2)})
```




```
sage: g = c_pol.plot(c_cart,     # draw a segment at phi=pi/4                #␣
↪needs sage.plot
....:                  fixed_coords={ph: pi/4})
```

```
>>> from sage.all import *
>>> g = c_pol.plot(c_cart,     # draw a segment at phi=pi/4                #␣
↪needs sage.plot
...                  fixed_coords={ph: pi/Integer(4)})
```

An example with the ambient chart lying in an another manifold (the plot is then performed via some manifold map passed as the argument `mapping`): 3D plot of the stereographic charts on the 2-sphere:


y
5
4
3
2
1
x
1
2
3
4
5

```
sage: S2 = Manifold(2, 'S^2', structure='topological')  # the 2-sphere
sage: U = S2.open_subset('U'); V = S2.open_subset('V')  # complement of the␣
↪North and South pole, respectively
sage: S2.declare_union(U,V)
sage: c_xy.<x,y> = U.chart()  # stereographic coordinates from the North pole
sage: c_uv.<u,v> = V.chart()  # stereographic coordinates from the South pole
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: R3 = Manifold(3, 'R^3', structure='topological')  # the Euclidean space␣
↪R^3
sage: c_cart.<X,Y,Z> = R3.chart()  # Cartesian coordinates on R^3
sage: Phi = S2.continuous_map(R3, {(c_xy, c_cart): [2*x/(1+x^2+y^2),
....:                         2*y/(1+x^2+y^2), (x^2+y^2-1)/(1+x^2+y^2)],
....:                         (c_uv, c_cart): [2*u/(1+u^2+v^2),
....:                         2*v/(1+u^2+v^2), (1-u^2-v^2)/(1+u^2+v^2)]},
....:                        name='Phi', latex_name=r'\Phi')  # Embedding of␣
↪S^2 in R^3
sage: g = c_xy.plot(c_cart, mapping=Phi); g                                 #␣
↪needs sage.plot
Graphics3d Object
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2', structure='topological')  # the 2-sphere
>>> U = S2.open_subset('U'); V = S2.open_subset('V')  # complement of the␣
↪North and South pole, respectively
>>> S2.declare_union(U,V)
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                 intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                 restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> R3 = Manifold(Integer(3), 'R^3', structure='topological')  # the␣
↪Euclidean space R^3
>>> c_cart = R3.chart(names=('X', 'Y', 'Z',)); (X, Y, Z,) = c_cart._first_
↪ngens(3)# Cartesian coordinates on R^3
>>> Phi = S2.continuous_map(R3, {(c_xy, c_cart): [Integer(2)*x/
↪(Integer(1)+x**Integer(2)+y**Integer(2)),
...                         Integer(2)*y/
↪(Integer(1)+x**Integer(2)+y**Integer(2)), (x**Integer(2)+y**Integer(2)-
↪Integer(1))/(Integer(1)+x**Integer(2)+y**Integer(2))],
...                         (c_uv, c_cart): [Integer(2)*u/
↪(Integer(1)+u**Integer(2)+v**Integer(2)),
...                         Integer(2)*v/
↪(Integer(1)+u**Integer(2)+v**Integer(2)), (Integer(1)-u**Integer(2)-
↪v**Integer(2))/(Integer(1)+u**Integer(2)+v**Integer(2))]},
```

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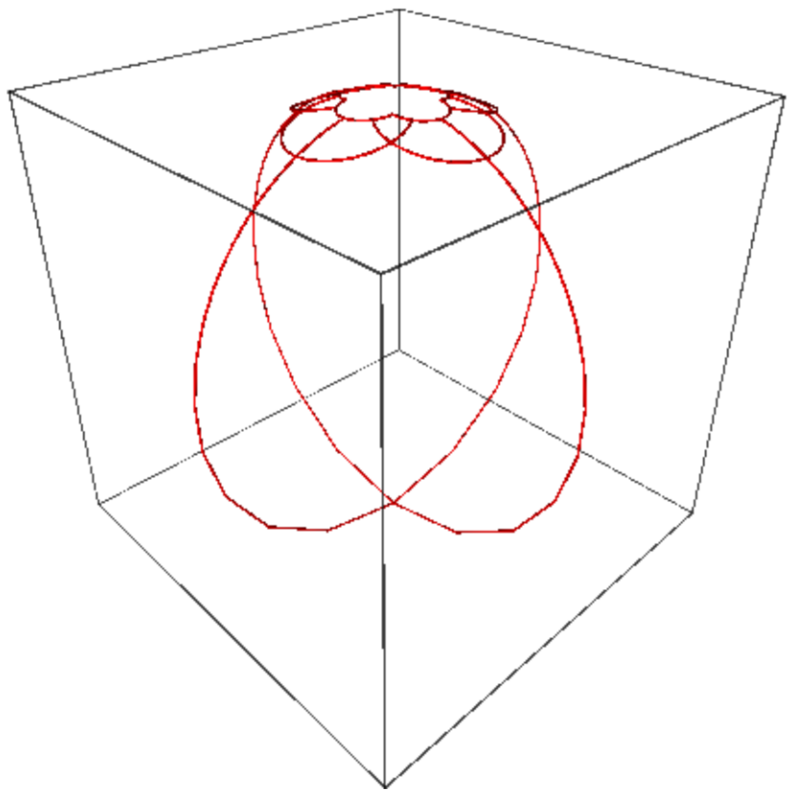
```
...                          name='Phi', latex_name=r'\Phi')  # Embedding of S^
↪2 in R^3
>>> g = c_xy.plot(c_cart, mapping=Phi); g                                     #␣
↪needs sage.plot
Graphics3d Object
```



NB: to get a better coverage of the whole sphere, one should increase the coordinate sampling via the argument `number_values` or the argument `steps` (only the default value, `number_values = 5`, is used here, which is pretty low).

The same plot without the `(X,Y,Z)` axes labels:

```
sage: g = c_xy.plot(c_cart, mapping=Phi, label_axes=False)                    #␣
↪needs sage.plot
```

```
>>> from sage.all import *
>>> g = c_xy.plot(c_cart, mapping=Phi, label_axes=False)                    #␣
↪needs sage.plot
```

The North and South stereographic charts on the same plot:

```
sage: g2 = c_uv.plot(c_cart, mapping=Phi, color='green')                     #␣
```

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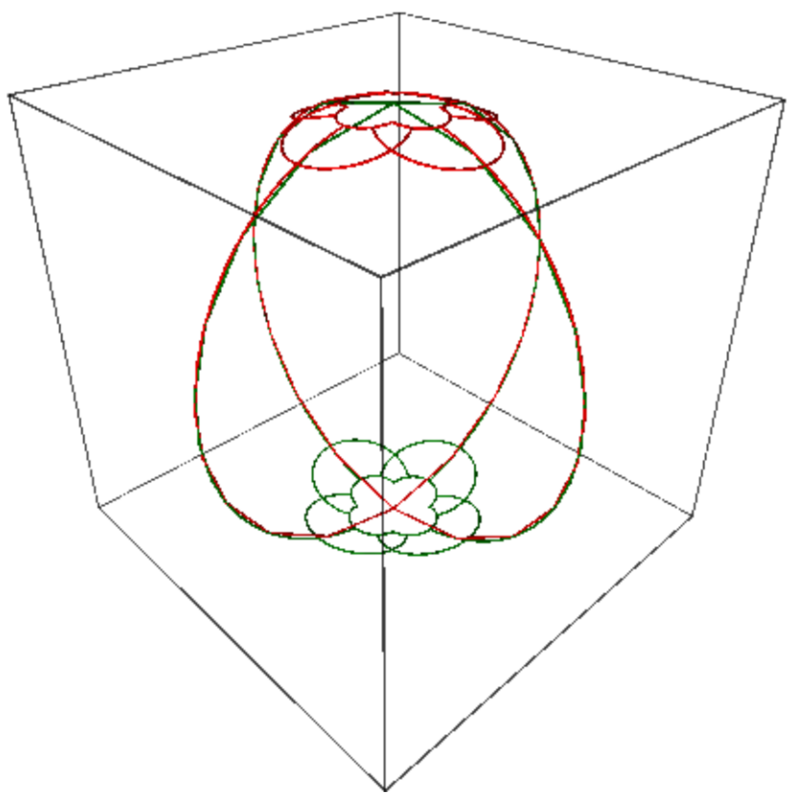
```
↪needs sage.plot
sage: g + g2                                                            #␣
↪needs sage.plot
Graphics3d Object
```

```
>>> from sage.all import *
>>> g2 = c_uv.plot(c_cart, mapping=Phi, color='green')                 #␣
↪needs sage.plot
>>> g + g2                                                             #␣
↪needs sage.plot
Graphics3d Object
```



South stereographic chart drawn in terms of the North one (we split the plot in four parts to avoid the singularity at $(u, v) = (0, 0)$):

```
sage: # long time, needs sage.plot
sage: W = U.intersection(V) # the subset common to both charts
sage: c_uvW = c_uv.restrict(W) # chart (W,(u,v))
sage: gSN1 = c_uvW.plot(c_xy, ranges={u:[-6.,-0.02], v:[-6.,-0.02]})
sage: gSN2 = c_uvW.plot(c_xy, ranges={u:[-6.,-0.02], v:[0.02,6.]})
sage: gSN3 = c_uvW.plot(c_xy, ranges={u:[0.02,6.], v:[-6.,-0.02]})
sage: gSN4 = c_uvW.plot(c_xy, ranges={u:[0.02,6.], v:[0.02,6.]})
```

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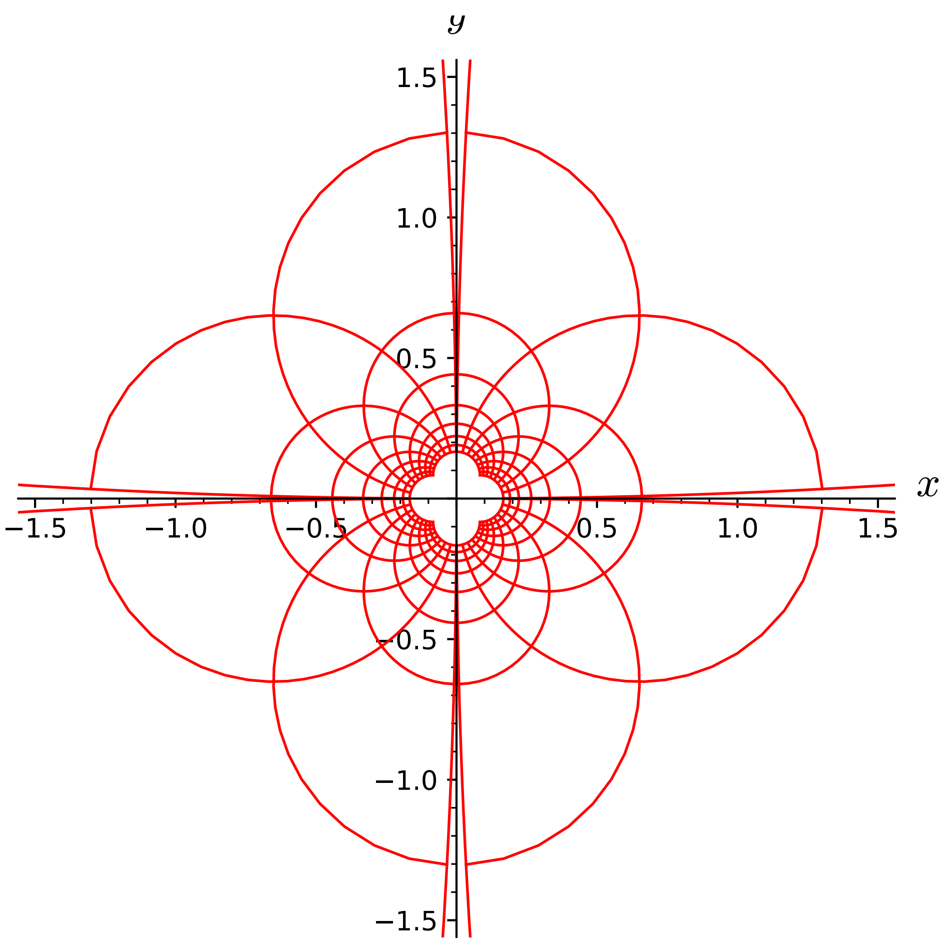
```
sage: show(gSN1+gSN2+gSN3+gSN4, xmin=-1.5, xmax=1.5, ymin=-1.5, ymax=1.5)
```

```
>>> from sage.all import *
>>> # long time, needs sage.plot
>>> W = U.intersection(V) # the subset common to both charts
>>> c_uvW = c_uv.restrict(W) # chart (W,(u,v))
>>> gSN1 = c_uvW.plot(c_xy, ranges={u:[-RealNumber('6.'),-RealNumber('0.02')],
↪ v:[-RealNumber('6.'),-RealNumber('0.02')]})
>>> gSN2 = c_uvW.plot(c_xy, ranges={u:[-RealNumber('6.'),-RealNumber('0.02')],
↪ v:[RealNumber('0.02'),RealNumber('6.')]})
>>> gSN3 = c_uvW.plot(c_xy, ranges={u:[RealNumber('0.02'),RealNumber('6.')],␣
↪v:[-RealNumber('6.'),-RealNumber('0.02')]})
>>> gSN4 = c_uvW.plot(c_xy, ranges={u:[RealNumber('0.02'),RealNumber('6.')],␣
↪v:[RealNumber('0.02'),RealNumber('6.')]})
>>> show(gSN1+gSN2+gSN3+gSN4, xmin=-RealNumber('1.5'), xmax=RealNumber('1.5'),
↪ ymin=-RealNumber('1.5'), ymax=RealNumber('1.5'))
```



The coordinate line $u = 1$ (red) and the coordinate line $v = 1$ (green) on the same plot:

```
sage: # long time, needs sage.plot
sage: gu1 = c_uvW.plot(c_xy, fixed_coords={u: 1}, max_range=20,
....:                  plot_points=300)
```

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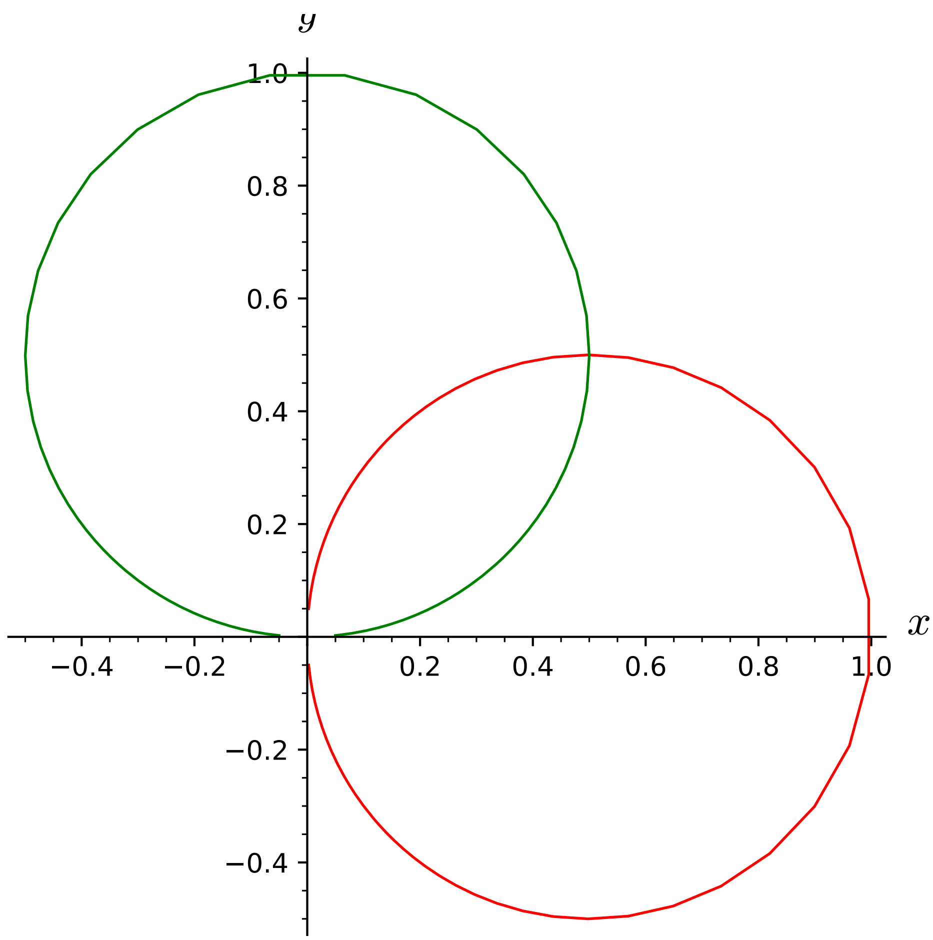
```
sage: gv1 = c_uvW.plot(c_xy, fixed_coords={v: 1}, max_range=20,
....:                  plot_points=300, color='green')
sage: gu1 + gv1
Graphics object consisting of 2 graphics primitives
```

```
>>> from sage.all import *
>>> # long time, needs sage.plot
>>> gu1 = c_uvW.plot(c_xy, fixed_coords={u: Integer(1)}, max_
↪range=Integer(20),
...                  plot_points=Integer(300))
>>> gv1 = c_uvW.plot(c_xy, fixed_coords={v: Integer(1)}, max_
↪range=Integer(20),
...                  plot_points=Integer(300), color='green')
>>> gu1 + gv1
Graphics object consisting of 2 graphics primitives
```



Note that we have set `max_range=20` to have a wider range for the coordinates u and v, i.e. to have $[-20, 20]$ instead of the default $[-8, 8]$.

A 3-dimensional chart plotted in terms of itself results in a 3D rectangular grid:

```
sage: # long time, needs sage.plot
```

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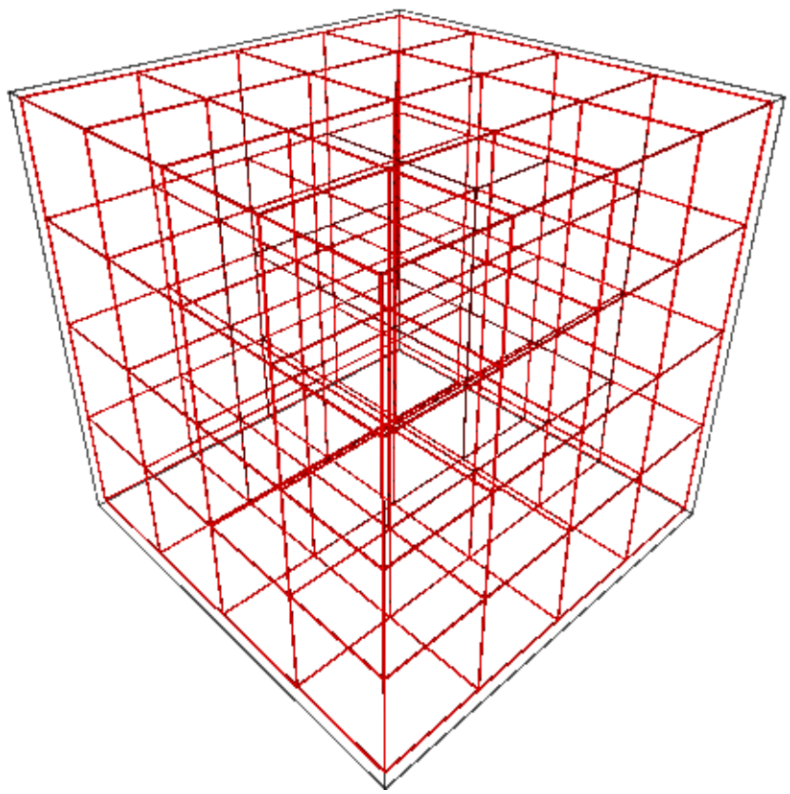
```
sage: g = c_cart.plot()  # equivalent to c_cart.plot(c_cart)
sage: g
Graphics3d Object
```

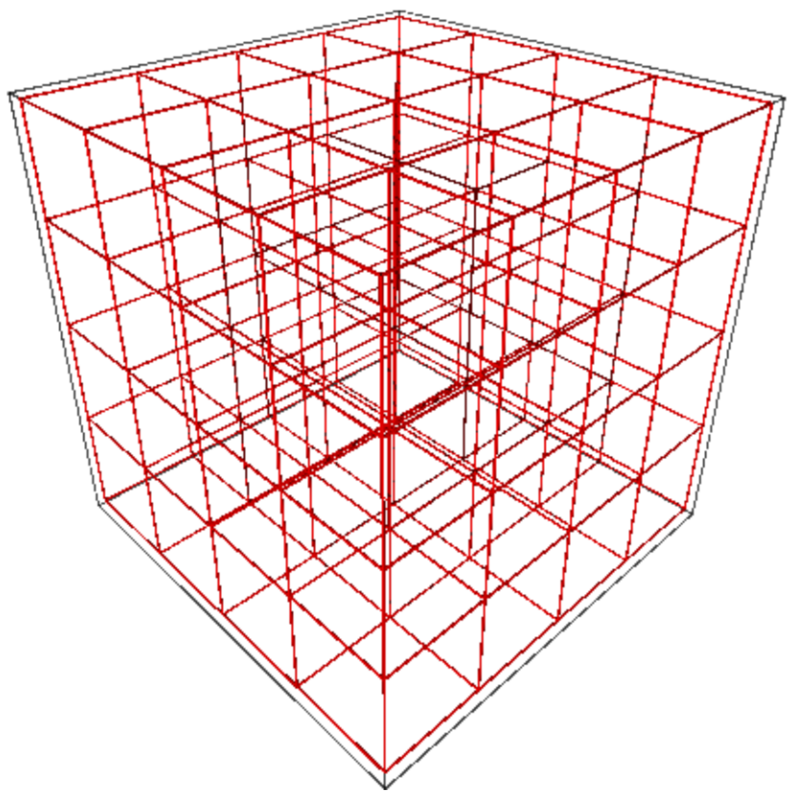
```
>>> from sage.all import *
>>> # long time, needs sage.plot
>>> g = c_cart.plot()  # equivalent to c_cart.plot(c_cart)
>>> g
Graphics3d Object
```



A 4-dimensional chart plotted in terms of itself (the plot is performed for at most 3 coordinates, which must be specified via the argument `ambient_coords`):

```
sage: # needs sage.plot
sage: M = Manifold(4, 'M', structure='topological')
sage: X.<t,x,y,z> = M.chart()
sage: g = X.plot(ambient_coords=(t,x,y))  # the coordinate z is not depicted
↪# long time
sage: g
↪# long time
Graphics3d Object
```

```
>>> from sage.all import *
>>> # needs sage.plot
>>> M = Manifold(Integer(4), 'M', structure='topological')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> g = X.plot(ambient_coords=(t,x,y))  # the coordinate z is not depicted  #␣
↪long time
>>> g                                                                        #␣
↪long time
Graphics3d Object
```



```
sage: # needs sage.plot
sage: g = X.plot(ambient_coords=(t,y))  # the coordinates x and z are not␣
↪depicted
sage: g
Graphics object consisting of 18 graphics primitives
```

```
>>> from sage.all import *
>>> # needs sage.plot
>>> g = X.plot(ambient_coords=(t,y))  # the coordinates x and z are not␣
↪depicted
>>> g
```

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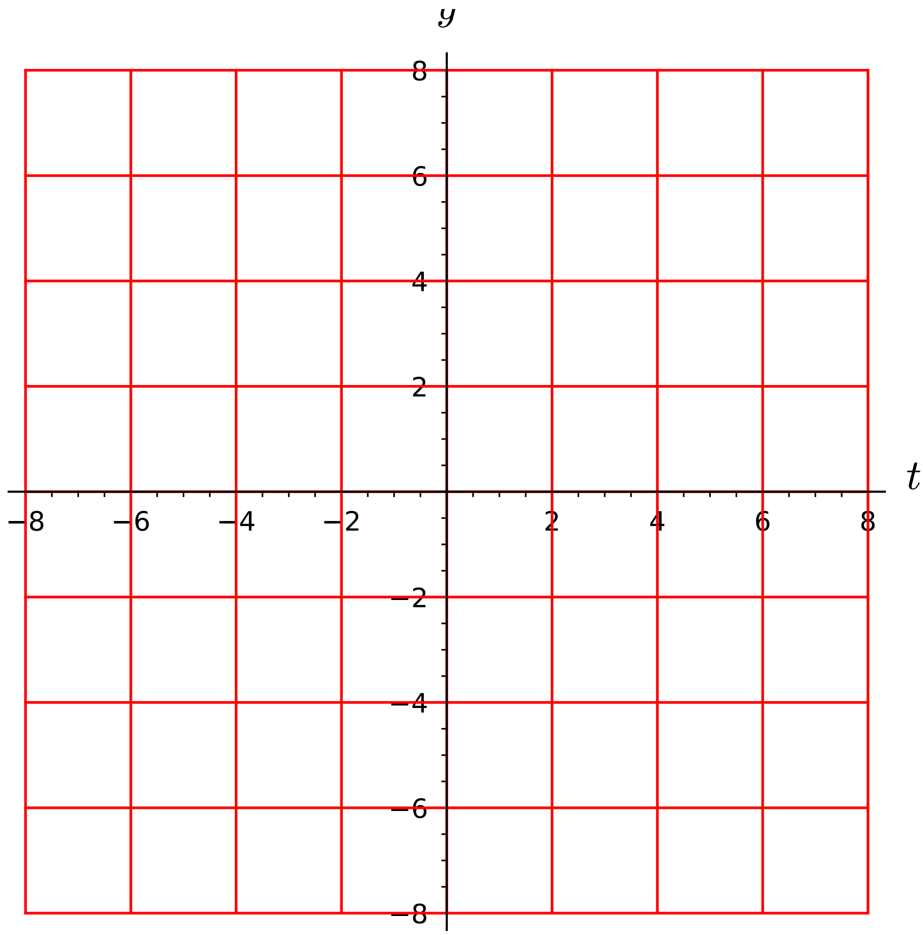
```
Graphics object consisting of 18 graphics primitives
```



Note that the default values of some arguments of the method `plot` are stored in the dictionary `plot.options`:

```
sage: X.plot.options  # random (dictionary output)
{'color': 'red', 'label_axes': True, 'max_range': 8,
 'plot_points': 75, 'style': '-', 'thickness': 1}
```

```
>>> from sage.all import *
>>> X.plot.options  # random (dictionary output)
{'color': 'red', 'label_axes': True, 'max_range': 8,
 'plot_points': 75, 'style': '-', 'thickness': 1}
```

so that they can be adjusted by the user:

```
sage: X.plot.options['color'] = 'blue'
```

```
>>> from sage.all import *
>>> X.plot.options['color'] = 'blue'
```

From now on, all chart plots will use blue as the default color. To restore the original default options, it suffices to type:

```
sage: X.plot.reset()
```

```
>>> from sage.all import *
>>> X.plot.reset()
```

restrict(*subset*, *restrictions=None*)

Return the restriction of the chart to some open subset of its domain.

If the current chart is (U,φ), a *restriction* (or *subchart*) is a chart (V,ψ) such that $V\subset U$ and $\psi=\varphi|_V$.

If such subchart has not been defined yet, it is constructed here.

The coordinates of the subchart bare the same names as the coordinates of the current chart.

INPUT:

- `subset` – open subset V of the chart domain U (must be an instance of `TopologicalManifold`)
- `restrictions` – (default: `None`) list of coordinate restrictions defining the subset V

A restriction can be any symbolic equality or inequality involving the coordinates, such as `x > y` or `x^2 + y^2 != 0`. The items of the list `restrictions` are combined with the `and` operator; if some restrictions are to be combined with the `or` operator instead, they have to be passed as a tuple in some single item of the list `restrictions`. For example:

```
restrictions = [x > y, (x != 0, y != 0), z^2 < x]
```

means `(x > y) and ((x != 0) or (y != 0)) and (z^2 < x)`. If the list `restrictions` contains only one item, this item can be passed as such, i.e. writing `x > y` instead of the single element list `[x > y]`.

OUTPUT:

- the chart (V,ψ) as a `RealChart`

EXAMPLES:

Cartesian coordinates on the unit open disc in $\mathbf{R}^2$ as a subchart of the global Cartesian coordinates:

```
sage: M = Manifold(2, 'R^2', structure='topological')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: D = M.open_subset('D') # the unit open disc
sage: c_cart_D = c_cart.restrict(D, x^2+y^2<1)
sage: p = M.point((1/2, 0))
sage: p in D
True
sage: q = M.point((1, 2))
sage: q in D
False
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', structure='topological')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#␣
↪Cartesian coordinates on R^2
>>> D = M.open_subset('D') # the unit open disc
>>> c_cart_D = c_cart.restrict(D, x**Integer(2)+y**Integer(2)<Integer(1))
>>> p = M.point((Integer(1)/Integer(2), Integer(0)))
>>> p in D
True
```

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```
>>> q = M.point((Integer(1), Integer(2)))
>>> q in D
False
```

Cartesian coordinates on the annulus $1 < \sqrt{x^2 + y^2} < 2$:

```
sage: A = M.open_subset('A')
sage: c_cart_A = c_cart.restrict(A, [x^2+y^2>1, x^2+y^2<4])
sage: p in A, q in A
(False, False)
sage: a = M.point((3/2,0))
sage: a in A
True
```

```
>>> from sage.all import *
>>> A = M.open_subset('A')
>>> c_cart_A = c_cart.restrict(A, [x**Integer(2)+y**Integer(2)>Integer(1),␣
↪x**Integer(2)+y**Integer(2)<Integer(4)])
>>> p in A, q in A
(False, False)
>>> a = M.point((Integer(3)/Integer(2),Integer(0)))
>>> a in A
True
```

valid_coordinates(**coordinates*, ***kwds*)

Check whether a tuple of coordinates can be the coordinates of a point in the chart domain.

INPUT:

- `*coordinates` – coordinate values
- `**kwds` – options:
  - `tolerance=0`, to set the absolute tolerance in the test of coordinate ranges
  - `parameters=None`, to set some numerical values to parameters

OUTPUT:

- `True` if the coordinate values are admissible in the chart range and `False` otherwise

EXAMPLES:

Cartesian coordinates on a square interior:

```
sage: forget()  # for doctest only
sage: M = Manifold(2, 'M', structure='topological')  # the square interior
sage: X.<x,y> = M.chart('x:(-2,2) y:(-2,2)')
sage: X.valid_coordinates(0,1)
True
sage: X.valid_coordinates(-3/2,5/4)
True
sage: X.valid_coordinates(0,3)
False
```

```
>>> from sage.all import *
>>> forget()  # for doctest only
>>> M = Manifold(Integer(2), 'M', structure='topological')  # the square␣
↪interior
>>> X = M.chart('x:(-2,2) y:(-2,2)', names=('x', 'y',)); (x, y,) = X._first_
↪ngens(2)
>>> X.valid_coordinates(Integer(0),Integer(1))
True
>>> X.valid_coordinates(-Integer(3)/Integer(2),Integer(5)/Integer(4))
True
>>> X.valid_coordinates(Integer(0),Integer(3))
False
```

The unit open disk inside the square:

```
sage: D = M.open_subset('D', coord_def={X: x^2+y^2<1})
sage: XD = X.restrict(D)
sage: XD.valid_coordinates(0,1)
False
sage: XD.valid_coordinates(-3/2,5/4)
False
sage: XD.valid_coordinates(-1/2,1/2)
True
sage: XD.valid_coordinates(0,0)
True
```

```
>>> from sage.all import *
>>> D = M.open_subset('D', coord_def={X: x**Integer(2)+y**Integer(2)
↪<Integer(1)})
>>> XD = X.restrict(D)
>>> XD.valid_coordinates(Integer(0),Integer(1))
False
>>> XD.valid_coordinates(-Integer(3)/Integer(2),Integer(5)/Integer(4))
False
>>> XD.valid_coordinates(-Integer(1)/Integer(2),Integer(1)/Integer(2))
True
>>> XD.valid_coordinates(Integer(0),Integer(0))
True
```

Another open subset of the square, defined by $x^2 + y^2 < 1$ or ($x > 0$ and $|y| < 1$):

```
sage: B = M.open_subset('B',
....:                   coord_def={X: (x^2+y^2<1,
....:                                  [x>0, abs(y)<1])})
sage: XB = X.restrict(B)
sage: XB.valid_coordinates(-1/2, 0)
True
sage: XB.valid_coordinates(-1/2, 3/2)
False
sage: XB.valid_coordinates(3/2, 1/2)
True
```

```
>>> from sage.all import *
>>> B = M.open_subset('B',
...                   coord_def={X: (x**Integer(2)+y**Integer(2)<Integer(1),
...                              [x>Integer(0), abs(y)<Integer(1)])})
>>> XB = X.restrict(B)
>>> XB.valid_coordinates(-Integer(1)/Integer(2), Integer(0))
True
>>> XB.valid_coordinates(-Integer(1)/Integer(2), Integer(3)/Integer(2))
False
>>> XB.valid_coordinates(Integer(3)/Integer(2), Integer(1)/Integer(2))
True
```

valid_coordinates_numerical(**coordinates*)

Check whether a tuple of float coordinates can be the coordinates of a point in the chart domain.

This version is optimized for float numbers, and cannot accept parameters nor tolerance. The chart restriction must also be specified in CNF (i.e. a list of tuples).

INPUT:

- `*coordinates` – coordinate values

OUTPUT:

- `True` if the coordinate values are admissible in the chart range and `False` otherwise

EXAMPLES:

Cartesian coordinates on a square interior:

```
sage: forget()  # for doctest only
sage: M = Manifold(2, 'M', structure='topological')  # the square interior
sage: X.<x,y> = M.chart('x:(-2,2) y:(-2,2)')
sage: X.valid_coordinates_numerical(0,1)
True
sage: X.valid_coordinates_numerical(-3/2,5/4)
True
sage: X.valid_coordinates_numerical(0,3)
False
```

```
>>> from sage.all import *
>>> forget()  # for doctest only
>>> M = Manifold(Integer(2), 'M', structure='topological')  # the square␣
↪interior
>>> X = M.chart('x:(-2,2) y:(-2,2)', names=('x', 'y',)); (x, y,) = X._first_
↪ngens(2)
>>> X.valid_coordinates_numerical(Integer(0),Integer(1))
True
>>> X.valid_coordinates_numerical(-Integer(3)/Integer(2),Integer(5)/
↪Integer(4))
True
>>> X.valid_coordinates_numerical(Integer(0),Integer(3))
False
```

The unit open disk inside the square:

```
sage: D = M.open_subset('D', coord_def={X: x^2+y^2<1})
sage: XD = X.restrict(D)
sage: XD.valid_coordinates_numerical(0,1)
False
sage: XD.valid_coordinates_numerical(-3/2,5/4)
False
sage: XD.valid_coordinates_numerical(-1/2,1/2)
True
sage: XD.valid_coordinates_numerical(0,0)
True
```

```
>>> from sage.all import *
>>> D = M.open_subset('D', coord_def={X: x**Integer(2)+y**Integer(2)
↪<Integer(1)})
>>> XD = X.restrict(D)
>>> XD.valid_coordinates_numerical(Integer(0),Integer(1))
False
>>> XD.valid_coordinates_numerical(-Integer(3)/Integer(2),Integer(5)/
↪Integer(4))
False
>>> XD.valid_coordinates_numerical(-Integer(1)/Integer(2),Integer(1)/
↪Integer(2))
True
>>> XD.valid_coordinates_numerical(Integer(0),Integer(0))
True
```

Another open subset of the square, defined by $x^2 + y^2 < 1$ or ($x > 0$ and $|y| < 1$):

```
sage: B = M.open_subset('B',coord_def={X: [(x^2+y^2<1, x>0),
....:                     (x^2+y^2<1,  abs(y)<1)]})
sage: XB = X.restrict(B)
sage: XB.valid_coordinates_numerical(-1/2, 0)
True
sage: XB.valid_coordinates_numerical(-1/2, 3/2)
False
sage: XB.valid_coordinates_numerical(3/2, 1/2)
True
```

```
>>> from sage.all import *
>>> B = M.open_subset('B',coord_def={X: [(x**Integer(2)+y**Integer(2)
↪<Integer(1), x>Integer(0)),
...                     (x**Integer(2)+y**Integer(2)<Integer(1),  abs(y)
↪<Integer(1))]})
>>> XB = X.restrict(B)
>>> XB.valid_coordinates_numerical(-Integer(1)/Integer(2), Integer(0))
True
>>> XB.valid_coordinates_numerical(-Integer(1)/Integer(2), Integer(3)/
↪Integer(2))
False
>>> XB.valid_coordinates_numerical(Integer(3)/Integer(2), Integer(1)/
↪Integer(2))
True
```

1.5.2 Chart Functions

In the context of a topological manifold M over a topological field K, a *chart function* is a function from a chart codomain to K. In other words, a chart function is a K-valued function of the coordinates associated to some chart. The internal coordinate expressions of chart functions and calculus on them are taken in charge by different calculus methods, at the choice of the user:

- Sage's default symbolic engine (Pynac + Maxima), implemented via the Symbolic Ring (`SR`)
- SymPy engine, denoted `sympy` hereafter

See *`CalculusMethod`* for details.

AUTHORS:

- Marco Mancini (2017) : initial version
- Eric Gourgoulhon (2015) : for a previous class implementing only SR calculus (CoordFunctionSymb)
- Florentin Jaffredo (2018) : series expansion with respect to a given parameter

class `sage.manifolds.chart_func.`**`ChartFunction`**(*parent*, *expression=None*, *calc_method=None*, *expansion_symbol=None*, *order=None*)

Bases: `AlgebraElement`, `ModuleElementWithMutability`

Function of coordinates of a given chart.

If (U, φ) is a chart on a topological manifold M of dimension n over a topological field K, a *chart function* associated to (U, φ) is a map

$$\begin{array}{llcl} f: & V \subset K^n & \longrightarrow & K \\ & (x^1, \ldots, x^n) & \longmapsto & f(x^1, \ldots, x^n), \end{array}$$

where V is the codomain of φ. In other words, f is a K-valued function of the coordinates associated to the chart (U, φ).

The chart function f can be represented by expressions pertaining to different calculus methods; the currently implemented ones are

- `SR` (Sage's Symbolic Ring)
- `SymPy`

See *`expr()`* for details.

INPUT:

- `parent` – the algebra of chart functions on the chart (U, φ)
- `expression` – (default: `None`) a symbolic expression representing $f(x^1, \ldots, x^n)$, where $(x^1, \ldots, x^n)$ are the coordinates of the chart (U, φ)
- `calc_method` – string (default: `None`); the calculus method with respect to which the internal expression of `self` must be initialized from `expression`. One of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy
  - `None`: the chart current calculus method is assumed
- `expansion_symbol` – (default: `None`) symbolic variable (the "small parameter") with respect to which the coordinate expression is expanded in power series (around the zero value of this variable)

- `order` – integer (default: `None`); the order of the expansion if `expansion_symbol` is not `None`; the *order* is defined as the degree of the polynomial representing the truncated power series in `expansion_symbol`

> **Warning**
>
> The value of `order` is $n-1$, where n is the order of the big O in the power series expansion

EXAMPLES:

A symbolic chart function on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x^2+3*y+1)
sage: type(f)
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_class'>
sage: f.display()
(x, y) ↦ x^2 + 3*y + 1
sage: f(x,y)
x^2 + 3*y + 1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x**Integer(2)+Integer(3)*y+Integer(1))
>>> type(f)
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_class'>
>>> f.display()
(x, y) ↦ x^2 + 3*y + 1
>>> f(x,y)
x^2 + 3*y + 1
```

The symbolic expression is returned when asking for the direct display of the function:

```
sage: f
x^2 + 3*y + 1
sage: latex(f)
x^{2} + 3 \, y + 1
```

```
>>> from sage.all import *
>>> f
x^2 + 3*y + 1
>>> latex(f)
x^{2} + 3 \, y + 1
```

A similar output is obtained by means of the method *expr()*:

```
sage: f.expr()
x^2 + 3*y + 1
```

```
>>> from sage.all import *
>>> f.expr()
x^2 + 3*y + 1
```

The expression returned by `expr()` is by default a Sage symbolic expression:

```
sage: type(f.expr())
<class 'sage.symbolic.expression.Expression'>
```

```
>>> from sage.all import *
>>> type(f.expr())
<class 'sage.symbolic.expression.Expression'>
```

A SymPy expression can also be asked for:

```
sage: f.expr('sympy')
x**2 + 3*y + 1
sage: type(f.expr('sympy'))
<class 'sympy.core.add.Add'>
```

```
>>> from sage.all import *
>>> f.expr('sympy')
x**2 + 3*y + 1
>>> type(f.expr('sympy'))
<class 'sympy.core.add.Add'>
```

The value of the function at specified coordinates is obtained by means of the standard parentheses notation:

```
sage: f(2,-1)
2
sage: var('a b')
(a, b)
sage: f(a,b)
a^2 + 3*b + 1
```

```
>>> from sage.all import *
>>> f(Integer(2),-Integer(1))
2
>>> var('a b')
(a, b)
>>> f(a,b)
a^2 + 3*b + 1
```

An unspecified chart function:

```
sage: g = X.function(function('G')(x, y))
sage: g
G(x, y)
sage: g.display()
(x, y) ↦ G(x, y)
sage: g.expr()
G(x, y)
sage: g(2,3)
G(2, 3)
```

```
>>> from sage.all import *
>>> g = X.function(function('G')(x, y))
```

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```
>>> g
G(x, y)
>>> g.display()
(x, y) ↦ G(x, y)
>>> g.expr()
G(x, y)
>>> g(Integer(2),Integer(3))
G(2, 3)
```

Coordinate functions can be compared to other values:

```
sage: f = X.function(x^2+3*y+1)
sage: f == 2
False
sage: f == x^2 + 3*y + 1
True
sage: g = X.function(x*y)
sage: f == g
False
sage: h = X.function(x^2+3*y+1)
sage: f == h
True
```

```
>>> from sage.all import *
>>> f = X.function(x**Integer(2)+Integer(3)*y+Integer(1))
>>> f == Integer(2)
False
>>> f == x**Integer(2) + Integer(3)*y + Integer(1)
True
>>> g = X.function(x*y)
>>> f == g
False
>>> h = X.function(x**Integer(2)+Integer(3)*y+Integer(1))
>>> f == h
True
```

A coercion by means of the restriction is implemented:

```
sage: D = M.open_subset('D')
sage: X_D = X.restrict(D, x^2+y^2<1)  # open disk
sage: c = X_D.function(x^2)
sage: c + f
2*x^2 + 3*y + 1
```

```
>>> from sage.all import *
>>> D = M.open_subset('D')
>>> X_D = X.restrict(D, x**Integer(2)+y**Integer(2)<Integer(1))  # open disk
>>> c = X_D.function(x**Integer(2))
>>> c + f
2*x^2 + 3*y + 1
```

Expansion to a given order with respect to a small parameter:

```
sage: t = var('t')  # the small parameter
sage: f = X.function(cos(t)*x*y, expansion_symbol=t, order=2)
```

```
>>> from sage.all import *
>>> t = var('t')  # the small parameter
>>> f = X.function(cos(t)*x*y, expansion_symbol=t, order=Integer(2))
```

The expansion is triggered by the call to *simplify()*:

```
sage: f
x*y*cos(t)
sage: f.simplify()
-1/2*t^2*x*y + x*y
```

```
>>> from sage.all import *
>>> f
x*y*cos(t)
>>> f.simplify()
-1/2*t^2*x*y + x*y
```

Differences between `ChartFunction` and callable symbolic expressions

Callable symbolic expressions are defined directly from symbolic expressions of the coordinates:

```
sage: f0(x,y) = x^2 + 3*y + 1
sage: type(f0)
<class 'sage.symbolic.expression.Expression'>
sage: f0
(x, y) |--> x^2 + 3*y + 1
sage: f0(x,y)
x^2 + 3*y + 1
```

```
>>> from sage.all import *
>>> __tmp__=var("x,y"); f0 = symbolic_expression(x**Integer(2) + Integer(3)*y +␣
↪Integer(1)).function(x,y)
>>> type(f0)
<class 'sage.symbolic.expression.Expression'>
>>> f0
(x, y) |--> x^2 + 3*y + 1
>>> f0(x,y)
x^2 + 3*y + 1
```

To get an output similar to that of `f0` for a chart function, we must use the method *display()*:

```
sage: f = X.function(x^2+3*y+1)
sage: f
x^2 + 3*y + 1
sage: f.display()
(x, y) ↦ x^2 + 3*y + 1
sage: f(x,y)
x^2 + 3*y + 1
```

```
>>> from sage.all import *
>>> f = X.function(x**Integer(2)+Integer(3)*y+Integer(1))
>>> f
x^2 + 3*y + 1
>>> f.display()
(x, y) ↦ x^2 + 3*y + 1
>>> f(x,y)
x^2 + 3*y + 1
```

More importantly, instances of *ChartFunction* differ from callable symbolic expression by the automatic simplifications in all operations. For instance, adding the two callable symbolic expressions:

```
sage: f0(x,y,z) = cos(x)^2 ; g0(x,y,z) = sin(x)^2
```

```
>>> from sage.all import *
>>> __tmp__=var("x,y,z"); f0 = symbolic_expression(cos(x)**Integer(2) ).
↪function(x,y,z); __tmp__=var("x,y,z"); g0 = symbolic_
↪expression(sin(x)**Integer(2)).function(x,y,z)
```

results in:

```
sage: f0 + g0
(x, y, z) |--> cos(x)^2 + sin(x)^2
```

```
>>> from sage.all import *
>>> f0 + g0
(x, y, z) |--> cos(x)^2 + sin(x)^2
```

To get 1, one has to call `simplify_trig()`:

```
sage: (f0 + g0).simplify_trig()
(x, y, z) |--> 1
```

```
>>> from sage.all import *
>>> (f0 + g0).simplify_trig()
(x, y, z) |--> 1
```

On the contrary, the sum of the corresponding *ChartFunction* instances is automatically simplified (see *simplify_chain_real()* and *simplify_chain_generic()* for details):

```
sage: f = X.function(cos(x)^2) ; g = X.function(sin(x)^2)
sage: f + g
1
```

```
>>> from sage.all import *
>>> f = X.function(cos(x)**Integer(2)) ; g = X.function(sin(x)**Integer(2))
>>> f + g
1
```

Another difference regards the display of partial derivatives: for callable symbolic functions, it involves `diff`:

```
sage: g = function('g')(x, y)
sage: f0(x,y) = diff(g, x) + diff(g, y)
```

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```
sage: f0
(x, y) |--> diff(g(x, y), x) + diff(g(x, y), y)
```

```
>>> from sage.all import *
>>> g = function('g')(x, y)
>>> __tmp__=var("x,y"); f0 = symbolic_expression(diff(g, x) + diff(g, y)).
↪function(x,y)
>>> f0
(x, y) |--> diff(g(x, y), x) + diff(g(x, y), y)
```

while for chart functions, the display is more "textbook" like:

```
sage: f = X.function(diff(g, x) + diff(g, y))
sage: f
d(g)/dx + d(g)/dy
```

```
>>> from sage.all import *
>>> f = X.function(diff(g, x) + diff(g, y))
>>> f
d(g)/dx + d(g)/dy
```

The difference is even more dramatic on LaTeX outputs:

```
sage: latex(f0)
\left( x, y \right) \ {\mapsto} \ \frac{\partial}{\partial x}g\left(x, y\right) +␣
↪\frac{\partial}{\partial y}g\left(x, y\right)
sage: latex(f)
\frac{\partial\,g}{\partial x} + \frac{\partial\,g}{\partial y}
```

```
>>> from sage.all import *
>>> latex(f0)
\left( x, y \right) \ {\mapsto} \ \frac{\partial}{\partial x}g\left(x, y\right) +␣
↪\frac{\partial}{\partial y}g\left(x, y\right)
>>> latex(f)
\frac{\partial\,g}{\partial x} + \frac{\partial\,g}{\partial y}
```

Note that this regards only the display of coordinate functions: internally, the `diff` notation is still used, as we can check by asking for the symbolic expression stored in `f`:

```
sage: f.expr()
diff(g(x, y), x) + diff(g(x, y), y)
```

```
>>> from sage.all import *
>>> f.expr()
diff(g(x, y), x) + diff(g(x, y), y)
```

One can switch to Pynac notation by changing the options:

```
sage: Manifold.options.textbook_output=False
sage: latex(f)
\frac{\partial}{\partial x}g\left(x, y\right) + \frac{\partial}{\partial y}g\
↪left(x, y\right)
```

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```
sage: Manifold.options._reset()
sage: latex(f)
\frac{\partial\,g}{\partial x} + \frac{\partial\,g}{\partial y}
```

```
>>> from sage.all import *
>>> Manifold.options.textbook_output=False
>>> latex(f)
\frac{\partial}{\partial x}g\left(x, y\right) + \frac{\partial}{\partial y}g\
↪left(x, y\right)
>>> Manifold.options._reset()
>>> latex(f)
\frac{\partial\,g}{\partial x} + \frac{\partial\,g}{\partial y}
```

Another difference between `ChartFunction` and callable symbolic expression is the possibility to switch off the display of the arguments of unspecified functions. Consider for instance:

```
sage: f = X.function(function('u')(x, y) * function('v')(x, y))
sage: f
u(x, y)*v(x, y)
sage: f0(x,y) = function('u')(x, y) * function('v')(x, y)
sage: f0
(x, y) |--> u(x, y)*v(x, y)
```

```
>>> from sage.all import *
>>> f = X.function(function('u')(x, y) * function('v')(x, y))
>>> f
u(x, y)*v(x, y)
>>> __tmp__=var("x,y"); f0 = symbolic_expression(function('u')(x, y) * function('v
↪')(x, y)).function(x,y)
>>> f0
(x, y) |--> u(x, y)*v(x, y)
```

If there is a clear understanding that u and v are functions of (x, y), the explicit mention of the latter can be cumbersome in lengthy tensor expressions. We can switch it off by:

```
sage: Manifold.options.omit_function_arguments=True
sage: f
u*v
```

```
>>> from sage.all import *
>>> Manifold.options.omit_function_arguments=True
>>> f
u*v
```

Note that neither the callable symbolic expression `f0` nor the internal expression of `f` is affected by the above command:

```
sage: f0
(x, y) |--> u(x, y)*v(x, y)
sage: f.expr()
u(x, y)*v(x, y)
```

```
>>> from sage.all import *
>>> f0
(x, y) |--> u(x, y)*v(x, y)
>>> f.expr()
u(x, y)*v(x, y)
```

We revert to the default behavior by:

```
sage: Manifold.options._reset()
sage: f
u(x, y)*v(x, y)
```

```
>>> from sage.all import *
>>> Manifold.options._reset()
>>> f
u(x, y)*v(x, y)
```

__call__(**coords*, ***options*)

Compute the value of the function at specified coordinates.

INPUT:

- `*coords` – list of coordinates $(x^1, \ldots, x^n)$, where the function f is to be evaluated
- `**options` – allows to pass `simplify=False` to disable the call of the simplification chain on the result

OUTPUT:

- the value $f(x^1, \ldots, x^n)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(sin(x*y))
sage: f.__call__(-2, 3)
-sin(6)
sage: f(-2, 3)
-sin(6)
sage: var('a b')
(a, b)
sage: f.__call__(a, b)
sin(a*b)
sage: f(a,b)
sin(a*b)
sage: f.__call__(pi, 1)
0
sage: f.__call__(pi, 1/2)
1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(sin(x*y))
```

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```
>>> f.__call__(-Integer(2), Integer(3))
-sin(6)
>>> f(-Integer(2), Integer(3))
-sin(6)
>>> var('a b')
(a, b)
>>> f.__call__(a, b)
sin(a*b)
>>> f(a,b)
sin(a*b)
>>> f.__call__(pi, Integer(1))
0
>>> f.__call__(pi, Integer(1)/Integer(2))
1
```

With SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f(-2,3)
-sin(6)
sage: type(f(-2,3))
<class 'sympy.core.mul.Mul'>
sage: f(a,b)
sin(a*b)
sage: type(f(a,b))
sin
sage: type(f(pi,1))
<class 'sympy.core.numbers.Zero'>
sage: f(pi, 1/2)
1
sage: type(f(pi, 1/2))
<class 'sympy.core.numbers.One'>
```

```
>>> from sage.all import *
>>> X.calculus_method().set('sympy')
>>> f(-Integer(2),Integer(3))
-sin(6)
>>> type(f(-Integer(2),Integer(3)))
<class 'sympy.core.mul.Mul'>
>>> f(a,b)
sin(a*b)
>>> type(f(a,b))
sin
>>> type(f(pi,Integer(1)))
<class 'sympy.core.numbers.Zero'>
>>> f(pi, Integer(1)/Integer(2))
1
>>> type(f(pi, Integer(1)/Integer(2)))
<class 'sympy.core.numbers.One'>
```

arccos()

Arc cosine of `self`.

OUTPUT:

- chart function $\arccos(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.arccos()
arccos(x*y)
sage: arccos(f)  # equivalent to f.arccos()
arccos(x*y)
sage: acos(f)  # equivalent to f.arccos()
arccos(x*y)
sage: arccos(f).display()
(x, y) ↦ arccos(x*y)
sage: arccos(X.zero_function()).display()
(x, y) ↦ 1/2*pi
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x*y)
>>> f.arccos()
arccos(x*y)
>>> arccos(f)  # equivalent to f.arccos()
arccos(x*y)
>>> acos(f)  # equivalent to f.arccos()
arccos(x*y)
>>> arccos(f).display()
(x, y) ↦ arccos(x*y)
>>> arccos(X.zero_function()).display()
(x, y) ↦ 1/2*pi
```

The same test with SymPy:

```
sage: M.set_calculus_method('sympy')
sage: f = X.function(x*y)
sage: f.arccos()
acos(x*y)
sage: arccos(f)  # equivalent to f.arccos()
acos(x*y)
sage: acos(f)  # equivalent to f.arccos()
acos(x*y)
sage: arccos(f).display()
(x, y) ↦ acos(x*y)
```

```
>>> from sage.all import *
>>> M.set_calculus_method('sympy')
>>> f = X.function(x*y)
>>> f.arccos()
acos(x*y)
>>> arccos(f)  # equivalent to f.arccos()
```

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```
acos(x*y)
>>> acos(f)  # equivalent to f.arccos()
acos(x*y)
>>> arccos(f).display()
(x, y) ↦ acos(x*y)
```

arccosh()

Inverse hyperbolic cosine of `self`.

OUTPUT:

- chart function $\operatorname{arccosh}(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.arccosh()
arccosh(x*y)
sage: arccosh(f)  # equivalent to f.arccosh()
arccosh(x*y)
sage: acosh(f)  # equivalent to f.arccosh()
arccosh(x*y)
sage: arccosh(f).display()
(x, y) ↦ arccosh(x*y)
sage: arccosh(X.function(1)) == X.zero_function()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x*y)
>>> f.arccosh()
arccosh(x*y)
>>> arccosh(f)  # equivalent to f.arccosh()
arccosh(x*y)
>>> acosh(f)  # equivalent to f.arccosh()
arccosh(x*y)
>>> arccosh(f).display()
(x, y) ↦ arccosh(x*y)
>>> arccosh(X.function(Integer(1))) == X.zero_function()
True
```

The same tests with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f.arccosh()
acosh(x*y)
sage: arccosh(f)  # equivalent to f.arccosh()
acosh(x*y)
sage: acosh(f)  # equivalent to f.arccosh()
acosh(x*y)
```

```
>>> from sage.all import *
>>> X.calculus_method().set('sympy')
>>> f.arccosh()
acosh(x*y)
>>> arccosh(f)  # equivalent to f.arccosh()
acosh(x*y)
>>> acosh(f)  # equivalent to f.arccosh()
acosh(x*y)
```

arcsin()

Arc sine of `self`.

OUTPUT:

- chart function $\arcsin(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.arcsin()
arcsin(x*y)
sage: arcsin(f)  # equivalent to f.arcsin()
arcsin(x*y)
sage: asin(f)  # equivalent to f.arcsin()
arcsin(x*y)
sage: arcsin(f).display()
(x, y) ↦ arcsin(x*y)
sage: arcsin(X.zero_function()) == X.zero_function()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x*y)
>>> f.arcsin()
arcsin(x*y)
>>> arcsin(f)  # equivalent to f.arcsin()
arcsin(x*y)
>>> asin(f)  # equivalent to f.arcsin()
arcsin(x*y)
>>> arcsin(f).display()
(x, y) ↦ arcsin(x*y)
>>> arcsin(X.zero_function()) == X.zero_function()
True
```

The same tests with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f.arcsin()
asin(x*y)
sage: arcsin(f)  # equivalent to f.arcsin()
asin(x*y)
```

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```
sage: asin(f)  # equivalent to f.arcsin()
asin(x*y)
```

```
>>> from sage.all import *
>>> X.calculus_method().set('sympy')
>>> f.arcsin()
asin(x*y)
>>> arcsin(f)  # equivalent to f.arcsin()
asin(x*y)
>>> asin(f)  # equivalent to f.arcsin()
asin(x*y)
```

arcsinh()

Inverse hyperbolic sine of `self`.

OUTPUT:

- chart function $\operatorname{arcsinh}(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.arcsinh()
arcsinh(x*y)
sage: arcsinh(f)  # equivalent to f.arcsinh()
arcsinh(x*y)
sage: asinh(f)  # equivalent to f.arcsinh()
arcsinh(x*y)
sage: arcsinh(f).display()
(x, y) ↦ arcsinh(x*y)
sage: arcsinh(X.zero_function()) == X.zero_function()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x*y)
>>> f.arcsinh()
arcsinh(x*y)
>>> arcsinh(f)  # equivalent to f.arcsinh()
arcsinh(x*y)
>>> asinh(f)  # equivalent to f.arcsinh()
arcsinh(x*y)
>>> arcsinh(f).display()
(x, y) ↦ arcsinh(x*y)
>>> arcsinh(X.zero_function()) == X.zero_function()
True
```

The same tests with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f.arcsinh()
asinh(x*y)
sage: arcsinh(f)  # equivalent to f.arcsinh()
asinh(x*y)
sage: asinh(f)  # equivalent to f.arcsinh()
asinh(x*y)
```

```
>>> from sage.all import *
>>> X.calculus_method().set('sympy')
>>> f.arcsinh()
asinh(x*y)
>>> arcsinh(f)  # equivalent to f.arcsinh()
asinh(x*y)
>>> asinh(f)  # equivalent to f.arcsinh()
asinh(x*y)
```

arctan()

Arc tangent of `self`.

OUTPUT:

- chart function $\arctan(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.arctan()
arctan(x*y)
sage: arctan(f)  # equivalent to f.arctan()
arctan(x*y)
sage: atan(f)  # equivalent to f.arctan()
arctan(x*y)
sage: arctan(f).display()
(x, y) ↦ arctan(x*y)
sage: arctan(X.zero_function()) == X.zero_function()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x*y)
>>> f.arctan()
arctan(x*y)
>>> arctan(f)  # equivalent to f.arctan()
arctan(x*y)
>>> atan(f)  # equivalent to f.arctan()
arctan(x*y)
>>> arctan(f).display()
(x, y) ↦ arctan(x*y)
>>> arctan(X.zero_function()) == X.zero_function()
True
```

The same tests with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f.arctan()
atan(x*y)
sage: arctan(f)  # equivalent to f.arctan()
atan(x*y)
sage: atan(f)  # equivalent to f.arctan()
atan(x*y)
```

```
>>> from sage.all import *
>>> X.calculus_method().set('sympy')
>>> f.arctan()
atan(x*y)
>>> arctan(f)  # equivalent to f.arctan()
atan(x*y)
>>> atan(f)  # equivalent to f.arctan()
atan(x*y)
```

arctanh()

Inverse hyperbolic tangent of `self`.

OUTPUT:

- chart function $\operatorname{arctanh}(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.arctanh()
arctanh(x*y)
sage: arctanh(f)  # equivalent to f.arctanh()
arctanh(x*y)
sage: atanh(f)  # equivalent to f.arctanh()
arctanh(x*y)
sage: arctanh(f).display()
(x, y) ↦ arctanh(x*y)
sage: arctanh(X.zero_function()) == X.zero_function()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x*y)
>>> f.arctanh()
arctanh(x*y)
>>> arctanh(f)  # equivalent to f.arctanh()
arctanh(x*y)
>>> atanh(f)  # equivalent to f.arctanh()
arctanh(x*y)
>>> arctanh(f).display()
(x, y) ↦ arctanh(x*y)
```

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```
>>> arctanh(X.zero_function()) == X.zero_function()
True
```

The same tests with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f.arctanh()
atanh(x*y)
sage: arctanh(f)  # equivalent to f.arctanh()
atanh(x*y)
sage: atanh(f)  # equivalent to f.arctanh()
atanh(x*y)
```

```
>>> from sage.all import *
>>> X.calculus_method().set('sympy')
>>> f.arctanh()
atanh(x*y)
>>> arctanh(f)  # equivalent to f.arctanh()
atanh(x*y)
>>> atanh(f)  # equivalent to f.arctanh()
atanh(x*y)
```

chart()

Return the chart with respect to which `self` is defined.

OUTPUT: a *Chart*

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(1+x+y^2)
sage: f.chart()
Chart (M, (x, y))
sage: f.chart() is X
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(Integer(1)+x+y**Integer(2))
>>> f.chart()
Chart (M, (x, y))
>>> f.chart() is X
True
```

collect(*s*)

Collect the coefficients of s in the expression of `self` into a group.

INPUT:

- `s` – the symbol whose coefficients will be collected

OUTPUT:

- `self` with the coefficients of `s` grouped in its expression

EXAMPLES:

Action on a 2-dimensional chart function:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x^2*y + x*y + (x*y)^2)
sage: f.display()
(x, y) ↦ x^2*y^2 + x^2*y + x*y
sage: f.collect(y)
x^2*y^2 + (x^2 + x)*y
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x**Integer(2)*y + x*y + (x*y)**Integer(2))
>>> f.display()
(x, y) ↦ x^2*y^2 + x^2*y + x*y
>>> f.collect(y)
x^2*y^2 + (x^2 + x)*y
```

The method `collect()` has changed the expression of `f`:

```
sage: f.display()
(x, y) ↦ x^2*y^2 + (x^2 + x)*y
```

```
>>> from sage.all import *
>>> f.display()
(x, y) ↦ x^2*y^2 + (x^2 + x)*y
```

The same test with SymPy

```
sage: X.calculus_method().set('sympy')
sage: f = X.function(x^2*y + x*y + (x*y)^2)
sage: f.display()
(x, y) ↦ x**2*y**2 + x**2*y + x*y
sage: f.collect(y)
x**2*y**2 + y*(x**2 + x)
```

```
>>> from sage.all import *
>>> X.calculus_method().set('sympy')
>>> f = X.function(x**Integer(2)*y + x*y + (x*y)**Integer(2))
>>> f.display()
(x, y) ↦ x**2*y**2 + x**2*y + x*y
>>> f.collect(y)
x**2*y**2 + y*(x**2 + x)
```

collect_common_factors()

Collect common factors in the expression of `self`.

This method does not perform a full factorization but only looks for factors which are already explicitly present.

OUTPUT:

- `self` with the common factors collected in its expression

EXAMPLES:

Action on a 2-dimensional chart function:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x/(x^2*y + x*y))
sage: f.display()
(x, y) ↦ x/(x^2*y + x*y)
sage: f.collect_common_factors()
1/((x + 1)*y)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x/(x**Integer(2)*y + x*y))
>>> f.display()
(x, y) ↦ x/(x^2*y + x*y)
>>> f.collect_common_factors()
1/((x + 1)*y)
```

The method `collect_common_factors()` has changed the expression of `f`:

```
sage: f.display()
(x, y) ↦ 1/((x + 1)*y)
```

```
>>> from sage.all import *
>>> f.display()
(x, y) ↦ 1/((x + 1)*y)
```

The same test with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: g = X.function(x/(x^2*y + x*y))
sage: g.display()
(x, y) ↦ x/(x**2*y + x*y)
sage: g.collect_common_factors()
1/(y*(x + 1))
```

```
>>> from sage.all import *
>>> X.calculus_method().set('sympy')
>>> g = X.function(x/(x**Integer(2)*y + x*y))
>>> g.display()
(x, y) ↦ x/(x**2*y + x*y)
>>> g.collect_common_factors()
1/(y*(x + 1))
```

copy()

Return an exact copy of the object.

OUTPUT: a `ChartFunctionSymb`

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x+y^2)
sage: g = f.copy(); g
y^2 + x
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x+y**Integer(2))
>>> g = f.copy(); g
y^2 + x
```

By construction, `g` is identical to `f`:

```
sage: type(g) == type(f)
True
sage: g == f
True
```

```
>>> from sage.all import *
>>> type(g) == type(f)
True
>>> g == f
True
```

but it is not the same object:

```
sage: g is f
False
```

```
>>> from sage.all import *
>>> g is f
False
```

cos()

Cosine of `self`.

OUTPUT:

- chart function $\cos(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.cos()
cos(x*y)
sage: cos(f)  # equivalent to f.cos()
cos(x*y)
sage: cos(f).display()
(x, y) ↦ cos(x*y)
```

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```
sage: cos(X.zero_function()).display()
(x, y) ↦ 1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x*y)
>>> f.cos()
cos(x*y)
>>> cos(f)  # equivalent to f.cos()
cos(x*y)
>>> cos(f).display()
(x, y) ↦ cos(x*y)
>>> cos(X.zero_function()).display()
(x, y) ↦ 1
```

The same tests with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f.cos()
cos(x*y)
sage: cos(f)  # equivalent to f.cos()
cos(x*y)
```

```
>>> from sage.all import *
>>> X.calculus_method().set('sympy')
>>> f.cos()
cos(x*y)
>>> cos(f)  # equivalent to f.cos()
cos(x*y)
```

cosh()

Hyperbolic cosine of `self`.

OUTPUT:

- chart function $\cosh(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.cosh()
cosh(x*y)
sage: cosh(f)  # equivalent to f.cosh()
cosh(x*y)
sage: cosh(f).display()
(x, y) ↦ cosh(x*y)
sage: cosh(X.zero_function()).display()
(x, y) ↦ 1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x*y)
>>> f.cosh()
cosh(x*y)
>>> cosh(f)  # equivalent to f.cosh()
cosh(x*y)
>>> cosh(f).display()
(x, y) ↦ cosh(x*y)
>>> cosh(X.zero_function()).display()
(x, y) ↦ 1
```

The same tests with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f.cosh()
cosh(x*y)
sage: cosh(f)  # equivalent to f.cosh()
cosh(x*y)
```

```
>>> from sage.all import *
>>> X.calculus_method().set('sympy')
>>> f.cosh()
cosh(x*y)
>>> cosh(f)  # equivalent to f.cosh()
cosh(x*y)
```

derivative(*coord*)

Partial derivative with respect to a coordinate.

INPUT:

- `coord` – either the coordinate x^i with respect to which the derivative of the chart function f is to be taken, or the index i labelling this coordinate (with the index convention defined on the chart domain via the parameter `start_index`)

OUTPUT:

- a *ChartFunction* representing the partial derivative $\frac{\partial f}{\partial x^i}$

EXAMPLES:

Partial derivatives of a 2-dimensional chart function:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart(calc_method='SR')
sage: f = X.function(x^2+3*y+1); f
x^2 + 3*y + 1
sage: f.derivative(x)
2*x
sage: f.derivative(y)
3
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(calc_method='SR', names=('x', 'y',)); (x, y,) = X._first_
↪ngens(2)
>>> f = X.function(x**Integer(2)+Integer(3)*y+Integer(1)); f
x^2 + 3*y + 1
>>> f.derivative(x)
2*x
>>> f.derivative(y)
3
```

An alias is `diff`:

```
sage: f.diff(x)
2*x
```

```
>>> from sage.all import *
>>> f.diff(x)
2*x
```

Each partial derivative is itself a chart function:

```
sage: type(f.diff(x))
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>
```

```
>>> from sage.all import *
>>> type(f.diff(x))
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>
```

The same result is returned by the function `diff`:

```
sage: diff(f, x)
2*x
```

```
>>> from sage.all import *
>>> diff(f, x)
2*x
```

An index can be used instead of the coordinate symbol:

```
sage: f.diff(0)
2*x
sage: diff(f, 1)
3
```

```
>>> from sage.all import *
>>> f.diff(Integer(0))
2*x
>>> diff(f, Integer(1))
3
```

The index range depends on the convention used on the chart's domain:

```
sage: M = Manifold(2, 'M', structure='topological', start_index=1)
sage: X.<x,y> = M.chart()
sage: f = X.function(x^2+3*y+1)
sage: f.diff(0)
Traceback (most recent call last):
...
ValueError: coordinate index out of range
sage: f.diff(1)
2*x
sage: f.diff(2)
3
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological', start_
↪index=Integer(1))
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x**Integer(2)+Integer(3)*y+Integer(1))
>>> f.diff(Integer(0))
Traceback (most recent call last):
...
ValueError: coordinate index out of range
>>> f.diff(Integer(1))
2*x
>>> f.diff(Integer(2))
3
```

The same test with SymPy:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart(calc_method='sympy')
sage: f = X.function(x^2+3*y+1); f
x**2 + 3*y + 1
sage: f.diff(x)
2*x
sage: f.diff(y)
3
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(calc_method='sympy', names=('x', 'y',)); (x, y,) = X._first_
↪ngens(2)
>>> f = X.function(x**Integer(2)+Integer(3)*y+Integer(1)); f
x**2 + 3*y + 1
>>> f.diff(x)
2*x
>>> f.diff(y)
3
```

diff(*coord*)

Partial derivative with respect to a coordinate.

INPUT:

- `coord` – either the coordinate x^i with respect to which the derivative of the chart function f is to be taken, or the index i labelling this coordinate (with the index convention defined on the chart domain via the parameter `start_index`)

OUTPUT:

- a *ChartFunction* representing the partial derivative $\frac{\partial f}{\partial x^i}$

EXAMPLES:

Partial derivatives of a 2-dimensional chart function:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart(calc_method='SR')
sage: f = X.function(x^2+3*y+1); f
x^2 + 3*y + 1
sage: f.derivative(x)
2*x
sage: f.derivative(y)
3
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(calc_method='SR', names=('x', 'y',)); (x, y,) = X._first_
↪ngens(2)
>>> f = X.function(x**Integer(2)+Integer(3)*y+Integer(1)); f
x^2 + 3*y + 1
>>> f.derivative(x)
2*x
>>> f.derivative(y)
3
```

An alias is `diff`:

```
sage: f.diff(x)
2*x
```

```
>>> from sage.all import *
>>> f.diff(x)
2*x
```

Each partial derivative is itself a chart function:

```
sage: type(f.diff(x))
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>
```

```
>>> from sage.all import *
>>> type(f.diff(x))
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>
```

The same result is returned by the function `diff`:

```
sage: diff(f, x)
2*x
```

```
>>> from sage.all import *
>>> diff(f, x)
2*x
```

An index can be used instead of the coordinate symbol:

```
sage: f.diff(0)
2*x
sage: diff(f, 1)
3
```

```
>>> from sage.all import *
>>> f.diff(Integer(0))
2*x
>>> diff(f, Integer(1))
3
```

The index range depends on the convention used on the chart's domain:

```
sage: M = Manifold(2, 'M', structure='topological', start_index=1)
sage: X.<x,y> = M.chart()
sage: f = X.function(x^2+3*y+1)
sage: f.diff(0)
Traceback (most recent call last):
...
ValueError: coordinate index out of range
sage: f.diff(1)
2*x
sage: f.diff(2)
3
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological', start_
↪index=Integer(1))
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x**Integer(2)+Integer(3)*y+Integer(1))
>>> f.diff(Integer(0))
Traceback (most recent call last):
...
ValueError: coordinate index out of range
>>> f.diff(Integer(1))
2*x
>>> f.diff(Integer(2))
3
```

The same test with SymPy:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart(calc_method='sympy')
sage: f = X.function(x^2+3*y+1); f
x**2 + 3*y + 1
sage: f.diff(x)
2*x
```

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```
sage: f.diff(y)
3
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(calc_method='sympy', names=('x', 'y',)); (x, y,) = X._first_
↪ngens(2)
>>> f = X.function(x**Integer(2)+Integer(3)*y+Integer(1)); f
x**2 + 3*y + 1
>>> f.diff(x)
2*x
>>> f.diff(y)
3
```

disp()

Display `self` in arrow notation. For display the standard `SR` representation is used.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

EXAMPLES:

Coordinate function on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(cos(x*y/2))
sage: f.display()
(x, y) ↦ cos(1/2*x*y)
sage: latex(f.display())
\left(x, y\right) \mapsto \cos\left(\frac{1}{2} \, x y\right)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(cos(x*y/Integer(2)))
>>> f.display()
(x, y) ↦ cos(1/2*x*y)
>>> latex(f.display())
\left(x, y\right) \mapsto \cos\left(\frac{1}{2} \, x y\right)
```

A shortcut is `disp()`:

```
sage: f.disp()
(x, y) ↦ cos(1/2*x*y)
```

```
>>> from sage.all import *
>>> f.disp()
(x, y) ↦ cos(1/2*x*y)
```

Display of the zero function:

```
sage: X.zero_function().display()
(x, y) ↦ 0
```

```
>>> from sage.all import *
>>> X.zero_function().display()
(x, y) ↦ 0
```

display()

Display `self` in arrow notation. For display the standard `SR` representation is used.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

EXAMPLES:

Coordinate function on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(cos(x*y/2))
sage: f.display()
(x, y) ↦ cos(1/2*x*y)
sage: latex(f.display())
\left(x, y\right) \mapsto \cos\left(\frac{1}{2} \, x y\right)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(cos(x*y/Integer(2)))
>>> f.display()
(x, y) ↦ cos(1/2*x*y)
>>> latex(f.display())
\left(x, y\right) \mapsto \cos\left(\frac{1}{2} \, x y\right)
```

A shortcut is `disp()`:

```
sage: f.disp()
(x, y) ↦ cos(1/2*x*y)
```

```
>>> from sage.all import *
>>> f.disp()
(x, y) ↦ cos(1/2*x*y)
```

Display of the zero function:

```
sage: X.zero_function().display()
(x, y) ↦ 0
```

```
>>> from sage.all import *
>>> X.zero_function().display()
(x, y) ↦ 0
```

exp()

Exponential of `self`.

OUTPUT:

- chart function $\exp(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x+y)
sage: f.exp()
e^(x + y)
sage: exp(f) # equivalent to f.exp()
e^(x + y)
sage: exp(f).display()
(x, y) ↦ e^(x + y)
sage: exp(X.zero_function())
1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x+y)
>>> f.exp()
e^(x + y)
>>> exp(f) # equivalent to f.exp()
e^(x + y)
>>> exp(f).display()
(x, y) ↦ e^(x + y)
>>> exp(X.zero_function())
1
```

The same test with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f = X.function(x+y)
sage: f.exp()
exp(x + y)
sage: exp(f) # equivalent to f.exp()
exp(x + y)
sage: exp(f).display()
(x, y) ↦ exp(x + y)
sage: exp(X.zero_function())
1
```

```
>>> from sage.all import *
>>> X.calculus_method().set('sympy')
>>> f = X.function(x+y)
>>> f.exp()
exp(x + y)
>>> exp(f) # equivalent to f.exp()
exp(x + y)
>>> exp(f).display()
(x, y) ↦ exp(x + y)
>>> exp(X.zero_function())
1
```

expand()

Expand the coordinate expression of `self`.

OUTPUT: `self` with its expression expanded

EXAMPLES:

Expanding a 2-dimensional chart function:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function((x - y)^2)
sage: f.display()
(x, y) ↦ (x - y)^2
sage: f.expand()
x^2 - 2*x*y + y^2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function((x - y)**Integer(2))
>>> f.display()
(x, y) ↦ (x - y)^2
>>> f.expand()
x^2 - 2*x*y + y^2
```

The method `expand()` has changed the expression of `f`:

```
sage: f.display()
(x, y) ↦ x^2 - 2*x*y + y^2
```

```
>>> from sage.all import *
>>> f.display()
(x, y) ↦ x^2 - 2*x*y + y^2
```

The same test with SymPy

```
sage: X.calculus_method().set('sympy')
sage: g = X.function((x - y)^2)
sage: g.expand()
x**2 - 2*x*y + y**2
```

```
>>> from sage.all import *
>>> X.calculus_method().set('sympy')
>>> g = X.function((x - y)**Integer(2))
>>> g.expand()
x**2 - 2*x*y + y**2
```

expr(*method=None*)

Return the symbolic expression of `self` in terms of the chart coordinates, as an object of a specified calculus method.

INPUT:

- `method` – string (default: `None`); the calculus method which the returned expression belongs to. One of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy

– `None`: the chart current calculus method is assumed

OUTPUT:

- a `Sage symbolic expression` if `method` is `'SR'`
- a SymPy object if `method` is `'sympy'`

EXAMPLES:

Chart function on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x^2+y)
sage: f.expr()
x^2 + y
sage: type(f.expr())
<class 'sage.symbolic.expression.Expression'>
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x**Integer(2)+y)
>>> f.expr()
x^2 + y
>>> type(f.expr())
<class 'sage.symbolic.expression.Expression'>
```

Asking for the SymPy expression:

```
sage: f.expr('sympy')
x**2 + y
sage: type(f.expr('sympy'))
<class 'sympy.core.add.Add'>
```

```
>>> from sage.all import *
>>> f.expr('sympy')
x**2 + y
>>> type(f.expr('sympy'))
<class 'sympy.core.add.Add'>
```

The default corresponds to the current calculus method, here the one based on the Symbolic Ring `SR`:

```
sage: f.expr() is f.expr('SR')
True
```

```
>>> from sage.all import *
>>> f.expr() is f.expr('SR')
True
```

If we change the current calculus method on chart `X`, we change the default:

```
sage: X.calculus_method().set('sympy')
sage: f.expr()
x**2 + y
```

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```
sage: f.expr() is f.expr('sympy')
True
sage: X.calculus_method().set('SR')  # revert back to SR
```

```
>>> from sage.all import *
>>> X.calculus_method().set('sympy')
>>> f.expr()
x**2 + y
>>> f.expr() is f.expr('sympy')
True
>>> X.calculus_method().set('SR')  # revert back to SR
```

Internally, the expressions corresponding to various calculus methods are stored in the dictionary `_express`:

```
sage: for method in sorted(f._express):
....:     print("'{}': {}".format(method, f._express[method]))
....:
'SR': x^2 + y
'sympy': x**2 + y
```

```
>>> from sage.all import *
>>> for method in sorted(f._express):
...     print("'{}': {}".format(method, f._express[method]))
....:
'SR': x^2 + y
'sympy': x**2 + y
```

The method `expr()` is useful for accessing to all the symbolic expression functionalities in Sage; for instance:

```
sage: var('a')
a
sage: f = X.function(a*x*y); f.display()
(x, y) ↦ a*x*y
sage: f.expr()
a*x*y
sage: f.expr().subs(a=2)
2*x*y
```

```
>>> from sage.all import *
>>> var('a')
a
>>> f = X.function(a*x*y); f.display()
(x, y) ↦ a*x*y
>>> f.expr()
a*x*y
>>> f.expr().subs(a=Integer(2))
2*x*y
```

Note that for substituting the value of a coordinate, the function call can be used as well:

```
sage: f(x,3)
3*a*x
```

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```
sage: bool( f(x,3) == f.expr().subs(y=3) )
True
```

```
>>> from sage.all import *
>>> f(x,Integer(3))
3*a*x
>>> bool( f(x,Integer(3)) == f.expr().subs(y=Integer(3)) )
True
```

factor()

Factorize the coordinate expression of `self`.

OUTPUT: `self` with its expression factorized

EXAMPLES:

Factorization of a 2-dimensional chart function:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x^2 + 2*x*y + y^2)
sage: f.display()
(x, y) ↦ x^2 + 2*x*y + y^2
sage: f.factor()
(x + y)^2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x**Integer(2) + Integer(2)*x*y + y**Integer(2))
>>> f.display()
(x, y) ↦ x^2 + 2*x*y + y^2
>>> f.factor()
(x + y)^2
```

The method `factor()` has changed the expression of `f`:

```
sage: f.display()
(x, y) ↦ (x + y)^2
```

```
>>> from sage.all import *
>>> f.display()
(x, y) ↦ (x + y)^2
```

The same test with SymPy

```
sage: X.calculus_method().set('sympy')
sage: g = X.function(x^2 + 2*x*y + y^2)
sage: g.display()
(x, y) ↦ x**2 + 2*x*y + y**2
sage: g.factor()
(x + y)**2
```

```
>>> from sage.all import *
>>> X.calculus_method().set('sympy')
>>> g = X.function(x**Integer(2) + Integer(2)*x*y + y**Integer(2))
>>> g.display()
(x, y) ↦ x**2 + 2*x*y + y**2
>>> g.factor()
(x + y)**2
```

is_trivial_one()

Check if `self` is trivially equal to one without any simplification.

This method is supposed to be fast as compared with `self == 1` and is intended to be used in library code where trying to obtain a mathematically correct result by applying potentially expensive rewrite rules is not desirable.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(1)
sage: f.is_trivial_one()
True
sage: f = X.function(float(1.0))
sage: f.is_trivial_one()
True
sage: f = X.function(x-x+1)
sage: f.is_trivial_one()
True
sage: X.one_function().is_trivial_one()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(Integer(1))
>>> f.is_trivial_one()
True
>>> f = X.function(float(RealNumber('1.0')))
>>> f.is_trivial_one()
True
>>> f = X.function(x-x+Integer(1))
>>> f.is_trivial_one()
True
>>> X.one_function().is_trivial_one()
True
```

No simplification is attempted, so that `False` is returned for non-trivial cases:

```
sage: f = X.function(cos(x)^2 + sin(x)^2)
sage: f.is_trivial_one()
False
```

```
>>> from sage.all import *
>>> f = X.function(cos(x)**Integer(2) + sin(x)**Integer(2))
>>> f.is_trivial_one()
False
```

On the contrary, the method `is_zero()` and the direct comparison to one involve some simplification algorithms and return `True`:

```
sage: (f - 1).is_zero()
True
sage: f == 1
True
```

```
>>> from sage.all import *
>>> (f - Integer(1)).is_zero()
True
>>> f == Integer(1)
True
```

is_trivial_zero()

Check if `self` is trivially equal to zero without any simplification.

This method is supposed to be fast as compared with `self.is_zero()` or `self == 0` and is intended to be used in library code where trying to obtain a mathematically correct result by applying potentially expensive rewrite rules is not desirable.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(0)
sage: f.is_trivial_zero()
True
sage: f = X.function(float(0.0))
sage: f.is_trivial_zero()
True
sage: f = X.function(x-x)
sage: f.is_trivial_zero()
True
sage: X.zero_function().is_trivial_zero()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(Integer(0))
>>> f.is_trivial_zero()
True
>>> f = X.function(float(RealNumber('0.0')))
>>> f.is_trivial_zero()
True
>>> f = X.function(x-x)
>>> f.is_trivial_zero()
```

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```
True
>>> X.zero_function().is_trivial_zero()
True
```

No simplification is attempted, so that `False` is returned for non-trivial cases:

```
sage: f = X.function(cos(x)^2 + sin(x)^2 - 1)
sage: f.is_trivial_zero()
False
```

```
>>> from sage.all import *
>>> f = X.function(cos(x)**Integer(2) + sin(x)**Integer(2) - Integer(1))
>>> f.is_trivial_zero()
False
```

On the contrary, the method `is_zero()` and the direct comparison to zero involve some simplification algorithms and return `True`:

```
sage: f.is_zero()
True
sage: f == 0
True
```

```
>>> from sage.all import *
>>> f.is_zero()
True
>>> f == Integer(0)
True
```

is_unit()

Return `True` iff `self` is not trivially zero since most chart functions are invertible and an actual computation would take too much time.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x^2+3*y+1)
sage: f.is_unit()
True
sage: zero = X.function(0)
sage: zero.is_unit()
False
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x**Integer(2)+Integer(3)*y+Integer(1))
>>> f.is_unit()
True
>>> zero = X.function(Integer(0))
```

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```
>>> zero.is_unit()
False
```

log(*base=None*)

Logarithm of `self`.

INPUT:

- `base` – (default: `None`) base of the logarithm; if `None`, the natural logarithm (i.e. logarithm to base e) is returned

OUTPUT:

- chart function $\log_a(f)$, where f is the current chart function and a is the base

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x+y)
sage: f.log()
log(x + y)
sage: log(f) # equivalent to f.log()
log(x + y)
sage: log(f).display()
(x, y) ↦ log(x + y)
sage: f.log(2)
log(x + y)/log(2)
sage: log(f, 2)
log(x + y)/log(2)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x+y)
>>> f.log()
log(x + y)
>>> log(f) # equivalent to f.log()
log(x + y)
>>> log(f).display()
(x, y) ↦ log(x + y)
>>> f.log(Integer(2))
log(x + y)/log(2)
>>> log(f, Integer(2))
log(x + y)/log(2)
```

The same test with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f = X.function(x+y)
sage: f.log()
log(x + y)
sage: log(f) # equivalent to f.log()
log(x + y)
```

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```
sage: log(f).display()
(x, y) ↦ log(x + y)
sage: f.log(2)
log(x + y)/log(2)
sage: log(f, 2)
log(x + y)/log(2)
```

```
>>> from sage.all import *
>>> X.calculus_method().set('sympy')
>>> f = X.function(x+y)
>>> f.log()
log(x + y)
>>> log(f) # equivalent to f.log()
log(x + y)
>>> log(f).display()
(x, y) ↦ log(x + y)
>>> f.log(Integer(2))
log(x + y)/log(2)
>>> log(f, Integer(2))
log(x + y)/log(2)
```

scalar_field(*name=None*, *latex_name=None*)

Construct the scalar field that has `self` as coordinate expression.

The domain of the scalar field is the open subset covered by the chart on which `self` is defined.

INPUT:

- `name` – (default: `None`) name given to the scalar field
- `latex_name` – (default: `None`) LaTeX symbol to denote the scalar field; if `None`, the LaTeX symbol is set to `name`

OUTPUT: a *ScalarField*

EXAMPLES:

Construction of a scalar field on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: fc = c_xy.function(x+2*y^3)
sage: f = fc.scalar_field() ; f
Scalar field on the 2-dimensional topological manifold M
sage: f.display()
M → ℝ
(x, y) ↦ 2*y^3 + x
sage: f.coord_function(c_xy) is fc
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> fc = c_xy.function(x+Integer(2)*y**Integer(3))
>>> f = fc.scalar_field() ; f
```

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```
Scalar field on the 2-dimensional topological manifold M
>>> f.display()
M → ℝ
(x, y) ↦ 2*y^3 + x
>>> f.coord_function(c_xy) is fc
True
```

set_expr(*calc_method*, *expression*)

Add an expression in a particular calculus method `self`. Some control is done to verify the consistency between the different representations of the same expression.

INPUT:

- `calc_method` – calculus method
- `expression` – symbolic expression

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(1+x^2)
sage: f._repr_()
'x^2 + 1'
sage: f.set_expr('sympy','x**2+1')
sage: f  # indirect doctest
x^2 + 1

sage: g = X.function(1+x^3)
sage: g._repr_()
'x^3 + 1'
sage: g.set_expr('sympy','x**2+y')
Traceback (most recent call last):
...
ValueError: Expressions are not equal
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(Integer(1)+x**Integer(2))
>>> f._repr_()
'x^2 + 1'
>>> f.set_expr('sympy','x**2+1')
>>> f  # indirect doctest
x^2 + 1

>>> g = X.function(Integer(1)+x**Integer(3))
>>> g._repr_()
'x^3 + 1'
>>> g.set_expr('sympy','x**2+y')
Traceback (most recent call last):
...
ValueError: Expressions are not equal
```

simplify()

Simplify the coordinate expression of `self`.

For details about the employed chain of simplifications for the `SR` calculus method, see *simplify_chain_real()* for chart functions on real manifolds and *simplify_chain_generic()* for the generic case.

If `self` has been defined with the small parameter `expansion_symbol` and some truncation order, the coordinate expression of `self` will be expanded in power series of that parameter and truncated to the given order.

OUTPUT: `self` with its coordinate expression simplified

EXAMPLES:

Simplification of a chart function on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(cos(x)^2 + sin(x)^2 + sqrt(x^2))
sage: f.display()
(x, y) ↦ cos(x)^2 + sin(x)^2 + abs(x)
sage: f.simplify()
abs(x) + 1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(cos(x)**Integer(2) + sin(x)**Integer(2) +␣
↪sqrt(x**Integer(2)))
>>> f.display()
(x, y) ↦ cos(x)^2 + sin(x)^2 + abs(x)
>>> f.simplify()
abs(x) + 1
```

The method `simplify()` has changed the expression of `f`:

```
sage: f.display()
(x, y) ↦ abs(x) + 1
```

```
>>> from sage.all import *
>>> f.display()
(x, y) ↦ abs(x) + 1
```

Another example:

```
sage: f = X.function((x^2-1)/(x+1)); f
(x^2 - 1)/(x + 1)
sage: f.simplify()
x - 1
```

```
>>> from sage.all import *
>>> f = X.function((x**Integer(2)-Integer(1))/(x+Integer(1))); f
(x^2 - 1)/(x + 1)
>>> f.simplify()
x - 1
```

Examples taking into account the declared range of a coordinate:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart('x:(1,+oo) y')
sage: f = X.function(sqrt(x^2-2*x+1)); f
sqrt(x^2 - 2*x + 1)
sage: f.simplify()
x - 1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart('x:(1,+oo) y', names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(sqrt(x**Integer(2)-Integer(2)*x+Integer(1))); f
sqrt(x^2 - 2*x + 1)
>>> f.simplify()
x - 1
```

```
sage: forget()  # to clear the previous assumption on x
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart('x:(-oo,0) y')
sage: f = X.function(sqrt(x^2-2*x+1)); f
sqrt(x^2 - 2*x + 1)
sage: f.simplify()
-x + 1
```

```
>>> from sage.all import *
>>> forget()  # to clear the previous assumption on x
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart('x:(-oo,0) y', names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(sqrt(x**Integer(2)-Integer(2)*x+Integer(1))); f
sqrt(x^2 - 2*x + 1)
>>> f.simplify()
-x + 1
```

The same tests with SymPy:

```
sage: forget()  # to clear the previous assumption on x
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart(calc_method='sympy')
sage: f = X.function(cos(x)^2 + sin(x)^2 + sqrt(x^2)); f
sin(x)**2 + cos(x)**2 + Abs(x)
sage: f.simplify()
Abs(x) + 1
```

```
>>> from sage.all import *
>>> forget()  # to clear the previous assumption on x
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(calc_method='sympy', names=('x', 'y',)); (x, y,) = X._first_
↪ngens(2)
>>> f = X.function(cos(x)**Integer(2) + sin(x)**Integer(2) +␣
↪sqrt(x**Integer(2))); f
sin(x)**2 + cos(x)**2 + Abs(x)
```

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```
>>> f.simplify()
Abs(x) + 1
```

```
sage: f = X.function((x^2-1)/(x+1)); f
(x**2 - 1)/(x + 1)
sage: f.simplify()
x - 1
```

```
>>> from sage.all import *
>>> f = X.function((x**Integer(2)-Integer(1))/(x+Integer(1))); f
(x**2 - 1)/(x + 1)
>>> f.simplify()
x - 1
```

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart('x:(1,+oo) y', calc_method='sympy')
sage: f = X.function(sqrt(x^2-2*x+1)); f
sqrt(x**2 - 2*x + 1)
sage: f.simplify()
x - 1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart('x:(1,+oo) y', calc_method='sympy', names=('x', 'y',)); (x, y,
↪) = X._first_ngens(2)
>>> f = X.function(sqrt(x**Integer(2)-Integer(2)*x+Integer(1))); f
sqrt(x**2 - 2*x + 1)
>>> f.simplify()
x - 1
```

```
sage: forget()  # to clear the previous assumption on x
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart('x:(-oo,0) y', calc_method='sympy')
sage: f = X.function(sqrt(x^2-2*x+1)); f
sqrt(x**2 - 2*x + 1)
sage: f.simplify()
1 - x
```

```
>>> from sage.all import *
>>> forget()  # to clear the previous assumption on x
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart('x:(-oo,0) y', calc_method='sympy', names=('x', 'y',)); (x, y,
↪) = X._first_ngens(2)
>>> f = X.function(sqrt(x**Integer(2)-Integer(2)*x+Integer(1))); f
sqrt(x**2 - 2*x + 1)
>>> f.simplify()
1 - x
```

Power series expansion with respect to a small parameter t (at the moment, this is implemented only for the `SR` calculus backend, hence the first line below):

```
sage: X.calculus_method().set('SR')
sage: t = var('t')
sage: f = X.function(exp(t*x), expansion_symbol=t, order=3)
```

```
>>> from sage.all import *
>>> X.calculus_method().set('SR')
>>> t = var('t')
>>> f = X.function(exp(t*x), expansion_symbol=t, order=Integer(3))
```

At this stage, f is not expanded in power series:

```
sage: f
e^(t*x)
```

```
>>> from sage.all import *
>>> f
e^(t*x)
```

Invoking `simplify()` triggers the expansion to the given order:

```
sage: f.simplify()
1/6*t^3*x^3 + 1/2*t^2*x^2 + t*x + 1
sage: f.display()
(x, y) ↦ 1/6*t^3*x^3 + 1/2*t^2*x^2 + t*x + 1
```

```
>>> from sage.all import *
>>> f.simplify()
1/6*t^3*x^3 + 1/2*t^2*x^2 + t*x + 1
>>> f.display()
(x, y) ↦ 1/6*t^3*x^3 + 1/2*t^2*x^2 + t*x + 1
```

sin()

Sine of `self`.

OUTPUT:

- chart function $\sin(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.sin()
sin(x*y)
sage: sin(f)  # equivalent to f.sin()
sin(x*y)
sage: sin(f).display()
(x, y) ↦ sin(x*y)
sage: sin(X.zero_function()) == X.zero_function()
True
sage: f = X.function(2-cos(x)^2+y)
sage: g = X.function(-sin(x)^2+y)
```

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```
sage: (f+g).simplify()
2*y + 1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x*y)
>>> f.sin()
sin(x*y)
>>> sin(f)  # equivalent to f.sin()
sin(x*y)
>>> sin(f).display()
(x, y) ↦ sin(x*y)
>>> sin(X.zero_function()) == X.zero_function()
True
>>> f = X.function(Integer(2)-cos(x)**Integer(2)+y)
>>> g = X.function(-sin(x)**Integer(2)+y)
>>> (f+g).simplify()
2*y + 1
```

The same tests with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f = X.function(x*y)
sage: f.sin()
sin(x*y)
sage: sin(f)  # equivalent to f.sin()
sin(x*y)
```

```
>>> from sage.all import *
>>> X.calculus_method().set('sympy')
>>> f = X.function(x*y)
>>> f.sin()
sin(x*y)
>>> sin(f)  # equivalent to f.sin()
sin(x*y)
```

sinh()

Hyperbolic sine of `self`.

OUTPUT:

- chart function $\sinh(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.sinh()
sinh(x*y)
sage: sinh(f)  # equivalent to f.sinh()
sinh(x*y)
```

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```
sage: sinh(f).display()
(x, y) ↦ sinh(x*y)
sage: sinh(X.zero_function()) == X.zero_function()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x*y)
>>> f.sinh()
sinh(x*y)
>>> sinh(f)  # equivalent to f.sinh()
sinh(x*y)
>>> sinh(f).display()
(x, y) ↦ sinh(x*y)
>>> sinh(X.zero_function()) == X.zero_function()
True
```

The same tests with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f.sinh()
sinh(x*y)
sage: sinh(f)  # equivalent to f.sinh()
sinh(x*y)
```

```
>>> from sage.all import *
>>> X.calculus_method().set('sympy')
>>> f.sinh()
sinh(x*y)
>>> sinh(f)  # equivalent to f.sinh()
sinh(x*y)
```

sqrt()

Square root of `self`.

OUTPUT:

- chart function $\sqrt{f}$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x+y)
sage: f.sqrt()
sqrt(x + y)
sage: sqrt(f)  # equivalent to f.sqrt()
sqrt(x + y)
sage: sqrt(f).display()
(x, y) ↦ sqrt(x + y)
sage: sqrt(X.zero_function()).display()
(x, y) ↦ 0
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x+y)
>>> f.sqrt()
sqrt(x + y)
>>> sqrt(f)  # equivalent to f.sqrt()
sqrt(x + y)
>>> sqrt(f).display()
(x, y) ↦ sqrt(x + y)
>>> sqrt(X.zero_function()).display()
(x, y) ↦ 0
```

tan()

Tangent of `self`.

OUTPUT:

- chart function $\tan(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.tan()
sin(x*y)/cos(x*y)
sage: tan(f)  # equivalent to f.tan()
sin(x*y)/cos(x*y)
sage: tan(f).display()
(x, y) ↦ sin(x*y)/cos(x*y)
sage: tan(X.zero_function()) == X.zero_function()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x*y)
>>> f.tan()
sin(x*y)/cos(x*y)
>>> tan(f)  # equivalent to f.tan()
sin(x*y)/cos(x*y)
>>> tan(f).display()
(x, y) ↦ sin(x*y)/cos(x*y)
>>> tan(X.zero_function()) == X.zero_function()
True
```

The same test with SymPy:

```
sage: M.set_calculus_method('sympy')
sage: g = X.function(x*y)
sage: g.tan()
tan(x*y)
sage: tan(g)  # equivalent to g.tan()
```

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```
tan(x*y)
sage: tan(g).display()
(x, y) ↦ tan(x*y)
```

```
>>> from sage.all import *
>>> M.set_calculus_method('sympy')
>>> g = X.function(x*y)
>>> g.tan()
tan(x*y)
>>> tan(g)  # equivalent to g.tan()
tan(x*y)
>>> tan(g).display()
(x, y) ↦ tan(x*y)
```

tanh()

Hyperbolic tangent of `self`.

OUTPUT:

- chart function $\tanh(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.tanh()
sinh(x*y)/cosh(x*y)
sage: tanh(f)  # equivalent to f.tanh()
sinh(x*y)/cosh(x*y)
sage: tanh(f).display()
(x, y) ↦ sinh(x*y)/cosh(x*y)
sage: tanh(X.zero_function()) == X.zero_function()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.function(x*y)
>>> f.tanh()
sinh(x*y)/cosh(x*y)
>>> tanh(f)  # equivalent to f.tanh()
sinh(x*y)/cosh(x*y)
>>> tanh(f).display()
(x, y) ↦ sinh(x*y)/cosh(x*y)
>>> tanh(X.zero_function()) == X.zero_function()
True
```

The same tests with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f.tanh()
tanh(x*y)
```

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```
sage: tanh(f)  # equivalent to f.tanh()
tanh(x*y)
```

```
>>> from sage.all import *
>>> X.calculus_method().set('sympy')
>>> f.tanh()
tanh(x*y)
>>> tanh(f)  # equivalent to f.tanh()
tanh(x*y)
```

class sage.manifolds.chart_func.**ChartFunctionRing**(*chart*)

Bases: Parent, UniqueRepresentation

Ring of all chart functions on a chart.

INPUT:

- chart – a coordinate chart, as an instance of class *Chart*

EXAMPLES:

The ring of all chart functions w.r.t. to a chart:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: FR = X.function_ring(); FR
Ring of chart functions on Chart (M, (x, y))
sage: type(FR)
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category'>
sage: FR.category()
Category of commutative algebras over Symbolic Ring
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> FR = X.function_ring(); FR
Ring of chart functions on Chart (M, (x, y))
>>> type(FR)
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category'>
>>> FR.category()
Category of commutative algebras over Symbolic Ring
```

Coercions by means of restrictions are implemented:

```
sage: FR_X = X.function_ring()
sage: D = M.open_subset('D')
sage: X_D = X.restrict(D, x^2+y^2<1)  # open disk
sage: FR_X_D = X_D.function_ring()
sage: FR_X_D.has_coerce_map_from(FR_X)
True
```

```
>>> from sage.all import *
>>> FR_X = X.function_ring()
>>> D = M.open_subset('D')
```

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```
>>> X_D = X.restrict(D, x**Integer(2)+y**Integer(2)<Integer(1))  # open disk
>>> FR_X_D = X_D.function_ring()
>>> FR_X_D.has_coerce_map_from(FR_X)
True
```

But only if the charts are compatible:

```
sage: Y.<t,z> = D.chart()
sage: FR_Y = Y.function_ring()
sage: FR_Y.has_coerce_map_from(FR_X)
False
```

```
>>> from sage.all import *
>>> Y = D.chart(names=('t', 'z',)); (t, z,) = Y._first_ngens(2)
>>> FR_Y = Y.function_ring()
>>> FR_Y.has_coerce_map_from(FR_X)
False
```

Element

alias of *ChartFunction*

is_field(*proof=True*)

Return `False` as `self` is not an integral domain.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: FR = X.function_ring()
sage: FR.is_integral_domain()
False
sage: FR.is_field()
False
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> FR = X.function_ring()
>>> FR.is_integral_domain()
False
>>> FR.is_field()
False
```

is_integral_domain(*proof=True*)

Return `False` as `self` is not an integral domain.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: FR = X.function_ring()
sage: FR.is_integral_domain()
False
```

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```
sage: FR.is_field()
False
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> FR = X.function_ring()
>>> FR.is_integral_domain()
False
>>> FR.is_field()
False
```

one()

Return the constant function 1 in `self`.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: FR = X.function_ring()
sage: FR.one()
1

sage: M = Manifold(2, 'M', structure='topological', field=Qp(5))
sage: X.<x,y> = M.chart()
sage: X.function_ring().one()
1 + O(5^20)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> FR = X.function_ring()
>>> FR.one()
1

>>> M = Manifold(Integer(2), 'M', structure='topological',␣
↪field=Qp(Integer(5)))
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> X.function_ring().one()
1 + O(5^20)
```

zero()

Return the constant function 0 in `self`.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: FR = X.function_ring()
sage: FR.zero()
0
```

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```
sage: M = Manifold(2, 'M', structure='topological', field=Qp(5))
sage: X.<x,y> = M.chart()
sage: X.function_ring().zero()
0
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> FR = X.function_ring()
>>> FR.zero()
0

>>> M = Manifold(Integer(2), 'M', structure='topological',␣
↪field=Qp(Integer(5)))
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> X.function_ring().zero()
0
```

class sage.manifolds.chart_func.**MultiCoordFunction**(*chart*, *expressions*)

Bases: `SageObject`, `Mutability`

Coordinate function to some Cartesian power of the base field.

If n and m are two positive integers and (U,φ) is a chart on a topological manifold M of dimension n over a topological field K, a *multi-coordinate function* associated to (U,φ) is a map

$$\begin{array}{llcl} f: & V \subset K^n & \longrightarrow & K^m \\ & (x^1,\ldots,x^n) & \longmapsto & (f_1(x^1,\ldots,x^n),\ldots,f_m(x^1,\ldots,x^n)), \end{array}$$

where V is the codomain of φ. In other words, f is a K^m-valued function of the coordinates associated to the chart (U,φ). Each component f_i $(1 \leq i \leq m)$ is a coordinate function and is therefore stored as a `ChartFunction`.

INPUT:

- `chart` – the chart (U,φ)
- `expressions` – list (or tuple) of length m of elements to construct the coordinate functions f_i $(1 \leq i \leq m)$; for symbolic coordinate functions, this must be symbolic expressions involving the chart coordinates, while for numerical coordinate functions, this must be data file names

EXAMPLES:

A function $f: V \subset \mathbf{R}^2 \longrightarrow \mathbf{R}^3$:

```
sage: forget()  # to clear the previous assumption on x
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.multifunction(x-y, x*y, cos(x)*exp(y)); f
Coordinate functions (x - y, x*y, cos(x)*e^y) on the Chart (M, (x, y))
sage: type(f)
<class 'sage.manifolds.chart_func.MultiCoordFunction'>
sage: f(x,y)
(x - y, x*y, cos(x)*e^y)
sage: latex(f)
\left(x - y, x y, \cos\left(x\right) e^{y}\right)
```

```
>>> from sage.all import *
>>> forget()  # to clear the previous assumption on x
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.multifunction(x-y, x*y, cos(x)*exp(y)); f
Coordinate functions (x - y, x*y, cos(x)*e^y) on the Chart (M, (x, y))
>>> type(f)
<class 'sage.manifolds.chart_func.MultiCoordFunction'>
>>> f(x,y)
(x - y, x*y, cos(x)*e^y)
>>> latex(f)
\left(x - y, x y, \cos\left(x\right) e^{y}\right)
```

Each real-valued function f_i ($1 \leq i \leq m$) composing f can be accessed via the square-bracket operator, by providing $i-1$ as an argument:

```
sage: f[0]
x - y
sage: f[1]
x*y
sage: f[2]
cos(x)*e^y
```

```
>>> from sage.all import *
>>> f[Integer(0)]
x - y
>>> f[Integer(1)]
x*y
>>> f[Integer(2)]
cos(x)*e^y
```

We can give a more verbose explanation of each function:

```
sage: f[0].display()
(x, y) ↦ x - y
```

```
>>> from sage.all import *
>>> f[Integer(0)].display()
(x, y) ↦ x - y
```

Each `f[i-1]` is an instance of *ChartFunction*:

```
sage: isinstance(f[0], sage.manifolds.chart_func.ChartFunction)
True
```

```
>>> from sage.all import *
>>> isinstance(f[Integer(0)], sage.manifolds.chart_func.ChartFunction)
True
```

A class *MultiCoordFunction* can represent a real-valued function (case $m = 1$), although one should rather employ the class *ChartFunction* for this purpose:

```
sage: g = X.multifunction(x*y^2)
sage: g(x,y)
(x*y^2,)
```

```
>>> from sage.all import *
>>> g = X.multifunction(x*y**Integer(2))
>>> g(x,y)
(x*y^2,)
```

Evaluating the functions at specified coordinates:

```
sage: f(1,2)
(-1, 2, cos(1)*e^2)
sage: var('a b')
(a, b)
sage: f(a,b)
(a - b, a*b, cos(a)*e^b)
sage: g(1,2)
(4,)
```

```
>>> from sage.all import *
>>> f(Integer(1),Integer(2))
(-1, 2, cos(1)*e^2)
>>> var('a b')
(a, b)
>>> f(a,b)
(a - b, a*b, cos(a)*e^b)
>>> g(Integer(1),Integer(2))
(4,)
```

chart()

Return the chart with respect to which `self` is defined.

OUTPUT: a *Chart*

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.multifunction(x-y, x*y, cos(x)*exp(y))
sage: f.chart()
Chart (M, (x, y))
sage: f.chart() is X
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.multifunction(x-y, x*y, cos(x)*exp(y))
>>> f.chart()
Chart (M, (x, y))
>>> f.chart() is X
True
```

expr(*method=None*)

Return a tuple of data, the item no. i being sufficient to reconstruct the coordinate function no. i.

In other words, if `f` is a multi-coordinate function, then `f.chart().multifunction(*(f.expr()))` results in a multi-coordinate function identical to `f`.

INPUT:

- `method` – string (default: `None`); the calculus method which the returned expressions belong to. One of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy
  - `None`: the chart current calculus method is assumed

OUTPUT:

- a tuple of the symbolic expressions of the chart functions composing `self`

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.multifunction(x-y, x*y, cos(x)*exp(y))
sage: f.expr()
(x - y, x*y, cos(x)*e^y)
sage: type(f.expr()[0])
<class 'sage.symbolic.expression.Expression'>
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.multifunction(x-y, x*y, cos(x)*exp(y))
>>> f.expr()
(x - y, x*y, cos(x)*e^y)
>>> type(f.expr()[Integer(0)])
<class 'sage.symbolic.expression.Expression'>
```

A SymPy output:

```
sage: f.expr('sympy')
(x - y, x*y, exp(y)*cos(x))
sage: type(f.expr('sympy')[0])
<class 'sympy.core.add.Add'>
```

```
>>> from sage.all import *
>>> f.expr('sympy')
(x - y, x*y, exp(y)*cos(x))
>>> type(f.expr('sympy')[Integer(0)])
<class 'sympy.core.add.Add'>
```

One shall always have:

```
sage: f.chart().multifunction(*(f.expr())) == f
True
```

```
>>> from sage.all import *
>>> f.chart().multifunction(*(f.expr())) == f
True
```

jacobian()

Return the Jacobian matrix of the system of coordinate functions.

`jacobian()` is a 2-dimensional array of size $m \times n$, where m is the number of functions and n the number of coordinates, the generic element being $J_{ij} = \frac{\partial f_i}{\partial x^j}$ with $1 \leq i \leq m$ (row index) and $1 \leq j \leq n$ (column index).

OUTPUT:

- Jacobian matrix as a 2-dimensional array `J` of coordinate functions with `J[i-1][j-1]` being $J_{ij} = \frac{\partial f_i}{\partial x^j}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$

EXAMPLES:

Jacobian of a set of 3 functions of 2 coordinates:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.multifunction(x-y, x*y, y^3*cos(x))
sage: f.jacobian()
[           1           -1]
[           y            x]
[ -y^3*sin(x)  3*y^2*cos(x)]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.multifunction(x-y, x*y, y**Integer(3)*cos(x))
>>> f.jacobian()
[           1           -1]
[           y            x]
[ -y^3*sin(x)  3*y^2*cos(x)]
```

Each element of the result is a *chart function*:

```
sage: type(f.jacobian()[2,0])
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>
sage: f.jacobian()[2,0].display()
(x, y) ↦ -y^3*sin(x)
```

```
>>> from sage.all import *
>>> type(f.jacobian()[Integer(2),Integer(0)])
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>
>>> f.jacobian()[Integer(2),Integer(0)].display()
(x, y) ↦ -y^3*sin(x)
```

Test of the computation:

```
sage: [[f.jacobian()[i,j] == f[i].diff(j) for j in range(2)] for i in␣
↪range(3)]
[[True, True], [True, True], [True, True]]
```

```
>>> from sage.all import *
>>> [[f.jacobian()[i,j] == f[i].diff(j) for j in range(Integer(2))] for i in␣
↪range(Integer(3))]
[[True, True], [True, True], [True, True]]
```

Test with `start_index = 1`:

```
sage: M = Manifold(2, 'M', structure='topological', start_index=1)
sage: X.<x,y> = M.chart()
sage: f = X.multifunction(x-y, x*y, y^3*cos(x))
sage: f.jacobian()
[           1           -1]
[           y            x]
[ -y^3*sin(x) 3*y^2*cos(x)]
sage: [[f.jacobian()[i,j] == f[i].diff(j+1) for j in range(2)]  # note the j+1
....:                                         for i in range(3)]
[[True, True], [True, True], [True, True]]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological', start_
↪index=Integer(1))
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.multifunction(x-y, x*y, y**Integer(3)*cos(x))
>>> f.jacobian()
[           1           -1]
[           y            x]
[ -y^3*sin(x) 3*y^2*cos(x)]
>>> [[f.jacobian()[i,j] == f[i].diff(j+Integer(1)) for j in␣
↪range(Integer(2))]  # note the j+1
...                                         for i in range(Integer(3))]
[[True, True], [True, True], [True, True]]
```

jacobian_det()

Return the Jacobian determinant of the system of functions.

The number m of coordinate functions must equal the number n of coordinates.

OUTPUT: a *ChartFunction* representing the determinant

EXAMPLES:

Jacobian determinant of a set of 2 functions of 2 coordinates:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.multifunction(x-y, x*y)
sage: f.jacobian_det()
x + y
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = X.multifunction(x-y, x*y)
>>> f.jacobian_det()
x + y
```

The output of `jacobian_det()` is an instance of `ChartFunction` and can therefore be called on specific values of the coordinates, e.g. $(x, y) = (1, 2)$:

```
sage: type(f.jacobian_det())
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>
sage: f.jacobian_det().display()
(x, y) ↦ x + y
sage: f.jacobian_det()(1,2)
3
```

```
>>> from sage.all import *
>>> type(f.jacobian_det())
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>
>>> f.jacobian_det().display()
(x, y) ↦ x + y
>>> f.jacobian_det()(Integer(1),Integer(2))
3
```

The result is cached:

```
sage: f.jacobian_det() is f.jacobian_det()
True
```

```
>>> from sage.all import *
>>> f.jacobian_det() is f.jacobian_det()
True
```

We verify the determinant of the Jacobian:

```
sage: f.jacobian_det() == det(matrix([[f[i].diff(j).expr() for j in range(2)]
....:                                    for i in range(2)]))
True
```

```
>>> from sage.all import *
>>> f.jacobian_det() == det(matrix([[f[i].diff(j).expr() for j in␣
↪range(Integer(2))]
...                                  for i in range(Integer(2))]))
True
```

An example using SymPy:

```
sage: M.set_calculus_method('sympy')
sage: g = X.multifunction(x*y^3, e^x)
sage: g.jacobian_det()
```

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```
-3*x*y**2*exp(x)
sage: type(g.jacobian_det().expr())
<class 'sympy.core.mul.Mul'>
```

```
>>> from sage.all import *
>>> M.set_calculus_method('sympy')
>>> g = X.multifunction(x*y**Integer(3), e**x)
>>> g.jacobian_det()
-3*x*y**2*exp(x)
>>> type(g.jacobian_det().expr())
<class 'sympy.core.mul.Mul'>
```

Jacobian determinant of a set of 3 functions of 3 coordinates:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: X.<x,y,z> = M.chart()
sage: f = X.multifunction(x*y+z^2, z^2*x+y^2*z, (x*y*z)^3)
sage: f.jacobian_det().display()
(x, y, z) ↦ 6*x^3*y^5*z^3 - 3*x^4*y^3*z^4 - 12*x^2*y^4*z^5 + 6*x^3*y^2*z^6
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> f = X.multifunction(x*y+z**Integer(2), z**Integer(2)*x+y**Integer(2)*z,␣
↪(x*y*z)**Integer(3))
>>> f.jacobian_det().display()
(x, y, z) ↦ 6*x^3*y^5*z^3 - 3*x^4*y^3*z^4 - 12*x^2*y^4*z^5 + 6*x^3*y^2*z^6
```

We verify the determinant of the Jacobian:

```
sage: f.jacobian_det() == det(matrix([[f[i].diff(j).expr() for j in range(3)]
....:                                     for i in range(3)]))
True
```

```
>>> from sage.all import *
>>> f.jacobian_det() == det(matrix([[f[i].diff(j).expr() for j in␣
↪range(Integer(3))]
...                                     for i in range(Integer(3))]))
True
```

set_immutable()

Set `self` and all chart functions of `self` immutable.

EXAMPLES:

Declare a coordinate function immutable:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: X.<x,y,z> = M.chart()
sage: f = X.multifunction(x+y+z, x*y*z)
sage: f.is_immutable()
False
```

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```
sage: f.set_immutable()
sage: f.is_immutable()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> f = X.multifunction(x+y+z, x*y*z)
>>> f.is_immutable()
False
>>> f.set_immutable()
>>> f.is_immutable()
True
```

The chart functions are now immutable, too:

```
sage: f[0].parent()
Ring of chart functions on Chart (M, (x, y, z))
sage: f[0].is_immutable()
True
```

```
>>> from sage.all import *
>>> f[Integer(0)].parent()
Ring of chart functions on Chart (M, (x, y, z))
>>> f[Integer(0)].is_immutable()
True
```

1.5.3 Coordinate calculus methods

The class `CalculusMethod` governs the calculus methods (symbolic and numerical) used for coordinate computations on manifolds.

AUTHORS:

- Marco Mancini (2017): initial version
- Eric Gourgoulhon (2019): add `set_simplify_function()` and various accessors

class `sage.manifolds.calculus_method.`**`CalculusMethod`**(*current=None*, *base_field_type='real'*)

Bases: `SageObject`

Control of calculus backends used on coordinate charts of manifolds.

This class stores the possible calculus methods and permits to switch between them, as well as to change the simplifying functions associated with them. For the moment, only two calculus backends are implemented:

- Sage's symbolic engine (Pynac + Maxima), implemented via the Symbolic Ring `SR`
- SymPy engine, denoted `sympy` hereafter

INPUT:

- `current` – (default: `None`) string defining the calculus method that will be considered as the active one, until it is changed by `set()`; must be one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy

  - None: the default calculus method ('SR')

- base_field_type – (default: 'real') base field type of the manifold (cf. *base_field_type()*)

EXAMPLES:

```
sage: from sage.manifolds.calculus_method import CalculusMethod
sage: cm = CalculusMethod()
```

```
>>> from sage.all import *
>>> from sage.manifolds.calculus_method import CalculusMethod
>>> cm = CalculusMethod()
```

In the display, the currently active method is pointed out with a star:

```
sage: cm
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
```

```
>>> from sage.all import *
>>> cm
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
```

It can be changed with *set()*:

```
sage: cm.set('sympy')
sage: cm
Available calculus methods (* = current):
 - SR (default)
 - sympy (*)
```

```
>>> from sage.all import *
>>> cm.set('sympy')
>>> cm
Available calculus methods (* = current):
 - SR (default)
 - sympy (*)
```

while *reset()* brings back to the default:

```
sage: cm.reset()
sage: cm
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
```

```
>>> from sage.all import *
>>> cm.reset()
>>> cm
Available calculus methods (* = current):
```

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```
- SR (default) (*)
- sympy
```

See *simplify_function()* for the default simplification algorithms associated with each calculus method and *set_simplify_function()* for introducing a new simplification algorithm.

current()

Return the active calculus method as a string.

OUTPUT:

String defining the calculus method; one of

- 'SR' – Sage's default symbolic engine (Symbolic Ring)
- 'sympy' – SymPy

EXAMPLES:

```
sage: from sage.manifolds.calculus_method import CalculusMethod
sage: cm = CalculusMethod(); cm
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
sage: cm.current()
'SR'
sage: cm.set('sympy')
sage: cm.current()
'sympy'
```

```
>>> from sage.all import *
>>> from sage.manifolds.calculus_method import CalculusMethod
>>> cm = CalculusMethod(); cm
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
>>> cm.current()
'SR'
>>> cm.set('sympy')
>>> cm.current()
'sympy'
```

is_trivial_zero(*expression*, *method=None*)

Check if an expression is trivially equal to zero without any simplification.

INPUT:

- expression – expression
- method – (default: None) string defining the calculus method to use; if None the current calculus method of self is used

OUTPUT: True is expression is trivially zero, False elsewhere

EXAMPLES:

```
sage: from sage.manifolds.calculus_method import CalculusMethod
sage: cm = CalculusMethod(base_field_type='real')
sage: f = sin(x) - sin(x)
sage: cm.is_trivial_zero(f)
True
sage: cm.is_trivial_zero(f._sympy_(), method='sympy')
True
```

```
>>> from sage.all import *
>>> from sage.manifolds.calculus_method import CalculusMethod
>>> cm = CalculusMethod(base_field_type='real')
>>> f = sin(x) - sin(x)
>>> cm.is_trivial_zero(f)
True
>>> cm.is_trivial_zero(f._sympy_(), method='sympy')
True
```

```
sage: f = sin(x)^2 + cos(x)^2 - 1
sage: cm.is_trivial_zero(f)
False
sage: cm.is_trivial_zero(f._sympy_(), method='sympy')
False
```

```
>>> from sage.all import *
>>> f = sin(x)**Integer(2) + cos(x)**Integer(2) - Integer(1)
>>> cm.is_trivial_zero(f)
False
>>> cm.is_trivial_zero(f._sympy_(), method='sympy')
False
```

reset()

Set the current calculus method to the default one.

EXAMPLES:

```
sage: from sage.manifolds.calculus_method import CalculusMethod
sage: cm = CalculusMethod(base_field_type='complex')
sage: cm
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
sage: cm.set('sympy')
sage: cm
Available calculus methods (* = current):
 - SR (default)
 - sympy (*)
sage: cm.reset()
sage: cm
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
```

```
>>> from sage.all import *
>>> from sage.manifolds.calculus_method import CalculusMethod
>>> cm = CalculusMethod(base_field_type='complex')
>>> cm
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
>>> cm.set('sympy')
>>> cm
Available calculus methods (* = current):
 - SR (default)
 - sympy (*)
>>> cm.reset()
>>> cm
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
```

set(*method*)

Set the currently active calculus method.

- `method` – string defining the calculus method

EXAMPLES:

```
sage: from sage.manifolds.calculus_method import CalculusMethod
sage: cm = CalculusMethod(base_field_type='complex')
sage: cm
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
sage: cm.set('sympy')
sage: cm
Available calculus methods (* = current):
 - SR (default)
 - sympy (*)
sage: cm.set('lala')
Traceback (most recent call last):
...
NotImplementedError: method lala not implemented
```

```
>>> from sage.all import *
>>> from sage.manifolds.calculus_method import CalculusMethod
>>> cm = CalculusMethod(base_field_type='complex')
>>> cm
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
>>> cm.set('sympy')
>>> cm
Available calculus methods (* = current):
 - SR (default)
 - sympy (*)
```

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```
>>> cm.set('lala')
Traceback (most recent call last):
...
NotImplementedError: method lala not implemented
```

set_simplify_function(*simplifying_func*, *method=None*)

Set the simplifying function associated to a given calculus method.

INPUT:

- `simplifying_func` – either the string `'default'` for restoring the default simplifying function or a function `f` of a single argument `expr` such that `f(expr)` returns an object of the same type as `expr` (hopefully the simplified version of `expr`), this type being
  - `Expression` if `method` = `'SR'`
  - a SymPy type if `method` = `'sympy'`
- `method` – (default: `None`) string defining the calculus method for which `simplifying_func` is provided; must be one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy
  - `None`: the currently active calculus method of `self` is assumed

EXAMPLES:

On a real manifold, the default simplifying function is `simplify_chain_real()` when the calculus method is `SR`:

```
sage: from sage.manifolds.calculus_method import CalculusMethod
sage: cm = CalculusMethod(base_field_type='real'); cm
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
sage: cm.simplify_function() is \
....: sage.manifolds.utilities.simplify_chain_real
True
```

```
>>> from sage.all import *
>>> from sage.manifolds.calculus_method import CalculusMethod
>>> cm = CalculusMethod(base_field_type='real'); cm
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
>>> cm.simplify_function() is sage.manifolds.utilities.simplify_chain_real
True
```

Let us change it to `simplify()`:

```
sage: cm.set_simplify_function(simplify)
sage: cm.simplify_function() is simplify
True
```

```
>>> from sage.all import *
>>> cm.set_simplify_function(simplify)
>>> cm.simplify_function() is simplify
True
```

Since SR is the current calculus method, the above is equivalent to:

```
sage: cm.set_simplify_function(simplify, method='SR')
sage: cm.simplify_function(method='SR') is simplify
True
```

```
>>> from sage.all import *
>>> cm.set_simplify_function(simplify, method='SR')
>>> cm.simplify_function(method='SR') is simplify
True
```

We revert to the default simplifying function by:

```
sage: cm.set_simplify_function('default')
```

```
>>> from sage.all import *
>>> cm.set_simplify_function('default')
```

Then we are back to:

```
sage: cm.simplify_function() is \
....: sage.manifolds.utilities.simplify_chain_real
True
```

```
>>> from sage.all import *
>>> cm.simplify_function() is sage.manifolds.utilities.simplify_chain_real
True
```

simplify(*expression*, *method=None*)

Apply the simplifying function associated with a given calculus method to a symbolic expression.

INPUT:

- `expression` – symbolic expression to simplify
- `method` – (default: `None`) string defining the calculus method to use; must be one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy
  - `None`: the current calculus method of `self` is used

OUTPUT: the simplified version of `expression`

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x, y> = M.chart()
sage: f = x^2 + sin(x)^2 + cos(x)^2
sage: from sage.manifolds.calculus_method import CalculusMethod
```

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```
sage: cm = CalculusMethod(base_field_type='real')
sage: cm.simplify(f)
x^2 + 1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = x**Integer(2) + sin(x)**Integer(2) + cos(x)**Integer(2)
>>> from sage.manifolds.calculus_method import CalculusMethod
>>> cm = CalculusMethod(base_field_type='real')
>>> cm.simplify(f)
x^2 + 1
```

Using a weaker simplifying function, like `simplify()`, does not succeed in this case:

```
sage: cm.set_simplify_function(simplify)
sage: cm.simplify(f)
x^2 + cos(x)^2 + sin(x)^2
```

```
>>> from sage.all import *
>>> cm.set_simplify_function(simplify)
>>> cm.simplify(f)
x^2 + cos(x)^2 + sin(x)^2
```

Back to the default simplifying function (*`simplify_chain_real()`* in the present case):

```
sage: cm.set_simplify_function('default')
sage: cm.simplify(f)
x^2 + 1
```

```
>>> from sage.all import *
>>> cm.set_simplify_function('default')
>>> cm.simplify(f)
x^2 + 1
```

A `SR` expression, such as `f`, cannot be simplified when the current calculus method is `sympy`:

```
sage: cm.set('sympy')
sage: cm.simplify(f)
Traceback (most recent call last):
...
AttributeError: 'sage.symbolic.expression.Expression' object has no attribute
↪'combsimp'...
```

```
>>> from sage.all import *
>>> cm.set('sympy')
>>> cm.simplify(f)
Traceback (most recent call last):
...
AttributeError: 'sage.symbolic.expression.Expression' object has no attribute
↪'combsimp'...
```

In the present case, one should either transform `f` to a SymPy object:

```
sage: cm.simplify(f._sympy_())
x**2 + 1
```

```
>>> from sage.all import *
>>> cm.simplify(f._sympy_())
x**2 + 1
```

or force the calculus method to be `'SR'`:

```
sage: cm.simplify(f, method='SR')
x^2 + 1
```

```
>>> from sage.all import *
>>> cm.simplify(f, method='SR')
x^2 + 1
```

simplify_function(*method=None*)

Return the simplifying function associated to a given calculus method.

The simplifying function is that used in all computations involved with the calculus method.

INPUT:

- `method` – (default: `None`) string defining the calculus method for which `simplifying_func` is provided; must be one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy
  - `None`: the currently active calculus method of `self` is assumed

OUTPUT: the simplifying function

EXAMPLES:

```
sage: from sage.manifolds.calculus_method import CalculusMethod
sage: cm = CalculusMethod()
sage: cm
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
sage: cm.simplify_function()  # random (memory address)
<function simplify_chain_real at 0x7f958d5b6758>
```

```
>>> from sage.all import *
>>> from sage.manifolds.calculus_method import CalculusMethod
>>> cm = CalculusMethod()
>>> cm
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
>>> cm.simplify_function()  # random (memory address)
<function simplify_chain_real at 0x7f958d5b6758>
```

The output stands for the function *simplify_chain_real()*:

```
sage: cm.simplify_function() is \
....: sage.manifolds.utilities.simplify_chain_real
True
```

```
>>> from sage.all import *
>>> cm.simplify_function() is sage.manifolds.utilities.simplify_chain_real
True
```

Since `SR` is the default calculus method, we have:

```
sage: cm.simplify_function() is cm.simplify_function(method='SR')
True
```

```
>>> from sage.all import *
>>> cm.simplify_function() is cm.simplify_function(method='SR')
True
```

The simplifying function associated with `sympy` is *simplify_chain_real_sympy()*:

```
sage: cm.simplify_function(method='sympy')  # random (memory address)
<function simplify_chain_real_sympy at 0x7f0b35a578c0>
sage: cm.simplify_function(method='sympy') is \
....: sage.manifolds.utilities.simplify_chain_real_sympy
True
```

```
>>> from sage.all import *
>>> cm.simplify_function(method='sympy')  # random (memory address)
<function simplify_chain_real_sympy at 0x7f0b35a578c0>
>>> cm.simplify_function(method='sympy') is sage.manifolds.utilities.simplify_
↪chain_real_sympy
True
```

On complex manifolds, the simplifying functions are *simplify_chain_generic()* and *simplify_chain_generic_sympy()* for respectively `SR` and `sympy`:

```
sage: cmc = CalculusMethod(base_field_type='complex')
sage: cmc.simplify_function(method='SR') is \
....: sage.manifolds.utilities.simplify_chain_generic
True
sage: cmc.simplify_function(method='sympy') is \
....: sage.manifolds.utilities.simplify_chain_generic_sympy
True
```

```
>>> from sage.all import *
>>> cmc = CalculusMethod(base_field_type='complex')
>>> cmc.simplify_function(method='SR') is sage.manifolds.utilities.simplify_
↪chain_generic
True
>>> cmc.simplify_function(method='sympy') is sage.manifolds.utilities.
↪simplify_chain_generic_sympy
True
```

Note that the simplifying functions can be customized via *set_simplify_function()*.

1.6 Scalar Fields

1.6.1 Algebra of Scalar Fields

The class `ScalarFieldAlgebra` implements the commutative algebra $C^0(M)$ of scalar fields on a topological manifold M over a topological field K. By *scalar field*, it is meant a continuous function $M \to K$. The set $C^0(M)$ is an algebra over K, whose ring product is the pointwise multiplication of K-valued functions, which is clearly commutative.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2014-2015): initial version
- Travis Scrimshaw (2016): review tweaks

REFERENCES:

- [Lee2011]
- [KN1963]

class sage.manifolds.scalarfield_algebra.**ScalarFieldAlgebra**(*domain*)

Bases: `UniqueRepresentation`, `Parent`

Commutative algebra of scalar fields on a topological manifold.

If M is a topological manifold over a topological field K, the commutative algebra of scalar fields on M is the set $C^0(M)$ of all continuous maps $M \to K$. The set $C^0(M)$ is an algebra over K, whose ring product is the pointwise multiplication of K-valued functions, which is clearly commutative.

If $K = \mathbf{R}$ or $K = \mathbf{C}$, the field K over which the algebra $C^0(M)$ is constructed is represented by the `Symbolic Ring` `SR`, since there is no exact representation of $\mathbf{R}$ nor $\mathbf{C}$.

INPUT:

- `domain` – the topological manifold M on which the scalar fields are defined

EXAMPLES:

Algebras of scalar fields on the sphere S^2 and on some open subsets of it:

```
sage: M = Manifold(2, 'M', structure='topological') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U')  # complement of the North pole
sage: c_xy.<x,y> = U.chart()  # stereographic coordinates from the North pole
sage: V = M.open_subset('V')  # complement of the South pole
sage: c_uv.<u,v> = V.chart()  # stereographic coordinates from the South pole
sage: M.declare_union(U,V)    # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                                intersection_name='W',
....:                                restrictions1= x^2+y^2!=0,
....:                                restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: CM = M.scalar_field_algebra(); CM
Algebra of scalar fields on the 2-dimensional topological manifold M
sage: W = U.intersection(V)  # S^2 minus the two poles
sage: CW = W.scalar_field_algebra(); CW
Algebra of scalar fields on the Open subset W of the
 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological') # the 2-dimensional␣
↪sphere S^2
>>> U = M.open_subset('U')  # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V')  # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)    # S^2 is the union of U and V
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                                 intersection_name='W',
...                                 restrictions1= x**Integer(2)+y**Integer(2)!
↪=Integer(0),
...                                 restrictions2= u**Integer(2)+v**Integer(2)!
↪=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> CM = M.scalar_field_algebra(); CM
Algebra of scalar fields on the 2-dimensional topological manifold M
>>> W = U.intersection(V)  # S^2 minus the two poles
>>> CW = W.scalar_field_algebra(); CW
Algebra of scalar fields on the Open subset W of the
 2-dimensional topological manifold M
```

$C^0(M)$ and $C^0(W)$ belong to the category of commutative algebras over **R** (represented here by `SymbolicRing`):

```
sage: CM.category()
Join of Category of commutative algebras over Symbolic Ring and Category of␣
↪homsets of topological spaces
sage: CM.base_ring()
Symbolic Ring
sage: CW.category()
Join of Category of commutative algebras over Symbolic Ring and Category of␣
↪homsets of topological spaces
sage: CW.base_ring()
Symbolic Ring
```

```
>>> from sage.all import *
>>> CM.category()
Join of Category of commutative algebras over Symbolic Ring and Category of␣
↪homsets of topological spaces
>>> CM.base_ring()
Symbolic Ring
>>> CW.category()
Join of Category of commutative algebras over Symbolic Ring and Category of␣
↪homsets of topological spaces
>>> CW.base_ring()
Symbolic Ring
```

The elements of $C^0(M)$ are scalar fields on M:

```
sage: CM.an_element()
Scalar field on the 2-dimensional topological manifold M
sage: CM.an_element().display()  # this sample element is a constant field
M → ℝ
on U: (x, y) ↦ 2
on V: (u, v) ↦ 2
```

```
>>> from sage.all import *
>>> CM.an_element()
Scalar field on the 2-dimensional topological manifold M
>>> CM.an_element().display()  # this sample element is a constant field
M → ℝ
on U: (x, y) ↦ 2
on V: (u, v) ↦ 2
```

Those of $C^0(W)$ are scalar fields on W:

```
sage: CW.an_element()
Scalar field on the Open subset W of the 2-dimensional topological
 manifold M
sage: CW.an_element().display()  # this sample element is a constant field
W → ℝ
(x, y) ↦ 2
(u, v) ↦ 2
```

```
>>> from sage.all import *
>>> CW.an_element()
Scalar field on the Open subset W of the 2-dimensional topological
 manifold M
>>> CW.an_element().display()  # this sample element is a constant field
W → ℝ
(x, y) ↦ 2
(u, v) ↦ 2
```

The zero element:

```
sage: CM.zero()
Scalar field zero on the 2-dimensional topological manifold M
sage: CM.zero().display()
zero: M → ℝ
on U: (x, y) ↦ 0
on V: (u, v) ↦ 0
```

```
>>> from sage.all import *
>>> CM.zero()
Scalar field zero on the 2-dimensional topological manifold M
>>> CM.zero().display()
zero: M → ℝ
on U: (x, y) ↦ 0
on V: (u, v) ↦ 0
```

```
sage: CW.zero()
Scalar field zero on the Open subset W of the 2-dimensional
 topological manifold M
sage: CW.zero().display()
zero: W → ℝ
   (x, y) ↦ 0
   (u, v) ↦ 0
```

```
>>> from sage.all import *
>>> CW.zero()
Scalar field zero on the Open subset W of the 2-dimensional
 topological manifold M
>>> CW.zero().display()
zero: W → ℝ
   (x, y) ↦ 0
   (u, v) ↦ 0
```

The unit element:

```
sage: CM.one()
Scalar field 1 on the 2-dimensional topological manifold M
sage: CM.one().display()
1: M → ℝ
on U: (x, y) ↦ 1
on V: (u, v) ↦ 1
```

```
>>> from sage.all import *
>>> CM.one()
Scalar field 1 on the 2-dimensional topological manifold M
>>> CM.one().display()
1: M → ℝ
on U: (x, y) ↦ 1
on V: (u, v) ↦ 1
```

```
sage: CW.one()
Scalar field 1 on the Open subset W of the 2-dimensional topological
 manifold M
sage: CW.one().display()
1: W → ℝ
  (x, y) ↦ 1
  (u, v) ↦ 1
```

```
>>> from sage.all import *
>>> CW.one()
Scalar field 1 on the Open subset W of the 2-dimensional topological
 manifold M
>>> CW.one().display()
1: W → ℝ
  (x, y) ↦ 1
  (u, v) ↦ 1
```

A generic element can be constructed by using a dictionary of the coordinate expressions defining the scalar field:

```
sage: f = CM({c_xy: atan(x^2+y^2), c_uv: pi/2 - atan(u^2+v^2)}); f
Scalar field on the 2-dimensional topological manifold M
sage: f.display()
M → ℝ
on U: (x, y) ↦ arctan(x^2 + y^2)
on V: (u, v) ↦ 1/2*pi - arctan(u^2 + v^2)
sage: f.parent()
Algebra of scalar fields on the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> f = CM({c_xy: atan(x**Integer(2)+y**Integer(2)), c_uv: pi/Integer(2) -
↪atan(u**Integer(2)+v**Integer(2))}); f
Scalar field on the 2-dimensional topological manifold M
>>> f.display()
M → ℝ
on U: (x, y) ↦ arctan(x^2 + y^2)
on V: (u, v) ↦ 1/2*pi - arctan(u^2 + v^2)
>>> f.parent()
Algebra of scalar fields on the 2-dimensional topological manifold M
```

Specific elements can also be constructed in this way:

```
sage: CM(0) == CM.zero()
True
sage: CM(1) == CM.one()
True
```

```
>>> from sage.all import *
>>> CM(Integer(0)) == CM.zero()
True
>>> CM(Integer(1)) == CM.one()
True
```

Note that the zero scalar field is cached:

```
sage: CM(0) is CM.zero()
True
```

```
>>> from sage.all import *
>>> CM(Integer(0)) is CM.zero()
True
```

Elements can also be constructed by means of the method `scalar_field()` acting on the domain (this allows one to set the name of the scalar field at the construction):

```
sage: f1 = M.scalar_field({c_xy: atan(x^2+y^2), c_uv: pi/2 - atan(u^2+v^2)},
....:                     name='f')
sage: f1.parent()
Algebra of scalar fields on the 2-dimensional topological manifold M
sage: f1 == f
True
sage: M.scalar_field(0, chart='all') == CM.zero()
True
```

```
>>> from sage.all import *
>>> f1 = M.scalar_field({c_xy: atan(x**Integer(2)+y**Integer(2)), c_uv: pi/
↪Integer(2) - atan(u**Integer(2)+v**Integer(2))},
...                     name='f')
>>> f1.parent()
Algebra of scalar fields on the 2-dimensional topological manifold M
>>> f1 == f
True
>>> M.scalar_field(Integer(0), chart='all') == CM.zero()
True
```

The algebra $C^0(M)$ coerces to $C^0(W)$ since W is an open subset of M:

```
sage: CW.has_coerce_map_from(CM)
True
```

```
>>> from sage.all import *
>>> CW.has_coerce_map_from(CM)
True
```

The reverse is of course false:

```
sage: CM.has_coerce_map_from(CW)
False
```

```
>>> from sage.all import *
>>> CM.has_coerce_map_from(CW)
False
```

The coercion map is nothing but the restriction to W of scalar fields on M:

```
sage: fW = CW(f) ; fW
Scalar field on the Open subset W of the
 2-dimensional topological manifold M
sage: fW.display()
W → ℝ
  (x, y) ↦ arctan(x^2 + y^2)
  (u, v) ↦ 1/2*pi - arctan(u^2 + v^2)
```

```
>>> from sage.all import *
>>> fW = CW(f) ; fW
Scalar field on the Open subset W of the
 2-dimensional topological manifold M
>>> fW.display()
W → ℝ
  (x, y) ↦ arctan(x^2 + y^2)
  (u, v) ↦ 1/2*pi - arctan(u^2 + v^2)
```

```
sage: CW(CM.one()) == CW.one()
True
```

```
>>> from sage.all import *
>>> CW(CM.one()) == CW.one()
True
```

The coercion map allows for the addition of elements of $C^0(W)$ with elements of $C^0(M)$, the result being an element of $C^0(W)$:

```
sage: s = fW + f
sage: s.parent()
Algebra of scalar fields on the Open subset W of the
 2-dimensional topological manifold M
sage: s.display()
W → ℝ
  (x, y) ↦ 2*arctan(x^2 + y^2)
  (u, v) ↦ pi - 2*arctan(u^2 + v^2)
```

```
>>> from sage.all import *
>>> s = fW + f
>>> s.parent()
Algebra of scalar fields on the Open subset W of the
 2-dimensional topological manifold M
>>> s.display()
W → ℝ
  (x, y) ↦ 2*arctan(x^2 + y^2)
  (u, v) ↦ pi - 2*arctan(u^2 + v^2)
```

Another coercion is that from the Symbolic Ring. Since the Symbolic Ring is the base ring for the algebra `CM`, the coercion of a symbolic expression `s` is performed by the operation `s*CM.one()`, which invokes the (reflected) multiplication operator. If the symbolic expression does not involve any chart coordinate, the outcome is a constant scalar field:

```
sage: h = CM(pi*sqrt(2)) ; h
Scalar field on the 2-dimensional topological manifold M
sage: h.display()
M → ℝ
on U: (x, y) ↦ sqrt(2)*pi
on V: (u, v) ↦ sqrt(2)*pi
sage: a = var('a')
sage: h = CM(a); h.display()
M → ℝ
on U: (x, y) ↦ a
on V: (u, v) ↦ a
```

```
>>> from sage.all import *
>>> h = CM(pi*sqrt(Integer(2))) ; h
Scalar field on the 2-dimensional topological manifold M
>>> h.display()
M → ℝ
on U: (x, y) ↦ sqrt(2)*pi
on V: (u, v) ↦ sqrt(2)*pi
>>> a = var('a')
>>> h = CM(a); h.display()
M → ℝ
```

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```
on U: (x, y) ↦ a
on V: (u, v) ↦ a
```

If the symbolic expression involves some coordinate of one of the manifold's charts, the outcome is initialized only on the chart domain:

```
sage: h = CM(a+x); h.display()
M → ℝ
on U: (x, y) ↦ a + x
on W: (u, v) ↦ (a*u^2 + a*v^2 + u)/(u^2 + v^2)
sage: h = CM(a+u); h.display()
M → ℝ
on W: (x, y) ↦ (a*x^2 + a*y^2 + x)/(x^2 + y^2)
on V: (u, v) ↦ a + u
```

```
>>> from sage.all import *
>>> h = CM(a+x); h.display()
M → ℝ
on U: (x, y) ↦ a + x
on W: (u, v) ↦ (a*u^2 + a*v^2 + u)/(u^2 + v^2)
>>> h = CM(a+u); h.display()
M → ℝ
on W: (x, y) ↦ (a*x^2 + a*y^2 + x)/(x^2 + y^2)
on V: (u, v) ↦ a + u
```

If the symbolic expression involves coordinates of different charts, the scalar field is created as a Python object, but is not initialized, in order to avoid any ambiguity:

```
sage: h = CM(x+u); h.display()
M → ℝ
```

```
>>> from sage.all import *
>>> h = CM(x+u); h.display()
M → ℝ
```

Element

alias of *ScalarField*

one()

Return the unit element of the algebra.

This is nothing but the constant scalar field 1 on the manifold, where 1 is the unit element of the base field.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: CM = M.scalar_field_algebra()
sage: h = CM.one(); h
Scalar field 1 on the 2-dimensional topological manifold M
sage: h.display()
1: M → ℝ
   (x, y) ↦ 1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> CM = M.scalar_field_algebra()
>>> h = CM.one(); h
Scalar field 1 on the 2-dimensional topological manifold M
>>> h.display()
1: M → ℝ
   (x, y) ↦ 1
```

The result is cached:

```
sage: CM.one() is h
True
```

```
>>> from sage.all import *
>>> CM.one() is h
True
```

zero()

Return the zero element of the algebra.

This is nothing but the constant scalar field 0 on the manifold, where 0 is the zero element of the base field.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: CM = M.scalar_field_algebra()
sage: z = CM.zero(); z
Scalar field zero on the 2-dimensional topological manifold M
sage: z.display()
zero: M → ℝ
   (x, y) ↦ 0
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> CM = M.scalar_field_algebra()
>>> z = CM.zero(); z
Scalar field zero on the 2-dimensional topological manifold M
>>> z.display()
zero: M → ℝ
   (x, y) ↦ 0
```

The result is cached:

```
sage: CM.zero() is z
True
```

```
>>> from sage.all import *
>>> CM.zero() is z
True
```

1.6.2 Scalar Fields

Given a topological manifold M over a topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$), a *scalar field* on M is a continuous map

$$f : M \longrightarrow K$$

Scalar fields are implemented by the class `ScalarField`.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015): initial version
- Travis Scrimshaw (2016): review tweaks
- Marco Mancini (2017): SymPy as an optional symbolic engine, alternative to SR
- Florentin Jaffredo (2018) : series expansion with respect to a given parameter
- Michael Jung (2019) : improve restrictions; make `display()` show all distinct expressions

REFERENCES:

- [Lee2011]
- [KN1963]

class sage.manifolds.scalarfield.**ScalarField**(*parent*, *coord_expression=None*, *chart=None*, *name=None*, *latex_name=None*)

Bases: `CommutativeAlgebraElement`, `ModuleElementWithMutability`

Scalar field on a topological manifold.

Given a topological manifold M over a topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$), a *scalar field on* M is a continuous map

$$f : M \longrightarrow K.$$

A scalar field on M is an element of the commutative algebra $C^0(M)$ (see `ScalarFieldAlgebra`).

INPUT:

- `parent` – the algebra of scalar fields containing the scalar field (must be an instance of class `ScalarFieldAlgebra`)
- `coord_expression` – (default: `None`) coordinate expression(s) of the scalar field; this can be either
  - a dictionary of coordinate expressions in various charts on the domain, with the charts as keys;
  - a single coordinate expression; if the argument `chart` is `'all'`, this expression is set to all the charts defined on the open set; otherwise, the expression is set in the specific chart provided by the argument `chart`
- `chart` – (default: `None`) chart defining the coordinates used in `coord_expression` when the latter is a single coordinate expression; if none is provided (default), the default chart of the open set is assumed. If `chart=='all'`, `coord_expression` is assumed to be independent of the chart (constant scalar field).
- `name` – (default: `None`) string; name (symbol) given to the scalar field
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the scalar field; if none is provided, the LaTeX symbol is set to `name`

If `coord_expression` is `None` or incomplete, coordinate expressions can be added after the creation of the object, by means of the methods `add_expr()`, `add_expr_by_continuation()` and `set_expr()`.

EXAMPLES:

A scalar field on the 2-sphere:

```
sage: M = Manifold(2, 'M', structure='topological') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                                intersection_name='W',
....:                                restrictions1= x^2+y^2!=0,
....:                                restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: f = M.scalar_field({c_xy: 1/(1+x^2+y^2), c_uv: (u^2+v^2)/(1+u^2+v^2)},
....:                    name='f') ; f
Scalar field f on the 2-dimensional topological manifold M
sage: f.display()
f: M → ℝ
on U: (x, y) ↦ 1/(x^2 + y^2 + 1)
on V: (u, v) ↦ (u^2 + v^2)/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological') # the 2-dimensional␣
↪sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                                intersection_name='W',
...                                restrictions1= x**Integer(2)+y**Integer(2)!
↪=Integer(0),
...                                restrictions2= u**Integer(2)+v**Integer(2)!
↪=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> f = M.scalar_field({c_xy: Integer(1)/(Integer(1)+x**Integer(2)+y**Integer(2)),
↪ c_uv: (u**Integer(2)+v**Integer(2))/(Integer(1)+u**Integer(2)+v**Integer(2))},
...                    name='f') ; f
Scalar field f on the 2-dimensional topological manifold M
>>> f.display()
f: M → ℝ
on U: (x, y) ↦ 1/(x^2 + y^2 + 1)
on V: (u, v) ↦ (u^2 + v^2)/(u^2 + v^2 + 1)
```

For scalar fields defined by a single coordinate expression, the latter can be passed instead of the dictionary over the charts:

```
sage: g = U.scalar_field(x*y, chart=c_xy, name='g') ; g
Scalar field g on the Open subset U of the 2-dimensional topological
 manifold M
```

```
>>> from sage.all import *
>>> g = U.scalar_field(x*y, chart=c_xy, name='g') ; g
Scalar field g on the Open subset U of the 2-dimensional topological
 manifold M
```

The above is indeed equivalent to:

```
sage: g = U.scalar_field({c_xy: x*y}, name='g') ; g
Scalar field g on the Open subset U of the 2-dimensional topological
 manifold M
```

```
>>> from sage.all import *
>>> g = U.scalar_field({c_xy: x*y}, name='g') ; g
Scalar field g on the Open subset U of the 2-dimensional topological
 manifold M
```

Since `c_xy` is the default chart of `U`, the argument `chart` can be skipped:

```
sage: g = U.scalar_field(x*y, name='g') ; g
Scalar field g on the Open subset U of the 2-dimensional topological
 manifold M
```

```
>>> from sage.all import *
>>> g = U.scalar_field(x*y, name='g') ; g
Scalar field g on the Open subset U of the 2-dimensional topological
 manifold M
```

The scalar field g is defined on U and has an expression in terms of the coordinates (u, v) on $W = U \cap V$:

```
sage: g.display()
g: U → ℝ
   (x, y) ↦ x*y
on W: (u, v) ↦ u*v/(u^4 + 2*u^2*v^2 + v^4)
```

```
>>> from sage.all import *
>>> g.display()
g: U → ℝ
   (x, y) ↦ x*y
on W: (u, v) ↦ u*v/(u^4 + 2*u^2*v^2 + v^4)
```

Scalar fields on M can also be declared with a single chart:

```
sage: f = M.scalar_field(1/(1+x^2+y^2), chart=c_xy, name='f') ; f
Scalar field f on the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> f = M.scalar_field(Integer(1)/(Integer(1)+x**Integer(2)+y**Integer(2)),␣
↪chart=c_xy, name='f') ; f
Scalar field f on the 2-dimensional topological manifold M
```

Their definition must then be completed by providing the expressions on other charts, via the method *add_expr()*, to get a global cover of the manifold:

```
sage: f.add_expr((u^2+v^2)/(1+u^2+v^2), chart=c_uv)
sage: f.display()
f: M → ℝ
on U: (x, y) ↦ 1/(x^2 + y^2 + 1)
on V: (u, v) ↦ (u^2 + v^2)/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> f.add_expr((u**Integer(2)+v**Integer(2))/
↪(Integer(1)+u**Integer(2)+v**Integer(2)), chart=c_uv)
>>> f.display()
f: M → ℝ
on U: (x, y) ↦ 1/(x^2 + y^2 + 1)
on V: (u, v) ↦ (u^2 + v^2)/(u^2 + v^2 + 1)
```

We can even first declare the scalar field without any coordinate expression and provide them subsequently:

```
sage: f = M.scalar_field(name='f')
sage: f.add_expr(1/(1+x^2+y^2), chart=c_xy)
sage: f.add_expr((u^2+v^2)/(1+u^2+v^2), chart=c_uv)
sage: f.display()
f: M → ℝ
on U: (x, y) ↦ 1/(x^2 + y^2 + 1)
on V: (u, v) ↦ (u^2 + v^2)/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> f = M.scalar_field(name='f')
>>> f.add_expr(Integer(1)/(Integer(1)+x**Integer(2)+y**Integer(2)), chart=c_xy)
>>> f.add_expr((u**Integer(2)+v**Integer(2))/
↪(Integer(1)+u**Integer(2)+v**Integer(2)), chart=c_uv)
>>> f.display()
f: M → ℝ
on U: (x, y) ↦ 1/(x^2 + y^2 + 1)
on V: (u, v) ↦ (u^2 + v^2)/(u^2 + v^2 + 1)
```

We may also use the method *add_expr_by_continuation()* to complete the coordinate definition using the analytic continuation from domains in which charts overlap:

```
sage: f = M.scalar_field(1/(1+x^2+y^2), chart=c_xy, name='f') ; f
Scalar field f on the 2-dimensional topological manifold M
sage: f.add_expr_by_continuation(c_uv, U.intersection(V))
sage: f.display()
f: M → ℝ
on U: (x, y) ↦ 1/(x^2 + y^2 + 1)
on V: (u, v) ↦ (u^2 + v^2)/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> f = M.scalar_field(Integer(1)/(Integer(1)+x**Integer(2)+y**Integer(2)),␣
↪chart=c_xy, name='f') ; f
Scalar field f on the 2-dimensional topological manifold M
>>> f.add_expr_by_continuation(c_uv, U.intersection(V))
>>> f.display()
f: M → ℝ
```

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```
on U: (x, y) ↦ 1/(x^2 + y^2 + 1)
on V: (u, v) ↦ (u^2 + v^2)/(u^2 + v^2 + 1)
```

A scalar field can also be defined by some unspecified function of the coordinates:

```
sage: h = U.scalar_field(function('H')(x, y), name='h') ; h
Scalar field h on the Open subset U of the 2-dimensional topological
 manifold M
sage: h.display()
h: U → ℝ
   (x, y) ↦ H(x, y)
on W: (u, v) ↦ H(u/(u^2 + v^2), v/(u^2 + v^2))
```

```
>>> from sage.all import *
>>> h = U.scalar_field(function('H')(x, y), name='h') ; h
Scalar field h on the Open subset U of the 2-dimensional topological
 manifold M
>>> h.display()
h: U → ℝ
   (x, y) ↦ H(x, y)
on W: (u, v) ↦ H(u/(u^2 + v^2), v/(u^2 + v^2))
```

We may use the argument `latex_name` to specify the LaTeX symbol denoting the scalar field if the latter is different from `name`:

```
sage: latex(f)
f
sage: f = M.scalar_field({c_xy: 1/(1+x^2+y^2), c_uv: (u^2+v^2)/(1+u^2+v^2)},
....:                    name='f', latex_name=r'\mathcal{F}')
sage: latex(f)
\mathcal{F}
```

```
>>> from sage.all import *
>>> latex(f)
f
>>> f = M.scalar_field({c_xy: Integer(1)/(Integer(1)+x**Integer(2)+y**Integer(2)),
↪ c_uv: (u**Integer(2)+v**Integer(2))/(Integer(1)+u**Integer(2)+v**Integer(2))},
...                    name='f', latex_name=r'\mathcal{F}')
>>> latex(f)
\mathcal{F}
```

The coordinate expression in a given chart is obtained via the method *expr()*, which returns a symbolic expression:

```
sage: f.expr(c_uv)
(u^2 + v^2)/(u^2 + v^2 + 1)
sage: type(f.expr(c_uv))
<class 'sage.symbolic.expression.Expression'>
```

```
>>> from sage.all import *
>>> f.expr(c_uv)
(u^2 + v^2)/(u^2 + v^2 + 1)
```

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```
>>> type(f.expr(c_uv))
<class 'sage.symbolic.expression.Expression'>
```

The method *coord_function()* returns instead a function of the chart coordinates, i.e. an instance of *ChartFunction*:

```
sage: f.coord_function(c_uv)
(u^2 + v^2)/(u^2 + v^2 + 1)
sage: type(f.coord_function(c_uv))
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_class'>
sage: f.coord_function(c_uv).display()
(u, v) ↦ (u^2 + v^2)/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> f.coord_function(c_uv)
(u^2 + v^2)/(u^2 + v^2 + 1)
>>> type(f.coord_function(c_uv))
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_class'>
>>> f.coord_function(c_uv).display()
(u, v) ↦ (u^2 + v^2)/(u^2 + v^2 + 1)
```

The value returned by the method *expr()* is actually the coordinate expression of the chart function:

```
sage: f.expr(c_uv) is f.coord_function(c_uv).expr()
True
```

```
>>> from sage.all import *
>>> f.expr(c_uv) is f.coord_function(c_uv).expr()
True
```

A constant scalar field is declared by setting the argument `chart` to `'all'`:

```
sage: c = M.scalar_field(2, chart='all', name='c') ; c
Scalar field c on the 2-dimensional topological manifold M
sage: c.display()
c: M → ℝ
on U: (x, y) ↦ 2
on V: (u, v) ↦ 2
```

```
>>> from sage.all import *
>>> c = M.scalar_field(Integer(2), chart='all', name='c') ; c
Scalar field c on the 2-dimensional topological manifold M
>>> c.display()
c: M → ℝ
on U: (x, y) ↦ 2
on V: (u, v) ↦ 2
```

A shortcut is to use the method *constant_scalar_field()*:

```
sage: c == M.constant_scalar_field(2)
True
```

```
>>> from sage.all import *
>>> c == M.constant_scalar_field(Integer(2))
True
```

The constant value can be some unspecified parameter:

```
sage: var('a')
a
sage: c = M.constant_scalar_field(a, name='c') ; c
Scalar field c on the 2-dimensional topological manifold M
sage: c.display()
c: M → ℝ
on U: (x, y) ↦ a
on V: (u, v) ↦ a
```

```
>>> from sage.all import *
>>> var('a')
a
>>> c = M.constant_scalar_field(a, name='c') ; c
Scalar field c on the 2-dimensional topological manifold M
>>> c.display()
c: M → ℝ
on U: (x, y) ↦ a
on V: (u, v) ↦ a
```

A special case of constant field is the zero scalar field:

```
sage: zer = M.constant_scalar_field(0) ; zer
Scalar field zero on the 2-dimensional topological manifold M
sage: zer.display()
zero: M → ℝ
on U: (x, y) ↦ 0
on V: (u, v) ↦ 0
```

```
>>> from sage.all import *
>>> zer = M.constant_scalar_field(Integer(0)) ; zer
Scalar field zero on the 2-dimensional topological manifold M
>>> zer.display()
zero: M → ℝ
on U: (x, y) ↦ 0
on V: (u, v) ↦ 0
```

It can be obtained directly by means of the function *zero_scalar_field()*:

```
sage: zer is M.zero_scalar_field()
True
```

```
>>> from sage.all import *
>>> zer is M.zero_scalar_field()
True
```

A third way is to get it as the zero element of the algebra $C^0(M)$ of scalar fields on M (see below):

```
sage: zer is M.scalar_field_algebra().zero()
True
```

```
>>> from sage.all import *
>>> zer is M.scalar_field_algebra().zero()
True
```

The constant scalar fields zero and one are immutable, and therefore their expressions cannot be changed:

```
sage: zer.is_immutable()
True
sage: zer.set_expr(x)
Traceback (most recent call last):
...
ValueError: the expressions of an immutable element cannot be
 changed
sage: one = M.one_scalar_field()
sage: one.is_immutable()
True
sage: one.set_expr(x)
Traceback (most recent call last):
...
ValueError: the expressions of an immutable element cannot be
 changed
```

```
>>> from sage.all import *
>>> zer.is_immutable()
True
>>> zer.set_expr(x)
Traceback (most recent call last):
...
ValueError: the expressions of an immutable element cannot be
 changed
>>> one = M.one_scalar_field()
>>> one.is_immutable()
True
>>> one.set_expr(x)
Traceback (most recent call last):
...
ValueError: the expressions of an immutable element cannot be
 changed
```

Other scalar fields can be declared immutable, too:

```
sage: c.is_immutable()
False
sage: c.set_immutable()
sage: c.is_immutable()
True
sage: c.set_expr(y^2)
Traceback (most recent call last):
...
ValueError: the expressions of an immutable element cannot be
```

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```
 changed
sage: c.set_name('b')
Traceback (most recent call last):
...
ValueError: the name of an immutable element cannot be changed
```

```
>>> from sage.all import *
>>> c.is_immutable()
False
>>> c.set_immutable()
>>> c.is_immutable()
True
>>> c.set_expr(y**Integer(2))
Traceback (most recent call last):
...
ValueError: the expressions of an immutable element cannot be
 changed
>>> c.set_name('b')
Traceback (most recent call last):
...
ValueError: the name of an immutable element cannot be changed
```

Immutable elements are hashable and can therefore be used as keys for dictionaries:

```
sage: {c: 1}[c]
1
```

```
>>> from sage.all import *
>>> {c: Integer(1)}[c]
1
```

By definition, a scalar field acts on the manifold's points, sending them to elements of the manifold's base field (real numbers in the present case):

```
sage: N = M.point((0,0), chart=c_uv) # the North pole
sage: S = M.point((0,0), chart=c_xy) # the South pole
sage: E = M.point((1,0), chart=c_xy) # a point at the equator
sage: f(N)
0
sage: f(S)
1
sage: f(E)
1/2
sage: h(E)
H(1, 0)
sage: c(E)
a
sage: zer(E)
0
```

```
>>> from sage.all import *
>>> N = M.point((Integer(0),Integer(0)), chart=c_uv) # the North pole
```

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```
>>> S = M.point((Integer(0),Integer(0)), chart=c_xy) # the South pole
>>> E = M.point((Integer(1),Integer(0)), chart=c_xy) # a point at the equator
>>> f(N)
0
>>> f(S)
1
>>> f(E)
1/2
>>> h(E)
H(1, 0)
>>> c(E)
a
>>> zer(E)
0
```

A scalar field can be compared to another scalar field:

```
sage: f == g
False
```

```
>>> from sage.all import *
>>> f == g
False
```

…to a symbolic expression:

```
sage: f == x*y
False
sage: g == x*y
True
sage: c == a
True
```

```
>>> from sage.all import *
>>> f == x*y
False
>>> g == x*y
True
>>> c == a
True
```

…to a number:

```
sage: f == 2
False
sage: zer == 0
True
```

```
>>> from sage.all import *
>>> f == Integer(2)
False
```

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```
>>> zer == Integer(0)
True
```

…to anything else:

```
sage: f == M
False
```

```
>>> from sage.all import *
>>> f == M
False
```

Standard mathematical functions are implemented:

```
sage: sqrt(f)
Scalar field sqrt(f) on the 2-dimensional topological manifold M
sage: sqrt(f).display()
sqrt(f): M → ℝ
on U: (x, y) ↦ 1/sqrt(x^2 + y^2 + 1)
on V: (u, v) ↦ sqrt(u^2 + v^2)/sqrt(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> sqrt(f)
Scalar field sqrt(f) on the 2-dimensional topological manifold M
>>> sqrt(f).display()
sqrt(f): M → ℝ
on U: (x, y) ↦ 1/sqrt(x^2 + y^2 + 1)
on V: (u, v) ↦ sqrt(u^2 + v^2)/sqrt(u^2 + v^2 + 1)
```

```
sage: tan(f)
Scalar field tan(f) on the 2-dimensional topological manifold M
sage: tan(f).display()
tan(f): M → ℝ
on U: (x, y) ↦ sin(1/(x^2 + y^2 + 1))/cos(1/(x^2 + y^2 + 1))
on V: (u, v) ↦ sin((u^2 + v^2)/(u^2 + v^2 + 1))/cos((u^2 + v^2)/(u^2 + v^2 + 1))
```

```
>>> from sage.all import *
>>> tan(f)
Scalar field tan(f) on the 2-dimensional topological manifold M
>>> tan(f).display()
tan(f): M → ℝ
on U: (x, y) ↦ sin(1/(x^2 + y^2 + 1))/cos(1/(x^2 + y^2 + 1))
on V: (u, v) ↦ sin((u^2 + v^2)/(u^2 + v^2 + 1))/cos((u^2 + v^2)/(u^2 + v^2 + 1))
```

Arithmetics of scalar fields

Scalar fields on M (resp. U) belong to the algebra $C^0(M)$ (resp. $C^0(U)$):

```
sage: f.parent()
Algebra of scalar fields on the 2-dimensional topological manifold M
sage: f.parent() is M.scalar_field_algebra()
```

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```
True
sage: g.parent()
Algebra of scalar fields on the Open subset U of the 2-dimensional
 topological manifold M
sage: g.parent() is U.scalar_field_algebra()
True
```

```
>>> from sage.all import *
>>> f.parent()
Algebra of scalar fields on the 2-dimensional topological manifold M
>>> f.parent() is M.scalar_field_algebra()
True
>>> g.parent()
Algebra of scalar fields on the Open subset U of the 2-dimensional
 topological manifold M
>>> g.parent() is U.scalar_field_algebra()
True
```

Consequently, scalar fields can be added:

```
sage: s = f + c ; s
Scalar field f+c on the 2-dimensional topological manifold M
sage: s.display()
f+c: M → ℝ
on U: (x, y) ↦ (a*x^2 + a*y^2 + a + 1)/(x^2 + y^2 + 1)
on V: (u, v) ↦ ((a + 1)*u^2 + (a + 1)*v^2 + a)/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> s = f + c ; s
Scalar field f+c on the 2-dimensional topological manifold M
>>> s.display()
f+c: M → ℝ
on U: (x, y) ↦ (a*x^2 + a*y^2 + a + 1)/(x^2 + y^2 + 1)
on V: (u, v) ↦ ((a + 1)*u^2 + (a + 1)*v^2 + a)/(u^2 + v^2 + 1)
```

and subtracted:

```
sage: s = f - c ; s
Scalar field f-c on the 2-dimensional topological manifold M
sage: s.display()
f-c: M → ℝ
on U: (x, y) ↦ -(a*x^2 + a*y^2 + a - 1)/(x^2 + y^2 + 1)
on V: (u, v) ↦ -((a - 1)*u^2 + (a - 1)*v^2 + a)/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> s = f - c ; s
Scalar field f-c on the 2-dimensional topological manifold M
>>> s.display()
f-c: M → ℝ
on U: (x, y) ↦ -(a*x^2 + a*y^2 + a - 1)/(x^2 + y^2 + 1)
on V: (u, v) ↦ -((a - 1)*u^2 + (a - 1)*v^2 + a)/(u^2 + v^2 + 1)
```

Some tests:

```
sage: f + zer == f
True
sage: f - f == zer
True
sage: f + (-f) == zer
True
sage: (f+c)-f == c
True
sage: (f-c)+c == f
True
```

```
>>> from sage.all import *
>>> f + zer == f
True
>>> f - f == zer
True
>>> f + (-f) == zer
True
>>> (f+c)-f == c
True
>>> (f-c)+c == f
True
```

We may add a number (interpreted as a constant scalar field) to a scalar field:

```
sage: s = f + 1 ; s
Scalar field f+1 on the 2-dimensional topological manifold M
sage: s.display()
f+1: M → ℝ
on U: (x, y) ↦ (x^2 + y^2 + 2)/(x^2 + y^2 + 1)
on V: (u, v) ↦ (2*u^2 + 2*v^2 + 1)/(u^2 + v^2 + 1)
sage: (f+1)-1 == f
True
```

```
>>> from sage.all import *
>>> s = f + Integer(1) ; s
Scalar field f+1 on the 2-dimensional topological manifold M
>>> s.display()
f+1: M → ℝ
on U: (x, y) ↦ (x^2 + y^2 + 2)/(x^2 + y^2 + 1)
on V: (u, v) ↦ (2*u^2 + 2*v^2 + 1)/(u^2 + v^2 + 1)
>>> (f+Integer(1))-Integer(1) == f
True
```

The number can represented by a symbolic variable:

```
sage: s = a + f ; s
Scalar field on the 2-dimensional topological manifold M
sage: s == c + f
True
```

```
>>> from sage.all import *
>>> s = a + f ; s
Scalar field on the 2-dimensional topological manifold M
>>> s == c + f
True
```

However if the symbolic variable is a chart coordinate, the addition is performed only on the chart domain:

```
sage: s = f + x; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M → ℝ
on U: (x, y) ↦ (x^3 + x*y^2 + x + 1)/(x^2 + y^2 + 1)
on W: (u, v) ↦ (u^4 + v^4 + u^3 + (2*u^2 + u)*v^2 + u)/(u^4 + v^4 + (2*u^2 + 1)*v^
↪2 + u^2)
sage: s = f + u; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M → ℝ
on W: (x, y) ↦ (x^3 + (x + 1)*y^2 + x^2 + x)/(x^4 + y^4 + (2*x^2 + 1)*y^2 + x^2)
on V: (u, v) ↦ (u^3 + (u + 1)*v^2 + u^2 + u)/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> s = f + x; s
Scalar field on the 2-dimensional topological manifold M
>>> s.display()
M → ℝ
on U: (x, y) ↦ (x^3 + x*y^2 + x + 1)/(x^2 + y^2 + 1)
on W: (u, v) ↦ (u^4 + v^4 + u^3 + (2*u^2 + u)*v^2 + u)/(u^4 + v^4 + (2*u^2 + 1)*v^
↪2 + u^2)
>>> s = f + u; s
Scalar field on the 2-dimensional topological manifold M
>>> s.display()
M → ℝ
on W: (x, y) ↦ (x^3 + (x + 1)*y^2 + x^2 + x)/(x^4 + y^4 + (2*x^2 + 1)*y^2 + x^2)
on V: (u, v) ↦ (u^3 + (u + 1)*v^2 + u^2 + u)/(u^2 + v^2 + 1)
```

The addition of two scalar fields with different domains is possible if the domain of one of them is a subset of the domain of the other; the domain of the result is then this subset:

```
sage: f.domain()
2-dimensional topological manifold M
sage: g.domain()
Open subset U of the 2-dimensional topological manifold M
sage: s = f + g ; s
Scalar field f+g on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.domain()
Open subset U of the 2-dimensional topological manifold M
sage: s.display()
f+g: U → ℝ
   (x, y) ↦ (x*y^3 + (x^3 + x)*y + 1)/(x^2 + y^2 + 1)
on W: (u, v) ↦ (u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6 + u*v^3
```

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```
 + (u^3 + u)*v)/(u^6 + v^6 + (3*u^2 + 1)*v^4 + u^4 + (3*u^4 + 2*u^2)*v^2)
```

```
>>> from sage.all import *
>>> f.domain()
2-dimensional topological manifold M
>>> g.domain()
Open subset U of the 2-dimensional topological manifold M
>>> s = f + g ; s
Scalar field f+g on the Open subset U of the 2-dimensional topological
 manifold M
>>> s.domain()
Open subset U of the 2-dimensional topological manifold M
>>> s.display()
f+g: U → ℝ
   (x, y) ↦ (x*y^3 + (x^3 + x)*y + 1)/(x^2 + y^2 + 1)
on W: (u, v) ↦ (u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6 + u*v^3
 + (u^3 + u)*v)/(u^6 + v^6 + (3*u^2 + 1)*v^4 + u^4 + (3*u^4 + 2*u^2)*v^2)
```

The operation actually performed is $f|_U + g$:

```
sage: s == f.restrict(U) + g
True
```

```
>>> from sage.all import *
>>> s == f.restrict(U) + g
True
```

In Sage framework, the addition of f and g is permitted because there is a *coercion* of the parent of f, namely $C^0(M)$, to the parent of g, namely $C^0(U)$ (see `ScalarFieldAlgebra`):

```
sage: CM = M.scalar_field_algebra()
sage: CU = U.scalar_field_algebra()
sage: CU.has_coerce_map_from(CM)
True
```

```
>>> from sage.all import *
>>> CM = M.scalar_field_algebra()
>>> CU = U.scalar_field_algebra()
>>> CU.has_coerce_map_from(CM)
True
```

The coercion map is nothing but the restriction to domain U:

```
sage: CU.coerce(f) == f.restrict(U)
True
```

```
>>> from sage.all import *
>>> CU.coerce(f) == f.restrict(U)
True
```

Since the algebra $C^0(M)$ is a vector space over **R**, scalar fields can be multiplied by a number, either an explicit one:

```
sage: s = 2*f ; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M → ℝ
on U: (x, y) ↦ 2/(x^2 + y^2 + 1)
on V: (u, v) ↦ 2*(u^2 + v^2)/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> s = Integer(2)*f ; s
Scalar field on the 2-dimensional topological manifold M
>>> s.display()
M → ℝ
on U: (x, y) ↦ 2/(x^2 + y^2 + 1)
on V: (u, v) ↦ 2*(u^2 + v^2)/(u^2 + v^2 + 1)
```

or a symbolic one:

```
sage: s = a*f ; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M → ℝ
on U: (x, y) ↦ a/(x^2 + y^2 + 1)
on V: (u, v) ↦ (u^2 + v^2)*a/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> s = a*f ; s
Scalar field on the 2-dimensional topological manifold M
>>> s.display()
M → ℝ
on U: (x, y) ↦ a/(x^2 + y^2 + 1)
on V: (u, v) ↦ (u^2 + v^2)*a/(u^2 + v^2 + 1)
```

However, if the symbolic variable is a chart coordinate, the multiplication is performed only in the corresponding chart:

```
sage: s = x*f; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M → ℝ
on U: (x, y) ↦ x/(x^2 + y^2 + 1)
on W: (u, v) ↦ u/(u^2 + v^2 + 1)
sage: s = u*f; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M → ℝ
on W: (x, y) ↦ x/(x^4 + y^4 + (2*x^2 + 1)*y^2 + x^2)
on V: (u, v) ↦ (u^2 + v^2)*u/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> s = x*f; s
Scalar field on the 2-dimensional topological manifold M
>>> s.display()
```

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```
M → ℝ
on U: (x, y) ↦ x/(x^2 + y^2 + 1)
on W: (u, v) ↦ u/(u^2 + v^2 + 1)
>>> s = u*f; s
Scalar field on the 2-dimensional topological manifold M
>>> s.display()
M → ℝ
on W: (x, y) ↦ x/(x^4 + y^4 + (2*x^2 + 1)*y^2 + x^2)
on V: (u, v) ↦ (u^2 + v^2)*u/(u^2 + v^2 + 1)
```

Some tests:

```
sage: 0*f == 0
True
sage: 0*f == zer
True
sage: 1*f == f
True
sage: (-2)*f == - f - f
True
```

```
>>> from sage.all import *
>>> Integer(0)*f == Integer(0)
True
>>> Integer(0)*f == zer
True
>>> Integer(1)*f == f
True
>>> (-Integer(2))*f == - f - f
True
```

The ring multiplication of the algebras $C^0(M)$ and $C^0(U)$ is the pointwise multiplication of functions:

```
sage: s = f*f ; s
Scalar field f*f on the 2-dimensional topological manifold M
sage: s.display()
f*f: M → ℝ
on U: (x, y) ↦ 1/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1)
on V: (u, v) ↦ (u^4 + 2*u^2*v^2 + v^4)/(u^4 + v^4 + 2*(u^2 + 1)*v^2
 + 2*u^2 + 1)
sage: s = g*h ; s
Scalar field g*h on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
g*h: U → ℝ
   (x, y) ↦ x*y*H(x, y)
on W: (u, v) ↦ u*v*H(u/(u^2 + v^2), v/(u^2 + v^2))/(u^4 + 2*u^2*v^2 + v^4)
```

```
>>> from sage.all import *
>>> s = f*f ; s
Scalar field f*f on the 2-dimensional topological manifold M
>>> s.display()
```

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```
f*f: M → ℝ
on U: (x, y) ↦ 1/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1)
on V: (u, v) ↦ (u^4 + 2*u^2*v^2 + v^4)/(u^4 + v^4 + 2*(u^2 + 1)*v^2
 + 2*u^2 + 1)
>>> s = g*h ; s
Scalar field g*h on the Open subset U of the 2-dimensional topological
 manifold M
>>> s.display()
g*h: U → ℝ
   (x, y) ↦ x*y*H(x, y)
on W: (u, v) ↦ u*v*H(u/(u^2 + v^2), v/(u^2 + v^2))/(u^4 + 2*u^2*v^2 + v^4)
```

Thanks to the coercion $C^0(M) \to C^0(U)$ mentioned above, it is possible to multiply a scalar field defined on M by a scalar field defined on U, the result being a scalar field defined on U:

```
sage: f.domain(), g.domain()
(2-dimensional topological manifold M,
 Open subset U of the 2-dimensional topological manifold M)
sage: s = f*g ; s
Scalar field f*g on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
f*g: U → ℝ
   (x, y) ↦ x*y/(x^2 + y^2 + 1)
on W: (u, v) ↦ u*v/(u^4 + v^4 + (2*u^2 + 1)*v^2 + u^2)
sage: s == f.restrict(U)*g
True
```

```
>>> from sage.all import *
>>> f.domain(), g.domain()
(2-dimensional topological manifold M,
 Open subset U of the 2-dimensional topological manifold M)
>>> s = f*g ; s
Scalar field f*g on the Open subset U of the 2-dimensional topological
 manifold M
>>> s.display()
f*g: U → ℝ
   (x, y) ↦ x*y/(x^2 + y^2 + 1)
on W: (u, v) ↦ u*v/(u^4 + v^4 + (2*u^2 + 1)*v^2 + u^2)
>>> s == f.restrict(U)*g
True
```

Scalar fields can be divided (pointwise division):

```
sage: s = f/c ; s
Scalar field f/c on the 2-dimensional topological manifold M
sage: s.display()
f/c: M → ℝ
on U: (x, y) ↦ 1/(a*x^2 + a*y^2 + a)
on V: (u, v) ↦ (u^2 + v^2)/(a*u^2 + a*v^2 + a)
sage: s = g/h ; s
Scalar field g/h on the Open subset U of the 2-dimensional topological
```

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```
 manifold M
sage: s.display()
g/h: U → ℝ
   (x, y) ↦ x*y/H(x, y)
on W: (u, v) ↦ u*v/((u^4 + 2*u^2*v^2 + v^4)*H(u/(u^2 + v^2), v/(u^2 + v^2)))
sage: s = f/g ; s
Scalar field f/g on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
f/g: U → ℝ
   (x, y) ↦ 1/(x*y^3 + (x^3 + x)*y)
on W: (u, v) ↦ (u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6)/(u*v^3 + (u^3 + u)*v)
sage: s == f.restrict(U)/g
True
```

```
>>> from sage.all import *
>>> s = f/c ; s
Scalar field f/c on the 2-dimensional topological manifold M
>>> s.display()
f/c: M → ℝ
on U: (x, y) ↦ 1/(a*x^2 + a*y^2 + a)
on V: (u, v) ↦ (u^2 + v^2)/(a*u^2 + a*v^2 + a)
>>> s = g/h ; s
Scalar field g/h on the Open subset U of the 2-dimensional topological
 manifold M
>>> s.display()
g/h: U → ℝ
   (x, y) ↦ x*y/H(x, y)
on W: (u, v) ↦ u*v/((u^4 + 2*u^2*v^2 + v^4)*H(u/(u^2 + v^2), v/(u^2 + v^2)))
>>> s = f/g ; s
Scalar field f/g on the Open subset U of the 2-dimensional topological
 manifold M
>>> s.display()
f/g: U → ℝ
   (x, y) ↦ 1/(x*y^3 + (x^3 + x)*y)
on W: (u, v) ↦ (u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6)/(u*v^3 + (u^3 + u)*v)
>>> s == f.restrict(U)/g
True
```

For scalar fields defined on a single chart domain, we may perform some arithmetics with symbolic expressions involving the chart coordinates:

```
sage: s = g + x^2 - y ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
U → ℝ
(x, y) ↦ x^2 + (x - 1)*y
on W: (u, v) ↦ -(v^3 - u^2 + (u^2 - u)*v)/(u^4 + 2*u^2*v^2 + v^4)
```

```
>>> from sage.all import *
>>> s = g + x**Integer(2) - y ; s
```

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```
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
>>> s.display()
U → ℝ
(x, y) ↦ x^2 + (x - 1)*y
on W: (u, v) ↦ -(v^3 - u^2 + (u^2 - u)*v)/(u^4 + 2*u^2*v^2 + v^4)
```

```
sage: s = g*x ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
U → ℝ
(x, y) ↦ x^2*y
on W: (u, v) ↦ u^2*v/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6)
```

```
>>> from sage.all import *
>>> s = g*x ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
>>> s.display()
U → ℝ
(x, y) ↦ x^2*y
on W: (u, v) ↦ u^2*v/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6)
```

```
sage: s = g/x ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
U → ℝ
(x, y) ↦ y
on W: (u, v) ↦ v/(u^2 + v^2)
sage: s = x/g ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
U → ℝ
(x, y) ↦ 1/y
on W: (u, v) ↦ (u^2 + v^2)/v
```

```
>>> from sage.all import *
>>> s = g/x ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
>>> s.display()
U → ℝ
(x, y) ↦ y
on W: (u, v) ↦ v/(u^2 + v^2)
>>> s = x/g ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
>>> s.display()
```

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```
U → ℝ
(x, y) ↦ 1/y
on W: (u, v) ↦ (u^2 + v^2)/v
```

Examples with SymPy as the symbolic engine

From now on, we ask that all symbolic calculus on manifold M are performed by SymPy:

```
sage: M.set_calculus_method('sympy')
```

```
>>> from sage.all import *
>>> M.set_calculus_method('sympy')
```

We define f as above:

```
sage: f = M.scalar_field({c_xy: 1/(1+x^2+y^2), c_uv: (u^2+v^2)/(1+u^2+v^2)},
....:                    name='f') ; f
Scalar field f on the 2-dimensional topological manifold M
sage: f.display()  # notice the SymPy display of exponents
f: M → ℝ
on U: (x, y) ↦ 1/(x**2 + y**2 + 1)
on V: (u, v) ↦ (u**2 + v**2)/(u**2 + v**2 + 1)
sage: type(f.coord_function(c_xy).expr())
<class 'sympy.core.power.Pow'>
```

```
>>> from sage.all import *
>>> f = M.scalar_field({c_xy: Integer(1)/(Integer(1)+x**Integer(2)+y**Integer(2)),
↪ c_uv: (u**Integer(2)+v**Integer(2))/(Integer(1)+u**Integer(2)+v**Integer(2))},
...                    name='f') ; f
Scalar field f on the 2-dimensional topological manifold M
>>> f.display()  # notice the SymPy display of exponents
f: M → ℝ
on U: (x, y) ↦ 1/(x**2 + y**2 + 1)
on V: (u, v) ↦ (u**2 + v**2)/(u**2 + v**2 + 1)
>>> type(f.coord_function(c_xy).expr())
<class 'sympy.core.power.Pow'>
```

The scalar field g defined on U:

```
sage: g = U.scalar_field({c_xy: x*y}, name='g')
sage: g.display() # again notice the SymPy display of exponents
g: U → ℝ
   (x, y) ↦ x*y
on W: (u, v) ↦ u*v/(u**4 + 2*u**2*v**2 + v**4)
```

```
>>> from sage.all import *
>>> g = U.scalar_field({c_xy: x*y}, name='g')
>>> g.display() # again notice the SymPy display of exponents
g: U → ℝ
   (x, y) ↦ x*y
on W: (u, v) ↦ u*v/(u**4 + 2*u**2*v**2 + v**4)
```

Definition on a single chart and subsequent completion:

```
sage: f = M.scalar_field(1/(1+x^2+y^2), chart=c_xy, name='f')
sage: f.add_expr((u^2+v^2)/(1+u^2+v^2), chart=c_uv)
sage: f.display()
f: M → ℝ
on U: (x, y) ↦ 1/(x**2 + y**2 + 1)
on V: (u, v) ↦ (u**2 + v**2)/(u**2 + v**2 + 1)
```

```
>>> from sage.all import *
>>> f = M.scalar_field(Integer(1)/(Integer(1)+x**Integer(2)+y**Integer(2)),␣
↪chart=c_xy, name='f')
>>> f.add_expr((u**Integer(2)+v**Integer(2))/
↪(Integer(1)+u**Integer(2)+v**Integer(2)), chart=c_uv)
>>> f.display()
f: M → ℝ
on U: (x, y) ↦ 1/(x**2 + y**2 + 1)
on V: (u, v) ↦ (u**2 + v**2)/(u**2 + v**2 + 1)
```

Definition without any coordinate expression and subsequent completion:

```
sage: f = M.scalar_field(name='f')
sage: f.add_expr(1/(1+x^2+y^2), chart=c_xy)
sage: f.add_expr((u^2+v^2)/(1+u^2+v^2), chart=c_uv)
sage: f.display()
f: M → ℝ
on U: (x, y) ↦ 1/(x**2 + y**2 + 1)
on V: (u, v) ↦ (u**2 + v**2)/(u**2 + v**2 + 1)
```

```
>>> from sage.all import *
>>> f = M.scalar_field(name='f')
>>> f.add_expr(Integer(1)/(Integer(1)+x**Integer(2)+y**Integer(2)), chart=c_xy)
>>> f.add_expr((u**Integer(2)+v**Integer(2))/
↪(Integer(1)+u**Integer(2)+v**Integer(2)), chart=c_uv)
>>> f.display()
f: M → ℝ
on U: (x, y) ↦ 1/(x**2 + y**2 + 1)
on V: (u, v) ↦ (u**2 + v**2)/(u**2 + v**2 + 1)
```

Use of `add_expr_by_continuation()`:

```
sage: f = M.scalar_field(1/(1+x^2+y^2), chart=c_xy, name='f')
sage: f.add_expr_by_continuation(c_uv, U.intersection(V))
sage: f.display()
f: M → ℝ
on U: (x, y) ↦ 1/(x**2 + y**2 + 1)
on V: (u, v) ↦ (u**2 + v**2)/(u**2 + v**2 + 1)
```

```
>>> from sage.all import *
>>> f = M.scalar_field(Integer(1)/(Integer(1)+x**Integer(2)+y**Integer(2)),␣
↪chart=c_xy, name='f')
>>> f.add_expr_by_continuation(c_uv, U.intersection(V))
>>> f.display()
```

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```
f: M → ℝ
on U: (x, y) ↦ 1/(x**2 + y**2 + 1)
on V: (u, v) ↦ (u**2 + v**2)/(u**2 + v**2 + 1)
```

A scalar field defined by some unspecified function of the coordinates:

```
sage: h = U.scalar_field(function('H')(x, y), name='h') ; h
Scalar field h on the Open subset U of the 2-dimensional topological
 manifold M
sage: h.display()
h: U → ℝ
   (x, y) ↦ H(x, y)
on W: (u, v) ↦ H(u/(u**2 + v**2), v/(u**2 + v**2))
```

```
>>> from sage.all import *
>>> h = U.scalar_field(function('H')(x, y), name='h') ; h
Scalar field h on the Open subset U of the 2-dimensional topological
 manifold M
>>> h.display()
h: U → ℝ
   (x, y) ↦ H(x, y)
on W: (u, v) ↦ H(u/(u**2 + v**2), v/(u**2 + v**2))
```

The coordinate expression in a given chart is obtained via the method *expr()*, which in the present context, returns a SymPy object:

```
sage: f.expr(c_uv)
(u**2 + v**2)/(u**2 + v**2 + 1)
sage: type(f.expr(c_uv))
<class 'sympy.core.mul.Mul'>
```

```
>>> from sage.all import *
>>> f.expr(c_uv)
(u**2 + v**2)/(u**2 + v**2 + 1)
>>> type(f.expr(c_uv))
<class 'sympy.core.mul.Mul'>
```

The method *coord_function()* returns instead a function of the chart coordinates, i.e. an instance of *ChartFunction*:

```
sage: f.coord_function(c_uv)
(u**2 + v**2)/(u**2 + v**2 + 1)
sage: type(f.coord_function(c_uv))
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_class'>
sage: f.coord_function(c_uv).display()
(u, v) ↦ (u**2 + v**2)/(u**2 + v**2 + 1)
```

```
>>> from sage.all import *
>>> f.coord_function(c_uv)
(u**2 + v**2)/(u**2 + v**2 + 1)
>>> type(f.coord_function(c_uv))
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_class'>
```

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```
>>> f.coord_function(c_uv).display()
(u, v) ↦ (u**2 + v**2)/(u**2 + v**2 + 1)
```

The value returned by the method *expr()* is actually the coordinate expression of the chart function:

```
sage: f.expr(c_uv) is f.coord_function(c_uv).expr()
True
```

```
>>> from sage.all import *
>>> f.expr(c_uv) is f.coord_function(c_uv).expr()
True
```

We may ask for the `SR` representation of the coordinate function:

```
sage: f.coord_function(c_uv).expr('SR')
(u^2 + v^2)/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> f.coord_function(c_uv).expr('SR')
(u^2 + v^2)/(u^2 + v^2 + 1)
```

A constant scalar field with SymPy representation:

```
sage: c = M.constant_scalar_field(2, name='c')
sage: c.display()
c: M → ℝ
on U: (x, y) ↦ 2
on V: (u, v) ↦ 2
sage: type(c.expr(c_xy))
<class 'sympy.core.numbers.Integer'>
```

```
>>> from sage.all import *
>>> c = M.constant_scalar_field(Integer(2), name='c')
>>> c.display()
c: M → ℝ
on U: (x, y) ↦ 2
on V: (u, v) ↦ 2
>>> type(c.expr(c_xy))
<class 'sympy.core.numbers.Integer'>
```

The constant value can be some unspecified parameter:

```
sage: var('a')
a
sage: c = M.constant_scalar_field(a, name='c')
sage: c.display()
c: M → ℝ
on U: (x, y) ↦ a
on V: (u, v) ↦ a
sage: type(c.expr(c_xy))
<class 'sympy.core.symbol.Symbol'>
```

```
>>> from sage.all import *
>>> var('a')
a
>>> c = M.constant_scalar_field(a, name='c')
>>> c.display()
c: M → ℝ
on U: (x, y) ↦ a
on V: (u, v) ↦ a
>>> type(c.expr(c_xy))
<class 'sympy.core.symbol.Symbol'>
```

The zero scalar field:

```
sage: zer = M.constant_scalar_field(0) ; zer
Scalar field zero on the 2-dimensional topological manifold M
sage: zer.display()
zero: M → ℝ
on U: (x, y) ↦ 0
on V: (u, v) ↦ 0
sage: type(zer.expr(c_xy))
<class 'sympy.core.numbers.Zero'>
sage: zer is M.zero_scalar_field()
True
```

```
>>> from sage.all import *
>>> zer = M.constant_scalar_field(Integer(0)) ; zer
Scalar field zero on the 2-dimensional topological manifold M
>>> zer.display()
zero: M → ℝ
on U: (x, y) ↦ 0
on V: (u, v) ↦ 0
>>> type(zer.expr(c_xy))
<class 'sympy.core.numbers.Zero'>
>>> zer is M.zero_scalar_field()
True
```

Action of scalar fields on manifold's points:

```
sage: N = M.point((0,0), chart=c_uv) # the North pole
sage: S = M.point((0,0), chart=c_xy) # the South pole
sage: E = M.point((1,0), chart=c_xy) # a point at the equator
sage: f(N)
0
sage: f(S)
1
sage: f(E)
1/2
sage: h(E)
H(1, 0)
sage: c(E)
a
sage: zer(E)
0
```

```
>>> from sage.all import *
>>> N = M.point((Integer(0),Integer(0)), chart=c_uv) # the North pole
>>> S = M.point((Integer(0),Integer(0)), chart=c_xy) # the South pole
>>> E = M.point((Integer(1),Integer(0)), chart=c_xy) # a point at the equator
>>> f(N)
0
>>> f(S)
1
>>> f(E)
1/2
>>> h(E)
H(1, 0)
>>> c(E)
a
>>> zer(E)
0
```

A scalar field can be compared to another scalar field:

```
sage: f == g
False
```

```
>>> from sage.all import *
>>> f == g
False
```

…to a symbolic expression:

```
sage: f == x*y
False
sage: g == x*y
True
sage: c == a
True
```

```
>>> from sage.all import *
>>> f == x*y
False
>>> g == x*y
True
>>> c == a
True
```

…to a number:

```
sage: f == 2
False
sage: zer == 0
True
```

```
>>> from sage.all import *
>>> f == Integer(2)
```

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```
False
>>> zer == Integer(0)
True
```

…to anything else:

```
sage: f == M
False
```

```
>>> from sage.all import *
>>> f == M
False
```

Standard mathematical functions are implemented:

```
sage: sqrt(f)
Scalar field sqrt(f) on the 2-dimensional topological manifold M
sage: sqrt(f).display()
sqrt(f): M → ℝ
on U: (x, y) ↦ 1/sqrt(x**2 + y**2 + 1)
on V: (u, v) ↦ sqrt(u**2 + v**2)/sqrt(u**2 + v**2 + 1)
```

```
>>> from sage.all import *
>>> sqrt(f)
Scalar field sqrt(f) on the 2-dimensional topological manifold M
>>> sqrt(f).display()
sqrt(f): M → ℝ
on U: (x, y) ↦ 1/sqrt(x**2 + y**2 + 1)
on V: (u, v) ↦ sqrt(u**2 + v**2)/sqrt(u**2 + v**2 + 1)
```

```
sage: tan(f)
Scalar field tan(f) on the 2-dimensional topological manifold M
sage: tan(f).display()
tan(f): M → ℝ
on U: (x, y) ↦ tan(1/(x**2 + y**2 + 1))
on V: (u, v) ↦ tan((u**2 + v**2)/(u**2 + v**2 + 1))
```

```
>>> from sage.all import *
>>> tan(f)
Scalar field tan(f) on the 2-dimensional topological manifold M
>>> tan(f).display()
tan(f): M → ℝ
on U: (x, y) ↦ tan(1/(x**2 + y**2 + 1))
on V: (u, v) ↦ tan((u**2 + v**2)/(u**2 + v**2 + 1))
```

Arithmetics of scalar fields with SymPy

Scalar fields on M (resp. U) belong to the algebra $C^0(M)$ (resp. $C^0(U)$):

```
sage: f.parent()
Algebra of scalar fields on the 2-dimensional topological manifold M
```

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```
sage: f.parent() is M.scalar_field_algebra()
True
sage: g.parent()
Algebra of scalar fields on the Open subset U of the 2-dimensional
 topological manifold M
sage: g.parent() is U.scalar_field_algebra()
True
```

```
>>> from sage.all import *
>>> f.parent()
Algebra of scalar fields on the 2-dimensional topological manifold M
>>> f.parent() is M.scalar_field_algebra()
True
>>> g.parent()
Algebra of scalar fields on the Open subset U of the 2-dimensional
 topological manifold M
>>> g.parent() is U.scalar_field_algebra()
True
```

Consequently, scalar fields can be added:

```
sage: s = f + c ; s
Scalar field f+c on the 2-dimensional topological manifold M
sage: s.display()
f+c: M → ℝ
on U: (x, y) ↦ (a*x**2 + a*y**2 + a + 1)/(x**2 + y**2 + 1)
on V: (u, v) ↦ (a*u**2 + a*v**2 + a + u**2 + v**2)/(u**2 + v**2 + 1)
```

```
>>> from sage.all import *
>>> s = f + c ; s
Scalar field f+c on the 2-dimensional topological manifold M
>>> s.display()
f+c: M → ℝ
on U: (x, y) ↦ (a*x**2 + a*y**2 + a + 1)/(x**2 + y**2 + 1)
on V: (u, v) ↦ (a*u**2 + a*v**2 + a + u**2 + v**2)/(u**2 + v**2 + 1)
```

and subtracted:

```
sage: s = f - c ; s
Scalar field f-c on the 2-dimensional topological manifold M
sage: s.display()
f-c: M → ℝ
on U: (x, y) ↦ (-a*x**2 - a*y**2 - a + 1)/(x**2 + y**2 + 1)
on V: (u, v) ↦ (-a*u**2 - a*v**2 - a + u**2 + v**2)/(u**2 + v**2 + 1)
```

```
>>> from sage.all import *
>>> s = f - c ; s
Scalar field f-c on the 2-dimensional topological manifold M
>>> s.display()
f-c: M → ℝ
on U: (x, y) ↦ (-a*x**2 - a*y**2 - a + 1)/(x**2 + y**2 + 1)
on V: (u, v) ↦ (-a*u**2 - a*v**2 - a + u**2 + v**2)/(u**2 + v**2 + 1)
```

Some tests:

```
sage: f + zer == f
True
sage: f - f == zer
True
sage: f + (-f) == zer
True
sage: (f+c)-f == c
True
sage: (f-c)+c == f
True
```

```
>>> from sage.all import *
>>> f + zer == f
True
>>> f - f == zer
True
>>> f + (-f) == zer
True
>>> (f+c)-f == c
True
>>> (f-c)+c == f
True
```

We may add a number (interpreted as a constant scalar field) to a scalar field:

```
sage: s = f + 1 ; s
Scalar field f+1 on the 2-dimensional topological manifold M
sage: s.display()
f+1: M → ℝ
on U: (x, y) ↦ (x**2 + y**2 + 2)/(x**2 + y**2 + 1)
on V: (u, v) ↦ (2*u**2 + 2*v**2 + 1)/(u**2 + v**2 + 1)
sage: (f+1)-1 == f
True
```

```
>>> from sage.all import *
>>> s = f + Integer(1) ; s
Scalar field f+1 on the 2-dimensional topological manifold M
>>> s.display()
f+1: M → ℝ
on U: (x, y) ↦ (x**2 + y**2 + 2)/(x**2 + y**2 + 1)
on V: (u, v) ↦ (2*u**2 + 2*v**2 + 1)/(u**2 + v**2 + 1)
>>> (f+Integer(1))-Integer(1) == f
True
```

The number can represented by a symbolic variable:

```
sage: s = a + f ; s
Scalar field on the 2-dimensional topological manifold M
sage: s == c + f
True
```

```
>>> from sage.all import *
>>> s = a + f ; s
Scalar field on the 2-dimensional topological manifold M
>>> s == c + f
True
```

However if the symbolic variable is a chart coordinate, the addition is performed only on the chart domain:

```
sage: s = f + x; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M → ℝ
on U: (x, y) ↦ (x**3 + x*y**2 + x + 1)/(x**2 + y**2 + 1)
on W: (u, v) ↦ (u**4 + u**3 + 2*u**2*v**2 + u*v**2 + u + v**4)/(u**4 +␣
↪2*u**2*v**2 + u**2 + v**4 + v**2)
sage: s = f + u; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M → ℝ
on W: (x, y) ↦ (x**3 + x**2 + x*y**2 + x + y**2)/(x**4 + 2*x**2*y**2 + x**2 +␣
↪y**4 + y**2)
on V: (u, v) ↦ (u**3 + u**2 + u*v**2 + u + v**2)/(u**2 + v**2 + 1)
```

```
>>> from sage.all import *
>>> s = f + x; s
Scalar field on the 2-dimensional topological manifold M
>>> s.display()
M → ℝ
on U: (x, y) ↦ (x**3 + x*y**2 + x + 1)/(x**2 + y**2 + 1)
on W: (u, v) ↦ (u**4 + u**3 + 2*u**2*v**2 + u*v**2 + u + v**4)/(u**4 +␣
↪2*u**2*v**2 + u**2 + v**4 + v**2)
>>> s = f + u; s
Scalar field on the 2-dimensional topological manifold M
>>> s.display()
M → ℝ
on W: (x, y) ↦ (x**3 + x**2 + x*y**2 + x + y**2)/(x**4 + 2*x**2*y**2 + x**2 +␣
↪y**4 + y**2)
on V: (u, v) ↦ (u**3 + u**2 + u*v**2 + u + v**2)/(u**2 + v**2 + 1)
```

The addition of two scalar fields with different domains is possible if the domain of one of them is a subset of the domain of the other; the domain of the result is then this subset:

```
sage: f.domain()
2-dimensional topological manifold M
sage: g.domain()
Open subset U of the 2-dimensional topological manifold M
sage: s = f + g ; s
Scalar field f+g on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.domain()
Open subset U of the 2-dimensional topological manifold M
sage: s.display()
f+g: U → ℝ
```

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```
   (x, y) ↦ (x**3*y + x*y**3 + x*y + 1)/(x**2 + y**2 + 1)
on W: (u, v) ↦ (u**6 + 3*u**4*v**2 + u**3*v + 3*u**2*v**4 + u*v**3 + u*v + v**6)/
↪(u**6 + 3*u**4*v**2 + u**4 + 3*u**2*v**4 + 2*u**2*v**2 + v**6 + v**4)
```

```
>>> from sage.all import *
>>> f.domain()
2-dimensional topological manifold M
>>> g.domain()
Open subset U of the 2-dimensional topological manifold M
>>> s = f + g ; s
Scalar field f+g on the Open subset U of the 2-dimensional topological
 manifold M
>>> s.domain()
Open subset U of the 2-dimensional topological manifold M
>>> s.display()
f+g: U → ℝ
   (x, y) ↦ (x**3*y + x*y**3 + x*y + 1)/(x**2 + y**2 + 1)
on W: (u, v) ↦ (u**6 + 3*u**4*v**2 + u**3*v + 3*u**2*v**4 + u*v**3 + u*v + v**6)/
↪(u**6 + 3*u**4*v**2 + u**4 + 3*u**2*v**4 + 2*u**2*v**2 + v**6 + v**4)
```

The operation actually performed is $f|_U + g$:

```
sage: s == f.restrict(U) + g
True
```

```
>>> from sage.all import *
>>> s == f.restrict(U) + g
True
```

Since the algebra $C^0(M)$ is a vector space over **R**, scalar fields can be multiplied by a number, either an explicit one:

```
sage: s = 2*f ; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M → ℝ
on U: (x, y) ↦ 2/(x**2 + y**2 + 1)
on V: (u, v) ↦ 2*(u**2 + v**2)/(u**2 + v**2 + 1)
```

```
>>> from sage.all import *
>>> s = Integer(2)*f ; s
Scalar field on the 2-dimensional topological manifold M
>>> s.display()
M → ℝ
on U: (x, y) ↦ 2/(x**2 + y**2 + 1)
on V: (u, v) ↦ 2*(u**2 + v**2)/(u**2 + v**2 + 1)
```

or a symbolic one:

```
sage: s = a*f ; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
```

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```
M → ℝ
on U: (x, y) ↦ a/(x**2 + y**2 + 1)
on V: (u, v) ↦ a*(u**2 + v**2)/(u**2 + v**2 + 1)
```

```
>>> from sage.all import *
>>> s = a*f ; s
Scalar field on the 2-dimensional topological manifold M
>>> s.display()
M → ℝ
on U: (x, y) ↦ a/(x**2 + y**2 + 1)
on V: (u, v) ↦ a*(u**2 + v**2)/(u**2 + v**2 + 1)
```

However, if the symbolic variable is a chart coordinate, the multiplication is performed only in the corresponding chart:

```
sage: s = x*f; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M → ℝ
on U: (x, y) ↦ x/(x**2 + y**2 + 1)
on W: (u, v) ↦ u/(u**2 + v**2 + 1)
sage: s = u*f; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M → ℝ
on W: (x, y) ↦ x/(x**4 + 2*x**2*y**2 + x**2 + y**4 + y**2)
on V: (u, v) ↦ u*(u**2 + v**2)/(u**2 + v**2 + 1)
```

```
>>> from sage.all import *
>>> s = x*f; s
Scalar field on the 2-dimensional topological manifold M
>>> s.display()
M → ℝ
on U: (x, y) ↦ x/(x**2 + y**2 + 1)
on W: (u, v) ↦ u/(u**2 + v**2 + 1)
>>> s = u*f; s
Scalar field on the 2-dimensional topological manifold M
>>> s.display()
M → ℝ
on W: (x, y) ↦ x/(x**4 + 2*x**2*y**2 + x**2 + y**4 + y**2)
on V: (u, v) ↦ u*(u**2 + v**2)/(u**2 + v**2 + 1)
```

Some tests:

```
sage: 0*f == 0
True
sage: 0*f == zer
True
sage: 1*f == f
True
sage: (-2)*f == - f - f
True
```

```
>>> from sage.all import *
>>> Integer(0)*f == Integer(0)
True
>>> Integer(0)*f == zer
True
>>> Integer(1)*f == f
True
>>> (-Integer(2))*f == - f - f
True
```

The ring multiplication of the algebras $C^0(M)$ and $C^0(U)$ is the pointwise multiplication of functions:

```
sage: s = f*f ; s
Scalar field f*f on the 2-dimensional topological manifold M
sage: s.display()
f*f: M → ℝ
on U: (x, y) ↦ 1/(x**4 + 2*x**2*y**2 + 2*x**2 + y**4 + 2*y**2 + 1)
on V: (u, v) ↦ (u**4 + 2*u**2*v**2 + v**4)/(u**4 + 2*u**2*v**2 + 2*u**2 + v**4 +
↪2*v**2 + 1)

sage: s = g*h ; s
Scalar field g*h on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
g*h: U → ℝ
   (x, y) ↦ x*y*H(x, y)
on W: (u, v) ↦ u*v*H(u/(u**2 + v**2), v/(u**2 + v**2))/(u**4 + 2*u**2*v**2 + v**4)
```

```
>>> from sage.all import *
>>> s = f*f ; s
Scalar field f*f on the 2-dimensional topological manifold M
>>> s.display()
f*f: M → ℝ
on U: (x, y) ↦ 1/(x**4 + 2*x**2*y**2 + 2*x**2 + y**4 + 2*y**2 + 1)
on V: (u, v) ↦ (u**4 + 2*u**2*v**2 + v**4)/(u**4 + 2*u**2*v**2 + 2*u**2 + v**4 +
↪2*v**2 + 1)

>>> s = g*h ; s
Scalar field g*h on the Open subset U of the 2-dimensional topological
 manifold M
>>> s.display()
g*h: U → ℝ
   (x, y) ↦ x*y*H(x, y)
on W: (u, v) ↦ u*v*H(u/(u**2 + v**2), v/(u**2 + v**2))/(u**4 + 2*u**2*v**2 + v**4)
```

Thanks to the coercion $C^0(M) \to C^0(U)$ mentioned above, it is possible to multiply a scalar field defined on M by a scalar field defined on U, the result being a scalar field defined on U:

```
sage: f.domain(), g.domain()
(2-dimensional topological manifold M,
 Open subset U of the 2-dimensional topological manifold M)
sage: s = f*g ; s
Scalar field f*g on the Open subset U of the 2-dimensional topological
```

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```
 manifold M
sage: s.display()
f*g: U → ℝ
   (x, y) ↦ x*y/(x**2 + y**2 + 1)
on W: (u, v) ↦ u*v/(u**4 + 2*u**2*v**2 + u**2 + v**4 + v**2)

sage: s == f.restrict(U)*g
True
```

```
>>> from sage.all import *
>>> f.domain(), g.domain()
(2-dimensional topological manifold M,
 Open subset U of the 2-dimensional topological manifold M)
>>> s = f*g ; s
Scalar field f*g on the Open subset U of the 2-dimensional topological
 manifold M
>>> s.display()
f*g: U → ℝ
   (x, y) ↦ x*y/(x**2 + y**2 + 1)
on W: (u, v) ↦ u*v/(u**4 + 2*u**2*v**2 + u**2 + v**4 + v**2)

>>> s == f.restrict(U)*g
True
```

Scalar fields can be divided (pointwise division):

```
sage: s = f/c ; s
Scalar field f/c on the 2-dimensional topological manifold M
sage: s.display()
f/c: M → ℝ
on U: (x, y) ↦ 1/(a*(x**2 + y**2 + 1))
on V: (u, v) ↦ (u**2 + v**2)/(a*(u**2 + v**2 + 1))
sage: s = g/h ; s
Scalar field g/h on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
g/h: U → ℝ
   (x, y) ↦ x*y/H(x, y)
on W: (u, v) ↦ u*v/((u**4 + 2*u**2*v**2 + v**4)*H(u/(u**2 + v**2), v/(u**2 +
↪v**2)))

sage: s = f/g ; s
Scalar field f/g on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
f/g: U → ℝ
   (x, y) ↦ 1/(x*y*(x**2 + y**2 + 1))
on W: (u, v) ↦ (u**6 + 3*u**4*v**2 + 3*u**2*v**4 + v**6)/(u*v*(u**2 + v**2 + 1))
sage: s == f.restrict(U)/g
True
```

```
>>> from sage.all import *
>>> s = f/c ; s
Scalar field f/c on the 2-dimensional topological manifold M
>>> s.display()
f/c: M → ℝ
on U: (x, y) ↦ 1/(a*(x**2 + y**2 + 1))
on V: (u, v) ↦ (u**2 + v**2)/(a*(u**2 + v**2 + 1))
>>> s = g/h ; s
Scalar field g/h on the Open subset U of the 2-dimensional topological
 manifold M
>>> s.display()
g/h: U → ℝ
   (x, y) ↦ x*y/H(x, y)
on W: (u, v) ↦ u*v/((u**4 + 2*u**2*v**2 + v**4)*H(u/(u**2 + v**2), v/(u**2 +␣
↪v**2)))

>>> s = f/g ; s
Scalar field f/g on the Open subset U of the 2-dimensional topological
 manifold M
>>> s.display()
f/g: U → ℝ
   (x, y) ↦ 1/(x*y*(x**2 + y**2 + 1))
on W: (u, v) ↦ (u**6 + 3*u**4*v**2 + 3*u**2*v**4 + v**6)/(u*v*(u**2 + v**2 + 1))
>>> s == f.restrict(U)/g
True
```

For scalar fields defined on a single chart domain, we may perform some arithmetics with symbolic expressions involving the chart coordinates:

```
sage: s = g + x^2 - y ; s
Scalar field on the Open subset U of the 2-dimensional topological manifold M
sage: s.display()
U → ℝ
(x, y) ↦ x**2 + x*y - y
on W: (u, v) ↦ (-u**2*v + u**2 + u*v - v**3)/(u**4 + 2*u**2*v**2 + v**4)
```

```
>>> from sage.all import *
>>> s = g + x**Integer(2) - y ; s
Scalar field on the Open subset U of the 2-dimensional topological manifold M
>>> s.display()
U → ℝ
(x, y) ↦ x**2 + x*y - y
on W: (u, v) ↦ (-u**2*v + u**2 + u*v - v**3)/(u**4 + 2*u**2*v**2 + v**4)
```

```
sage: s = g*x ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
U → ℝ
(x, y) ↦ x**2*y
on W: (u, v) ↦ u**2*v/(u**6 + 3*u**4*v**2 + 3*u**2*v**4 + v**6)
```

```
>>> from sage.all import *
>>> s = g*x ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
>>> s.display()
U → ℝ
(x, y) ↦ x**2*y
on W: (u, v) ↦ u**2*v/(u**6 + 3*u**4*v**2 + 3*u**2*v**4 + v**6)
```

```
sage: s = g/x ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
U → ℝ
(x, y) ↦ y
on W: (u, v) ↦ v/(u**2 + v**2)
sage: s = x/g ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
U → ℝ
(x, y) ↦ 1/y
on W: (u, v) ↦ u**2/v + v
```

```
>>> from sage.all import *
>>> s = g/x ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
>>> s.display()
U → ℝ
(x, y) ↦ y
on W: (u, v) ↦ v/(u**2 + v**2)
>>> s = x/g ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
>>> s.display()
U → ℝ
(x, y) ↦ 1/y
on W: (u, v) ↦ u**2/v + v
```

The test suite is passed:

```
sage: TestSuite(f).run()
sage: TestSuite(zer).run()
```

```
>>> from sage.all import *
>>> TestSuite(f).run()
>>> TestSuite(zer).run()
```

add_expr(*coord_expression*, *chart=None*)

Add some coordinate expression to the scalar field.

The previous expressions with respect to other charts are kept. To clear them, use `set_expr()` instead.

INPUT:

- `coord_expression` – coordinate expression of the scalar field
- `chart` – (default: `None`) chart in which `coord_expression` is defined; if `None`, the default chart of the scalar field's domain is assumed

Warning

If the scalar field has already expressions in other charts, it is the user's responsibility to make sure that the expression to be added is consistent with them.

EXAMPLES:

Adding scalar field expressions on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(x^2 + 2*x*y +1)
sage: f._express
{Chart (M, (x, y)): x^2 + 2*x*y + 1}
sage: f.add_expr(3*y)
sage: f._express  # the (x,y) expression has been changed:
{Chart (M, (x, y)): 3*y}
sage: c_uv.<u,v> = M.chart()
sage: f.add_expr(cos(u)-sin(v), c_uv)
sage: f._express # random (dict. output); f has now 2 expressions:
{Chart (M, (x, y)): 3*y, Chart (M, (u, v)): cos(u) - sin(v)}
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> f = M.scalar_field(x**Integer(2) + Integer(2)*x*y +Integer(1))
>>> f._express
{Chart (M, (x, y)): x^2 + 2*x*y + 1}
>>> f.add_expr(Integer(3)*y)
>>> f._express  # the (x,y) expression has been changed:
{Chart (M, (x, y)): 3*y}
>>> c_uv = M.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> f.add_expr(cos(u)-sin(v), c_uv)
>>> f._express # random (dict. output); f has now 2 expressions:
{Chart (M, (x, y)): 3*y, Chart (M, (u, v)): cos(u) - sin(v)}
```

Since zero and one are special elements, their expressions cannot be changed:

```
sage: z = M.zero_scalar_field()
sage: z.add_expr(cos(u)-sin(v), c_uv)
Traceback (most recent call last):
...
ValueError: the expressions of an immutable element cannot be
 changed
sage: one = M.one_scalar_field()
sage: one.add_expr(cos(u)-sin(v), c_uv)
Traceback (most recent call last):
```

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```
...
ValueError: the expressions of an immutable element cannot be
 changed
```

```
>>> from sage.all import *
>>> z = M.zero_scalar_field()
>>> z.add_expr(cos(u)-sin(v), c_uv)
Traceback (most recent call last):
...
ValueError: the expressions of an immutable element cannot be
 changed
>>> one = M.one_scalar_field()
>>> one.add_expr(cos(u)-sin(v), c_uv)
Traceback (most recent call last):
...
ValueError: the expressions of an immutable element cannot be
 changed
```

add_expr_by_continuation(*chart*, *subdomain*)

Set coordinate expression in a chart by continuation of the coordinate expression in a subchart.

The continuation is performed by demanding that the coordinate expression is identical to that in the restriction of the chart to a given subdomain.

INPUT:

- `chart` – coordinate chart $(U,(x^i))$ in which the expression of the scalar field is to set
- `subdomain` – open subset $V \subset U$ in which the expression in terms of the restriction of the coordinate chart $(U,(x^i))$ to V is already known or can be evaluated by a change of coordinates.

EXAMPLES:

Scalar field on the sphere S^2:

```
sage: M = Manifold(2, 'S^2', structure='topological')
sage: U = M.open_subset('U') ; V = M.open_subset('V') # the complement of␣
↪resp. N pole and S pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart() # stereographic␣
↪coordinates
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W =  U.intersection(V)  # S^2 minus the two poles
sage: f = M.scalar_field(atan(x^2+y^2), chart=c_xy, name='f')
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', structure='topological')
>>> U = M.open_subset('U') ; V = M.open_subset('V') # the complement of resp.␣
↪N pole and S pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
```

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```
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)# stereographic␣
↪coordinates
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                 intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                 restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W =  U.intersection(V)  # S^2 minus the two poles
>>> f = M.scalar_field(atan(x**Integer(2)+y**Integer(2)), chart=c_xy, name='f
↪')
```

The scalar field has been defined only on the domain covered by the chart `c_xy`, i.e. U:

```
sage: f.display()
f: S^2 → ℝ
on U: (x, y) ↦ arctan(x^2 + y^2)
on W: (u, v) ↦ arctan(1/(u^2 + v^2))
```

```
>>> from sage.all import *
>>> f.display()
f: S^2 → ℝ
on U: (x, y) ↦ arctan(x^2 + y^2)
on W: (u, v) ↦ arctan(1/(u^2 + v^2))
```

We note that on $W = U \cap V$, the expression of f in terms of coordinates (u, v) can be deduced from that in the coordinates (x, y) thanks to the transition map between the two charts:

```
sage: f.display(c_uv.restrict(W))
f: S^2 → ℝ
on W: (u, v) ↦ arctan(1/(u^2 + v^2))
```

```
>>> from sage.all import *
>>> f.display(c_uv.restrict(W))
f: S^2 → ℝ
on W: (u, v) ↦ arctan(1/(u^2 + v^2))
```

We use this fact to extend the definition of f to the open subset V, covered by the chart `c_uv`:

```
sage: f.add_expr_by_continuation(c_uv, W)
```

```
>>> from sage.all import *
>>> f.add_expr_by_continuation(c_uv, W)
```

Then, f is known on the whole sphere:

```
sage: f.display()
f: S^2 → ℝ
on U: (x, y) ↦ arctan(x^2 + y^2)
on V: (u, v) ↦ arctan(1/(u^2 + v^2))
```

```
>>> from sage.all import *
>>> f.display()
f: S^2 → ℝ
on U: (x, y) ↦ arctan(x^2 + y^2)
on V: (u, v) ↦ arctan(1/(u^2 + v^2))
```

arccos()

Arc cosine of the scalar field.

OUTPUT: the scalar field arccos f, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
sage: g = arccos(f) ; g
Scalar field arccos(f) on the 2-dimensional topological manifold M
sage: latex(g)
\arccos\left(\Phi\right)
sage: g.display()
arccos(f): M → ℝ
   (x, y) ↦ arccos(x*y)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
>>> g = arccos(f) ; g
Scalar field arccos(f) on the 2-dimensional topological manifold M
>>> latex(g)
\arccos\left(\Phi\right)
>>> g.display()
arccos(f): M → ℝ
   (x, y) ↦ arccos(x*y)
```

The notation `acos` can be used as well:

```
sage: acos(f)
Scalar field arccos(f) on the 2-dimensional topological manifold M
sage: acos(f) == g
True
```

```
>>> from sage.all import *
>>> acos(f)
Scalar field arccos(f) on the 2-dimensional topological manifold M
>>> acos(f) == g
True
```

Some tests:

```
sage: cos(g) == f
True
```

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```
sage: arccos(M.constant_scalar_field(1)) == M.zero_scalar_field()
True
sage: arccos(M.zero_scalar_field()) == M.constant_scalar_field(pi/2)
True
```

```
>>> from sage.all import *
>>> cos(g) == f
True
>>> arccos(M.constant_scalar_field(Integer(1))) == M.zero_scalar_field()
True
>>> arccos(M.zero_scalar_field()) == M.constant_scalar_field(pi/Integer(2))
True
```

arccosh()

Inverse hyperbolic cosine of the scalar field.

OUTPUT:

- the scalar field $\operatorname{arccosh} f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
sage: g = arccosh(f) ; g
Scalar field arccosh(f) on the 2-dimensional topological manifold M
sage: latex(g)
\,\mathrm{arccosh}\left(\Phi\right)
sage: g.display()
arccosh(f): M → ℝ
   (x, y) ↦ arccosh(x*y)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
>>> g = arccosh(f) ; g
Scalar field arccosh(f) on the 2-dimensional topological manifold M
>>> latex(g)
\,\mathrm{arccosh}\left(\Phi\right)
>>> g.display()
arccosh(f): M → ℝ
   (x, y) ↦ arccosh(x*y)
```

The notation `acosh` can be used as well:

```
sage: acosh(f)
Scalar field arccosh(f) on the 2-dimensional topological manifold M
sage: acosh(f) == g
True
```

```
>>> from sage.all import *
>>> acosh(f)
Scalar field arccosh(f) on the 2-dimensional topological manifold M
>>> acosh(f) == g
True
```

Some tests:

```
sage: cosh(g) == f
True
sage: arccosh(M.constant_scalar_field(1)) == M.zero_scalar_field()
True
```

```
>>> from sage.all import *
>>> cosh(g) == f
True
>>> arccosh(M.constant_scalar_field(Integer(1))) == M.zero_scalar_field()
True
```

arcsin()

Arc sine of the scalar field.

OUTPUT: the scalar field $\arcsin f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
sage: g = arcsin(f) ; g
Scalar field arcsin(f) on the 2-dimensional topological manifold M
sage: latex(g)
\arcsin\left(\Phi\right)
sage: g.display()
arcsin(f): M → ℝ
   (x, y) ↦ arcsin(x*y)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
>>> g = arcsin(f) ; g
Scalar field arcsin(f) on the 2-dimensional topological manifold M
>>> latex(g)
\arcsin\left(\Phi\right)
>>> g.display()
arcsin(f): M → ℝ
   (x, y) ↦ arcsin(x*y)
```

The notation `asin` can be used as well:

```
sage: asin(f)
Scalar field arcsin(f) on the 2-dimensional topological manifold M
```

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```
sage: asin(f) == g
True
```

```
>>> from sage.all import *
>>> asin(f)
Scalar field arcsin(f) on the 2-dimensional topological manifold M
>>> asin(f) == g
True
```

Some tests:

```
sage: sin(g) == f
True
sage: arcsin(M.zero_scalar_field()) == M.zero_scalar_field()
True
sage: arcsin(M.constant_scalar_field(1)) == M.constant_scalar_field(pi/2)
True
```

```
>>> from sage.all import *
>>> sin(g) == f
True
>>> arcsin(M.zero_scalar_field()) == M.zero_scalar_field()
True
>>> arcsin(M.constant_scalar_field(Integer(1))) == M.constant_scalar_field(pi/
↪Integer(2))
True
```

arcsinh()

Inverse hyperbolic sine of the scalar field.

OUTPUT:

- the scalar field arcsinh f, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
sage: g = arcsinh(f) ; g
Scalar field arcsinh(f) on the 2-dimensional topological manifold M
sage: latex(g)
\,\mathrm{arcsinh}\left(\Phi\right)
sage: g.display()
arcsinh(f): M → ℝ
   (x, y) ↦ arcsinh(x*y)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
>>> g = arcsinh(f) ; g
Scalar field arcsinh(f) on the 2-dimensional topological manifold M
```

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```
>>> latex(g)
\,\mathrm{arcsinh}\left(\Phi\right)
>>> g.display()
arcsinh(f): M → ℝ
   (x, y) ↦ arcsinh(x*y)
```

The notation `asinh` can be used as well:

```
sage: asinh(f)
Scalar field arcsinh(f) on the 2-dimensional topological manifold M
sage: asinh(f) == g
True
```

```
>>> from sage.all import *
>>> asinh(f)
Scalar field arcsinh(f) on the 2-dimensional topological manifold M
>>> asinh(f) == g
True
```

Some tests:

```
sage: sinh(g) == f
True
sage: arcsinh(M.zero_scalar_field()) == M.zero_scalar_field()
True
```

```
>>> from sage.all import *
>>> sinh(g) == f
True
>>> arcsinh(M.zero_scalar_field()) == M.zero_scalar_field()
True
```

arctan()

Arc tangent of the scalar field.

OUTPUT: the scalar field $\arctan f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
sage: g = arctan(f) ; g
Scalar field arctan(f) on the 2-dimensional topological manifold M
sage: latex(g)
\arctan\left(\Phi\right)
sage: g.display()
arctan(f): M → ℝ
   (x, y) ↦ arctan(x*y)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
```

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```
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
>>> g = arctan(f) ; g
Scalar field arctan(f) on the 2-dimensional topological manifold M
>>> latex(g)
\arctan\left(\Phi\right)
>>> g.display()
arctan(f): M → ℝ
   (x, y) ↦ arctan(x*y)
```

The notation `atan` can be used as well:

```
sage: atan(f)
Scalar field arctan(f) on the 2-dimensional topological manifold M
sage: atan(f) == g
True
```

```
>>> from sage.all import *
>>> atan(f)
Scalar field arctan(f) on the 2-dimensional topological manifold M
>>> atan(f) == g
True
```

Some tests:

```
sage: tan(g) == f
True
sage: arctan(M.zero_scalar_field()) == M.zero_scalar_field()
True
sage: arctan(M.constant_scalar_field(1)) == M.constant_scalar_field(pi/4)
True
```

```
>>> from sage.all import *
>>> tan(g) == f
True
>>> arctan(M.zero_scalar_field()) == M.zero_scalar_field()
True
>>> arctan(M.constant_scalar_field(Integer(1))) == M.constant_scalar_field(pi/
↪Integer(4))
True
```

arctanh()

Inverse hyperbolic tangent of the scalar field.

OUTPUT:

- the scalar field arctanh f, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
```

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```
sage: g = arctanh(f) ; g
Scalar field arctanh(f) on the 2-dimensional topological manifold M
sage: latex(g)
\,\mathrm{arctanh}\left(\Phi\right)
sage: g.display()
arctanh(f): M → ℝ
   (x, y) ↦ arctanh(x*y)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
>>> g = arctanh(f) ; g
Scalar field arctanh(f) on the 2-dimensional topological manifold M
>>> latex(g)
\,\mathrm{arctanh}\left(\Phi\right)
>>> g.display()
arctanh(f): M → ℝ
   (x, y) ↦ arctanh(x*y)
```

The notation `atanh` can be used as well:

```
sage: atanh(f)
Scalar field arctanh(f) on the 2-dimensional topological manifold M
sage: atanh(f) == g
True
```

```
>>> from sage.all import *
>>> atanh(f)
Scalar field arctanh(f) on the 2-dimensional topological manifold M
>>> atanh(f) == g
True
```

Some tests:

```
sage: tanh(g) == f
True
sage: arctanh(M.zero_scalar_field()) == M.zero_scalar_field()
True
sage: arctanh(M.constant_scalar_field(1/2)) == M.constant_scalar_field(log(3)/
↪2)
True
```

```
>>> from sage.all import *
>>> tanh(g) == f
True
>>> arctanh(M.zero_scalar_field()) == M.zero_scalar_field()
True
>>> arctanh(M.constant_scalar_field(Integer(1)/Integer(2))) == M.constant_
↪scalar_field(log(Integer(3))/Integer(2))
True
```

codomain()

Return the codomain of the scalar field.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(x+2*y)
sage: f.codomain()
Real Field with 53 bits of precision
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> f = M.scalar_field(x+Integer(2)*y)
>>> f.codomain()
Real Field with 53 bits of precision
```

common_charts(*other*)

Find common charts for the expressions of the scalar field and `other`.

INPUT:

- `other` – a scalar field

OUTPUT:

- list of common charts; if no common chart is found, `None` is returned (instead of an empty list)

EXAMPLES:

Search for common charts on a 2-dimensional manifold with 2 overlapping domains:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: U = M.open_subset('U')
sage: c_xy.<x,y> = U.chart()
sage: V = M.open_subset('V')
sage: c_uv.<u,v> = V.chart()
sage: M.declare_union(U,V)   # M is the union of U and V
sage: f = U.scalar_field(x^2)
sage: g = M.scalar_field(x+y)
sage: f.common_charts(g)
[Chart (U, (x, y))]
sage: g.add_expr(u, c_uv)
sage: f._express
{Chart (U, (x, y)): x^2}
sage: g._express  # random (dictionary output)
{Chart (U, (x, y)): x + y, Chart (V, (u, v)): u}
sage: f.common_charts(g)
[Chart (U, (x, y))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> U = M.open_subset('U')
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> V = M.open_subset('V')
```

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```
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> M.declare_union(U,V)   # M is the union of U and V
>>> f = U.scalar_field(x**Integer(2))
>>> g = M.scalar_field(x+y)
>>> f.common_charts(g)
[Chart (U, (x, y))]
>>> g.add_expr(u, c_uv)
>>> f._express
{Chart (U, (x, y)): x^2}
>>> g._express  # random (dictionary output)
{Chart (U, (x, y)): x + y, Chart (V, (u, v)): u}
>>> f.common_charts(g)
[Chart (U, (x, y))]
```

Common charts found as subcharts: the subcharts are introduced via a transition map between charts c_xy and c_uv on the intersecting subdomain $W = U \cap V$:

```
sage: trans = c_xy.transition_map(c_uv, (x+y, x-y), 'W', x<0, u+v<0)
sage: M.atlas()
[Chart (U, (x, y)), Chart (V, (u, v)), Chart (W, (x, y)),
 Chart (W, (u, v))]
sage: c_xy_W = M.atlas()[2]
sage: c_uv_W = M.atlas()[3]
sage: trans.inverse()
Change of coordinates from Chart (W, (u, v)) to Chart (W, (x, y))
sage: f.common_charts(g)
[Chart (U, (x, y))]
sage: f.expr(c_xy_W)
x^2
sage: f._express  # random (dictionary output)
{Chart (U, (x, y)): x^2, Chart (W, (x, y)): x^2}
sage: g._express  # random (dictionary output)
{Chart (U, (x, y)): x + y, Chart (V, (u, v)): u}
sage: g.common_charts(f)  # c_xy_W is not returned because it is subchart of
↪'xy'
[Chart (U, (x, y))]
sage: f.expr(c_uv_W)
1/4*u^2 + 1/2*u*v + 1/4*v^2
sage: f._express  # random (dictionary output)
{Chart (U, (x, y)): x^2, Chart (W, (x, y)): x^2,
 Chart (W, (u, v)): 1/4*u^2 + 1/2*u*v + 1/4*v^2}
sage: g._express  # random (dictionary output)
{Chart (U, (x, y)): x + y, Chart (V, (u, v)): u}
sage: f.common_charts(g)
[Chart (U, (x, y)), Chart (W, (u, v))]
sage: # the expressions have been updated on the subcharts
sage: g._express #  random (dictionary output)
{Chart (U, (x, y)): x + y, Chart (V, (u, v)): u,
 Chart (W, (u, v)): u}
```

```
>>> from sage.all import *
>>> trans = c_xy.transition_map(c_uv, (x+y, x-y), 'W', x<Integer(0), u+v
```

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```
↪<Integer(0))
>>> M.atlas()
[Chart (U, (x, y)), Chart (V, (u, v)), Chart (W, (x, y)),
 Chart (W, (u, v))]
>>> c_xy_W = M.atlas()[Integer(2)]
>>> c_uv_W = M.atlas()[Integer(3)]
>>> trans.inverse()
Change of coordinates from Chart (W, (u, v)) to Chart (W, (x, y))
>>> f.common_charts(g)
[Chart (U, (x, y))]
>>> f.expr(c_xy_W)
x^2
>>> f._express  # random (dictionary output)
{Chart (U, (x, y)): x^2, Chart (W, (x, y)): x^2}
>>> g._express  # random (dictionary output)
{Chart (U, (x, y)): x + y, Chart (V, (u, v)): u}
>>> g.common_charts(f)  # c_xy_W is not returned because it is subchart of 'xy
↪'
[Chart (U, (x, y))]
>>> f.expr(c_uv_W)
1/4*u^2 + 1/2*u*v + 1/4*v^2
>>> f._express  # random (dictionary output)
{Chart (U, (x, y)): x^2, Chart (W, (x, y)): x^2,
 Chart (W, (u, v)): 1/4*u^2 + 1/2*u*v + 1/4*v^2}
>>> g._express  # random (dictionary output)
{Chart (U, (x, y)): x + y, Chart (V, (u, v)): u}
>>> f.common_charts(g)
[Chart (U, (x, y)), Chart (W, (u, v))]
>>> # the expressions have been updated on the subcharts
>>> g._express #  random (dictionary output)
{Chart (U, (x, y)): x + y, Chart (V, (u, v)): u,
 Chart (W, (u, v)): u}
```

Common charts found by computing some coordinate changes:

```
sage: W = U.intersection(V)
sage: f = W.scalar_field(x^2, c_xy_W)
sage: g = W.scalar_field(u+1, c_uv_W)
sage: f._express
{Chart (W, (x, y)): x^2}
sage: g._express
{Chart (W, (u, v)): u + 1}
sage: f.common_charts(g)
[Chart (W, (x, y)), Chart (W, (u, v))]
sage: f._express # random (dictionary output)
{Chart (W, (u, v)): 1/4*u^2 + 1/2*u*v + 1/4*v^2,
 Chart (W, (x, y)): x^2}
sage: g._express # random (dictionary output)
{Chart (W, (u, v)): u + 1, Chart (W, (x, y)): x + y + 1}
```

```
>>> from sage.all import *
>>> W = U.intersection(V)
```

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```
>>> f = W.scalar_field(x**Integer(2), c_xy_W)
>>> g = W.scalar_field(u+Integer(1), c_uv_W)
>>> f._express
{Chart (W, (x, y)): x^2}
>>> g._express
{Chart (W, (u, v)): u + 1}
>>> f.common_charts(g)
[Chart (W, (x, y)), Chart (W, (u, v))]
>>> f._express # random (dictionary output)
{Chart (W, (u, v)): 1/4*u^2 + 1/2*u*v + 1/4*v^2,
 Chart (W, (x, y)): x^2}
>>> g._express # random (dictionary output)
{Chart (W, (u, v)): u + 1, Chart (W, (x, y)): x + y + 1}
```

coord_function(*chart=None*, *from_chart=None*)

Return the function of the coordinates representing the scalar field in a given chart.

INPUT:

- `chart` – (default: `None`) chart with respect to which the coordinate expression is to be returned; if `None`, the default chart of the scalar field's domain will be used
- `from_chart` – (default: `None`) chart from which the required expression is computed if it is not known already in the chart `chart`; if `None`, a chart is picked in the known expressions

OUTPUT:

- instance of `ChartFunction` representing the coordinate function of the scalar field in the given chart

EXAMPLES:

Coordinate function on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(x*y^2)
sage: f.coord_function()
x*y^2
sage: f.coord_function(c_xy)  # equivalent form (since c_xy is the default␣
↪chart)
x*y^2
sage: type(f.coord_function())
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> f = M.scalar_field(x*y**Integer(2))
>>> f.coord_function()
x*y^2
>>> f.coord_function(c_xy)  # equivalent form (since c_xy is the default␣
↪chart)
x*y^2
>>> type(f.coord_function())
```

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```
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>
```

Expression via a change of coordinates:

```
sage: c_uv.<u,v> = M.chart()
sage: c_uv.transition_map(c_xy, [u+v, u-v])
Change of coordinates from Chart (M, (u, v)) to Chart (M, (x, y))
sage: f._express # at this stage, f is expressed only in terms of (x,y)␣
↪coordinates
{Chart (M, (x, y)): x*y^2}
sage: f.coord_function(c_uv) # forces the computation of the expression of f␣
↪in terms of (u,v) coordinates
u^3 - u^2*v - u*v^2 + v^3
sage: f.coord_function(c_uv) == (u+v)*(u-v)^2  # check
True
sage: f._express  # random (dict. output); f has now 2 coordinate expressions:
{Chart (M, (x, y)): x*y^2, Chart (M, (u, v)): u^3 - u^2*v - u*v^2 + v^3}
```

```
>>> from sage.all import *
>>> c_uv = M.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> c_uv.transition_map(c_xy, [u+v, u-v])
Change of coordinates from Chart (M, (u, v)) to Chart (M, (x, y))
>>> f._express # at this stage, f is expressed only in terms of (x,y)␣
↪coordinates
{Chart (M, (x, y)): x*y^2}
>>> f.coord_function(c_uv) # forces the computation of the expression of f in␣
↪terms of (u,v) coordinates
u^3 - u^2*v - u*v^2 + v^3
>>> f.coord_function(c_uv) == (u+v)*(u-v)**Integer(2)  # check
True
>>> f._express  # random (dict. output); f has now 2 coordinate expressions:
{Chart (M, (x, y)): x*y^2, Chart (M, (u, v)): u^3 - u^2*v - u*v^2 + v^3}
```

Usage in a physical context (simple Lorentz transformation - boost in `x` direction, with relative velocity `v` between `o1` and `o2` frames):

```
sage: M = Manifold(2, 'M', structure='topological')
sage: o1.<t,x> = M.chart()
sage: o2.<T,X> = M.chart()
sage: f = M.scalar_field(x^2 - t^2)
sage: f.coord_function(o1)
-t^2 + x^2
sage: v = var('v'); gam = 1/sqrt(1-v^2)
sage: o2.transition_map(o1, [gam*(T - v*X), gam*(X - v*T)])
Change of coordinates from Chart (M, (T, X)) to Chart (M, (t, x))
sage: f.coord_function(o2)
-T^2 + X^2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> o1 = M.chart(names=('t', 'x',)); (t, x,) = o1._first_ngens(2)
```

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```
>>> o2 = M.chart(names=('T', 'X',)); (T, X,) = o2._first_ngens(2)
>>> f = M.scalar_field(x**Integer(2) - t**Integer(2))
>>> f.coord_function(o1)
-t^2 + x^2
>>> v = var('v'); gam = Integer(1)/sqrt(Integer(1)-v**Integer(2))
>>> o2.transition_map(o1, [gam*(T - v*X), gam*(X - v*T)])
Change of coordinates from Chart (M, (T, X)) to Chart (M, (t, x))
>>> f.coord_function(o2)
-T^2 + X^2
```

copy(*name=None*, *latex_name=None*)

Return an exact copy of the scalar field.

INPUT:

- `name` – (default: `None`) name given to the copy
- `latex_name` – (default: `None`) LaTeX symbol to denote the copy; if none is provided, the LaTeX symbol is set to `name`

EXAMPLES:

Copy on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(x*y^2)
sage: g = f.copy()
sage: type(g)
<class 'sage.manifolds.scalarfield_algebra.ScalarFieldAlgebra_with_category.
↪element_class'>
sage: g.expr()
x*y^2
sage: g == f
True
sage: g is f
False
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> f = M.scalar_field(x*y**Integer(2))
>>> g = f.copy()
>>> type(g)
<class 'sage.manifolds.scalarfield_algebra.ScalarFieldAlgebra_with_category.
↪element_class'>
>>> g.expr()
x*y^2
>>> g == f
True
>>> g is f
False
```

copy_from(*other*)

Make `self` a copy of `other`.

INPUT:

- `other` – other scalar field, in the same module as `self`

Note

While the derived quantities are not copied, the name is kept.

Warning

All previous defined expressions and restrictions will be deleted!

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(x*y^2, name='f')
sage: f.display()
f: M → ℝ
   (x, y) ↦ x*y^2
sage: g = M.scalar_field(name='g')
sage: g.copy_from(f)
sage: g.display()
g: M → ℝ
   (x, y) ↦ x*y^2
sage: f == g
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> f = M.scalar_field(x*y**Integer(2), name='f')
>>> f.display()
f: M → ℝ
   (x, y) ↦ x*y^2
>>> g = M.scalar_field(name='g')
>>> g.copy_from(f)
>>> g.display()
g: M → ℝ
   (x, y) ↦ x*y^2
>>> f == g
True
```

While the original scalar field is modified, the copy is not:

```
sage: f.set_expr(x-y)
sage: f.display()
f: M → ℝ
   (x, y) ↦ x - y
sage: g.display()
g: M → ℝ
```

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```
    (x, y) ↦ x*y^2
sage: f == g
False
```

```
>>> from sage.all import *
>>> f.set_expr(x-y)
>>> f.display()
f: M → ℝ
    (x, y) ↦ x - y
>>> g.display()
g: M → ℝ
    (x, y) ↦ x*y^2
>>> f == g
False
```

cos()

Cosine of the scalar field.

OUTPUT: the scalar field $\cos f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
sage: g = cos(f) ; g
Scalar field cos(f) on the 2-dimensional topological manifold M
sage: latex(g)
\cos\left(\Phi\right)
sage: g.display()
cos(f): M → ℝ
    (x, y) ↦ cos(x*y)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
>>> g = cos(f) ; g
Scalar field cos(f) on the 2-dimensional topological manifold M
>>> latex(g)
\cos\left(\Phi\right)
>>> g.display()
cos(f): M → ℝ
    (x, y) ↦ cos(x*y)
```

Some tests:

```
sage: cos(M.zero_scalar_field()) == M.constant_scalar_field(1)
True
sage: cos(M.constant_scalar_field(pi/2)) == M.zero_scalar_field()
True
```

```
>>> from sage.all import *
>>> cos(M.zero_scalar_field()) == M.constant_scalar_field(Integer(1))
True
>>> cos(M.constant_scalar_field(pi/Integer(2))) == M.zero_scalar_field()
True
```

cosh()

Hyperbolic cosine of the scalar field.

OUTPUT: the scalar field $\cosh f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
sage: g = cosh(f) ; g
Scalar field cosh(f) on the 2-dimensional topological manifold M
sage: latex(g)
\cosh\left(\Phi\right)
sage: g.display()
cosh(f): M → ℝ
   (x, y) ↦ cosh(x*y)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
>>> g = cosh(f) ; g
Scalar field cosh(f) on the 2-dimensional topological manifold M
>>> latex(g)
\cosh\left(\Phi\right)
>>> g.display()
cosh(f): M → ℝ
   (x, y) ↦ cosh(x*y)
```

Some test:

```
sage: cosh(M.zero_scalar_field()) == M.constant_scalar_field(1)
True
```

```
>>> from sage.all import *
>>> cosh(M.zero_scalar_field()) == M.constant_scalar_field(Integer(1))
True
```

disp(*chart=None*)

Display the expression of the scalar field in a given chart.

Without any argument, this function displays all known, distinct expressions.

INPUT:

- `chart` – (default: `None`) chart with respect to which the coordinate expression is to be displayed; if `None`, the display is performed in all the greatest charts in which the coordinate expression is known

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

EXAMPLES:

Various displays:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(sqrt(x+1), name='f')
sage: f.display()
f: M → ℝ
   (x, y) ↦ sqrt(x + 1)
sage: latex(f.display())
\begin{array}{llcl} f:& M & \longrightarrow & \mathbb{R} \\ & \left(x, y\
↪right) & \longmapsto & \sqrt{x + 1} \end{array}
sage: g = M.scalar_field(function('G')(x, y), name='g')
sage: g.display()
g: M → ℝ
   (x, y) ↦ G(x, y)
sage: latex(g.display())
\begin{array}{llcl} g:& M & \longrightarrow & \mathbb{R} \\ & \left(x, y\
↪right) & \longmapsto & G\left(x, y\right) \end{array}
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> f = M.scalar_field(sqrt(x+Integer(1)), name='f')
>>> f.display()
f: M → ℝ
   (x, y) ↦ sqrt(x + 1)
>>> latex(f.display())
\begin{array}{llcl} f:& M & \longrightarrow & \mathbb{R} \\ & \left(x, y\
↪right) & \longmapsto & \sqrt{x + 1} \end{array}
>>> g = M.scalar_field(function('G')(x, y), name='g')
>>> g.display()
g: M → ℝ
   (x, y) ↦ G(x, y)
>>> latex(g.display())
\begin{array}{llcl} g:& M & \longrightarrow & \mathbb{R} \\ & \left(x, y\
↪right) & \longmapsto & G\left(x, y\right) \end{array}
```

A shortcut of `display()` is `disp()`:

```
sage: f.disp()
f: M → ℝ
   (x, y) ↦ sqrt(x + 1)
```

```
>>> from sage.all import *
>>> f.disp()
f: M → ℝ
   (x, y) ↦ sqrt(x + 1)
```

In case the scalar field is piecewise-defined, the `display()` command still outputs all expressions. Each expression displayed corresponds to the chart on the greatest domain where this particular expression is known:

```
sage: U = M.open_subset('U')
sage: f.set_expr(y^2, c_xy.restrict(U))
sage: f.display()
f: M → ℝ
on U: (x, y) ↦ y^2
sage: latex(f.display())
\begin{array}{llcl} f:& M & \longrightarrow & \mathbb{R} \\ \text{on}\ U : & \
↪left(x, y\right) & \longmapsto & y^{2} \end{array}
```

```
>>> from sage.all import *
>>> U = M.open_subset('U')
>>> f.set_expr(y**Integer(2), c_xy.restrict(U))
>>> f.display()
f: M → ℝ
on U: (x, y) ↦ y^2
>>> latex(f.display())
\begin{array}{llcl} f:& M & \longrightarrow & \mathbb{R} \\ \text{on}\ U : & \
↪left(x, y\right) & \longmapsto & y^{2} \end{array}
```

display(*chart=None*)

Display the expression of the scalar field in a given chart.

Without any argument, this function displays all known, distinct expressions.

INPUT:

- `chart` – (default: `None`) chart with respect to which the coordinate expression is to be displayed; if `None`, the display is performed in all the greatest charts in which the coordinate expression is known

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

EXAMPLES:

Various displays:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(sqrt(x+1), name='f')
sage: f.display()
f: M → ℝ
   (x, y) ↦ sqrt(x + 1)
sage: latex(f.display())
\begin{array}{llcl} f:& M & \longrightarrow & \mathbb{R} \\ & \left(x, y\
↪right) & \longmapsto & \sqrt{x + 1} \end{array}
sage: g = M.scalar_field(function('G')(x, y), name='g')
sage: g.display()
g: M → ℝ
   (x, y) ↦ G(x, y)
sage: latex(g.display())
\begin{array}{llcl} g:& M & \longrightarrow & \mathbb{R} \\ & \left(x, y\
↪right) & \longmapsto & G\left(x, y\right) \end{array}
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
```

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```
>>> f = M.scalar_field(sqrt(x+Integer(1)), name='f')
>>> f.display()
f: M → ℝ
   (x, y) ↦ sqrt(x + 1)
>>> latex(f.display())
\begin{array}{llcl} f:& M & \longrightarrow & \mathbb{R} \\ & \left(x, y\
↪right) & \longmapsto & \sqrt{x + 1} \end{array}
>>> g = M.scalar_field(function('G')(x, y), name='g')
>>> g.display()
g: M → ℝ
   (x, y) ↦ G(x, y)
>>> latex(g.display())
\begin{array}{llcl} g:& M & \longrightarrow & \mathbb{R} \\ & \left(x, y\
↪right) & \longmapsto & G\left(x, y\right) \end{array}
```

A shortcut of `display()` is `disp()`:

```
sage: f.disp()
f: M → ℝ
   (x, y) ↦ sqrt(x + 1)
```

```
>>> from sage.all import *
>>> f.disp()
f: M → ℝ
   (x, y) ↦ sqrt(x + 1)
```

In case the scalar field is piecewise-defined, the `display()` command still outputs all expressions. Each expression displayed corresponds to the chart on the greatest domain where this particular expression is known:

```
sage: U = M.open_subset('U')
sage: f.set_expr(y^2, c_xy.restrict(U))
sage: f.display()
f: M → ℝ
on U: (x, y) ↦ y^2
sage: latex(f.display())
\begin{array}{llcl} f:& M & \longrightarrow & \mathbb{R} \\ \text{on}\ U : & \
↪left(x, y\right) & \longmapsto & y^{2} \end{array}
```

```
>>> from sage.all import *
>>> U = M.open_subset('U')
>>> f.set_expr(y**Integer(2), c_xy.restrict(U))
>>> f.display()
f: M → ℝ
on U: (x, y) ↦ y^2
>>> latex(f.display())
\begin{array}{llcl} f:& M & \longrightarrow & \mathbb{R} \\ \text{on}\ U : & \
↪left(x, y\right) & \longmapsto & y^{2} \end{array}
```

domain()

Return the open subset on which the scalar field is defined.

OUTPUT:

- instance of class `TopologicalManifold` representing the manifold's open subset on which the scalar field is defined

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(x+2*y)
sage: f.domain()
2-dimensional topological manifold M
sage: U = M.open_subset('U', coord_def={c_xy: x<0})
sage: g = f.restrict(U)
sage: g.domain()
Open subset U of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> f = M.scalar_field(x+Integer(2)*y)
>>> f.domain()
2-dimensional topological manifold M
>>> U = M.open_subset('U', coord_def={c_xy: x<Integer(0)})
>>> g = f.restrict(U)
>>> g.domain()
Open subset U of the 2-dimensional topological manifold M
```

exp()

Exponential of the scalar field.

OUTPUT: the scalar field $\exp f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x+y}, name='f', latex_name=r"\Phi")
sage: g = exp(f) ; g
Scalar field exp(f) on the 2-dimensional topological manifold M
sage: g.display()
exp(f): M → ℝ
   (x, y) ↦ e^(x + y)
sage: latex(g)
\exp\left(\Phi\right)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: x+y}, name='f', latex_name=r"\Phi")
>>> g = exp(f) ; g
Scalar field exp(f) on the 2-dimensional topological manifold M
>>> g.display()
exp(f): M → ℝ
   (x, y) ↦ e^(x + y)
>>> latex(g)
\exp\left(\Phi\right)
```

Automatic simplifications occur:

```
sage: f = M.scalar_field({X: 2*ln(1+x^2)}, name='f')
sage: exp(f).display()
exp(f): M → ℝ
   (x, y) ↦ x^4 + 2*x^2 + 1
```

```
>>> from sage.all import *
>>> f = M.scalar_field({X: Integer(2)*ln(Integer(1)+x**Integer(2))}, name='f')
>>> exp(f).display()
exp(f): M → ℝ
   (x, y) ↦ x^4 + 2*x^2 + 1
```

The inverse function is `log()`:

```
sage: log(exp(f)) == f
True
```

```
>>> from sage.all import *
>>> log(exp(f)) == f
True
```

Some tests:

```
sage: exp(M.zero_scalar_field()) == M.constant_scalar_field(1)
True
sage: exp(M.constant_scalar_field(1)) == M.constant_scalar_field(e)
True
```

```
>>> from sage.all import *
>>> exp(M.zero_scalar_field()) == M.constant_scalar_field(Integer(1))
True
>>> exp(M.constant_scalar_field(Integer(1))) == M.constant_scalar_field(e)
True
```

expr(*chart=None*, *from_chart=None*)

Return the coordinate expression of the scalar field in a given chart.

INPUT:

- `chart` – (default: `None`) chart with respect to which the coordinate expression is required; if `None`, the default chart of the scalar field's domain will be used
- `from_chart` – (default: `None`) chart from which the required expression is computed if it is not known already in the chart `chart`; if `None`, a chart is picked in `self._express`

OUTPUT:

- the coordinate expression of the scalar field in the given chart, either as a Sage's symbolic expression or as a SymPy object, depending on the symbolic calculus method used on the chart

EXAMPLES:

Expression of a scalar field on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(x*y^2)
sage: f.expr()
x*y^2
sage: f.expr(c_xy)  # equivalent form (since c_xy is the default chart)
x*y^2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> f = M.scalar_field(x*y**Integer(2))
>>> f.expr()
x*y^2
>>> f.expr(c_xy)  # equivalent form (since c_xy is the default chart)
x*y^2
```

Expression via a change of coordinates:

```
sage: c_uv.<u,v> = M.chart()
sage: c_uv.transition_map(c_xy, [u+v, u-v])
Change of coordinates from Chart (M, (u, v)) to Chart (M, (x, y))
sage: f._express # at this stage, f is expressed only in terms of (x,y)␣
↪coordinates
{Chart (M, (x, y)): x*y^2}
sage: f.expr(c_uv) # forces the computation of the expression of f in terms␣
↪of (u,v) coordinates
u^3 - u^2*v - u*v^2 + v^3
sage: bool( f.expr(c_uv) == (u+v)*(u-v)^2 ) # check
True
sage: f._express  # random (dict. output); f has now 2 coordinate expressions:
{Chart (M, (x, y)): x*y^2, Chart (M, (u, v)): u^3 - u^2*v - u*v^2 + v^3}
```

```
>>> from sage.all import *
>>> c_uv = M.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> c_uv.transition_map(c_xy, [u+v, u-v])
Change of coordinates from Chart (M, (u, v)) to Chart (M, (x, y))
>>> f._express # at this stage, f is expressed only in terms of (x,y)␣
↪coordinates
{Chart (M, (x, y)): x*y^2}
>>> f.expr(c_uv) # forces the computation of the expression of f in terms of␣
↪(u,v) coordinates
u^3 - u^2*v - u*v^2 + v^3
>>> bool( f.expr(c_uv) == (u+v)*(u-v)**Integer(2) ) # check
True
>>> f._express  # random (dict. output); f has now 2 coordinate expressions:
{Chart (M, (x, y)): x*y^2, Chart (M, (u, v)): u^3 - u^2*v - u*v^2 + v^3}
```

Note that the object returned by `expr()` depends on the symbolic backend used for coordinate computations:

```
sage: type(f.expr())
<class 'sage.symbolic.expression.Expression'>
sage: M.set_calculus_method('sympy')
```

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```
sage: type(f.expr())
<class 'sympy.core.mul.Mul'>
sage: f.expr()  # note the SymPy exponent notation
x*y**2
```

```
>>> from sage.all import *
>>> type(f.expr())
<class 'sage.symbolic.expression.Expression'>
>>> M.set_calculus_method('sympy')
>>> type(f.expr())
<class 'sympy.core.mul.Mul'>
>>> f.expr()  # note the SymPy exponent notation
x*y**2
```

is_trivial_one()

Check if `self` is trivially equal to one without any simplification.

This method is supposed to be fast as compared with `self == 1` and is intended to be used in library code where trying to obtain a mathematically correct result by applying potentially expensive rewrite rules is not desirable.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: 1})
sage: f.is_trivial_one()
True
sage: f = M.scalar_field(1)
sage: f.is_trivial_one()
True
sage: M.one_scalar_field().is_trivial_one()
True
sage: f = M.scalar_field({X: x+y})
sage: f.is_trivial_one()
False
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: Integer(1)})
>>> f.is_trivial_one()
True
>>> f = M.scalar_field(Integer(1))
>>> f.is_trivial_one()
True
>>> M.one_scalar_field().is_trivial_one()
True
>>> f = M.scalar_field({X: x+y})
>>> f.is_trivial_one()
False
```

Scalar field defined by means of two charts:

```
sage: U1 = M.open_subset('U1'); X1.<x1,y1> = U1.chart()
sage: U2 = M.open_subset('U2'); X2.<x2,y2> = U2.chart()
sage: f = M.scalar_field({X1: 1, X2: 1})
sage: f.is_trivial_one()
True
sage: f = M.scalar_field({X1: 0, X2: 1})
sage: f.is_trivial_one()
False
```

```
>>> from sage.all import *
>>> U1 = M.open_subset('U1'); X1 = U1.chart(names=('x1', 'y1',)); (x1, y1,) =␣
↪X1._first_ngens(2)
>>> U2 = M.open_subset('U2'); X2 = U2.chart(names=('x2', 'y2',)); (x2, y2,) =␣
↪X2._first_ngens(2)
>>> f = M.scalar_field({X1: Integer(1), X2: Integer(1)})
>>> f.is_trivial_one()
True
>>> f = M.scalar_field({X1: Integer(0), X2: Integer(1)})
>>> f.is_trivial_one()
False
```

No simplification is attempted, so that `False` is returned for non-trivial cases:

```
sage: f = M.scalar_field({X: cos(x)^2 + sin(x)^2})
sage: f.is_trivial_one()
False
```

```
>>> from sage.all import *
>>> f = M.scalar_field({X: cos(x)**Integer(2) + sin(x)**Integer(2)})
>>> f.is_trivial_one()
False
```

On the contrary, the method `is_zero()` and the direct comparison to one involve some simplification algorithms and return `True`:

```
sage: (f - 1).is_zero()
True
sage: f == 1
True
```

```
>>> from sage.all import *
>>> (f - Integer(1)).is_zero()
True
>>> f == Integer(1)
True
```

is_trivial_zero()

Check if `self` is trivially equal to zero without any simplification.

This method is supposed to be fast as compared with `self.is_zero()` or `self == 0` and is intended to be used in library code where trying to obtain a mathematically correct result by applying potentially expensive rewrite rules is not desirable.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: 0})
sage: f.is_trivial_zero()
True
sage: f = M.scalar_field(0)
sage: f.is_trivial_zero()
True
sage: M.zero_scalar_field().is_trivial_zero()
True
sage: f = M.scalar_field({X: x+y})
sage: f.is_trivial_zero()
False
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: Integer(0)})
>>> f.is_trivial_zero()
True
>>> f = M.scalar_field(Integer(0))
>>> f.is_trivial_zero()
True
>>> M.zero_scalar_field().is_trivial_zero()
True
>>> f = M.scalar_field({X: x+y})
>>> f.is_trivial_zero()
False
```

Scalar field defined by means of two charts:

```
sage: U1 = M.open_subset('U1'); X1.<x1,y1> = U1.chart()
sage: U2 = M.open_subset('U2'); X2.<x2,y2> = U2.chart()
sage: f = M.scalar_field({X1: 0, X2: 0})
sage: f.is_trivial_zero()
True
sage: f = M.scalar_field({X1: 0, X2: 1})
sage: f.is_trivial_zero()
False
```

```
>>> from sage.all import *
>>> U1 = M.open_subset('U1'); X1 = U1.chart(names=('x1', 'y1',)); (x1, y1,) =
→X1._first_ngens(2)
>>> U2 = M.open_subset('U2'); X2 = U2.chart(names=('x2', 'y2',)); (x2, y2,) =
→X2._first_ngens(2)
>>> f = M.scalar_field({X1: Integer(0), X2: Integer(0)})
>>> f.is_trivial_zero()
True
>>> f = M.scalar_field({X1: Integer(0), X2: Integer(1)})
>>> f.is_trivial_zero()
False
```

No simplification is attempted, so that `False` is returned for non-trivial cases:

```
sage: f = M.scalar_field({X: cos(x)^2 + sin(x)^2 - 1})
sage: f.is_trivial_zero()
False
```

```
>>> from sage.all import *
>>> f = M.scalar_field({X: cos(x)**Integer(2) + sin(x)**Integer(2) -␣
↪Integer(1)})
>>> f.is_trivial_zero()
False
```

On the contrary, the method `is_zero()` and the direct comparison to zero involve some simplification algorithms and return `True`:

```
sage: f.is_zero()
True
sage: f == 0
True
```

```
>>> from sage.all import *
>>> f.is_zero()
True
>>> f == Integer(0)
True
```

is_unit()

Return `True` iff `self` is not trivially zero in at least one of the given expressions since most scalar fields are invertible and a complete computation would take too much time.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top')
sage: one = M.scalar_field_algebra().one()
sage: one.is_unit()
True
sage: zero = M.scalar_field_algebra().zero()
sage: zero.is_unit()
False
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> one = M.scalar_field_algebra().one()
>>> one.is_unit()
True
>>> zero = M.scalar_field_algebra().zero()
>>> zero.is_unit()
False
```

log()

Natural logarithm of the scalar field.

OUTPUT: the scalar field $\ln f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x+y}, name='f', latex_name=r"\Phi")
sage: g = log(f) ; g
Scalar field ln(f) on the 2-dimensional topological manifold M
sage: g.display()
ln(f): M → ℝ
   (x, y) ↦ log(x + y)
sage: latex(g)
\ln\left(\Phi\right)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: x+y}, name='f', latex_name=r"\Phi")
>>> g = log(f) ; g
Scalar field ln(f) on the 2-dimensional topological manifold M
>>> g.display()
ln(f): M → ℝ
   (x, y) ↦ log(x + y)
>>> latex(g)
\ln\left(\Phi\right)
```

The inverse function is `exp()`:

```
sage: exp(log(f)) == f
True
```

```
>>> from sage.all import *
>>> exp(log(f)) == f
True
```

preimage(*codomain_subset*, *name=None*, *latex_name=None*)

Return the preimage of `codomain_subset`.

An alias is `pullback()`.

INPUT:

- `codomain_subset` – an instance of `RealSet`
- `name` – string; name (symbol) given to the subset
- `latex_name` – string (default: `None`); LaTeX symbol to denote the subset; if none are provided, it is set to `name`

OUTPUT:

- either a `TopologicalManifold` or a `ManifoldSubsetPullback`

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x+y}, name='f')
sage: L = f.pullback(RealSet.point(1)); latex(L)
```

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```
f^{-1}(\{1\})
sage: M((-1, 1)) in L
False
sage: M((0, 1)) in L
True

sage: M.zero_scalar_field().preimage(RealSet.point(0)) is M
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: x+y}, name='f')
>>> L = f.pullback(RealSet.point(Integer(1))); latex(L)
f^{-1}(\{1\})
>>> M((-Integer(1), Integer(1))) in L
False
>>> M((Integer(0), Integer(1))) in L
True

>>> M.zero_scalar_field().preimage(RealSet.point(Integer(0))) is M
True
```

pullback(*codomain_subset*, *name=None*, *latex_name=None*)

Return the preimage of `codomain_subset`.

An alias is *pullback()*.

INPUT:

- `codomain_subset` – an instance of `RealSet`
- `name` – string; name (symbol) given to the subset
- `latex_name` – string (default: `None`); LaTeX symbol to denote the subset; if none are provided, it is set to `name`

OUTPUT:

- either a *TopologicalManifold* or a *ManifoldSubsetPullback*

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x+y}, name='f')
sage: L = f.pullback(RealSet.point(1)); latex(L)
f^{-1}(\{1\})
sage: M((-1, 1)) in L
False
sage: M((0, 1)) in L
True

sage: M.zero_scalar_field().preimage(RealSet.point(0)) is M
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: x+y}, name='f')
>>> L = f.pullback(RealSet.point(Integer(1))); latex(L)
f^{-1}(\{1\})
>>> M((-Integer(1), Integer(1))) in L
False
>>> M((Integer(0), Integer(1))) in L
True

>>> M.zero_scalar_field().preimage(RealSet.point(Integer(0))) is M
True
```

restrict (*subdomain*)

Restriction of the scalar field to an open subset of its domain of definition.

INPUT:

- `subdomain` – an open subset of the scalar field's domain

OUTPUT:

- instance of *ScalarField* representing the restriction of the scalar field to `subdomain`

EXAMPLES:

Restriction of a scalar field defined on $\mathbf{R}^2$ to the unit open disc:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()  # Cartesian coordinates
sage: U = M.open_subset('U', coord_def={X: x^2+y^2 < 1}) # U unit open disc
sage: f = M.scalar_field(cos(x*y), name='f')
sage: f_U = f.restrict(U) ; f_U
Scalar field f on the Open subset U of the 2-dimensional
 topological manifold M
sage: f_U.display()
f: U → ℝ
   (x, y) ↦ cos(x*y)
sage: f.parent()
Algebra of scalar fields on the 2-dimensional topological
 manifold M
sage: f_U.parent()
Algebra of scalar fields on the Open subset U of the 2-dimensional
 topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)# Cartesian␣
↪coordinates
>>> U = M.open_subset('U', coord_def={X: x**Integer(2)+y**Integer(2) <␣
↪Integer(1)}) # U unit open disc
>>> f = M.scalar_field(cos(x*y), name='f')
>>> f_U = f.restrict(U) ; f_U
Scalar field f on the Open subset U of the 2-dimensional
```

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```
 topological manifold M
>>> f_U.display()
f: U → ℝ
   (x, y) ↦ cos(x*y)
>>> f.parent()
Algebra of scalar fields on the 2-dimensional topological
 manifold M
>>> f_U.parent()
Algebra of scalar fields on the Open subset U of the 2-dimensional
 topological manifold M
```

The restriction to the whole domain is the identity:

```
sage: f.restrict(M) is f
True
sage: f_U.restrict(U) is f_U
True
```

```
>>> from sage.all import *
>>> f.restrict(M) is f
True
>>> f_U.restrict(U) is f_U
True
```

Restriction of the zero scalar field:

```
sage: M.zero_scalar_field().restrict(U)
Scalar field zero on the Open subset U of the 2-dimensional
 topological manifold M
sage: M.zero_scalar_field().restrict(U) is U.zero_scalar_field()
True
```

```
>>> from sage.all import *
>>> M.zero_scalar_field().restrict(U)
Scalar field zero on the Open subset U of the 2-dimensional
 topological manifold M
>>> M.zero_scalar_field().restrict(U) is U.zero_scalar_field()
True
```

set_calc_order(*symbol*, *order*, *truncate=False*)

Trigger a power series expansion with respect to a small parameter in computations involving the scalar field.

This property is propagated by usual operations. The internal representation must be `SR` for this to take effect.

If the small parameter is ϵ and f is `self`, the power series expansion to order n is

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots + \epsilon^n f_n + O(\epsilon^{n+1}),$$

where $f_0, f_1, \ldots, f_n$ are $n + 1$ scalar fields that do not depend upon ϵ.

INPUT:

- `symbol` – symbolic variable (the "small parameter" ϵ) with respect to which the coordinate expressions of `self` in various charts are expanded in power series (around the zero value of this variable)

- `order` – integer; the order n of the expansion, defined as the degree of the polynomial representing the truncated power series in `symbol`

> **Warning**
>
> The order of the big O in the power series expansion is $n + 1$, where n is `order`.

- `truncate` – boolean (default: `False`); determines whether the coordinate expressions of `self` are replaced by their expansions to the given order

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: t = var('t')  # the small parameter
sage: f = M.scalar_field(exp(-t*x))
sage: f.expr()
e^(-t*x)
sage: f.set_calc_order(t, 2, truncate=True)
sage: f.expr()
1/2*t^2*x^2 - t*x + 1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> t = var('t')  # the small parameter
>>> f = M.scalar_field(exp(-t*x))
>>> f.expr()
e^(-t*x)
>>> f.set_calc_order(t, Integer(2), truncate=True)
>>> f.expr()
1/2*t^2*x^2 - t*x + 1
```

set_expr(*coord_expression*, *chart=None*)

Set the coordinate expression of the scalar field.

The expressions with respect to other charts are deleted, in order to avoid any inconsistency. To keep them, use *add_expr()* instead.

INPUT:

- `coord_expression` – coordinate expression of the scalar field
- `chart` – (default: `None`) chart in which `coord_expression` is defined; if `None`, the default chart of the scalar field's domain is assumed

EXAMPLES:

Setting scalar field expressions on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(x^2 + 2*x*y +1)
sage: f._express
{Chart (M, (x, y)): x^2 + 2*x*y + 1}
```

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```
sage: f.set_expr(3*y)
sage: f._express  # the (x,y) expression has been changed:
{Chart (M, (x, y)): 3*y}
sage: c_uv.<u,v> = M.chart()
sage: f.set_expr(cos(u)-sin(v), c_uv)
sage: f._express # the (x,y) expression has been lost:
{Chart (M, (u, v)): cos(u) - sin(v)}
sage: f.set_expr(3*y)
sage: f._express # the (u,v) expression has been lost:
{Chart (M, (x, y)): 3*y}
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> f = M.scalar_field(x**Integer(2) + Integer(2)*x*y +Integer(1))
>>> f._express
{Chart (M, (x, y)): x^2 + 2*x*y + 1}
>>> f.set_expr(Integer(3)*y)
>>> f._express  # the (x,y) expression has been changed:
{Chart (M, (x, y)): 3*y}
>>> c_uv = M.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> f.set_expr(cos(u)-sin(v), c_uv)
>>> f._express # the (x,y) expression has been lost:
{Chart (M, (u, v)): cos(u) - sin(v)}
>>> f.set_expr(Integer(3)*y)
>>> f._express # the (u,v) expression has been lost:
{Chart (M, (x, y)): 3*y}
```

Since zero and one are special elements, their expressions cannot be changed:

```
sage: z = M.zero_scalar_field()
sage: z.set_expr(3*y)
Traceback (most recent call last):
...
ValueError: the expressions of an immutable element cannot be
 changed
sage: one = M.one_scalar_field()
sage: one.set_expr(3*y)
Traceback (most recent call last):
...
ValueError: the expressions of an immutable element cannot be
 changed
```

```
>>> from sage.all import *
>>> z = M.zero_scalar_field()
>>> z.set_expr(Integer(3)*y)
Traceback (most recent call last):
...
ValueError: the expressions of an immutable element cannot be
 changed
>>> one = M.one_scalar_field()
>>> one.set_expr(Integer(3)*y)
```

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```
Traceback (most recent call last):
...
ValueError: the expressions of an immutable element cannot be
 changed
```

set_immutable()

Set `self` and all restrictions of `self` immutable.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: U = M.open_subset('U', coord_def={X: x^2+y^2<1})  # disk
sage: V = M.open_subset('U', coord_def={X: x>0})  # half plane
sage: f = M.scalar_field(x^2, name='f')
sage: fU = f.restrict(U)
sage: f.set_immutable()
sage: fU.is_immutable()
True
sage: f.restrict(V).is_immutable()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> U = M.open_subset('U', coord_def={X: x**Integer(2)+y**Integer(2)
↪<Integer(1)})  # disk
>>> V = M.open_subset('U', coord_def={X: x>Integer(0)})  # half plane
>>> f = M.scalar_field(x**Integer(2), name='f')
>>> fU = f.restrict(U)
>>> f.set_immutable()
>>> fU.is_immutable()
True
>>> f.restrict(V).is_immutable()
True
```

set_name(*name=None*, *latex_name=None*)

Set (or change) the text name and LaTeX name of the scalar field.

INPUT:

- `name` – (string; default: `None`) name given to the scalar field
- `latex_name` – (string; default: `None`) LaTeX symbol to denote the scalar field; if `None` while `name` is provided, the LaTeX symbol is set to `name`

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x+y})
sage: f = M.scalar_field({X: x+y}); f
Scalar field on the 2-dimensional topological manifold M
sage: f.set_name('f'); f
```

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```
Scalar field f on the 2-dimensional topological manifold M
sage: latex(f)
f
sage: f.set_name('f', latex_name=r'\Phi'); f
Scalar field f on the 2-dimensional topological manifold M
sage: latex(f)
\Phi
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: x+y})
>>> f = M.scalar_field({X: x+y}); f
Scalar field on the 2-dimensional topological manifold M
>>> f.set_name('f'); f
Scalar field f on the 2-dimensional topological manifold M
>>> latex(f)
f
>>> f.set_name('f', latex_name=r'\Phi'); f
Scalar field f on the 2-dimensional topological manifold M
>>> latex(f)
\Phi
```

set_restriction(*rst*)

Define a restriction of `self` to some subdomain.

INPUT:

- `rst` – *ScalarField* defined on a subdomain of the domain of `self`

EXAMPLES:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: f = M.scalar_field(name='f')
sage: g = U.scalar_field(x^2+y)
sage: f.set_restriction(g)
sage: f.display()
f: M → R
on U: (x, y) ↦ x^2 + y
sage: f.restrict(U) == g
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
```

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```
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2) #␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)    # S^2 is the union of U and V
>>> f = M.scalar_field(name='f')
>>> g = U.scalar_field(x**Integer(2)+y)
>>> f.set_restriction(g)
>>> f.display()
f: M → ℝ
on U: (x, y) ↦ x^2 + y
>>> f.restrict(U) == g
True
```

sin()

Sine of the scalar field.

OUTPUT: the scalar field $\sin f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
sage: g = sin(f) ; g
Scalar field sin(f) on the 2-dimensional topological manifold M
sage: latex(g)
\sin\left(\Phi\right)
sage: g.display()
sin(f): M → ℝ
   (x, y) ↦ sin(x*y)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
>>> g = sin(f) ; g
Scalar field sin(f) on the 2-dimensional topological manifold M
>>> latex(g)
\sin\left(\Phi\right)
>>> g.display()
sin(f): M → ℝ
   (x, y) ↦ sin(x*y)
```

Some tests:

```
sage: sin(M.zero_scalar_field()) == M.zero_scalar_field()
True
sage: sin(M.constant_scalar_field(pi/2)) == M.constant_scalar_field(1)
True
```

```
>>> from sage.all import *
>>> sin(M.zero_scalar_field()) == M.zero_scalar_field()
True
>>> sin(M.constant_scalar_field(pi/Integer(2))) == M.constant_scalar_
```

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```
↪field(Integer(1))
True
```

sinh()

Hyperbolic sine of the scalar field.

OUTPUT: the scalar field $\sinh f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
sage: g = sinh(f) ; g
Scalar field sinh(f) on the 2-dimensional topological manifold M
sage: latex(g)
\sinh\left(\Phi\right)
sage: g.display()
sinh(f): M → ℝ
   (x, y) ↦ sinh(x*y)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
>>> g = sinh(f) ; g
Scalar field sinh(f) on the 2-dimensional topological manifold M
>>> latex(g)
\sinh\left(\Phi\right)
>>> g.display()
sinh(f): M → ℝ
   (x, y) ↦ sinh(x*y)
```

Some test:

```
sage: sinh(M.zero_scalar_field()) == M.zero_scalar_field()
True
```

```
>>> from sage.all import *
>>> sinh(M.zero_scalar_field()) == M.zero_scalar_field()
True
```

sqrt()

Square root of the scalar field.

OUTPUT: the scalar field $\sqrt{f}$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: 1+x^2+y^2}, name='f',
....:                    latex_name=r"\Phi")
sage: g = sqrt(f) ; g
```

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```
Scalar field sqrt(f) on the 2-dimensional topological manifold M
sage: latex(g)
\sqrt{\Phi}
sage: g.display()
sqrt(f): M → ℝ
   (x, y) ↦ sqrt(x^2 + y^2 + 1)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: Integer(1)+x**Integer(2)+y**Integer(2)}, name='f',
...                    latex_name=r"\Phi")
>>> g = sqrt(f) ; g
Scalar field sqrt(f) on the 2-dimensional topological manifold M
>>> latex(g)
\sqrt{\Phi}
>>> g.display()
sqrt(f): M → ℝ
   (x, y) ↦ sqrt(x^2 + y^2 + 1)
```

Some tests:

```
sage: g^2 == f
True
sage: sqrt(M.zero_scalar_field()) == M.zero_scalar_field()
True
```

```
>>> from sage.all import *
>>> g**Integer(2) == f
True
>>> sqrt(M.zero_scalar_field()) == M.zero_scalar_field()
True
```

tan()

Tangent of the scalar field.

OUTPUT: the scalar field $\tan f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
sage: g = tan(f) ; g
Scalar field tan(f) on the 2-dimensional topological manifold M
sage: latex(g)
\tan\left(\Phi\right)
sage: g.display()
tan(f): M → ℝ
   (x, y) ↦ sin(x*y)/cos(x*y)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
>>> g = tan(f) ; g
Scalar field tan(f) on the 2-dimensional topological manifold M
>>> latex(g)
\tan\left(\Phi\right)
>>> g.display()
tan(f): M → ℝ
   (x, y) ↦ sin(x*y)/cos(x*y)
```

Some tests:

```
sage: tan(f) == sin(f) / cos(f)
True
sage: tan(M.zero_scalar_field()) == M.zero_scalar_field()
True
sage: tan(M.constant_scalar_field(pi/4)) == M.constant_scalar_field(1)
True
```

```
>>> from sage.all import *
>>> tan(f) == sin(f) / cos(f)
True
>>> tan(M.zero_scalar_field()) == M.zero_scalar_field()
True
>>> tan(M.constant_scalar_field(pi/Integer(4))) == M.constant_scalar_
↪field(Integer(1))
True
```

tanh()

Hyperbolic tangent of the scalar field.

OUTPUT: the scalar field $\tanh f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
sage: g = tanh(f) ; g
Scalar field tanh(f) on the 2-dimensional topological manifold M
sage: latex(g)
\tanh\left(\Phi\right)
sage: g.display()
tanh(f): M → ℝ
   (x, y) ↦ sinh(x*y)/cosh(x*y)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
>>> g = tanh(f) ; g
```

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```
Scalar field tanh(f) on the 2-dimensional topological manifold M
>>> latex(g)
\tanh\left(\Phi\right)
>>> g.display()
tanh(f): M → ℝ
   (x, y) ↦ sinh(x*y)/cosh(x*y)
```

Some tests:

```
sage: tanh(f) == sinh(f) / cosh(f)
True
sage: tanh(M.zero_scalar_field()) == M.zero_scalar_field()
True
```

```
>>> from sage.all import *
>>> tanh(f) == sinh(f) / cosh(f)
True
>>> tanh(M.zero_scalar_field()) == M.zero_scalar_field()
True
```

1.7 Continuous Maps

1.7.1 Set of Morphisms between Topological Manifolds

The class *TopologicalManifoldHomset* implements sets of morphisms between two topological manifolds over the same topological field K, a morphism being a *continuous map* for the category of topological manifolds.

AUTHORS:

- Eric Gourgoulhon (2015): initial version
- Travis Scrimshaw (2016): review tweaks

REFERENCES:

- [Lee2011]
- [KN1963]

class sage.manifolds.manifold_homset.**TopologicalManifoldHomset**(*domain*, *codomain*, *name=None*, *latex_name=None*)

Bases: `UniqueRepresentation`, `Homset`

Set of continuous maps between two topological manifolds.

Given two topological manifolds M and N over a topological field K, the class *TopologicalManifoldHomset* implements the set $\mathrm{Hom}(M,N)$ of morphisms (i.e. continuous maps) $M \to N$.

This is a Sage *parent* class, whose *element* class is *ContinuousMap*.

INPUT:

- `domain` – *TopologicalManifold*; the domain topological manifold M of the morphisms
- `codomain` – *TopologicalManifold*; the codomain topological manifold N of the morphisms
- `name` – (default: `None`) string; the name of `self`; if `None`, `Hom(M,N)` will be used
- `latex_name` – (default: `None`) string; LaTeX symbol to denote `self`; if `None`, $\mathrm{Hom}(M,N)$ will be used

EXAMPLES:

Set of continuous maps between a 2-dimensional manifold and a 3-dimensional one:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: N = Manifold(3, 'N', structure='topological')
sage: Y.<u,v,w> = N.chart()
sage: H = Hom(M, N) ; H
Set of Morphisms from 2-dimensional topological manifold M to
 3-dimensional topological manifold N in Category of manifolds over
 Real Field with 53 bits of precision
sage: type(H)
<class 'sage.manifolds.manifold_homset.TopologicalManifoldHomset_with_category'>
sage: H.category()
Category of homsets of topological spaces
sage: latex(H)
\mathrm{Hom}\left(M,N\right)
sage: H.domain()
2-dimensional topological manifold M
sage: H.codomain()
3-dimensional topological manifold N
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> N = Manifold(Integer(3), 'N', structure='topological')
>>> Y = N.chart(names=('u', 'v', 'w',)); (u, v, w,) = Y._first_ngens(3)
>>> H = Hom(M, N) ; H
Set of Morphisms from 2-dimensional topological manifold M to
 3-dimensional topological manifold N in Category of manifolds over
 Real Field with 53 bits of precision
>>> type(H)
<class 'sage.manifolds.manifold_homset.TopologicalManifoldHomset_with_category'>
>>> H.category()
Category of homsets of topological spaces
>>> latex(H)
\mathrm{Hom}\left(M,N\right)
>>> H.domain()
2-dimensional topological manifold M
>>> H.codomain()
3-dimensional topological manifold N
```

An element of `H` is a continuous map from `M` to `N`:

```
sage: H.Element
<class 'sage.manifolds.continuous_map.ContinuousMap'>
sage: f = H.an_element() ; f
Continuous map from the 2-dimensional topological manifold M to the
 3-dimensional topological manifold N
sage: f.display()
M → N
   (x, y) ↦ (u, v, w) = (0, 0, 0)
```

```
>>> from sage.all import *
>>> H.Element
<class 'sage.manifolds.continuous_map.ContinuousMap'>
>>> f = H.an_element() ; f
Continuous map from the 2-dimensional topological manifold M to the
 3-dimensional topological manifold N
>>> f.display()
M → N
   (x, y) ↦ (u, v, w) = (0, 0, 0)
```

The test suite is passed:

```
sage: TestSuite(H).run()
```

```
>>> from sage.all import *
>>> TestSuite(H).run()
```

When the codomain coincides with the domain, the homset is a set of *endomorphisms* in the category of topological manifolds:

```
sage: E = Hom(M, M) ; E
Set of Morphisms from 2-dimensional topological manifold M to
 2-dimensional topological manifold M in Category of manifolds over
 Real Field with 53 bits of precision
sage: E.category()
Category of endsets of topological spaces
sage: E.is_endomorphism_set()
True
sage: E is End(M)
True
```

```
>>> from sage.all import *
>>> E = Hom(M, M) ; E
Set of Morphisms from 2-dimensional topological manifold M to
 2-dimensional topological manifold M in Category of manifolds over
 Real Field with 53 bits of precision
>>> E.category()
Category of endsets of topological spaces
>>> E.is_endomorphism_set()
True
>>> E is End(M)
True
```

In this case, the homset is a monoid for the law of morphism composition:

```
sage: E in Monoids()
True
```

```
>>> from sage.all import *
>>> E in Monoids()
True
```

This was of course not the case of `H = Hom(M, N)`:

```
sage: H in Monoids()
False
```

```
>>> from sage.all import *
>>> H in Monoids()
False
```

The identity element of the monoid is of course the identity map of `M`:

```
sage: E.one()
Identity map Id_M of the 2-dimensional topological manifold M
sage: E.one() is M.identity_map()
True
sage: E.one().display()
Id_M: M → M
   (x, y) ↦ (x, y)
```

```
>>> from sage.all import *
>>> E.one()
Identity map Id_M of the 2-dimensional topological manifold M
>>> E.one() is M.identity_map()
True
>>> E.one().display()
Id_M: M → M
   (x, y) ↦ (x, y)
```

The test suite is passed by `E`:

```
sage: TestSuite(E).run()
```

```
>>> from sage.all import *
>>> TestSuite(E).run()
```

This test suite includes more tests than in the case of `H`, since `E` has some extra structure (monoid).

Element

alias of *ContinuousMap*

one()

Return the identity element of `self` considered as a monoid (case of a set of endomorphisms).

This applies only when the codomain of the homset is equal to its domain, i.e. when the homset is of the type $\mathrm{Hom}(M, M)$. Indeed, $\mathrm{Hom}(M, M)$ equipped with the law of morphisms composition is a monoid, whose identity element is nothing but the identity map of M.

OUTPUT:

- the identity map of M, as an instance of *ContinuousMap*

EXAMPLES:

The identity map of a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
```

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```
sage: H = Hom(M, M) ; H
Set of Morphisms from 2-dimensional topological manifold M to
 2-dimensional topological manifold M in Category of manifolds over
 Real Field with 53 bits of precision
sage: H in Monoids()
True
sage: H.one()
Identity map Id_M of the 2-dimensional topological manifold M
sage: H.one().parent() is H
True
sage: H.one().display()
Id_M: M → M
   (x, y) ↦ (x, y)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> H = Hom(M, M) ; H
Set of Morphisms from 2-dimensional topological manifold M to
 2-dimensional topological manifold M in Category of manifolds over
 Real Field with 53 bits of precision
>>> H in Monoids()
True
>>> H.one()
Identity map Id_M of the 2-dimensional topological manifold M
>>> H.one().parent() is H
True
>>> H.one().display()
Id_M: M → M
   (x, y) ↦ (x, y)
```

The identity map is cached:

```
sage: H.one() is H.one()
True
```

```
>>> from sage.all import *
>>> H.one() is H.one()
True
```

If the homset is not a set of endomorphisms, the identity element is meaningless:

```
sage: N = Manifold(3, 'N', structure='topological')
sage: Y.<u,v,w> = N.chart()
sage: Hom(M, N).one()
Traceback (most recent call last):
...
TypeError: Set of Morphisms
 from 2-dimensional topological manifold M
 to 3-dimensional topological manifold N
 in Category of manifolds over Real Field with 53 bits of precision
 is not a monoid
```

```
>>> from sage.all import *
>>> N = Manifold(Integer(3), 'N', structure='topological')
>>> Y = N.chart(names=('u', 'v', 'w',)); (u, v, w,) = Y._first_ngens(3)
>>> Hom(M, N).one()
Traceback (most recent call last):
...
TypeError: Set of Morphisms
 from 2-dimensional topological manifold M
 to 3-dimensional topological manifold N
 in Category of manifolds over Real Field with 53 bits of precision
 is not a monoid
```

1.7.2 Continuous Maps Between Topological Manifolds

ContinuousMap implements continuous maps from a topological manifold M to some topological manifold N over the same topological field K as M.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015): initial version
- Travis Scrimshaw (2016): review tweaks

REFERENCES:

- Chap. 1 of [KN1963]
- [Lee2011]

class sage.manifolds.continuous_map.**ContinuousMap**(*parent*, *coord_functions=None*, *name=None*, *latex_name=None*, *is_isomorphism=False*, *is_identity=False*)

Bases: Morphism

Continuous map between two topological manifolds.

This class implements continuous maps of the type

$$\Phi : M \longrightarrow N,$$

where M and N are topological manifolds over the same topological field K.

Continuous maps are the morphisms of the category of topological manifolds. The set of all continuous maps from M to N is therefore the homset between M and N, which is denoted by $\mathrm{Hom}(M, N)$.

The class *ContinuousMap* is a Sage *element* class, whose *parent* class is *TopologicalManifoldHomset*.

INPUT:

- `parent` – homset $\mathrm{Hom}(M, N)$ to which the continuous map belongs
- `coord_functions` – dictionary of the coordinate expressions (as lists or tuples of the coordinates of the image expressed in terms of the coordinates of the considered point) with the pairs of charts `(chart1, chart2)` as keys (`chart1` being a chart on M and `chart2` a chart on N)
- `name` – (default: `None`) name given to `self`
- `latex_name` – (default: `None`) LaTeX symbol to denote the continuous map; if `None`, the LaTeX symbol is set to `name`
- `is_isomorphism` – boolean (default: `False`); determines whether the constructed object is a isomorphism (i.e. a homeomorphism). If set to `True`, then the manifolds M and N must have the same dimension.

- `is_identity` – boolean (default: `False`); determines whether the constructed object is the identity map. If set to `True`, then N must be M and the entry `coord_functions` is not used.

> **Note**
>
> If the information passed by means of the argument `coord_functions` is not sufficient to fully specify the continuous map, further coordinate expressions, in other charts, can be subsequently added by means of the method `add_expr()`.

EXAMPLES:

The standard embedding of the sphere S^2 into $\mathbf{R}^3$:

```
sage: M = Manifold(2, 'S^2', structure='topological') # the 2-dimensional sphere␣
↪S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                                intersection_name='W',
....:                                restrictions1=x^2+y^2!=0,
....:                                restrictions2=u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: N = Manifold(3, 'R^3', latex_name=r'\RR^3', structure='topological')  # R^3
sage: c_cart.<X,Y,Z> = N.chart()  # Cartesian coordinates on R^3
sage: Phi = M.continuous_map(N,
....:   {(c_xy, c_cart): [2*x/(1+x^2+y^2), 2*y/(1+x^2+y^2), (x^2+y^2-1)/(1+x^2+y^
↪2)],
....:    (c_uv, c_cart): [2*u/(1+u^2+v^2), 2*v/(1+u^2+v^2), (1-u^2-v^2)/(1+u^2+v^
↪2)]},
....:   name='Phi', latex_name=r'\Phi')
sage: Phi
Continuous map Phi from the 2-dimensional topological manifold S^2
 to the 3-dimensional topological manifold R^3
sage: Phi.parent()
Set of Morphisms from 2-dimensional topological manifold S^2
 to 3-dimensional topological manifold R^3
 in Category of manifolds over Real Field with 53 bits of precision
sage: Phi.parent() is Hom(M, N)
True
sage: type(Phi)
<class 'sage.manifolds.manifold_homset.TopologicalManifoldHomset_with_category.
↪element_class'>
sage: Phi.display()
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 + y^2 -
↪ 1)/(x^2 + y^2 + 1))
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^2 + v^2␣
↪- 1)/(u^2 + v^2 + 1))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', structure='topological') # the 2-dimensional␣
↪sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                                intersection_name='W',
...                                restrictions1=x**Integer(2)+y**Integer(2)!
↪=Integer(0),
...                                restrictions2=u**Integer(2)+v**Integer(2)!
↪=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> N = Manifold(Integer(3), 'R^3', latex_name=r'\RR^3', structure='topological')␣
↪ # R^3
>>> c_cart = N.chart(names=('X', 'Y', 'Z',)); (X, Y, Z,) = c_cart._first_ngens(3)
↪# Cartesian coordinates on R^3
>>> Phi = M.continuous_map(N,
...   {(c_xy, c_cart): [Integer(2)*x/(Integer(1)+x**Integer(2)+y**Integer(2)),␣
↪Integer(2)*y/(Integer(1)+x**Integer(2)+y**Integer(2)),␣
↪(x**Integer(2)+y**Integer(2)-Integer(1))/
↪(Integer(1)+x**Integer(2)+y**Integer(2))],
...    (c_uv, c_cart): [Integer(2)*u/(Integer(1)+u**Integer(2)+v**Integer(2)),␣
↪Integer(2)*v/(Integer(1)+u**Integer(2)+v**Integer(2)), (Integer(1)-
↪u**Integer(2)-v**Integer(2))/(Integer(1)+u**Integer(2)+v**Integer(2))]},
...   name='Phi', latex_name=r'\Phi')
>>> Phi
Continuous map Phi from the 2-dimensional topological manifold S^2
 to the 3-dimensional topological manifold R^3
>>> Phi.parent()
Set of Morphisms from 2-dimensional topological manifold S^2
 to 3-dimensional topological manifold R^3
 in Category of manifolds over Real Field with 53 bits of precision
>>> Phi.parent() is Hom(M, N)
True
>>> type(Phi)
<class 'sage.manifolds.manifold_homset.TopologicalManifoldHomset_with_category.
↪element_class'>
>>> Phi.display()
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 + y^2 -
↪ 1)/(x^2 + y^2 + 1))
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^2 + v^2␣
↪- 1)/(u^2 + v^2 + 1))
```

It is possible to create the map using *continuous_map()* with only in a single pair of charts. The argument `coord_functions` is then a mere list of coordinate expressions (and not a dictionary) and the arguments `chart1` and `chart2` have to be provided if the charts differ from the default ones on the domain and/or codomain:

```
sage: Phi1 = M.continuous_map(N, [2*x/(1+x^2+y^2), 2*y/(1+x^2+y^2), (x^2+y^2-1)/
↪(1+x^2+y^2)],
....:                          chart1=c_xy, chart2=c_cart,
....:                          name='Phi', latex_name=r'\Phi')
```

```
>>> from sage.all import *
>>> Phi1 = M.continuous_map(N, [Integer(2)*x/
↪(Integer(1)+x**Integer(2)+y**Integer(2)), Integer(2)*y/
↪(Integer(1)+x**Integer(2)+y**Integer(2)), (x**Integer(2)+y**Integer(2)-
↪Integer(1))/(Integer(1)+x**Integer(2)+y**Integer(2))],
...                          chart1=c_xy, chart2=c_cart,
...                          name='Phi', latex_name=r'\Phi')
```

Since `c_xy` and `c_cart` are the default charts on respectively `M` and `N`, they can be omitted, so that the above declaration is equivalent to:

```
sage: Phi1 = M.continuous_map(N, [2*x/(1+x^2+y^2), 2*y/(1+x^2+y^2), (x^2+y^2-1)/
↪(1+x^2+y^2)],
....:                          name='Phi', latex_name=r'\Phi')
```

```
>>> from sage.all import *
>>> Phi1 = M.continuous_map(N, [Integer(2)*x/
↪(Integer(1)+x**Integer(2)+y**Integer(2)), Integer(2)*y/
↪(Integer(1)+x**Integer(2)+y**Integer(2)), (x**Integer(2)+y**Integer(2)-
↪Integer(1))/(Integer(1)+x**Integer(2)+y**Integer(2))],
...                          name='Phi', latex_name=r'\Phi')
```

With such a declaration, the continuous map `Phi1` is only partially defined on the manifold S^2 as it is known in only one chart:

```
sage: Phi1.display()
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 + y^2 -
↪ 1)/(x^2 + y^2 + 1))
```

```
>>> from sage.all import *
>>> Phi1.display()
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 + y^2 -
↪ 1)/(x^2 + y^2 + 1))
```

The definition can be completed by using *add_expr()*:

```
sage: Phi1.add_expr(c_uv, c_cart, [2*u/(1+u^2+v^2), 2*v/(1+u^2+v^2), (1-u^2-v^2)/
↪(1+u^2+v^2)])
sage: Phi1.display()
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 + y^2 -
↪ 1)/(x^2 + y^2 + 1))
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^2 + v^2␣
↪- 1)/(u^2 + v^2 + 1))
```

```
>>> from sage.all import *
>>> Phi1.add_expr(c_uv, c_cart, [Integer(2)*u/
↪(Integer(1)+u**Integer(2)+v**Integer(2)), Integer(2)*v/
↪(Integer(1)+u**Integer(2)+v**Integer(2)), (Integer(1)-u**Integer(2)-
↪v**Integer(2))/(Integer(1)+u**Integer(2)+v**Integer(2))])
>>> Phi1.display()
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 + y^2 -
↪ 1)/(x^2 + y^2 + 1))
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^2 + v^2␣
↪- 1)/(u^2 + v^2 + 1))
```

At this stage, `Phi1` and `Phi` are fully equivalent:

```
sage: Phi1 == Phi
True
```

```
>>> from sage.all import *
>>> Phi1 == Phi
True
```

The map acts on points:

```
sage: np = M.point((0,0), chart=c_uv)  # the North pole
sage: Phi(np)
Point on the 3-dimensional topological manifold R^3
sage: Phi(np).coord()  # Cartesian coordinates
(0, 0, 1)
sage: sp = M.point((0,0), chart=c_xy)  # the South pole
sage: Phi(sp).coord()  # Cartesian coordinates
(0, 0, -1)
```

```
>>> from sage.all import *
>>> np = M.point((Integer(0),Integer(0)), chart=c_uv)  # the North pole
>>> Phi(np)
Point on the 3-dimensional topological manifold R^3
>>> Phi(np).coord()  # Cartesian coordinates
(0, 0, 1)
>>> sp = M.point((Integer(0),Integer(0)), chart=c_xy)  # the South pole
>>> Phi(sp).coord()  # Cartesian coordinates
(0, 0, -1)
```

The test suite is passed:

```
sage: TestSuite(Phi).run()
sage: TestSuite(Phi1).run()
```

```
>>> from sage.all import *
>>> TestSuite(Phi).run()
>>> TestSuite(Phi1).run()
```

Continuous maps can be composed by means of the operator `*`. Let us introduce the map $\mathbf{R}^3 \to \mathbf{R}^2$ corresponding to the projection from the point $(X, Y, Z) = (0, 0, 1)$ onto the equatorial plane $Z = 0$:

```
sage: P = Manifold(2, 'R^2', latex_name=r'\RR^2', structure='topological') # R^2␣
↪(equatorial plane)
sage: cP.<xP, yP> = P.chart()
sage: Psi = N.continuous_map(P, (X/(1-Z), Y/(1-Z)), name='Psi',
....:                         latex_name=r'\Psi')
sage: Psi
Continuous map Psi from the 3-dimensional topological manifold R^3
 to the 2-dimensional topological manifold R^2
sage: Psi.display()
Psi: R^3 → R^2
   (X, Y, Z) ↦ (xP, yP) = (-X/(Z - 1), -Y/(Z - 1))
```

```
>>> from sage.all import *
>>> P = Manifold(Integer(2), 'R^2', latex_name=r'\RR^2', structure='topological')
↪# R^2 (equatorial plane)
>>> cP = P.chart(names=('xP', 'yP',)); (xP, yP,) = cP._first_ngens(2)
>>> Psi = N.continuous_map(P, (X/(Integer(1)-Z), Y/(Integer(1)-Z)), name='Psi',
...                         latex_name=r'\Psi')
>>> Psi
Continuous map Psi from the 3-dimensional topological manifold R^3
 to the 2-dimensional topological manifold R^2
>>> Psi.display()
Psi: R^3 → R^2
   (X, Y, Z) ↦ (xP, yP) = (-X/(Z - 1), -Y/(Z - 1))
```

Then we compose `Psi` with `Phi`, thereby getting a map $S^2 \to \mathbf{R}^2$:

```
sage: ster = Psi * Phi ; ster
Continuous map from the 2-dimensional topological manifold S^2
 to the 2-dimensional topological manifold R^2
```

```
>>> from sage.all import *
>>> ster = Psi * Phi ; ster
Continuous map from the 2-dimensional topological manifold S^2
 to the 2-dimensional topological manifold R^2
```

Let us test on the South pole (`sp`) that `ster` is indeed the composite of `Psi` and `Phi`:

```
sage: ster(sp) == Psi(Phi(sp))
True
```

```
>>> from sage.all import *
>>> ster(sp) == Psi(Phi(sp))
True
```

Actually `ster` is the stereographic projection from the North pole, as its coordinate expression reveals:

```
sage: ster.display()
S^2 → R^2
on U: (x, y) ↦ (xP, yP) = (x, y)
on V: (u, v) ↦ (xP, yP) = (u/(u^2 + v^2), v/(u^2 + v^2))
```

```
>>> from sage.all import *
>>> ster.display()
S^2 → R^2
on U: (x, y) ↦ (xP, yP) = (x, y)
on V: (u, v) ↦ (xP, yP) = (u/(u^2 + v^2), v/(u^2 + v^2))
```

If the codomain of a continuous map is 1-dimensional, the map can be defined by a single symbolic expression for each pair of charts and not by a list/tuple with a single element:

```
sage: N = Manifold(1, 'N', structure='topological')
sage: c_N = N.chart('X')
sage: Phi = M.continuous_map(N, {(c_xy, c_N): x^2+y^2,
....:                            (c_uv, c_N): 1/(u^2+v^2)})

sage: Psi = M.continuous_map(N, {(c_xy, c_N): [x^2+y^2],
....:                            (c_uv, c_N): [1/(u^2+v^2)]})
sage: Phi == Psi
True
```

```
>>> from sage.all import *
>>> N = Manifold(Integer(1), 'N', structure='topological')
>>> c_N = N.chart('X')
>>> Phi = M.continuous_map(N, {(c_xy, c_N): x**Integer(2)+y**Integer(2),
...                            (c_uv, c_N): Integer(1)/
↪(u**Integer(2)+v**Integer(2))})

>>> Psi = M.continuous_map(N, {(c_xy, c_N): [x**Integer(2)+y**Integer(2)],
...                            (c_uv, c_N): [Integer(1)/
↪(u**Integer(2)+v**Integer(2))]})
>>> Phi == Psi
True
```

Next we construct an example of continuous map $\mathbf{R} \to \mathbf{R}^2$:

```
sage: R = Manifold(1, 'R', structure='topological')  # field R
sage: T.<t> = R.chart()  # canonical chart on R
sage: R2 = Manifold(2, 'R^2', structure='topological')  # R^2
sage: c_xy.<x,y> = R2.chart() # Cartesian coordinates on R^2
sage: Phi = R.continuous_map(R2, [cos(t), sin(t)], name='Phi'); Phi
Continuous map Phi from the 1-dimensional topological manifold R
 to the 2-dimensional topological manifold R^2
sage: Phi.parent()
Set of Morphisms from 1-dimensional topological manifold R
 to 2-dimensional topological manifold R^2
 in Category of manifolds over Real Field with 53 bits of precision
sage: Phi.parent() is Hom(R, R2)
True
sage: Phi.display()
Phi: R → R^2
   t ↦ (x, y) = (cos(t), sin(t))
```

```
>>> from sage.all import *
>>> R = Manifold(Integer(1), 'R', structure='topological')  # field R
```

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```
>>> T = R.chart(names=('t',)); (t,) = T._first_ngens(1)# canonical chart on R
>>> R2 = Manifold(Integer(2), 'R^2', structure='topological')  # R^2
>>> c_xy = R2.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)# Cartesian␣
↪coordinates on R^2
>>> Phi = R.continuous_map(R2, [cos(t), sin(t)], name='Phi'); Phi
Continuous map Phi from the 1-dimensional topological manifold R
 to the 2-dimensional topological manifold R^2
>>> Phi.parent()
Set of Morphisms from 1-dimensional topological manifold R
 to 2-dimensional topological manifold R^2
 in Category of manifolds over Real Field with 53 bits of precision
>>> Phi.parent() is Hom(R, R2)
True
>>> Phi.display()
Phi: R → R^2
   t ↦ (x, y) = (cos(t), sin(t))
```

An example of homeomorphism between the unit open disk and the Euclidean plane $\mathbf{R}^2$:

```
sage: D = R2.open_subset('D', coord_def={c_xy: x^2+y^2<1}) # the open unit disk
sage: Phi = D.homeomorphism(R2, [x/sqrt(1-x^2-y^2), y/sqrt(1-x^2-y^2)],
....:                            name='Phi', latex_name=r'\Phi')
sage: Phi
Homeomorphism Phi from the Open subset D of the 2-dimensional
 topological manifold R^2 to the 2-dimensional topological manifold R^2
sage: Phi.parent()
Set of Morphisms from Open subset D of the 2-dimensional topological
 manifold R^2 to 2-dimensional topological manifold R^2 in Category of
 manifolds over Real Field with 53 bits of precision
sage: Phi.parent() is Hom(D, R2)
True
sage: Phi.display()
Phi: D → R^2
   (x, y) ↦ (x, y) = (x/sqrt(-x^2 - y^2 + 1), y/sqrt(-x^2 - y^2 + 1))
```

```
>>> from sage.all import *
>>> D = R2.open_subset('D', coord_def={c_xy: x**Integer(2)+y**Integer(2)
↪<Integer(1)}) # the open unit disk
>>> Phi = D.homeomorphism(R2, [x/sqrt(Integer(1)-x**Integer(2)-y**Integer(2)), y/
↪sqrt(Integer(1)-x**Integer(2)-y**Integer(2))],
...                          name='Phi', latex_name=r'\Phi')
>>> Phi
Homeomorphism Phi from the Open subset D of the 2-dimensional
 topological manifold R^2 to the 2-dimensional topological manifold R^2
>>> Phi.parent()
Set of Morphisms from Open subset D of the 2-dimensional topological
 manifold R^2 to 2-dimensional topological manifold R^2 in Category of
 manifolds over Real Field with 53 bits of precision
>>> Phi.parent() is Hom(D, R2)
True
>>> Phi.display()
Phi: D → R^2
```

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```
    (x, y) ↦ (x, y) = (x/sqrt(-x^2 - y^2 + 1), y/sqrt(-x^2 - y^2 + 1))
```

The image of a point:

```
sage: p = D.point((1/2,0))
sage: q = Phi(p) ; q
Point on the 2-dimensional topological manifold R^2
sage: q.coord()
(1/3*sqrt(3), 0)
```

```
>>> from sage.all import *
>>> p = D.point((Integer(1)/Integer(2),Integer(0)))
>>> q = Phi(p) ; q
Point on the 2-dimensional topological manifold R^2
>>> q.coord()
(1/3*sqrt(3), 0)
```

The inverse homeomorphism is computed by `inverse()`:

```
sage: Phi.inverse()
Homeomorphism Phi^(-1) from the 2-dimensional topological manifold R^2
 to the Open subset D of the 2-dimensional topological manifold R^2
sage: Phi.inverse().display()
Phi^(-1): R^2 → D
    (x, y) ↦ (x, y) = (x/sqrt(x^2 + y^2 + 1), y/sqrt(x^2 + y^2 + 1))
```

```
>>> from sage.all import *
>>> Phi.inverse()
Homeomorphism Phi^(-1) from the 2-dimensional topological manifold R^2
 to the Open subset D of the 2-dimensional topological manifold R^2
>>> Phi.inverse().display()
Phi^(-1): R^2 → D
    (x, y) ↦ (x, y) = (x/sqrt(x^2 + y^2 + 1), y/sqrt(x^2 + y^2 + 1))
```

Equivalently, one may use the notations `^(-1)` or `~` to get the inverse:

```
sage: Phi^(-1) is Phi.inverse()
True
sage: ~Phi is Phi.inverse()
True
```

```
>>> from sage.all import *
>>> Phi**(-Integer(1)) is Phi.inverse()
True
>>> ~Phi is Phi.inverse()
True
```

Check that `~Phi` is indeed the inverse of `Phi`:

```
sage: (~Phi)(q) == p
True
sage: Phi * ~Phi == R2.identity_map()
```

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```
True
sage: ~Phi * Phi == D.identity_map()
True
```

```
>>> from sage.all import *
>>> (~Phi)(q) == p
True
>>> Phi * ~Phi == R2.identity_map()
True
>>> ~Phi * Phi == D.identity_map()
True
```

The coordinate expression of the inverse homeomorphism:

```
sage: (~Phi).display()
Phi^(-1): R^2 → D
   (x, y) ↦ (x, y) = (x/sqrt(x^2 + y^2 + 1), y/sqrt(x^2 + y^2 + 1))
```

```
>>> from sage.all import *
>>> (~Phi).display()
Phi^(-1): R^2 → D
   (x, y) ↦ (x, y) = (x/sqrt(x^2 + y^2 + 1), y/sqrt(x^2 + y^2 + 1))
```

A special case of homeomorphism: the identity map of the open unit disk:

```
sage: id = D.identity_map() ; id
Identity map Id_D of the Open subset D of the 2-dimensional topological
 manifold R^2
sage: latex(id)
\mathrm{Id}_{D}
sage: id.parent()
Set of Morphisms from Open subset D of the 2-dimensional topological
 manifold R^2 to Open subset D of the 2-dimensional topological
 manifold R^2 in Join of Category of subobjects of sets and Category of
 manifolds over Real Field with 53 bits of precision
sage: id.parent() is Hom(D, D)
True
sage: id is Hom(D,D).one()  # the identity element of the monoid Hom(D,D)
True
```

```
>>> from sage.all import *
>>> id = D.identity_map() ; id
Identity map Id_D of the Open subset D of the 2-dimensional topological
 manifold R^2
>>> latex(id)
\mathrm{Id}_{D}
>>> id.parent()
Set of Morphisms from Open subset D of the 2-dimensional topological
 manifold R^2 to Open subset D of the 2-dimensional topological
 manifold R^2 in Join of Category of subobjects of sets and Category of
 manifolds over Real Field with 53 bits of precision
>>> id.parent() is Hom(D, D)
```

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```
True
>>> id is Hom(D,D).one()   # the identity element of the monoid Hom(D,D)
True
```

The identity map acting on a point:

```
sage: id(p)
Point on the 2-dimensional topological manifold R^2
sage: id(p) == p
True
sage: id(p) is p
True
```

```
>>> from sage.all import *
>>> id(p)
Point on the 2-dimensional topological manifold R^2
>>> id(p) == p
True
>>> id(p) is p
True
```

The coordinate expression of the identity map:

```
sage: id.display()
Id_D: D → D
   (x, y) ↦ (x, y)
```

```
>>> from sage.all import *
>>> id.display()
Id_D: D → D
   (x, y) ↦ (x, y)
```

The identity map is its own inverse:

```
sage: id^(-1) is id
True
sage: ~id is id
True
```

```
>>> from sage.all import *
>>> id**(-Integer(1)) is id
True
>>> ~id is id
True
```

add_expr(*chart1*, *chart2*, *coord_functions*)

Set a new coordinate representation of `self`.

The previous expressions with respect to other charts are kept. To clear them, use *set_expr()* instead.

INPUT:

- `chart1` – chart for the coordinates on the map's domain

- `chart2` – chart for the coordinates on the map's codomain
- `coord_functions` – the coordinate symbolic expression of the map in the above charts: list (or tuple) of the coordinates of the image expressed in terms of the coordinates of the considered point; if the dimension of the arrival manifold is 1, a single coordinate expression can be passed instead of a tuple with a single element

Warning

If the map has already expressions in other charts, it is the user's responsibility to make sure that the expression to be added is consistent with them.

EXAMPLES:

Polar representation of a planar rotation initially defined in Cartesian coordinates:

```
sage: M = Manifold(2, 'R^2', latex_name=r'\RR^2', structure='topological')  #␣
↪the Euclidean plane R^2
sage: c_xy.<x,y> = M.chart() # Cartesian coordinate on R^2
sage: U = M.open_subset('U', coord_def={c_xy: (y!=0, x<0)}) # the complement␣
↪of the segment y=0 and x>0
sage: c_cart = c_xy.restrict(U) # Cartesian coordinates on U
sage: c_spher.<r,ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi') # spherical␣
↪coordinates on U
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', latex_name=r'\RR^2', structure=
↪'topological')  # the Euclidean plane R^2
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪Cartesian coordinate on R^2
>>> U = M.open_subset('U', coord_def={c_xy: (y!=Integer(0), x<Integer(0))}) #␣
↪the complement of the segment y=0 and x>0
>>> c_cart = c_xy.restrict(U) # Cartesian coordinates on U
>>> c_spher = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi', names=('r', 'ph',)); (r,␣
↪ph,) = c_spher._first_ngens(2)# spherical coordinates on U
```

We construct the links between spherical coordinates and Cartesian ones:

```
sage: ch_cart_spher = c_cart.transition_map(c_spher, [sqrt(x*x+y*y), atan2(y,
↪x)])
sage: ch_cart_spher.set_inverse(r*cos(ph), r*sin(ph))
Check of the inverse coordinate transformation:
  x == x  *passed*
  y == y  *passed*
  r == r  *passed*
  ph == arctan2(r*sin(ph), r*cos(ph))  **failed**
NB: a failed report can reflect a mere lack of simplification.
sage: rot = U.continuous_map(U, ((x - sqrt(3)*y)/2, (sqrt(3)*x + y)/2),
....:                          name='R')
sage: rot.display(c_cart, c_cart)
R: U → U
   (x, y) ↦ (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
```

```
>>> from sage.all import *
>>> ch_cart_spher = c_cart.transition_map(c_spher, [sqrt(x*x+y*y), atan2(y,
↪x)])
>>> ch_cart_spher.set_inverse(r*cos(ph), r*sin(ph))
Check of the inverse coordinate transformation:
  x == x  *passed*
  y == y  *passed*
  r == r  *passed*
  ph == arctan2(r*sin(ph), r*cos(ph))  **failed**
NB: a failed report can reflect a mere lack of simplification.
>>> rot = U.continuous_map(U, ((x - sqrt(Integer(3))*y)/Integer(2),␣
↪(sqrt(Integer(3))*x + y)/Integer(2)),
...                         name='R')
>>> rot.display(c_cart, c_cart)
R: U → U
   (x, y) ↦ (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
```

If we calculate the expression in terms of spherical coordinates, via the method `display()`, we notice some difficulties in `arctan2` simplifications:

```
sage: rot.display(c_spher, c_spher)
R: U → U
   (r, ph) ↦ (r, arctan2(1/2*(sqrt(3)*cos(ph) + sin(ph))*r, -1/
↪2*(sqrt(3)*sin(ph) - cos(ph))*r))
```

```
>>> from sage.all import *
>>> rot.display(c_spher, c_spher)
R: U → U
   (r, ph) ↦ (r, arctan2(1/2*(sqrt(3)*cos(ph) + sin(ph))*r, -1/
↪2*(sqrt(3)*sin(ph) - cos(ph))*r))
```

Therefore, we use the method `add_expr()` to set the spherical-coordinate expression by hand:

```
sage: rot.add_expr(c_spher, c_spher, (r, ph+pi/3))
sage: rot.display(c_spher, c_spher)
R: U → U
   (r, ph) ↦ (r, 1/3*pi + ph)
```

```
>>> from sage.all import *
>>> rot.add_expr(c_spher, c_spher, (r, ph+pi/Integer(3)))
>>> rot.display(c_spher, c_spher)
R: U → U
   (r, ph) ↦ (r, 1/3*pi + ph)
```

The call to `add_expr()` has not deleted the expression in terms of Cartesian coordinates, as we can check by printing the internal dictionary `_coord_expression`, which stores the various internal representations of the continuous map:

```
sage: rot._coord_expression # random (dictionary output)
{(Chart (U, (x, y)), Chart (U, (x, y))):
 Coordinate functions (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
  on the Chart (U, (x, y)),
```

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```
 (Chart (U, (r, ph)), Chart (U, (r, ph))):
  Coordinate functions (r, 1/3*pi + ph) on the Chart (U, (r, ph))}
```

```
>>> from sage.all import *
>>> rot._coord_expression # random (dictionary output)
{(Chart (U, (x, y)), Chart (U, (x, y))):
 Coordinate functions (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
  on the Chart (U, (x, y)),
 (Chart (U, (r, ph)), Chart (U, (r, ph))):
  Coordinate functions (r, 1/3*pi + ph) on the Chart (U, (r, ph))}
```

If, on the contrary, we use `set_expr()`, the expression in Cartesian coordinates is lost:

```
sage: rot.set_expr(c_spher, c_spher, (r, ph+pi/3))
sage: rot._coord_expression
{(Chart (U, (r, ph)), Chart (U, (r, ph))):
 Coordinate functions (r, 1/3*pi + ph) on the Chart (U, (r, ph))}
```

```
>>> from sage.all import *
>>> rot.set_expr(c_spher, c_spher, (r, ph+pi/Integer(3)))
>>> rot._coord_expression
{(Chart (U, (r, ph)), Chart (U, (r, ph))):
 Coordinate functions (r, 1/3*pi + ph) on the Chart (U, (r, ph))}
```

It is recovered (thanks to the known change of coordinates) by a call to `display()`:

```
sage: rot.display(c_cart, c_cart)
R: U → U
   (x, y) ↦ (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)

sage: rot._coord_expression # random (dictionary output)
{(Chart (U, (x, y)), Chart (U, (x, y))):
 Coordinate functions (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
  on the Chart (U, (x, y)),
 (Chart (U, (r, ph)), Chart (U, (r, ph))):
 Coordinate functions (r, 1/3*pi + ph) on the Chart (U, (r, ph))}
```

```
>>> from sage.all import *
>>> rot.display(c_cart, c_cart)
R: U → U
   (x, y) ↦ (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)

>>> rot._coord_expression # random (dictionary output)
{(Chart (U, (x, y)), Chart (U, (x, y))):
 Coordinate functions (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
  on the Chart (U, (x, y)),
 (Chart (U, (r, ph)), Chart (U, (r, ph))):
 Coordinate functions (r, 1/3*pi + ph) on the Chart (U, (r, ph))}
```

The rotation can be applied to a point by means of either coordinate system:

```
sage: p = M.point((1,2))  #  p defined by its Cartesian coord.
sage: q = rot(p)  # q is computed by means of Cartesian coord.
sage: p1 = M.point((sqrt(5), arctan(2)), chart=c_spher) # p1 is defined only␣
↪in terms of c_spher
sage: q1 = rot(p1) # computation by means of spherical coordinates
sage: q1 == q
True
```

```
>>> from sage.all import *
>>> p = M.point((Integer(1),Integer(2)))  #  p defined by its Cartesian coord.
>>> q = rot(p)  # q is computed by means of Cartesian coord.
>>> p1 = M.point((sqrt(Integer(5)), arctan(Integer(2))), chart=c_spher) # p1␣
↪is defined only in terms of c_spher
>>> q1 = rot(p1) # computation by means of spherical coordinates
>>> q1 == q
True
```

add_expression(*chart1*, *chart2*, *coord_functions*)

Set a new coordinate representation of `self`.

The previous expressions with respect to other charts are kept. To clear them, use *set_expr()* instead.

INPUT:

- `chart1` – chart for the coordinates on the map's domain
- `chart2` – chart for the coordinates on the map's codomain
- `coord_functions` – the coordinate symbolic expression of the map in the above charts: list (or tuple) of the coordinates of the image expressed in terms of the coordinates of the considered point; if the dimension of the arrival manifold is 1, a single coordinate expression can be passed instead of a tuple with a single element

Warning

If the map has already expressions in other charts, it is the user's responsibility to make sure that the expression to be added is consistent with them.

EXAMPLES:

Polar representation of a planar rotation initially defined in Cartesian coordinates:

```
sage: M = Manifold(2, 'R^2', latex_name=r'\RR^2', structure='topological')  #␣
↪the Euclidean plane R^2
sage: c_xy.<x,y> = M.chart() # Cartesian coordinate on R^2
sage: U = M.open_subset('U', coord_def={c_xy: (y!=0, x<0)}) # the complement␣
↪of the segment y=0 and x>0
sage: c_cart = c_xy.restrict(U) # Cartesian coordinates on U
sage: c_spher.<r,ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi') # spherical␣
↪coordinates on U
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', latex_name=r'\RR^2', structure=
↪'topological')  # the Euclidean plane R^2
```

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```
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪Cartesian coordinate on R^2
>>> U = M.open_subset('U', coord_def={c_xy: (y!=Integer(0), x<Integer(0))}) #␣
↪the complement of the segment y=0 and x>0
>>> c_cart = c_xy.restrict(U) # Cartesian coordinates on U
>>> c_spher = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi', names=('r', 'ph',)); (r,␣
↪ph,) = c_spher._first_ngens(2)# spherical coordinates on U
```

We construct the links between spherical coordinates and Cartesian ones:

```
sage: ch_cart_spher = c_cart.transition_map(c_spher, [sqrt(x*x+y*y), atan2(y,
↪x)])
sage: ch_cart_spher.set_inverse(r*cos(ph), r*sin(ph))
Check of the inverse coordinate transformation:
  x == x  *passed*
  y == y  *passed*
  r == r  *passed*
  ph == arctan2(r*sin(ph), r*cos(ph))  **failed**
NB: a failed report can reflect a mere lack of simplification.
sage: rot = U.continuous_map(U, ((x - sqrt(3)*y)/2, (sqrt(3)*x + y)/2),
....:                         name='R')
sage: rot.display(c_cart, c_cart)
R: U → U
   (x, y) ↦ (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
```

```
>>> from sage.all import *
>>> ch_cart_spher = c_cart.transition_map(c_spher, [sqrt(x*x+y*y), atan2(y,
↪x)])
>>> ch_cart_spher.set_inverse(r*cos(ph), r*sin(ph))
Check of the inverse coordinate transformation:
  x == x  *passed*
  y == y  *passed*
  r == r  *passed*
  ph == arctan2(r*sin(ph), r*cos(ph))  **failed**
NB: a failed report can reflect a mere lack of simplification.
>>> rot = U.continuous_map(U, ((x - sqrt(Integer(3))*y)/Integer(2),␣
↪(sqrt(Integer(3))*x + y)/Integer(2)),
...                         name='R')
>>> rot.display(c_cart, c_cart)
R: U → U
   (x, y) ↦ (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
```

If we calculate the expression in terms of spherical coordinates, via the method `display()`, we notice some difficulties in `arctan2` simplifications:

```
sage: rot.display(c_spher, c_spher)
R: U → U
   (r, ph) ↦ (r, arctan2(1/2*(sqrt(3)*cos(ph) + sin(ph))*r, -1/
↪2*(sqrt(3)*sin(ph) - cos(ph))*r))
```

```
>>> from sage.all import *
>>> rot.display(c_spher, c_spher)
```

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```
R: U → U
   (r, ph) ↦ (r, arctan2(1/2*(sqrt(3)*cos(ph) + sin(ph))*r, -1/
↪2*(sqrt(3)*sin(ph) - cos(ph))*r))
```

Therefore, we use the method `add_expr()` to set the spherical-coordinate expression by hand:

```
sage: rot.add_expr(c_spher, c_spher, (r, ph+pi/3))
sage: rot.display(c_spher, c_spher)
R: U → U
   (r, ph) ↦ (r, 1/3*pi + ph)
```

```
>>> from sage.all import *
>>> rot.add_expr(c_spher, c_spher, (r, ph+pi/Integer(3)))
>>> rot.display(c_spher, c_spher)
R: U → U
   (r, ph) ↦ (r, 1/3*pi + ph)
```

The call to `add_expr()` has not deleted the expression in terms of Cartesian coordinates, as we can check by printing the internal dictionary `_coord_expression`, which stores the various internal representations of the continuous map:

```
sage: rot._coord_expression # random (dictionary output)
{(Chart (U, (x, y)), Chart (U, (x, y))):
 Coordinate functions (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
  on the Chart (U, (x, y)),
 (Chart (U, (r, ph)), Chart (U, (r, ph))):
  Coordinate functions (r, 1/3*pi + ph) on the Chart (U, (r, ph))}
```

```
>>> from sage.all import *
>>> rot._coord_expression # random (dictionary output)
{(Chart (U, (x, y)), Chart (U, (x, y))):
 Coordinate functions (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
  on the Chart (U, (x, y)),
 (Chart (U, (r, ph)), Chart (U, (r, ph))):
  Coordinate functions (r, 1/3*pi + ph) on the Chart (U, (r, ph))}
```

If, on the contrary, we use `set_expr()`, the expression in Cartesian coordinates is lost:

```
sage: rot.set_expr(c_spher, c_spher, (r, ph+pi/3))
sage: rot._coord_expression
{(Chart (U, (r, ph)), Chart (U, (r, ph))):
 Coordinate functions (r, 1/3*pi + ph) on the Chart (U, (r, ph))}
```

```
>>> from sage.all import *
>>> rot.set_expr(c_spher, c_spher, (r, ph+pi/Integer(3)))
>>> rot._coord_expression
{(Chart (U, (r, ph)), Chart (U, (r, ph))):
 Coordinate functions (r, 1/3*pi + ph) on the Chart (U, (r, ph))}
```

It is recovered (thanks to the known change of coordinates) by a call to `display()`:

```
sage: rot.display(c_cart, c_cart)
R: U → U
   (x, y) ↦ (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)

sage: rot._coord_expression # random (dictionary output)
{(Chart (U, (x, y)), Chart (U, (x, y))):
 Coordinate functions (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
  on the Chart (U, (x, y)),
 (Chart (U, (r, ph)), Chart (U, (r, ph))):
 Coordinate functions (r, 1/3*pi + ph) on the Chart (U, (r, ph))}
```

```
>>> from sage.all import *
>>> rot.display(c_cart, c_cart)
R: U → U
   (x, y) ↦ (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)

>>> rot._coord_expression # random (dictionary output)
{(Chart (U, (x, y)), Chart (U, (x, y))):
 Coordinate functions (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
  on the Chart (U, (x, y)),
 (Chart (U, (r, ph)), Chart (U, (r, ph))):
 Coordinate functions (r, 1/3*pi + ph) on the Chart (U, (r, ph))}
```

The rotation can be applied to a point by means of either coordinate system:

```
sage: p = M.point((1,2))  #  p defined by its Cartesian coord.
sage: q = rot(p)  # q is computed by means of Cartesian coord.
sage: p1 = M.point((sqrt(5), arctan(2)), chart=c_spher) # p1 is defined only␣
↪in terms of c_spher
sage: q1 = rot(p1) # computation by means of spherical coordinates
sage: q1 == q
True
```

```
>>> from sage.all import *
>>> p = M.point((Integer(1),Integer(2)))  #  p defined by its Cartesian coord.
>>> q = rot(p)  # q is computed by means of Cartesian coord.
>>> p1 = M.point((sqrt(Integer(5)), arctan(Integer(2))), chart=c_spher) # p1␣
↪is defined only in terms of c_spher
>>> q1 = rot(p1) # computation by means of spherical coordinates
>>> q1 == q
True
```

coord_functions(*chart1=None*, *chart2=None*)

Return the functions of the coordinates representing `self` in a given pair of charts.

If these functions are not already known, they are computed from known ones by means of change-of-chart formulas.

INPUT:

- `chart1` – (default: `None`) chart on the domain of `self`; if `None`, the domain's default chart is assumed
- `chart2` – (default: `None`) chart on the codomain of `self`; if `None`, the codomain's default chart is assumed

OUTPUT:

- a `MultiCoordFunction` representing the continuous map in the above two charts

EXAMPLES:

Continuous map from a 2-dimensional manifold to a 3-dimensional one:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: N = Manifold(3, 'N', structure='topological')
sage: c_uv.<u,v> = M.chart()
sage: c_xyz.<x,y,z> = N.chart()
sage: Phi = M.continuous_map(N, (u*v, u/v, u+v), name='Phi',
....:                        latex_name=r'\Phi')
sage: Phi.display()
Phi: M → N
   (u, v) ↦ (x, y, z) = (u*v, u/v, u + v)
sage: Phi.coord_functions(c_uv, c_xyz)
Coordinate functions (u*v, u/v, u + v) on the Chart (M, (u, v))
sage: Phi.coord_functions() # equivalent to above since 'uv' and 'xyz' are␣
↪default charts
Coordinate functions (u*v, u/v, u + v) on the Chart (M, (u, v))
sage: type(Phi.coord_functions())
<class 'sage.manifolds.chart_func.MultiCoordFunction'>
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> N = Manifold(Integer(3), 'N', structure='topological')
>>> c_uv = M.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> c_xyz = N.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> Phi = M.continuous_map(N, (u*v, u/v, u+v), name='Phi',
...                        latex_name=r'\Phi')
>>> Phi.display()
Phi: M → N
   (u, v) ↦ (x, y, z) = (u*v, u/v, u + v)
>>> Phi.coord_functions(c_uv, c_xyz)
Coordinate functions (u*v, u/v, u + v) on the Chart (M, (u, v))
>>> Phi.coord_functions() # equivalent to above since 'uv' and 'xyz' are␣
↪default charts
Coordinate functions (u*v, u/v, u + v) on the Chart (M, (u, v))
>>> type(Phi.coord_functions())
<class 'sage.manifolds.chart_func.MultiCoordFunction'>
```

Coordinate representation in other charts:

```
sage: c_UV.<U,V> = M.chart()  # new chart on M
sage: ch_uv_UV = c_uv.transition_map(c_UV, [u-v, u+v])
sage: ch_uv_UV.inverse()(U,V)
(1/2*U + 1/2*V, -1/2*U + 1/2*V)
sage: c_XYZ.<X,Y,Z> = N.chart() # new chart on N
sage: ch_xyz_XYZ = c_xyz.transition_map(c_XYZ,
....:                                   [2*x-3*y+z, y+z-x, -x+2*y-z])
sage: ch_xyz_XYZ.inverse()(X,Y,Z)
(3*X + Y + 4*Z, 2*X + Y + 3*Z, X + Y + Z)
```

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```
sage: Phi.coord_functions(c_UV, c_xyz)
Coordinate functions (-1/4*U^2 + 1/4*V^2, -(U + V)/(U - V), V) on
 the Chart (M, (U, V))
sage: Phi.coord_functions(c_uv, c_XYZ)
Coordinate functions (((2*u + 1)*v^2 + u*v - 3*u)/v,
 -((u - 1)*v^2 - u*v - u)/v, -((u + 1)*v^2 + u*v - 2*u)/v) on the
 Chart (M, (u, v))
sage: Phi.coord_functions(c_UV, c_XYZ)
Coordinate functions
 (-1/2*(U^3 - (U - 2)*V^2 + V^3 - (U^2 + 2*U + 6)*V - 6*U)/(U - V),
  1/4*(U^3 - (U + 4)*V^2 + V^3 - (U^2 - 4*U + 4)*V - 4*U)/(U - V),
  1/4*(U^3 - (U - 4)*V^2 + V^3 - (U^2 + 4*U + 8)*V - 8*U)/(U - V))
 on the Chart (M, (U, V))
```

```
>>> from sage.all import *
>>> c_UV = M.chart(names=('U', 'V',)); (U, V,) = c_UV._first_ngens(2)# new␣
↪chart on M
>>> ch_uv_UV = c_uv.transition_map(c_UV, [u-v, u+v])
>>> ch_uv_UV.inverse()(U,V)
(1/2*U + 1/2*V, -1/2*U + 1/2*V)
>>> c_XYZ = N.chart(names=('X', 'Y', 'Z',)); (X, Y, Z,) = c_XYZ._first_
↪ngens(3)# new chart on N
>>> ch_xyz_XYZ = c_xyz.transition_map(c_XYZ,
...                                    [Integer(2)*x-Integer(3)*y+z, y+z-x, -
↪x+Integer(2)*y-z])
>>> ch_xyz_XYZ.inverse()(X,Y,Z)
(3*X + Y + 4*Z, 2*X + Y + 3*Z, X + Y + Z)
>>> Phi.coord_functions(c_UV, c_xyz)
Coordinate functions (-1/4*U^2 + 1/4*V^2, -(U + V)/(U - V), V) on
 the Chart (M, (U, V))
>>> Phi.coord_functions(c_uv, c_XYZ)
Coordinate functions (((2*u + 1)*v^2 + u*v - 3*u)/v,
 -((u - 1)*v^2 - u*v - u)/v, -((u + 1)*v^2 + u*v - 2*u)/v) on the
 Chart (M, (u, v))
>>> Phi.coord_functions(c_UV, c_XYZ)
Coordinate functions
 (-1/2*(U^3 - (U - 2)*V^2 + V^3 - (U^2 + 2*U + 6)*V - 6*U)/(U - V),
  1/4*(U^3 - (U + 4)*V^2 + V^3 - (U^2 - 4*U + 4)*V - 4*U)/(U - V),
  1/4*(U^3 - (U - 4)*V^2 + V^3 - (U^2 + 4*U + 8)*V - 8*U)/(U - V))
 on the Chart (M, (U, V))
```

Coordinate representation with respect to a subchart in the domain:

```
sage: A = M.open_subset('A', coord_def={c_uv: u>0})
sage: Phi.coord_functions(c_uv.restrict(A), c_xyz)
Coordinate functions (u*v, u/v, u + v) on the Chart (A, (u, v))
```

```
>>> from sage.all import *
>>> A = M.open_subset('A', coord_def={c_uv: u>Integer(0)})
>>> Phi.coord_functions(c_uv.restrict(A), c_xyz)
Coordinate functions (u*v, u/v, u + v) on the Chart (A, (u, v))
```

Coordinate representation with respect to a superchart in the codomain:

```
sage: B = N.open_subset('B', coord_def={c_xyz: x<0})
sage: c_xyz_B = c_xyz.restrict(B)
sage: Phi1 = M.continuous_map(B, {(c_uv, c_xyz_B): (u*v, u/v, u+v)})
sage: Phi1.coord_functions(c_uv, c_xyz_B) # definition charts
Coordinate functions (u*v, u/v, u + v) on the Chart (M, (u, v))
sage: Phi1.coord_functions(c_uv, c_xyz) # c_xyz = superchart of c_xyz_B
Coordinate functions (u*v, u/v, u + v) on the Chart (M, (u, v))
```

```
>>> from sage.all import *
>>> B = N.open_subset('B', coord_def={c_xyz: x<Integer(0)})
>>> c_xyz_B = c_xyz.restrict(B)
>>> Phi1 = M.continuous_map(B, {(c_uv, c_xyz_B): (u*v, u/v, u+v)})
>>> Phi1.coord_functions(c_uv, c_xyz_B) # definition charts
Coordinate functions (u*v, u/v, u + v) on the Chart (M, (u, v))
>>> Phi1.coord_functions(c_uv, c_xyz) # c_xyz = superchart of c_xyz_B
Coordinate functions (u*v, u/v, u + v) on the Chart (M, (u, v))
```

Coordinate representation with respect to a pair `(subchart, superchart)`:

```
sage: Phi1.coord_functions(c_uv.restrict(A), c_xyz)
Coordinate functions (u*v, u/v, u + v) on the Chart (A, (u, v))
```

```
>>> from sage.all import *
>>> Phi1.coord_functions(c_uv.restrict(A), c_xyz)
Coordinate functions (u*v, u/v, u + v) on the Chart (A, (u, v))
```

Same example with SymPy as the symbolic calculus engine:

```
sage: M.set_calculus_method('sympy')
sage: N.set_calculus_method('sympy')
sage: Phi = M.continuous_map(N, (u*v, u/v, u+v), name='Phi',
....:                        latex_name=r'\Phi')
sage: Phi.coord_functions(c_uv, c_xyz)
Coordinate functions (u*v, u/v, u + v) on the Chart (M, (u, v))
sage: Phi.coord_functions(c_UV, c_xyz)
Coordinate functions (-U**2/4 + V**2/4, (-U - V)/(U - V), V) on the Chart (M,
↪(U, V))
sage: Phi.coord_functions(c_UV, c_XYZ)
Coordinate functions ((-U**3 + U**2*V + U*V**2 + 2*U*V + 6*U - V**3
 - 2*V**2 + 6*V)/(2*(U - V)), (U**3/4 - U**2*V/4 - U*V**2/4 + U*V
 - U + V**3/4 - V**2 - V)/(U - V), (U**3 - U**2*V - U*V**2 - 4*U*V
 - 8*U + V**3 + 4*V**2 - 8*V)/(4*(U - V))) on the Chart (M, (U, V))
```

```
>>> from sage.all import *
>>> M.set_calculus_method('sympy')
>>> N.set_calculus_method('sympy')
>>> Phi = M.continuous_map(N, (u*v, u/v, u+v), name='Phi',
...                        latex_name=r'\Phi')
>>> Phi.coord_functions(c_uv, c_xyz)
Coordinate functions (u*v, u/v, u + v) on the Chart (M, (u, v))
>>> Phi.coord_functions(c_UV, c_xyz)
```

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```
Coordinate functions (-U**2/4 + V**2/4, (-U - V)/(U - V), V) on the Chart (M,␣
↪(U, V))
>>> Phi.coord_functions(c_UV, c_XYZ)
Coordinate functions ((-U**3 + U**2*V + U*V**2 + 2*U*V + 6*U - V**3
 - 2*V**2 + 6*V)/(2*(U - V)), (U**3/4 - U**2*V/4 - U*V**2/4 + U*V
 - U + V**3/4 - V**2 - V)/(U - V), (U**3 - U**2*V - U*V**2 - 4*U*V
 - 8*U + V**3 + 4*V**2 - 8*V)/(4*(U - V))) on the Chart (M, (U, V))
```

disp(*chart1=None*, *chart2=None*)

Display the expression of `self` in one or more pair of charts.

If the expression is not known already, it is computed from some expression in other charts by means of change-of-coordinate formulas.

INPUT:

- `chart1` – (default: `None`) chart on the domain of `self`; if `None`, the display is performed on all the charts on the domain in which the map is known or computable via some change of coordinates
- `chart2` – (default: `None`) chart on the codomain of `self`; if `None`, the display is performed on all the charts on the codomain in which the map is known or computable via some change of coordinates

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

EXAMPLES:

A simple reparametrization:

```
sage: R.<t> = manifolds.RealLine()
sage: I = R.open_interval(0, 2*pi)
sage: J = R.open_interval(2*pi, 6*pi)
sage: h = J.continuous_map(I, ((t-2*pi)/2,), name='h')
sage: h.display()
h: (2*pi, 6*pi) → (0, 2*pi)
   t ↦ t = -pi + 1/2*t
sage: latex(h.display())
\begin{array}{llcl} h:& \left(2 \, \pi, 6 \, \pi\right) &
 \longrightarrow & \left(0, 2 \, \pi\right) \\ & t & \longmapsto &
 t = -\pi + \frac{1}{2} \, t \end{array}
```

```
>>> from sage.all import *
>>> R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1)
>>> I = R.open_interval(Integer(0), Integer(2)*pi)
>>> J = R.open_interval(Integer(2)*pi, Integer(6)*pi)
>>> h = J.continuous_map(I, ((t-Integer(2)*pi)/Integer(2),), name='h')
>>> h.display()
h: (2*pi, 6*pi) → (0, 2*pi)
   t ↦ t = -pi + 1/2*t
>>> latex(h.display())
\begin{array}{llcl} h:& \left(2 \, \pi, 6 \, \pi\right) &
 \longrightarrow & \left(0, 2 \, \pi\right) \\ & t & \longmapsto &
 t = -\pi + \frac{1}{2} \, t \end{array}
```

Standard embedding of the sphere S^2 in $\mathbf{R}^3$:

```
sage: M = Manifold(2, 'S^2', structure='topological') # the 2-dimensional␣
↪sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: N = Manifold(3, 'R^3', latex_name=r'\RR^3', structure='topological')  #␣
↪R^3
sage: c_cart.<X,Y,Z> = N.chart()  # Cartesian coordinates on R^3
sage: Phi = M.continuous_map(N,
....:   {(c_xy, c_cart): [2*x/(1+x^2+y^2), 2*y/(1+x^2+y^2), (x^2+y^2-1)/(1+x^
↪2+y^2)],
....:    (c_uv, c_cart): [2*u/(1+u^2+v^2), 2*v/(1+u^2+v^2), (1-u^2-v^2)/(1+u^
↪2+v^2)]},
....:   name='Phi', latex_name=r'\Phi')
sage: Phi.display(c_xy, c_cart)
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 +␣
↪y^2 - 1)/(x^2 + y^2 + 1))
sage: Phi.display(c_uv, c_cart)
Phi: S^2 → R^3
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^2 +␣
↪v^2 - 1)/(u^2 + v^2 + 1))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', structure='topological') # the 2-
↪dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> N = Manifold(Integer(3), 'R^3', latex_name=r'\RR^3', structure=
↪'topological')  # R^3
>>> c_cart = N.chart(names=('X', 'Y', 'Z',)); (X, Y, Z,) = c_cart._first_
↪ngens(3)# Cartesian coordinates on R^3
>>> Phi = M.continuous_map(N,
...   {(c_xy, c_cart): [Integer(2)*x/(Integer(1)+x**Integer(2)+y**Integer(2)),
↪ Integer(2)*y/(Integer(1)+x**Integer(2)+y**Integer(2)),␣
↪(x**Integer(2)+y**Integer(2)-Integer(1))/
↪(Integer(1)+x**Integer(2)+y**Integer(2))],
...    (c_uv, c_cart): [Integer(2)*u/(Integer(1)+u**Integer(2)+v**Integer(2)),
↪ Integer(2)*v/(Integer(1)+u**Integer(2)+v**Integer(2)), (Integer(1)-
↪u**Integer(2)-v**Integer(2))/(Integer(1)+u**Integer(2)+v**Integer(2))]},
...   name='Phi', latex_name=r'\Phi')
>>> Phi.display(c_xy, c_cart)
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 +␣
↪y^2 - 1)/(x^2 + y^2 + 1))
```

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```
>>> Phi.display(c_uv, c_cart)
Phi: S^2 → R^3
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^2 +
↪v^2 - 1)/(u^2 + v^2 + 1))
```

The LaTeX output of that embedding is:

```
sage: latex(Phi.display(c_xy, c_cart))
\begin{array}{llcl} \Phi:& S^2 & \longrightarrow & \RR^3
 \\ \text{on}\ U : & \left(x, y\right) & \longmapsto
 & \left(X, Y, Z\right) = \left(\frac{2 \, x}{x^{2} + y^{2} + 1},
   \frac{2 \, y}{x^{2} + y^{2} + 1},
   \frac{x^{2} + y^{2} - 1}{x^{2} + y^{2} + 1}\right)
 \end{array}
```

```
>>> from sage.all import *
>>> latex(Phi.display(c_xy, c_cart))
\begin{array}{llcl} \Phi:& S^2 & \longrightarrow & \RR^3
 \\ \text{on}\ U : & \left(x, y\right) & \longmapsto
 & \left(X, Y, Z\right) = \left(\frac{2 \, x}{x^{2} + y^{2} + 1},
   \frac{2 \, y}{x^{2} + y^{2} + 1},
   \frac{x^{2} + y^{2} - 1}{x^{2} + y^{2} + 1}\right)
 \end{array}
```

If the argument `chart2` is not specified, the display is performed on all the charts on the codomain in which the map is known or computable via some change of coordinates (here only one chart: `c_cart`):

```
sage: Phi.display(c_xy)
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 +
↪y^2 - 1)/(x^2 + y^2 + 1))
```

```
>>> from sage.all import *
>>> Phi.display(c_xy)
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 +
↪y^2 - 1)/(x^2 + y^2 + 1))
```

Similarly, if the argument `chart1` is omitted, the display is performed on all the charts on the domain of `Phi` in which the map is known or computable via some change of coordinates:

```
sage: Phi.display(chart2=c_cart)
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 +
↪y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^2 +
↪v^2 - 1)/(u^2 + v^2 + 1))
```

```
>>> from sage.all import *
>>> Phi.display(chart2=c_cart)
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 +
```

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```
↪y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^2 +␣
↪v^2 - 1)/(u^2 + v^2 + 1))
```

If neither `chart1` nor `chart2` is specified, the display is performed on all the pair of charts in which `Phi` is known or computable via some change of coordinates:

```
sage: Phi.display()
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 +␣
↪y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^2 +␣
↪v^2 - 1)/(u^2 + v^2 + 1))
```

```
>>> from sage.all import *
>>> Phi.display()
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 +␣
↪y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^2 +␣
↪v^2 - 1)/(u^2 + v^2 + 1))
```

If a chart covers entirely the map's domain, the mention "on …" is omitted:

```
sage: Phi.restrict(U).display()
Phi: U → R^3
   (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 + y^2␣
↪- 1)/(x^2 + y^2 + 1))
```

```
>>> from sage.all import *
>>> Phi.restrict(U).display()
Phi: U → R^3
   (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 + y^2␣
↪- 1)/(x^2 + y^2 + 1))
```

A shortcut of `display()` is `disp()`:

```
sage: Phi.disp()
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 +␣
↪y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^2 +␣
↪v^2 - 1)/(u^2 + v^2 + 1))
```

```
>>> from sage.all import *
>>> Phi.disp()
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 +␣
↪y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^2 +␣
↪v^2 - 1)/(u^2 + v^2 + 1))
```

Display when SymPy is the symbolic engine:

```
sage: M.set_calculus_method('sympy')
sage: N.set_calculus_method('sympy')
sage: Phi.display(c_xy, c_cart)
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x**2 + y**2 + 1),
 2*y/(x**2 + y**2 + 1), (x**2 + y**2 - 1)/(x**2 + y**2 + 1))
sage: latex(Phi.display(c_xy, c_cart))
\begin{array}{llcl} \Phi:& S^2 & \longrightarrow & \RR^3
 \\ \text{on}\ U : & \left(x, y\right) & \longmapsto
 & \left(X, Y, Z\right) = \left(\frac{2 x}{x^{2} + y^{2} + 1},
   \frac{2 y}{x^{2} + y^{2} + 1},
   \frac{x^{2} + y^{2} - 1}{x^{2} + y^{2} + 1}\right)
 \end{array}
```

```
>>> from sage.all import *
>>> M.set_calculus_method('sympy')
>>> N.set_calculus_method('sympy')
>>> Phi.display(c_xy, c_cart)
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x**2 + y**2 + 1),
 2*y/(x**2 + y**2 + 1), (x**2 + y**2 - 1)/(x**2 + y**2 + 1))
>>> latex(Phi.display(c_xy, c_cart))
\begin{array}{llcl} \Phi:& S^2 & \longrightarrow & \RR^3
 \\ \text{on}\ U : & \left(x, y\right) & \longmapsto
 & \left(X, Y, Z\right) = \left(\frac{2 x}{x^{2} + y^{2} + 1},
   \frac{2 y}{x^{2} + y^{2} + 1},
   \frac{x^{2} + y^{2} - 1}{x^{2} + y^{2} + 1}\right)
 \end{array}
```

display(*chart1=None*, *chart2=None*)

Display the expression of `self` in one or more pair of charts.

If the expression is not known already, it is computed from some expression in other charts by means of change-of-coordinate formulas.

INPUT:

- `chart1` – (default: `None`) chart on the domain of `self`; if `None`, the display is performed on all the charts on the domain in which the map is known or computable via some change of coordinates
- `chart2` – (default: `None`) chart on the codomain of `self`; if `None`, the display is performed on all the charts on the codomain in which the map is known or computable via some change of coordinates

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

EXAMPLES:

A simple reparametrization:

```
sage: R.<t> = manifolds.RealLine()
sage: I = R.open_interval(0, 2*pi)
sage: J = R.open_interval(2*pi, 6*pi)
sage: h = J.continuous_map(I, ((t-2*pi)/2,), name='h')
sage: h.display()
h: (2*pi, 6*pi) → (0, 2*pi)
   t ↦ t = -pi + 1/2*t
```

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```
sage: latex(h.display())
\begin{array}{llcl} h:& \left(2 \, \pi, 6 \, \pi\right) &
 \longrightarrow & \left(0, 2 \, \pi\right) \\ & t & \longmapsto &
 t = -\pi + \frac{1}{2} \, t \end{array}
```

```
>>> from sage.all import *
>>> R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1)
>>> I = R.open_interval(Integer(0), Integer(2)*pi)
>>> J = R.open_interval(Integer(2)*pi, Integer(6)*pi)
>>> h = J.continuous_map(I, ((t-Integer(2)*pi)/Integer(2),), name='h')
>>> h.display()
h: (2*pi, 6*pi) → (0, 2*pi)
   t ↦ t = -pi + 1/2*t
>>> latex(h.display())
\begin{array}{llcl} h:& \left(2 \, \pi, 6 \, \pi\right) &
 \longrightarrow & \left(0, 2 \, \pi\right) \\ & t & \longmapsto &
 t = -\pi + \frac{1}{2} \, t \end{array}
```

Standard embedding of the sphere S^2 in $\mathbf{R}^3$:

```
sage: M = Manifold(2, 'S^2', structure='topological') # the 2-dimensional␣
↪sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: N = Manifold(3, 'R^3', latex_name=r'\RR^3', structure='topological')  #␣
↪R^3
sage: c_cart.<X,Y,Z> = N.chart()  # Cartesian coordinates on R^3
sage: Phi = M.continuous_map(N,
....:   {(c_xy, c_cart): [2*x/(1+x^2+y^2), 2*y/(1+x^2+y^2), (x^2+y^2-1)/(1+x^
↪2+y^2)],
....:    (c_uv, c_cart): [2*u/(1+u^2+v^2), 2*v/(1+u^2+v^2), (1-u^2-v^2)/(1+u^
↪2+v^2)]},
....:   name='Phi', latex_name=r'\Phi')
sage: Phi.display(c_xy, c_cart)
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 +␣
↪y^2 - 1)/(x^2 + y^2 + 1))
sage: Phi.display(c_uv, c_cart)
Phi: S^2 → R^3
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^2 +␣
↪v^2 - 1)/(u^2 + v^2 + 1))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', structure='topological') # the 2-
↪dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
```

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```
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2) #␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> N = Manifold(Integer(3), 'R^3', latex_name=r'\RR^3', structure=
↪'topological')  # R^3
>>> c_cart = N.chart(names=('X', 'Y', 'Z',)); (X, Y, Z,) = c_cart._first_
↪ngens(3)# Cartesian coordinates on R^3
>>> Phi = M.continuous_map(N,
...   {(c_xy, c_cart): [Integer(2)*x/(Integer(1)+x**Integer(2)+y**Integer(2)),
↪ Integer(2)*y/(Integer(1)+x**Integer(2)+y**Integer(2)),␣
↪(x**Integer(2)+y**Integer(2)-Integer(1))/
↪(Integer(1)+x**Integer(2)+y**Integer(2))],
...    (c_uv, c_cart): [Integer(2)*u/(Integer(1)+u**Integer(2)+v**Integer(2)),
↪ Integer(2)*v/(Integer(1)+u**Integer(2)+v**Integer(2)), (Integer(1)-
↪u**Integer(2)-v**Integer(2))/(Integer(1)+u**Integer(2)+v**Integer(2))]},
...   name='Phi', latex_name=r'\Phi')
>>> Phi.display(c_xy, c_cart)
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 +␣
↪y^2 - 1)/(x^2 + y^2 + 1))
>>> Phi.display(c_uv, c_cart)
Phi: S^2 → R^3
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^2 +␣
↪v^2 - 1)/(u^2 + v^2 + 1))
```

The LaTeX output of that embedding is:

```
sage: latex(Phi.display(c_xy, c_cart))
\begin{array}{llcl} \Phi:& S^2 & \longrightarrow & \RR^3
 \\ \text{on}\ U : & \left(x, y\right) & \longmapsto
 & \left(X, Y, Z\right) = \left(\frac{2 \, x}{x^{2} + y^{2} + 1},
   \frac{2 \, y}{x^{2} + y^{2} + 1},
   \frac{x^{2} + y^{2} - 1}{x^{2} + y^{2} + 1}\right)
 \end{array}
```

```
>>> from sage.all import *
>>> latex(Phi.display(c_xy, c_cart))
\begin{array}{llcl} \Phi:& S^2 & \longrightarrow & \RR^3
 \\ \text{on}\ U : & \left(x, y\right) & \longmapsto
 & \left(X, Y, Z\right) = \left(\frac{2 \, x}{x^{2} + y^{2} + 1},
   \frac{2 \, y}{x^{2} + y^{2} + 1},
   \frac{x^{2} + y^{2} - 1}{x^{2} + y^{2} + 1}\right)
 \end{array}
```

If the argument `chart2` is not specified, the display is performed on all the charts on the codomain in which the map is known or computable via some change of coordinates (here only one chart: `c_cart`):

```
sage: Phi.display(c_xy)
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 +␣
↪y^2 - 1)/(x^2 + y^2 + 1))
```

```
>>> from sage.all import *
>>> Phi.display(c_xy)
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 +␣
↪y^2 - 1)/(x^2 + y^2 + 1))
```

Similarly, if the argument `chart1` is omitted, the display is performed on all the charts on the domain of `Phi` in which the map is known or computable via some change of coordinates:

```
sage: Phi.display(chart2=c_cart)
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 +␣
↪y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^2 +␣
↪v^2 - 1)/(u^2 + v^2 + 1))
```

```
>>> from sage.all import *
>>> Phi.display(chart2=c_cart)
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 +␣
↪y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^2 +␣
↪v^2 - 1)/(u^2 + v^2 + 1))
```

If neither `chart1` nor `chart2` is specified, the display is performed on all the pair of charts in which `Phi` is known or computable via some change of coordinates:

```
sage: Phi.display()
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 +␣
↪y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^2 +␣
↪v^2 - 1)/(u^2 + v^2 + 1))
```

```
>>> from sage.all import *
>>> Phi.display()
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 +␣
↪y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^2 +␣
↪v^2 - 1)/(u^2 + v^2 + 1))
```

If a chart covers entirely the map's domain, the mention "on …" is omitted:

```
sage: Phi.restrict(U).display()
Phi: U → R^3
   (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 + y^2␣
↪- 1)/(x^2 + y^2 + 1))
```

```
>>> from sage.all import *
>>> Phi.restrict(U).display()
Phi: U → R^3
```

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```
   (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 + y^2␣
↪- 1)/(x^2 + y^2 + 1))
```

A shortcut of `display()` is `disp()`:

```
sage: Phi.disp()
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 +␣
↪y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^2 +␣
↪v^2 - 1)/(u^2 + v^2 + 1))
```

```
>>> from sage.all import *
>>> Phi.disp()
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 +␣
↪y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^2 +␣
↪v^2 - 1)/(u^2 + v^2 + 1))
```

Display when SymPy is the symbolic engine:

```
sage: M.set_calculus_method('sympy')
sage: N.set_calculus_method('sympy')
sage: Phi.display(c_xy, c_cart)
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x**2 + y**2 + 1),
 2*y/(x**2 + y**2 + 1), (x**2 + y**2 - 1)/(x**2 + y**2 + 1))
sage: latex(Phi.display(c_xy, c_cart))
\begin{array}{llcl} \Phi:& S^2 & \longrightarrow & \RR^3
 \\ \text{on}\ U : & \left(x, y\right) & \longmapsto
 & \left(X, Y, Z\right) = \left(\frac{2 x}{x^{2} + y^{2} + 1},
   \frac{2 y}{x^{2} + y^{2} + 1},
   \frac{x^{2} + y^{2} - 1}{x^{2} + y^{2} + 1}\right)
 \end{array}
```

```
>>> from sage.all import *
>>> M.set_calculus_method('sympy')
>>> N.set_calculus_method('sympy')
>>> Phi.display(c_xy, c_cart)
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x**2 + y**2 + 1),
 2*y/(x**2 + y**2 + 1), (x**2 + y**2 - 1)/(x**2 + y**2 + 1))
>>> latex(Phi.display(c_xy, c_cart))
\begin{array}{llcl} \Phi:& S^2 & \longrightarrow & \RR^3
 \\ \text{on}\ U : & \left(x, y\right) & \longmapsto
 & \left(X, Y, Z\right) = \left(\frac{2 x}{x^{2} + y^{2} + 1},
   \frac{2 y}{x^{2} + y^{2} + 1},
   \frac{x^{2} + y^{2} - 1}{x^{2} + y^{2} + 1}\right)
 \end{array}
```

expr(*chart1=None*, *chart2=None*)

Return the expression of `self` in terms of specified coordinates.

If the expression is not already known, it is computed from some known expression by means of change-of-chart formulas.

INPUT:

- `chart1` – (default: `None`) chart on the map's domain; if `None`, the domain's default chart is assumed
- `chart2` – (default: `None`) chart on the map's codomain; if `None`, the codomain's default chart is assumed

OUTPUT:

- symbolic expression representing the continuous map in the above two charts

EXAMPLES:

Continuous map from a 2-dimensional manifold to a 3-dimensional one:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: N = Manifold(3, 'N', structure='topological')
sage: c_uv.<u,v> = M.chart()
sage: c_xyz.<x,y,z> = N.chart()
sage: Phi = M.continuous_map(N, (u*v, u/v, u+v), name='Phi',
....:                        latex_name=r'\Phi')
sage: Phi.display()
Phi: M → N
   (u, v) ↦ (x, y, z) = (u*v, u/v, u + v)
sage: Phi.expr(c_uv, c_xyz)
(u*v, u/v, u + v)
sage: Phi.expr()  # equivalent to above since 'uv' and 'xyz' are default
↪charts
(u*v, u/v, u + v)
sage: type(Phi.expr()[0])
<class 'sage.symbolic.expression.Expression'>
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> N = Manifold(Integer(3), 'N', structure='topological')
>>> c_uv = M.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> c_xyz = N.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> Phi = M.continuous_map(N, (u*v, u/v, u+v), name='Phi',
...                        latex_name=r'\Phi')
>>> Phi.display()
Phi: M → N
   (u, v) ↦ (x, y, z) = (u*v, u/v, u + v)
>>> Phi.expr(c_uv, c_xyz)
(u*v, u/v, u + v)
>>> Phi.expr()  # equivalent to above since 'uv' and 'xyz' are default charts
(u*v, u/v, u + v)
>>> type(Phi.expr()[Integer(0)])
<class 'sage.symbolic.expression.Expression'>
```

Expressions in other charts:

```
sage: c_UV.<U,V> = M.chart()  # new chart on M
sage: ch_uv_UV = c_uv.transition_map(c_UV, [u-v, u+v])
sage: ch_uv_UV.inverse()(U,V)
```

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```
(1/2*U + 1/2*V, -1/2*U + 1/2*V)
sage: c_XYZ.<X,Y,Z> = N.chart() # new chart on N
sage: ch_xyz_XYZ = c_xyz.transition_map(c_XYZ,
....:                                  [2*x-3*y+z, y+z-x, -x+2*y-z])
sage: ch_xyz_XYZ.inverse()(X,Y,Z)
(3*X + Y + 4*Z, 2*X + Y + 3*Z, X + Y + Z)
sage: Phi.expr(c_UV, c_xyz)
(-1/4*U^2 + 1/4*V^2, -(U + V)/(U - V), V)
sage: Phi.expr(c_uv, c_XYZ)
(((2*u + 1)*v^2 + u*v - 3*u)/v,
 -((u - 1)*v^2 - u*v - u)/v,
 -((u + 1)*v^2 + u*v - 2*u)/v)
sage: Phi.expr(c_UV, c_XYZ)
 (-1/2*(U^3 - (U - 2)*V^2 + V^3 - (U^2 + 2*U + 6)*V - 6*U)/(U - V),
  1/4*(U^3 - (U + 4)*V^2 + V^3 - (U^2 - 4*U + 4)*V - 4*U)/(U - V),
  1/4*(U^3 - (U - 4)*V^2 + V^3 - (U^2 + 4*U + 8)*V - 8*U)/(U - V))
```

```
>>> from sage.all import *
>>> c_UV = M.chart(names=('U', 'V',)); (U, V,) = c_UV._first_ngens(2)# new␣
↪chart on M
>>> ch_uv_UV = c_uv.transition_map(c_UV, [u-v, u+v])
>>> ch_uv_UV.inverse()(U,V)
(1/2*U + 1/2*V, -1/2*U + 1/2*V)
>>> c_XYZ = N.chart(names=('X', 'Y', 'Z',)); (X, Y, Z,) = c_XYZ._first_
↪ngens(3)# new chart on N
>>> ch_xyz_XYZ = c_xyz.transition_map(c_XYZ,
...                                  [Integer(2)*x-Integer(3)*y+z, y+z-x, -
↪x+Integer(2)*y-z])
>>> ch_xyz_XYZ.inverse()(X,Y,Z)
(3*X + Y + 4*Z, 2*X + Y + 3*Z, X + Y + Z)
>>> Phi.expr(c_UV, c_xyz)
(-1/4*U^2 + 1/4*V^2, -(U + V)/(U - V), V)
>>> Phi.expr(c_uv, c_XYZ)
(((2*u + 1)*v^2 + u*v - 3*u)/v,
 -((u - 1)*v^2 - u*v - u)/v,
 -((u + 1)*v^2 + u*v - 2*u)/v)
>>> Phi.expr(c_UV, c_XYZ)
 (-1/2*(U^3 - (U - 2)*V^2 + V^3 - (U^2 + 2*U + 6)*V - 6*U)/(U - V),
  1/4*(U^3 - (U + 4)*V^2 + V^3 - (U^2 - 4*U + 4)*V - 4*U)/(U - V),
  1/4*(U^3 - (U - 4)*V^2 + V^3 - (U^2 + 4*U + 8)*V - 8*U)/(U - V))
```

A rotation in some Euclidean plane:

```
sage: M = Manifold(2, 'M', structure='topological') # the plane (minus a␣
↪segment to have global regular spherical coordinates)
sage: c_spher.<r,ph> = M.chart(r'r:(0,+oo) ph:(0,2*pi):\phi') # spherical␣
↪coordinates on the plane
sage: rot = M.continuous_map(M, (r, ph+pi/3), name='R') # pi/3 rotation␣
↪around r=0
sage: rot.expr()
(r, 1/3*pi + ph)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological') # the plane (minus␣
↪a segment to have global regular spherical coordinates)
>>> c_spher = M.chart(r'r:(0,+oo) ph:(0,2*pi):\phi', names=('r', 'ph',)); (r,␣
↪ph,) = c_spher._first_ngens(2)# spherical coordinates on the plane
>>> rot = M.continuous_map(M, (r, ph+pi/Integer(3)), name='R') # pi/3␣
↪rotation around r=0
>>> rot.expr()
(r, 1/3*pi + ph)
```

Expression of the rotation in terms of Cartesian coordinates:

```
sage: c_cart.<x,y> = M.chart() # Declaration of Cartesian coordinates
sage: ch_spher_cart = c_spher.transition_map(c_cart,
....:                   [r*cos(ph), r*sin(ph)]) # relation to spherical␣
↪coordinates
sage: ch_spher_cart.set_inverse(sqrt(x^2+y^2), atan2(y,x))
Check of the inverse coordinate transformation:
  r == r  *passed*
  ph == arctan2(r*sin(ph), r*cos(ph))  **failed**
  x == x  *passed*
  y == y  *passed*
NB: a failed report can reflect a mere lack of simplification.
sage: rot.expr(c_cart, c_cart)
(-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
```

```
>>> from sage.all import *
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#␣
↪Declaration of Cartesian coordinates
>>> ch_spher_cart = c_spher.transition_map(c_cart,
...                   [r*cos(ph), r*sin(ph)]) # relation to spherical␣
↪coordinates
>>> ch_spher_cart.set_inverse(sqrt(x**Integer(2)+y**Integer(2)), atan2(y,x))
Check of the inverse coordinate transformation:
  r == r  *passed*
  ph == arctan2(r*sin(ph), r*cos(ph))  **failed**
  x == x  *passed*
  y == y  *passed*
NB: a failed report can reflect a mere lack of simplification.
>>> rot.expr(c_cart, c_cart)
(-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
```

expression(*chart1=None*, *chart2=None*)

Return the expression of `self` in terms of specified coordinates.

If the expression is not already known, it is computed from some known expression by means of change-of-chart formulas.

INPUT:

- `chart1` – (default: `None`) chart on the map's domain; if `None`, the domain's default chart is assumed
- `chart2` – (default: `None`) chart on the map's codomain; if `None`, the codomain's default chart is assumed

OUTPUT:

- symbolic expression representing the continuous map in the above two charts

EXAMPLES:

Continuous map from a 2-dimensional manifold to a 3-dimensional one:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: N = Manifold(3, 'N', structure='topological')
sage: c_uv.<u,v> = M.chart()
sage: c_xyz.<x,y,z> = N.chart()
sage: Phi = M.continuous_map(N, (u*v, u/v, u+v), name='Phi',
....:                        latex_name=r'\Phi')
sage: Phi.display()
Phi: M → N
   (u, v) ↦ (x, y, z) = (u*v, u/v, u + v)
sage: Phi.expr(c_uv, c_xyz)
(u*v, u/v, u + v)
sage: Phi.expr()  # equivalent to above since 'uv' and 'xyz' are default␣
↪charts
(u*v, u/v, u + v)
sage: type(Phi.expr()[0])
<class 'sage.symbolic.expression.Expression'>
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> N = Manifold(Integer(3), 'N', structure='topological')
>>> c_uv = M.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> c_xyz = N.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> Phi = M.continuous_map(N, (u*v, u/v, u+v), name='Phi',
...                        latex_name=r'\Phi')
>>> Phi.display()
Phi: M → N
   (u, v) ↦ (x, y, z) = (u*v, u/v, u + v)
>>> Phi.expr(c_uv, c_xyz)
(u*v, u/v, u + v)
>>> Phi.expr()  # equivalent to above since 'uv' and 'xyz' are default charts
(u*v, u/v, u + v)
>>> type(Phi.expr()[Integer(0)])
<class 'sage.symbolic.expression.Expression'>
```

Expressions in other charts:

```
sage: c_UV.<U,V> = M.chart()  # new chart on M
sage: ch_uv_UV = c_uv.transition_map(c_UV, [u-v, u+v])
sage: ch_uv_UV.inverse()(U,V)
(1/2*U + 1/2*V, -1/2*U + 1/2*V)
sage: c_XYZ.<X,Y,Z> = N.chart() # new chart on N
sage: ch_xyz_XYZ = c_xyz.transition_map(c_XYZ,
....:                                   [2*x-3*y+z, y+z-x, -x+2*y-z])
sage: ch_xyz_XYZ.inverse()(X,Y,Z)
(3*X + Y + 4*Z, 2*X + Y + 3*Z, X + Y + Z)
sage: Phi.expr(c_UV, c_xyz)
(-1/4*U^2 + 1/4*V^2, -(U + V)/(U - V), V)
```

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```
sage: Phi.expr(c_uv, c_XYZ)
(((2*u + 1)*v^2 + u*v - 3*u)/v,
 -((u - 1)*v^2 - u*v - u)/v,
 -((u + 1)*v^2 + u*v - 2*u)/v)
sage: Phi.expr(c_UV, c_XYZ)
 (-1/2*(U^3 - (U - 2)*V^2 + V^3 - (U^2 + 2*U + 6)*V - 6*U)/(U - V),
  1/4*(U^3 - (U + 4)*V^2 + V^3 - (U^2 - 4*U + 4)*V - 4*U)/(U - V),
  1/4*(U^3 - (U - 4)*V^2 + V^3 - (U^2 + 4*U + 8)*V - 8*U)/(U - V))
```

```
>>> from sage.all import *
>>> c_UV = M.chart(names=('U', 'V',)); (U, V,) = c_UV._first_ngens(2)# new␣
↪chart on M
>>> ch_uv_UV = c_uv.transition_map(c_UV, [u-v, u+v])
>>> ch_uv_UV.inverse()(U,V)
(1/2*U + 1/2*V, -1/2*U + 1/2*V)
>>> c_XYZ = N.chart(names=('X', 'Y', 'Z',)); (X, Y, Z,) = c_XYZ._first_
↪ngens(3)# new chart on N
>>> ch_xyz_XYZ = c_xyz.transition_map(c_XYZ,
...                                   [Integer(2)*x-Integer(3)*y+z, y+z-x, -
↪x+Integer(2)*y-z])
>>> ch_xyz_XYZ.inverse()(X,Y,Z)
(3*X + Y + 4*Z, 2*X + Y + 3*Z, X + Y + Z)
>>> Phi.expr(c_UV, c_xyz)
(-1/4*U^2 + 1/4*V^2, -(U + V)/(U - V), V)
>>> Phi.expr(c_uv, c_XYZ)
(((2*u + 1)*v^2 + u*v - 3*u)/v,
 -((u - 1)*v^2 - u*v - u)/v,
 -((u + 1)*v^2 + u*v - 2*u)/v)
>>> Phi.expr(c_UV, c_XYZ)
 (-1/2*(U^3 - (U - 2)*V^2 + V^3 - (U^2 + 2*U + 6)*V - 6*U)/(U - V),
  1/4*(U^3 - (U + 4)*V^2 + V^3 - (U^2 - 4*U + 4)*V - 4*U)/(U - V),
  1/4*(U^3 - (U - 4)*V^2 + V^3 - (U^2 + 4*U + 8)*V - 8*U)/(U - V))
```

A rotation in some Euclidean plane:

```
sage: M = Manifold(2, 'M', structure='topological') # the plane (minus a␣
↪segment to have global regular spherical coordinates)
sage: c_spher.<r,ph> = M.chart(r'r:(0,+oo) ph:(0,2*pi):\phi') # spherical␣
↪coordinates on the plane
sage: rot = M.continuous_map(M, (r, ph+pi/3), name='R') # pi/3 rotation␣
↪around r=0
sage: rot.expr()
(r, 1/3*pi + ph)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological') # the plane (minus␣
↪a segment to have global regular spherical coordinates)
>>> c_spher = M.chart(r'r:(0,+oo) ph:(0,2*pi):\phi', names=('r', 'ph',)); (r,␣
↪ph,) = c_spher._first_ngens(2)# spherical coordinates on the plane
>>> rot = M.continuous_map(M, (r, ph+pi/Integer(3)), name='R') # pi/3␣
↪rotation around r=0
>>> rot.expr()
```

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```
(r, 1/3*pi + ph)
```

Expression of the rotation in terms of Cartesian coordinates:

```
sage: c_cart.<x,y> = M.chart() # Declaration of Cartesian coordinates
sage: ch_spher_cart = c_spher.transition_map(c_cart,
....:                     [r*cos(ph), r*sin(ph)]) # relation to spherical␣
↪coordinates
sage: ch_spher_cart.set_inverse(sqrt(x^2+y^2), atan2(y,x))
Check of the inverse coordinate transformation:
  r == r  *passed*
  ph == arctan2(r*sin(ph), r*cos(ph))  **failed**
  x == x  *passed*
  y == y  *passed*
NB: a failed report can reflect a mere lack of simplification.
sage: rot.expr(c_cart, c_cart)
(-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
```

```
>>> from sage.all import *
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#␣
↪Declaration of Cartesian coordinates
>>> ch_spher_cart = c_spher.transition_map(c_cart,
...                     [r*cos(ph), r*sin(ph)]) # relation to spherical␣
↪coordinates
>>> ch_spher_cart.set_inverse(sqrt(x**Integer(2)+y**Integer(2)), atan2(y,x))
Check of the inverse coordinate transformation:
  r == r  *passed*
  ph == arctan2(r*sin(ph), r*cos(ph))  **failed**
  x == x  *passed*
  y == y  *passed*
NB: a failed report can reflect a mere lack of simplification.
>>> rot.expr(c_cart, c_cart)
(-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
```

image(*subset=None*, *inverse=None*)

Return the image of `self` or the image of `subset` under `self`.

INPUT:

- `inverse` – (default: `None`) continuous map from `map.codomain()` to `map.domain()`, which once restricted to the image of Φ is the inverse of Φ onto its image if the latter exists (NB: no check of this is performed)
- `subset` – (default: the domain of `map`) a subset of the domain of `self`

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: N = Manifold(1, 'N', ambient=M, structure='topological')
sage: CM.<x,y> = M.chart()
sage: CN.<u> = N.chart(coord_restrictions=lambda u: [u > -1, u < 1])
sage: Phi = N.continuous_map(M, {(CN,CM): [u, u^2]}, name='Phi')
sage: Phi.image()
Image of the Continuous map Phi
```

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```
  from the 1-dimensional topological submanifold N
    immersed in the 2-dimensional topological manifold M
  to the 2-dimensional topological manifold M

sage: S = N.subset('S')
sage: Phi_S = Phi.image(S); Phi_S
Image of the Subset S of the
 1-dimensional topological submanifold N
  immersed in the 2-dimensional topological manifold M
 under the Continuous map Phi
  from the 1-dimensional topological submanifold N
   immersed in the 2-dimensional topological manifold M
 to the 2-dimensional topological manifold M
sage: Phi_S.is_subset(M)
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> N = Manifold(Integer(1), 'N', ambient=M, structure='topological')
>>> CM = M.chart(names=('x', 'y',)); (x, y,) = CM._first_ngens(2)
>>> CN = N.chart(coord_restrictions=lambda u: [u > -Integer(1), u <␣
↪Integer(1)], names=('u',)); (u,) = CN._first_ngens(1)
>>> Phi = N.continuous_map(M, {(CN,CM): [u, u**Integer(2)]}, name='Phi')
>>> Phi.image()
Image of the Continuous map Phi
  from the 1-dimensional topological submanifold N
    immersed in the 2-dimensional topological manifold M
  to the 2-dimensional topological manifold M

>>> S = N.subset('S')
>>> Phi_S = Phi.image(S); Phi_S
Image of the Subset S of the
 1-dimensional topological submanifold N
  immersed in the 2-dimensional topological manifold M
 under the Continuous map Phi
  from the 1-dimensional topological submanifold N
   immersed in the 2-dimensional topological manifold M
 to the 2-dimensional topological manifold M
>>> Phi_S.is_subset(M)
True
```

inverse()

Return the inverse of `self` if it is an isomorphism.

OUTPUT: the inverse isomorphism

EXAMPLES:

The inverse of a rotation in the Euclidean plane:

```
sage: M = Manifold(2, 'R^2', latex_name=r'\RR^2', structure='topological')
sage: c_cart.<x,y> = M.chart()
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', latex_name=r'\RR^2', structure=
↪'topological')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)
```

A pi/3 rotation around the origin:

```
sage: rot = M.homeomorphism(M, ((x - sqrt(3)*y)/2, (sqrt(3)*x + y)/2),
....:                       name='R')
sage: rot.inverse()
Homeomorphism R^(-1) of the 2-dimensional topological manifold R^2
sage: rot.inverse().display()
R^(-1): R^2 → R^2
   (x, y) ↦ (1/2*sqrt(3)*y + 1/2*x, -1/2*sqrt(3)*x + 1/2*y)
```

```
>>> from sage.all import *
>>> rot = M.homeomorphism(M, ((x - sqrt(Integer(3))*y)/Integer(2),
↪(sqrt(Integer(3))*x + y)/Integer(2)),
...                       name='R')
>>> rot.inverse()
Homeomorphism R^(-1) of the 2-dimensional topological manifold R^2
>>> rot.inverse().display()
R^(-1): R^2 → R^2
   (x, y) ↦ (1/2*sqrt(3)*y + 1/2*x, -1/2*sqrt(3)*x + 1/2*y)
```

Checking that applying successively the homeomorphism and its inverse results in the identity:

```
sage: (a, b) = var('a b')
sage: p = M.point((a,b)) # a generic point on M
sage: q = rot(p)
sage: p1 = rot.inverse()(q)
sage: p1 == p
True
```

```
>>> from sage.all import *
>>> (a, b) = var('a b')
>>> p = M.point((a,b)) # a generic point on M
>>> q = rot(p)
>>> p1 = rot.inverse()(q)
>>> p1 == p
True
```

The result is cached:

```
sage: rot.inverse() is rot.inverse()
True
```

```
>>> from sage.all import *
>>> rot.inverse() is rot.inverse()
True
```

The notations `^(-1)` or `~` can also be used for the inverse:

```
sage: rot^(-1) is rot.inverse()
True
sage: ~rot is rot.inverse()
True
```

```
>>> from sage.all import *
>>> rot**(-Integer(1)) is rot.inverse()
True
>>> ~rot is rot.inverse()
True
```

An example with multiple charts: the equatorial symmetry on the 2-sphere:

```
sage: M = Manifold(2, 'M', structure='topological') # the 2-dimensional␣
↪sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                                intersection_name='W',
....:                                restrictions1=x^2+y^2!=0,
....:                                restrictions2=u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: s = M.homeomorphism(M, {(c_xy, c_uv): [x, y], (c_uv, c_xy): [u, v]},
....:                     name='s')
sage: s.display()
s: M → M
on U: (x, y) ↦ (u, v) = (x, y)
on V: (u, v) ↦ (x, y) = (u, v)
sage: si = s.inverse(); si
Homeomorphism s^(-1) of the 2-dimensional topological manifold M
sage: si.display()
s^(-1): M → M
on U: (x, y) ↦ (u, v) = (x, y)
on V: (u, v) ↦ (x, y) = (u, v)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological') # the 2-
↪dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                                intersection_name='W',
...                                restrictions1=x**Integer(2)+y**Integer(2)!
```

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```
↪=Integer(0),
...                                    restrictions2=u**Integer(2)+v**Integer(2)!
↪=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> s = M.homeomorphism(M, {(c_xy, c_uv): [x, y], (c_uv, c_xy): [u, v]},
...                     name='s')
>>> s.display()
s: M → M
on U: (x, y) ↦ (u, v) = (x, y)
on V: (u, v) ↦ (x, y) = (u, v)
>>> si = s.inverse(); si
Homeomorphism s^(-1) of the 2-dimensional topological manifold M
>>> si.display()
s^(-1): M → M
on U: (x, y) ↦ (u, v) = (x, y)
on V: (u, v) ↦ (x, y) = (u, v)
```

The equatorial symmetry is of course an involution:

```
sage: si == s
True
```

```
>>> from sage.all import *
>>> si == s
True
```

is_identity()

Check whether `self` is an identity map.

EXAMPLES:

Tests on continuous maps of a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: M.identity_map().is_identity()  # obviously...
True
sage: Hom(M, M).one().is_identity()  # a variant of the obvious
True
sage: a = M.continuous_map(M, coord_functions={(X,X): (x, y)})
sage: a.is_identity()
True
sage: a = M.continuous_map(M, coord_functions={(X,X): (x, y+1)})
sage: a.is_identity()
False
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> M.identity_map().is_identity()  # obviously...
True
>>> Hom(M, M).one().is_identity()  # a variant of the obvious
True
```

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```
>>> a = M.continuous_map(M, coord_functions={(X,X): (x, y)})
>>> a.is_identity()
True
>>> a = M.continuous_map(M, coord_functions={(X,X): (x, y+Integer(1))})
>>> a.is_identity()
False
```

Of course, if the codomain of the map does not coincide with its domain, the outcome is `False`:

```
sage: N = Manifold(2, 'N', structure='topological')
sage: Y.<u,v> = N.chart()
sage: a = M.continuous_map(N, {(X,Y): (x, y)})
sage: a.display()
M → N
   (x, y) ↦ (u, v) = (x, y)
sage: a.is_identity()
False
```

```
>>> from sage.all import *
>>> N = Manifold(Integer(2), 'N', structure='topological')
>>> Y = N.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)
>>> a = M.continuous_map(N, {(X,Y): (x, y)})
>>> a.display()
M → N
   (x, y) ↦ (u, v) = (x, y)
>>> a.is_identity()
False
```

preimage(*codomain_subset*, *name=None*, *latex_name=None*)

Return the preimage of `codomain_subset` under `self`.

An alias is *pullback()*.

INPUT:

- `codomain_subset` – an instance of *ManifoldSubset*
- `name` – string; name (symbol) given to the subset
- `latex_name` – string (default: `None`); LaTeX symbol to denote the subset; if none are provided, it is set to `name`

OUTPUT:

- either a *TopologicalManifold* or a *ManifoldSubsetPullback*

EXAMPLES:

```
sage: R = Manifold(1, 'R', structure='topological')  # field R
sage: T.<t> = R.chart()  # canonical chart on R
sage: R2 = Manifold(2, 'R^2', structure='topological')  # R^2
sage: c_xy.<x,y> = R2.chart() # Cartesian coordinates on R^2
sage: Phi = R.continuous_map(R2, [cos(t), sin(t)], name='Phi'); Phi
Continuous map Phi
 from the 1-dimensional topological manifold R
 to the 2-dimensional topological manifold R^2
```

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```
sage: Q1 = R2.open_subset('Q1', coord_def={c_xy: [x>0, y>0]}); Q1
Open subset Q1 of the 2-dimensional topological manifold R^2
sage: Phi_inv_Q1 = Phi.preimage(Q1); Phi_inv_Q1
Subset Phi_inv_Q1 of the 1-dimensional topological manifold R
sage: R.point([pi/4]) in Phi_inv_Q1
True
sage: R.point([0]) in Phi_inv_Q1
False
sage: R.point([3*pi/4]) in Phi_inv_Q1
False
```

```
>>> from sage.all import *
>>> R = Manifold(Integer(1), 'R', structure='topological')  # field R
>>> T = R.chart(names=('t',)); (t,) = T._first_ngens(1)# canonical chart on R
>>> R2 = Manifold(Integer(2), 'R^2', structure='topological')  # R^2
>>> c_xy = R2.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪Cartesian coordinates on R^2
>>> Phi = R.continuous_map(R2, [cos(t), sin(t)], name='Phi'); Phi
Continuous map Phi
 from the 1-dimensional topological manifold R
 to the 2-dimensional topological manifold R^2
>>> Q1 = R2.open_subset('Q1', coord_def={c_xy: [x>Integer(0), y>Integer(0)]});
↪ Q1
Open subset Q1 of the 2-dimensional topological manifold R^2
>>> Phi_inv_Q1 = Phi.preimage(Q1); Phi_inv_Q1
Subset Phi_inv_Q1 of the 1-dimensional topological manifold R
>>> R.point([pi/Integer(4)]) in Phi_inv_Q1
True
>>> R.point([Integer(0)]) in Phi_inv_Q1
False
>>> R.point([Integer(3)*pi/Integer(4)]) in Phi_inv_Q1
False
```

The identity map is handled specially:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: M.identity_map().preimage(M)
2-dimensional topological manifold M
sage: M.identity_map().preimage(M) is M
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> M.identity_map().preimage(M)
2-dimensional topological manifold M
>>> M.identity_map().preimage(M) is M
True
```

Another trivial case:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: D1 = M.open_subset('D1', coord_def={X: x^2+y^2<1}) # the open unit disk
sage: D2 = M.open_subset('D2', coord_def={X: x^2+y^2<4})
sage: f = Hom(D1,D2)({(X.restrict(D1), X.restrict(D2)): (2*x, 2*y)}, name='f')
sage: f.preimage(D2)
Open subset D1 of the 2-dimensional topological manifold M
sage: f.preimage(M)
Open subset D1 of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> D1 = M.open_subset('D1', coord_def={X: x**Integer(2)+y**Integer(2)
↪<Integer(1)}) # the open unit disk
>>> D2 = M.open_subset('D2', coord_def={X: x**Integer(2)+y**Integer(2)
↪<Integer(4)})
>>> f = Hom(D1,D2)({(X.restrict(D1), X.restrict(D2)): (Integer(2)*x,␣
↪Integer(2)*y)}, name='f')
>>> f.preimage(D2)
Open subset D1 of the 2-dimensional topological manifold M
>>> f.preimage(M)
Open subset D1 of the 2-dimensional topological manifold M
```

pullback(*codomain_subset*, *name=None*, *latex_name=None*)

Return the preimage of `codomain_subset` under `self`.

An alias is *pullback()*.

INPUT:

- `codomain_subset` – an instance of *ManifoldSubset*
- `name` – string; name (symbol) given to the subset
- `latex_name` – string (default: `None`); LaTeX symbol to denote the subset; if none are provided, it is set to `name`

OUTPUT:

- either a *TopologicalManifold* or a *ManifoldSubsetPullback*

EXAMPLES:

```
sage: R = Manifold(1, 'R', structure='topological')  # field R
sage: T.<t> = R.chart()  # canonical chart on R
sage: R2 = Manifold(2, 'R^2', structure='topological')  # R^2
sage: c_xy.<x,y> = R2.chart() # Cartesian coordinates on R^2
sage: Phi = R.continuous_map(R2, [cos(t), sin(t)], name='Phi'); Phi
Continuous map Phi
 from the 1-dimensional topological manifold R
 to the 2-dimensional topological manifold R^2
sage: Q1 = R2.open_subset('Q1', coord_def={c_xy: [x>0, y>0]}); Q1
Open subset Q1 of the 2-dimensional topological manifold R^2
sage: Phi_inv_Q1 = Phi.preimage(Q1); Phi_inv_Q1
Subset Phi_inv_Q1 of the 1-dimensional topological manifold R
```

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```
sage: R.point([pi/4]) in Phi_inv_Q1
True
sage: R.point([0]) in Phi_inv_Q1
False
sage: R.point([3*pi/4]) in Phi_inv_Q1
False
```

```
>>> from sage.all import *
>>> R = Manifold(Integer(1), 'R', structure='topological')  # field R
>>> T = R.chart(names=('t',)); (t,) = T._first_ngens(1)# canonical chart on R
>>> R2 = Manifold(Integer(2), 'R^2', structure='topological')  # R^2
>>> c_xy = R2.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪Cartesian coordinates on R^2
>>> Phi = R.continuous_map(R2, [cos(t), sin(t)], name='Phi'); Phi
Continuous map Phi
 from the 1-dimensional topological manifold R
 to the 2-dimensional topological manifold R^2
>>> Q1 = R2.open_subset('Q1', coord_def={c_xy: [x>Integer(0), y>Integer(0)]});
↪ Q1
Open subset Q1 of the 2-dimensional topological manifold R^2
>>> Phi_inv_Q1 = Phi.preimage(Q1); Phi_inv_Q1
Subset Phi_inv_Q1 of the 1-dimensional topological manifold R
>>> R.point([pi/Integer(4)]) in Phi_inv_Q1
True
>>> R.point([Integer(0)]) in Phi_inv_Q1
False
>>> R.point([Integer(3)*pi/Integer(4)]) in Phi_inv_Q1
False
```

The identity map is handled specially:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: M.identity_map().preimage(M)
2-dimensional topological manifold M
sage: M.identity_map().preimage(M) is M
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> M.identity_map().preimage(M)
2-dimensional topological manifold M
>>> M.identity_map().preimage(M) is M
True
```

Another trivial case:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: D1 = M.open_subset('D1', coord_def={X: x^2+y^2<1}) # the open unit disk
sage: D2 = M.open_subset('D2', coord_def={X: x^2+y^2<4})
```

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```
sage: f = Hom(D1,D2)({(X.restrict(D1), X.restrict(D2)): (2*x, 2*y)}, name='f')
sage: f.preimage(D2)
Open subset D1 of the 2-dimensional topological manifold M
sage: f.preimage(M)
Open subset D1 of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> D1 = M.open_subset('D1', coord_def={X: x**Integer(2)+y**Integer(2)
↪<Integer(1)}) # the open unit disk
>>> D2 = M.open_subset('D2', coord_def={X: x**Integer(2)+y**Integer(2)
↪<Integer(4)})
>>> f = Hom(D1,D2)({(X.restrict(D1), X.restrict(D2)): (Integer(2)*x,␣
↪Integer(2)*y)}, name='f')
>>> f.preimage(D2)
Open subset D1 of the 2-dimensional topological manifold M
>>> f.preimage(M)
Open subset D1 of the 2-dimensional topological manifold M
```

restrict(*subdomain*, *subcodomain=None*)

Restriction of `self` to some open subset of its domain of definition.

INPUT:

- `subdomain` – *TopologicalManifold*; an open subset of the domain of `self`
- `subcodomain` – (default: `None`) an open subset of the codomain of `self`; if `None`, the codomain of `self` is assumed

OUTPUT:

- a *ContinuousMap* that is the restriction of `self` to `subdomain`

EXAMPLES:

Restriction to an annulus of a homeomorphism between the open unit disk and $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'R^2', structure='topological')  # R^2
sage: c_xy.<x,y> = M.chart()  # Cartesian coord. on R^2
sage: D = M.open_subset('D', coord_def={c_xy: x^2+y^2<1}) # the open unit disk
sage: Phi = D.continuous_map(M, [x/sqrt(1-x^2-y^2), y/sqrt(1-x^2-y^2)],
....:                         name='Phi', latex_name=r'\Phi')
sage: Phi.display()
Phi: D → R^2
   (x, y) ↦ (x, y) = (x/sqrt(-x^2 - y^2 + 1), y/sqrt(-x^2 - y^2 + 1))
sage: c_xy_D = c_xy.restrict(D)
sage: U = D.open_subset('U', coord_def={c_xy_D: x^2+y^2>1/2}) # the annulus 1/
↪2 < r < 1
sage: Phi.restrict(U)
Continuous map Phi
 from the Open subset U of the 2-dimensional topological manifold R^2
 to the 2-dimensional topological manifold R^2
sage: Phi.restrict(U).parent()
Set of Morphisms from Open subset U of the 2-dimensional topological
```

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```
 manifold R^2 to 2-dimensional topological manifold R^2 in Category
 of manifolds over Real Field with 53 bits of precision
sage: Phi.domain()
Open subset D of the 2-dimensional topological manifold R^2
sage: Phi.restrict(U).domain()
Open subset U of the 2-dimensional topological manifold R^2
sage: Phi.restrict(U).display()
Phi: U → R^2
   (x, y) ↦ (x, y) = (x/sqrt(-x^2 - y^2 + 1), y/sqrt(-x^2 - y^2 + 1))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', structure='topological')  # R^2
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#
↪Cartesian coord. on R^2
>>> D = M.open_subset('D', coord_def={c_xy: x**Integer(2)+y**Integer(2)
↪<Integer(1)}) # the open unit disk
>>> Phi = D.continuous_map(M, [x/sqrt(Integer(1)-x**Integer(2)-y**Integer(2)),
↪ y/sqrt(Integer(1)-x**Integer(2)-y**Integer(2))],
...                        name='Phi', latex_name=r'\Phi')
>>> Phi.display()
Phi: D → R^2
   (x, y) ↦ (x, y) = (x/sqrt(-x^2 - y^2 + 1), y/sqrt(-x^2 - y^2 + 1))
>>> c_xy_D = c_xy.restrict(D)
>>> U = D.open_subset('U', coord_def={c_xy_D: x**Integer(2)+y**Integer(2)>
↪Integer(1)/Integer(2)}) # the annulus 1/2 < r < 1
>>> Phi.restrict(U)
Continuous map Phi
 from the Open subset U of the 2-dimensional topological manifold R^2
 to the 2-dimensional topological manifold R^2
>>> Phi.restrict(U).parent()
Set of Morphisms from Open subset U of the 2-dimensional topological
 manifold R^2 to 2-dimensional topological manifold R^2 in Category
 of manifolds over Real Field with 53 bits of precision
>>> Phi.domain()
Open subset D of the 2-dimensional topological manifold R^2
>>> Phi.restrict(U).domain()
Open subset U of the 2-dimensional topological manifold R^2
>>> Phi.restrict(U).display()
Phi: U → R^2
   (x, y) ↦ (x, y) = (x/sqrt(-x^2 - y^2 + 1), y/sqrt(-x^2 - y^2 + 1))
```

The result is cached:

```
sage: Phi.restrict(U) is Phi.restrict(U)
True
```

```
>>> from sage.all import *
>>> Phi.restrict(U) is Phi.restrict(U)
True
```

The restriction of the identity map:

```
sage: id = D.identity_map() ; id
Identity map Id_D of the Open subset D of the 2-dimensional
 topological manifold R^2
sage: id.restrict(U)
Identity map Id_U of the Open subset U of the 2-dimensional
 topological manifold R^2
sage: id.restrict(U) is U.identity_map()
True
```

```
>>> from sage.all import *
>>> id = D.identity_map() ; id
Identity map Id_D of the Open subset D of the 2-dimensional
 topological manifold R^2
>>> id.restrict(U)
Identity map Id_U of the Open subset U of the 2-dimensional
 topological manifold R^2
>>> id.restrict(U) is U.identity_map()
True
```

The codomain can be restricted (i.e. made tighter):

```
sage: Phi = D.continuous_map(M, [x/sqrt(1+x^2+y^2), y/sqrt(1+x^2+y^2)])
sage: Phi
Continuous map from
 the Open subset D of the 2-dimensional topological manifold R^2
 to the 2-dimensional topological manifold R^2
sage: Phi.restrict(D, subcodomain=D)
Continuous map from the Open subset D of the 2-dimensional
 topological manifold R^2 to itself
```

```
>>> from sage.all import *
>>> Phi = D.continuous_map(M, [x/sqrt(Integer(1)+x**Integer(2)+y**Integer(2)),
↪ y/sqrt(Integer(1)+x**Integer(2)+y**Integer(2))])
>>> Phi
Continuous map from
 the Open subset D of the 2-dimensional topological manifold R^2
 to the 2-dimensional topological manifold R^2
>>> Phi.restrict(D, subcodomain=D)
Continuous map from the Open subset D of the 2-dimensional
 topological manifold R^2 to itself
```

set_expr(*chart1*, *chart2*, *coord_functions*)

Set a new coordinate representation of `self`.

The expressions with respect to other charts are deleted, in order to avoid any inconsistency. To keep them, use *add_expr()* instead.

INPUT:

- `chart1` – chart for the coordinates on the domain of `self`
- `chart2` – chart for the coordinates on the codomain of `self`
- `coord_functions` – the coordinate symbolic expression of the map in the above charts: list (or tuple) of the coordinates of the image expressed in terms of the coordinates of the considered point; if the

dimension of the arrival manifold is 1, a single coordinate expression can be passed instead of a tuple with a single element

EXAMPLES:

Polar representation of a planar rotation initially defined in Cartesian coordinates:

```
sage: M = Manifold(2, 'R^2', latex_name=r'\RR^2', structure='topological')  #␣
↪the Euclidean plane R^2
sage: c_xy.<x,y> = M.chart() # Cartesian coordinate on R^2
sage: U = M.open_subset('U', coord_def={c_xy: (y!=0, x<0)}) # the complement␣
↪of the segment y=0 and x>0
sage: c_cart = c_xy.restrict(U) # Cartesian coordinates on U
sage: c_spher.<r,ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi') # spherical␣
↪coordinates on U
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', latex_name=r'\RR^2', structure=
↪'topological')  # the Euclidean plane R^2
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪Cartesian coordinate on R^2
>>> U = M.open_subset('U', coord_def={c_xy: (y!=Integer(0), x<Integer(0))}) #␣
↪the complement of the segment y=0 and x>0
>>> c_cart = c_xy.restrict(U) # Cartesian coordinates on U
>>> c_spher = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi', names=('r', 'ph',)); (r,␣
↪ph,) = c_spher._first_ngens(2)# spherical coordinates on U
```

Links between spherical coordinates and Cartesian ones:

```
sage: ch_cart_spher = c_cart.transition_map(c_spher,
....:                                        [sqrt(x*x+y*y), atan2(y,x)])
sage: ch_cart_spher.set_inverse(r*cos(ph), r*sin(ph))
Check of the inverse coordinate transformation:
  x == x  *passed*
  y == y  *passed*
  r == r  *passed*
  ph == arctan2(r*sin(ph), r*cos(ph))  **failed**
NB: a failed report can reflect a mere lack of simplification.
sage: rot = U.continuous_map(U, ((x - sqrt(3)*y)/2, (sqrt(3)*x + y)/2),
....:                        name='R')
sage: rot.display(c_cart, c_cart)
R: U → U
   (x, y) ↦ (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
```

```
>>> from sage.all import *
>>> ch_cart_spher = c_cart.transition_map(c_spher,
...                                        [sqrt(x*x+y*y), atan2(y,x)])
>>> ch_cart_spher.set_inverse(r*cos(ph), r*sin(ph))
Check of the inverse coordinate transformation:
  x == x  *passed*
  y == y  *passed*
  r == r  *passed*
  ph == arctan2(r*sin(ph), r*cos(ph))  **failed**
NB: a failed report can reflect a mere lack of simplification.
```

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```
>>> rot = U.continuous_map(U, ((x - sqrt(Integer(3))*y)/Integer(2),␣
↪(sqrt(Integer(3))*x + y)/Integer(2)),
...                         name='R')
>>> rot.display(c_cart, c_cart)
R: U → U
   (x, y) ↦ (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
```

Let us use the method *set_expr()* to set the spherical-coordinate expression by hand:

```
sage: rot.set_expr(c_spher, c_spher, (r, ph+pi/3))
sage: rot.display(c_spher, c_spher)
R: U → U
   (r, ph) ↦ (r, 1/3*pi + ph)
```

```
>>> from sage.all import *
>>> rot.set_expr(c_spher, c_spher, (r, ph+pi/Integer(3)))
>>> rot.display(c_spher, c_spher)
R: U → U
   (r, ph) ↦ (r, 1/3*pi + ph)
```

The expression in Cartesian coordinates has been erased:

```
sage: rot._coord_expression
{(Chart (U, (r, ph)),
  Chart (U, (r, ph))): Coordinate functions (r, 1/3*pi + ph)
  on the Chart (U, (r, ph))}
```

```
>>> from sage.all import *
>>> rot._coord_expression
{(Chart (U, (r, ph)),
  Chart (U, (r, ph))): Coordinate functions (r, 1/3*pi + ph)
  on the Chart (U, (r, ph))}
```

It is recovered (thanks to the known change of coordinates) by a call to *display()*:

```
sage: rot.display(c_cart, c_cart)
R: U → U
   (x, y) ↦ (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)

sage: rot._coord_expression  # random (dictionary output)
{(Chart (U, (x, y)),
  Chart (U, (x, y))): Coordinate functions (-1/2*sqrt(3)*y + 1/2*x,
  1/2*sqrt(3)*x + 1/2*y) on the Chart (U, (x, y)),
 (Chart (U, (r, ph)),
  Chart (U, (r, ph))): Coordinate functions (r, 1/3*pi + ph)
  on the Chart (U, (r, ph))}
```

```
>>> from sage.all import *
>>> rot.display(c_cart, c_cart)
R: U → U
   (x, y) ↦ (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
```

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```
>>> rot._coord_expression  # random (dictionary output)
{(Chart (U, (x, y)),
  Chart (U, (x, y))): Coordinate functions (-1/2*sqrt(3)*y + 1/2*x,
  1/2*sqrt(3)*x + 1/2*y) on the Chart (U, (x, y)),
 (Chart (U, (r, ph)),
  Chart (U, (r, ph))): Coordinate functions (r, 1/3*pi + ph)
  on the Chart (U, (r, ph))}
```

set_expression(*chart1*, *chart2*, *coord_functions*)

Set a new coordinate representation of `self`.

The expressions with respect to other charts are deleted, in order to avoid any inconsistency. To keep them, use *add_expr()* instead.

INPUT:

- `chart1` – chart for the coordinates on the domain of `self`
- `chart2` – chart for the coordinates on the codomain of `self`
- `coord_functions` – the coordinate symbolic expression of the map in the above charts: list (or tuple) of the coordinates of the image expressed in terms of the coordinates of the considered point; if the dimension of the arrival manifold is 1, a single coordinate expression can be passed instead of a tuple with a single element

EXAMPLES:

Polar representation of a planar rotation initially defined in Cartesian coordinates:

```
sage: M = Manifold(2, 'R^2', latex_name=r'\RR^2', structure='topological')  #␣
↪the Euclidean plane R^2
sage: c_xy.<x,y> = M.chart() # Cartesian coordinate on R^2
sage: U = M.open_subset('U', coord_def={c_xy: (y!=0, x<0)}) # the complement␣
↪of the segment y=0 and x>0
sage: c_cart = c_xy.restrict(U) # Cartesian coordinates on U
sage: c_spher.<r,ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi') # spherical␣
↪coordinates on U
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', latex_name=r'\RR^2', structure=
↪'topological')  # the Euclidean plane R^2
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪Cartesian coordinate on R^2
>>> U = M.open_subset('U', coord_def={c_xy: (y!=Integer(0), x<Integer(0))}) #␣
↪the complement of the segment y=0 and x>0
>>> c_cart = c_xy.restrict(U) # Cartesian coordinates on U
>>> c_spher = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi', names=('r', 'ph',)); (r,␣
↪ph,) = c_spher._first_ngens(2)# spherical coordinates on U
```

Links between spherical coordinates and Cartesian ones:

```
sage: ch_cart_spher = c_cart.transition_map(c_spher,
....:                                       [sqrt(x*x+y*y), atan2(y,x)])
sage: ch_cart_spher.set_inverse(r*cos(ph), r*sin(ph))
Check of the inverse coordinate transformation:
```

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```
  x == x  *passed*
  y == y  *passed*
  r == r  *passed*
  ph == arctan2(r*sin(ph), r*cos(ph))  **failed**
NB: a failed report can reflect a mere lack of simplification.
sage: rot = U.continuous_map(U, ((x - sqrt(3)*y)/2, (sqrt(3)*x + y)/2),
....:                        name='R')
sage: rot.display(c_cart, c_cart)
R: U → U
   (x, y) ↦ (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
```

```
>>> from sage.all import *
>>> ch_cart_spher = c_cart.transition_map(c_spher,
...                                       [sqrt(x*x+y*y), atan2(y,x)])
>>> ch_cart_spher.set_inverse(r*cos(ph), r*sin(ph))
Check of the inverse coordinate transformation:
  x == x  *passed*
  y == y  *passed*
  r == r  *passed*
  ph == arctan2(r*sin(ph), r*cos(ph))  **failed**
NB: a failed report can reflect a mere lack of simplification.
>>> rot = U.continuous_map(U, ((x - sqrt(Integer(3))*y)/Integer(2),␣
↪(sqrt(Integer(3))*x + y)/Integer(2)),
...                        name='R')
>>> rot.display(c_cart, c_cart)
R: U → U
   (x, y) ↦ (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
```

Let us use the method `set_expr()` to set the spherical-coordinate expression by hand:

```
sage: rot.set_expr(c_spher, c_spher, (r, ph+pi/3))
sage: rot.display(c_spher, c_spher)
R: U → U
   (r, ph) ↦ (r, 1/3*pi + ph)
```

```
>>> from sage.all import *
>>> rot.set_expr(c_spher, c_spher, (r, ph+pi/Integer(3)))
>>> rot.display(c_spher, c_spher)
R: U → U
   (r, ph) ↦ (r, 1/3*pi + ph)
```

The expression in Cartesian coordinates has been erased:

```
sage: rot._coord_expression
{(Chart (U, (r, ph)),
  Chart (U, (r, ph))): Coordinate functions (r, 1/3*pi + ph)
  on the Chart (U, (r, ph))}
```

```
>>> from sage.all import *
>>> rot._coord_expression
{(Chart (U, (r, ph)),
```

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```
 Chart (U, (r, ph))): Coordinate functions (r, 1/3*pi + ph)
 on the Chart (U, (r, ph))}
```

It is recovered (thanks to the known change of coordinates) by a call to `display()`:

```
sage: rot.display(c_cart, c_cart)
R: U → U
   (x, y) ↦ (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)

sage: rot._coord_expression  # random (dictionary output)
{(Chart (U, (x, y)),
  Chart (U, (x, y))): Coordinate functions (-1/2*sqrt(3)*y + 1/2*x,
  1/2*sqrt(3)*x + 1/2*y) on the Chart (U, (x, y)),
 (Chart (U, (r, ph)),
  Chart (U, (r, ph))): Coordinate functions (r, 1/3*pi + ph)
  on the Chart (U, (r, ph))}
```

```
>>> from sage.all import *
>>> rot.display(c_cart, c_cart)
R: U → U
   (x, y) ↦ (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)

>>> rot._coord_expression  # random (dictionary output)
{(Chart (U, (x, y)),
  Chart (U, (x, y))): Coordinate functions (-1/2*sqrt(3)*y + 1/2*x,
  1/2*sqrt(3)*x + 1/2*y) on the Chart (U, (x, y)),
 (Chart (U, (r, ph)),
  Chart (U, (r, ph))): Coordinate functions (r, 1/3*pi + ph)
  on the Chart (U, (r, ph))}
```

1.7.3 Images of Manifold Subsets under Continuous Maps as Subsets of the Codomain

`ImageManifoldSubset` implements the image of a continuous map Φ from a manifold M to some manifold N as a subset $\Phi(M)$ of N, or more generally, the image $\Phi(S)$ of a subset $S \subseteq M$ as a subset of N.

class `sage.manifolds.continuous_map_image.`**`ImageManifoldSubset`**(*map*, *inverse=None*, *name=None*, *latex_name=None*, *domain_subset=None*)

Bases: `ManifoldSubset`

Subset of a topological manifold that is a continuous image of a manifold subset.

INPUT:

- `map` – continuous map Φ
- `inverse` – (default: `None`) continuous map from `map.codomain()` to `map.domain()`, which once restricted to the image of Φ is the inverse of Φ onto its image if the latter exists (NB: no check of this is performed)
- `name` – (default: computed from the names of the map and the subset) string; name (symbol) given to the subset

- `latex_name` – string (default: `None`); LaTeX symbol to denote the subset; if none is provided, it is set to `name`
- `domain_subset` – (default: the domain of `map`) a subset of the domain of `map`

1.8 Submanifolds of topological manifolds

Given a topological manifold M over a topological field K, a *topological submanifold of* M is defined by a topological manifold N over the same field K of dimension lower than the dimension of M and a topological embedding ϕ from N to M (i.e. ϕ is a homeomorphism onto its image).

In the case where the map ϕ is only an embedding locally, it is called an *topological immersion*, and defines an *immersed submanifold*.

The global embedding property cannot be checked in sage, so the immersed or embedded aspect of the manifold must be declared by the user, by calling either `set_embedding()` or `set_immersion()` while declaring the map ϕ.

The map $\phi : N \to M$ can also depend on one or multiple parameters. As long as ϕ remains injective in these parameters, it represents a *foliation*. The *dimension* of the foliation is defined as the number of parameters.

AUTHORS:

- Florentin Jaffredo (2018): initial version
- Eric Gourgoulhon (2018-2019): add documentation
- Matthias Koeppe (2021): open subsets of submanifolds

REFERENCES:

- J. M. Lee: *Introduction to Smooth Manifolds* [Lee2013]

class sage.manifolds.topological_submanifold.**TopologicalSubmanifold**(*n*, *name*, *field*, *structure*, *ambient=None*, *base_manifold=None*, *latex_name=None*, *start_index=0*, *category=None*, *unique_tag=None*)

Bases: `TopologicalManifold`

Submanifold of a topological manifold.

Given a topological manifold M over a topological field K, a *topological submanifold of* M is defined by a topological manifold N over the same field K of dimension lower than the dimension of M and a topological embedding ϕ from N to M (i.e. ϕ is an homeomorphism onto its image).

In the case where ϕ is only a topological immersion (i.e. is only locally an embedding), one says that N is an *immersed submanifold*.

The map ϕ can also depend on one or multiple parameters. As long as ϕ remains injective in these parameters, it represents a *foliation*. The *dimension* of the foliation is defined as the number of parameters.

INPUT:

- `n` – positive integer; dimension of the submanifold
- `name` – string; name (symbol) given to the submanifold
- `field` – field K on which the submanifold is defined; allowed values are
  - `'real'` or an object of type `RealField` (e.g., `RR`) for a manifold over $\mathbf{R}$

  - 'complex' or an object of type ComplexField (e.g., CC) for a manifold over **C**
  - an object in the category of topological fields (see Fields and TopologicalSpaces) for other types of manifolds

- structure – manifold structure (see *TopologicalStructure* or *RealTopologicalStructure*)
- ambient – (default: None) codomain M of the immersion ϕ; must be a topological manifold. If None, it is set to self
- base_manifold – (default: None) if not None, must be a topological manifold; the created object is then an open subset of base_manifold
- latex_name – (default: None) string; LaTeX symbol to denote the submanifold; if none are provided, it is set to name
- start_index – (default: 0) integer; lower value of the range of indices used for "indexed objects" on the submanifold, e.g., coordinates in a chart
- category – (default: None) to specify the category; if None, Manifolds(field) is assumed (see the category Manifolds)
- unique_tag – (default: None) tag used to force the construction of a new object when all the other arguments have been used previously (without unique_tag, the UniqueRepresentation behavior inherited from *ManifoldSubset* via *TopologicalManifold* would return the previously constructed object corresponding to these arguments)

EXAMPLES:

Let N be a 2-dimensional submanifold of a 3-dimensional manifold M:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: N = Manifold(2, 'N', ambient=M, structure='topological')
sage: N
2-dimensional topological submanifold N immersed in the 3-dimensional
 topological manifold M
sage: CM.<x,y,z> = M.chart()
sage: CN.<u,v> = N.chart()
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological')
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='topological')
>>> N
2-dimensional topological submanifold N immersed in the 3-dimensional
 topological manifold M
>>> CM = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = CM._first_ngens(3)
>>> CN = N.chart(names=('u', 'v',)); (u, v,) = CN._first_ngens(2)
```

Let us define a 1-dimensional foliation indexed by t:

```
sage: t = var('t')
sage: phi = N.continuous_map(M, {(CN,CM): [u, v, t+u^2+v^2]})
sage: phi.display()
N → M
   (u, v) ↦ (x, y, z) = (u, v, u^2 + v^2 + t)
```

```
>>> from sage.all import *
>>> t = var('t')
```

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```
>>> phi = N.continuous_map(M, {(CN,CM): [u, v, t+u**Integer(2)+v**Integer(2)]})
>>> phi.display()
N → M
   (u, v) ↦ (x, y, z) = (u, v, u^2 + v^2 + t)
```

The foliation inverse maps are needed for computing the adapted chart on the ambient manifold:

```
sage: phi_inv = M.continuous_map(N, {(CM, CN): [x, y]})
sage: phi_inv.display()
M → N
   (x, y, z) ↦ (u, v) = (x, y)
sage: phi_inv_t = M.scalar_field({CM: z-x^2-y^2})
sage: phi_inv_t.display()
M → ℝ
(x, y, z) ↦ -x^2 - y^2 + z
```

```
>>> from sage.all import *
>>> phi_inv = M.continuous_map(N, {(CM, CN): [x, y]})
>>> phi_inv.display()
M → N
   (x, y, z) ↦ (u, v) = (x, y)
>>> phi_inv_t = M.scalar_field({CM: z-x**Integer(2)-y**Integer(2)})
>>> phi_inv_t.display()
M → ℝ
(x, y, z) ↦ -x^2 - y^2 + z
```

ϕ can then be declared as an embedding $N \to M$:

```
sage: N.set_embedding(phi, inverse=phi_inv, var=t,
....:                 t_inverse={t: phi_inv_t})
```

```
>>> from sage.all import *
>>> N.set_embedding(phi, inverse=phi_inv, var=t,
...                 t_inverse={t: phi_inv_t})
```

The foliation can also be used to find new charts on the ambient manifold that are adapted to the foliation, i.e. in which the expression of the immersion is trivial. At the same time, the appropriate coordinate changes are computed:

```
sage: N.adapted_chart()
[Chart (M, (u_M, v_M, t_M))]
sage: M.atlas()
[Chart (M, (x, y, z)), Chart (M, (u_M, v_M, t_M))]
sage: len(M.coord_changes())
2
```

```
>>> from sage.all import *
>>> N.adapted_chart()
[Chart (M, (u_M, v_M, t_M))]
>>> M.atlas()
[Chart (M, (x, y, z)), Chart (M, (u_M, v_M, t_M))]
```

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```
>>> len(M.coord_changes())
2
```

The foliation parameters are always added as the last coordinates.

See also

manifold

adapted_chart(*postscript=None*, *latex_postscript=None*)

Create charts and changes of charts in the ambient manifold adapted to the foliation.

A manifold M of dimension m can be foliated by submanifolds N of dimension n. The corresponding embedding needs $m - n$ free parameters to describe the whole manifold.

A chart adapted to the foliation is a set of coordinates $(x_1, \ldots, x_n, t_1, \ldots, t_{m-n})$ on M such that $(x_1, \ldots, x_n)$ are coordinates on N and $(t_1, \ldots, t_{m-n})$ are the $m - n$ free parameters of the foliation.

Provided that an embedding with free variables is already defined, this function constructs such charts and coordinates changes whenever it is possible.

If there are restrictions of the coordinates on the starting chart, these restrictions are also propagated.

INPUT:

- `postscript` – (default: `None`) string defining the name of the coordinates of the adapted chart. This string will be appended to the names of the coordinates $(x_1, \ldots, x_n)$ and of the parameters $(t_1, \ldots, t_{m-n})$. If `None`, `"_" + self.ambient()._name` is used
- `latex_postscript` – (default: `None`) string defining the LaTeX name of the coordinates of the adapted chart. This string will be appended to the LaTeX names of the coordinates $(x_1, \ldots, x_n)$ and of the parameters $(t_1, \ldots, t_{m-n})$, If `None`, `"_" + self.ambient()._latex_()` is used

OUTPUT: list of adapted charts on M created from the charts of `self`

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='topological',
....:              latex_name=r"\mathcal{M}")
sage: N = Manifold(2, 'N', ambient=M, structure='topological')
sage: N
2-dimensional topological submanifold N immersed in the
 3-dimensional topological manifold M
sage: CM.<x,y,z> = M.chart()
sage: CN.<u,v> = N.chart()
sage: t = var('t')
sage: phi = N.continuous_map(M, {(CN,CM): [u,v,t+u^2+v^2]})
sage: phi_inv = M.continuous_map(N, {(CM,CN): [x,y]})
sage: phi_inv_t = M.scalar_field({CM: z-x^2-y^2})
sage: N.set_embedding(phi, inverse=phi_inv, var=t,
....:                  t_inverse={t:phi_inv_t})
sage: N.adapted_chart()
[Chart (M, (u_M, v_M, t_M))]
sage: latex(_)
\left[\left(\mathcal{M},({{u}_{\mathcal{M}}}, {{v}_{\mathcal{M}}},
 {{t}_{\mathcal{M}}})\right)\right]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological',
...                 latex_name=r"\mathcal{M}")
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='topological')
>>> N
2-dimensional topological submanifold N immersed in the
 3-dimensional topological manifold M
>>> CM = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = CM._first_ngens(3)
>>> CN = N.chart(names=('u', 'v',)); (u, v,) = CN._first_ngens(2)
>>> t = var('t')
>>> phi = N.continuous_map(M, {(CN,CM): [u,v,t+u**Integer(2)+v**Integer(2)]})
>>> phi_inv = M.continuous_map(N, {(CM,CN): [x,y]})
>>> phi_inv_t = M.scalar_field({CM: z-x**Integer(2)-y**Integer(2)})
>>> N.set_embedding(phi, inverse=phi_inv, var=t,
...                    t_inverse={t:phi_inv_t})
>>> N.adapted_chart()
[Chart (M, (u_M, v_M, t_M))]
>>> latex(_)
\left[\left(\mathcal{M},({{u}_{\mathcal{M}}}, {{v}_{\mathcal{M}}},
 {{t}_{\mathcal{M}}})\right)\right]
```

The adapted chart has been added to the atlas of `M`:

```
sage: M.atlas()
[Chart (M, (x, y, z)), Chart (M, (u_M, v_M, t_M))]
sage: N.atlas()
[Chart (N, (u, v))]
```

```
>>> from sage.all import *
>>> M.atlas()
[Chart (M, (x, y, z)), Chart (M, (u_M, v_M, t_M))]
>>> N.atlas()
[Chart (N, (u, v))]
```

The names of the adapted coordinates can be customized:

```
sage: N.adapted_chart(postscript='1', latex_postscript='_1')
[Chart (M, (u1, v1, t1))]
sage: latex(_)
\left[\left(\mathcal{M},({{u}_1}, {{v}_1}, {{t}_1})\right)\right]
```

```
>>> from sage.all import *
>>> N.adapted_chart(postscript='1', latex_postscript='_1')
[Chart (M, (u1, v1, t1))]
>>> latex(_)
\left[\left(\mathcal{M},({{u}_1}, {{v}_1}, {{t}_1})\right)\right]
```

ambient()

Return the manifold in which `self` is immersed or embedded.

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: N = Manifold(2, 'N', ambient=M, structure='topological')
sage: N.ambient()
3-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological')
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='topological')
>>> N.ambient()
3-dimensional topological manifold M
```

as_subset()

Return `self` as a subset of the ambient manifold.

`self` must be an embedded submanifold.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: N = Manifold(1, 'N', ambient=M, structure='topological')
sage: CM.<x,y> = M.chart()
sage: CN.<u> = N.chart(coord_restrictions=lambda u: [u > -1, u < 1])
sage: phi = N.continuous_map(M, {(CN,CM): [u, u^2]})
sage: N.set_embedding(phi)
sage: N
1-dimensional topological submanifold N
  embedded in the 2-dimensional topological manifold M
sage: N.as_subset()
Image of the Continuous map
  from the 1-dimensional topological submanifold N
    embedded in the 2-dimensional topological manifold M
  to the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> N = Manifold(Integer(1), 'N', ambient=M, structure='topological')
>>> CM = M.chart(names=('x', 'y',)); (x, y,) = CM._first_ngens(2)
>>> CN = N.chart(coord_restrictions=lambda u: [u > -Integer(1), u <␣
↪Integer(1)], names=('u',)); (u,) = CN._first_ngens(1)
>>> phi = N.continuous_map(M, {(CN,CM): [u, u**Integer(2)]})
>>> N.set_embedding(phi)
>>> N
1-dimensional topological submanifold N
  embedded in the 2-dimensional topological manifold M
>>> N.as_subset()
Image of the Continuous map
  from the 1-dimensional topological submanifold N
    embedded in the 2-dimensional topological manifold M
  to the 2-dimensional topological manifold M
```

declare_embedding()

Declare that the immersion provided by *set_immersion()* is in fact an embedding.

A *topological embedding* is a continuous map that is a homeomorphism onto its image. A *differentiable* *embedding* is a topological embedding that is also a differentiable immersion.

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: N = Manifold(2, 'N', ambient=M, structure='topological')
sage: N
2-dimensional topological submanifold N immersed in the
 3-dimensional topological manifold M
sage: CM.<x,y,z> = M.chart()
sage: CN.<u,v> = N.chart()
sage: t = var('t')
sage: phi = N.continuous_map(M, {(CN,CM): [u,v,t+u^2+v^2]})
sage: phi_inv = M.continuous_map(N, {(CM,CN): [x,y]})
sage: phi_inv_t = M.scalar_field({CM: z-x^2-y^2})
sage: N.set_immersion(phi, inverse=phi_inv, var=t,
....:                 t_inverse={t: phi_inv_t})
sage: N._immersed
True
sage: N._embedded
False
sage: N.declare_embedding()
sage: N._immersed
True
sage: N._embedded
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological')
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='topological')
>>> N
2-dimensional topological submanifold N immersed in the
 3-dimensional topological manifold M
>>> CM = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = CM._first_ngens(3)
>>> CN = N.chart(names=('u', 'v',)); (u, v,) = CN._first_ngens(2)
>>> t = var('t')
>>> phi = N.continuous_map(M, {(CN,CM): [u,v,t+u**Integer(2)+v**Integer(2)]})
>>> phi_inv = M.continuous_map(N, {(CM,CN): [x,y]})
>>> phi_inv_t = M.scalar_field({CM: z-x**Integer(2)-y**Integer(2)})
>>> N.set_immersion(phi, inverse=phi_inv, var=t,
...                 t_inverse={t: phi_inv_t})
>>> N._immersed
True
>>> N._embedded
False
>>> N.declare_embedding()
>>> N._immersed
True
>>> N._embedded
True
```

embedding()

Return the embedding of `self` into the ambient manifold.

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: N = Manifold(2, 'N', ambient=M, structure='topological')
sage: CM.<x,y,z> = M.chart()
sage: CN.<u,v> = N.chart()
sage: t = var('t')
sage: phi = N.continuous_map(M, {(CN,CM): [u,v,t+u^2+v^2]})
sage: phi_inv = M.continuous_map(N, {(CM,CN): [x,y]})
sage: phi_inv_t = M.scalar_field({CM: z-x^2-y^2})
sage: N.set_embedding(phi, inverse=phi_inv, var=t,
....:                 t_inverse={t: phi_inv_t})
sage: N.embedding()
Continuous map from the 2-dimensional topological submanifold N
 embedded in the 3-dimensional topological manifold M to the
 3-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological')
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='topological')
>>> CM = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = CM._first_ngens(3)
>>> CN = N.chart(names=('u', 'v',)); (u, v,) = CN._first_ngens(2)
>>> t = var('t')
>>> phi = N.continuous_map(M, {(CN,CM): [u,v,t+u**Integer(2)+v**Integer(2)]})
>>> phi_inv = M.continuous_map(N, {(CM,CN): [x,y]})
>>> phi_inv_t = M.scalar_field({CM: z-x**Integer(2)-y**Integer(2)})
>>> N.set_embedding(phi, inverse=phi_inv, var=t,
...                 t_inverse={t: phi_inv_t})
>>> N.embedding()
Continuous map from the 2-dimensional topological submanifold N
 embedded in the 3-dimensional topological manifold M to the
 3-dimensional topological manifold M
```

immersion()

Return the immersion of `self` into the ambient manifold.

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: N = Manifold(2, 'N', ambient=M, structure='topological')
sage: CM.<x,y,z> = M.chart()
sage: CN.<u,v> = N.chart()
sage: t = var('t')
sage: phi = N.continuous_map(M, {(CN,CM): [u,v,t+u^2+v^2]})
sage: phi_inv = M.continuous_map(N, {(CM,CN): [x,y]})
sage: phi_inv_t = M.scalar_field({CM: z-x^2-y^2})
sage: N.set_immersion(phi, inverse=phi_inv, var=t,
....:                 t_inverse={t: phi_inv_t})
sage: N.immersion()
Continuous map from the 2-dimensional topological submanifold N
 immersed in the 3-dimensional topological manifold M to the
 3-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological')
```

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```
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='topological')
>>> CM = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = CM._first_ngens(3)
>>> CN = N.chart(names=('u', 'v',)); (u, v,) = CN._first_ngens(2)
>>> t = var('t')
>>> phi = N.continuous_map(M, {(CN,CM): [u,v,t+u**Integer(2)+v**Integer(2)]})
>>> phi_inv = M.continuous_map(N, {(CM,CN): [x,y]})
>>> phi_inv_t = M.scalar_field({CM: z-x**Integer(2)-y**Integer(2)})
>>> N.set_immersion(phi, inverse=phi_inv, var=t,
...                 t_inverse={t: phi_inv_t})
>>> N.immersion()
Continuous map from the 2-dimensional topological submanifold N
 immersed in the 3-dimensional topological manifold M to the
 3-dimensional topological manifold M
```

open_subset(*name*, *latex_name=None*, *coord_def={}*, *supersets=None*)

Create an open subset of the manifold.

An open subset is a set that is (i) included in the manifold and (ii) open with respect to the manifold's topology. It is a topological manifold by itself.

As `self` is a submanifold of its ambient manifold, the new open subset is also considered a submanifold of that. Hence the returned object is an instance of *TopologicalSubmanifold*.

INPUT:

- `name` – name given to the open subset
- `latex_name` – (default: `None`) LaTeX symbol to denote the subset; if none are provided, it is set to `name`
- `coord_def` – (default: {}) definition of the subset in terms of coordinates; `coord_def` must a be dictionary with keys charts on the manifold and values the symbolic expressions formed by the coordinates to define the subset
- `supersets` – (default: only `self`) list of sets that the new open subset is a subset of

OUTPUT: the open subset, as an instance of *TopologicalSubmanifold*

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: N = Manifold(2, 'N', ambient=M, structure='topological'); N
2-dimensional topological submanifold N immersed in the
 3-dimensional topological manifold M
sage: S = N.subset('S'); S
Subset S of the
 2-dimensional topological submanifold N immersed in the
  3-dimensional topological manifold M
sage: O = N.subset('O', is_open=True); O  # indirect doctest
Open subset O of the
 2-dimensional topological submanifold N immersed in the
  3-dimensional topological manifold M

sage: phi = N.continuous_map(M)
sage: N.set_embedding(phi)
sage: N
```

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```
2-dimensional topological submanifold N embedded in the
 3-dimensional topological manifold M
sage: S = N.subset('S'); S
Subset S of the
 2-dimensional topological submanifold N embedded in the
  3-dimensional topological manifold M
sage: O = N.subset('O', is_open=True); O  # indirect doctest
Open subset O of the
 2-dimensional topological submanifold N embedded in the
  3-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological')
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='topological'); N
2-dimensional topological submanifold N immersed in the
 3-dimensional topological manifold M
>>> S = N.subset('S'); S
Subset S of the
 2-dimensional topological submanifold N immersed in the
  3-dimensional topological manifold M
>>> O = N.subset('O', is_open=True); O  # indirect doctest
Open subset O of the
 2-dimensional topological submanifold N immersed in the
  3-dimensional topological manifold M

>>> phi = N.continuous_map(M)
>>> N.set_embedding(phi)
>>> N
2-dimensional topological submanifold N embedded in the
 3-dimensional topological manifold M
>>> S = N.subset('S'); S
Subset S of the
 2-dimensional topological submanifold N embedded in the
  3-dimensional topological manifold M
>>> O = N.subset('O', is_open=True); O  # indirect doctest
Open subset O of the
 2-dimensional topological submanifold N embedded in the
  3-dimensional topological manifold M
```

plot(*param*, *u*, *v*, *chart1=None*, *chart2=None*, ***kwargs*)

Plot an embedding.

Plot the embedding defined by the foliation and a set of values for the free parameters. This function can only plot 2-dimensional surfaces embedded in 3-dimensional manifolds. It ultimately calls `ParametricSurface`.

INPUT:

- `param` – dictionary of values indexed by the free variables appearing in the foliation
- `u` – iterable of the values taken by the first coordinate of the surface to plot
- `v` – iterable of the values taken by the second coordinate of the surface to plot
- `chart1` – (default: `None`) chart in which `u` and `v` are considered. By default, the default chart of the submanifold is used

- `chart2` – (default: `None`) chart in the codomain of the embedding. By default, the default chart of the codomain is used
- `**kwargs` – other arguments as used in `ParametricSurface`

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: N = Manifold(2, 'N', ambient = M, structure='topological')
sage: CM.<x,y,z> = M.chart()
sage: CN.<u,v> = N.chart()
sage: t = var('t')
sage: phi = N.continuous_map(M, {(CN,CM): [u,v,t+u^2+v^2]})
sage: phi_inv = M.continuous_map(N, {(CM,CN): [x,y]})
sage: phi_inv_t = M.scalar_field({CM: z-x^2-y^2})
sage: N.set_embedding(phi, inverse=phi_inv, var=t,
....:                 t_inverse = {t:phi_inv_t})
sage: N.adapted_chart()
[Chart (M, (u_M, v_M, t_M))]
sage: P0 = N.plot({t:0}, srange(-1, 1, 0.1), srange(-1, 1, 0.1),
....:             CN, CM, opacity=0.3, mesh=True)
sage: P1 = N.plot({t:1}, srange(-1, 1, 0.1), srange(-1, 1, 0.1),
....:             CN, CM, opacity=0.3, mesh=True)
sage: P2 = N.plot({t:2}, srange(-1, 1, 0.1), srange(-1, 1, 0.1),
....:             CN, CM, opacity=0.3, mesh=True)
sage: P3 = N.plot({t:3}, srange(-1, 1, 0.1), srange(-1, 1, 0.1),
....:             CN, CM, opacity=0.3, mesh=True)
sage: P0 + P1 + P2 + P3
Graphics3d Object
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological')
>>> N = Manifold(Integer(2), 'N', ambient = M, structure='topological')
>>> CM = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = CM._first_ngens(3)
>>> CN = N.chart(names=('u', 'v',)); (u, v,) = CN._first_ngens(2)
>>> t = var('t')
>>> phi = N.continuous_map(M, {(CN,CM): [u,v,t+u**Integer(2)+v**Integer(2)]})
>>> phi_inv = M.continuous_map(N, {(CM,CN): [x,y]})
>>> phi_inv_t = M.scalar_field({CM: z-x**Integer(2)-y**Integer(2)})
>>> N.set_embedding(phi, inverse=phi_inv, var=t,
...                 t_inverse = {t:phi_inv_t})
>>> N.adapted_chart()
[Chart (M, (u_M, v_M, t_M))]
>>> P0 = N.plot({t:Integer(0)}, srange(-Integer(1), Integer(1), RealNumber('0.
↪1')), srange(-Integer(1), Integer(1), RealNumber('0.1')),
...             CN, CM, opacity=RealNumber('0.3'), mesh=True)
>>> P1 = N.plot({t:Integer(1)}, srange(-Integer(1), Integer(1), RealNumber('0.
↪1')), srange(-Integer(1), Integer(1), RealNumber('0.1')),
...             CN, CM, opacity=RealNumber('0.3'), mesh=True)
>>> P2 = N.plot({t:Integer(2)}, srange(-Integer(1), Integer(1), RealNumber('0.
↪1')), srange(-Integer(1), Integer(1), RealNumber('0.1')),
...             CN, CM, opacity=RealNumber('0.3'), mesh=True)
>>> P3 = N.plot({t:Integer(3)}, srange(-Integer(1), Integer(1), RealNumber('0.
↪1')), srange(-Integer(1), Integer(1), RealNumber('0.1')),
```

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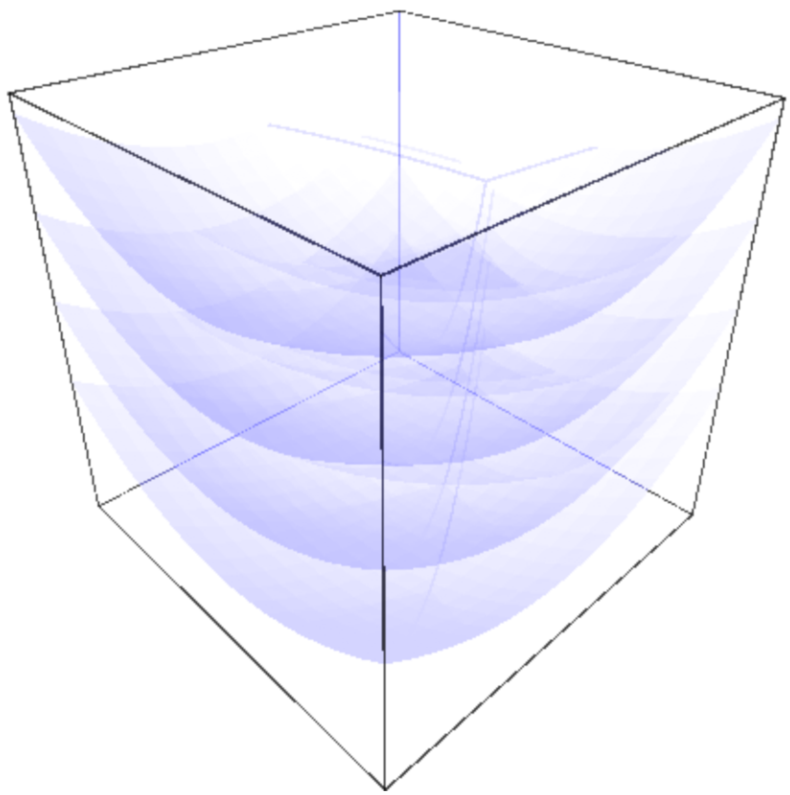
```
...                 CN, CM, opacity=RealNumber('0.3'), mesh=True)
>>> P0 + P1 + P2 + P3
Graphics3d Object
```



See also:

ParametricSurface

set_embedding(*phi*, *inverse=None*, *var=None*, *t_inverse=None*)

Register the embedding of an embedded submanifold.

A *topological embedding* is a continuous map that is a homeomorphism onto its image. A *differentiable embedding* is a topological embedding that is also a differentiable immersion.

INPUT:

- `phi` – continuous map ϕ from `self` to `self.ambient()`
- `inverse` – (default: `None`) continuous map from `self.ambient()` to `self`, which once restricted to the image of ϕ is the inverse of ϕ onto its image (NB: no check of this is performed)
- `var` – (default: `None`) list of parameters involved in the definition of ϕ (case of foliation); if ϕ depends on a single parameter `t`, one can write `var=t` as a shortcut for `var=[t]`

- `t_inverse` – (default: `None`) dictionary of scalar fields on `self.ambient()` providing the values of the parameters involved in the definition of ϕ (case of foliation), the keys being the parameters

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: N = Manifold(2, 'N', ambient=M, structure='topological')
sage: N
2-dimensional topological submanifold N immersed in the
 3-dimensional topological manifold M
sage: CM.<x,y,z> = M.chart()
sage: CN.<u,v> = N.chart()
sage: t = var('t')
sage: phi = N.continuous_map(M, {(CN,CM): [u,v,t+u^2+v^2]})
sage: phi.display()
N → M
   (u, v) ↦ (x, y, z) = (u, v, u^2 + v^2 + t)
sage: phi_inv = M.continuous_map(N, {(CM,CN): [x,y]})
sage: phi_inv.display()
M → N
    (x, y, z) ↦ (u, v) = (x, y)
sage: phi_inv_t = M.scalar_field({CM: z-x^2-y^2})
sage: phi_inv_t.display()
M → ℝ
(x, y, z) ↦ -x^2 - y^2 + z
sage: N.set_embedding(phi, inverse=phi_inv, var=t,
....:                 t_inverse={t: phi_inv_t})
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological')
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='topological')
>>> N
2-dimensional topological submanifold N immersed in the
 3-dimensional topological manifold M
>>> CM = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = CM._first_ngens(3)
>>> CN = N.chart(names=('u', 'v',)); (u, v,) = CN._first_ngens(2)
>>> t = var('t')
>>> phi = N.continuous_map(M, {(CN,CM): [u,v,t+u**Integer(2)+v**Integer(2)]})
>>> phi.display()
N → M
   (u, v) ↦ (x, y, z) = (u, v, u^2 + v^2 + t)
>>> phi_inv = M.continuous_map(N, {(CM,CN): [x,y]})
>>> phi_inv.display()
M → N
    (x, y, z) ↦ (u, v) = (x, y)
>>> phi_inv_t = M.scalar_field({CM: z-x**Integer(2)-y**Integer(2)})
>>> phi_inv_t.display()
M → ℝ
(x, y, z) ↦ -x^2 - y^2 + z
>>> N.set_embedding(phi, inverse=phi_inv, var=t,
...                 t_inverse={t: phi_inv_t})
```

Now `N` appears as an embedded submanifold:

```
sage: N
2-dimensional topological submanifold N embedded in the
 3-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> N
2-dimensional topological submanifold N embedded in the
 3-dimensional topological manifold M
```

set_immersion(*phi*, *inverse=None*, *var=None*, *t_inverse=None*)

Register the immersion of the immersed submanifold.

A *topological immersion* is a continuous map that is locally a topological embedding (i.e. a homeomorphism onto its image). A *differentiable immersion* is a differentiable map whose differential is injective at each point.

If an inverse of the immersion onto its image exists, it can be registered at the same time. If the immersion depends on parameters, they must also be declared here.

INPUT:

- `phi` – continuous map ϕ from `self` to `self.ambient()`
- `inverse` – (default: `None`) continuous map from `self.ambient()` to `self`, which once restricted to the image of ϕ is the inverse of ϕ onto its image if the latter exists (NB: no check of this is performed)
- `var` – (default: `None`) list of parameters involved in the definition of ϕ (case of foliation); if ϕ depends on a single parameter `t`, one can write `var=t` as a shortcut for `var=[t]`
- `t_inverse` – (default: `None`) dictionary of scalar fields on `self.ambient()` providing the values of the parameters involved in the definition of ϕ (case of foliation), the keys being the parameters

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: N = Manifold(2, 'N', ambient=M, structure='topological')
sage: N
2-dimensional topological submanifold N immersed in the
 3-dimensional topological manifold M
sage: CM.<x,y,z> = M.chart()
sage: CN.<u,v> = N.chart()
sage: t = var('t')
sage: phi = N.continuous_map(M, {(CN,CM): [u,v,t+u^2+v^2]})
sage: phi.display()
N → M
   (u, v) ↦ (x, y, z) = (u, v, u^2 + v^2 + t)
sage: phi_inv = M.continuous_map(N, {(CM,CN): [x,y]})
sage: phi_inv.display()
M → N
    (x, y, z) ↦ (u, v) = (x, y)
sage: phi_inv_t = M.scalar_field({CM: z-x^2-y^2})
sage: phi_inv_t.display()
M → ℝ
(x, y, z) ↦ -x^2 - y^2 + z
sage: N.set_immersion(phi, inverse=phi_inv, var=t,
....:                 t_inverse={t: phi_inv_t})
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological')
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='topological')
>>> N
2-dimensional topological submanifold N immersed in the
 3-dimensional topological manifold M
>>> CM = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = CM._first_ngens(3)
>>> CN = N.chart(names=('u', 'v',)); (u, v,) = CN._first_ngens(2)
>>> t = var('t')
>>> phi = N.continuous_map(M, {(CN,CM): [u,v,t+u**Integer(2)+v**Integer(2)]})
>>> phi.display()
N → M
   (u, v) ↦ (x, y, z) = (u, v, u^2 + v^2 + t)
>>> phi_inv = M.continuous_map(N, {(CM,CN): [x,y]})
>>> phi_inv.display()
M → N
    (x, y, z) ↦ (u, v) = (x, y)
>>> phi_inv_t = M.scalar_field({CM: z-x**Integer(2)-y**Integer(2)})
>>> phi_inv_t.display()
M → ℝ
(x, y, z) ↦ -x^2 - y^2 + z
>>> N.set_immersion(phi, inverse=phi_inv, var=t,
...                 t_inverse={t: phi_inv_t})
```

1.9 Topological Vector Bundles

1.9.1 Topological Vector Bundle

Let K be a topological field. A *vector bundle* of rank n over the field K and over a topological manifold B (base space) is a topological manifold E (total space) together with a continuous and surjective map $\pi : E \to B$ such that for every point $p \in B$, we have:

- the set $E_p = \pi^{-1}(p)$ has the vector space structure of K^n,
- there is a neighborhood $U \subset B$ of p and a homeomorphism (trivialization) $\varphi : \pi^{-1}(p) \to U \times K^n$ such that φ is compatible with the fibers, namely $\pi \circ \varphi^{-1} = \mathrm{pr}_1$, and $v \mapsto \varphi^{-1}(q, v)$ is a linear isomorphism between K^n and E_q for any $q \in U$.

AUTHORS:

- Michael Jung (2019) : initial version

REFERENCES:

- [Lee2013]
- [Mil1974]

class sage.manifolds.vector_bundle.**TopologicalVectorBundle**(*rank*, *name*, *base_space*, *field='real'*, *latex_name=None*, *category=None*, *unique_tag=None*)

Bases: CategoryObject, UniqueRepresentation

An instance of this class is a topological vector bundle $E \to B$ over a topological field K.

INPUT:

- `rank` – positive integer; rank of the vector bundle
- `name` – string representation given to the total space
- `base_space` – the base space (topological manifold) over which the vector bundle is defined
- `field` – field K which gives the fibers the structure of a vector space over K; allowed values are
  - `'real'` or an object of type `RealField` (e.g., `RR`) for a vector bundle over **R**
  - `'complex'` or an object of type `ComplexField` (e.g., `CC`) for a vector bundle over **C**
  - an object in the category of topological fields (see `Fields` and `TopologicalSpaces`) for other types of topological fields
- `latex_name` – (default: `None`) LaTeX representation given to the total space
- `category` – (default: `None`) to specify the category; if `None`, `VectorBundles(base_space, c_field)` is assumed (see the category `VectorBundles`)
- `unique_tag` – (default: `None`) tag used to force the construction of a new object when all the other arguments have been used previously (without `unique_tag`, the `UniqueRepresentation` behavior would return the previously constructed object corresponding to these arguments)

EXAMPLES:

A real line bundle over some 4-dimensional topological manifold:

```
sage: M = Manifold(4, 'M', structure='top')
sage: E = M.vector_bundle(1, 'E'); E
Topological real vector bundle E -> M of rank 1 over the base space
 4-dimensional topological manifold M
sage: E.base_space()
4-dimensional topological manifold M
sage: E.base_ring()
Real Field with 53 bits of precision
sage: E.rank()
1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='top')
>>> E = M.vector_bundle(Integer(1), 'E'); E
Topological real vector bundle E -> M of rank 1 over the base space
 4-dimensional topological manifold M
>>> E.base_space()
4-dimensional topological manifold M
>>> E.base_ring()
Real Field with 53 bits of precision
>>> E.rank()
1
```

For a more sophisticated example, let us define a non-trivial 2-manifold to work with:

```
sage: M = Manifold(2, 'M', structure='top')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                   intersection_name='W', restrictions1= x>0,
```

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```
....:                     restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: E = M.vector_bundle(2, 'E'); E
Topological real vector bundle E -> M of rank 2 over the base space
 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv = V.
↪chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
...                     intersection_name='W', restrictions1= x>Integer(0),
...                     restrictions2= u+v>Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V)
>>> E = M.vector_bundle(Integer(2), 'E'); E
Topological real vector bundle E -> M of rank 2 over the base space
 2-dimensional topological manifold M
```

Now, there a two ways to go. Most effortlessly, we define trivializations similar to charts (see `Trivialization`):

```
sage: phi_U = E.trivialization('phi_U', domain=U); phi_U
Trivialization (phi_U, E|_U)
sage: phi_V = E.trivialization('phi_V', domain=V); phi_V
Trivialization (phi_V, E|_V)
sage: transf = phi_U.transition_map(phi_V, [[0,x],[x,0]]) # transition map␣
↪between trivializations
sage: fU = phi_U.frame(); fU
Trivialization frame (E|_U, ((phi_U^*e_1),(phi_U^*e_2)))
sage: fV = phi_V.frame(); fV
Trivialization frame (E|_V, ((phi_V^*e_1),(phi_V^*e_2)))
sage: E.changes_of_frame() # random
{(Local frame (E|_W, ((phi_U^*e_1),(phi_U^*e_2))),
 Local frame (E|_W, ((phi_V^*e_1),(phi_V^*e_2)))): Automorphism
 phi_U^(-1)*phi_V^(-1) of the Free module C^0(W;E) of sections on
 the Open subset W of the 2-dimensional topological manifold M with
 values in the real vector bundle E of rank 2,
 (Local frame (E|_W, ((phi_V^*e_1),(phi_V^*e_2))),
 Local frame (E|_W, ((phi_U^*e_1),(phi_U^*e_2)))): Automorphism
 phi_U^(-1)*phi_V of the Free module C^0(W;E) of sections on the
 Open subset W of the 2-dimensional topological manifold M with
 values in the real vector bundle E of rank 2}
```

```
>>> from sage.all import *
>>> phi_U = E.trivialization('phi_U', domain=U); phi_U
Trivialization (phi_U, E|_U)
>>> phi_V = E.trivialization('phi_V', domain=V); phi_V
Trivialization (phi_V, E|_V)
>>> transf = phi_U.transition_map(phi_V, [[Integer(0),x],[x,Integer(0)]]) #␣
```

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```
↪transition map between trivializations
>>> fU = phi_U.frame(); fU
Trivialization frame (E|_U, ((phi_U^*e_1),(phi_U^*e_2)))
>>> fV = phi_V.frame(); fV
Trivialization frame (E|_V, ((phi_V^*e_1),(phi_V^*e_2)))
>>> E.changes_of_frame() # random
{(Local frame (E|_W, ((phi_U^*e_1),(phi_U^*e_2))),
 Local frame (E|_W, ((phi_V^*e_1),(phi_V^*e_2)))): Automorphism
 phi_U^(-1)*phi_V^(-1) of the Free module C^0(W;E) of sections on
 the Open subset W of the 2-dimensional topological manifold M with
 values in the real vector bundle E of rank 2,
 (Local frame (E|_W, ((phi_V^*e_1),(phi_V^*e_2))),
 Local frame (E|_W, ((phi_U^*e_1),(phi_U^*e_2)))): Automorphism
 phi_U^(-1)*phi_V of the Free module C^0(W;E) of sections on the
 Open subset W of the 2-dimensional topological manifold M with
 values in the real vector bundle E of rank 2}
```

Then, the atlas of E consists of all known trivializations defined on E:

```
sage: E.atlas() # a shallow copy of the atlas
[Trivialization (phi_U, E|_U), Trivialization (phi_V, E|_V)]
```

```
>>> from sage.all import *
>>> E.atlas() # a shallow copy of the atlas
[Trivialization (phi_U, E|_U), Trivialization (phi_V, E|_V)]
```

Or we just define frames, an automorphism on the free section module over the intersection domain W and declare the change of frame manually (for more details consult `LocalFrame`):

```
sage: eU = E.local_frame('eU', domain=U); eU
Local frame (E|_U, (eU_0,eU_1))
sage: eUW = eU.restrict(W) # to trivialize E|_W
sage: eV = E.local_frame('eV', domain=V); eV
Local frame (E|_V, (eV_0,eV_1))
sage: eVW = eV.restrict(W)
sage: a = E.section_module(domain=W).automorphism(); a
Automorphism of the Free module C^0(W;E) of sections on the Open
 subset W of the 2-dimensional topological manifold M with values in
 the real vector bundle E of rank 2
sage: a[eUW,:] = [[0,x],[x,0]]
sage: E.set_change_of_frame(eUW, eVW, a)
sage: E.change_of_frame(eUW, eVW)
Automorphism of the Free module C^0(W;E) of sections on the Open
 subset W of the 2-dimensional topological manifold M with values in
 the real vector bundle E of rank 2
```

```
>>> from sage.all import *
>>> eU = E.local_frame('eU', domain=U); eU
Local frame (E|_U, (eU_0,eU_1))
>>> eUW = eU.restrict(W) # to trivialize E|_W
>>> eV = E.local_frame('eV', domain=V); eV
Local frame (E|_V, (eV_0,eV_1))
```

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```
>>> eVW = eV.restrict(W)
>>> a = E.section_module(domain=W).automorphism(); a
Automorphism of the Free module C^0(W;E) of sections on the Open
 subset W of the 2-dimensional topological manifold M with values in
 the real vector bundle E of rank 2
>>> a[eUW,:] = [[Integer(0),x],[x,Integer(0)]]
>>> E.set_change_of_frame(eUW, eVW, a)
>>> E.change_of_frame(eUW, eVW)
Automorphism of the Free module C^0(W;E) of sections on the Open
 subset W of the 2-dimensional topological manifold M with values in
 the real vector bundle E of rank 2
```

Now, the list of all known frames defined on E can be displayed via `frames()`:

```
sage: E.frames() # a shallow copy of all known frames on E
[Trivialization frame (E|_U, ((phi_U^*e_1),(phi_U^*e_2))),
 Trivialization frame (E|_V, ((phi_V^*e_1),(phi_V^*e_2))),
 Local frame (E|_W, ((phi_U^*e_1),(phi_U^*e_2))),
 Local frame (E|_W, ((phi_V^*e_1),(phi_V^*e_2))),
 Local frame (E|_U, (eU_0,eU_1)),
 Local frame (E|_W, (eU_0,eU_1)),
 Local frame (E|_V, (eV_0,eV_1)),
 Local frame (E|_W, (eV_0,eV_1))]
```

```
>>> from sage.all import *
>>> E.frames() # a shallow copy of all known frames on E
[Trivialization frame (E|_U, ((phi_U^*e_1),(phi_U^*e_2))),
 Trivialization frame (E|_V, ((phi_V^*e_1),(phi_V^*e_2))),
 Local frame (E|_W, ((phi_U^*e_1),(phi_U^*e_2))),
 Local frame (E|_W, ((phi_V^*e_1),(phi_V^*e_2))),
 Local frame (E|_U, (eU_0,eU_1)),
 Local frame (E|_W, (eU_0,eU_1)),
 Local frame (E|_V, (eV_0,eV_1)),
 Local frame (E|_W, (eV_0,eV_1))]
```

By definition E is a manifold, in this case of dimension 4 (notice that the induced charts are not implemented, yet):

```
sage: E.total_space()
4-dimensional topological manifold E
```

```
>>> from sage.all import *
>>> E.total_space()
4-dimensional topological manifold E
```

The method `section()` returns a section while the method `section_module()` returns the section module on the corresponding domain:

```
sage: s = E.section(name='s'); s
Section s on the 2-dimensional topological manifold M with values in
 the real vector bundle E of rank 2
sage: s in E.section_module()
True
```

```
>>> from sage.all import *
>>> s = E.section(name='s'); s
Section s on the 2-dimensional topological manifold M with values in
 the real vector bundle E of rank 2
>>> s in E.section_module()
True
```

atlas()

Return the list of trivializations that have been defined for `self`.

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: U = M.open_subset('U')
sage: V = M.open_subset('V')
sage: E = M.vector_bundle(2, 'E')
sage: phi_U = E.trivialization('phi_U', domain=U)
sage: phi_V = E.trivialization('phi_V', domain=V)
sage: phi_M = E.trivialization('phi_M')
sage: E.atlas()
[Trivialization (phi_U, E|_U),
 Trivialization (phi_V, E|_V),
 Trivialization (phi_M, E|_M)]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> U = M.open_subset('U')
>>> V = M.open_subset('V')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> phi_U = E.trivialization('phi_U', domain=U)
>>> phi_V = E.trivialization('phi_V', domain=V)
>>> phi_M = E.trivialization('phi_M')
>>> E.atlas()
[Trivialization (phi_U, E|_U),
 Trivialization (phi_V, E|_V),
 Trivialization (phi_M, E|_M)]
```

base_field()

Return the field on which the fibers are defined.

OUTPUT: a topological field

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: E = M.vector_bundle(2, 'E', field=CC)
sage: E.base_field()
Complex Field with 53 bits of precision
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological')
>>> E = M.vector_bundle(Integer(2), 'E', field=CC)
>>> E.base_field()
Complex Field with 53 bits of precision
```

base_field_type()

Return the type of topological field on which the fibers are defined.

OUTPUT:

A string describing the field, with three possible values:

- 'real' for the real field **R**
- 'complex' for the complex field **C**
- 'neither_real_nor_complex' for a field different from **R** and **C**

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top')
sage: E = M.vector_bundle(2, 'E', field=CC)
sage: E.base_field_type()
'complex'
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> E = M.vector_bundle(Integer(2), 'E', field=CC)
>>> E.base_field_type()
'complex'
```

base_space()

Return the base space of the vector bundle.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top')
sage: E = M.vector_bundle(2, 'E')
sage: E.base_space()
2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> E.base_space()
2-dimensional topological manifold M
```

change_of_frame(*frame1*, *frame2*)

Return a change of local frames defined on self.

INPUT:

- frame1 – local frame 1
- frame2 – local frame 2

OUTPUT:

- a FreeModuleAutomorphism representing, at each point, the vector space automorphism P that relates frame 1, (e_i) say, to frame 2, (f_i) say, according to $f_i = P(e_i)$

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: X.<x,y,z> = M.chart()
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e')
sage: a = E.section_module().automorphism() # Now, the section module is free
sage: a[:] = [[sqrt(3)/2, -1/2], [1/2, sqrt(3)/2]]
sage: f = e.new_frame(a, 'f')
sage: E.change_of_frame(e, f)
Automorphism of the Free module C^0(M;E) of sections on the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 2
sage: a == E.change_of_frame(e, f)
True
sage: a.inverse() == E.change_of_frame(f, e)
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e')
>>> a = E.section_module().automorphism() # Now, the section module is free
>>> a[:] = [[sqrt(Integer(3))/Integer(2), -Integer(1)/Integer(2)],
↪[Integer(1)/Integer(2), sqrt(Integer(3))/Integer(2)]]
>>> f = e.new_frame(a, 'f')
>>> E.change_of_frame(e, f)
Automorphism of the Free module C^0(M;E) of sections on the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 2
>>> a == E.change_of_frame(e, f)
True
>>> a.inverse() == E.change_of_frame(f, e)
True
```

changes_of_frame()

Return all the changes of local frames defined on `self`.

OUTPUT:

- dictionary of vector bundle automorphisms representing the changes of frames, the keys being the pair of frames

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: c_xyz.<x,y,z> = M.chart()
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e'); e
Local frame (E|_M, (e_0,e_1))
sage: auto_group = E.section_module().general_linear_group()
sage: e_to_f = auto_group([[0,1],[1,0]]); e_to_f
Automorphism of the Free module C^0(M;E) of sections on the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 2
```

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```
sage: f_in_e = auto_group([[0,1],[1,0]])
sage: f = e.new_frame(f_in_e, 'f'); f
Local frame (E|_M, (f_0,f_1))
sage: E.changes_of_frame() # random
{(Local frame (E|_M, (f_0,f_1)),
 Local frame (E|_M, (e_0,e_1))): Automorphism of the Free module
 C^0(M;E) of sections on the 3-dimensional topological manifold M
 with values in the real vector bundle E of rank 2,
 (Local frame (E|_M, (e_0,e_1)),
 Local frame (E|_M, (f_0,f_1))): Automorphism of the Free module
 C^0(M;E) of sections on the 3-dimensional topological manifold M
 with values in the real vector bundle E of rank 2}
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e'); e
Local frame (E|_M, (e_0,e_1))
>>> auto_group = E.section_module().general_linear_group()
>>> e_to_f = auto_group([[Integer(0),Integer(1)],[Integer(1),Integer(0)]]); e_
↪to_f
Automorphism of the Free module C^0(M;E) of sections on the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 2
>>> f_in_e = auto_group([[Integer(0),Integer(1)],[Integer(1),Integer(0)]])
>>> f = e.new_frame(f_in_e, 'f'); f
Local frame (E|_M, (f_0,f_1))
>>> E.changes_of_frame() # random
{(Local frame (E|_M, (f_0,f_1)),
 Local frame (E|_M, (e_0,e_1))): Automorphism of the Free module
 C^0(M;E) of sections on the 3-dimensional topological manifold M
 with values in the real vector bundle E of rank 2,
 (Local frame (E|_M, (e_0,e_1)),
 Local frame (E|_M, (f_0,f_1))): Automorphism of the Free module
 C^0(M;E) of sections on the 3-dimensional topological manifold M
 with values in the real vector bundle E of rank 2}
```

coframes()

Return the list of coframes defined on `self`.

OUTPUT: list of coframes defined on `self`

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: E = M.vector_bundle(2, 'E')
sage: U = M.open_subset('U'); V = M.open_subset('V')
sage: phi_U = E.trivialization('phi_U', domain=U)
sage: e = E.local_frame('e', domain=V)
sage: E.coframes()
[Trivialization coframe (E|_U, ((phi_U^*e^1),(phi_U^*e^2))),
```

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```
 Local coframe (E|_V, (e^0,e^1))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> U = M.open_subset('U'); V = M.open_subset('V')
>>> phi_U = E.trivialization('phi_U', domain=U)
>>> e = E.local_frame('e', domain=V)
>>> E.coframes()
[Trivialization coframe (E|_U, ((phi_U^*e^1),(phi_U^*e^2))),
 Local coframe (E|_V, (e^0,e^1))]
```

default_frame()

Return the default frame of on `self`.

OUTPUT:

- a local frame as an instance of *LocalFrame*

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e')
sage: E.default_frame()
Local frame (E|_M, (e_0,e_1))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e')
>>> E.default_frame()
Local frame (E|_M, (e_0,e_1))
```

fiber(*point*)

Return the vector bundle fiber over a point.

INPUT:

- `point` – *ManifoldPoint*; point p of the base space of `self`

OUTPUT:

- instance of *VectorBundleFiber* representing the fiber over p

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: X.<x,y,z> = M.chart()
sage: p = M((0,2,1), name='p'); p
Point p on the 3-dimensional topological manifold M
sage: E = M.vector_bundle(2, 'E'); E
Topological real vector bundle E -> M of rank 2 over the base space
 3-dimensional topological manifold M
sage: E.fiber(p)
Fiber of E at Point p on the 3-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> p = M((Integer(0),Integer(2),Integer(1)), name='p'); p
Point p on the 3-dimensional topological manifold M
>>> E = M.vector_bundle(Integer(2), 'E'); E
Topological real vector bundle E -> M of rank 2 over the base space
 3-dimensional topological manifold M
>>> E.fiber(p)
Fiber of E at Point p on the 3-dimensional topological manifold M
```

frames()

Return the list of local frames defined on `self`.

OUTPUT: list of local frames defined on `self`

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: E = M.vector_bundle(2, 'E')
sage: U = M.open_subset('U'); V = M.open_subset('V')
sage: phi_U = E.trivialization('phi_U', domain=U)
sage: e = E.local_frame('e', domain=V)
sage: E.frames()
[Trivialization frame (E|_U, ((phi_U^*e_1),(phi_U^*e_2))),
 Local frame (E|_V, (e_0,e_1))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> U = M.open_subset('U'); V = M.open_subset('V')
>>> phi_U = E.trivialization('phi_U', domain=U)
>>> e = E.local_frame('e', domain=V)
>>> E.frames()
[Trivialization frame (E|_U, ((phi_U^*e_1),(phi_U^*e_2))),
 Local frame (E|_V, (e_0,e_1))]
```

has_orientation()

Check whether `self` admits an obvious or by user set orientation.

See also

Consult *orientation()* for details about orientations.

Note

Notice that if *has_orientation()* returns `False` this does not necessarily mean that the vector bundle admits no orientation. It just means that the user has to set an orientation manually in that case, see *set_orientation()*.

EXAMPLES:

The trivial case:

```
sage: M = Manifold(3, 'M', structure='top')
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e')
sage: E.has_orientation()  # trivial case
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e')
>>> E.has_orientation()  # trivial case
True
```

Non-trivial case:

```
sage: M = Manifold(3, 'M', structure='top')
sage: U = M.open_subset('U'); V = M.open_subset('V')
sage: M.declare_union(U, V)
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e', domain=U)
sage: f = E.local_frame('f', domain=V)
sage: E.has_orientation()
False
sage: E.set_orientation([e, f])
sage: E.has_orientation()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> U = M.open_subset('U'); V = M.open_subset('V')
>>> M.declare_union(U, V)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e', domain=U)
>>> f = E.local_frame('f', domain=V)
>>> E.has_orientation()
False
>>> E.set_orientation([e, f])
>>> E.has_orientation()
True
```

irange(*start=None*)

Single index generator.

INPUT:

- `start` – (default: `None`) initial value i_0 of the index; if none are provided, the value returned by `sage.manifolds.manifold.Manifold.start_index()` is assumed

OUTPUT:

- an iterable index, starting from i_0 and ending at $i_0 + n - 1$, where n is the vector bundle's dimension

EXAMPLES:

Index range on a 4-dimensional vector bundle over a 5-dimensional manifold:

```
sage: M = Manifold(5, 'M', structure='topological')
sage: E = M.vector_bundle(4, 'E')
sage: list(E.irange())
[0, 1, 2, 3]
sage: list(E.irange(2))
[2, 3]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(5), 'M', structure='topological')
>>> E = M.vector_bundle(Integer(4), 'E')
>>> list(E.irange())
[0, 1, 2, 3]
>>> list(E.irange(Integer(2)))
[2, 3]
```

Index range on a 4-dimensional vector bundle over a 5-dimensional manifold with starting index=1:

```
sage: M = Manifold(5, 'M', structure='topological', start_index=1)
sage: E = M.vector_bundle(4, 'E')
sage: list(E.irange())
[1, 2, 3, 4]
sage: list(E.irange(2))
[2, 3, 4]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(5), 'M', structure='topological', start_
↪index=Integer(1))
>>> E = M.vector_bundle(Integer(4), 'E')
>>> list(E.irange())
[1, 2, 3, 4]
>>> list(E.irange(Integer(2)))
[2, 3, 4]
```

In general, one has always:

```
sage: next(E.irange()) == M.start_index()
True
```

```
>>> from sage.all import *
>>> next(E.irange()) == M.start_index()
True
```

is_manifestly_trivial()

Return `True` if `self` is manifestly a trivial bundle, i.e. there exists a frame or a trivialization defined on the whole base space.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top')
sage: E = M.vector_bundle(1, 'E')
sage: U = M.open_subset('U')
sage: V = M.open_subset('V')
sage: M.declare_union(U, V)
```

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```
sage: phi_U = E.trivialization('phi_U', domain=U); phi_U
Trivialization (phi_U, E|_U)
sage: phi_V = E.trivialization('phi_V', domain=V); phi_V
Trivialization (phi_V, E|_V)
sage: E.is_manifestly_trivial()
False
sage: E.trivialization('phi_M', M)
Trivialization (phi_M, E|_M)
sage: E.is_manifestly_trivial()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> E = M.vector_bundle(Integer(1), 'E')
>>> U = M.open_subset('U')
>>> V = M.open_subset('V')
>>> M.declare_union(U, V)
>>> phi_U = E.trivialization('phi_U', domain=U); phi_U
Trivialization (phi_U, E|_U)
>>> phi_V = E.trivialization('phi_V', domain=V); phi_V
Trivialization (phi_V, E|_V)
>>> E.is_manifestly_trivial()
False
>>> E.trivialization('phi_M', M)
Trivialization (phi_M, E|_M)
>>> E.is_manifestly_trivial()
True
```

local_frame(**args*, ***kwargs*)

Define a local frame on `self`.

A *local frame* is a section on a subset $U \subset M$ in E that provides, at each point p of the base space, a vector basis of the fiber E_p at p.

See also

LocalFrame for complete documentation.

INPUT:

- `symbol` – either a string, to be used as a common base for the symbols of the sections constituting the local frame, or a list/tuple of strings, representing the individual symbols of the sections
- `sections` – tuple or list of n linearly independent sections on `self` (n being the rank of `self`) defining the local frame; can be omitted if the local frame is created from scratch
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the sections constituting the local frame, or a list/tuple of strings, representing the individual LaTeX symbols of the sections; if `None`, `symbol` is used in place of `latex_symbol`
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the sections of the frame; if `None`, the indices will be generated as integers within the range declared on `self`

- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the sections; if `None`, `indices` is used instead
- `symbol_dual` – (default: `None`) same as `symbol` but for the dual coframe; if `None`, `symbol` must be a string and is used for the common base of the symbols of the elements of the dual coframe
- `latex_symbol_dual` – (default: `None`) same as `latex_symbol` but for the dual coframe
- `domain` – (default: `None`) domain on which the local frame is defined; if `None`, the whole base space is assumed

OUTPUT:

- a `LocalFrame` representing the defined local frame

EXAMPLES:

Defining a local frame from two linearly independent sections on a real rank-2 vector bundle:

```
sage: M = Manifold(3, 'M', structure='top')
sage: U = M.open_subset('U')
sage: X.<x,y,z> = U.chart()
sage: E = M.vector_bundle(2, 'E')
sage: phi = E.trivialization('phi', domain=U)
sage: s0 = E.section(name='s_0', domain=U)
sage: s0[:] = 1+z^2, -2
sage: s1 = E.section(name='s_1', domain=U)
sage: s1[:] = 1, 1+x^2
sage: e = E.local_frame('e', (s0, s1), domain=U); e
Local frame (E|_U, (e_0,e_1))
sage: (e[0], e[1]) == (s0, s1)
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> U = M.open_subset('U')
>>> X = U.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> phi = E.trivialization('phi', domain=U)
>>> s0 = E.section(name='s_0', domain=U)
>>> s0[:] = Integer(1)+z**Integer(2), -Integer(2)
>>> s1 = E.section(name='s_1', domain=U)
>>> s1[:] = Integer(1), Integer(1)+x**Integer(2)
>>> e = E.local_frame('e', (s0, s1), domain=U); e
Local frame (E|_U, (e_0,e_1))
>>> (e[Integer(0)], e[Integer(1)]) == (s0, s1)
True
```

If the sections are not linearly independent, an error is raised:

```
sage: e = E.local_frame('z', (s0, -s0), domain=U)
Traceback (most recent call last):
...
ValueError: the provided sections are not linearly independent
```

```
>>> from sage.all import *
>>> e = E.local_frame('z', (s0, -s0), domain=U)
```

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```
Traceback (most recent call last):
...
ValueError: the provided sections are not linearly independent
```

It is also possible to create a local frame from scratch, without connecting it to previously defined local frames or sections (this can still be performed later via the method *set_change_of_frame()*):

```
sage: f = E.local_frame('f', domain=U); f
Local frame (E|_U, (f_0,f_1))
```

```
>>> from sage.all import *
>>> f = E.local_frame('f', domain=U); f
Local frame (E|_U, (f_0,f_1))
```

For a global frame, the argument `domain` is omitted:

```
sage: g = E.local_frame('g'); g
Local frame (E|_M, (g_0,g_1))
```

```
>>> from sage.all import *
>>> g = E.local_frame('g'); g
Local frame (E|_M, (g_0,g_1))
```

See also

For more options, in particular for the choice of symbols and indices, see *LocalFrame*.

orientation()

Get the orientation of `self` if available.

An *orientation* on a vector bundle is a choice of local frames whose

1. union of domains cover the base space,
2. changes of frames are pairwise orientation preserving, i.e. have positive determinant.

A vector bundle endowed with an orientation is called *orientable*.

The trivial case corresponds to `self` being trivial, i.e. `self` can be covered by one frame. In that case, if no preferred orientation has been set before, one of those frames (usually the default frame) is set automatically to the preferred orientation and returned here.

EXAMPLES:

The trivial case is covered automatically:

```
sage: M = Manifold(3, 'M', structure='top')
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e'); e
Local frame (E|_M, (e_0,e_1))
sage: E.orientation()  # trivial case
[Local frame (E|_M, (e_0,e_1))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e'); e
Local frame (E|_M, (e_0,e_1))
>>> E.orientation()  # trivial case
[Local frame (E|_M, (e_0,e_1))]
```

The orientation can also be set by the user:

```
sage: f = E.local_frame('f'); f
Local frame (E|_M, (f_0,f_1))
sage: E.set_orientation(f)
sage: E.orientation()
[Local frame (E|_M, (f_0,f_1))]
```

```
>>> from sage.all import *
>>> f = E.local_frame('f'); f
Local frame (E|_M, (f_0,f_1))
>>> E.set_orientation(f)
>>> E.orientation()
[Local frame (E|_M, (f_0,f_1))]
```

In case of the non-trivial case, the orientation must be set manually, otherwise no orientation is returned:

```
sage: M = Manifold(3, 'M', structure='top')
sage: U = M.open_subset('U'); V = M.open_subset('V')
sage: M.declare_union(U, V)
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e', domain=U); e
Local frame (E|_U, (e_0,e_1))
sage: f = E.local_frame('f', domain=V); f
Local frame (E|_V, (f_0,f_1))
sage: E.orientation()
[]
sage: E.set_orientation([e, f])
sage: E.orientation()
[Local frame (E|_U, (e_0,e_1)),
 Local frame (E|_V, (f_0,f_1))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> U = M.open_subset('U'); V = M.open_subset('V')
>>> M.declare_union(U, V)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e', domain=U); e
Local frame (E|_U, (e_0,e_1))
>>> f = E.local_frame('f', domain=V); f
Local frame (E|_V, (f_0,f_1))
>>> E.orientation()
[]
>>> E.set_orientation([e, f])
>>> E.orientation()
```

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```
[Local frame (E|_U, (e_0,e_1)),
 Local frame (E|_V, (f_0,f_1))]
```

rank()

Return the rank of the vector bundle.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top')
sage: E = M.vector_bundle(3, 'E')
sage: E.rank()
3
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> E = M.vector_bundle(Integer(3), 'E')
>>> E.rank()
3
```

section(**comp*, ***kwargs*)

Return a continuous section of `self`.

INPUT:

- `domain` – (default: `None`) domain on which the section shall be defined; if `None`, the base space is assumed
- `name` – (default: `None`) name of the local section
- `latex_name` – (default``None``) latex representation of the local section

OUTPUT:

- an instance of `Section` representing a continuous section of M with values on E

EXAMPLES:

A section on a non-trivial rank 2 vector bundle over a non-trivial 2-manifold:

```
sage: M = Manifold(2, 'M', structure='top')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                    intersection_name='W', restrictions1= x>0,
....:                    restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: E = M.vector_bundle(2, 'E') # define the vector bundle
sage: phi_U = E.trivialization('phi_U', domain=U) # define trivializations
sage: phi_V = E.trivialization('phi_V', domain=V)
sage: transf = phi_U.transition_map(phi_V, [[0,x],[x,0]]) # transition map␣
↪between trivializations
sage: fU = phi_U.frame(); fV = phi_V.frame() # define induced frames
sage: s = E.section(name='s'); s
Section s on the 2-dimensional topological manifold M with values in the
 real vector bundle E of rank 2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
...                    intersection_name='W', restrictions1= x>Integer(0),
...                    restrictions2= u+v>Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V)
>>> E = M.vector_bundle(Integer(2), 'E') # define the vector bundle
>>> phi_U = E.trivialization('phi_U', domain=U) # define trivializations
>>> phi_V = E.trivialization('phi_V', domain=V)
>>> transf = phi_U.transition_map(phi_V, [[Integer(0),x],[x,Integer(0)]]) #␣
↪transition map between trivializations
>>> fU = phi_U.frame(); fV = phi_V.frame() # define induced frames
>>> s = E.section(name='s'); s
Section s on the 2-dimensional topological manifold M with values in the
 real vector bundle E of rank 2
```

section_module(*domain=None*, *force_free=False*)

Return the section module of continuous sections on `self`.

See *SectionModule* for a complete documentation.

INPUT:

- `domain` – (default: `None`) the domain on which the module is defined; if `None` the base space is assumed
- `force_free` – boolean (default: `False`); if set to `True`, force the construction of a *free* module (this implies that E is trivial)

OUTPUT:

- a *SectionModule* (or if E is trivial, a *SectionFreeModule*) representing the module of continuous sections on U taking values in E

EXAMPLES:

Module of sections on the Möbius bundle over the real-projective space $M = \mathbf{R}P^1$:

```
sage: M = Manifold(1, 'RP^1', structure='top', start_index=1)
sage: U = M.open_subset('U')  # the complement of one point
sage: c_u.<u> =  U.chart() # [1:u] in homogeneous coord.
sage: V = M.open_subset('V') # the complement of the point u=0
sage: M.declare_union(U,V)   # [v:1] in homogeneous coord.
sage: c_v.<v> = V.chart()
sage: u_to_v = c_u.transition_map(c_v, (1/u),
....:                             intersection_name='W',
....:                             restrictions1 = u!=0,
....:                             restrictions2 = v!=0)
sage: v_to_u = u_to_v.inverse()
sage: W = U.intersection(V)
sage: E = M.vector_bundle(1, 'E')
sage: phi_U = E.trivialization('phi_U', latex_name=r'\varphi_U',
....:                          domain=U)
```

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```
sage: phi_V = E.trivialization('phi_V', latex_name=r'\varphi_V',
....:                           domain=V)
sage: transf = phi_U.transition_map(phi_V, [[u]])
sage: C0 = E.section_module(); C0
Module C^0(RP^1;E) of sections on the 1-dimensional topological
 manifold RP^1 with values in the real vector bundle E of rank 1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(1), 'RP^1', structure='top', start_index=Integer(1))
>>> U = M.open_subset('U')  # the complement of one point
>>> c_u = U.chart(names=('u',)); (u,) = c_u._first_ngens(1)# [1:u] in␣
↪homogeneous coord.
>>> V = M.open_subset('V') # the complement of the point u=0
>>> M.declare_union(U,V)   # [v:1] in homogeneous coord.
>>> c_v = V.chart(names=('v',)); (v,) = c_v._first_ngens(1)
>>> u_to_v = c_u.transition_map(c_v, (Integer(1)/u),
...                              intersection_name='W',
...                              restrictions1 = u!=Integer(0),
...                              restrictions2 = v!=Integer(0))
>>> v_to_u = u_to_v.inverse()
>>> W = U.intersection(V)
>>> E = M.vector_bundle(Integer(1), 'E')
>>> phi_U = E.trivialization('phi_U', latex_name=r'\varphi_U',
...                          domain=U)
>>> phi_V = E.trivialization('phi_V', latex_name=r'\varphi_V',
...                          domain=V)
>>> transf = phi_U.transition_map(phi_V, [[u]])
>>> C0 = E.section_module(); C0
Module C^0(RP^1;E) of sections on the 1-dimensional topological
 manifold RP^1 with values in the real vector bundle E of rank 1
```

$C^0(\mathbf{R}P^1; E)$ is a module over the algebra $C^0(\mathbf{R}P^1)$:

```
sage: C0.category()
Category of modules over Algebra of scalar fields on the
 1-dimensional topological manifold RP^1
sage: C0.base_ring() is M.scalar_field_algebra()
True
```

```
>>> from sage.all import *
>>> C0.category()
Category of modules over Algebra of scalar fields on the
 1-dimensional topological manifold RP^1
>>> C0.base_ring() is M.scalar_field_algebra()
True
```

However, $C^0(\mathbf{R}P^1; E)$ is not a free module:

```
sage: isinstance(C0, FiniteRankFreeModule)
False
```

```
>>> from sage.all import *
>>> isinstance(C0, FiniteRankFreeModule)
False
```

since the Möbius bundle is not trivial:

```
sage: E.is_manifestly_trivial()
False
```

```
>>> from sage.all import *
>>> E.is_manifestly_trivial()
False
```

The section module over U, on the other hand, is a free module since $E|_U$ admits a trivialization and therefore has a local frame:

```
sage: C0_U = E.section_module(domain=U)
sage: isinstance(C0_U, FiniteRankFreeModule)
True
```

```
>>> from sage.all import *
>>> C0_U = E.section_module(domain=U)
>>> isinstance(C0_U, FiniteRankFreeModule)
True
```

The elements of $C^0(U)$ are sections on U:

```
sage: C0_U.an_element()
Section on the Open subset U of the 1-dimensional topological
 manifold RP^1 with values in the real vector bundle E of rank 1
sage: C0_U.an_element().display(phi_U.frame())
2 (phi_U^*e_1)
```

```
>>> from sage.all import *
>>> C0_U.an_element()
Section on the Open subset U of the 1-dimensional topological
 manifold RP^1 with values in the real vector bundle E of rank 1
>>> C0_U.an_element().display(phi_U.frame())
2 (phi_U^*e_1)
```

set_change_of_frame(*frame1*, *frame2*, *change_of_frame*, *compute_inverse=True*)

Relate two vector frames by an automorphism.

This updates the internal dictionary `self._frame_changes`.

INPUT:

- `frame1` – frame 1, denoted (e_i) below
- `frame2` – frame 2, denoted (f_i) below
- `change_of_frame` – instance of class `FreeModuleAutomorphism` describing the automorphism P that relates the basis (e_i) to the basis (f_i) according to $f_i = P(e_i)$
- `compute_inverse` – boolean (default: `True`); if set to True, the inverse automorphism is computed and the change from basis (f_i) to (e_i) is set to it in the internal dictionary `self._frame_changes`

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: c_xyz.<x,y,z> = M.chart()
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e')
sage: f = E.local_frame('f')
sage: a = E.section_module().automorphism()
sage: a[e,:] = [[1,2],[0,3]]
sage: E.set_change_of_frame(e, f, a)
sage: f[0].display(e)
f_0 = e_0
sage: f[1].display(e)
f_1 = 2 e_0 + 3 e_1
sage: e[0].display(f)
e_0 = f_0
sage: e[1].display(f)
e_1 = -2/3 f_0 + 1/3 f_1
sage: E.change_of_frame(e,f)[e,:]
[1 2]
[0 3]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e')
>>> f = E.local_frame('f')
>>> a = E.section_module().automorphism()
>>> a[e,:] = [[Integer(1),Integer(2)],[Integer(0),Integer(3)]]
>>> E.set_change_of_frame(e, f, a)
>>> f[Integer(0)].display(e)
f_0 = e_0
>>> f[Integer(1)].display(e)
f_1 = 2 e_0 + 3 e_1
>>> e[Integer(0)].display(f)
e_0 = f_0
>>> e[Integer(1)].display(f)
e_1 = -2/3 f_0 + 1/3 f_1
>>> E.change_of_frame(e,f)[e,:]
[1 2]
[0 3]
```

set_default_frame(*frame*)

Set the default frame of `self`.

INPUT:

- `frame` – a local frame defined on `self` as an instance of *LocalFrame*

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: E = M.vector_bundle(2, 'E')
```

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```
sage: e = E.local_frame('e')
sage: E.default_frame()
Local frame (E|_M, (e_0,e_1))
sage: f = E.local_frame('f')
sage: E.set_default_frame(f)
sage: E.default_frame()
Local frame (E|_M, (f_0,f_1))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e')
>>> E.default_frame()
Local frame (E|_M, (e_0,e_1))
>>> f = E.local_frame('f')
>>> E.set_default_frame(f)
>>> E.default_frame()
Local frame (E|_M, (f_0,f_1))
```

set_orientation(*orientation*)

Set the preferred orientation of `self`.

INPUT:

- `orientation` – a local frame or a list of local frames whose domains cover the base space

> **Warning**
>
> It is the user's responsibility that the orientation set here is indeed an orientation. There is no check going on in the background. See *orientation()* for the definition of an orientation.

EXAMPLES:

Set an orientation on a vector bundle:

```
sage: M = Manifold(3, 'M', structure='top')
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e'); e
Local frame (E|_M, (e_0,e_1))
sage: f = E.local_frame('f'); f
Local frame (E|_M, (f_0,f_1))
sage: E.set_orientation(f)
sage: E.orientation()
[Local frame (E|_M, (f_0,f_1))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e'); e
Local frame (E|_M, (e_0,e_1))
>>> f = E.local_frame('f'); f
Local frame (E|_M, (f_0,f_1))
```

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```
>>> E.set_orientation(f)
>>> E.orientation()
[Local frame (E|_M, (f_0,f_1))]
```

Set an orientation in the non-trivial case:

```
sage: M = Manifold(3, 'M', structure='top')
sage: U = M.open_subset('U'); V = M.open_subset('V')
sage: M.declare_union(U, V)
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e', domain=U); e
Local frame (E|_U, (e_0,e_1))
sage: f = E.local_frame('f', domain=V); f
Local frame (E|_V, (f_0,f_1))
sage: E.orientation()
[]
sage: E.set_orientation([e, f])
sage: E.orientation()
[Local frame (E|_U, (e_0,e_1)),
 Local frame (E|_V, (f_0,f_1))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> U = M.open_subset('U'); V = M.open_subset('V')
>>> M.declare_union(U, V)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e', domain=U); e
Local frame (E|_U, (e_0,e_1))
>>> f = E.local_frame('f', domain=V); f
Local frame (E|_V, (f_0,f_1))
>>> E.orientation()
[]
>>> E.set_orientation([e, f])
>>> E.orientation()
[Local frame (E|_U, (e_0,e_1)),
 Local frame (E|_V, (f_0,f_1))]
```

total_space()

Return the total space of `self`.

> **Note**
>
> At this stage, the total space does not come with induced charts.

OUTPUT:

- the total space of `self` as an instance of *TopologicalManifold*

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: E = M.vector_bundle(2, 'E')
```

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```
sage: E.total_space()
6-dimensional topological manifold E
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> E.total_space()
6-dimensional topological manifold E
```

transition(*triv1*, *triv2*)

Return the transition map between two trivializations defined over the manifold.

The transition map must have been defined previously, for instance by the method `transition_map()`.

INPUT:

- `triv1` – trivialization 1
- `triv2` – trivialization 2

OUTPUT:

- instance of `TransitionMap` representing the transition map from trivialization 1 to trivialization 2

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: U = M.open_subset('U')
sage: V = M.open_subset('V')
sage: X_UV = X.restrict(U.intersection(V))
sage: E = M.vector_bundle(2, 'E')
sage: phi_U = E.trivialization('phi_U', domain=U)
sage: phi_V = E.trivialization('phi_V', domain=V)
sage: phi_U_to_phi_V = phi_U.transition_map(phi_V, 1)
sage: E.transition(phi_V, phi_U)
Transition map from Trivialization (phi_V, E|_V) to Trivialization
 (phi_U, E|_U)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> U = M.open_subset('U')
>>> V = M.open_subset('V')
>>> X_UV = X.restrict(U.intersection(V))
>>> E = M.vector_bundle(Integer(2), 'E')
>>> phi_U = E.trivialization('phi_U', domain=U)
>>> phi_V = E.trivialization('phi_V', domain=V)
>>> phi_U_to_phi_V = phi_U.transition_map(phi_V, Integer(1))
>>> E.transition(phi_V, phi_U)
Transition map from Trivialization (phi_V, E|_V) to Trivialization
 (phi_U, E|_U)
```

transitions()

Return the transition maps defined over subsets of the base space.

OUTPUT: dictionary of transition maps, with pairs of trivializations as keys

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: U = M.open_subset('U')
sage: V = M.open_subset('V')
sage: X_UV = X.restrict(U.intersection(V))
sage: E = M.vector_bundle(2, 'E')
sage: phi_U = E.trivialization('phi_U', domain=U)
sage: phi_V = E.trivialization('phi_U', domain=V)
sage: phi_U_to_phi_V = phi_U.transition_map(phi_V, 1)
sage: E.transitions() # random
{(Trivialization (phi_U, E|_U),
 Trivialization (phi_U, E|_V)): Transition map from Trivialization
 (phi_U, E|_U) to Trivialization (phi_U, E|_V),
 (Trivialization (phi_U, E|_V),
 Trivialization (phi_U, E|_U)): Transition map from Trivialization
 (phi_U, E|_V) to Trivialization (phi_U, E|_U)}
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> U = M.open_subset('U')
>>> V = M.open_subset('V')
>>> X_UV = X.restrict(U.intersection(V))
>>> E = M.vector_bundle(Integer(2), 'E')
>>> phi_U = E.trivialization('phi_U', domain=U)
>>> phi_V = E.trivialization('phi_U', domain=V)
>>> phi_U_to_phi_V = phi_U.transition_map(phi_V, Integer(1))
>>> E.transitions() # random
{(Trivialization (phi_U, E|_U),
 Trivialization (phi_U, E|_V)): Transition map from Trivialization
 (phi_U, E|_U) to Trivialization (phi_U, E|_V),
 (Trivialization (phi_U, E|_V),
 Trivialization (phi_U, E|_U)): Transition map from Trivialization
 (phi_U, E|_V) to Trivialization (phi_U, E|_U)}
```

trivialization(*name*, *domain=None*, *latex_name=None*)

Return a trivialization of `self` over the domain `domain`.

INPUT:

- `domain` – (default: `None`) domain on which the trivialization is defined; if `None` the base space is assumed
- `name` – (default: `None`) name given to the trivialization
- `latex_name` – (default: `None`) LaTeX name given to the trivialization

OUTPUT:

- a *`Trivialization`* representing a trivialization of E

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: U = M.open_subset('U')
sage: E = M.vector_bundle(2, 'E')
sage: phi = E.trivialization('phi', domain=U); phi
Trivialization (phi, E|_U)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> U = M.open_subset('U')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> phi = E.trivialization('phi', domain=U); phi
Trivialization (phi, E|_U)
```

1.9.2 Vector Bundle Fibers

The class `VectorBundleFiber` implements fibers over a vector bundle.

AUTHORS:

- Michael Jung (2019): initial version

class sage.manifolds.vector_bundle_fiber.**VectorBundleFiber**(*vector_bundle*, *point*)

Bases: `FiniteRankFreeModule`

Fiber of a given vector bundle at a given point.

Let $\pi : E \to M$ be a vector bundle of rank n over the field K (see `TopologicalVectorBundle`) and $p \in M$. The fiber E_p at p is defined via $E_p := \pi^{-1}(p)$ and takes the structure of an n-dimensional vector space over the field K.

INPUT:

- `vector_bundle` – `TopologicalVectorBundle`; vector bundle E on which the fiber is defined
- `point` – `ManifoldPoint`; point p at which the fiber is defined

EXAMPLES:

A vector bundle fiber in a trivial rank 2 vector bundle over a 4-dimensional topological manifold:

```
sage: M = Manifold(4, 'M', structure='top')
sage: X.<x,y,z,t> = M.chart()
sage: p = M((0,0,0,0), name='p')
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e')
sage: Ep = E.fiber(p); Ep
Fiber of E at Point p on the 4-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='top')
>>> X = M.chart(names=('x', 'y', 'z', 't',)); (x, y, z, t,) = X._first_ngens(4)
>>> p = M((Integer(0),Integer(0),Integer(0),Integer(0)), name='p')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e')
>>> Ep = E.fiber(p); Ep
Fiber of E at Point p on the 4-dimensional topological manifold M
```

Fibers are free modules of finite rank over `SymbolicRing` (actually vector spaces of finite dimension over the vector bundle field K, here $K = \mathbf{R}$):

```
sage: Ep.base_ring()
Symbolic Ring
sage: Ep.category()
Category of finite dimensional vector spaces over Symbolic Ring
sage: Ep.rank()
2
sage: dim(Ep)
2
```

```
>>> from sage.all import *
>>> Ep.base_ring()
Symbolic Ring
>>> Ep.category()
Category of finite dimensional vector spaces over Symbolic Ring
>>> Ep.rank()
2
>>> dim(Ep)
2
```

The fiber is automatically endowed with bases deduced from the local frames around the point:

```
sage: Ep.bases()
[Basis (e_0,e_1) on the Fiber of E at Point p on the 4-dimensional
 topological manifold M]
sage: E.frames()
[Local frame (E|_M, (e_0,e_1))]
```

```
>>> from sage.all import *
>>> Ep.bases()
[Basis (e_0,e_1) on the Fiber of E at Point p on the 4-dimensional
 topological manifold M]
>>> E.frames()
[Local frame (E|_M, (e_0,e_1))]
```

At this stage, only one basis has been defined in the fiber, but new bases can be added from local frames on the vector bundle by means of the method `at()`:

```
sage: aut = E.section_module().automorphism()
sage: aut[:] = [[-1, x], [y, 2]]
sage: f = e.new_frame(aut, 'f')
sage: fp = f.at(p); fp
Basis (f_0,f_1) on the Fiber of E at Point p on the 4-dimensional
 topological manifold M
sage: Ep.bases()
[Basis (e_0,e_1) on the Fiber of E at Point p on the 4-dimensional
 topological manifold M,
 Basis (f_0,f_1) on the Fiber of E at Point p on the 4-dimensional
 topological manifold M]
```

```
>>> from sage.all import *
>>> aut = E.section_module().automorphism()
>>> aut[:] = [[-Integer(1), x], [y, Integer(2)]]
>>> f = e.new_frame(aut, 'f')
>>> fp = f.at(p); fp
Basis (f_0,f_1) on the Fiber of E at Point p on the 4-dimensional
 topological manifold M
>>> Ep.bases()
[Basis (e_0,e_1) on the Fiber of E at Point p on the 4-dimensional
 topological manifold M,
 Basis (f_0,f_1) on the Fiber of E at Point p on the 4-dimensional
 topological manifold M]
```

The changes of bases are applied to the fibers:

```
sage: f[1].display(e) # second component of frame f
f_1 = x e_0 + 2 e_1
sage: ep = e.at(p)
sage: fp[1].display(ep) # second component of frame f at p
f_1 = 2 e_1
```

```
>>> from sage.all import *
>>> f[Integer(1)].display(e) # second component of frame f
f_1 = x e_0 + 2 e_1
>>> ep = e.at(p)
>>> fp[Integer(1)].display(ep) # second component of frame f at p
f_1 = 2 e_1
```

All the bases defined on `Ep` are on the same footing. Accordingly the fiber is not in the category of modules with a distinguished basis:

```
sage: Ep in ModulesWithBasis(SR)
False
```

```
>>> from sage.all import *
>>> Ep in ModulesWithBasis(SR)
False
```

It is simply in the category of modules:

```
sage: Ep in Modules(SR)
True
```

```
>>> from sage.all import *
>>> Ep in Modules(SR)
True
```

Since the base ring is a field, it is actually in the category of vector spaces:

```
sage: Ep in VectorSpaces(SR)
True
```

```
>>> from sage.all import *
>>> Ep in VectorSpaces(SR)
True
```

A typical element:

```
sage: v = Ep.an_element(); v
Vector in the fiber of E at Point p on the 4-dimensional topological
 manifold M
sage: v.display()
e_0 + 2 e_1
sage: v.parent()
Fiber of E at Point p on the 4-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> v = Ep.an_element(); v
Vector in the fiber of E at Point p on the 4-dimensional topological
 manifold M
>>> v.display()
e_0 + 2 e_1
>>> v.parent()
Fiber of E at Point p on the 4-dimensional topological manifold M
```

The zero vector:

```
sage: Ep.zero()
Vector zero in the fiber of E at Point p on the 4-dimensional
 topological manifold M
sage: Ep.zero().display()
zero = 0
sage: Ep.zero().parent()
Fiber of E at Point p on the 4-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> Ep.zero()
Vector zero in the fiber of E at Point p on the 4-dimensional
 topological manifold M
>>> Ep.zero().display()
zero = 0
>>> Ep.zero().parent()
Fiber of E at Point p on the 4-dimensional topological manifold M
```

Fibers are unique:

```
sage: E.fiber(p) is Ep
True
sage: p1 = M.point((0,0,0,0))
sage: E.fiber(p1) is Ep
True
```

```
>>> from sage.all import *
>>> E.fiber(p) is Ep
```

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```
True
>>> p1 = M.point((Integer(0),Integer(0),Integer(0),Integer(0)))
>>> E.fiber(p1) is Ep
True
```

even if points are different instances:

```
sage: p1 is p
False
```

```
>>> from sage.all import *
>>> p1 is p
False
```

but `p1` and `p` share the same fiber because they compare equal:

```
sage: p1 == p
True
```

```
>>> from sage.all import *
>>> p1 == p
True
```

See also

`FiniteRankFreeModule` for more documentation.

Element

alias of *VectorBundleFiberElement*

base_point()

Return the manifold point over which `self` is defined.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top')
sage: X.<x,y> = M.chart()
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e')
sage: p = M.point((3,-2), name='p')
sage: Ep = E.fiber(p)
sage: Ep.base_point()
Point p on the 2-dimensional topological manifold M
sage: p is Ep.base_point()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e')
```

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```
>>> p = M.point((Integer(3),-Integer(2)), name='p')
>>> Ep = E.fiber(p)
>>> Ep.base_point()
Point p on the 2-dimensional topological manifold M
>>> p is Ep.base_point()
True
```

construction()

dim()

Return the vector space dimension of `self`.

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: X.<x,y,z> = M.chart()
sage: p = M((0,0,0), name='p')
sage: E = M.vector_bundle(2, 'E')
sage: Ep = E.fiber(p)
sage: Ep.dim()
2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> p = M((Integer(0),Integer(0),Integer(0)), name='p')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> Ep = E.fiber(p)
>>> Ep.dim()
2
```

dimension()

Return the vector space dimension of `self`.

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: X.<x,y,z> = M.chart()
sage: p = M((0,0,0), name='p')
sage: E = M.vector_bundle(2, 'E')
sage: Ep = E.fiber(p)
sage: Ep.dim()
2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> p = M((Integer(0),Integer(0),Integer(0)), name='p')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> Ep = E.fiber(p)
>>> Ep.dim()
2
```

1.9.3 Vector Bundle Fiber Elements

The class `VectorBundleFiberElement` implements vectors in the fiber of a vector bundle.

AUTHORS:

- Michael Jung (2019): initial version

class sage.manifolds.vector_bundle_fiber_element.**VectorBundleFiberElement**(*parent*, *name=None*, *latex_name=None*)

Bases: `FiniteRankFreeModuleElement`

Vector in a fiber of a vector bundle at the given point.

INPUT:

- `parent` – `VectorBundleFiber`; the fiber to which the vector belongs
- `name` – (default: `None`) string; symbol given to the vector
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the vector; if `None`, `name` will be used

EXAMPLES:

A vector v in a fiber of a rank 2 vector bundle:

```
sage: M = Manifold(2, 'M', structure='top')
sage: X.<x,y> = M.chart()
sage: p = M((1,-1), name='p')
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e')
sage: Ep = E.fiber(p)
sage: v = Ep((-2,1), name='v'); v
Vector v in the fiber of E at Point p on the 2-dimensional topological
 manifold M
sage: v.display()
v = -2 e_0 + e_1
sage: v.parent()
Fiber of E at Point p on the 2-dimensional topological manifold M
sage: v in Ep
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> p = M((Integer(1),-Integer(1)), name='p')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e')
>>> Ep = E.fiber(p)
>>> v = Ep((-Integer(2),Integer(1)), name='v'); v
Vector v in the fiber of E at Point p on the 2-dimensional topological
 manifold M
>>> v.display()
v = -2 e_0 + e_1
>>> v.parent()
Fiber of E at Point p on the 2-dimensional topological manifold M
>>> v in Ep
True
```

See also

FiniteRankFreeModuleElement for more documentation.

1.9.4 Trivializations

The class *Trivialization* implements trivializations on vector bundles. The corresponding transition maps between two trivializations are represented by *TransitionMap*.

AUTHORS:

- Michael Jung (2019) : initial version

class sage.manifolds.trivialization.**TransitionMap**(*triv1*, *triv2*, *transf*, *compute_inverse=True*)

Bases: SageObject

Transition map between two trivializations.

Given a vector bundle $\pi : E \to M$ of class C^k and rank n over the field K, and two trivializations $\varphi_U : \pi^{-1}(U) \to U \times K^n$ and $\varphi_V : \pi^{-1}(V) \to V \times K^n$, the transition map from φ_U to φ_V is given by the composition

$$\varphi_V \circ \varphi_U^{-1} : U \cap V \times K^n \to U \cap V \times K^n.$$

This composition is of the form

$$(p, v) \mapsto (p, g(p)v),$$

where $p \mapsto g(p)$ is a C^k family of invertible $n \times n$ matrices.

INPUT:

- triv1 – trivialization 1
- triv2 – trivialization 2
- transf – the transformation between both trivializations in form of a matrix of scalar fields (*ScalarField*) or coordinate functions (*ChartFunction*), or a bundle automorphism (FreeModuleAutomorphism)
- compute_inverse – boolean (default: True); determines whether the inverse shall be computed or not

EXAMPLES:

Transition map of two trivializations on a real rank 2 vector bundle of the 2-sphere:

```
sage: S2 = Manifold(2, 'S^2', structure='top')
sage: U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the North␣
↪and South pole, respectively
sage: S2.declare_union(U,V)
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: W = U.intersection(V)
sage: uv_to_xy = xy_to_uv.inverse()
sage: E = S2.vector_bundle(2, 'E')
sage: phi_U = E.trivialization('phi_U', domain=U)
sage: phi_V = E.trivialization('phi_V', domain=V)
sage: phi_U_to_phi_V = phi_U.transition_map(phi_V, [[0,1],[1,0]])
```

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```
sage: phi_U_to_phi_V
Transition map from Trivialization (phi_U, E|_U) to Trivialization
 (phi_V, E|_V)
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2', structure='top')
>>> U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the North␣
↪and South pole, respectively
>>> S2.declare_union(U,V)
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                    intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                    restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> W = U.intersection(V)
>>> uv_to_xy = xy_to_uv.inverse()
>>> E = S2.vector_bundle(Integer(2), 'E')
>>> phi_U = E.trivialization('phi_U', domain=U)
>>> phi_V = E.trivialization('phi_V', domain=V)
>>> phi_U_to_phi_V = phi_U.transition_map(phi_V, [[Integer(0),Integer(1)],
↪[Integer(1),Integer(0)]])
>>> phi_U_to_phi_V
Transition map from Trivialization (phi_U, E|_U) to Trivialization
 (phi_V, E|_V)
```

automorphism()

Return the automorphism connecting both trivializations.

The family of matrices $p \mapsto g(p)$ given by the transition map induce a bundle automorphism

$$\varphi_U^{-1} \circ \varphi_V : \pi^{-1}(U \cap V) \to \pi^{-1}(U \cap V)$$

correlating the local frames induced by the trivializations in the following way:

$$(\varphi_U^{-1} \circ \varphi_V)(\varphi_V^* e_i) = \varphi_U^* e_i.$$

Then, for each point $p \in M$, the matrix $g(p)$ is the representation of the induced automorphism on the fiber $E_p = \pi^{-1}(p)$ in the basis $((\varphi_V^* e_i)(p))_{i=1,\ldots,n}$.

EXAMPLES:

```
sage: S2 = Manifold(2, 'S^2', structure='top')
sage: U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the␣
↪North and South pole, respectively
sage: S2.declare_union(U,V)
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
```

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```
....:                    restrictions2= u^2+v^2!=0)
sage: W = U.intersection(V)
sage: uv_to_xy = xy_to_uv.inverse()
sage: E = S2.vector_bundle(2, 'E')
sage: phi_U = E.trivialization('phi_U', latex_name=r'\varphi_U',
....:                              domain=U); phi_U
Trivialization (phi_U, E|_U)
sage: phi_V = E.trivialization('phi_V', latex_name=r'\varphi_V',
....:                              domain=V); phi_V
Trivialization (phi_V, E|_V)
sage: phi_U_to_phi_V = phi_U.transition_map(phi_V, [[0,1],[1,0]])
sage: aut = phi_U_to_phi_V.automorphism(); aut
Automorphism phi_U^(-1)*phi_V of the Free module C^0(W;E) of
 sections on the Open subset W of the 2-dimensional topological
 manifold S^2 with values in the real vector bundle E of rank 2
sage: aut.display(phi_U.frame().restrict(W))
phi_U^(-1)*phi_V = (phi_U^*e_1)⊗(phi_U^*e^2) +
 (phi_U^*e_2)⊗(phi_U^*e^1)
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2', structure='top')
>>> U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the␣
↪North and South pole, respectively
>>> S2.declare_union(U,V)
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                    intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                    restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> W = U.intersection(V)
>>> uv_to_xy = xy_to_uv.inverse()
>>> E = S2.vector_bundle(Integer(2), 'E')
>>> phi_U = E.trivialization('phi_U', latex_name=r'\varphi_U',
...                              domain=U); phi_U
Trivialization (phi_U, E|_U)
>>> phi_V = E.trivialization('phi_V', latex_name=r'\varphi_V',
...                              domain=V); phi_V
Trivialization (phi_V, E|_V)
>>> phi_U_to_phi_V = phi_U.transition_map(phi_V, [[Integer(0),Integer(1)],
↪[Integer(1),Integer(0)]])
>>> aut = phi_U_to_phi_V.automorphism(); aut
Automorphism phi_U^(-1)*phi_V of the Free module C^0(W;E) of
 sections on the Open subset W of the 2-dimensional topological
 manifold S^2 with values in the real vector bundle E of rank 2
>>> aut.display(phi_U.frame().restrict(W))
phi_U^(-1)*phi_V = (phi_U^*e_1)⊗(phi_U^*e^2) +
 (phi_U^*e_2)⊗(phi_U^*e^1)
```

det()

Return the determinant of `self`.

OUTPUT: an instance of *ScalarField*

EXAMPLES:

```
sage: S2 = Manifold(2, 'S^2', structure='top')
sage: U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the␣
↪North and South pole, respectively
sage: S2.declare_union(U,V)
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                    intersection_name='W', restrictions1= x^2+y^2!=0,
....:                    restrictions2= u^2+v^2!=0)
sage: W = U.intersection(V)
sage: uv_to_xy = xy_to_uv.inverse()
sage: E = S2.vector_bundle(2, 'E')
sage: phi_U = E.trivialization('phi_U', latex_name=r'\varphi_U',
....:                          domain=U); phi_U
Trivialization (phi_U, E|_U)
sage: phi_V = E.trivialization('phi_V', latex_name=r'\varphi_V',
....:                          domain=V); phi_V
Trivialization (phi_V, E|_V)
sage: phi_U_to_phi_V = phi_U.transition_map(phi_V, [[0,1],[1,0]])
sage: det = phi_U_to_phi_V.det(); det
Scalar field det(phi_U^(-1)*phi_V) on the Open subset W of the
 2-dimensional topological manifold S^2
sage: det.display()
det(phi_U^(-1)*phi_V): W → ℝ
     (x, y) ↦ -1
     (u, v) ↦ -1
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2', structure='top')
>>> U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the␣
↪North and South pole, respectively
>>> S2.declare_union(U,V)
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                    intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                    restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> W = U.intersection(V)
>>> uv_to_xy = xy_to_uv.inverse()
>>> E = S2.vector_bundle(Integer(2), 'E')
>>> phi_U = E.trivialization('phi_U', latex_name=r'\varphi_U',
...                          domain=U); phi_U
Trivialization (phi_U, E|_U)
```

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```
>>> phi_V = E.trivialization('phi_V', latex_name=r'\varphi_V',
...                          domain=V); phi_V
Trivialization (phi_V, E|_V)
>>> phi_U_to_phi_V = phi_U.transition_map(phi_V, [[Integer(0),Integer(1)],
↪[Integer(1),Integer(0)]])
>>> det = phi_U_to_phi_V.det(); det
Scalar field det(phi_U^(-1)*phi_V) on the Open subset W of the
 2-dimensional topological manifold S^2
>>> det.display()
det(phi_U^(-1)*phi_V): W → ℝ
     (x, y) ↦ -1
     (u, v) ↦ -1
```

inverse()

Return the inverse transition map.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top')
sage: X.<x,y> = M.chart()
sage: U = M.open_subset('U'); V = M.open_subset('V')
sage: XU = X.restrict(U); XV = X.restrict(U)
sage: W = U.intersection(V)
sage: XW = X.restrict(W)
sage: E = M.vector_bundle(2, 'E')
sage: phi_U = E.trivialization('phi_U', domain=U)
sage: phi_V = E.trivialization('phi_V', domain=V)
sage: phi_U_to_phi_V = phi_U.transition_map(phi_V, [[1,1],[-1,1]],
....:                                         compute_inverse=False)
sage: phi_V_to_phi_U = phi_U_to_phi_V.inverse(); phi_V_to_phi_U
Transition map from Trivialization (phi_V, E|_V) to Trivialization (phi_U, E|_
↪U)
sage: phi_V_to_phi_U.automorphism() == phi_U_to_phi_V.automorphism().inverse()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> U = M.open_subset('U'); V = M.open_subset('V')
>>> XU = X.restrict(U); XV = X.restrict(U)
>>> W = U.intersection(V)
>>> XW = X.restrict(W)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> phi_U = E.trivialization('phi_U', domain=U)
>>> phi_V = E.trivialization('phi_V', domain=V)
>>> phi_U_to_phi_V = phi_U.transition_map(phi_V, [[Integer(1),Integer(1)],[-
↪Integer(1),Integer(1)]],
...                                         compute_inverse=False)
>>> phi_V_to_phi_U = phi_U_to_phi_V.inverse(); phi_V_to_phi_U
Transition map from Trivialization (phi_V, E|_V) to Trivialization (phi_U, E|_
↪U)
>>> phi_V_to_phi_U.automorphism() == phi_U_to_phi_V.automorphism().inverse()
```

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```
True
```

matrix()

Return the matrix representation the transition map.

EXAMPLES:

Local trivializations on a real rank 2 vector bundle over the 2-sphere:

```
sage: S2 = Manifold(2, 'S^2', structure='top')
sage: U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the␣
↪North and South pole, respectively
sage: S2.declare_union(U,V)
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                   intersection_name='W', restrictions1= x^2+y^2!=0,
....:                   restrictions2= u^2+v^2!=0)
sage: W = U.intersection(V)
sage: uv_to_xy = xy_to_uv.inverse()
sage: E = S2.vector_bundle(2, 'E')
sage: phi_U = E.trivialization('phi_U', latex_name=r'\varphi_U',
....:                          domain=U); phi_U
Trivialization (phi_U, E|_U)
sage: phi_V = E.trivialization('phi_V', latex_name=r'\varphi_V',
....:                          domain=V); phi_V
Trivialization (phi_V, E|_V)
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2', structure='top')
>>> U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the␣
↪North and South pole, respectively
>>> S2.declare_union(U,V)
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                   intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                   restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> W = U.intersection(V)
>>> uv_to_xy = xy_to_uv.inverse()
>>> E = S2.vector_bundle(Integer(2), 'E')
>>> phi_U = E.trivialization('phi_U', latex_name=r'\varphi_U',
...                          domain=U); phi_U
Trivialization (phi_U, E|_U)
>>> phi_V = E.trivialization('phi_V', latex_name=r'\varphi_V',
...                          domain=V); phi_V
Trivialization (phi_V, E|_V)
```

The input is coerced into a bundle automorphism. From there, the matrix can be recovered:

```
sage: phi_U_to_phi_V = phi_U.transition_map(phi_V, [[0,1],[1,0]])
sage: matrix = phi_U_to_phi_V.matrix(); matrix
[Scalar field zero on the Open subset W of the 2-dimensional
 topological manifold S^2      Scalar field 1 on the Open subset
 W of the 2-dimensional topological manifold S^2]
 [     Scalar field 1 on the Open subset W of the 2-dimensional
 topological manifold S^2 Scalar field zero on the Open subset W of
 the 2-dimensional topological manifold S^2]
```

```
>>> from sage.all import *
>>> phi_U_to_phi_V = phi_U.transition_map(phi_V, [[Integer(0),Integer(1)],
↪[Integer(1),Integer(0)]])
>>> matrix = phi_U_to_phi_V.matrix(); matrix
[Scalar field zero on the Open subset W of the 2-dimensional
 topological manifold S^2      Scalar field 1 on the Open subset
 W of the 2-dimensional topological manifold S^2]
 [     Scalar field 1 on the Open subset W of the 2-dimensional
 topological manifold S^2 Scalar field zero on the Open subset W of
 the 2-dimensional topological manifold S^2]
```

Let us check the matrix components:

```
sage: matrix[0,0].display()
zero: W → ℝ
    (x, y) ↦ 0
    (u, v) ↦ 0
sage: matrix[0,1].display()
1: W → ℝ
 (x, y) ↦ 1
 (u, v) ↦ 1
sage: matrix[1,0].display()
1: W → ℝ
 (x, y) ↦ 1
 (u, v) ↦ 1
sage: matrix[1,1].display()
zero: W → ℝ
    (x, y) ↦ 0
    (u, v) ↦ 0
```

```
>>> from sage.all import *
>>> matrix[Integer(0),Integer(0)].display()
zero: W → ℝ
    (x, y) ↦ 0
    (u, v) ↦ 0
>>> matrix[Integer(0),Integer(1)].display()
1: W → ℝ
 (x, y) ↦ 1
 (u, v) ↦ 1
>>> matrix[Integer(1),Integer(0)].display()
1: W → ℝ
 (x, y) ↦ 1
 (u, v) ↦ 1
```

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```
>>> matrix[Integer(1),Integer(1)].display()
zero: W → ℝ
    (x, y) ↦ 0
    (u, v) ↦ 0
```

class sage.manifolds.trivialization.**Trivialization**(*vector_bundle*, *name*, *domain*, *latex_name=None*)

Bases: UniqueRepresentation, SageObject

A local trivialization of a given vector bundle.

Let $\pi : E \to M$ be a vector bundle of rank n and class C^k over the field K (see *TopologicalVectorBundle* or *DifferentiableVectorBundle*). A *local trivialization* over an open subset $U \subset M$ is a C^k-diffeomorphism $\varphi : \pi^{-1}(U) \to U \times K^n$ such that $\pi \circ \varphi^{-1} = \mathrm{pr}_1$ and $v \mapsto \varphi^{-1}(q, v)$ is a linear isomorphism for any $q \in U$.

Note

Notice that frames and trivializations are equivalent concepts (for further details see *LocalFrame*). However, in order to facilitate applications and being consistent with the implementations of charts, trivializations are introduced separately.

EXAMPLES:

Local trivializations on a real rank 2 vector bundle over the 2-sphere:

```
sage: S2 = Manifold(2, 'S^2', structure='top')
sage: U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the North␣
↪and South pole, respectively
sage: S2.declare_union(U,V)
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                    intersection_name='W', restrictions1= x^2+y^2!=0,
....:                    restrictions2= u^2+v^2!=0)
sage: W = U.intersection(V)
sage: uv_to_xy = xy_to_uv.inverse()
sage: E = S2.vector_bundle(2, 'E')
sage: phi_U = E.trivialization('phi_U', latex_name=r'\varphi_U',
....:                          domain=U); phi_U
Trivialization (phi_U, E|_U)
sage: phi_V = E.trivialization('phi_V', latex_name=r'\varphi_V',
....:                          domain=V); phi_V
Trivialization (phi_V, E|_V)
sage: phi_U_to_phi_V = phi_U.transition_map(phi_V, [[0,1],[1,0]]); phi_U_to_phi_V
Transition map from Trivialization (phi_U, E|_U) to Trivialization
 (phi_V, E|_V)
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2', structure='top')
>>> U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the North␣
↪and South pole, respectively
>>> S2.declare_union(U,V)
```

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```
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                     intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                     restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> W = U.intersection(V)
>>> uv_to_xy = xy_to_uv.inverse()
>>> E = S2.vector_bundle(Integer(2), 'E')
>>> phi_U = E.trivialization('phi_U', latex_name=r'\varphi_U',
...                              domain=U); phi_U
Trivialization (phi_U, E|_U)
>>> phi_V = E.trivialization('phi_V', latex_name=r'\varphi_V',
...                              domain=V); phi_V
Trivialization (phi_V, E|_V)
>>> phi_U_to_phi_V = phi_U.transition_map(phi_V, [[Integer(0),Integer(1)],
↪[Integer(1),Integer(0)]]); phi_U_to_phi_V
Transition map from Trivialization (phi_U, E|_U) to Trivialization
 (phi_V, E|_V)
```

The LaTeX output gives the following:

```
sage: latex(phi_U)
\varphi_U : E |_{U} \to U \times \Bold{R}^2
sage: latex(phi_V)
\varphi_V : E |_{V} \to V \times \Bold{R}^2
```

```
>>> from sage.all import *
>>> latex(phi_U)
\varphi_U : E |_{U} \to U \times \Bold{R}^2
>>> latex(phi_V)
\varphi_V : E |_{V} \to V \times \Bold{R}^2
```

The trivializations are part of the vector bundle atlas:

```
sage: E.atlas()
[Trivialization (phi_U, E|_U), Trivialization (phi_V, E|_V)]
```

```
>>> from sage.all import *
>>> E.atlas()
[Trivialization (phi_U, E|_U), Trivialization (phi_V, E|_V)]
```

Each trivialization induces a local trivialization frame:

```
sage: fU = phi_U.frame(); fU
Trivialization frame (E|_U, ((phi_U^*e_1),(phi_U^*e_2)))
sage: fV = phi_V.frame(); fV
Trivialization frame (E|_V, ((phi_V^*e_1),(phi_V^*e_2)))
```

```
>>> from sage.all import *
>>> fU = phi_U.frame(); fU
Trivialization frame (E|_U, ((phi_U^*e_1),(phi_U^*e_2)))
>>> fV = phi_V.frame(); fV
Trivialization frame (E|_V, ((phi_V^*e_1),(phi_V^*e_2)))
```

and the transition map connects these two frames via a bundle automorphism:

```
sage: aut = phi_U_to_phi_V.automorphism(); aut
Automorphism phi_U^(-1)*phi_V of the Free module C^0(W;E) of sections on
 the Open subset W of the 2-dimensional topological manifold S^2 with
 values in the real vector bundle E of rank 2
sage: aut.display(fU.restrict(W))
phi_U^(-1)*phi_V = (phi_U^*e_1)⊗(phi_U^*e^2) + (phi_U^*e_2)⊗(phi_U^*e^1)
sage: aut.display(fV.restrict(W))
phi_U^(-1)*phi_V = (phi_V^*e_1)⊗(phi_V^*e^2) + (phi_V^*e_2)⊗(phi_V^*e^1)
```

```
>>> from sage.all import *
>>> aut = phi_U_to_phi_V.automorphism(); aut
Automorphism phi_U^(-1)*phi_V of the Free module C^0(W;E) of sections on
 the Open subset W of the 2-dimensional topological manifold S^2 with
 values in the real vector bundle E of rank 2
>>> aut.display(fU.restrict(W))
phi_U^(-1)*phi_V = (phi_U^*e_1)⊗(phi_U^*e^2) + (phi_U^*e_2)⊗(phi_U^*e^1)
>>> aut.display(fV.restrict(W))
phi_U^(-1)*phi_V = (phi_V^*e_1)⊗(phi_V^*e^2) + (phi_V^*e_2)⊗(phi_V^*e^1)
```

The automorphisms are listed in the frame changes of the vector bundle:

```
sage: E.changes_of_frame() # random
{(Local frame (E|_W, ((phi_U^*e_1),(phi_U^*e_2))),
 Local frame (E|_W, ((phi_V^*e_1),(phi_V^*e_2)))): Automorphism
 phi_U^(-1)*phi_V^(-1) of the Free module C^0(W;E) of sections on the
 Open subset W of the 2-dimensional topological manifold S^2 with values
 in the real vector bundle E of rank 2,
 (Local frame (E|_W, ((phi_V^*e_1),(phi_V^*e_2))),
 Local frame (E|_W, ((phi_U^*e_1),(phi_U^*e_2)))): Automorphism
 phi_U^(-1)*phi_V of the Free module C^0(W;E) of sections on the Open
 subset W of the 2-dimensional topological manifold S^2 with values in
 the real vector bundle E of rank 2}
```

```
>>> from sage.all import *
>>> E.changes_of_frame() # random
{(Local frame (E|_W, ((phi_U^*e_1),(phi_U^*e_2))),
 Local frame (E|_W, ((phi_V^*e_1),(phi_V^*e_2)))): Automorphism
 phi_U^(-1)*phi_V^(-1) of the Free module C^0(W;E) of sections on the
 Open subset W of the 2-dimensional topological manifold S^2 with values
 in the real vector bundle E of rank 2,
 (Local frame (E|_W, ((phi_V^*e_1),(phi_V^*e_2))),
 Local frame (E|_W, ((phi_U^*e_1),(phi_U^*e_2)))): Automorphism
 phi_U^(-1)*phi_V of the Free module C^0(W;E) of sections on the Open
 subset W of the 2-dimensional topological manifold S^2 with values in
 the real vector bundle E of rank 2}
```

Let us check the components of `fU` with respect to the frame `fV`:

```
sage: fU[0].comp(fV.restrict(W))[:]
[0, 1]
sage: fU[1].comp(fV.restrict(W))[:]
[1, 0]
```

```
>>> from sage.all import *
>>> fU[Integer(0)].comp(fV.restrict(W))[:]
[0, 1]
>>> fU[Integer(1)].comp(fV.restrict(W))[:]
[1, 0]
```

base_space()

Return the manifold on which the trivialization is defined.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top')
sage: U = M.open_subset('U')
sage: E = M.vector_bundle(2, 'E')
sage: phi = E.trivialization('phi', domain=U)
sage: phi.base_space()
2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> U = M.open_subset('U')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> phi = E.trivialization('phi', domain=U)
>>> phi.base_space()
2-dimensional topological manifold M
```

coframe()

Return the standard coframe induced by `self`.

> **See also**
>
> *LocalCoFrame*

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top')
sage: E = M.vector_bundle(2, 'E')
sage: phi = E.trivialization('phi')
sage: phi.coframe()
Trivialization coframe (E|_M, ((phi^*e^1),(phi^*e^2)))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> phi = E.trivialization('phi')
```

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```
>>> phi.coframe()
Trivialization coframe (E|_M, ((phi^*e^1),(phi^*e^2)))
```

domain()

Return the domain on which the trivialization is defined.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top')
sage: U = M.open_subset('U')
sage: E = M.vector_bundle(2, 'E')
sage: phi = E.trivialization('phi', domain=U)
sage: phi.domain()
Open subset U of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> U = M.open_subset('U')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> phi = E.trivialization('phi', domain=U)
>>> phi.domain()
Open subset U of the 2-dimensional topological manifold M
```

frame()

Return the standard frame induced by `self`. If ψ is a trivialization then the corresponding frame can be obtained by the maps $p \mapsto \psi^{-1}(p, e_i)$, where $(e_1, \ldots, e_n)$ is the standard basis of K^n. We briefly denote $(\psi^* e_i)$ instead of $\psi^{-1}(\cdot, e_i)$.

See also

LocalFrame

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top')
sage: E = M.vector_bundle(2, 'E')
sage: phi = E.trivialization('phi')
sage: phi.frame()
Trivialization frame (E|_M, ((phi^*e_1),(phi^*e_2)))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> phi = E.trivialization('phi')
>>> phi.frame()
Trivialization frame (E|_M, ((phi^*e_1),(phi^*e_2)))
```

transition_map(*other*, *transf*, *compute_inverse=True*)

Return the transition map between `self` and `other`.

INPUT:

- `other` – the trivialization where the transition map from `self` goes to

- `transf` – transformation of the transition map
- `intersection_name` – (default: `None`) name to be given to the subset $U \cap V$ if the latter differs from U or V

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top')
sage: X.<x,y> = M.chart()
sage: U = M.open_subset('U'); V = M.open_subset('V')
sage: XU = X.restrict(U); XV = X.restrict(U)
sage: W = U.intersection(V)
sage: XW = X.restrict(W)
sage: E = M.vector_bundle(2, 'E')
sage: phi_U = E.trivialization('phi_U', domain=U)
sage: phi_V = E.trivialization('phi_V', domain=V)
sage: phi_U.transition_map(phi_V, 1)
Transition map from Trivialization (phi_U, E|_U) to Trivialization
 (phi_V, E|_V)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> U = M.open_subset('U'); V = M.open_subset('V')
>>> XU = X.restrict(U); XV = X.restrict(U)
>>> W = U.intersection(V)
>>> XW = X.restrict(W)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> phi_U = E.trivialization('phi_U', domain=U)
>>> phi_V = E.trivialization('phi_V', domain=V)
>>> phi_U.transition_map(phi_V, Integer(1))
Transition map from Trivialization (phi_U, E|_U) to Trivialization
 (phi_V, E|_V)
```

vector_bundle()

Return the vector bundle on which the trivialization is defined.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top')
sage: U = M.open_subset('U')
sage: E = M.vector_bundle(2, 'E')
sage: phi = E.trivialization('phi', domain=U)
sage: phi.vector_bundle()
Topological real vector bundle E -> M of rank 2 over the base space
 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> U = M.open_subset('U')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> phi = E.trivialization('phi', domain=U)
>>> phi.vector_bundle()
Topological real vector bundle E -> M of rank 2 over the base space
 2-dimensional topological manifold M
```

1.9.5 Local Frames

The class `LocalFrame` implements local frames on vector bundles (see `TopologicalVectorBundle` or `DifferentiableVectorBundle`).

For $k = 0, 1, \ldots$, a *local frame* on a vector bundle $E \to M$ of class C^k and rank n is a local section $(e_1, \ldots, e_n) : U \to E^n$ of class C^k defined on some subset U of the base space M, such that $e(p)$ is a basis of the fiber E_p for any $p \in U$.

AUTHORS:

- Michael Jung (2019): initial version

EXAMPLES:

Defining a global frame on a topological vector bundle of rank 3:

```
sage: M = Manifold(3, 'M', structure='top')
sage: E = M.vector_bundle(3, 'E')
sage: e = E.local_frame('e'); e
Local frame (E|_M, (e_0,e_1,e_2))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> E = M.vector_bundle(Integer(3), 'E')
>>> e = E.local_frame('e'); e
Local frame (E|_M, (e_0,e_1,e_2))
```

This frame is now the default frame of the corresponding section module and saved in the vector bundle:

```
sage: e in E.frames()
True
sage: sec_module = E.section_module(); sec_module
Free module C^0(M;E) of sections on the 3-dimensional topological manifold M
 with values in the real vector bundle E of rank 3
sage: sec_module.default_basis()
Local frame (E|_M, (e_0,e_1,e_2))
```

```
>>> from sage.all import *
>>> e in E.frames()
True
>>> sec_module = E.section_module(); sec_module
Free module C^0(M;E) of sections on the 3-dimensional topological manifold M
 with values in the real vector bundle E of rank 3
>>> sec_module.default_basis()
Local frame (E|_M, (e_0,e_1,e_2))
```

However, the default frame can be changed:

```
sage: sec_module.set_default_basis(e)
sage: sec_module.default_basis()
Local frame (E|_M, (e_0,e_1,e_2))
```

```
>>> from sage.all import *
>>> sec_module.set_default_basis(e)
>>> sec_module.default_basis()
Local frame (E|_M, (e_0,e_1,e_2))
```

The elements of a local frame are local sections in the vector bundle:

```
sage: for vec in e:
....:     print(vec)
Section e_0 on the 3-dimensional topological manifold M with values in the
 real vector bundle E of rank 3
 Section e_1 on the 3-dimensional topological manifold M with values in the
 real vector bundle E of rank 3
 Section e_2 on the 3-dimensional topological manifold M with values in the
 real vector bundle E of rank 3
```

```
>>> from sage.all import *
>>> for vec in e:
...     print(vec)
Section e_0 on the 3-dimensional topological manifold M with values in the
 real vector bundle E of rank 3
 Section e_1 on the 3-dimensional topological manifold M with values in the
 real vector bundle E of rank 3
 Section e_2 on the 3-dimensional topological manifold M with values in the
 real vector bundle E of rank 3
```

Each element of a vector frame can be accessed by its index:

```
sage: e[0]
Section e_0 on the 3-dimensional topological manifold M with values in the
 real vector bundle E of rank 3
```

```
>>> from sage.all import *
>>> e[Integer(0)]
Section e_0 on the 3-dimensional topological manifold M with values in the
 real vector bundle E of rank 3
```

The slice operator `:` can be used to access to more than one element:

```
sage: e[0:2]
(Section e_0 on the 3-dimensional topological manifold M with values in the
 real vector bundle E of rank 3,
 Section e_1 on the 3-dimensional topological manifold M with values in the
 real vector bundle E of rank 3)
sage: e[:]
(Section e_0 on the 3-dimensional topological manifold M with values in the
 real vector bundle E of rank 3,
 Section e_1 on the 3-dimensional topological manifold M with values in the
 real vector bundle E of rank 3,
 Section e_2 on the 3-dimensional topological manifold M with values in the
 real vector bundle E of rank 3)
```

```
>>> from sage.all import *
>>> e[Integer(0):Integer(2)]
(Section e_0 on the 3-dimensional topological manifold M with values in the
 real vector bundle E of rank 3,
 Section e_1 on the 3-dimensional topological manifold M with values in the
 real vector bundle E of rank 3)
```

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```
>>> e[:]
(Section e_0 on the 3-dimensional topological manifold M with values in the
 real vector bundle E of rank 3,
 Section e_1 on the 3-dimensional topological manifold M with values in the
 real vector bundle E of rank 3,
 Section e_2 on the 3-dimensional topological manifold M with values in the
 real vector bundle E of rank 3)
```

The index range depends on the starting index defined on the manifold:

```
sage: M = Manifold(3, 'M', structure='top', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: U = M.open_subset('U')
sage: c_xyz_U = c_xyz.restrict(U)
sage: E = M.vector_bundle(3, 'E')
sage: e = E.local_frame('e', domain=U); e
Local frame (E|_U, (e_1,e_2,e_3))
sage: [e[i] for i in M.irange()]
[Section e_1 on the Open subset U of the 3-dimensional topological manifold
 M with values in the real vector bundle E of rank 3,
 Section e_2 on the Open subset U of the 3-dimensional topological manifold
 M with values in the real vector bundle E of rank 3,
 Section e_3 on the Open subset U of the 3-dimensional topological manifold
 M with values in the real vector bundle E of rank 3]
sage: e[1], e[2], e[3]
(Section e_1 on the Open subset U of the 3-dimensional topological manifold
 M with values in the real vector bundle E of rank 3,
 Section e_2 on the Open subset U of the 3-dimensional topological manifold
 M with values in the real vector bundle E of rank 3,
 Section e_3 on the Open subset U of the 3-dimensional topological manifold
 M with values in the real vector bundle E of rank 3)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top', start_index=Integer(1))
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_ngens(3)
>>> U = M.open_subset('U')
>>> c_xyz_U = c_xyz.restrict(U)
>>> E = M.vector_bundle(Integer(3), 'E')
>>> e = E.local_frame('e', domain=U); e
Local frame (E|_U, (e_1,e_2,e_3))
>>> [e[i] for i in M.irange()]
[Section e_1 on the Open subset U of the 3-dimensional topological manifold
 M with values in the real vector bundle E of rank 3,
 Section e_2 on the Open subset U of the 3-dimensional topological manifold
 M with values in the real vector bundle E of rank 3,
 Section e_3 on the Open subset U of the 3-dimensional topological manifold
 M with values in the real vector bundle E of rank 3]
>>> e[Integer(1)], e[Integer(2)], e[Integer(3)]
(Section e_1 on the Open subset U of the 3-dimensional topological manifold
 M with values in the real vector bundle E of rank 3,
 Section e_2 on the Open subset U of the 3-dimensional topological manifold
 M with values in the real vector bundle E of rank 3,
```

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```
Section e_3 on the Open subset U of the 3-dimensional topological manifold
M with values in the real vector bundle E of rank 3)
```

Let us check that the local sections `e[i]` are indeed the frame vectors from their components with respect to the frame e:

```
sage: e[1].comp(e)[:]
[1, 0, 0]
sage: e[2].comp(e)[:]
[0, 1, 0]
sage: e[3].comp(e)[:]
[0, 0, 1]
```

```
>>> from sage.all import *
>>> e[Integer(1)].comp(e)[:]
[1, 0, 0]
>>> e[Integer(2)].comp(e)[:]
[0, 1, 0]
>>> e[Integer(3)].comp(e)[:]
[0, 0, 1]
```

Defining a local frame on a vector bundle, the dual coframe is automatically created, which, by default, bares the same name (here e):

```
sage: E.coframes()
[Local coframe (E|_U, (e^1,e^2,e^3))]
sage: e_dual = E.coframes()[0] ; e_dual
Local coframe (E|_U, (e^1,e^2,e^3))
sage: e_dual is e.coframe()
True
```

```
>>> from sage.all import *
>>> E.coframes()
[Local coframe (E|_U, (e^1,e^2,e^3))]
>>> e_dual = E.coframes()[Integer(0)] ; e_dual
Local coframe (E|_U, (e^1,e^2,e^3))
>>> e_dual is e.coframe()
True
```

Let us check that the coframe (e^i) is indeed the dual of the vector frame (e_i):

```
sage: e_dual[1](e[1]) # linear form e^1 applied to local section e_1
Scalar field e^1(e_1) on the Open subset U of the 3-dimensional topological
 manifold M
sage: e_dual[1](e[1]).expr() # the explicit expression of e^1(e_1)
1
sage: e_dual[1](e[1]).expr(), e_dual[1](e[2]).expr(), e_dual[1](e[3]).expr()
(1, 0, 0)
sage: e_dual[2](e[1]).expr(), e_dual[2](e[2]).expr(), e_dual[2](e[3]).expr()
(0, 1, 0)
sage: e_dual[3](e[1]).expr(), e_dual[3](e[2]).expr(), e_dual[3](e[3]).expr()
(0, 0, 1)
```

```
>>> from sage.all import *
>>> e_dual[Integer(1)](e[Integer(1)]) # linear form e^1 applied to local section e_1
Scalar field e^1(e_1) on the Open subset U of the 3-dimensional topological
 manifold M
>>> e_dual[Integer(1)](e[Integer(1)]).expr() # the explicit expression of e^1(e_1)
1
>>> e_dual[Integer(1)](e[Integer(1)]).expr(), e_dual[Integer(1)](e[Integer(2)]).
↪expr(), e_dual[Integer(1)](e[Integer(3)]).expr()
(1, 0, 0)
>>> e_dual[Integer(2)](e[Integer(1)]).expr(), e_dual[Integer(2)](e[Integer(2)]).
↪expr(), e_dual[Integer(2)](e[Integer(3)]).expr()
(0, 1, 0)
>>> e_dual[Integer(3)](e[Integer(1)]).expr(), e_dual[Integer(3)](e[Integer(2)]).
↪expr(), e_dual[Integer(3)](e[Integer(3)]).expr()
(0, 0, 1)
```

Via bundle automorphisms, a new frame can be created from an existing one:

```
sage: sec_module_U = E.section_module(domain=U)
sage: change_frame = sec_module_U.automorphism()
sage: change_frame[:] = [[0,1,0],[0,0,1],[1,0,0]]
sage: f = e.new_frame(change_frame, 'f'); f
Local frame (E|_U, (f_1,f_2,f_3))
```

```
>>> from sage.all import *
>>> sec_module_U = E.section_module(domain=U)
>>> change_frame = sec_module_U.automorphism()
>>> change_frame[:] = [[Integer(0),Integer(1),Integer(0)],[Integer(0),Integer(0),
↪Integer(1)],[Integer(1),Integer(0),Integer(0)]]
>>> f = e.new_frame(change_frame, 'f'); f
Local frame (E|_U, (f_1,f_2,f_3))
```

A copy of this automorphism and its inverse is now part of the vector bundle's frame changes:

```
sage: E.change_of_frame(e, f)
Automorphism of the Free module C^0(U;E) of sections on the Open subset U of
 the 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 3
sage: E.change_of_frame(e, f) == change_frame
True
sage: E.change_of_frame(f, e) == change_frame.inverse()
True
```

```
>>> from sage.all import *
>>> E.change_of_frame(e, f)
Automorphism of the Free module C^0(U;E) of sections on the Open subset U of
 the 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 3
>>> E.change_of_frame(e, f) == change_frame
True
>>> E.change_of_frame(f, e) == change_frame.inverse()
True
```

Let us check the components of f with respect to the frame e:

```
sage: f[1].comp(e)[:]
[0, 0, 1]
sage: f[2].comp(e)[:]
[1, 0, 0]
sage: f[3].comp(e)[:]
[0, 1, 0]
```

```
>>> from sage.all import *
>>> f[Integer(1)].comp(e)[:]
[0, 0, 1]
>>> f[Integer(2)].comp(e)[:]
[1, 0, 0]
>>> f[Integer(3)].comp(e)[:]
[0, 1, 0]
```

class sage.manifolds.local_frame.**LocalCoFrame**(*frame*, *symbol*, *latex_symbol=None*, *indices=None*, *latex_indices=None*)

Bases: FreeModuleCoBasis

Local coframe on a vector bundle.

A *local coframe* on a vector bundle $E \to M$ of class C^k is a local section $e^* : U \to E^n$ of class C^k on some subset U of the base space M, such that $e^*(p)$ is a basis of the fiber E^*_p of the dual bundle for any $p \in U$.

INPUT:

- `frame` – the local frame dual to the coframe
- `symbol` – either a string, to be used as a common base for the symbols of the linear forms constituting the coframe, or a tuple of strings, representing the individual symbols of the linear forms
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the linear forms constituting the coframe, or a tuple of strings, representing the individual LaTeX symbols of the linear forms; if `None`, `symbol` is used in place of `latex_symbol`
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the linear forms of the coframe; if `None`, the indices will be generated as integers within the range declared on the coframe's domain
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the linear forms of the coframe; if `None`, `indices` is used instead

EXAMPLES:

Local coframe on a topological vector bundle of rank 3:

```
sage: M = Manifold(3, 'M', structure='top', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: E = M.vector_bundle(3, 'E')
sage: e = E.local_frame('e')
sage: from sage.manifolds.local_frame import LocalCoFrame
sage: f = LocalCoFrame(e, 'f'); f
Local coframe (E|_M, (f^1,f^2,f^3))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top', start_index=Integer(1))
```

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```
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> E = M.vector_bundle(Integer(3), 'E')
>>> e = E.local_frame('e')
>>> from sage.manifolds.local_frame import LocalCoFrame
>>> f = LocalCoFrame(e, 'f'); f
Local coframe (E|_M, (f^1,f^2,f^3))
```

The local coframe can also be obtained by using the method `dual_basis()` or `coframe()`:

```
sage: e_dual = e.dual_basis(); e_dual
Local coframe (E|_M, (e^1,e^2,e^3))
sage: e_dual is e.coframe()
True
sage: e_dual is f
False
sage: e_dual[:] == f[:]
True
sage: f[1].display(e)
f^1 = e^1
```

```
>>> from sage.all import *
>>> e_dual = e.dual_basis(); e_dual
Local coframe (E|_M, (e^1,e^2,e^3))
>>> e_dual is e.coframe()
True
>>> e_dual is f
False
>>> e_dual[:] == f[:]
True
>>> f[Integer(1)].display(e)
f^1 = e^1
```

The consisted linear forms can be obtained via the operator `[]`:

```
sage: f[1], f[2], f[3]
(Linear form f^1 on the Free module C^0(M;E) of sections on the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 3,
 Linear form f^2 on the Free module C^0(M;E) of sections on the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 3,
 Linear form f^3 on the Free module C^0(M;E) of sections on the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 3)
```

```
>>> from sage.all import *
>>> f[Integer(1)], f[Integer(2)], f[Integer(3)]
(Linear form f^1 on the Free module C^0(M;E) of sections on the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 3,
 Linear form f^2 on the Free module C^0(M;E) of sections on the
 3-dimensional topological manifold M with values in the real vector
```

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```
bundle E of rank 3,
Linear form f^3 on the Free module C^0(M;E) of sections on the
3-dimensional topological manifold M with values in the real vector
bundle E of rank 3)
```

Checking that f is the dual of e:

```
sage: f[1](e[1]).expr(), f[1](e[2]).expr(), f[1](e[3]).expr()
(1, 0, 0)
sage: f[2](e[1]).expr(), f[2](e[2]).expr(), f[2](e[3]).expr()
(0, 1, 0)
sage: f[3](e[1]).expr(), f[3](e[2]).expr(), f[3](e[3]).expr()
(0, 0, 1)
```

```
>>> from sage.all import *
>>> f[Integer(1)](e[Integer(1)]).expr(), f[Integer(1)](e[Integer(2)]).expr(),␣
↪f[Integer(1)](e[Integer(3)]).expr()
(1, 0, 0)
>>> f[Integer(2)](e[Integer(1)]).expr(), f[Integer(2)](e[Integer(2)]).expr(),␣
↪f[Integer(2)](e[Integer(3)]).expr()
(0, 1, 0)
>>> f[Integer(3)](e[Integer(1)]).expr(), f[Integer(3)](e[Integer(2)]).expr(),␣
↪f[Integer(3)](e[Integer(3)]).expr()
(0, 0, 1)
```

at (*point*)

Return the value of `self` at a given point on the base space, this value being a basis of the dual vector bundle at this point.

INPUT:

- `point` – *ManifoldPoint*; point p in the domain U of the coframe (denoted f hereafter)

OUTPUT:

- `FreeModuleCoBasis` representing the basis $f(p)$ of the vector space E^*_p, dual to the vector bundle fiber E_p

EXAMPLES:

Cobasis of a vector bundle fiber:

```
sage: M = Manifold(2, 'M', structure='top', start_index=1)
sage: X.<x,y> = M.chart()
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e')
sage: e_dual = e.coframe(); e_dual
Local coframe (E|_M, (e^1,e^2))
sage: p = M.point((-1,2), name='p')
sage: e_dual_p = e_dual.at(p) ; e_dual_p
Dual basis (e^1,e^2) on the Fiber of E at Point p on the
2-dimensional topological manifold M
sage: type(e_dual_p)
<class 'sage.tensor.modules.free_module_basis.FreeModuleCoBasis_with_category
↪'>
```

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```
sage: e_dual_p[1]
Linear form e^1 on the Fiber of E at Point p on the 2-dimensional
 topological manifold M
sage: e_dual_p[2]
Linear form e^2 on the Fiber of E at Point p on the 2-dimensional
 topological manifold M
sage: e_dual_p is e.at(p).dual_basis()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e')
>>> e_dual = e.coframe(); e_dual
Local coframe (E|_M, (e^1,e^2))
>>> p = M.point((-Integer(1),Integer(2)), name='p')
>>> e_dual_p = e_dual.at(p) ; e_dual_p
Dual basis (e^1,e^2) on the Fiber of E at Point p on the
2-dimensional topological manifold M
>>> type(e_dual_p)
<class 'sage.tensor.modules.free_module_basis.FreeModuleCoBasis_with_category
↪'>
>>> e_dual_p[Integer(1)]
Linear form e^1 on the Fiber of E at Point p on the 2-dimensional
 topological manifold M
>>> e_dual_p[Integer(2)]
Linear form e^2 on the Fiber of E at Point p on the 2-dimensional
 topological manifold M
>>> e_dual_p is e.at(p).dual_basis()
True
```

set_name(*symbol*, *latex_symbol=None*, *indices=None*, *latex_indices=None*, *index_position='up'*, *include_domain=True*)

Set (or change) the text name and LaTeX name of `self`.

INPUT:

- `symbol` – either a string, to be used as a common base for the symbols of the linear forms constituting the coframe, or a list/tuple of strings, representing the individual symbols of the linear forms
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the linear forms constituting the coframe, or a list/tuple of strings, representing the individual LaTeX symbols of the linear forms; if `None`, `symbol` is used in place of `latex_symbol`
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the linear forms of the coframe; if `None`, the indices will be generated as integers within the range declared on `self`
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the linear forms; if `None`, `indices` is used instead
- `index_position` – (default: `'up'`) determines the position of the indices labelling the linear forms of the coframe; can be either `'down'` or `'up'`

- `include_domain` – boolean (default: `True`); determining whether the name of the domain is included in the beginning of the coframe name

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e').coframe(); e
Local coframe (E|_M, (e^0,e^1))
sage: e.set_name('f'); e
Local coframe (E|_M, (f^0,f^1))
sage: e.set_name('e', latex_symbol=r'\epsilon')
sage: latex(e)
\left(E|_{M}, \left(\epsilon^{0},\epsilon^{1}\right)\right)
sage: e.set_name('e', include_domain=False); e
Local coframe (e^0,e^1)
sage: e.set_name(['a', 'b'], latex_symbol=[r'\alpha', r'\beta']); e
Local coframe (E|_M, (a,b))
sage: latex(e)
\left(E|_{M}, \left(\alpha,\beta\right)\right)
sage: e.set_name('e', indices=['x','y'],
....:            latex_indices=[r'\xi', r'\zeta']); e
Local coframe (E|_M, (e^x,e^y))
sage: latex(e)
\left(E|_{M}, \left(e^{\xi},e^{\zeta}\right)\right)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e').coframe(); e
Local coframe (E|_M, (e^0,e^1))
>>> e.set_name('f'); e
Local coframe (E|_M, (f^0,f^1))
>>> e.set_name('e', latex_symbol=r'\epsilon')
>>> latex(e)
\left(E|_{M}, \left(\epsilon^{0},\epsilon^{1}\right)\right)
>>> e.set_name('e', include_domain=False); e
Local coframe (e^0,e^1)
>>> e.set_name(['a', 'b'], latex_symbol=[r'\alpha', r'\beta']); e
Local coframe (E|_M, (a,b))
>>> latex(e)
\left(E|_{M}, \left(\alpha,\beta\right)\right)
>>> e.set_name('e', indices=['x','y'],
...            latex_indices=[r'\xi', r'\zeta']); e
Local coframe (E|_M, (e^x,e^y))
>>> latex(e)
\left(E|_{M}, \left(e^{\xi},e^{\zeta}\right)\right)
```

class `sage.manifolds.local_frame.`**`LocalFrame`**(*section_module*, *symbol*, *latex_symbol=None*, *indices=None*, *latex_indices=None*, *symbol_dual=None*, *latex_symbol_dual=None*)

Bases: `FreeModuleBasis`

Local frame on a vector bundle.

A *local frame* on a vector bundle $E \to M$ of class C^k is a local section $(e_1, \ldots, e_n) : U \to E^n$ of class C^k defined on some subset U of the base space M, such that $e(p)$ is a basis of the fiber E_p for any $p \in U$.

For each instantiation of a local frame, a local coframe is automatically created, as an instance of the class `LocalCoFrame`. It is returned by the method `coframe()`.

INPUT:

- `section_module` – free module of local sections over U in the given vector bundle $E \to M$
- `symbol` – either a string, to be used as a common base for the symbols of the local sections constituting the local frame, or a tuple of strings, representing the individual symbols of the local sections
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the local sections constituting the local frame, or a tuple of strings, representing the individual LaTeX symbols of the local sections; if `None`, `symbol` is used in place of `latex_symbol`
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the local sections of the frame; if `None`, the indices will be generated as integers within the range declared on the local frame's domain
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the local sections; if `None`, `indices` is used instead
- `symbol_dual` – (default: `None`) same as `symbol` but for the dual coframe; if `None`, `symbol` must be a string and is used for the common base of the symbols of the elements of the dual coframe
- `latex_symbol_dual` – (default: `None`) same as `latex_symbol` but for the dual coframe

EXAMPLES:

Defining a local frame on a 3-dimensional vector bundle over a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1, structure='top')
sage: E = M.vector_bundle(3, 'E')
sage: e = E.local_frame('e'); e
Local frame (E|_M, (e_1,e_2,e_3))
sage: latex(e)
\left(E|_{M}, \left(e_{1},e_{2},e_{3}\right)\right)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1), structure='top')
>>> E = M.vector_bundle(Integer(3), 'E')
>>> e = E.local_frame('e'); e
Local frame (E|_M, (e_1,e_2,e_3))
>>> latex(e)
\left(E|_{M}, \left(e_{1},e_{2},e_{3}\right)\right)
```

The individual elements of the vector frame are accessed via square brackets, with the possibility to invoke the slice operator ':' to get more than a single element:

```
sage: e[2]
Section e_2 on the 3-dimensional topological manifold M with values in
 the real vector bundle E of rank 3
sage: e[1:3]
(Section e_1 on the 3-dimensional topological manifold M with values in
 the real vector bundle E of rank 3,
 Section e_2 on the 3-dimensional topological manifold M with values in
 the real vector bundle E of rank 3)
```

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```
sage: e[:]
(Section e_1 on the 3-dimensional topological manifold M with values in
 the real vector bundle E of rank 3,
 Section e_2 on the 3-dimensional topological manifold M with values in
 the real vector bundle E of rank 3,
 Section e_3 on the 3-dimensional topological manifold M with values in
 the real vector bundle E of rank 3)
```

```
>>> from sage.all import *
>>> e[Integer(2)]
Section e_2 on the 3-dimensional topological manifold M with values in
 the real vector bundle E of rank 3
>>> e[Integer(1):Integer(3)]
(Section e_1 on the 3-dimensional topological manifold M with values in
 the real vector bundle E of rank 3,
 Section e_2 on the 3-dimensional topological manifold M with values in
 the real vector bundle E of rank 3)
>>> e[:]
(Section e_1 on the 3-dimensional topological manifold M with values in
 the real vector bundle E of rank 3,
 Section e_2 on the 3-dimensional topological manifold M with values in
 the real vector bundle E of rank 3,
 Section e_3 on the 3-dimensional topological manifold M with values in
 the real vector bundle E of rank 3)
```

The LaTeX symbol can be specified:

```
sage: eps = E.local_frame('eps', latex_symbol=r'\epsilon')
sage: latex(eps)
\left(E|_{M}, \left(\epsilon_{1},\epsilon_{2},\epsilon_{3}\right)\right)
```

```
>>> from sage.all import *
>>> eps = E.local_frame('eps', latex_symbol=r'\epsilon')
>>> latex(eps)
\left(E|_{M}, \left(\epsilon_{1},\epsilon_{2},\epsilon_{3}\right)\right)
```

By default, the elements of the local frame are labelled by integers within the range specified at the manifold declaration. It is however possible to fully customize the labels, via the argument `indices`:

```
sage: u = E.local_frame('u', indices=('x', 'y', 'z')) ; u
Local frame (E|_M, (u_x,u_y,u_z))
sage: u[1]
Section u_x on the 3-dimensional topological manifold M with values in
 the real vector bundle E of rank 3
sage: u.coframe()
Local coframe (E|_M, (u^x,u^y,u^z))
```

```
>>> from sage.all import *
>>> u = E.local_frame('u', indices=('x', 'y', 'z')) ; u
Local frame (E|_M, (u_x,u_y,u_z))
>>> u[Integer(1)]
Section u_x on the 3-dimensional topological manifold M with values in
```

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```
 the real vector bundle E of rank 3
>>> u.coframe()
Local coframe (E|_M, (u^x,u^y,u^z))
```

The LaTeX format of the indices can be adjusted:

```
sage: v = E.local_frame('v', indices=('a', 'b', 'c'),
....:                     latex_indices=(r'\alpha', r'\beta', r'\gamma'))
sage: v
Local frame (E|_M, (v_a,v_b,v_c))
sage: latex(v)
\left(E|_{M}, \left(v_{\alpha},v_{\beta},v_{\gamma}\right)\right)
sage: latex(v.coframe())
\left(E|_{M}, \left(v^{\alpha},v^{\beta},v^{\gamma}\right)\right)
```

```
>>> from sage.all import *
>>> v = E.local_frame('v', indices=('a', 'b', 'c'),
...                     latex_indices=(r'\alpha', r'\beta', r'\gamma'))
>>> v
Local frame (E|_M, (v_a,v_b,v_c))
>>> latex(v)
\left(E|_{M}, \left(v_{\alpha},v_{\beta},v_{\gamma}\right)\right)
>>> latex(v.coframe())
\left(E|_{M}, \left(v^{\alpha},v^{\beta},v^{\gamma}\right)\right)
```

The symbol of each element of the local frame can also be freely chosen, by providing a tuple of symbols as the first argument of `local_frame`; it is then mandatory to specify as well some symbols for the dual coframe:

```
sage: h = E.local_frame(('a', 'b', 'c'), symbol_dual=('A', 'B', 'C')); h
Local frame (E|_M, (a,b,c))
sage: h[1]
Section a on the 3-dimensional topological manifold M with values in the
 real vector bundle E of rank 3
sage: h.coframe()
Local coframe (E|_M, (A,B,C))
sage: h.coframe()[1]
Linear form A on the Free module C^0(M;E) of sections on the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 3
```

```
>>> from sage.all import *
>>> h = E.local_frame(('a', 'b', 'c'), symbol_dual=('A', 'B', 'C')); h
Local frame (E|_M, (a,b,c))
>>> h[Integer(1)]
Section a on the 3-dimensional topological manifold M with values in the
 real vector bundle E of rank 3
>>> h.coframe()
Local coframe (E|_M, (A,B,C))
>>> h.coframe()[Integer(1)]
Linear form A on the Free module C^0(M;E) of sections on the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 3
```

Local frames are bases of free modules formed by local sections:

```
sage: N = Manifold(2, 'N', structure='top', start_index=1)
sage: X.<x,y> = N.chart()
sage: U = N.open_subset('U')
sage: F = N.vector_bundle(2, 'F')
sage: f = F.local_frame('f', domain=U)
sage: f.module()
Free module C^0(U;F) of sections on the Open subset U of the
 2-dimensional topological manifold N with values in the real vector
 bundle F of rank 2
sage: f.module().base_ring()
Algebra of scalar fields on the Open subset U of the 2-dimensional
 topological manifold N
sage: f.module() is F.section_module(domain=f.domain())
True
sage: f in F.section_module(domain=U).bases()
True
```

```
>>> from sage.all import *
>>> N = Manifold(Integer(2), 'N', structure='top', start_index=Integer(1))
>>> X = N.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> U = N.open_subset('U')
>>> F = N.vector_bundle(Integer(2), 'F')
>>> f = F.local_frame('f', domain=U)
>>> f.module()
Free module C^0(U;F) of sections on the Open subset U of the
 2-dimensional topological manifold N with values in the real vector
 bundle F of rank 2
>>> f.module().base_ring()
Algebra of scalar fields on the Open subset U of the 2-dimensional
 topological manifold N
>>> f.module() is F.section_module(domain=f.domain())
True
>>> f in F.section_module(domain=U).bases()
True
```

The value of the local frame at a given point is a basis of the corresponding fiber:

```
sage: X_U = X.restrict(U) # We need coordinates on the subset
sage: p = N((0,1), name='p') ; p
Point p on the 2-dimensional topological manifold N
sage: f.at(p)
Basis (f_1,f_2) on the Fiber of F at Point p on the 2-dimensional
 topological manifold N
```

```
>>> from sage.all import *
>>> X_U = X.restrict(U) # We need coordinates on the subset
>>> p = N((Integer(0),Integer(1)), name='p') ; p
Point p on the 2-dimensional topological manifold N
>>> f.at(p)
Basis (f_1,f_2) on the Fiber of F at Point p on the 2-dimensional
 topological manifold N
```

at (*point*)

Return the value of `self` at a given point, this value being a basis of the vector bundle fiber at the point.

INPUT:

- `point` – *ManifoldPoint*; point p in the domain U of the local frame (denoted e hereafter)

OUTPUT:

- `FreeModuleBasis` representing the basis $e(p)$ of the vector bundle fiber E_p

EXAMPLES:

Basis of a fiber of a trivial vector bundle:

```
sage: M = Manifold(2, 'M', structure='top')
sage: X.<x,y> = M.chart()
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e'); e
Local frame (E|_M, (e_0,e_1))
sage: p = M.point((-1,2), name='p')
sage: ep = e.at(p) ; ep
Basis (e_0,e_1) on the Fiber of E at Point p on the 2-dimensional
 topological manifold M
sage: type(ep)
<class 'sage.tensor.modules.free_module_basis.FreeModuleBasis_with_category'>
sage: ep[0]
Vector e_0 in the fiber of E at Point p on the 2-dimensional
 topological manifold M
sage: ep[1]
Vector e_1 in the fiber of E at Point p on the 2-dimensional
 topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e'); e
Local frame (E|_M, (e_0,e_1))
>>> p = M.point((-Integer(1),Integer(2)), name='p')
>>> ep = e.at(p) ; ep
Basis (e_0,e_1) on the Fiber of E at Point p on the 2-dimensional
 topological manifold M
>>> type(ep)
<class 'sage.tensor.modules.free_module_basis.FreeModuleBasis_with_category'>
>>> ep[Integer(0)]
Vector e_0 in the fiber of E at Point p on the 2-dimensional
 topological manifold M
>>> ep[Integer(1)]
Vector e_1 in the fiber of E at Point p on the 2-dimensional
 topological manifold M
```

Note that the symbols used to denote the vectors are same as those for the vector fields of the frame. At this stage, `ep` is the unique basis on fiber at `p`:

```
sage: Ep = E.fiber(p)
sage: Ep.bases()
[Basis (e_0,e_1) on the Fiber of E at Point p on the 2-dimensional
 topological manifold M]
```

```
>>> from sage.all import *
>>> Ep = E.fiber(p)
>>> Ep.bases()
[Basis (e_0,e_1) on the Fiber of E at Point p on the 2-dimensional
 topological manifold M]
```

Let us consider another local frame:

```
sage: aut = E.section_module().automorphism()
sage: aut[:] = [[1+y^2, 0], [0, 2]]
sage: f = e.new_frame(aut, 'f') ; f
Local frame (E|_M, (f_0,f_1))
sage: fp = f.at(p) ; fp
Basis (f_0,f_1) on the Fiber of E at Point p on the 2-dimensional
 topological manifold M
```

```
>>> from sage.all import *
>>> aut = E.section_module().automorphism()
>>> aut[:] = [[Integer(1)+y**Integer(2), Integer(0)], [Integer(0),␣
↪Integer(2)]]
>>> f = e.new_frame(aut, 'f') ; f
Local frame (E|_M, (f_0,f_1))
>>> fp = f.at(p) ; fp
Basis (f_0,f_1) on the Fiber of E at Point p on the 2-dimensional
 topological manifold M
```

There are now two bases on the fiber:

```
sage: Ep.bases()
[Basis (e_0,e_1) on the Fiber of E at Point p on the 2-dimensional
 topological manifold M,
 Basis (f_0,f_1) on the Fiber of E at Point p on the 2-dimensional
 topological manifold M]
```

```
>>> from sage.all import *
>>> Ep.bases()
[Basis (e_0,e_1) on the Fiber of E at Point p on the 2-dimensional
 topological manifold M,
 Basis (f_0,f_1) on the Fiber of E at Point p on the 2-dimensional
 topological manifold M]
```

Moreover, the changes of bases in the tangent space have been computed from the known relation between the frames `e` and `f` (via the automorphism `aut` defined above):

```
sage: Ep.change_of_basis(ep, fp)
Automorphism of the Fiber of E at Point p on the 2-dimensional
 topological manifold M
```

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```
sage: Ep.change_of_basis(ep, fp).display()
5 e_0⊗e^0 + 2 e_1⊗e^1
sage: Ep.change_of_basis(fp, ep)
Automorphism of the Fiber of E at Point p on the 2-dimensional
 topological manifold M
sage: Ep.change_of_basis(fp, ep).display()
1/5 e_0⊗e^0 + 1/2 e_1⊗e^1
```

```
>>> from sage.all import *
>>> Ep.change_of_basis(ep, fp)
Automorphism of the Fiber of E at Point p on the 2-dimensional
 topological manifold M
>>> Ep.change_of_basis(ep, fp).display()
5 e_0⊗e^0 + 2 e_1⊗e^1
>>> Ep.change_of_basis(fp, ep)
Automorphism of the Fiber of E at Point p on the 2-dimensional
 topological manifold M
>>> Ep.change_of_basis(fp, ep).display()
1/5 e_0⊗e^0 + 1/2 e_1⊗e^1
```

The dual bases:

```
sage: e.coframe()
Local coframe (E|_M, (e^0,e^1))
sage: ep.dual_basis()
Dual basis (e^0,e^1) on the Fiber of E at Point p on the
 2-dimensional topological manifold M
sage: ep.dual_basis() is e.coframe().at(p)
True
sage: f.coframe()
Local coframe (E|_M, (f^0,f^1))
sage: fp.dual_basis()
Dual basis (f^0,f^1) on the Fiber of E at Point p on the
 2-dimensional topological manifold M
sage: fp.dual_basis() is f.coframe().at(p)
True
```

```
>>> from sage.all import *
>>> e.coframe()
Local coframe (E|_M, (e^0,e^1))
>>> ep.dual_basis()
Dual basis (e^0,e^1) on the Fiber of E at Point p on the
 2-dimensional topological manifold M
>>> ep.dual_basis() is e.coframe().at(p)
True
>>> f.coframe()
Local coframe (E|_M, (f^0,f^1))
>>> fp.dual_basis()
Dual basis (f^0,f^1) on the Fiber of E at Point p on the
 2-dimensional topological manifold M
>>> fp.dual_basis() is f.coframe().at(p)
True
```

base_space()

Return the base space on which the overlying vector bundle is defined.

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: U = M.open_subset('U')
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e', domain=U)
sage: e.base_space()
3-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> U = M.open_subset('U')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e', domain=U)
>>> e.base_space()
3-dimensional topological manifold M
```

coframe()

Return the coframe of `self`.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top')
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e'); e
Local frame (E|_M, (e_0,e_1))
sage: e.coframe()
Local coframe (E|_M, (e^0,e^1))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e'); e
Local frame (E|_M, (e_0,e_1))
>>> e.coframe()
Local coframe (E|_M, (e^0,e^1))
```

domain()

Return the domain on which `self` is defined.

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: U = M.open_subset('U')
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e', domain=U); e
Local frame (E|_U, (e_0,e_1))
sage: e.domain()
Open subset U of the 3-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> U = M.open_subset('U')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e', domain=U); e
Local frame (E|_U, (e_0,e_1))
>>> e.domain()
Open subset U of the 3-dimensional topological manifold M
```

new_frame(*change_of_frame*, *symbol*, *latex_symbol=None*, *indices=None*, *latex_indices=None*, *symbol_dual=None*, *latex_symbol_dual=None*)

Define a new local frame from `self`.

The new local frame is defined from vector bundle automorphisms; its module is the same as that of the current frame.

INPUT:

- `change_of_frame` – `FreeModuleAutomorphism`; vector bundle automorphisms P that relates the current frame (e_i) to the new frame (f_i) according to $f_i = P(e_i)$
- `symbol` – either a string, to be used as a common base for the symbols of the sections constituting the local frame, or a list/tuple of strings, representing the individual symbols of the sections
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the sections constituting the local frame, or a list/tuple of strings, representing the individual LaTeX symbols of the sections; if `None`, `symbol` is used in place of `latex_symbol`
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the sections of the frame; if `None`, the indices will be generated as integers within the range declared on `self`
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the sections; if `None`, `indices` is used instead
- `symbol_dual` – (default: `None`) same as `symbol` but for the dual coframe; if `None`, `symbol` must be a string and is used for the common base of the symbols of the elements of the dual coframe
- `latex_symbol_dual` – (default: `None`) same as `latex_symbol` but for the dual coframe

OUTPUT:

- the new frame (f_i), as an instance of *LocalFrame*

EXAMPLES:

Orthogonal transformation of a frame on the 2-dimensional trivial vector bundle over the Euclidean plane:

```
sage: M = Manifold(2, 'R^2', structure='top', start_index=1)
sage: c_cart.<x,y> = M.chart()
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e'); e
Local frame (E|_R^2, (e_1,e_2))
sage: orth = E.section_module().automorphism()
sage: orth[:] = [[sqrt(3)/2, -1/2], [1/2, sqrt(3)/2]]
sage: f = e.new_frame(orth, 'f')
sage: f[1][:]
[1/2*sqrt(3), 1/2]
sage: f[2][:]
```

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```
[-1/2, 1/2*sqrt(3)]
sage: a =  E.change_of_frame(e,f)
sage: a[:]
[1/2*sqrt(3)        -1/2]
[        1/2 1/2*sqrt(3)]
sage: a == orth
True
sage: a is orth
False
sage: a._components # random (dictionary output)
{Local frame (E|_D_0, (e_1,e_2)): 2-indices components w.r.t.
 Local frame (E|_D_0, (e_1,e_2)),
 Local frame (E|_D_0, (f_1,f_2)): 2-indices components w.r.t.
 Local frame (E|_D_0, (f_1,f_2))}
sage: a.comp(f)[:]
[1/2*sqrt(3)        -1/2]
 [        1/2 1/2*sqrt(3)]
sage: a1 = E.change_of_frame(f,e)
sage: a1[:]
[1/2*sqrt(3)         1/2]
[       -1/2 1/2*sqrt(3)]
sage: a1 == orth.inverse()
True
sage: a1 is orth.inverse()
False
sage: e[1].comp(f)[:]
[1/2*sqrt(3), -1/2]
sage: e[2].comp(f)[:]
[1/2, 1/2*sqrt(3)]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', structure='top', start_index=Integer(1))
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e'); e
Local frame (E|_R^2, (e_1,e_2))
>>> orth = E.section_module().automorphism()
>>> orth[:] = [[sqrt(Integer(3))/Integer(2), -Integer(1)/Integer(2)],
↪[Integer(1)/Integer(2), sqrt(Integer(3))/Integer(2)]]
>>> f = e.new_frame(orth, 'f')
>>> f[Integer(1)][:]
[1/2*sqrt(3), 1/2]
>>> f[Integer(2)][:]
[-1/2, 1/2*sqrt(3)]
>>> a =  E.change_of_frame(e,f)
>>> a[:]
[1/2*sqrt(3)        -1/2]
[        1/2 1/2*sqrt(3)]
>>> a == orth
True
>>> a is orth
False
```

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```
>>> a._components # random (dictionary output)
{Local frame (E|_D_0, (e_1,e_2)): 2-indices components w.r.t.
 Local frame (E|_D_0, (e_1,e_2)),
 Local frame (E|_D_0, (f_1,f_2)): 2-indices components w.r.t.
 Local frame (E|_D_0, (f_1,f_2))}
>>> a.comp(f)[:]
[1/2*sqrt(3)        -1/2]
 [        1/2 1/2*sqrt(3)]
>>> a1 = E.change_of_frame(f,e)
>>> a1[:]
[1/2*sqrt(3)         1/2]
[        -1/2 1/2*sqrt(3)]
>>> a1 == orth.inverse()
True
>>> a1 is orth.inverse()
False
>>> e[Integer(1)].comp(f)[:]
[1/2*sqrt(3), -1/2]
>>> e[Integer(2)].comp(f)[:]
[1/2, 1/2*sqrt(3)]
```

restrict (*subdomain*)

Return the restriction of `self` to some open subset of its domain.

If the restriction has not been defined yet, it is constructed here.

INPUT:

- `subdomain` – open subset V of the current frame domain U

OUTPUT: the restriction of the current frame to V as a *LocalFrame*

EXAMPLES:

Restriction of a frame defined on $\mathbf{R}^2$ to the unit disk:

```
sage: M = Manifold(2, 'R^2', structure='top', start_index=1)
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e'); e
Local frame (E|_R^2, (e_1,e_2))
sage: a = E.section_module().automorphism()
sage: a[:] = [[1-y^2,0], [1+x^2, 2]]
sage: f = e.new_frame(a, 'f'); f
Local frame (E|_R^2, (f_1,f_2))
sage: U = M.open_subset('U', coord_def={c_cart: x^2+y^2<1})
sage: e_U = e.restrict(U); e_U
Local frame (E|_U, (e_1,e_2))
sage: f_U = f.restrict(U) ; f_U
Local frame (E|_U, (f_1,f_2))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', structure='top', start_index=Integer(1))
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#
↪Cartesian coordinates on R^2
```

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```
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e'); e
Local frame (E|_R^2, (e_1,e_2))
>>> a = E.section_module().automorphism()
>>> a[:] = [[Integer(1)-y**Integer(2),Integer(0)], [Integer(1)+x**Integer(2),
↪Integer(2)]]
>>> f = e.new_frame(a, 'f'); f
Local frame (E|_R^2, (f_1,f_2))
>>> U = M.open_subset('U', coord_def={c_cart: x**Integer(2)+y**Integer(2)
↪<Integer(1)})
>>> e_U = e.restrict(U); e_U
Local frame (E|_U, (e_1,e_2))
>>> f_U = f.restrict(U) ; f_U
Local frame (E|_U, (f_1,f_2))
```

The vectors of the restriction have the same symbols as those of the original frame:

```
sage: f_U[1].display()
f_1 = (-y^2 + 1) e_1 + (x^2 + 1) e_2
sage: f_U[2].display()
f_2 = 2 e_2
```

```
>>> from sage.all import *
>>> f_U[Integer(1)].display()
f_1 = (-y^2 + 1) e_1 + (x^2 + 1) e_2
>>> f_U[Integer(2)].display()
f_2 = 2 e_2
```

Actually, the components are the restrictions of the original frame vectors:

```
sage: f_U[1] is f[1].restrict(U)
True
sage: f_U[2] is f[2].restrict(U)
True
```

```
>>> from sage.all import *
>>> f_U[Integer(1)] is f[Integer(1)].restrict(U)
True
>>> f_U[Integer(2)] is f[Integer(2)].restrict(U)
True
```

set_name(*symbol*, *latex_symbol=None*, *indices=None*, *latex_indices=None*, *index_position='down'*, *include_domain=True*)

Set (or change) the text name and LaTeX name of `self`.

INPUT:

- `symbol` – either a string, to be used as a common base for the symbols of the local sections constituting the local frame, or a list/tuple of strings, representing the individual symbols of the local sections
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the local sections constituting the local frame, or a list/tuple of strings, representing the individual LaTeX symbols of the local sections; if `None`, `symbol` is used in place of `latex_symbol`

- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the local sections of the frame; if `None`, the indices will be generated as integers within the range declared on `self`
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the local sections; if `None`, `indices` is used instead
- `index_position` – (default: `'down'`) determines the position of the indices labelling the local sections of the frame; can be either `'down'` or `'up'`
- `include_domain` – boolean (default: `True`); determining whether the name of the domain is included in the beginning of the vector frame name

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e'); e
Local frame (E|_M, (e_0,e_1))
sage: e.set_name('f'); e
Local frame (E|_M, (f_0,f_1))
sage: e.set_name('e', include_domain=False); e
Local frame (e_0,e_1)
sage: e.set_name(['a', 'b']); e
Local frame (E|_M, (a,b))
sage: e.set_name('e', indices=['x', 'y']); e
Local frame (E|_M, (e_x,e_y))
sage: e.set_name('e', latex_symbol=r'\epsilon')
sage: latex(e)
\left(E|_{M}, \left(\epsilon_{0},\epsilon_{1}\right)\right)
sage: e.set_name('e', latex_symbol=[r'\alpha', r'\beta'])
sage: latex(e)
\left(E|_{M}, \left(\alpha,\beta\right)\right)
sage: e.set_name('e', latex_symbol='E',
....:            latex_indices=[r'\alpha', r'\beta'])
sage: latex(e)
\left(E|_{M}, \left(E_{\alpha},E_{\beta}\right)\right)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e'); e
Local frame (E|_M, (e_0,e_1))
>>> e.set_name('f'); e
Local frame (E|_M, (f_0,f_1))
>>> e.set_name('e', include_domain=False); e
Local frame (e_0,e_1)
>>> e.set_name(['a', 'b']); e
Local frame (E|_M, (a,b))
>>> e.set_name('e', indices=['x', 'y']); e
Local frame (E|_M, (e_x,e_y))
>>> e.set_name('e', latex_symbol=r'\epsilon')
>>> latex(e)
\left(E|_{M}, \left(\epsilon_{0},\epsilon_{1}\right)\right)
>>> e.set_name('e', latex_symbol=[r'\alpha', r'\beta'])
```

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```
>>> latex(e)
\left(E|_{M}, \left(\alpha,\beta\right)\right)
>>> e.set_name('e', latex_symbol='E',
...             latex_indices=[r'\alpha', r'\beta'])
>>> latex(e)
\left(E|_{M}, \left(E_{\alpha},E_{\beta}\right)\right)
```

vector_bundle()

Return the vector bundle on which `self` is defined.

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: U = M.open_subset('U')
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e', domain=U)
sage: e.vector_bundle()
Topological real vector bundle E -> M of rank 2 over the base space
3-dimensional topological manifold M
sage: e.vector_bundle() is E
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> U = M.open_subset('U')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e', domain=U)
>>> e.vector_bundle()
Topological real vector bundle E -> M of rank 2 over the base space
3-dimensional topological manifold M
>>> e.vector_bundle() is E
True
```

class sage.manifolds.local_frame.**TrivializationCoFrame**(*triv_frame*, *symbol*, *latex_symbol=None*, *indices=None*, *latex_indices=None*)

Bases: `LocalCoFrame`

Trivialization coframe on a vector bundle.

A *trivialization coframe* is the coframe of the trivialization frame induced by a trivialization (see: `TrivializationFrame`).

More precisely, a *trivialization frame* on a vector bundle $E \to M$ of class C^k and rank n over the topological field K and over a topological manifold M is a local coframe induced by a local trivialization $\varphi : E|_U \to U \times K^n$ of the domain $U \in M$. Namely, the local dual sections

$$\varphi^* e^i := \varphi(\,\cdot\,, e^i)$$

on U induce a local frame $(\varphi^* e^1, \dots, \varphi^* e^n)$, where $(e^1, \dots, e^n)$ is the dual of the standard basis of K^n.

INPUT:

- `triv_frame` – trivialization frame dual to the trivialization coframe
- `symbol` – either a string, to be used as a common base for the symbols of the dual sections constituting the coframe, or a tuple of strings, representing the individual symbols of the dual sections

- latex_symbol – (default: None) either a string, to be used as a common base for the LaTeX symbols of the dual sections constituting the coframe, or a tuple of strings, representing the individual LaTeX symbols of the dual sections; if None, symbol is used in place of latex_symbol
- indices – (default: None; used only if symbol is a single string) tuple of strings representing the indices labelling the dual sections of the coframe; if None, the indices will be generated as integers within the range declared on the local frame's domain
- latex_indices – (default: None) tuple of strings representing the indices for the LaTeX symbols of the dual sections of the coframe; if None, indices is used instead

EXAMPLES:

Trivialization coframe on a trivial vector bundle of rank 3:

```
sage: M = Manifold(3, 'M', start_index=1, structure='top')
sage: X.<x,y,z> = M.chart()
sage: E = M.vector_bundle(3, 'E')
sage: phi = E.trivialization('phi'); phi
Trivialization (phi, E|_M)
sage: E.frames()
[Trivialization frame (E|_M, ((phi^*e_1),(phi^*e_2),(phi^*e_3)))]
sage: E.coframes()
[Trivialization coframe (E|_M, ((phi^*e^1),(phi^*e^2),(phi^*e^3)))]
sage: f = E.coframes()[0] ; f
Trivialization coframe (E|_M, ((phi^*e^1),(phi^*e^2),(phi^*e^3)))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1), structure='top')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> E = M.vector_bundle(Integer(3), 'E')
>>> phi = E.trivialization('phi'); phi
Trivialization (phi, E|_M)
>>> E.frames()
[Trivialization frame (E|_M, ((phi^*e_1),(phi^*e_2),(phi^*e_3)))]
>>> E.coframes()
[Trivialization coframe (E|_M, ((phi^*e^1),(phi^*e^2),(phi^*e^3)))]
>>> f = E.coframes()[Integer(0)] ; f
Trivialization coframe (E|_M, ((phi^*e^1),(phi^*e^2),(phi^*e^3)))
```

The linear forms composing the coframe are obtained via the operator []:

```
sage: f[1]
Linear form (phi^*e^1) on the Free module C^0(M;E) of sections on the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 3
sage: f[2]
Linear form (phi^*e^2) on the Free module C^0(M;E) of sections on the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 3
sage: f[3]
Linear form (phi^*e^3) on the Free module C^0(M;E) of sections on the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 3
sage: f[1][:]
```

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```
[1, 0, 0]
sage: f[2][:]
[0, 1, 0]
sage: f[3][:]
[0, 0, 1]
```

```
>>> from sage.all import *
>>> f[Integer(1)]
Linear form (phi^*e^1) on the Free module C^0(M;E) of sections on the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 3
>>> f[Integer(2)]
Linear form (phi^*e^2) on the Free module C^0(M;E) of sections on the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 3
>>> f[Integer(3)]
Linear form (phi^*e^3) on the Free module C^0(M;E) of sections on the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 3
>>> f[Integer(1)][:]
[1, 0, 0]
>>> f[Integer(2)][:]
[0, 1, 0]
>>> f[Integer(3)][:]
[0, 0, 1]
```

The coframe is the dual of the trivialization frame:

```
sage: e = phi.frame() ; e
Trivialization frame (E|_M, ((phi^*e_1),(phi^*e_2),(phi^*e_3)))
sage: f[1](e[1]).expr(), f[1](e[2]).expr(), f[1](e[3]).expr()
(1, 0, 0)
sage: f[2](e[1]).expr(), f[2](e[2]).expr(), f[2](e[3]).expr()
(0, 1, 0)
sage: f[3](e[1]).expr(), f[3](e[2]).expr(), f[3](e[3]).expr()
(0, 0, 1)
```

```
>>> from sage.all import *
>>> e = phi.frame() ; e
Trivialization frame (E|_M, ((phi^*e_1),(phi^*e_2),(phi^*e_3)))
>>> f[Integer(1)](e[Integer(1)]).expr(), f[Integer(1)](e[Integer(2)]).expr(),␣
→f[Integer(1)](e[Integer(3)]).expr()
(1, 0, 0)
>>> f[Integer(2)](e[Integer(1)]).expr(), f[Integer(2)](e[Integer(2)]).expr(),␣
→f[Integer(2)](e[Integer(3)]).expr()
(0, 1, 0)
>>> f[Integer(3)](e[Integer(1)]).expr(), f[Integer(3)](e[Integer(2)]).expr(),␣
→f[Integer(3)](e[Integer(3)]).expr()
(0, 0, 1)
```

class sage.manifolds.local_frame.**TrivializationFrame**(*trivialization*)

Bases: *LocalFrame*

Trivialization frame on a topological vector bundle.

A *trivialization frame* on a topological vector bundle $E \to M$ of rank n over the topological field K and over a topological manifold M is a local frame induced by a local trivialization $\varphi : E|_U \to U \times K^n$ of the domain $U \in M$. More precisely, the local sections

$$\varphi^* e_i := \varphi(\,\cdot\,, e_i)$$

on U induce a local frame $(\varphi^* e_1, \dots, \varphi^* e_n)$, where $(e_1, \dots, e_n)$ is the standard basis of K^n.

INPUT:

- `trivialization` – the trivialization defined on the vector bundle

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: U = M.open_subset('U')
sage: E = M.vector_bundle(2, 'E')
sage: phi_U = E.trivialization('phi_U', domain=U)
sage: phi_U.frame()
Trivialization frame (E|_U, ((phi_U^*e_1),(phi_U^*e_2)))
sage: latex(phi_U.frame())
\left(E|_{U}, \left(\left(phi_U^* e_{ 1 }\right),\left(phi_U^* e_{ 2 }\right)\
↪right)\right)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> U = M.open_subset('U')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> phi_U = E.trivialization('phi_U', domain=U)
>>> phi_U.frame()
Trivialization frame (E|_U, ((phi_U^*e_1),(phi_U^*e_2)))
>>> latex(phi_U.frame())
\left(E|_{U}, \left(\left(phi_U^* e_{ 1 }\right),\left(phi_U^* e_{ 2 }\right)\
↪right)\right)
```

trivialization()

Return the underlying trivialization of `self`.

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: U = M.open_subset('U')
sage: E = M.vector_bundle(2, 'E')
sage: phi_U = E.trivialization('phi_U', domain=U)
sage: e = phi_U.frame()
sage: e.trivialization()
Trivialization (phi_U, E|_U)
sage: e.trivialization() is phi_U
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> U = M.open_subset('U')
>>> E = M.vector_bundle(Integer(2), 'E')
```

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```
>>> phi_U = E.trivialization('phi_U', domain=U)
>>> e = phi_U.frame()
>>> e.trivialization()
Trivialization (phi_U, E|_U)
>>> e.trivialization() is phi_U
True
```

1.9.6 Section Modules

The set of sections over a vector bundle $E \to M$ of class C^k on a domain $U \in M$ is a module over the algebra $C^k(U)$ of scalar fields on U.

Depending on the domain, there are two classes of section modules:

- `SectionModule` for local sections over a non-trivial part of a topological vector bundle
- `SectionFreeModule` for local sections over a trivial part of a topological vector bundle

AUTHORS:

- Michael Jung (2019): initial version

class sage.manifolds.section_module.**SectionFreeModule**(*vbundle*, *domain*)

Bases: `FiniteRankFreeModule`

Free module of sections over a vector bundle $E \to M$ of class C^k on a domain $U \in M$ which admits a trivialization or local frame.

The *section module* $C^k(U; E)$ is the set of all C^k-maps, called *sections*, of type

$$s : U \longrightarrow E$$

such that

$$\forall p \in U, \ s(p) \in E_p,$$

where E_p is the vector bundle fiber of E at the point p.

Since the domain U admits a local frame, the corresponding vector bundle $E|_U \to U$ is trivial and $C^k(U; E)$ is a free module over $C^k(U)$.

> **Note**
>
> If $E|_U$ is not trivial, the class `SectionModule` should be used instead, for $C^k(U; E)$ is no longer a free module.

INPUT:

- `vbundle` – vector bundle E on which the sections takes its values
- `domain` – (default: `None`) subdomain U of the base space on which the sections are defined

EXAMPLES:

Module of sections on the 2-rank trivial vector bundle over the Euclidean plane $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'R^2', structure='top')
sage: c_cart.<x,y> = M.chart()
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e') # Trivializes the vector bundle
sage: C0 = E.section_module(); C0
Free module C^0(R^2;E) of sections on the 2-dimensional topological
 manifold R^2 with values in the real vector bundle E of rank 2
sage: C0.category()
Category of finite dimensional modules over Algebra of scalar fields on
 the 2-dimensional topological manifold R^2
sage: C0.base_ring() is M.scalar_field_algebra()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', structure='top')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e') # Trivializes the vector bundle
>>> C0 = E.section_module(); C0
Free module C^0(R^2;E) of sections on the 2-dimensional topological
 manifold R^2 with values in the real vector bundle E of rank 2
>>> C0.category()
Category of finite dimensional modules over Algebra of scalar fields on
 the 2-dimensional topological manifold R^2
>>> C0.base_ring() is M.scalar_field_algebra()
True
```

The vector bundle admits a global frame and is therefore trivial:

```
sage: E.is_manifestly_trivial()
True
```

```
>>> from sage.all import *
>>> E.is_manifestly_trivial()
True
```

Since the vector bundle is trivial, its section module of global sections is a free module:

```
sage: isinstance(C0, FiniteRankFreeModule)
True
```

```
>>> from sage.all import *
>>> isinstance(C0, FiniteRankFreeModule)
True
```

Some elements are:

```
sage: C0.an_element().display()
2 e_0 + 2 e_1
sage: C0.zero().display()
zero = 0
sage: s = C0([-y,x]); s
```

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```
Section on the 2-dimensional topological manifold R^2 with values in the
 real vector bundle E of rank 2
sage: s.display()
-y e_0 + x e_1
```

```
>>> from sage.all import *
>>> C0.an_element().display()
2 e_0 + 2 e_1
>>> C0.zero().display()
zero = 0
>>> s = C0([-y,x]); s
Section on the 2-dimensional topological manifold R^2 with values in the
 real vector bundle E of rank 2
>>> s.display()
-y e_0 + x e_1
```

The rank of the free module equals the rank of the vector bundle:

```
sage: C0.rank()
2
```

```
>>> from sage.all import *
>>> C0.rank()
2
```

The basis is given by the definition above:

```
sage: C0.bases()
[Local frame (E|_R^2, (e_0,e_1))]
```

```
>>> from sage.all import *
>>> C0.bases()
[Local frame (E|_R^2, (e_0,e_1))]
```

The test suite is passed as well:

```
sage: TestSuite(C0).run()
```

```
>>> from sage.all import *
>>> TestSuite(C0).run()
```

Element

alias of *TrivialSection*

base_space()

Return the base space of the sections in this module.

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: U = M.open_subset('U')
sage: E = U.vector_bundle(2, 'E')
```

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```
sage: C0 = E.section_module(force_free=True); C0
Free module C^0(U;E) of sections on the Open subset U of the
 3-dimensional topological manifold M with values in the real
 vector bundle E of rank 2
sage: C0.base_space()
Open subset U of the 3-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> U = M.open_subset('U')
>>> E = U.vector_bundle(Integer(2), 'E')
>>> C0 = E.section_module(force_free=True); C0
Free module C^0(U;E) of sections on the Open subset U of the
 3-dimensional topological manifold M with values in the real
 vector bundle E of rank 2
>>> C0.base_space()
Open subset U of the 3-dimensional topological manifold M
```

basis(*symbol=None*, *latex_symbol=None*, *from_frame=None*, *indices=None*, *latex_indices=None*, *symbol_dual=None*, *latex_symbol_dual=None*)

Define a basis of `self`.

A basis of the section module is actually a local frame on the differentiable manifold U over which the section module is defined.

If the basis specified by the given symbol already exists, it is simply returned. If no argument is provided the module's default basis is returned.

INPUT:

- `symbol` – (default: `None`) either a string, to be used as a common base for the symbols of the elements of the basis, or a tuple of strings, representing the individual symbols of the elements of the basis
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the elements of the basis, or a tuple of strings, representing the individual LaTeX symbols of the elements of the basis; if `None`, `symbol` is used in place of `latex_symbol`
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the elements of the basis; if `None`, the indices will be generated as integers within the range declared on `self`
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the elements of the basis; if `None`, `indices` is used instead
- `symbol_dual` – (default: `None`) same as `symbol` but for the dual basis; if `None`, `symbol` must be a string and is used for the common base of the symbols of the elements of the dual basis
- `latex_symbol_dual` – (default: `None`) same as `latex_symbol` but for the dual basis

OUTPUT:

- a *LocalFrame* representing a basis on `self`

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top')
sage: E = M.vector_bundle(2, 'E')
```

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```
sage: C0 = E.section_module(force_free=True)
sage: e = C0.basis('e'); e
Local frame (E|_M, (e_0,e_1))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> C0 = E.section_module(force_free=True)
>>> e = C0.basis('e'); e
Local frame (E|_M, (e_0,e_1))
```

See `LocalFrame` for more examples and documentation.

default_frame()

Return the default basis of the free module `self`.

The *default basis* is simply a basis whose name can be skipped in methods requiring a basis as an argument. By default, it is the first basis introduced on the module. It can be changed by the method `set_default_basis()`.

OUTPUT:

- instance of `FreeModuleBasis`

EXAMPLES:

At the module construction, no default basis is assumed:

```
sage: M = FiniteRankFreeModule(ZZ, 2, name='M', start_index=1)
sage: M.default_basis()
No default basis has been defined on the
 Rank-2 free module M over the Integer Ring
```

```
>>> from sage.all import *
>>> M = FiniteRankFreeModule(ZZ, Integer(2), name='M', start_index=Integer(1))
>>> M.default_basis()
No default basis has been defined on the
 Rank-2 free module M over the Integer Ring
```

The first defined basis becomes the default one:

```
sage: e = M.basis('e') ; e
Basis (e_1,e_2) on the Rank-2 free module M over the Integer Ring
sage: M.default_basis()
Basis (e_1,e_2) on the Rank-2 free module M over the Integer Ring
sage: f =  M.basis('f') ; f
Basis (f_1,f_2) on the Rank-2 free module M over the Integer Ring
sage: M.default_basis()
Basis (e_1,e_2) on the Rank-2 free module M over the Integer Ring
```

```
>>> from sage.all import *
>>> e = M.basis('e') ; e
Basis (e_1,e_2) on the Rank-2 free module M over the Integer Ring
>>> M.default_basis()
```

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```
Basis (e_1,e_2) on the Rank-2 free module M over the Integer Ring
>>> f =  M.basis('f') ; f
Basis (f_1,f_2) on the Rank-2 free module M over the Integer Ring
>>> M.default_basis()
Basis (e_1,e_2) on the Rank-2 free module M over the Integer Ring
```

domain()

Return the domain of the section module.

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: U = M.open_subset('U')
sage: E = M.vector_bundle(2, 'E')
sage: C0_U = E.section_module(domain=U, force_free=True); C0_U
Free module C^0(U;E) of sections on the Open subset U of the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 2
sage: C0_U.domain()
Open subset U of the 3-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> U = M.open_subset('U')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> C0_U = E.section_module(domain=U, force_free=True); C0_U
Free module C^0(U;E) of sections on the Open subset U of the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 2
>>> C0_U.domain()
Open subset U of the 3-dimensional topological manifold M
```

set_default_frame(*basis*)

Set the default basis of `self`.

The *default basis* is simply a basis whose name can be skipped in methods requiring a basis as an argument. By default, it is the first basis introduced on the module.

INPUT:

- `basis` – instance of `FreeModuleBasis` representing a basis on `self`

EXAMPLES:

Changing the default basis on a rank-3 free module:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1)
sage: e = M.basis('e') ; e
Basis (e_1,e_2,e_3) on the Rank-3 free module M over the Integer Ring
sage: f =  M.basis('f') ; f
Basis (f_1,f_2,f_3) on the Rank-3 free module M over the Integer Ring
sage: M.default_basis()
Basis (e_1,e_2,e_3) on the Rank-3 free module M over the Integer Ring
sage: M.set_default_basis(f)
```

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```
sage: M.default_basis()
Basis (f_1,f_2,f_3) on the Rank-3 free module M over the Integer Ring
```

```
>>> from sage.all import *
>>> M = FiniteRankFreeModule(ZZ, Integer(3), name='M', start_index=Integer(1))
>>> e = M.basis('e') ; e
Basis (e_1,e_2,e_3) on the Rank-3 free module M over the Integer Ring
>>> f =  M.basis('f') ; f
Basis (f_1,f_2,f_3) on the Rank-3 free module M over the Integer Ring
>>> M.default_basis()
Basis (e_1,e_2,e_3) on the Rank-3 free module M over the Integer Ring
>>> M.set_default_basis(f)
>>> M.default_basis()
Basis (f_1,f_2,f_3) on the Rank-3 free module M over the Integer Ring
```

vector_bundle()

Return the overlying vector bundle on which the section module is defined.

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: E = M.vector_bundle(2, 'E')
sage: C0 = E.section_module(force_free=True); C0
Free module C^0(M;E) of sections on the 3-dimensional topological
 manifold M with values in the real vector bundle E of rank 2
sage: C0.vector_bundle()
Topological real vector bundle E -> M of rank 2 over the base space
 3-dimensional topological manifold M
sage: E is C0.vector_bundle()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> C0 = E.section_module(force_free=True); C0
Free module C^0(M;E) of sections on the 3-dimensional topological
 manifold M with values in the real vector bundle E of rank 2
>>> C0.vector_bundle()
Topological real vector bundle E -> M of rank 2 over the base space
 3-dimensional topological manifold M
>>> E is C0.vector_bundle()
True
```

class sage.manifolds.section_module.**SectionModule**(*vbundle*, *domain*)

Bases: UniqueRepresentation, Parent

Module of sections over a vector bundle $E \to M$ of class C^k on a domain $U \in M$.

The *section module* $C^k(U;E)$ is the set of all C^k-maps, called *sections*, of type

$$s : U \longrightarrow E$$

such that

$$\forall p \in U,\ s(p) \in E_p,$$

where E_p is the vector bundle fiber of E at the point p.

$C^k(U;E)$ is a module over $C^k(U)$, the algebra of C^k scalar fields on U.

INPUT:

- `vbundle` – vector bundle E on which the sections takes its values
- `domain` – (default: `None`) subdomain U of the base space on which the sections are defined

EXAMPLES:

Module of sections on the Möbius bundle:

```
sage: M = Manifold(1, 'RP^1', structure='top', start_index=1)
sage: U = M.open_subset('U')  # the complement of one point
sage: c_u.<u> =  U.chart() # [1:u] in homogeneous coord.
sage: V = M.open_subset('V') # the complement of the point u=0
sage: M.declare_union(U,V)   # [v:1] in homogeneous coord.
sage: c_v.<v> = V.chart()
sage: u_to_v = c_u.transition_map(c_v, (1/u),
....:                             intersection_name='W',
....:                             restrictions1 = u!=0,
....:                             restrictions2 = v!=0)
sage: v_to_u = u_to_v.inverse()
sage: W = U.intersection(V)
sage: E = M.vector_bundle(1, 'E')
sage: phi_U = E.trivialization('phi_U', latex_name=r'\varphi_U',
....:                          domain=U)
sage: phi_V = E.trivialization('phi_V', latex_name=r'\varphi_V',
....:                          domain=V)
sage: transf = phi_U.transition_map(phi_V, [[u]])
sage: C0 = E.section_module(); C0
Module C^0(RP^1;E) of sections on the 1-dimensional topological manifold
 RP^1 with values in the real vector bundle E of rank 1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(1), 'RP^1', structure='top', start_index=Integer(1))
>>> U = M.open_subset('U')  # the complement of one point
>>> c_u = U.chart(names=('u',)); (u,) = c_u._first_ngens(1)# [1:u] in homogeneous␣
↪coord.
>>> V = M.open_subset('V') # the complement of the point u=0
>>> M.declare_union(U,V)   # [v:1] in homogeneous coord.
>>> c_v = V.chart(names=('v',)); (v,) = c_v._first_ngens(1)
>>> u_to_v = c_u.transition_map(c_v, (Integer(1)/u),
...                             intersection_name='W',
...                             restrictions1 = u!=Integer(0),
...                             restrictions2 = v!=Integer(0))
>>> v_to_u = u_to_v.inverse()
>>> W = U.intersection(V)
>>> E = M.vector_bundle(Integer(1), 'E')
>>> phi_U = E.trivialization('phi_U', latex_name=r'\varphi_U',
...                          domain=U)
>>> phi_V = E.trivialization('phi_V', latex_name=r'\varphi_V',
...                          domain=V)
>>> transf = phi_U.transition_map(phi_V, [[u]])
```

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```
>>> C0 = E.section_module(); C0
Module C^0(RP^1;E) of sections on the 1-dimensional topological manifold
 RP^1 with values in the real vector bundle E of rank 1
```

$C^0(\mathbf{R}P^1; E)$ is a module over the algebra $C^0(\mathbf{R}P^1)$:

```
sage: C0.category()
Category of modules over Algebra of scalar fields on the 1-dimensional
 topological manifold RP^1
sage: C0.base_ring() is M.scalar_field_algebra()
True
```

```
>>> from sage.all import *
>>> C0.category()
Category of modules over Algebra of scalar fields on the 1-dimensional
 topological manifold RP^1
>>> C0.base_ring() is M.scalar_field_algebra()
True
```

However, $C^0(\mathbf{R}P^1; E)$ is not a free module:

```
sage: isinstance(C0, FiniteRankFreeModule)
False
```

```
>>> from sage.all import *
>>> isinstance(C0, FiniteRankFreeModule)
False
```

since the Möbius bundle is not trivial:

```
sage: E.is_manifestly_trivial()
False
```

```
>>> from sage.all import *
>>> E.is_manifestly_trivial()
False
```

The section module over U, on the other hand, is a free module since $E|_U$ admits a trivialization and therefore has a local frame:

```
sage: C0_U = E.section_module(domain=U)
sage: isinstance(C0_U, FiniteRankFreeModule)
True
```

```
>>> from sage.all import *
>>> C0_U = E.section_module(domain=U)
>>> isinstance(C0_U, FiniteRankFreeModule)
True
```

The zero element of the module:

```
sage: z = C0.zero() ; z
Section zero on the 1-dimensional topological manifold RP^1 with values
 in the real vector bundle E of rank 1
sage: z.display(phi_U.frame())
zero = 0
sage: z.display(phi_V.frame())
zero = 0
```

```
>>> from sage.all import *
>>> z = C0.zero() ; z
Section zero on the 1-dimensional topological manifold RP^1 with values
 in the real vector bundle E of rank 1
>>> z.display(phi_U.frame())
zero = 0
>>> z.display(phi_V.frame())
zero = 0
```

The module $C^0(M;E)$ coerces to any module of sections defined on a subdomain of M, for instance $C^0(U;E)$:

```
sage: C0_U.has_coerce_map_from(C0)
True
sage: C0_U.coerce_map_from(C0)
Coercion map:
  From: Module C^0(RP^1;E) of sections on the 1-dimensional topological
   manifold RP^1 with values in the real vector bundle E of rank 1
  To:   Free module C^0(U;E) of sections on the Open subset U of the
   1-dimensional topological manifold RP^1 with values in the real vector
   bundle E of rank 1
```

```
>>> from sage.all import *
>>> C0_U.has_coerce_map_from(C0)
True
>>> C0_U.coerce_map_from(C0)
Coercion map:
  From: Module C^0(RP^1;E) of sections on the 1-dimensional topological
   manifold RP^1 with values in the real vector bundle E of rank 1
  To:   Free module C^0(U;E) of sections on the Open subset U of the
   1-dimensional topological manifold RP^1 with values in the real vector
   bundle E of rank 1
```

The conversion map is actually the restriction of sections defined on M to U.

Element

alias of *Section*

base_space()

Return the base space of the sections in this module.

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: U = M.open_subset('U')
sage: E = U.vector_bundle(2, 'E')
sage: C0 = E.section_module(); C0
```

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```
Module C^0(U;E) of sections on the Open subset U of the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 2
sage: C0.base_space()
Open subset U of the 3-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> U = M.open_subset('U')
>>> E = U.vector_bundle(Integer(2), 'E')
>>> C0 = E.section_module(); C0
Module C^0(U;E) of sections on the Open subset U of the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 2
>>> C0.base_space()
Open subset U of the 3-dimensional topological manifold M
```

default_frame()

Return the default frame defined on `self`.

EXAMPLES:

Get the default local frame of a non-trivial section module:

```
sage: M = Manifold(3, 'M', structure='top')
sage: U = M.open_subset('U')
sage: V = M.open_subset('V')
sage: M.declare_union(U, V)
sage: E = M.vector_bundle(2, 'E')
sage: C0 = E.section_module()
sage: e = E.local_frame('e', domain=U)
sage: C0.default_frame()
Local frame (E|_U, (e_0,e_1))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> U = M.open_subset('U')
>>> V = M.open_subset('V')
>>> M.declare_union(U, V)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> C0 = E.section_module()
>>> e = E.local_frame('e', domain=U)
>>> C0.default_frame()
Local frame (E|_U, (e_0,e_1))
```

The local frame is indeed the same, and not a copy:

```
sage: e is C0.default_frame()
True
```

```
>>> from sage.all import *
>>> e is C0.default_frame()
True
```

domain()

Return the domain of the section module.

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: U = M.open_subset('U')
sage: E = M.vector_bundle(2, 'E')
sage: C0_U = E.section_module(domain=U); C0_U
Module C^0(U;E) of sections on the Open subset U of the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 2
sage: C0_U.domain()
Open subset U of the 3-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> U = M.open_subset('U')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> C0_U = E.section_module(domain=U); C0_U
Module C^0(U;E) of sections on the Open subset U of the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 2
>>> C0_U.domain()
Open subset U of the 3-dimensional topological manifold M
```

set_default_frame(*basis*)

Set the default local frame on `self`.

EXAMPLES:

Set a default frame of a non-trivial section module:

```
sage: M = Manifold(3, 'M', structure='top')
sage: U = M.open_subset('U')
sage: V = M.open_subset('V')
sage: M.declare_union(U, V)
sage: E = M.vector_bundle(2, 'E')
sage: C0 = E.section_module(); C0
Module C^0(M;E) of sections on the 3-dimensional topological
 manifold M with values in the real vector bundle E of rank 2
sage: e = E.local_frame('e', domain=U)
sage: C0.set_default_frame(e)
sage: C0.default_frame()
Local frame (E|_U, (e_0,e_1))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> U = M.open_subset('U')
>>> V = M.open_subset('V')
>>> M.declare_union(U, V)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> C0 = E.section_module(); C0
Module C^0(M;E) of sections on the 3-dimensional topological
```

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```
 manifold M with values in the real vector bundle E of rank 2
>>> e = E.local_frame('e', domain=U)
>>> C0.set_default_frame(e)
>>> C0.default_frame()
Local frame (E|_U, (e_0,e_1))
```

The local frame is indeed the same, and not a copy:

```
sage: e is C0.default_frame()
True
```

```
>>> from sage.all import *
>>> e is C0.default_frame()
True
```

Notice, that the local frame is defined on a subset and is not part of the section module $C^k(M;E)$:

```
sage: C0.default_frame().domain()
Open subset U of the 3-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> C0.default_frame().domain()
Open subset U of the 3-dimensional topological manifold M
```

vector_bundle()

Return the overlying vector bundle on which the section module is defined.

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: E = M.vector_bundle(2, 'E')
sage: C0 = E.section_module(); C0
Module C^0(M;E) of sections on the 3-dimensional topological
 manifold M with values in the real vector bundle E of rank 2
sage: C0.vector_bundle()
Topological real vector bundle E -> M of rank 2 over the base space
 3-dimensional topological manifold M
sage: E is C0.vector_bundle()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> C0 = E.section_module(); C0
Module C^0(M;E) of sections on the 3-dimensional topological
 manifold M with values in the real vector bundle E of rank 2
>>> C0.vector_bundle()
Topological real vector bundle E -> M of rank 2 over the base space
 3-dimensional topological manifold M
>>> E is C0.vector_bundle()
True
```

zero()

Return the zero of `self`.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top')
sage: X.<x,y> = M.chart()
sage: E = M.vector_bundle(2, 'E')
sage: C0 = E.section_module()
sage: z = C0.zero(); z
Section zero on the 2-dimensional topological manifold M with values
 in the real vector bundle E of rank 2
sage: z == 0
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> C0 = E.section_module()
>>> z = C0.zero(); z
Section zero on the 2-dimensional topological manifold M with values
 in the real vector bundle E of rank 2
>>> z == Integer(0)
True
```

1.9.7 Sections

The class *`Section`* implements sections on vector bundles. The derived class *`TrivialSection`* is devoted to sections on trivial parts of a vector bundle.

AUTHORS:

- Michael Jung (2019): initial version

class `sage.manifolds.section.`**Section**(*section_module*, *name=None*, *latex_name=None*)

Bases: `ModuleElementWithMutability`

Section in a vector bundle.

An instance of this class is a section in a vector bundle $E \to M$ of class C^k, where $E|_U$ is not manifestly trivial. More precisely, a *(local) section* on a subset $U \in M$ is a map of class C^k

$$s : U \longrightarrow E$$

such that

$$\forall p \in U,\ s(p) \in E_p$$

where E_p denotes the vector bundle fiber of E over the point $p \in U$.

If $E|_U$ is trivial, the class *`TrivialSection`* should be used instead.

This is a Sage *element* class, the corresponding *parent* class being *`SectionModule`*.

INPUT:

- `section_module` – module $C^k(U;E)$ of sections on E over U (cf. *`SectionModule`*)

- `name` – (default: `None`) name given to the section
- `latex_name` – (default: `None`) LaTeX symbol to denote the section; if none is provided, the LaTeX symbol is set to `name`

EXAMPLES:

A section on a non-trivial rank 2 vector bundle over a non-trivial 2-manifold:

```
sage: M = Manifold(2, 'M', structure='top')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                     intersection_name='W', restrictions1= x>0,
....:                     restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: E = M.vector_bundle(2, 'E') # define the vector bundle
sage: phi_U = E.trivialization('phi_U', domain=U) # define trivializations
sage: phi_V = E.trivialization('phi_V', domain=V)
sage: transf = phi_U.transition_map(phi_V, [[0,x],[x,0]]) # transition map␣
↪between trivializations
sage: fU = phi_U.frame(); fV = phi_V.frame() # define induced frames
sage: s = E.section(name='s'); s
Section s on the 2-dimensional topological manifold M with values in the
 real vector bundle E of rank 2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv = V.
↪chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
...                     intersection_name='W', restrictions1= x>Integer(0),
...                     restrictions2= u+v>Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V)
>>> E = M.vector_bundle(Integer(2), 'E') # define the vector bundle
>>> phi_U = E.trivialization('phi_U', domain=U) # define trivializations
>>> phi_V = E.trivialization('phi_V', domain=V)
>>> transf = phi_U.transition_map(phi_V, [[Integer(0),x],[x,Integer(0)]]) #␣
↪transition map between trivializations
>>> fU = phi_U.frame(); fV = phi_V.frame() # define induced frames
>>> s = E.section(name='s'); s
Section s on the 2-dimensional topological manifold M with values in the
 real vector bundle E of rank 2
```

The parent of s is not a free module, since E is not trivial:

```
sage: isinstance(s.parent(), FiniteRankFreeModule)
False
```

```
>>> from sage.all import *
>>> isinstance(s.parent(), FiniteRankFreeModule)
False
```

To fully define s, we have to specify its components in some local frames defined on the trivial parts of E. The components consist of scalar fields defined on the corresponding domain. Let us start with $E|_U$:

```
sage: s[fU,:] = [x^2, 1-y]
sage: s.display(fU)
s = x^2 (phi_U^*e_1) + (-y + 1) (phi_U^*e_2)
```

```
>>> from sage.all import *
>>> s[fU,:] = [x**Integer(2), Integer(1)-y]
>>> s.display(fU)
s = x^2 (phi_U^*e_1) + (-y + 1) (phi_U^*e_2)
```

To set the components of s on V consistently, we copy the expressions of the components in the common subset W:

```
sage: fUW = fU.restrict(W); fVW = fV.restrict(W)
sage: c_uvW = c_uv.restrict(W)
sage: s[fV,0] = s[fVW,0,c_uvW].expr()   # long time
sage: s[fV,1] = s[fVW,1,c_uvW].expr()   # long time
```

```
>>> from sage.all import *
>>> fUW = fU.restrict(W); fVW = fV.restrict(W)
>>> c_uvW = c_uv.restrict(W)
>>> s[fV,Integer(0)] = s[fVW,Integer(0),c_uvW].expr()   # long time
>>> s[fV,Integer(1)] = s[fVW,Integer(1),c_uvW].expr()   # long time
```

Actually, the operation above can be performed in a single line by means of the method `add_comp_by_continuation()`:

```
sage: s.add_comp_by_continuation(fV, W, chart=c_uv)
```

```
>>> from sage.all import *
>>> s.add_comp_by_continuation(fV, W, chart=c_uv)
```

At this stage, s is fully defined, having components in frames `fU` and `fV` and the union of the domains of `fU` and `fV` being the whole manifold:

```
sage: s.display(fV)
s = (-1/4*u^2 + 1/4*v^2 + 1/2*u + 1/2*v) (phi_V^*e_1)
    + (1/8*u^3 + 3/8*u^2*v + 3/8*u*v^2 + 1/8*v^3) (phi_V^*e_2)
```

```
>>> from sage.all import *
>>> s.display(fV)
s = (-1/4*u^2 + 1/4*v^2 + 1/2*u + 1/2*v) (phi_V^*e_1)
    + (1/8*u^3 + 3/8*u^2*v + 3/8*u*v^2 + 1/8*v^3) (phi_V^*e_2)
```

Sections can be pointwisely added:

```
sage: t = E.section([x,y], frame=fU, name='t'); t
Section t on the 2-dimensional topological manifold M with values in the
 real vector bundle E of rank 2
sage: t.add_comp_by_continuation(fV, W, chart=c_uv)
sage: t.display(fV)
t = (1/4*u^2 - 1/4*v^2) (phi_V^*e_1) + (1/4*u^2 + 1/2*u*v + 1/4*v^2) (phi_V^*e_2)
sage: a = s + t; a
Section s+t on the 2-dimensional topological manifold M with values
 in the real vector bundle E of rank 2
sage: a.display(fU)
s+t = (x^2 + x) (phi_U^*e_1) + (phi_U^*e_2)
sage: a.display(fV)
s+t = (1/2*u + 1/2*v) (phi_V^*e_1) + (1/8*u^3 + 1/8*(3*u + 2)*v^2
      + 1/8*v^3 + 1/4*u^2 + 1/8*(3*u^2 + 4*u)*v) (phi_V^*e_2)
```

```
>>> from sage.all import *
>>> t = E.section([x,y], frame=fU, name='t'); t
Section t on the 2-dimensional topological manifold M with values in the
 real vector bundle E of rank 2
>>> t.add_comp_by_continuation(fV, W, chart=c_uv)
>>> t.display(fV)
t = (1/4*u^2 - 1/4*v^2) (phi_V^*e_1) + (1/4*u^2 + 1/2*u*v + 1/4*v^2) (phi_V^*e_2)
>>> a = s + t; a
Section s+t on the 2-dimensional topological manifold M with values
 in the real vector bundle E of rank 2
>>> a.display(fU)
s+t = (x^2 + x) (phi_U^*e_1) + (phi_U^*e_2)
>>> a.display(fV)
s+t = (1/2*u + 1/2*v) (phi_V^*e_1) + (1/8*u^3 + 1/8*(3*u + 2)*v^2
      + 1/8*v^3 + 1/4*u^2 + 1/8*(3*u^2 + 4*u)*v) (phi_V^*e_2)
```

and multiplied by scalar fields:

```
sage: f = M.scalar_field(y^2-x^2, name='f')
sage: f.add_expr_by_continuation(c_uv, W)
sage: f.display()
f: M → ℝ
on U: (x, y) ↦ -x^2 + y^2
on V: (u, v) ↦ -u*v
sage: b = f*s; b
Section f*s on the 2-dimensional topological manifold M with values
 in the real vector bundle E of rank 2
sage: b.display(fU)
f*s = (-x^4 + x^2*y^2) (phi_U^*e_1) + (x^2*y - y^3 - x^2 + y^2) (phi_U^*e_2)
sage: b.display(fV)
f*s = (-1/4*u*v^3 - 1/2*u*v^2 + 1/4*(u^3 - 2*u^2)*v) (phi_V^*e_1)
      + (-1/8*u^4*v - 3/8*u^3*v^2 - 3/8*u^2*v^3 - 1/8*u*v^4) (phi_V^*e_2)
```

```
>>> from sage.all import *
>>> f = M.scalar_field(y**Integer(2)-x**Integer(2), name='f')
>>> f.add_expr_by_continuation(c_uv, W)
>>> f.display()
```

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```
f: M → ℝ
on U: (x, y) ↦ -x^2 + y^2
on V: (u, v) ↦ -u*v
>>> b = f*s; b
Section f*s on the 2-dimensional topological manifold M with values
 in the real vector bundle E of rank 2
>>> b.display(fU)
f*s = (-x^4 + x^2*y^2) (phi_U^*e_1) + (x^2*y - y^3 - x^2 + y^2) (phi_U^*e_2)
>>> b.display(fV)
f*s = (-1/4*u*v^3 - 1/2*u*v^2 + 1/4*(u^3 - 2*u^2)*v) (phi_V^*e_1)
      + (-1/8*u^4*v - 3/8*u^3*v^2 - 3/8*u^2*v^3 - 1/8*u*v^4) (phi_V^*e_2)
```

The domain on which the section should be defined, can be stated via the `domain` option in *section()*:

```
sage: cU = E.section([1,x], domain=U, name='c'); cU
Section c on the Open subset U of the 2-dimensional topological manifold
 M with values in the real vector bundle E of rank 2
sage: cU.display()
c = (phi_U^*e_1) + x (phi_U^*e_2)
```

```
>>> from sage.all import *
>>> cU = E.section([Integer(1),x], domain=U, name='c'); cU
Section c on the Open subset U of the 2-dimensional topological manifold
 M with values in the real vector bundle E of rank 2
>>> cU.display()
c = (phi_U^*e_1) + x (phi_U^*e_2)
```

Since $E|_U$ is trivial, `cU` now belongs to the free module:

```
sage: isinstance(cU.parent(), FiniteRankFreeModule)
True
```

```
>>> from sage.all import *
>>> isinstance(cU.parent(), FiniteRankFreeModule)
True
```

Omitting the `domain` option, the section is defined on the whole base space:

```
sage: c = E.section(name='c'); c
Section c on the 2-dimensional topological manifold M with values in the
 real vector bundle E of rank 2
```

```
>>> from sage.all import *
>>> c = E.section(name='c'); c
Section c on the 2-dimensional topological manifold M with values in the
 real vector bundle E of rank 2
```

Via *set_restriction()*, `cU` can be defined as the restriction of `c` to U:

```
sage: c.set_restriction(cU)
sage: c.display(fU)
c = (phi_U^*e_1) + x (phi_U^*e_2)
```

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```
sage: c.restrict(U) == cU
True
```

```
>>> from sage.all import *
>>> c.set_restriction(cU)
>>> c.display(fU)
c = (phi_U^*e_1) + x (phi_U^*e_2)
>>> c.restrict(U) == cU
True
```

Notice that the zero section is immutable, and therefore its components cannot be changed:

```
sage: zer = E.section_module().zero()
sage: zer.is_immutable()
True
sage: zer.set_comp()
Traceback (most recent call last):
...
ValueError: the components of an immutable element cannot be
 changed
```

```
>>> from sage.all import *
>>> zer = E.section_module().zero()
>>> zer.is_immutable()
True
>>> zer.set_comp()
Traceback (most recent call last):
...
ValueError: the components of an immutable element cannot be
 changed
```

Other sections can be declared immutable, too:

```
sage: c.is_immutable()
False
sage: c.set_immutable()
sage: c.is_immutable()
True
sage: c.set_comp()
Traceback (most recent call last):
...
ValueError: the components of an immutable element cannot be
 changed
sage: c.set_name('b')
Traceback (most recent call last):
...
ValueError: the name of an immutable element cannot be changed
```

```
>>> from sage.all import *
>>> c.is_immutable()
False
>>> c.set_immutable()
```

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```
>>> c.is_immutable()
True
>>> c.set_comp()
Traceback (most recent call last):
...
ValueError: the components of an immutable element cannot be
 changed
>>> c.set_name('b')
Traceback (most recent call last):
...
ValueError: the name of an immutable element cannot be changed
```

add_comp(*basis=None*)

Return the components of `self` in a given local frame for assignment.

The components with respect to other frames having the same domain as the provided local frame are kept. To delete them, use the method *set_comp()* instead.

INPUT:

- `basis` – (default: `None`) local frame in which the components are defined; if `None`, the components are assumed to refer to the section domain's default frame

OUTPUT:

- components in the given frame, as a `Components`; if such components did not exist previously, they are created

EXAMPLES:

```
sage: S2 = Manifold(2, 'S^2', structure='top', start_index=1)
sage: U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the
↪North and South pole, respectively
sage: S2.declare_union(U,V)
sage: stereoN.<x,y> = U.chart() # stereographic coordinates from the North
↪pole
sage: stereoS.<u,v> = V.chart() # stereographic coordinates from the South
↪pole
sage: xy_to_uv = stereoN.transition_map(stereoS, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: W = U.intersection(V)
sage: uv_to_xy = xy_to_uv.inverse()
sage: E = S2.vector_bundle(2, 'E') # define vector bundle
sage: phi_U = E.trivialization('phi_U', domain=U) # define trivializations
sage: phi_V = E.trivialization('phi_V', domain=V)
sage: transf = phi_U.transition_map(phi_V, [[0,1],[1,0]])
sage: fN = phi_U.frame(); fS = phi_V.frame() # get induced frames
sage: s = E.section(name='s')
sage: s.add_comp(fS)
1-index components w.r.t. Trivialization frame (E|_V, ((phi_V^*e_1),(phi_V^*e_
↪2)))
sage: s.add_comp(fS)[1] = u+v
sage: s.display(fS)
s = (u + v) (phi_V^*e_1)
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2', structure='top', start_index=Integer(1))
>>> U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the␣
↪North and South pole, respectively
>>> S2.declare_union(U,V)
>>> stereoN = U.chart(names=('x', 'y',)); (x, y,) = stereoN._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> stereoS = V.chart(names=('u', 'v',)); (u, v,) = stereoS._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> xy_to_uv = stereoN.transition_map(stereoS, (x/
↪(x**Integer(2)+y**Integer(2)), y/(x**Integer(2)+y**Integer(2))),
...                intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> W = U.intersection(V)
>>> uv_to_xy = xy_to_uv.inverse()
>>> E = S2.vector_bundle(Integer(2), 'E') # define vector bundle
>>> phi_U = E.trivialization('phi_U', domain=U) # define trivializations
>>> phi_V = E.trivialization('phi_V', domain=V)
>>> transf = phi_U.transition_map(phi_V, [[Integer(0),Integer(1)],[Integer(1),
↪Integer(0)]])
>>> fN = phi_U.frame(); fS = phi_V.frame() # get induced frames
>>> s = E.section(name='s')
>>> s.add_comp(fS)
1-index components w.r.t. Trivialization frame (E|_V, ((phi_V^*e_1),(phi_V^*e_
↪2)))
>>> s.add_comp(fS)[Integer(1)] = u+v
>>> s.display(fS)
s = (u + v) (phi_V^*e_1)
```

Setting the components in a new frame:

```
sage: e = E.local_frame('e', domain=V)
sage: s.add_comp(e)
1-index components w.r.t. Local frame (E|_V, (e_1,e_2))
sage: s.add_comp(e)[1] = u*v
sage: s.display(e)
s = u*v e_1
```

```
>>> from sage.all import *
>>> e = E.local_frame('e', domain=V)
>>> s.add_comp(e)
1-index components w.r.t. Local frame (E|_V, (e_1,e_2))
>>> s.add_comp(e)[Integer(1)] = u*v
>>> s.display(e)
s = u*v e_1
```

The components with respect to `fS` are kept:

```
sage: s.display(fS)
s = (u + v) (phi_V^*e_1)
```

```
>>> from sage.all import *
>>> s.display(fS)
s = (u + v) (phi_V^*e_1)
```

add_comp_by_continuation(*frame*, *subdomain*, *chart=None*)

Set components with respect to a local frame by continuation of the coordinate expression of the components in a subframe.

The continuation is performed by demanding that the components have the same coordinate expression as those on the restriction of the frame to a given subdomain.

INPUT:

- `frame` – local frame e in which the components are to be set
- `subdomain` – open subset of e's domain in which the components are known or can be evaluated from other components
- `chart` – (default: `None`) coordinate chart on e's domain in which the extension of the expression of the components is to be performed; if `None`, the default's chart of e's domain is assumed

EXAMPLES:

Components of a vector field on the sphere S^2:

```
sage: S2 = Manifold(2, 'S^2', structure='top', start_index=1)
sage: U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the␣
↪North and South pole, respectively
sage: S2.declare_union(U,V)
sage: stereoN.<x,y> = U.chart() # stereographic coordinates from the North␣
↪pole
sage: stereoS.<u,v> = V.chart() # stereographic coordinates from the South␣
↪pole
sage: xy_to_uv = stereoN.transition_map(stereoS,
....:                                    (x/(x^2+y^2), y/(x^2+y^2)),
....:                                    intersection_name='W',
....:                                    restrictions1= x^2+y^2!=0,
....:                                    restrictions2= u^2+v^2!=0)
sage: W = U.intersection(V)
sage: uv_to_xy = xy_to_uv.inverse()
sage: E = S2.vector_bundle(2, 'E') # define vector bundle
sage: phi_U = E.trivialization('phi_U', domain=U) # define trivializations
sage: phi_V = E.trivialization('phi_V', domain=V)
sage: transf = phi_U.transition_map(phi_V, [[0,1],[1,0]])
sage: fN = phi_U.frame(); fS = phi_V.frame() # get induced frames
sage: a = E.section({fN: [x, 2+y]}, name='a')
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2', structure='top', start_index=Integer(1))
>>> U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the␣
↪North and South pole, respectively
>>> S2.declare_union(U,V)
>>> stereoN = U.chart(names=('x', 'y',)); (x, y,) = stereoN._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> stereoS = V.chart(names=('u', 'v',)); (u, v,) = stereoS._first_ngens(2)#␣
↪stereographic coordinates from the South pole
```

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```
>>> xy_to_uv = stereoN.transition_map(stereoS,
...                                  (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                                  intersection_name='W',
...                                  restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                                  restrictions2=␣
↪u**Integer(2)+v**Integer(2)!=Integer(0))
>>> W = U.intersection(V)
>>> uv_to_xy = xy_to_uv.inverse()
>>> E = S2.vector_bundle(Integer(2), 'E') # define vector bundle
>>> phi_U = E.trivialization('phi_U', domain=U) # define trivializations
>>> phi_V = E.trivialization('phi_V', domain=V)
>>> transf = phi_U.transition_map(phi_V, [[Integer(0),Integer(1)],[Integer(1),
↪Integer(0)]])
>>> fN = phi_U.frame(); fS = phi_V.frame() # get induced frames
>>> a = E.section({fN: [x, Integer(2)+y]}, name='a')
```

At this stage, the section has been defined only on the open subset `U` (through its components in the frame `fN`):

```
sage: a.display(fN)
a = x (phi_U^*e_1) + (y + 2) (phi_U^*e_2)
```

```
>>> from sage.all import *
>>> a.display(fN)
a = x (phi_U^*e_1) + (y + 2) (phi_U^*e_2)
```

The components with respect to the restriction of `fS` to the common subdomain `W`, in terms of the `(u,v)` coordinates, are obtained by a change-of-frame formula on `W`:

```
sage: a.display(fS.restrict(W), stereoS.restrict(W))
a = (2*u^2 + 2*v^2 + v)/(u^2 + v^2) (phi_V^*e_1) + u/(u^2 + v^2)
 (phi_V^*e_2)
```

```
>>> from sage.all import *
>>> a.display(fS.restrict(W), stereoS.restrict(W))
a = (2*u^2 + 2*v^2 + v)/(u^2 + v^2) (phi_V^*e_1) + u/(u^2 + v^2)
 (phi_V^*e_2)
```

The continuation consists in extending the definition of the vector field to the whole open subset `V` by demanding that the components in the frame eV have the same coordinate expression as the above one:

```
sage: a.add_comp_by_continuation(fS, W, chart=stereoS)
```

```
>>> from sage.all import *
>>> a.add_comp_by_continuation(fS, W, chart=stereoS)
```

We have then:

```
sage: a.display(fS)
a = (2*u^2 + 2*v^2 + v)/(u^2 + v^2) (phi_V^*e_1) + u/(u^2 + v^2)
```

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```
(phi_V^*e_2)
```

```
>>> from sage.all import *
>>> a.display(fS)
a = (2*u^2 + 2*v^2 + v)/(u^2 + v^2) (phi_V^*e_1) + u/(u^2 + v^2)
 (phi_V^*e_2)
```

and a is defined on the entire manifold S^2.

add_expr_from_subdomain(*frame*, *subdomain*)

Add an expression to an existing component from a subdomain.

INPUT:

- `frame` – local frame e in which the components are to be set
- `subdomain` – open subset of e's domain in which the components have additional expressions

EXAMPLES:

We are going to consider a section on the trivial rank 2 vector bundle over the 2-sphere:

```
sage: S2 = Manifold(2, 'S^2', structure='top', start_index=1)
sage: U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the␣
↪North and South pole, respectively
sage: S2.declare_union(U,V)
sage: stereoN.<x,y> = U.chart() # stereographic coordinates from the North␣
↪pole
sage: stereoS.<u,v> = V.chart() # stereographic coordinates from the South␣
↪pole
sage: xy_to_uv = stereoN.transition_map(stereoS,
....:                (x/(x^2+y^2), y/(x^2+y^2)),
....:                intersection_name='W', restrictions1= x^2+y^2!=0,
....:                restrictions2= u^2+v^2!=0)
sage: W = U.intersection(V)
sage: uv_to_xy = xy_to_uv.inverse()
sage: E = S2.vector_bundle(2, 'E') # define vector bundle
sage: e = E.local_frame('e') # frame to trivialize E
sage: eU = e.restrict(U); eV = e.restrict(V); eW = e.restrict(W) # this step␣
↪is essential since U, V and W must be trivial
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2', structure='top', start_index=Integer(1))
>>> U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the␣
↪North and South pole, respectively
>>> S2.declare_union(U,V)
>>> stereoN = U.chart(names=('x', 'y',)); (x, y,) = stereoN._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> stereoS = V.chart(names=('u', 'v',)); (u, v,) = stereoS._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> xy_to_uv = stereoN.transition_map(stereoS,
...                (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                intersection_name='W', restrictions1=␣
```

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```
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                 restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> W = U.intersection(V)
>>> uv_to_xy = xy_to_uv.inverse()
>>> E = S2.vector_bundle(Integer(2), 'E') # define vector bundle
>>> e = E.local_frame('e') # frame to trivialize E
>>> eU = e.restrict(U); eV = e.restrict(V); eW = e.restrict(W) # this step is␣
↪essential since U, V and W must be trivial
```

To define a section `s` on S^2, we first set the components on `U`:

```
sage: s = E.section(name='s')
sage: sU = s.restrict(U)
sage: sU[:] = [x, y]
```

```
>>> from sage.all import *
>>> s = E.section(name='s')
>>> sU = s.restrict(U)
>>> sU[:] = [x, y]
```

But because `E` is trivial, these components can be extended with respect to the global frame `e` onto S^2:

```
sage: s.add_comp_by_continuation(e, U)
```

```
>>> from sage.all import *
>>> s.add_comp_by_continuation(e, U)
```

One can see that `s` is not yet fully defined: the components (scalar fields) do not have values on the whole manifold:

```
sage: sorted(s._components.values())[0]._comp[(1,)].display()
S^2 → R
on U: (x, y) ↦ x
on W: (u, v) ↦ u/(u^2 + v^2)
```

```
>>> from sage.all import *
>>> sorted(s._components.values())[Integer(0)]._comp[(Integer(1),)].display()
S^2 → R
on U: (x, y) ↦ x
on W: (u, v) ↦ u/(u^2 + v^2)
```

To fix that, we extend the components from `W` to `V` first, using `add_comp_by_continuation()`:

```
sage: s.add_comp_by_continuation(eV, W, stereoS)
```

```
>>> from sage.all import *
>>> s.add_comp_by_continuation(eV, W, stereoS)
```

Then, the expression on the subdomain `V` is added to the components on S^2 already known by:

```
sage: s.add_expr_from_subdomain(e, V)
```

```
>>> from sage.all import *
>>> s.add_expr_from_subdomain(e, V)
```

The definition of `s` is now complete:

```
sage: sorted(s._components.values())[0]._comp[(2,)].display()
S^2 → ℝ
on U: (x, y) ↦ y
on V: (u, v) ↦ v/(u^2 + v^2)
```

```
>>> from sage.all import *
>>> sorted(s._components.values())[Integer(0)]._comp[(Integer(2),)].display()
S^2 → ℝ
on U: (x, y) ↦ y
on V: (u, v) ↦ v/(u^2 + v^2)
```

at(*point*)

Value of `self` at a point of its domain.

If the current section is

$$s : U \longrightarrow E,$$

then for any point $p \in U$, $s(p)$ is a vector in the fiber E_p of E at p.

INPUT:

- `point` – *ManifoldPoint*; point p in the domain of the section U

OUTPUT:

- *VectorBundleFiberElement* representing the vector $s(p)$ in the fiber E_p of E at p.

EXAMPLES:

Vector on a rank 2 vector bundle fiber over a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='top')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y),
....:                    intersection_name='W', restrictions1= x>0,
....:                    restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: E = M.vector_bundle(2, 'E') # define vector bundle
sage: phi_U = E.trivialization('phi_U', domain=U) # define trivializations
sage: phi_V = E.trivialization('phi_V', domain=V)
sage: transf = phi_U.transition_map(phi_V, [[0,x],[x,0]])
sage: fU = phi_U.frame(); fV = phi_V.frame() # get induced frames
sage: s = E.section({fU: [1+y, x]}, name='s')
sage: s.add_comp_by_continuation(fV, W, chart=c_uv)
sage: s.display(fU)
s = (y + 1) (phi_U^*e_1) + x (phi_U^*e_2)
sage: s.display(fV)
```

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```
s = (1/4*u^2 + 1/2*u*v + 1/4*v^2) (phi_V^*e_1) + (1/4*u^2 - 1/4*v^2
 + 1/2*u + 1/2*v) (phi_V^*e_2)
sage: p = M.point((2,3), chart=c_xy, name='p')
sage: sp = s.at(p) ; sp
Vector s in the fiber of E at Point p on the 2-dimensional
 topological manifold M
sage: sp.parent()
Fiber of E at Point p on the 2-dimensional topological manifold M
sage: sp.display(fU.at(p))
s = 4 (phi_U^*e_1) + 2 (phi_U^*e_2)
sage: sp.display(fV.at(p))
s = 4 (phi_V^*e_1) + 8 (phi_V^*e_2)
sage: p.coord(c_uv) # to check the above expression
(5, -1)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> transf = c_xy.transition_map(c_uv, (x+y, x-y),
...                    intersection_name='W', restrictions1= x>Integer(0),
...                    restrictions2= u+v>Integer(0))
>>> inv = transf.inverse()
>>> W = U.intersection(V)
>>> E = M.vector_bundle(Integer(2), 'E') # define vector bundle
>>> phi_U = E.trivialization('phi_U', domain=U) # define trivializations
>>> phi_V = E.trivialization('phi_V', domain=V)
>>> transf = phi_U.transition_map(phi_V, [[Integer(0),x],[x,Integer(0)]])
>>> fU = phi_U.frame(); fV = phi_V.frame() # get induced frames
>>> s = E.section({fU: [Integer(1)+y, x]}, name='s')
>>> s.add_comp_by_continuation(fV, W, chart=c_uv)
>>> s.display(fU)
s = (y + 1) (phi_U^*e_1) + x (phi_U^*e_2)
>>> s.display(fV)
s = (1/4*u^2 + 1/2*u*v + 1/4*v^2) (phi_V^*e_1) + (1/4*u^2 - 1/4*v^2
 + 1/2*u + 1/2*v) (phi_V^*e_2)
>>> p = M.point((Integer(2),Integer(3)), chart=c_xy, name='p')
>>> sp = s.at(p) ; sp
Vector s in the fiber of E at Point p on the 2-dimensional
 topological manifold M
>>> sp.parent()
Fiber of E at Point p on the 2-dimensional topological manifold M
>>> sp.display(fU.at(p))
s = 4 (phi_U^*e_1) + 2 (phi_U^*e_2)
>>> sp.display(fV.at(p))
s = 4 (phi_V^*e_1) + 8 (phi_V^*e_2)
>>> p.coord(c_uv) # to check the above expression
(5, -1)
```

base_module()

Return the section module on which `self` acts as a section.

OUTPUT:

- instance of `SectionModule`

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: U = M.open_subset('U')
sage: E = M.vector_bundle(2, 'E')
sage: s = E.section(domain=U)
sage: s.base_module()
Module C^0(U;E) of sections on the Open subset U of the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> U = M.open_subset('U')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> s = E.section(domain=U)
>>> s.base_module()
Module C^0(U;E) of sections on the Open subset U of the
 3-dimensional topological manifold M with values in the real vector
 bundle E of rank 2
```

comp(*basis=None*, *from_basis=None*)

Return the components in a given local frame.

If the components are not known already, they are computed by the change-of-basis formula from components in another local frame.

INPUT:

- `basis` – (default: `None`) local frame in which the components are required; if none is provided, the components are assumed to refer to the section module's default frame on the corresponding domain
- `from_basis` – (default: `None`) local frame from which the required components are computed, via the change-of-basis formula, if they are not known already in the basis `basis`

OUTPUT:

- components in the local frame `basis`, as a `Components`

EXAMPLES:

Components of a section defined on a rank 2 vector bundle over two open subsets:

```
sage: M = Manifold(2, 'M', structure='top')
sage: X.<x, y> = M.chart()
sage: U = M.open_subset('U'); V = M.open_subset('V')
sage: M.declare_union(U, V)
sage: XU = X.restrict(U); XV = X.restrict(V)
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e', domain=U); e
Local frame (E|_U, (e_0,e_1))
sage: f = E.local_frame('f', domain=V); f
Local frame (E|_V, (f_0,f_1))
```

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```
sage: s = E.section(name='s')
sage: s[e,:] = - x + y^3, 2+x
sage: s[f,0] = x^2
sage: s[f,1] = x+y
sage: s.comp(e)
1-index components w.r.t. Local frame (E|_U, (e_0,e_1))
sage: s.comp(e)[:]
[y^3 - x, x + 2]
sage: s.comp(f)
1-index components w.r.t. Local frame (E|_V, (f_0,f_1))
sage: s.comp(f)[:]
[x^2, x + y]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> U = M.open_subset('U'); V = M.open_subset('V')
>>> M.declare_union(U, V)
>>> XU = X.restrict(U); XV = X.restrict(V)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e', domain=U); e
Local frame (E|_U, (e_0,e_1))
>>> f = E.local_frame('f', domain=V); f
Local frame (E|_V, (f_0,f_1))
>>> s = E.section(name='s')
>>> s[e,:] = - x + y**Integer(3), Integer(2)+x
>>> s[f,Integer(0)] = x**Integer(2)
>>> s[f,Integer(1)] = x+y
>>> s.comp(e)
1-index components w.r.t. Local frame (E|_U, (e_0,e_1))
>>> s.comp(e)[:]
[y^3 - x, x + 2]
>>> s.comp(f)
1-index components w.r.t. Local frame (E|_V, (f_0,f_1))
>>> s.comp(f)[:]
[x^2, x + y]
```

Since `e` is the default frame of `E|_U`, the argument `e` can be omitted after restricting:

```
sage: e is E.section_module(domain=U).default_frame()
True
sage: s.restrict(U).comp() is s.comp(e)
True
```

```
>>> from sage.all import *
>>> e is E.section_module(domain=U).default_frame()
True
>>> s.restrict(U).comp() is s.comp(e)
True
```

copy(*name=None*, *latex_name=None*)

Return an exact copy of `self`.

INPUT:

- `name` – (default: `None`) name given to the copy
- `latex_name` – (default: `None`) LaTeX symbol to denote the copy; if none is provided, the LaTeX symbol is set to `name`

Note

The name and the derived quantities are not copied.

EXAMPLES:

Copy of a section on a rank 2 vector bundle over a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='top')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                     intersection_name='W', restrictions1= x>0,
....:                     restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: E = M.vector_bundle(2, 'E') # define vector bundle
sage: phi_U = E.trivialization('phi_U', domain=U) # define trivializations
sage: phi_V = E.trivialization('phi_V', domain=V)
sage: transf = phi_U.transition_map(phi_V, [[0,x],[x,0]])
sage: fU = phi_U.frame(); fV = phi_V.frame()
sage: s = E.section(name='s')
sage: s[fU,:] = [2, 1-y]
sage: s.add_comp_by_continuation(fV, U.intersection(V), c_uv)
sage: t = s.copy(); t
Section on the 2-dimensional topological manifold M with values in
 the real vector bundle E of rank 2
sage: t.display(fU)
2 (phi_U^*e_1) + (-y + 1) (phi_U^*e_2)
sage: t == s
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
...                     intersection_name='W', restrictions1= x>Integer(0),
...                     restrictions2= u+v>Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V)
>>> E = M.vector_bundle(Integer(2), 'E') # define vector bundle
>>> phi_U = E.trivialization('phi_U', domain=U) # define trivializations
>>> phi_V = E.trivialization('phi_V', domain=V)
```

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```
>>> transf = phi_U.transition_map(phi_V, [[Integer(0),x],[x,Integer(0)]])
>>> fU = phi_U.frame(); fV = phi_V.frame()
>>> s = E.section(name='s')
>>> s[fU,:] = [Integer(2), Integer(1)-y]
>>> s.add_comp_by_continuation(fV, U.intersection(V), c_uv)
>>> t = s.copy(); t
Section on the 2-dimensional topological manifold M with values in
 the real vector bundle E of rank 2
>>> t.display(fU)
2 (phi_U^*e_1) + (-y + 1) (phi_U^*e_2)
>>> t == s
True
```

If the original section is modified, the copy is not:

```
sage: s[fU,0] = -1
sage: s.display(fU)
s = -(phi_U^*e_1) + (-y + 1) (phi_U^*e_2)
sage: t.display(fU)
2 (phi_U^*e_1) + (-y + 1) (phi_U^*e_2)
sage: t == s
False
```

```
>>> from sage.all import *
>>> s[fU,Integer(0)] = -Integer(1)
>>> s.display(fU)
s = -(phi_U^*e_1) + (-y + 1) (phi_U^*e_2)
>>> t.display(fU)
2 (phi_U^*e_1) + (-y + 1) (phi_U^*e_2)
>>> t == s
False
```

copy_from(*other*)

Make `self` a copy of `other`.

INPUT:

- `other` – other section, in the same module as `self`

Note

While the derived quantities are not copied, the name is kept.

Warning

All previous defined components and restrictions will be deleted!

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                    intersection_name='W', restrictions1= x>0,
....:                    restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: E = M.vector_bundle(2, 'E') # define vector bundle
sage: phi_U = E.trivialization('phi_U', domain=U) # define trivializations
sage: phi_V = E.trivialization('phi_V', domain=V)
sage: transf = phi_U.transition_map(phi_V, [[0,x],[x,0]])
sage: fU = phi_U.frame(); fV = phi_V.frame()
sage: s = E.section(name='s')
sage: s[fU,:] = [2, 1-y]
sage: s.add_comp_by_continuation(fV, U.intersection(V), c_uv)
sage: t = E.section(name='t')
sage: t.copy_from(s)
sage: t.display(fU)
t = 2 (phi_U^*e_1) + (-y + 1) (phi_U^*e_2)
sage: s == t
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
...                    intersection_name='W', restrictions1= x>Integer(0),
...                    restrictions2= u+v>Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V)
>>> E = M.vector_bundle(Integer(2), 'E') # define vector bundle
>>> phi_U = E.trivialization('phi_U', domain=U) # define trivializations
>>> phi_V = E.trivialization('phi_V', domain=V)
>>> transf = phi_U.transition_map(phi_V, [[Integer(0),x],[x,Integer(0)]])
>>> fU = phi_U.frame(); fV = phi_V.frame()
>>> s = E.section(name='s')
>>> s[fU,:] = [Integer(2), Integer(1)-y]
>>> s.add_comp_by_continuation(fV, U.intersection(V), c_uv)
>>> t = E.section(name='t')
>>> t.copy_from(s)
>>> t.display(fU)
t = 2 (phi_U^*e_1) + (-y + 1) (phi_U^*e_2)
>>> s == t
True
```

If the original section is modified, the copy is not:

```
sage: s[fU,0] = -1
sage: s.display(fU)
s = -(phi_U^*e_1) + (-y + 1) (phi_U^*e_2)
sage: t.display(fU)
t = 2 (phi_U^*e_1) + (-y + 1) (phi_U^*e_2)
sage: s == t
False
```

```
>>> from sage.all import *
>>> s[fU,Integer(0)] = -Integer(1)
>>> s.display(fU)
s = -(phi_U^*e_1) + (-y + 1) (phi_U^*e_2)
>>> t.display(fU)
t = 2 (phi_U^*e_1) + (-y + 1) (phi_U^*e_2)
>>> s == t
False
```

disp(*frame=None*, *chart=None*)

Display the section in terms of its expansion with respect to a given local frame.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- `frame` – (default: `None`) local frame with respect to which the section is expanded; if `frame` is `None` and `chart` is not `None`, the default frame in the corresponding section module is assumed
- `chart` – (default: `None`) chart with respect to which the components of the section in the selected frame are expressed; if `None`, the default chart of the local frame domain is assumed

EXAMPLES:

Display of section on a rank 2 vector bundle over the 2-sphere:

```
sage: S2 = Manifold(2, 'S^2', structure='top', start_index=1)
sage: U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the
↪North and South pole, respectively
sage: S2.declare_union(U,V)
sage: stereoN.<x,y> = U.chart() # stereographic coordinates from the North
↪pole
sage: stereoS.<u,v> = V.chart() # stereographic coordinates from the South
↪pole
sage: xy_to_uv = stereoN.transition_map(stereoS,
....:                                     (x/(x^2+y^2), y/(x^2+y^2)),
....:                                     intersection_name='W',
....:                                     restrictions1= x^2+y^2!=0,
....:                                     restrictions2= u^2+v^2!=0)
sage: W = U.intersection(V)
sage: uv_to_xy = xy_to_uv.inverse()
sage: E = S2.vector_bundle(2, 'E') # define vector bundle
sage: phi_U = E.trivialization('phi_U', domain=U) # define trivializations
sage: phi_V = E.trivialization('phi_V', domain=V)
sage: transf = phi_U.transition_map(phi_V, [[0,1],[1,0]])
sage: fN = phi_U.frame(); fS = phi_V.frame() # get induced frames
sage: s = E.section(name='s')
```

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```
sage: s[fN,:] = [x, y]
sage: s.add_comp_by_continuation(fS, W, stereoS)
sage: s.display(fN)
s = x (phi_U^*e_1) + y (phi_U^*e_2)
sage: s.display(fS)
s = v/(u^2 + v^2) (phi_V^*e_1) + u/(u^2 + v^2) (phi_V^*e_2)
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2', structure='top', start_index=Integer(1))
>>> U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the␣
↪North and South pole, respectively
>>> S2.declare_union(U,V)
>>> stereoN = U.chart(names=('x', 'y',)); (x, y,) = stereoN._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> stereoS = V.chart(names=('u', 'v',)); (u, v,) = stereoS._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> xy_to_uv = stereoN.transition_map(stereoS,
...                                   (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                                   intersection_name='W',
...                                   restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                                   restrictions2=␣
↪u**Integer(2)+v**Integer(2)!=Integer(0))
>>> W = U.intersection(V)
>>> uv_to_xy = xy_to_uv.inverse()
>>> E = S2.vector_bundle(Integer(2), 'E') # define vector bundle
>>> phi_U = E.trivialization('phi_U', domain=U) # define trivializations
>>> phi_V = E.trivialization('phi_V', domain=V)
>>> transf = phi_U.transition_map(phi_V, [[Integer(0),Integer(1)],[Integer(1),
↪Integer(0)]])
>>> fN = phi_U.frame(); fS = phi_V.frame() # get induced frames
>>> s = E.section(name='s')
>>> s[fN,:] = [x, y]
>>> s.add_comp_by_continuation(fS, W, stereoS)
>>> s.display(fN)
s = x (phi_U^*e_1) + y (phi_U^*e_2)
>>> s.display(fS)
s = v/(u^2 + v^2) (phi_V^*e_1) + u/(u^2 + v^2) (phi_V^*e_2)
```

Since `fN` is the default frame on `E|_U`, the argument `fN` can be omitted after restricting:

```
sage: fN is E.section_module(domain=U).default_frame()
True
sage: s.restrict(U).display()
s = x (phi_U^*e_1) + y (phi_U^*e_2)
```

```
>>> from sage.all import *
>>> fN is E.section_module(domain=U).default_frame()
True
>>> s.restrict(U).display()
s = x (phi_U^*e_1) + y (phi_U^*e_2)
```

Similarly, since `fS` is `V`'s default frame, the argument `fS` can be omitted when considering the restriction of `s` to `V`:

```
sage: s.restrict(V).display()
s = v/(u^2 + v^2) (phi_V^*e_1) + u/(u^2 + v^2) (phi_V^*e_2)
```

```
>>> from sage.all import *
>>> s.restrict(V).display()
s = v/(u^2 + v^2) (phi_V^*e_1) + u/(u^2 + v^2) (phi_V^*e_2)
```

The second argument comes into play whenever the frame's domain is covered by two distinct charts. Since `stereoN.restrict(W)` is the default chart on `W`, the second argument can be omitted for the expression in this chart:

```
sage: s.display(fS.restrict(W))
s = y (phi_V^*e_1) + x (phi_V^*e_2)
```

```
>>> from sage.all import *
>>> s.display(fS.restrict(W))
s = y (phi_V^*e_1) + x (phi_V^*e_2)
```

To get the expression in the other chart, the second argument must be used:

```
sage: s.display(fN.restrict(W), stereoS.restrict(W))
s = u/(u^2 + v^2) (phi_U^*e_1) + v/(u^2 + v^2) (phi_U^*e_2)
```

```
>>> from sage.all import *
>>> s.display(fN.restrict(W), stereoS.restrict(W))
s = u/(u^2 + v^2) (phi_U^*e_1) + v/(u^2 + v^2) (phi_U^*e_2)
```

One can ask for the display with respect to a frame in which `s` has not been initialized yet (this will automatically trigger the use of the change-of-frame formula for tensors):

```
sage: a = E.section_module(domain=U).automorphism()
sage: a[:] = [[1+x^2,0],[0,1+y^2]]
sage: e = fN.new_frame(a, 'e')
sage: [e[i].display() for i in S2.irange()]
[e_1 = (x^2 + 1) (phi_U^*e_1), e_2 = (y^2 + 1) (phi_U^*e_2)]
sage: s.display(e)
s = x/(x^2 + 1) e_1 + y/(y^2 + 1) e_2
```

```
>>> from sage.all import *
>>> a = E.section_module(domain=U).automorphism()
>>> a[:] = [[Integer(1)+x**Integer(2),Integer(0)],[Integer(0),
↪Integer(1)+y**Integer(2)]]
>>> e = fN.new_frame(a, 'e')
>>> [e[i].display() for i in S2.irange()]
[e_1 = (x^2 + 1) (phi_U^*e_1), e_2 = (y^2 + 1) (phi_U^*e_2)]
>>> s.display(e)
s = x/(x^2 + 1) e_1 + y/(y^2 + 1) e_2
```

A shortcut of `display()` is `disp()`:

```
sage: s.disp(fS)
s = v/(u^2 + v^2) (phi_V^*e_1) + u/(u^2 + v^2) (phi_V^*e_2)
```

```
>>> from sage.all import *
>>> s.disp(fS)
s = v/(u^2 + v^2) (phi_V^*e_1) + u/(u^2 + v^2) (phi_V^*e_2)
```

display(*frame=None*, *chart=None*)

Display the section in terms of its expansion with respect to a given local frame.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- `frame` – (default: `None`) local frame with respect to which the section is expanded; if `frame` is `None` and `chart` is not `None`, the default frame in the corresponding section module is assumed
- `chart` – (default: `None`) chart with respect to which the components of the section in the selected frame are expressed; if `None`, the default chart of the local frame domain is assumed

EXAMPLES:

Display of section on a rank 2 vector bundle over the 2-sphere:

```
sage: S2 = Manifold(2, 'S^2', structure='top', start_index=1)
sage: U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the␣
↪North and South pole, respectively
sage: S2.declare_union(U,V)
sage: stereoN.<x,y> = U.chart() # stereographic coordinates from the North␣
↪pole
sage: stereoS.<u,v> = V.chart() # stereographic coordinates from the South␣
↪pole
sage: xy_to_uv = stereoN.transition_map(stereoS,
....:                                   (x/(x^2+y^2), y/(x^2+y^2)),
....:                                   intersection_name='W',
....:                                   restrictions1= x^2+y^2!=0,
....:                                   restrictions2= u^2+v^2!=0)
sage: W = U.intersection(V)
sage: uv_to_xy = xy_to_uv.inverse()
sage: E = S2.vector_bundle(2, 'E') # define vector bundle
sage: phi_U = E.trivialization('phi_U', domain=U) # define trivializations
sage: phi_V = E.trivialization('phi_V', domain=V)
sage: transf = phi_U.transition_map(phi_V, [[0,1],[1,0]])
sage: fN = phi_U.frame(); fS = phi_V.frame() # get induced frames
sage: s = E.section(name='s')
sage: s[fN,:] = [x, y]
sage: s.add_comp_by_continuation(fS, W, stereoS)
sage: s.display(fN)
s = x (phi_U^*e_1) + y (phi_U^*e_2)
sage: s.display(fS)
s = v/(u^2 + v^2) (phi_V^*e_1) + u/(u^2 + v^2) (phi_V^*e_2)
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2', structure='top', start_index=Integer(1))
>>> U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the␣
```

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```
↪North and South pole, respectively
>>> S2.declare_union(U,V)
>>> stereoN = U.chart(names=('x', 'y',)); (x, y,) = stereoN._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> stereoS = V.chart(names=('u', 'v',)); (u, v,) = stereoS._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> xy_to_uv = stereoN.transition_map(stereoS,
...                                    (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                                    intersection_name='W',
...                                    restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                                    restrictions2=␣
↪u**Integer(2)+v**Integer(2)!=Integer(0))
>>> W = U.intersection(V)
>>> uv_to_xy = xy_to_uv.inverse()
>>> E = S2.vector_bundle(Integer(2), 'E') # define vector bundle
>>> phi_U = E.trivialization('phi_U', domain=U) # define trivializations
>>> phi_V = E.trivialization('phi_V', domain=V)
>>> transf = phi_U.transition_map(phi_V, [[Integer(0),Integer(1)],[Integer(1),
↪Integer(0)]])
>>> fN = phi_U.frame(); fS = phi_V.frame() # get induced frames
>>> s = E.section(name='s')
>>> s[fN,:] = [x, y]
>>> s.add_comp_by_continuation(fS, W, stereoS)
>>> s.display(fN)
s = x (phi_U^*e_1) + y (phi_U^*e_2)
>>> s.display(fS)
s = v/(u^2 + v^2) (phi_V^*e_1) + u/(u^2 + v^2) (phi_V^*e_2)
```

Since `fN` is the default frame on `E|_U`, the argument `fN` can be omitted after restricting:

```
sage: fN is E.section_module(domain=U).default_frame()
True
sage: s.restrict(U).display()
s = x (phi_U^*e_1) + y (phi_U^*e_2)
```

```
>>> from sage.all import *
>>> fN is E.section_module(domain=U).default_frame()
True
>>> s.restrict(U).display()
s = x (phi_U^*e_1) + y (phi_U^*e_2)
```

Similarly, since `fS` is `V`'s default frame, the argument `fS` can be omitted when considering the restriction of `s` to `V`:

```
sage: s.restrict(V).display()
s = v/(u^2 + v^2) (phi_V^*e_1) + u/(u^2 + v^2) (phi_V^*e_2)
```

```
>>> from sage.all import *
>>> s.restrict(V).display()
s = v/(u^2 + v^2) (phi_V^*e_1) + u/(u^2 + v^2) (phi_V^*e_2)
```

The second argument comes into play whenever the frame's domain is covered by two distinct charts. Since `stereoN.restrict(W)` is the default chart on `W`, the second argument can be omitted for the expression in this chart:

```
sage: s.display(fS.restrict(W))
s = y (phi_V^*e_1) + x (phi_V^*e_2)
```

```
>>> from sage.all import *
>>> s.display(fS.restrict(W))
s = y (phi_V^*e_1) + x (phi_V^*e_2)
```

To get the expression in the other chart, the second argument must be used:

```
sage: s.display(fN.restrict(W), stereoS.restrict(W))
s = u/(u^2 + v^2) (phi_U^*e_1) + v/(u^2 + v^2) (phi_U^*e_2)
```

```
>>> from sage.all import *
>>> s.display(fN.restrict(W), stereoS.restrict(W))
s = u/(u^2 + v^2) (phi_U^*e_1) + v/(u^2 + v^2) (phi_U^*e_2)
```

One can ask for the display with respect to a frame in which `s` has not been initialized yet (this will automatically trigger the use of the change-of-frame formula for tensors):

```
sage: a = E.section_module(domain=U).automorphism()
sage: a[:] = [[1+x^2,0],[0,1+y^2]]
sage: e = fN.new_frame(a, 'e')
sage: [e[i].display() for i in S2.irange()]
[e_1 = (x^2 + 1) (phi_U^*e_1), e_2 = (y^2 + 1) (phi_U^*e_2)]
sage: s.display(e)
s = x/(x^2 + 1) e_1 + y/(y^2 + 1) e_2
```

```
>>> from sage.all import *
>>> a = E.section_module(domain=U).automorphism()
>>> a[:] = [[Integer(1)+x**Integer(2),Integer(0)],[Integer(0),
↪Integer(1)+y**Integer(2)]]
>>> e = fN.new_frame(a, 'e')
>>> [e[i].display() for i in S2.irange()]
[e_1 = (x^2 + 1) (phi_U^*e_1), e_2 = (y^2 + 1) (phi_U^*e_2)]
>>> s.display(e)
s = x/(x^2 + 1) e_1 + y/(y^2 + 1) e_2
```

A shortcut of `display()` is `disp()`:

```
sage: s.disp(fS)
s = v/(u^2 + v^2) (phi_V^*e_1) + u/(u^2 + v^2) (phi_V^*e_2)
```

```
>>> from sage.all import *
>>> s.disp(fS)
s = v/(u^2 + v^2) (phi_V^*e_1) + u/(u^2 + v^2) (phi_V^*e_2)
```

display_comp(*frame=None*, *chart=None*, *only_nonzero=True*)

Display the section components with respect to a given frame, one per line.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- `frame` – (default: `None`) local frame with respect to which the section components are defined; if `None`, then the default frame on the section module is used
- `chart` – (default: `None`) chart specifying the coordinate expression of the components; if `None`, the default chart of the section domain is used
- `only_nonzero` – boolean (default: `True`); if `True`, only nonzero components are displayed

EXAMPLES:

Display of the components of a section defined on two open subsets:

```
sage: M = Manifold(2, 'M', structure='top')
sage: U = M.open_subset('U')
sage: c_xy.<x, y> = U.chart()
sage: V = M.open_subset('V')
sage: c_uv.<u, v> = V.chart()
sage: M.declare_union(U,V)   # M is the union of U and V
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e', domain=U)
sage: f = E.local_frame('f', domain=V)
sage: s = E.section(name='s')
sage: s[e,0] = - x + y^3
sage: s[e,1] = 2+x
sage: s[f,1] = - u*v
sage: s.display_comp(e)
s^0 = y^3 - x
s^1 = x + 2
sage: s.display_comp(f)
s^1 = -u*v
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> U = M.open_subset('U')
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> V = M.open_subset('V')
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> M.declare_union(U,V)   # M is the union of U and V
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e', domain=U)
>>> f = E.local_frame('f', domain=V)
>>> s = E.section(name='s')
>>> s[e,Integer(0)] = - x + y**Integer(3)
>>> s[e,Integer(1)] = Integer(2)+x
>>> s[f,Integer(1)] = - u*v
>>> s.display_comp(e)
s^0 = y^3 - x
s^1 = x + 2
>>> s.display_comp(f)
s^1 = -u*v
```

See documentation of `sage.manifolds.section.TrivialSection.display_comp()` for more options.

domain()

Return the manifold on which `self` is defined.

OUTPUT:

- instance of class *TopologicalManifold*

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: U = M.open_subset('U')
sage: E = M.vector_bundle(2, 'E')
sage: C0_U = E.section_module(domain=U, force_free=True)
sage: z = C0_U.zero()
sage: z.domain()
Open subset U of the 3-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> U = M.open_subset('U')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> C0_U = E.section_module(domain=U, force_free=True)
>>> z = C0_U.zero()
>>> z.domain()
Open subset U of the 3-dimensional topological manifold M
```

restrict(*subdomain*)

Return the restriction of `self` to some subdomain.

If the restriction has not been defined yet, it is constructed here.

INPUT:

- `subdomain` – *DifferentiableManifold*; open subset U of the section domain S

OUTPUT: *Section* representing the restriction

EXAMPLES:

Restrictions of a section on a rank 2 vector bundle over the 2-sphere:

```
sage: S2 = Manifold(2, 'S^2', structure='top', start_index=1)
sage: U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the␣
↪North and South pole, respectively
sage: S2.declare_union(U,V)
sage: stereoN.<x,y> = U.chart() # stereographic coordinates from the North␣
↪pole
sage: stereoS.<u,v> = V.chart() # stereographic coordinates from the South␣
↪pole
sage: xy_to_uv = stereoN.transition_map(stereoS,
....:                                   (x/(x^2+y^2), y/(x^2+y^2)),
....:                                   intersection_name='W',
....:                                   restrictions1= x^2+y^2!=0,
....:                                   restrictions2= u^2+v^2!=0)
sage: W = U.intersection(V)
sage: uv_to_xy = xy_to_uv.inverse()
sage: E = S2.vector_bundle(2, 'E') # define vector bundle
```

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```
sage: phi_U = E.trivialization('phi_U', domain=U) # define trivializations
sage: phi_V = E.trivialization('phi_V', domain=V)
sage: transf = phi_U.transition_map(phi_V, [[0,x],[y,0]])
sage: fN = phi_U.frame(); fS = phi_V.frame() # get induced frames
sage: fN_W = fN.restrict(W); fS_W = fS.restrict(W) # restrict them
sage: stereoN_W = stereoN.restrict(W) # restrict charts, too
sage: stereoS_W = stereoS.restrict(W)
sage: s = E.section({fN: [1, 0]}, name='s')
sage: s.display(fN)
s = (phi_U^*e_1)
sage: sU = s.restrict(U) ; sU
Section s on the Open subset U of the 2-dimensional topological
 manifold S^2 with values in the real vector bundle E of rank 2
sage: sU.display() # fN is the default frame on U
s = (phi_U^*e_1)
sage: sU == fN[1]
True
sage: sW = s.restrict(W) ; sW
Section s on the Open subset W of the 2-dimensional topological
 manifold S^2 with values in the real vector bundle E of rank 2
sage: sW.display(fN_W)
s = (phi_U^*e_1)
sage: sW.display(fS_W, stereoN_W)
s = y (phi_V^*e_2)
sage: sW.display(fS_W, stereoS_W)
s = v/(u^2 + v^2) (phi_V^*e_2)
sage: sW == fN_W[1]
True
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2', structure='top', start_index=Integer(1))
>>> U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the␣
↪North and South pole, respectively
>>> S2.declare_union(U,V)
>>> stereoN = U.chart(names=('x', 'y',)); (x, y,) = stereoN._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> stereoS = V.chart(names=('u', 'v',)); (u, v,) = stereoS._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> xy_to_uv = stereoN.transition_map(stereoS,
...                                    (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                                    intersection_name='W',
...                                    restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                                    restrictions2=␣
↪u**Integer(2)+v**Integer(2)!=Integer(0))
>>> W = U.intersection(V)
>>> uv_to_xy = xy_to_uv.inverse()
>>> E = S2.vector_bundle(Integer(2), 'E') # define vector bundle
>>> phi_U = E.trivialization('phi_U', domain=U) # define trivializations
>>> phi_V = E.trivialization('phi_V', domain=V)
>>> transf = phi_U.transition_map(phi_V, [[Integer(0),x],[y,Integer(0)]])
```

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```
>>> fN = phi_U.frame(); fS = phi_V.frame() # get induced frames
>>> fN_W = fN.restrict(W); fS_W = fS.restrict(W) # restrict them
>>> stereoN_W = stereoN.restrict(W) # restrict charts, too
>>> stereoS_W = stereoS.restrict(W)
>>> s = E.section({fN: [Integer(1), Integer(0)]}, name='s')
>>> s.display(fN)
s = (phi_U^*e_1)
>>> sU = s.restrict(U) ; sU
Section s on the Open subset U of the 2-dimensional topological
 manifold S^2 with values in the real vector bundle E of rank 2
>>> sU.display() # fN is the default frame on U
s = (phi_U^*e_1)
>>> sU == fN[Integer(1)]
True
>>> sW = s.restrict(W) ; sW
Section s on the Open subset W of the 2-dimensional topological
 manifold S^2 with values in the real vector bundle E of rank 2
>>> sW.display(fN_W)
s = (phi_U^*e_1)
>>> sW.display(fS_W, stereoN_W)
s = y (phi_V^*e_2)
>>> sW.display(fS_W, stereoS_W)
s = v/(u^2 + v^2) (phi_V^*e_2)
>>> sW == fN_W[Integer(1)]
True
```

At this stage, defining the restriction of `s` to the open subset `V` fully specifies `s`:

```
sage: s.restrict(V)[1] = sW[fS_W, 1, stereoS_W].expr()  # note that fS is the default frame on V
sage: s.restrict(V)[2] = sW[fS_W, 2, stereoS_W].expr()
sage: s.display(fS, stereoS)
s = v/(u^2 + v^2) (phi_V^*e_2)
sage: s.restrict(U).display()
s = (phi_U^*e_1)
sage: s.restrict(V).display()
s = v/(u^2 + v^2) (phi_V^*e_2)
```

```
>>> from sage.all import *
>>> s.restrict(V)[Integer(1)] = sW[fS_W, Integer(1), stereoS_W].expr()  # note that fS is the default frame on V
>>> s.restrict(V)[Integer(2)] = sW[fS_W, Integer(2), stereoS_W].expr()
>>> s.display(fS, stereoS)
s = v/(u^2 + v^2) (phi_V^*e_2)
>>> s.restrict(U).display()
s = (phi_U^*e_1)
>>> s.restrict(V).display()
s = v/(u^2 + v^2) (phi_V^*e_2)
```

The restriction of the section to its own domain is of course itself:

```
sage: s.restrict(S2) is s
```

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```
True
sage: sU.restrict(U) is sU
True
```

```
>>> from sage.all import *
>>> s.restrict(S2) is s
True
>>> sU.restrict(U) is sU
True
```

set_comp(*basis=None*)

Return the components of `self` in a given local frame for assignment.

The components with respect to other frames having the same domain as the provided local frame are deleted, in order to avoid any inconsistency. To keep them, use the method `add_comp()` instead.

INPUT:

- `basis` – (default: `None`) local frame in which the components are defined; if none is provided, the components are assumed to refer to the section domain's default frame

OUTPUT:

- components in the given frame, as a `Components`; if such components did not exist previously, they are created

EXAMPLES:

```
sage: S2 = Manifold(2, 'S^2', structure='top', start_index=1)
sage: U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the␣
↪North and South pole, respectively
sage: S2.declare_union(U,V)
sage: stereoN.<x,y> = U.chart() # stereographic coordinates from the North␣
↪pole
sage: stereoS.<u,v> = V.chart() # stereographic coordinates from the South␣
↪pole
sage: xy_to_uv = stereoN.transition_map(stereoS,
....:                                   (x/(x^2+y^2), y/(x^2+y^2)),
....:                                   intersection_name='W',
....:                                   restrictions1= x^2+y^2!=0,
....:                                   restrictions2= u^2+v^2!=0)
sage: W = U.intersection(V)
sage: uv_to_xy = xy_to_uv.inverse()
sage: E = S2.vector_bundle(2, 'E') # define vector bundle
sage: phi_U = E.trivialization('phi_U', domain=U) # define trivializations
sage: phi_V = E.trivialization('phi_V', domain=V)
sage: transf = phi_U.transition_map(phi_V, [[0,x],[y,0]])
sage: fN = phi_U.frame(); fS = phi_V.frame() # get induced frames
sage: s = E.section(name='s')
sage: s.set_comp(fS)
1-index components w.r.t. Trivialization frame (E|_V, ((phi_V^*e_1),(phi_V^*e_
↪2)))
sage: s.set_comp(fS)[1] = u+v
sage: s.display(fS)
s = (u + v) (phi_V^*e_1)
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2', structure='top', start_index=Integer(1))
>>> U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the␣
↪North and South pole, respectively
>>> S2.declare_union(U,V)
>>> stereoN = U.chart(names=('x', 'y',)); (x, y,) = stereoN._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> stereoS = V.chart(names=('u', 'v',)); (u, v,) = stereoS._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> xy_to_uv = stereoN.transition_map(stereoS,
...                                    (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                                    intersection_name='W',
...                                    restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                                    restrictions2=␣
↪u**Integer(2)+v**Integer(2)!=Integer(0))
>>> W = U.intersection(V)
>>> uv_to_xy = xy_to_uv.inverse()
>>> E = S2.vector_bundle(Integer(2), 'E') # define vector bundle
>>> phi_U = E.trivialization('phi_U', domain=U) # define trivializations
>>> phi_V = E.trivialization('phi_V', domain=V)
>>> transf = phi_U.transition_map(phi_V, [[Integer(0),x],[y,Integer(0)]])
>>> fN = phi_U.frame(); fS = phi_V.frame() # get induced frames
>>> s = E.section(name='s')
>>> s.set_comp(fS)
1-index components w.r.t. Trivialization frame (E|_V, ((phi_V^*e_1),(phi_V^*e_
↪2)))
>>> s.set_comp(fS)[Integer(1)] = u+v
>>> s.display(fS)
s = (u + v) (phi_V^*e_1)
```

Setting the components in a new frame (e):

```
sage: e = E.local_frame('e', domain=V)
sage: s.set_comp(e)
1-index components w.r.t. Local frame (E|_V, (e_1,e_2))
sage: s.set_comp(e)[1] = u*v
sage: s.display(e)
s = u*v e_1
```

```
>>> from sage.all import *
>>> e = E.local_frame('e', domain=V)
>>> s.set_comp(e)
1-index components w.r.t. Local frame (E|_V, (e_1,e_2))
>>> s.set_comp(e)[Integer(1)] = u*v
>>> s.display(e)
s = u*v e_1
```

Since the frames e and fS are defined on the same domain, the components w.r.t. fS have been erased:

```
sage: s.display(phi_V.frame())
Traceback (most recent call last):
```

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```
...
ValueError: no basis could be found for computing the components in
 the Trivialization frame (E|_V, ((phi_V^*e_1),(phi_V^*e_2)))
```

```
>>> from sage.all import *
>>> s.display(phi_V.frame())
Traceback (most recent call last):
...
ValueError: no basis could be found for computing the components in
 the Trivialization frame (E|_V, ((phi_V^*e_1),(phi_V^*e_2)))
```

set_immutable()

Set `self` and all restrictions of `self` immutable.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: U = M.open_subset('U', coord_def={X: x^2+y^2<1})
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e')
sage: s = E.section([1+y,x], name='s')
sage: sU = s.restrict(U)
sage: s.set_immutable()
sage: s.is_immutable()
True
sage: sU.is_immutable()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> U = M.open_subset('U', coord_def={X: x**Integer(2)+y**Integer(2)
↪<Integer(1)})
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e')
>>> s = E.section([Integer(1)+y,x], name='s')
>>> sU = s.restrict(U)
>>> s.set_immutable()
>>> s.is_immutable()
True
>>> sU.is_immutable()
True
```

set_name(*name=None*, *latex_name=None*)

Set (or change) the text name and LaTeX name of `self`.

INPUT:

- `name` – string (default: `None`); name given to the section
- `latex_name` – string (default: `None`); LaTeX symbol to denote the section; if `None` while `name` is provided, the LaTeX symbol is set to `name`

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='top')
sage: E = M.vector_bundle(2, 'E')
sage: s = E.section(); s
Section on the 3-dimensional topological manifold M with values in
 the real vector bundle E of rank 2
sage: s.set_name(name='s')
sage: s
Section s on the 3-dimensional topological manifold M with values in
 the real vector bundle E of rank 2
sage: latex(s)
s
sage: s.set_name(latex_name=r'\sigma')
sage: latex(s)
\sigma
sage: s.set_name(name='a')
sage: s
Section a on the 3-dimensional topological manifold M with values in
 the real vector bundle E of rank 2
sage: latex(s)
a
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='top')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> s = E.section(); s
Section on the 3-dimensional topological manifold M with values in
 the real vector bundle E of rank 2
>>> s.set_name(name='s')
>>> s
Section s on the 3-dimensional topological manifold M with values in
 the real vector bundle E of rank 2
>>> latex(s)
s
>>> s.set_name(latex_name=r'\sigma')
>>> latex(s)
\sigma
>>> s.set_name(name='a')
>>> s
Section a on the 3-dimensional topological manifold M with values in
 the real vector bundle E of rank 2
>>> latex(s)
a
```

set_restriction(*rst*)

Define a restriction of `self` to some subdomain.

INPUT:

- `rst` – *Section* defined on a subdomain of the domain of `self`

EXAMPLES:

```
sage: S2 = Manifold(2, 'S^2', structure='top')
sage: U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the
```

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```
↪North and South pole, respectively
sage: S2.declare_union(U,V)
sage: stereoN.<x,y> = U.chart() # stereographic coordinates from the North␣
↪pole
sage: stereoS.<u,v> = V.chart() # stereographic coordinates from the South␣
↪pole
sage: xy_to_uv = stereoN.transition_map(stereoS,
....:                                   (x/(x^2+y^2), y/(x^2+y^2)),
....:                                   intersection_name='W',
....:                                   restrictions1= x^2+y^2!=0,
....:                                   restrictions2= u^2+v^2!=0)
sage: W = U.intersection(V)
sage: uv_to_xy = xy_to_uv.inverse()
sage: E = S2.vector_bundle(2, 'E')
sage: phi_U = E.trivialization('phi_U', domain=U)
sage: phi_V = E.trivialization('phi_V', domain=V)
sage: s = E.section(name='s')
sage: sU = E.section(domain=U, name='s')
sage: sU[:] = x+y, x
sage: s.set_restriction(sU)
sage: s.display(phi_U.frame())
s = (x + y) (phi_U^*e_1) + x (phi_U^*e_2)
sage: s.restrict(U) == sU
True
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2', structure='top')
>>> U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the␣
↪North and South pole, respectively
>>> S2.declare_union(U,V)
>>> stereoN = U.chart(names=('x', 'y',)); (x, y,) = stereoN._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> stereoS = V.chart(names=('u', 'v',)); (u, v,) = stereoS._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> xy_to_uv = stereoN.transition_map(stereoS,
...                                   (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                                   intersection_name='W',
...                                   restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                                   restrictions2=␣
↪u**Integer(2)+v**Integer(2)!=Integer(0))
>>> W = U.intersection(V)
>>> uv_to_xy = xy_to_uv.inverse()
>>> E = S2.vector_bundle(Integer(2), 'E')
>>> phi_U = E.trivialization('phi_U', domain=U)
>>> phi_V = E.trivialization('phi_V', domain=V)
>>> s = E.section(name='s')
>>> sU = E.section(domain=U, name='s')
>>> sU[:] = x+y, x
>>> s.set_restriction(sU)
>>> s.display(phi_U.frame())
```

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```
s = (x + y) (phi_U^*e_1) + x (phi_U^*e_2)
>>> s.restrict(U) == sU
True
```

class sage.manifolds.section.**TrivialSection**(*section_module*, *name=None*, *latex_name=None*)

Bases: `FiniteRankFreeModuleElement`, *Section*

Section in a trivial vector bundle.

An instance of this class is a section in a vector bundle $E \to M$ of class C^k, where $E|_U$ is manifestly trivial. More precisely, a *(local) section* on a subset $U \in M$ is a map of class C^k

$$s : U \longrightarrow E$$

such that

$$\forall p \in U, \ s(p) \in E_p$$

where E_p denotes the vector bundle fiber of E over the point $p \in U$. E being trivial means E being homeomorphic to $E \times F$, for F is the typical fiber of E, namely the underlying topological vector space. By this means, s can be seen as a map of class $C^k(U;E)$

$$s : U \longrightarrow F,$$

so that the set of all sections $C^k(U;E)$ becomes a *free* module over the algebra of scalar fields on U.

> **Note**
>
> If $E|_U$ is not manifestly trivial, the class *Section* should be used instead.

This is a Sage *element* class, the corresponding *parent* class being *SectionFreeModule*.

INPUT:

- `section_module` – free module $C^k(U;E)$ of sections on E over U (cf. *SectionFreeModule*)
- `name` – (default: `None`) name given to the section
- `latex_name` – (default: `None`) LaTeX symbol to denote the section; if none is provided, the LaTeX symbol is set to `name`

EXAMPLES:

A section on a trivial rank 3 vector bundle over the 3-sphere:

```
sage: M = Manifold(3, 'S^3', structure='top')
sage: U = M.open_subset('U') ; V = M.open_subset('V') # complement of the North␣
↪and South pole, respectively
sage: M.declare_union(U,V)
sage: stereoN.<x,y,z> = U.chart() # stereographic coordinates from the North pole
sage: stereoS.<u,v,t> = V.chart() # stereographic coordinates from the South pole
sage: xyz_to_uvt = stereoN.transition_map(stereoS,
....:             (x/(x^2+y^2+z^2), y/(x^2+y^2+z^2), z/(x^2+y^2+z^2)),
....:             intersection_name='W',
....:             restrictions1= x^2+y^2+z^2!=0,
```

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```
....:              restrictions2= u^2+v^2+t^2!=0)
sage: W = U.intersection(V)
sage: uvt_to_xyz = xyz_to_uvt.inverse()
sage: E = M.vector_bundle(3, 'E')
sage: e = E.local_frame('e') # Trivializes E
sage: s = E.section(name='s'); s
Section s on the 3-dimensional topological manifold S^3 with values in
 the real vector bundle E of rank 3
sage: s[e,:] = z^2, x-y, 1-x
sage: s.display()
s = z^2 e_0 + (x - y) e_1 + (-x + 1) e_2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'S^3', structure='top')
>>> U = M.open_subset('U') ; V = M.open_subset('V') # complement of the North and␣
↪South pole, respectively
>>> M.declare_union(U,V)
>>> stereoN = U.chart(names=('x', 'y', 'z',)); (x, y, z,) = stereoN._first_
↪ngens(3)# stereographic coordinates from the North pole
>>> stereoS = V.chart(names=('u', 'v', 't',)); (u, v, t,) = stereoS._first_
↪ngens(3)# stereographic coordinates from the South pole
>>> xyz_to_uvt = stereoN.transition_map(stereoS,
...             (x/(x**Integer(2)+y**Integer(2)+z**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2)+z**Integer(2)), z/
↪(x**Integer(2)+y**Integer(2)+z**Integer(2))),
...             intersection_name='W',
...             restrictions1= x**Integer(2)+y**Integer(2)+z**Integer(2)!
↪=Integer(0),
...             restrictions2= u**Integer(2)+v**Integer(2)+t**Integer(2)!
↪=Integer(0))
>>> W = U.intersection(V)
>>> uvt_to_xyz = xyz_to_uvt.inverse()
>>> E = M.vector_bundle(Integer(3), 'E')
>>> e = E.local_frame('e') # Trivializes E
>>> s = E.section(name='s'); s
Section s on the 3-dimensional topological manifold S^3 with values in
 the real vector bundle E of rank 3
>>> s[e,:] = z**Integer(2), x-y, Integer(1)-x
>>> s.display()
s = z^2 e_0 + (x - y) e_1 + (-x + 1) e_2
```

Since E is trivial, s is now element of a free section module:

```
sage: s.parent()
Free module C^0(S^3;E) of sections on the 3-dimensional topological
 manifold S^3 with values in the real vector bundle E of rank 3
sage: isinstance(s.parent(), FiniteRankFreeModule)
True
```

```
>>> from sage.all import *
>>> s.parent()
Free module C^0(S^3;E) of sections on the 3-dimensional topological
```

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```
manifold S^3 with values in the real vector bundle E of rank 3
>>> isinstance(s.parent(), FiniteRankFreeModule)
True
```

add_comp(*basis=None*)

Return the components of the section in a given local frame for assignment.

The components with respect to other frames on the same domain are kept. To delete them, use the method `set_comp()` instead.

INPUT:

- `basis` – (default: `None`) local frame in which the components are defined; if none is provided, the components are assumed to refer to the section module's default frame

OUTPUT:

- components in the given frame, as an instance of the class `Components`; if such components did not exist previously, they are created

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top')
sage: X.<x,y> = M.chart()
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e') # makes E trivial
sage: s = E.section(name='s')
sage: s.add_comp(e)
1-index components w.r.t. Local frame (E|_M, (e_0,e_1))
sage: s.add_comp(e)[0] = 2
sage: s.display(e)
s = 2 e_0
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e') # makes E trivial
>>> s = E.section(name='s')
>>> s.add_comp(e)
1-index components w.r.t. Local frame (E|_M, (e_0,e_1))
>>> s.add_comp(e)[Integer(0)] = Integer(2)
>>> s.display(e)
s = 2 e_0
```

Adding components with respect to a new frame (`f`):

```
sage: f = E.local_frame('f')
sage: s.add_comp(f)
1-index components w.r.t. Local frame (E|_M, (f_0,f_1))
sage: s.add_comp(f)[0] = x
sage: s.display(f)
s = x f_0
```

```
>>> from sage.all import *
>>> f = E.local_frame('f')
>>> s.add_comp(f)
1-index components w.r.t. Local frame (E|_M, (f_0,f_1))
>>> s.add_comp(f)[Integer(0)] = x
>>> s.display(f)
s = x f_0
```

The components with respect to the frame `e` are kept:

```
sage: s.display(e)
s = 2 e_0
```

```
>>> from sage.all import *
>>> s.display(e)
s = 2 e_0
```

Adding components in a frame defined on a subdomain:

```
sage: U = M.open_subset('U', coord_def={X: x>0})
sage: g = E.local_frame('g', domain=U)
sage: s.add_comp(g)
1-index components w.r.t. Local frame (E|_U, (g_0,g_1))
sage: s.add_comp(g)[0] = 1+y
sage: s.display(g)
s = (y + 1) g_0
```

```
>>> from sage.all import *
>>> U = M.open_subset('U', coord_def={X: x>Integer(0)})
>>> g = E.local_frame('g', domain=U)
>>> s.add_comp(g)
1-index components w.r.t. Local frame (E|_U, (g_0,g_1))
>>> s.add_comp(g)[Integer(0)] = Integer(1)+y
>>> s.display(g)
s = (y + 1) g_0
```

The components previously defined are kept:

```
sage: s.display(e)
s = 2 e_0
sage: s.display(f)
s = x f_0
```

```
>>> from sage.all import *
>>> s.display(e)
s = 2 e_0
>>> s.display(f)
s = x f_0
```

at (*point*)

Value of `self` at a point of its domain.

If the current section is

$$s : U \longrightarrow E,$$

then for any point $p \in U$, $s(p)$ is a vector in the fiber E_p of E at the point $p \in U$.

INPUT:

- `point` – *ManifoldPoint* point p in the domain of the section U

OUTPUT:

- `FreeModuleTensor` representing the vector $s(p)$ in the vector space E_p

EXAMPLES:

Vector in a tangent space of a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='top')
sage: X.<x,y> = M.chart()
sage: p = M.point((-2,3), name='p')
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e') # makes E trivial
sage: s = E.section(y, x^2, name='s')
sage: s.display()
s = y e_0 + x^2 e_1
sage: sp = s.at(p) ; sp
Vector s in the fiber of E at Point p on the 2-dimensional
 topological manifold M
sage: sp.parent()
Fiber of E at Point p on the 2-dimensional topological manifold M
sage: sp.display()
s = 3 e_0 + 4 e_1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> p = M.point((-Integer(2),Integer(3)), name='p')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e') # makes E trivial
>>> s = E.section(y, x**Integer(2), name='s')
>>> s.display()
s = y e_0 + x^2 e_1
>>> sp = s.at(p) ; sp
Vector s in the fiber of E at Point p on the 2-dimensional
 topological manifold M
>>> sp.parent()
Fiber of E at Point p on the 2-dimensional topological manifold M
>>> sp.display()
s = 3 e_0 + 4 e_1
```

comp(*basis=None*, *from_basis=None*)

Return the components in a given local frame.

If the components are not known already, they are computed by the tensor change-of-basis formula from components in another local frame.

INPUT:

- `basis` – (default: `None`) local frame in which the components are required; if none is provided, the components are assumed to refer to the section module's default frame
- `from_basis` – (default: `None`) local frame from which the required components are computed, via the tensor change-of-basis formula, if they are not known already in the basis `basis`

OUTPUT:

- components in the local frame `basis`, as an instance of the class `Components`

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top', start_index=1)
sage: X.<x,y> = M.chart()
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e') # makes E trivial
sage: s = E.section(name='s')
sage: s[1] = x*y
sage: s.comp(e)
1-index components w.r.t. Local frame (E|_M, (e_1,e_2))
sage: s.comp()  # the default frame is e
1-index components w.r.t. Local frame (E|_M, (e_1,e_2))
sage: s.comp()[:]
[x*y, 0]
sage: f = E.local_frame('f')
sage: s[f, 1] = x-3
sage: s.comp(f)
1-index components w.r.t. Local frame (E|_M, (f_1,f_2))
sage: s.comp(f)[:]
[x - 3, 0]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e') # makes E trivial
>>> s = E.section(name='s')
>>> s[Integer(1)] = x*y
>>> s.comp(e)
1-index components w.r.t. Local frame (E|_M, (e_1,e_2))
>>> s.comp()  # the default frame is e
1-index components w.r.t. Local frame (E|_M, (e_1,e_2))
>>> s.comp()[:]
[x*y, 0]
>>> f = E.local_frame('f')
>>> s[f, Integer(1)] = x-Integer(3)
>>> s.comp(f)
1-index components w.r.t. Local frame (E|_M, (f_1,f_2))
>>> s.comp(f)[:]
[x - 3, 0]
```

display_comp(*frame=None*, *chart=None*, *only_nonzero=False*)

Display the section components with respect to a given frame, one per line.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- `frame` – (default: `None`) local frame with respect to which the section components are defined; if `None`, then the default basis of the section module on which the section is defined is used
- `chart` – (default: `None`) chart specifying the coordinate expression of the components; if `None`, the default chart of the section module domain is used
- `only_nonzero` – boolean (default: `False`); if `True`, only nonzero components are displayed

EXAMPLES:

Display of the components of a section on a rank 4 vector bundle over a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='top')
sage: X.<x,y> = M.chart()
sage: E = M.vector_bundle(3, 'E')
sage: e = E.local_frame('e') # makes E trivial
sage: s = E.section(name='s')
sage: s[0], s[2] = x+y, x*y
sage: s.display_comp()
s^0 = x + y
s^1 = 0
s^2 = x*y
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> E = M.vector_bundle(Integer(3), 'E')
>>> e = E.local_frame('e') # makes E trivial
>>> s = E.section(name='s')
>>> s[Integer(0)], s[Integer(2)] = x+y, x*y
>>> s.display_comp()
s^0 = x + y
s^1 = 0
s^2 = x*y
```

By default, the vanishing components are displayed, too; to see only non-vanishing components, the argument `only_nonzero` must be set to `True`:

```
sage: s.display_comp(only_nonzero=True)
s^0 = x + y
s^2 = x*y
```

```
>>> from sage.all import *
>>> s.display_comp(only_nonzero=True)
s^0 = x + y
s^2 = x*y
```

Display in a frame different from the default one:

```
sage: a = E.section_module().automorphism()
sage: a[:] = [[1+y^2, 0, 0], [0, 2+x^2, 0], [0, 0, 1]]
sage: f = e.new_frame(a, 'f')
sage: s.display_comp(frame=f)
s^0 = (x + y)/(y^2 + 1)
s^1 = 0
s^2 = x*y
```

```
>>> from sage.all import *
>>> a = E.section_module().automorphism()
>>> a[:] = [[Integer(1)+y**Integer(2), Integer(0), Integer(0)], [Integer(0),␣
↪Integer(2)+x**Integer(2), Integer(0)], [Integer(0), Integer(0), Integer(1)]]
>>> f = e.new_frame(a, 'f')
>>> s.display_comp(frame=f)
s^0 = (x + y)/(y^2 + 1)
s^1 = 0
s^2 = x*y
```

Display with respect to a chart different from the default one:

```
sage: Y.<u,v> = M.chart()
sage: X_to_Y = X.transition_map(Y, [x+y, x-y])
sage: Y_to_X = X_to_Y.inverse()
sage: s.display_comp(chart=Y)
s^0 = u
s^1 = 0
s^2 = 1/4*u^2 - 1/4*v^2
```

```
>>> from sage.all import *
>>> Y = M.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)
>>> X_to_Y = X.transition_map(Y, [x+y, x-y])
>>> Y_to_X = X_to_Y.inverse()
>>> s.display_comp(chart=Y)
s^0 = u
s^1 = 0
s^2 = 1/4*u^2 - 1/4*v^2
```

Display of the components with respect to a specific frame, expressed in terms of a specific chart:

```
sage: s.display_comp(frame=f, chart=Y)
s^0 = 4*u/(u^2 - 2*u*v + v^2 + 4)
s^1 = 0
s^2 = 1/4*u^2 - 1/4*v^2
```

```
>>> from sage.all import *
>>> s.display_comp(frame=f, chart=Y)
s^0 = 4*u/(u^2 - 2*u*v + v^2 + 4)
s^1 = 0
s^2 = 1/4*u^2 - 1/4*v^2
```

restrict(*subdomain*)

Return the restriction of `self` to some subdomain.

If the restriction has not been defined yet, it is constructed here.

INPUT:

- `subdomain` – *DifferentiableManifold*; open subset U of the section module domain S

OUTPUT: instance of *TrivialSection* representing the restriction

EXAMPLES:

Restriction of a section defined over $\mathbf{R}^2$ to a disk:

```
sage: M = Manifold(2, 'R^2')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e') # makes E trivial
sage: s = E.section(x+y, -1+x^2, name='s')
sage: D = M.open_subset('D') # the unit open disc
sage: e_D = e.restrict(D)
sage: c_cart_D = c_cart.restrict(D, x^2+y^2<1)
sage: s_D = s.restrict(D) ; s_D
Section s on the Open subset D of the 2-dimensional differentiable
 manifold R^2 with values in the real vector bundle E of rank 2
sage: s_D.display(e_D)
s = (x + y) e_0 + (x^2 - 1) e_1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#
↪Cartesian coordinates on R^2
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e') # makes E trivial
>>> s = E.section(x+y, -Integer(1)+x**Integer(2), name='s')
>>> D = M.open_subset('D') # the unit open disc
>>> e_D = e.restrict(D)
>>> c_cart_D = c_cart.restrict(D, x**Integer(2)+y**Integer(2)<Integer(1))
>>> s_D = s.restrict(D) ; s_D
Section s on the Open subset D of the 2-dimensional differentiable
 manifold R^2 with values in the real vector bundle E of rank 2
>>> s_D.display(e_D)
s = (x + y) e_0 + (x^2 - 1) e_1
```

The symbolic expressions of the components with respect to Cartesian coordinates are equal:

```
sage: bool( s_D[1].expr() == s[1].expr() )
True
```

```
>>> from sage.all import *
>>> bool( s_D[Integer(1)].expr() == s[Integer(1)].expr() )
True
```

but neither the chart functions representing the components (they are defined on different charts):

```
sage: s_D[1] == s[1]
False
```

```
>>> from sage.all import *
>>> s_D[Integer(1)] == s[Integer(1)]
False
```

nor the scalar fields representing the components (they are defined on different open subsets):

```
sage: s_D[[1]] == s[[1]]
False
```

```
>>> from sage.all import *
>>> s_D[[Integer(1)]] == s[[Integer(1)]]
False
```

The restriction of the section to its own domain is of course itself:

```
sage: s.restrict(M) is s
True
```

```
>>> from sage.all import *
>>> s.restrict(M) is s
True
```

set_comp(*basis=None*)

Return the components of the section in a given local frame for assignment.

The components with respect to other frames on the same domain are deleted, in order to avoid any inconsistency. To keep them, use the method `add_comp()` instead.

INPUT:

- `basis` – (default: `None`) local frame in which the components are defined; if none is provided, the components are assumed to refer to the section module's default frame

OUTPUT:

- components in the given frame, as an instance of the class `Components`; if such components did not exist previously, they are created

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top')
sage: X.<x,y> = M.chart()
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e') # makes E trivial
sage: s = E.section(name='s')
sage: s.set_comp(e)
1-index components w.r.t. Local frame (E|_M, (e_0,e_1))
sage: s.set_comp(e)[0] = 2
sage: s.display(e)
s = 2 e_0
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='top')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e') # makes E trivial
>>> s = E.section(name='s')
>>> s.set_comp(e)
1-index components w.r.t. Local frame (E|_M, (e_0,e_1))
>>> s.set_comp(e)[Integer(0)] = Integer(2)
>>> s.display(e)
s = 2 e_0
```

Setting components in a new frame (`f`):

```
sage: f = E.local_frame('f')
sage: s.set_comp(f)
1-index components w.r.t. Local frame (E|_M, (f_0,f_1))
sage: s.set_comp(f)[0] = x
sage: s.display(f)
s = x f_0
```

```
>>> from sage.all import *
>>> f = E.local_frame('f')
>>> s.set_comp(f)
1-index components w.r.t. Local frame (E|_M, (f_0,f_1))
>>> s.set_comp(f)[Integer(0)] = x
>>> s.display(f)
s = x f_0
```

The components with respect to the frame `e` have be erased:

```
sage: s.display(e)
Traceback (most recent call last):
...
ValueError: no basis could be found for computing the components
 in the Local frame (E|_M, (e_0,e_1))
```

```
>>> from sage.all import *
>>> s.display(e)
Traceback (most recent call last):
...
ValueError: no basis could be found for computing the components
 in the Local frame (E|_M, (e_0,e_1))
```

Setting components in a frame defined on a subdomain deletes previously defined components as well:

```
sage: U = M.open_subset('U', coord_def={X: x>0})
sage: g = E.local_frame('g', domain=U)
sage: s.set_comp(g)
1-index components w.r.t. Local frame (E|_U, (g_0,g_1))
sage: s.set_comp(g)[0] = 1+y
sage: s.display(g)
s = (y + 1) g_0
sage: s.display(f)
Traceback (most recent call last):
...
ValueError: no basis could be found for computing the components
 in the Local frame (E|_M, (f_0,f_1))
```

```
>>> from sage.all import *
>>> U = M.open_subset('U', coord_def={X: x>Integer(0)})
>>> g = E.local_frame('g', domain=U)
>>> s.set_comp(g)
1-index components w.r.t. Local frame (E|_U, (g_0,g_1))
>>> s.set_comp(g)[Integer(0)] = Integer(1)+y
>>> s.display(g)
```

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```
s = (y + 1) g_0
>>> s.display(f)
Traceback (most recent call last):
...
ValueError: no basis could be found for computing the components
 in the Local frame (E|_M, (f_0,f_1))
```

1.10 Families of Manifold Objects

The class *ManifoldObjectFiniteFamily* is a subclass of FiniteFamily that provides an associative container of manifold objects, indexed by their _name attributes.

ManifoldObjectFiniteFamily instances are totally ordered according to their lexicographically ordered element names.

The subclass *ManifoldSubsetFiniteFamily* customizes the print representation further.

AUTHORS:

- Matthias Koeppe (2021): initial version

class sage.manifolds.family.**ManifoldObjectFiniteFamily**(*objects=()*, *keys=None*)

Bases: FiniteFamily

Finite family of manifold objects, indexed by their names.

The class *ManifoldObjectFiniteFamily* inherits from FiniteFamily. Therefore it is an associative container.

It provides specialized __repr__ and _latex_ methods.

ManifoldObjectFiniteFamily instances are totally ordered according to their lexicographically ordered element names.

EXAMPLES:

```
sage: from sage.manifolds.family import ManifoldObjectFiniteFamily
sage: M = Manifold(2, 'M', structure='topological')
sage: A = M.subset('A')
sage: B = M.subset('B')
sage: C = B.subset('C')
sage: F = ManifoldObjectFiniteFamily([A, B, C]); F
Set {A, B, C} of objects of the 2-dimensional topological manifold M
sage: latex(F)
\{A, B, C\}
sage: F['B']
Subset B of the 2-dimensional topological manifold M
```

```
>>> from sage.all import *
>>> from sage.manifolds.family import ManifoldObjectFiniteFamily
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> A = M.subset('A')
>>> B = M.subset('B')
>>> C = B.subset('C')
>>> F = ManifoldObjectFiniteFamily([A, B, C]); F
```

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```
Set {A, B, C} of objects of the 2-dimensional topological manifold M
>>> latex(F)
\{A, B, C\}
>>> F['B']
Subset B of the 2-dimensional topological manifold M
```

All objects must have the same base manifold:

```
sage: N = Manifold(2, 'N', structure='topological')
sage: ManifoldObjectFiniteFamily([M, N])
Traceback (most recent call last):
...
TypeError: all objects must have the same manifold
```

```
>>> from sage.all import *
>>> N = Manifold(Integer(2), 'N', structure='topological')
>>> ManifoldObjectFiniteFamily([M, N])
Traceback (most recent call last):
...
TypeError: all objects must have the same manifold
```

class sage.manifolds.family.**ManifoldSubsetFiniteFamily**(*objects=()*, *keys=None*)

Bases: *ManifoldObjectFiniteFamily*

Finite family of subsets of a topological manifold, indexed by their names.

The class *ManifoldSubsetFiniteFamily* inherits from *ManifoldObjectFiniteFamily*. It provides an associative container with specialized `__repr__` and `_latex_` methods.

ManifoldSubsetFiniteFamily instances are totally ordered according to their lexicographically ordered element (subset) names.

EXAMPLES:

```
sage: from sage.manifolds.family import ManifoldSubsetFiniteFamily
sage: M = Manifold(2, 'M', structure='topological')
sage: A = M.subset('A')
sage: B = M.subset('B')
sage: C = B.subset('C')
sage: ManifoldSubsetFiniteFamily([A, B, C])
Set {A, B, C} of subsets of the 2-dimensional topological manifold M
sage: latex(_)
\{A, B, C\}
```

```
>>> from sage.all import *
>>> from sage.manifolds.family import ManifoldSubsetFiniteFamily
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> A = M.subset('A')
>>> B = M.subset('B')
>>> C = B.subset('C')
>>> ManifoldSubsetFiniteFamily([A, B, C])
Set {A, B, C} of subsets of the 2-dimensional topological manifold M
>>> latex(_)
\{A, B, C\}
```

All subsets must have the same base manifold:

```
sage: N = Manifold(2, 'N', structure='topological')
sage: ManifoldSubsetFiniteFamily([M, N])
Traceback (most recent call last):
...
TypeError: all open subsets must have the same manifold
```

```
>>> from sage.all import *
>>> N = Manifold(Integer(2), 'N', structure='topological')
>>> ManifoldSubsetFiniteFamily([M, N])
Traceback (most recent call last):
...
TypeError: all open subsets must have the same manifold
```

classmethod from_subsets_or_families(**subsets_or_families*)

Construct a `ManifoldSubsetFiniteFamily` from given subsets or iterables of subsets.

EXAMPLES:

```
sage: from sage.manifolds.family import ManifoldSubsetFiniteFamily
sage: M = Manifold(2, 'M', structure='topological')
sage: A = M.subset('A')
sage: Bs = (M.subset(f'B{i}') for i in range(5))
sage: Cs = ManifoldSubsetFiniteFamily([M.subset('C0'), M.subset('C1')])
sage: ManifoldSubsetFiniteFamily.from_subsets_or_families(A, Bs, Cs)
Set {A, B0, B1, B2, B3, B4, C0, C1} of subsets of the 2-dimensional␣
↪topological manifold M
```

```
>>> from sage.all import *
>>> from sage.manifolds.family import ManifoldSubsetFiniteFamily
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> A = M.subset('A')
>>> Bs = (M.subset(f'B{i}') for i in range(Integer(5)))
>>> Cs = ManifoldSubsetFiniteFamily([M.subset('C0'), M.subset('C1')])
>>> ManifoldSubsetFiniteFamily.from_subsets_or_families(A, Bs, Cs)
Set {A, B0, B1, B2, B3, B4, C0, C1} of subsets of the 2-dimensional␣
↪topological manifold M
```

1.11 Topological Closures of Manifold Subsets

ManifoldSubsetClosure implements the topological closure of a manifold subset in the topology of the manifold.

class sage.manifolds.subsets.closure.**ManifoldSubsetClosure**(*subset*, *name=None*, *latex_name=None*)

Bases: *ManifoldSubset*

Topological closure of a manifold subset in the topology of the manifold.

INPUT:

- `subset` – a *ManifoldSubset*
- `name` – (default: computed from the name of the subset) string; name (symbol) given to the closure

- `latex_name` – string (default: `None`); LaTeX symbol to denote the subset. If none is provided, it is set to `name`.

EXAMPLES:

```
sage: M = Manifold(2, 'R^2', structure='topological')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: D = M.open_subset('D', coord_def={c_cart: x^2+y^2<1}); D
Open subset D of the 2-dimensional topological manifold R^2
sage: cl_D = D.closure()
sage: cl_D
Topological closure cl_D of the Open subset D of the 2-dimensional
 topological manifold R^2
sage: latex(cl_D)
\mathop{\mathrm{cl}}(D)
sage: type(cl_D)
<class 'sage.manifolds.subsets.closure.ManifoldSubsetClosure_with_category'>
sage: cl_D.category()
Category of subobjects of sets
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', structure='topological')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#␣
↪Cartesian coordinates on R^2
>>> D = M.open_subset('D', coord_def={c_cart: x**Integer(2)+y**Integer(2)
↪<Integer(1)}); D
Open subset D of the 2-dimensional topological manifold R^2
>>> cl_D = D.closure()
>>> cl_D
Topological closure cl_D of the Open subset D of the 2-dimensional
 topological manifold R^2
>>> latex(cl_D)
\mathop{\mathrm{cl}}(D)
>>> type(cl_D)
<class 'sage.manifolds.subsets.closure.ManifoldSubsetClosure_with_category'>
>>> cl_D.category()
Category of subobjects of sets
```

The closure of the subset D is a subset of every closed superset of D:

```
sage: S = D.superset('S')
sage: S.declare_closed()
sage: cl_D.is_subset(S)
True
```

```
>>> from sage.all import *
>>> S = D.superset('S')
>>> S.declare_closed()
>>> cl_D.is_subset(S)
True
```

is_closed()

Return if `self` is a closed set.

This implementation of the method always returns `True`.

EXAMPLES:

```
sage: from sage.manifolds.subsets.closure import ManifoldSubsetClosure
sage: M = Manifold(2, 'R^2', structure='topological')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: D = M.open_subset('D', coord_def={c_cart: x^2+y^2<1}); D
Open subset D of the 2-dimensional topological manifold R^2
sage: cl_D = D.closure(); cl_D  # indirect doctest
Topological closure cl_D of the Open subset D of the 2-dimensional␣
↪topological manifold R^2
sage: cl_D.is_closed()
True
```

```
>>> from sage.all import *
>>> from sage.manifolds.subsets.closure import ManifoldSubsetClosure
>>> M = Manifold(Integer(2), 'R^2', structure='topological')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#␣
↪Cartesian coordinates on R^2
>>> D = M.open_subset('D', coord_def={c_cart: x**Integer(2)+y**Integer(2)
↪<Integer(1)}); D
Open subset D of the 2-dimensional topological manifold R^2
>>> cl_D = D.closure(); cl_D  # indirect doctest
Topological closure cl_D of the Open subset D of the 2-dimensional␣
↪topological manifold R^2
>>> cl_D.is_closed()
True
```

1.12 Manifold Subsets Defined as Pullbacks of Subsets under Continuous Maps

class sage.manifolds.subsets.pullback.**ManifoldSubsetPullback**(*map*, *codomain_subset*, *inverse*, *name*, *latex_name*)

Bases: *ManifoldSubset*

Manifold subset defined as a pullback of a subset under a continuous map.

INPUT:

- map – an instance of *ContinuousMap*, *ScalarField*, or *Chart*
- codomain_subset – an instance of *ManifoldSubset*, RealSet, or ConvexSet_base

EXAMPLES:

```
sage: from sage.manifolds.subsets.pullback import ManifoldSubsetPullback
sage: M = Manifold(2, 'R^2', structure='topological')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
```

```
>>> from sage.all import *
>>> from sage.manifolds.subsets.pullback import ManifoldSubsetPullback
>>> M = Manifold(Integer(2), 'R^2', structure='topological')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#␣
↪Cartesian coordinates on R^2
```

Pulling back a real interval under a scalar field:

```
sage: r_squared = M.scalar_field(x^2+y^2)
sage: r_squared.set_immutable()
sage: cl_I = RealSet([1, 4]); cl_I
[1, 4]
sage: cl_O = ManifoldSubsetPullback(r_squared, cl_I); cl_O
Subset f_inv_[1, 4] of the 2-dimensional topological manifold R^2
sage: M.point((0, 0)) in cl_O
False
sage: M.point((0, 1)) in cl_O
True
```

```
>>> from sage.all import *
>>> r_squared = M.scalar_field(x**Integer(2)+y**Integer(2))
>>> r_squared.set_immutable()
>>> cl_I = RealSet([Integer(1), Integer(4)]); cl_I
[1, 4]
>>> cl_O = ManifoldSubsetPullback(r_squared, cl_I); cl_O
Subset f_inv_[1, 4] of the 2-dimensional topological manifold R^2
>>> M.point((Integer(0), Integer(0))) in cl_O
False
>>> M.point((Integer(0), Integer(1))) in cl_O
True
```

Pulling back an open real interval gives an open subset:

```
sage: I = RealSet((1, 4)); I
(1, 4)
sage: O = ManifoldSubsetPullback(r_squared, I); O
Open subset f_inv_(1, 4) of the 2-dimensional topological manifold R^2
sage: M.point((1, 0)) in O
False
sage: M.point((1, 1)) in O
True
```

```
>>> from sage.all import *
>>> I = RealSet((Integer(1), Integer(4))); I
(1, 4)
>>> O = ManifoldSubsetPullback(r_squared, I); O
Open subset f_inv_(1, 4) of the 2-dimensional topological manifold R^2
>>> M.point((Integer(1), Integer(0))) in O
False
>>> M.point((Integer(1), Integer(1))) in O
True
```

Pulling back a polytope under a chart:

```
sage: # needs sage.geometry.polyhedron
sage: P = Polyhedron(vertices=[[0, 0], [1, 2], [2, 1]]); P
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
sage: S = ManifoldSubsetPullback(c_cart, P); S
Subset x_y_inv_P of the 2-dimensional topological manifold R^2
```

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```
sage: M((1, 2)) in S
True
sage: M((2, 0)) in S
False
```

```
>>> from sage.all import *
>>> # needs sage.geometry.polyhedron
>>> P = Polyhedron(vertices=[[Integer(0), Integer(0)], [Integer(1), Integer(2)],
↪[Integer(2), Integer(1)]]); P
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
>>> S = ManifoldSubsetPullback(c_cart, P); S
Subset x_y_inv_P of the 2-dimensional topological manifold R^2
>>> M((Integer(1), Integer(2))) in S
True
>>> M((Integer(2), Integer(0))) in S
False
```

Pulling back the interior of a polytope under a chart:

```
sage: # needs sage.geometry.polyhedron
sage: int_P = P.interior(); int_P
Relative interior of a
 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
sage: int_S = ManifoldSubsetPullback(c_cart, int_P, name='int_S'); int_S
Open subset int_S of the 2-dimensional topological manifold R^2
sage: M((0, 0)) in int_S
False
sage: M((1, 1)) in int_S
True
```

```
>>> from sage.all import *
>>> # needs sage.geometry.polyhedron
>>> int_P = P.interior(); int_P
Relative interior of a
 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
>>> int_S = ManifoldSubsetPullback(c_cart, int_P, name='int_S'); int_S
Open subset int_S of the 2-dimensional topological manifold R^2
>>> M((Integer(0), Integer(0))) in int_S
False
>>> M((Integer(1), Integer(1))) in int_S
True
```

Using the embedding map of a submanifold:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: N = Manifold(2, 'N', ambient=M, structure='topological'); N
2-dimensional topological submanifold N
 immersed in the 3-dimensional topological manifold M
sage: CM.<x,y,z> = M.chart()
sage: CN.<u,v> = N.chart()
sage: t = var('t')
sage: phi = N.continuous_map(M, {(CN,CM): [u,v,t+u^2+v^2]})
```

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```
sage: phi_inv = M.continuous_map(N, {(CM,CN): [x,y]})
sage: phi_inv_t = M.scalar_field({CM: z-x^2-y^2})
sage: N.set_immersion(phi, inverse=phi_inv, var=t,
....:                 t_inverse={t: phi_inv_t})
sage: N.declare_embedding()

sage: from sage.manifolds.subsets.pullback import ManifoldSubsetPullback
sage: S = M.open_subset('S', coord_def={CM: z<1})
sage: phi_without_t = N.continuous_map(M, {(CN, CM): [expr.subs(t=0)
....:                                      for expr in phi.expr()]})
sage: phi_without_t
Continuous map
 from the 2-dimensional topological submanifold N
  embedded in the 3-dimensional topological manifold M
 to the 3-dimensional topological manifold M
sage: phi_without_t.expr()
(u, v, u^2 + v^2)
sage: D = ManifoldSubsetPullback(phi_without_t, S); D
Subset f_inv_S of the 2-dimensional topological submanifold N
 embedded in the 3-dimensional topological manifold M
sage: N.point((2,0)) in D
False
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological')
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='topological'); N
2-dimensional topological submanifold N
 immersed in the 3-dimensional topological manifold M
>>> CM = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = CM._first_ngens(3)
>>> CN = N.chart(names=('u', 'v',)); (u, v,) = CN._first_ngens(2)
>>> t = var('t')
>>> phi = N.continuous_map(M, {(CN,CM): [u,v,t+u**Integer(2)+v**Integer(2)]})
>>> phi_inv = M.continuous_map(N, {(CM,CN): [x,y]})
>>> phi_inv_t = M.scalar_field({CM: z-x**Integer(2)-y**Integer(2)})
>>> N.set_immersion(phi, inverse=phi_inv, var=t,
...                 t_inverse={t: phi_inv_t})
>>> N.declare_embedding()

>>> from sage.manifolds.subsets.pullback import ManifoldSubsetPullback
>>> S = M.open_subset('S', coord_def={CM: z<Integer(1)})
>>> phi_without_t = N.continuous_map(M, {(CN, CM): [expr.subs(t=Integer(0))
...                                      for expr in phi.expr()]})
>>> phi_without_t
Continuous map
 from the 2-dimensional topological submanifold N
  embedded in the 3-dimensional topological manifold M
 to the 3-dimensional topological manifold M
>>> phi_without_t.expr()
(u, v, u^2 + v^2)
>>> D = ManifoldSubsetPullback(phi_without_t, S); D
Subset f_inv_S of the 2-dimensional topological submanifold N
 embedded in the 3-dimensional topological manifold M
```

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```
>>> N.point((Integer(2),Integer(0))) in D
False
```

closure(*name=None*, *latex_name=None*)

Return the topological closure of `self` in the manifold.

Because `self` is a pullback of some subset under a continuous map, the closure of `self` is the pullback of the closure.

EXAMPLES:

```
sage: from sage.manifolds.subsets.pullback import ManifoldSubsetPullback
sage: M = Manifold(2, 'R^2', structure='topological')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: r_squared = M.scalar_field(x^2+y^2)
sage: r_squared.set_immutable()
sage: I = RealSet.open_closed(1, 2); I
(1, 2]
sage: O = ManifoldSubsetPullback(r_squared, I); O
Subset f_inv_(1, 2] of the 2-dimensional topological manifold R^2
sage: latex(O)
f^{-1}((1, 2])
sage: cl_O = O.closure(); cl_O
Subset f_inv_[1, 2] of the 2-dimensional topological manifold R^2
sage: cl_O.is_closed()
True
```

```
>>> from sage.all import *
>>> from sage.manifolds.subsets.pullback import ManifoldSubsetPullback
>>> M = Manifold(Integer(2), 'R^2', structure='topological')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#␣
↪Cartesian coordinates on R^2
>>> r_squared = M.scalar_field(x**Integer(2)+y**Integer(2))
>>> r_squared.set_immutable()
>>> I = RealSet.open_closed(Integer(1), Integer(2)); I
(1, 2]
>>> O = ManifoldSubsetPullback(r_squared, I); O
Subset f_inv_(1, 2] of the 2-dimensional topological manifold R^2
>>> latex(O)
f^{-1}((1, 2])
>>> cl_O = O.closure(); cl_O
Subset f_inv_[1, 2] of the 2-dimensional topological manifold R^2
>>> cl_O.is_closed()
True
```

is_closed()

Return if `self` is (known to be) a closed subset of the manifold.

EXAMPLES:

```
sage: from sage.manifolds.subsets.pullback import ManifoldSubsetPullback
sage: M = Manifold(2, 'R^2', structure='topological')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
```

```
>>> from sage.all import *
>>> from sage.manifolds.subsets.pullback import ManifoldSubsetPullback
>>> M = Manifold(Integer(2), 'R^2', structure='topological')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#␣
↪Cartesian coordinates on R^2
```

The pullback of a closed real interval under a scalar field is closed:

```
sage: r_squared = M.scalar_field(x^2+y^2)
sage: r_squared.set_immutable()
sage: cl_I = RealSet([1, 2]); cl_I
[1, 2]
sage: cl_O = ManifoldSubsetPullback(r_squared, cl_I); cl_O
Subset f_inv_[1, 2] of the 2-dimensional topological manifold R^2
sage: cl_O.is_closed()
True
```

```
>>> from sage.all import *
>>> r_squared = M.scalar_field(x**Integer(2)+y**Integer(2))
>>> r_squared.set_immutable()
>>> cl_I = RealSet([Integer(1), Integer(2)]); cl_I
[1, 2]
>>> cl_O = ManifoldSubsetPullback(r_squared, cl_I); cl_O
Subset f_inv_[1, 2] of the 2-dimensional topological manifold R^2
>>> cl_O.is_closed()
True
```

The pullback of a (closed convex) polyhedron under a chart is closed:

```
sage: # needs sage.geometry.polyhedron
sage: P = Polyhedron(vertices=[[0, 0], [1, 2], [3, 4]]); P
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
sage: McP = ManifoldSubsetPullback(c_cart, P, name='McP'); McP
Subset McP of the 2-dimensional topological manifold R^2
sage: McP.is_closed()
True
```

```
>>> from sage.all import *
>>> # needs sage.geometry.polyhedron
>>> P = Polyhedron(vertices=[[Integer(0), Integer(0)], [Integer(1),␣
↪Integer(2)], [Integer(3), Integer(4)]]); P
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
>>> McP = ManifoldSubsetPullback(c_cart, P, name='McP'); McP
Subset McP of the 2-dimensional topological manifold R^2
>>> McP.is_closed()
True
```

The pullback of real vector subspaces under a chart is closed:

```
sage: V = span([[1, 2]], RR); V
Vector space of degree 2 and dimension 1 over Real Field with 53 bits of␣
↪precision
Basis matrix:
```

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```
[1.00000000000000 2.00000000000000]
sage: McV = ManifoldSubsetPullback(c_cart, V, name='McV'); McV
Subset McV of the 2-dimensional topological manifold R^2
sage: McV.is_closed()
True
```

```
>>> from sage.all import *
>>> V = span([[Integer(1), Integer(2)]], RR); V
Vector space of degree 2 and dimension 1 over Real Field with 53 bits of␣
→precision
Basis matrix:
[1.00000000000000 2.00000000000000]
>>> McV = ManifoldSubsetPullback(c_cart, V, name='McV'); McV
Subset McV of the 2-dimensional topological manifold R^2
>>> McV.is_closed()
True
```

The pullback of point lattices under a chart is closed:

```
sage: W = span([[1, 0], [3, 5]], ZZ); W
Free module of degree 2 and rank 2 over Integer Ring
Echelon basis matrix:
[1 0]
[0 5]
sage: McW = ManifoldSubsetPullback(c_cart, W, name='McW'); McW
Subset McW of the 2-dimensional topological manifold R^2
sage: McW.is_closed()
True
```

```
>>> from sage.all import *
>>> W = span([[Integer(1), Integer(0)], [Integer(3), Integer(5)]], ZZ); W
Free module of degree 2 and rank 2 over Integer Ring
Echelon basis matrix:
[1 0]
[0 5]
>>> McW = ManifoldSubsetPullback(c_cart, W, name='McW'); McW
Subset McW of the 2-dimensional topological manifold R^2
>>> McW.is_closed()
True
```

The pullback of finite sets is closed:

```
sage: F = Family([vector(QQ, [1, 2], immutable=True), vector(QQ, [2, 3],␣
→immutable=True)])
sage: McF = ManifoldSubsetPullback(c_cart, F, name='McF'); McF
Subset McF of the 2-dimensional topological manifold R^2
sage: McF.is_closed()
True
```

```
>>> from sage.all import *
>>> F = Family([vector(QQ, [Integer(1), Integer(2)], immutable=True),␣
→vector(QQ, [Integer(2), Integer(3)], immutable=True)])
```

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```
>>> McF = ManifoldSubsetPullback(c_cart, F, name='McF'); McF
Subset McF of the 2-dimensional topological manifold R^2
>>> McF.is_closed()
True
```

is_open()

Return if `self` is (known to be) an open set.

This version of the method always returns `False`.

Because the map is continuous, the pullback is open if the `codomain_subset` is open.

However, the design of *ManifoldSubset* requires that open subsets are instances of the subclass *sage.manifolds.manifold.TopologicalManifold*. The constructor of *ManifoldSubsetPullback* delegates to a subclass of *sage.manifolds.manifold.TopologicalManifold* for some open subsets.

EXAMPLES:

```
sage: from sage.manifolds.subsets.pullback import ManifoldSubsetPullback
sage: M = Manifold(2, 'R^2', structure='topological')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2

sage: # needs sage.geometry.polyhedron
sage: P = Polyhedron(vertices=[[0, 0], [1, 2], [3, 4]]); P
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
sage: P.is_open()
False
sage: McP = ManifoldSubsetPullback(c_cart, P, name='McP'); McP
Subset McP of the 2-dimensional topological manifold R^2
sage: McP.is_open()
False
```

```
>>> from sage.all import *
>>> from sage.manifolds.subsets.pullback import ManifoldSubsetPullback
>>> M = Manifold(Integer(2), 'R^2', structure='topological')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#␣
↪Cartesian coordinates on R^2

>>> # needs sage.geometry.polyhedron
>>> P = Polyhedron(vertices=[[Integer(0), Integer(0)], [Integer(1),␣
↪Integer(2)], [Integer(3), Integer(4)]]); P
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
>>> P.is_open()
False
>>> McP = ManifoldSubsetPullback(c_cart, P, name='McP'); McP
Subset McP of the 2-dimensional topological manifold R^2
>>> McP.is_open()
False
```

some_elements()

Generate some elements of `self`.

EXAMPLES:

```
sage: # needs sage.geometry.polyhedron
sage: from sage.manifolds.subsets.pullback import ManifoldSubsetPullback
sage: M = Manifold(3, 'R^3', structure='topological')
sage: c_cart.<x,y,z> = M.chart() # Cartesian coordinates on R^3
sage: Cube = polytopes.cube(); Cube
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 8 vertices
sage: McCube = ManifoldSubsetPullback(c_cart, Cube, name='McCube'); McCube
Subset McCube of the 3-dimensional topological manifold R^3
sage: L = list(McCube.some_elements()); L
[Point on the 3-dimensional topological manifold R^3,
 Point on the 3-dimensional topological manifold R^3,
 Point on the 3-dimensional topological manifold R^3,
 Point on the 3-dimensional topological manifold R^3,
 Point on the 3-dimensional topological manifold R^3,
 Point on the 3-dimensional topological manifold R^3]
sage: list(p.coordinates(c_cart) for p in L)
[(0, 0, 0),
 (1, -1, -1),
 (1, 0, -1),
 (1, 1/2, 0),
 (1, -1/4, 1/2),
 (0, -5/8, 3/4)]

sage: # needs sage.geometry.polyhedron
sage: Empty = Polyhedron(ambient_dim=3)
sage: McEmpty = ManifoldSubsetPullback(c_cart, Empty, name='McEmpty')
sage: McEmpty
Subset McEmpty of the 3-dimensional topological manifold R^3
sage: list(McEmpty.some_elements())
[]
```

```
>>> from sage.all import *
>>> # needs sage.geometry.polyhedron
>>> from sage.manifolds.subsets.pullback import ManifoldSubsetPullback
>>> M = Manifold(Integer(3), 'R^3', structure='topological')
>>> c_cart = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_cart._first_
↪ngens(3)# Cartesian coordinates on R^3
>>> Cube = polytopes.cube(); Cube
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 8 vertices
>>> McCube = ManifoldSubsetPullback(c_cart, Cube, name='McCube'); McCube
Subset McCube of the 3-dimensional topological manifold R^3
>>> L = list(McCube.some_elements()); L
[Point on the 3-dimensional topological manifold R^3,
 Point on the 3-dimensional topological manifold R^3,
 Point on the 3-dimensional topological manifold R^3,
 Point on the 3-dimensional topological manifold R^3,
 Point on the 3-dimensional topological manifold R^3,
 Point on the 3-dimensional topological manifold R^3]
>>> list(p.coordinates(c_cart) for p in L)
[(0, 0, 0),
 (1, -1, -1),
 (1, 0, -1),
```

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```
 (1, 1/2, 0),
 (1, -1/4, 1/2),
 (0, -5/8, 3/4)]

>>> # needs sage.geometry.polyhedron
>>> Empty = Polyhedron(ambient_dim=Integer(3))
>>> McEmpty = ManifoldSubsetPullback(c_cart, Empty, name='McEmpty')
>>> McEmpty
Subset McEmpty of the 3-dimensional topological manifold R^3
>>> list(McEmpty.some_elements())
[]
```

CHAPTER

TWO

DIFFERENTIABLE MANIFOLDS

2.1 Differentiable Manifolds

Given a non-discrete topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$; see however [Ser1992] for $K = \mathbf{Q}_p$ and [Ber2008] for other fields), a *differentiable manifold over* K is a topological manifold M over K equipped with an atlas whose transitions maps are of class C^k (i.e. k-times continuously differentiable) for a fixed positive integer k (possibly $k = \infty$). M is then called a C^k*-manifold over* K.

Note that

- if the mention of K is omitted, then $K = \mathbf{R}$ is assumed;
- if $K = \mathbf{C}$, any C^k-manifold with $k \geq 1$ is actually a C^∞-manifold (even an analytic manifold);
- if $K = \mathbf{R}$, any C^k-manifold with $k \geq 1$ admits a compatible C^∞-structure (Whitney's smoothing theorem).

Differentiable manifolds are implemented via the class `DifferentiableManifold`. Open subsets of differentiable manifolds are also implemented via `DifferentiableManifold`, since they are differentiable manifolds by themselves.

The user interface is provided by the generic function `Manifold()`, with the argument `structure` set to `'differentiable'` and the argument `diff_degree` set to k, or the argument `structure` set to `'smooth'` (the default value).

Example 1: the 2-sphere as a differentiable manifold of dimension 2 over $\mathbf{R}$

One starts by declaring S^2 as a 2-dimensional differentiable manifold:

```
sage: M = Manifold(2, 'S^2')
sage: M
2-dimensional differentiable manifold S^2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2')
>>> M
2-dimensional differentiable manifold S^2
```

Since the base topological field has not been specified in the argument list of `Manifold`, $\mathbf{R}$ is assumed:

```
sage: M.base_field()
Real Field with 53 bits of precision
sage: dim(M)
2
```

```
>>> from sage.all import *
>>> M.base_field()
Real Field with 53 bits of precision
>>> dim(M)
2
```

By default, the created object is a smooth manifold:

```
sage: M.diff_degree()
 +Infinity
```

```
>>> from sage.all import *
>>> M.diff_degree()
 +Infinity
```

Let us consider the complement of a point, the "North pole" say; this is an open subset of S^2, which we call U:

```
sage: U = M.open_subset('U'); U
Open subset U of the 2-dimensional differentiable manifold S^2
```

```
>>> from sage.all import *
>>> U = M.open_subset('U'); U
Open subset U of the 2-dimensional differentiable manifold S^2
```

A standard chart on U is provided by the stereographic projection from the North pole to the equatorial plane:

```
sage: stereoN.<x,y> = U.chart(); stereoN
Chart (U, (x, y))
```

```
>>> from sage.all import *
>>> stereoN = U.chart(names=('x', 'y',)); (x, y,) = stereoN._first_ngens(2); stereoN
Chart (U, (x, y))
```

Thanks to the operator `<x,y>` on the left-hand side, the coordinates declared in a chart (here x and y), are accessible by their names; they are Sage's symbolic variables:

```
sage: y
y
sage: type(y)
<class 'sage.symbolic.expression.Expression'>
```

```
>>> from sage.all import *
>>> y
y
>>> type(y)
<class 'sage.symbolic.expression.Expression'>
```

The South pole is the point of coordinates $(x, y) = (0, 0)$ in the above chart:

```
sage: S = U.point((0,0), chart=stereoN, name='S'); S
Point S on the 2-dimensional differentiable manifold S^2
```

```
>>> from sage.all import *
>>> S = U.point((Integer(0),Integer(0)), chart=stereoN, name='S'); S
Point S on the 2-dimensional differentiable manifold S^2
```

Let us call V the open subset that is the complement of the South pole and let us introduce on it the chart induced by the stereographic projection from the South pole to the equatorial plane:

```
sage: V = M.open_subset('V'); V
Open subset V of the 2-dimensional differentiable manifold S^2
sage: stereoS.<u,v> = V.chart(); stereoS
Chart (V, (u, v))
```

```
>>> from sage.all import *
>>> V = M.open_subset('V'); V
Open subset V of the 2-dimensional differentiable manifold S^2
>>> stereoS = V.chart(names=('u', 'v',)); (u, v,) = stereoS._first_ngens(2); stereoS
Chart (V, (u, v))
```

The North pole is the point of coordinates $(u, v) = (0, 0)$ in this chart:

```
sage: N = V.point((0,0), chart=stereoS, name='N'); N
Point N on the 2-dimensional differentiable manifold S^2
```

```
>>> from sage.all import *
>>> N = V.point((Integer(0),Integer(0)), chart=stereoS, name='N'); N
Point N on the 2-dimensional differentiable manifold S^2
```

To fully construct the manifold, we declare that it is the union of U and V:

```
sage: M.declare_union(U,V)
```

```
>>> from sage.all import *
>>> M.declare_union(U,V)
```

and we provide the transition map between the charts `stereoN` = $(U, (x, y))$ and `stereoS` = $(V, (u, v))$, denoting by W the intersection of U and V (W is the subset of U defined by $x^2 + y^2 \neq 0$, as well as the subset of V defined by $u^2 + v^2 \neq 0$):

```
sage: stereoN_to_S = stereoN.transition_map(stereoS,
....:                [x/(x^2+y^2), y/(x^2+y^2)], intersection_name='W',
....:                restrictions1= x^2+y^2!=0, restrictions2= u^2+v^2!=0)
sage: stereoN_to_S
Change of coordinates from Chart (W, (x, y)) to Chart (W, (u, v))
sage: stereoN_to_S.display()
u = x/(x^2 + y^2)
v = y/(x^2 + y^2)
```

```
>>> from sage.all import *
>>> stereoN_to_S = stereoN.transition_map(stereoS,
...                [x/(x**Integer(2)+y**Integer(2)), y/(x**Integer(2)+y**Integer(2))],
↪ intersection_name='W',
...                restrictions1= x**Integer(2)+y**Integer(2)!=Integer(0),␣
```

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```
↪restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> stereoN_to_S
Change of coordinates from Chart (W, (x, y)) to Chart (W, (u, v))
>>> stereoN_to_S.display()
u = x/(x^2 + y^2)
v = y/(x^2 + y^2)
```

We give the name `W` to the Python variable representing $W = U \cap V$:

```
sage: W = U.intersection(V)
```

```
>>> from sage.all import *
>>> W = U.intersection(V)
```

The inverse of the transition map is computed by the method `inverse()`:

```
sage: stereoN_to_S.inverse()
Change of coordinates from Chart (W, (u, v)) to Chart (W, (x, y))
sage: stereoN_to_S.inverse().display()
x = u/(u^2 + v^2)
y = v/(u^2 + v^2)
```

```
>>> from sage.all import *
>>> stereoN_to_S.inverse()
Change of coordinates from Chart (W, (u, v)) to Chart (W, (x, y))
>>> stereoN_to_S.inverse().display()
x = u/(u^2 + v^2)
y = v/(u^2 + v^2)
```

At this stage, we have four open subsets on S^2:

```
sage: M.subset_family()
Set {S^2, U, V, W} of open subsets of the 2-dimensional differentiable manifold S^2
```

```
>>> from sage.all import *
>>> M.subset_family()
Set {S^2, U, V, W} of open subsets of the 2-dimensional differentiable manifold S^2
```

W is the open subset that is the complement of the two poles:

```
sage: N in W or S in W
False
```

```
>>> from sage.all import *
>>> N in W or S in W
False
```

The North pole lies in V and the South pole in U:

```
sage: N in V, N in U
(True, False)
```

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```
sage: S in U, S in V
(True, False)
```

```
>>> from sage.all import *
>>> N in V, N in U
(True, False)
>>> S in U, S in V
(True, False)
```

The manifold's (user) atlas contains four charts, two of them being restrictions of charts to a smaller domain:

```
sage: M.atlas()
[Chart (U, (x, y)), Chart (V, (u, v)), Chart (W, (x, y)), Chart (W, (u, v))]
```

```
>>> from sage.all import *
>>> M.atlas()
[Chart (U, (x, y)), Chart (V, (u, v)), Chart (W, (x, y)), Chart (W, (u, v))]
```

Let us consider the point of coordinates (1,2) in the chart `stereoN`:

```
sage: p = M.point((1,2), chart=stereoN, name='p'); p
Point p on the 2-dimensional differentiable manifold S^2
sage: p.parent()
2-dimensional differentiable manifold S^2
sage: p in W
True
```

```
>>> from sage.all import *
>>> p = M.point((Integer(1),Integer(2)), chart=stereoN, name='p'); p
Point p on the 2-dimensional differentiable manifold S^2
>>> p.parent()
2-dimensional differentiable manifold S^2
>>> p in W
True
```

The coordinates of p in the chart `stereoS` are computed by letting the chart act on the point:

```
sage: stereoS(p)
(1/5, 2/5)
```

```
>>> from sage.all import *
>>> stereoS(p)
(1/5, 2/5)
```

Given the definition of p, we have of course:

```
sage: stereoN(p)
(1, 2)
```

```
>>> from sage.all import *
>>> stereoN(p)
(1, 2)
```

Similarly:

```
sage: stereoS(N)
(0, 0)
sage: stereoN(S)
(0, 0)
```

```
>>> from sage.all import *
>>> stereoS(N)
(0, 0)
>>> stereoN(S)
(0, 0)
```

A differentiable scalar field on the sphere:

```
sage: f = M.scalar_field({stereoN: atan(x^2+y^2), stereoS: pi/2-atan(u^2+v^2)},
....:                    name='f')
sage: f
Scalar field f on the 2-dimensional differentiable manifold S^2
sage: f.display()
f: S^2 → R
on U: (x, y) ↦ arctan(x^2 + y^2)
on V: (u, v) ↦ 1/2*pi - arctan(u^2 + v^2)
sage: f(p)
arctan(5)
sage: f(N)
1/2*pi
sage: f(S)
0
sage: f.parent()
Algebra of differentiable scalar fields on the 2-dimensional differentiable
 manifold S^2
sage: f.parent().category()
Join of Category of commutative algebras over Symbolic Ring and Category of homsets␣
↪of topological spaces
```

```
>>> from sage.all import *
>>> f = M.scalar_field({stereoN: atan(x**Integer(2)+y**Integer(2)), stereoS: pi/
↪Integer(2)-atan(u**Integer(2)+v**Integer(2))},
...                    name='f')
>>> f
Scalar field f on the 2-dimensional differentiable manifold S^2
>>> f.display()
f: S^2 → R
on U: (x, y) ↦ arctan(x^2 + y^2)
on V: (u, v) ↦ 1/2*pi - arctan(u^2 + v^2)
>>> f(p)
arctan(5)
>>> f(N)
1/2*pi
>>> f(S)
0
>>> f.parent()
```

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```
Algebra of differentiable scalar fields on the 2-dimensional differentiable
 manifold S^2
>>> f.parent().category()
Join of Category of commutative algebras over Symbolic Ring and Category of homsets␣
↪of topological spaces
```

A differentiable manifold has a default vector frame, which, unless otherwise specified, is the coordinate frame associated with the first defined chart:

```
sage: M.default_frame()
Coordinate frame (U, (∂/∂x,∂/∂y))
sage: latex(M.default_frame())
\left(U, \left(\frac{\partial}{\partial x },\frac{\partial}{\partial y }\right)\right)
sage: M.default_frame() is stereoN.frame()
True
```

```
>>> from sage.all import *
>>> M.default_frame()
Coordinate frame (U, (∂/∂x,∂/∂y))
>>> latex(M.default_frame())
\left(U, \left(\frac{\partial}{\partial x },\frac{\partial}{\partial y }\right)\right)
>>> M.default_frame() is stereoN.frame()
True
```

A vector field on the sphere:

```
sage: w = M.vector_field(name='w')
sage: w[stereoN.frame(), :] = [x, y]
sage: w.add_comp_by_continuation(stereoS.frame(), W, stereoS)
sage: w.display() # display in the default frame (stereoN.frame())
w = x ∂/∂x + y ∂/∂y
sage: w.display(stereoS.frame())
w = -u ∂/∂u - v ∂/∂v
sage: w.parent()
Module X(S^2) of vector fields on the 2-dimensional differentiable
 manifold S^2
sage: w.parent().category()
Category of modules over Algebra of differentiable scalar fields on the
 2-dimensional differentiable manifold S^2
```

```
>>> from sage.all import *
>>> w = M.vector_field(name='w')
>>> w[stereoN.frame(), :] = [x, y]
>>> w.add_comp_by_continuation(stereoS.frame(), W, stereoS)
>>> w.display() # display in the default frame (stereoN.frame())
w = x ∂/∂x + y ∂/∂y
>>> w.display(stereoS.frame())
w = -u ∂/∂u - v ∂/∂v
>>> w.parent()
Module X(S^2) of vector fields on the 2-dimensional differentiable
 manifold S^2
>>> w.parent().category()
```

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```
Category of modules over Algebra of differentiable scalar fields on the
 2-dimensional differentiable manifold S^2
```

Vector fields act on scalar fields:

```
sage: w(f)
Scalar field w(f) on the 2-dimensional differentiable manifold S^2
sage: w(f).display()
w(f): S^2 → ℝ
on U: (x, y) ↦ 2*(x^2 + y^2)/(x^4 + 2*x^2*y^2 + y^4 + 1)
on V: (u, v) ↦ 2*(u^2 + v^2)/(u^4 + 2*u^2*v^2 + v^4 + 1)
sage: w(f) == f.differential()(w)
True
```

```
>>> from sage.all import *
>>> w(f)
Scalar field w(f) on the 2-dimensional differentiable manifold S^2
>>> w(f).display()
w(f): S^2 → ℝ
on U: (x, y) ↦ 2*(x^2 + y^2)/(x^4 + 2*x^2*y^2 + y^4 + 1)
on V: (u, v) ↦ 2*(u^2 + v^2)/(u^4 + 2*u^2*v^2 + v^4 + 1)
>>> w(f) == f.differential()(w)
True
```

The value of the vector field at point p is a vector tangent to the sphere:

```
sage: w.at(p)
Tangent vector w at Point p on the 2-dimensional differentiable manifold S^2
sage: w.at(p).display()
w = ∂/∂x + 2 ∂/∂y
sage: w.at(p).parent()
Tangent space at Point p on the 2-dimensional differentiable manifold S^2
```

```
>>> from sage.all import *
>>> w.at(p)
Tangent vector w at Point p on the 2-dimensional differentiable manifold S^2
>>> w.at(p).display()
w = ∂/∂x + 2 ∂/∂y
>>> w.at(p).parent()
Tangent space at Point p on the 2-dimensional differentiable manifold S^2
```

A 1-form on the sphere:

```
sage: df = f.differential() ; df
1-form df on the 2-dimensional differentiable manifold S^2
sage: df.display()
df = 2*x/(x^4 + 2*x^2*y^2 + y^4 + 1) dx + 2*y/(x^4 + 2*x^2*y^2 + y^4 + 1) dy
sage: df.display(stereoS.frame())
df = -2*u/(u^4 + 2*u^2*v^2 + v^4 + 1) du - 2*v/(u^4 + 2*u^2*v^2 + v^4 + 1) dv
sage: df.parent()
Module Omega^1(S^2) of 1-forms on the 2-dimensional differentiable
 manifold S^2
```

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```
sage: df.parent().category()
Category of modules over Algebra of differentiable scalar fields on the
 2-dimensional differentiable manifold S^2
```

```
>>> from sage.all import *
>>> df = f.differential() ; df
1-form df on the 2-dimensional differentiable manifold S^2
>>> df.display()
df = 2*x/(x^4 + 2*x^2*y^2 + y^4 + 1) dx + 2*y/(x^4 + 2*x^2*y^2 + y^4 + 1) dy
>>> df.display(stereoS.frame())
df = -2*u/(u^4 + 2*u^2*v^2 + v^4 + 1) du - 2*v/(u^4 + 2*u^2*v^2 + v^4 + 1) dv
>>> df.parent()
Module Omega^1(S^2) of 1-forms on the 2-dimensional differentiable
 manifold S^2
>>> df.parent().category()
Category of modules over Algebra of differentiable scalar fields on the
 2-dimensional differentiable manifold S^2
```

The value of the 1-form at point p is a linear form on the tangent space at p:

```
sage: df.at(p)
Linear form df on the Tangent space at Point p on the 2-dimensional
 differentiable manifold S^2
sage: df.at(p).display()
df = 1/13 dx + 2/13 dy
sage: df.at(p).parent()
Dual of the Tangent space at Point p on the 2-dimensional differentiable
 manifold S^2
```

```
>>> from sage.all import *
>>> df.at(p)
Linear form df on the Tangent space at Point p on the 2-dimensional
 differentiable manifold S^2
>>> df.at(p).display()
df = 1/13 dx + 2/13 dy
>>> df.at(p).parent()
Dual of the Tangent space at Point p on the 2-dimensional differentiable
 manifold S^2
```

Example 2: the Riemann sphere as a differentiable manifold of dimension 1 over $\mathbf{C}$

We declare the Riemann sphere $\mathbf{C}^*$ as a 1-dimensional differentiable manifold over $\mathbf{C}$:

```
sage: M = Manifold(1, 'ℂ*', field='complex'); M
1-dimensional complex manifold ℂ*
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(1), 'ℂ*', field='complex'); M
1-dimensional complex manifold ℂ*
```

We introduce a first open subset, which is actually $\mathbf{C} = \mathbf{C}^* \setminus \{\infty\}$ if we interpret $\mathbf{C}^*$ as the Alexandroff one-point compactification of $\mathbf{C}$:

```
sage: U = M.open_subset('U')
```

```
>>> from sage.all import *
>>> U = M.open_subset('U')
```

A natural chart on U is then nothing but the identity map of $\mathbf{C}$, hence we denote the associated coordinate by z:

```
sage: Z.<z> = U.chart()
```

```
>>> from sage.all import *
>>> Z = U.chart(names=('z',)); (z,) = Z._first_ngens(1)
```

The origin of the complex plane is the point of coordinate $z = 0$:

```
sage: O = U.point((0,), chart=Z, name='O'); O
Point O on the 1-dimensional complex manifold ℂ*
```

```
>>> from sage.all import *
>>> O = U.point((Integer(0),), chart=Z, name='O'); O
Point O on the 1-dimensional complex manifold ℂ*
```

Another open subset of $\mathbf{C}^*$ is $V = \mathbf{C}^* \setminus \{O\}$:

```
sage: V = M.open_subset('V')
```

```
>>> from sage.all import *
>>> V = M.open_subset('V')
```

We define a chart on V such that the point at infinity is the point of coordinate 0 in this chart:

```
sage: W.<w> = V.chart(); W
Chart (V, (w,))
sage: inf = M.point((0,), chart=W, name='inf', latex_name=r'\infty')
sage: inf
Point inf on the 1-dimensional complex manifold ℂ*
```

```
>>> from sage.all import *
>>> W = V.chart(names=('w',)); (w,) = W._first_ngens(1); W
Chart (V, (w,))
>>> inf = M.point((Integer(0),), chart=W, name='inf', latex_name=r'\infty')
>>> inf
Point inf on the 1-dimensional complex manifold ℂ*
```

To fully construct the Riemann sphere, we declare that it is the union of U and V:

```
sage: M.declare_union(U,V)
```

```
>>> from sage.all import *
>>> M.declare_union(U,V)
```

and we provide the transition map between the two charts as $w = 1/z$ on on $A = U \cap V$:

```
sage: Z_to_W = Z.transition_map(W, 1/z, intersection_name='A',
....:                           restrictions1= z!=0, restrictions2= w!=0)
sage: Z_to_W
Change of coordinates from Chart (A, (z,)) to Chart (A, (w,))
sage: Z_to_W.display()
w = 1/z
sage: Z_to_W.inverse()
Change of coordinates from Chart (A, (w,)) to Chart (A, (z,))
sage: Z_to_W.inverse().display()
z = 1/w
```

```
>>> from sage.all import *
>>> Z_to_W = Z.transition_map(W, Integer(1)/z, intersection_name='A',
...                           restrictions1= z!=Integer(0), restrictions2= w!
↪=Integer(0))
>>> Z_to_W
Change of coordinates from Chart (A, (z,)) to Chart (A, (w,))
>>> Z_to_W.display()
w = 1/z
>>> Z_to_W.inverse()
Change of coordinates from Chart (A, (w,)) to Chart (A, (z,))
>>> Z_to_W.inverse().display()
z = 1/w
```

Let consider the complex number i as a point of the Riemann sphere:

```
sage: i = M((I,), chart=Z, name='i'); i
Point i on the 1-dimensional complex manifold ℂ*
```

```
>>> from sage.all import *
>>> i = M((I,), chart=Z, name='i'); i
Point i on the 1-dimensional complex manifold ℂ*
```

Its coordinates with respect to the charts `Z` and `W` are:

```
sage: Z(i)
(I,)
sage: W(i)
(-I,)
```

```
>>> from sage.all import *
>>> Z(i)
(I,)
>>> W(i)
(-I,)
```

and we have:

```
sage: i in U
True
sage: i in V
True
```

```
>>> from sage.all import *
>>> i in U
True
>>> i in V
True
```

The following subsets and charts have been defined:

```
sage: M.subset_family()
Set {A, U, V, ℂ*} of open subsets of the 1-dimensional complex manifold ℂ*
sage: M.atlas()
[Chart (U, (z,)), Chart (V, (w,)), Chart (A, (z,)), Chart (A, (w,))]
```

```
>>> from sage.all import *
>>> M.subset_family()
Set {A, U, V, ℂ*} of open subsets of the 1-dimensional complex manifold ℂ*
>>> M.atlas()
[Chart (U, (z,)), Chart (V, (w,)), Chart (A, (z,)), Chart (A, (w,))]
```

A constant map $\mathbf{C}^* \to \mathbf{C}$:

```
sage: f = M.constant_scalar_field(3+2*I, name='f'); f
Scalar field f on the 1-dimensional complex manifold ℂ*
sage: f.display()
f: ℂ* → ℂ
on U: z ↦ 2*I + 3
on V: w ↦ 2*I + 3
sage: f(O)
2*I + 3
sage: f(i)
2*I + 3
sage: f(inf)
2*I + 3
sage: f.parent()
Algebra of differentiable scalar fields on the 1-dimensional complex
 manifold ℂ*
sage: f.parent().category()
Join of Category of commutative algebras over Symbolic Ring and Category of homsets␣
↪of topological spaces
```

```
>>> from sage.all import *
>>> f = M.constant_scalar_field(Integer(3)+Integer(2)*I, name='f'); f
Scalar field f on the 1-dimensional complex manifold ℂ*
>>> f.display()
f: ℂ* → ℂ
on U: z ↦ 2*I + 3
on V: w ↦ 2*I + 3
>>> f(O)
2*I + 3
>>> f(i)
2*I + 3
>>> f(inf)
2*I + 3
```

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```
>>> f.parent()
Algebra of differentiable scalar fields on the 1-dimensional complex
 manifold ℂ*
>>> f.parent().category()
Join of Category of commutative algebras over Symbolic Ring and Category of homsets␣
↪of topological spaces
```

A vector field on the Riemann sphere:

```
sage: v = M.vector_field(name='v')
sage: v[Z.frame(), 0] = z^2
sage: v.add_comp_by_continuation(W.frame(), U.intersection(V), W)
sage: v.display(Z.frame())
v = z^2 ∂/∂z
sage: v.display(W.frame())
v = -∂/∂w
sage: v.parent()
Module X(ℂ*) of vector fields on the 1-dimensional complex manifold ℂ*
```

```
>>> from sage.all import *
>>> v = M.vector_field(name='v')
>>> v[Z.frame(), Integer(0)] = z**Integer(2)
>>> v.add_comp_by_continuation(W.frame(), U.intersection(V), W)
>>> v.display(Z.frame())
v = z^2 ∂/∂z
>>> v.display(W.frame())
v = -∂/∂w
>>> v.parent()
Module X(ℂ*) of vector fields on the 1-dimensional complex manifold ℂ*
```

The vector field v acting on the scalar field f:

```
sage: v(f)
Scalar field zero on the 1-dimensional complex manifold ℂ*
```

```
>>> from sage.all import *
>>> v(f)
Scalar field zero on the 1-dimensional complex manifold ℂ*
```

Since f is constant, $v(f)$ is vanishing:

```
sage: v(f).display()
zero: ℂ* → ℂ
on U: z ↦ 0
on V: w ↦ 0
```

```
>>> from sage.all import *
>>> v(f).display()
zero: ℂ* → ℂ
on U: z ↦ 0
on V: w ↦ 0
```

The value of the vector field v at the point ∞ is a vector tangent to the Riemann sphere:

```
sage: v.at(inf)
Tangent vector v at Point inf on the 1-dimensional complex manifold ℂ*
sage: v.at(inf).display()
v = -∂/∂w
sage: v.at(inf).parent()
Tangent space at Point inf on the 1-dimensional complex manifold ℂ*
```

```
>>> from sage.all import *
>>> v.at(inf)
Tangent vector v at Point inf on the 1-dimensional complex manifold ℂ*
>>> v.at(inf).display()
v = -∂/∂w
>>> v.at(inf).parent()
Tangent space at Point inf on the 1-dimensional complex manifold ℂ*
```

AUTHORS:

- Eric Gourgoulhon (2015): initial version
- Travis Scrimshaw (2016): review tweaks
- Michael Jung (2020): tensor bundles and orientability
- Matthias Koeppe (2021): refactoring of subsets code

REFERENCES:

- [Lee2013]
- [KN1963]
- [Huy2005]
- [Ser1992]
- [Ber2008]
- [BG1988]

class sage.manifolds.differentiable.manifold.**DifferentiableManifold**(*n*, *name*, *field*, *structure*, *base_manifold=None*, *diff_degree=+Infinity*, *latex_name=None*, *start_index=0*, *category=None*, *unique_tag=None*)

Bases: *TopologicalManifold*

Differentiable manifold over a topological field K.

Given a non-discrete topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$; see however [Ser1992] for $K = \mathbf{Q}_p$ and [Ber2008] for other fields), a *differentiable manifold over* K is a topological manifold M over K equipped with an atlas whose transitions maps are of class C^k (i.e. k-times continuously differentiable) for a fixed positive integer k (possibly $k = \infty$). M is then called a C^k*-manifold over* K.

Note that

- if the mention of K is omitted, then $K = \mathbf{R}$ is assumed;
- if $K = \mathbf{C}$, any C^k-manifold with $k \geq 1$ is actually a C^∞-manifold (even an analytic manifold);
- if $K = \mathbf{R}$, any C^k-manifold with $k \geq 1$ admits a compatible C^∞-structure (Whitney's smoothing theorem).

INPUT:

- `n` – positive integer; dimension of the manifold
- `name` – string; name (symbol) given to the manifold
- `field` – field K on which the manifold is defined; allowed values are
  - `'real'` or an object of type `RealField` (e.g., `RR`) for a manifold over **R**
  - `'complex'` or an object of type `ComplexField` (e.g., `CC`) for a manifold over **C**
  - an object in the category of topological fields (see `Fields` and `TopologicalSpaces`) for other types of manifolds
- `structure` – manifold structure (see `DifferentialStructure` or `RealDifferentialStructure`)
- `base_manifold` – (default: `None`) if not `None`, must be a differentiable manifold; the created object is then an open subset of `base_manifold`
- `diff_degree` – (default: `infinity`) degree k of differentiability
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the manifold; if none is provided, it is set to `name`
- `start_index` – (default: 0) integer; lower value of the range of indices used for "indexed objects" on the manifold, e.g. coordinates in a chart
- `category` – (default: `None`) to specify the category; if `None`, `Manifolds(field).Differentiable()` (or `Manifolds(field).Smooth()` if `diff_degree` = `infinity`) is assumed (see the category `Manifolds`)
- `unique_tag` – (default: `None`) tag used to force the construction of a new object when all the other arguments have been used previously (without `unique_tag`, the `UniqueRepresentation` behavior inherited from `ManifoldSubset`, via `TopologicalManifold`, would return the previously constructed object corresponding to these arguments).

EXAMPLES:

A 4-dimensional differentiable manifold (over **R**):

```
sage: M = Manifold(4, 'M', latex_name=r'\mathcal{M}'); M
4-dimensional differentiable manifold M
sage: type(M)
<class 'sage.manifolds.differentiable.manifold.DifferentiableManifold_with_
↪category'>
sage: latex(M)
\mathcal{M}
sage: dim(M)
4
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', latex_name=r'\mathcal{M}'); M
4-dimensional differentiable manifold M
>>> type(M)
<class 'sage.manifolds.differentiable.manifold.DifferentiableManifold_with_
↪category'>
>>> latex(M)
\mathcal{M}
>>> dim(M)
4
```

Since the base field has not been specified, **R** has been assumed:

```
sage: M.base_field()
Real Field with 53 bits of precision
```

```
>>> from sage.all import *
>>> M.base_field()
Real Field with 53 bits of precision
```

Since the degree of differentiability has not been specified, the default value, C^∞, has been assumed:

```
sage: M.diff_degree()
+Infinity
```

```
>>> from sage.all import *
>>> M.diff_degree()
+Infinity
```

The input parameter `start_index` defines the range of indices on the manifold:

```
sage: M = Manifold(4, 'M')
sage: list(M.irange())
[0, 1, 2, 3]
sage: M = Manifold(4, 'M', start_index=1)
sage: list(M.irange())
[1, 2, 3, 4]
sage: list(Manifold(4, 'M', start_index=-2).irange())
[-2, -1, 0, 1]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M')
>>> list(M.irange())
[0, 1, 2, 3]
>>> M = Manifold(Integer(4), 'M', start_index=Integer(1))
>>> list(M.irange())
[1, 2, 3, 4]
>>> list(Manifold(Integer(4), 'M', start_index=-Integer(2)).irange())
[-2, -1, 0, 1]
```

A complex manifold:

```
sage: N = Manifold(3, 'N', field='complex'); N
3-dimensional complex manifold N
```

```
>>> from sage.all import *
>>> N = Manifold(Integer(3), 'N', field='complex'); N
3-dimensional complex manifold N
```

A differentiable manifold over $\mathbf{Q}_5$, the field of 5-adic numbers:

```
sage: N = Manifold(2, 'N', field=Qp(5)); N
2-dimensional differentiable manifold N over the 5-adic Field with
 capped relative precision 20
```

```
>>> from sage.all import *
>>> N = Manifold(Integer(2), 'N', field=Qp(Integer(5))); N
2-dimensional differentiable manifold N over the 5-adic Field with
 capped relative precision 20
```

A differentiable manifold is of course a topological manifold:

```
sage: isinstance(M, sage.manifolds.manifold.TopologicalManifold)
True
sage: isinstance(N, sage.manifolds.manifold.TopologicalManifold)
True
```

```
>>> from sage.all import *
>>> isinstance(M, sage.manifolds.manifold.TopologicalManifold)
True
>>> isinstance(N, sage.manifolds.manifold.TopologicalManifold)
True
```

A differentiable manifold is a Sage *parent* object, in the category of differentiable (here smooth) manifolds over a given topological field (see `Manifolds`):

```
sage: isinstance(M, Parent)
True
sage: M.category()
Category of smooth manifolds over Real Field with 53 bits of precision
sage: from sage.categories.manifolds import Manifolds
sage: M.category() is Manifolds(RR).Smooth()
True
sage: M.category() is Manifolds(M.base_field()).Smooth()
True
sage: M in Manifolds(RR).Smooth()
True
sage: N in Manifolds(Qp(5)).Smooth()
True
```

```
>>> from sage.all import *
>>> isinstance(M, Parent)
True
>>> M.category()
Category of smooth manifolds over Real Field with 53 bits of precision
>>> from sage.categories.manifolds import Manifolds
>>> M.category() is Manifolds(RR).Smooth()
True
>>> M.category() is Manifolds(M.base_field()).Smooth()
True
>>> M in Manifolds(RR).Smooth()
True
>>> N in Manifolds(Qp(Integer(5))).Smooth()
True
```

The corresponding Sage *elements* are points:

```
sage: X.<t, x, y, z> = M.chart()
sage: p = M.an_element(); p
Point on the 4-dimensional differentiable manifold M
sage: p.parent()
4-dimensional differentiable manifold M
sage: M.is_parent_of(p)
True
sage: p in M
True
```

```
>>> from sage.all import *
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_ngens(4)
>>> p = M.an_element(); p
Point on the 4-dimensional differentiable manifold M
>>> p.parent()
4-dimensional differentiable manifold M
>>> M.is_parent_of(p)
True
>>> p in M
True
```

The manifold's points are instances of class *ManifoldPoint*:

```
sage: isinstance(p, sage.manifolds.point.ManifoldPoint)
True
```

```
>>> from sage.all import *
>>> isinstance(p, sage.manifolds.point.ManifoldPoint)
True
```

Since an open subset of a differentiable manifold M is itself a differentiable manifold, open subsets of M have all attributes of manifolds:

```
sage: U = M.open_subset('U', coord_def={X: t>0}); U
Open subset U of the 4-dimensional differentiable manifold M
sage: U.category()
Join of Category of subobjects of sets and Category of smooth manifolds
 over Real Field with 53 bits of precision
sage: U.base_field() == M.base_field()
True
sage: dim(U) == dim(M)
True
```

```
>>> from sage.all import *
>>> U = M.open_subset('U', coord_def={X: t>Integer(0)}); U
Open subset U of the 4-dimensional differentiable manifold M
>>> U.category()
Join of Category of subobjects of sets and Category of smooth manifolds
 over Real Field with 53 bits of precision
>>> U.base_field() == M.base_field()
True
>>> dim(U) == dim(M)
True
```

The manifold passes all the tests of the test suite relative to its category:

```
sage: TestSuite(M).run()
```

```
>>> from sage.all import *
>>> TestSuite(M).run()
```

affine_connection(*name*, *latex_name=None*)

Define an affine connection on the manifold.

See *AffineConnection* for a complete documentation.

INPUT:

- `name` – name given to the affine connection
- `latex_name` – (default: `None`) LaTeX symbol to denote the affine connection

OUTPUT:

- the affine connection, as an instance of *AffineConnection*

EXAMPLES:

Affine connection on an open subset of a 3-dimensional smooth manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: A = M.open_subset('A', latex_name=r'\mathcal{A}')
sage: nab = A.affine_connection('nabla', r'\nabla') ; nab
Affine connection nabla on the Open subset A of the 3-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> A = M.open_subset('A', latex_name=r'\mathcal{A}')
>>> nab = A.affine_connection('nabla', r'\nabla') ; nab
Affine connection nabla on the Open subset A of the 3-dimensional
 differentiable manifold M
```

See also

AffineConnection for more examples.

automorphism_field(**comp*, ***kwargs*)

Define a field of automorphisms (invertible endomorphisms in each tangent space) on `self`.

Via the argument `dest_map`, it is possible to let the field take its values on another manifold. More precisely, if M is the current manifold, N a differentiable manifold and $\Phi : M \to N$ a differentiable map, a *field of automorphisms along* M *with values on* N is a differentiable map

$$t : M \longrightarrow T^{(1,1)}N$$

($T^{(1,1)}N$ being the tensor bundle of type $(1, 1)$ over N) such that

$$\forall p \in M, \; t(p) \in \mathrm{GL}\left(T_{\Phi(p)}N\right),$$

where $\mathrm{GL}\left(T_{\Phi(p)}N\right)$ is the general linear group of the tangent space $T_{\Phi(p)}N$.

The standard case of a field of automorphisms *on* M corresponds to $N = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in N (M is then an open interval of $\mathbf{R}$).

See also:

`AutomorphismField` and `AutomorphismFieldParal` for a complete documentation.

INPUT:

- `comp` – (optional) either the components of the field of automorphisms with respect to the vector frame specified by the argument `frame` or a dictionary of components, the keys of which are vector frames or pairs `(f, c)` where `f` is a vector frame and `c` the chart in which the components are expressed
- `frame` – (default: `None`; unused if `comp` is not given or is a dictionary) vector frame in which the components are given; if `None`, the default vector frame of `self` is assumed
- `chart` – (default: `None`; unused if `comp` is not given or is a dictionary) coordinate chart in which the components are expressed; if `None`, the default chart on the domain of `frame` is assumed
- `name` – (default: `None`) name given to the field
- `latex_name` – (default: `None`) LaTeX symbol to denote the field; if none is provided, the LaTeX symbol is set to `name`
- `dest_map` – (default: `None`) the destination map $\Phi : M \to N$; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of a field of automorphisms *on* M), otherwise `dest_map` must be a `DiffMap`

OUTPUT:

- a `AutomorphismField` (or if N is parallelizable, a `AutomorphismFieldParal`) representing the defined field of automorphisms

EXAMPLES:

A field of automorphisms on a 2-dimensional manifold:

```
sage: M = Manifold(2,'M')
sage: X.<x,y> = M.chart()
sage: a = M.automorphism_field([[1+x^2, 0], [0, 1+y^2]], name='A')
sage: a
Field of tangent-space automorphisms A on the 2-dimensional
 differentiable manifold M
sage: a.parent()
General linear group of the Free module X(M) of vector fields on
 the 2-dimensional differentiable manifold M
sage: a(X.frame()[0]).display()
A(∂/∂x) = (x^2 + 1) ∂/∂x
sage: a(X.frame()[1]).display()
A(∂/∂y) = (y^2 + 1) ∂/∂y
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2),'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> a = M.automorphism_field([[Integer(1)+x**Integer(2), Integer(0)],
→[Integer(0), Integer(1)+y**Integer(2)]], name='A')
>>> a
```

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```
Field of tangent-space automorphisms A on the 2-dimensional
 differentiable manifold M
>>> a.parent()
General linear group of the Free module X(M) of vector fields on
 the 2-dimensional differentiable manifold M
>>> a(X.frame()[Integer(0)]).display()
A(∂/∂x) = (x^2 + 1) ∂/∂x
>>> a(X.frame()[Integer(1)]).display()
A(∂/∂y) = (y^2 + 1) ∂/∂y
```

For more examples, see `AutomorphismField` and `AutomorphismFieldParal`.

automorphism_field_group(*dest_map=None*)

Return the group of tangent-space automorphism fields defined on `self`, possibly with values in another manifold, as a module over the algebra of scalar fields defined on `self`.

If M is the current manifold and Φ a differentiable map $\Phi : M \to N$, where N is a differentiable manifold, this method called with `dest_map` being Φ returns the general linear group $\mathrm{GL}(\mathfrak{X}(M, \Phi))$ of the module $\mathfrak{X}(M, \Phi)$ of vector fields along M with values in $\Phi(M) \subset N$.

INPUT:

- `dest_map` – (default: `None`) destination map, i.e. a differentiable map $\Phi : M \to N$, where M is the current manifold and N a differentiable manifold; if `None`, it is assumed that $N = M$ and that Φ is the identity map, otherwise `dest_map` must be a `DiffMap`

OUTPUT:

- a `AutomorphismFieldParalGroup` (if N is parallelizable) or a `AutomorphismFieldGroup` (if N is not parallelizable) representing $\mathrm{GL}(\mathfrak{X}(U, \Phi))$

EXAMPLES:

Group of tangent-space automorphism fields of a 2-dimensional differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: M.automorphism_field_group()
General linear group of the Module X(M) of vector fields on the
 2-dimensional differentiable manifold M
sage: M.automorphism_field_group().category()
Category of groups
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> M.automorphism_field_group()
General linear group of the Module X(M) of vector fields on the
 2-dimensional differentiable manifold M
>>> M.automorphism_field_group().category()
Category of groups
```

See also

For more examples, see `AutomorphismFieldParalGroup` and `AutomorphismFieldGroup`.

change_of_frame(*frame1*, *frame2*)

Return a change of vector frames defined on `self`.

INPUT:

- `frame1` – vector frame 1
- `frame2` – vector frame 2

OUTPUT:

- a *AutomorphismField* representing, at each point, the vector space automorphism P that relates frame 1, (e_i) say, to frame 2, (n_i) say, according to $n_i = P(e_i)$

EXAMPLES:

Change of vector frames induced by a change of coordinates:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: c_uv.<u,v> = M.chart()
sage: c_xy.transition_map(c_uv, (x+y, x-y))
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
sage: M.change_of_frame(c_xy.frame(), c_uv.frame())
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
sage: M.change_of_frame(c_xy.frame(), c_uv.frame())[:]
[ 1/2  1/2]
[ 1/2 -1/2]
sage: M.change_of_frame(c_uv.frame(), c_xy.frame())
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
sage: M.change_of_frame(c_uv.frame(), c_xy.frame())[:]
[ 1  1]
[ 1 -1]
sage: M.change_of_frame(c_uv.frame(), c_xy.frame()) == \
....:       M.change_of_frame(c_xy.frame(), c_uv.frame()).inverse()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> c_uv = M.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> c_xy.transition_map(c_uv, (x+y, x-y))
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
>>> M.change_of_frame(c_xy.frame(), c_uv.frame())
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
>>> M.change_of_frame(c_xy.frame(), c_uv.frame())[:]
[ 1/2  1/2]
[ 1/2 -1/2]
>>> M.change_of_frame(c_uv.frame(), c_xy.frame())
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
>>> M.change_of_frame(c_uv.frame(), c_xy.frame())[:]
[ 1  1]
```

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```
[ 1 -1]
>>> M.change_of_frame(c_uv.frame(), c_xy.frame()) ==       M.change_of_
↪frame(c_xy.frame(), c_uv.frame()).inverse()
True
```

In the present example, the manifold M is parallelizable, so that the module $X(M)$ of vector fields on M is free. A change of frame on M is then identical to a change of basis in $X(M)$:

```
sage: XM = M.vector_field_module() ; XM
Free module X(M) of vector fields on the 2-dimensional
 differentiable manifold M
sage: XM.print_bases()
Bases defined on the Free module X(M) of vector fields on the
 2-dimensional differentiable manifold M:
 - (M, (∂/∂x,∂/∂y)) (default basis)
 - (M, (∂/∂u,∂/∂v))
sage: XM.change_of_basis(c_xy.frame(), c_uv.frame())
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
sage: M.change_of_frame(c_xy.frame(), c_uv.frame()) is \
....:  XM.change_of_basis(c_xy.frame(), c_uv.frame())
True
```

```
>>> from sage.all import *
>>> XM = M.vector_field_module() ; XM
Free module X(M) of vector fields on the 2-dimensional
 differentiable manifold M
>>> XM.print_bases()
Bases defined on the Free module X(M) of vector fields on the
 2-dimensional differentiable manifold M:
 - (M, (∂/∂x,∂/∂y)) (default basis)
 - (M, (∂/∂u,∂/∂v))
>>> XM.change_of_basis(c_xy.frame(), c_uv.frame())
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
>>> M.change_of_frame(c_xy.frame(), c_uv.frame()) is  XM.change_of_basis(c_xy.
↪frame(), c_uv.frame())
True
```

changes_of_frame()

Return all the changes of vector frames defined on `self`.

OUTPUT:

- dictionary of fields of tangent-space automorphisms representing the changes of frames, the keys being the pair of frames

EXAMPLES:

Let us consider a first vector frame on a 2-dimensional differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
```

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```
sage: e = X.frame(); e
Coordinate frame (M, (∂/∂x,∂/∂y))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> e = X.frame(); e
Coordinate frame (M, (∂/∂x,∂/∂y))
```

At this stage, the dictionary of changes of frame is empty:

```
sage: M.changes_of_frame()
{}
```

```
>>> from sage.all import *
>>> M.changes_of_frame()
{}
```

We introduce a second frame on the manifold, relating it to frame `e` by a field of tangent space automorphisms:

```
sage: a = M.automorphism_field(name='a')
sage: a[:] = [[-y, x], [1, 2]]
sage: f = e.new_frame(a, 'f'); f
Vector frame (M, (f_0,f_1))
```

```
>>> from sage.all import *
>>> a = M.automorphism_field(name='a')
>>> a[:] = [[-y, x], [Integer(1), Integer(2)]]
>>> f = e.new_frame(a, 'f'); f
Vector frame (M, (f_0,f_1))
```

Then we have:

```
sage: M.changes_of_frame()  # random (dictionary output)
{(Coordinate frame (M, (∂/∂x,∂/∂y)),
  Vector frame (M, (f_0,f_1))): Field of tangent-space
   automorphisms on the 2-dimensional differentiable manifold M,
 (Vector frame (M, (f_0,f_1)),
  Coordinate frame (M, (∂/∂x,∂/∂y))): Field of tangent-space
   automorphisms on the 2-dimensional differentiable manifold M}
```

```
>>> from sage.all import *
>>> M.changes_of_frame()  # random (dictionary output)
{(Coordinate frame (M, (∂/∂x,∂/∂y)),
  Vector frame (M, (f_0,f_1))): Field of tangent-space
   automorphisms on the 2-dimensional differentiable manifold M,
 (Vector frame (M, (f_0,f_1)),
  Coordinate frame (M, (∂/∂x,∂/∂y))): Field of tangent-space
   automorphisms on the 2-dimensional differentiable manifold M}
```

Some checks:

```
sage: M.changes_of_frame()[(e,f)] == a
True
sage: M.changes_of_frame()[(f,e)] == a^(-1)
True
```

```
>>> from sage.all import *
>>> M.changes_of_frame()[(e,f)] == a
True
>>> M.changes_of_frame()[(f,e)] == a**(-Integer(1))
True
```

coframes()

Return the list of coframes defined on open subsets of `self`.

OUTPUT: list of coframes defined on open subsets of `self`

EXAMPLES:

Coframes on subsets of $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'R^2')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: M.coframes()
[Coordinate coframe (R^2, (dx,dy))]
sage: e = M.vector_frame('e')
sage: M.coframes()
[Coordinate coframe (R^2, (dx,dy)), Coframe (R^2, (e^0,e^1))]
sage: U = M.open_subset('U', coord_def={c_cart: x^2+y^2<1}) # unit disk
sage: U.coframes()
[Coordinate coframe (U, (dx,dy))]
sage: e.restrict(U)
Vector frame (U, (e_0,e_1))
sage: U.coframes()
[Coordinate coframe (U, (dx,dy)), Coframe (U, (e^0,e^1))]
sage: M.coframes()
[Coordinate coframe (R^2, (dx,dy)),
 Coframe (R^2, (e^0,e^1)),
 Coordinate coframe (U, (dx,dy)),
 Coframe (U, (e^0,e^1))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#
↪Cartesian coordinates on R^2
>>> M.coframes()
[Coordinate coframe (R^2, (dx,dy))]
>>> e = M.vector_frame('e')
>>> M.coframes()
[Coordinate coframe (R^2, (dx,dy)), Coframe (R^2, (e^0,e^1))]
>>> U = M.open_subset('U', coord_def={c_cart: x**Integer(2)+y**Integer(2)
↪<Integer(1)}) # unit disk
>>> U.coframes()
[Coordinate coframe (U, (dx,dy))]
>>> e.restrict(U)
```

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```
Vector frame (U, (e_0,e_1))
>>> U.coframes()
[Coordinate coframe (U, (dx,dy)), Coframe (U, (e^0,e^1))]
>>> M.coframes()
[Coordinate coframe (R^2, (dx,dy)),
 Coframe (R^2, (e^0,e^1)),
 Coordinate coframe (U, (dx,dy)),
 Coframe (U, (e^0,e^1))]
```

cotangent_bundle(*dest_map=None*)

Return the cotangent bundle possibly along a destination map with base space `self`.

See also

`TensorBundle` for complete documentation.

INPUT:

- `dest_map` – (default: `None`) destination map $\Phi : M \rightarrow N$ (type: `DiffMap`) from which the cotangent bundle is pulled back; if `None`, it is assumed that $N = M$ and Φ is the identity map of M (case of the standard tangent bundle over M)

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: cTM = M.cotangent_bundle(); cTM
Cotangent bundle T*M over the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> cTM = M.cotangent_bundle(); cTM
Cotangent bundle T*M over the 2-dimensional differentiable
 manifold M
```

curve(*coord_expression, param, chart=None, name=None, latex_name=None*)

Define a differentiable curve in the manifold.

See also

`DifferentiableCurve` for details.

INPUT:

- `coord_expression` – either
  - (i) a dictionary whose keys are charts on the manifold and values the coordinate expressions (as lists or tuples) of the curve in the given chart
  - (ii) a single coordinate expression in a given chart on the manifold, the latter being provided by the argument `chart`

  in both cases, if the dimension of the manifold is 1, a single coordinate expression can be passed instead of a tuple with a single element

- param – tuple of the type (t, t_min, t_max), where
  - t is the curve parameter used in coord_expression;
  - t_min is its minimal value;
  - t_max its maximal value;

  if t_min=-Infinity and t_max=+Infinity, they can be omitted and t can be passed for param instead of the tuple (t, t_min, t_max)
- chart – (default: None) chart on the manifold used for case (ii) above; if None the default chart of the manifold is assumed
- name – (default: None) string; symbol given to the curve
- latex_name – (default: None) string; LaTeX symbol to denote the curve; if none is provided, name will be used

OUTPUT: *DifferentiableCurve*

EXAMPLES:

The lemniscate of Gerono in the 2-dimensional Euclidean plane:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: R.<t> = manifolds.RealLine()
sage: c = M.curve([sin(t), sin(2*t)/2], (t, 0, 2*pi), name='c') ; c
Curve c in the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1)
>>> c = M.curve([sin(t), sin(Integer(2)*t)/Integer(2)], (t, Integer(0),
↪Integer(2)*pi), name='c') ; c
Curve c in the 2-dimensional differentiable manifold M
```

The same definition with the coordinate expression passed as a dictionary:

```
sage: c = M.curve({X: [sin(t), sin(2*t)/2]}, (t, 0, 2*pi), name='c') ; c
Curve c in the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> c = M.curve({X: [sin(t), sin(Integer(2)*t)/Integer(2)]}, (t, Integer(0),
↪Integer(2)*pi), name='c') ; c
Curve c in the 2-dimensional differentiable manifold M
```

An example of definition with t_min and t_max omitted: a helix in $\mathbf{R}^3$:

```
sage: R3 = Manifold(3, 'R^3')
sage: X.<x,y,z> = R3.chart()
sage: c = R3.curve([cos(t), sin(t), t], t, name='c') ; c
Curve c in the 3-dimensional differentiable manifold R^3
sage: c.domain() # check that t is unbounded
Real number line ℝ
```

```
>>> from sage.all import *
>>> R3 = Manifold(Integer(3), 'R^3')
>>> X = R3.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> c = R3.curve([cos(t), sin(t), t], t, name='c') ; c
Curve c in the 3-dimensional differentiable manifold R^3
>>> c.domain() # check that t is unbounded
Real number line ℝ
```

See also:

`DifferentiableCurve` for more examples, including plots.

de_rham_complex(*dest_map=None*)

Return the set of mixed forms defined on `self`, possibly with values in another manifold, as a graded algebra.

See also:

`MixedFormAlgebra` for complete documentation.

INPUT:

- `dest_map` – (default: `None`) destination map, i.e. a differentiable map $\Phi : M \rightarrow N$, where M is the current manifold and N a differentiable manifold; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of mixed forms *on* M), otherwise `dest_map` must be a `DiffMap`

OUTPUT:

- a `MixedFormAlgebra` representing the graded algebra $\Omega^*(M, \Phi)$ of mixed forms on M taking values on $\Phi(M) \subset N$

EXAMPLES:

Graded algebra of mixed forms on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: M.mixed_form_algebra()
Graded algebra Omega^*(M) of mixed differential forms on the
 2-dimensional differentiable manifold M
sage: M.mixed_form_algebra().category()
Join of Category of graded algebras over Symbolic Ring and Category of chain␣
↪complexes over Symbolic Ring
sage: M.mixed_form_algebra().base_ring()
Symbolic Ring
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> M.mixed_form_algebra()
Graded algebra Omega^*(M) of mixed differential forms on the
 2-dimensional differentiable manifold M
>>> M.mixed_form_algebra().category()
```

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```
Join of Category of graded algebras over Symbolic Ring and Category of chain
→complexes over Symbolic Ring
>>> M.mixed_form_algebra().base_ring()
Symbolic Ring
```

The outcome is cached:

```
sage: M.mixed_form_algebra() is M.mixed_form_algebra()
True
```

```
>>> from sage.all import *
>>> M.mixed_form_algebra() is M.mixed_form_algebra()
True
```

default_frame()

Return the default vector frame defined on `self`.

By *vector frame*, it is meant a field on the manifold that provides, at each point p, a vector basis of the tangent space at p.

Unless changed via `set_default_frame()`, the default frame is the first one defined on the manifold, usually implicitly as the coordinate basis associated with the first chart defined on the manifold.

OUTPUT:

- a `VectorFrame` representing the default vector frame

EXAMPLES:

The default vector frame is often the coordinate frame associated with the first chart defined on the manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: M.default_frame()
Coordinate frame (M, (∂/∂x,∂/∂y))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> M.default_frame()
Coordinate frame (M, (∂/∂x,∂/∂y))
```

degenerate_metric(*name*, *latex_name=None*, *dest_map=None*)

Define a degenerate (or null or lightlike) metric on the manifold.

A *degenerate metric* is a field of degenerate symmetric bilinear forms acting in the tangent spaces.

See `DegenerateMetric` for a complete documentation.

INPUT:

- `name` – name given to the metric
- `latex_name` – (default: `None`) LaTeX symbol to denote the metric; if `None`, it is formed from `name`
- `dest_map` – (default: `None`) instance of class `DiffMap` representing the destination map $\Phi : U \to M$, where U is the current manifold; if `None`, the identity map is assumed (case of a metric tensor field *on* U)

OUTPUT:

- instance of `DegenerateMetric` representing the defined degenerate metric.

EXAMPLES:

Lightlike cone:

```
sage: M = Manifold(3, 'M'); X.<x,y,z> = M.chart()
sage: g = M.degenerate_metric('g'); g
degenerate metric g on the 3-dimensional differentiable manifold M
sage: det(g)
Scalar field zero on the 3-dimensional differentiable manifold M
sage: g.parent()
Free module T^(0,2)(M) of type-(0,2) tensors fields on the
3-dimensional differentiable manifold M
sage: g[0,0], g[0,1], g[0,2] = (y^2 + z^2)/(x^2 + y^2 + z^2), \
....: - x*y/(x^2 + y^2 + z^2), - x*z/(x^2 + y^2 + z^2)
sage: g[1,1], g[1,2], g[2,2] = (x^2 + z^2)/(x^2 + y^2 + z^2), \
....: - y*z/(x^2 + y^2 + z^2), (x^2 + y^2)/(x^2 + y^2 + z^2)
sage: g.disp()
g = (y^2 + z^2)/(x^2 + y^2 + z^2) dx⊗dx - x*y/(x^2 + y^2 + z^2) dx⊗dy
- x*z/(x^2 + y^2 + z^2) dx⊗dz - x*y/(x^2 + y^2 + z^2) dy⊗dx
+ (x^2 + z^2)/(x^2 + y^2 + z^2) dy⊗dy - y*z/(x^2 + y^2 + z^2) dy⊗dz
- x*z/(x^2 + y^2 + z^2) dz⊗dx - y*z/(x^2 + y^2 + z^2) dz⊗dy
+ (x^2 + y^2)/(x^2 + y^2 + z^2) dz⊗dz
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M'); X = M.chart(names=('x', 'y', 'z',)); (x, y,
↪ z,) = X._first_ngens(3)
>>> g = M.degenerate_metric('g'); g
degenerate metric g on the 3-dimensional differentiable manifold M
>>> det(g)
Scalar field zero on the 3-dimensional differentiable manifold M
>>> g.parent()
Free module T^(0,2)(M) of type-(0,2) tensors fields on the
3-dimensional differentiable manifold M
>>> g[Integer(0),Integer(0)], g[Integer(0),Integer(1)], g[Integer(0),
↪Integer(2)] = (y**Integer(2) + z**Integer(2))/(x**Integer(2) +␣
↪y**Integer(2) + z**Integer(2)), - x*y/(x**Integer(2) + y**Integer(2) +␣
↪z**Integer(2)), - x*z/(x**Integer(2) + y**Integer(2) + z**Integer(2))
>>> g[Integer(1),Integer(1)], g[Integer(1),Integer(2)], g[Integer(2),
↪Integer(2)] = (x**Integer(2) + z**Integer(2))/(x**Integer(2) +␣
↪y**Integer(2) + z**Integer(2)), - y*z/(x**Integer(2) + y**Integer(2) +␣
↪z**Integer(2)), (x**Integer(2) + y**Integer(2))/(x**Integer(2) +␣
↪y**Integer(2) + z**Integer(2))
>>> g.disp()
g = (y^2 + z^2)/(x^2 + y^2 + z^2) dx⊗dx - x*y/(x^2 + y^2 + z^2) dx⊗dy
- x*z/(x^2 + y^2 + z^2) dx⊗dz - x*y/(x^2 + y^2 + z^2) dy⊗dx
+ (x^2 + z^2)/(x^2 + y^2 + z^2) dy⊗dy - y*z/(x^2 + y^2 + z^2) dy⊗dz
- x*z/(x^2 + y^2 + z^2) dz⊗dx - y*z/(x^2 + y^2 + z^2) dz⊗dy
+ (x^2 + y^2)/(x^2 + y^2 + z^2) dz⊗dz
```

See also:

`DegenerateMetric` for more examples.

diff_degree()

Return the manifold's degree of differentiability.

The degree of differentiability is the integer k (possibly $k = \infty$) such that the manifold is a C^k-manifold over its base field.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: M.diff_degree()
+Infinity
sage: M = Manifold(2, 'M', structure='differentiable', diff_degree=3)
sage: M.diff_degree()
3
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> M.diff_degree()
+Infinity
>>> M = Manifold(Integer(2), 'M', structure='differentiable', diff_
↪degree=Integer(3))
>>> M.diff_degree()
3
```

diff_form(**args*, ***kwargs*)

Define a differential form on `self`.

Via the argument `dest_map`, it is possible to let the differential form take its values on another manifold. More precisely, if M is the current manifold, N a differentiable manifold, $\Phi: M \to N$ a differentiable map and p a nonnegative integer, a *differential form of degree* p (or p*-form*) *along* M *with values on* N is a differentiable map

$$t: M \longrightarrow T^{(0,p)}N$$

($T^{(0,p)}N$ being the tensor bundle of type $(0, p)$ over N) such that

$$\forall x \in M, \quad t(x) \in \Lambda^p(T^*_{\Phi(x)}N),$$

where $\Lambda^p(T^*_{\Phi(x)}N)$ is the p-th exterior power of the dual of the tangent space $T_{\Phi(x)}N$.

The standard case of a differential form *on* M corresponds to $N = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in N (M is then an open interval of $\mathbf{R}$).

For $p = 1$, one can use the method `one_form()` instead.

See also:

`DiffForm` and `DiffFormParal` for a complete documentation.

INPUT:

- `degree` – the degree p of the differential form (i.e. its tensor rank)
- `comp` – (optional) either the components of the differential form with respect to the vector frame specified by the argument `frame` or a dictionary of components, the keys of which are vector frames or pairs `(f, c)` where `f` is a vector frame and `c` the chart in which the components are expressed
- `frame` – (default: `None`; unused if `comp` is not given or is a dictionary) vector frame in which the components are given; if `None`, the default vector frame of `self` is assumed
- `chart` – (default: `None`; unused if `comp` is not given or is a dictionary) coordinate chart in which the components are expressed; if `None`, the default chart on the domain of `frame` is assumed
- `name` – (default: `None`) name given to the differential form
- `latex_name` – (default: `None`) LaTeX symbol to denote the differential form; if none is provided, the LaTeX symbol is set to `name`
- `dest_map` – (default: `None`) the destination map $\Phi : M \to N$; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of a differential form *on* M), otherwise `dest_map` must be a `DiffMap`

OUTPUT:

- the p-form as a `DiffForm` (or if N is parallelizable, a `DiffFormParal`)

EXAMPLES:

A 2-form on a 3-dimensional differentiable manifold:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: f = M.diff_form(2, name='F'); f
2-form F on the 3-dimensional differentiable manifold M
sage: f[0,1], f[1,2] = x+y, x*z
sage: f.display()
F = (x + y) dx∧dy + x*z dy∧dz
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> f = M.diff_form(Integer(2), name='F'); f
2-form F on the 3-dimensional differentiable manifold M
>>> f[Integer(0),Integer(1)], f[Integer(1),Integer(2)] = x+y, x*z
>>> f.display()
F = (x + y) dx∧dy + x*z dy∧dz
```

For more examples, see `DiffForm` and `DiffFormParal`.

diff_form_module(*degree*, *dest_map=None*)

Return the set of differential forms of a given degree defined on `self`, possibly with values in another manifold, as a module over the algebra of scalar fields defined on `self`.

See also

`DiffFormModule` for complete documentation.

INPUT:

- `degree` – positive integer; the degree p of the differential forms

- `dest_map` – (default: `None`) destination map, i.e. a differentiable map $\Phi: M \to N$, where M is the current manifold and N a differentiable manifold; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of differential forms *on* M), otherwise `dest_map` must be a `DiffMap`

OUTPUT:

- a `DiffFormModule` (or if N is parallelizable, a `DiffFormFreeModule`) representing the module $\Omega^p(M, \Phi)$ of p-forms on M taking values on $\Phi(M) \subset N$

EXAMPLES:

Module of 2-forms on a 3-dimensional parallelizable manifold:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: M.diff_form_module(2)
Free module Omega^2(M) of 2-forms on the 3-dimensional
 differentiable manifold M
sage: M.diff_form_module(2).category()
Category of finite dimensional modules over Algebra of
 differentiable scalar fields on the 3-dimensional
 differentiable manifold M
sage: M.diff_form_module(2).base_ring()
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
sage: M.diff_form_module(2).rank()
3
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> M.diff_form_module(Integer(2))
Free module Omega^2(M) of 2-forms on the 3-dimensional
 differentiable manifold M
>>> M.diff_form_module(Integer(2)).category()
Category of finite dimensional modules over Algebra of
 differentiable scalar fields on the 3-dimensional
 differentiable manifold M
>>> M.diff_form_module(Integer(2)).base_ring()
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
>>> M.diff_form_module(Integer(2)).rank()
3
```

The outcome is cached:

```
sage: M.diff_form_module(2) is M.diff_form_module(2)
True
```

```
>>> from sage.all import *
>>> M.diff_form_module(Integer(2)) is M.diff_form_module(Integer(2))
True
```

diff_map(*codomain*, *coord_functions=None*, *chart1=None*, *chart2=None*, *name=None*, *latex_name=None*)

Define a differentiable map between the current differentiable manifold and a differentiable manifold over the same topological field.

See *DiffMap* for a complete documentation.

INPUT:

- `codomain` – the map codomain (a differentiable manifold over the same topological field as the current differentiable manifold)
- `coord_functions` – (default: `None`) if not `None`, must be either
  - (i) a dictionary of the coordinate expressions (as lists (or tuples) of the coordinates of the image expressed in terms of the coordinates of the considered point) with the pairs of charts (chart1, chart2) as keys (chart1 being a chart on the current manifold and chart2 a chart on `codomain`)
  - (ii) a single coordinate expression in a given pair of charts, the latter being provided by the arguments `chart1` and `chart2`

  In both cases, if the dimension of the arrival manifold is 1, a single coordinate expression can be passed instead of a tuple with a single element
- `chart1` – (default: `None`; used only in case (ii) above) chart on the current manifold defining the start coordinates involved in `coord_functions` for case (ii); if none is provided, the coordinates are assumed to refer to the manifold's default chart
- `chart2` – (default: `None`; used only in case (ii) above) chart on `codomain` defining the arrival coordinates involved in `coord_functions` for case (ii); if none is provided, the coordinates are assumed to refer to the default chart of `codomain`
- `name` – (default: `None`) name given to the differentiable map
- `latex_name` – (default: `None`) LaTeX symbol to denote the differentiable map; if none is provided, the LaTeX symbol is set to `name`

OUTPUT:

- the differentiable map, as an instance of *DiffMap*

EXAMPLES:

A differentiable map between an open subset of S^2 covered by regular spherical coordinates and $\mathbf{R}^3$:

```
sage: M = Manifold(2, 'S^2')
sage: U = M.open_subset('U')
sage: c_spher.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: N = Manifold(3, 'R^3', r'\RR^3')
sage: c_cart.<x,y,z> = N.chart()  # Cartesian coord. on R^3
sage: Phi = U.diff_map(N, (sin(th)*cos(ph), sin(th)*sin(ph), cos(th)),
....:                  name='Phi', latex_name=r'\Phi')
sage: Phi
Differentiable map Phi from the Open subset U of the 2-dimensional
 differentiable manifold S^2 to the 3-dimensional differentiable
 manifold R^3
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2')
>>> U = M.open_subset('U')
>>> c_spher = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi', names=('th', 'ph',
↪)); (th, ph,) = c_spher._first_ngens(2)
>>> N = Manifold(Integer(3), 'R^3', r'\RR^3')
>>> c_cart = N.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_cart._first_
↪ngens(3)# Cartesian coord. on R^3
```

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```
>>> Phi = U.diff_map(N, (sin(th)*cos(ph), sin(th)*sin(ph), cos(th)),
...                   name='Phi', latex_name=r'\Phi')
>>> Phi
Differentiable map Phi from the Open subset U of the 2-dimensional
 differentiable manifold S^2 to the 3-dimensional differentiable
 manifold R^3
```

The same definition, but with a dictionary with pairs of charts as keys (case (i) above):

```
sage: Phi1 = U.diff_map(N,
....:        {(c_spher, c_cart): (sin(th)*cos(ph), sin(th)*sin(ph),
....:         cos(th))}, name='Phi', latex_name=r'\Phi')
sage: Phi1 == Phi
True
```

```
>>> from sage.all import *
>>> Phi1 = U.diff_map(N,
...        {(c_spher, c_cart): (sin(th)*cos(ph), sin(th)*sin(ph),
...         cos(th))}, name='Phi', latex_name=r'\Phi')
>>> Phi1 == Phi
True
```

The differentiable map acting on a point:

```
sage: p = U.point((pi/2, pi)) ; p
Point on the 2-dimensional differentiable manifold S^2
sage: Phi(p)
Point on the 3-dimensional differentiable manifold R^3
sage: Phi(p).coord(c_cart)
(-1, 0, 0)
sage: Phi1(p) == Phi(p)
True
```

```
>>> from sage.all import *
>>> p = U.point((pi/Integer(2), pi)) ; p
Point on the 2-dimensional differentiable manifold S^2
>>> Phi(p)
Point on the 3-dimensional differentiable manifold R^3
>>> Phi(p).coord(c_cart)
(-1, 0, 0)
>>> Phi1(p) == Phi(p)
True
```

See the documentation of class `DiffMap` for more examples.

diffeomorphism(*codomain=None*, *coord_functions=None*, *chart1=None*, *chart2=None*, *name=None*, *latex_name=None*)

Define a diffeomorphism between the current manifold and another one.

See `DiffMap` for a complete documentation.

INPUT:

- `codomain` – (default: `None`) codomain of the diffeomorphism (the arrival manifold or some subset of it). If `None`, the current manifold is taken.
- `coord_functions` – (default: `None`) if not `None`, must be either
  - (i) a dictionary of the coordinate expressions (as lists (or tuples) of the coordinates of the image expressed in terms of the coordinates of the considered point) with the pairs of charts (chart1, chart2) as keys (chart1 being a chart on the current manifold and chart2 a chart on `codomain`)
  - (ii) a single coordinate expression in a given pair of charts, the latter being provided by the arguments `chart1` and `chart2`

  In both cases, if the dimension of the arrival manifold is 1, a single coordinate expression can be passed instead of a tuple with a single element
- `chart1` – (default: `None`; used only in case (ii) above) chart on the current manifold defining the start coordinates involved in `coord_functions` for case (ii); if none is provided, the coordinates are assumed to refer to the manifold's default chart
- `chart2` – (default: `None`; used only in case (ii) above) chart on `codomain` defining the arrival coordinates involved in `coord_functions` for case (ii); if none is provided, the coordinates are assumed to refer to the default chart of `codomain`
- `name` – (default: `None`) name given to the diffeomorphism
- `latex_name` – (default: `None`) LaTeX symbol to denote the diffeomorphism; if none is provided, the LaTeX symbol is set to `name`

OUTPUT:

- the diffeomorphism, as an instance of `DiffMap`

EXAMPLES:

Diffeomorphism between the open unit disk in $\mathbf{R}^2$ and $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'M')  # the open unit disk
sage: forget()  # for doctests only
sage: c_xy.<x,y> = M.chart('x:(-1,1) y:(-1,1)', coord_restrictions=lambda x,
↪y: x^2+y^2<1)
....:     # Cartesian coord on M
sage: N = Manifold(2, 'N')  # R^2
sage: c_XY.<X,Y> = N.chart()  # canonical coordinates on R^2
sage: Phi = M.diffeomorphism(N, [x/sqrt(1-x^2-y^2), y/sqrt(1-x^2-y^2)],
....:                        name='Phi', latex_name=r'\Phi')
sage: Phi
Diffeomorphism Phi from the 2-dimensional differentiable manifold M
 to the 2-dimensional differentiable manifold N
sage: Phi.display()
Phi: M → N
   (x, y) ↦ (X, Y) = (x/sqrt(-x^2 - y^2 + 1), y/sqrt(-x^2 - y^2 + 1))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')  # the open unit disk
>>> forget()  # for doctests only
>>> c_xy = M.chart('x:(-1,1) y:(-1,1)', coord_restrictions=lambda x,y:␣
↪x**Integer(2)+y**Integer(2)<Integer(1), names=('x', 'y',)); (x, y,) = c_xy._
↪first_ngens(2)
...     # Cartesian coord on M
```

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```
>>> N = Manifold(Integer(2), 'N')   # R^2
>>> c_XY = N.chart(names=('X', 'Y',)); (X, Y,) = c_XY._first_ngens(2)#␣
↪canonical coordinates on R^2
>>> Phi = M.diffeomorphism(N, [x/sqrt(Integer(1)-x**Integer(2)-y**Integer(2)),
↪ y/sqrt(Integer(1)-x**Integer(2)-y**Integer(2))],
...                            name='Phi', latex_name=r'\Phi')
>>> Phi
Diffeomorphism Phi from the 2-dimensional differentiable manifold M
 to the 2-dimensional differentiable manifold N
>>> Phi.display()
Phi: M → N
   (x, y) ↦ (X, Y) = (x/sqrt(-x^2 - y^2 + 1), y/sqrt(-x^2 - y^2 + 1))
```

The inverse diffeomorphism:

```
sage: Phi^(-1)
Diffeomorphism Phi^(-1) from the 2-dimensional differentiable
 manifold N to the 2-dimensional differentiable manifold M
sage: (Phi^(-1)).display()
Phi^(-1): N → M
   (X, Y) ↦ (x, y) = (X/sqrt(X^2 + Y^2 + 1), Y/sqrt(X^2 + Y^2 + 1))
```

```
>>> from sage.all import *
>>> Phi**(-Integer(1))
Diffeomorphism Phi^(-1) from the 2-dimensional differentiable
 manifold N to the 2-dimensional differentiable manifold M
>>> (Phi**(-Integer(1))).display()
Phi^(-1): N → M
   (X, Y) ↦ (x, y) = (X/sqrt(X^2 + Y^2 + 1), Y/sqrt(X^2 + Y^2 + 1))
```

See the documentation of class `DiffMap` for more examples.

frames()

Return the list of vector frames defined on open subsets of `self`.

OUTPUT: list of vector frames defined on open subsets of `self`

EXAMPLES:

Vector frames on subsets of $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'R^2')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: M.frames()
[Coordinate frame (R^2, (∂/∂x,∂/∂y))]
sage: e = M.vector_frame('e')
sage: M.frames()
[Coordinate frame (R^2, (∂/∂x,∂/∂y)),
 Vector frame (R^2, (e_0,e_1))]
sage: U = M.open_subset('U', coord_def={c_cart: x^2+y^2<1}) # unit disk
sage: U.frames()
[Coordinate frame (U, (∂/∂x,∂/∂y))]
sage: M.frames()
[Coordinate frame (R^2, (∂/∂x,∂/∂y)),
```

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```
 Vector frame (R^2, (e_0,e_1)),
 Coordinate frame (U, (∂/∂x,∂/∂y))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#␣
↪Cartesian coordinates on R^2
>>> M.frames()
[Coordinate frame (R^2, (∂/∂x,∂/∂y))]
>>> e = M.vector_frame('e')
>>> M.frames()
[Coordinate frame (R^2, (∂/∂x,∂/∂y)),
 Vector frame (R^2, (e_0,e_1))]
>>> U = M.open_subset('U', coord_def={c_cart: x**Integer(2)+y**Integer(2)
↪<Integer(1)}) # unit disk
>>> U.frames()
[Coordinate frame (U, (∂/∂x,∂/∂y))]
>>> M.frames()
[Coordinate frame (R^2, (∂/∂x,∂/∂y)),
 Vector frame (R^2, (e_0,e_1)),
 Coordinate frame (U, (∂/∂x,∂/∂y))]
```

integrated_autoparallel_curve(*affine_connection*, *curve_param*, *initial_tangent_vector*, *chart=None*, *name=None*, *latex_name=None*, *verbose=False*, *across_charts=False*)

Construct an autoparallel curve on the manifold with respect to a given affine connection.

See also

IntegratedAutoparallelCurve for details.

INPUT:

- `affine_connection` – *AffineConnection*; affine connection with respect to which the curve is autoparallel
- `curve_param` – tuple of the type `(t, t_min, t_max)`, where
  - `t` is the symbolic variable to be used as the parameter of the curve (the equations defining an instance of *IntegratedAutoparallelCurve* are such that `t` will actually be an affine parameter of the curve);
  - `t_min` is its minimal (finite) value;
  - `t_max` its maximal (finite) value.
- `initial_tangent_vector` – *TangentVector*; initial tangent vector of the curve
- `chart` – (default: `None`) chart on the manifold in which the equations are given ; if `None` the default chart of the manifold is assumed
- `name` – (default: `None`) string; symbol given to the curve
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the curve; if none is provided, `name` will be used

OUTPUT: *IntegratedAutoparallelCurve*

EXAMPLES:

Autoparallel curves associated with the Mercator projection of the 2-sphere $\mathbb{S}^2$:

```
sage: S2 = Manifold(2, 'S^2', start_index=1)
sage: polar.<th,ph> = S2.chart('th ph')
sage: epolar = polar.frame()
sage: ch_basis = S2.automorphism_field()
sage: ch_basis[1,1], ch_basis[2,2] = 1, 1/sin(th)
sage: epolar_ON=S2.default_frame().new_frame(ch_basis,'epolar_ON')
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2', start_index=Integer(1))
>>> polar = S2.chart('th ph', names=('th', 'ph',)); (th, ph,) = polar._first_
↪ngens(2)
>>> epolar = polar.frame()
>>> ch_basis = S2.automorphism_field()
>>> ch_basis[Integer(1),Integer(1)], ch_basis[Integer(2),Integer(2)] =␣
↪Integer(1), Integer(1)/sin(th)
>>> epolar_ON=S2.default_frame().new_frame(ch_basis,'epolar_ON')
```

Set the affine connection associated with Mercator projection; it is metric compatible but it has non-vanishing torsion:

```
sage: nab = S2.affine_connection('nab')
sage: nab.set_coef(epolar_ON)[:]
[[[0, 0], [0, 0]], [[0, 0], [0, 0]]]
sage: g = S2.metric('g')
sage: g[1,1], g[2,2] = 1, (sin(th))^2
sage: nab(g)[:]
[[[0, 0], [0, 0]], [[0, 0], [0, 0]]]
sage: nab.torsion()[:]
[[[0, 0], [0, 0]], [[0, cos(th)/sin(th)], [-cos(th)/sin(th), 0]]]
```

```
>>> from sage.all import *
>>> nab = S2.affine_connection('nab')
>>> nab.set_coef(epolar_ON)[:]
[[[0, 0], [0, 0]], [[0, 0], [0, 0]]]
>>> g = S2.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)] = Integer(1),␣
↪(sin(th))**Integer(2)
>>> nab(g)[:]
[[[0, 0], [0, 0]], [[0, 0], [0, 0]]]
>>> nab.torsion()[:]
[[[0, 0], [0, 0]], [[0, cos(th)/sin(th)], [-cos(th)/sin(th), 0]]]
```

Declare an integrated autoparallel curve with respect to this connection:

```
sage: p = S2.point((pi/4, 0), name='p')
sage: Tp = S2.tangent_space(p)
sage: v = Tp((1,1), basis=epolar_ON.at(p))
sage: t = var('t')
sage: c = S2.integrated_autoparallel_curve(nab, (t, 0, 2.3),
....:                               v, chart=polar, name='c')
```

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```
sage: sys = c.system(verbose=True)
Autoparallel curve c in the 2-dimensional differentiable
 manifold S^2 equipped with Affine connection nab on the
 2-dimensional differentiable manifold S^2, and integrated
 over the Real interval (0, 2.30000000000000) as a solution to the
 following equations, written with respect to
 Chart (S^2, (th, ph)):

Initial point: Point p on the 2-dimensional differentiable
 manifold S^2 with coordinates [1/4*pi, 0] with respect to
 Chart (S^2, (th, ph))
Initial tangent vector: Tangent vector at Point p on the
 2-dimensional differentiable manifold S^2 with
 components [1, sqrt(2)] with respect to
 Chart (S^2, (th, ph))

d(th)/dt = Dth
d(ph)/dt = Dph
d(Dth)/dt = 0
d(Dph)/dt = -Dph*Dth*cos(th)/sin(th)

sage: sol = c.solve()
sage: interp = c.interpolate()
sage: p = c(1.3, verbose=True)
Evaluating point coordinates from the interpolation
 associated with the key 'cubic spline-interp-odeint'
 by default...
sage: p
Point on the 2-dimensional differentiable manifold S^2
sage: polar(p)     # abs tol 1e-12
(2.0853981633974477, 1.4203177070475606)
sage: tgt_vec = c.tangent_vector_eval_at(1.3, verbose=True)
Evaluating tangent vector components from the interpolation
 associated with the key 'cubic spline-interp-odeint'
 by default...
sage: tgt_vec[:]    # abs tol 1e-12
[1.000000000000011, 1.148779968412235]
```

```
>>> from sage.all import *
>>> p = S2.point((pi/Integer(4), Integer(0)), name='p')
>>> Tp = S2.tangent_space(p)
>>> v = Tp((Integer(1),Integer(1)), basis=epolar_ON.at(p))
>>> t = var('t')
>>> c = S2.integrated_autoparallel_curve(nab, (t, Integer(0), RealNumber('2.3
↪')),
...                                v, chart=polar, name='c')
>>> sys = c.system(verbose=True)
Autoparallel curve c in the 2-dimensional differentiable
 manifold S^2 equipped with Affine connection nab on the
 2-dimensional differentiable manifold S^2, and integrated
 over the Real interval (0, 2.30000000000000) as a solution to the
 following equations, written with respect to
```

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```
 Chart (S^2, (th, ph)):
<BLANKLINE>
Initial point: Point p on the 2-dimensional differentiable
 manifold S^2 with coordinates [1/4*pi, 0] with respect to
 Chart (S^2, (th, ph))
Initial tangent vector: Tangent vector at Point p on the
 2-dimensional differentiable manifold S^2 with
 components [1, sqrt(2)] with respect to
 Chart (S^2, (th, ph))
<BLANKLINE>
d(th)/dt = Dth
d(ph)/dt = Dph
d(Dth)/dt = 0
d(Dph)/dt = -Dph*Dth*cos(th)/sin(th)
<BLANKLINE>
>>> sol = c.solve()
>>> interp = c.interpolate()
>>> p = c(RealNumber('1.3'), verbose=True)
Evaluating point coordinates from the interpolation
 associated with the key 'cubic spline-interp-odeint'
 by default...
>>> p
Point on the 2-dimensional differentiable manifold S^2
>>> polar(p)     # abs tol 1e-12
(2.0853981633974477, 1.4203177070475606)
>>> tgt_vec = c.tangent_vector_eval_at(RealNumber('1.3'), verbose=True)
Evaluating tangent vector components from the interpolation
 associated with the key 'cubic spline-interp-odeint'
 by default...
>>> tgt_vec[:]    # abs tol 1e-12
[1.0000000000000011, 1.148779968412235]
```

integrated_curve(*equations_rhs*, *velocities*, *curve_param*, *initial_tangent_vector*, *chart=None*, *name=None*, *latex_name=None*, *verbose=False*, *across_charts=False*)

Construct a curve defined by a system of second order differential equations in the coordinate functions.

See also

IntegratedCurve for details.

INPUT:

- `equations_rhs` – list of the right-hand sides of the equations on the velocities only
- `velocities` – list of the symbolic expressions used in `equations_rhs` to denote the velocities
- `curve_param` – tuple of the type `(t, t_min, t_max)`, where
  - `t` is the symbolic variable used in `equations_rhs` to denote the parameter of the curve;
  - `t_min` is its minimal (finite) value;
  - `t_max` its maximal (finite) value.
- `initial_tangent_vector` – *TangentVector*; initial tangent vector of the curve

- `chart` – (default: `None`) chart on the manifold in which the equations are given; if `None` the default chart of the manifold is assumed
- `name` – (default: `None`) string; symbol given to the curve
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the curve; if none is provided, `name` will be used

OUTPUT: *IntegratedCurve*

EXAMPLES:

Trajectory of a particle of unit mass and unit charge in a unit, uniform, stationary magnetic field:

```
sage: M = Manifold(3, 'M')
sage: X.<x1,x2,x3> = M.chart()
sage: t = var('t')
sage: D = X.symbolic_velocities()
sage: eqns = [D[1], -D[0], SR(0)]
sage: p = M.point((0,0,0), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,0,1))
sage: c = M.integrated_curve(eqns, D, (t,0,6), v, name='c'); c
Integrated curve c in the 3-dimensional differentiable
 manifold M
sage: sys = c.system(verbose=True)
Curve c in the 3-dimensional differentiable manifold M
 integrated over the Real interval (0, 6) as a solution to
 the following system, written with respect to
 Chart (M, (x1, x2, x3)):

Initial point: Point p on the 3-dimensional differentiable
 manifold M with coordinates [0, 0, 0] with respect to
 Chart (M, (x1, x2, x3))
Initial tangent vector: Tangent vector at Point p on the
 3-dimensional differentiable manifold M with
 components [1, 0, 1] with respect to Chart (M, (x1, x2, x3))

d(x1)/dt = Dx1
d(x2)/dt = Dx2
d(x3)/dt = Dx3
d(Dx1)/dt = Dx2
d(Dx2)/dt = -Dx1
d(Dx3)/dt = 0

sage: sol = c.solve()
sage: interp = c.interpolate()
sage: p = c(1.3, verbose=True)
Evaluating point coordinates from the interpolation
 associated with the key 'cubic spline-interp-odeint'
 by default...
sage: p
Point on the 3-dimensional differentiable manifold M
sage: p.coordinates()     # abs tol 1e-12
(0.9635581599167499, -0.7325011788437327, 1.3)
sage: tgt_vec = c.tangent_vector_eval_at(3.7, verbose=True)
```

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```
Evaluating tangent vector components from the interpolation
 associated with the key 'cubic spline-interp-odeint'
 by default...
sage: tgt_vec[:]     # abs tol 1e-12
[-0.8481007454066425, 0.5298350137284363, 1.0]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x1', 'x2', 'x3',)); (x1, x2, x3,) = X._first_ngens(3)
>>> t = var('t')
>>> D = X.symbolic_velocities()
>>> eqns = [D[Integer(1)], -D[Integer(0)], SR(Integer(0))]
>>> p = M.point((Integer(0),Integer(0),Integer(0)), name='p')
>>> Tp = M.tangent_space(p)
>>> v = Tp((Integer(1),Integer(0),Integer(1)))
>>> c = M.integrated_curve(eqns, D, (t,Integer(0),Integer(6)), v, name='c'); c
Integrated curve c in the 3-dimensional differentiable
 manifold M
>>> sys = c.system(verbose=True)
Curve c in the 3-dimensional differentiable manifold M
 integrated over the Real interval (0, 6) as a solution to
 the following system, written with respect to
 Chart (M, (x1, x2, x3)):
<BLANKLINE>
Initial point: Point p on the 3-dimensional differentiable
 manifold M with coordinates [0, 0, 0] with respect to
 Chart (M, (x1, x2, x3))
Initial tangent vector: Tangent vector at Point p on the
 3-dimensional differentiable manifold M with
 components [1, 0, 1] with respect to Chart (M, (x1, x2, x3))
<BLANKLINE>
d(x1)/dt = Dx1
d(x2)/dt = Dx2
d(x3)/dt = Dx3
d(Dx1)/dt = Dx2
d(Dx2)/dt = -Dx1
d(Dx3)/dt = 0
<BLANKLINE>
>>> sol = c.solve()
>>> interp = c.interpolate()
>>> p = c(RealNumber('1.3'), verbose=True)
Evaluating point coordinates from the interpolation
 associated with the key 'cubic spline-interp-odeint'
 by default...
>>> p
Point on the 3-dimensional differentiable manifold M
>>> p.coordinates()     # abs tol 1e-12
(0.9635581599167499, -0.7325011788437327, 1.3)
>>> tgt_vec = c.tangent_vector_eval_at(RealNumber('3.7'), verbose=True)
Evaluating tangent vector components from the interpolation
 associated with the key 'cubic spline-interp-odeint'
 by default...
```

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```
>>> tgt_vec[:]     # abs tol 1e-12
[-0.8481007454066425, 0.5298350137284363, 1.0]
```

integrated_geodesic(*metric*, *curve_param*, *initial_tangent_vector*, *chart=None*, *name=None*, *latex_name=None*, *verbose=False*, *across_charts=False*)

Construct a geodesic on the manifold with respect to a given metric.

See also

IntegratedGeodesic for details.

INPUT:

- `metric` – *PseudoRiemannianMetric* metric with respect to which the curve is a geodesic
- `curve_param` – tuple of the type `(t, t_min, t_max)`, where
  - `t` is the symbolic variable to be used as the parameter of the curve (the equations defining an instance of *IntegratedGeodesic* are such that `t` will actually be an affine parameter of the curve);
  - `t_min` is its minimal (finite) value;
  - `t_max` its maximal (finite) value.
- `initial_tangent_vector` – *TangentVector*; initial tangent vector of the curve
- `chart` – (default: `None`) chart on the manifold in which the equations are given; if `None` the default chart of the manifold is assumed
- `name` – (default: `None`) string; symbol given to the curve
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the curve; if none is provided, `name` will be used

OUTPUT: *IntegratedGeodesic*

EXAMPLES:

Geodesics of the unit 2-sphere $\mathbb{S}^2$:

```
sage: S2 = Manifold(2, 'S^2', start_index=1)
sage: polar.<th,ph> = S2.chart('th ph')
sage: epolar = polar.frame()
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2', start_index=Integer(1))
>>> polar = S2.chart('th ph', names=('th', 'ph',)); (th, ph,) = polar._first_
↪ngens(2)
>>> epolar = polar.frame()
```

Set the standard metric tensor g on $\mathbb{S}^2$:

```
sage: g = S2.metric('g')
sage: g[1,1], g[2,2] = 1, (sin(th))^2
```

```
>>> from sage.all import *
>>> g = S2.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)] = Integer(1),␣
↪(sin(th))**Integer(2)
```

Declare an integrated geodesic with respect to this metric:

```
sage: p = S2.point((pi/4, 0), name='p')
sage: Tp = S2.tangent_space(p)
sage: v = Tp((1, 1), basis=epolar.at(p))
sage: t = var('t')
sage: c = S2.integrated_geodesic(g, (t, 0, 6), v,
....:                                   chart=polar, name='c')
sage: sys = c.system(verbose=True)
Geodesic c in the 2-dimensional differentiable manifold S^2
 equipped with Riemannian metric g on the 2-dimensional
 differentiable manifold S^2, and integrated over the Real
 interval (0, 6) as a solution to the following geodesic
 equations, written with respect to Chart (S^2, (th, ph)):

Initial point: Point p on the 2-dimensional differentiable
manifold S^2 with coordinates [1/4*pi, 0] with respect to
Chart (S^2, (th, ph))
Initial tangent vector: Tangent vector at Point p on the
2-dimensional differentiable manifold S^2 with
components [1, 1] with respect to Chart (S^2, (th, ph))

d(th)/dt = Dth
d(ph)/dt = Dph
d(Dth)/dt = Dph^2*cos(th)*sin(th)
d(Dph)/dt = -2*Dph*Dth*cos(th)/sin(th)

sage: sol = c.solve()
sage: interp = c.interpolate()
sage: p = c(1.3, verbose=True)
Evaluating point coordinates from the interpolation
 associated with the key 'cubic spline-interp-odeint'
 by default...
sage: p
Point on the 2-dimensional differentiable manifold S^2
sage: p.coordinates()     # abs tol 1e-12
(2.2047435672397526, 0.7986602654406825)
sage: tgt_vec = c.tangent_vector_eval_at(3.7, verbose=True)
Evaluating tangent vector components from the interpolation
 associated with the key 'cubic spline-interp-odeint'
 by default...
sage: tgt_vec[:]     # abs tol 1e-12
[-1.0907409234671228, 0.6205670379855032]
```

```
>>> from sage.all import *
>>> p = S2.point((pi/Integer(4), Integer(0)), name='p')
>>> Tp = S2.tangent_space(p)
>>> v = Tp((Integer(1), Integer(1)), basis=epolar.at(p))
```

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```
>>> t = var('t')
>>> c = S2.integrated_geodesic(g, (t, Integer(0), Integer(6)), v,
...                                    chart=polar, name='c')
>>> sys = c.system(verbose=True)
Geodesic c in the 2-dimensional differentiable manifold S^2
 equipped with Riemannian metric g on the 2-dimensional
 differentiable manifold S^2, and integrated over the Real
 interval (0, 6) as a solution to the following geodesic
 equations, written with respect to Chart (S^2, (th, ph)):
<BLANKLINE>
Initial point: Point p on the 2-dimensional differentiable
manifold S^2 with coordinates [1/4*pi, 0] with respect to
Chart (S^2, (th, ph))
Initial tangent vector: Tangent vector at Point p on the
2-dimensional differentiable manifold S^2 with
components [1, 1] with respect to Chart (S^2, (th, ph))
<BLANKLINE>
d(th)/dt = Dth
d(ph)/dt = Dph
d(Dth)/dt = Dph^2*cos(th)*sin(th)
d(Dph)/dt = -2*Dph*Dth*cos(th)/sin(th)
<BLANKLINE>
>>> sol = c.solve()
>>> interp = c.interpolate()
>>> p = c(RealNumber('1.3'), verbose=True)
Evaluating point coordinates from the interpolation
 associated with the key 'cubic spline-interp-odeint'
 by default...
>>> p
Point on the 2-dimensional differentiable manifold S^2
>>> p.coordinates()     # abs tol 1e-12
(2.2047435672397526, 0.7986602654406825)
>>> tgt_vec = c.tangent_vector_eval_at(RealNumber('3.7'), verbose=True)
Evaluating tangent vector components from the interpolation
 associated with the key 'cubic spline-interp-odeint'
 by default...
>>> tgt_vec[:]    # abs tol 1e-12
[-1.0907409234671228, 0.6205670379855032]
```

is_manifestly_parallelizable()

Return `True` if `self` is known to be a parallelizable and `False` otherwise.

If `False` is returned, either the manifold is not parallelizable or no vector frame has been defined on it yet.

EXAMPLES:

A just created manifold is a priori not manifestly parallelizable:

```
sage: M = Manifold(2, 'M')
sage: M.is_manifestly_parallelizable()
False
```

```
>>> from sage.all import *
```

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```
>>> M = Manifold(Integer(2), 'M')
>>> M.is_manifestly_parallelizable()
False
```

Defining a vector frame on it makes it parallelizable:

```
sage: e = M.vector_frame('e')
sage: M.is_manifestly_parallelizable()
True
```

```
>>> from sage.all import *
>>> e = M.vector_frame('e')
>>> M.is_manifestly_parallelizable()
True
```

Defining a coordinate chart on the whole manifold also makes it parallelizable:

```
sage: N = Manifold(4, 'N')
sage: X.<t,x,y,z> = N.chart()
sage: N.is_manifestly_parallelizable()
True
```

```
>>> from sage.all import *
>>> N = Manifold(Integer(4), 'N')
>>> X = N.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> N.is_manifestly_parallelizable()
True
```

lorentzian_metric(*name*, *signature='positive'*, *latex_name=None*, *dest_map=None*)

Define a Lorentzian metric on the manifold.

A *Lorentzian metric* is a field of nondegenerate symmetric bilinear forms acting in the tangent spaces, with signature $(-,+,\cdots,+)$ or $(+,-,\cdots,-)$.

See *PseudoRiemannianMetric* for a complete documentation.

INPUT:

- `name` – name given to the metric
- `signature` – (default: `'positive'`) sign of the metric signature:
  - if set to 'positive', the signature is n-2, where n is the manifold's dimension, i.e. $(-,+,\cdots,+)$
  - if set to 'negative', the signature is -n+2, i.e. $(+,-,\cdots,-)$
- `latex_name` – (default: `None`) LaTeX symbol to denote the metric; if `None`, it is formed from `name`
- `dest_map` – (default: `None`) instance of class *DiffMap* representing the destination map $\Phi : U \to M$, where U is the current manifold; if `None`, the identity map is assumed (case of a metric tensor field *on* U)

OUTPUT:

- instance of *PseudoRiemannianMetric* representing the defined Lorentzian metric.

EXAMPLES:

Metric of Minkowski spacetime:

```
sage: M = Manifold(4, 'M')
sage: X.<t,x,y,z> = M.chart()
sage: g = M.lorentzian_metric('g'); g
Lorentzian metric g on the 4-dimensional differentiable manifold M
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1, 1, 1, 1
sage: g.display()
g = -dt⊗dt + dx⊗dx + dy⊗dy + dz⊗dz
sage: g.signature()
2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> g = M.lorentzian_metric('g'); g
Lorentzian metric g on the 4-dimensional differentiable manifold M
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1), Integer(1), Integer(1),
↪ Integer(1)
>>> g.display()
g = -dt⊗dt + dx⊗dx + dy⊗dy + dz⊗dz
>>> g.signature()
2
```

Choice of a negative signature:

```
sage: g = M.lorentzian_metric('g', signature='negative'); g
Lorentzian metric g on the 4-dimensional differentiable manifold M
sage: g[0,0], g[1,1], g[2,2], g[3,3] = 1, -1, -1, -1
sage: g.display()
g = dt⊗dt - dx⊗dx - dy⊗dy - dz⊗dz
sage: g.signature()
-2
```

```
>>> from sage.all import *
>>> g = M.lorentzian_metric('g', signature='negative'); g
Lorentzian metric g on the 4-dimensional differentiable manifold M
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = Integer(1), -Integer(1), -
↪Integer(1), -Integer(1)
>>> g.display()
g = dt⊗dt - dx⊗dx - dy⊗dy - dz⊗dz
>>> g.signature()
-2
```

metric(*name*, *signature=None*, *latex_name=None*, *dest_map=None*)

Define a pseudo-Riemannian metric on the manifold.

A *pseudo-Riemannian metric* is a field of nondegenerate symmetric bilinear forms acting in the tangent spaces. See `PseudoRiemannianMetric` for a complete documentation.

INPUT:

- `name` – name given to the metric
- `signature` – (default: `None`) signature S of the metric as a single integer: $S = n_+ - n_-$, where n_+ (resp. n_-) is the number of positive terms (resp. number of negative terms) in any diagonal writing of the metric components; if `signature` is not provided, S is set to the manifold's dimension (Riemannian signature)
- `latex_name` – (default: `None`) LaTeX symbol to denote the metric; if `None`, it is formed from `name`
- `dest_map` – (default: `None`) instance of class `DiffMap` representing the destination map $\Phi : U \to M$, where U is the current manifold; if `None`, the identity map is assumed (case of a metric tensor field *on* U)

OUTPUT:

- instance of `PseudoRiemannianMetric` representing the defined pseudo-Riemannian metric.

EXAMPLES:

Metric on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: g = M.metric('g'); g
Riemannian metric g on the 3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> g = M.metric('g'); g
Riemannian metric g on the 3-dimensional differentiable manifold M
```

See also

`PseudoRiemannianMetric` for more examples.

mixed_form(*comp=None*, *name=None*, *latex_name=None*, *dest_map=None*)

Define a mixed form on `self`.

Via the argument `dest_map`, it is possible to let the mixed form take its values on another manifold. More precisely, if M is the current manifold, N a differentiable manifold, $\Phi : M \to N$ a differentiable map, a *mixed form along* Φ can be considered as a differentiable map

$$a : M \longrightarrow \bigoplus_{k=0}^{n} T^{(0,k)} N$$

($T^{(0,k)}N$ being the tensor bundle of type $(0, k)$ over N, $\oplus$ being the Whitney sum and n being the dimension of N) such that

$$\forall x \in M, \quad a(x) \in \bigoplus_{k=0}^{n} \Lambda^k(T^*_{\Phi(x)}N),$$

where $\Lambda^k(T^*_{\Phi(x)}N)$ is the k-th exterior power of the dual of the tangent space $T_{\Phi(x)}N$.

The standard case of a mixed form *on* M corresponds to $N = M$ and $\Phi = \mathrm{Id}_M$.

See also:

`MixedForm` for complete documentation.

INPUT:

- `comp` – (default: `None`) homogeneous components of the mixed form as a list; if none is provided, the components are set to innocent unnamed differential forms
- `name` – (default: `None`) name given to the differential form
- `latex_name` – (default: `None`) LaTeX symbol to denote the differential form; if none is provided, the LaTeX symbol is set to `name`
- `dest_map` – (default: `None`) the destination map $\Phi : M \to N$; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of a differential form *on* M), otherwise `dest_map` must be a `DiffMap`

OUTPUT:

- the mixed form as a `MixedForm`

EXAMPLES:

A mixed form on an open subset of a 3-dimensional differentiable manifold:

```
sage: M = Manifold(3, 'M')
sage: U = M.open_subset('U', latex_name=r'\mathcal{U}'); U
Open subset U of the 3-dimensional differentiable manifold M
sage: c_xyz.<x,y,z> = U.chart()
sage: f = U.mixed_form(name='F'); f
Mixed differential form F on the Open subset U of the 3-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> U = M.open_subset('U', latex_name=r'\mathcal{U}'); U
Open subset U of the 3-dimensional differentiable manifold M
>>> c_xyz = U.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> f = U.mixed_form(name='F'); f
Mixed differential form F on the Open subset U of the 3-dimensional
 differentiable manifold M
```

See the documentation of class `MixedForm` for more examples.

mixed_form_algebra(*dest_map=None*)

Return the set of mixed forms defined on `self`, possibly with values in another manifold, as a graded algebra.

See also:

`MixedFormAlgebra` for complete documentation.

INPUT:

- `dest_map` – (default: `None`) destination map, i.e. a differentiable map $\Phi : M \to N$, where M is the current manifold and N a differentiable manifold; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of mixed forms *on* M), otherwise `dest_map` must be a *DiffMap*

OUTPUT:

- a *MixedFormAlgebra* representing the graded algebra $\Omega^*(M, \Phi)$ of mixed forms on M taking values on $\Phi(M) \subset N$

EXAMPLES:

Graded algebra of mixed forms on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: M.mixed_form_algebra()
Graded algebra Omega^*(M) of mixed differential forms on the
 2-dimensional differentiable manifold M
sage: M.mixed_form_algebra().category()
Join of Category of graded algebras over Symbolic Ring and Category of chain␣
↪complexes over Symbolic Ring
sage: M.mixed_form_algebra().base_ring()
Symbolic Ring
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> M.mixed_form_algebra()
Graded algebra Omega^*(M) of mixed differential forms on the
 2-dimensional differentiable manifold M
>>> M.mixed_form_algebra().category()
Join of Category of graded algebras over Symbolic Ring and Category of chain␣
↪complexes over Symbolic Ring
>>> M.mixed_form_algebra().base_ring()
Symbolic Ring
```

The outcome is cached:

```
sage: M.mixed_form_algebra() is M.mixed_form_algebra()
True
```

```
>>> from sage.all import *
>>> M.mixed_form_algebra() is M.mixed_form_algebra()
True
```

multivector_field(**args*, ***kwargs*)

Define a multivector field on `self`.

Via the argument `dest_map`, it is possible to let the multivector field take its values on another manifold. More precisely, if M is the current manifold, N a differentiable manifold, $\Phi : M \to N$ a differentiable map and p a nonnegative integer, a *multivector field of degree* p (or p*-vector field*) *along* M *with values on* N is a differentiable map

$$t : M \longrightarrow T^{(p,0)}N$$

($T^{(p,0)}N$ being the tensor bundle of type $(p, 0)$ over N) such that

$$\forall x \in M, \quad t(x) \in \Lambda^p(T_{\Phi(x)}N),$$

where $\Lambda^p(T_{\Phi(x)}N)$ is the p-th exterior power of the tangent vector space $T_{\Phi(x)}N$.

The standard case of a p-vector field *on* M corresponds to $N = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in N (M is then an open interval of $\mathbf{R}$).

For $p = 1$, one can use the method `vector_field()` instead.

See also

`MultivectorField` and `MultivectorFieldParal` for a complete documentation.

INPUT:

- `degree` – the degree p of the multivector field (i.e. its tensor rank)
- `comp` – (optional) either the components of the multivector field with respect to the vector frame specified by the argument `frame` or a dictionary of components, the keys of which are vector frames or pairs `(f, c)` where `f` is a vector frame and `c` the chart in which the components are expressed
- `frame` – (default: `None`; unused if `comp` is not given or is a dictionary) vector frame in which the components are given; if `None`, the default vector frame of `self` is assumed
- `chart` – (default: `None`; unused if `comp` is not given or is a dictionary) coordinate chart in which the components are expressed; if `None`, the default chart on the domain of `frame` is assumed
- `name` – (default: `None`) name given to the multivector field
- `latex_name` – (default: `None`) LaTeX symbol to denote the multivector field; if none is provided, the LaTeX symbol is set to `name`
- `dest_map` – (default: `None`) the destination map $\Phi : M \to N$; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of a multivector field *on* M), otherwise `dest_map` must be a `DiffMap`

OUTPUT:

- the p-vector field as a `MultivectorField` (or if N is parallelizable, a `MultivectorFieldParal`)

EXAMPLES:

A 2-vector field on a 3-dimensional differentiable manifold:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: h = M.multivector_field(2, name='H'); h
2-vector field H on the 3-dimensional differentiable manifold M
sage: h[0,1], h[0,2], h[1,2] = x+y, x*z, -3
sage: h.display()
H = (x + y) ∂/∂x∧∂/∂y + x*z ∂/∂x∧∂/∂z - 3 ∂/∂y∧∂/∂z
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> h = M.multivector_field(Integer(2), name='H'); h
2-vector field H on the 3-dimensional differentiable manifold M
>>> h[Integer(0),Integer(1)], h[Integer(0),Integer(2)], h[Integer(1),
↪Integer(2)] = x+y, x*z, -Integer(3)
>>> h.display()
H = (x + y) ∂/∂x∧∂/∂y + x*z ∂/∂x∧∂/∂z - 3 ∂/∂y∧∂/∂z
```

For more examples, see `MultivectorField` and `MultivectorFieldParal`.

multivector_module(*degree*, *dest_map=None*)

Return the set of multivector fields of a given degree defined on `self`, possibly with values in another manifold, as a module over the algebra of scalar fields defined on `self`.

See also

`MultivectorModule` for complete documentation.

INPUT:

- `degree` – positive integer; the degree p of the multivector fields
- `dest_map` – (default: `None`) destination map, i.e. a differentiable map $\Phi: M \to N$, where M is the current manifold and N a differentiable manifold; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of multivector fields *on* M), otherwise `dest_map` must be a `DiffMap`

OUTPUT:

- a `MultivectorModule` (or if N is parallelizable, a `MultivectorFreeModule`) representing the module $\Omega^p(M, \Phi)$ of p-forms on M taking values on $\Phi(M) \subset N$

EXAMPLES:

Module of 2-vector fields on a 3-dimensional parallelizable manifold:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: M.multivector_module(2)
Free module A^2(M) of 2-vector fields on the 3-dimensional
 differentiable manifold M
sage: M.multivector_module(2).category()
Category of finite dimensional modules over Algebra of
 differentiable scalar fields on the 3-dimensional
 differentiable manifold M
sage: M.multivector_module(2).base_ring()
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
sage: M.multivector_module(2).rank()
3
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> M.multivector_module(Integer(2))
Free module A^2(M) of 2-vector fields on the 3-dimensional
 differentiable manifold M
>>> M.multivector_module(Integer(2)).category()
Category of finite dimensional modules over Algebra of
 differentiable scalar fields on the 3-dimensional
 differentiable manifold M
>>> M.multivector_module(Integer(2)).base_ring()
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
```

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```
>>> M.multivector_module(Integer(2)).rank()
3
```

The outcome is cached:

```
sage: M.multivector_module(2) is M.multivector_module(2)
True
```

```
>>> from sage.all import *
>>> M.multivector_module(Integer(2)) is M.multivector_module(Integer(2))
True
```

one_form(**comp*, ***kwargs*)

Define a 1-form on the manifold.

Via the argument `dest_map`, it is possible to let the 1-form take its values on another manifold. More precisely, if M is the current manifold, N a differentiable manifold and $\Phi : M \to N$ a differentiable map, a *1-form along* M *with values on* N is a differentiable map

$$t : M \longrightarrow T^*N$$

(T^*N being the cotangent bundle of N) such that

$$\forall p \in M, \quad t(p) \in T^*_{\Phi(p)}N,$$

where $T^*_{\Phi(p)}$ is the dual of the tangent space $T_{\Phi(p)}N$.

The standard case of a 1-form *on* M corresponds to $N = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in N (M is then an open interval of **R**).

See also

`DiffForm` and `DiffFormParal` for a complete documentation.

INPUT:

- `comp` – (optional) either the components of 1-form with respect to the vector frame specified by the argument `frame` or a dictionary of components, the keys of which are vector frames or pairs `(f, c)` where `f` is a vector frame and `c` the chart in which the components are expressed
- `frame` – (default: `None`; unused if `comp` is not given or is a dictionary) vector frame in which the components are given; if `None`, the default vector frame of `self` is assumed
- `chart` – (default: `None`; unused if `comp` is not given or is a dictionary) coordinate chart in which the components are expressed; if `None`, the default chart on the domain of `frame` is assumed
- `name` – (default: `None`) name given to the 1-form
- `latex_name` – (default: `None`) LaTeX symbol to denote the 1-form; if none is provided, the LaTeX symbol is set to `name`
- `dest_map` – (default: `None`) the destination map $\Phi : M \to N$; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of a 1-form *on* M), otherwise `dest_map` must be a `DiffMap`

OUTPUT:

- the 1-form as a `DiffForm` (or if N is parallelizable, a `DiffFormParal`)

EXAMPLES:

A 1-form on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: om = M.one_form(-y, 2+x, name='omega', latex_name=r'\omega')
sage: om
1-form omega on the 2-dimensional differentiable manifold M
sage: om.display()
omega = -y dx + (x + 2) dy
sage: om.parent()
Free module Omega^1(M) of 1-forms on the 2-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> om = M.one_form(-y, Integer(2)+x, name='omega', latex_name=r'\omega')
>>> om
1-form omega on the 2-dimensional differentiable manifold M
>>> om.display()
omega = -y dx + (x + 2) dy
>>> om.parent()
Free module Omega^1(M) of 1-forms on the 2-dimensional
 differentiable manifold M
```

For more examples, see *DiffForm* and *DiffFormParal*.

open_subset(*name*, *latex_name=None*, *coord_def={}*, *supersets=None*)

Create an open subset of the manifold.

An open subset is a set that is (i) included in the manifold and (ii) open with respect to the manifold's topology. It is a differentiable manifold by itself. Hence the returned object is an instance of *DifferentiableManifold*.

INPUT:

- `name` – name given to the open subset
- `latex_name` – (default: `None`) LaTeX symbol to denote the subset; if none is provided, it is set to `name`
- `coord_def` – (default: `{}`) definition of the subset in terms of coordinates; `coord_def` must a be dictionary with keys charts in the manifold's atlas and values the symbolic expressions formed by the coordinates to define the subset.
- `supersets` – (default: only `self`) list of sets that the new open subset is a subset of

OUTPUT: the open subset, as an instance of *DifferentiableManifold*

EXAMPLES:

Creating an open subset of a differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: A = M.open_subset('A'); A
Open subset A of the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> A = M.open_subset('A'); A
Open subset A of the 2-dimensional differentiable manifold M
```

As an open subset of a differentiable manifold, `A` is itself a differentiable manifold, on the same topological field and of the same dimension as `M`:

```
sage: A.category()
Join of Category of subobjects of sets and Category of smooth
 manifolds over Real Field with 53 bits of precision
sage: A.base_field() == M.base_field()
True
sage: dim(A) == dim(M)
True
```

```
>>> from sage.all import *
>>> A.category()
Join of Category of subobjects of sets and Category of smooth
 manifolds over Real Field with 53 bits of precision
>>> A.base_field() == M.base_field()
True
>>> dim(A) == dim(M)
True
```

Creating an open subset of `A`:

```
sage: B = A.open_subset('B'); B
Open subset B of the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> B = A.open_subset('B'); B
Open subset B of the 2-dimensional differentiable manifold M
```

We have then:

```
sage: A.subset_family()
Set {A, B} of open subsets of the 2-dimensional differentiable manifold M
sage: B.is_subset(A)
True
sage: B.is_subset(M)
True
```

```
>>> from sage.all import *
>>> A.subset_family()
Set {A, B} of open subsets of the 2-dimensional differentiable manifold M
>>> B.is_subset(A)
True
>>> B.is_subset(M)
True
```

Defining an open subset by some coordinate restrictions: the open unit disk in of the Euclidean plane:

```
sage: X.<x,y> = M.chart() # Cartesian coordinates on M
sage: U = M.open_subset('U', coord_def={X: x^2+y^2<1}); U
Open subset U of the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)# Cartesian␣
↪coordinates on M
>>> U = M.open_subset('U', coord_def={X: x**Integer(2)+y**Integer(2)
↪<Integer(1)}); U
Open subset U of the 2-dimensional differentiable manifold M
```

Since the argument `coord_def` has been set, `U` is automatically endowed with a chart, which is the restriction of `X` to `U`:

```
sage: U.atlas()
[Chart (U, (x, y))]
sage: U.default_chart()
Chart (U, (x, y))
sage: U.default_chart() is X.restrict(U)
True
```

```
>>> from sage.all import *
>>> U.atlas()
[Chart (U, (x, y))]
>>> U.default_chart()
Chart (U, (x, y))
>>> U.default_chart() is X.restrict(U)
True
```

A point in `U`:

```
sage: p = U.an_element(); p
Point on the 2-dimensional differentiable manifold M
sage: X(p)   # the coordinates (x,y) of p
(0, 0)
sage: p in U
True
```

```
>>> from sage.all import *
>>> p = U.an_element(); p
Point on the 2-dimensional differentiable manifold M
>>> X(p)   # the coordinates (x,y) of p
(0, 0)
>>> p in U
True
```

Checking whether various points, defined by their coordinates with respect to chart `X`, are in `U`:

```
sage: M((0,1/2)) in U
True
sage: M((0,1)) in U
False
```

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```
sage: M((1/2,1)) in U
False
sage: M((-1/2,1/3)) in U
True
```

```
>>> from sage.all import *
>>> M((Integer(0),Integer(1)/Integer(2))) in U
True
>>> M((Integer(0),Integer(1))) in U
False
>>> M((Integer(1)/Integer(2),Integer(1))) in U
False
>>> M((-Integer(1)/Integer(2),Integer(1)/Integer(3))) in U
True
```

orientation()

Get the preferred orientation of `self` if available.

An *orientation* on a differentiable manifold is an atlas of charts whose transition maps are pairwise orientation preserving, i.e. whose Jacobian determinants are pairwise positive.

A differentiable manifold with an orientation is called *orientable*.

A differentiable manifold is orientable if and only if the tangent bundle is orientable in terms of a vector bundle, see `orientation()`.

> **Note**
>
> In contrast to topological manifolds, see `orientation()`, differentiable manifolds preferably use the notion of orientability in terms of the tangent bundle.

The trivial case corresponds to the manifold being parallelizable, i.e. admitting a frame covering the whole manifold. In that case, if no preferred orientation has been manually set before, one of those frames (usually the default frame) is set to the preferred orientation on `self` and returned here.

EXAMPLES:

In case one frame already covers the manifold, an orientation is readily obtained:

```
sage: M = Manifold(3, 'M')
sage: c.<x,y,z> = M.chart()
sage: M.orientation()
[Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> c = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c._first_ngens(3)
>>> M.orientation()
[Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z))]
```

However, orientations are usually not easy to obtain:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U'); V = M.open_subset('V')
sage: M.declare_union(U, V)
sage: c_xy.<x,y> = U.chart(); c_uv.<u,v> = V.chart()
sage: M.orientation()
[]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U'); V = M.open_subset('V')
>>> M.declare_union(U, V)
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> M.orientation()
[]
```

In that case, the orientation can be set by the user; either in terms of charts or in terms of frames:

```
sage: M.set_orientation([c_xy, c_uv])
sage: M.orientation()
[Coordinate frame (U, (∂/∂x,∂/∂y)),
 Coordinate frame (V, (∂/∂u,∂/∂v))]
sage: M.set_orientation([c_xy.frame(), c_uv.frame()])
sage: M.orientation()
[Coordinate frame (U, (∂/∂x,∂/∂y)),
 Coordinate frame (V, (∂/∂u,∂/∂v))]
```

```
>>> from sage.all import *
>>> M.set_orientation([c_xy, c_uv])
>>> M.orientation()
[Coordinate frame (U, (∂/∂x,∂/∂y)),
 Coordinate frame (V, (∂/∂u,∂/∂v))]
>>> M.set_orientation([c_xy.frame(), c_uv.frame()])
>>> M.orientation()
[Coordinate frame (U, (∂/∂x,∂/∂y)),
 Coordinate frame (V, (∂/∂u,∂/∂v))]
```

The orientation on submanifolds are inherited from the ambient manifold:

```
sage: W = U.intersection(V, name='W')
sage: W.orientation()
[Vector frame (W, (∂/∂x,∂/∂y))]
```

```
>>> from sage.all import *
>>> W = U.intersection(V, name='W')
>>> W.orientation()
[Vector frame (W, (∂/∂x,∂/∂y))]
```

poisson_tensor(*name=None*, *latex_name=None*)

Construct a Poisson tensor on the current manifold.

OUTPUT:

- instance of `PoissonTensorField`

EXAMPLES:

Standard Poisson tensor on $\mathbf{R}^2$:

```
sage: M.<q, p> = EuclideanSpace(2)
sage: poisson = M.poisson_tensor('varpi')
sage: poisson.set_comp()[1,2] = -1
sage: poisson.display()
varpi = -e_q∧e_p
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(Integer(2), names=('q', 'p',)); (q, p,) = M._first_
↪ngens(2)
>>> poisson = M.poisson_tensor('varpi')
>>> poisson.set_comp()[Integer(1),Integer(2)] = -Integer(1)
>>> poisson.display()
varpi = -e_q∧e_p
```

riemannian_metric(*name*, *latex_name=None*, *dest_map=None*)

Define a Riemannian metric on the manifold.

A *Riemannian metric* is a field of positive definite symmetric bilinear forms acting in the tangent spaces.

See `PseudoRiemannianMetric` for a complete documentation.

INPUT:

- `name` – name given to the metric
- `latex_name` – (default: `None`) LaTeX symbol to denote the metric; if `None`, it is formed from `name`
- `dest_map` – (default: `None`) instance of class `DiffMap` representing the destination map $\Phi : U \to M$, where U is the current manifold; if `None`, the identity map is assumed (case of a metric tensor field *on* U)

OUTPUT:

- instance of `PseudoRiemannianMetric` representing the defined Riemannian metric.

EXAMPLES:

Metric of the hyperbolic plane H^2:

```
sage: H2 = Manifold(2, 'H^2', start_index=1)
sage: X.<x,y> = H2.chart('x y:(0,+oo)')  # Poincaré half-plane coord.
sage: g = H2.riemannian_metric('g')
sage: g[1,1], g[2,2] = 1/y^2, 1/y^2
sage: g
Riemannian metric g on the 2-dimensional differentiable manifold H^2
sage: g.display()
g = y^(-2) dx⊗dx + y^(-2) dy⊗dy
sage: g.signature()
2
```

```
>>> from sage.all import *
>>> H2 = Manifold(Integer(2), 'H^2', start_index=Integer(1))
>>> X = H2.chart('x y:(0,+oo)', names=('x', 'y',)); (x, y,) = X._first_
↪ngens(2)# Poincaré half-plane coord.
```

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```
>>> g = H2.riemannian_metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)] = Integer(1)/
↪y**Integer(2), Integer(1)/y**Integer(2)
>>> g
Riemannian metric g on the 2-dimensional differentiable manifold H^2
>>> g.display()
g = y^(-2) dx⊗dx + y^(-2) dy⊗dy
>>> g.signature()
2
```

See also

`PseudoRiemannianMetric` for more examples.

set_change_of_frame(*frame1*, *frame2*, *change_of_frame*, *compute_inverse=True*)

Relate two vector frames by an automorphism.

This updates the internal dictionary `self._frame_changes`.

INPUT:

- `frame1` – frame 1, denoted (e_i) below
- `frame2` – frame 2, denoted (f_i) below
- `change_of_frame` – instance of class `AutomorphismFieldParal` describing the automorphism P that relates the basis (e_i) to the basis (f_i) according to $f_i = P(e_i)$
- `compute_inverse` – boolean (default: `True`); if set to True, the inverse automorphism is computed and the change from basis (f_i) to (e_i) is set to it in the internal dictionary `self._frame_changes`

EXAMPLES:

Connecting two vector frames on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: e = M.vector_frame('e')
sage: f = M.vector_frame('f')
sage: a = M.automorphism_field()
sage: a[e,:] = [[1,2],[0,3]]
sage: M.set_change_of_frame(e, f, a)
sage: f[0].display(e)
f_0 = e_0
sage: f[1].display(e)
f_1 = 2 e_0 + 3 e_1
sage: e[0].display(f)
e_0 = f_0
sage: e[1].display(f)
e_1 = -2/3 f_0 + 1/3 f_1
sage: M.change_of_frame(e,f)[e,:]
[1 2]
[0 3]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> e = M.vector_frame('e')
>>> f = M.vector_frame('f')
>>> a = M.automorphism_field()
>>> a[e,:] = [[Integer(1),Integer(2)],[Integer(0),Integer(3)]]
>>> M.set_change_of_frame(e, f, a)
>>> f[Integer(0)].display(e)
f_0 = e_0
>>> f[Integer(1)].display(e)
f_1 = 2 e_0 + 3 e_1
>>> e[Integer(0)].display(f)
e_0 = f_0
>>> e[Integer(1)].display(f)
e_1 = -2/3 f_0 + 1/3 f_1
>>> M.change_of_frame(e,f)[e,:]
[1 2]
[0 3]
```

set_default_frame(*frame*)

Changing the default vector frame on `self`.

INPUT:

- `frame` – *VectorFrame* a vector frame defined on some subset of `self`

EXAMPLES:

Changing the default frame on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: e = M.vector_frame('e')
sage: M.default_frame()
Coordinate frame (M, (∂/∂x,∂/∂y))
sage: M.set_default_frame(e)
sage: M.default_frame()
Vector frame (M, (e_0,e_1))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> e = M.vector_frame('e')
>>> M.default_frame()
Coordinate frame (M, (∂/∂x,∂/∂y))
>>> M.set_default_frame(e)
>>> M.default_frame()
Vector frame (M, (e_0,e_1))
```

set_orientation(*orientation*)

Set the preferred orientation of `self`.

INPUT:

- `orientation` – either a chart / list of charts, or a vector frame / list of vector frames, covering `self`

Warning: It is the user's responsibility that the orientation set here is indeed an orientation. There is no check going on in the background. See *orientation()* for the definition of an orientation.

EXAMPLES:

Set an orientation on a manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart(); c_uv.<u,v> = M.chart()
sage: M.set_orientation(c_uv)
sage: M.orientation()
[Coordinate frame (M, (∂/∂u,∂/∂v))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪M.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> M.set_orientation(c_uv)
>>> M.orientation()
[Coordinate frame (M, (∂/∂u,∂/∂v))]
```

Instead of a chart, a vector frame can be given, too:

```
sage: M.set_orientation(c_xy.frame())
sage: M.orientation()
[Coordinate frame (M, (∂/∂x,∂/∂y))]
```

```
>>> from sage.all import *
>>> M.set_orientation(c_xy.frame())
>>> M.orientation()
[Coordinate frame (M, (∂/∂x,∂/∂y))]
```

Set an orientation in the non-trivial case:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U'); V = M.open_subset('V')
sage: M.declare_union(U, V)
sage: c_xy.<x,y> = U.chart(); c_uv.<u,v> = V.chart()
sage: M.set_orientation([c_xy, c_uv])
sage: M.orientation()
[Coordinate frame (U, (∂/∂x,∂/∂y)),
 Coordinate frame (V, (∂/∂u,∂/∂v))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U'); V = M.open_subset('V')
>>> M.declare_union(U, V)
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> M.set_orientation([c_xy, c_uv])
```

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```
>>> M.orientation()
[Coordinate frame (U, (∂/∂x,∂/∂y)),
 Coordinate frame (V, (∂/∂u,∂/∂v))]
```

Again, the vector frame notion can be used instead:

```
sage: M.set_orientation([c_xy.frame(), c_uv.frame()])
sage: M.orientation()
[Coordinate frame (U, (∂/∂x,∂/∂y)),
 Coordinate frame (V, (∂/∂u,∂/∂v))]
```

```
>>> from sage.all import *
>>> M.set_orientation([c_xy.frame(), c_uv.frame()])
>>> M.orientation()
[Coordinate frame (U, (∂/∂x,∂/∂y)),
 Coordinate frame (V, (∂/∂u,∂/∂v))]
```

sym_bilin_form_field(**comp*, ***kwargs*)

Define a field of symmetric bilinear forms on `self`.

Via the argument `dest_map`, it is possible to let the field take its values on another manifold. More precisely, if M is the current manifold, N a differentiable manifold and $\Phi: M \to N$ a differentiable map, a *field of symmetric bilinear forms along* M *with values on* N is a differentiable map

$$t: M \longrightarrow T^{(0,2)}N$$

($T^{(0,2)}N$ being the tensor bundle of type $(0,2)$ over N) such that

$$\forall p \in M, \ t(p) \in S(T_{\Phi(p)}N),$$

where $S(T_{\Phi(p)}N)$ is the space of symmetric bilinear forms on the tangent space $T_{\Phi(p)}N$.

The standard case of fields of symmetric bilinear forms *on* M corresponds to $N = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in N (M is then an open interval of $\mathbf{R}$).

INPUT:

- `comp` – (optional) either the components of the field of symmetric bilinear forms with respect to the vector frame specified by the argument `frame` or a dictionary of components, the keys of which are vector frames or pairs `(f, c)` where `f` is a vector frame and `c` the chart in which the components are expressed
- `frame` – (default: `None`; unused if `comp` is not given or is a dictionary) vector frame in which the components are given; if `None`, the default vector frame of `self` is assumed
- `chart` – (default: `None`; unused if `comp` is not given or is a dictionary) coordinate chart in which the components are expressed; if `None`, the default chart on the domain of `frame` is assumed
- `name` – (default: `None`) name given to the field
- `latex_name` – (default: `None`) LaTeX symbol to denote the field; if none is provided, the LaTeX symbol is set to `name`
- `dest_map` – (default: `None`) the destination map $\Phi: M \to N$; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of a field *on* M), otherwise `dest_map` must be an instance of instance of class `DiffMap`

OUTPUT:

- a *TensorField* (or if N is parallelizable, a *TensorFieldParal*) of tensor type $(0,2)$ and symmetric representing the defined field of symmetric bilinear forms

EXAMPLES:

A field of symmetric bilinear forms on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: t = M.sym_bilin_form_field(name='T'); t
Field of symmetric bilinear forms T on the 2-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> t = M.sym_bilin_form_field(name='T'); t
Field of symmetric bilinear forms T on the 2-dimensional
 differentiable manifold M
```

Such a object is a tensor field of rank 2 and type $(0,2)$:

```
sage: t.parent()
Free module T^(0,2)(M) of type-(0,2) tensors fields on the
 2-dimensional differentiable manifold M
sage: t.tensor_rank()
2
sage: t.tensor_type()
(0, 2)
```

```
>>> from sage.all import *
>>> t.parent()
Free module T^(0,2)(M) of type-(0,2) tensors fields on the
 2-dimensional differentiable manifold M
>>> t.tensor_rank()
2
>>> t.tensor_type()
(0, 2)
```

The LaTeX symbol is deduced from the name or can be specified when creating the object:

```
sage: latex(t)
T
sage: om = M.sym_bilin_form_field(name='Omega', latex_name=r'\Omega')
sage: latex(om)
\Omega
```

```
>>> from sage.all import *
>>> latex(t)
T
>>> om = M.sym_bilin_form_field(name='Omega', latex_name=r'\Omega')
>>> latex(om)
\Omega
```

Setting the components in the manifold's default vector frame:

```
sage: t[0,0], t[0,1], t[1,1] = -1, x, x*y
```

```
>>> from sage.all import *
>>> t[Integer(0),Integer(0)], t[Integer(0),Integer(1)], t[Integer(1),
↪Integer(1)] = -Integer(1), x, x*y
```

The unset components are either zero or deduced by symmetry:

```
sage: t[1, 0]
x
sage: t[:]
[ -1    x]
[  x x*y]
```

```
>>> from sage.all import *
>>> t[Integer(1), Integer(0)]
x
>>> t[:]
[ -1    x]
[  x x*y]
```

One can also set the components while defining the field of symmetric bilinear forms:

```
sage: t = M.sym_bilin_form_field([[-1, x], [x, x*y]], name='T')
```

```
>>> from sage.all import *
>>> t = M.sym_bilin_form_field([[-Integer(1), x], [x, x*y]], name='T')
```

A symmetric bilinear form acts on vector pairs:

```
sage: v1 = M.vector_field(y, x, name='V_1')
sage: v2 = M.vector_field(x+y, 2, name='V_2')
sage: s = t(v1,v2) ; s
Scalar field T(V_1,V_2) on the 2-dimensional differentiable
 manifold M
sage: s.expr()
x^3 + (3*x^2 + x)*y - y^2
sage: s.expr() - t[0,0]*v1[0]*v2[0] - \
....: t[0,1]*(v1[0]*v2[1]+v1[1]*v2[0]) - t[1,1]*v1[1]*v2[1]
0
sage: latex(s)
T\left(V_1,V_2\right)
```

```
>>> from sage.all import *
>>> v1 = M.vector_field(y, x, name='V_1')
>>> v2 = M.vector_field(x+y, Integer(2), name='V_2')
>>> s = t(v1,v2) ; s
Scalar field T(V_1,V_2) on the 2-dimensional differentiable
 manifold M
>>> s.expr()
x^3 + (3*x^2 + x)*y - y^2
>>> s.expr() - t[Integer(0),Integer(0)]*v1[Integer(0)]*v2[Integer(0)] -␣
```

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```
↪t[Integer(0),
↪Integer(1)]*(v1[Integer(0)]*v2[Integer(1)]+v1[Integer(1)]*v2[Integer(0)]) -␣
↪t[Integer(1),Integer(1)]*v1[Integer(1)]*v2[Integer(1)]
0
>>> latex(s)
T\left(V_1,V_2\right)
```

Adding two symmetric bilinear forms results in another symmetric bilinear form:

```
sage: a = M.sym_bilin_form_field([[1, 2], [2, 3]])
sage: b = M.sym_bilin_form_field([[-1, 4], [4, 5]])
sage: s = a + b ; s
Field of symmetric bilinear forms on the 2-dimensional
 differentiable manifold M
sage: s[:]
[0 6]
[6 8]
```

```
>>> from sage.all import *
>>> a = M.sym_bilin_form_field([[Integer(1), Integer(2)], [Integer(2),␣
↪Integer(3)]])
>>> b = M.sym_bilin_form_field([[-Integer(1), Integer(4)], [Integer(4),␣
↪Integer(5)]])
>>> s = a + b ; s
Field of symmetric bilinear forms on the 2-dimensional
 differentiable manifold M
>>> s[:]
[0 6]
[6 8]
```

But adding a symmetric bilinear from with a non-symmetric bilinear form results in a generic type $(0, 2)$ tensor:

```
sage: c = M.tensor_field(0, 2, [[-2, -3], [1,7]])
sage: s1 = a + c ; s1
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
sage: s1[:]
[-1 -1]
[ 3 10]
sage: s2 = c + a ; s2
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
sage: s2[:]
[-1 -1]
[ 3 10]
```

```
>>> from sage.all import *
>>> c = M.tensor_field(Integer(0), Integer(2), [[-Integer(2), -Integer(3)],␣
↪[Integer(1),Integer(7)]])
>>> s1 = a + c ; s1
Tensor field of type (0,2) on the 2-dimensional differentiable
```

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```
 manifold M
>>> s1[:]
[-1 -1]
[ 3 10]
>>> s2 = c + a ; s2
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
>>> s2[:]
[-1 -1]
[ 3 10]
```

symplectic_form(*name=None*, *latex_name=None*)

Construct a symplectic form on the current manifold.

OUTPUT:

- instance of *SymplecticForm*

EXAMPLES:

Standard symplectic form on $\mathbf{R}^2$:

```
sage: M.<q, p> = EuclideanSpace(2)
sage: omega = M.symplectic_form('omega', r'\omega')
sage: omega.set_comp()[1,2] = -1
sage: omega.display()
omega = -dq∧dp
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(Integer(2), names=('q', 'p',)); (q, p,) = M._first_
↪ngens(2)
>>> omega = M.symplectic_form('omega', r'\omega')
>>> omega.set_comp()[Integer(1),Integer(2)] = -Integer(1)
>>> omega.display()
omega = -dq∧dp
```

tangent_bundle(*dest_map=None*)

Return the tangent bundle possibly along a destination map with base space `self`.

See also

TensorBundle for complete documentation.

INPUT:

- `dest_map` – (default: `None`) destination map $\Phi : M \to N$ (type: *DiffMap*) from which the tangent bundle is pulled back; if `None`, it is assumed that $N = M$ and Φ is the identity map of M (case of the standard tangent bundle over M)

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: TM = M.tangent_bundle(); TM
Tangent bundle TM over the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> TM = M.tangent_bundle(); TM
Tangent bundle TM over the 2-dimensional differentiable manifold M
```

tangent_identity_field(*dest_map=None*)

Return the field of identity maps in the tangent spaces on `self`.

Via the argument `dest_map`, it is possible to let the field take its values on another manifold. More precisely, if M is the current manifold, N a differentiable manifold and $\Phi : M \to N$ a differentiable map, a *field of identity maps along* M *with values on* N is a differentiable map

$$t : M \longrightarrow T^{(1,1)}N$$

($T^{(1,1)}N$ being the tensor bundle of type $(1,1)$ over N) such that

$$\forall p \in M, \ t(p) = \mathrm{Id}_{T_{\Phi(p)}N},$$

where $\mathrm{Id}_{T_{\Phi(p)}N}$ is the identity map of the tangent space $T_{\Phi(p)}N$.

The standard case of a field of identity maps *on* M corresponds to $N = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in N (M is then an open interval of **R**).

INPUT:

- `name` – (string; default: 'Id') name given to the field of identity maps
- `latex_name` – (string; default: `None`) LaTeX symbol to denote the field of identity map; if none is provided, the LaTeX symbol is set to 'mathrm{Id}' if `name` is 'Id' and to `name` otherwise
- `dest_map` – (default: `None`) the destination map $\Phi : M \to N$; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of a field of identity maps *on* M), otherwise `dest_map` must be a *DiffMap*

OUTPUT:

- a *AutomorphismField* (or if N is parallelizable, a *AutomorphismFieldParal*) representing the field of identity maps

EXAMPLES:

Field of tangent-space identity maps on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: a = M.tangent_identity_field(); a
Field of tangent-space identity maps on the 3-dimensional
 differentiable manifold M
sage: a.comp()
Kronecker delta of size 3x3
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> a = M.tangent_identity_field(); a
Field of tangent-space identity maps on the 3-dimensional
 differentiable manifold M
```

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```
>>> a.comp()
Kronecker delta of size 3x3
```

For more examples, see `AutomorphismField`.

tangent_space(*point*, *base_ring=None*)

Tangent space to `self` at a given point.

INPUT:

- `point` – `ManifoldPoint`; point p on the manifold
- `base_ring` – (default: the symbolic ring) the base ring

OUTPUT:

- `TangentSpace` representing the tangent vector space $T_p M$, where M is the current manifold

EXAMPLES:

A tangent space to a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: p = M.point((2, -3), name='p')
sage: Tp = M.tangent_space(p); Tp
Tangent space at Point p on the 2-dimensional differentiable
 manifold M
sage: Tp.category()
Category of finite dimensional vector spaces over Symbolic Ring
sage: dim(Tp)
2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> p = M.point((Integer(2), -Integer(3)), name='p')
>>> Tp = M.tangent_space(p); Tp
Tangent space at Point p on the 2-dimensional differentiable
 manifold M
>>> Tp.category()
Category of finite dimensional vector spaces over Symbolic Ring
>>> dim(Tp)
2
```

See also

`TangentSpace` for more examples.

tangent_vector(**args*, ***kwargs*)

Define a tangent vector at a given point of `self`.

INPUT:

- `point` – `ManifoldPoint`; point p on `self`

- `comp` – components of the vector with respect to the basis specified by the argument `basis`, either as an iterable or as a sequence of n components, n being the dimension of `self` (see examples below)
- `basis` – (default: `None`) `FreeModuleBasis`; basis of the tangent space at p with respect to which the components are defined; if `None`, the default basis of the tangent space is used
- `name` – (default: `None`) string; symbol given to the vector
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the vector; if `None`, `name` will be used

OUTPUT:

- `TangentVector` representing the tangent vector at point p

EXAMPLES:

Vector at a point p of the Euclidean plane:

```
sage: E.<x,y>= EuclideanSpace()
sage: p = E((1, 2), name='p')
sage: v = E.tangent_vector(p, -1, 3, name='v'); v
Vector v at Point p on the Euclidean plane E^2
sage: v.display()
v = -e_x + 3 e_y
sage: v.parent()
Tangent space at Point p on the Euclidean plane E^2
sage: v in E.tangent_space(p)
True
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(names=('x', 'y',)); (x, y,) = E._first_ngens(2)
>>> p = E((Integer(1), Integer(2)), name='p')
>>> v = E.tangent_vector(p, -Integer(1), Integer(3), name='v'); v
Vector v at Point p on the Euclidean plane E^2
>>> v.display()
v = -e_x + 3 e_y
>>> v.parent()
Tangent space at Point p on the Euclidean plane E^2
>>> v in E.tangent_space(p)
True
```

An alias of `tangent_vector` is `vector`:

```
sage: v = E.vector(p, -1, 3, name='v'); v
Vector v at Point p on the Euclidean plane E^2
```

```
>>> from sage.all import *
>>> v = E.vector(p, -Integer(1), Integer(3), name='v'); v
Vector v at Point p on the Euclidean plane E^2
```

The components can be passed as a tuple or a list:

```
sage: v1 = E.vector(p, (-1, 3)); v1
Vector at Point p on the Euclidean plane E^2
sage: v1 == v
True
```

```
>>> from sage.all import *
>>> v1 = E.vector(p, (-Integer(1), Integer(3))); v1
Vector at Point p on the Euclidean plane E^2
>>> v1 == v
True
```

or as an object created by the `vector` function:

```
sage: v2 = E.vector(p, vector([-1, 3])); v2
Vector at Point p on the Euclidean plane E^2
sage: v2 == v
True
```

```
>>> from sage.all import *
>>> v2 = E.vector(p, vector([-Integer(1), Integer(3)])); v2
Vector at Point p on the Euclidean plane E^2
>>> v2 == v
True
```

Example of use with the options `basis` and `latex_name`:

```
sage: polar_basis = E.polar_frame().at(p)
sage: polar_basis
Basis (e_r,e_ph) on the Tangent space at Point p on the Euclidean plane E^2
sage: v = E.vector(p, 2, -1, basis=polar_basis, name='v',
....:              latex_name=r'\vec{v}')
sage: v
Vector v at Point p on the Euclidean plane E^2
sage: v.display(polar_basis)
v = 2 e_r - e_ph
sage: v.display()
v = 4/5*sqrt(5) e_x + 3/5*sqrt(5) e_y
sage: latex(v)
\vec{v}
```

```
>>> from sage.all import *
>>> polar_basis = E.polar_frame().at(p)
>>> polar_basis
Basis (e_r,e_ph) on the Tangent space at Point p on the Euclidean plane E^2
>>> v = E.vector(p, Integer(2), -Integer(1), basis=polar_basis, name='v',
...              latex_name=r'\vec{v}')
>>> v
Vector v at Point p on the Euclidean plane E^2
>>> v.display(polar_basis)
v = 2 e_r - e_ph
>>> v.display()
v = 4/5*sqrt(5) e_x + 3/5*sqrt(5) e_y
>>> latex(v)
\vec{v}
```

tensor_bundle(*k*, *l*, *dest_map=None*)

Return a tensor bundle of type (k, l) defined over `self`, possibly along a destination map.

INPUT:

- k – the contravariant rank of the tensor bundle
- l – the covariant rank of the tensor bundle
- dest_map – (default: None) destination map $\Phi : M \to N$ (type: *DiffMap*) from which the tensor bundle is pulled back; if None, it is assumed that $N = M$ and Φ is the identity map of M (case of the standard tangent bundle over M)

OUTPUT:

- a *TensorBundle* representing a tensor bundle of type-(k, l) over self

EXAMPLES:

A tensor bundle over a parallelizable 2-dimensional differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()   # makes M parallelizable
sage: M.tensor_bundle(1, 2)
Tensor bundle T^(1,2)M over the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)# makes M␣
↪parallelizable
>>> M.tensor_bundle(Integer(1), Integer(2))
Tensor bundle T^(1,2)M over the 2-dimensional differentiable
 manifold M
```

The special case of the tangent bundle as tensor bundle of type (1,0):

```
sage: M.tensor_bundle(1,0)
Tangent bundle TM over the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M.tensor_bundle(Integer(1),Integer(0))
Tangent bundle TM over the 2-dimensional differentiable manifold M
```

The result is cached:

```
sage: M.tensor_bundle(1, 2) is M.tensor_bundle(1, 2)
True
```

```
>>> from sage.all import *
>>> M.tensor_bundle(Integer(1), Integer(2)) is M.tensor_bundle(Integer(1),␣
↪Integer(2))
True
```

See also

TensorBundle for more examples and documentation.

tensor_field(**args*, ***kwargs*)

Define a tensor field on self.

Via the argument `dest_map`, it is possible to let the tensor field take its values on another manifold. More precisely, if M is the current manifold, N a differentiable manifold, $\Phi: M \to N$ a differentiable map and (k, l) a pair of nonnegative integers, a *tensor field of type* (k, l) *along* M *with values on* N is a differentiable map

$$t: M \longrightarrow T^{(k,l)}N$$

($T^{(k,l)}N$ being the tensor bundle of type (k, l) over N) such that

$$\forall p \in M, \ t(p) \in T^{(k,l)}(T_{\Phi(p)}N),$$

where $T^{(k,l)}(T_{\Phi(p)}N)$ is the space of tensors of type (k, l) on the tangent space $T_{\Phi(p)}N$.

The standard case of tensor fields *on* M corresponds to $N = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in N (M is then an open interval of $\mathbf{R}$).

See also:

`TensorField` and `TensorFieldParal` for a complete documentation.

INPUT:

- `k` – the contravariant rank k, the tensor type being (k, l)
- `l` – the covariant rank l, the tensor type being (k, l)
- `comp` – (optional) either the components of the tensor field with respect to the vector frame specified by the argument `frame` or a dictionary of components, the keys of which are vector frames or pairs `(f, c)` where `f` is a vector frame and `c` the chart in which the components are expressed
- `frame` – (default: `None`; unused if `comp` is not given or is a dictionary) vector frame in which the components are given; if `None`, the default vector frame of `self` is assumed
- `chart` – (default: `None`; unused if `comp` is not given or is a dictionary) coordinate chart in which the components are expressed; if `None`, the default chart on the domain of `frame` is assumed
- `name` – (default: `None`) name given to the tensor field
- `latex_name` – (default: `None`) LaTeX symbol to denote the tensor field; if `None`, the LaTeX symbol is set to `name`
- `sym` – (default: `None`) a symmetry or a list of symmetries among the tensor arguments: each symmetry is described by a tuple containing the positions of the involved arguments, with the convention `position=0` for the first argument; for instance:
  - `sym = (0,1)` for a symmetry between the 1st and 2nd arguments
  - `sym = [(0,2), (1,3,4)]` for a symmetry between the 1st and 3rd arguments and a symmetry between the 2nd, 4th and 5th arguments
- `antisym` – (default: `None`) antisymmetry or list of antisymmetries among the arguments, with the same convention as for `sym`
- `dest_map` – (default: `None`) the destination map $\Phi: M \to N$; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of a tensor field *on* M), otherwise `dest_map` must be a `DiffMap`

OUTPUT:

- a `TensorField` (or if N is parallelizable, a `TensorFieldParal`) representing the defined tensor field

EXAMPLES:

A tensor field of type $(2, 0)$ on a 2-dimensional differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: t = M.tensor_field(2, 0, [[1+x, -y], [0, x*y]], name='T'); t
Tensor field T of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: t.display()
T = (x + 1) ∂/∂x⊗∂/∂x - y ∂/∂x⊗∂/∂y + x*y ∂/∂y⊗∂/∂y
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> t = M.tensor_field(Integer(2), Integer(0), [[Integer(1)+x, -y],
↪[Integer(0), x*y]], name='T'); t
Tensor field T of type (2,0) on the 2-dimensional differentiable
 manifold M
>>> t.display()
T = (x + 1) ∂/∂x⊗∂/∂x - y ∂/∂x⊗∂/∂y + x*y ∂/∂y⊗∂/∂y
```

The type $(2,0)$ tensor fields on M form the set $\mathcal{T}^{(2,0)}(M)$, which is a module over the algebra $C^k(M)$ of differentiable scalar fields on M:

```
sage: t.parent()
Free module T^(2,0)(M) of type-(2,0) tensors fields on the
 2-dimensional differentiable manifold M
sage: t in M.tensor_field_module((2,0))
True
```

```
>>> from sage.all import *
>>> t.parent()
Free module T^(2,0)(M) of type-(2,0) tensors fields on the
 2-dimensional differentiable manifold M
>>> t in M.tensor_field_module((Integer(2),Integer(0)))
True
```

For more examples, see `TensorField` and `TensorFieldParal`.

tensor_field_module(*tensor_type*, *dest_map=None*)

Return the set of tensor fields of a given type defined on `self`, possibly with values in another manifold, as a module over the algebra of scalar fields defined on `self`.

See also

`TensorFieldModule` for a complete documentation.

INPUT:

- `tensor_type` – pair (k,l) with k being the contravariant rank and l the covariant rank
- `dest_map` – (default: `None`) destination map, i.e. a differentiable map $\Phi: M \to N$, where M is the current manifold and N a differentiable manifold; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of tensor fields *on* M), otherwise `dest_map` must be a `DiffMap`

OUTPUT:

- a *TensorFieldModule* (or if N is parallelizable, a *TensorFieldFreeModule*) representing the module $\mathcal{T}^{(k,l)}(M,\Phi)$ of type-(k,l) tensor fields on M taking values on $\Phi(M) \subset N$

EXAMPLES:

Module of type-$(2,1)$ tensor fields on a 3-dimensional open subset of a differentiable manifold:

```
sage: M = Manifold(3, 'M')
sage: U = M.open_subset('U')
sage: c_xyz.<x,y,z> = U.chart()
sage: TU = U.tensor_field_module((2,1)) ; TU
Free module T^(2,1)(U) of type-(2,1) tensors fields on the Open
 subset U of the 3-dimensional differentiable manifold M
sage: TU.category()
Category of tensor products of finite dimensional modules
 over Algebra of differentiable scalar fields
  on the Open subset U of the 3-dimensional differentiable manifold M
sage: TU.base_ring()
Algebra of differentiable scalar fields on the Open subset U of
 the 3-dimensional differentiable manifold M
sage: TU.base_ring() is U.scalar_field_algebra()
True
sage: TU.an_element()
Tensor field of type (2,1) on the Open subset U of the
 3-dimensional differentiable manifold M
sage: TU.an_element().display()
2 ∂/∂x⊗∂/∂x⊗dx
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> U = M.open_subset('U')
>>> c_xyz = U.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> TU = U.tensor_field_module((Integer(2),Integer(1))) ; TU
Free module T^(2,1)(U) of type-(2,1) tensors fields on the Open
 subset U of the 3-dimensional differentiable manifold M
>>> TU.category()
Category of tensor products of finite dimensional modules
 over Algebra of differentiable scalar fields
  on the Open subset U of the 3-dimensional differentiable manifold M
>>> TU.base_ring()
Algebra of differentiable scalar fields on the Open subset U of
 the 3-dimensional differentiable manifold M
>>> TU.base_ring() is U.scalar_field_algebra()
True
>>> TU.an_element()
Tensor field of type (2,1) on the Open subset U of the
 3-dimensional differentiable manifold M
>>> TU.an_element().display()
2 ∂/∂x⊗∂/∂x⊗dx
```

vector(**args*, ***kwargs*)

Define a tangent vector at a given point of `self`.

INPUT:

- `point` – `ManifoldPoint`; point p on `self`
- `comp` – components of the vector with respect to the basis specified by the argument `basis`, either as an iterable or as a sequence of n components, n being the dimension of `self` (see examples below)
- `basis` – (default: `None`) `FreeModuleBasis`; basis of the tangent space at p with respect to which the components are defined; if `None`, the default basis of the tangent space is used
- `name` – (default: `None`) string; symbol given to the vector
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the vector; if `None`, `name` will be used

OUTPUT:

- `TangentVector` representing the tangent vector at point p

EXAMPLES:

Vector at a point p of the Euclidean plane:

```
sage: E.<x,y>= EuclideanSpace()
sage: p = E((1, 2), name='p')
sage: v = E.tangent_vector(p, -1, 3, name='v'); v
Vector v at Point p on the Euclidean plane E^2
sage: v.display()
v = -e_x + 3 e_y
sage: v.parent()
Tangent space at Point p on the Euclidean plane E^2
sage: v in E.tangent_space(p)
True
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(names=('x', 'y',)); (x, y,) = E._first_ngens(2)
>>> p = E((Integer(1), Integer(2)), name='p')
>>> v = E.tangent_vector(p, -Integer(1), Integer(3), name='v'); v
Vector v at Point p on the Euclidean plane E^2
>>> v.display()
v = -e_x + 3 e_y
>>> v.parent()
Tangent space at Point p on the Euclidean plane E^2
>>> v in E.tangent_space(p)
True
```

An alias of `tangent_vector` is `vector`:

```
sage: v = E.vector(p, -1, 3, name='v'); v
Vector v at Point p on the Euclidean plane E^2
```

```
>>> from sage.all import *
>>> v = E.vector(p, -Integer(1), Integer(3), name='v'); v
Vector v at Point p on the Euclidean plane E^2
```

The components can be passed as a tuple or a list:

```
sage: v1 = E.vector(p, (-1, 3)); v1
Vector at Point p on the Euclidean plane E^2
sage: v1 == v
True
```

```
>>> from sage.all import *
>>> v1 = E.vector(p, (-Integer(1), Integer(3))); v1
Vector at Point p on the Euclidean plane E^2
>>> v1 == v
True
```

or as an object created by the `vector` function:

```
sage: v2 = E.vector(p, vector([-1, 3])); v2
Vector at Point p on the Euclidean plane E^2
sage: v2 == v
True
```

```
>>> from sage.all import *
>>> v2 = E.vector(p, vector([-Integer(1), Integer(3)])); v2
Vector at Point p on the Euclidean plane E^2
>>> v2 == v
True
```

Example of use with the options `basis` and `latex_name`:

```
sage: polar_basis = E.polar_frame().at(p)
sage: polar_basis
Basis (e_r,e_ph) on the Tangent space at Point p on the Euclidean plane E^2
sage: v = E.vector(p, 2, -1, basis=polar_basis, name='v',
....:              latex_name=r'\vec{v}')
sage: v
Vector v at Point p on the Euclidean plane E^2
sage: v.display(polar_basis)
v = 2 e_r - e_ph
sage: v.display()
v = 4/5*sqrt(5) e_x + 3/5*sqrt(5) e_y
sage: latex(v)
\vec{v}
```

```
>>> from sage.all import *
>>> polar_basis = E.polar_frame().at(p)
>>> polar_basis
Basis (e_r,e_ph) on the Tangent space at Point p on the Euclidean plane E^2
>>> v = E.vector(p, Integer(2), -Integer(1), basis=polar_basis, name='v',
...              latex_name=r'\vec{v}')
>>> v
Vector v at Point p on the Euclidean plane E^2
>>> v.display(polar_basis)
v = 2 e_r - e_ph
>>> v.display()
v = 4/5*sqrt(5) e_x + 3/5*sqrt(5) e_y
>>> latex(v)
\vec{v}
```

vector_bundle(*rank*, *name*, *field='real'*, *latex_name=None*)

Return a differentiable vector bundle over the given field with given rank over this differentiable manifold of the same differentiability class as the manifold.

INPUT:

- `rank` – rank of the vector bundle
- `name` – name given to the total space
- `field` – (default: `'real'`) topological field giving the vector space structure to the fibers
- `latex_name` – (optional) LaTeX name for the total space

OUTPUT:

- a differentiable vector bundle as an instance of `DifferentiableVectorBundle`

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: M.vector_bundle(2, 'E')
Differentiable real vector bundle E -> M of rank 2 over the base
 space 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> M.vector_bundle(Integer(2), 'E')
Differentiable real vector bundle E -> M of rank 2 over the base
 space 2-dimensional differentiable manifold M
```

vector_field(**comp*, ***kwargs*)

Define a vector field on `self`.

Via the argument `dest_map`, it is possible to let the vector field take its values on another manifold. More precisely, if M is the current manifold, N a differentiable manifold and $\Phi : M \to N$ a differentiable map, a *vector field along M with values on N* is a differentiable map

$$v : M \longrightarrow TN$$

(TN being the tangent bundle of N) such that

$$\forall p \in M, \ v(p) \in T_{\Phi(p)}N,$$

where $T_{\Phi(p)}N$ is the tangent space to N at the point $\Phi(p)$.

The standard case of vector fields *on* M corresponds to $N = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in N (M is then an open interval of $\mathbf{R}$).

See also

`VectorField` and `VectorFieldParal` for a complete documentation.

INPUT:

- `comp` – (optional) either the components of the vector field with respect to the vector frame specified by the argument `frame` or a dictionary of components, the keys of which are vector frames or pairs `(f, c)` where `f` is a vector frame and `c` the chart in which the components are expressed
- `frame` – (default: `None`; unused if `comp` is not given or is a dictionary) vector frame in which the components are given; if `None`, the default vector frame of `self` is assumed
- `chart` – (default: `None`; unused if `comp` is not given or is a dictionary) coordinate chart in which the components are expressed; if `None`, the default chart on the domain of `frame` is assumed

- `name` – (default: `None`) name given to the vector field
- `latex_name` – (default: `None`) LaTeX symbol to denote the vector field; if none is provided, the LaTeX symbol is set to `name`
- `dest_map` – (default: `None`) the destination map $\Phi : M \to N$; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of a vector field *on* M), otherwise `dest_map` must be a `DiffMap`

OUTPUT:

- a `VectorField` (or if N is parallelizable, a `VectorFieldParal`) representing the defined vector field

EXAMPLES:

A vector field on a open subset of a 3-dimensional differentiable manifold:

```
sage: M = Manifold(3, 'M')
sage: U = M.open_subset('U')
sage: c_xyz.<x,y,z> = U.chart()
sage: v = U.vector_field(y, -x*z, 1+y, name='v'); v
Vector field v on the Open subset U of the 3-dimensional
 differentiable manifold M
sage: v.display()
v = y ∂/∂x - x*z ∂/∂y + (y + 1) ∂/∂z
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> U = M.open_subset('U')
>>> c_xyz = U.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> v = U.vector_field(y, -x*z, Integer(1)+y, name='v'); v
Vector field v on the Open subset U of the 3-dimensional
 differentiable manifold M
>>> v.display()
v = y ∂/∂x - x*z ∂/∂y + (y + 1) ∂/∂z
```

The vector fields on U form the set $\mathfrak{X}(U)$, which is a module over the algebra $C^k(U)$ of differentiable scalar fields on U:

```
sage: v.parent()
Free module X(U) of vector fields on the Open subset U of the
 3-dimensional differentiable manifold M
sage: v in U.vector_field_module()
True
```

```
>>> from sage.all import *
>>> v.parent()
Free module X(U) of vector fields on the Open subset U of the
 3-dimensional differentiable manifold M
>>> v in U.vector_field_module()
True
```

For more examples, see `VectorField` and `VectorFieldParal`.

vector_field_module(*dest_map=None*, *force_free=False*)

Return the set of vector fields defined on `self`, possibly with values in another differentiable manifold, as a module over the algebra of scalar fields defined on the manifold.

See `VectorFieldModule` for a complete documentation.

INPUT:

- `dest_map` – (default: `None`) destination map, i.e. a differentiable map $\Phi : M \to N$, where M is the current manifold and N a differentiable manifold; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of vector fields *on* M), otherwise `dest_map` must be a `DiffMap`
- `force_free` – boolean (default: `False`); if set to `True`, force the construction of a *free* module (this implies that N is parallelizable)

OUTPUT:

- a `VectorFieldModule` (or if N is parallelizable, a `VectorFieldFreeModule`) representing the $C^k(M)$-module $\mathfrak{X}(M,\Phi)$ of vector fields on M taking values on $\Phi(M) \subset N$

EXAMPLES:

Vector field module $\mathfrak{X}(U) := \mathfrak{X}(U, \mathrm{Id}_U)$ of the complement U of the two poles on the sphere $\mathbb{S}^2$:

```
sage: S2 = Manifold(2, 'S^2')
sage: U = S2.open_subset('U')  # the complement of the two poles
sage: spher_coord.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi') #
↪spherical coordinates
sage: XU = U.vector_field_module() ; XU
Free module X(U) of vector fields on the Open subset U of
 the 2-dimensional differentiable manifold S^2
sage: XU.category()
Category of finite dimensional modules over Algebra of
 differentiable scalar fields on the Open subset U of
 the 2-dimensional differentiable manifold S^2
sage: XU.base_ring()
Algebra of differentiable scalar fields on the Open subset U of
 the 2-dimensional differentiable manifold S^2
sage: XU.base_ring() is U.scalar_field_algebra()
True
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2')
>>> U = S2.open_subset('U')  # the complement of the two poles
>>> spher_coord = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi', names=('th',
↪'ph',)); (th, ph,) = spher_coord._first_ngens(2)# spherical coordinates
>>> XU = U.vector_field_module() ; XU
Free module X(U) of vector fields on the Open subset U of
 the 2-dimensional differentiable manifold S^2
>>> XU.category()
Category of finite dimensional modules over Algebra of
 differentiable scalar fields on the Open subset U of
 the 2-dimensional differentiable manifold S^2
>>> XU.base_ring()
Algebra of differentiable scalar fields on the Open subset U of
 the 2-dimensional differentiable manifold S^2
>>> XU.base_ring() is U.scalar_field_algebra()
True
```

$\mathfrak{X}(U)$ is a free module because U is parallelizable (being a chart domain):

```
sage: U.is_manifestly_parallelizable()
True
```

```
>>> from sage.all import *
>>> U.is_manifestly_parallelizable()
True
```

Its rank is the manifold's dimension:

```
sage: XU.rank()
2
```

```
>>> from sage.all import *
>>> XU.rank()
2
```

The elements of $\mathfrak{X}(U)$ are vector fields on U:

```
sage: XU.an_element()
Vector field on the Open subset U of the 2-dimensional
 differentiable manifold S^2
sage: XU.an_element().display()
2 ∂/∂th + 2 ∂/∂ph
```

```
>>> from sage.all import *
>>> XU.an_element()
Vector field on the Open subset U of the 2-dimensional
 differentiable manifold S^2
>>> XU.an_element().display()
2 ∂/∂th + 2 ∂/∂ph
```

Vector field module $\mathfrak{X}(U, \Phi)$ of the $\mathbf{R}^3$-valued vector fields along U, associated with the embedding Φ of $\mathbb{S}^2$ into $\mathbf{R}^3$:

```
sage: R3 = Manifold(3, 'R^3')
sage: cart_coord.<x, y, z> = R3.chart()
sage: Phi = U.diff_map(R3,
....:       [sin(th)*cos(ph), sin(th)*sin(ph), cos(th)], name='Phi')
sage: XU_R3 = U.vector_field_module(dest_map=Phi) ; XU_R3
Free module X(U,Phi) of vector fields along the Open subset U of
 the 2-dimensional differentiable manifold S^2 mapped into the
 3-dimensional differentiable manifold R^3
sage: XU_R3.base_ring()
Algebra of differentiable scalar fields on the Open subset U of the
 2-dimensional differentiable manifold S^2
```

```
>>> from sage.all import *
>>> R3 = Manifold(Integer(3), 'R^3')
>>> cart_coord = R3.chart(names=('x', 'y', 'z',)); (x, y, z,) = cart_coord._
↪first_ngens(3)
>>> Phi = U.diff_map(R3,
...       [sin(th)*cos(ph), sin(th)*sin(ph), cos(th)], name='Phi')
```

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```
>>> XU_R3 = U.vector_field_module(dest_map=Phi) ; XU_R3
Free module X(U,Phi) of vector fields along the Open subset U of
 the 2-dimensional differentiable manifold S^2 mapped into the
 3-dimensional differentiable manifold R^3
>>> XU_R3.base_ring()
Algebra of differentiable scalar fields on the Open subset U of the
 2-dimensional differentiable manifold S^2
```

$\mathfrak{X}(U, \Phi)$ is a free module because $\mathbf{R}^3$ is parallelizable and its rank is 3:

```
sage: XU_R3.rank()
3
```

```
>>> from sage.all import *
>>> XU_R3.rank()
3
```

Without any information on the manifold, the vector field module is not free by default:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: isinstance(XM, FiniteRankFreeModule)
False
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> XM = M.vector_field_module()
>>> isinstance(XM, FiniteRankFreeModule)
False
```

In particular, declaring a coordinate chart on `M` would yield an error:

```
sage: X.<x,y> = M.chart()
Traceback (most recent call last):
...
ValueError: the Module X(M) of vector fields on the 2-dimensional
 differentiable manifold M has already been constructed as a
 non-free module, which implies that the 2-dimensional
 differentiable manifold M is not parallelizable and hence cannot
 be the domain of a coordinate chart
```

```
>>> from sage.all import *
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
Traceback (most recent call last):
...
ValueError: the Module X(M) of vector fields on the 2-dimensional
 differentiable manifold M has already been constructed as a
 non-free module, which implies that the 2-dimensional
 differentiable manifold M is not parallelizable and hence cannot
 be the domain of a coordinate chart
```

Similarly, one cannot declare a vector frame on M:

```
sage: e = M.vector_frame('e')
Traceback (most recent call last):
...
ValueError: the Module X(M) of vector fields on the 2-dimensional
 differentiable manifold M has already been constructed as a
 non-free module and therefore cannot have a basis
```

```
>>> from sage.all import *
>>> e = M.vector_frame('e')
Traceback (most recent call last):
...
ValueError: the Module X(M) of vector fields on the 2-dimensional
 differentiable manifold M has already been constructed as a
 non-free module and therefore cannot have a basis
```

One shall use the keyword `force_free=True` to construct a free module before declaring the chart:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module(force_free=True)
sage: X.<x,y> = M.chart()  # OK
sage: e = M.vector_frame('e')  # OK
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> XM = M.vector_field_module(force_free=True)
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)# OK
>>> e = M.vector_frame('e')  # OK
```

If one declares the chart or the vector frame before asking for the vector field module, the latter is initialized as a free module, without the need to specify `force_free=True`. Indeed, the information that M is the domain of a chart or a vector frame implies that M is parallelizable and is therefore sufficient to assert that $\mathfrak{X}(M)$ is a free module over $C^k(M)$:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: XM = M.vector_field_module()
sage: isinstance(XM, FiniteRankFreeModule)
True
sage: M.is_manifestly_parallelizable()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> XM = M.vector_field_module()
>>> isinstance(XM, FiniteRankFreeModule)
True
>>> M.is_manifestly_parallelizable()
True
```

vector_frame(**args*, ***kwargs*)

Define a vector frame on `self`.

A *vector frame* is a field on the manifold that provides, at each point p of the manifold, a vector basis of the tangent space at p (or at $\Phi(p)$ when `dest_map` is not `None`, see below).

The vector frame can be defined from a set of n linearly independent vector fields, n being the dimension of `self`.

See also

VectorFrame for complete documentation.

INPUT:

- `symbol` – either a string, to be used as a common base for the symbols of the vector fields constituting the vector frame, or a list/tuple of strings, representing the individual symbols of the vector fields; can be omitted only if `from_frame` is not `None` (see below)
- `vector_fields` – tuple or list of n linearly independent vector fields on the manifold `self` (n being the dimension of `self`) defining the vector frame; can be omitted if the vector frame is created from scratch or if `from_frame` is not `None`
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the vector fields constituting the vector frame, or a list/tuple of strings, representing the individual LaTeX symbols of the vector fields; if `None`, `symbol` is used in place of `latex_symbol`
- `dest_map` – (default: `None`) *DiffMap*; destination map $\Phi : U \rightarrow M$, where U is `self` and M is a differentiable manifold; for each $p \in U$, the vector frame evaluated at p is a basis of the tangent space $T_{\Phi(p)}M$; if `dest_map` is `None`, the identity map is assumed (case of a vector frame *on* U)
- `from_frame` – (default: `None`) vector frame $\tilde{e}$ on the codomain M of the destination map Φ; the returned frame e is then such that for all $p \in U$, we have $e(p) = \tilde{e}(\Phi(p))$
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the vector fields of the frame; if `None`, the indices will be generated as integers within the range declared on `self`
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the vector fields; if `None`, `indices` is used instead
- `symbol_dual` – (default: `None`) same as `symbol` but for the dual coframe; if `None`, `symbol` must be a string and is used for the common base of the symbols of the elements of the dual coframe
- `latex_symbol_dual` – (default: `None`) same as `latex_symbol` but for the dual coframe

OUTPUT:

- a *VectorFrame* representing the defined vector frame

EXAMPLES:

Defining a vector frame from two linearly independent vector fields on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: e0 = M.vector_field(1+x^2, 1+y^2)
sage: e1 = M.vector_field(2, -x*y)
sage: e = M.vector_frame('e', (e0, e1)); e
Vector frame (M, (e_0,e_1))
sage: e[0].display()
e_0 = (x^2 + 1) ∂/∂x + (y^2 + 1) ∂/∂y
```

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```
sage: e[1].display()
e_1 = 2 ∂/∂x - x*y ∂/∂y
sage: (e[0], e[1]) == (e0, e1)
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> e0 = M.vector_field(Integer(1)+x**Integer(2), Integer(1)+y**Integer(2))
>>> e1 = M.vector_field(Integer(2), -x*y)
>>> e = M.vector_frame('e', (e0, e1)); e
Vector frame (M, (e_0,e_1))
>>> e[Integer(0)].display()
e_0 = (x^2 + 1) ∂/∂x + (y^2 + 1) ∂/∂y
>>> e[Integer(1)].display()
e_1 = 2 ∂/∂x - x*y ∂/∂y
>>> (e[Integer(0)], e[Integer(1)]) == (e0, e1)
True
```

If the vector fields are not linearly independent, an error is raised:

```
sage: z = M.vector_frame('z', (e0, -e0))
Traceback (most recent call last):
...
ValueError: the provided vector fields are not linearly
 independent
```

```
>>> from sage.all import *
>>> z = M.vector_frame('z', (e0, -e0))
Traceback (most recent call last):
...
ValueError: the provided vector fields are not linearly
 independent
```

Another example, involving a pair vector fields along a curve:

```
sage: R.<t> = manifolds.RealLine()
sage: c = M.curve([sin(t), sin(2*t)/2], (t, 0, 2*pi), name='c')
sage: I = c.domain(); I
Real interval (0, 2*pi)
sage: v = c.tangent_vector_field()
sage: v.display()
c' = cos(t) ∂/∂x + (2*cos(t)^2 - 1) ∂/∂y
sage: w = I.vector_field(1-2*cos(t)^2, cos(t), dest_map=c)
sage: u = I.vector_frame('u', (v, w))
sage: u[0].display()
u_0 = cos(t) ∂/∂x + (2*cos(t)^2 - 1) ∂/∂y
sage: u[1].display()
u_1 = (-2*cos(t)^2 + 1) ∂/∂x + cos(t) ∂/∂y
sage: (u[0], u[1]) == (v, w)
True
```

```
>>> from sage.all import *
>>> R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1)
>>> c = M.curve([sin(t), sin(Integer(2)*t)/Integer(2)], (t, Integer(0),␣
↪Integer(2)*pi), name='c')
>>> I = c.domain(); I
Real interval (0, 2*pi)
>>> v = c.tangent_vector_field()
>>> v.display()
c' = cos(t) ∂/∂x + (2*cos(t)^2 - 1) ∂/∂y
>>> w = I.vector_field(Integer(1)-Integer(2)*cos(t)**Integer(2), cos(t), dest_
↪map=c)
>>> u = I.vector_frame('u', (v, w))
>>> u[Integer(0)].display()
u_0 = cos(t) ∂/∂x + (2*cos(t)^2 - 1) ∂/∂y
>>> u[Integer(1)].display()
u_1 = (-2*cos(t)^2 + 1) ∂/∂x + cos(t) ∂/∂y
>>> (u[Integer(0)], u[Integer(1)]) == (v, w)
True
```

It is also possible to create a vector frame from scratch, without connecting it to previously defined vector frames or vector fields (this can still be performed later via the method `set_change_of_frame()`):

```
sage: f = M.vector_frame('f'); f
Vector frame (M, (f_0,f_1))
sage: f[0]
Vector field f_0 on the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> f = M.vector_frame('f'); f
Vector frame (M, (f_0,f_1))
>>> f[Integer(0)]
Vector field f_0 on the 2-dimensional differentiable manifold M
```

Thanks to the keywords `dest_map` and `from_frame`, one can also define a vector frame from one preexisting on another manifold, via a differentiable map (here provided by the curve `c`):

```
sage: fc = I.vector_frame(dest_map=c, from_frame=f); fc
Vector frame ((0, 2*pi), (f_0,f_1)) with values on the
 2-dimensional differentiable manifold M
sage: fc[0]
Vector field f_0 along the Real interval (0, 2*pi) with values on
 the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> fc = I.vector_frame(dest_map=c, from_frame=f); fc
Vector frame ((0, 2*pi), (f_0,f_1)) with values on the
 2-dimensional differentiable manifold M
>>> fc[Integer(0)]
Vector field f_0 along the Real interval (0, 2*pi) with values on
 the 2-dimensional differentiable manifold M
```

Note that the symbol for `fc`, namely f, is inherited from `f`, the original vector frame.

See also

For more options, in particular for the choice of symbols and indices, see `VectorFrame`.

2.2 Coordinate Charts on Differentiable Manifolds

The class `DiffChart` implements coordinate charts on a differentiable manifold over a topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$).

The subclass `RealDiffChart` is devoted to the case $K = \mathbf{R}$, for which the concept of coordinate range is meaningful. Moreover, `RealDiffChart` is endowed with some plotting capabilities (cf. method `plot()`).

Transition maps between charts are implemented via the class `DiffCoordChange`.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015) : initial version

REFERENCES:

- Chap. 1 of [Lee2013]

class sage.manifolds.differentiable.chart.**DiffChart**(*domain*, *coordinates*, *calc_method=None*, *periods=None*, *coord_restrictions=None*)

Bases: `Chart`

Chart on a differentiable manifold.

Given a differentiable manifold M of dimension n over a topological field K, a *chart* is a member (U, φ) of the manifold's differentiable atlas; U is then an open subset of M and $\varphi : U \to V \subset K^n$ is a homeomorphism from U to an open subset V of K^n.

The components $(x^1, \ldots, x^n)$ of φ, defined by $\varphi(p) = (x^1(p), \ldots, x^n(p)) \in K^n$ for any point $p \in U$, are called the *coordinates* of the chart (U, φ).

INPUT:

- `domain` – open subset U on which the chart is defined
- `coordinates` – (default: '' (empty string)) single string defining the coordinate symbols, with `' '` (whitespace) as a separator; each item has at most three fields, separated by a colon (`:`):
  1. the coordinate symbol (a letter or a few letters)
  2. (optional) the period of the coordinate if the coordinate is periodic; the period field must be written as `period=T`, where `T` is the period (see examples below)
  3. (optional) the LaTeX spelling of the coordinate; if not provided the coordinate symbol given in the first field will be used

  The order of fields 2 and 3 does not matter and each of them can be omitted. If it contains any LaTeX expression, the string `coordinates` must be declared with the prefix 'r' (for "raw") to allow for a proper treatment of LaTeX's backslash character (see examples below). If no period and no LaTeX spelling are to be set for any coordinate, the argument `coordinates` can be omitted when the shortcut operator `<,>` is used to declare the chart (see examples below).
- `calc_method` – (default: `None`) string defining the calculus method for computations involving coordinates of the chart; must be one of
  - `'SR'` – Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'` – SymPy

- None – the default of *CalculusMethod* will be used

- names – (default: None) unused argument, except if coordinates is not provided; it must then be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator <,> is used).
- coord_restrictions – additional restrictions on the coordinates. A restriction can be any symbolic equality or inequality involving the coordinates, such as x > y or x^2 + y^2 != 0. The items of the list (or set or frozenset) coord_restrictions are combined with the and operator; if some restrictions are to be combined with the or operator instead, they have to be passed as a tuple in some single item of the list (or set or frozenset) coord_restrictions. For example:

```
coord_restrictions=[x > y, (x != 0, y != 0), z^2 < x]
```

means (x > y) and ((x != 0) or (y != 0)) and (z^2 < x). If the list coord_restrictions contains only one item, this item can be passed as such, i.e. writing x > y instead of the single element list [x > y]. If the chart variables have not been declared as variables yet, coord_restrictions must be lambda-quoted.

EXAMPLES:

A chart on a complex 2-dimensional differentiable manifold:

```
sage: M = Manifold(2, 'M', field='complex')
sage: X = M.chart('x y'); X
Chart (M, (x, y))
sage: latex(X)
\left(M,(x, y)\right)
sage: type(X)
<class 'sage.manifolds.differentiable.chart.DiffChart'>
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', field='complex')
>>> X = M.chart('x y'); X
Chart (M, (x, y))
>>> latex(X)
\left(M,(x, y)\right)
>>> type(X)
<class 'sage.manifolds.differentiable.chart.DiffChart'>
```

To manipulate the coordinates (x, y) as global variables, one has to set:

```
sage: x,y = X[:]
```

```
>>> from sage.all import *
>>> x,y = X[:]
```

However, a shortcut is to use the declarator <x,y> in the left-hand side of the chart declaration (there is then no need to pass the string 'x y' to chart()):

```
sage: M = Manifold(2, 'M', field='complex')
sage: X.<x,y> = M.chart(); X
Chart (M, (x, y))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', field='complex')
```

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```
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2); X
Chart (M, (x, y))
```

The coordinates are then immediately accessible:

```
sage: y
y
sage: x is X[0] and y is X[1]
True
```

```
>>> from sage.all import *
>>> y
y
>>> x is X[Integer(0)] and y is X[Integer(1)]
True
```

The trick is performed by Sage preparser:

```
sage: preparse("X.<x,y> = M.chart()")
"X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)"
```

```
>>> from sage.all import *
>>> preparse("X.<x,y> = M.chart()")
"X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)"
```

Note that x and y declared in <x,y> are mere Python variable names and do not have to coincide with the coordinate symbols; for instance, one may write:

```
sage: M = Manifold(2, 'M', field='complex')
sage: X.<x1,y1> = M.chart('x y'); X
Chart (M, (x, y))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', field='complex')
>>> X = M.chart('x y', names=('x1', 'y1',)); (x1, y1,) = X._first_ngens(2); X
Chart (M, (x, y))
```

Then y is not known as a global Python variable and the coordinate y is accessible only through the global variable y1:

```
sage: y1
y
sage: latex(y1)
y
sage: y1 is X[1]
True
```

```
>>> from sage.all import *
>>> y1
y
>>> latex(y1)
y
```

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```
>>> y1 is X[Integer(1)]
True
```

However, having the name of the Python variable coincide with the coordinate symbol is quite convenient; so it is recommended to declare:

```
sage: M = Manifold(2, 'M', field='complex')
sage: X.<x,y> = M.chart()
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', field='complex')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
```

In the above example, the chart X covers entirely the manifold M:

```
sage: X.domain()
2-dimensional complex manifold M
```

```
>>> from sage.all import *
>>> X.domain()
2-dimensional complex manifold M
```

Of course, one may declare a chart only on an open subset of M:

```
sage: U = M.open_subset('U')
sage: Y.<z1, z2> = U.chart(r'z1:\zeta_1 z2:\zeta_2'); Y
Chart (U, (z1, z2))
sage: Y.domain()
Open subset U of the 2-dimensional complex manifold M
```

```
>>> from sage.all import *
>>> U = M.open_subset('U')
>>> Y = U.chart(r'z1:\zeta_1 z2:\zeta_2', names=('z1', 'z2',)); (z1, z2,) = Y._
↪first_ngens(2); Y
Chart (U, (z1, z2))
>>> Y.domain()
Open subset U of the 2-dimensional complex manifold M
```

In the above declaration, we have also specified some LaTeX writing of the coordinates different from the text one:

```
sage: latex(z1)
{\zeta_1}
```

```
>>> from sage.all import *
>>> latex(z1)
{\zeta_1}
```

Note the prefix `r` in front of the string `r'z1:\zeta_1 z2:\zeta_2'`; it makes sure that the backslash character is treated as an ordinary character, to be passed to the LaTeX interpreter.

Periodic coordinates are declared through the keyword `period=` in the coordinate field:

```
sage: N = Manifold(2, 'N', field='complex')
sage: XN.<Z1,Z2> = N.chart('Z1:period=1+2*I Z2')
sage: XN.periods()
(2*I + 1, None)
```

```
>>> from sage.all import *
>>> N = Manifold(Integer(2), 'N', field='complex')
>>> XN = N.chart('Z1:period=1+2*I Z2', names=('Z1', 'Z2',)); (Z1, Z2,) = XN._
↪first_ngens(2)
>>> XN.periods()
(2*I + 1, None)
```

Coordinates are Sage symbolic variables (see `sage.symbolic.expression`):

```
sage: type(z1)
<class 'sage.symbolic.expression.Expression'>
```

```
>>> from sage.all import *
>>> type(z1)
<class 'sage.symbolic.expression.Expression'>
```

In addition to the Python variable name provided in the operator `<.,.>`, the coordinates are accessible by their indices:

```
sage: Y[0], Y[1]
(z1, z2)
```

```
>>> from sage.all import *
>>> Y[Integer(0)], Y[Integer(1)]
(z1, z2)
```

The index range is that declared during the creation of the manifold. By default, it starts at 0, but this can be changed via the parameter `start_index`:

```
sage: M1 = Manifold(2, 'M_1', field='complex', start_index=1)
sage: Z.<u,v> = M1.chart()
sage: Z[1], Z[2]
(u, v)
```

```
>>> from sage.all import *
>>> M1 = Manifold(Integer(2), 'M_1', field='complex', start_index=Integer(1))
>>> Z = M1.chart(names=('u', 'v',)); (u, v,) = Z._first_ngens(2)
>>> Z[Integer(1)], Z[Integer(2)]
(u, v)
```

The full set of coordinates is obtained by means of the operator `[:]`:

```
sage: Y[:]
(z1, z2)
```

```
>>> from sage.all import *
>>> Y[:]
(z1, z2)
```

Each constructed chart is automatically added to the manifold's user atlas:

```
sage: M.atlas()
[Chart (M, (x, y)), Chart (U, (z1, z2))]
```

```
>>> from sage.all import *
>>> M.atlas()
[Chart (M, (x, y)), Chart (U, (z1, z2))]
```

and to the atlas of the chart's domain:

```
sage: U.atlas()
[Chart (U, (z1, z2))]
```

```
>>> from sage.all import *
>>> U.atlas()
[Chart (U, (z1, z2))]
```

Manifold subsets have a *default chart*, which, unless changed via the method `set_default_chart()`, is the first defined chart on the subset (or on a open subset of it):

```
sage: M.default_chart()
Chart (M, (x, y))
sage: U.default_chart()
Chart (U, (z1, z2))
```

```
>>> from sage.all import *
>>> M.default_chart()
Chart (M, (x, y))
>>> U.default_chart()
Chart (U, (z1, z2))
```

The default charts are not privileged charts on the manifold, but rather charts whose name can be skipped in the argument list of functions having an optional `chart=` argument.

The action of the chart map φ on a point is obtained by means of the call operator, i.e. the operator `()`:

```
sage: p = M.point((1+i, 2), chart=X); p
Point on the 2-dimensional complex manifold M
sage: X(p)
(I + 1, 2)
sage: X(p) == p.coord(X)
True
```

```
>>> from sage.all import *
>>> p = M.point((Integer(1)+i, Integer(2)), chart=X); p
Point on the 2-dimensional complex manifold M
>>> X(p)
(I + 1, 2)
>>> X(p) == p.coord(X)
True
```

A vector frame is naturally associated to each chart:

```
sage: X.frame()
Coordinate frame (M, (∂/∂x,∂/∂y))
sage: Y.frame()
Coordinate frame (U, (∂/∂z1,∂/∂z2))
```

```
>>> from sage.all import *
>>> X.frame()
Coordinate frame (M, (∂/∂x,∂/∂y))
>>> Y.frame()
Coordinate frame (U, (∂/∂z1,∂/∂z2))
```

as well as a dual frame (basis of 1-forms):

```
sage: X.coframe()
Coordinate coframe (M, (dx,dy))
sage: Y.coframe()
Coordinate coframe (U, (dz1,dz2))
```

```
>>> from sage.all import *
>>> X.coframe()
Coordinate coframe (M, (dx,dy))
>>> Y.coframe()
Coordinate coframe (U, (dz1,dz2))
```

See also

RealDiffChart for charts on differentiable manifolds over **R**.

coframe()

Return the coframe (basis of coordinate differentials) associated with `self`.

OUTPUT:

- a *CoordCoFrame* representing the coframe

EXAMPLES:

Coordinate coframe associated with some chart on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: c_xy.coframe()
Coordinate coframe (M, (dx,dy))
sage: type(c_xy.coframe())
<class 'sage.manifolds.differentiable.vectorframe.CoordCoFrame_with_category'>
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> c_xy.coframe()
Coordinate coframe (M, (dx,dy))
>>> type(c_xy.coframe())
<class 'sage.manifolds.differentiable.vectorframe.CoordCoFrame_with_category'>
```

Check that `c_xy.coframe()` is indeed the coordinate coframe associated with the coordinates (x, y):

```
sage: dx = c_xy.coframe()[0] ; dx
1-form dx on the 2-dimensional differentiable manifold M
sage: dy = c_xy.coframe()[1] ; dy
1-form dy on the 2-dimensional differentiable manifold M
sage: ex = c_xy.frame()[0] ; ex
Vector field ∂/∂x on the 2-dimensional differentiable manifold M
sage: ey = c_xy.frame()[1] ; ey
Vector field ∂/∂y on the 2-dimensional differentiable manifold M
sage: dx(ex).display()
dx(∂/∂x): M → ℝ
   (x, y) ↦ 1
sage: dx(ey).display()
dx(∂/∂y): M → ℝ
   (x, y) ↦ 0
sage: dy(ex).display()
dy(∂/∂x): M → ℝ
   (x, y) ↦ 0
sage: dy(ey).display()
dy(∂/∂y): M → ℝ
   (x, y) ↦ 1
```

```
>>> from sage.all import *
>>> dx = c_xy.coframe()[Integer(0)] ; dx
1-form dx on the 2-dimensional differentiable manifold M
>>> dy = c_xy.coframe()[Integer(1)] ; dy
1-form dy on the 2-dimensional differentiable manifold M
>>> ex = c_xy.frame()[Integer(0)] ; ex
Vector field ∂/∂x on the 2-dimensional differentiable manifold M
>>> ey = c_xy.frame()[Integer(1)] ; ey
Vector field ∂/∂y on the 2-dimensional differentiable manifold M
>>> dx(ex).display()
dx(∂/∂x): M → ℝ
   (x, y) ↦ 1
>>> dx(ey).display()
dx(∂/∂y): M → ℝ
   (x, y) ↦ 0
>>> dy(ex).display()
dy(∂/∂x): M → ℝ
   (x, y) ↦ 0
>>> dy(ey).display()
dy(∂/∂y): M → ℝ
   (x, y) ↦ 1
```

frame()

Return the vector frame (coordinate frame) associated with `self`.

OUTPUT:

- a *CoordFrame* representing the coordinate frame

EXAMPLES:

Coordinate frame associated with some chart on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: c_xy.frame()
Coordinate frame (M, (∂/∂x,∂/∂y))
sage: type(c_xy.frame())
<class 'sage.manifolds.differentiable.vectorframe.CoordFrame_with_category'>
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> c_xy.frame()
Coordinate frame (M, (∂/∂x,∂/∂y))
>>> type(c_xy.frame())
<class 'sage.manifolds.differentiable.vectorframe.CoordFrame_with_category'>
```

Check that `c_xy.frame()` is indeed the coordinate frame associated with the coordinates (x, y):

```
sage: ex = c_xy.frame()[0] ; ex
Vector field ∂/∂x on the 2-dimensional differentiable manifold M
sage: ey = c_xy.frame()[1] ; ey
Vector field ∂/∂y on the 2-dimensional differentiable manifold M
sage: ex(M.scalar_field(x)).display()
1: M → ℝ
   (x, y) ↦ 1
sage: ex(M.scalar_field(y)).display()
zero: M → ℝ
   (x, y) ↦ 0
sage: ey(M.scalar_field(x)).display()
zero: M → ℝ
   (x, y) ↦ 0
sage: ey(M.scalar_field(y)).display()
1: M → ℝ
   (x, y) ↦ 1
```

```
>>> from sage.all import *
>>> ex = c_xy.frame()[Integer(0)] ; ex
Vector field ∂/∂x on the 2-dimensional differentiable manifold M
>>> ey = c_xy.frame()[Integer(1)] ; ey
Vector field ∂/∂y on the 2-dimensional differentiable manifold M
>>> ex(M.scalar_field(x)).display()
1: M → ℝ
   (x, y) ↦ 1
>>> ex(M.scalar_field(y)).display()
zero: M → ℝ
   (x, y) ↦ 0
>>> ey(M.scalar_field(x)).display()
zero: M → ℝ
   (x, y) ↦ 0
>>> ey(M.scalar_field(y)).display()
1: M → ℝ
   (x, y) ↦ 1
```

restrict(*subset*, *restrictions=None*)

Return the restriction of `self` to some subset.

If the current chart is (U, φ), a *restriction* (or *subchart*) is a chart (V, ψ) such that $V \subset U$ and $\psi = \varphi|_V$.

If such subchart has not been defined yet, it is constructed here.

The coordinates of the subchart bare the same names as the coordinates of the original chart.

INPUT:

- `subset` – open subset V of the chart domain U
- `restrictions` – (default: `None`) list of coordinate restrictions defining the subset V

A restriction can be any symbolic equality or inequality involving the coordinates, such as `x > y` or `x^2 + y^2 != 0`. The items of the list `restrictions` are combined with the `and` operator; if some restrictions are to be combined with the `or` operator instead, they have to be passed as a tuple in some single item of the list `restrictions`. For example:

```
restrictions = [x > y, (x != 0, y != 0), z^2 < x]
```

means `(x > y) and ((x != 0) or (y != 0)) and (z^2 < x)`. If the list `restrictions` contains only one item, this item can be passed as such, i.e. writing `x > y` instead of the single element list `[x > y]`.

OUTPUT:

- a `DiffChart` (V, ψ)

EXAMPLES:

Coordinates on the unit open ball of $\mathbf{C}^2$ as a subchart of the global coordinates of $\mathbf{C}^2$:

```
sage: M = Manifold(2, 'C^2', field='complex')
sage: X.<z1, z2> = M.chart()
sage: B = M.open_subset('B')
sage: X_B = X.restrict(B, abs(z1)^2 + abs(z2)^2 < 1); X_B
Chart (B, (z1, z2))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'C^2', field='complex')
>>> X = M.chart(names=('z1', 'z2',)); (z1, z2,) = X._first_ngens(2)
>>> B = M.open_subset('B')
>>> X_B = X.restrict(B, abs(z1)**Integer(2) + abs(z2)**Integer(2) <␣
↪Integer(1)); X_B
Chart (B, (z1, z2))
```

symbolic_velocities(*left='D'*, *right=None*)

Return a list of symbolic variables ready to be used by the user as the derivatives of the coordinate functions with respect to a curve parameter (i.e. the velocities along the curve). It may actually serve to denote anything else than velocities, with a name including the coordinate functions. The choice of strings provided as 'left' and 'right' arguments is not entirely free since it must comply with Python prescriptions.

INPUT:

- `left` – (default: `D`) string to concatenate to the left of each coordinate functions of the chart
- `right` – (default: `None`) string to concatenate to the right of each coordinate functions of the chart

OUTPUT: list of symbolic expressions with the desired names

EXAMPLES:

Symbolic derivatives of the Cartesian coordinates of the 3-dimensional Euclidean space:

```
sage: R3 = Manifold(3, 'R3', start_index=1)
sage: cart.<X,Y,Z> = R3.chart()
sage: D = cart.symbolic_velocities(); D
[DX, DY, DZ]
sage: D = cart.symbolic_velocities(left='d', right='/dt'); D
Traceback (most recent call last):
...
ValueError: The name "dX/dt" is not a valid Python
 identifier.
sage: D = cart.symbolic_velocities(left='d', right='_dt'); D
[dX_dt, dY_dt, dZ_dt]
sage: D = cart.symbolic_velocities(left='', right="'"); D
Traceback (most recent call last):
...
ValueError: The name "X'" is not a valid Python
 identifier.
sage: D = cart.symbolic_velocities(left='', right='_dot'); D
[X_dot, Y_dot, Z_dot]
sage: R.<t> = manifolds.RealLine()
sage: canon_chart = R.default_chart()
sage: D = canon_chart.symbolic_velocities() ; D
[Dt]
```

```
>>> from sage.all import *
>>> R3 = Manifold(Integer(3), 'R3', start_index=Integer(1))
>>> cart = R3.chart(names=('X', 'Y', 'Z',)); (X, Y, Z,) = cart._first_ngens(3)
>>> D = cart.symbolic_velocities(); D
[DX, DY, DZ]
>>> D = cart.symbolic_velocities(left='d', right='/dt'); D
Traceback (most recent call last):
...
ValueError: The name "dX/dt" is not a valid Python
 identifier.
>>> D = cart.symbolic_velocities(left='d', right='_dt'); D
[dX_dt, dY_dt, dZ_dt]
>>> D = cart.symbolic_velocities(left='', right="'"); D
Traceback (most recent call last):
...
ValueError: The name "X'" is not a valid Python
 identifier.
>>> D = cart.symbolic_velocities(left='', right='_dot'); D
[X_dot, Y_dot, Z_dot]
>>> R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1)
>>> canon_chart = R.default_chart()
>>> D = canon_chart.symbolic_velocities() ; D
[Dt]
```

transition_map(*other*, *transformations*, *intersection_name=None*, *restrictions1=None*, *restrictions2=None*)

Construct the transition map between the current chart, (U, φ) say, and another one, (V, ψ) say.

If n is the manifold's dimension, the *transition map* is the map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \subset K^n \to \psi(U \cap V) \subset K^n,$$

where K is the manifold's base field. In other words, the transition map expresses the coordinates $(y^1, \ldots, y^n)$ of (V, ψ) in terms of the coordinates $(x^1, \ldots, x^n)$ of (U, φ) on the open subset where the two charts intersect, i.e. on $U \cap V$.

By definition, the transition map $\psi \circ \varphi^{-1}$ must be of class C^k, where k is the degree of differentiability of the manifold (cf. `diff_degree()`).

INPUT:

- `other` – the chart (V, ψ)
- `transformations` – tuple (or list) $(Y_1, \ldots, Y_2)$, where Y_i is the symbolic expression of the coordinate y^i in terms of the coordinates $(x^1, \ldots, x^n)$
- `intersection_name` – (default: `None`) name to be given to the subset $U \cap V$ if the latter differs from U or V
- `restrictions1` – (default: `None`) list of conditions on the coordinates of the current chart that define $U \cap V$ if the latter differs from U. `restrictions1` must be a list of of symbolic equalities or inequalities involving the coordinates, such as x>y or x^2+y^2 != 0. The items of the list `restrictions1` are combined with the `and` operator; if some restrictions are to be combined with the `or` operator instead, they have to be passed as a tuple in some single item of the list `restrictions1`. For example, `restrictions1` = [x>y, (x!=0, y!=0), z^2<x] means (x>y) and ((x!=0) or (y!=0)) and (z^2<x). If the list `restrictions1` contains only one item, this item can be passed as such, i.e. writing x>y instead of the single-element list [x>y].
- `restrictions2` – (default: `None`) list of conditions on the coordinates of the chart (V, ψ) that define $U \cap V$ if the latter differs from V (see `restrictions1` for the syntax)

OUTPUT:

- The transition map $\psi \circ \varphi^{-1}$ defined on $U \cap V$, as an instance of `DiffCoordChange`.

EXAMPLES:

Transition map between two stereographic charts on the circle S^1:

```
sage: M = Manifold(1, 'S^1')
sage: U = M.open_subset('U') # Complement of the North pole
sage: cU.<x> = U.chart() # Stereographic chart from the North pole
sage: V = M.open_subset('V') # Complement of the South pole
sage: cV.<y> = V.chart() # Stereographic chart from the South pole
sage: M.declare_union(U,V)   # S^1 is the union of U and V
sage: trans = cU.transition_map(cV, 1/x, intersection_name='W',
....:                           restrictions1= x!=0, restrictions2 = y!=0)
sage: trans
Change of coordinates from Chart (W, (x,)) to Chart (W, (y,))
sage: trans.display()
y = 1/x
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(1), 'S^1')
>>> U = M.open_subset('U') # Complement of the North pole
>>> cU = U.chart(names=('x',)); (x,) = cU._first_ngens(1)# Stereographic␣
→chart from the North pole
>>> V = M.open_subset('V') # Complement of the South pole
>>> cV = V.chart(names=('y',)); (y,) = cV._first_ngens(1)# Stereographic␣
→chart from the South pole
>>> M.declare_union(U,V)   # S^1 is the union of U and V
```

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```
>>> trans = cU.transition_map(cV, Integer(1)/x, intersection_name='W',
...                           restrictions1= x!=Integer(0), restrictions2 = y!
↪=Integer(0))
>>> trans
Change of coordinates from Chart (W, (x,)) to Chart (W, (y,))
>>> trans.display()
y = 1/x
```

The subset W, intersection of U and V, has been created by `transition_map()`:

```
sage: F = M.subset_family(); F
Set {S^1, U, V, W} of open subsets of the 1-dimensional differentiable␣
↪manifold S^1
sage: W = F['W']
sage: W is U.intersection(V)
True
sage: M.atlas()
[Chart (U, (x,)), Chart (V, (y,)), Chart (W, (x,)), Chart (W, (y,))]
```

```
>>> from sage.all import *
>>> F = M.subset_family(); F
Set {S^1, U, V, W} of open subsets of the 1-dimensional differentiable␣
↪manifold S^1
>>> W = F['W']
>>> W is U.intersection(V)
True
>>> M.atlas()
[Chart (U, (x,)), Chart (V, (y,)), Chart (W, (x,)), Chart (W, (y,))]
```

Transition map between the polar chart and the Cartesian one on $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'R^2')
sage: c_cart.<x,y> = M.chart()
sage: U = M.open_subset('U') # the complement of the half line {y=0, x >= 0}
sage: c_spher.<r,phi> = U.chart(r'r:(0,+oo) phi:(0,2*pi):\phi')
sage: trans = c_spher.transition_map(c_cart, (r*cos(phi), r*sin(phi)),
....:                                restrictions2=(y!=0, x<0))
sage: trans
Change of coordinates from Chart (U, (r, phi)) to Chart (U, (x, y))
sage: trans.display()
x = r*cos(phi)
y = r*sin(phi)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)
>>> U = M.open_subset('U') # the complement of the half line {y=0, x >= 0}
>>> c_spher = U.chart(r'r:(0,+oo) phi:(0,2*pi):\phi', names=('r', 'phi',));␣
↪(r, phi,) = c_spher._first_ngens(2)
>>> trans = c_spher.transition_map(c_cart, (r*cos(phi), r*sin(phi)),
...                                restrictions2=(y!=Integer(0), x
↪<Integer(0)))
```

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```
>>> trans
Change of coordinates from Chart (U, (r, phi)) to Chart (U, (x, y))
>>> trans.display()
x = r*cos(phi)
y = r*sin(phi)
```

In this case, no new subset has been created since $U \cap M = U$:

```
sage: M.subset_family()
Set {R^2, U} of open subsets of the 2-dimensional differentiable manifold R^2
```

```
>>> from sage.all import *
>>> M.subset_family()
Set {R^2, U} of open subsets of the 2-dimensional differentiable manifold R^2
```

but a new chart has been created: $(U, (x, y))$:

```
sage: M.atlas()
[Chart (R^2, (x, y)), Chart (U, (r, phi)), Chart (U, (x, y))]
```

```
>>> from sage.all import *
>>> M.atlas()
[Chart (R^2, (x, y)), Chart (U, (r, phi)), Chart (U, (x, y))]
```

class `sage.manifolds.differentiable.chart.`**`DiffCoordChange`**(*chart1*, *chart2*, **transformations*)

Bases: `CoordChange`

Transition map between two charts of a differentiable manifold.

Giving two coordinate charts (U, φ) and (V, ψ) on a differentiable manifold M of dimension n over a topological field K, the *transition map from* (U, φ) *to* (V, ψ) is the map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \subset K^n \to \psi(U \cap V) \subset K^n,$$

In other words, the transition map $\psi \circ \varphi^{-1}$ expresses the coordinates $(y^1, \ldots, y^n)$ of (V, ψ) in terms of the coordinates $(x^1, \ldots, x^n)$ of (U, φ) on the open subset where the two charts intersect, i.e. on $U \cap V$.

By definition, the transition map $\psi \circ \varphi^{-1}$ must be of class C^k, where k is the degree of differentiability of the manifold (cf. `diff_degree()`).

INPUT:

- `chart1` – chart (U, φ)
- `chart2` – chart (V, ψ)
- `transformations` – tuple (or list) $(Y_1, \ldots, Y_2)$, where Y_i is the symbolic expression of the coordinate y^i in terms of the coordinates $(x^1, \ldots, x^n)$

EXAMPLES:

Transition map on a 2-dimensional differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: Y.<u,v> = M.chart()
```

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```
sage: X_to_Y = X.transition_map(Y, [x+y, x-y])
sage: X_to_Y
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
sage: type(X_to_Y)
<class 'sage.manifolds.differentiable.chart.DiffCoordChange'>
sage: X_to_Y.display()
u = x + y
v = x - y
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> Y = M.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)
>>> X_to_Y = X.transition_map(Y, [x+y, x-y])
>>> X_to_Y
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
>>> type(X_to_Y)
<class 'sage.manifolds.differentiable.chart.DiffCoordChange'>
>>> X_to_Y.display()
u = x + y
v = x - y
```

jacobian()

Return the Jacobian matrix of `self`.

If `self` corresponds to the change of coordinates

$$y^i = Y^i(x^1, \ldots, x^n) \qquad 1 \leq i \leq n$$

the Jacobian matrix J is given by

$$J_{ij} = \frac{\partial Y^i}{\partial x^j}$$

where i is the row index and j the column one.

OUTPUT:

- Jacobian matrix J, the elements J_{ij} of which being coordinate functions (cf. *ChartFunction*)

EXAMPLES:

Jacobian matrix of a 2-dimensional transition map:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: Y.<u,v> = M.chart()
sage: X_to_Y = X.transition_map(Y, [x+y^2, 3*x-y])
sage: X_to_Y.jacobian()
[  1 2*y]
[  3  -1]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
```

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```
>>> Y = M.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)
>>> X_to_Y = X.transition_map(Y, [x+y**Integer(2), Integer(3)*x-y])
>>> X_to_Y.jacobian()
[  1 2*y]
[  3  -1]
```

Each element of the Jacobian matrix is a coordinate function:

```
sage: parent(X_to_Y.jacobian()[0,0])
Ring of chart functions on Chart (M, (x, y))
```

```
>>> from sage.all import *
>>> parent(X_to_Y.jacobian()[Integer(0),Integer(0)])
Ring of chart functions on Chart (M, (x, y))
```

jacobian_det()

Return the Jacobian determinant of `self`.

The Jacobian determinant is the determinant of the Jacobian matrix (see *jacobian()*).

OUTPUT:

- determinant of the Jacobian matrix J as a coordinate function (cf. *ChartFunction*)

EXAMPLES:

Jacobian determinant of a 2-dimensional transition map:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: Y.<u,v> = M.chart()
sage: X_to_Y = X.transition_map(Y, [x+y^2, 3*x-y])
sage: X_to_Y.jacobian_det()
-6*y - 1
sage: X_to_Y.jacobian_det() == det(X_to_Y.jacobian())
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> Y = M.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)
>>> X_to_Y = X.transition_map(Y, [x+y**Integer(2), Integer(3)*x-y])
>>> X_to_Y.jacobian_det()
-6*y - 1
>>> X_to_Y.jacobian_det() == det(X_to_Y.jacobian())
True
```

The Jacobian determinant is a coordinate function:

```
sage: parent(X_to_Y.jacobian_det())
Ring of chart functions on Chart (M, (x, y))
```

```
>>> from sage.all import *
>>> parent(X_to_Y.jacobian_det())
Ring of chart functions on Chart (M, (x, y))
```

class sage.manifolds.differentiable.chart.**RealDiffChart**(*domain*, *coordinates*, *calc_method=None*, *bounds=None*, *periods=None*, *coord_restrictions=None*)

Bases: *DiffChart*, *RealChart*

Chart on a differentiable manifold over **R**.

Given a differentiable manifold M of dimension n over **R**, a *chart* is a member (U, φ) of the manifold's differentiable atlas; U is then an open subset of M and $\varphi : U \to V \subset \mathbf{R}^n$ is a homeomorphism from U to an open subset V of $\mathbf{R}^n$.

The components $(x^1, \ldots, x^n)$ of φ, defined by $\varphi(p) = (x^1(p), \ldots, x^n(p)) \in \mathbf{R}^n$ for any point $p \in U$, are called the *coordinates* of the chart (U, φ).

INPUT:

- `domain` – open subset U on which the chart is defined
- `coordinates` – (default: '' (empty string)) single string defining the coordinate symbols, with `' '` (whitespace) as a separator; each item has at most four fields, separated by a colon (`:`):
  1. the coordinate symbol (a letter or a few letters)
  2. (optional) the interval I defining the coordinate range: if not provided, the coordinate is assumed to span all **R**; otherwise I must be provided in the form `(a,b)` (or equivalently `]a,b[`); the bounds `a` and `b` can be `+/-Infinity`, `Inf`, `infinity`, `inf` or `oo`; for *singular* coordinates, non-open intervals such as `[a,b]` and `(a,b]` (or equivalently `]a,b]`) are allowed; note that the interval declaration must not contain any whitespace
  3. (optional) indicator of the periodic character of the coordinate, either as `period=T`, where `T` is the period, or as the keyword `periodic` (the value of the period is then deduced from the interval I declared in field 2; see examples below)
  4. (optional) the LaTeX spelling of the coordinate; if not provided the coordinate symbol given in the first field will be used

  The order of fields 2 to 4 does not matter and each of them can be omitted. If it contains any LaTeX expression, the string `coordinates` must be declared with the prefix 'r' (for "raw") to allow for a proper treatment of LaTeX's backslash character (see examples below). If interval range, no period and no LaTeX spelling are to be set for any coordinate, the argument `coordinates` can be omitted when the shortcut operator `<,>` is used to declare the chart (see examples below).
- `calc_method` – (default: `None`) string defining the calculus method for computations involving coordinates of the chart; must be one of
  - `'SR'` – Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'` – SymPy
  - `None` – the default of *CalculusMethod* will be used
- `names` – (default: `None`) unused argument, except if `coordinates` is not provided; it must then be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used).
- `coord_restrictions` – additional restrictions on the coordinates. A restriction can be any symbolic equality or inequality involving the coordinates, such as `x > y` or `x^2 + y^2 != 0`. The items of the list (or set or frozenset) `coord_restrictions` are combined with the `and` operator; if some restrictions are to be

combined with the `or` operator instead, they have to be passed as a tuple in some single item of the list (or set or frozenset) `coord_restrictions`. For example:

```
coord_restrictions=[x > y, (x != 0, y != 0), z^2 < x]
```

means `(x > y) and ((x != 0) or (y != 0)) and (z^2 < x)`. If the list `coord_restrictions` contains only one item, this item can be passed as such, i.e. writing `x > y` instead of the single element list `[x > y]`. If the chart variables have not been declared as variables yet, `coord_restrictions` must be `lambda`-quoted.

EXAMPLES:

Cartesian coordinates on $\mathbf{R}^3$:

```
sage: M = Manifold(3, 'R^3', r'\RR^3', start_index=1)
sage: c_cart = M.chart('x y z'); c_cart
Chart (R^3, (x, y, z))
sage: type(c_cart)
<class 'sage.manifolds.differentiable.chart.RealDiffChart'>
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'R^3', r'\RR^3', start_index=Integer(1))
>>> c_cart = M.chart('x y z'); c_cart
Chart (R^3, (x, y, z))
>>> type(c_cart)
<class 'sage.manifolds.differentiable.chart.RealDiffChart'>
```

To have the coordinates accessible as global variables, one has to set:

```
sage: (x,y,z) = c_cart[:]
```

```
>>> from sage.all import *
>>> (x,y,z) = c_cart[:]
```

However, a shortcut is to use the declarator `<x,y,z>` in the left-hand side of the chart declaration (there is then no need to pass the string `'x y z'` to `chart()`):

```
sage: M = Manifold(3, 'R^3', r'\RR^3', start_index=1)
sage: c_cart.<x,y,z> = M.chart(); c_cart
Chart (R^3, (x, y, z))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'R^3', r'\RR^3', start_index=Integer(1))
>>> c_cart = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_cart._first_ngens(3);
↪ c_cart
Chart (R^3, (x, y, z))
```

The coordinates are then immediately accessible:

```
sage: y
y
sage: y is c_cart[2]
True
```

```
>>> from sage.all import *
>>> y
y
>>> y is c_cart[Integer(2)]
True
```

The trick is performed by Sage preparser:

```
sage: preparse("c_cart.<x,y,z> = M.chart()")
"c_cart = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_cart._first_ngens(3)"
```

```
>>> from sage.all import *
>>> preparse("c_cart.<x,y,z> = M.chart()")
"c_cart = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_cart._first_ngens(3)"
```

Note that `x, y, z` declared in `<x,y,z>` are mere Python variable names and do not have to coincide with the coordinate symbols; for instance, one may write:

```
sage: M = Manifold(3, 'R^3', r'\RR^3', start_index=1)
sage: c_cart.<x1,y1,z1> = M.chart('x y z'); c_cart
Chart (R^3, (x, y, z))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'R^3', r'\RR^3', start_index=Integer(1))
>>> c_cart = M.chart('x y z', names=('x1', 'y1', 'z1',)); (x1, y1, z1,) = c_cart._
↪first_ngens(3); c_cart
Chart (R^3, (x, y, z))
```

Then `y` is not known as a global variable and the coordinate y is accessible only through the global variable `y1`:

```
sage: y1
y
sage: y1 is c_cart[2]
True
```

```
>>> from sage.all import *
>>> y1
y
>>> y1 is c_cart[Integer(2)]
True
```

However, having the name of the Python variable coincide with the coordinate symbol is quite convenient; so it is recommended to declare:

```
sage: forget()   # for doctests only
sage: M = Manifold(3, 'R^3', r'\RR^3', start_index=1)
sage: c_cart.<x,y,z> = M.chart()
```

```
>>> from sage.all import *
>>> forget()   # for doctests only
>>> M = Manifold(Integer(3), 'R^3', r'\RR^3', start_index=Integer(1))
>>> c_cart = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_cart._first_ngens(3)
```

Spherical coordinates on the subset U of $\mathbf{R}^3$ that is the complement of the half-plane $\{y = 0, x \geq 0\}$:

```
sage: U = M.open_subset('U')
sage: c_spher.<r,th,ph> = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: c_spher
Chart (U, (r, th, ph))
```

```
>>> from sage.all import *
>>> U = M.open_subset('U')
>>> c_spher = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi', names=('r',
→'th', 'ph',)); (r, th, ph,) = c_spher._first_ngens(3)
>>> c_spher
Chart (U, (r, th, ph))
```

Note the prefix 'r' for the string defining the coordinates in the arguments of `chart`.

Coordinates are Sage symbolic variables (see `sage.symbolic.expression`):

```
sage: type(th)
<class 'sage.symbolic.expression.Expression'>
sage: latex(th)
{\theta}
sage: assumptions(th)
[th is real, th > 0, th < pi]
```

```
>>> from sage.all import *
>>> type(th)
<class 'sage.symbolic.expression.Expression'>
>>> latex(th)
{\theta}
>>> assumptions(th)
[th is real, th > 0, th < pi]
```

Coordinate are also accessible by their indices:

```
sage: x1 = c_spher[1]; x2 = c_spher[2]; x3 = c_spher[3]
sage: [x1, x2, x3]
[r, th, ph]
sage: (x1, x2, x3) == (r, th, ph)
True
```

```
>>> from sage.all import *
>>> x1 = c_spher[Integer(1)]; x2 = c_spher[Integer(2)]; x3 = c_spher[Integer(3)]
>>> [x1, x2, x3]
[r, th, ph]
>>> (x1, x2, x3) == (r, th, ph)
True
```

The full set of coordinates is obtained by means of the operator [:]:

```
sage: c_cart[:]
(x, y, z)
sage: c_spher[:]
(r, th, ph)
```

```
>>> from sage.all import *
>>> c_cart[:]
(x, y, z)
>>> c_spher[:]
(r, th, ph)
```

Let us check that the declared coordinate ranges have been taken into account:

```
sage: c_cart.coord_range()
x: (-oo, +oo); y: (-oo, +oo); z: (-oo, +oo)
sage: c_spher.coord_range()
r: (0, +oo); th: (0, pi); ph: (0, 2*pi)
sage: bool(th>0 and th<pi)
True
sage: assumptions()  # list all current symbolic assumptions
[x is real, y is real, z is real, r is real, r > 0, th is real,
 th > 0, th < pi, ph is real, ph > 0, ph < 2*pi]
```

```
>>> from sage.all import *
>>> c_cart.coord_range()
x: (-oo, +oo); y: (-oo, +oo); z: (-oo, +oo)
>>> c_spher.coord_range()
r: (0, +oo); th: (0, pi); ph: (0, 2*pi)
>>> bool(th>Integer(0) and th<pi)
True
>>> assumptions()  # list all current symbolic assumptions
[x is real, y is real, z is real, r is real, r > 0, th is real,
 th > 0, th < pi, ph is real, ph > 0, ph < 2*pi]
```

The coordinate ranges are used for simplifications:

```
sage: simplify(abs(r)) # r has been declared to lie in the interval (0,+oo)
r
sage: simplify(abs(x)) # no positive range has been declared for x
abs(x)
```

```
>>> from sage.all import *
>>> simplify(abs(r)) # r has been declared to lie in the interval (0,+oo)
r
>>> simplify(abs(x)) # no positive range has been declared for x
abs(x)
```

A coordinate can be declared periodic by adding the keyword `periodic` to its range:

```
sage: V = M.open_subset('V')
sage: c_spher1.<r,th,ph1> = \
....: V.chart(r'r:(0,+oo) th:(0,pi):\theta ph1:(0,2*pi):periodic:\phi_1')
sage: c_spher1.periods()
(None, None, 2*pi)
sage: c_spher1.coord_range()
r: (0, +oo); th: (0, pi); ph1: [0, 2*pi] (periodic)
```

```
>>> from sage.all import *
>>> V = M.open_subset('V')
>>> c_spher1 = V.chart(r'r:(0,+oo) th:(0,pi):\theta ph1:(0,2*pi):periodic:\phi_1',
↪ names=('r', 'th', 'ph1',)); (r, th, ph1,) = c_spher1._first_ngens(3)
>>> c_spher1.periods()
(None, None, 2*pi)
>>> c_spher1.coord_range()
r: (0, +oo); th: (0, pi); ph1: [0, 2*pi] (periodic)
```

It is equivalent to give the period as `period=2*pi`, skipping the coordinate range:

```
sage: c_spher2.<r,th,ph2> = \
....: V.chart(r'r:(0,+oo) th:(0,pi):\theta ph2:period=2*pi:\phi_2')
sage: c_spher2.periods()
(None, None, 2*pi)
sage: c_spher2.coord_range()
r: (0, +oo); th: (0, pi); ph2: [0, 2*pi] (periodic)
```

```
>>> from sage.all import *
>>> c_spher2 = V.chart(r'r:(0,+oo) th:(0,pi):\theta ph2:period=2*pi:\phi_2',
↪names=('r', 'th', 'ph2',)); (r, th, ph2,) = c_spher2._first_ngens(3)
>>> c_spher2.periods()
(None, None, 2*pi)
>>> c_spher2.coord_range()
r: (0, +oo); th: (0, pi); ph2: [0, 2*pi] (periodic)
```

Each constructed chart is automatically added to the manifold's user atlas:

```
sage: M.atlas()
[Chart (R^3, (x, y, z)), Chart (U, (r, th, ph)),
 Chart (V, (r, th, ph1)), Chart (V, (r, th, ph2))]
```

```
>>> from sage.all import *
>>> M.atlas()
[Chart (R^3, (x, y, z)), Chart (U, (r, th, ph)),
 Chart (V, (r, th, ph1)), Chart (V, (r, th, ph2))]
```

and to the atlas of its domain:

```
sage: U.atlas()
[Chart (U, (r, th, ph))]
```

```
>>> from sage.all import *
>>> U.atlas()
[Chart (U, (r, th, ph))]
```

Manifold subsets have a *default chart*, which, unless changed via the method `set_default_chart()`, is the first defined chart on the subset (or on a open subset of it):

```
sage: M.default_chart()
Chart (R^3, (x, y, z))
sage: U.default_chart()
Chart (U, (r, th, ph))
```

```
>>> from sage.all import *
>>> M.default_chart()
Chart (R^3, (x, y, z))
>>> U.default_chart()
Chart (U, (r, th, ph))
```

The default charts are not privileged charts on the manifold, but rather charts whose name can be skipped in the argument list of functions having an optional `chart=` argument.

The action of the chart map φ on a point is obtained by means of the call operator, i.e. the operator `()`:

```
sage: p = M.point((1,0,-2)); p
Point on the 3-dimensional differentiable manifold R^3
sage: c_cart(p)
(1, 0, -2)
sage: c_cart(p) == p.coord(c_cart)
True
sage: q = M.point((2,pi/2,pi/3), chart=c_spher) # point defined by its spherical␣
↪coordinates
sage: c_spher(q)
(2, 1/2*pi, 1/3*pi)
sage: c_spher(q) == q.coord(c_spher)
True
sage: a = U.point((1,pi/2,pi)) # the default coordinates on U are the spherical␣
↪ones
sage: c_spher(a)
(1, 1/2*pi, pi)
sage: c_spher(a) == a.coord(c_spher)
True
```

```
>>> from sage.all import *
>>> p = M.point((Integer(1),Integer(0),-Integer(2))); p
Point on the 3-dimensional differentiable manifold R^3
>>> c_cart(p)
(1, 0, -2)
>>> c_cart(p) == p.coord(c_cart)
True
>>> q = M.point((Integer(2),pi/Integer(2),pi/Integer(3)), chart=c_spher) # point␣
↪defined by its spherical coordinates
>>> c_spher(q)
(2, 1/2*pi, 1/3*pi)
>>> c_spher(q) == q.coord(c_spher)
True
>>> a = U.point((Integer(1),pi/Integer(2),pi)) # the default coordinates on U are␣
↪the spherical ones
>>> c_spher(a)
(1, 1/2*pi, pi)
>>> c_spher(a) == a.coord(c_spher)
True
```

Cartesian coordinates on U as an example of chart construction with coordinate restrictions: since U is the complement of the half-plane $\{y = 0, x \geq 0\}$, we must have $y \neq 0$ or $x < 0$ on U. Accordingly, we set:

```
sage: c_cartU.<x,y,z> = U.chart(coord_restrictions=lambda x,y,z: (y!=0, x<0))
....:     # the tuple (y!=0, x<0) means y!=0 or x<0
....:     #            [y!=0, x<0] would have meant y!=0 AND x<0
sage: U.atlas()
[Chart (U, (r, th, ph)), Chart (U, (x, y, z))]
sage: M.atlas()
[Chart (R^3, (x, y, z)), Chart (U, (r, th, ph)),
 Chart (V, (r, th, ph1)), Chart (V, (r, th, ph2)),
 Chart (U, (x, y, z))]
sage: c_cartU.valid_coordinates(-1,0,2)
True
sage: c_cartU.valid_coordinates(1,0,2)
False
sage: c_cart.valid_coordinates(1,0,2)
True
```

```
>>> from sage.all import *
>>> c_cartU = U.chart(coord_restrictions=lambda x,y,z: (y!=Integer(0), x
↪<Integer(0)), names=('x', 'y', 'z',)); (x, y, z,) = c_cartU._first_ngens(3)
...     # the tuple (y!=0, x<0) means y!=0 or x<0
...     #            [y!=0, x<0] would have meant y!=0 AND x<0
>>> U.atlas()
[Chart (U, (r, th, ph)), Chart (U, (x, y, z))]
>>> M.atlas()
[Chart (R^3, (x, y, z)), Chart (U, (r, th, ph)),
 Chart (V, (r, th, ph1)), Chart (V, (r, th, ph2)),
 Chart (U, (x, y, z))]
>>> c_cartU.valid_coordinates(-Integer(1),Integer(0),Integer(2))
True
>>> c_cartU.valid_coordinates(Integer(1),Integer(0),Integer(2))
False
>>> c_cart.valid_coordinates(Integer(1),Integer(0),Integer(2))
True
```

A vector frame is naturally associated to each chart:

```
sage: c_cart.frame()
Coordinate frame (R^3, (∂/∂x,∂/∂y,∂/∂z))
sage: c_spher.frame()
Coordinate frame (U, (∂/∂r,∂/∂th,∂/∂ph))
```

```
>>> from sage.all import *
>>> c_cart.frame()
Coordinate frame (R^3, (∂/∂x,∂/∂y,∂/∂z))
>>> c_spher.frame()
Coordinate frame (U, (∂/∂r,∂/∂th,∂/∂ph))
```

as well as a dual frame (basis of 1-forms):

```
sage: c_cart.coframe()
Coordinate coframe (R^3, (dx,dy,dz))
sage: c_spher.coframe()
Coordinate coframe (U, (dr,dth,dph))
```

```
>>> from sage.all import *
>>> c_cart.coframe()
Coordinate coframe (R^3, (dx,dy,dz))
>>> c_spher.coframe()
Coordinate coframe (U, (dr,dth,dph))
```

Chart grids can be drawn in 2D or 3D graphics thanks to the method `plot()`.

restrict(*subset*, *restrictions=None*)

Return the restriction of the chart to some subset.

If the current chart is (U, φ), a *restriction* (or *subchart*) is a chart (V, ψ) such that $V \subset U$ and $\psi = \varphi|_V$.

If such subchart has not been defined yet, it is constructed here.

The coordinates of the subchart bare the same names as the coordinates of the original chart.

INPUT:

- `subset` – open subset V of the chart domain U
- `restrictions` – (default: `None`) list of coordinate restrictions defining the subset V

A restriction can be any symbolic equality or inequality involving the coordinates, such as `x > y` or `x^2 + y^2 != 0`. The items of the list `restrictions` are combined with the `and` operator; if some restrictions are to be combined with the `or` operator instead, they have to be passed as a tuple in some single item of the list `restrictions`. For example:

```
restrictions = [x > y, (x != 0, y != 0), z^2 < x]
```

means `(x > y) and ((x != 0) or (y != 0)) and (z^2 < x)`. If the list `restrictions` contains only one item, this item can be passed as such, i.e. writing `x > y` instead of the single element list `[x > y]`.

OUTPUT:

- a `RealDiffChart` (V, ψ)

EXAMPLES:

Cartesian coordinates on the unit open disc in $\mathbf{R}^2$ as a subchart of the global Cartesian coordinates:

```
sage: M = Manifold(2, 'R^2')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: D = M.open_subset('D') # the unit open disc
sage: c_cart_D = c_cart.restrict(D, x^2+y^2<1)
sage: p = M.point((1/2, 0))
sage: p in D
True
sage: q = M.point((1, 2))
sage: q in D
False
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#
↪Cartesian coordinates on R^2
>>> D = M.open_subset('D') # the unit open disc
>>> c_cart_D = c_cart.restrict(D, x**Integer(2)+y**Integer(2)<Integer(1))
```

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```
>>> p = M.point((Integer(1)/Integer(2), Integer(0)))
>>> p in D
True
>>> q = M.point((Integer(1), Integer(2)))
>>> q in D
False
```

Cartesian coordinates on the annulus $1 < \sqrt{x^2 + y^2} < 2$:

```
sage: A = M.open_subset('A')
sage: c_cart_A = c_cart.restrict(A, [x^2+y^2>1, x^2+y^2<4])
sage: p in A, q in A
(False, False)
sage: a = M.point((3/2,0))
sage: a in A
True
```

```
>>> from sage.all import *
>>> A = M.open_subset('A')
>>> c_cart_A = c_cart.restrict(A, [x**Integer(2)+y**Integer(2)>Integer(1),␣
↪x**Integer(2)+y**Integer(2)<Integer(4)])
>>> p in A, q in A
(False, False)
>>> a = M.point((Integer(3)/Integer(2),Integer(0)))
>>> a in A
True
```

2.3 The Real Line and Open Intervals

The class `OpenInterval` implement open intervals as 1-dimensional differentiable manifolds over **R**. The derived class `RealLine` is devoted to **R** itself, as the open interval $(-\infty, +\infty)$.

AUTHORS:

- Eric Gourgoulhon (2015): initial version
- Travis Scrimshaw (2016): review tweaks

REFERENCES:

- [Lee2013]

class sage.manifolds.differentiable.examples.real_line.**OpenInterval**(*lower*, *upper*, *ambient_interval=None*, *name=None*, *latex_name=None*, *coordinate=None*, *names=None*, *start_index=0*)

Bases: `DifferentiableManifold`

Open interval as a 1-dimensional differentiable manifold over **R**.

INPUT:

- `lower` – lower bound of the interval (possibly `-Infinity`)

- `upper` – upper bound of the interval (possibly `+Infinity`)
- `ambient_interval` – (default: `None`) another open interval, to which the constructed interval is a subset of
- `name` – (default: `None`) string; name (symbol) given to the interval; if `None`, the name is constructed from `lower` and `upper`
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the interval; if `None`, the LaTeX symbol is constructed from `lower` and `upper` if `name` is `None`, otherwise, it is set to `name`
- `coordinate` – (default: `None`) string defining the symbol of the canonical coordinate set on the interval; if none is provided and `names` is `None`, the symbol 't' is used
- `names` – (default: `None`) used only when `coordinate` is `None`: it must be a single-element tuple containing the canonical coordinate symbol (this is guaranteed if the shortcut `<names>` is used, see examples below)
- `start_index` – (default: 0) unique value of the index for vectors and forms on the interval manifold

EXAMPLES:

The interval $(0, \pi)$:

```
sage: I = manifolds.OpenInterval(0, pi); I
Real interval (0, pi)
sage: latex(I)
\left(0, \pi\right)
```

```
>>> from sage.all import *
>>> I = manifolds.OpenInterval(Integer(0), pi); I
Real interval (0, pi)
>>> latex(I)
\left(0, \pi\right)
```

`I` is a 1-dimensional smooth manifold over **R**:

```
sage: I.category()
Category of smooth connected manifolds over Real Field with 53 bits of
 precision
sage: I.base_field()
Real Field with 53 bits of precision
sage: dim(I)
1
```

```
>>> from sage.all import *
>>> I.category()
Category of smooth connected manifolds over Real Field with 53 bits of
 precision
>>> I.base_field()
Real Field with 53 bits of precision
>>> dim(I)
1
```

It is infinitely differentiable (smooth manifold):

```
sage: I.diff_degree()
+Infinity
```

```
>>> from sage.all import *
>>> I.diff_degree()
+Infinity
```

The instance is unique (as long as the constructor arguments are the same):

```
sage: I is manifolds.OpenInterval(0, pi)
True
sage: I is manifolds.OpenInterval(0, pi, name='I')
False
```

```
>>> from sage.all import *
>>> I is manifolds.OpenInterval(Integer(0), pi)
True
>>> I is manifolds.OpenInterval(Integer(0), pi, name='I')
False
```

The display of the interval can be customized:

```
sage: I  # default display
Real interval (0, pi)
sage: latex(I)  # default LaTeX display
\left(0, \pi\right)
sage: I1 = manifolds.OpenInterval(0, pi, name='I'); I1
Real interval I
sage: latex(I1)
I
sage: I2 = manifolds.OpenInterval(0, pi, name='I', latex_name=r'\mathcal{I}'); I2
Real interval I
sage: latex(I2)
\mathcal{I}
```

```
>>> from sage.all import *
>>> I  # default display
Real interval (0, pi)
>>> latex(I)  # default LaTeX display
\left(0, \pi\right)
>>> I1 = manifolds.OpenInterval(Integer(0), pi, name='I'); I1
Real interval I
>>> latex(I1)
I
>>> I2 = manifolds.OpenInterval(Integer(0), pi, name='I', latex_name=r'\mathcal{I}
↪'); I2
Real interval I
>>> latex(I2)
\mathcal{I}
```

`I` is endowed with a canonical chart:

```
sage: I.canonical_chart()
Chart ((0, pi), (t,))
sage: I.canonical_chart() is I.default_chart()
True
```

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```
sage: I.atlas()
[Chart ((0, pi), (t,))]
```

```
>>> from sage.all import *
>>> I.canonical_chart()
Chart ((0, pi), (t,))
>>> I.canonical_chart() is I.default_chart()
True
>>> I.atlas()
[Chart ((0, pi), (t,))]
```

The canonical coordinate is returned by the method `canonical_coordinate()`:

```
sage: I.canonical_coordinate()
t
sage: t = I.canonical_coordinate()
sage: type(t)
<class 'sage.symbolic.expression.Expression'>
```

```
>>> from sage.all import *
>>> I.canonical_coordinate()
t
>>> t = I.canonical_coordinate()
>>> type(t)
<class 'sage.symbolic.expression.Expression'>
```

However, it can be obtained in the same step as the interval construction by means of the shortcut `I.<names>`:

```
sage: I.<t> = manifolds.OpenInterval(0, pi)
sage: t
t
sage: type(t)
<class 'sage.symbolic.expression.Expression'>
```

```
>>> from sage.all import *
>>> I = manifolds.OpenInterval(Integer(0), pi, names=('t',)); (t,) = I._first_
↪ngens(1)
>>> t
t
>>> type(t)
<class 'sage.symbolic.expression.Expression'>
```

The trick is performed by the Sage preparser:

```
sage: preparse("I.<t> = manifolds.OpenInterval(0, pi)")
"I = manifolds.OpenInterval(Integer(0), pi, names=('t',)); (t,) = I._first_
↪ngens(1)"
```

```
>>> from sage.all import *
>>> preparse("I.<t> = manifolds.OpenInterval(0, pi)")
"I = manifolds.OpenInterval(Integer(0), pi, names=('t',)); (t,) = I._first_
↪ngens(1)"
```

In particular the shortcut can be used to set a canonical coordinate symbol different from `'t'`:

```
sage: J.<x> = manifolds.OpenInterval(0, pi)
sage: J.canonical_chart()
Chart ((0, pi), (x,))
sage: J.canonical_coordinate()
x
```

```
>>> from sage.all import *
>>> J = manifolds.OpenInterval(Integer(0), pi, names=('x',)); (x,) = J._first_
↪ngens(1)
>>> J.canonical_chart()
Chart ((0, pi), (x,))
>>> J.canonical_coordinate()
x
```

The LaTeX symbol of the canonical coordinate can be adjusted via the same syntax as a chart declaration (see *RealChart*):

```
sage: J.<x> = manifolds.OpenInterval(0, pi, coordinate=r'x:\xi')
sage: latex(x)
{\xi}
sage: latex(J.canonical_chart())
\left(\left(0, \pi\right),({\xi})\right)
```

```
>>> from sage.all import *
>>> J = manifolds.OpenInterval(Integer(0), pi, coordinate=r'x:\xi', names=('x',));
↪ (x,) = J._first_ngens(1)
>>> latex(x)
{\xi}
>>> latex(J.canonical_chart())
\left(\left(0, \pi\right),({\xi})\right)
```

An element of the open interval `I`:

```
sage: x = I.an_element(); x
Point on the Real interval (0, pi)
sage: x.coord() # coordinates in the default chart = canonical chart
(1/2*pi,)
```

```
>>> from sage.all import *
>>> x = I.an_element(); x
Point on the Real interval (0, pi)
>>> x.coord() # coordinates in the default chart = canonical chart
(1/2*pi,)
```

As for any manifold subset, a specific element of `I` can be created by providing a tuple containing its coordinate(s) in a given chart:

```
sage: x = I((2,)) # (2,) = tuple of coordinates in the canonical chart
sage: x
Point on the Real interval (0, pi)
```

```
>>> from sage.all import *
>>> x = I((Integer(2),)) # (2,) = tuple of coordinates in the canonical chart
>>> x
Point on the Real interval (0, pi)
```

But for convenience, it can also be created directly from the coordinate:

```
sage: x = I(2); x
Point on the Real interval (0, pi)
sage: x.coord()
(2,)
sage: I(2) == I((2,))
True
```

```
>>> from sage.all import *
>>> x = I(Integer(2)); x
Point on the Real interval (0, pi)
>>> x.coord()
(2,)
>>> I(Integer(2)) == I((Integer(2),))
True
```

By default, the coordinates passed for the element `x` are those relative to the canonical chart:

```
sage: I(2) ==  I((2,), chart=I.canonical_chart())
True
```

```
>>> from sage.all import *
>>> I(Integer(2)) ==  I((Integer(2),), chart=I.canonical_chart())
True
```

The lower and upper bounds of the interval `I`:

```
sage: I.lower_bound()
0
sage: I.upper_bound()
pi
```

```
>>> from sage.all import *
>>> I.lower_bound()
0
>>> I.upper_bound()
pi
```

One of the endpoint can be infinite:

```
sage: J = manifolds.OpenInterval(1, +oo); J
Real interval (1, +Infinity)
sage: J.an_element().coord()
(2,)
```

```
>>> from sage.all import *
>>> J = manifolds.OpenInterval(Integer(1), +oo); J
Real interval (1, +Infinity)
>>> J.an_element().coord()
(2,)
```

The construction of a subinterval can be performed via the argument `ambient_interval` of `OpenInterval`:

```
sage: J = manifolds.OpenInterval(0, 1, ambient_interval=I); J
Real interval (0, 1)
```

```
>>> from sage.all import *
>>> J = manifolds.OpenInterval(Integer(0), Integer(1), ambient_interval=I); J
Real interval (0, 1)
```

However, it is recommended to use the method *open_interval()* instead:

```
sage: J = I.open_interval(0, 1); J
Real interval (0, 1)
sage: J.is_subset(I)
True
sage: J.manifold() is I
True
```

```
>>> from sage.all import *
>>> J = I.open_interval(Integer(0), Integer(1)); J
Real interval (0, 1)
>>> J.is_subset(I)
True
>>> J.manifold() is I
True
```

A subinterval of a subinterval:

```
sage: K = J.open_interval(1/2, 1); K
Real interval (1/2, 1)
sage: K.is_subset(J)
True
sage: K.is_subset(I)
True
sage: K.manifold() is I
True
```

```
>>> from sage.all import *
>>> K = J.open_interval(Integer(1)/Integer(2), Integer(1)); K
Real interval (1/2, 1)
>>> K.is_subset(J)
True
>>> K.is_subset(I)
True
>>> K.manifold() is I
True
```

We have:

```
sage: list(I.subset_family())
[Real interval (0, 1), Real interval (0, pi), Real interval (1/2, 1)]
sage: list(J.subset_family())
[Real interval (0, 1), Real interval (1/2, 1)]
sage: list(K.subset_family())
[Real interval (1/2, 1)]
```

```
>>> from sage.all import *
>>> list(I.subset_family())
[Real interval (0, 1), Real interval (0, pi), Real interval (1/2, 1)]
>>> list(J.subset_family())
[Real interval (0, 1), Real interval (1/2, 1)]
>>> list(K.subset_family())
[Real interval (1/2, 1)]
```

As any open subset of a manifold, open subintervals are created in a category of subobjects of smooth manifolds:

```
sage: J.category()
Join of Category of subobjects of sets and Category of smooth manifolds
 over Real Field with 53 bits of precision and Category of connected
 manifolds over Real Field with 53 bits of precision
sage: K.category()
Join of Category of subobjects of sets and Category of smooth manifolds
 over Real Field with 53 bits of precision and Category of connected
 manifolds over Real Field with 53 bits of precision
```

```
>>> from sage.all import *
>>> J.category()
Join of Category of subobjects of sets and Category of smooth manifolds
 over Real Field with 53 bits of precision and Category of connected
 manifolds over Real Field with 53 bits of precision
>>> K.category()
Join of Category of subobjects of sets and Category of smooth manifolds
 over Real Field with 53 bits of precision and Category of connected
 manifolds over Real Field with 53 bits of precision
```

On the contrary, `I`, which has not been created as a subinterval, is in the category of smooth manifolds (see `Manifolds`):

```
sage: I.category()
Category of smooth connected manifolds over Real Field with 53 bits of
 precision
```

```
>>> from sage.all import *
>>> I.category()
Category of smooth connected manifolds over Real Field with 53 bits of
 precision
```

and we have:

```
sage: J.category() is I.category().Subobjects()
True
```

```
>>> from sage.all import *
>>> J.category() is I.category().Subobjects()
True
```

All intervals are parents:

```
sage: x = J(1/2); x
Point on the Real interval (0, pi)
sage: x.parent() is J
True
sage: y = K(3/4); y
Point on the Real interval (0, pi)
sage: y.parent() is K
True
```

```
>>> from sage.all import *
>>> x = J(Integer(1)/Integer(2)); x
Point on the Real interval (0, pi)
>>> x.parent() is J
True
>>> y = K(Integer(3)/Integer(4)); y
Point on the Real interval (0, pi)
>>> y.parent() is K
True
```

We have:

```
sage: x in I, x in J, x in K
(True, True, False)
sage: y in I, y in J, y in K
(True, True, True)
```

```
>>> from sage.all import *
>>> x in I, x in J, x in K
(True, True, False)
>>> y in I, y in J, y in K
(True, True, True)
```

The canonical chart of subintervals is inherited from the canonical chart of the parent interval:

```
sage: XI = I.canonical_chart(); XI
Chart ((0, pi), (t,))
sage: XI.coord_range()
t: (0, pi)
sage: XJ = J.canonical_chart(); XJ
Chart ((0, 1), (t,))
sage: XJ.coord_range()
t: (0, 1)
sage: XK = K.canonical_chart(); XK
Chart ((1/2, 1), (t,))
sage: XK.coord_range()
t: (1/2, 1)
```

```
>>> from sage.all import *
>>> XI = I.canonical_chart(); XI
Chart ((0, pi), (t,))
>>> XI.coord_range()
t: (0, pi)
>>> XJ = J.canonical_chart(); XJ
Chart ((0, 1), (t,))
>>> XJ.coord_range()
t: (0, 1)
>>> XK = K.canonical_chart(); XK
Chart ((1/2, 1), (t,))
>>> XK.coord_range()
t: (1/2, 1)
```

canonical_chart()

Return the canonical chart defined on `self`.

OUTPUT: *RealDiffChart*

EXAMPLES:

Canonical chart on the interval $(0, \pi)$:

```
sage: I = manifolds.OpenInterval(0, pi)
sage: I.canonical_chart()
Chart ((0, pi), (t,))
sage: I.canonical_chart().coord_range()
t: (0, pi)
```

```
>>> from sage.all import *
>>> I = manifolds.OpenInterval(Integer(0), pi)
>>> I.canonical_chart()
Chart ((0, pi), (t,))
>>> I.canonical_chart().coord_range()
t: (0, pi)
```

The symbol used for the coordinate of the canonical chart is that defined during the construction of the interval:

```
sage: I.<x> = manifolds.OpenInterval(0, pi)
sage: I.canonical_chart()
Chart ((0, pi), (x,))
```

```
>>> from sage.all import *
>>> I = manifolds.OpenInterval(Integer(0), pi, names=('x',)); (x,) = I._first_
↪ngens(1)
>>> I.canonical_chart()
Chart ((0, pi), (x,))
```

canonical_coordinate()

Return the canonical coordinate defined on the interval.

OUTPUT: the symbolic variable representing the canonical coordinate

EXAMPLES:

Canonical coordinate on the interval $(0, \pi)$:

```
sage: I = manifolds.OpenInterval(0, pi)
sage: I.canonical_coordinate()
t
sage: type(I.canonical_coordinate())
<class 'sage.symbolic.expression.Expression'>
sage: I.canonical_coordinate().is_real()
True
```

```
>>> from sage.all import *
>>> I = manifolds.OpenInterval(Integer(0), pi)
>>> I.canonical_coordinate()
t
>>> type(I.canonical_coordinate())
<class 'sage.symbolic.expression.Expression'>
>>> I.canonical_coordinate().is_real()
True
```

The canonical coordinate is the first (unique) coordinate of the canonical chart:

```
sage: I.canonical_coordinate() is I.canonical_chart()[0]
True
```

```
>>> from sage.all import *
>>> I.canonical_coordinate() is I.canonical_chart()[Integer(0)]
True
```

Its default symbol is t; but it can be customized during the creation of the interval:

```
sage: I = manifolds.OpenInterval(0, pi, coordinate='x')
sage: I.canonical_coordinate()
x
sage: I.<x> = manifolds.OpenInterval(0, pi)
sage: I.canonical_coordinate()
x
```

```
>>> from sage.all import *
>>> I = manifolds.OpenInterval(Integer(0), pi, coordinate='x')
>>> I.canonical_coordinate()
x
>>> I = manifolds.OpenInterval(Integer(0), pi, names=('x',)); (x,) = I._first_
↪ngens(1)
>>> I.canonical_coordinate()
x
```

inf()

Return the lower bound (infimum) of the interval.

EXAMPLES:

```
sage: I = manifolds.OpenInterval(1/4, 3)
sage: I.lower_bound()
1/4
sage: J = manifolds.OpenInterval(-oo, 2)
```

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```
sage: J.lower_bound()
-Infinity
```

```
>>> from sage.all import *
>>> I = manifolds.OpenInterval(Integer(1)/Integer(4), Integer(3))
>>> I.lower_bound()
1/4
>>> J = manifolds.OpenInterval(-oo, Integer(2))
>>> J.lower_bound()
-Infinity
```

An alias of `lower_bound()` is `inf()`:

```
sage: I.inf()
1/4
sage: J.inf()
-Infinity
```

```
>>> from sage.all import *
>>> I.inf()
1/4
>>> J.inf()
-Infinity
```

lower_bound()

Return the lower bound (infimum) of the interval.

EXAMPLES:

```
sage: I = manifolds.OpenInterval(1/4, 3)
sage: I.lower_bound()
1/4
sage: J = manifolds.OpenInterval(-oo, 2)
sage: J.lower_bound()
-Infinity
```

```
>>> from sage.all import *
>>> I = manifolds.OpenInterval(Integer(1)/Integer(4), Integer(3))
>>> I.lower_bound()
1/4
>>> J = manifolds.OpenInterval(-oo, Integer(2))
>>> J.lower_bound()
-Infinity
```

An alias of `lower_bound()` is `inf()`:

```
sage: I.inf()
1/4
sage: J.inf()
-Infinity
```

```
>>> from sage.all import *
>>> I.inf()
1/4
>>> J.inf()
-Infinity
```

open_interval(*lower*, *upper*, *name=None*, *latex_name=None*)

Define an open subinterval of `self`.

INPUT:

- `lower` – lower bound of the subinterval (possibly `-Infinity`)
- `upper` – upper bound of the subinterval (possibly `+Infinity`)
- `name` – (default: `None`) string; name (symbol) given to the subinterval; if `None`, the name is constructed from `lower` and `upper`
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the subinterval; if `None`, the LaTeX symbol is constructed from `lower` and `upper` if `name` is `None`, otherwise, it is set to `name`

OUTPUT:

- *OpenInterval* representing the open interval (`lower`, `upper`)

EXAMPLES:

The interval $(0, \pi)$ as a subinterval of $(-4, 4)$:

```
sage: I = manifolds.OpenInterval(-4, 4)
sage: J = I.open_interval(0, pi); J
Real interval (0, pi)
sage: J.is_subset(I)
True
sage: list(I.subset_family())
[Real interval (-4, 4), Real interval (0, pi)]
```

```
>>> from sage.all import *
>>> I = manifolds.OpenInterval(-Integer(4), Integer(4))
>>> J = I.open_interval(Integer(0), pi); J
Real interval (0, pi)
>>> J.is_subset(I)
True
>>> list(I.subset_family())
[Real interval (-4, 4), Real interval (0, pi)]
```

`J` is considered as an open submanifold of `I`:

```
sage: J.manifold() is I
True
```

```
>>> from sage.all import *
>>> J.manifold() is I
True
```

The subinterval $(-4, 4)$ is `I` itself:

```
sage: I.open_interval(-4, 4) is I
True
```

```
>>> from sage.all import *
>>> I.open_interval(-Integer(4), Integer(4)) is I
True
```

sup()

Return the upper bound (supremum) of the interval.

EXAMPLES:

```
sage: I = manifolds.OpenInterval(1/4, 3)
sage: I.upper_bound()
3
sage: J = manifolds.OpenInterval(1, +oo)
sage: J.upper_bound()
+Infinity
```

```
>>> from sage.all import *
>>> I = manifolds.OpenInterval(Integer(1)/Integer(4), Integer(3))
>>> I.upper_bound()
3
>>> J = manifolds.OpenInterval(Integer(1), +oo)
>>> J.upper_bound()
+Infinity
```

An alias of `upper_bound()` is `sup()`:

```
sage: I.sup()
3
sage: J.sup()
+Infinity
```

```
>>> from sage.all import *
>>> I.sup()
3
>>> J.sup()
+Infinity
```

upper_bound()

Return the upper bound (supremum) of the interval.

EXAMPLES:

```
sage: I = manifolds.OpenInterval(1/4, 3)
sage: I.upper_bound()
3
sage: J = manifolds.OpenInterval(1, +oo)
sage: J.upper_bound()
+Infinity
```

```
>>> from sage.all import *
>>> I = manifolds.OpenInterval(Integer(1)/Integer(4), Integer(3))
>>> I.upper_bound()
3
>>> J = manifolds.OpenInterval(Integer(1), +oo)
>>> J.upper_bound()
+Infinity
```

An alias of `upper_bound()` is `sup()`:

```
sage: I.sup()
3
sage: J.sup()
+Infinity
```

```
>>> from sage.all import *
>>> I.sup()
3
>>> J.sup()
+Infinity
```

class sage.manifolds.differentiable.examples.real_line.**RealLine**(*name='ℝ'*, *latex_name='\\Bold{R}'*, *coordinate=None*, *names=None*, *start_index=0*)

Bases: `OpenInterval`

Field of real numbers, as a differentiable manifold of dimension 1 (real line) with a canonical coordinate chart.

INPUT:

- `name` – (default: `'R'`) string; name (symbol) given to the real line
- `latex_name` – (default: `r'\Bold{R}'`) string; LaTeX symbol to denote the real line
- `coordinate` – (default: `None`) string defining the symbol of the canonical coordinate set on the real line; if none is provided and `names` is `None`, the symbol 't' is used
- `names` – (default: `None`) used only when `coordinate` is `None`: it must be a single-element tuple containing the canonical coordinate symbol (this is guaranteed if the shortcut `<names>` is used, see examples below)
- `start_index` – (default: 0) unique value of the index for vectors and forms on the real line manifold

EXAMPLES:

Constructing the real line without any argument:

```
sage: R = manifolds.RealLine() ; R
Real number line ℝ
sage: latex(R)
\Bold{R}
```

```
>>> from sage.all import *
>>> R = manifolds.RealLine() ; R
Real number line ℝ
>>> latex(R)
\Bold{R}
```

R is a 1-dimensional real smooth manifold:

```
sage: R.category()
Category of smooth connected manifolds over Real Field with 53 bits of
 precision
sage: isinstance(R, sage.manifolds.differentiable.manifold.DifferentiableManifold)
True
sage: dim(R)
1
```

```
>>> from sage.all import *
>>> R.category()
Category of smooth connected manifolds over Real Field with 53 bits of
 precision
>>> isinstance(R, sage.manifolds.differentiable.manifold.DifferentiableManifold)
True
>>> dim(R)
1
```

It is endowed with a canonical chart:

```
sage: R.canonical_chart()
Chart (ℝ, (t,))
sage: R.canonical_chart() is R.default_chart()
True
sage: R.atlas()
[Chart (ℝ, (t,))]
```

```
>>> from sage.all import *
>>> R.canonical_chart()
Chart (ℝ, (t,))
>>> R.canonical_chart() is R.default_chart()
True
>>> R.atlas()
[Chart (ℝ, (t,))]
```

The instance is unique (as long as the constructor arguments are the same):

```
sage: R is manifolds.RealLine()
True
sage: R is manifolds.RealLine(latex_name='R')
False
```

```
>>> from sage.all import *
>>> R is manifolds.RealLine()
True
>>> R is manifolds.RealLine(latex_name='R')
False
```

The canonical coordinate is returned by the method `canonical_coordinate()`:

```
sage: R.canonical_coordinate()
t
```

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```
sage: t = R.canonical_coordinate()
sage: type(t)
<class 'sage.symbolic.expression.Expression'>
```

```
>>> from sage.all import *
>>> R.canonical_coordinate()
t
>>> t = R.canonical_coordinate()
>>> type(t)
<class 'sage.symbolic.expression.Expression'>
```

However, it can be obtained in the same step as the real line construction by means of the shortcut `R.<names>`:

```
sage: R.<t> = manifolds.RealLine()
sage: t
t
sage: type(t)
<class 'sage.symbolic.expression.Expression'>
```

```
>>> from sage.all import *
>>> R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1)
>>> t
t
>>> type(t)
<class 'sage.symbolic.expression.Expression'>
```

The trick is performed by Sage preparser:

```
sage: preparse("R.<t> = manifolds.RealLine()")
"R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1)"
```

```
>>> from sage.all import *
>>> preparse("R.<t> = manifolds.RealLine()")
"R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1)"
```

In particular the shortcut is to be used to set a canonical coordinate symbol different from 't':

```
sage: R.<x> = manifolds.RealLine()
sage: R.canonical_chart()
Chart (ℝ, (x,))
sage: R.atlas()
[Chart (ℝ, (x,))]
sage: R.canonical_coordinate()
x
```

```
>>> from sage.all import *
>>> R = manifolds.RealLine(names=('x',)); (x,) = R._first_ngens(1)
>>> R.canonical_chart()
Chart (ℝ, (x,))
>>> R.atlas()
[Chart (ℝ, (x,))]
```

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```
>>> R.canonical_coordinate()
x
```

The LaTeX symbol of the canonical coordinate can be adjusted via the same syntax as a chart declaration (see `RealChart`):

```
sage: R.<x> = manifolds.RealLine(coordinate=r'x:\xi')
sage: latex(x)
{\xi}
sage: latex(R.canonical_chart())
\left(\Bold{R},({\xi})\right)
```

```
>>> from sage.all import *
>>> R = manifolds.RealLine(coordinate=r'x:\xi', names=('x',)); (x,) = R._first_
↪ngens(1)
>>> latex(x)
{\xi}
>>> latex(R.canonical_chart())
\left(\Bold{R},({\xi})\right)
```

The LaTeX symbol of the real line itself can also be customized:

```
sage: R.<x> = manifolds.RealLine(latex_name=r'\mathbb{R}')
sage: latex(R)
\mathbb{R}
```

```
>>> from sage.all import *
>>> R = manifolds.RealLine(latex_name=r'\mathbb{R}', names=('x',)); (x,) = R._
↪first_ngens(1)
>>> latex(R)
\mathbb{R}
```

Elements of the real line can be constructed directly from a number:

```
sage: p = R(2) ; p
Point on the Real number line ℝ
sage: p.coord()
(2,)
sage: p = R(1.742) ; p
Point on the Real number line ℝ
sage: p.coord()
(1.74200000000000,)
```

```
>>> from sage.all import *
>>> p = R(Integer(2)) ; p
Point on the Real number line ℝ
>>> p.coord()
(2,)
>>> p = R(RealNumber('1.742')) ; p
Point on the Real number line ℝ
>>> p.coord()
(1.74200000000000,)
```

Symbolic variables can also be used:

```
sage: p = R(pi, name='pi') ; p
Point pi on the Real number line ℝ
sage: p.coord()
(pi,)
sage: a = var('a')
sage: p = R(a) ; p
Point on the Real number line ℝ
sage: p.coord()
(a,)
```

```
>>> from sage.all import *
>>> p = R(pi, name='pi') ; p
Point pi on the Real number line ℝ
>>> p.coord()
(pi,)
>>> a = var('a')
>>> p = R(a) ; p
Point on the Real number line ℝ
>>> p.coord()
(a,)
```

The real line is considered as the open interval $(-\infty, +\infty)$:

```
sage: isinstance(R, sage.manifolds.differentiable.examples.real_line.OpenInterval)
True
sage: R.lower_bound()
-Infinity
sage: R.upper_bound()
+Infinity
```

```
>>> from sage.all import *
>>> isinstance(R, sage.manifolds.differentiable.examples.real_line.OpenInterval)
True
>>> R.lower_bound()
-Infinity
>>> R.upper_bound()
+Infinity
```

A real interval can be created from `R` means of the method *open_interval()*:

```
sage: I = R.open_interval(0, 1); I
Real interval (0, 1)
sage: I.manifold()
Real number line ℝ
sage: list(R.subset_family())
[Real interval (0, 1), Real number line ℝ]
```

```
>>> from sage.all import *
>>> I = R.open_interval(Integer(0), Integer(1)); I
Real interval (0, 1)
>>> I.manifold()
```

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```
Real number line ℝ
>>> list(R.subset_family())
[Real interval (0, 1), Real number line ℝ]
```

2.4 Scalar Fields

2.4.1 Algebra of Differentiable Scalar Fields

The class *DiffScalarFieldAlgebra* implements the commutative algebra $C^k(M)$ of differentiable scalar fields on a differentiable manifold M of class C^k over a topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$). By *differentiable scalar field*, it is meant a function $M \to K$ that is k-times continuously differentiable. $C^k(M)$ is an algebra over K, whose ring product is the pointwise multiplication of K-valued functions, which is clearly commutative.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2014-2015): initial version

REFERENCES:

- [KN1963]
- [Lee2013]
- [ONe1983]

class sage.manifolds.differentiable.scalarfield_algebra.**DiffScalarFieldAlgebra**(*domain*)

Bases: *ScalarFieldAlgebra*

Commutative algebra of differentiable scalar fields on a differentiable manifold.

If M is a differentiable manifold of class C^k over a topological field K, the *commutative algebra of scalar fields on* M is the set $C^k(M)$ of all k-times continuously differentiable maps $M \to K$. The set $C^k(M)$ is an algebra over K, whose ring product is the pointwise multiplication of K-valued functions, which is clearly commutative.

If $K = \mathbf{R}$ or $K = \mathbf{C}$, the field K over which the algebra $C^k(M)$ is constructed is represented by Sage's Symbolic Ring `SR`, since there is no exact representation of **R** nor **C** in Sage.

Via its base class *ScalarFieldAlgebra*, the class *DiffScalarFieldAlgebra* inherits from `Parent`, with the category set to `CommutativeAlgebras`. The corresponding *element* class is *DiffScalarField*.

INPUT:

- `domain` – the differentiable manifold M on which the scalar fields are defined (must be an instance of class *DifferentiableManifold*)

EXAMPLES:

Algebras of scalar fields on the sphere S^2 and on some open subset of it:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
```

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```
sage: uv_to_xy = xy_to_uv.inverse()
sage: CM = M.scalar_field_algebra() ; CM
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
sage: W = U.intersection(V)  # S^2 minus the two poles
sage: CW = W.scalar_field_algebra() ; CW
Algebra of differentiable scalar fields on the Open subset W of the
 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                  intersection_name='W', restrictions1=
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                  restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> CM = M.scalar_field_algebra() ; CM
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
>>> W = U.intersection(V)  # S^2 minus the two poles
>>> CW = W.scalar_field_algebra() ; CW
Algebra of differentiable scalar fields on the Open subset W of the
 2-dimensional differentiable manifold M
```

$C^k(M)$ and $C^k(W)$ belong to the category of commutative algebras over **R** (represented here by Sage's Symbolic Ring):

```
sage: CM.category()
Join of Category of commutative algebras over Symbolic Ring and Category of
↪homsets of topological spaces
sage: CM.base_ring()
Symbolic Ring
sage: CW.category()
Join of Category of commutative algebras over Symbolic Ring and Category of
↪homsets of topological spaces
sage: CW.base_ring()
Symbolic Ring
```

```
>>> from sage.all import *
>>> CM.category()
Join of Category of commutative algebras over Symbolic Ring and Category of
↪homsets of topological spaces
>>> CM.base_ring()
Symbolic Ring
```

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```
>>> CW.category()
Join of Category of commutative algebras over Symbolic Ring and Category of␣
↪homsets of topological spaces
>>> CW.base_ring()
Symbolic Ring
```

The elements of $C^k(M)$ are scalar fields on M:

```
sage: CM.an_element()
Scalar field on the 2-dimensional differentiable manifold M
sage: CM.an_element().display()  # this sample element is a constant field
M → ℝ
on U: (x, y) ↦ 2
on V: (u, v) ↦ 2
```

```
>>> from sage.all import *
>>> CM.an_element()
Scalar field on the 2-dimensional differentiable manifold M
>>> CM.an_element().display()  # this sample element is a constant field
M → ℝ
on U: (x, y) ↦ 2
on V: (u, v) ↦ 2
```

Those of $C^k(W)$ are scalar fields on W:

```
sage: CW.an_element()
Scalar field on the Open subset W of the 2-dimensional differentiable
 manifold M
sage: CW.an_element().display()  # this sample element is a constant field
W → ℝ
(x, y) ↦ 2
(u, v) ↦ 2
```

```
>>> from sage.all import *
>>> CW.an_element()
Scalar field on the Open subset W of the 2-dimensional differentiable
 manifold M
>>> CW.an_element().display()  # this sample element is a constant field
W → ℝ
(x, y) ↦ 2
(u, v) ↦ 2
```

The zero element:

```
sage: CM.zero()
Scalar field zero on the 2-dimensional differentiable manifold M
sage: CM.zero().display()
zero: M → ℝ
on U: (x, y) ↦ 0
on V: (u, v) ↦ 0
```

```
>>> from sage.all import *
>>> CM.zero()
Scalar field zero on the 2-dimensional differentiable manifold M
>>> CM.zero().display()
zero: M → ℝ
on U: (x, y) ↦ 0
on V: (u, v) ↦ 0
```

```
sage: CW.zero()
Scalar field zero on the Open subset W of the 2-dimensional
 differentiable manifold M
sage: CW.zero().display()
zero: W → ℝ
   (x, y) ↦ 0
   (u, v) ↦ 0
```

```
>>> from sage.all import *
>>> CW.zero()
Scalar field zero on the Open subset W of the 2-dimensional
 differentiable manifold M
>>> CW.zero().display()
zero: W → ℝ
   (x, y) ↦ 0
   (u, v) ↦ 0
```

The unit element:

```
sage: CM.one()
Scalar field 1 on the 2-dimensional differentiable manifold M
sage: CM.one().display()
1: M → ℝ
on U: (x, y) ↦ 1
on V: (u, v) ↦ 1
```

```
>>> from sage.all import *
>>> CM.one()
Scalar field 1 on the 2-dimensional differentiable manifold M
>>> CM.one().display()
1: M → ℝ
on U: (x, y) ↦ 1
on V: (u, v) ↦ 1
```

```
sage: CW.one()
Scalar field 1 on the Open subset W of the 2-dimensional differentiable
 manifold M
sage: CW.one().display()
1: W → ℝ
(x, y) ↦ 1
(u, v) ↦ 1
```

```
>>> from sage.all import *
>>> CW.one()
```

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```
Scalar field 1 on the Open subset W of the 2-dimensional differentiable
 manifold M
>>> CW.one().display()
1: W → ℝ
(x, y) ↦ 1
(u, v) ↦ 1
```

A generic element can be constructed as for any parent in Sage, namely by means of the `__call__` operator on the parent (here with the dictionary of the coordinate expressions defining the scalar field):

```
sage: f = CM({c_xy: atan(x^2+y^2), c_uv: pi/2 - atan(u^2+v^2)}); f
Scalar field on the 2-dimensional differentiable manifold M
sage: f.display()
M → ℝ
on U: (x, y) ↦ arctan(x^2 + y^2)
on V: (u, v) ↦ 1/2*pi - arctan(u^2 + v^2)
sage: f.parent()
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> f = CM({c_xy: atan(x**Integer(2)+y**Integer(2)), c_uv: pi/Integer(2) -␣
↪atan(u**Integer(2)+v**Integer(2))}); f
Scalar field on the 2-dimensional differentiable manifold M
>>> f.display()
M → ℝ
on U: (x, y) ↦ arctan(x^2 + y^2)
on V: (u, v) ↦ 1/2*pi - arctan(u^2 + v^2)
>>> f.parent()
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
```

Specific elements can also be constructed in this way:

```
sage: CM(0) == CM.zero()
True
sage: CM(1) == CM.one()
True
```

```
>>> from sage.all import *
>>> CM(Integer(0)) == CM.zero()
True
>>> CM(Integer(1)) == CM.one()
True
```

Note that the zero scalar field is cached:

```
sage: CM(0) is CM.zero()
True
```

```
>>> from sage.all import *
>>> CM(Integer(0)) is CM.zero()
```

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```
True
```

Elements can also be constructed by means of the method *scalar_field()* acting on the domain (this allows one to set the name of the scalar field at the construction):

```
sage: f1 = M.scalar_field({c_xy: atan(x^2+y^2), c_uv: pi/2 - atan(u^2+v^2)},
....:                     name='f')
sage: f1.parent()
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
sage: f1 == f
True
sage: M.scalar_field(0, chart='all') == CM.zero()
True
```

```
>>> from sage.all import *
>>> f1 = M.scalar_field({c_xy: atan(x**Integer(2)+y**Integer(2)), c_uv: pi/
↪Integer(2) - atan(u**Integer(2)+v**Integer(2))},
...                     name='f')
>>> f1.parent()
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
>>> f1 == f
True
>>> M.scalar_field(Integer(0), chart='all') == CM.zero()
True
```

The algebra $C^k(M)$ coerces to $C^k(W)$ since W is an open subset of M:

```
sage: CW.has_coerce_map_from(CM)
True
```

```
>>> from sage.all import *
>>> CW.has_coerce_map_from(CM)
True
```

The reverse is of course false:

```
sage: CM.has_coerce_map_from(CW)
False
```

```
>>> from sage.all import *
>>> CM.has_coerce_map_from(CW)
False
```

The coercion map is nothing but the restriction to W of scalar fields on M:

```
sage: fW = CW(f) ; fW
Scalar field on the Open subset W of the 2-dimensional differentiable
 manifold M
sage: fW.display()
W → ℝ
```

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```
(x, y) ↦ arctan(x^2 + y^2)
(u, v) ↦ 1/2*pi - arctan(u^2 + v^2)
```

```
>>> from sage.all import *
>>> fW = CW(f) ; fW
Scalar field on the Open subset W of the 2-dimensional differentiable
 manifold M
>>> fW.display()
W → ℝ
(x, y) ↦ arctan(x^2 + y^2)
(u, v) ↦ 1/2*pi - arctan(u^2 + v^2)
```

```
sage: CW(CM.one()) == CW.one()
True
```

```
>>> from sage.all import *
>>> CW(CM.one()) == CW.one()
True
```

The coercion map allows for the addition of elements of $C^k(W)$ with elements of $C^k(M)$, the result being an element of $C^k(W)$:

```
sage: s = fW + f
sage: s.parent()
Algebra of differentiable scalar fields on the Open subset W of the
 2-dimensional differentiable manifold M
sage: s.display()
W → ℝ
(x, y) ↦ 2*arctan(x^2 + y^2)
(u, v) ↦ pi - 2*arctan(u^2 + v^2)
```

```
>>> from sage.all import *
>>> s = fW + f
>>> s.parent()
Algebra of differentiable scalar fields on the Open subset W of the
 2-dimensional differentiable manifold M
>>> s.display()
W → ℝ
(x, y) ↦ 2*arctan(x^2 + y^2)
(u, v) ↦ pi - 2*arctan(u^2 + v^2)
```

Another coercion is that from the Symbolic Ring, the parent of all symbolic expressions (cf. `SymbolicRing`). Since the Symbolic Ring is the base ring for the algebra `CM`, the coercion of a symbolic expression `s` is performed by the operation `s*CM.one()`, which invokes the reflected multiplication operator `sage.manifolds.scalarfield.ScalarField._rmul_()`. If the symbolic expression does not involve any chart coordinate, the outcome is a constant scalar field:

```
sage: h = CM(pi*sqrt(2)) ; h
Scalar field on the 2-dimensional differentiable manifold M
sage: h.display()
M → ℝ
```

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```
on U: (x, y) ↦ sqrt(2)*pi
on V: (u, v) ↦ sqrt(2)*pi
sage: a = var('a')
sage: h = CM(a); h.display()
M → ℝ
on U: (x, y) ↦ a
on V: (u, v) ↦ a
```

```
>>> from sage.all import *
>>> h = CM(pi*sqrt(Integer(2))) ; h
Scalar field on the 2-dimensional differentiable manifold M
>>> h.display()
M → ℝ
on U: (x, y) ↦ sqrt(2)*pi
on V: (u, v) ↦ sqrt(2)*pi
>>> a = var('a')
>>> h = CM(a); h.display()
M → ℝ
on U: (x, y) ↦ a
on V: (u, v) ↦ a
```

If the symbolic expression involves some coordinate of one of the manifold's charts, the outcome is initialized only on the chart domain:

```
sage: h = CM(a+x); h.display()
M → ℝ
on U: (x, y) ↦ a + x
on W: (u, v) ↦ (a*u^2 + a*v^2 + u)/(u^2 + v^2)
sage: h = CM(a+u); h.display()
M → ℝ
on W: (x, y) ↦ (a*x^2 + a*y^2 + x)/(x^2 + y^2)
on V: (u, v) ↦ a + u
```

```
>>> from sage.all import *
>>> h = CM(a+x); h.display()
M → ℝ
on U: (x, y) ↦ a + x
on W: (u, v) ↦ (a*u^2 + a*v^2 + u)/(u^2 + v^2)
>>> h = CM(a+u); h.display()
M → ℝ
on W: (x, y) ↦ (a*x^2 + a*y^2 + x)/(x^2 + y^2)
on V: (u, v) ↦ a + u
```

If the symbolic expression involves coordinates of different charts, the scalar field is created as a Python object, but is not initialized, in order to avoid any ambiguity:

```
sage: h = CM(x+u); h.display()
M → ℝ
```

```
>>> from sage.all import *
>>> h = CM(x+u); h.display()
M → ℝ
```

TESTS OF THE ALGEBRA LAWS:

Ring laws:

```
sage: h = CM(pi*sqrt(2))
sage: s = f + h ; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M → ℝ
on U: (x, y) ↦ sqrt(2)*pi + arctan(x^2 + y^2)
on V: (u, v) ↦ 1/2*pi*(2*sqrt(2) + 1) - arctan(u^2 + v^2)
```

```
>>> from sage.all import *
>>> h = CM(pi*sqrt(Integer(2)))
>>> s = f + h ; s
Scalar field on the 2-dimensional differentiable manifold M
>>> s.display()
M → ℝ
on U: (x, y) ↦ sqrt(2)*pi + arctan(x^2 + y^2)
on V: (u, v) ↦ 1/2*pi*(2*sqrt(2) + 1) - arctan(u^2 + v^2)
```

```
sage: s = f - h ; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M → ℝ
on U: (x, y) ↦ -sqrt(2)*pi + arctan(x^2 + y^2)
on V: (u, v) ↦ -1/2*pi*(2*sqrt(2) - 1) - arctan(u^2 + v^2)
```

```
>>> from sage.all import *
>>> s = f - h ; s
Scalar field on the 2-dimensional differentiable manifold M
>>> s.display()
M → ℝ
on U: (x, y) ↦ -sqrt(2)*pi + arctan(x^2 + y^2)
on V: (u, v) ↦ -1/2*pi*(2*sqrt(2) - 1) - arctan(u^2 + v^2)
```

```
sage: s = f*h ; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M → ℝ
on U: (x, y) ↦ sqrt(2)*pi*arctan(x^2 + y^2)
on V: (u, v) ↦ 1/2*sqrt(2)*(pi^2 - 2*pi*arctan(u^2 + v^2))
```

```
>>> from sage.all import *
>>> s = f*h ; s
Scalar field on the 2-dimensional differentiable manifold M
>>> s.display()
M → ℝ
on U: (x, y) ↦ sqrt(2)*pi*arctan(x^2 + y^2)
on V: (u, v) ↦ 1/2*sqrt(2)*(pi^2 - 2*pi*arctan(u^2 + v^2))
```

```
sage: s = f/h ; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M → ℝ
on U: (x, y) ↦ 1/2*sqrt(2)*arctan(x^2 + y^2)/pi
on V: (u, v) ↦ 1/4*sqrt(2)*(pi - 2*arctan(u^2 + v^2))/pi
```

```
>>> from sage.all import *
>>> s = f/h ; s
Scalar field on the 2-dimensional differentiable manifold M
>>> s.display()
M → ℝ
on U: (x, y) ↦ 1/2*sqrt(2)*arctan(x^2 + y^2)/pi
on V: (u, v) ↦ 1/4*sqrt(2)*(pi - 2*arctan(u^2 + v^2))/pi
```

```
sage: f*(h+f) == f*h + f*f
True
```

```
>>> from sage.all import *
>>> f*(h+f) == f*h + f*f
True
```

Ring laws with coercion:

```
sage: f - fW == CW.zero()
True
sage: f/fW == CW.one()
True
sage: s = f*fW ; s
Scalar field on the Open subset W of the 2-dimensional differentiable
 manifold M
sage: s.display()
W → ℝ
(x, y) ↦ arctan(x^2 + y^2)^2
(u, v) ↦ 1/4*pi^2 - pi*arctan(u^2 + v^2) + arctan(u^2 + v^2)^2
sage: s/f == fW
True
```

```
>>> from sage.all import *
>>> f - fW == CW.zero()
True
>>> f/fW == CW.one()
True
>>> s = f*fW ; s
Scalar field on the Open subset W of the 2-dimensional differentiable
 manifold M
>>> s.display()
W → ℝ
(x, y) ↦ arctan(x^2 + y^2)^2
(u, v) ↦ 1/4*pi^2 - pi*arctan(u^2 + v^2) + arctan(u^2 + v^2)^2
>>> s/f == fW
True
```

Multiplication by a number:

```
sage: s = 2*f ; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M → ℝ
on U: (x, y) ↦ 2*arctan(x^2 + y^2)
on V: (u, v) ↦ pi - 2*arctan(u^2 + v^2)
```

```
>>> from sage.all import *
>>> s = Integer(2)*f ; s
Scalar field on the 2-dimensional differentiable manifold M
>>> s.display()
M → ℝ
on U: (x, y) ↦ 2*arctan(x^2 + y^2)
on V: (u, v) ↦ pi - 2*arctan(u^2 + v^2)
```

```
sage: 0*f == CM.zero()
True
sage: 1*f == f
True
sage: 2*(f/2) == f
True
sage: (f+2*f)/3 == f
True
sage: 1/3*(f+2*f) == f
True
```

```
>>> from sage.all import *
>>> Integer(0)*f == CM.zero()
True
>>> Integer(1)*f == f
True
>>> Integer(2)*(f/Integer(2)) == f
True
>>> (f+Integer(2)*f)/Integer(3) == f
True
>>> Integer(1)/Integer(3)*(f+Integer(2)*f) == f
True
```

The Sage test suite for algebras is passed:

```
sage: TestSuite(CM).run()
```

```
>>> from sage.all import *
>>> TestSuite(CM).run()
```

It is passed also for $C^k(W)$:

```
sage: TestSuite(CW).run()
```

```
>>> from sage.all import *
>>> TestSuite(CW).run()
```

Element

alias of *DiffScalarField*

2.4.2 Differentiable Scalar Fields

Given a differentiable manifold M of class C^k over a topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$), a *differentiable scalar field* on M is a map

$$f : M \longrightarrow K$$

of class C^k.

Differentiable scalar fields are implemented by the class *DiffScalarField*.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015): initial version
- Eric Gourgoulhon (2018): operators gradient, Laplacian and d'Alembertian

REFERENCES:

- [KN1963]
- [Lee2013]
- [ONe1983]

class sage.manifolds.differentiable.scalarfield.**DiffScalarField**(*parent*, *coord_expression=None*, *chart=None*, *name=None*, *latex_name=None*)

Bases: *ScalarField*

Differentiable scalar field on a differentiable manifold.

Given a differentiable manifold M of class C^k over a topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$), a *differentiable scalar field* defined on M is a map

$$f : M \longrightarrow K$$

that is k-times continuously differentiable.

The class *DiffScalarField* is a Sage *element* class, whose *parent* class is *DiffScalarFieldAlgebra*. It inherits from the class *ScalarField* devoted to generic continuous scalar fields on topological manifolds.

INPUT:

- `parent` – the algebra of scalar fields containing the scalar field (must be an instance of class *DiffScalarFieldAlgebra*)
- `coord_expression` – (default: `None`) coordinate expression(s) of the scalar field; this can be either
  - a dictionary of coordinate expressions in various charts on the domain, with the charts as keys;
  - a single coordinate expression; if the argument `chart` is `'all'`, this expression is set to all the charts defined on the open set; otherwise, the expression is set in the specific chart provided by the argument `chart`

  NB: If `coord_expression` is `None` or incomplete, coordinate expressions can be added after the creation of the object, by means of the methods *add_expr()*, *add_expr_by_continuation()* and *set_expr()*

- `chart` – (default: `None`) chart defining the coordinates used in `coord_expression` when the latter is a single coordinate expression; if none is provided (default), the default chart of the open set is assumed. If `chart=='all'`, `coord_expression` is assumed to be independent of the chart (constant scalar field).
- `name` – (default: `None`) string; name (symbol) given to the scalar field
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the scalar field; if none is provided, the LaTeX symbol is set to `name`

EXAMPLES:

A scalar field on the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                                intersection_name='W',
....:                                restrictions1= x^2+y^2!=0,
....:                                restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: f = M.scalar_field({c_xy: 1/(1+x^2+y^2), c_uv: (u^2+v^2)/(1+u^2+v^2)},
....:                    name='f') ; f
Scalar field f on the 2-dimensional differentiable manifold M
sage: f.display()
f: M → ℝ
on U: (x, y) ↦ 1/(x^2 + y^2 + 1)
on V: (u, v) ↦ (u^2 + v^2)/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                                intersection_name='W',
...                                restrictions1= x**Integer(2)+y**Integer(2)!
↪=Integer(0),
...                                restrictions2= u**Integer(2)+v**Integer(2)!
↪=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> f = M.scalar_field({c_xy: Integer(1)/(Integer(1)+x**Integer(2)+y**Integer(2)),
↪ c_uv: (u**Integer(2)+v**Integer(2))/(Integer(1)+u**Integer(2)+v**Integer(2))},
...                    name='f') ; f
Scalar field f on the 2-dimensional differentiable manifold M
>>> f.display()
f: M → ℝ
```

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```
on U: (x, y) ↦ 1/(x^2 + y^2 + 1)
on V: (u, v) ↦ (u^2 + v^2)/(u^2 + v^2 + 1)
```

For scalar fields defined by a single coordinate expression, the latter can be passed instead of the dictionary over the charts:

```
sage: g = U.scalar_field(x*y, chart=c_xy, name='g') ; g
Scalar field g on the Open subset U of the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> g = U.scalar_field(x*y, chart=c_xy, name='g') ; g
Scalar field g on the Open subset U of the 2-dimensional differentiable
 manifold M
```

The above is indeed equivalent to:

```
sage: g = U.scalar_field({c_xy: x*y}, name='g') ; g
Scalar field g on the Open subset U of the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> g = U.scalar_field({c_xy: x*y}, name='g') ; g
Scalar field g on the Open subset U of the 2-dimensional differentiable
 manifold M
```

Since `c_xy` is the default chart of `U`, the argument `chart` can be skipped:

```
sage: g = U.scalar_field(x*y, name='g') ; g
Scalar field g on the Open subset U of the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> g = U.scalar_field(x*y, name='g') ; g
Scalar field g on the Open subset U of the 2-dimensional differentiable
 manifold M
```

The scalar field g is defined on U and has an expression in terms of the coordinates (u, v) on $W = U \cap V$:

```
sage: g.display()
g: U → ℝ
   (x, y) ↦ x*y
on W: (u, v) ↦ u*v/(u^4 + 2*u^2*v^2 + v^4)
```

```
>>> from sage.all import *
>>> g.display()
g: U → ℝ
   (x, y) ↦ x*y
on W: (u, v) ↦ u*v/(u^4 + 2*u^2*v^2 + v^4)
```

Scalar fields on M can also be declared with a single chart:

```
sage: f = M.scalar_field(1/(1+x^2+y^2), chart=c_xy, name='f') ; f
Scalar field f on the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> f = M.scalar_field(Integer(1)/(Integer(1)+x**Integer(2)+y**Integer(2)),␣
↪chart=c_xy, name='f') ; f
Scalar field f on the 2-dimensional differentiable manifold M
```

Their definition must then be completed by providing the expressions on other charts, via the method `add_expr()`, to get a global cover of the manifold:

```
sage: f.add_expr((u^2+v^2)/(1+u^2+v^2), chart=c_uv)
sage: f.display()
f: M → ℝ
on U: (x, y) ↦ 1/(x^2 + y^2 + 1)
on V: (u, v) ↦ (u^2 + v^2)/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> f.add_expr((u**Integer(2)+v**Integer(2))/
↪(Integer(1)+u**Integer(2)+v**Integer(2)), chart=c_uv)
>>> f.display()
f: M → ℝ
on U: (x, y) ↦ 1/(x^2 + y^2 + 1)
on V: (u, v) ↦ (u^2 + v^2)/(u^2 + v^2 + 1)
```

We can even first declare the scalar field without any coordinate expression and provide them subsequently:

```
sage: f = M.scalar_field(name='f')
sage: f.add_expr(1/(1+x^2+y^2), chart=c_xy)
sage: f.add_expr((u^2+v^2)/(1+u^2+v^2), chart=c_uv)
sage: f.display()
f: M → ℝ
on U: (x, y) ↦ 1/(x^2 + y^2 + 1)
on V: (u, v) ↦ (u^2 + v^2)/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> f = M.scalar_field(name='f')
>>> f.add_expr(Integer(1)/(Integer(1)+x**Integer(2)+y**Integer(2)), chart=c_xy)
>>> f.add_expr((u**Integer(2)+v**Integer(2))/
↪(Integer(1)+u**Integer(2)+v**Integer(2)), chart=c_uv)
>>> f.display()
f: M → ℝ
on U: (x, y) ↦ 1/(x^2 + y^2 + 1)
on V: (u, v) ↦ (u^2 + v^2)/(u^2 + v^2 + 1)
```

We may also use the method `add_expr_by_continuation()` to complete the coordinate definition using the analytic continuation from domains in which charts overlap:

```
sage: f = M.scalar_field(1/(1+x^2+y^2), chart=c_xy, name='f') ; f
Scalar field f on the 2-dimensional differentiable manifold M
sage: f.add_expr_by_continuation(c_uv, U.intersection(V))
sage: f.display()
```

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```
f: M → ℝ
on U: (x, y) ↦ 1/(x^2 + y^2 + 1)
on V: (u, v) ↦ (u^2 + v^2)/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> f = M.scalar_field(Integer(1)/(Integer(1)+x**Integer(2)+y**Integer(2)),␣
↪chart=c_xy, name='f') ; f
Scalar field f on the 2-dimensional differentiable manifold M
>>> f.add_expr_by_continuation(c_uv, U.intersection(V))
>>> f.display()
f: M → ℝ
on U: (x, y) ↦ 1/(x^2 + y^2 + 1)
on V: (u, v) ↦ (u^2 + v^2)/(u^2 + v^2 + 1)
```

A scalar field can also be defined by some unspecified function of the coordinates:

```
sage: h = U.scalar_field(function('H')(x, y), name='h') ; h
Scalar field h on the Open subset U of the 2-dimensional differentiable
 manifold M
sage: h.display()
h: U → ℝ
   (x, y) ↦ H(x, y)
on W: (u, v) ↦ H(u/(u^2 + v^2), v/(u^2 + v^2))
```

```
>>> from sage.all import *
>>> h = U.scalar_field(function('H')(x, y), name='h') ; h
Scalar field h on the Open subset U of the 2-dimensional differentiable
 manifold M
>>> h.display()
h: U → ℝ
   (x, y) ↦ H(x, y)
on W: (u, v) ↦ H(u/(u^2 + v^2), v/(u^2 + v^2))
```

We may use the argument `latex_name` to specify the LaTeX symbol denoting the scalar field if the latter is different from `name`:

```
sage: latex(f)
f
sage: f = M.scalar_field({c_xy: 1/(1+x^2+y^2), c_uv: (u^2+v^2)/(1+u^2+v^2)},
....:                    name='f', latex_name=r'\mathcal{F}')
sage: latex(f)
\mathcal{F}
```

```
>>> from sage.all import *
>>> latex(f)
f
>>> f = M.scalar_field({c_xy: Integer(1)/(Integer(1)+x**Integer(2)+y**Integer(2)),
↪ c_uv: (u**Integer(2)+v**Integer(2))/(Integer(1)+u**Integer(2)+v**Integer(2))},
...                    name='f', latex_name=r'\mathcal{F}')
>>> latex(f)
\mathcal{F}
```

The coordinate expression in a given chart is obtained via the method *expr()*, which returns a symbolic expression:

```
sage: f.expr(c_uv)
(u^2 + v^2)/(u^2 + v^2 + 1)
sage: type(f.expr(c_uv))
<class 'sage.symbolic.expression.Expression'>
```

```
>>> from sage.all import *
>>> f.expr(c_uv)
(u^2 + v^2)/(u^2 + v^2 + 1)
>>> type(f.expr(c_uv))
<class 'sage.symbolic.expression.Expression'>
```

The method *coord_function()* returns instead a function of the chart coordinates, i.e. an instance of *ChartFunction*:

```
sage: f.coord_function(c_uv)
(u^2 + v^2)/(u^2 + v^2 + 1)
sage: type(f.coord_function(c_uv))
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_class'>
sage: f.coord_function(c_uv).display()
(u, v) ↦ (u^2 + v^2)/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> f.coord_function(c_uv)
(u^2 + v^2)/(u^2 + v^2 + 1)
>>> type(f.coord_function(c_uv))
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_class'>
>>> f.coord_function(c_uv).display()
(u, v) ↦ (u^2 + v^2)/(u^2 + v^2 + 1)
```

The value returned by the method *expr()* is actually the coordinate expression of the chart function:

```
sage: f.expr(c_uv) is f.coord_function(c_uv).expr()
True
```

```
>>> from sage.all import *
>>> f.expr(c_uv) is f.coord_function(c_uv).expr()
True
```

A constant scalar field is declared by setting the argument `chart` to `'all'`:

```
sage: c = M.scalar_field(2, chart='all', name='c') ; c
Scalar field c on the 2-dimensional differentiable manifold M
sage: c.display()
c: M → ℝ
on U: (x, y) ↦ 2
on V: (u, v) ↦ 2
```

```
>>> from sage.all import *
>>> c = M.scalar_field(Integer(2), chart='all', name='c') ; c
Scalar field c on the 2-dimensional differentiable manifold M
>>> c.display()
```

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```
c: M → ℝ
on U: (x, y) ↦ 2
on V: (u, v) ↦ 2
```

A shortcut is to use the method `constant_scalar_field()`:

```
sage: c == M.constant_scalar_field(2)
True
```

```
>>> from sage.all import *
>>> c == M.constant_scalar_field(Integer(2))
True
```

The constant value can be some unspecified parameter:

```
sage: var('a')
a
sage: c = M.constant_scalar_field(a, name='c') ; c
Scalar field c on the 2-dimensional differentiable manifold M
sage: c.display()
c: M → ℝ
on U: (x, y) ↦ a
on V: (u, v) ↦ a
```

```
>>> from sage.all import *
>>> var('a')
a
>>> c = M.constant_scalar_field(a, name='c') ; c
Scalar field c on the 2-dimensional differentiable manifold M
>>> c.display()
c: M → ℝ
on U: (x, y) ↦ a
on V: (u, v) ↦ a
```

A special case of constant field is the zero scalar field:

```
sage: zer = M.constant_scalar_field(0) ; zer
Scalar field zero on the 2-dimensional differentiable manifold M
sage: zer.display()
zero: M → ℝ
on U: (x, y) ↦ 0
on V: (u, v) ↦ 0
```

```
>>> from sage.all import *
>>> zer = M.constant_scalar_field(Integer(0)) ; zer
Scalar field zero on the 2-dimensional differentiable manifold M
>>> zer.display()
zero: M → ℝ
on U: (x, y) ↦ 0
on V: (u, v) ↦ 0
```

It can be obtained directly by means of the function `zero_scalar_field()`:

```
sage: zer is M.zero_scalar_field()
True
```

```
>>> from sage.all import *
>>> zer is M.zero_scalar_field()
True
```

A third way is to get it as the zero element of the algebra $C^k(M)$ of scalar fields on M (see below):

```
sage: zer is M.scalar_field_algebra().zero()
True
```

```
>>> from sage.all import *
>>> zer is M.scalar_field_algebra().zero()
True
```

By definition, a scalar field acts on the manifold's points, sending them to elements of the manifold's base field (real numbers in the present case):

```
sage: N = M.point((0,0), chart=c_uv) # the North pole
sage: S = M.point((0,0), chart=c_xy) # the South pole
sage: E = M.point((1,0), chart=c_xy) # a point at the equator
sage: f(N)
0
sage: f(S)
1
sage: f(E)
1/2
sage: h(E)
H(1, 0)
sage: c(E)
a
sage: zer(E)
0
```

```
>>> from sage.all import *
>>> N = M.point((Integer(0),Integer(0)), chart=c_uv) # the North pole
>>> S = M.point((Integer(0),Integer(0)), chart=c_xy) # the South pole
>>> E = M.point((Integer(1),Integer(0)), chart=c_xy) # a point at the equator
>>> f(N)
0
>>> f(S)
1
>>> f(E)
1/2
>>> h(E)
H(1, 0)
>>> c(E)
a
>>> zer(E)
0
```

A scalar field can be compared to another scalar field:

```
sage: f == g
False
```

```
>>> from sage.all import *
>>> f == g
False
```

…to a symbolic expression:

```
sage: f == x*y
False
sage: g == x*y
True
sage: c == a
True
```

```
>>> from sage.all import *
>>> f == x*y
False
>>> g == x*y
True
>>> c == a
True
```

…to a number:

```
sage: f == 2
False
sage: zer == 0
True
```

```
>>> from sage.all import *
>>> f == Integer(2)
False
>>> zer == Integer(0)
True
```

…to anything else:

```
sage: f == M
False
```

```
>>> from sage.all import *
>>> f == M
False
```

Standard mathematical functions are implemented:

```
sage: sqrt(f)
Scalar field sqrt(f) on the 2-dimensional differentiable manifold M
sage: sqrt(f).display()
sqrt(f): M → ℝ
```

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```
on U: (x, y) ↦ 1/sqrt(x^2 + y^2 + 1)
on V: (u, v) ↦ sqrt(u^2 + v^2)/sqrt(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> sqrt(f)
Scalar field sqrt(f) on the 2-dimensional differentiable manifold M
>>> sqrt(f).display()
sqrt(f): M → ℝ
on U: (x, y) ↦ 1/sqrt(x^2 + y^2 + 1)
on V: (u, v) ↦ sqrt(u^2 + v^2)/sqrt(u^2 + v^2 + 1)
```

```
sage: tan(f)
Scalar field tan(f) on the 2-dimensional differentiable manifold M
sage: tan(f).display()
tan(f): M → ℝ
on U: (x, y) ↦ sin(1/(x^2 + y^2 + 1))/cos(1/(x^2 + y^2 + 1))
on V: (u, v) ↦ sin((u^2 + v^2)/(u^2 + v^2 + 1))/cos((u^2 + v^2)/(u^2 + v^2 + 1))
```

```
>>> from sage.all import *
>>> tan(f)
Scalar field tan(f) on the 2-dimensional differentiable manifold M
>>> tan(f).display()
tan(f): M → ℝ
on U: (x, y) ↦ sin(1/(x^2 + y^2 + 1))/cos(1/(x^2 + y^2 + 1))
on V: (u, v) ↦ sin((u^2 + v^2)/(u^2 + v^2 + 1))/cos((u^2 + v^2)/(u^2 + v^2 + 1))
```

Arithmetics of scalar fields

Scalar fields on M (resp. U) belong to the algebra $C^k(M)$ (resp. $C^k(U)$):

```
sage: f.parent()
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
sage: f.parent() is M.scalar_field_algebra()
True
sage: g.parent()
Algebra of differentiable scalar fields on the Open subset U of the
 2-dimensional differentiable manifold M
sage: g.parent() is U.scalar_field_algebra()
True
```

```
>>> from sage.all import *
>>> f.parent()
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
>>> f.parent() is M.scalar_field_algebra()
True
>>> g.parent()
Algebra of differentiable scalar fields on the Open subset U of the
 2-dimensional differentiable manifold M
```

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```
>>> g.parent() is U.scalar_field_algebra()
True
```

Consequently, scalar fields can be added:

```
sage: s = f + c ; s
Scalar field f+c on the 2-dimensional differentiable manifold M
sage: s.display()
f+c: M → ℝ
on U: (x, y) ↦ (a*x^2 + a*y^2 + a + 1)/(x^2 + y^2 + 1)
on V: (u, v) ↦ ((a + 1)*u^2 + (a + 1)*v^2 + a)/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> s = f + c ; s
Scalar field f+c on the 2-dimensional differentiable manifold M
>>> s.display()
f+c: M → ℝ
on U: (x, y) ↦ (a*x^2 + a*y^2 + a + 1)/(x^2 + y^2 + 1)
on V: (u, v) ↦ ((a + 1)*u^2 + (a + 1)*v^2 + a)/(u^2 + v^2 + 1)
```

and subtracted:

```
sage: s = f - c ; s
Scalar field f-c on the 2-dimensional differentiable manifold M
sage: s.display()
f-c: M → ℝ
on U: (x, y) ↦ -(a*x^2 + a*y^2 + a - 1)/(x^2 + y^2 + 1)
on V: (u, v) ↦ -((a - 1)*u^2 + (a - 1)*v^2 + a)/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> s = f - c ; s
Scalar field f-c on the 2-dimensional differentiable manifold M
>>> s.display()
f-c: M → ℝ
on U: (x, y) ↦ -(a*x^2 + a*y^2 + a - 1)/(x^2 + y^2 + 1)
on V: (u, v) ↦ -((a - 1)*u^2 + (a - 1)*v^2 + a)/(u^2 + v^2 + 1)
```

Some tests:

```
sage: f + zer == f
True
sage: f - f == zer
True
sage: f + (-f) == zer
True
sage: (f+c)-f == c
True
sage: (f-c)+c == f
True
```

```
>>> from sage.all import *
>>> f + zer == f
```

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```
True
>>> f - f == zer
True
>>> f + (-f) == zer
True
>>> (f+c)-f == c
True
>>> (f-c)+c == f
True
```

We may add a number (interpreted as a constant scalar field) to a scalar field:

```
sage: s = f + 1 ; s
Scalar field f+1 on the 2-dimensional differentiable manifold M
sage: s.display()
f+1: M → ℝ
on U: (x, y) ↦ (x^2 + y^2 + 2)/(x^2 + y^2 + 1)
on V: (u, v) ↦ (2*u^2 + 2*v^2 + 1)/(u^2 + v^2 + 1)
sage: (f+1)-1 == f
True
```

```
>>> from sage.all import *
>>> s = f + Integer(1) ; s
Scalar field f+1 on the 2-dimensional differentiable manifold M
>>> s.display()
f+1: M → ℝ
on U: (x, y) ↦ (x^2 + y^2 + 2)/(x^2 + y^2 + 1)
on V: (u, v) ↦ (2*u^2 + 2*v^2 + 1)/(u^2 + v^2 + 1)
>>> (f+Integer(1))-Integer(1) == f
True
```

The number can represented by a symbolic variable:

```
sage: s = a + f ; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s == c + f
True
```

```
>>> from sage.all import *
>>> s = a + f ; s
Scalar field on the 2-dimensional differentiable manifold M
>>> s == c + f
True
```

However if the symbolic variable is a chart coordinate, the addition is performed only on the chart domain:

```
sage: s = f + x; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M → ℝ
on U: (x, y) ↦ (x^3 + x*y^2 + x + 1)/(x^2 + y^2 + 1)
on W: (u, v) ↦ (u^4 + v^4 + u^3 + (2*u^2 + u)*v^2 + u)/(u^4 + v^4 + (2*u^2 + 1)*v^
```

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```
↪2 + u^2)
sage: s = f + u; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M → ℝ
on W: (x, y) ↦ (x^3 + (x + 1)*y^2 + x^2 + x)/(x^4 + y^4 + (2*x^2 + 1)*y^2 + x^2)
on V: (u, v) ↦ (u^3 + (u + 1)*v^2 + u^2 + u)/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> s = f + x; s
Scalar field on the 2-dimensional differentiable manifold M
>>> s.display()
M → ℝ
on U: (x, y) ↦ (x^3 + x*y^2 + x + 1)/(x^2 + y^2 + 1)
on W: (u, v) ↦ (u^4 + v^4 + u^3 + (2*u^2 + u)*v^2 + u)/(u^4 + v^4 + (2*u^2 + 1)*v^
↪2 + u^2)
>>> s = f + u; s
Scalar field on the 2-dimensional differentiable manifold M
>>> s.display()
M → ℝ
on W: (x, y) ↦ (x^3 + (x + 1)*y^2 + x^2 + x)/(x^4 + y^4 + (2*x^2 + 1)*y^2 + x^2)
on V: (u, v) ↦ (u^3 + (u + 1)*v^2 + u^2 + u)/(u^2 + v^2 + 1)
```

The addition of two scalar fields with different domains is possible if the domain of one of them is a subset of the domain of the other; the domain of the result is then this subset:

```
sage: f.domain()
2-dimensional differentiable manifold M
sage: g.domain()
Open subset U of the 2-dimensional differentiable manifold M
sage: s = f + g ; s
Scalar field f+g on the Open subset U of the 2-dimensional
 differentiable manifold M
sage: s.domain()
Open subset U of the 2-dimensional differentiable manifold M
sage: s.display()
f+g: U → ℝ
   (x, y) ↦ (x*y^3 + (x^3 + x)*y + 1)/(x^2 + y^2 + 1)
on W: (u, v) ↦ (u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6 + u*v^3
 + (u^3 + u)*v)/(u^6 + v^6 + (3*u^2 + 1)*v^4 + u^4 + (3*u^4 + 2*u^2)*v^2)
```

```
>>> from sage.all import *
>>> f.domain()
2-dimensional differentiable manifold M
>>> g.domain()
Open subset U of the 2-dimensional differentiable manifold M
>>> s = f + g ; s
Scalar field f+g on the Open subset U of the 2-dimensional
 differentiable manifold M
>>> s.domain()
Open subset U of the 2-dimensional differentiable manifold M
>>> s.display()
```

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```
f+g: U → ℝ
   (x, y) ↦ (x*y^3 + (x^3 + x)*y + 1)/(x^2 + y^2 + 1)
on W: (u, v) ↦ (u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6 + u*v^3
 + (u^3 + u)*v)/(u^6 + v^6 + (3*u^2 + 1)*v^4 + u^4 + (3*u^4 + 2*u^2)*v^2)
```

The operation actually performed is $f|_U + g$:

```
sage: s == f.restrict(U) + g
True
```

```
>>> from sage.all import *
>>> s == f.restrict(U) + g
True
```

In Sage framework, the addition of f and g is permitted because there is a *coercion* of the parent of f, namely $C^k(M)$, to the parent of g, namely $C^k(U)$ (see `DiffScalarFieldAlgebra`):

```
sage: CM = M.scalar_field_algebra()
sage: CU = U.scalar_field_algebra()
sage: CU.has_coerce_map_from(CM)
True
```

```
>>> from sage.all import *
>>> CM = M.scalar_field_algebra()
>>> CU = U.scalar_field_algebra()
>>> CU.has_coerce_map_from(CM)
True
```

The coercion map is nothing but the restriction to domain U:

```
sage: CU.coerce(f) == f.restrict(U)
True
```

```
>>> from sage.all import *
>>> CU.coerce(f) == f.restrict(U)
True
```

Since the algebra $C^k(M)$ is a vector space over $\mathbf{R}$, scalar fields can be multiplied by a number, either an explicit one:

```
sage: s = 2*f ; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M → ℝ
on U: (x, y) ↦ 2/(x^2 + y^2 + 1)
on V: (u, v) ↦ 2*(u^2 + v^2)/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> s = Integer(2)*f ; s
Scalar field on the 2-dimensional differentiable manifold M
>>> s.display()
M → ℝ
```

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```
on U: (x, y) ↦ 2/(x^2 + y^2 + 1)
on V: (u, v) ↦ 2*(u^2 + v^2)/(u^2 + v^2 + 1)
```

or a symbolic one:

```
sage: s = a*f ; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M → ℝ
on U: (x, y) ↦ a/(x^2 + y^2 + 1)
on V: (u, v) ↦ (u^2 + v^2)*a/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> s = a*f ; s
Scalar field on the 2-dimensional differentiable manifold M
>>> s.display()
M → ℝ
on U: (x, y) ↦ a/(x^2 + y^2 + 1)
on V: (u, v) ↦ (u^2 + v^2)*a/(u^2 + v^2 + 1)
```

However, if the symbolic variable is a chart coordinate, the multiplication is performed only in the corresponding chart:

```
sage: s = x*f; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M → ℝ
on U: (x, y) ↦ x/(x^2 + y^2 + 1)
on W: (u, v) ↦ u/(u^2 + v^2 + 1)
sage: s = u*f; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M → ℝ
on W: (x, y) ↦ x/(x^4 + y^4 + (2*x^2 + 1)*y^2 + x^2)
on V: (u, v) ↦ (u^2 + v^2)*u/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> s = x*f; s
Scalar field on the 2-dimensional differentiable manifold M
>>> s.display()
M → ℝ
on U: (x, y) ↦ x/(x^2 + y^2 + 1)
on W: (u, v) ↦ u/(u^2 + v^2 + 1)
>>> s = u*f; s
Scalar field on the 2-dimensional differentiable manifold M
>>> s.display()
M → ℝ
on W: (x, y) ↦ x/(x^4 + y^4 + (2*x^2 + 1)*y^2 + x^2)
on V: (u, v) ↦ (u^2 + v^2)*u/(u^2 + v^2 + 1)
```

Some tests:

```
sage: 0*f == 0
True
sage: 0*f == zer
True
sage: 1*f == f
True
sage: (-2)*f == - f - f
True
```

```
>>> from sage.all import *
>>> Integer(0)*f == Integer(0)
True
>>> Integer(0)*f == zer
True
>>> Integer(1)*f == f
True
>>> (-Integer(2))*f == - f - f
True
```

The ring multiplication of the algebras $C^k(M)$ and $C^k(U)$ is the pointwise multiplication of functions:

```
sage: s = f*f ; s
Scalar field f*f on the 2-dimensional differentiable manifold M
sage: s.display()
f*f: M → ℝ
on U: (x, y) ↦ 1/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1)
on V: (u, v) ↦ (u^4 + 2*u^2*v^2 + v^4)/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1)
sage: s = g*h ; s
Scalar field g*h on the Open subset U of the 2-dimensional
 differentiable manifold M
sage: s.display()
g*h: U → ℝ
   (x, y) ↦ x*y*H(x, y)
on W: (u, v) ↦ u*v*H(u/(u^2 + v^2), v/(u^2 + v^2))/(u^4 + 2*u^2*v^2 + v^4)
```

```
>>> from sage.all import *
>>> s = f*f ; s
Scalar field f*f on the 2-dimensional differentiable manifold M
>>> s.display()
f*f: M → ℝ
on U: (x, y) ↦ 1/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1)
on V: (u, v) ↦ (u^4 + 2*u^2*v^2 + v^4)/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1)
>>> s = g*h ; s
Scalar field g*h on the Open subset U of the 2-dimensional
 differentiable manifold M
>>> s.display()
g*h: U → ℝ
   (x, y) ↦ x*y*H(x, y)
on W: (u, v) ↦ u*v*H(u/(u^2 + v^2), v/(u^2 + v^2))/(u^4 + 2*u^2*v^2 + v^4)
```

Thanks to the coercion $C^k(M) \to C^k(U)$ mentioned above, it is possible to multiply a scalar field defined on M by a scalar field defined on U, the result being a scalar field defined on U:

```
sage: f.domain(), g.domain()
(2-dimensional differentiable manifold M,
 Open subset U of the 2-dimensional differentiable manifold M)
sage: s = f*g ; s
Scalar field f*g on the Open subset U of the 2-dimensional
 differentiable manifold M
sage: s.display()
f*g: U → ℝ
   (x, y) ↦ x*y/(x^2 + y^2 + 1)
on W: (u, v) ↦ u*v/(u^4 + v^4 + (2*u^2 + 1)*v^2 + u^2)
sage: s == f.restrict(U)*g
True
```

```
>>> from sage.all import *
>>> f.domain(), g.domain()
(2-dimensional differentiable manifold M,
 Open subset U of the 2-dimensional differentiable manifold M)
>>> s = f*g ; s
Scalar field f*g on the Open subset U of the 2-dimensional
 differentiable manifold M
>>> s.display()
f*g: U → ℝ
   (x, y) ↦ x*y/(x^2 + y^2 + 1)
on W: (u, v) ↦ u*v/(u^4 + v^4 + (2*u^2 + 1)*v^2 + u^2)
>>> s == f.restrict(U)*g
True
```

Scalar fields can be divided (pointwise division):

```
sage: s = f/c ; s
Scalar field f/c on the 2-dimensional differentiable manifold M
sage: s.display()
f/c: M → ℝ
on U: (x, y) ↦ 1/(a*x^2 + a*y^2 + a)
on V: (u, v) ↦ (u^2 + v^2)/(a*u^2 + a*v^2 + a)
sage: s = g/h ; s
Scalar field g/h on the Open subset U of the 2-dimensional
 differentiable manifold M
sage: s.display()
g/h: U → ℝ
   (x, y) ↦ x*y/H(x, y)
on W: (u, v) ↦ u*v/((u^4 + 2*u^2*v^2 + v^4)*H(u/(u^2 + v^2), v/(u^2 + v^2)))
sage: s = f/g ; s
Scalar field f/g on the Open subset U of the 2-dimensional
 differentiable manifold M
sage: s.display()
f/g: U → ℝ
   (x, y) ↦ 1/(x*y^3 + (x^3 + x)*y)
on W: (u, v) ↦ (u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6)/(u*v^3 + (u^3 + u)*v)
sage: s == f.restrict(U)/g
True
```

```
>>> from sage.all import *
>>> s = f/c ; s
Scalar field f/c on the 2-dimensional differentiable manifold M
>>> s.display()
f/c: M → ℝ
on U: (x, y) ↦ 1/(a*x^2 + a*y^2 + a)
on V: (u, v) ↦ (u^2 + v^2)/(a*u^2 + a*v^2 + a)
>>> s = g/h ; s
Scalar field g/h on the Open subset U of the 2-dimensional
 differentiable manifold M
>>> s.display()
g/h: U → ℝ
   (x, y) ↦ x*y/H(x, y)
on W: (u, v) ↦ u*v/((u^4 + 2*u^2*v^2 + v^4)*H(u/(u^2 + v^2), v/(u^2 + v^2)))
>>> s = f/g ; s
Scalar field f/g on the Open subset U of the 2-dimensional
 differentiable manifold M
>>> s.display()
f/g: U → ℝ
   (x, y) ↦ 1/(x*y^3 + (x^3 + x)*y)
on W: (u, v) ↦ (u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6)/(u*v^3 + (u^3 + u)*v)
>>> s == f.restrict(U)/g
True
```

For scalar fields defined on a single chart domain, we may perform some arithmetics with symbolic expressions involving the chart coordinates:

```
sage: s = g + x^2 - y ; s
Scalar field on the Open subset U of the 2-dimensional differentiable
 manifold M
sage: s.display()
U → ℝ
(x, y) ↦ x^2 + (x - 1)*y
on W: (u, v) ↦ -(v^3 - u^2 + (u^2 - u)*v)/(u^4 + 2*u^2*v^2 + v^4)
```

```
>>> from sage.all import *
>>> s = g + x**Integer(2) - y ; s
Scalar field on the Open subset U of the 2-dimensional differentiable
 manifold M
>>> s.display()
U → ℝ
(x, y) ↦ x^2 + (x - 1)*y
on W: (u, v) ↦ -(v^3 - u^2 + (u^2 - u)*v)/(u^4 + 2*u^2*v^2 + v^4)
```

```
sage: s = g*x ; s
Scalar field on the Open subset U of the 2-dimensional differentiable
 manifold M
sage: s.display()
U → ℝ
(x, y) ↦ x^2*y
on W: (u, v) ↦ u^2*v/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6)
```

```
>>> from sage.all import *
>>> s = g*x ; s
Scalar field on the Open subset U of the 2-dimensional differentiable
 manifold M
>>> s.display()
U → ℝ
(x, y) ↦ x^2*y
on W: (u, v) ↦ u^2*v/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6)
```

```
sage: s = g/x ; s
Scalar field on the Open subset U of the 2-dimensional differentiable
 manifold M
sage: s.display()
U → ℝ
(x, y) ↦ y
on W: (u, v) ↦ v/(u^2 + v^2)
sage: s = x/g ; s
Scalar field on the Open subset U of the 2-dimensional differentiable
 manifold M
sage: s.display()
U → ℝ
(x, y) ↦ 1/y
on W: (u, v) ↦ (u^2 + v^2)/v
```

```
>>> from sage.all import *
>>> s = g/x ; s
Scalar field on the Open subset U of the 2-dimensional differentiable
 manifold M
>>> s.display()
U → ℝ
(x, y) ↦ y
on W: (u, v) ↦ v/(u^2 + v^2)
>>> s = x/g ; s
Scalar field on the Open subset U of the 2-dimensional differentiable
 manifold M
>>> s.display()
U → ℝ
(x, y) ↦ 1/y
on W: (u, v) ↦ (u^2 + v^2)/v
```

The test suite is passed:

```
sage: TestSuite(f).run()
sage: TestSuite(zer).run()
```

```
>>> from sage.all import *
>>> TestSuite(f).run()
>>> TestSuite(zer).run()
```

bracket(*other*)

Return the Schouten-Nijenhuis bracket of `self`, considered as a multivector field of degree 0, with a multivector field.

See `bracket()` for details.

INPUT:

- `other` – a multivector field of degree p

OUTPUT:

- if $p = 0$, a zero scalar field
- if $p = 1$, an instance of `DiffScalarField` representing the Schouten-Nijenhuis bracket `[self, other]`
- if $p \geq 2$, an instance of `MultivectorField` representing the Schouten-Nijenhuis bracket `[self, other]`

EXAMPLES:

The Schouten-Nijenhuis bracket of two scalar fields is identically zero:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x+y^2}, name='f')
sage: g = M.scalar_field({X: y-x}, name='g')
sage: s = f.bracket(g); s
Scalar field zero on the 2-dimensional differentiable manifold M
sage: s.display()
zero: M → ℝ
   (x, y) ↦ 0
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: x+y**Integer(2)}, name='f')
>>> g = M.scalar_field({X: y-x}, name='g')
>>> s = f.bracket(g); s
Scalar field zero on the 2-dimensional differentiable manifold M
>>> s.display()
zero: M → ℝ
   (x, y) ↦ 0
```

while the Schouten-Nijenhuis bracket of a scalar field f with a multivector field a is equal to minus the interior product of the differential of f with a:

```
sage: a = M.multivector_field(2, name='a')
sage: a[0,1] = x*y ; a.display()
a = x*y ∂/∂x∧∂/∂y
sage: s = f.bracket(a); s
Vector field -i_df a on the 2-dimensional differentiable manifold M
sage: s.display()
-i_df a = 2*x*y^2 ∂/∂x - x*y ∂/∂y
```

```
>>> from sage.all import *
>>> a = M.multivector_field(Integer(2), name='a')
>>> a[Integer(0),Integer(1)] = x*y ; a.display()
a = x*y ∂/∂x∧∂/∂y
>>> s = f.bracket(a); s
```

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```
Vector field -i_df a on the 2-dimensional differentiable manifold M
>>> s.display()
-i_df a = 2*x*y^2 ∂/∂x - x*y ∂/∂y
```

See `bracket()` for other examples.

dalembertian(*metric=None*)

Return the d'Alembertian of `self` with respect to a given Lorentzian metric.

The *d'Alembertian* of a scalar field f with respect to a Lorentzian metric g is nothing but the Laplacian (see `laplacian()`) of f with respect to that metric:

$$\Box f = g^{ij}\nabla_i\nabla_j f = \nabla_i\nabla^i f$$

where ∇ is the Levi-Civita connection of g.

Note

If the metric g is not Lorentzian, the name *d'Alembertian* is not appropriate and one should use `laplacian()` instead.

INPUT:

- `metric` – (default: `None`) the Lorentzian metric g involved in the definition of the d'Alembertian; if none is provided, the domain of `self` is supposed to be endowed with a default Lorentzian metric (i.e. is supposed to be Lorentzian manifold, see `PseudoRiemannianManifold`) and the latter is used to define the d'Alembertian

OUTPUT:

- instance of `DiffScalarField` representing the d'Alembertian of `self`

EXAMPLES:

d'Alembertian of a scalar field in Minkowski spacetime:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1, 1, 1, 1
sage: f = M.scalar_field(t + x^2 + t^2*y^3 - x*z^4, name='f')
sage: s = f.dalembertian(); s
Scalar field Box(f) on the 4-dimensional Lorentzian manifold M
sage: s.display()
Box(f): M → ℝ
   (t, x, y, z) ↦ 6*t^2*y - 2*y^3 - 12*x*z^2 + 2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1), Integer(1), Integer(1),
```

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```
↪ Integer(1)
>>> f = M.scalar_field(t + x**Integer(2) + t**Integer(2)*y**Integer(3) -␣
↪x*z**Integer(4), name='f')
>>> s = f.dalembertian(); s
Scalar field Box(f) on the 4-dimensional Lorentzian manifold M
>>> s.display()
Box(f): M → ℝ
   (t, x, y, z) ↦ 6*t^2*y - 2*y^3 - 12*x*z^2 + 2
```

The function `dalembertian()` from the `operators` module can be used instead of the method `dalembertian()`:

```
sage: from sage.manifolds.operators import dalembertian
sage: dalembertian(f) == s
True
```

```
>>> from sage.all import *
>>> from sage.manifolds.operators import dalembertian
>>> dalembertian(f) == s
True
```

degree()

Return the degree of `self`, considered as a differential form or a multivector field, i.e. zero.

This trivial method is provided for consistency with the exterior calculus scheme, cf. the methods `degree()` (differential forms) and `degree()` (multivector fields).

OUTPUT: 0

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x+y^2})
sage: f.degree()
0
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: x+y**Integer(2)})
>>> f.degree()
0
```

derivative()

Return the differential of `self`.

OUTPUT:

- a `DiffForm` (or of `DiffFormParal` if the scalar field's domain is parallelizable) representing the 1-form that is the differential of the scalar field

EXAMPLES:

Differential of a scalar field on a 3-dimensional differentiable manifold:

```
sage: M = Manifold(3, 'M')
sage: c_xyz.<x,y,z> = M.chart()
sage: f = M.scalar_field(cos(x)*z^3 + exp(y)*z^2, name='f')
sage: df = f.differential() ; df
1-form df on the 3-dimensional differentiable manifold M
sage: df.display()
df = -z^3*sin(x) dx + z^2*e^y dy + (3*z^2*cos(x) + 2*z*e^y) dz
sage: latex(df)
\mathrm{d}f
sage: df.parent()
Free module Omega^1(M) of 1-forms on the 3-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> f = M.scalar_field(cos(x)*z**Integer(3) + exp(y)*z**Integer(2), name='f')
>>> df = f.differential() ; df
1-form df on the 3-dimensional differentiable manifold M
>>> df.display()
df = -z^3*sin(x) dx + z^2*e^y dy + (3*z^2*cos(x) + 2*z*e^y) dz
>>> latex(df)
\mathrm{d}f
>>> df.parent()
Free module Omega^1(M) of 1-forms on the 3-dimensional
 differentiable manifold M
```

The result is cached, i.e. is not recomputed unless `f` is changed:

```
sage: f.differential() is df
True
```

```
>>> from sage.all import *
>>> f.differential() is df
True
```

Instead of invoking the method *differential()*, one may apply the function `diff` to the scalar field:

```
sage: diff(f) is f.differential()
True
```

```
>>> from sage.all import *
>>> diff(f) is f.differential()
True
```

Since the exterior derivative of a scalar field (considered a 0-form) is nothing but its differential, `exterior_derivative()` is an alias of `differential()`:

```
sage: df = f.exterior_derivative() ; df
1-form df on the 3-dimensional differentiable manifold M
sage: df.display()
```

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```
df = -z^3*sin(x) dx + z^2*e^y dy + (3*z^2*cos(x) + 2*z*e^y) dz
sage: latex(df)
\mathrm{d}f
```

```
>>> from sage.all import *
>>> df = f.exterior_derivative() ; df
1-form df on the 3-dimensional differentiable manifold M
>>> df.display()
df = -z^3*sin(x) dx + z^2*e^y dy + (3*z^2*cos(x) + 2*z*e^y) dz
>>> latex(df)
\mathrm{d}f
```

Differential computed on a chart that is not the default one:

```
sage: c_uvw.<u,v,w> = M.chart()
sage: g = M.scalar_field(u*v^2*w^3, c_uvw, name='g')
sage: dg = g.differential() ; dg
1-form dg on the 3-dimensional differentiable manifold M
sage: dg._components
{Coordinate frame (M, (∂/∂u,∂/∂v,∂/∂w)): 1-index components w.r.t.
 Coordinate frame (M, (∂/∂u,∂/∂v,∂/∂w))}
sage: dg.comp(c_uvw.frame())[:, c_uvw]
[v^2*w^3, 2*u*v*w^3, 3*u*v^2*w^2]
sage: dg.display(c_uvw)
dg = v^2*w^3 du + 2*u*v*w^3 dv + 3*u*v^2*w^2 dw
```

```
>>> from sage.all import *
>>> c_uvw = M.chart(names=('u', 'v', 'w',)); (u, v, w,) = c_uvw._first_
↪ngens(3)
>>> g = M.scalar_field(u*v**Integer(2)*w**Integer(3), c_uvw, name='g')
>>> dg = g.differential() ; dg
1-form dg on the 3-dimensional differentiable manifold M
>>> dg._components
{Coordinate frame (M, (∂/∂u,∂/∂v,∂/∂w)): 1-index components w.r.t.
 Coordinate frame (M, (∂/∂u,∂/∂v,∂/∂w))}
>>> dg.comp(c_uvw.frame())[:, c_uvw]
[v^2*w^3, 2*u*v*w^3, 3*u*v^2*w^2]
>>> dg.display(c_uvw)
dg = v^2*w^3 du + 2*u*v*w^3 dv + 3*u*v^2*w^2 dw
```

The exterior derivative is nilpotent:

```
sage: ddf = df.exterior_derivative() ; ddf
2-form ddf on the 3-dimensional differentiable manifold M
sage: ddf == 0
True
sage: ddf[:] # for the incredule
[0 0 0]
[0 0 0]
[0 0 0]
sage: ddg = dg.exterior_derivative() ; ddg
2-form ddg on the 3-dimensional differentiable manifold M
```

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```
sage: ddg == 0
True
```

```
>>> from sage.all import *
>>> ddf = df.exterior_derivative() ; ddf
2-form ddf on the 3-dimensional differentiable manifold M
>>> ddf == Integer(0)
True
>>> ddf[:] # for the incredule
[0 0 0]
[0 0 0]
[0 0 0]
>>> ddg = dg.exterior_derivative() ; ddg
2-form ddg on the 3-dimensional differentiable manifold M
>>> ddg == Integer(0)
True
```

differential()

Return the differential of `self`.

OUTPUT:

- a `DiffForm` (or of `DiffFormParal` if the scalar field's domain is parallelizable) representing the 1-form that is the differential of the scalar field

EXAMPLES:

Differential of a scalar field on a 3-dimensional differentiable manifold:

```
sage: M = Manifold(3, 'M')
sage: c_xyz.<x,y,z> = M.chart()
sage: f = M.scalar_field(cos(x)*z^3 + exp(y)*z^2, name='f')
sage: df = f.differential() ; df
1-form df on the 3-dimensional differentiable manifold M
sage: df.display()
df = -z^3*sin(x) dx + z^2*e^y dy + (3*z^2*cos(x) + 2*z*e^y) dz
sage: latex(df)
\mathrm{d}f
sage: df.parent()
Free module Omega^1(M) of 1-forms on the 3-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> f = M.scalar_field(cos(x)*z**Integer(3) + exp(y)*z**Integer(2), name='f')
>>> df = f.differential() ; df
1-form df on the 3-dimensional differentiable manifold M
>>> df.display()
df = -z^3*sin(x) dx + z^2*e^y dy + (3*z^2*cos(x) + 2*z*e^y) dz
>>> latex(df)
\mathrm{d}f
```

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```
>>> df.parent()
Free module Omega^1(M) of 1-forms on the 3-dimensional
 differentiable manifold M
```

The result is cached, i.e. is not recomputed unless `f` is changed:

```
sage: f.differential() is df
True
```

```
>>> from sage.all import *
>>> f.differential() is df
True
```

Instead of invoking the method `differential()`, one may apply the function `diff` to the scalar field:

```
sage: diff(f) is f.differential()
True
```

```
>>> from sage.all import *
>>> diff(f) is f.differential()
True
```

Since the exterior derivative of a scalar field (considered a 0-form) is nothing but its differential, `exterior_derivative()` is an alias of `differential()`:

```
sage: df = f.exterior_derivative() ; df
1-form df on the 3-dimensional differentiable manifold M
sage: df.display()
df = -z^3*sin(x) dx + z^2*e^y dy + (3*z^2*cos(x) + 2*z*e^y) dz
sage: latex(df)
\mathrm{d}f
```

```
>>> from sage.all import *
>>> df = f.exterior_derivative() ; df
1-form df on the 3-dimensional differentiable manifold M
>>> df.display()
df = -z^3*sin(x) dx + z^2*e^y dy + (3*z^2*cos(x) + 2*z*e^y) dz
>>> latex(df)
\mathrm{d}f
```

Differential computed on a chart that is not the default one:

```
sage: c_uvw.<u,v,w> = M.chart()
sage: g = M.scalar_field(u*v^2*w^3, c_uvw, name='g')
sage: dg = g.differential() ; dg
1-form dg on the 3-dimensional differentiable manifold M
sage: dg._components
{Coordinate frame (M, (∂/∂u,∂/∂v,∂/∂w)): 1-index components w.r.t.
 Coordinate frame (M, (∂/∂u,∂/∂v,∂/∂w))}
sage: dg.comp(c_uvw.frame())[:, c_uvw]
[v^2*w^3, 2*u*v*w^3, 3*u*v^2*w^2]
```

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```
sage: dg.display(c_uvw)
dg = v^2*w^3 du + 2*u*v*w^3 dv + 3*u*v^2*w^2 dw
```

```
>>> from sage.all import *
>>> c_uvw = M.chart(names=('u', 'v', 'w',)); (u, v, w,) = c_uvw._first_
↪ngens(3)
>>> g = M.scalar_field(u*v**Integer(2)*w**Integer(3), c_uvw, name='g')
>>> dg = g.differential() ; dg
1-form dg on the 3-dimensional differentiable manifold M
>>> dg._components
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 Coordinate frame (M, (∂/∂u,∂/∂v,∂/∂w))}
>>> dg.comp(c_uvw.frame())[:, c_uvw]
[v^2*w^3, 2*u*v*w^3, 3*u*v^2*w^2]
>>> dg.display(c_uvw)
dg = v^2*w^3 du + 2*u*v*w^3 dv + 3*u*v^2*w^2 dw
```

The exterior derivative is nilpotent:

```
sage: ddf = df.exterior_derivative() ; ddf
2-form ddf on the 3-dimensional differentiable manifold M
sage: ddf == 0
True
sage: ddf[:] # for the incredule
[0 0 0]
[0 0 0]
[0 0 0]
sage: ddg = dg.exterior_derivative() ; ddg
2-form ddg on the 3-dimensional differentiable manifold M
sage: ddg == 0
True
```

```
>>> from sage.all import *
>>> ddf = df.exterior_derivative() ; ddf
2-form ddf on the 3-dimensional differentiable manifold M
>>> ddf == Integer(0)
True
>>> ddf[:] # for the incredule
[0 0 0]
[0 0 0]
[0 0 0]
>>> ddg = dg.exterior_derivative() ; ddg
2-form ddg on the 3-dimensional differentiable manifold M
>>> ddg == Integer(0)
True
```

exterior_derivative()

Return the differential of `self`.

OUTPUT:

- a *DiffForm* (or of *DiffFormParal* if the scalar field's domain is parallelizable) representing the 1-form that is the differential of the scalar field

EXAMPLES:

Differential of a scalar field on a 3-dimensional differentiable manifold:

```
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sage: f = M.scalar_field(cos(x)*z^3 + exp(y)*z^2, name='f')
sage: df = f.differential() ; df
1-form df on the 3-dimensional differentiable manifold M
sage: df.display()
df = -z^3*sin(x) dx + z^2*e^y dy + (3*z^2*cos(x) + 2*z*e^y) dz
sage: latex(df)
\mathrm{d}f
sage: df.parent()
Free module Omega^1(M) of 1-forms on the 3-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> f = M.scalar_field(cos(x)*z**Integer(3) + exp(y)*z**Integer(2), name='f')
>>> df = f.differential() ; df
1-form df on the 3-dimensional differentiable manifold M
>>> df.display()
df = -z^3*sin(x) dx + z^2*e^y dy + (3*z^2*cos(x) + 2*z*e^y) dz
>>> latex(df)
\mathrm{d}f
>>> df.parent()
Free module Omega^1(M) of 1-forms on the 3-dimensional
 differentiable manifold M
```

The result is cached, i.e. is not recomputed unless `f` is changed:

```
sage: f.differential() is df
True
```

```
>>> from sage.all import *
>>> f.differential() is df
True
```

Instead of invoking the method *differential()*, one may apply the function `diff` to the scalar field:

```
sage: diff(f) is f.differential()
True
```

```
>>> from sage.all import *
>>> diff(f) is f.differential()
True
```

Since the exterior derivative of a scalar field (considered a 0-form) is nothing but its differential, `exterior_derivative()` is an alias of `differential()`:

```
sage: df = f.exterior_derivative() ; df
1-form df on the 3-dimensional differentiable manifold M
sage: df.display()
df = -z^3*sin(x) dx + z^2*e^y dy + (3*z^2*cos(x) + 2*z*e^y) dz
sage: latex(df)
\mathrm{d}f
```

```
>>> from sage.all import *
>>> df = f.exterior_derivative() ; df
1-form df on the 3-dimensional differentiable manifold M
>>> df.display()
df = -z^3*sin(x) dx + z^2*e^y dy + (3*z^2*cos(x) + 2*z*e^y) dz
>>> latex(df)
\mathrm{d}f
```

Differential computed on a chart that is not the default one:

```
sage: c_uvw.<u,v,w> = M.chart()
sage: g = M.scalar_field(u*v^2*w^3, c_uvw, name='g')
sage: dg = g.differential() ; dg
1-form dg on the 3-dimensional differentiable manifold M
sage: dg._components
{Coordinate frame (M, (∂/∂u,∂/∂v,∂/∂w)): 1-index components w.r.t.
 Coordinate frame (M, (∂/∂u,∂/∂v,∂/∂w))}
sage: dg.comp(c_uvw.frame())[:, c_uvw]
[v^2*w^3, 2*u*v*w^3, 3*u*v^2*w^2]
sage: dg.display(c_uvw)
dg = v^2*w^3 du + 2*u*v*w^3 dv + 3*u*v^2*w^2 dw
```

```
>>> from sage.all import *
>>> c_uvw = M.chart(names=('u', 'v', 'w',)); (u, v, w,) = c_uvw._first_
↪ngens(3)
>>> g = M.scalar_field(u*v**Integer(2)*w**Integer(3), c_uvw, name='g')
>>> dg = g.differential() ; dg
1-form dg on the 3-dimensional differentiable manifold M
>>> dg._components
{Coordinate frame (M, (∂/∂u,∂/∂v,∂/∂w)): 1-index components w.r.t.
 Coordinate frame (M, (∂/∂u,∂/∂v,∂/∂w))}
>>> dg.comp(c_uvw.frame())[:, c_uvw]
[v^2*w^3, 2*u*v*w^3, 3*u*v^2*w^2]
>>> dg.display(c_uvw)
dg = v^2*w^3 du + 2*u*v*w^3 dv + 3*u*v^2*w^2 dw
```

The exterior derivative is nilpotent:

```
sage: ddf = df.exterior_derivative() ; ddf
2-form ddf on the 3-dimensional differentiable manifold M
sage: ddf == 0
True
sage: ddf[:] # for the incredule
[0 0 0]
[0 0 0]
[0 0 0]
```

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```
sage: ddg = dg.exterior_derivative() ; ddg
2-form ddg on the 3-dimensional differentiable manifold M
sage: ddg == 0
True
```

```
>>> from sage.all import *
>>> ddf = df.exterior_derivative() ; ddf
2-form ddf on the 3-dimensional differentiable manifold M
>>> ddf == Integer(0)
True
>>> ddf[:] # for the incredule
[0 0 0]
[0 0 0]
[0 0 0]
>>> ddg = dg.exterior_derivative() ; ddg
2-form ddg on the 3-dimensional differentiable manifold M
>>> ddg == Integer(0)
True
```

gradient(*metric=None*)

Return the gradient of `self` (with respect to a given metric).

The *gradient* of a scalar field f with respect to a metric g is the vector field grad f whose components in any coordinate frame are

$$(\mathrm{grad}\, f)^i = g^{ij} \frac{\partial F}{\partial x^j}$$

where the x^j's are the coordinates with respect to which the frame is defined and F is the chart function representing f in these coordinates: $f(p) = F(x^1(p), \ldots, x^n(p))$ for any point p in the chart domain. In other words, the gradient of f is the vector field that is the g-dual of the differential of f.

INPUT:

- `metric` – (default: `None`) the pseudo-Riemannian metric g involved in the definition of the gradient; if none is provided, the domain of `self` is supposed to be endowed with a default metric (i.e. is supposed to be pseudo-Riemannian manifold, see *PseudoRiemannianManifold*) and the latter is used to define the gradient

OUTPUT:

- instance of *VectorField* representing the gradient of `self`

EXAMPLES:

Gradient of a scalar field in the Euclidean plane:

```
sage: M.<x,y> = EuclideanSpace()
sage: f = M.scalar_field(cos(x*y), name='f')
sage: v = f.gradient(); v
Vector field grad(f) on the Euclidean plane E^2
sage: v.display()
grad(f) = -y*sin(x*y) e_x - x*sin(x*y) e_y
sage: v[:]
[-y*sin(x*y), -x*sin(x*y)]
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y',)); (x, y,) = M._first_ngens(2)
>>> f = M.scalar_field(cos(x*y), name='f')
>>> v = f.gradient(); v
Vector field grad(f) on the Euclidean plane E^2
>>> v.display()
grad(f) = -y*sin(x*y) e_x - x*sin(x*y) e_y
>>> v[:]
[-y*sin(x*y), -x*sin(x*y)]
```

Gradient in polar coordinates:

```
sage: M.<r,phi> = EuclideanSpace(coordinates='polar')
sage: f = M.scalar_field(r*cos(phi), name='f')
sage: f.gradient().display()
grad(f) = cos(phi) e_r - sin(phi) e_phi
sage: f.gradient()[:]
[cos(phi), -sin(phi)]
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(coordinates='polar', names=('r', 'phi',)); (r, phi,) =␣
↪M._first_ngens(2)
>>> f = M.scalar_field(r*cos(phi), name='f')
>>> f.gradient().display()
grad(f) = cos(phi) e_r - sin(phi) e_phi
>>> f.gradient()[:]
[cos(phi), -sin(phi)]
```

Note that `(e_r, e_phi)` is the orthonormal vector frame associated with polar coordinates (see *polar_frame()*); the gradient expressed in the coordinate frame is:

```
sage: f.gradient().display(M.polar_coordinates().frame())
grad(f) = cos(phi) ∂/∂r - sin(phi)/r ∂/∂phi
```

```
>>> from sage.all import *
>>> f.gradient().display(M.polar_coordinates().frame())
grad(f) = cos(phi) ∂/∂r - sin(phi)/r ∂/∂phi
```

The function *grad()* from the *operators* module can be used instead of the method *gradient()*:

```
sage: from sage.manifolds.operators import grad
sage: grad(f) == f.gradient()
True
```

```
>>> from sage.all import *
>>> from sage.manifolds.operators import grad
>>> grad(f) == f.gradient()
True
```

The gradient can be taken with respect to a metric tensor that is not the default one:

```
sage: h = M.lorentzian_metric('h')
sage: h[1,1], h[2,2] = -1, 1/(1+r^2)
```

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```
sage: h.display(M.polar_coordinates().frame())
h = -dr⊗dr + r^2/(r^2 + 1) dphi⊗dphi
sage: v = f.gradient(h); v
Vector field grad_h(f) on the Euclidean plane E^2
sage: v.display()
grad_h(f) = -cos(phi) e_r + (-r^2*sin(phi) - sin(phi)) e_phi
```

```
>>> from sage.all import *
>>> h = M.lorentzian_metric('h')
>>> h[Integer(1),Integer(1)], h[Integer(2),Integer(2)] = -Integer(1),␣
↪Integer(1)/(Integer(1)+r**Integer(2))
>>> h.display(M.polar_coordinates().frame())
h = -dr⊗dr + r^2/(r^2 + 1) dphi⊗dphi
>>> v = f.gradient(h); v
Vector field grad_h(f) on the Euclidean plane E^2
>>> v.display()
grad_h(f) = -cos(phi) e_r + (-r^2*sin(phi) - sin(phi)) e_phi
```

hodge_dual(*nondegenerate_tensor*)

Compute the Hodge dual of the scalar field with respect to some non-degenerate bilinear form (Riemannian metric or symplectic form).

If M is the domain of the scalar field (denoted by f), n is the dimension of M and g is a non-degenerate bilinear form on M, the *Hodge dual* of f w.r.t. g is the n-form $*f$ defined by

$$*f = f\epsilon,$$

where ϵ is the volume n-form associated with g (see `volume_form()`).

INPUT:

- `nondegenerate_tensor` – a non-degenerate bilinear form defined on the same manifold as the current differential form; must be an instance of `PseudoRiemannianMetric` or `SymplecticForm`.

OUTPUT: the n-form $*f$

EXAMPLES:

Hodge dual of a scalar field in the Euclidean space R^3:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: g = M.metric('g')
sage: g[1,1], g[2,2], g[3,3] = 1, 1, 1
sage: f = M.scalar_field(function('F')(x,y,z), name='f')
sage: sf = f.hodge_dual(g) ; sf
3-form *f on the 3-dimensional differentiable manifold M
sage: sf.display()
*f = F(x, y, z) dx∧dy∧dz
sage: ssf = sf.hodge_dual(g) ; ssf
Scalar field **f on the 3-dimensional differentiable manifold M
sage: ssf.display()
**f: M → ℝ
   (x, y, z) ↦ F(x, y, z)
sage: ssf == f # must hold for a Riemannian metric
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> g = M.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)], g[Integer(3),
↪Integer(3)] = Integer(1), Integer(1), Integer(1)
>>> f = M.scalar_field(function('F')(x,y,z), name='f')
>>> sf = f.hodge_dual(g) ; sf
3-form *f on the 3-dimensional differentiable manifold M
>>> sf.display()
*f = F(x, y, z) dx∧dy∧dz
>>> ssf = sf.hodge_dual(g) ; ssf
Scalar field **f on the 3-dimensional differentiable manifold M
>>> ssf.display()
**f: M → ℝ
   (x, y, z) ↦ F(x, y, z)
>>> ssf == f # must hold for a Riemannian metric
True
```

Instead of calling the method `hodge_dual()` on the scalar field, one can invoke the method `hodge_star()` of the metric:

```
sage: f.hodge_dual(g) == g.hodge_star(f)
True
```

```
>>> from sage.all import *
>>> f.hodge_dual(g) == g.hodge_star(f)
True
```

laplacian(*metric=None*)

Return the Laplacian of `self` with respect to a given metric (Laplace-Beltrami operator).

The *Laplacian* of a scalar field f with respect to a metric g is the scalar field

$$\Delta f = g^{ij}\nabla_i\nabla_j f = \nabla_i\nabla^i f$$

where ∇ is the Levi-Civita connection of g. Δ is also called the *Laplace-Beltrami operator*.

INPUT:

- `metric` – (default: `None`) the pseudo-Riemannian metric g involved in the definition of the Laplacian; if none is provided, the domain of `self` is supposed to be endowed with a default metric (i.e. is supposed to be pseudo-Riemannian manifold, see `PseudoRiemannianManifold`) and the latter is used to define the Laplacian

OUTPUT:

- instance of `DiffScalarField` representing the Laplacian of `self`

EXAMPLES:

Laplacian of a scalar field on the Euclidean plane:

```
sage: M.<x,y> = EuclideanSpace()
sage: f = M.scalar_field(function('F')(x,y), name='f')
sage: s = f.laplacian(); s
```

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```
Scalar field Delta(f) on the Euclidean plane E^2
sage: s.display()
Delta(f): E^2 → ℝ
   (x, y) ↦ d^2(F)/dx^2 + d^2(F)/dy^2
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y',)); (x, y,) = M._first_ngens(2)
>>> f = M.scalar_field(function('F')(x,y), name='f')
>>> s = f.laplacian(); s
Scalar field Delta(f) on the Euclidean plane E^2
>>> s.display()
Delta(f): E^2 → ℝ
   (x, y) ↦ d^2(F)/dx^2 + d^2(F)/dy^2
```

The function `laplacian()` from the `operators` module can be used instead of the method `laplacian()`:

```
sage: from sage.manifolds.operators import laplacian
sage: laplacian(f) == s
True
```

```
>>> from sage.all import *
>>> from sage.manifolds.operators import laplacian
>>> laplacian(f) == s
True
```

The Laplacian can be taken with respect to a metric tensor that is not the default one:

```
sage: h = M.lorentzian_metric('h')
sage: h[1,1], h[2,2] = -1, 1/(1+x^2+y^2)
sage: s = f.laplacian(h); s
Scalar field Delta_h(f) on the Euclidean plane E^2
sage: s.display()
Delta_h(f): E^2 → ℝ
   (x, y) ↦ (y^4*d^2(F)/dy^2 + y^3*d(F)/dy
   + (2*(x^2 + 1)*d^2(F)/dy^2 - d^2(F)/dx^2)*y^2
   + (x^2 + 1)*y*d(F)/dy + x*d(F)/dx - (x^2 + 1)*d^2(F)/dx^2
   + (x^4 + 2*x^2 + 1)*d^2(F)/dy^2)/(x^2 + y^2 + 1)
```

```
>>> from sage.all import *
>>> h = M.lorentzian_metric('h')
>>> h[Integer(1),Integer(1)], h[Integer(2),Integer(2)] = -Integer(1),␣
↪Integer(1)/(Integer(1)+x**Integer(2)+y**Integer(2))
>>> s = f.laplacian(h); s
Scalar field Delta_h(f) on the Euclidean plane E^2
>>> s.display()
Delta_h(f): E^2 → ℝ
   (x, y) ↦ (y^4*d^2(F)/dy^2 + y^3*d(F)/dy
   + (2*(x^2 + 1)*d^2(F)/dy^2 - d^2(F)/dx^2)*y^2
   + (x^2 + 1)*y*d(F)/dy + x*d(F)/dx - (x^2 + 1)*d^2(F)/dx^2
   + (x^4 + 2*x^2 + 1)*d^2(F)/dy^2)/(x^2 + y^2 + 1)
```

The Laplacian of f is equal to the divergence of the gradient of f:

$$\Delta f = \mathrm{div}(\mathrm{grad}\, f)$$

Let us check this formula:

```
sage: s == f.gradient(h).div(h)
True
```

```
>>> from sage.all import *
>>> s == f.gradient(h).div(h)
True
```

lie_der(*vector*)

Compute the Lie derivative with respect to a vector field.

In the present case (scalar field), the Lie derivative is equal to the scalar field resulting from the action of the vector field on the scalar field.

INPUT:

- `vector` – vector field with respect to which the Lie derivative is to be taken

OUTPUT:

- the scalar field that is the Lie derivative of the scalar field with respect to `vector`

EXAMPLES:

Lie derivative on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(x^2*cos(y))
sage: v = M.vector_field(name='v')
sage: v[:] = (-y, x)
sage: f.lie_derivative(v)
Scalar field on the 2-dimensional differentiable manifold M
sage: f.lie_derivative(v).expr()
-x^3*sin(y) - 2*x*y*cos(y)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> f = M.scalar_field(x**Integer(2)*cos(y))
>>> v = M.vector_field(name='v')
>>> v[:] = (-y, x)
>>> f.lie_derivative(v)
Scalar field on the 2-dimensional differentiable manifold M
>>> f.lie_derivative(v).expr()
-x^3*sin(y) - 2*x*y*cos(y)
```

The result is cached:

```
sage: f.lie_derivative(v) is f.lie_derivative(v)
True
```

```
>>> from sage.all import *
>>> f.lie_derivative(v) is f.lie_derivative(v)
True
```

An alias is `lie_der`:

```
sage: f.lie_der(v) is f.lie_derivative(v)
True
```

```
>>> from sage.all import *
>>> f.lie_der(v) is f.lie_derivative(v)
True
```

Alternative expressions of the Lie derivative of a scalar field:

```
sage: f.lie_der(v) == v(f)   # the vector acting on f
True
sage: f.lie_der(v) == f.differential()(v)   # the differential of f acting on␣
↪the vector
True
```

```
>>> from sage.all import *
>>> f.lie_der(v) == v(f)   # the vector acting on f
True
>>> f.lie_der(v) == f.differential()(v)   # the differential of f acting on␣
↪the vector
True
```

A vanishing Lie derivative:

```
sage: f.set_expr(x^2 + y^2)
sage: f.lie_der(v).display()
M → ℝ
(x, y) ↦ 0
```

```
>>> from sage.all import *
>>> f.set_expr(x**Integer(2) + y**Integer(2))
>>> f.lie_der(v).display()
M → ℝ
(x, y) ↦ 0
```

lie_derivative(*vector*)

Compute the Lie derivative with respect to a vector field.

In the present case (scalar field), the Lie derivative is equal to the scalar field resulting from the action of the vector field on the scalar field.

INPUT:

- `vector` – vector field with respect to which the Lie derivative is to be taken

OUTPUT:

- the scalar field that is the Lie derivative of the scalar field with respect to `vector`

EXAMPLES:

Lie derivative on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(x^2*cos(y))
sage: v = M.vector_field(name='v')
sage: v[:] = (-y, x)
sage: f.lie_derivative(v)
Scalar field on the 2-dimensional differentiable manifold M
sage: f.lie_derivative(v).expr()
-x^3*sin(y) - 2*x*y*cos(y)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> f = M.scalar_field(x**Integer(2)*cos(y))
>>> v = M.vector_field(name='v')
>>> v[:] = (-y, x)
>>> f.lie_derivative(v)
Scalar field on the 2-dimensional differentiable manifold M
>>> f.lie_derivative(v).expr()
-x^3*sin(y) - 2*x*y*cos(y)
```

The result is cached:

```
sage: f.lie_derivative(v) is f.lie_derivative(v)
True
```

```
>>> from sage.all import *
>>> f.lie_derivative(v) is f.lie_derivative(v)
True
```

An alias is `lie_der`:

```
sage: f.lie_der(v) is f.lie_derivative(v)
True
```

```
>>> from sage.all import *
>>> f.lie_der(v) is f.lie_derivative(v)
True
```

Alternative expressions of the Lie derivative of a scalar field:

```
sage: f.lie_der(v) == v(f)  # the vector acting on f
True
sage: f.lie_der(v) == f.differential()(v)  # the differential of f acting on␣
↪the vector
True
```

```
>>> from sage.all import *
>>> f.lie_der(v) == v(f)  # the vector acting on f
```

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```
True
>>> f.lie_der(v) == f.differential()(v)  # the differential of f acting on␣
↪the vector
True
```

A vanishing Lie derivative:

```
sage: f.set_expr(x^2 + y^2)
sage: f.lie_der(v).display()
M → ℝ
(x, y) ↦ 0
```

```
>>> from sage.all import *
>>> f.set_expr(x**Integer(2) + y**Integer(2))
>>> f.lie_der(v).display()
M → ℝ
(x, y) ↦ 0
```

tensor_type()

Return the tensor type of `self`, when the latter is considered as a tensor field on the manifold. This is always $(0, 0)$.

OUTPUT:

- always $(0, 0)$

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(x+2*y)
sage: f.tensor_type()
(0, 0)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> f = M.scalar_field(x+Integer(2)*y)
>>> f.tensor_type()
(0, 0)
```

wedge(*other*)

Return the exterior product of `self`, considered as a differential form of degree 0 or a multivector field of degree 0, with `other`.

See *wedge()* (exterior product of differential forms) or *wedge()* (exterior product of multivector fields) for details.

For a scalar field f and a p-form (or p-vector field) a, the exterior product reduces to the standard product on the left by an element of the base ring of the module of p-forms (or p-vector fields): $f \wedge a = fa$.

INPUT:

- `other` – a differential form or a multivector field a

OUTPUT: the product fa, where f is `self`

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x+y^2}, name='f')
sage: a = M.diff_form(2, name='a')
sage: a[0,1] = x*y
sage: s = f.wedge(a); s
2-form f*a on the 2-dimensional differentiable manifold M
sage: s.display()
f*a = (x*y^3 + x^2*y) dx∧dy
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field({X: x+y**Integer(2)}, name='f')
>>> a = M.diff_form(Integer(2), name='a')
>>> a[Integer(0),Integer(1)] = x*y
>>> s = f.wedge(a); s
2-form f*a on the 2-dimensional differentiable manifold M
>>> s.display()
f*a = (x*y^3 + x^2*y) dx∧dy
```

2.5 Differentiable Maps and Curves

2.5.1 Sets of Morphisms between Differentiable Manifolds

The class `DifferentiableManifoldHomset` implements sets of morphisms between two differentiable manifolds over the same topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$), a morphism being a *differentiable map* for the category of differentiable manifolds.

The subclass `DifferentiableCurveSet` is devoted to the specific case of differential curves, i.e. morphisms whose domain is an open interval of $\mathbf{R}$.

The subclass `IntegratedCurveSet` is devoted to differentiable curves that are defined as a solution to a system of second order differential equations.

The subclass `IntegratedAutoparallelCurveSet` is devoted to differentiable curves that are defined as autoparallel curves with respect to a certain affine connection.

The subclass `IntegratedGeodesicSet` is devoted to differentiable curves that are defined as geodesics with respect to a certain metric.

AUTHORS:

- Eric Gourgoulhon (2015): initial version
- Travis Scrimshaw (2016): review tweaks
- Karim Van Aelst (2017): sets of integrated curves

REFERENCES:

- [Lee2013]
- [KN1963]

class sage.manifolds.differentiable.manifold_homset.**DifferentiableCurveSet**(*domain*, *codomain*, *name=None*, *latex_name=None*)

Bases: `DifferentiableManifoldHomset`

Set of differentiable curves in a differentiable manifold.

Given an open interval I of $\mathbf{R}$ (possibly $I = \mathbf{R}$) and a differentiable manifold M over $\mathbf{R}$, this is the set $\mathrm{Hom}(I, M)$ of morphisms (i.e. differentiable curves) $I \to M$.

INPUT:

- `domain` – `OpenInterval` if an open interval $I \subset \mathbf{R}$ (domain of the morphisms), or `RealLine` if $I = \mathbf{R}$
- `codomain` – `DifferentiableManifold`; differentiable manifold M (codomain of the morphisms)
- `name` – (default: `None`) string; name given to the set of curves; if `None`, `Hom(I, M)` will be used
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the set of curves; if `None`, $\mathrm{Hom}(I, M)$ will be used

EXAMPLES:

Set of curves $\mathbf{R} \longrightarrow M$, where M is a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: R.<t> = manifolds.RealLine() ; R
Real number line ℝ
sage: H = Hom(R, M) ; H
Set of Morphisms from Real number line ℝ to 2-dimensional
 differentiable manifold M in Category of smooth manifolds over Real
 Field with 53 bits of precision
sage: H.category()
Category of homsets of topological spaces
sage: latex(H)
\mathrm{Hom}\left(\Bold{R},M\right)
sage: H.domain()
Real number line ℝ
sage: H.codomain()
2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1); R
Real number line ℝ
>>> H = Hom(R, M) ; H
Set of Morphisms from Real number line ℝ to 2-dimensional
 differentiable manifold M in Category of smooth manifolds over Real
 Field with 53 bits of precision
>>> H.category()
Category of homsets of topological spaces
>>> latex(H)
\mathrm{Hom}\left(\Bold{R},M\right)
>>> H.domain()
```

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```
Real number line ℝ
>>> H.codomain()
2-dimensional differentiable manifold M
```

An element of `H` is a curve in `M`:

```
sage: c = H.an_element(); c
Curve in the 2-dimensional differentiable manifold M
sage: c.display()
ℝ → M
   t ↦ (x, y) = (1/(t^2 + 1) - 1/2, 0)
```

```
>>> from sage.all import *
>>> c = H.an_element(); c
Curve in the 2-dimensional differentiable manifold M
>>> c.display()
ℝ → M
   t ↦ (x, y) = (1/(t^2 + 1) - 1/2, 0)
```

The test suite is passed:

```
sage: TestSuite(H).run()
```

```
>>> from sage.all import *
>>> TestSuite(H).run()
```

The set of curves $(0,1) \longrightarrow U$, where U is an open subset of M:

```
sage: I = R.open_interval(0, 1) ; I
Real interval (0, 1)
sage: U = M.open_subset('U', coord_def={X: x^2+y^2<1}) ; U
Open subset U of the 2-dimensional differentiable manifold M
sage: H = Hom(I, U) ; H
Set of Morphisms from Real interval (0, 1) to Open subset U of the
 2-dimensional differentiable manifold M in Join of Category of
 subobjects of sets and Category of smooth manifolds over Real Field
 with 53 bits of precision
```

```
>>> from sage.all import *
>>> I = R.open_interval(Integer(0), Integer(1)) ; I
Real interval (0, 1)
>>> U = M.open_subset('U', coord_def={X: x**Integer(2)+y**Integer(2)<Integer(1)})
↪; U
Open subset U of the 2-dimensional differentiable manifold M
>>> H = Hom(I, U) ; H
Set of Morphisms from Real interval (0, 1) to Open subset U of the
 2-dimensional differentiable manifold M in Join of Category of
 subobjects of sets and Category of smooth manifolds over Real Field
 with 53 bits of precision
```

An element of `H` is a curve in `U`:

```
sage: c = H.an_element() ; c
Curve in the Open subset U of the 2-dimensional differentiable
 manifold M
sage: c.display()
(0, 1) → U
   t ↦ (x, y) = (1/(t^2 + 1) - 1/2, 0)
```

```
>>> from sage.all import *
>>> c = H.an_element() ; c
Curve in the Open subset U of the 2-dimensional differentiable
 manifold M
>>> c.display()
(0, 1) → U
   t ↦ (x, y) = (1/(t^2 + 1) - 1/2, 0)
```

The set of curves $\mathbf{R} \longrightarrow \mathbf{R}$ is a set of (manifold) endomorphisms:

```
sage: E = Hom(R, R) ; E
Set of Morphisms from Real number line ℝ to Real number line ℝ in
 Category of smooth connected manifolds over Real Field with 53 bits of
 precision
sage: E.category()
Category of endsets of topological spaces
sage: E.is_endomorphism_set()
True
sage: E is End(R)
True
```

```
>>> from sage.all import *
>>> E = Hom(R, R) ; E
Set of Morphisms from Real number line ℝ to Real number line ℝ in
 Category of smooth connected manifolds over Real Field with 53 bits of
 precision
>>> E.category()
Category of endsets of topological spaces
>>> E.is_endomorphism_set()
True
>>> E is End(R)
True
```

It is a monoid for the law of morphism composition:

```
sage: E in Monoids()
True
```

```
>>> from sage.all import *
>>> E in Monoids()
True
```

The identity element of the monoid is the identity map of $\mathbf{R}$:

```
sage: E.one()
Identity map Id_ℝ of the Real number line ℝ
```

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```
sage: E.one() is R.identity_map()
True
sage: E.one().display()
Id_ℝ: ℝ → ℝ
   t ↦ t
```

```
>>> from sage.all import *
>>> E.one()
Identity map Id_ℝ of the Real number line ℝ
>>> E.one() is R.identity_map()
True
>>> E.one().display()
Id_ℝ: ℝ → ℝ
   t ↦ t
```

A "typical" element of the monoid:

```
sage: E.an_element().display()
ℝ → ℝ
   t ↦ 1/(t^2 + 1) - 1/2
```

```
>>> from sage.all import *
>>> E.an_element().display()
ℝ → ℝ
   t ↦ 1/(t^2 + 1) - 1/2
```

The test suite is passed by `E`:

```
sage: TestSuite(E).run()
```

```
>>> from sage.all import *
>>> TestSuite(E).run()
```

Similarly, the set of curves $I \longrightarrow I$ is a monoid, whose elements are (manifold) endomorphisms:

```
sage: EI = Hom(I, I) ; EI
Set of Morphisms from Real interval (0, 1) to Real interval (0, 1) in
 Join of Category of subobjects of sets and Category of smooth manifolds
 over Real Field with 53 bits of precision and Category of connected
 manifolds over Real Field with 53 bits of precision
sage: EI.category()
Category of endsets of subobjects of sets and topological spaces
sage: EI is End(I)
True
sage: EI in Monoids()
True
```

```
>>> from sage.all import *
>>> EI = Hom(I, I) ; EI
Set of Morphisms from Real interval (0, 1) to Real interval (0, 1) in
 Join of Category of subobjects of sets and Category of smooth manifolds
```

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```
 over Real Field with 53 bits of precision and Category of connected
 manifolds over Real Field with 53 bits of precision
>>> EI.category()
Category of endsets of subobjects of sets and topological spaces
>>> EI is End(I)
True
>>> EI in Monoids()
True
```

The identity element and a "typical" element of this monoid:

```
sage: EI.one()
Identity map Id_(0, 1) of the Real interval (0, 1)
sage: EI.one().display()
Id_(0, 1): (0, 1) → (0, 1)
   t ↦ t
sage: EI.an_element().display()
(0, 1) → (0, 1)
   t ↦ 1/2/(t^2 + 1) + 1/4
```

```
>>> from sage.all import *
>>> EI.one()
Identity map Id_(0, 1) of the Real interval (0, 1)
>>> EI.one().display()
Id_(0, 1): (0, 1) → (0, 1)
   t ↦ t
>>> EI.an_element().display()
(0, 1) → (0, 1)
   t ↦ 1/2/(t^2 + 1) + 1/4
```

The test suite is passed by `EI`:

```
sage: TestSuite(EI).run()
```

```
>>> from sage.all import *
>>> TestSuite(EI).run()
```

Element

alias of `DifferentiableCurve`

class `sage.manifolds.differentiable.manifold_homset.`**`DifferentiableManifoldHomset`**(*domain*, *codomain*, *name=None*, *latex_name=None*)

Bases: `TopologicalManifoldHomset`

Set of differentiable maps between two differentiable manifolds.

Given two differentiable manifolds M and N over a topological field K, the class `DifferentiableManifoldHomset` implements the set $\mathrm{Hom}(M, N)$ of morphisms (i.e. differentiable maps) $M \to N$.

This is a Sage *parent* class, whose *element* class is `DiffMap`.

INPUT:

- `domain` – differentiable manifold M (domain of the morphisms), as an instance of *DifferentiableManifold*
- `codomain` – differentiable manifold N (codomain of the morphisms), as an instance of *DifferentiableManifold*
- `name` – (default: `None`) string; name given to the homset; if `None`, Hom(M,N) will be used
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the homset; if `None`, $\mathrm{Hom}(M, N)$ will be used

EXAMPLES:

Set of differentiable maps between a 2-dimensional differentiable manifold and a 3-dimensional one:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: N = Manifold(3, 'N')
sage: Y.<u,v,w> = N.chart()
sage: H = Hom(M, N) ; H
Set of Morphisms from 2-dimensional differentiable manifold M to
 3-dimensional differentiable manifold N in Category of smooth
 manifolds over Real Field with 53 bits of precision
sage: type(H)
<class 'sage.manifolds.differentiable.manifold_homset.
↪DifferentiableManifoldHomset_with_category'>
sage: H.category()
Category of homsets of topological spaces
sage: latex(H)
\mathrm{Hom}\left(M,N\right)
sage: H.domain()
2-dimensional differentiable manifold M
sage: H.codomain()
3-dimensional differentiable manifold N
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> N = Manifold(Integer(3), 'N')
>>> Y = N.chart(names=('u', 'v', 'w',)); (u, v, w,) = Y._first_ngens(3)
>>> H = Hom(M, N) ; H
Set of Morphisms from 2-dimensional differentiable manifold M to
 3-dimensional differentiable manifold N in Category of smooth
 manifolds over Real Field with 53 bits of precision
>>> type(H)
<class 'sage.manifolds.differentiable.manifold_homset.
↪DifferentiableManifoldHomset_with_category'>
>>> H.category()
Category of homsets of topological spaces
>>> latex(H)
\mathrm{Hom}\left(M,N\right)
>>> H.domain()
2-dimensional differentiable manifold M
>>> H.codomain()
3-dimensional differentiable manifold N
```

An element of `H` is a differentiable map from `M` to `N`:

```
sage: H.Element
<class 'sage.manifolds.differentiable.diff_map.DiffMap'>
sage: f = H.an_element() ; f
Differentiable map from the 2-dimensional differentiable manifold M to the
 3-dimensional differentiable manifold N
sage: f.display()
M → N
   (x, y) ↦ (u, v, w) = (0, 0, 0)
```

```
>>> from sage.all import *
>>> H.Element
<class 'sage.manifolds.differentiable.diff_map.DiffMap'>
>>> f = H.an_element() ; f
Differentiable map from the 2-dimensional differentiable manifold M to the
 3-dimensional differentiable manifold N
>>> f.display()
M → N
   (x, y) ↦ (u, v, w) = (0, 0, 0)
```

The test suite is passed:

```
sage: TestSuite(H).run()
```

```
>>> from sage.all import *
>>> TestSuite(H).run()
```

When the codomain coincides with the domain, the homset is a set of *endomorphisms* in the category of differentiable manifolds:

```
sage: E = Hom(M, M) ; E
Set of Morphisms from 2-dimensional differentiable manifold M to
 2-dimensional differentiable manifold M in Category of smooth
 manifolds over Real Field with 53 bits of precision
sage: E.category()
Category of endsets of topological spaces
sage: E.is_endomorphism_set()
True
sage: E is End(M)
True
```

```
>>> from sage.all import *
>>> E = Hom(M, M) ; E
Set of Morphisms from 2-dimensional differentiable manifold M to
 2-dimensional differentiable manifold M in Category of smooth
 manifolds over Real Field with 53 bits of precision
>>> E.category()
Category of endsets of topological spaces
>>> E.is_endomorphism_set()
True
>>> E is End(M)
True
```

In this case, the homset is a monoid for the law of morphism composition:

```
sage: E in Monoids()
True
```

```
>>> from sage.all import *
>>> E in Monoids()
True
```

This was of course not the case for `H = Hom(M, N)`:

```
sage: H in Monoids()
False
```

```
>>> from sage.all import *
>>> H in Monoids()
False
```

The identity element of the monoid is of course the identity map of `M`:

```
sage: E.one()
Identity map Id_M of the 2-dimensional differentiable manifold M
sage: E.one() is M.identity_map()
True
sage: E.one().display()
Id_M: M → M
   (x, y) ↦ (x, y)
```

```
>>> from sage.all import *
>>> E.one()
Identity map Id_M of the 2-dimensional differentiable manifold M
>>> E.one() is M.identity_map()
True
>>> E.one().display()
Id_M: M → M
   (x, y) ↦ (x, y)
```

The test suite is passed by `E`:

```
sage: TestSuite(E).run()
```

```
>>> from sage.all import *
>>> TestSuite(E).run()
```

This test suite includes more tests than in the case of `H`, since `E` has some extra structure (monoid).

Element

alias of *DiffMap*

class sage.manifolds.differentiable.manifold_homset.**IntegratedAutoparallelCurveSet**(*domain*, *codomain*, *name=None*, *latex_name=None*)

Bases: `IntegratedCurveSet`

Set of integrated autoparallel curves in a differentiable manifold.

INPUT:

- `domain` – `OpenInterval` open interval $I \subset \mathbf{R}$ with finite boundaries (domain of the morphisms)
- `codomain` – `DifferentiableManifold`; differentiable manifold M (codomain of the morphisms)
- `name` – (default: `None`) string; name given to the set of integrated autoparallel curves; if `None`, `Hom_autoparallel(I, M)` will be used
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the set of integrated autoparallel curves; if `None`, $\mathrm{Hom}_{\mathrm{autoparallel}}(I, M)$ will be used

EXAMPLES:

This parent class needs to be imported:

```
sage: from sage.manifolds.differentiable.manifold_homset import␣
↪IntegratedAutoparallelCurveSet
```

```
>>> from sage.all import *
>>> from sage.manifolds.differentiable.manifold_homset import␣
↪IntegratedAutoparallelCurveSet
```

Integrated autoparallel curves are only allowed to be defined on an interval with finite bounds. This forbids to define an instance of this parent class whose domain has infinite bounds:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: R.<t> = manifolds.RealLine()
sage: H = IntegratedAutoparallelCurveSet(R, M)
Traceback (most recent call last):
...
ValueError: both boundaries of the interval defining the domain
 of a Homset of integrated autoparallel curves need to be finite
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1)
>>> H = IntegratedAutoparallelCurveSet(R, M)
Traceback (most recent call last):
...
ValueError: both boundaries of the interval defining the domain
 of a Homset of integrated autoparallel curves need to be finite
```

An instance whose domain is an interval with finite bounds allows to build a curve that is autoparallel with respect to a connection defined on the codomain:

```
sage: I = R.open_interval(-1, 2)
sage: H = IntegratedAutoparallelCurveSet(I, M) ; H
Set of Morphisms from Real interval (-1, 2) to 2-dimensional
 differentiable manifold M in Category of homsets of topological spaces
```

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```
 which actually are integrated autoparallel curves with respect to a
 certain affine connection
sage: nab = M.affine_connection('nabla')
sage: nab[0,1,0], nab[0,0,1] = 1,2
sage: nab.torsion()[:]
[[[0, -1], [1, 0]], [[0, 0], [0, 0]]]
sage: t = var('t')
sage: p = M.point((3,4))
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,2))
sage: c = H(nab, t, v, name='c') ; c
Integrated autoparallel curve c in the 2-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> I = R.open_interval(-Integer(1), Integer(2))
>>> H = IntegratedAutoparallelCurveSet(I, M) ; H
Set of Morphisms from Real interval (-1, 2) to 2-dimensional
 differentiable manifold M in Category of homsets of topological spaces
 which actually are integrated autoparallel curves with respect to a
 certain affine connection
>>> nab = M.affine_connection('nabla')
>>> nab[Integer(0),Integer(1),Integer(0)], nab[Integer(0),Integer(0),Integer(1)]␣
↪= Integer(1),Integer(2)
>>> nab.torsion()[:]
[[[0, -1], [1, 0]], [[0, 0], [0, 0]]]
>>> t = var('t')
>>> p = M.point((Integer(3),Integer(4)))
>>> Tp = M.tangent_space(p)
>>> v = Tp((Integer(1),Integer(2)))
>>> c = H(nab, t, v, name='c') ; c
Integrated autoparallel curve c in the 2-dimensional
 differentiable manifold M
```

A “typical” element of `H` is an autoparallel curve in `M`:

```
sage: d = H.an_element(); d
Integrated autoparallel curve in the 2-dimensional
 differentiable manifold M
sage: sys = d.system(verbose=True)
Autoparallel curve in the 2-dimensional differentiable manifold
 M equipped with Affine connection nab on the 2-dimensional
 differentiable manifold M, and integrated over the Real
 interval (-1, 2) as a solution to the following equations,
 written with respect to Chart (M, (x, y)):

Initial point: Point on the 2-dimensional differentiable
 manifold M with coordinates [0, -1/2] with respect to
 Chart (M, (x, y))
Initial tangent vector: Tangent vector at Point on the
 2-dimensional differentiable manifold M with components
 [-1/6/(e^(-1) - 1), 1/3] with respect to Chart (M, (x, y))
```

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```
d(x)/dt = Dx
d(y)/dt = Dy
d(Dx)/dt = -Dx*Dy
d(Dy)/dt = 0
```

```
>>> from sage.all import *
>>> d = H.an_element(); d
Integrated autoparallel curve in the 2-dimensional
 differentiable manifold M
>>> sys = d.system(verbose=True)
Autoparallel curve in the 2-dimensional differentiable manifold
 M equipped with Affine connection nab on the 2-dimensional
 differentiable manifold M, and integrated over the Real
 interval (-1, 2) as a solution to the following equations,
 written with respect to Chart (M, (x, y)):
<BLANKLINE>
Initial point: Point on the 2-dimensional differentiable
 manifold M with coordinates [0, -1/2] with respect to
 Chart (M, (x, y))
Initial tangent vector: Tangent vector at Point on the
 2-dimensional differentiable manifold M with components
 [-1/6/(e^(-1) - 1), 1/3] with respect to Chart (M, (x, y))
<BLANKLINE>
d(x)/dt = Dx
d(y)/dt = Dy
d(Dx)/dt = -Dx*Dy
d(Dy)/dt = 0
<BLANKLINE>
```

The test suite is passed:

```
sage: TestSuite(H).run()
```

```
>>> from sage.all import *
>>> TestSuite(H).run()
```

For any open interval J with finite bounds (a, b), all curves are autoparallel with respect to any connection. Therefore, the set of autoparallel curves $J \longrightarrow J$ is a set of numerical (manifold) endomorphisms that is a monoid for the law of morphism composition:

```
sage: [a,b] = var('a b')
sage: J = R.open_interval(a, b)
sage: H = IntegratedAutoparallelCurveSet(J, J); H
Set of Morphisms from Real interval (a, b) to Real interval
 (a, b) in Category of endsets of subobjects of sets and
 topological spaces which actually are integrated autoparallel
 curves with respect to a certain affine connection
sage: H.category()
Category of endsets of subobjects of sets and topological spaces
sage: H in Monoids()
True
```

```
>>> from sage.all import *
>>> [a,b] = var('a b')
>>> J = R.open_interval(a, b)
>>> H = IntegratedAutoparallelCurveSet(J, J); H
Set of Morphisms from Real interval (a, b) to Real interval
 (a, b) in Category of endsets of subobjects of sets and
 topological spaces which actually are integrated autoparallel
 curves with respect to a certain affine connection
>>> H.category()
Category of endsets of subobjects of sets and topological spaces
>>> H in Monoids()
True
```

Although it is a monoid, no identity map is implemented via the `one` method of this class or its subclass devoted to geodesics. This is justified by the lack of relevance of the identity map within the framework of this parent class and its subclass, whose purpose is mainly devoted to numerical issues (therefore, the user is left free to set a numerical version of the identity if needed):

```
sage: H.one()
Traceback (most recent call last):
...
ValueError: the identity is not implemented for integrated
 curves and associated subclasses
```

```
>>> from sage.all import *
>>> H.one()
Traceback (most recent call last):
...
ValueError: the identity is not implemented for integrated
 curves and associated subclasses
```

A "typical" element of the monoid:

```
sage: g = H.an_element() ; g
Integrated autoparallel curve in the Real interval (a, b)
sage: sys = g.system(verbose=True)
Autoparallel curve in the Real interval (a, b) equipped with
 Affine connection nab on the Real interval (a, b), and
 integrated over the Real interval (a, b) as a solution to the
 following equations, written with respect to Chart ((a, b), (t,)):

Initial point: Point on the Real number line ℝ with coordinates
 [0] with respect to Chart ((a, b), (t,))
Initial tangent vector: Tangent vector at Point on the Real
 number line ℝ with components
 [-(e^(1/2) - 1)/(a - b)] with respect to
 Chart ((a, b), (t,))

d(t)/ds = Dt
d(Dt)/ds = -Dt^2
```

```
>>> from sage.all import *
>>> g = H.an_element() ; g
```

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```
Integrated autoparallel curve in the Real interval (a, b)
>>> sys = g.system(verbose=True)
Autoparallel curve in the Real interval (a, b) equipped with
 Affine connection nab on the Real interval (a, b), and
 integrated over the Real interval (a, b) as a solution to the
 following equations, written with respect to Chart ((a, b), (t,)):
<BLANKLINE>
Initial point: Point on the Real number line ℝ with coordinates
 [0] with respect to Chart ((a, b), (t,))
Initial tangent vector: Tangent vector at Point on the Real
 number line ℝ with components
 [-(e^(1/2) - 1)/(a - b)] with respect to
 Chart ((a, b), (t,))
<BLANKLINE>
d(t)/ds = Dt
d(Dt)/ds = -Dt^2
<BLANKLINE>
```

The test suite is passed, tests `_test_one` and `_test_prod` being skipped for reasons mentioned above:

```
sage: TestSuite(H).run(skip=["_test_one", "_test_prod"])
```

```
>>> from sage.all import *
>>> TestSuite(H).run(skip=["_test_one", "_test_prod"])
```

Element

alias of *IntegratedAutoparallelCurve*

class sage.manifolds.differentiable.manifold_homset.**IntegratedCurveSet**(*domain*, *codomain*, *name=None*, *latex_name=None*)

Bases: *DifferentiableCurveSet*

Set of integrated curves in a differentiable manifold.

INPUT:

- `domain` – *OpenInterval* open interval $I \subset \mathbf{R}$ with finite boundaries (domain of the morphisms)
- `codomain` – *DifferentiableManifold*; differentiable manifold M (codomain of the morphisms)
- `name` – (default: `None`) string; name given to the set of integrated curves; if `None`, `Hom_integrated(I, M)` will be used
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the set of integrated curves; if `None`, $\mathrm{Hom}_{\mathrm{integrated}}(I, M)$ will be used

EXAMPLES:

This parent class needs to be imported:

```
sage: from sage.manifolds.differentiable.manifold_homset import IntegratedCurveSet
```

```
>>> from sage.all import *
>>> from sage.manifolds.differentiable.manifold_homset import IntegratedCurveSet
```

Integrated curves are only allowed to be defined on an interval with finite bounds. This forbids to define an instance of this parent class whose domain has infinite bounds:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: R.<t> = manifolds.RealLine()
sage: H = IntegratedCurveSet(R, M)
Traceback (most recent call last):
...
ValueError: both boundaries of the interval defining the domain
 of a Homset of integrated curves need to be finite
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1)
>>> H = IntegratedCurveSet(R, M)
Traceback (most recent call last):
...
ValueError: both boundaries of the interval defining the domain
 of a Homset of integrated curves need to be finite
```

An instance whose domain is an interval with finite bounds allows to build an integrated curve defined on the interval:

```
sage: I = R.open_interval(-1, 2)
sage: H = IntegratedCurveSet(I, M) ; H
Set of Morphisms from Real interval (-1, 2) to 2-dimensional
 differentiable manifold M in Category of homsets of topological spaces
 which actually are integrated curves
sage: eqns_rhs = [1,1]
sage: vels = X.symbolic_velocities()
sage: t = var('t')
sage: p = M.point((3,4))
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,2))
sage: c = H(eqns_rhs, vels, t, v, name='c') ; c
Integrated curve c in the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> I = R.open_interval(-Integer(1), Integer(2))
>>> H = IntegratedCurveSet(I, M) ; H
Set of Morphisms from Real interval (-1, 2) to 2-dimensional
 differentiable manifold M in Category of homsets of topological spaces
 which actually are integrated curves
>>> eqns_rhs = [Integer(1),Integer(1)]
>>> vels = X.symbolic_velocities()
>>> t = var('t')
>>> p = M.point((Integer(3),Integer(4)))
>>> Tp = M.tangent_space(p)
>>> v = Tp((Integer(1),Integer(2)))
>>> c = H(eqns_rhs, vels, t, v, name='c') ; c
```

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```
Integrated curve c in the 2-dimensional differentiable
 manifold M
```

A “typical” element of `H` is a curve in `M`:

```
sage: d = H.an_element(); d
Integrated curve in the 2-dimensional differentiable manifold M
sage: sys = d.system(verbose=True)
Curve in the 2-dimensional differentiable manifold M integrated
 over the Real interval (-1, 2) as a solution to the following
 system, written with respect to Chart (M, (x, y)):

Initial point: Point on the 2-dimensional differentiable
 manifold M with coordinates [0, 0] with respect to Chart (M, (x, y))
Initial tangent vector: Tangent vector at Point on the
 2-dimensional differentiable manifold M with components
 [1/4, 0] with respect to Chart (M, (x, y))

d(x)/dt = Dx
d(y)/dt = Dy
d(Dx)/dt = -1/4*sin(t + 1)
d(Dy)/dt = 0
```

```
>>> from sage.all import *
>>> d = H.an_element(); d
Integrated curve in the 2-dimensional differentiable manifold M
>>> sys = d.system(verbose=True)
Curve in the 2-dimensional differentiable manifold M integrated
 over the Real interval (-1, 2) as a solution to the following
 system, written with respect to Chart (M, (x, y)):
<BLANKLINE>
Initial point: Point on the 2-dimensional differentiable
 manifold M with coordinates [0, 0] with respect to Chart (M, (x, y))
Initial tangent vector: Tangent vector at Point on the
 2-dimensional differentiable manifold M with components
 [1/4, 0] with respect to Chart (M, (x, y))
<BLANKLINE>
d(x)/dt = Dx
d(y)/dt = Dy
d(Dx)/dt = -1/4*sin(t + 1)
d(Dy)/dt = 0
<BLANKLINE>
```

The test suite is passed:

```
sage: TestSuite(H).run()
```

```
>>> from sage.all import *
>>> TestSuite(H).run()
```

More generally, an instance of this class may be defined with abstract bounds (a, b):

```
sage: [a,b] = var('a b')
sage: J = R.open_interval(a, b)
sage: H = IntegratedCurveSet(J, M) ; H
Set of Morphisms from Real interval (a, b) to 2-dimensional
 differentiable manifold M in Category of homsets of topological spaces
 which actually are integrated curves
```

```
>>> from sage.all import *
>>> [a,b] = var('a b')
>>> J = R.open_interval(a, b)
>>> H = IntegratedCurveSet(J, M) ; H
Set of Morphisms from Real interval (a, b) to 2-dimensional
 differentiable manifold M in Category of homsets of topological spaces
 which actually are integrated curves
```

A "typical" element of `H` is a curve in `M`:

```
sage: f = H.an_element(); f
Integrated curve in the 2-dimensional differentiable manifold M
sage: sys = f.system(verbose=True)
Curve in the 2-dimensional differentiable manifold M integrated
 over the Real interval (a, b) as a solution to the following
 system, written with respect to Chart (M, (x, y)):

Initial point: Point on the 2-dimensional differentiable
 manifold M with coordinates [0, 0] with respect to Chart (M, (x, y))
Initial tangent vector: Tangent vector at Point on the
 2-dimensional differentiable manifold M with components
 [1/4, 0] with respect to Chart (M, (x, y))

d(x)/dt = Dx
d(y)/dt = Dy
d(Dx)/dt = -1/4*sin(-a + t)
d(Dy)/dt = 0
```

```
>>> from sage.all import *
>>> f = H.an_element(); f
Integrated curve in the 2-dimensional differentiable manifold M
>>> sys = f.system(verbose=True)
Curve in the 2-dimensional differentiable manifold M integrated
 over the Real interval (a, b) as a solution to the following
 system, written with respect to Chart (M, (x, y)):
<BLANKLINE>
Initial point: Point on the 2-dimensional differentiable
 manifold M with coordinates [0, 0] with respect to Chart (M, (x, y))
Initial tangent vector: Tangent vector at Point on the
 2-dimensional differentiable manifold M with components
 [1/4, 0] with respect to Chart (M, (x, y))
<BLANKLINE>
d(x)/dt = Dx
d(y)/dt = Dy
d(Dx)/dt = -1/4*sin(-a + t)
```

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```
d(Dy)/dt = 0
<BLANKLINE>
```

Yet, even in the case of abstract bounds, considering any of them to be infinite is still prohibited since no numerical integration could be performed:

```
sage: f.solve(parameters_values={a:-1, b:+oo})
Traceback (most recent call last):
...
ValueError: both boundaries of the interval need to be finite
```

```
>>> from sage.all import *
>>> f.solve(parameters_values={a:-Integer(1), b:+oo})
Traceback (most recent call last):
...
ValueError: both boundaries of the interval need to be finite
```

The set of integrated curves $J \longrightarrow J$ is a set of numerical (manifold) endomorphisms:

```
sage: H = IntegratedCurveSet(J, J); H
Set of Morphisms from Real interval (a, b) to Real interval
 (a, b) in Category of endsets of subobjects of sets and
 topological spaces which actually are integrated curves
sage: H.category()
Category of endsets of subobjects of sets and topological spaces
```

```
>>> from sage.all import *
>>> H = IntegratedCurveSet(J, J); H
Set of Morphisms from Real interval (a, b) to Real interval
 (a, b) in Category of endsets of subobjects of sets and
 topological spaces which actually are integrated curves
>>> H.category()
Category of endsets of subobjects of sets and topological spaces
```

It is a monoid for the law of morphism composition:

```
sage: H in Monoids()
True
```

```
>>> from sage.all import *
>>> H in Monoids()
True
```

Although it is a monoid, no identity map is implemented via the `one` method of this class or any of its subclasses. This is justified by the lack of relevance of the identity map within the framework of this parent class and its subclasses, whose purpose is mainly devoted to numerical issues (therefore, the user is left free to set a numerical version of the identity if needed):

```
sage: H.one()
Traceback (most recent call last):
...
```

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```
ValueError: the identity is not implemented for integrated
 curves and associated subclasses
```

```
>>> from sage.all import *
>>> H.one()
Traceback (most recent call last):
...
ValueError: the identity is not implemented for integrated
 curves and associated subclasses
```

A “typical” element of the monoid:

```
sage: g = H.an_element() ; g
Integrated curve in the Real interval (a, b)
sage: sys = g.system(verbose=True)
Curve in the Real interval (a, b) integrated over the Real
 interval (a, b) as a solution to the following system, written
 with respect to Chart ((a, b), (t,)):

Initial point: Point on the Real number line ℝ with coordinates
 [0] with respect to Chart ((a, b), (t,))
Initial tangent vector: Tangent vector at Point on the Real
 number line ℝ with components [1/4] with respect to
 Chart ((a, b), (t,))

d(t)/ds = Dt
d(Dt)/ds = -1/4*sin(-a + s)
```

```
>>> from sage.all import *
>>> g = H.an_element() ; g
Integrated curve in the Real interval (a, b)
>>> sys = g.system(verbose=True)
Curve in the Real interval (a, b) integrated over the Real
 interval (a, b) as a solution to the following system, written
 with respect to Chart ((a, b), (t,)):
<BLANKLINE>
Initial point: Point on the Real number line ℝ with coordinates
 [0] with respect to Chart ((a, b), (t,))
Initial tangent vector: Tangent vector at Point on the Real
 number line ℝ with components [1/4] with respect to
 Chart ((a, b), (t,))
<BLANKLINE>
d(t)/ds = Dt
d(Dt)/ds = -1/4*sin(-a + s)
<BLANKLINE>
```

The test suite is passed, tests `_test_one` and `_test_prod` being skipped for reasons mentioned above:

```
sage: TestSuite(H).run(skip=["_test_one", "_test_prod"])
```

```
>>> from sage.all import *
>>> TestSuite(H).run(skip=["_test_one", "_test_prod"])
```

Element

alias of *IntegratedCurve*

one()

Raise an error refusing to provide the identity element. This overrides the `one` method of class *TopologicalManifoldHomset*, which would actually raise an error as well, due to lack of option `is_identity` in `element_constructor` method of `self`.

class sage.manifolds.differentiable.manifold_homset.**IntegratedGeodesicSet**(*domain*, *codomain*, *name=None*, *latex_name=None*)

Bases: *IntegratedAutoparallelCurveSet*

Set of integrated geodesic in a differentiable manifold.

INPUT:

- `domain` – *OpenInterval* open interval $I \subset \mathbf{R}$ with finite boundaries (domain of the morphisms)
- `codomain` – *DifferentiableManifold*; differentiable manifold M (codomain of the morphisms)
- `name` – (default: `None`) string; name given to the set of integrated geodesics; if `None`, `Hom_geodesic(I, M)` will be used
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the set of integrated geodesics; if `None`, $\mathrm{Hom}_{\mathrm{geodesic}}(I, M)$ will be used

EXAMPLES:

This parent class needs to be imported:

```
sage: from sage.manifolds.differentiable.manifold_homset import␣
↪IntegratedGeodesicSet
```

```
>>> from sage.all import *
>>> from sage.manifolds.differentiable.manifold_homset import␣
↪IntegratedGeodesicSet
```

Integrated geodesics are only allowed to be defined on an interval with finite bounds. This forbids to define an instance of this parent class whose domain has infinite bounds:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: R.<t> = manifolds.RealLine()
sage: H = IntegratedGeodesicSet(R, M)
Traceback (most recent call last):
...
ValueError: both boundaries of the interval defining the domain
 of a Homset of integrated geodesics need to be finite
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1)
>>> H = IntegratedGeodesicSet(R, M)
Traceback (most recent call last):
```

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```
...
ValueError: both boundaries of the interval defining the domain
 of a Homset of integrated geodesics need to be finite
```

An instance whose domain is an interval with finite bounds allows to build a geodesic with respect to a metric defined on the codomain:

```
sage: I = R.open_interval(-1, 2)
sage: H = IntegratedGeodesicSet(I, M) ; H
Set of Morphisms from Real interval (-1, 2) to 2-dimensional
 differentiable manifold M in Category of homsets of topological spaces
 which actually are integrated geodesics with respect to a certain
 metric
sage: g = M.metric('g')
sage: g[0,0], g[1,1], g[0,1] = 1, 1, 2
sage: t = var('t')
sage: p = M.point((3,4))
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,2))
sage: c = H(g, t, v, name='c') ; c
Integrated geodesic c in the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> I = R.open_interval(-Integer(1), Integer(2))
>>> H = IntegratedGeodesicSet(I, M) ; H
Set of Morphisms from Real interval (-1, 2) to 2-dimensional
 differentiable manifold M in Category of homsets of topological spaces
 which actually are integrated geodesics with respect to a certain
 metric
>>> g = M.metric('g')
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(0),Integer(1)]␣
↪= Integer(1), Integer(1), Integer(2)
>>> t = var('t')
>>> p = M.point((Integer(3),Integer(4)))
>>> Tp = M.tangent_space(p)
>>> v = Tp((Integer(1),Integer(2)))
>>> c = H(g, t, v, name='c') ; c
Integrated geodesic c in the 2-dimensional differentiable
 manifold M
```

A "typical" element of `H` is a geodesic in `M`:

```
sage: d = H.an_element(); d
Integrated geodesic in the 2-dimensional differentiable
 manifold M
sage: sys = d.system(verbose=True)
Geodesic in the 2-dimensional differentiable manifold M equipped
 with Riemannian metric g on the 2-dimensional differentiable
 manifold M, and integrated over the Real interval (-1, 2) as a
 solution to the following geodesic equations, written
 with respect to Chart (M, (x, y)):
```

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```
Initial point: Point on the 2-dimensional differentiable
 manifold M with coordinates [0, 0] with respect to
 Chart (M, (x, y))
Initial tangent vector: Tangent vector at Point on the
 2-dimensional differentiable manifold M with components
 [1/3*e^(1/2) - 1/3, 0] with respect to Chart (M, (x, y))

d(x)/dt = Dx
d(y)/dt = Dy
d(Dx)/dt = -Dx^2
d(Dy)/dt = 0
```

```
>>> from sage.all import *
>>> d = H.an_element(); d
Integrated geodesic in the 2-dimensional differentiable
 manifold M
>>> sys = d.system(verbose=True)
Geodesic in the 2-dimensional differentiable manifold M equipped
 with Riemannian metric g on the 2-dimensional differentiable
 manifold M, and integrated over the Real interval (-1, 2) as a
 solution to the following geodesic equations, written
 with respect to Chart (M, (x, y)):
<BLANKLINE>
Initial point: Point on the 2-dimensional differentiable
 manifold M with coordinates [0, 0] with respect to
 Chart (M, (x, y))
Initial tangent vector: Tangent vector at Point on the
 2-dimensional differentiable manifold M with components
 [1/3*e^(1/2) - 1/3, 0] with respect to Chart (M, (x, y))
<BLANKLINE>
d(x)/dt = Dx
d(y)/dt = Dy
d(Dx)/dt = -Dx^2
d(Dy)/dt = 0
```

The test suite is passed:

```
sage: TestSuite(H).run()
```

```
>>> from sage.all import *
>>> TestSuite(H).run()
```

For any open interval J with finite bounds (a, b), all curves are geodesics with respect to any metric. Therefore, the set of geodesics $J \longrightarrow J$ is a set of numerical (manifold) endomorphisms that is a monoid for the law of morphism composition:

```
sage: [a,b] = var('a b')
sage: J = R.open_interval(a, b)
sage: H = IntegratedGeodesicSet(J, J); H
Set of Morphisms from Real interval (a, b) to Real interval
 (a, b) in Category of endsets of subobjects of sets and
```

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```
 topological spaces which actually are integrated geodesics
 with respect to a certain metric
sage: H.category()
Category of endsets of subobjects of sets and topological spaces
sage: H in Monoids()
True
```

```
>>> from sage.all import *
>>> [a,b] = var('a b')
>>> J = R.open_interval(a, b)
>>> H = IntegratedGeodesicSet(J, J); H
Set of Morphisms from Real interval (a, b) to Real interval
 (a, b) in Category of endsets of subobjects of sets and
 topological spaces which actually are integrated geodesics
 with respect to a certain metric
>>> H.category()
Category of endsets of subobjects of sets and topological spaces
>>> H in Monoids()
True
```

Although it is a monoid, no identity map is implemented via the `one` method of this class. This is justified by the lack of relevance of the identity map within the framework of this parent class, whose purpose is mainly devoted to numerical issues (therefore, the user is left free to set a numerical version of the identity if needed):

```
sage: H.one()
Traceback (most recent call last):
...
ValueError: the identity is not implemented for integrated
 curves and associated subclasses
```

```
>>> from sage.all import *
>>> H.one()
Traceback (most recent call last):
...
ValueError: the identity is not implemented for integrated
 curves and associated subclasses
```

A "typical" element of the monoid:

```
sage: g = H.an_element() ; g
Integrated geodesic in the Real interval (a, b)
sage: sys = g.system(verbose=True)
Geodesic in the Real interval (a, b) equipped with Riemannian
 metric g on the Real interval (a, b), and integrated over the
 Real interval (a, b) as a solution to the following geodesic
 equations, written with respect to Chart ((a, b), (t,)):

Initial point: Point on the Real number line ℝ with coordinates
 [0] with respect to Chart ((a, b), (t,))
Initial tangent vector: Tangent vector at Point on the Real
 number line ℝ with components [-(e^(1/2) - 1)/(a - b)]
 with respect to Chart ((a, b), (t,))
```

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```
d(t)/ds = Dt
d(Dt)/ds = -Dt^2
```

```
>>> from sage.all import *
>>> g = H.an_element() ; g
Integrated geodesic in the Real interval (a, b)
>>> sys = g.system(verbose=True)
Geodesic in the Real interval (a, b) equipped with Riemannian
 metric g on the Real interval (a, b), and integrated over the
 Real interval (a, b) as a solution to the following geodesic
 equations, written with respect to Chart ((a, b), (t,)):
<BLANKLINE>
Initial point: Point on the Real number line ℝ with coordinates
 [0] with respect to Chart ((a, b), (t,))
Initial tangent vector: Tangent vector at Point on the Real
 number line ℝ with components [-(e^(1/2) - 1)/(a - b)]
 with respect to Chart ((a, b), (t,))
<BLANKLINE>
d(t)/ds = Dt
d(Dt)/ds = -Dt^2
<BLANKLINE>
```

The test suite is passed, tests `_test_one` and `_test_prod` being skipped for reasons mentioned above:

```
sage: TestSuite(H).run(skip=["_test_one", "_test_prod"])
```

```
>>> from sage.all import *
>>> TestSuite(H).run(skip=["_test_one", "_test_prod"])
```

Element

alias of `IntegratedGeodesic`

2.5.2 Differentiable Maps between Differentiable Manifolds

The class `DiffMap` implements differentiable maps from a differentiable manifold M to a differentiable manifold N over the same topological field K as M (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$):

$$\Phi : M \longrightarrow N$$

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015): initial version
- Marco Mancini (2018): pullback parallelization

REFERENCES:

- Chap. 1 of [KN1963]
- Chaps. 2 and 3 of [Lee2013]

class `sage.manifolds.differentiable.diff_map.DiffMap`(*parent*, *coord_functions=None*, *name=None*, *latex_name=None*, *is_isomorphism=False*, *is_identity=False*)

Bases: `ContinuousMap`

Differentiable map between two differentiable manifolds.

This class implements differentiable maps of the type

$$\Phi : M \longrightarrow N$$

where M and N are differentiable manifolds over the same topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$).

Differentiable maps are the *morphisms* of the *category* of differentiable manifolds. The set of all differentiable maps from M to N is therefore the homset between M and N, which is denoted by $\mathrm{Hom}(M, N)$.

The class `DiffMap` is a Sage *element* class, whose *parent* class is `DifferentiableManifoldHomset`. It inherits from the class `ContinuousMap` since a differentiable map is obviously a continuous one.

INPUT:

- `parent` – homset $\mathrm{Hom}(M, N)$ to which the differentiable map belongs
- `coord_functions` – (default: `None`) if not `None`, must be a dictionary of the coordinate expressions (as lists (or tuples) of the coordinates of the image expressed in terms of the coordinates of the considered point) with the pairs of charts (chart1, chart2) as keys (chart1 being a chart on M and chart2 a chart on N). If the dimension of the map's codomain is 1, a single coordinate expression can be passed instead of a tuple with a single element
- `name` – (default: `None`) name given to the differentiable map
- `latex_name` – (default: `None`) LaTeX symbol to denote the differentiable map; if `None`, the LaTeX symbol is set to `name`
- `is_isomorphism` – boolean (default: `False`); determines whether the constructed object is a isomorphism (i.e. a diffeomorphism). If set to `True`, then the manifolds M and N must have the same dimension.
- `is_identity` – boolean (default: `False`); determines whether the constructed object is the identity map. If set to `True`, then N must be M and the entry `coord_functions` is not used.

Note

If the information passed by means of the argument `coord_functions` is not sufficient to fully specify the differentiable map, further coordinate expressions, in other charts, can be subsequently added by means of the method `add_expr()`

EXAMPLES:

The standard embedding of the sphere S^2 into $\mathbf{R}^3$:

```
sage: M = Manifold(2, 'S^2') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
```

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```
sage: N = Manifold(3, 'R^3', r'\RR^3')  # R^3
sage: c_cart.<X,Y,Z> = N.chart()  # Cartesian coordinates on R^3
sage: Phi = M.diff_map(N,
....: {(c_xy, c_cart): [2*x/(1+x^2+y^2), 2*y/(1+x^2+y^2), (x^2+y^2-1)/(1+x^2+y^
↪2)],
....:  (c_uv, c_cart): [2*u/(1+u^2+v^2), 2*v/(1+u^2+v^2), (1-u^2-v^2)/(1+u^2+v^
↪2)]},
....: name='Phi', latex_name=r'\Phi')
sage: Phi
Differentiable map Phi from the 2-dimensional differentiable manifold
 S^2 to the 3-dimensional differentiable manifold R^3
sage: Phi.parent()
Set of Morphisms from 2-dimensional differentiable manifold S^2 to
 3-dimensional differentiable manifold R^3 in Category of smooth
 manifolds over Real Field with 53 bits of precision
sage: Phi.parent() is Hom(M, N)
True
sage: type(Phi)
<class 'sage.manifolds.differentiable.manifold_homset.
↪DifferentiableManifoldHomset_with_category.element_class'>
sage: Phi.display()
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1),
                             (x^2 + y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1),
                             -(u^2 + v^2 - 1)/(u^2 + v^2 + 1))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                     intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                     restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> N = Manifold(Integer(3), 'R^3', r'\RR^3')  # R^3
>>> c_cart = N.chart(names=('X', 'Y', 'Z',)); (X, Y, Z,) = c_cart._first_ngens(3)
↪# Cartesian coordinates on R^3
>>> Phi = M.diff_map(N,
... {(c_xy, c_cart): [Integer(2)*x/(Integer(1)+x**Integer(2)+y**Integer(2)),␣
↪Integer(2)*y/(Integer(1)+x**Integer(2)+y**Integer(2)),␣
↪(x**Integer(2)+y**Integer(2)-Integer(1))/
↪(Integer(1)+x**Integer(2)+y**Integer(2))],
...  (c_uv, c_cart): [Integer(2)*u/(Integer(1)+u**Integer(2)+v**Integer(2)),␣
↪Integer(2)*v/(Integer(1)+u**Integer(2)+v**Integer(2)), (Integer(1)-
```

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```
↪u**Integer(2)-v**Integer(2))/(Integer(1)+u**Integer(2)+v**Integer(2))]},
... name='Phi', latex_name=r'\Phi')
>>> Phi
Differentiable map Phi from the 2-dimensional differentiable manifold
 S^2 to the 3-dimensional differentiable manifold R^3
>>> Phi.parent()
Set of Morphisms from 2-dimensional differentiable manifold S^2 to
 3-dimensional differentiable manifold R^3 in Category of smooth
 manifolds over Real Field with 53 bits of precision
>>> Phi.parent() is Hom(M, N)
True
>>> type(Phi)
<class 'sage.manifolds.differentiable.manifold_homset.
↪DifferentiableManifoldHomset_with_category.element_class'>
>>> Phi.display()
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1),
                             (x^2 + y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1),
                             -(u^2 + v^2 - 1)/(u^2 + v^2 + 1))
```

It is possible to create the map via the method *diff_map()* only in a single pair of charts: the argument `coord_functions` is then a mere list of coordinate expressions (and not a dictionary) and the arguments `chart1` and `chart2` have to be provided if the charts differ from the default ones on the domain and/or the codomain:

```
sage: Phi1 = M.diff_map(N, [2*x/(1+x^2+y^2), 2*y/(1+x^2+y^2),
....:                        (x^2+y^2-1)/(1+x^2+y^2)],
....:                   chart1=c_xy, chart2=c_cart, name='Phi',
....:                   latex_name=r'\Phi')
```

```
>>> from sage.all import *
>>> Phi1 = M.diff_map(N, [Integer(2)*x/(Integer(1)+x**Integer(2)+y**Integer(2)),␣
↪Integer(2)*y/(Integer(1)+x**Integer(2)+y**Integer(2)),
...                        (x**Integer(2)+y**Integer(2)-Integer(1))/
↪(Integer(1)+x**Integer(2)+y**Integer(2))],
...                   chart1=c_xy, chart2=c_cart, name='Phi',
...                   latex_name=r'\Phi')
```

Since `c_xy` and `c_cart` are the default charts on respectively `M` and `N`, they can be omitted, so that the above declaration is equivalent to:

```
sage: Phi1 = M.diff_map(N, [2*x/(1+x^2+y^2), 2*y/(1+x^2+y^2),
....:                        (x^2+y^2-1)/(1+x^2+y^2)],
....:                   name='Phi', latex_name=r'\Phi')
```

```
>>> from sage.all import *
>>> Phi1 = M.diff_map(N, [Integer(2)*x/(Integer(1)+x**Integer(2)+y**Integer(2)),␣
↪Integer(2)*y/(Integer(1)+x**Integer(2)+y**Integer(2)),
...                        (x**Integer(2)+y**Integer(2)-Integer(1))/
↪(Integer(1)+x**Integer(2)+y**Integer(2))],
...                   name='Phi', latex_name=r'\Phi')
```

With such a declaration, the differentiable map is only partially defined on the manifold S^2, being known in only one chart:

```
sage: Phi1.display()
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1),
                             (x^2 + y^2 - 1)/(x^2 + y^2 + 1))
```

```
>>> from sage.all import *
>>> Phi1.display()
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1),
                             (x^2 + y^2 - 1)/(x^2 + y^2 + 1))
```

The definition can be completed by means of the method `add_expr()`:

```
sage: Phi1.add_expr(c_uv, c_cart, [2*u/(1+u^2+v^2), 2*v/(1+u^2+v^2),
....:                              (1-u^2-v^2)/(1+u^2+v^2)])
sage: Phi1.display()
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1),
                             (x^2 + y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1),
                            -(u^2 + v^2 - 1)/(u^2 + v^2 + 1))
```

```
>>> from sage.all import *
>>> Phi1.add_expr(c_uv, c_cart, [Integer(2)*u/
↪(Integer(1)+u**Integer(2)+v**Integer(2)), Integer(2)*v/
↪(Integer(1)+u**Integer(2)+v**Integer(2)),
...                              (Integer(1)-u**Integer(2)-v**Integer(2))/
↪(Integer(1)+u**Integer(2)+v**Integer(2))])
>>> Phi1.display()
Phi: S^2 → R^3
on U: (x, y) ↦ (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1),
                             (x^2 + y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) ↦ (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1),
                            -(u^2 + v^2 - 1)/(u^2 + v^2 + 1))
```

At this stage, `Phi1` and `Phi` are fully equivalent:

```
sage: Phi1 == Phi
True
```

```
>>> from sage.all import *
>>> Phi1 == Phi
True
```

The test suite is passed:

```
sage: TestSuite(Phi).run()
sage: TestSuite(Phi1).run()
```

```
>>> from sage.all import *
>>> TestSuite(Phi).run()
```

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```
>>> TestSuite(Phi1).run()
```

The map acts on points:

```
sage: np = M.point((0,0), chart=c_uv, name='N')  # the North pole
sage: Phi(np)
Point Phi(N) on the 3-dimensional differentiable manifold R^3
sage: Phi(np).coord() # Cartesian coordinates
(0, 0, 1)
sage: sp = M.point((0,0), chart=c_xy, name='S')  # the South pole
sage: Phi(sp).coord() # Cartesian coordinates
(0, 0, -1)
```

```
>>> from sage.all import *
>>> np = M.point((Integer(0),Integer(0)), chart=c_uv, name='N')  # the North pole
>>> Phi(np)
Point Phi(N) on the 3-dimensional differentiable manifold R^3
>>> Phi(np).coord() # Cartesian coordinates
(0, 0, 1)
>>> sp = M.point((Integer(0),Integer(0)), chart=c_xy, name='S')  # the South pole
>>> Phi(sp).coord() # Cartesian coordinates
(0, 0, -1)
```

The differential $\mathrm{d}\Phi$ of the map Φ at the North pole and at the South pole:

```
sage: Phi.differential(np)
Generic morphism:
  From: Tangent space at Point N on the 2-dimensional differentiable manifold S^2
  To:   Tangent space at Point Phi(N) on the 3-dimensional differentiable␣
↪manifold R^3
sage: Phi.differential(sp)
Generic morphism:
  From: Tangent space at Point S on the 2-dimensional differentiable manifold S^2
  To:   Tangent space at Point Phi(S) on the 3-dimensional differentiable␣
↪manifold R^3
```

```
>>> from sage.all import *
>>> Phi.differential(np)
Generic morphism:
  From: Tangent space at Point N on the 2-dimensional differentiable manifold S^2
  To:   Tangent space at Point Phi(N) on the 3-dimensional differentiable␣
↪manifold R^3
>>> Phi.differential(sp)
Generic morphism:
  From: Tangent space at Point S on the 2-dimensional differentiable manifold S^2
  To:   Tangent space at Point Phi(S) on the 3-dimensional differentiable␣
↪manifold R^3
```

The matrix of the linear map $\mathrm{d}\Phi_N$ with respect to the default bases of $T_N S^2$ and $T_{\Phi(N)}\mathbf{R}^3$:

```
sage: Phi.differential(np).matrix()
[2 0]
```

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```
[0 2]
[0 0]
```

```
>>> from sage.all import *
>>> Phi.differential(np).matrix()
[2 0]
[0 2]
[0 0]
```

the default bases being:

```
sage: Phi.differential(np).domain().default_basis()
Basis (∂/∂u,∂/∂v) on the Tangent space at Point N on the 2-dimensional
 differentiable manifold S^2
sage: Phi.differential(np).codomain().default_basis()
Basis (∂/∂X,∂/∂Y,∂/∂Z) on the Tangent space at Point Phi(N) on the
 3-dimensional differentiable manifold R^3
```

```
>>> from sage.all import *
>>> Phi.differential(np).domain().default_basis()
Basis (∂/∂u,∂/∂v) on the Tangent space at Point N on the 2-dimensional
 differentiable manifold S^2
>>> Phi.differential(np).codomain().default_basis()
Basis (∂/∂X,∂/∂Y,∂/∂Z) on the Tangent space at Point Phi(N) on the
 3-dimensional differentiable manifold R^3
```

A convenient way to display the matrix of the differential:

```
sage: Phi.differential(np).display()
     ∂/∂u ∂/∂v
∂/∂X⎛   2    0⎞
∂/∂Y⎜   0    2⎟
∂/∂Z⎝   0    0⎠
```

```
>>> from sage.all import *
>>> Phi.differential(np).display()
     ∂/∂u ∂/∂v
∂/∂X⎛   2    0⎞
∂/∂Y⎜   0    2⎟
∂/∂Z⎝   0    0⎠
```

Differentiable maps can be composed by means of the operator `*`: let us introduce the map $\mathbf{R}^3 \to \mathbf{R}^2$ corresponding to the projection from the point $(X, Y, Z) = (0, 0, 1)$ onto the equatorial plane $Z = 0$:

```
sage: P = Manifold(2, 'R^2', r'\RR^2') # R^2 (equatorial plane)
sage: cP.<xP, yP> = P.chart()
sage: Psi = N.diff_map(P, (X/(1-Z), Y/(1-Z)), name='Psi',
....:                  latex_name=r'\Psi')
sage: Psi
Differentiable map Psi from the 3-dimensional differentiable manifold
 R^3 to the 2-dimensional differentiable manifold R^2
sage: Psi.display()
```

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```
Psi: R^3 → R^2
   (X, Y, Z) ↦ (xP, yP) = (-X/(Z - 1), -Y/(Z - 1))
```

```
>>> from sage.all import *
>>> P = Manifold(Integer(2), 'R^2', r'\RR^2') # R^2 (equatorial plane)
>>> cP = P.chart(names=('xP', 'yP',)); (xP, yP,) = cP._first_ngens(2)
>>> Psi = N.diff_map(P, (X/(Integer(1)-Z), Y/(Integer(1)-Z)), name='Psi',
...                  latex_name=r'\Psi')
>>> Psi
Differentiable map Psi from the 3-dimensional differentiable manifold
 R^3 to the 2-dimensional differentiable manifold R^2
>>> Psi.display()
Psi: R^3 → R^2
   (X, Y, Z) ↦ (xP, yP) = (-X/(Z - 1), -Y/(Z - 1))
```

Then we compose `Psi` with `Phi`, thereby getting a map $S^2 \to \mathbf{R}^2$:

```
sage: ster = Psi*Phi ; ster
Differentiable map from the 2-dimensional differentiable manifold S^2
 to the 2-dimensional differentiable manifold R^2
```

```
>>> from sage.all import *
>>> ster = Psi*Phi ; ster
Differentiable map from the 2-dimensional differentiable manifold S^2
 to the 2-dimensional differentiable manifold R^2
```

Let us test on the South pole (`sp`) that `ster` is indeed the composite of `Psi` and `Phi`:

```
sage: ster(sp) == Psi(Phi(sp))
True
```

```
>>> from sage.all import *
>>> ster(sp) == Psi(Phi(sp))
True
```

Actually `ster` is the stereographic projection from the North pole, as its coordinate expression reveals:

```
sage: ster.display()
S^2 → R^2
on U: (x, y) ↦ (xP, yP) = (x, y)
on V: (u, v) ↦ (xP, yP) = (u/(u^2 + v^2), v/(u^2 + v^2))
```

```
>>> from sage.all import *
>>> ster.display()
S^2 → R^2
on U: (x, y) ↦ (xP, yP) = (x, y)
on V: (u, v) ↦ (xP, yP) = (u/(u^2 + v^2), v/(u^2 + v^2))
```

If its codomain is 1-dimensional, a differentiable map must be defined by a single symbolic expression for each pair of charts, and not by a list/tuple with a single element:

```
sage: N = Manifold(1, 'N')
sage: c_N = N.chart('X')
sage: Phi = M.diff_map(N, {(c_xy, c_N): x^2+y^2,
....: (c_uv, c_N): 1/(u^2+v^2)})  # not ...[1/(u^2+v^2)] or (1/(u^2+v^2),)
```

```
>>> from sage.all import *
>>> N = Manifold(Integer(1), 'N')
>>> c_N = N.chart('X')
>>> Phi = M.diff_map(N, {(c_xy, c_N): x**Integer(2)+y**Integer(2),
... (c_uv, c_N): Integer(1)/(u**Integer(2)+v**Integer(2))})  # not ...[1/(u^2+v^
→2)] or (1/(u^2+v^2),)
```

An example of differentiable map $\mathbf{R} \rightarrow \mathbf{R}^2$:

```
sage: R = Manifold(1, 'R')  # field R
sage: T.<t> = R.chart()  # canonical chart on R
sage: R2 = Manifold(2, 'R^2')  # R^2
sage: c_xy.<x,y> = R2.chart() # Cartesian coordinates on R^2
sage: Phi = R.diff_map(R2, [cos(t), sin(t)], name='Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable manifold R
 to the 2-dimensional differentiable manifold R^2
sage: Phi.parent()
Set of Morphisms from 1-dimensional differentiable manifold R to
 2-dimensional differentiable manifold R^2 in Category of smooth
 manifolds over Real Field with 53 bits of precision
sage: Phi.parent() is Hom(R, R2)
True
sage: Phi.display()
Phi: R → R^2
   t ↦ (x, y) = (cos(t), sin(t))
```

```
>>> from sage.all import *
>>> R = Manifold(Integer(1), 'R')  # field R
>>> T = R.chart(names=('t',)); (t,) = T._first_ngens(1)# canonical chart on R
>>> R2 = Manifold(Integer(2), 'R^2')  # R^2
>>> c_xy = R2.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)# Cartesian␣
→coordinates on R^2
>>> Phi = R.diff_map(R2, [cos(t), sin(t)], name='Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable manifold R
 to the 2-dimensional differentiable manifold R^2
>>> Phi.parent()
Set of Morphisms from 1-dimensional differentiable manifold R to
 2-dimensional differentiable manifold R^2 in Category of smooth
 manifolds over Real Field with 53 bits of precision
>>> Phi.parent() is Hom(R, R2)
True
>>> Phi.display()
Phi: R → R^2
   t ↦ (x, y) = (cos(t), sin(t))
```

An example of diffeomorphism between the unit open disk and the Euclidean plane $\mathbf{R}^2$:

```
sage: D = R2.open_subset('D', coord_def={c_xy: x^2+y^2<1}) # the open unit disk
sage: Phi = D.diffeomorphism(R2, [x/sqrt(1-x^2-y^2), y/sqrt(1-x^2-y^2)],
....:                         name='Phi', latex_name=r'\Phi')
sage: Phi
Diffeomorphism Phi from the Open subset D of the 2-dimensional
 differentiable manifold R^2 to the 2-dimensional differentiable
 manifold R^2
sage: Phi.parent()
Set of Morphisms from Open subset D of the 2-dimensional differentiable
 manifold R^2 to 2-dimensional differentiable manifold R^2 in Category
 of smooth manifolds over Real Field with 53 bits of precision
sage: Phi.parent() is Hom(D, R2)
True
sage: Phi.display()
Phi: D → R^2
   (x, y) ↦ (x, y) = (x/sqrt(-x^2 - y^2 + 1), y/sqrt(-x^2 - y^2 + 1))
```

```
>>> from sage.all import *
>>> D = R2.open_subset('D', coord_def={c_xy: x**Integer(2)+y**Integer(2)
↪<Integer(1)}) # the open unit disk
>>> Phi = D.diffeomorphism(R2, [x/sqrt(Integer(1)-x**Integer(2)-y**Integer(2)), y/
↪sqrt(Integer(1)-x**Integer(2)-y**Integer(2))],
...                         name='Phi', latex_name=r'\Phi')
>>> Phi
Diffeomorphism Phi from the Open subset D of the 2-dimensional
 differentiable manifold R^2 to the 2-dimensional differentiable
 manifold R^2
>>> Phi.parent()
Set of Morphisms from Open subset D of the 2-dimensional differentiable
 manifold R^2 to 2-dimensional differentiable manifold R^2 in Category
 of smooth manifolds over Real Field with 53 bits of precision
>>> Phi.parent() is Hom(D, R2)
True
>>> Phi.display()
Phi: D → R^2
   (x, y) ↦ (x, y) = (x/sqrt(-x^2 - y^2 + 1), y/sqrt(-x^2 - y^2 + 1))
```

The image of a point:

```
sage: p = D.point((1/2,0))
sage: q = Phi(p) ; q
Point on the 2-dimensional differentiable manifold R^2
sage: q.coord()
(1/3*sqrt(3), 0)
```

```
>>> from sage.all import *
>>> p = D.point((Integer(1)/Integer(2),Integer(0)))
>>> q = Phi(p) ; q
Point on the 2-dimensional differentiable manifold R^2
>>> q.coord()
(1/3*sqrt(3), 0)
```

The inverse diffeomorphism is computed by means of the method `inverse()`:

```
sage: Phi.inverse()
Diffeomorphism Phi^(-1) from the 2-dimensional differentiable manifold R^2
 to the Open subset D of the 2-dimensional differentiable manifold R^2
sage: Phi.inverse().display()
Phi^(-1): R^2 → D
   (x, y) ↦ (x, y) = (x/sqrt(x^2 + y^2 + 1), y/sqrt(x^2 + y^2 + 1))
```

```
>>> from sage.all import *
>>> Phi.inverse()
Diffeomorphism Phi^(-1) from the 2-dimensional differentiable manifold R^2
 to the Open subset D of the 2-dimensional differentiable manifold R^2
>>> Phi.inverse().display()
Phi^(-1): R^2 → D
   (x, y) ↦ (x, y) = (x/sqrt(x^2 + y^2 + 1), y/sqrt(x^2 + y^2 + 1))
```

Equivalently, one may use the notations `^(-1)` or `~` to get the inverse:

```
sage: Phi^(-1) is Phi.inverse()
True
sage: ~Phi is Phi.inverse()
True
```

```
>>> from sage.all import *
>>> Phi**(-Integer(1)) is Phi.inverse()
True
>>> ~Phi is Phi.inverse()
True
```

Check that `~Phi` is indeed the inverse of `Phi`:

```
sage: (~Phi)(q) == p
True
sage: Phi * ~Phi == R2.identity_map()
True
sage: ~Phi * Phi == D.identity_map()
True
```

```
>>> from sage.all import *
>>> (~Phi)(q) == p
True
>>> Phi * ~Phi == R2.identity_map()
True
>>> ~Phi * Phi == D.identity_map()
True
```

The coordinate expression of the inverse diffeomorphism:

```
sage: (~Phi).display()
Phi^(-1): R^2 → D
   (x, y) ↦ (x, y) = (x/sqrt(x^2 + y^2 + 1), y/sqrt(x^2 + y^2 + 1))
```

```
>>> from sage.all import *
>>> (~Phi).display()
Phi^(-1): R^2 → D
   (x, y) ↦ (x, y) = (x/sqrt(x^2 + y^2 + 1), y/sqrt(x^2 + y^2 + 1))
```

A special case of diffeomorphism: the identity map of the open unit disk:

```
sage: id = D.identity_map() ; id
Identity map Id_D of the Open subset D of the 2-dimensional
 differentiable manifold R^2
sage: latex(id)
\mathrm{Id}_{D}
sage: id.parent()
Set of Morphisms from Open subset D of the 2-dimensional differentiable
 manifold R^2 to Open subset D of the 2-dimensional differentiable
 manifold R^2 in Join of Category of subobjects of sets and Category of
 smooth manifolds over Real Field with 53 bits of precision
sage: id.parent() is Hom(D, D)
True
sage: id is Hom(D,D).one()  # the identity element of the monoid Hom(D,D)
True
```

```
>>> from sage.all import *
>>> id = D.identity_map() ; id
Identity map Id_D of the Open subset D of the 2-dimensional
 differentiable manifold R^2
>>> latex(id)
\mathrm{Id}_{D}
>>> id.parent()
Set of Morphisms from Open subset D of the 2-dimensional differentiable
 manifold R^2 to Open subset D of the 2-dimensional differentiable
 manifold R^2 in Join of Category of subobjects of sets and Category of
 smooth manifolds over Real Field with 53 bits of precision
>>> id.parent() is Hom(D, D)
True
>>> id is Hom(D,D).one()  # the identity element of the monoid Hom(D,D)
True
```

The identity map acting on a point:

```
sage: id(p)
Point on the 2-dimensional differentiable manifold R^2
sage: id(p) == p
True
sage: id(p) is p
True
```

```
>>> from sage.all import *
>>> id(p)
Point on the 2-dimensional differentiable manifold R^2
>>> id(p) == p
True
```

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```
>>> id(p) is p
True
```

The coordinate expression of the identity map:

```
sage: id.display()
Id_D: D → D
   (x, y) ↦ (x, y)
```

```
>>> from sage.all import *
>>> id.display()
Id_D: D → D
   (x, y) ↦ (x, y)
```

The identity map is its own inverse:

```
sage: id^(-1) is id
True
sage: ~id is id
True
```

```
>>> from sage.all import *
>>> id**(-Integer(1)) is id
True
>>> ~id is id
True
```

differential(*point*)

Return the differential of `self` at a given point.

If the differentiable map `self` is

$$\Phi : M \longrightarrow N,$$

where M and N are differentiable manifolds, the *differential* of Φ at a point $p \in M$ is the tangent space linear map:

$$\mathrm{d}\Phi_p : T_p M \longrightarrow T_{\Phi(p)} N$$

defined by

$$\forall v \in T_p M, \quad \mathrm{d}\Phi_p(v) : \begin{array}{ccc} C^k(N) & \longrightarrow & \mathbb{R} \\ f & \longmapsto & v(f \circ \Phi) \end{array}$$

INPUT:

- `point` – point p in the domain M of the differentiable map Φ

OUTPUT:

- $\mathrm{d}\Phi_p$, the differential of Φ at p, as a `FiniteRankFreeModuleMorphism`

EXAMPLES:

Differential of a differentiable map between a 2-dimensional manifold and a 3-dimensional one:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: N = Manifold(3, 'N')
sage: Y.<u,v,w> = N.chart()
sage: Phi = M.diff_map(N, {(X,Y): (x-2*y, x*y, x^2-y^3)}, name='Phi',
....:                  latex_name = r'\Phi')
sage: p = M.point((2,-1), name='p')
sage: dPhip = Phi.differential(p) ; dPhip
Generic morphism:
  From: Tangent space at Point p on the 2-dimensional differentiable manifold␣
↪M
  To:   Tangent space at Point Phi(p) on the 3-dimensional differentiable␣
↪manifold N
sage: latex(dPhip)
{\mathrm{d}\Phi}_{p}
sage: dPhip.parent()
Set of Morphisms from Tangent space at Point p on the 2-dimensional
 differentiable manifold M to Tangent space at Point Phi(p) on the
 3-dimensional differentiable manifold N in Category of finite
 dimensional vector spaces over Symbolic Ring
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> N = Manifold(Integer(3), 'N')
>>> Y = N.chart(names=('u', 'v', 'w',)); (u, v, w,) = Y._first_ngens(3)
>>> Phi = M.diff_map(N, {(X,Y): (x-Integer(2)*y, x*y, x**Integer(2)-
↪y**Integer(3))}, name='Phi',
...                  latex_name = r'\Phi')
>>> p = M.point((Integer(2),-Integer(1)), name='p')
>>> dPhip = Phi.differential(p) ; dPhip
Generic morphism:
  From: Tangent space at Point p on the 2-dimensional differentiable manifold␣
↪M
  To:   Tangent space at Point Phi(p) on the 3-dimensional differentiable␣
↪manifold N
>>> latex(dPhip)
{\mathrm{d}\Phi}_{p}
>>> dPhip.parent()
Set of Morphisms from Tangent space at Point p on the 2-dimensional
 differentiable manifold M to Tangent space at Point Phi(p) on the
 3-dimensional differentiable manifold N in Category of finite
 dimensional vector spaces over Symbolic Ring
```

The matrix of $\mathrm{d}\Phi_p$ w.r.t. to the default bases of $T_p M$ and $T_{\Phi(p)} N$:

```
sage: dPhip.matrix()
[ 1 -2]
[-1  2]
[ 4 -3]
```

```
>>> from sage.all import *
```

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```
>>> dPhip.matrix()
[ 1 -2]
[-1  2]
[ 4 -3]
```

differential_functions(*chart1=None*, *chart2=None*)

Return the coordinate expression of the differential of the differentiable map with respect to a pair of charts.

If the differentiable map is

$$\Phi : M \longrightarrow N,$$

where M and N are differentiable manifolds, the *differential* of Φ at a point $p \in M$ is the tangent space linear map:

$$\mathrm{d}\Phi_p : T_pM \longrightarrow T_{\Phi(p)}N$$

defined by

$$\begin{array}{llcl} \forall v \in T_pM, & \mathrm{d}\Phi_p(v) : & C^k(N) & \longrightarrow & \mathbb{R}, \\ & & f & \longmapsto & v(f \circ \Phi). \end{array}$$

If the coordinate expression of Φ is

$$y^i = Y^i(x^1, \ldots, x^n), \quad 1 \leq i \leq m,$$

where $(x^1, \ldots, x^n)$ are coordinates of a chart on M and $(y^1, \ldots, y^m)$ are coordinates of a chart on $\Phi(M)$, the expression of the differential of Φ with respect to these coordinates is

$$J_{ij} = \frac{\partial Y^i}{\partial x^j} \quad 1 \leq i \leq m, \qquad 1 \leq j \leq n.$$

$\left. J_{ij} \right|_p$ is then the matrix of the linear map $\mathrm{d}\Phi_p$ with respect to the bases of T_pM and $T_{\Phi(p)}N$ associated to the above charts:

$$\mathrm{d}\Phi_p \left(\left. \frac{\partial}{\partial x^j} \right|_p \right) = \left. J_{ij} \right|_p \left. \frac{\partial}{\partial y^i} \right|_{\Phi(p)}.$$

INPUT:

- `chart1` – (default: `None`) chart on the domain M of Φ (coordinates denoted by (x^j) above); if `None`, the domain's default chart is assumed
- `chart2` – (default: `None`) chart on the codomain of Φ (coordinates denoted by (y^i) above); if `None`, the codomain's default chart is assumed

OUTPUT:

- the functions J_{ij} as a double array, J_{ij} being the element `[i][j]` represented by a *ChartFunction*

To get symbolic expressions, use the method *jacobian_matrix()* instead.

EXAMPLES:

Differential functions of a map between a 2-dimensional manifold and a 3-dimensional one:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: N = Manifold(3, 'N')
sage: Y.<u,v,w> = N.chart()
sage: Phi = M.diff_map(N, {(X,Y): (x-2*y, x*y, x^2-y^3)}, name='Phi',
....:                  latex_name = r'\Phi')
sage: J = Phi.differential_functions(X, Y) ; J
[     1      -2]
[     y       x]
[   2*x -3*y^2]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> N = Manifold(Integer(3), 'N')
>>> Y = N.chart(names=('u', 'v', 'w',)); (u, v, w,) = Y._first_ngens(3)
>>> Phi = M.diff_map(N, {(X,Y): (x-Integer(2)*y, x*y, x**Integer(2)-
↪y**Integer(3))}, name='Phi',
...                  latex_name = r'\Phi')
>>> J = Phi.differential_functions(X, Y) ; J
[     1      -2]
[     y       x]
[   2*x -3*y^2]
```

The result is cached:

```
sage: Phi.differential_functions(X, Y) is J
True
```

```
>>> from sage.all import *
>>> Phi.differential_functions(X, Y) is J
True
```

The elements of `J` are functions of the coordinates of the chart `X`:

```
sage: J[2][0]
2*x
sage: type(J[2][0])
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>

sage: J[2][0].display()
(x, y) ↦ 2*x
```

```
>>> from sage.all import *
>>> J[Integer(2)][Integer(0)]
2*x
>>> type(J[Integer(2)][Integer(0)])
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>

>>> J[Integer(2)][Integer(0)].display()
(x, y) ↦ 2*x
```

In contrast, the method *jacobian_matrix()* leads directly to symbolic expressions:

```
sage: JJ = Phi.jacobian_matrix(X,Y) ; JJ
[      1      -2]
[      y       x]
[    2*x -3*y^2]
sage: JJ[2,0]
2*x
sage: type(JJ[2,0])
<class 'sage.symbolic.expression.Expression'>
sage: bool( JJ[2,0] == J[2][0].expr() )
True
```

```
>>> from sage.all import *
>>> JJ = Phi.jacobian_matrix(X,Y) ; JJ
[      1      -2]
[      y       x]
[    2*x -3*y^2]
>>> JJ[Integer(2),Integer(0)]
2*x
>>> type(JJ[Integer(2),Integer(0)])
<class 'sage.symbolic.expression.Expression'>
>>> bool( JJ[Integer(2),Integer(0)] == J[Integer(2)][Integer(0)].expr() )
True
```

jacobian_matrix(*chart1=None*, *chart2=None*)

Return the Jacobian matrix resulting from the coordinate expression of the differentiable map with respect to a pair of charts.

If Φ is the current differentiable map and its coordinate expression is

$$y^i = Y^i(x^1, \ldots, x^n), \quad 1 \leq i \leq m,$$

where $(x^1, \ldots, x^n)$ are coordinates of a chart X on the domain of Φ and $(y^1, \ldots, y^m)$ are coordinates of a chart Y on the codomain of Φ, the *Jacobian matrix* of the differentiable map Φ w.r.t. to charts X and Y is

$$J = \left(\frac{\partial Y^i}{\partial x^j} \right)_{\substack{1\leq i\leq m \\ 1\leq j\leq n}},$$

where i is the row index and j the column one.

INPUT:

- `chart1` – (default: `None`) chart X on the domain of Φ; if none is provided, the domain's default chart is assumed
- `chart2` – (default: `None`) chart Y on the codomain of Φ; if none is provided, the codomain's default chart is assumed

OUTPUT: the matrix J defined above

EXAMPLES:

Jacobian matrix of a map between a 2-dimensional manifold and a 3-dimensional one:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
```

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```
sage: N = Manifold(3, 'N')
sage: Y.<u,v,w> = N.chart()
sage: Phi = M.diff_map(N, {(X,Y): (x-2*y, x*y, x^2-y^3)}, name='Phi',
....:                  latex_name = r'\Phi')
sage: Phi.display()
Phi: M → N
   (x, y) ↦ (u, v, w) = (x - 2*y, x*y, -y^3 + x^2)
sage: J = Phi.jacobian_matrix(X, Y) ; J
[     1     -2]
[     y      x]
[   2*x -3*y^2]
sage: J.parent()
Full MatrixSpace of 3 by 2 dense matrices over Symbolic Ring
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> N = Manifold(Integer(3), 'N')
>>> Y = N.chart(names=('u', 'v', 'w',)); (u, v, w,) = Y._first_ngens(3)
>>> Phi = M.diff_map(N, {(X,Y): (x-Integer(2)*y, x*y, x**Integer(2)-
↪y**Integer(3))}, name='Phi',
...                  latex_name = r'\Phi')
>>> Phi.display()
Phi: M → N
   (x, y) ↦ (u, v, w) = (x - 2*y, x*y, -y^3 + x^2)
>>> J = Phi.jacobian_matrix(X, Y) ; J
[     1     -2]
[     y      x]
[   2*x -3*y^2]
>>> J.parent()
Full MatrixSpace of 3 by 2 dense matrices over Symbolic Ring
```

pullback(*tensor_or_codomain_subset*, *name=None*, *latex_name=None*)

Pullback operator associated with `self`.

In what follows, let Φ denote a differentiable map `self`, M its domain and N its codomain.

INPUT:

One of the following:

- `tensor_or_codomain_subset` – one of the following:
  - a *TensorField*; a fully covariant tensor field T on N, i.e. a tensor field of type $(0, p)$, with p a positive or zero integer; the case $p = 0$ corresponds to a scalar field
  - a *ManifoldSubset* S

OUTPUT:

- (if the input is a tensor field T) a *TensorField* representing a fully covariant tensor field on M that is the pullback of T by Φ
- (if the input is a manifold subset S) a *ManifoldSubset* that is the preimage $\Phi^{-1}(S)$; same as `preimage()`

EXAMPLES:

Pullback on S^2 of a scalar field defined on R^3:

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: U = M.open_subset('U') # the complement of a meridian (domain of␣
↪spherical coordinates)
sage: c_spher.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi') #␣
↪spherical coord. on U
sage: N = Manifold(3, 'R^3', r'\RR^3', start_index=1)
sage: c_cart.<x,y,z> = N.chart() # Cartesian coord. on R^3
sage: Phi = U.diff_map(N, (sin(th)*cos(ph), sin(th)*sin(ph), cos(th)),
....:                  name='Phi', latex_name=r'\Phi')
sage: f = N.scalar_field(x*y*z, name='f') ; f
Scalar field f on the 3-dimensional differentiable manifold R^3
sage: f.display()
f: R^3 → ℝ
   (x, y, z) ↦ x*y*z
sage: pf = Phi.pullback(f) ; pf
Scalar field Phi^*(f) on the Open subset U of the 2-dimensional
 differentiable manifold S^2
sage: pf.display()
Phi^*(f): U → ℝ
   (th, ph) ↦ cos(ph)*cos(th)*sin(ph)*sin(th)^2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', start_index=Integer(1))
>>> U = M.open_subset('U') # the complement of a meridian (domain of␣
↪spherical coordinates)
>>> c_spher = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi', names=('th', 'ph',
↪)); (th, ph,) = c_spher._first_ngens(2)# spherical coord. on U
>>> N = Manifold(Integer(3), 'R^3', r'\RR^3', start_index=Integer(1))
>>> c_cart = N.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_cart._first_
↪ngens(3)# Cartesian coord. on R^3
>>> Phi = U.diff_map(N, (sin(th)*cos(ph), sin(th)*sin(ph), cos(th)),
...                  name='Phi', latex_name=r'\Phi')
>>> f = N.scalar_field(x*y*z, name='f') ; f
Scalar field f on the 3-dimensional differentiable manifold R^3
>>> f.display()
f: R^3 → ℝ
   (x, y, z) ↦ x*y*z
>>> pf = Phi.pullback(f) ; pf
Scalar field Phi^*(f) on the Open subset U of the 2-dimensional
 differentiable manifold S^2
>>> pf.display()
Phi^*(f): U → ℝ
   (th, ph) ↦ cos(ph)*cos(th)*sin(ph)*sin(th)^2
```

Pullback on S^2 of the standard Euclidean metric on R^3:

```
sage: g = N.sym_bilin_form_field(name='g')
sage: g[1,1], g[2,2], g[3,3] = 1, 1, 1
sage: g.display()
g = dx⊗dx + dy⊗dy + dz⊗dz
sage: pg = Phi.pullback(g) ; pg
```

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```
Field of symmetric bilinear forms Phi^*(g) on the Open subset U of
 the 2-dimensional differentiable manifold S^2
sage: pg.display()
Phi^*(g) = dth⊗dth + sin(th)^2 dph⊗dph
```

```
>>> from sage.all import *
>>> g = N.sym_bilin_form_field(name='g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)], g[Integer(3),
↪Integer(3)] = Integer(1), Integer(1), Integer(1)
>>> g.display()
g = dx⊗dx + dy⊗dy + dz⊗dz
>>> pg = Phi.pullback(g) ; pg
Field of symmetric bilinear forms Phi^*(g) on the Open subset U of
 the 2-dimensional differentiable manifold S^2
>>> pg.display()
Phi^*(g) = dth⊗dth + sin(th)^2 dph⊗dph
```

Parallel computation:

```
sage: Parallelism().set('tensor', nproc=2)
sage: pg = Phi.pullback(g) ; pg
Field of symmetric bilinear forms Phi^*(g) on the Open subset U of
 the 2-dimensional differentiable manifold S^2
sage: pg.display()
Phi^*(g) = dth⊗dth + sin(th)^2 dph⊗dph
sage: Parallelism().set('tensor', nproc=1)  # switch off parallelization
```

```
>>> from sage.all import *
>>> Parallelism().set('tensor', nproc=Integer(2))
>>> pg = Phi.pullback(g) ; pg
Field of symmetric bilinear forms Phi^*(g) on the Open subset U of
 the 2-dimensional differentiable manifold S^2
>>> pg.display()
Phi^*(g) = dth⊗dth + sin(th)^2 dph⊗dph
>>> Parallelism().set('tensor', nproc=Integer(1))  # switch off␣
↪parallelization
```

Pullback on S^2 of a 3-form on R^3:

```
sage: a = N.diff_form(3, name='A')
sage: a[1,2,3] = f
sage: a.display()
A = x*y*z dx∧dy∧dz
sage: pa = Phi.pullback(a) ; pa
3-form Phi^*(A) on the Open subset U of the 2-dimensional
 differentiable manifold S^2
sage: pa.display() # should be zero (as any 3-form on a 2-dimensional␣
↪manifold)
Phi^*(A) = 0
```

```
>>> from sage.all import *
>>> a = N.diff_form(Integer(3), name='A')
```

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```
>>> a[Integer(1),Integer(2),Integer(3)] = f
>>> a.display()
A = x*y*z dx∧dy∧dz
>>> pa = Phi.pullback(a) ; pa
3-form Phi^*(A) on the Open subset U of the 2-dimensional
 differentiable manifold S^2
>>> pa.display() # should be zero (as any 3-form on a 2-dimensional manifold)
Phi^*(A) = 0
```

pushforward(*tensor*)

Pushforward operator associated with `self`.

In what follows, let Φ denote the differentiable map, M its domain and N its codomain.

INPUT:

- `tensor` – *TensorField*; a fully contrariant tensor field T on M, i.e. a tensor field of type $(p, 0)$, with p a positive integer

OUTPUT:

- a *TensorField* representing a fully contravariant tensor field along M with values in N, which is the pushforward of T by Φ

EXAMPLES:

Pushforward of a vector field on the 2-sphere S^2 to the Euclidean 3-space $\mathbf{R}^3$, via the standard embedding of S^2:

```
sage: S2 = Manifold(2, 'S^2', start_index=1)
sage: U = S2.open_subset('U')  # domain of spherical coordinates
sage: spher.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: R3 = Manifold(3, 'R^3', start_index=1)
sage: cart.<x,y,z> = R3.chart()
sage: Phi = U.diff_map(R3, {(spher, cart): [sin(th)*cos(ph),
....:   sin(th)*sin(ph), cos(th)]}, name='Phi', latex_name=r'\Phi')
sage: v = U.vector_field(name='v')
sage: v[:] = 0, 1
sage: v.display()
v = ∂/∂ph
sage: pv = Phi.pushforward(v); pv
Vector field Phi_*(v) along the Open subset U of the 2-dimensional
 differentiable manifold S^2 with values on the 3-dimensional
 differentiable manifold R^3
sage: pv.display()
Phi_*(v) = -sin(ph)*sin(th) ∂/∂x + cos(ph)*sin(th) ∂/∂y
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2', start_index=Integer(1))
>>> U = S2.open_subset('U')  # domain of spherical coordinates
>>> spher = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi', names=('th', 'ph',
↪)); (th, ph,) = spher._first_ngens(2)
>>> R3 = Manifold(Integer(3), 'R^3', start_index=Integer(1))
>>> cart = R3.chart(names=('x', 'y', 'z',)); (x, y, z,) = cart._first_ngens(3)
>>> Phi = U.diff_map(R3, {(spher, cart): [sin(th)*cos(ph),
```

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```
...     sin(th)*sin(ph), cos(th)]}, name='Phi', latex_name=r'\Phi')
>>> v = U.vector_field(name='v')
>>> v[:] = Integer(0), Integer(1)
>>> v.display()
v = ∂/∂ph
>>> pv = Phi.pushforward(v); pv
Vector field Phi_*(v) along the Open subset U of the 2-dimensional
 differentiable manifold S^2 with values on the 3-dimensional
 differentiable manifold R^3
>>> pv.display()
Phi_*(v) = -sin(ph)*sin(th) ∂/∂x + cos(ph)*sin(th) ∂/∂y
```

Pushforward of a vector field on the real line to the $\mathbf{R}^3$, via a helix embedding:

```
sage: R.<t> = manifolds.RealLine()
sage: Psi = R.diff_map(R3, [cos(t), sin(t), t], name='Psi',
....:                  latex_name=r'\Psi')
sage: u = R.vector_field(name='u')
sage: u[0] = 1
sage: u.display()
u = ∂/∂t
sage: pu = Psi.pushforward(u); pu
Vector field Psi_*(u) along the Real number line ℝ with values on
 the 3-dimensional differentiable manifold R^3
sage: pu.display()
Psi_*(u) = -sin(t) ∂/∂x + cos(t) ∂/∂y + ∂/∂z
```

```
>>> from sage.all import *
>>> R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1)
>>> Psi = R.diff_map(R3, [cos(t), sin(t), t], name='Psi',
...                  latex_name=r'\Psi')
>>> u = R.vector_field(name='u')
>>> u[Integer(0)] = Integer(1)
>>> u.display()
u = ∂/∂t
>>> pu = Psi.pushforward(u); pu
Vector field Psi_*(u) along the Real number line ℝ with values on
 the 3-dimensional differentiable manifold R^3
>>> pu.display()
Psi_*(u) = -sin(t) ∂/∂x + cos(t) ∂/∂y + ∂/∂z
```

2.5.3 Curves in Manifolds

Given a differentiable manifold M, a *differentiable curve* in M is a differentiable mapping

$$\gamma : I \longrightarrow M,$$

where I is an interval of $\mathbf{R}$.

Differentiable curves are implemented by `DifferentiableCurve`.

AUTHORS:

- Eric Gourgoulhon (2015): initial version

- Travis Scrimshaw (2016): review tweaks

REFERENCES:

- Chap. 1 of [KN1963]
- Chap. 3 of [Lee2013]

class sage.manifolds.differentiable.curve.**DifferentiableCurve**(*parent*, *coord_expression=None*, *name=None*, *latex_name=None*, *is_isomorphism=False*, *is_identity=False*)

Bases: *DiffMap*

Curve in a differentiable manifold.

Given a differentiable manifold M, a *differentiable curve* in M is a differentiable map

$$\gamma : I \longrightarrow M,$$

where I is an interval of $\mathbf{R}$.

INPUT:

- `parent` – *DifferentiableCurveSet* the set of curves $\mathrm{Hom}(I, M)$ to which the curve belongs
- `coord_expression` – (default: `None`) dictionary (possibly empty) of the functions of the curve parameter t expressing the curve in various charts of M, the keys of the dictionary being the charts and the values being lists or tuples of n symbolic expressions of t, where n is the dimension of M
- `name` – (default: `None`) string; symbol given to the curve
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the curve; if none is provided, `name` will be used
- `is_isomorphism` – boolean (default: `False`); determines whether the constructed object is a diffeomorphism; if set to `True`, then M must have dimension one
- `is_identity` – boolean (default: `False`); determines whether the constructed object is the identity map; if set to `True`, then M must be the interval I

EXAMPLES:

The lemniscate of Gerono in the 2-dimensional Euclidean plane:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: t = var('t')
sage: c = M.curve({X: [sin(t), sin(2*t)/2]}, (t, 0, 2*pi), name='c') ; c
Curve c in the 2-dimensional differentiable manifold M
sage: type(c)
<class 'sage.manifolds.differentiable.manifold_homset.DifferentiableCurveSet_with_
↪category.element_class'>
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> t = var('t')
>>> c = M.curve({X: [sin(t), sin(Integer(2)*t)/Integer(2)]}, (t, Integer(0),␣
↪Integer(2)*pi), name='c') ; c
```

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```
Curve c in the 2-dimensional differentiable manifold M
>>> type(c)
<class 'sage.manifolds.differentiable.manifold_homset.DifferentiableCurveSet_with_
↪category.element_class'>
```

Instead of declaring the parameter t as a symbolic variable by means of `var('t')`, it is equivalent to get it as the canonical coordinate of the real number line (see `RealLine`):

```
sage: R.<t> = manifolds.RealLine()
sage: c = M.curve({X: [sin(t), sin(2*t)/2]}, (t, 0, 2*pi), name='c') ; c
Curve c in the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1)
>>> c = M.curve({X: [sin(t), sin(Integer(2)*t)/Integer(2)]}, (t, Integer(0),␣
↪Integer(2)*pi), name='c') ; c
Curve c in the 2-dimensional differentiable manifold M
```

A graphical view of the curve is provided by the method `plot()`:

```
sage: c.plot(aspect_ratio=1)                                                    #␣
↪needs sage.plot
Graphics object consisting of 1 graphics primitive
```

```
>>> from sage.all import *
>>> c.plot(aspect_ratio=Integer(1))                                              ␣
↪     # needs sage.plot
Graphics object consisting of 1 graphics primitive
```

Curves are considered as (manifold) morphisms from real intervals to differentiable manifolds:

```
sage: c.parent()
Set of Morphisms from Real interval (0, 2*pi) to 2-dimensional
 differentiable manifold M in Category of smooth manifolds over Real
 Field with 53 bits of precision
sage: I = R.open_interval(0, 2*pi)
sage: c.parent() is Hom(I, M)
True
sage: c.domain()
Real interval (0, 2*pi)
sage: c.domain() is I
True
sage: c.codomain()
2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> c.parent()
Set of Morphisms from Real interval (0, 2*pi) to 2-dimensional
 differentiable manifold M in Category of smooth manifolds over Real
 Field with 53 bits of precision
>>> I = R.open_interval(Integer(0), Integer(2)*pi)
>>> c.parent() is Hom(I, M)
```

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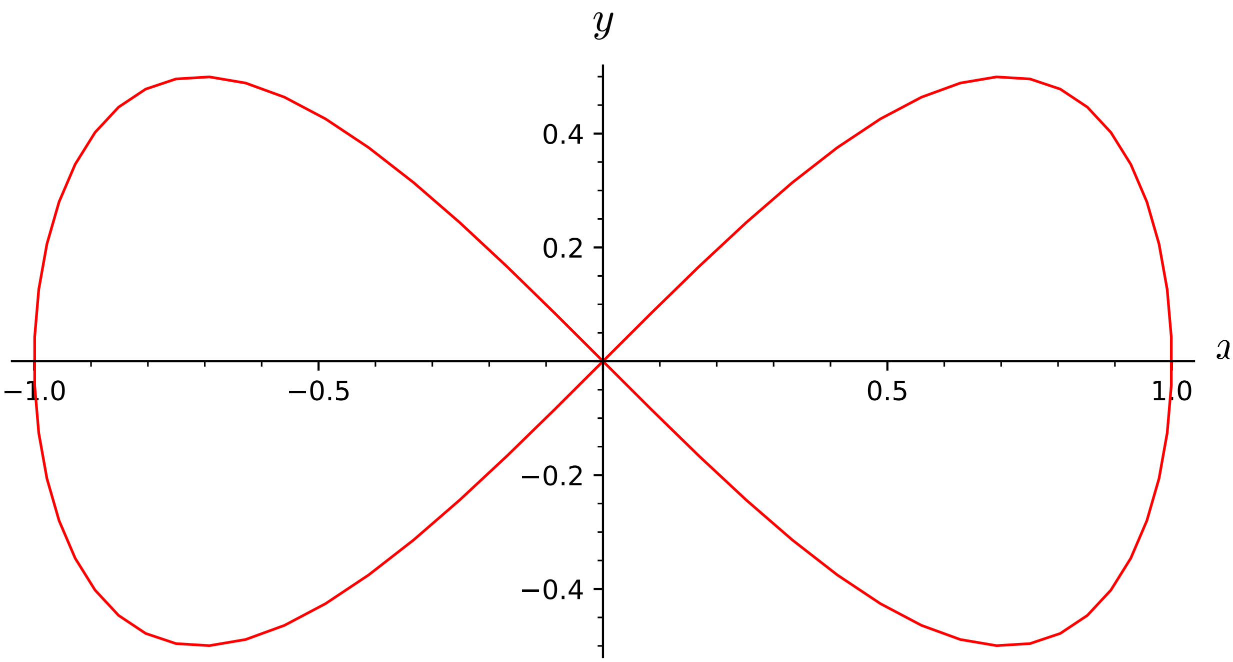
y
0.4
0.2
−0.2
−0.4
x
−1.0
−0.5
0.5
1.0

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```
True
>>> c.domain()
Real interval (0, 2*pi)
>>> c.domain() is I
True
>>> c.codomain()
2-dimensional differentiable manifold M
```

Accordingly, all methods of `DiffMap` are available for them. In particular, the method `display()` shows the coordinate representations in various charts of manifold `M`:

```
sage: c.display()
c: (0, 2*pi) → M
   t ↦ (x, y) = (sin(t), 1/2*sin(2*t))
```

```
>>> from sage.all import *
>>> c.display()
c: (0, 2*pi) → M
   t ↦ (x, y) = (sin(t), 1/2*sin(2*t))
```

Another map method is using the usual call syntax, which returns the image of a point in the curve's domain:

```
sage: t0 = pi/2
sage: I(t0)
Point on the Real number line ℝ
sage: c(I(t0))
Point on the 2-dimensional differentiable manifold M
sage: c(I(t0)).coord(X)
(1, 0)
```

```
>>> from sage.all import *
>>> t0 = pi/Integer(2)
>>> I(t0)
Point on the Real number line ℝ
>>> c(I(t0))
Point on the 2-dimensional differentiable manifold M
>>> c(I(t0)).coord(X)
(1, 0)
```

For curves, the value of the parameter, instead of the corresponding point in the real line manifold, can be passed directly:

```
sage: c(t0)
Point c(1/2*pi) on the 2-dimensional differentiable manifold M
sage: c(t0).coord(X)
(1, 0)
sage: c(t0) == c(I(t0))
True
```

```
>>> from sage.all import *
>>> c(t0)
Point c(1/2*pi) on the 2-dimensional differentiable manifold M
```

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```
>>> c(t0).coord(X)
(1, 0)
>>> c(t0) == c(I(t0))
True
```

Instead of a dictionary of coordinate expressions, the curve can be defined by a single coordinate expression in a given chart:

```
sage: c = M.curve([sin(t), sin(2*t)/2], (t, 0, 2*pi), chart=X, name='c') ; c
Curve c in the 2-dimensional differentiable manifold M
sage: c.display()
c: (0, 2*pi) → M
   t ↦ (x, y) = (sin(t), 1/2*sin(2*t))
```

```
>>> from sage.all import *
>>> c = M.curve([sin(t), sin(Integer(2)*t)/Integer(2)], (t, Integer(0),␣
↪Integer(2)*pi), chart=X, name='c') ; c
Curve c in the 2-dimensional differentiable manifold M
>>> c.display()
c: (0, 2*pi) → M
   t ↦ (x, y) = (sin(t), 1/2*sin(2*t))
```

Since `X` is the default chart on `M`, it can be omitted:

```
sage: c = M.curve([sin(t), sin(2*t)/2], (t, 0, 2*pi), name='c') ; c
Curve c in the 2-dimensional differentiable manifold M
sage: c.display()
c: (0, 2*pi) → M
   t ↦ (x, y) = (sin(t), 1/2*sin(2*t))
```

```
>>> from sage.all import *
>>> c = M.curve([sin(t), sin(Integer(2)*t)/Integer(2)], (t, Integer(0),␣
↪Integer(2)*pi), name='c') ; c
Curve c in the 2-dimensional differentiable manifold M
>>> c.display()
c: (0, 2*pi) → M
   t ↦ (x, y) = (sin(t), 1/2*sin(2*t))
```

Note that a curve in M can also be created as a differentiable map $I \to M$:

```
sage: c1 = I.diff_map(M, coord_functions={X: [sin(t), sin(2*t)/2]},
....:                 name='c') ; c1
Curve c in the 2-dimensional differentiable manifold M
sage: c1.parent() is c.parent()
True
sage: c1 == c
True
```

```
>>> from sage.all import *
>>> c1 = I.diff_map(M, coord_functions={X: [sin(t), sin(Integer(2)*t)/Integer(2)]}
↪,
...                 name='c') ; c1
```

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```
Curve c in the 2-dimensional differentiable manifold M
>>> c1.parent() is c.parent()
True
>>> c1 == c
True
```

LaTeX symbols representing a curve:

```
sage: c = M.curve([sin(t), sin(2*t)/2], (t, 0, 2*pi))
sage: latex(c)
\text{Curve in the 2-dimensional differentiable manifold M}
sage: c = M.curve([sin(t), sin(2*t)/2], (t, 0, 2*pi), name='c')
sage: latex(c)
c
sage: c = M.curve([sin(t), sin(2*t)/2], (t, 0, 2*pi), name='c',
....:             latex_name=r'\gamma')
sage: latex(c)
\gamma
```

```
>>> from sage.all import *
>>> c = M.curve([sin(t), sin(Integer(2)*t)/Integer(2)], (t, Integer(0),␣
↪Integer(2)*pi))
>>> latex(c)
\text{Curve in the 2-dimensional differentiable manifold M}
>>> c = M.curve([sin(t), sin(Integer(2)*t)/Integer(2)], (t, Integer(0),␣
↪Integer(2)*pi), name='c')
>>> latex(c)
c
>>> c = M.curve([sin(t), sin(Integer(2)*t)/Integer(2)], (t, Integer(0),␣
↪Integer(2)*pi), name='c',
...             latex_name=r'\gamma')
>>> latex(c)
\gamma
```

The curve's tangent vector field (velocity vector):

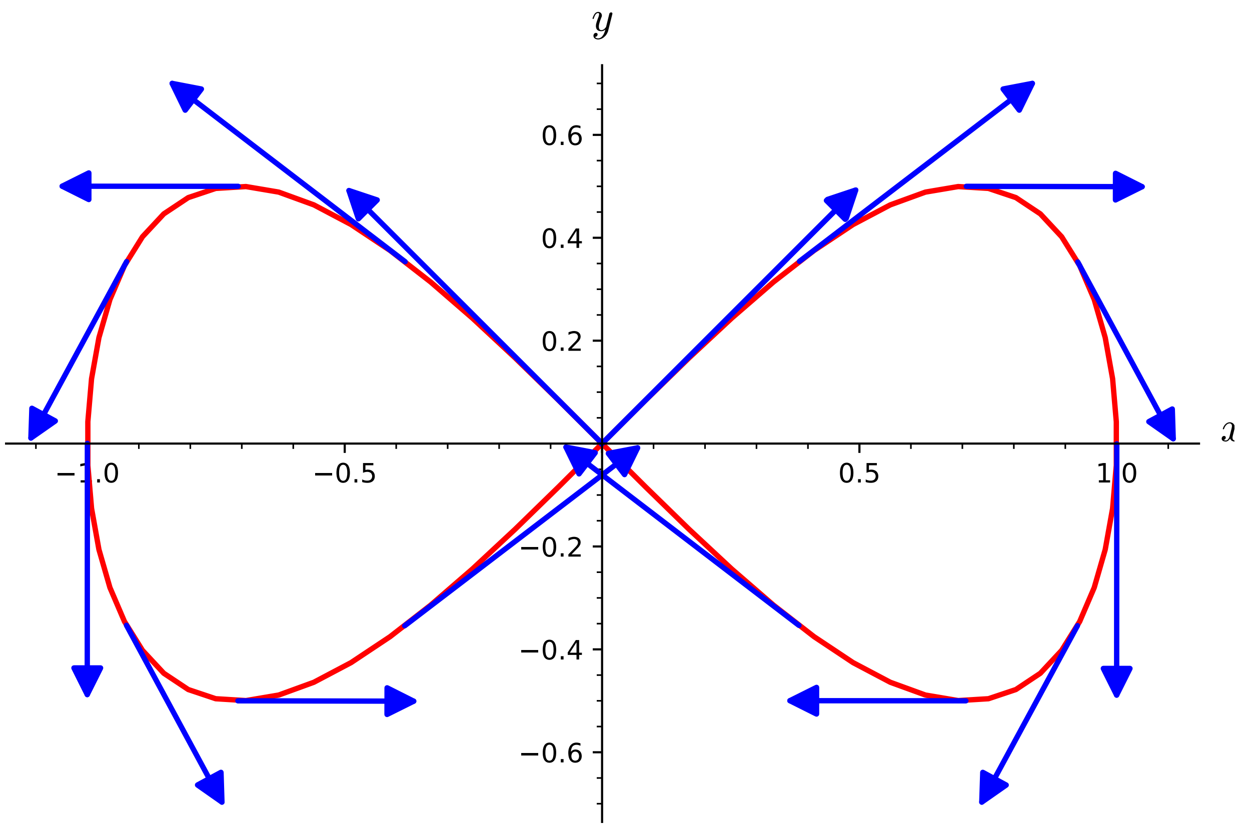
```
sage: v = c.tangent_vector_field() ; v
Vector field c' along the Real interval (0, 2*pi) with values on the
 2-dimensional differentiable manifold M
sage: v.display()
c' = cos(t) ∂/∂x + (2*cos(t)^2 - 1) ∂/∂y
```

```
>>> from sage.all import *
>>> v = c.tangent_vector_field() ; v
Vector field c' along the Real interval (0, 2*pi) with values on the
 2-dimensional differentiable manifold M
>>> v.display()
c' = cos(t) ∂/∂x + (2*cos(t)^2 - 1) ∂/∂y
```

Plot of the curve and its tangent vector field:

```
sage: show(c.plot(thickness=2, aspect_ratio=1) +
....:      v.plot(chart=X, number_values=17, scale=0.5))
```

```
>>> from sage.all import *
>>> show(c.plot(thickness=Integer(2), aspect_ratio=Integer(1)) +
...      v.plot(chart=X, number_values=Integer(17), scale=RealNumber('0.5')))
```



Value of the tangent vector field at $t = \pi$:

```
sage: v.at(R(pi))
Tangent vector c' at Point on the 2-dimensional differentiable
 manifold M
sage: v.at(R(pi)) in M.tangent_space(c(R(pi)))
True
sage: v.at(R(pi)).display()
c' = -∂/∂x + ∂/∂y
```

```
>>> from sage.all import *
>>> v.at(R(pi))
Tangent vector c' at Point on the 2-dimensional differentiable
 manifold M
>>> v.at(R(pi)) in M.tangent_space(c(R(pi)))
```

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```
True
>>> v.at(R(pi)).display()
c' = -∂/∂x + ∂/∂y
```

Curves $\mathbf{R} \to \mathbf{R}$ can be composed: the operator $\circ$ is given by `*`:

```
sage: f = R.curve(t^2, (t,-oo,+oo))
sage: g = R.curve(cos(t), (t,-oo,+oo))
sage: s = g*f ; s
Differentiable map from the Real number line ℝ to itself
sage: s.display()
ℝ → ℝ
   t ↦ cos(t^2)
sage: s = f*g ; s
Differentiable map from the Real number line ℝ to itself
sage: s.display()
ℝ → ℝ
   t ↦ cos(t)^2
```

```
>>> from sage.all import *
>>> f = R.curve(t**Integer(2), (t,-oo,+oo))
>>> g = R.curve(cos(t), (t,-oo,+oo))
>>> s = g*f ; s
Differentiable map from the Real number line ℝ to itself
>>> s.display()
ℝ → ℝ
   t ↦ cos(t^2)
>>> s = f*g ; s
Differentiable map from the Real number line ℝ to itself
>>> s.display()
ℝ → ℝ
   t ↦ cos(t)^2
```

Curvature and torsion of a curve in a Riemannian manifold

Let us consider a helix C in the Euclidean space $\mathbb{E}^3$ parametrized by its arc length s:

```
sage: E.<x,y,z> = EuclideanSpace()
sage: R.<s> = manifolds.RealLine()
sage: C = E.curve((2*cos(s/3), 2*sin(s/3), sqrt(5)*s/3), (s, 0, 6*pi),
....:             name='C')
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(names=('x', 'y', 'z',)); (x, y, z,) = E._first_ngens(3)
>>> R = manifolds.RealLine(names=('s',)); (s,) = R._first_ngens(1)
>>> C = E.curve((Integer(2)*cos(s/Integer(3)), Integer(2)*sin(s/Integer(3)),␣
↪sqrt(Integer(5))*s/Integer(3)), (s, Integer(0), Integer(6)*pi),
...             name='C')
```

Its tangent vector field is:

```
sage: T = C.tangent_vector_field()
sage: T.display()
C' = -2/3*sin(1/3*s) e_x + 2/3*cos(1/3*s) e_y + 1/3*sqrt(5) e_z
```

```
>>> from sage.all import *
>>> T = C.tangent_vector_field()
>>> T.display()
C' = -2/3*sin(1/3*s) e_x + 2/3*cos(1/3*s) e_y + 1/3*sqrt(5) e_z
```

Since C is parametrized by its arc length s, T is a unit vector (with respect to the Euclidean metric of $\mathbb{E}^3$):

```
sage: norm(T)
Scalar field |C'| on the Real interval (0, 6*pi)
sage: norm(T).display()
|C'|: (0, 6*pi) → ℝ
   s ↦ 1
```

```
>>> from sage.all import *
>>> norm(T)
Scalar field |C'| on the Real interval (0, 6*pi)
>>> norm(T).display()
|C'|: (0, 6*pi) → ℝ
   s ↦ 1
```

Vector fields along C are defined by the method `vector_field()` of the domain of C with the keyword argument `dest_map` set to C. For instance the derivative vector $T' = \mathrm{d}T/\mathrm{d}s$ is:

```
sage: I = C.domain(); I
Real interval (0, 6*pi)
sage: Tp = I.vector_field([diff(T[i], s) for i in E.irange()], dest_map=C,
....:                     name="T'")
sage: Tp.display()
T' = -2/9*cos(1/3*s) e_x - 2/9*sin(1/3*s) e_y
```

```
>>> from sage.all import *
>>> I = C.domain(); I
Real interval (0, 6*pi)
>>> Tp = I.vector_field([diff(T[i], s) for i in E.irange()], dest_map=C,
...                     name="T'")
>>> Tp.display()
T' = -2/9*cos(1/3*s) e_x - 2/9*sin(1/3*s) e_y
```

The norm of T' is the curvature of the helix:

```
sage: kappa = norm(Tp)
sage: kappa
Scalar field |T'| on the Real interval (0, 6*pi)
sage: kappa.expr()
2/9
```

```
>>> from sage.all import *
>>> kappa = norm(Tp)
```

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```
>>> kappa
Scalar field |T'| on the Real interval (0, 6*pi)
>>> kappa.expr()
2/9
```

The unit normal vector along C is:

```
sage: N = Tp / kappa
sage: N.display()
-cos(1/3*s) e_x - sin(1/3*s) e_y
```

```
>>> from sage.all import *
>>> N = Tp / kappa
>>> N.display()
-cos(1/3*s) e_x - sin(1/3*s) e_y
```

while the binormal vector along C is $B = T \times N$:

```
sage: B = T.cross_product(N)
sage: B
Vector field along the Real interval (0, 6*pi) with values on the
 Euclidean space E^3
sage: B.display()
1/3*sqrt(5)*sin(1/3*s) e_x - 1/3*sqrt(5)*cos(1/3*s) e_y + 2/3 e_z
```

```
>>> from sage.all import *
>>> B = T.cross_product(N)
>>> B
Vector field along the Real interval (0, 6*pi) with values on the
 Euclidean space E^3
>>> B.display()
1/3*sqrt(5)*sin(1/3*s) e_x - 1/3*sqrt(5)*cos(1/3*s) e_y + 2/3 e_z
```

The three vector fields (T, N, B) form the **Frenet-Serret frame** along C:

```
sage: FS = I.vector_frame(('T', 'N', 'B'), (T, N, B),
....:                     symbol_dual=('t', 'n', 'b'))
sage: FS
Vector frame ((0, 6*pi), (T,N,B)) with values on the Euclidean space E^3
```

```
>>> from sage.all import *
>>> FS = I.vector_frame(('T', 'N', 'B'), (T, N, B),
...                     symbol_dual=('t', 'n', 'b'))
>>> FS
Vector frame ((0, 6*pi), (T,N,B)) with values on the Euclidean space E^3
```

The Frenet-Serret frame is orthonormal:

```
sage: matrix([[u.dot(v).expr() for v in FS] for u in FS])
[1 0 0]
[0 1 0]
[0 0 1]
```

```
>>> from sage.all import *
>>> matrix([[u.dot(v).expr() for v in FS] for u in FS])
[1 0 0]
[0 1 0]
[0 0 1]
```

The derivative vectors N' and B':

```
sage: Np = I.vector_field([diff(N[i], s) for i in E.irange()],
....:                     dest_map=C, name="N'")
sage: Np.display()
N' = 1/3*sin(1/3*s) e_x - 1/3*cos(1/3*s) e_y
sage: Bp = I.vector_field([diff(B[i], s) for i in E.irange()],
....:                     dest_map=C, name="B'")
sage: Bp.display()
B' = 1/9*sqrt(5)*cos(1/3*s) e_x + 1/9*sqrt(5)*sin(1/3*s) e_y
```

```
>>> from sage.all import *
>>> Np = I.vector_field([diff(N[i], s) for i in E.irange()],
...                     dest_map=C, name="N'")
>>> Np.display()
N' = 1/3*sin(1/3*s) e_x - 1/3*cos(1/3*s) e_y
>>> Bp = I.vector_field([diff(B[i], s) for i in E.irange()],
...                     dest_map=C, name="B'")
>>> Bp.display()
B' = 1/9*sqrt(5)*cos(1/3*s) e_x + 1/9*sqrt(5)*sin(1/3*s) e_y
```

The Frenet-Serret formulas:

```
sage: for v in (Tp, Np, Bp):
....:     v.display(FS)
....:
T' = 2/9 N
N' = -2/9 T + 1/9*sqrt(5) B
B' = -1/9*sqrt(5) N
```

```
>>> from sage.all import *
>>> for v in (Tp, Np, Bp):
...     v.display(FS)
....:
T' = 2/9 N
N' = -2/9 T + 1/9*sqrt(5) B
B' = -1/9*sqrt(5) N
```

The torsion of C is obtained as the third component of N' in the Frenet-Serret frame:

```
sage: tau = Np[FS, 3]
sage: tau
1/9*sqrt(5)
```

```
>>> from sage.all import *
>>> tau = Np[FS, Integer(3)]
```

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```
>>> tau
1/9*sqrt(5)
```

coord_expr(*chart=None*)

Return the coordinate functions expressing the curve in a given chart.

INPUT:

- `chart` – (default: `None`) chart on the curve's codomain; if `None`, the codomain's default chart is assumed

OUTPUT: symbolic expression representing the curve in the above chart

EXAMPLES:

Cartesian and polar expression of a curve in the Euclidean plane:

```
sage: M = Manifold(2, 'R^2', r'\RR^2')  # the Euclidean plane R^2
sage: c_xy.<x,y> = M.chart() # Cartesian coordinate on R^2
sage: U = M.open_subset('U', coord_def={c_xy: (y!=0, x<0)}) # the complement␣
↪of the segment y=0 and x>0
sage: c_cart = c_xy.restrict(U) # Cartesian coordinates on U
sage: c_spher.<r,ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi') # spherical␣
↪coordinates on U
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', r'\RR^2')  # the Euclidean plane R^2
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪Cartesian coordinate on R^2
>>> U = M.open_subset('U', coord_def={c_xy: (y!=Integer(0), x<Integer(0))}) #␣
↪the complement of the segment y=0 and x>0
>>> c_cart = c_xy.restrict(U) # Cartesian coordinates on U
>>> c_spher = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi', names=('r', 'ph',)); (r,␣
↪ph,) = c_spher._first_ngens(2)# spherical coordinates on U
```

Links between spherical coordinates and Cartesian ones:

```
sage: ch_cart_spher = c_cart.transition_map(c_spher, [sqrt(x*x+y*y), atan2(y,
↪x)])
sage: ch_cart_spher.set_inverse(r*cos(ph), r*sin(ph))
Check of the inverse coordinate transformation:
  x == x  *passed*
  y == y  *passed*
  r == r  *passed*
  ph == arctan2(r*sin(ph), r*cos(ph))  **failed**
NB: a failed report can reflect a mere lack of simplification.
sage: R.<t> = manifolds.RealLine()
sage: c = U.curve({c_spher: (1,t)}, (t, 0, 2*pi), name='c')
sage: c.coord_expr(c_spher)
(1, t)
sage: c.coord_expr(c_cart)
(cos(t), sin(t))
```

```
>>> from sage.all import *
>>> ch_cart_spher = c_cart.transition_map(c_spher, [sqrt(x*x+y*y), atan2(y,
```

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```
↪x)])
>>> ch_cart_spher.set_inverse(r*cos(ph), r*sin(ph))
Check of the inverse coordinate transformation:
  x == x  *passed*
  y == y  *passed*
  r == r  *passed*
  ph == arctan2(r*sin(ph), r*cos(ph))  **failed**
NB: a failed report can reflect a mere lack of simplification.
>>> R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1)
>>> c = U.curve({c_spher: (Integer(1),t)}, (t, Integer(0), Integer(2)*pi),
↪name='c')
>>> c.coord_expr(c_spher)
(1, t)
>>> c.coord_expr(c_cart)
(cos(t), sin(t))
```

Since `c_cart` is the default chart on `U`, it can be omitted:

```
sage: c.coord_expr()
(cos(t), sin(t))
```

```
>>> from sage.all import *
>>> c.coord_expr()
(cos(t), sin(t))
```

Cartesian expression of a cardioid:

```
sage: c = U.curve({c_spher: (2*(1+cos(t)), t)}, (t, 0, 2*pi), name='c')
sage: c.coord_expr(c_cart)
(2*cos(t)^2 + 2*cos(t), 2*(cos(t) + 1)*sin(t))
```

```
>>> from sage.all import *
>>> c = U.curve({c_spher: (Integer(2)*(Integer(1)+cos(t)), t)}, (t,
↪Integer(0), Integer(2)*pi), name='c')
>>> c.coord_expr(c_cart)
(2*cos(t)^2 + 2*cos(t), 2*(cos(t) + 1)*sin(t))
```

plot(*chart=None*, *ambient_coords=None*, *mapping=None*, *prange=None*, *include_end_point=(True, True)*, *end_point_offset=(0.001, 0.001)*, *parameters=None*, *color='red'*, *style='-'*, *label_axes=True*, *thickness=1*, *plot_points=75*, *max_range=8*, *aspect_ratio='automatic'*, ***kwds*)

Plot the current curve in a Cartesian graph based on the coordinates of some ambient chart.

The curve is drawn in terms of two (2D graphics) or three (3D graphics) coordinates of a given chart, called hereafter the *ambient chart*. The ambient chart's domain must overlap with the curve's codomain or with the codomain of the composite curve $\Phi \circ c$, where c is the current curve and Φ some manifold differential map (argument `mapping` below).

INPUT:

- `chart` – (default: `None`) the ambient chart (see above); if `None`, the default chart of the codomain of the curve (or of the curve composed with Φ) is used
- `ambient_coords` – (default: `None`) tuple containing the 2 or 3 coordinates of the ambient chart in terms of which the plot is performed; if `None`, all the coordinates of the ambient chart are considered

- `mapping` – (default: `None`) differentiable mapping Φ (instance of `DiffMap`) providing the link between the curve and the ambient chart `chart` (cf. above); if `None`, the ambient chart is supposed to be defined on the codomain of the curve.
- `prange` – (default: `None`) range of the curve parameter for the plot; if `None`, the entire parameter range declared during the curve construction is considered (with -Infinity replaced by `-max_range` and +Infinity by `max_range`)
- `include_end_point` – (default: `(True, True)`) determines whether the end points of `prange` are included in the plot
- `end_point_offset` – (default: `(0.001, 0.001)`) offsets from the end points when they are not included in the plot: if `include_end_point[0] == False`, the minimal value of the curve parameter used for the plot is `prange[0] + end_point_offset[0]`, while if `include_end_point[1] == False`, the maximal value is `prange[1] - end_point_offset[1]`.
- `max_range` – (default: 8) numerical value substituted to +Infinity if the latter is the upper bound of the parameter range; similarly `-max_range` is the numerical valued substituted for -Infinity
- `parameters` – (default: `None`) dictionary giving the numerical values of the parameters that may appear in the coordinate expression of the curve
- `color` – (default: `'red'`) color of the drawn curve
- `style` – (default: `'-'`) color of the drawn curve; NB: `style` is effective only for 2D plots
- `thickness` – (default: 1) thickness of the drawn curve
- `plot_points` – (default: 75) number of points to plot the curve
- `label_axes` – boolean (default: `True`); determining whether the labels of the coordinate axes of `chart` shall be added to the graph; can be set to `False` if the graph is 3D and must be superposed with another graph.
- `aspect_ratio` – (default: `'automatic'`) aspect ratio of the plot; the default value (`'automatic'`) applies only for 2D plots; for 3D plots, the default value is `1` instead

OUTPUT:

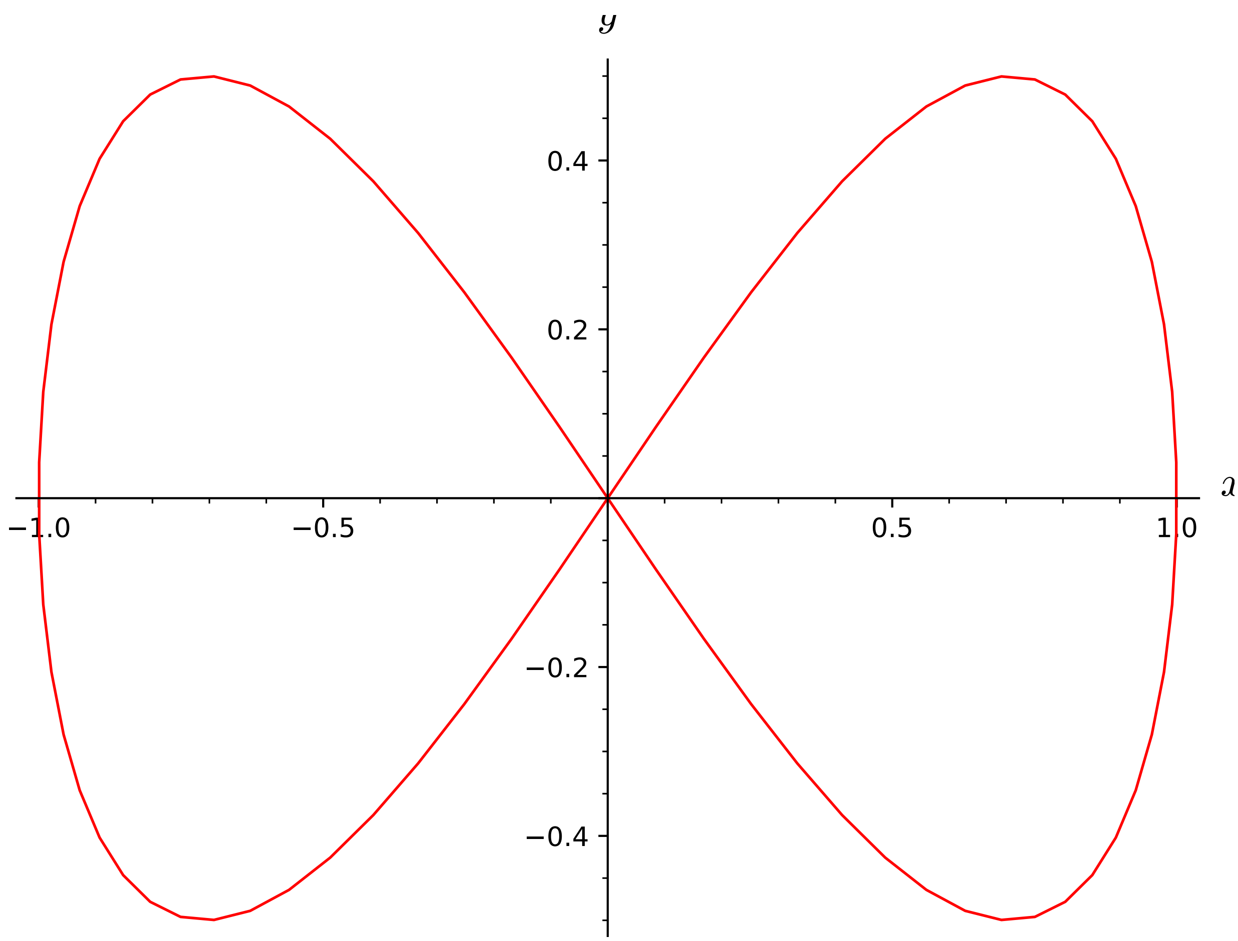
- a graphic object, either an instance of `Graphics` for a 2D plot (i.e. based on 2 coordinates of `chart`) or an instance of `Graphics3d` for a 3D plot (i.e. based on 3 coordinates of `chart`)

EXAMPLES:

Plot of the lemniscate of Gerono:

```
sage: R2 = Manifold(2, 'R^2')
sage: X.<x,y> = R2.chart()
sage: R.<t> = manifolds.RealLine()
sage: c = R2.curve([sin(t), sin(2*t)/2], (t, 0, 2*pi), name='c')
sage: c.plot()  # 2D plot
Graphics object consisting of 1 graphics primitive
```

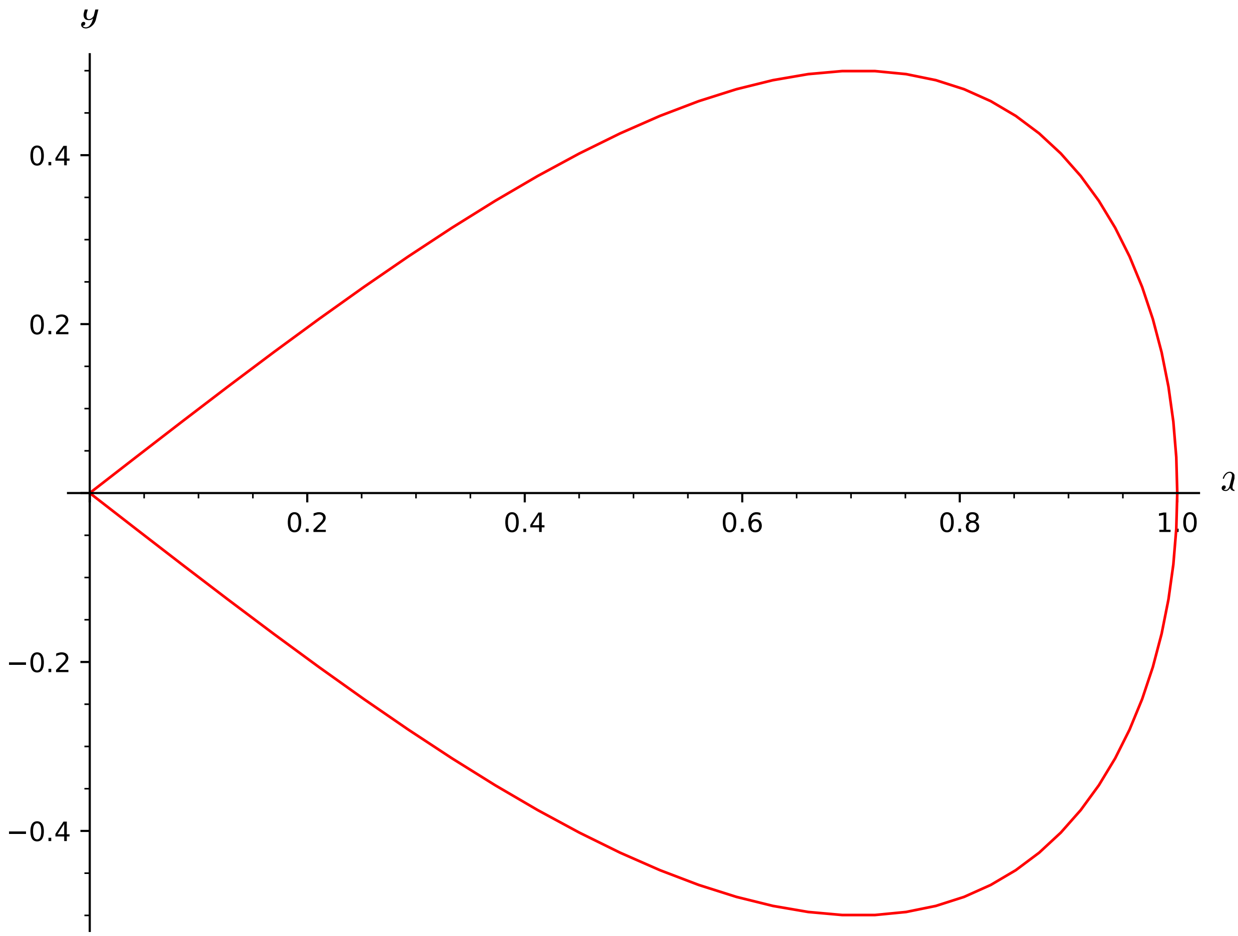
```
>>> from sage.all import *
>>> R2 = Manifold(Integer(2), 'R^2')
>>> X = R2.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1)
>>> c = R2.curve([sin(t), sin(Integer(2)*t)/Integer(2)], (t, Integer(0),␣
↪Integer(2)*pi), name='c')
>>> c.plot()  # 2D plot
Graphics object consisting of 1 graphics primitive
```


y
x
0.4
0.2
−0.2
−0.4
−1.0
−0.5
0.5
1.0

Plot for a subinterval of the curve's domain:

```
sage: c.plot(prange=(0,pi))
Graphics object consisting of 1 graphics primitive
```

```
>>> from sage.all import *
>>> c.plot(prange=(Integer(0),pi))
Graphics object consisting of 1 graphics primitive
```

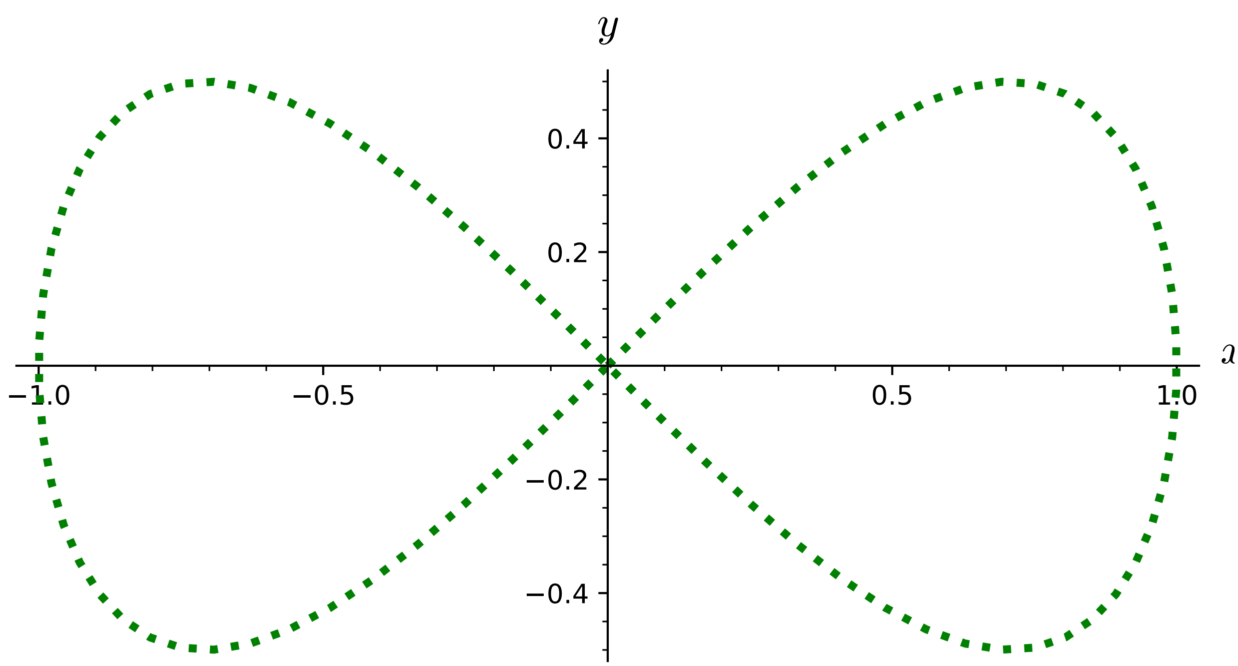


Plot with various options:

```
sage: c.plot(color='green', style=':', thickness=3, aspect_ratio=1)
Graphics object consisting of 1 graphics primitive
```

```
>>> from sage.all import *
>>> c.plot(color='green', style=':', thickness=Integer(3), aspect_
↪ratio=Integer(1))
Graphics object consisting of 1 graphics primitive
```

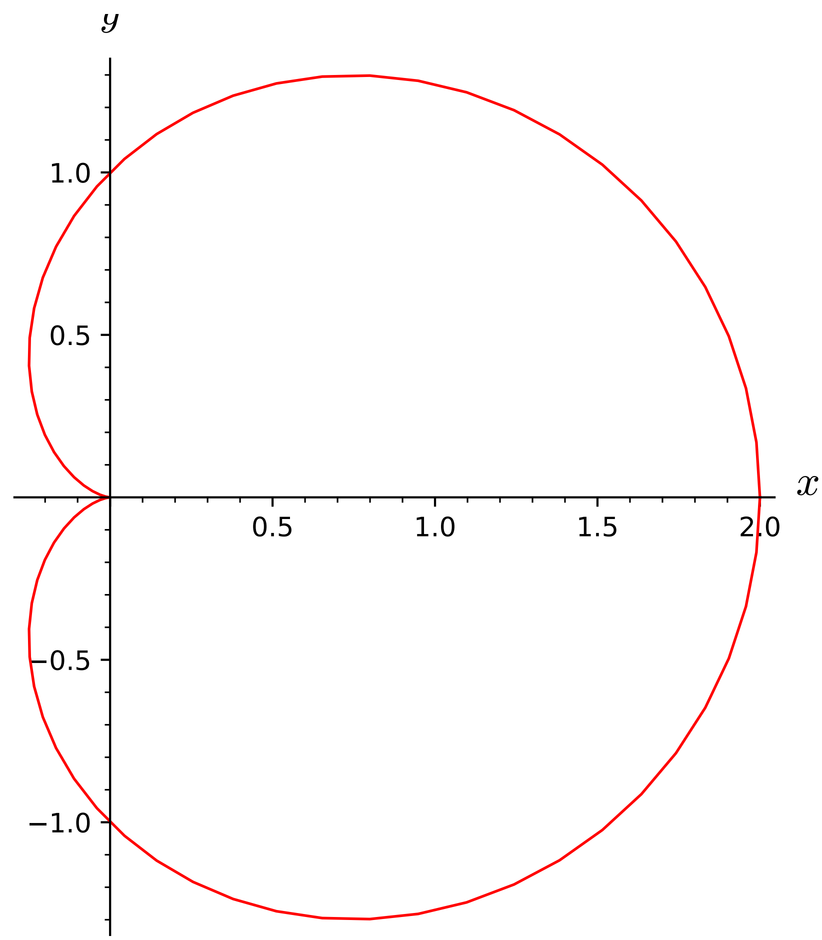
Cardioid defined in terms of polar coordinates and plotted with respect to Cartesian coordinates, as an example of use of the optional argument `chart`:



y
0.4
0.2
−0.2
−0.4
x
−1.0
−0.5
0.5
1.0

```
sage: E.<r,ph> = EuclideanSpace(coordinates='polar')
sage: c = E.curve((1 + cos(ph), ph), (ph, 0, 2*pi))
sage: c.plot(chart=E.cartesian_coordinates(), aspect_ratio=1)
Graphics object consisting of 1 graphics primitive
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(coordinates='polar', names=('r', 'ph',)); (r, ph,) = E.
↪_first_ngens(2)
>>> c = E.curve((Integer(1) + cos(ph), ph), (ph, Integer(0), Integer(2)*pi))
>>> c.plot(chart=E.cartesian_coordinates(), aspect_ratio=Integer(1))
Graphics object consisting of 1 graphics primitive
```



Plot via a mapping to another manifold: loxodrome of a sphere viewed in $\mathbf{R}^3$:

```
sage: S2 = Manifold(2, 'S^2')
sage: U = S2.open_subset('U')
sage: XS.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: R3 = Manifold(3, 'R^3')
sage: X3.<x,y,z> = R3.chart()
sage: F = S2.diff_map(R3, {(XS, X3): [sin(th)*cos(ph),
....:                          sin(th)*sin(ph), cos(th)]}, name='F')
sage: F.display()
```

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```
F: S^2 → R^3
on U: (th, ph) ↦ (x, y, z) = (cos(ph)*sin(th), sin(ph)*sin(th), cos(th))
sage: c = S2.curve([2*atan(exp(-t/10)), t], (t, -oo, +oo), name='c')
sage: graph_c = c.plot(mapping=F, max_range=40,
....:                  plot_points=200, thickness=2, label_axes=False)  # 3D␣
↪plot
sage: graph_S2 = XS.plot(X3, mapping=F, number_values=11, color='black') #␣
↪plot of the sphere
sage: show(graph_c + graph_S2) # the loxodrome + the sphere
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2')
>>> U = S2.open_subset('U')
>>> XS = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi', names=('th', 'ph',));␣
↪(th, ph,) = XS._first_ngens(2)
>>> R3 = Manifold(Integer(3), 'R^3')
>>> X3 = R3.chart(names=('x', 'y', 'z',)); (x, y, z,) = X3._first_ngens(3)
>>> F = S2.diff_map(R3, {(XS, X3): [sin(th)*cos(ph),
...                      sin(th)*sin(ph), cos(th)]}, name='F')
>>> F.display()
F: S^2 → R^3
on U: (th, ph) ↦ (x, y, z) = (cos(ph)*sin(th), sin(ph)*sin(th), cos(th))
>>> c = S2.curve([Integer(2)*atan(exp(-t/Integer(10))), t], (t, -oo, +oo),␣
↪name='c')
>>> graph_c = c.plot(mapping=F, max_range=Integer(40),
...                  plot_points=Integer(200), thickness=Integer(2), label_
↪axes=False)  # 3D plot
>>> graph_S2 = XS.plot(X3, mapping=F, number_values=Integer(11), color='black
↪') # plot of the sphere
>>> show(graph_c + graph_S2) # the loxodrome + the sphere
```

Example of use of the argument `parameters`: we define a curve with some symbolic parameters `a` and `b`:

```
sage: a, b = var('a b')
sage: c = R2.curve([a*cos(t) + b, a*sin(t)], (t, 0, 2*pi), name='c')
```

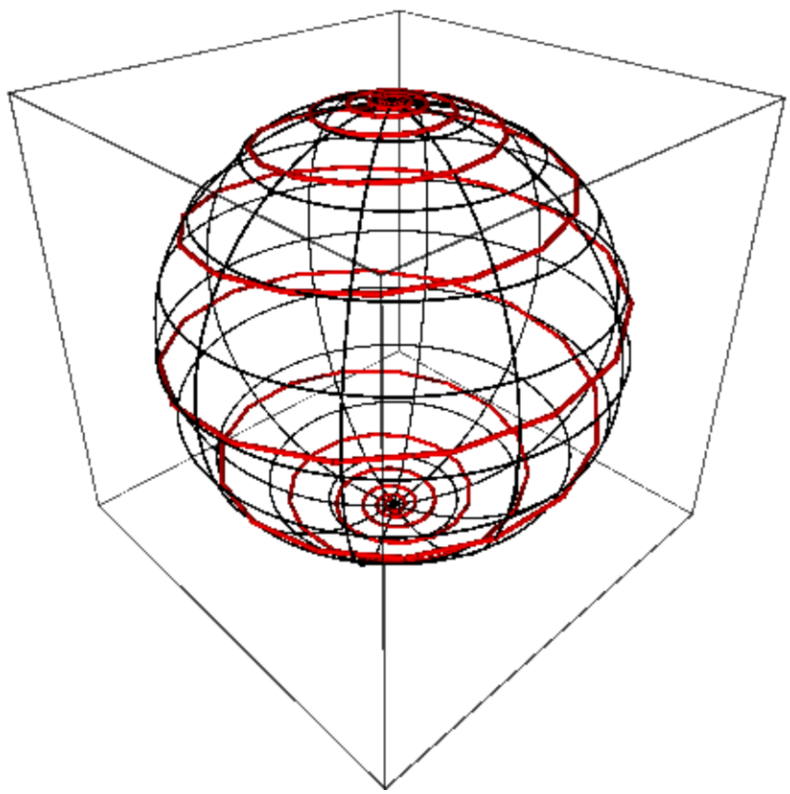
```
>>> from sage.all import *
>>> a, b = var('a b')
>>> c = R2.curve([a*cos(t) + b, a*sin(t)], (t, Integer(0), Integer(2)*pi),␣
↪name='c')
```

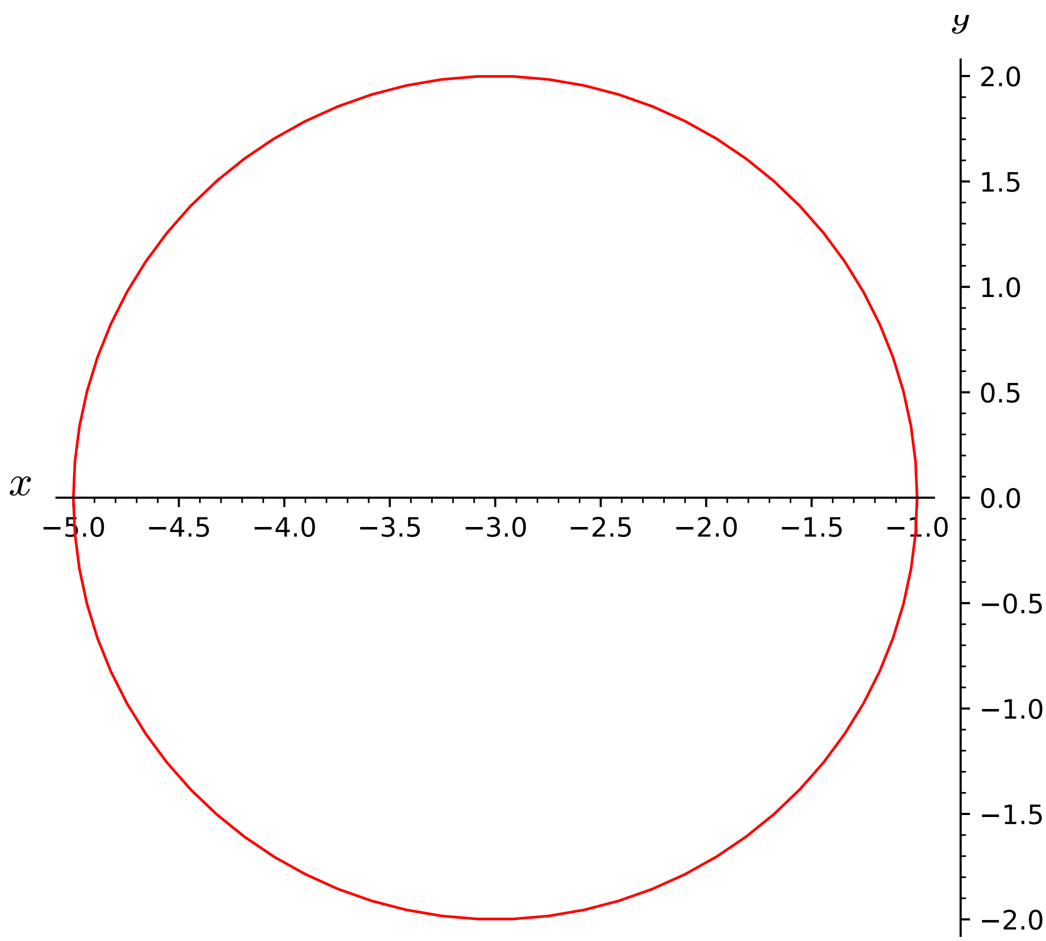
To make a plot, we set specific values for `a` and `b` by means of the Python dictionary `parameters`:

```
sage: c.plot(parameters={a: 2, b: -3}, aspect_ratio=1)
Graphics object consisting of 1 graphics primitive
```

```
>>> from sage.all import *
>>> c.plot(parameters={a: Integer(2), b: -Integer(3)}, aspect_
↪ratio=Integer(1))
Graphics object consisting of 1 graphics primitive
```

tangent_vector_field(*name=None*, *latex_name=None*)





y
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
x
−5.0
−4.5
−4.0
−3.5
−3.0
−2.5
−2.0
−1.5
−1.0

Return the tangent vector field to the curve (velocity vector).

INPUT:

- `name` – (default: `None`) string; symbol given to the tangent vector field; if none is provided, the primed curve symbol (if any) will be used
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the tangent vector field; if `None` then (i) if `name` is `None` as well, the primed curve LaTeX symbol (if any) will be used or (ii) if `name` is not `None`, `name` will be used

OUTPUT:

- the tangent vector field, as an instance of *VectorField*

EXAMPLES:

Tangent vector field to a circle curve in $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'R^2')
sage: X.<x,y> = M.chart()
sage: R.<t> = manifolds.RealLine()
sage: c = M.curve([cos(t), sin(t)], (t, 0, 2*pi), name='c')
sage: v = c.tangent_vector_field() ; v
Vector field c' along the Real interval (0, 2*pi) with values on
 the 2-dimensional differentiable manifold R^2
sage: v.display()
c' = -sin(t) ∂/∂x + cos(t) ∂/∂y
sage: latex(v)
{c'}
sage: v.parent()
Free module X((0, 2*pi),c) of vector fields along the Real interval
 (0, 2*pi) mapped into the 2-dimensional differentiable manifold R^2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1)
>>> c = M.curve([cos(t), sin(t)], (t, Integer(0), Integer(2)*pi), name='c')
>>> v = c.tangent_vector_field() ; v
Vector field c' along the Real interval (0, 2*pi) with values on
 the 2-dimensional differentiable manifold R^2
>>> v.display()
c' = -sin(t) ∂/∂x + cos(t) ∂/∂y
>>> latex(v)
{c'}
>>> v.parent()
Free module X((0, 2*pi),c) of vector fields along the Real interval
 (0, 2*pi) mapped into the 2-dimensional differentiable manifold R^2
```

Value of the tangent vector field for some specific value of the curve parameter ($t = \pi$):

```
sage: R(pi) in c.domain()  # pi in (0, 2*pi)
True
sage: vp = v.at(R(pi)) ; vp
Tangent vector c' at Point on the 2-dimensional differentiable
```

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```
 manifold R^2
sage: vp.parent() is M.tangent_space(c(R(pi)))
True
sage: vp.display()
c' = -∂/∂y
```

```
>>> from sage.all import *
>>> R(pi) in c.domain()  # pi in (0, 2*pi)
True
>>> vp = v.at(R(pi)) ; vp
Tangent vector c' at Point on the 2-dimensional differentiable
 manifold R^2
>>> vp.parent() is M.tangent_space(c(R(pi)))
True
>>> vp.display()
c' = -∂/∂y
```

Tangent vector field to a curve in a non-parallelizable manifold (the 2-sphere S^2): first, we introduce the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                    intersection_name='W', restrictions1= x^2+y^2!=0,
....:                    restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: A = W.open_subset('A', coord_def={c_xy.restrict(W): (y!=0, x<0)})
sage: c_spher.<th,ph> = A.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi') #␣
↪spherical coordinates
sage: spher_to_xy = c_spher.transition_map(c_xy.restrict(A),
....:              (sin(th)*cos(ph)/(1-cos(th)), sin(th)*sin(ph)/(1-cos(th))) )
sage: spher_to_xy.set_inverse(2*atan(1/sqrt(x^2+y^2)), atan2(y, x),␣
↪check=False)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                    intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
```

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```
...                    restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V)
>>> A = W.open_subset('A', coord_def={c_xy.restrict(W): (y!=Integer(0), x
↪<Integer(0))})
>>> c_spher = A.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi', names=('th', 'ph',
↪)); (th, ph,) = c_spher._first_ngens(2)# spherical coordinates
>>> spher_to_xy = c_spher.transition_map(c_xy.restrict(A),
...            (sin(th)*cos(ph)/(Integer(1)-cos(th)), sin(th)*sin(ph)/
↪(Integer(1)-cos(th))) )
>>> spher_to_xy.set_inverse(Integer(2)*atan(Integer(1)/
↪sqrt(x**Integer(2)+y**Integer(2))), atan2(y, x), check=False)
```

Then we define a curve (a loxodrome) by its expression in terms of spherical coordinates and evaluate the tangent vector field:

```
sage: R.<t> = manifolds.RealLine()
sage: c = M.curve({c_spher: [2*atan(exp(-t/10)), t]}, (t, -oo, +oo),
....:             name='c') ; c
Curve c in the 2-dimensional differentiable manifold M
sage: vc = c.tangent_vector_field() ; vc
Vector field c' along the Real number line ℝ with values on
 the 2-dimensional differentiable manifold M
sage: vc.parent()
Module X(ℝ,c) of vector fields along the Real number line ℝ
 mapped into the 2-dimensional differentiable manifold M
sage: vc.display(c_spher.frame().along(c.restrict(R,A)))
c' = -1/5*e^(1/10*t)/(e^(1/5*t) + 1) ∂/∂th + ∂/∂ph
```

```
>>> from sage.all import *
>>> R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1)
>>> c = M.curve({c_spher: [Integer(2)*atan(exp(-t/Integer(10))), t]}, (t, -oo,
↪ +oo),
...             name='c') ; c
Curve c in the 2-dimensional differentiable manifold M
>>> vc = c.tangent_vector_field() ; vc
Vector field c' along the Real number line ℝ with values on
 the 2-dimensional differentiable manifold M
>>> vc.parent()
Module X(ℝ,c) of vector fields along the Real number line ℝ
 mapped into the 2-dimensional differentiable manifold M
>>> vc.display(c_spher.frame().along(c.restrict(R,A)))
c' = -1/5*e^(1/10*t)/(e^(1/5*t) + 1) ∂/∂th + ∂/∂ph
```

2.5.4 Integrated Curves and Geodesics in Manifolds

Given a differentiable manifold M, an *integrated curve* in M is a differentiable curve constructed as a solution to a system of second order differential equations.

Integrated curves are implemented by the class `IntegratedCurve`, from which the classes `IntegratedAutoparallelCurve` and `IntegratedGeodesic` inherit.

Example: a geodesic in the hyperbolic plane

First declare the hyperbolic plane as a 2-dimensional Riemannian manifold M and introduce the chart X corresponding to the Poincaré half-plane model:

```
sage: M = Manifold(2, 'M', structure='Riemannian')
sage: X.<x,y> = M.chart('x y:(0,+oo)')
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='Riemannian')
>>> X = M.chart('x y:(0,+oo)', names=('x', 'y',)); (x, y,) = X._first_ngens(2)
```

Then set the metric to be the hyperbolic one:

```
sage: g = M.metric()
sage: g[0,0], g[1,1] = 1/y^2, 1/y^2
sage: g.display()
g = y^(-2) dx⊗dx + y^(-2) dy⊗dy
```

```
>>> from sage.all import *
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)] = Integer(1)/y**Integer(2),␣
↪Integer(1)/y**Integer(2)
>>> g.display()
g = y^(-2) dx⊗dx + y^(-2) dy⊗dy
```

Pick an initial point and an initial tangent vector:

```
sage: p = M((0,1), name='p')
sage: v = M.tangent_space(p)((1,3/2), name='v')
sage: v.display()
v = ∂/∂x + 3/2 ∂/∂y
```

```
>>> from sage.all import *
>>> p = M((Integer(0),Integer(1)), name='p')
>>> v = M.tangent_space(p)((Integer(1),Integer(3)/Integer(2)), name='v')
>>> v.display()
v = ∂/∂x + 3/2 ∂/∂y
```

Declare a geodesic with such initial conditions, denoting by t the corresponding affine parameter:

```
sage: t = var('t')
sage: c = M.integrated_geodesic(g, (t, 0, 10), v, name='c')
```

```
>>> from sage.all import *
>>> t = var('t')
>>> c = M.integrated_geodesic(g, (t, Integer(0), Integer(10)), v, name='c')
```

Numerically integrate the geodesic (see `solve()` for all possible options, including the choice of the numerical algorithm):

```
sage: sol = c.solve()                                                          #␣
↪needs scipy
```

```
>>> from sage.all import *
>>> sol = c.solve()                                                          #␣
↪needs scipy
```

Plot the geodesic after interpolating the solution `sol`:

```
sage: interp = c.interpolate()

sage: # needs sage.plot
sage: graph = c.plot_integrated()
sage: p_plot = p.plot(size=30, label_offset=-0.07, fontsize=20)
sage: v_plot = v.plot(label_offset=0.05, fontsize=20)
sage: graph + p_plot + v_plot
Graphics object consisting of 5 graphics primitives
```

```
>>> from sage.all import *
>>> interp = c.interpolate()

>>> # needs sage.plot
>>> graph = c.plot_integrated()
>>> p_plot = p.plot(size=Integer(30), label_offset=-RealNumber('0.07'),␣
↪fontsize=Integer(20))
>>> v_plot = v.plot(label_offset=RealNumber('0.05'), fontsize=Integer(20))
>>> graph + p_plot + v_plot
Graphics object consisting of 5 graphics primitives
```

c is a differentiable curve in M and inherits from the properties of `DifferentiableCurve`:

```
sage: c.domain()
Real interval (0, 10)
sage: c.codomain()
2-dimensional Riemannian manifold M
sage: c.display()
c: (0, 10) → M
```

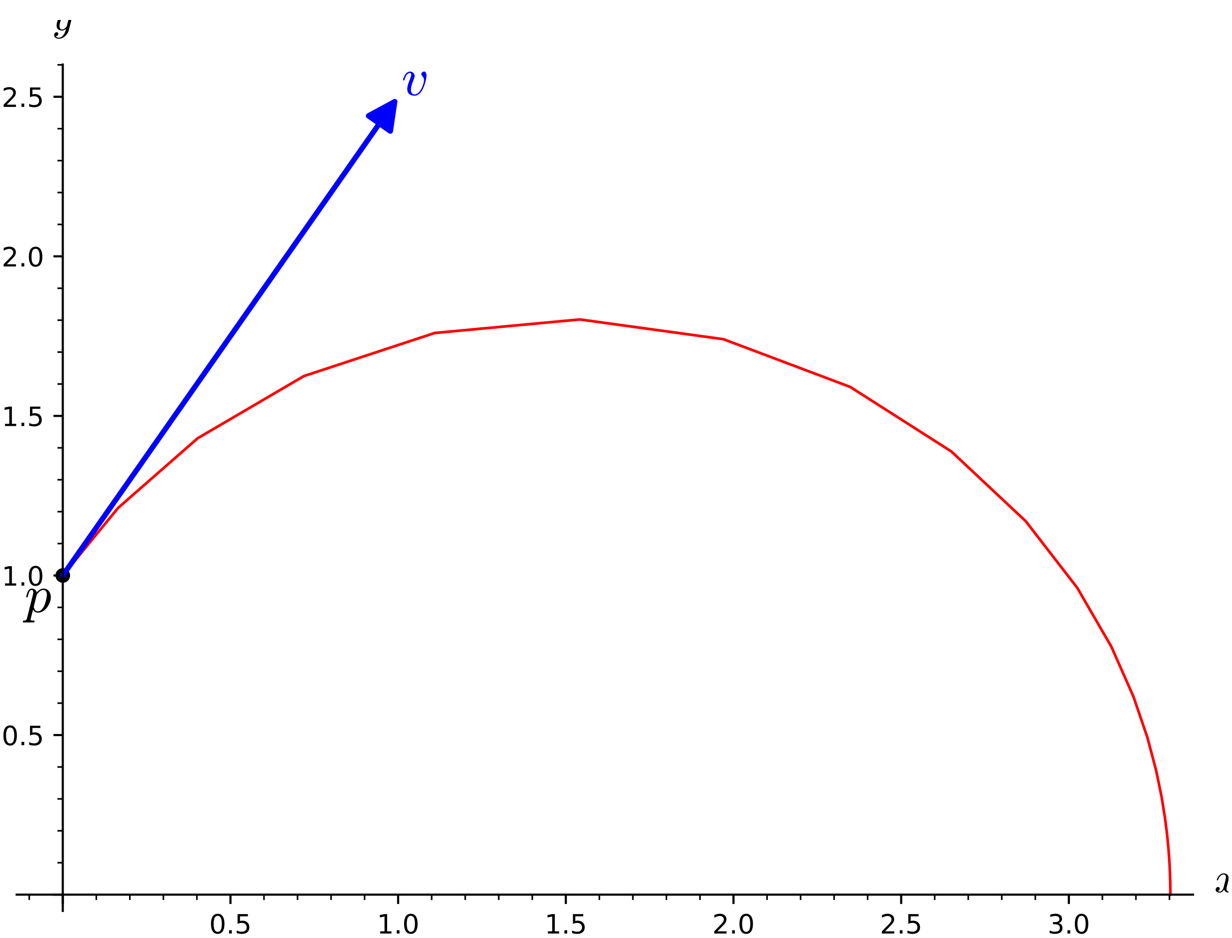
```
>>> from sage.all import *
>>> c.domain()
Real interval (0, 10)
>>> c.codomain()
2-dimensional Riemannian manifold M
>>> c.display()
c: (0, 10) → M
```

In particular, its value at $t = 1$ is:

```
sage: c(1)
Point on the 2-dimensional Riemannian manifold M
```

```
>>> from sage.all import *
>>> c(Integer(1))
Point on the 2-dimensional Riemannian manifold M
```

which corresponds to the following (x, y) coordinates:



y
v
2.5
2.0
1.5
1.0
p
0.5
x
0.5
1.0
1.5
2.0
2.5
3.0

```
sage: X(c(1))  # abs tol 1e-12
(2.4784140715580136, 1.5141683866138937)
```

```
>>> from sage.all import *
>>> X(c(Integer(1)))  # abs tol 1e-12
(2.4784140715580136, 1.5141683866138937)
```

AUTHORS:

- Karim Van Aelst (2017): initial version
- Florentin Jaffredo (2018): integration over multiple charts, use of `fast_callable` to improve the computation speed

class `sage.manifolds.differentiable.integrated_curve.`**`IntegratedAutoparallelCurve`**(*parent*, *affine_connection*, *curve_parameter*, *initial_tangent_vector*, *chart=None*, *name=None*, *latex_name=None*, *verbose=False*, *across_charts=False*)

Bases: *IntegratedCurve*

Autoparallel curve on the manifold with respect to a given affine connection.

INPUT:

- `parent` – *IntegratedAutoparallelCurveSet* the set of curves $\mathrm{Hom}_{\mathrm{autoparallel}}(I, M)$ to which the curve belongs
- `affine_connection` – *AffineConnection* affine connection with respect to which the curve is autoparallel
- `curve_parameter` – symbolic expression to be used as the parameter of the curve (the equations defining an instance of IntegratedAutoparallelCurve are such that `t` will actually be an affine parameter of the curve)
- `initial_tangent_vector` – *TangentVector* initial tangent vector of the curve
- `chart` – (default: `None`) chart on the manifold in terms of which the equations are expressed; if `None` the default chart of the manifold is assumed
- `name` – (default: `None`) string; symbol given to the curve
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the curve; if none is provided, `name` will be used

EXAMPLES:

Autoparallel curves associated with the Mercator projection of the unit 2-sphere $\mathbb{S}^2$.

See also

https://idontgetoutmuch.wordpress.com/2016/11/24/mercator-a-connection-with-torsion/ for more details about Mercator projection.

On the Mercator projection, the lines of longitude all appear vertical and then all parallel with respect to each other. Likewise, all the lines of latitude appear horizontal and parallel with respect to each other. These curves may be recovered as autoparallel curves of a certain connection ∇ to be made explicit.

Start with declaring the standard polar coordinates (θ, ϕ) on $\mathbb{S}^2$ and the corresponding coordinate frame (e_θ, e_ϕ):

```
sage: S2 = Manifold(2, 'S^2', start_index=1)
sage: polar.<th,ph>=S2.chart()
sage: epolar = polar.frame()
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2', start_index=Integer(1))
>>> polar = S2.chart(names=('th', 'ph',)); (th, ph,) = polar._first_ngens(2)
>>> epolar = polar.frame()
```

Normalizing e_ϕ provides an orthonormal basis:

```
sage: ch_basis = S2.automorphism_field()
sage: ch_basis[1,1], ch_basis[2,2] = 1, 1/sin(th)
sage: epolar_ON = epolar.new_frame(ch_basis,'epolar_ON')
```

```
>>> from sage.all import *
>>> ch_basis = S2.automorphism_field()
>>> ch_basis[Integer(1),Integer(1)], ch_basis[Integer(2),Integer(2)] = Integer(1),
↪ Integer(1)/sin(th)
>>> epolar_ON = epolar.new_frame(ch_basis,'epolar_ON')
```

Denote $(\hat{e}_\theta, \hat{e}_\phi)$ such an orthonormal frame field. In any point, the vector field $\hat{e}_\theta$ is normalized and tangent to the line of longitude through the point. Likewise, $\hat{e}_\phi$ is normalized and tangent to the line of latitude.

Now, set an affine connection with respect to such fields that are parallelly transported in all directions, that is: $\nabla \hat{e}_\theta = \nabla \hat{e}_\phi = 0$. This is equivalent to setting all the connection coefficients to zero with respect to this frame:

```
sage: nab = S2.affine_connection('nab')
sage: nab.set_coef(frame=epolar_ON)[:]
[[[0, 0], [0, 0]], [[0, 0], [0, 0]]]
```

```
>>> from sage.all import *
>>> nab = S2.affine_connection('nab')
>>> nab.set_coef(frame=epolar_ON)[:]
[[[0, 0], [0, 0]], [[0, 0], [0, 0]]]
```

This connection is such that two vectors are parallel if their angles to a given meridian are the same. Check that this connection is compatible with the Euclidean metric tensor g induced on $\mathbb{S}^2$:

```
sage: g = S2.metric('g')
sage: g[1,1], g[2,2] = 1, (sin(th))^2
sage: nab(g)[:]
[[[0, 0], [0, 0]], [[0, 0], [0, 0]]]
```

```
>>> from sage.all import *
>>> g = S2.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)] = Integer(1),
↪(sin(th))**Integer(2)
>>> nab(g)[:]
[[[0, 0], [0, 0]], [[0, 0], [0, 0]]]
```

Yet, this connection is not the Levi-Civita connection, which implies that it has non-vanishing torsion:

```
sage: nab.torsion()[:]
[[[0, 0], [0, 0]], [[0, cos(th)/sin(th)], [-cos(th)/sin(th), 0]]]
```

```
>>> from sage.all import *
>>> nab.torsion()[:]
[[[0, 0], [0, 0]], [[0, cos(th)/sin(th)], [-cos(th)/sin(th), 0]]]
```

Set generic initial conditions for the autoparallel curves to compute:

```
sage: [th0, ph0, v_th0, v_ph0] = var('th0 ph0 v_th0 v_ph0')
sage: p = S2.point((th0, ph0), name='p')
sage: Tp = S2.tangent_space(p)
sage: v = Tp((v_th0, v_ph0), basis=epolar_ON.at(p))
```

```
>>> from sage.all import *
>>> [th0, ph0, v_th0, v_ph0] = var('th0 ph0 v_th0 v_ph0')
>>> p = S2.point((th0, ph0), name='p')
>>> Tp = S2.tangent_space(p)
>>> v = Tp((v_th0, v_ph0), basis=epolar_ON.at(p))
```

Note here that the components `(v_th0, v_ph0)` of the initial tangent vector `v` refer to the basis `epolar_ON` $= (\hat{e}_\theta, \hat{e}_\phi)$ and not the coordinate basis `epolar` $= (e_\theta, e_\phi)$. This is merely to help picture the aspect of the tangent vector in the usual embedding of $\mathbb{S}^2$ in $\mathbb{R}^3$ thanks to using an orthonormal frame, since providing the components with respect to the coordinate basis would require multiplying the second component (i.e. the ϕ component) in order to picture the vector in the same way. This subtlety will need to be taken into account later when the numerical curve will be compared to the analytical solution.

Now, declare the corresponding integrated autoparallel curve and display the differential system it satisfies:

```
sage: [t, tmin, tmax] = var('t tmin tmax')
sage: c = S2.integrated_autoparallel_curve(nab, (t, tmin, tmax),
....:                                        v, chart=polar, name='c')
sage: sys = c.system(verbose=True)
Autoparallel curve c in the 2-dimensional differentiable
 manifold S^2 equipped with Affine connection nab on the
 2-dimensional differentiable manifold S^2, and integrated over
 the Real interval (tmin, tmax) as a solution to the following
 equations, written with respect to Chart (S^2, (th, ph)):

Initial point: Point p on the 2-dimensional differentiable
 manifold S^2 with coordinates [th0, ph0] with respect to
 Chart (S^2, (th, ph))
Initial tangent vector: Tangent vector at Point p on the
 2-dimensional differentiable manifold S^2 with
```

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```
 components [v_th0, v_ph0/sin(th0)] with respect to Chart (S^2, (th, ph))

d(th)/dt = Dth
d(ph)/dt = Dph
d(Dth)/dt = 0
d(Dph)/dt = -Dph*Dth*cos(th)/sin(th)
```

```
>>> from sage.all import *
>>> [t, tmin, tmax] = var('t tmin tmax')
>>> c = S2.integrated_autoparallel_curve(nab, (t, tmin, tmax),
...                                       v, chart=polar, name='c')
>>> sys = c.system(verbose=True)
Autoparallel curve c in the 2-dimensional differentiable
 manifold S^2 equipped with Affine connection nab on the
 2-dimensional differentiable manifold S^2, and integrated over
 the Real interval (tmin, tmax) as a solution to the following
 equations, written with respect to Chart (S^2, (th, ph)):
<BLANKLINE>
Initial point: Point p on the 2-dimensional differentiable
 manifold S^2 with coordinates [th0, ph0] with respect to
 Chart (S^2, (th, ph))
Initial tangent vector: Tangent vector at Point p on the
 2-dimensional differentiable manifold S^2 with
 components [v_th0, v_ph0/sin(th0)] with respect to Chart (S^2, (th, ph))
<BLANKLINE>
d(th)/dt = Dth
d(ph)/dt = Dph
d(Dth)/dt = 0
d(Dph)/dt = -Dph*Dth*cos(th)/sin(th)
<BLANKLINE>
```

Set a dictionary providing the parameter range and the initial conditions for a line of latitude and a line of longitude:

```
sage: dict_params={'latit':{tmin:0,tmax:3,th0:pi/4,ph0:0.1,v_th0:0,v_ph0:1},
....:   'longi':{tmin:0,tmax:3,th0:0.1,ph0:0.1,v_th0:1,v_ph0:0}}
```

```
>>> from sage.all import *
>>> dict_params={'latit':{tmin:Integer(0),tmax:Integer(3),th0:pi/Integer(4),
↪ph0:RealNumber('0.1'),v_th0:Integer(0),v_ph0:Integer(1)},
...   'longi':{tmin:Integer(0),tmax:Integer(3),th0:RealNumber('0.1'),
↪ph0:RealNumber('0.1'),v_th0:Integer(1),v_ph0:Integer(0)}}
```

Declare the Mercator coordinates (ξ, ζ) and the corresponding coordinate change from the polar coordinates:

```
sage: mercator.<xi,ze> = S2.chart(r'xi:(-oo,oo):\xi ze:(0,2*pi):\zeta')
sage: polar.transition_map(mercator, (log(tan(th/2)), ph))
Change of coordinates from Chart (S^2, (th, ph)) to Chart
 (S^2, (xi, ze))
```

```
>>> from sage.all import *
>>> mercator = S2.chart(r'xi:(-oo,oo):\xi ze:(0,2*pi):\zeta', names=('xi', 'ze',
↪)); (xi, ze,) = mercator._first_ngens(2)
```

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```
>>> polar.transition_map(mercator, (log(tan(th/Integer(2))), ph))
Change of coordinates from Chart (S^2, (th, ph)) to Chart
 (S^2, (xi, ze))
```

Ask for the identity map in terms of these charts in order to add this coordinate change to its dictionary of expressions. This is required to plot the curve with respect to the Mercator chart:

```
sage: identity = S2.identity_map()
sage: identity.coord_functions(polar, mercator)
Coordinate functions (log(sin(1/2*th)/cos(1/2*th)), ph) on the
 Chart (S^2, (th, ph))
```

```
>>> from sage.all import *
>>> identity = S2.identity_map()
>>> identity.coord_functions(polar, mercator)
Coordinate functions (log(sin(1/2*th)/cos(1/2*th)), ph) on the
 Chart (S^2, (th, ph))
```

Solve, interpolate and prepare the plot for the solutions corresponding to the two initial conditions previously set:

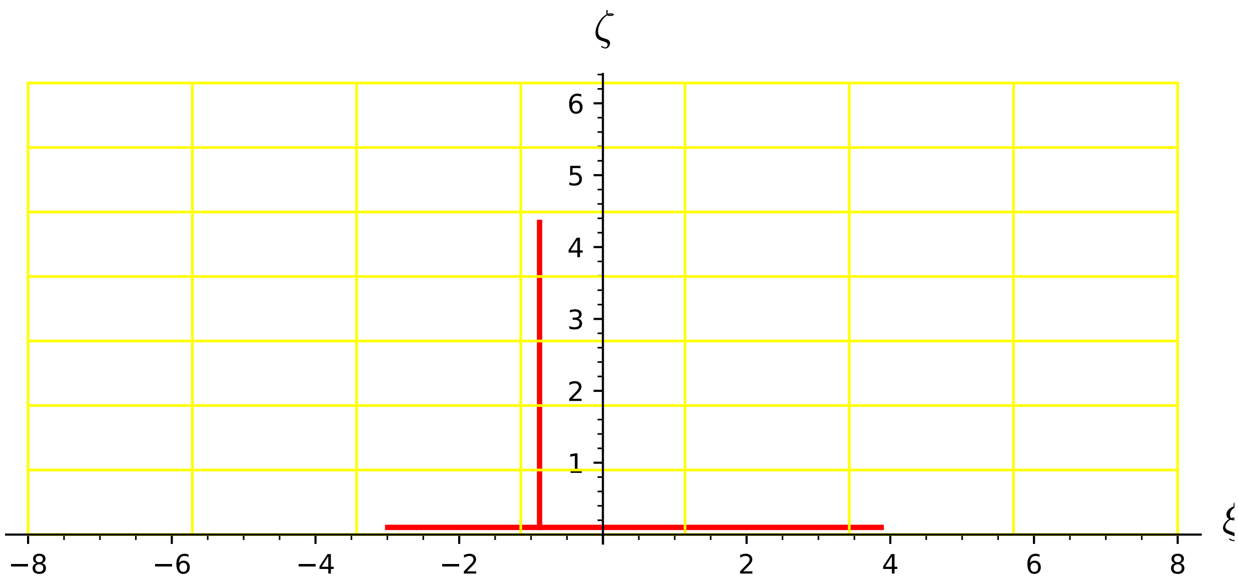
```
sage: graph2D_mercator = Graphics()
sage: for key in dict_params:
....:     sol = c.solve(solution_key='sol-'+key,
....:                         parameters_values=dict_params[key])
....:     interp = c.interpolate(solution_key='sol-'+key,
....:                            interpolation_key='interp-'+key)
....:     graph2D_mercator+=c.plot_integrated(interpolation_key='interp-'+key,
....:                               chart=mercator, thickness=2)
```

```
>>> from sage.all import *
>>> graph2D_mercator = Graphics()
>>> for key in dict_params:
...     sol = c.solve(solution_key='sol-'+key,
...                         parameters_values=dict_params[key])
...     interp = c.interpolate(solution_key='sol-'+key,
...                            interpolation_key='interp-'+key)
...     graph2D_mercator+=c.plot_integrated(interpolation_key='interp-'+key,
...                               chart=mercator, thickness=Integer(2))
```

Prepare a grid of Mercator coordinates lines, and plot the curves over it:

```
sage: graph2D_mercator_coords=mercator.plot(chart=mercator,
....:                                 number_values=8,color='yellow')
sage: graph2D_mercator + graph2D_mercator_coords
Graphics object consisting of 18 graphics primitives
```

```
>>> from sage.all import *
>>> graph2D_mercator_coords=mercator.plot(chart=mercator,
...                                 number_values=Integer(8),color='yellow')
>>> graph2D_mercator + graph2D_mercator_coords
Graphics object consisting of 18 graphics primitives
```



ζ
6
5
4
3
2
1
ξ
−8
−6
−4
−2
2
4
6
8

The resulting curves are horizontal and vertical as expected. It is easier to check that these are latitude and longitude lines respectively when plotting them on $\mathbb{S}^2$. To do so, use $\mathbb{R}^3$ as the codomain of the standard map embedding $(\mathbb{S}^2, (\theta, \phi))$ in the 3-dimensional Euclidean space:

```
sage: R3 = Manifold(3, 'R3', start_index=1)
sage: cart.<X,Y,Z> = R3.chart()
sage: euclid_embedding = S2.diff_map(R3,
....:  {(polar, cart):[sin(th)*cos(ph),sin(th)*sin(ph),cos(th)]})
```

```
>>> from sage.all import *
>>> R3 = Manifold(Integer(3), 'R3', start_index=Integer(1))
>>> cart = R3.chart(names=('X', 'Y', 'Z',)); (X, Y, Z,) = cart._first_ngens(3)
>>> euclid_embedding = S2.diff_map(R3,
...  {(polar, cart):[sin(th)*cos(ph),sin(th)*sin(ph),cos(th)]})
```

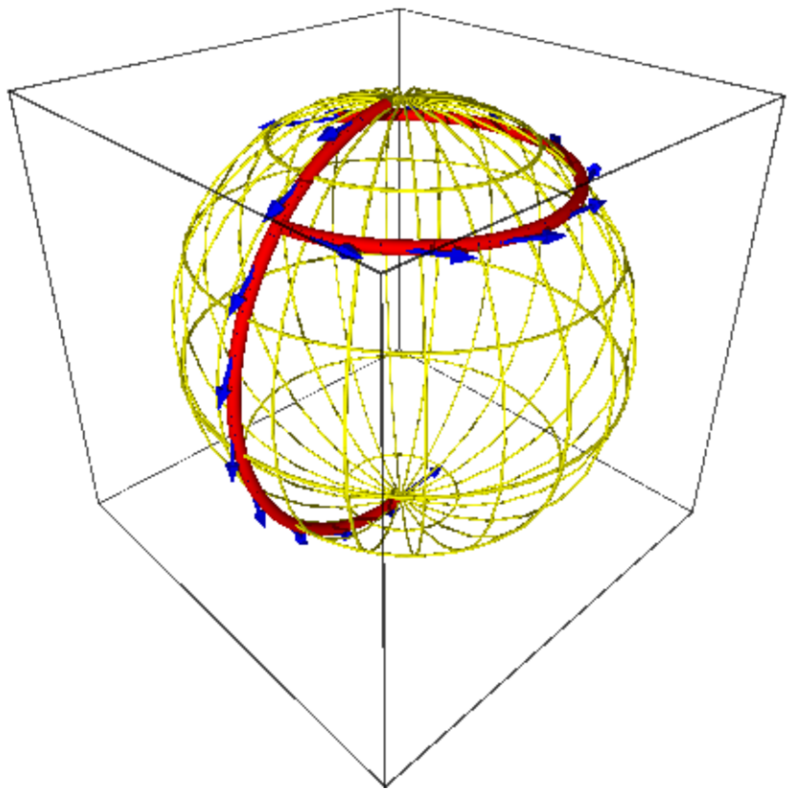
Plot the resulting curves on the grid of polar coordinates lines on $\mathbb{S}^2$:

```
sage: graph3D_embedded_curves = Graphics()
sage: for key in dict_params:
....:     graph3D_embedded_curves += c.plot_integrated(interpolation_key='interp-
↪'+key,
....:             mapping=euclid_embedding, thickness=5,
....:             display_tangent=True, scale=0.4, width_tangent=0.5)
sage: graph3D_embedded_polar_coords = polar.plot(chart=cart,
....:                          mapping=euclid_embedding,
....:                          number_values=15, color='yellow')
sage: graph3D_embedded_curves + graph3D_embedded_polar_coords
Graphics3d Object
```

```
>>> from sage.all import *
>>> graph3D_embedded_curves = Graphics()
>>> for key in dict_params:
...     graph3D_embedded_curves += c.plot_integrated(interpolation_key='interp-
↪'+key,
...             mapping=euclid_embedding, thickness=Integer(5),
...             display_tangent=True, scale=RealNumber('0.4'), width_
↪tangent=RealNumber('0.5'))
>>> graph3D_embedded_polar_coords = polar.plot(chart=cart,
...                          mapping=euclid_embedding,
...                          number_values=Integer(15), color='yellow')
>>> graph3D_embedded_curves + graph3D_embedded_polar_coords
Graphics3d Object
```

Finally, one may plot a general autoparallel curve with respect to ∇ that is neither a line of latitude or longitude. The vectors tangent to such a curve make an angle different from 0 or $\pi/2$ with the lines of latitude and longitude. Then, compute a curve such that both components of its initial tangent vectors are nonzero:

```
sage: sol = c.solve(solution_key='sol-angle',
....:  parameters_values={tmin:0,tmax:2,th0:pi/4,ph0:0.1,v_th0:1,v_ph0:8})
sage: interp = c.interpolate(solution_key='sol-angle',
....:                          interpolation_key='interp-angle')
```

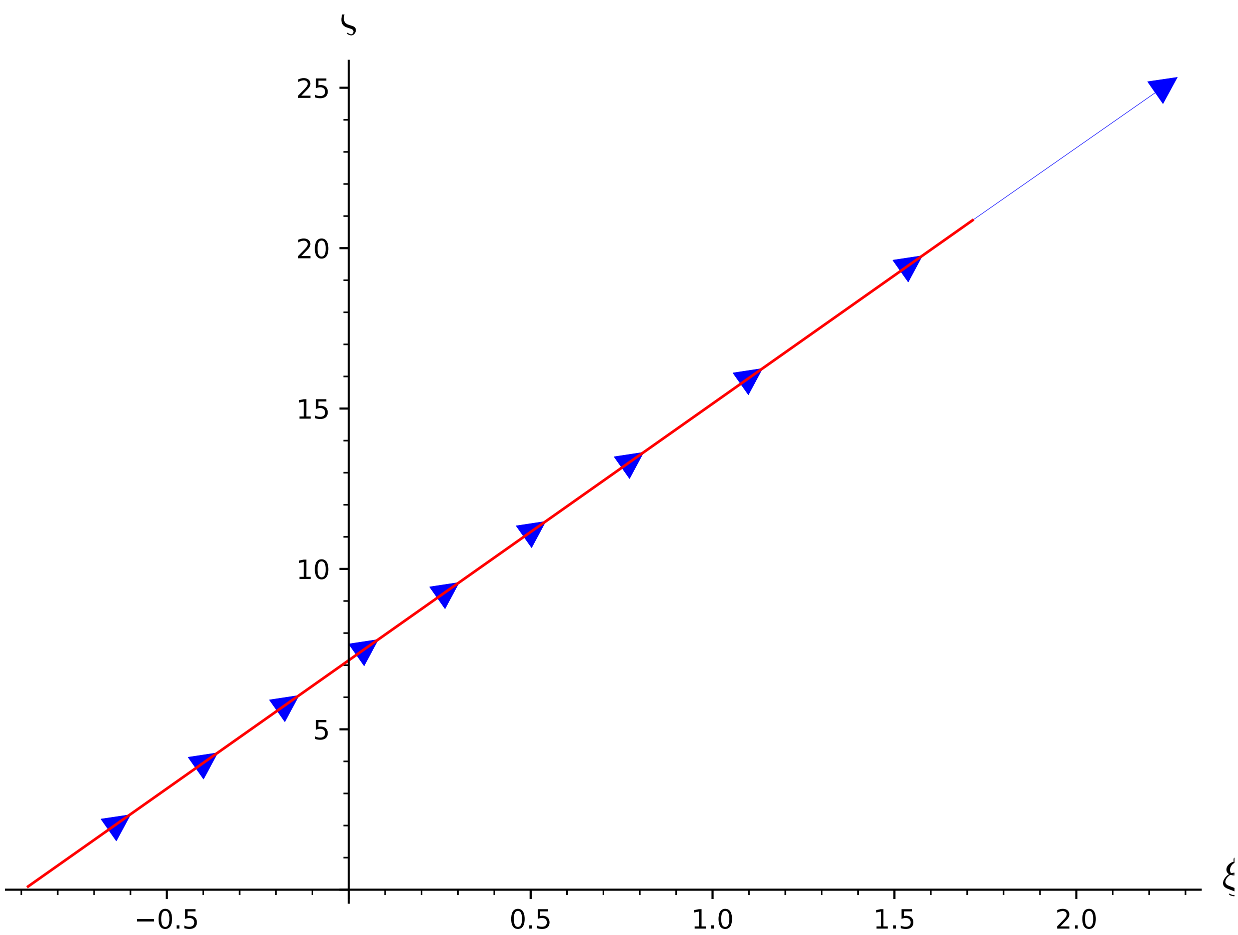


```
>>> from sage.all import *
>>> sol = c.solve(solution_key='sol-angle',
...  parameters_values={tmin:Integer(0),tmax:Integer(2),th0:pi/Integer(4),
↪ph0:RealNumber('0.1'),v_th0:Integer(1),v_ph0:Integer(8)})
>>> interp = c.interpolate(solution_key='sol-angle',
...                        interpolation_key='interp-angle')
```

Plot the resulting curve in the Mercator plane. This generates a straight line, as expected:

```
sage: c.plot_integrated(interpolation_key='interp-angle',
....:         chart=mercator, thickness=1, display_tangent=True,
....:         scale=0.2, width_tangent=0.2)
Graphics object consisting of 11 graphics primitives
```

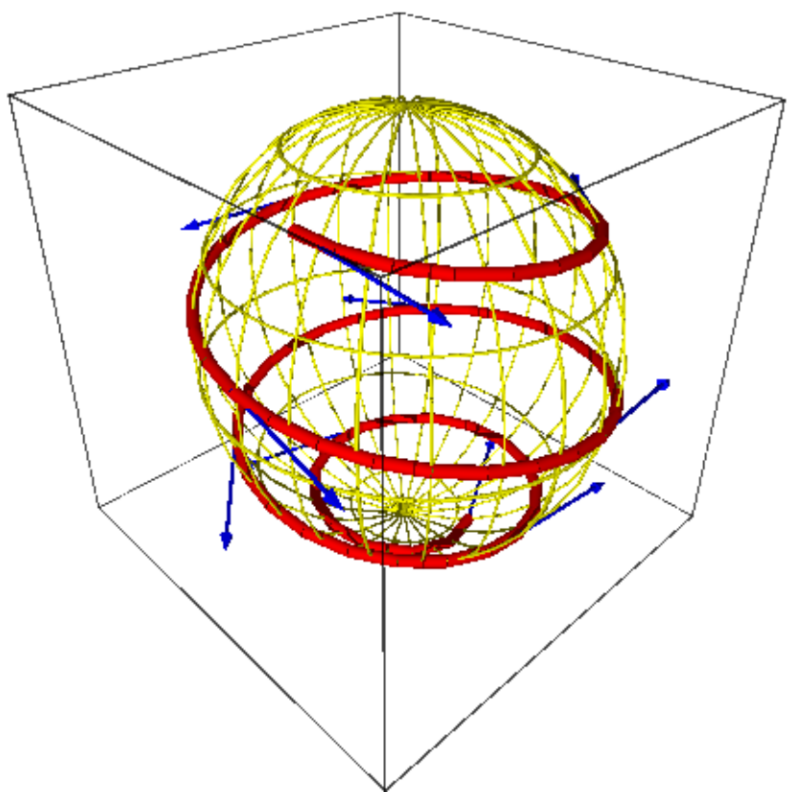
```
>>> from sage.all import *
>>> c.plot_integrated(interpolation_key='interp-angle',
...         chart=mercator, thickness=Integer(1), display_tangent=True,
...         scale=RealNumber('0.2'), width_tangent=RealNumber('0.2'))
Graphics object consisting of 11 graphics primitives
```



One may eventually plot such a curve on $\mathbb{S}^2$:

```
sage: graph3D_embedded_angle_curve=c.plot_integrated(interpolation_key='interp-
↪angle',
....:         mapping=euclid_embedding, thickness=5,
....:         display_tangent=True, scale=0.1, width_tangent=0.5)
sage: graph3D_embedded_angle_curve + graph3D_embedded_polar_coords
Graphics3d Object
```

```
>>> from sage.all import *
>>> graph3D_embedded_angle_curve=c.plot_integrated(interpolation_key='interp-angle
↪',
...         mapping=euclid_embedding, thickness=Integer(5),
...         display_tangent=True, scale=RealNumber('0.1'), width_
↪tangent=RealNumber('0.5'))
>>> graph3D_embedded_angle_curve + graph3D_embedded_polar_coords
Graphics3d Object
```



All the curves presented are loxodromes, and the differential system defining them (displayed above) may be solved analytically, providing the following expressions:

$$\theta(t) = \theta_0 + \dot{\theta}_0(t - t_0),$$

$$\phi(t) = \phi_0 - \frac{1}{\tan\alpha}\left(\ln\tan\frac{\theta_0 + \dot{\theta}_0(t - t_0)}{2} - \ln\tan\frac{\theta_0}{2}\right),$$

where α is the angle between the curve and any latitude line it crosses; then, one finds $\tan \alpha = -\dot{\theta}_0/(\dot{\phi}_0 \sin \theta_0)$ (then $\tan \alpha \leq 0$ when the initial tangent vector points towards the southeast).

In order to use these expressions to compare with the result provided by the numerical integration, remember that the components `(v_th0, v_ph0)` of the initial tangent vector `v` refer to the basis `epolar_ON` $= (\hat{e}_\theta, \hat{e}_\phi)$ and not the coordinate basis `epolar` $= (e_\theta, e_\phi)$. Therefore, the following relations hold: `v_ph0` $= \dot{\phi}_0 \sin \theta_0$ (and not merely $\dot{\phi}_0$), while `v_th0` clearly is $\dot{\theta}_0$.

With this in mind, plot an analytical curve to compare with a numerical solution:

```
sage: graph2D_mercator_angle_curve=c.plot_integrated(interpolation_key='interp-
↪angle',
....:                                    chart=mercator, thickness=1)
sage: expr_ph = ph0+v_ph0/v_th0*(ln(tan((v_th0*t+th0)/2))-ln(tan(th0/2)))
sage: c_loxo = S2.curve({polar:[th0+v_th0*t, expr_ph]}, (t,0,2),
....:                                                 name='c_loxo')
```

```
>>> from sage.all import *
>>> graph2D_mercator_angle_curve=c.plot_integrated(interpolation_key='interp-angle
↪',
...                                    chart=mercator, thickness=Integer(1))
>>> expr_ph = ph0+v_ph0/v_th0*(ln(tan((v_th0*t+th0)/Integer(2)))-ln(tan(th0/
↪Integer(2))))
>>> c_loxo = S2.curve({polar:[th0+v_th0*t, expr_ph]}, (t,Integer(0),Integer(2)),
...                                                 name='c_loxo')
```

Ask for the expression of the loxodrome in terms of the Mercator chart in order to add it to its dictionary of expressions. It is a particularly long expression, and there is no particular need to display it, which is why it may simply be affected to an arbitrary variable `expr_mercator`, which will never be used again. But adding the expression to the dictionary is required to plot the curve with respect to the Mercator chart:

```
sage: expr_mercator = c_loxo.expression(chart2=mercator)
```

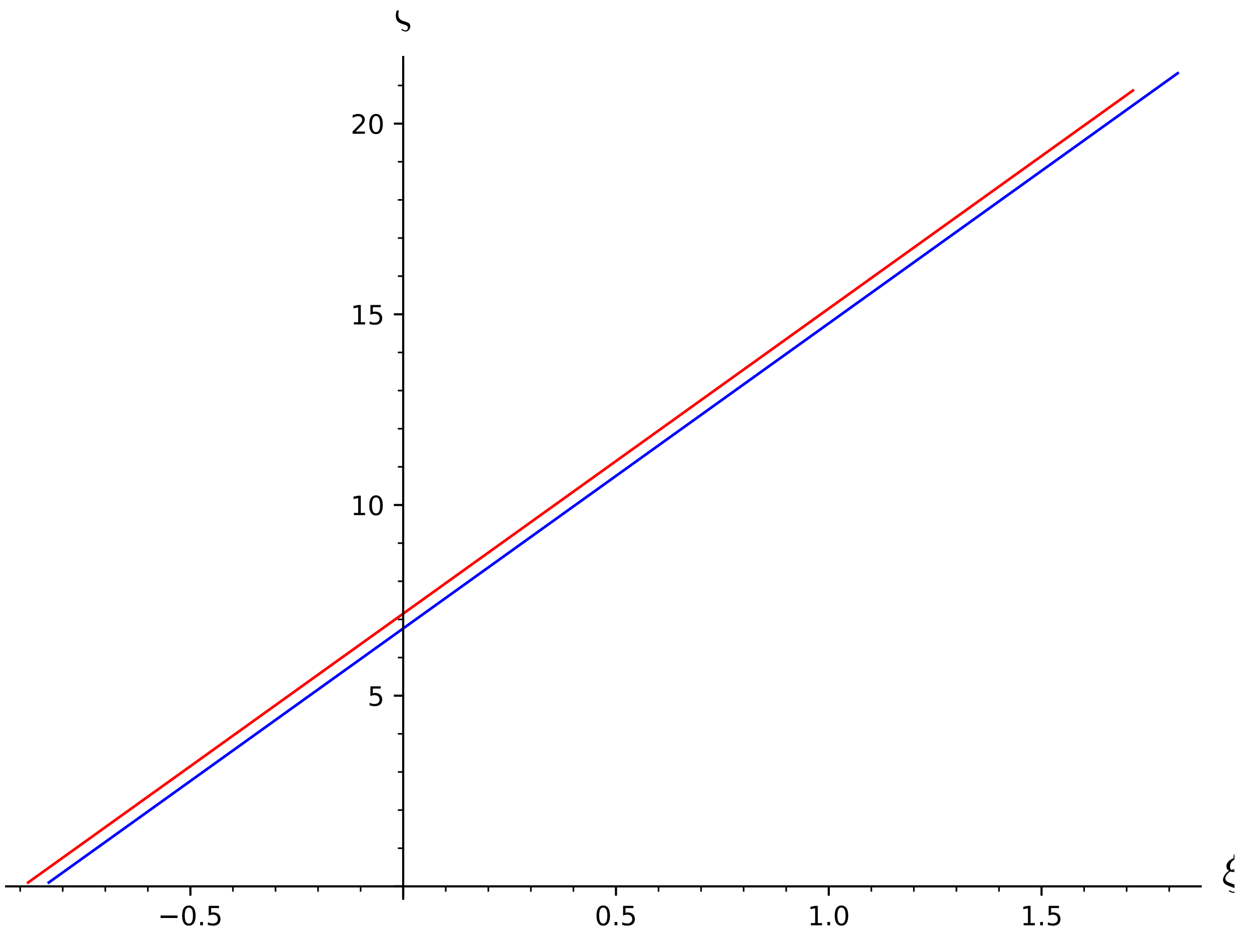
```
>>> from sage.all import *
>>> expr_mercator = c_loxo.expression(chart2=mercator)
```

Plot the curves (for clarity, set a 2 degrees shift in the initial value of θ_0 so that the curves do not overlap):

```
sage: graph2D_mercator_loxo = c_loxo.plot(chart=mercator,
....:  parameters={th0:pi/4+2*pi/180, ph0:0.1, v_th0:1, v_ph0:8},
....:  thickness=1, color='blue')
sage: graph2D_mercator_angle_curve + graph2D_mercator_loxo
Graphics object consisting of 2 graphics primitives
```

```
>>> from sage.all import *
>>> graph2D_mercator_loxo = c_loxo.plot(chart=mercator,
...  parameters={th0:pi/Integer(4)+Integer(2)*pi/Integer(180), ph0:RealNumber('0.1
↪'), v_th0:Integer(1), v_ph0:Integer(8)},
...  thickness=Integer(1), color='blue')
>>> graph2D_mercator_angle_curve + graph2D_mercator_loxo
Graphics object consisting of 2 graphics primitives
```

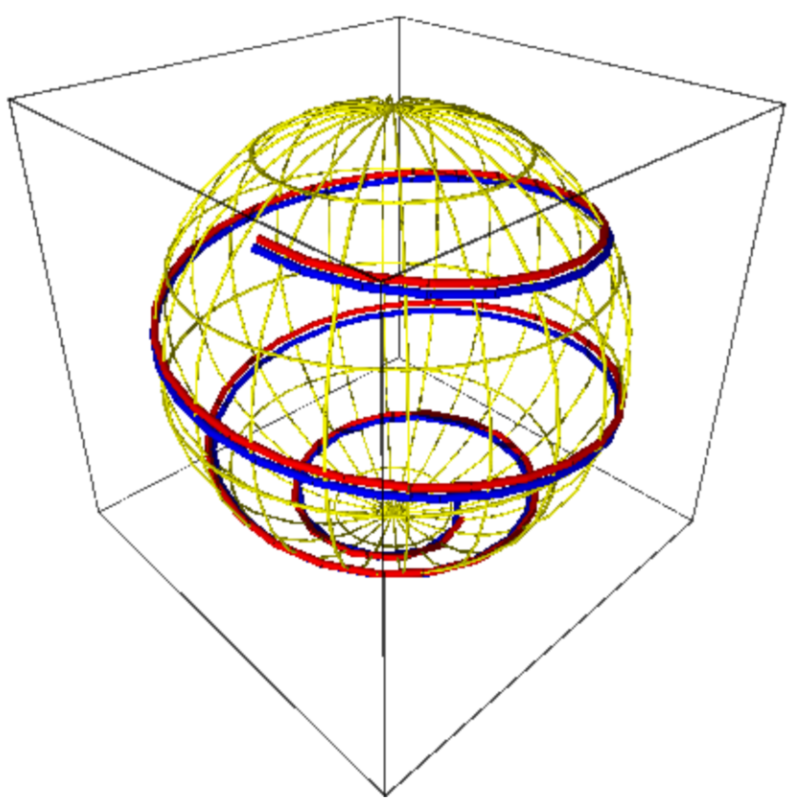
Both curves do have the same aspect. One may eventually compare these curves on $\mathbb{S}^2$:



ς
20
15
10
5
ξ
−0.5
0.5
1.0
1.5

```
sage: graph3D_embedded_angle_curve=c.plot_integrated(interpolation_key='interp-
↪angle',
....:                      mapping=euclid_embedding, thickness=3)
sage: graph3D_embedded_loxo = c_loxo.plot(mapping=euclid_embedding,
....:  parameters={th0:pi/4+2*pi/180, ph0:0.1, v_th0:1, v_ph0:8},
....:  thickness=3, color = 'blue')
sage: (graph3D_embedded_angle_curve + graph3D_embedded_loxo
....:  + graph3D_embedded_polar_coords)
Graphics3d Object
```

```
>>> from sage.all import *
>>> graph3D_embedded_angle_curve=c.plot_integrated(interpolation_key='interp-angle
↪',
...                      mapping=euclid_embedding, thickness=Integer(3))
>>> graph3D_embedded_loxo = c_loxo.plot(mapping=euclid_embedding,
...  parameters={th0:pi/Integer(4)+Integer(2)*pi/Integer(180), ph0:RealNumber('0.1
↪'), v_th0:Integer(1), v_ph0:Integer(8)},
...  thickness=Integer(3), color = 'blue')
>>> (graph3D_embedded_angle_curve + graph3D_embedded_loxo
...  + graph3D_embedded_polar_coords)
Graphics3d Object
```



system(*verbose=False*)

Provide a detailed description of the system defining the autoparallel curve and returns the system defining it: chart, equations and initial conditions.

INPUT:

- `verbose` – boolean (default: `False`); prints a detailed description of the curve

OUTPUT:

- list containing the
  - the equations
  - the initial conditions
  - the chart

EXAMPLES:

System defining an autoparallel curve:

```
sage: M = Manifold(3, 'M')
sage: X.<x1,x2,x3> = M.chart()
sage: [t, A, B] = var('t A B')
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[X.frame(),0,0,1],nab[X.frame(),2,1,2]=A*x1^2,B*x2*x3
sage: p = M.point((0,0,0), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,0,1))
sage: c = M.integrated_autoparallel_curve(nab, (t, 0, 5), v)
sage: sys = c.system(verbose=True)
Autoparallel curve in the 3-dimensional differentiable
 manifold M equipped with Affine connection nabla on the
 3-dimensional differentiable manifold M, and integrated
 over the Real interval (0, 5) as a solution to the
 following equations, written with respect to
 Chart (M, (x1, x2, x3)):

Initial point: Point p on the 3-dimensional differentiable
 manifold M with coordinates [0, 0, 0] with respect to
 Chart (M, (x1, x2, x3))
Initial tangent vector: Tangent vector at Point p on the
 3-dimensional differentiable manifold M with
 components [1, 0, 1] with respect to Chart (M, (x1, x2, x3))

d(x1)/dt = Dx1
d(x2)/dt = Dx2
d(x3)/dt = Dx3
d(Dx1)/dt = -A*Dx1*Dx2*x1^2
d(Dx2)/dt = 0
d(Dx3)/dt = -B*Dx2*Dx3*x2*x3

sage: sys_bis = c.system()
sage: sys_bis == sys
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
```

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```
>>> X = M.chart(names=('x1', 'x2', 'x3',)); (x1, x2, x3,) = X._first_ngens(3)
>>> [t, A, B] = var('t A B')
>>> nab = M.affine_connection('nabla', r'\nabla')
>>> nab[X.frame(),Integer(0),Integer(0),Integer(1)],nab[X.frame(),Integer(2),
↪Integer(1),Integer(2)]=A*x1**Integer(2),B*x2*x3
>>> p = M.point((Integer(0),Integer(0),Integer(0)), name='p')
>>> Tp = M.tangent_space(p)
>>> v = Tp((Integer(1),Integer(0),Integer(1)))
>>> c = M.integrated_autoparallel_curve(nab, (t, Integer(0), Integer(5)), v)
>>> sys = c.system(verbose=True)
Autoparallel curve in the 3-dimensional differentiable
 manifold M equipped with Affine connection nabla on the
 3-dimensional differentiable manifold M, and integrated
 over the Real interval (0, 5) as a solution to the
 following equations, written with respect to
 Chart (M, (x1, x2, x3)):
<BLANKLINE>
Initial point: Point p on the 3-dimensional differentiable
 manifold M with coordinates [0, 0, 0] with respect to
 Chart (M, (x1, x2, x3))
Initial tangent vector: Tangent vector at Point p on the
 3-dimensional differentiable manifold M with
 components [1, 0, 1] with respect to Chart (M, (x1, x2, x3))
<BLANKLINE>
d(x1)/dt = Dx1
d(x2)/dt = Dx2
d(x3)/dt = Dx3
d(Dx1)/dt = -A*Dx1*Dx2*x1^2
d(Dx2)/dt = 0
d(Dx3)/dt = -B*Dx2*Dx3*x2*x3
<BLANKLINE>
>>> sys_bis = c.system()
>>> sys_bis == sys
True
```

class `sage.manifolds.differentiable.integrated_curve.`**`IntegratedCurve`**(*parent*, *equations_rhs*, *velocities*, *curve_parameter*, *initial_tangent_vector*, *chart=None*, *name=None*, *latex_name=None*, *verbose=False*, *across_charts=False*)

Bases: `DifferentiableCurve`

Given a chart with coordinates denoted $(x_1, \ldots, x_n)$, an instance of `IntegratedCurve` is a curve $t \mapsto (x_1(t), \ldots, x_n(t))$ constructed as a solution to a system of second order differential equations satisfied by the coordinate curves $t \mapsto x_i(t)$.

INPUT:

- `parent` – `IntegratedCurveSet` the set of curves $\mathrm{Hom}_{\mathrm{integrated}}(I, M)$ to which the curve belongs

- `equations_rhs` – list of the right-hand sides of the equations on the velocities only (the term *velocity* referring to the derivatives dx_i/dt of the coordinate curves)
- `velocities` – list of the symbolic expressions used in `equations_rhs` to denote the velocities
- `curve_parameter` – symbolic expression used in `equations_rhs` to denote the parameter of the curve (denoted t in the descriptions above)
- `initial_tangent_vector` – *TangentVector* initial tangent vector of the curve
- `chart` – (default: `None`) chart on the manifold in which the equations are given; if `None` the default chart of the manifold is assumed
- `name` – (default: `None`) string; symbol given to the curve
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the curve; if none is provided, `name` will be used

EXAMPLES:

Motion of a charged particle in an axial magnetic field linearly increasing in time and exponentially decreasing in space:

$$\mathbf{B}(t,\mathbf{x}) = \frac{B_0 t}{T} \exp\left(-\frac{x_1^2+x_2^2}{L^2}\right)\mathbf{e_3}.$$

Equations of motion are:

$$\begin{aligned}\ddot{x}_1(t) &= \frac{qB(t,\mathbf{x}(t))}{m}\dot{x}_2(t),\\ \ddot{x}_2(t) &= -\frac{qB(t,\mathbf{x}(t))}{m}\dot{x}_1(t),\\ \ddot{x}_3(t) &= 0.\end{aligned}$$

Start with declaring a chart on a 3-dimensional manifold and the symbolic expressions denoting the velocities and the various parameters:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x1,x2,x3> = M.chart()
sage: var('t B_0 m q L T')
(t, B_0, m, q, L, T)
sage: B = B_0*t/T*exp(-(x1^2 + x2^2)/L^2)
sage: D = X.symbolic_velocities(); D
[Dx1, Dx2, Dx3]
sage: eqns = [q*B/m*D[1], -q*B/m*D[0], 0]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x1', 'x2', 'x3',)); (x1, x2, x3,) = X._first_ngens(3)
>>> var('t B_0 m q L T')
(t, B_0, m, q, L, T)
>>> B = B_0*t/T*exp(-(x1**Integer(2) + x2**Integer(2))/L**Integer(2))
>>> D = X.symbolic_velocities(); D
[Dx1, Dx2, Dx3]
>>> eqns = [q*B/m*D[Integer(1)], -q*B/m*D[Integer(0)], Integer(0)]
```

Set the initial conditions:

```
sage: p = M.point((0,0,0), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,0,1))
```

```
>>> from sage.all import *
>>> p = M.point((Integer(0),Integer(0),Integer(0)), name='p')
>>> Tp = M.tangent_space(p)
>>> v = Tp((Integer(1),Integer(0),Integer(1)))
```

Declare an integrated curve and display information relative to it:

```
sage: c = M.integrated_curve(eqns, D, (t, 0, 5), v, name='c',
....:                                             verbose=True)
The curve was correctly set.
Parameters appearing in the differential system defining the
 curve are [B_0, L, T, m, q].
sage: c
Integrated curve c in the 3-dimensional differentiable
 manifold M
sage: sys = c.system(verbose=True)
Curve c in the 3-dimensional differentiable manifold M
 integrated over the Real interval (0, 5) as a solution to the
 following system, written with respect to
 Chart (M, (x1, x2, x3)):

Initial point: Point p on the 3-dimensional differentiable
 manifold M with coordinates [0, 0, 0] with respect to
 Chart (M, (x1, x2, x3))
Initial tangent vector: Tangent vector at Point p on
 the 3-dimensional differentiable manifold M with
 components [1, 0, 1] with respect to Chart (M, (x1, x2, x3))

d(x1)/dt = Dx1
d(x2)/dt = Dx2
d(x3)/dt = Dx3
d(Dx1)/dt = B_0*Dx2*q*t*e^(-(x1^2 + x2^2)/L^2)/(T*m)
d(Dx2)/dt = -B_0*Dx1*q*t*e^(-(x1^2 + x2^2)/L^2)/(T*m)
d(Dx3)/dt = 0
```

```
>>> from sage.all import *
>>> c = M.integrated_curve(eqns, D, (t, Integer(0), Integer(5)), v, name='c',
...                                             verbose=True)
The curve was correctly set.
Parameters appearing in the differential system defining the
 curve are [B_0, L, T, m, q].
>>> c
Integrated curve c in the 3-dimensional differentiable
 manifold M
>>> sys = c.system(verbose=True)
Curve c in the 3-dimensional differentiable manifold M
 integrated over the Real interval (0, 5) as a solution to the
 following system, written with respect to
```

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```
 Chart (M, (x1, x2, x3)):
<BLANKLINE>
Initial point: Point p on the 3-dimensional differentiable
 manifold M with coordinates [0, 0, 0] with respect to
 Chart (M, (x1, x2, x3))
Initial tangent vector: Tangent vector at Point p on
 the 3-dimensional differentiable manifold M with
 components [1, 0, 1] with respect to Chart (M, (x1, x2, x3))
<BLANKLINE>
d(x1)/dt = Dx1
d(x2)/dt = Dx2
d(x3)/dt = Dx3
d(Dx1)/dt = B_0*Dx2*q*t*e^(-(x1^2 + x2^2)/L^2)/(T*m)
d(Dx2)/dt = -B_0*Dx1*q*t*e^(-(x1^2 + x2^2)/L^2)/(T*m)
d(Dx3)/dt = 0
<BLANKLINE>
```

Generate a solution of the system and an interpolation of this solution:

```
sage: sol = c.solve(step=0.2,                                              #␣
↪needs scipy
....:           parameters_values={B_0:1, m:1, q:1, L:10, T:1},
....:           solution_key='carac time 1', verbose=True)
Performing numerical integration with method 'odeint'...
Numerical integration completed.

Checking all points are in the chart domain...
All points are in the chart domain.

The resulting list of points was associated with the key
 'carac time 1' (if this key already referred to a former
 numerical solution, such a solution was erased).
sage: interp = c.interpolate(solution_key='carac time 1',                  #␣
↪needs scipy
....:                  interpolation_key='interp 1', verbose=True)
Performing cubic spline interpolation by default...
Interpolation completed and associated with the key 'interp 1'
 (if this key already referred to a former interpolation,
 such an interpolation was erased).
```

```
>>> from sage.all import *
>>> sol = c.solve(step=RealNumber('0.2'),                                   ␣
↪          # needs scipy
...           parameters_values={B_0:Integer(1), m:Integer(1), q:Integer(1),␣
↪L:Integer(10), T:Integer(1)},
...           solution_key='carac time 1', verbose=True)
Performing numerical integration with method 'odeint'...
Numerical integration completed.
<BLANKLINE>
Checking all points are in the chart domain...
All points are in the chart domain.
<BLANKLINE>
```

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```
The resulting list of points was associated with the key
 'carac time 1' (if this key already referred to a former
 numerical solution, such a solution was erased).
>>> interp = c.interpolate(solution_key='carac time 1',                 #␣
↪needs scipy
...                    interpolation_key='interp 1', verbose=True)
Performing cubic spline interpolation by default...
Interpolation completed and associated with the key 'interp 1'
 (if this key already referred to a former interpolation,
 such an interpolation was erased).
```

Such an interpolation is required to evaluate the curve and the vector tangent to the curve for any value of the curve parameter:

```
sage: # needs scipy
sage: p = c(1.9, verbose=True)
Evaluating point coordinates from the interpolation associated
 with the key 'interp 1' by default...
sage: p
Point on the 3-dimensional differentiable manifold M
sage: p.coordinates()     # abs tol 1e-12
(1.377689074756845, -0.900114533011232, 1.9)
sage: v2 = c.tangent_vector_eval_at(4.3, verbose=True)
Evaluating tangent vector components from the interpolation
 associated with the key 'interp 1' by default...
sage: v2
Tangent vector at Point on the 3-dimensional differentiable
 manifold M
sage: v2[:]     # abs tol 1e-12
[-0.9425156073651124, -0.33724314284285434, 1.0]
```

```
>>> from sage.all import *
>>> # needs scipy
>>> p = c(RealNumber('1.9'), verbose=True)
Evaluating point coordinates from the interpolation associated
 with the key 'interp 1' by default...
>>> p
Point on the 3-dimensional differentiable manifold M
>>> p.coordinates()     # abs tol 1e-12
(1.377689074756845, -0.900114533011232, 1.9)
>>> v2 = c.tangent_vector_eval_at(RealNumber('4.3'), verbose=True)
Evaluating tangent vector components from the interpolation
 associated with the key 'interp 1' by default...
>>> v2
Tangent vector at Point on the 3-dimensional differentiable
 manifold M
>>> v2[:]     # abs tol 1e-12
[-0.9425156073651124, -0.33724314284285434, 1.0]
```

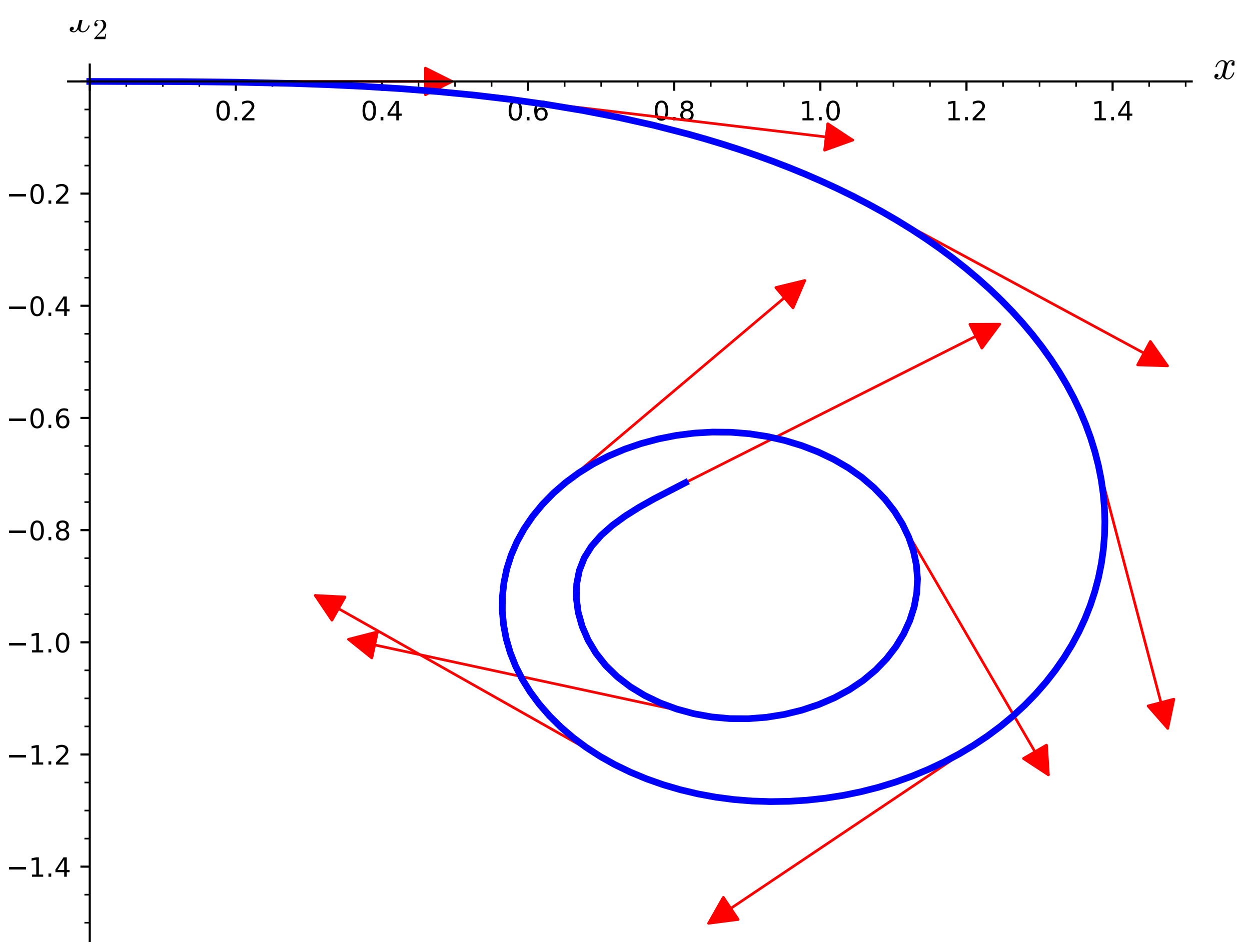
Plotting a numerical solution (with or without its tangent vector field) also requires the solution to be interpolated at least once:

```
sage: c_plot_2d_1 = c.plot_integrated(ambient_coords=[x1, x2],                # needs scipy
....:                 interpolation_key='interp 1', thickness=2.5,
....:                 display_tangent=True, plot_points=200,
....:                 plot_points_tangent=10, scale=0.5,
....:                 color='blue', color_tangent='red',
....:                 verbose=True)
A tiny final offset equal to 0.000251256281407035 was introduced
 for the last point in order to safely compute it from the
 interpolation.
sage: c_plot_2d_1                                                             # needs scipy sage.plot
Graphics object consisting of 11 graphics primitives
```

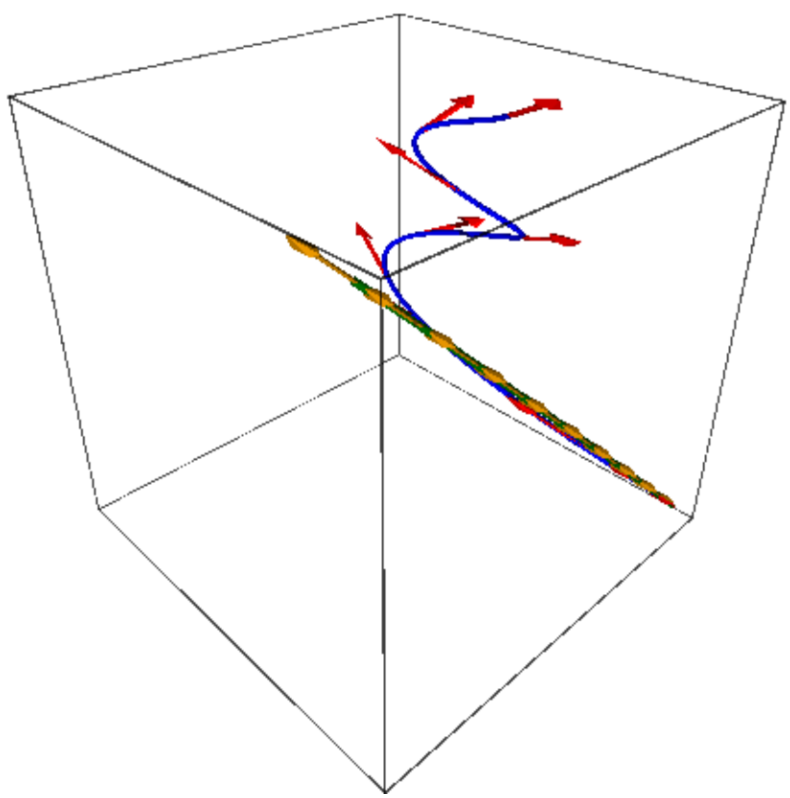
```
>>> from sage.all import *
>>> c_plot_2d_1 = c.plot_integrated(ambient_coords=[x1, x2],                # needs scipy
...                 interpolation_key='interp 1', thickness=RealNumber('2.5'),
...                 display_tangent=True, plot_points=Integer(200),
...                 plot_points_tangent=Integer(10), scale=RealNumber('0.5'),
...                 color='blue', color_tangent='red',
...                 verbose=True)
A tiny final offset equal to 0.000251256281407035 was introduced
 for the last point in order to safely compute it from the
 interpolation.
>>> c_plot_2d_1                                                             # needs scipy sage.plot
Graphics object consisting of 11 graphics primitives
```

An instance of `IntegratedCurve` may store several numerical solutions and interpolations:

```
sage: # needs scipy
sage: sol = c.solve(step=0.2,
....:         parameters_values={B_0:1, m:1, q:1, L:10, T:100},
....:         solution_key='carac time 100')
sage: interp = c.interpolate(solution_key='carac time 100',
....:                        interpolation_key='interp 100')
sage: c_plot_3d_100 = c.plot_integrated(interpolation_key='interp 100',      # needs sage.plot
....:                   thickness=2.5, display_tangent=True,
....:                   plot_points=200, plot_points_tangent=10,
....:                   scale=0.5, color='green',
....:                   color_tangent='orange')
sage: c_plot_3d_1 = c.plot_integrated(interpolation_key='interp 1',          # needs sage.plot
....:                   thickness=2.5, display_tangent=True,
....:                   plot_points=200, plot_points_tangent=10,
....:                   scale=0.5, color='blue',
....:                   color_tangent='red')
sage: c_plot_3d_1 + c_plot_3d_100                                            # needs sage.plot
Graphics3d Object
```


x_2
x
0.2
0.4
0.6
0.8
1.0
1.2
1.4
−0.2
−0.4
−0.6
−0.8
−1.0
−1.2
−1.4

```
>>> from sage.all import *
>>> # needs scipy
>>> sol = c.solve(step=RealNumber('0.2'),
...         parameters_values={B_0:Integer(1), m:Integer(1), q:Integer(1),␣
↪L:Integer(10), T:Integer(100)},
...         solution_key='carac time 100')
>>> interp = c.interpolate(solution_key='carac time 100',
...                         interpolation_key='interp 100')
>>> c_plot_3d_100 = c.plot_integrated(interpolation_key='interp 100',          #␣
↪needs sage.plot
...                   thickness=RealNumber('2.5'), display_tangent=True,
...                   plot_points=Integer(200), plot_points_tangent=Integer(10),
...                   scale=RealNumber('0.5'), color='green',
...                   color_tangent='orange')
>>> c_plot_3d_1 = c.plot_integrated(interpolation_key='interp 1',              #␣
↪needs sage.plot
...                   thickness=RealNumber('2.5'), display_tangent=True,
...                   plot_points=Integer(200), plot_points_tangent=Integer(10),
...                   scale=RealNumber('0.5'), color='blue',
...                   color_tangent='red')
>>> c_plot_3d_1 + c_plot_3d_100                                                #␣
↪needs sage.plot
Graphics3d Object
```



interpolate(*solution_key=None*, *method=None*, *interpolation_key=None*, *verbose=False*)

Interpolate the chosen numerical solution using the given interpolation method.

INPUT:

- `solution_key` – (default: `None`) key which the numerical solution to interpolate is associated to ; a default value is chosen if none is provided
- `method` – (default: `None`) interpolation scheme to use; algorithms available are
  - `'cubic spline'`, which makes use of `GSL` via `Spline`
- `interpolation_key` – (default: `None`) key which the resulting interpolation will be associated to ; a default value is given if none is provided
- `verbose` – boolean (default: `False`); prints information about the interpolation in progress

OUTPUT: built interpolation object

EXAMPLES:

Interpolating a numerical solution previously computed:

```
sage: M = Manifold(3, 'M')
sage: X.<x1,x2,x3> = M.chart()
sage: [t, B_0, m, q, L, T] = var('t B_0 m q L T')
sage: B = B_0*t/T*exp(-(x1^2 + x2^2)/L^2)
sage: D = X.symbolic_velocities()
sage: eqns = [q*B/m*D[1], -q*B/m*D[0], 0]
sage: p = M.point((0,0,0), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,0,1))
sage: c = M.integrated_curve(eqns, D, (t,0,5), v, name='c')

sage: # needs scipy
sage: sol = c.solve(method='odeint',
....:         solution_key='sol_T1',
....:         parameters_values={B_0:1, m:1, q:1, L:10, T:1})
sage: interp = c.interpolate(method='cubic spline',
....:                         solution_key='sol_T1',
....:                         interpolation_key='interp_T1',
....:                         verbose=True)
Interpolation completed and associated with the key
 'interp_T1' (if this key already referred to a former
 interpolation, such an interpolation was erased).
sage: interp = c.interpolate(verbose=True)
Interpolating the numerical solution associated with the
 key 'sol_T1' by default...
Performing cubic spline interpolation by default...
Resulting interpolation will be associated with the key
 'cubic spline-interp-sol_T1' by default.
Interpolation completed and associated with the key
 'cubic spline-interp-sol_T1' (if this key already referred
 to a former interpolation, such an interpolation was
 erased).
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x1', 'x2', 'x3',)); (x1, x2, x3,) = X._first_ngens(3)
>>> [t, B_0, m, q, L, T] = var('t B_0 m q L T')
>>> B = B_0*t/T*exp(-(x1**Integer(2) + x2**Integer(2))/L**Integer(2))
>>> D = X.symbolic_velocities()
>>> eqns = [q*B/m*D[Integer(1)], -q*B/m*D[Integer(0)], Integer(0)]
>>> p = M.point((Integer(0),Integer(0),Integer(0)), name='p')
>>> Tp = M.tangent_space(p)
>>> v = Tp((Integer(1),Integer(0),Integer(1)))
>>> c = M.integrated_curve(eqns, D, (t,Integer(0),Integer(5)), v, name='c')

>>> # needs scipy
>>> sol = c.solve(method='odeint',
...       solution_key='sol_T1',
...       parameters_values={B_0:Integer(1), m:Integer(1), q:Integer(1),␣
↪L:Integer(10), T:Integer(1)})
>>> interp = c.interpolate(method='cubic spline',
...                        solution_key='sol_T1',
...                        interpolation_key='interp_T1',
...                        verbose=True)
Interpolation completed and associated with the key
 'interp_T1' (if this key already referred to a former
 interpolation, such an interpolation was erased).
>>> interp = c.interpolate(verbose=True)
Interpolating the numerical solution associated with the
 key 'sol_T1' by default...
Performing cubic spline interpolation by default...
Resulting interpolation will be associated with the key
 'cubic spline-interp-sol_T1' by default.
Interpolation completed and associated with the key
 'cubic spline-interp-sol_T1' (if this key already referred
 to a former interpolation, such an interpolation was
 erased).
```

interpolation(*interpolation_key=None*, *verbose=False*)

Return the interpolation object associated with the given key.

INPUT:

- `interpolation_key` – (default: `None`) key which the requested interpolation is associated to; a default value is chosen if none is provided
- `verbose` – boolean (default: `False`); prints information about the interpolation object returned

OUTPUT: requested interpolation object

EXAMPLES:

Requesting an interpolation object previously computed:

```
sage: M = Manifold(3, 'M')
sage: X.<x1,x2,x3> = M.chart()
sage: [t, B_0, m, q, L, T] = var('t B_0 m q L T')
sage: B = B_0*t/T*exp(-(x1^2 + x2^2)/L^2)
sage: D = X.symbolic_velocities()
```

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```
sage: eqns = [q*B/m*D[1], -q*B/m*D[0], 0]
sage: p = M.point((0,0,0), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,0,1))
sage: c = M.integrated_curve(eqns, D, (t,0,5), v, name='c')

sage: # needs scipy
sage: sol = c.solve(method='odeint',
....:         solution_key='sol_T1',
....:         parameters_values={B_0:1, m:1, q:1, L:10, T:1})
sage: interp = c.interpolate(method='cubic spline',
....:                         solution_key='sol_T1',
....:                         interpolation_key='interp_T1')
sage: default_interp = c.interpolation(verbose=True)
Returning the interpolation associated with the key
 'interp_T1' by default...
sage: default_interp == interp
True
sage: interp_mute = c.interpolation()
sage: interp_mute == interp
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x1', 'x2', 'x3',)); (x1, x2, x3,) = X._first_ngens(3)
>>> [t, B_0, m, q, L, T] = var('t B_0 m q L T')
>>> B = B_0*t/T*exp(-(x1**Integer(2) + x2**Integer(2))/L**Integer(2))
>>> D = X.symbolic_velocities()
>>> eqns = [q*B/m*D[Integer(1)], -q*B/m*D[Integer(0)], Integer(0)]
>>> p = M.point((Integer(0),Integer(0),Integer(0)), name='p')
>>> Tp = M.tangent_space(p)
>>> v = Tp((Integer(1),Integer(0),Integer(1)))
>>> c = M.integrated_curve(eqns, D, (t,Integer(0),Integer(5)), v, name='c')

>>> # needs scipy
>>> sol = c.solve(method='odeint',
...         solution_key='sol_T1',
...         parameters_values={B_0:Integer(1), m:Integer(1), q:Integer(1),
↪L:Integer(10), T:Integer(1)})
>>> interp = c.interpolate(method='cubic spline',
...                         solution_key='sol_T1',
...                         interpolation_key='interp_T1')
>>> default_interp = c.interpolation(verbose=True)
Returning the interpolation associated with the key
 'interp_T1' by default...
>>> default_interp == interp
True
>>> interp_mute = c.interpolation()
>>> interp_mute == interp
True
```

plot_integrated(*chart=None*, *ambient_coords=None*, *mapping=None*, *prange=None*, *interpolation_key=None*, *include_end_point=(True, True)*, *end_point_offset=(0.001, 0.001)*, *verbose=False*, *color='red'*, *style='-'*, *label_axes=True*, *display_tangent=False*, *color_tangent='blue'*, *across_charts=False*, *thickness=1*, *plot_points=75*, *aspect_ratio='automatic'*, *plot_points_tangent=10*, *width_tangent=1*, *scale=1*, ***kwds*)

Plot the 2D or 3D projection of `self` onto the space of the chosen two or three ambient coordinates, based on the interpolation of a numerical solution previously computed.

See also

`plot` for complete information about the input.

ADDITIONAL INPUT:

- `interpolation_key` – (default: `None`) key associated to the interpolation object used for the plot; a default value is chosen if none is provided
- `verbose` – boolean (default: `False`); prints information about the interpolation object used and the plotting in progress
- `display_tangent` – boolean (default: `False`); determines whether some tangent vectors should also be plotted
- `color_tangent` – (default: `blue`) color of the tangent vectors when these are plotted
- `plot_points_tangent` – (default: 10) number of tangent vectors to display when these are plotted
- `width_tangent` – (default: 1) sets the width of the arrows representing the tangent vectors
- `scale` – (default: 1) scale applied to the tangent vectors before displaying them

EXAMPLES:

Trajectory of a particle of unit mass and unit charge in an unit, axial, uniform, stationary magnetic field:

```
sage: M = Manifold(3, 'M')
sage: X.<x1,x2,x3> = M.chart()
sage: var('t')
t
sage: D = X.symbolic_velocities()
sage: eqns = [D[1], -D[0], 0]
sage: p = M.point((0,0,0), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,0,1))
sage: c = M.integrated_curve(eqns, D, (t,0,6), v, name='c')

sage: # needs scipy
sage: sol = c.solve()
sage: interp = c.interpolate()
sage: c_plot_2d = c.plot_integrated(ambient_coords=[x1, x2],
....:                 thickness=2.5,
....:                 display_tangent=True, plot_points=200,
....:                 plot_points_tangent=10, scale=0.5,
....:                 color='blue', color_tangent='red',
....:                 verbose=True)
Plotting from the interpolation associated with the key
 'cubic spline-interp-odeint' by default...
```

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```
A tiny final offset equal to 0.000301507537688442 was
 introduced for the last point in order to safely compute it
 from the interpolation.
sage: c_plot_2d
Graphics object consisting of 11 graphics primitives
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x1', 'x2', 'x3',)); (x1, x2, x3,) = X._first_ngens(3)
>>> var('t')
t
>>> D = X.symbolic_velocities()
>>> eqns = [D[Integer(1)], -D[Integer(0)], Integer(0)]
>>> p = M.point((Integer(0),Integer(0),Integer(0)), name='p')
>>> Tp = M.tangent_space(p)
>>> v = Tp((Integer(1),Integer(0),Integer(1)))
>>> c = M.integrated_curve(eqns, D, (t,Integer(0),Integer(6)), v, name='c')

>>> # needs scipy
>>> sol = c.solve()
>>> interp = c.interpolate()
>>> c_plot_2d = c.plot_integrated(ambient_coords=[x1, x2],
...                 thickness=RealNumber('2.5'),
...                 display_tangent=True, plot_points=Integer(200),
...                 plot_points_tangent=Integer(10), scale=RealNumber('0.5'),
...                 color='blue', color_tangent='red',
...                 verbose=True)
Plotting from the interpolation associated with the key
 'cubic spline-interp-odeint' by default...
A tiny final offset equal to 0.000301507537688442 was
 introduced for the last point in order to safely compute it
 from the interpolation.
>>> c_plot_2d
Graphics object consisting of 11 graphics primitives
```

solution(*solution_key=None*, *verbose=False*)

Return the solution (list of points) associated with the given key.

INPUT:

- `solution_key` – (default: `None`) key which the requested numerical solution is associated to; a default value is chosen if none is provided
- `verbose` – boolean (default: `False`); prints information about the solution returned

OUTPUT: list of the numerical points of the solution requested

EXAMPLES:

Requesting a numerical solution previously computed:

```
sage: M = Manifold(3, 'M')
sage: X.<x1,x2,x3> = M.chart()
sage: [t, B_0, m, q, L, T] = var('t B_0 m q L T')
sage: B = B_0*t/T*exp(-(x1^2 + x2^2)/L^2)
```

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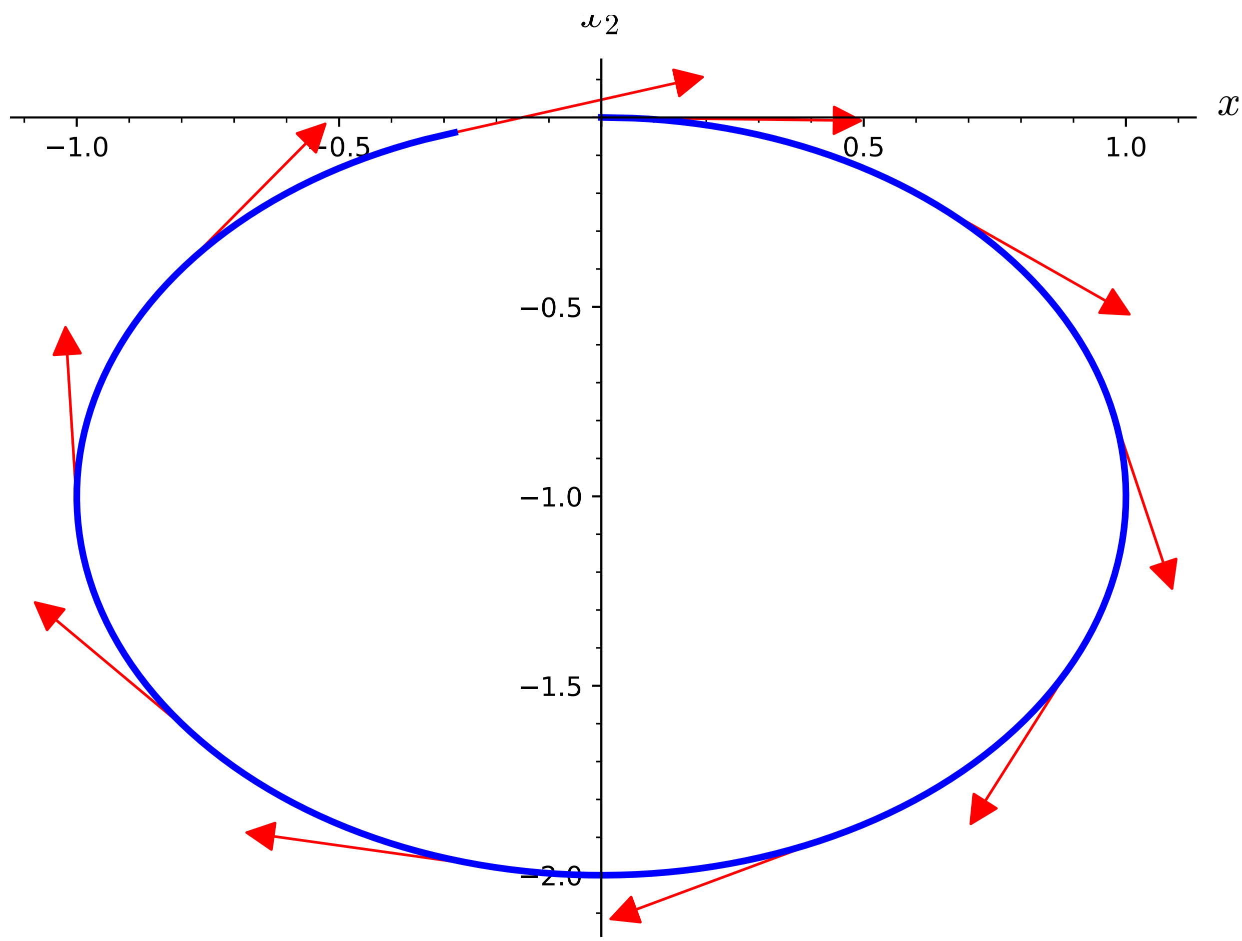

x_2
x
−1.0
−0.5
0.5
1.0
−0.5
−1.0
−1.5
−2.0

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```
sage: D = X.symbolic_velocities()
sage: eqns = [q*B/m*D[1], -q*B/m*D[0], 0]
sage: p = M.point((0,0,0), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,0,1))
sage: c = M.integrated_curve(eqns, D, (t,0,5), v, name='c')

sage: # needs scipy
sage: sol = c.solve(solution_key='sol_T1',
....:         parameters_values={B_0:1, m:1, q:1, L:10, T:1})
sage: sol_bis = c.solution(verbose=True)
Returning the numerical solution associated with the key
 'sol_T1' by default...
sage: sol_bis == sol
True
sage: sol_ter = c.solution(solution_key='sol_T1')
sage: sol_ter == sol
True
sage: sol_mute = c.solution()
sage: sol_mute == sol
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x1', 'x2', 'x3',)); (x1, x2, x3,) = X._first_ngens(3)
>>> [t, B_0, m, q, L, T] = var('t B_0 m q L T')
>>> B = B_0*t/T*exp(-(x1**Integer(2) + x2**Integer(2))/L**Integer(2))
>>> D = X.symbolic_velocities()
>>> eqns = [q*B/m*D[Integer(1)], -q*B/m*D[Integer(0)], Integer(0)]
>>> p = M.point((Integer(0),Integer(0),Integer(0)), name='p')
>>> Tp = M.tangent_space(p)
>>> v = Tp((Integer(1),Integer(0),Integer(1)))
>>> c = M.integrated_curve(eqns, D, (t,Integer(0),Integer(5)), v, name='c')

>>> # needs scipy
>>> sol = c.solve(solution_key='sol_T1',
...         parameters_values={B_0:Integer(1), m:Integer(1), q:Integer(1),␣
→L:Integer(10), T:Integer(1)})
>>> sol_bis = c.solution(verbose=True)
Returning the numerical solution associated with the key
 'sol_T1' by default...
>>> sol_bis == sol
True
>>> sol_ter = c.solution(solution_key='sol_T1')
>>> sol_ter == sol
True
>>> sol_mute = c.solution()
>>> sol_mute == sol
True
```

solve(*step=None*, *method='odeint'*, *solution_key=None*, *parameters_values=None*, *verbose=False*, ***control_param*)

Integrate the curve numerically over the domain of definition.

INPUT:

- `step` – (default: `None`) step of integration; default value is a hundredth of the domain of integration if none is provided
- `method` – (default: `'odeint'`) numerical scheme to use for the integration of the curve; available algorithms are:
  - `'odeint'` – makes use of `scipy.integrate.odeint()` via Sage solver `desolve_odeint()`; `odeint` invokes the LSODA algorithm of the ODEPACK suite, which automatically selects between implicit Adams method (for non-stiff problems) and a method based on backward differentiation formulas (BDF) (for stiff problems).
  - `'rk4_maxima'` – 4th order classical Runge-Kutta, which makes use of Maxima's dynamics package via Sage solver `desolve_system_rk4()` (quite slow)
  - `'dopri5'` – Dormand-Prince Runge-Kutta of order (4)5 provided by `scipy.integrate.ode`
  - `'dop853'` – Dormand-Prince Runge-Kutta of order 8(5,3) provided by `scipy.integrate.ode`

  and those provided by `GSL` via Sage class `ode_solver`:
  - `'rk2'` – embedded Runge-Kutta (2,3)
  - `'rk4'` – 4th order classical Runge-Kutta
  - `'rkf45'` – Runge-Kutta-Felhberg (4,5)
  - `'rkck'` – embedded Runge-Kutta-Cash-Karp (4,5)
  - `'rk8pd'` – Runge-Kutta Prince-Dormand (8,9)
  - `'rk2imp'` – implicit 2nd order Runge-Kutta at Gaussian points
  - `'rk4imp'` – implicit 4th order Runge-Kutta at Gaussian points
  - `'gear1'` – $M = 1$ implicit Gear
  - `'gear2'` – $M = 2$ implicit Gear
  - `'bsimp'` – implicit Bulirsch-Stoer (requires Jacobian)
- `solution_key` – (default: `None`) key which the resulting numerical solution will be associated to; a default value is given if none is provided
- `parameters_values` – (default: `None`) list of numerical values of the parameters present in the system defining the curve, to be substituted in the equations before integration
- `verbose` – boolean (default: `False`); prints information about the computation in progress
- `**control_param` – extra control parameters to be passed to the chosen solver; see the example with `rtol` and `atol` below

OUTPUT: list of the numerical points of the computed solution

EXAMPLES:

Computing a numerical solution:

```
sage: M = Manifold(3, 'M')
sage: X.<x1,x2,x3> = M.chart()
sage: [t, B_0, m, q, L, T] = var('t B_0 m q L T')
sage: B = B_0*t/T*exp(-(x1^2 + x2^2)/L^2)
sage: D = X.symbolic_velocities()
```

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```
sage: eqns = [q*B/m*D[1], -q*B/m*D[0], 0]
sage: p = M.point((0,0,0), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,0,1))
sage: c = M.integrated_curve(eqns, D, (t,0,5), v, name='c')
sage: sol = c.solve(parameters_values={B_0:1, m:1, q:1, L:10, T:1},          #␣
↪needs scipy
....:                  verbose=True)
Performing numerical integration with method 'odeint'...
Resulting list of points will be associated with the key
 'odeint' by default.
Numerical integration completed.

Checking all points are in the chart domain...
All points are in the chart domain.

The resulting list of points was associated with the key
 'odeint' (if this key already referred to a former
 numerical solution, such a solution was erased).
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x1', 'x2', 'x3',)); (x1, x2, x3,) = X._first_ngens(3)
>>> [t, B_0, m, q, L, T] = var('t B_0 m q L T')
>>> B = B_0*t/T*exp(-(x1**Integer(2) + x2**Integer(2))/L**Integer(2))
>>> D = X.symbolic_velocities()
>>> eqns = [q*B/m*D[Integer(1)], -q*B/m*D[Integer(0)], Integer(0)]
>>> p = M.point((Integer(0),Integer(0),Integer(0)), name='p')
>>> Tp = M.tangent_space(p)
>>> v = Tp((Integer(1),Integer(0),Integer(1)))
>>> c = M.integrated_curve(eqns, D, (t,Integer(0),Integer(5)), v, name='c')
>>> sol = c.solve(parameters_values={B_0:Integer(1), m:Integer(1),␣
↪q:Integer(1), L:Integer(10), T:Integer(1)},          # needs scipy
...                  verbose=True)
Performing numerical integration with method 'odeint'...
Resulting list of points will be associated with the key
 'odeint' by default.
Numerical integration completed.
<BLANKLINE>
Checking all points are in the chart domain...
All points are in the chart domain.
<BLANKLINE>
The resulting list of points was associated with the key
 'odeint' (if this key already referred to a former
 numerical solution, such a solution was erased).
```

The first 3 points of the solution, in the form `[t, x1, x2, x3]`:

```
sage: sol[:3]  # abs tol 1e-12                                               #␣
↪needs scipy
[[0.0, 0.0, 0.0, 0.0],
 [0.05, 0.04999999218759271, -2.083327338392213e-05, 0.05],
```

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```
 [0.1, 0.09999975001847655, -0.00016666146190783666, 0.1]]
```

```
>>> from sage.all import *
>>> sol[:Integer(3)]  # abs tol 1e-12                                    ␣
↪     # needs scipy
[[0.0, 0.0, 0.0, 0.0],
 [0.05, 0.04999999218759271, -2.083327338392213e-05, 0.05],
 [0.1, 0.09999975001847655, -0.00016666146190783666, 0.1]]
```

The default is `verbose=False`:

```
sage: sol_mute = c.solve(parameters_values={B_0:1, m:1, q:1,                  #␣
↪needs scipy
....:                                       L:10, T:1})
sage: sol_mute == sol                                                          #␣
↪needs scipy
True
```

```
>>> from sage.all import *
>>> sol_mute = c.solve(parameters_values={B_0:Integer(1), m:Integer(1),␣
↪q:Integer(1),                # needs scipy
...                                     L:Integer(10), T:Integer(1)})
>>> sol_mute == sol                                                          #␣
↪needs scipy
True
```

Specifying the relative and absolute error tolerance parameters to be used in `desolve_odeint()`:

```
sage: sol = c.solve(parameters_values={B_0:1, m:1, q:1, L:10, T:1},           #␣
↪needs scipy
....:               rtol=1e-12, atol=1e-12)
```

```
>>> from sage.all import *
>>> sol = c.solve(parameters_values={B_0:Integer(1), m:Integer(1),␣
↪q:Integer(1), L:Integer(10), T:Integer(1)},          # needs scipy
...               rtol=RealNumber('1e-12'), atol=RealNumber('1e-12'))
```

Using a numerical method different from the default one:

```
sage: sol = c.solve(parameters_values={B_0:1, m:1, q:1, L:10, T:1},           #␣
↪needs scipy
....:               method='rk8pd')
```

```
>>> from sage.all import *
>>> sol = c.solve(parameters_values={B_0:Integer(1), m:Integer(1),␣
↪q:Integer(1), L:Integer(10), T:Integer(1)},          # needs scipy
...               method='rk8pd')
```

solve_across_charts(*charts=None*, *step=None*, *solution_key=None*, *parameters_values=None*, *verbose=False*, ***control_param*)

Integrate the curve numerically over the domain of integration, with the ability to switch chart mid-integration.

The only supported solver is `scipy.integrate.ode`, because it supports basic event handling, needed to detect when the curve is reaching the frontier of the chart. This is an adaptive step solver. So the `step` is not the step of integration but instead the step used to peak at the current chart, and switch if needed.

INPUT:

- `step` – (default: `None`) step of chart checking; default value is a hundredth of the domain of integration if none is provided. If your curve can't find a new frame on exiting the current frame, consider reducing this parameter.
- `charts` – (default: `None`) list of chart allowed. The integration stops once it leaves those charts. By default the whole atlas is taken (only the top-charts).
- `solution_key` – (default: `None`) key which the resulting numerical solution will be associated to; a default value is given if none is provided
- `parameters_values` – (default: `None`) list of numerical values of the parameters present in the system defining the curve, to be substituted in the equations before integration
- `verbose` – boolean (default: `False`); prints information about the computation in progress
- `**control_param` – extra control parameters to be passed to the solver

OUTPUT: list of the numerical points of the computed solution

EXAMPLES:

Let us use `solve_across_charts()` to integrate a geodesic of the Euclidean plane (a straight line) in polar coordinates.

In pure polar coordinates (r, θ), artefacts can appear near the origin because of the fast variation of θ, resulting in the direction of the geodesic being different before and after getting close to the origin.

The solution to this problem is to switch to Cartesian coordinates near $(0, 0)$ to avoid any singularity.

First let's declare the plane as a 2-dimensional manifold, with two charts P en C (for "Polar" and "Cartesian") and their transition maps:

```
sage: M = Manifold(2, 'M', structure='Riemannian')
sage: C.<x,y> = M.chart(coord_restrictions=lambda x,y: x**2+y**2 < 3**2)
sage: P.<r,th> = M.chart(coord_restrictions=lambda r, th: r > 2)
sage: P_to_C = P.transition_map(C,(r*cos(th), r*sin(th)))
sage: C_to_P = C.transition_map(P,(sqrt(x**2+y**2), atan2(y,x)))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='Riemannian')
>>> C = M.chart(coord_restrictions=lambda x,y: x**Integer(2)+y**Integer(2) <␣
↪Integer(3)**Integer(2), names=('x', 'y',)); (x, y,) = C._first_ngens(2)
>>> P = M.chart(coord_restrictions=lambda r, th: r > Integer(2), names=('r',
↪'th',)); (r, th,) = P._first_ngens(2)
>>> P_to_C = P.transition_map(C,(r*cos(th), r*sin(th)))
>>> C_to_P = C.transition_map(P,(sqrt(x**Integer(2)+y**Integer(2)), atan2(y,
↪x)))
```

Here we added restrictions on those charts, to avoid any singularity. The intersection is the donut region $2 < r < 3$.

We still have to define the metric. This is done in the Cartesian frame. The metric in the polar frame is computed automatically:

```
sage: g = M.metric()
sage: g[0,0,C]=1
sage: g[1,1,C]=1
sage: g[P.frame(), : ,P]
[   1    0]
[   0 r^2]
```

```
>>> from sage.all import *
>>> g = M.metric()
>>> g[Integer(0),Integer(0),C]=Integer(1)
>>> g[Integer(1),Integer(1),C]=Integer(1)
>>> g[P.frame(), : ,P]
[   1    0]
[   0 r^2]
```

To visualize our manifold, let's declare a mapping between every chart and the Cartesian chart, and then plot each chart in term of this mapping:

```
sage: phi = M.diff_map(M, {(C,C): [x, y], (P,C): [r*cos(th), r*sin(th)]})
sage: fig = P.plot(number_values=9, chart=C, mapping=phi,                  #␣
↪needs sage.plot
....:              color='grey', ranges= {r:(2, 6), th:(0,2*pi)})
sage: fig += C.plot(number_values=13, chart=C, mapping=phi,                #␣
↪needs sage.plot
....:               color='grey', ranges= {x:(-3, 3), y:(-3, 3)})
```

```
>>> from sage.all import *
>>> phi = M.diff_map(M, {(C,C): [x, y], (P,C): [r*cos(th), r*sin(th)]})
>>> fig = P.plot(number_values=Integer(9), chart=C, mapping=phi,                ␣
↪     # needs sage.plot
...              color='grey', ranges= {r:(Integer(2), Integer(6)),␣
↪th:(Integer(0),Integer(2)*pi)})
>>> fig += C.plot(number_values=Integer(13), chart=C, mapping=phi,              ␣
↪     # needs sage.plot
...               color='grey', ranges= {x:(-Integer(3), Integer(3)), y:(-
↪Integer(3), Integer(3))})
```

There is a clear non-empty intersection between the two charts. This is the key point to successfully switch chart during the integration. Indeed, at least 2 points must fall in the intersection.

Geodesic integration

Let's define the time as t, the initial point as p, and the initial velocity vector as v (define as a member of the tangent space T_p). The chosen geodesic should enter the central region from the left and leave it to the right:

```
sage: t = var('t')
sage: p = M((5,pi+0.3), P)
sage: Tp = M.tangent_space(p)
sage: v = Tp((-1,-0.03), P.frame().at(p))
```

```
>>> from sage.all import *
>>> t = var('t')
```

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```
>>> p = M((Integer(5),pi+RealNumber('0.3')), P)
>>> Tp = M.tangent_space(p)
>>> v = Tp((-Integer(1),-RealNumber('0.03')), P.frame().at(p))
```

While creating the integrated geodesic, we need to specify the optional argument `across_chart=True`, to prepare the compiled version of the changes of charts:

```
sage: c = M.integrated_geodesic(g, (t, 0, 10), v, across_charts=True)
```

```
>>> from sage.all import *
>>> c = M.integrated_geodesic(g, (t, Integer(0), Integer(10)), v, across_
↪charts=True)
```

The integration is done as usual, but using the method *solve_across_charts()* instead of *solve()*. This forces the use of `scipy.integrate.ode` as the solver, because of event handling support.

The argument `verbose=True` will cause the solver to write a small message each time it is switching chart:

```
sage: sol = c.solve_across_charts(step=0.1, verbose=True)
Performing numerical integration with method 'ode'.
Integration will take place on the whole manifold domain.
Resulting list of points will be associated with the key 'ode_multichart' by␣
↪default.
   ...
Exiting chart, trying to switch to another chart.
New chart found. Resuming integration.
Exiting chart, trying to switch to another chart.
New chart found. Resuming integration.
Integration successful.
```

```
>>> from sage.all import *
>>> sol = c.solve_across_charts(step=RealNumber('0.1'), verbose=True)
Performing numerical integration with method 'ode'.
Integration will take place on the whole manifold domain.
Resulting list of points will be associated with the key 'ode_multichart' by␣
↪default.
   ...
Exiting chart, trying to switch to another chart.
New chart found. Resuming integration.
Exiting chart, trying to switch to another chart.
New chart found. Resuming integration.
Integration successful.
```

As expected, two changes of chart occur.

The returned solution is a list of pairs `(chart, solution)`, where each solution is given on a unique chart, and the last point of a solution is the first of the next.

The following code prints the corresponding charts:

```
sage: for chart, solution in sol:
....:     print(chart)
Chart (M, (r, th))
```

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```
Chart (M, (x, y))
Chart (M, (r, th))
```

```
>>> from sage.all import *
>>> for chart, solution in sol:
...     print(chart)
Chart (M, (r, th))
Chart (M, (x, y))
Chart (M, (r, th))
```

The interpolation is done as usual:

```
sage: interp = c.interpolate()
```

```
>>> from sage.all import *
>>> interp = c.interpolate()
```

To plot the result, you must first be sure that the mapping encompasses all the chart, which is the case here. You must also specify `across_charts=True` in order to call *plot_integrated()* again on each part. Finally, `color` can be a list, which will be cycled through:

```
sage: fig += c.plot_integrated(mapping=phi, color=["green","red"],              #␣
↪needs sage.plot
....: thickness=3, plot_points=100, across_charts=True)
sage: fig                                                                       #␣
↪needs sage.plot
Graphics object consisting of 43 graphics primitives
```

```
>>> from sage.all import *
>>> fig += c.plot_integrated(mapping=phi, color=["green","red"],              #␣
↪needs sage.plot
... thickness=Integer(3), plot_points=Integer(100), across_charts=True)
>>> fig                                                                       #␣
↪needs sage.plot
Graphics object consisting of 43 graphics primitives
```

solve_analytical(*verbose=False*)

Solve the differential system defining `self` analytically.

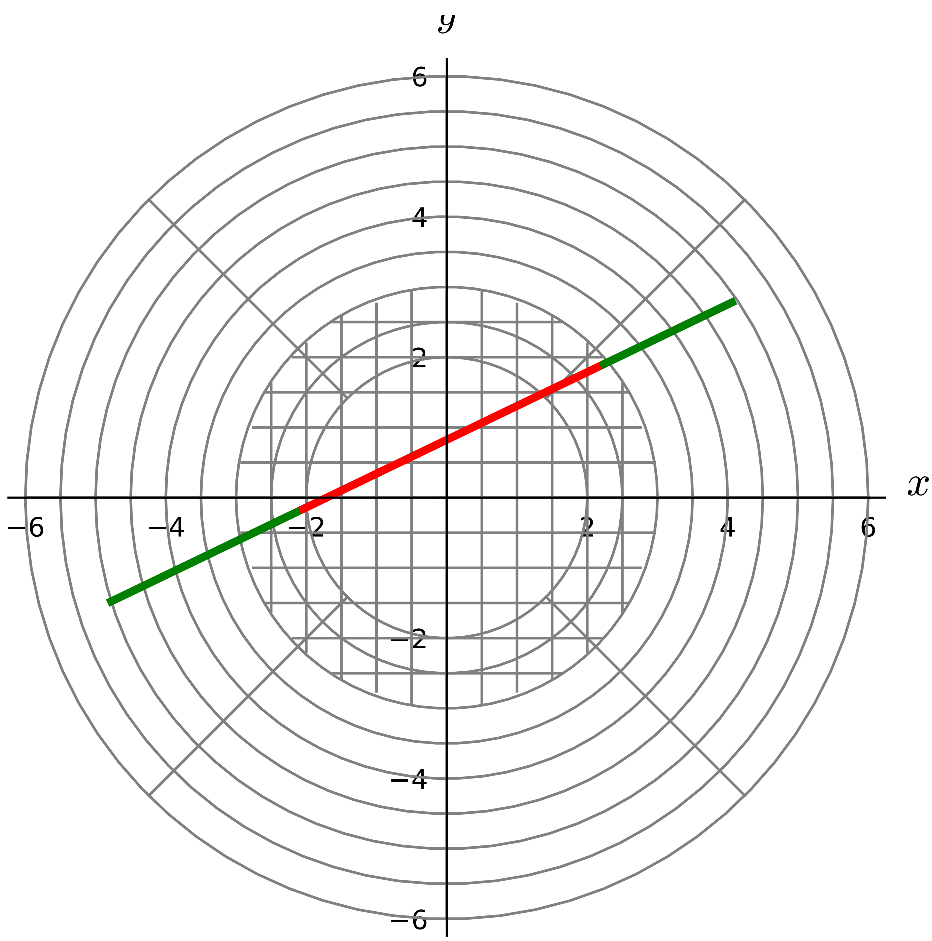
Solve analytically the differential system defining a curve using Maxima via Sage solver `desolve_system`. In case of success, the analytical expressions are added to the dictionary of expressions representing the curve. Pay attention to the fact that `desolve_system` only considers initial conditions given at an initial parameter value equal to zero, although the parameter range may not contain zero. Yet, assuming that it does, values of the coordinates functions at such zero initial parameter value are denoted by the name of the coordinate function followed by the string `'_0'`.

OUTPUT:

- list of the analytical expressions of the coordinate functions (when the differential system could be solved analytically), or boolean `False` (in case the differential system could not be solved analytically)

EXAMPLES:

Analytical expression of the trajectory of a charged particle in a uniform, stationary magnetic field:



y
x
6
4
2
−2
−4
−6
−6
−4
−2
2
4
6

```
sage: M = Manifold(3, 'M')
sage: X.<x1,x2,x3> = M.chart()
sage: [t, B_0, m, q] = var('t B_0 m q')
sage: D = X.symbolic_velocities()
sage: eqns = [q*B_0/m*D[1], -q*B_0/m*D[0], 0]
sage: p = M.point((0,0,0), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,0,1))
sage: c = M.integrated_curve(eqns, D, (t,0,5), v, name='c')
sage: sys = c.system(verbose=True)
Curve c in the 3-dimensional differentiable manifold M
 integrated over the Real interval (0, 5) as a solution to
 the following system, written with respect to
 Chart (M, (x1, x2, x3)):

Initial point: Point p on the 3-dimensional differentiable
 manifold M with coordinates [0, 0, 0] with respect to
 Chart (M, (x1, x2, x3))
Initial tangent vector: Tangent vector at Point p on the
 3-dimensional differentiable manifold M with components
 [1, 0, 1] with respect to Chart (M, (x1, x2, x3))

d(x1)/dt = Dx1
d(x2)/dt = Dx2
d(x3)/dt = Dx3
d(Dx1)/dt = B_0*Dx2*q/m
d(Dx2)/dt = -B_0*Dx1*q/m
d(Dx3)/dt = 0

sage: sol = c.solve_analytical()
sage: c.expr()
((B_0*q*x1_0 - Dx2_0*m*cos(B_0*q*t/m) +
   Dx1_0*m*sin(B_0*q*t/m) + Dx2_0*m)/(B_0*q),
 (B_0*q*x2_0 + Dx1_0*m*cos(B_0*q*t/m) +
  Dx2_0*m*sin(B_0*q*t/m) - Dx1_0*m)/(B_0*q),
 Dx3_0*t + x3_0)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x1', 'x2', 'x3',)); (x1, x2, x3,) = X._first_ngens(3)
>>> [t, B_0, m, q] = var('t B_0 m q')
>>> D = X.symbolic_velocities()
>>> eqns = [q*B_0/m*D[Integer(1)], -q*B_0/m*D[Integer(0)], Integer(0)]
>>> p = M.point((Integer(0),Integer(0),Integer(0)), name='p')
>>> Tp = M.tangent_space(p)
>>> v = Tp((Integer(1),Integer(0),Integer(1)))
>>> c = M.integrated_curve(eqns, D, (t,Integer(0),Integer(5)), v, name='c')
>>> sys = c.system(verbose=True)
Curve c in the 3-dimensional differentiable manifold M
 integrated over the Real interval (0, 5) as a solution to
 the following system, written with respect to
 Chart (M, (x1, x2, x3)):
```

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```
<BLANKLINE>
Initial point: Point p on the 3-dimensional differentiable
 manifold M with coordinates [0, 0, 0] with respect to
 Chart (M, (x1, x2, x3))
Initial tangent vector: Tangent vector at Point p on the
 3-dimensional differentiable manifold M with components
 [1, 0, 1] with respect to Chart (M, (x1, x2, x3))
<BLANKLINE>
d(x1)/dt = Dx1
d(x2)/dt = Dx2
d(x3)/dt = Dx3
d(Dx1)/dt = B_0*Dx2*q/m
d(Dx2)/dt = -B_0*Dx1*q/m
d(Dx3)/dt = 0
<BLANKLINE>
>>> sol = c.solve_analytical()
>>> c.expr()
((B_0*q*x1_0 - Dx2_0*m*cos(B_0*q*t/m) +
   Dx1_0*m*sin(B_0*q*t/m) + Dx2_0*m)/(B_0*q),
 (B_0*q*x2_0 + Dx1_0*m*cos(B_0*q*t/m) +
  Dx2_0*m*sin(B_0*q*t/m) - Dx1_0*m)/(B_0*q),
 Dx3_0*t + x3_0)
```

system(*verbose=False*)

Provide a detailed description of the system defining the curve and return the system defining it: chart, equations and initial conditions.

INPUT:

- `verbose` – boolean (default: `False`); prints a detailed description of the curve

OUTPUT: list containing

- the equations
- the initial conditions
- the chart

EXAMPLES:

System defining an integrated curve:

```
sage: M = Manifold(3, 'M')
sage: X.<x1,x2,x3> = M.chart()
sage: [t, B_0, m, q, L, T] = var('t B_0 m q L T')
sage: B = B_0*t/T*exp(-(x1^2 + x2^2)/L^2)
sage: D = X.symbolic_velocities()
sage: eqns = [q*B/m*D[1], -q*B/m*D[0], 0]
sage: p = M.point((0,0,0), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,0,1))
sage: c = M.integrated_curve(eqns, D, (t,0,5), v, name='c')
sage: sys = c.system(verbose=True)
Curve c in the 3-dimensional differentiable manifold M
 integrated over the Real interval (0, 5) as a solution to
```

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```
 the following system, written with respect to
 Chart (M, (x1, x2, x3)):

Initial point: Point p on the 3-dimensional differentiable
 manifold M with coordinates [0, 0, 0] with respect to
 Chart (M, (x1, x2, x3))
Initial tangent vector: Tangent vector at Point p on the
 3-dimensional differentiable manifold M with
 components [1, 0, 1] with respect to Chart (M, (x1, x2, x3))

d(x1)/dt = Dx1
d(x2)/dt = Dx2
d(x3)/dt = Dx3
d(Dx1)/dt = B_0*Dx2*q*t*e^(-(x1^2 + x2^2)/L^2)/(T*m)
d(Dx2)/dt = -B_0*Dx1*q*t*e^(-(x1^2 + x2^2)/L^2)/(T*m)
d(Dx3)/dt = 0

sage: sys_mute = c.system()
sage: sys_mute == sys
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x1', 'x2', 'x3',)); (x1, x2, x3,) = X._first_ngens(3)
>>> [t, B_0, m, q, L, T] = var('t B_0 m q L T')
>>> B = B_0*t/T*exp(-(x1**Integer(2) + x2**Integer(2))/L**Integer(2))
>>> D = X.symbolic_velocities()
>>> eqns = [q*B/m*D[Integer(1)], -q*B/m*D[Integer(0)], Integer(0)]
>>> p = M.point((Integer(0),Integer(0),Integer(0)), name='p')
>>> Tp = M.tangent_space(p)
>>> v = Tp((Integer(1),Integer(0),Integer(1)))
>>> c = M.integrated_curve(eqns, D, (t,Integer(0),Integer(5)), v, name='c')
>>> sys = c.system(verbose=True)
Curve c in the 3-dimensional differentiable manifold M
 integrated over the Real interval (0, 5) as a solution to
 the following system, written with respect to
 Chart (M, (x1, x2, x3)):
<BLANKLINE>
Initial point: Point p on the 3-dimensional differentiable
 manifold M with coordinates [0, 0, 0] with respect to
 Chart (M, (x1, x2, x3))
Initial tangent vector: Tangent vector at Point p on the
 3-dimensional differentiable manifold M with
 components [1, 0, 1] with respect to Chart (M, (x1, x2, x3))
<BLANKLINE>
d(x1)/dt = Dx1
d(x2)/dt = Dx2
d(x3)/dt = Dx3
d(Dx1)/dt = B_0*Dx2*q*t*e^(-(x1^2 + x2^2)/L^2)/(T*m)
d(Dx2)/dt = -B_0*Dx1*q*t*e^(-(x1^2 + x2^2)/L^2)/(T*m)
d(Dx3)/dt = 0
<BLANKLINE>
```

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```
>>> sys_mute = c.system()
>>> sys_mute == sys
True
```

tangent_vector_eval_at (*t*, *interpolation_key=None*, *verbose=False*)

Return the vector tangent to `self` at the given curve parameter with components evaluated from the given interpolation.

INPUT:

- `t` – curve parameter value at which the tangent vector is evaluated
- `interpolation_key` – (default: `None`) key which the interpolation requested to compute the tangent vector is associated to; a default value is chosen if none is provided
- `verbose` – boolean (default: `False`); prints information about the interpolation used

OUTPUT:

- `TangentVector` tangent vector with numerical components

EXAMPLES:

Evaluating a vector tangent to the curve:

```
sage: M = Manifold(3, 'M')
sage: X.<x1,x2,x3> = M.chart()
sage: [t, B_0, m, q, L, T] = var('t B_0 m q L T')
sage: B = B_0*t/T*exp(-(x1^2 + x2^2)/L^2)
sage: D = X.symbolic_velocities()
sage: eqns = [q*B/m*D[1], -q*B/m*D[0], 0]
sage: p = M.point((0,0,0), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,0,1))
sage: c = M.integrated_curve(eqns, D, (t,0,5), v, name='c')

sage: # needs scipy
sage: sol = c.solve(method='odeint',
....:         solution_key='sol_T1',
....:         parameters_values={B_0:1, m:1, q:1, L:10, T:1})
sage: interp = c.interpolate(method='cubic spline',
....:                          solution_key='sol_T1',
....:                          interpolation_key='interp_T1')
sage: tg_vec = c.tangent_vector_eval_at(1.22, verbose=True)
Evaluating tangent vector components from the interpolation
 associated with the key 'interp_T1' by default...
sage: tg_vec
Tangent vector at Point on the 3-dimensional differentiable
 manifold M
sage: tg_vec[:]     # abs tol 1e-12
[0.7392640422917979, -0.6734182509826023, 1.0]
sage: tg_vec_mute = c.tangent_vector_eval_at(1.22,
....:                          interpolation_key='interp_T1')
sage: tg_vec_mute == tg_vec
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x1', 'x2', 'x3',)); (x1, x2, x3,) = X._first_ngens(3)
>>> [t, B_0, m, q, L, T] = var('t B_0 m q L T')
>>> B = B_0*t/T*exp(-(x1**Integer(2) + x2**Integer(2))/L**Integer(2))
>>> D = X.symbolic_velocities()
>>> eqns = [q*B/m*D[Integer(1)], -q*B/m*D[Integer(0)], Integer(0)]
>>> p = M.point((Integer(0),Integer(0),Integer(0)), name='p')
>>> Tp = M.tangent_space(p)
>>> v = Tp((Integer(1),Integer(0),Integer(1)))
>>> c = M.integrated_curve(eqns, D, (t,Integer(0),Integer(5)), v, name='c')

>>> # needs scipy
>>> sol = c.solve(method='odeint',
...         solution_key='sol_T1',
...         parameters_values={B_0:Integer(1), m:Integer(1), q:Integer(1),␣
↪L:Integer(10), T:Integer(1)})
>>> interp = c.interpolate(method='cubic spline',
...                          solution_key='sol_T1',
...                          interpolation_key='interp_T1')
>>> tg_vec = c.tangent_vector_eval_at(RealNumber('1.22'), verbose=True)
Evaluating tangent vector components from the interpolation
 associated with the key 'interp_T1' by default...
>>> tg_vec
Tangent vector at Point on the 3-dimensional differentiable
 manifold M
>>> tg_vec[:]     # abs tol 1e-12
[0.7392640422917979, -0.6734182509826023, 1.0]
>>> tg_vec_mute = c.tangent_vector_eval_at(RealNumber('1.22'),
...                          interpolation_key='interp_T1')
>>> tg_vec_mute == tg_vec
True
```

class sage.manifolds.differentiable.integrated_curve.**IntegratedGeodesic**(*parent*, *metric*, *curve_parameter*, *initial_tangent_vector*, *chart=None*, *name=None*, *latex_name=None*, *verbose=False*, *across_charts=False*)

Bases: *IntegratedAutoparallelCurve*

Geodesic on the manifold with respect to a given metric.

INPUT:

- parent – *IntegratedGeodesicSet* the set of curves $\mathrm{Hom_{geodesic}}(I, M)$ to which the curve belongs
- metric – *PseudoRiemannianMetric* metric with respect to which the curve is a geodesic
- curve_parameter – symbolic expression to be used as the parameter of the curve (the equations defining an instance of IntegratedGeodesic are such that t will actually be an affine parameter of the curve);
- initial_tangent_vector – *TangentVector* initial tangent vector of the curve

- `chart` – (default: `None`) chart on the manifold in terms of which the equations are expressed; if `None` the default chart of the manifold is assumed
- `name` – (default: `None`) string; symbol given to the curve
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the curve; if none is provided, `name` will be used

EXAMPLES:

Geodesics of the unit 2-sphere $\mathbb{S}^2$. Start with declaring the standard polar coordinates (θ, ϕ) on $\mathbb{S}^2$ and the corresponding coordinate frame (e_θ, e_ϕ):

```
sage: S2 = Manifold(2, 'S^2', structure='Riemannian', start_index=1)
sage: polar.<th,ph>=S2.chart('th ph')
sage: epolar = polar.frame()
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2', structure='Riemannian', start_
↪index=Integer(1))
>>> polar = S2.chart('th ph', names=('th', 'ph',)); (th, ph,) = polar._first_
↪ngens(2)
>>> epolar = polar.frame()
```

Set the standard round metric:

```
sage: g = S2.metric()
sage: g[1,1], g[2,2] = 1, (sin(th))^2
```

```
>>> from sage.all import *
>>> g = S2.metric()
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)] = Integer(1),␣
↪(sin(th))**Integer(2)
```

Set generic initial conditions for the geodesics to compute:

```
sage: [th0, ph0, v_th0, v_ph0] = var('th0 ph0 v_th0 v_ph0')
sage: p = S2.point((th0, ph0), name='p')
sage: Tp = S2.tangent_space(p)
sage: v = Tp((v_th0, v_ph0), basis=epolar.at(p))
```

```
>>> from sage.all import *
>>> [th0, ph0, v_th0, v_ph0] = var('th0 ph0 v_th0 v_ph0')
>>> p = S2.point((th0, ph0), name='p')
>>> Tp = S2.tangent_space(p)
>>> v = Tp((v_th0, v_ph0), basis=epolar.at(p))
```

Declare the corresponding integrated geodesic and display the differential system it satisfies:

```
sage: [t, tmin, tmax] = var('t tmin tmax')
sage: c = S2.integrated_geodesic(g, (t, tmin, tmax), v,
....:                                 chart=polar, name='c')
sage: sys = c.system(verbose=True)
Geodesic c in the 2-dimensional Riemannian manifold S^2
 equipped with Riemannian metric g on the 2-dimensional
```

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```
 Riemannian manifold S^2, and integrated over the Real
 interval (tmin, tmax) as a solution to the following geodesic
 equations, written with respect to Chart (S^2, (th, ph)):

Initial point: Point p on the 2-dimensional Riemannian
manifold S^2 with coordinates [th0, ph0] with respect to
Chart (S^2, (th, ph))
Initial tangent vector: Tangent vector at Point p on the
2-dimensional Riemannian manifold S^2 with
components [v_th0, v_ph0] with respect to Chart (S^2, (th, ph))

d(th)/dt = Dth
d(ph)/dt = Dph
d(Dth)/dt = Dph^2*cos(th)*sin(th)
d(Dph)/dt = -2*Dph*Dth*cos(th)/sin(th)
```

```
>>> from sage.all import *
>>> [t, tmin, tmax] = var('t tmin tmax')
>>> c = S2.integrated_geodesic(g, (t, tmin, tmax), v,
...                            chart=polar, name='c')
>>> sys = c.system(verbose=True)
Geodesic c in the 2-dimensional Riemannian manifold S^2
 equipped with Riemannian metric g on the 2-dimensional
 Riemannian manifold S^2, and integrated over the Real
 interval (tmin, tmax) as a solution to the following geodesic
 equations, written with respect to Chart (S^2, (th, ph)):
<BLANKLINE>
Initial point: Point p on the 2-dimensional Riemannian
manifold S^2 with coordinates [th0, ph0] with respect to
Chart (S^2, (th, ph))
Initial tangent vector: Tangent vector at Point p on the
2-dimensional Riemannian manifold S^2 with
components [v_th0, v_ph0] with respect to Chart (S^2, (th, ph))
<BLANKLINE>
d(th)/dt = Dth
d(ph)/dt = Dph
d(Dth)/dt = Dph^2*cos(th)*sin(th)
d(Dph)/dt = -2*Dph*Dth*cos(th)/sin(th)
<BLANKLINE>
```

Set a dictionary providing the parameter range and the initial conditions for various geodesics:

```
sage: dict_params={'equat':{tmin:0,tmax:3,th0:pi/2,ph0:0.1,v_th0:0,v_ph0:1},
....:   'longi':{tmin:0,tmax:3,th0:0.1,ph0:0.1,v_th0:1,v_ph0:0},
....:   'angle':{tmin:0,tmax:3,th0:pi/4,ph0:0.1,v_th0:1,v_ph0:1}}
```

```
>>> from sage.all import *
>>> dict_params={'equat':{tmin:Integer(0),tmax:Integer(3),th0:pi/Integer(2),
↪ph0:RealNumber('0.1'),v_th0:Integer(0),v_ph0:Integer(1)},
...   'longi':{tmin:Integer(0),tmax:Integer(3),th0:RealNumber('0.1'),
↪ph0:RealNumber('0.1'),v_th0:Integer(1),v_ph0:Integer(0)},
...   'angle':{tmin:Integer(0),tmax:Integer(3),th0:pi/Integer(4),ph0:RealNumber(
```

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```
↪'0.1'),v_th0:Integer(1),v_ph0:Integer(1)}}
```

Use $\mathbb{R}^3$ as the codomain of the standard map embedding $(\mathbb{S}^2, (\theta, \phi))$ in the 3-dimensional Euclidean space:

```
sage: R3 = Manifold(3, 'R3', start_index=1)
sage: cart.<X,Y,Z> = R3.chart()
sage: euclid_embedding = S2.diff_map(R3,
....:  {(polar, cart):[sin(th)*cos(ph),sin(th)*sin(ph),cos(th)]})
```

```
>>> from sage.all import *
>>> R3 = Manifold(Integer(3), 'R3', start_index=Integer(1))
>>> cart = R3.chart(names=('X', 'Y', 'Z',)); (X, Y, Z,) = cart._first_ngens(3)
>>> euclid_embedding = S2.diff_map(R3,
...  {(polar, cart):[sin(th)*cos(ph),sin(th)*sin(ph),cos(th)]})
```

Solve, interpolate and prepare the plot for the solutions corresponding to the three initial conditions previously set:

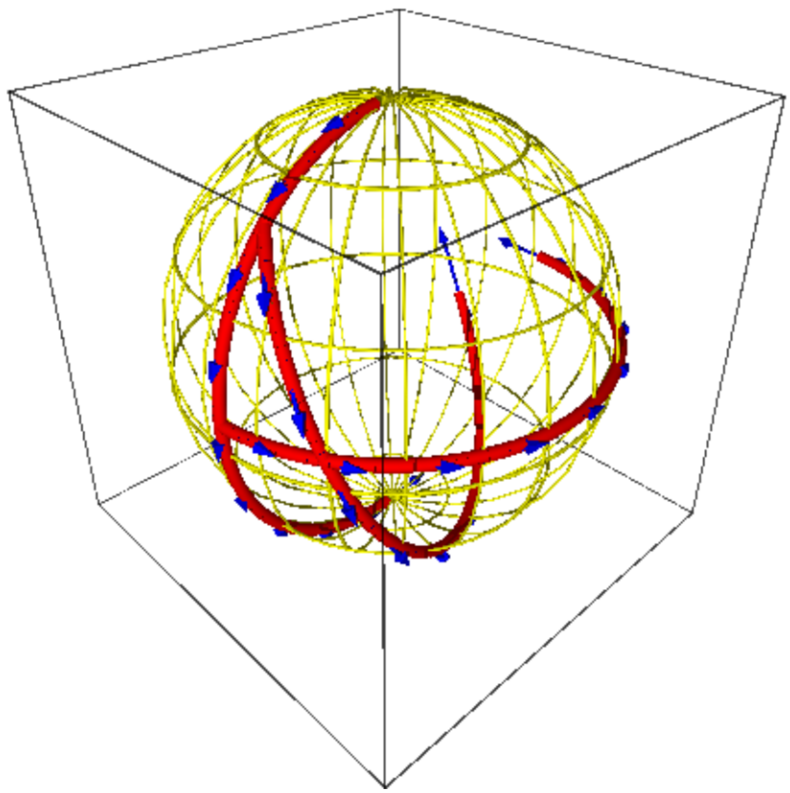
```
sage: # needs scipy sage.plot
sage: graph3D_embedded_geods = Graphics()
sage: for key in dict_params:
....:     sol = c.solve(solution_key='sol-'+key,
....:                        parameters_values=dict_params[key])
....:     interp = c.interpolate(solution_key='sol-'+key,
....:                           interpolation_key='interp-'+key)
....:     graph3D_embedded_geods += c.plot_integrated(interpolation_key='interp-
↪'+key,
....:                      mapping=euclid_embedding, thickness=5,
....:                      display_tangent=True, scale=0.3,
....:                      width_tangent=0.5)
```

```
>>> from sage.all import *
>>> # needs scipy sage.plot
>>> graph3D_embedded_geods = Graphics()
>>> for key in dict_params:
...     sol = c.solve(solution_key='sol-'+key,
...                        parameters_values=dict_params[key])
...     interp = c.interpolate(solution_key='sol-'+key,
...                           interpolation_key='interp-'+key)
...     graph3D_embedded_geods += c.plot_integrated(interpolation_key='interp-
↪'+key,
...                      mapping=euclid_embedding, thickness=Integer(5),
...                      display_tangent=True, scale=RealNumber('0.3'),
...                      width_tangent=RealNumber('0.5'))
```

Plot the resulting geodesics on the grid of polar coordinates lines on $\mathbb{S}^2$ and check that these are great circles:

```
sage: # needs scipy sage.plot
sage: graph3D_embedded_polar_coords = polar.plot(chart=cart,
....:                          mapping=euclid_embedding,
....:                          number_values=15, color='yellow')
sage: graph3D_embedded_geods + graph3D_embedded_polar_coords
Graphics3d Object
```

```
>>> from sage.all import *
>>> # needs scipy sage.plot
>>> graph3D_embedded_polar_coords = polar.plot(chart=cart,
...                               mapping=euclid_embedding,
...                               number_values=Integer(15), color='yellow')
>>> graph3D_embedded_geods + graph3D_embedded_polar_coords
Graphics3d Object
```



system(*verbose=False*)

Return the system defining the geodesic: chart, equations and initial conditions.

INPUT:

- `verbose` – boolean (default: `False`); prints a detailed description of the curve

OUTPUT: list containing

- the equations
- the initial equations
- the chart

EXAMPLES:

System defining a geodesic:

```
sage: S2 = Manifold(2, 'S^2',structure='Riemannian')
sage: X.<theta,phi> = S2.chart()
sage: t, A = var('t A')
sage: g = S2.metric()
sage: g[0,0] = A
sage: g[1,1] = A*sin(theta)^2
sage: p = S2.point((pi/2,0), name='p')
sage: Tp = S2.tangent_space(p)
sage: v = Tp((1/sqrt(2),1/sqrt(2)))
sage: c = S2.integrated_geodesic(g, (t, 0, pi), v, name='c')
sage: sys = c.system(verbose=True)
Geodesic c in the 2-dimensional Riemannian manifold S^2
 equipped with Riemannian metric g on the 2-dimensional
 Riemannian manifold S^2, and integrated over the Real
 interval (0, pi) as a solution to the following geodesic
 equations, written with respect to Chart (S^2, (theta, phi)):

Initial point: Point p on the 2-dimensional Riemannian
 manifold S^2 with coordinates [1/2*pi, 0] with respect to
 Chart (S^2, (theta, phi))
Initial tangent vector: Tangent vector at Point p on the
 2-dimensional Riemannian manifold S^2 with
 components [1/2*sqrt(2), 1/2*sqrt(2)] with respect to
 Chart (S^2, (theta, phi))

d(theta)/dt = Dtheta
d(phi)/dt = Dphi
d(Dtheta)/dt = Dphi^2*cos(theta)*sin(theta)
d(Dphi)/dt = -2*Dphi*Dtheta*cos(theta)/sin(theta)

sage: sys_bis = c.system()
sage: sys_bis == sys
True
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2',structure='Riemannian')
>>> X = S2.chart(names=('theta', 'phi',)); (theta, phi,) = X._first_ngens(2)
>>> t, A = var('t A')
>>> g = S2.metric()
>>> g[Integer(0),Integer(0)] = A
>>> g[Integer(1),Integer(1)] = A*sin(theta)**Integer(2)
>>> p = S2.point((pi/Integer(2),Integer(0)), name='p')
>>> Tp = S2.tangent_space(p)
>>> v = Tp((Integer(1)/sqrt(Integer(2)),Integer(1)/sqrt(Integer(2))))
>>> c = S2.integrated_geodesic(g, (t, Integer(0), pi), v, name='c')
>>> sys = c.system(verbose=True)
Geodesic c in the 2-dimensional Riemannian manifold S^2
 equipped with Riemannian metric g on the 2-dimensional
 Riemannian manifold S^2, and integrated over the Real
 interval (0, pi) as a solution to the following geodesic
 equations, written with respect to Chart (S^2, (theta, phi)):
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```

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```
Initial point: Point p on the 2-dimensional Riemannian
 manifold S^2 with coordinates [1/2*pi, 0] with respect to
 Chart (S^2, (theta, phi))
Initial tangent vector: Tangent vector at Point p on the
 2-dimensional Riemannian manifold S^2 with
 components [1/2*sqrt(2), 1/2*sqrt(2)] with respect to
 Chart (S^2, (theta, phi))
<BLANKLINE>
d(theta)/dt = Dtheta
d(phi)/dt = Dphi
d(Dtheta)/dt = Dphi^2*cos(theta)*sin(theta)
d(Dphi)/dt = -2*Dphi*Dtheta*cos(theta)/sin(theta)
<BLANKLINE>
>>> sys_bis = c.system()
>>> sys_bis == sys
True
```

2.6 Tangent Spaces

2.6.1 Tangent Spaces

The class *TangentSpace* implements tangent vector spaces to a differentiable manifold.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2014-2015): initial version
- Travis Scrimshaw (2016): review tweaks

REFERENCES:

- Chap. 3 of [Lee2013]

class sage.manifolds.differentiable.tangent_space.**TangentSpace**(*point: ManifoldPoint*, *base_ring=None*)

Bases: FiniteRankFreeModule

Tangent space to a differentiable manifold at a given point.

Let M be a differentiable manifold of dimension n over a topological field K and $p \in M$. The tangent space T_pM is an n-dimensional vector space over K (without a distinguished basis).

INPUT:

- point – *ManifoldPoint*; point p at which the tangent space is defined

EXAMPLES:

Tangent space on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: p = M.point((-1,2), name='p')
sage: Tp = M.tangent_space(p) ; Tp
Tangent space at Point p on the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> p = M.point((-Integer(1),Integer(2)), name='p')
>>> Tp = M.tangent_space(p) ; Tp
Tangent space at Point p on the 2-dimensional differentiable manifold M
```

Tangent spaces are free modules of finite rank over `SymbolicRing` (actually vector spaces of finite dimension over the manifold base field K, with $K = \mathbf{R}$ here):

```
sage: Tp.base_ring()
Symbolic Ring
sage: Tp.category()
Category of finite dimensional vector spaces over Symbolic Ring
sage: Tp.rank()
2
sage: dim(Tp)
2
```

```
>>> from sage.all import *
>>> Tp.base_ring()
Symbolic Ring
>>> Tp.category()
Category of finite dimensional vector spaces over Symbolic Ring
>>> Tp.rank()
2
>>> dim(Tp)
2
```

The tangent space is automatically endowed with bases deduced from the vector frames around the point:

```
sage: Tp.bases()
[Basis (∂/∂x,∂/∂y) on the Tangent space at Point p on the 2-dimensional
 differentiable manifold M]
sage: M.frames()
[Coordinate frame (M, (∂/∂x,∂/∂y))]
```

```
>>> from sage.all import *
>>> Tp.bases()
[Basis (∂/∂x,∂/∂y) on the Tangent space at Point p on the 2-dimensional
 differentiable manifold M]
>>> M.frames()
[Coordinate frame (M, (∂/∂x,∂/∂y))]
```

At this stage, only one basis has been defined in the tangent space, but new bases can be added from vector frames on the manifold by means of the method `at()`, for instance, from the frame associated with some new coordinates:

```
sage: c_uv.<u,v> = M.chart()
sage: c_uv.frame().at(p)
Basis (∂/∂u,∂/∂v) on the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
sage: Tp.bases()
[Basis (∂/∂x,∂/∂y) on the Tangent space at Point p on the 2-dimensional
```

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```
differentiable manifold M,
 Basis (∂/∂u,∂/∂v) on the Tangent space at Point p on the 2-dimensional
 differentiable manifold M]
```

```
>>> from sage.all import *
>>> c_uv = M.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> c_uv.frame().at(p)
Basis (∂/∂u,∂/∂v) on the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
>>> Tp.bases()
[Basis (∂/∂x,∂/∂y) on the Tangent space at Point p on the 2-dimensional
 differentiable manifold M,
 Basis (∂/∂u,∂/∂v) on the Tangent space at Point p on the 2-dimensional
 differentiable manifold M]
```

All the bases defined on `Tp` are on the same footing. Accordingly the tangent space is not in the category of modules with a distinguished basis:

```
sage: Tp in ModulesWithBasis(SR)
False
```

```
>>> from sage.all import *
>>> Tp in ModulesWithBasis(SR)
False
```

It is simply in the category of modules:

```
sage: Tp in Modules(SR)
True
```

```
>>> from sage.all import *
>>> Tp in Modules(SR)
True
```

Since the base ring is a field, it is actually in the category of vector spaces:

```
sage: Tp in VectorSpaces(SR)
True
```

```
>>> from sage.all import *
>>> Tp in VectorSpaces(SR)
True
```

A typical element:

```
sage: v = Tp.an_element() ; v
Tangent vector at Point p on the
 2-dimensional differentiable manifold M
sage: v.display()
∂/∂x + 2 ∂/∂y
sage: v.parent()
```

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```
Tangent space at Point p on the
 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> v = Tp.an_element() ; v
Tangent vector at Point p on the
 2-dimensional differentiable manifold M
>>> v.display()
∂/∂x + 2 ∂/∂y
>>> v.parent()
Tangent space at Point p on the
 2-dimensional differentiable manifold M
```

The zero vector:

```
sage: Tp.zero()
Tangent vector zero at Point p on the
 2-dimensional differentiable manifold M
sage: Tp.zero().display()
zero = 0
sage: Tp.zero().parent()
Tangent space at Point p on the
 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> Tp.zero()
Tangent vector zero at Point p on the
 2-dimensional differentiable manifold M
>>> Tp.zero().display()
zero = 0
>>> Tp.zero().parent()
Tangent space at Point p on the
 2-dimensional differentiable manifold M
```

Tangent spaces are unique:

```
sage: M.tangent_space(p) is Tp
True
sage: p1 = M.point((-1,2))
sage: M.tangent_space(p1) is Tp
True
```

```
>>> from sage.all import *
>>> M.tangent_space(p) is Tp
True
>>> p1 = M.point((-Integer(1),Integer(2)))
>>> M.tangent_space(p1) is Tp
True
```

even if points are not:

```
sage: p1 is p
False
```

```
>>> from sage.all import *
>>> p1 is p
False
```

Actually `p1` and `p` share the same tangent space because they compare equal:

```
sage: p1 == p
True
```

```
>>> from sage.all import *
>>> p1 == p
True
```

The tangent-space uniqueness holds even if the points are created in different coordinate systems:

```
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y))
sage: uv_to_xv = xy_to_uv.inverse()
sage: p2 = M.point((1, -3), chart=c_uv, name='p_2')
sage: p2 is p
False
sage: M.tangent_space(p2) is Tp
True
sage: p2 == p
True
```

```
>>> from sage.all import *
>>> xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y))
>>> uv_to_xv = xy_to_uv.inverse()
>>> p2 = M.point((Integer(1), -Integer(3)), chart=c_uv, name='p_2')
>>> p2 is p
False
>>> M.tangent_space(p2) is Tp
True
>>> p2 == p
True
```

An isomorphism of the tangent space with an inner product space with distinguished basis:

```
sage: g = M.metric('g')
sage: g[:] = ((1, 0), (0, 1))
sage: Q_Tp_xy = g[c_xy.frame(),:](*p.coordinates(c_xy)); Q_Tp_xy
[1 0]
[0 1]
sage: W_Tp_xy = VectorSpace(SR, 2, inner_product_matrix=Q_Tp_xy)
sage: Tp.bases()[0]
Basis (∂/∂x,∂/∂y) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
sage: phi_Tp_xy = Tp.isomorphism_with_fixed_basis(Tp.bases()[0], codomain=W_Tp_xy)
sage: phi_Tp_xy
```

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```
Generic morphism:
 From: Tangent space at Point p on the 2-dimensional differentiable manifold M
 To:   Ambient quadratic space of dimension 2 over Symbolic Ring
       Inner product matrix:
       [1 0]
       [0 1]

sage: Q_Tp_uv = g[c_uv.frame(),:](*p.coordinates(c_uv)); Q_Tp_uv
[1/2   0]
[  0 1/2]
sage: W_Tp_uv = VectorSpace(SR, 2, inner_product_matrix=Q_Tp_uv)
sage: Tp.bases()[1]
Basis (∂/∂u,∂/∂v) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
sage: phi_Tp_uv = Tp.isomorphism_with_fixed_basis(Tp.bases()[1], codomain=W_Tp_uv)
sage: phi_Tp_uv
Generic morphism:
 From: Tangent space at Point p on the 2-dimensional differentiable manifold M
 To:   Ambient quadratic space of dimension 2 over Symbolic Ring
       Inner product matrix:
       [1/2   0]
       [  0 1/2]

sage: t1, t2 = Tp.tensor((1,0)), Tp.tensor((1,0))
sage: t1[:] = (8, 15)
sage: t2[:] = (47, 11)
sage: t1[Tp.bases()[0],:]
[8, 15]
sage: phi_Tp_xy(t1), phi_Tp_xy(t2)
((8, 15), (47, 11))
sage: phi_Tp_xy(t1).inner_product(phi_Tp_xy(t2))
541

sage: Tp_xy_to_uv = M.change_of_frame(c_xy.frame(), c_uv.frame()).at(p); Tp_xy_to_
↪uv
Automorphism of the Tangent space at Point p on the
 2-dimensional differentiable manifold M
sage: Tp.set_change_of_basis(Tp.bases()[0], Tp.bases()[1], Tp_xy_to_uv)
sage: t1[Tp.bases()[1],:]
[23, -7]
sage: phi_Tp_uv(t1), phi_Tp_uv(t2)
((23, -7), (58, 36))
sage: phi_Tp_uv(t1).inner_product(phi_Tp_uv(t2))
541
```

```
>>> from sage.all import *
>>> g = M.metric('g')
>>> g[:] = ((Integer(1), Integer(0)), (Integer(0), Integer(1)))
>>> Q_Tp_xy = g[c_xy.frame(),:](*p.coordinates(c_xy)); Q_Tp_xy
[1 0]
[0 1]
>>> W_Tp_xy = VectorSpace(SR, Integer(2), inner_product_matrix=Q_Tp_xy)
```

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```
>>> Tp.bases()[Integer(0)]
Basis (∂/∂x,∂/∂y) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
>>> phi_Tp_xy = Tp.isomorphism_with_fixed_basis(Tp.bases()[Integer(0)],␣
↪codomain=W_Tp_xy)
>>> phi_Tp_xy
Generic morphism:
 From: Tangent space at Point p on the 2-dimensional differentiable manifold M
 To:   Ambient quadratic space of dimension 2 over Symbolic Ring
       Inner product matrix:
       [1 0]
       [0 1]

>>> Q_Tp_uv = g[c_uv.frame(),:](*p.coordinates(c_uv)); Q_Tp_uv
[1/2   0]
[  0 1/2]
>>> W_Tp_uv = VectorSpace(SR, Integer(2), inner_product_matrix=Q_Tp_uv)
>>> Tp.bases()[Integer(1)]
Basis (∂/∂u,∂/∂v) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
>>> phi_Tp_uv = Tp.isomorphism_with_fixed_basis(Tp.bases()[Integer(1)],␣
↪codomain=W_Tp_uv)
>>> phi_Tp_uv
Generic morphism:
 From: Tangent space at Point p on the 2-dimensional differentiable manifold M
 To:   Ambient quadratic space of dimension 2 over Symbolic Ring
       Inner product matrix:
       [1/2   0]
       [  0 1/2]

>>> t1, t2 = Tp.tensor((Integer(1),Integer(0))), Tp.tensor((Integer(1),
↪Integer(0)))
>>> t1[:] = (Integer(8), Integer(15))
>>> t2[:] = (Integer(47), Integer(11))
>>> t1[Tp.bases()[Integer(0)],:]
[8, 15]
>>> phi_Tp_xy(t1), phi_Tp_xy(t2)
((8, 15), (47, 11))
>>> phi_Tp_xy(t1).inner_product(phi_Tp_xy(t2))
541

>>> Tp_xy_to_uv = M.change_of_frame(c_xy.frame(), c_uv.frame()).at(p); Tp_xy_to_uv
Automorphism of the Tangent space at Point p on the
 2-dimensional differentiable manifold M
>>> Tp.set_change_of_basis(Tp.bases()[Integer(0)], Tp.bases()[Integer(1)], Tp_xy_
↪to_uv)
>>> t1[Tp.bases()[Integer(1)],:]
[23, -7]
>>> phi_Tp_uv(t1), phi_Tp_uv(t2)
((23, -7), (58, 36))
>>> phi_Tp_uv(t1).inner_product(phi_Tp_uv(t2))
541
```

See also

FiniteRankFreeModule for more documentation.

Element

alias of *TangentVector*

base_point()

Return the manifold point at which self is defined.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: p = M.point((1,-2), name='p')
sage: Tp = M.tangent_space(p)
sage: Tp.base_point()
Point p on the 2-dimensional differentiable manifold M
sage: Tp.base_point() is p
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> p = M.point((Integer(1),-Integer(2)), name='p')
>>> Tp = M.tangent_space(p)
>>> Tp.base_point()
Point p on the 2-dimensional differentiable manifold M
>>> Tp.base_point() is p
True
```

construction()

dim()

Return the vector space dimension of self.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: p = M.point((1,-2), name='p')
sage: Tp = M.tangent_space(p)
sage: Tp.dimension()
2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> p = M.point((Integer(1),-Integer(2)), name='p')
>>> Tp = M.tangent_space(p)
>>> Tp.dimension()
2
```

A shortcut is dim():

```
sage: Tp.dim()
2
```

```
>>> from sage.all import *
>>> Tp.dim()
2
```

One can also use the global function `dim`:

```
sage: dim(Tp)
2
```

```
>>> from sage.all import *
>>> dim(Tp)
2
```

dimension()

Return the vector space dimension of `self`.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: p = M.point((1,-2), name='p')
sage: Tp = M.tangent_space(p)
sage: Tp.dimension()
2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> p = M.point((Integer(1),-Integer(2)), name='p')
>>> Tp = M.tangent_space(p)
>>> Tp.dimension()
2
```

A shortcut is `dim()`:

```
sage: Tp.dim()
2
```

```
>>> from sage.all import *
>>> Tp.dim()
2
```

One can also use the global function `dim`:

```
sage: dim(Tp)
2
```

```
>>> from sage.all import *
>>> dim(Tp)
2
```

2.6.2 Tangent Vectors

The class *TangentVector* implements tangent vectors to a differentiable manifold.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2014-2015): initial version
- Travis Scrimshaw (2016): review tweaks

REFERENCES:

- Chap. 3 of [Lee2013]

class sage.manifolds.differentiable.tangent_vector.**TangentVector**(*parent*, *name=None*, *latex_name=None*)

Bases: FiniteRankFreeModuleElement

Tangent vector to a differentiable manifold at a given point.

INPUT:

- parent – *TangentSpace*; the tangent space to which the vector belongs
- name – (default: None) string; symbol given to the vector
- latex_name – (default: None) string; LaTeX symbol to denote the vector; if None, name will be used

EXAMPLES:

A tangent vector v on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: p = M.point((2,3), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((-2,1), name='v') ; v
Tangent vector v at Point p on the 2-dimensional differentiable
 manifold M
sage: v.display()
v = -2 ∂/∂x + ∂/∂y
sage: v.parent()
Tangent space at Point p on the 2-dimensional differentiable manifold M
sage: v in Tp
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> p = M.point((Integer(2),Integer(3)), name='p')
>>> Tp = M.tangent_space(p)
>>> v = Tp((-Integer(2),Integer(1)), name='v') ; v
Tangent vector v at Point p on the 2-dimensional differentiable
 manifold M
>>> v.display()
v = -2 ∂/∂x + ∂/∂y
>>> v.parent()
Tangent space at Point p on the 2-dimensional differentiable manifold M
>>> v in Tp
True
```

Tangent vectors can also be constructed via the manifold method *tangent_vector()*:

```
sage: v = M.tangent_vector(p, (-2, 1), name='v'); v
Tangent vector v at Point p on the 2-dimensional differentiable
 manifold M
sage: v.display()
v = -2 ∂/∂x + ∂/∂y
```

```
>>> from sage.all import *
>>> v = M.tangent_vector(p, (-Integer(2), Integer(1)), name='v'); v
Tangent vector v at Point p on the 2-dimensional differentiable
 manifold M
>>> v.display()
v = -2 ∂/∂x + ∂/∂y
```

or via the method *at()* of vector fields:

```
sage: vf = M.vector_field(x - 4*y/3, (x-y)^2, name='v')
sage: v = vf.at(p); v
Tangent vector v at Point p on the 2-dimensional differentiable
 manifold M
sage: v.display()
v = -2 ∂/∂x + ∂/∂y
```

```
>>> from sage.all import *
>>> vf = M.vector_field(x - Integer(4)*y/Integer(3), (x-y)**Integer(2), name='v')
>>> v = vf.at(p); v
Tangent vector v at Point p on the 2-dimensional differentiable
 manifold M
>>> v.display()
v = -2 ∂/∂x + ∂/∂y
```

By definition, a tangent vector at $p \in M$ is a *derivation at* p on the space $C^\infty(M)$ of smooth scalar fields on M. Indeed let us consider a generic scalar field f:

```
sage: f = M.scalar_field(function('F')(x,y), name='f')
sage: f.display()
f: M → ℝ
   (x, y) ↦ F(x, y)
```

```
>>> from sage.all import *
>>> f = M.scalar_field(function('F')(x,y), name='f')
>>> f.display()
f: M → ℝ
   (x, y) ↦ F(x, y)
```

The tangent vector v maps f to the real number $v^i \left.\frac{\partial F}{\partial x^i}\right|_p$:

```
sage: v(f)
-2*D[0](F)(2, 3) + D[1](F)(2, 3)
sage: vdf(x, y) = v[0]*diff(f.expr(), x) + v[1]*diff(f.expr(), y)
sage: X(p)
(2, 3)
```

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```
sage: bool( v(f) == vdf(*X(p)) )
True
```

```
>>> from sage.all import *
>>> v(f)
-2*D[0](F)(2, 3) + D[1](F)(2, 3)
>>> __tmp__=var("x,y"); vdf = symbolic_expression(v[Integer(0)]*diff(f.expr(), x)␣
↪+ v[Integer(1)]*diff(f.expr(), y)).function(x,y)
>>> X(p)
(2, 3)
>>> bool( v(f) == vdf(*X(p)) )
True
```

and if g is a second scalar field on M:

```
sage: g = M.scalar_field(function('G')(x,y), name='g')
```

```
>>> from sage.all import *
>>> g = M.scalar_field(function('G')(x,y), name='g')
```

then the product fg is also a scalar field on M:

```
sage: (f*g).display()
f*g: M → ℝ
   (x, y) ↦ F(x, y)*G(x, y)
```

```
>>> from sage.all import *
>>> (f*g).display()
f*g: M → ℝ
   (x, y) ↦ F(x, y)*G(x, y)
```

and we have the derivation law $v(fg) = v(f)g(p) + f(p)v(g)$:

```
sage: bool( v(f*g) == v(f)*g(p) + f(p)*v(g) )
True
```

```
>>> from sage.all import *
>>> bool( v(f*g) == v(f)*g(p) + f(p)*v(g) )
True
```

See also

`FiniteRankFreeModuleElement` for more documentation.

plot(*chart=None*, *ambient_coords=None*, *mapping=None*, *color='blue'*, *print_label=True*, *label=None*, *label_color=None*, *fontsize=10*, *label_offset=0.1*, *parameters=None*, *scale=1*, ***extra_options*)

Plot the vector in a Cartesian graph based on the coordinates of some ambient chart.

The vector is drawn in terms of two (2D graphics) or three (3D graphics) coordinates of a given chart, called hereafter the *ambient chart*. The vector's base point p (or its image $\Phi(p)$ by some differentiable mapping Φ) must lie in the ambient chart's domain. If Φ is different from the identity mapping, the vector actually depicted is $\mathrm{d}\Phi_p(v)$, where v is the current vector (`self`) (see the example of a vector tangent to the 2-sphere below, where $\Phi : S^2 \to \mathbf{R}^3$).

INPUT:

- `chart` – (default: `None`) the ambient chart (see above); if `None`, it is set to the default chart of the open set containing the point at which the vector (or the vector image via the differential $\mathrm{d}\Phi_p$ of `mapping`) is defined
- `ambient_coords` – (default: `None`) tuple containing the 2 or 3 coordinates of the ambient chart in terms of which the plot is performed; if `None`, all the coordinates of the ambient chart are considered
- `mapping` – (default: `None`) `DiffMap`; differentiable mapping Φ providing the link between the point p at which the vector is defined and the ambient chart `chart`: the domain of `chart` must contain $\Phi(p)$; if `None`, the identity mapping is assumed
- `scale` – (default: 1) value by which the length of the arrow representing the vector is multiplied
- `color` – (default: `'blue'`) color of the arrow representing the vector
- `print_label` – boolean (default: `True`); determines whether a label is printed next to the arrow representing the vector
- `label` – (string; default: `None`) label printed next to the arrow representing the vector; if `None`, the vector's symbol is used, if any
- `label_color` – (default: `None`) color to print the label; if `None`, the value of `color` is used
- `fontsize` – (default: 10) size of the font used to print the label
- `label_offset` – (default: 0.1) determines the separation between the vector arrow and the label
- `parameters` – (default: `None`) dictionary giving the numerical values of the parameters that may appear in the coordinate expression of `self` (see example below)
- `**extra_options` – extra options for the arrow plot, like `linestyle`, `width` or `arrowsize` (see `arrow2d()` and `arrow3d()` for details)

OUTPUT:

- a graphic object, either an instance of `Graphics` for a 2D plot (i.e. based on 2 coordinates of `chart`) or an instance of `Graphics3d` for a 3D plot (i.e. based on 3 coordinates of `chart`)

EXAMPLES:

Vector tangent to a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: p = M((2,2), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((2, 1), name='v') ; v
Tangent vector v at Point p on the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> p = M((Integer(2),Integer(2)), name='p')
>>> Tp = M.tangent_space(p)
>>> v = Tp((Integer(2), Integer(1)), name='v') ; v
```

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```
Tangent vector v at Point p on the 2-dimensional differentiable
 manifold M
```

Plot of the vector alone (arrow + label):

```
sage: v.plot()                                                          #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

```
>>> from sage.all import *
>>> v.plot()                                                          #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

Plot atop of the chart grid:

```
sage: X.plot() + v.plot()                                               #␣
↪needs sage.plot
Graphics object consisting of 20 graphics primitives
```

```
>>> from sage.all import *
>>> X.plot() + v.plot()                                               #␣
↪needs sage.plot
Graphics object consisting of 20 graphics primitives
```

Plots with various options:

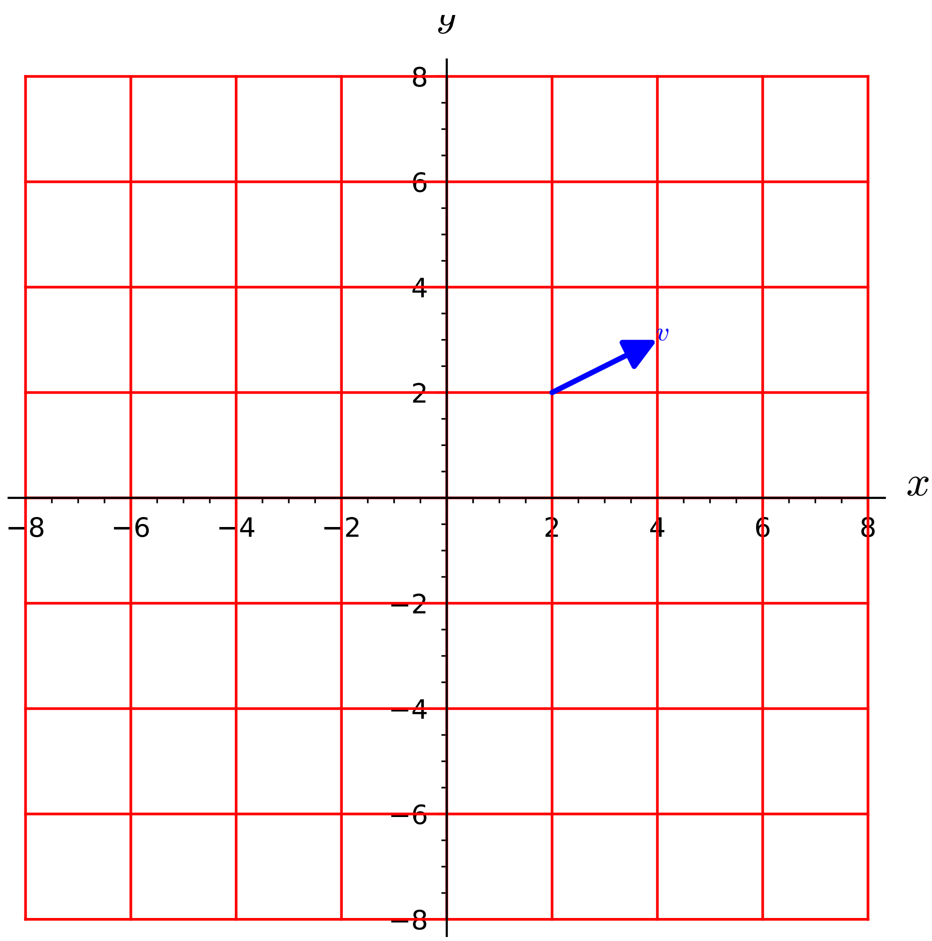
```
sage: X.plot() + v.plot(color='green', scale=2, label='V')              #␣
↪needs sage.plot
Graphics object consisting of 20 graphics primitives
```

```
>>> from sage.all import *
>>> X.plot() + v.plot(color='green', scale=Integer(2), label='V')       ␣
↪      # needs sage.plot
Graphics object consisting of 20 graphics primitives
```

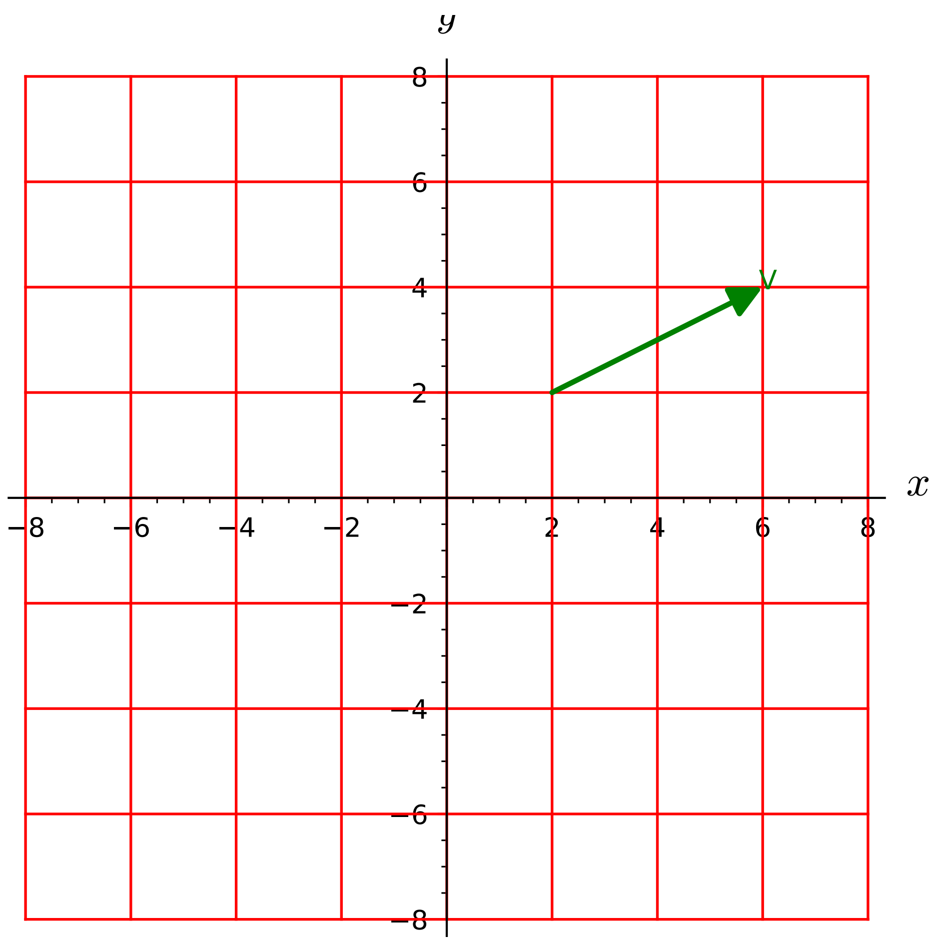
```
sage: X.plot() + v.plot(print_label=False)                              #␣
↪needs sage.plot
Graphics object consisting of 19 graphics primitives
```

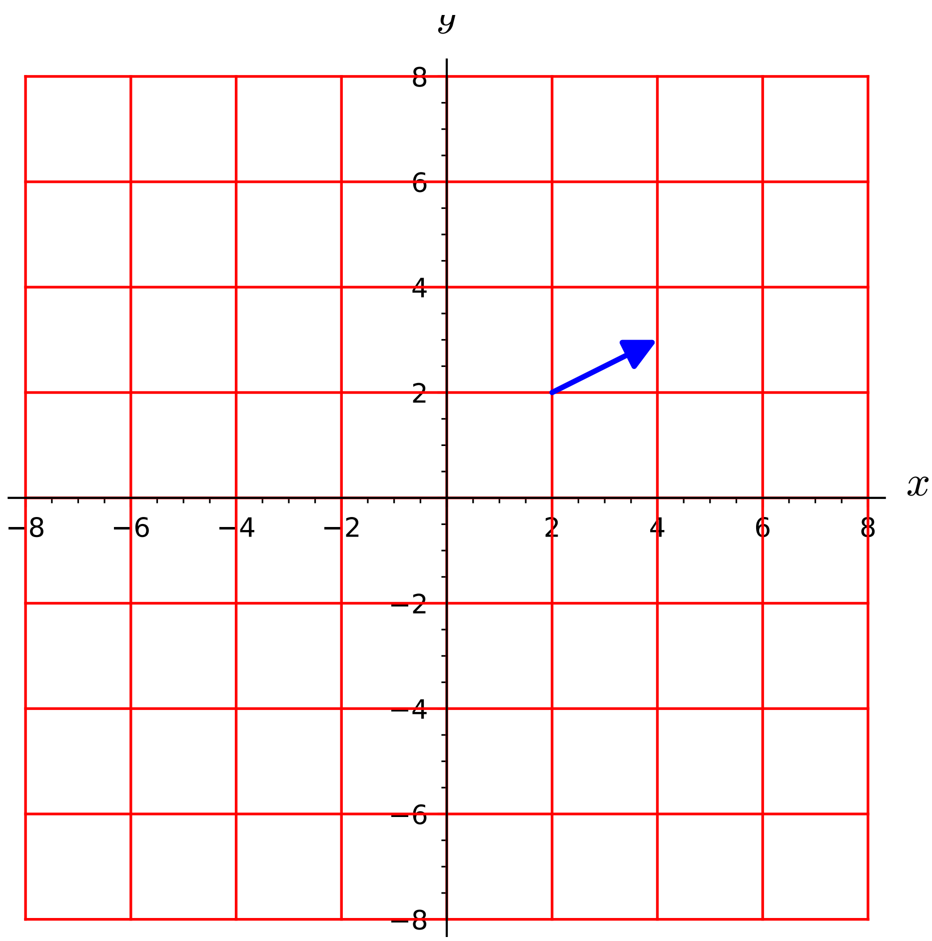
```
>>> from sage.all import *
>>> X.plot() + v.plot(print_label=False)                              #␣
↪needs sage.plot
Graphics object consisting of 19 graphics primitives
```

```
sage: X.plot() + v.plot(color='green', label_color='black',             #␣
↪needs sage.plot
....:                   fontsize=20, label_offset=0.2)
Graphics object consisting of 20 graphics primitives
```



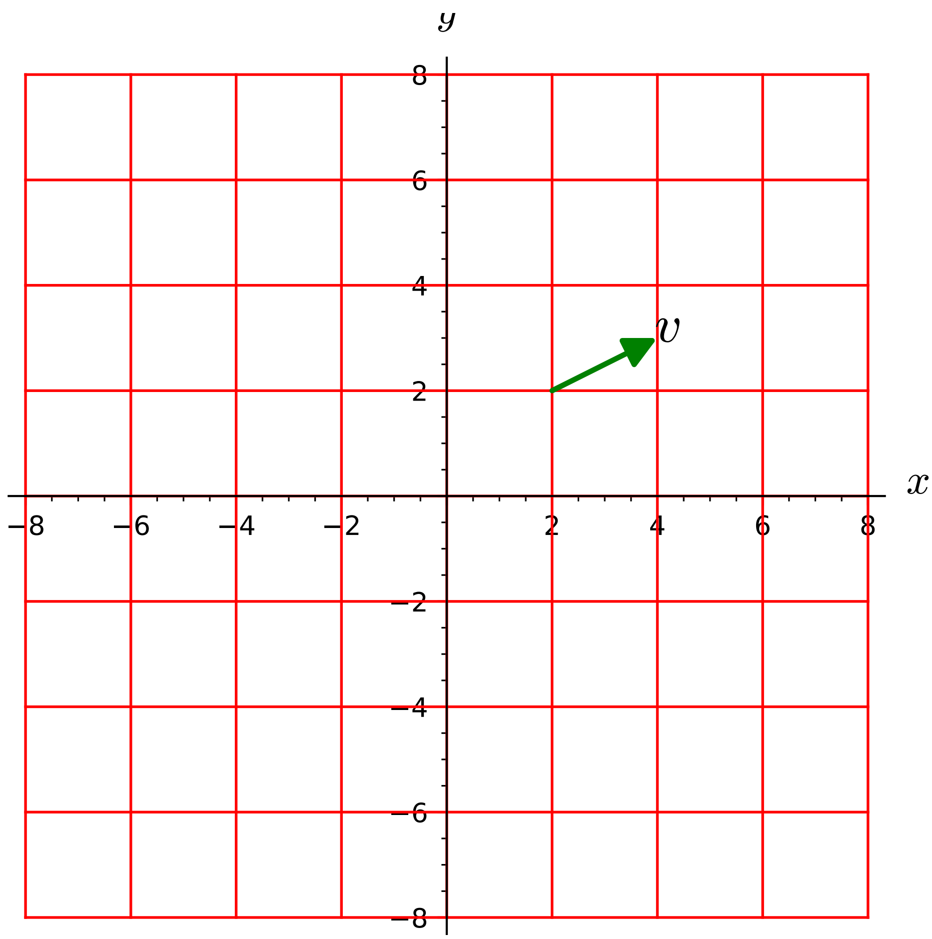
y
8
6
4
2
v
x
−8
−6
−4
−2
2
4
6
8
−2
−4
−6
−8


y
x
v
8
6
4
2
−2
−4
−6
−8
−8
−6
−4
−2
2
4
6
8



y
8
6
4
2
x
−8
−6
−4
−2
2
4
6
8
−2
−4
−6
−8

```
>>> from sage.all import *
>>> X.plot() + v.plot(color='green', label_color='black',                    #␣
↪needs sage.plot
...                     fontsize=Integer(20), label_offset=RealNumber('0.2'))
Graphics object consisting of 20 graphics primitives
```




```
sage: X.plot() + v.plot(linestyle=':', width=4, arrowsize=8,                  #␣
↪needs sage.plot
....:                   fontsize=20)
Graphics object consisting of 20 graphics primitives
```

```
>>> from sage.all import *
>>> X.plot() + v.plot(linestyle=':', width=Integer(4), arrowsize=Integer(8), ␣
↪               # needs sage.plot
...                     fontsize=Integer(20))
Graphics object consisting of 20 graphics primitives
```

Plot with specific values of some free parameters:

```
sage: var('a b')
(a, b)
sage: v = Tp((1+a, -b^2), name='v') ; v.display()
```

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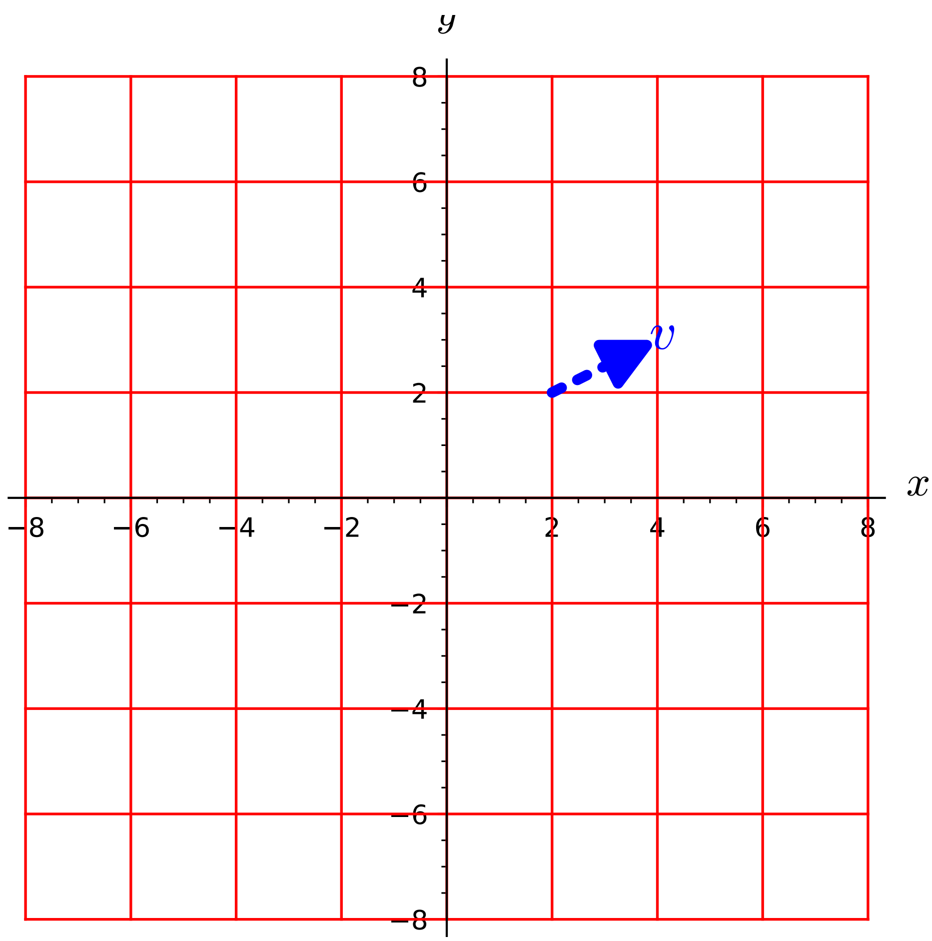
y
x
v
−8
−6
−4
−2
2
4
6
8
8
6
4
2
−2
−4
−6
−8

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```
v = (a + 1) ∂/∂x - b^2 ∂/∂y
sage: X.plot() + v.plot(parameters={a: -2, b: 3})                              #␣
↪needs sage.plot
Graphics object consisting of 20 graphics primitives
```

```
>>> from sage.all import *
>>> var('a b')
(a, b)
>>> v = Tp((Integer(1)+a, -b**Integer(2)), name='v') ; v.display()
v = (a + 1) ∂/∂x - b^2 ∂/∂y
>>> X.plot() + v.plot(parameters={a: -Integer(2), b: Integer(3)})             ␣
↪              # needs sage.plot
Graphics object consisting of 20 graphics primitives
```

Special case of the zero vector:

```
sage: v = Tp.zero() ; v
Tangent vector zero at Point p on the 2-dimensional differentiable
 manifold M
sage: X.plot() + v.plot()                                                      #␣
↪needs sage.plot
Graphics object consisting of 19 graphics primitives
```

```
>>> from sage.all import *
>>> v = Tp.zero() ; v
Tangent vector zero at Point p on the 2-dimensional differentiable
 manifold M
>>> X.plot() + v.plot()                                                      #␣
↪needs sage.plot
Graphics object consisting of 19 graphics primitives
```

Vector tangent to a 4-dimensional manifold:

```
sage: M = Manifold(4, 'M')
sage: X.<t,x,y,z> = M.chart()
sage: p = M((0,1,2,3), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((5,4,3,2), name='v') ; v
Tangent vector v at Point p on the 4-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> p = M((Integer(0),Integer(1),Integer(2),Integer(3)), name='p')
>>> Tp = M.tangent_space(p)
>>> v = Tp((Integer(5),Integer(4),Integer(3),Integer(2)), name='v') ; v
Tangent vector v at Point p on the 4-dimensional differentiable
 manifold M
```

We cannot make a 4D plot directly:

```
sage: v.plot()                                                              #␣
↪needs sage.plot
Traceback (most recent call last):
...
ValueError: the number of coordinates involved in the plot must
 be either 2 or 3, not 4
```

```
>>> from sage.all import *
>>> v.plot()                                                                #␣
↪needs sage.plot
Traceback (most recent call last):
...
ValueError: the number of coordinates involved in the plot must
 be either 2 or 3, not 4
```

Rather, we have to select some chart coordinates for the plot, via the argument `ambient_coords`. For instance, for a 2-dimensional plot in terms of the coordinates (x, y):

```
sage: v.plot(ambient_coords=(x,y))                                          #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

```
>>> from sage.all import *
>>> v.plot(ambient_coords=(x,y))                                            #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

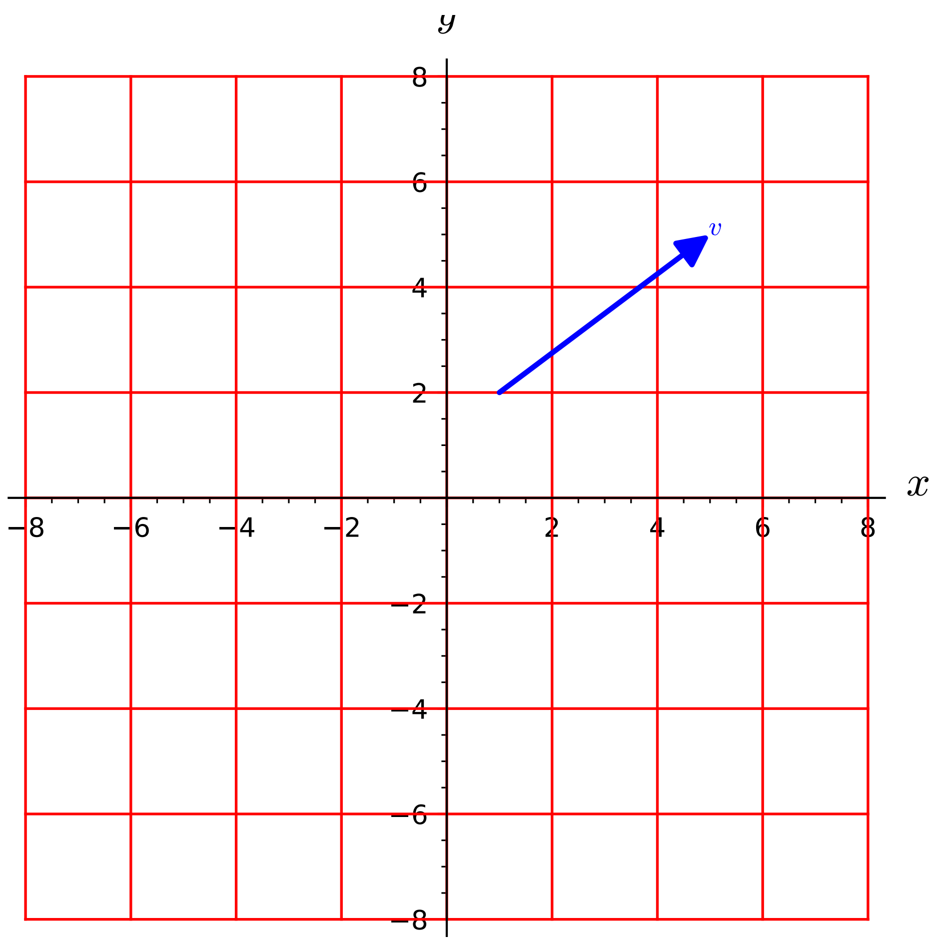
This plot involves only the components v^x and v^y of v. Similarly, for a 3-dimensional plot in terms of the coordinates (t, x, y):

```
sage: g = v.plot(ambient_coords=(t,x,z))                                    #␣
↪needs sage.plot
sage: print(g)                                                              #␣
↪needs sage.plot
Graphics3d Object
```

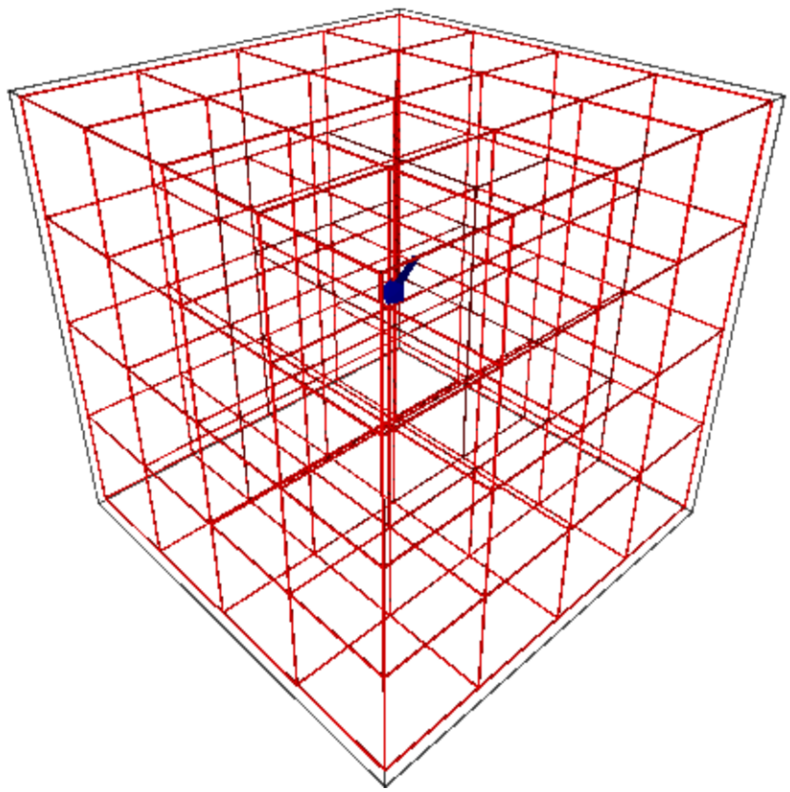
```
>>> from sage.all import *
>>> g = v.plot(ambient_coords=(t,x,z))                                      #␣
↪needs sage.plot
>>> print(g)                                                                #␣
↪needs sage.plot
Graphics3d Object
```

This plot involves only the components v^t, v^x and v^z of v. A nice 3D view atop the coordinate grid is obtained via:

```
sage: (X.plot(ambient_coords=(t,x,z))  # long time                          #␣
↪needs sage.plot
....:  + v.plot(ambient_coords=(t,x,z),
....:           label_offset=0.5, width=6))
Graphics3d Object
```


y
x
v
8
6
4
2
−2
−4
−6
−8
−8
−6
−4
−2
2
4
6
8

```
>>> from sage.all import *
>>> (X.plot(ambient_coords=(t,x,z))  # long time                              #␣
↪needs sage.plot
...  + v.plot(ambient_coords=(t,x,z),
...           label_offset=RealNumber('0.5'), width=Integer(6)))
Graphics3d Object
```



An example of plot via a differential mapping: plot of a vector tangent to a 2-sphere viewed in $\mathbf{R}^3$:

```
sage: S2 = Manifold(2, 'S^2')
sage: U = S2.open_subset('U') # the open set covered by spherical coord.
sage: XS.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: R3 = Manifold(3, 'R^3')
sage: X3.<x,y,z> = R3.chart()
sage: F = S2.diff_map(R3, {(XS, X3): [sin(th)*cos(ph),
....:                                  sin(th)*sin(ph),
....:                                  cos(th)]}, name='F')
sage: F.display() # the standard embedding of S^2 into R^3
F: S^2 → R^3
on U: (th, ph) ↦ (x, y, z) = (cos(ph)*sin(th), sin(ph)*sin(th), cos(th))
sage: p = U.point((pi/4, 7*pi/4), name='p')
sage: v = XS.frame()[1].at(p) ; v  # the coordinate vector ∂/∂phi at p
Tangent vector ∂/∂ph at Point p on the 2-dimensional differentiable
```

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```
 manifold S^2
sage: graph_v = v.plot(mapping=F)                                              #␣
↪needs sage.plot
sage: graph_S2 = XS.plot(chart=X3, mapping=F, number_values=9)   # long time,␣
↪needs sage.plot
sage: graph_v + graph_S2                                         # long time,␣
↪needs sage.plot
Graphics3d Object
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2')
>>> U = S2.open_subset('U') # the open set covered by spherical coord.
>>> XS = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi', names=('th', 'ph',));␣
↪(th, ph,) = XS._first_ngens(2)
>>> R3 = Manifold(Integer(3), 'R^3')
>>> X3 = R3.chart(names=('x', 'y', 'z',)); (x, y, z,) = X3._first_ngens(3)
>>> F = S2.diff_map(R3, {(XS, X3): [sin(th)*cos(ph),
...                                  sin(th)*sin(ph),
...                                  cos(th)]}, name='F')
>>> F.display() # the standard embedding of S^2 into R^3
F: S^2 → R^3
on U: (th, ph) ↦ (x, y, z) = (cos(ph)*sin(th), sin(ph)*sin(th), cos(th))
>>> p = U.point((pi/Integer(4), Integer(7)*pi/Integer(4)), name='p')
>>> v = XS.frame()[Integer(1)].at(p) ; v  # the coordinate vector ∂/∂phi at p
Tangent vector ∂/∂ph at Point p on the 2-dimensional differentiable
 manifold S^2
>>> graph_v = v.plot(mapping=F)                                              #␣
↪needs sage.plot
>>> graph_S2 = XS.plot(chart=X3, mapping=F, number_values=Integer(9))  # long␣
↪time, needs sage.plot
>>> graph_v + graph_S2                                         # long time,␣
↪needs sage.plot
Graphics3d Object
```

2.7 Vector Fields

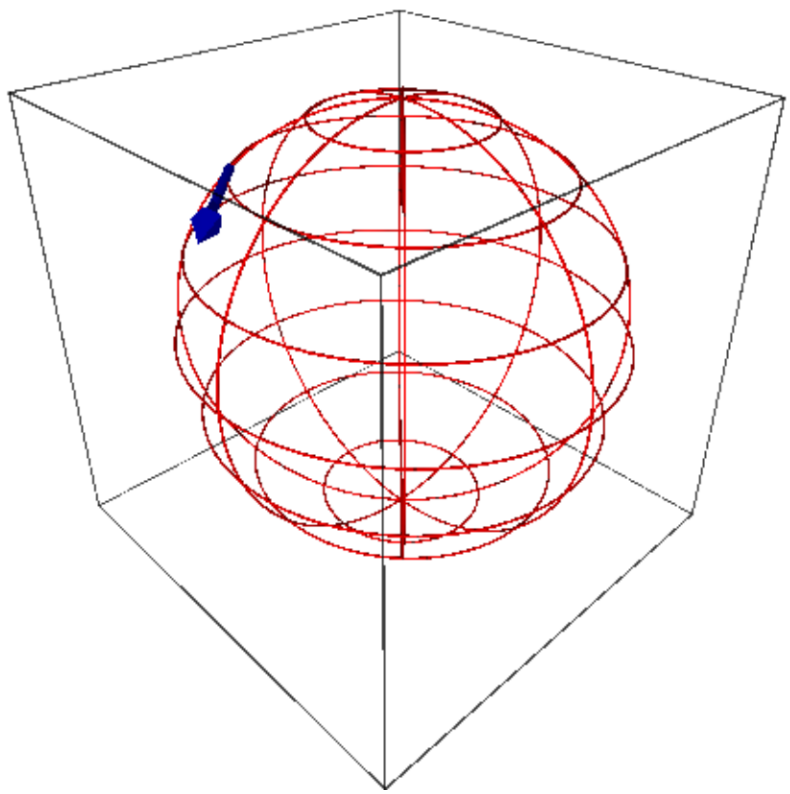
2.7.1 Vector Field Modules

The set of vector fields along a differentiable manifold U with values on a differentiable manifold M via a differentiable map $\Phi : U \to M$ (possibly $U = M$ and $\Phi = \mathrm{Id}_M$) is a module over the algebra $C^k(U)$ of differentiable scalar fields on U. If Φ is the identity map, this module is considered a Lie algebroid under the Lie bracket $[\ ,\ ]$ (cf. Wikipedia article Lie_algebroid). It is a free module if and only if M is parallelizable. Accordingly, there are two classes for vector field modules:

- `VectorFieldModule` for vector fields with values on a generic (in practice, not parallelizable) differentiable manifold M.
- `VectorFieldFreeModule` for vector fields with values on a parallelizable manifold M.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2014-2015): initial version
- Travis Scrimshaw (2016): structure of Lie algebroid (Issue #20771)



REFERENCES:

- [KN1963]
- [Lee2013]
- [ONe1983]

class sage.manifolds.differentiable.vectorfield_module.**VectorFieldFreeModule**(*domain*, *dest_map=None*)

Bases: FiniteRankFreeModule

Free module of vector fields along a differentiable manifold U with values on a parallelizable manifold M, via a differentiable map $U \to M$.

Given a differentiable map

$$\Phi : U \longrightarrow M$$

the *vector field module* $\mathfrak{X}(U,\Phi)$ is the set of all vector fields of the type

$$v : U \longrightarrow TM$$

(where TM is the tangent bundle of M) such that

$$\forall p \in U, \ v(p) \in T_{\Phi(p)}M,$$

where $T_{\Phi(p)}M$ is the tangent space to M at the point $\Phi(p)$.

Since M is parallelizable, the set $\mathfrak{X}(U,\Phi)$ is a free module over $C^k(U)$, the ring (algebra) of differentiable scalar fields on U (see *DiffScalarFieldAlgebra*). In fact, it carries the structure of a finite-dimensional Lie algebroid (cf. Wikipedia article Lie_algebroid).

The standard case of vector fields *on* a differentiable manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$; we then denote $\mathfrak{X}(M, \mathrm{Id}_M)$ by merely $\mathfrak{X}(M)$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note

If M is not parallelizable, the class *VectorFieldModule* should be used instead, for $\mathfrak{X}(U,\Phi)$ is no longer a free module.

INPUT:

- `domain` – differentiable manifold U along which the vector fields are defined
- `dest_map` – (default: `None`) destination map $\Phi : U \to M$ (type: *DiffMap*); if `None`, it is assumed that $U = M$ and Φ is the identity map of M (case of vector fields *on* M)

EXAMPLES:

Module of vector fields on $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'R^2')
sage: cart.<x,y> = M.chart()   # Cartesian coordinates on R^2
sage: XM = M.vector_field_module() ; XM
Free module X(R^2) of vector fields on the 2-dimensional differentiable
 manifold R^2
sage: XM.category()
```

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```
Category of finite dimensional modules
 over Algebra of differentiable scalar fields
 on the 2-dimensional differentiable manifold R^2
sage: XM.base_ring() is M.scalar_field_algebra()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2')
>>> cart = M.chart(names=('x', 'y',)); (x, y,) = cart._first_ngens(2)# Cartesian␣
↪coordinates on R^2
>>> XM = M.vector_field_module() ; XM
Free module X(R^2) of vector fields on the 2-dimensional differentiable
 manifold R^2
>>> XM.category()
Category of finite dimensional modules
 over Algebra of differentiable scalar fields
 on the 2-dimensional differentiable manifold R^2
>>> XM.base_ring() is M.scalar_field_algebra()
True
```

Since $\mathbf{R}^2$ is obviously parallelizable, XM is a free module:

```
sage: isinstance(XM, FiniteRankFreeModule)
True
```

```
>>> from sage.all import *
>>> isinstance(XM, FiniteRankFreeModule)
True
```

Some elements:

```
sage: XM.an_element().display()
2 ∂/∂x + 2 ∂/∂y
sage: XM.zero().display()
zero = 0
sage: v = XM([-y,x]) ; v
Vector field on the 2-dimensional differentiable manifold R^2
sage: v.display()
-y ∂/∂x + x ∂/∂y
```

```
>>> from sage.all import *
>>> XM.an_element().display()
2 ∂/∂x + 2 ∂/∂y
>>> XM.zero().display()
zero = 0
>>> v = XM([-y,x]) ; v
Vector field on the 2-dimensional differentiable manifold R^2
>>> v.display()
-y ∂/∂x + x ∂/∂y
```

An example of module of vector fields with a destination map Φ different from the identity map, namely a mapping $\Phi : I \to \mathbf{R}^2$, where I is an open interval of $\mathbf{R}$:

```
sage: I = Manifold(1, 'I')
sage: canon.<t> = I.chart('t:(0,2*pi)')
sage: Phi = I.diff_map(M, coord_functions=[cos(t), sin(t)], name='Phi',
....:                  latex_name=r'\Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable manifold
 I to the 2-dimensional differentiable manifold R^2
sage: Phi.display()
Phi: I → R^2
   t ↦ (x, y) = (cos(t), sin(t))
sage: XIM = I.vector_field_module(dest_map=Phi) ; XIM
Free module X(I,Phi) of vector fields along the 1-dimensional
 differentiable manifold I mapped into the 2-dimensional differentiable
 manifold R^2
sage: XIM.category()
Category of finite dimensional modules
 over Algebra of differentiable scalar fields
 on the 1-dimensional differentiable manifold I
```

```
>>> from sage.all import *
>>> I = Manifold(Integer(1), 'I')
>>> canon = I.chart('t:(0,2*pi)', names=('t',)); (t,) = canon._first_ngens(1)
>>> Phi = I.diff_map(M, coord_functions=[cos(t), sin(t)], name='Phi',
...                  latex_name=r'\Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable manifold
 I to the 2-dimensional differentiable manifold R^2
>>> Phi.display()
Phi: I → R^2
   t ↦ (x, y) = (cos(t), sin(t))
>>> XIM = I.vector_field_module(dest_map=Phi) ; XIM
Free module X(I,Phi) of vector fields along the 1-dimensional
 differentiable manifold I mapped into the 2-dimensional differentiable
 manifold R^2
>>> XIM.category()
Category of finite dimensional modules
 over Algebra of differentiable scalar fields
 on the 1-dimensional differentiable manifold I
```

The rank of the free module $\mathfrak{X}(I, \Phi)$ is the dimension of the manifold $\mathbf{R}^2$, namely two:

```
sage: XIM.rank()
2
```

```
>>> from sage.all import *
>>> XIM.rank()
2
```

A basis of it is induced by the coordinate vector frame of $\mathbf{R}^2$:

```
sage: XIM.bases()
[Vector frame (I, (∂/∂x,∂/∂y)) with values on the 2-dimensional
 differentiable manifold R^2]
```

```
>>> from sage.all import *
>>> XIM.bases()
[Vector frame (I, (∂/∂x,∂/∂y)) with values on the 2-dimensional
 differentiable manifold R^2]
```

Some elements of this module:

```
sage: XIM.an_element().display()
2 ∂/∂x + 2 ∂/∂y
sage: v = XIM([t, t^2]) ; v
Vector field along the 1-dimensional differentiable manifold I with
 values on the 2-dimensional differentiable manifold R^2
sage: v.display()
t ∂/∂x + t^2 ∂/∂y
```

```
>>> from sage.all import *
>>> XIM.an_element().display()
2 ∂/∂x + 2 ∂/∂y
>>> v = XIM([t, t**Integer(2)]) ; v
Vector field along the 1-dimensional differentiable manifold I with
 values on the 2-dimensional differentiable manifold R^2
>>> v.display()
t ∂/∂x + t^2 ∂/∂y
```

The test suite is passed:

```
sage: TestSuite(XIM).run()
```

```
>>> from sage.all import *
>>> TestSuite(XIM).run()
```

Let us introduce an open subset of $J \subset I$ and the vector field module corresponding to the restriction of Φ to it:

```
sage: J = I.open_subset('J', coord_def={canon: t<pi})
sage: XJM = J.vector_field_module(dest_map=Phi.restrict(J)); XJM
Free module X(J,Phi) of vector fields along the Open subset J of the
 1-dimensional differentiable manifold I mapped into the 2-dimensional
 differentiable manifold R^2
```

```
>>> from sage.all import *
>>> J = I.open_subset('J', coord_def={canon: t<pi})
>>> XJM = J.vector_field_module(dest_map=Phi.restrict(J)); XJM
Free module X(J,Phi) of vector fields along the Open subset J of the
 1-dimensional differentiable manifold I mapped into the 2-dimensional
 differentiable manifold R^2
```

We have then:

```
sage: XJM.default_basis()
Vector frame (J, (∂/∂x,∂/∂y)) with values on the 2-dimensional
 differentiable manifold R^2
sage: XJM.default_basis() is XIM.default_basis().restrict(J)
True
```

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```
sage: v.restrict(J)
Vector field along the Open subset J of the 1-dimensional
 differentiable manifold I with values on the 2-dimensional
 differentiable manifold R^2
sage: v.restrict(J).display()
t ∂/∂x + t^2 ∂/∂y
```

```
>>> from sage.all import *
>>> XJM.default_basis()
Vector frame (J, (∂/∂x,∂/∂y)) with values on the 2-dimensional
 differentiable manifold R^2
>>> XJM.default_basis() is XIM.default_basis().restrict(J)
True
>>> v.restrict(J)
Vector field along the Open subset J of the 1-dimensional
 differentiable manifold I with values on the 2-dimensional
 differentiable manifold R^2
>>> v.restrict(J).display()
t ∂/∂x + t^2 ∂/∂y
```

Let us now consider the module of vector fields on the circle S^1; we start by constructing the S^1 manifold:

```
sage: M = Manifold(1, 'S^1')
sage: U = M.open_subset('U')  # the complement of one point
sage: c_t.<t> =  U.chart('t:(0,2*pi)') # the standard angle coordinate
sage: V = M.open_subset('V') # the complement of the point t=pi
sage: M.declare_union(U,V)   # S^1 is the union of U and V
sage: c_u.<u> = V.chart('u:(0,2*pi)') # the angle t-pi
sage: t_to_u = c_t.transition_map(c_u, (t-pi,), intersection_name='W',
....:                      restrictions1 = t!=pi, restrictions2 = u!=pi)
sage: u_to_t = t_to_u.inverse()
sage: W = U.intersection(V)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(1), 'S^1')
>>> U = M.open_subset('U')  # the complement of one point
>>> c_t = U.chart('t:(0,2*pi)', names=('t',)); (t,) = c_t._first_ngens(1)# the␣
↪standard angle coordinate
>>> V = M.open_subset('V') # the complement of the point t=pi
>>> M.declare_union(U,V)   # S^1 is the union of U and V
>>> c_u = V.chart('u:(0,2*pi)', names=('u',)); (u,) = c_u._first_ngens(1)# the␣
↪angle t-pi
>>> t_to_u = c_t.transition_map(c_u, (t-pi,), intersection_name='W',
...                      restrictions1 = t!=pi, restrictions2 = u!=pi)
>>> u_to_t = t_to_u.inverse()
>>> W = U.intersection(V)
```

S^1 cannot be covered by a single chart, so it cannot be covered by a coordinate frame. It is however parallelizable and we introduce a global vector frame as follows. We notice that on their common subdomain, W, the coordinate vectors $\partial/\partial t$ and $\partial/\partial u$ coincide, as we can check explicitly:

```
sage: c_t.frame()[0].display(c_u.frame().restrict(W))
∂/∂t = ∂/∂u
```

```
>>> from sage.all import *
>>> c_t.frame()[Integer(0)].display(c_u.frame().restrict(W))
∂/∂t = ∂/∂u
```

Therefore, we can extend $\partial/\partial t$ to all V and hence to all S^1, to form a vector field on S^1 whose components w.r.t. both $\partial/\partial t$ and $\partial/\partial u$ are 1:

```
sage: e = M.vector_frame('e')
sage: U.set_change_of_frame(e.restrict(U), c_t.frame(),
....:                       U.tangent_identity_field())
sage: V.set_change_of_frame(e.restrict(V), c_u.frame(),
....:                       V.tangent_identity_field())
sage: e[0].display(c_t.frame())
e_0 = ∂/∂t
sage: e[0].display(c_u.frame())
e_0 = ∂/∂u
```

```
>>> from sage.all import *
>>> e = M.vector_frame('e')
>>> U.set_change_of_frame(e.restrict(U), c_t.frame(),
...                       U.tangent_identity_field())
>>> V.set_change_of_frame(e.restrict(V), c_u.frame(),
...                       V.tangent_identity_field())
>>> e[Integer(0)].display(c_t.frame())
e_0 = ∂/∂t
>>> e[Integer(0)].display(c_u.frame())
e_0 = ∂/∂u
```

Equipped with the frame e, the manifold S^1 is manifestly parallelizable:

```
sage: M.is_manifestly_parallelizable()
True
```

```
>>> from sage.all import *
>>> M.is_manifestly_parallelizable()
True
```

Consequently, the module of vector fields on S^1 is a free module:

```
sage: XM = M.vector_field_module() ; XM
Free module X(S^1) of vector fields on the 1-dimensional differentiable
 manifold S^1
sage: isinstance(XM, FiniteRankFreeModule)
True
sage: XM.category()
Category of finite dimensional modules
 over Algebra of differentiable scalar fields
 on the 1-dimensional differentiable manifold S^1
sage: XM.base_ring() is M.scalar_field_algebra()
True
```

```
>>> from sage.all import *
>>> XM = M.vector_field_module() ; XM
Free module X(S^1) of vector fields on the 1-dimensional differentiable
 manifold S^1
>>> isinstance(XM, FiniteRankFreeModule)
True
>>> XM.category()
Category of finite dimensional modules
 over Algebra of differentiable scalar fields
 on the 1-dimensional differentiable manifold S^1
>>> XM.base_ring() is M.scalar_field_algebra()
True
```

The zero element:

```
sage: z = XM.zero() ; z
Vector field zero on the 1-dimensional differentiable manifold S^1
sage: z.display()
zero = 0
sage: z.display(c_t.frame())
zero = 0
```

```
>>> from sage.all import *
>>> z = XM.zero() ; z
Vector field zero on the 1-dimensional differentiable manifold S^1
>>> z.display()
zero = 0
>>> z.display(c_t.frame())
zero = 0
```

The module $\mathfrak{X}(S^1)$ coerces to any module of vector fields defined on a subdomain of S^1, for instance $\mathfrak{X}(U)$:

```
sage: XU = U.vector_field_module() ; XU
Free module X(U) of vector fields on the Open subset U of the
 1-dimensional differentiable manifold S^1
sage: XU.has_coerce_map_from(XM)
True
sage: XU.coerce_map_from(XM)
Coercion map:
  From: Free module X(S^1) of vector fields on the 1-dimensional
   differentiable manifold S^1
  To:   Free module X(U) of vector fields on the Open subset U of the
   1-dimensional differentiable manifold S^1
```

```
>>> from sage.all import *
>>> XU = U.vector_field_module() ; XU
Free module X(U) of vector fields on the Open subset U of the
 1-dimensional differentiable manifold S^1
>>> XU.has_coerce_map_from(XM)
True
>>> XU.coerce_map_from(XM)
Coercion map:
  From: Free module X(S^1) of vector fields on the 1-dimensional
```

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```
differentiable manifold S^1
To:   Free module X(U) of vector fields on the Open subset U of the
 1-dimensional differentiable manifold S^1
```

The conversion map is actually the restriction of vector fields defined on S^1 to U.

The Sage test suite for modules is passed:

```
sage: TestSuite(XM).run()
```

```
>>> from sage.all import *
>>> TestSuite(XM).run()
```

Element

alias of *VectorFieldParal*

ambient_domain()

Return the manifold in which the vector fields of `self` take their values.

If the module is $\mathfrak{X}(U,\Phi)$, returns the codomain M of Φ.

OUTPUT:

- a *DifferentiableManifold* representing the manifold in which the vector fields of `self` take their values

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()  # makes M parallelizable
sage: XM = M.vector_field_module()
sage: XM.ambient_domain()
3-dimensional differentiable manifold M
sage: U = Manifold(2, 'U')
sage: Y.<u,v> = U.chart()
sage: Phi = U.diff_map(M, {(Y,X): [u+v, u-v, u*v]}, name='Phi')
sage: XU = U.vector_field_module(dest_map=Phi)
sage: XU.ambient_domain()
3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)#
↪makes M parallelizable
>>> XM = M.vector_field_module()
>>> XM.ambient_domain()
3-dimensional differentiable manifold M
>>> U = Manifold(Integer(2), 'U')
>>> Y = U.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)
>>> Phi = U.diff_map(M, {(Y,X): [u+v, u-v, u*v]}, name='Phi')
>>> XU = U.vector_field_module(dest_map=Phi)
>>> XU.ambient_domain()
3-dimensional differentiable manifold M
```

basis(*symbol=None*, *latex_symbol=None*, *from_frame=None*, *indices=None*, *latex_indices=None*, *symbol_dual=None*, *latex_symbol_dual=None*)

Define a basis of `self`.

A basis of the vector field module is actually a vector frame along the differentiable manifold U over which the vector field module is defined.

If the basis specified by the given symbol already exists, it is simply returned. If no argument is provided the module's default basis is returned.

INPUT:

- `symbol` – (default: `None`) either a string, to be used as a common base for the symbols of the elements of the basis, or a tuple of strings, representing the individual symbols of the elements of the basis
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the elements of the basis, or a tuple of strings, representing the individual LaTeX symbols of the elements of the basis; if `None`, `symbol` is used in place of `latex_symbol`
- `from_frame` – (default: `None`) vector frame $\tilde{e}$ on the codomain M of the destination map Φ of `self`; the returned basis e is then such that for all $p \in U$, we have $e(p) = \tilde{e}(\Phi(p))$
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the elements of the basis; if `None`, the indices will be generated as integers within the range declared on `self`
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the elements of the basis; if `None`, `indices` is used instead
- `symbol_dual` – (default: `None`) same as `symbol` but for the dual basis; if `None`, `symbol` must be a string and is used for the common base of the symbols of the elements of the dual basis
- `latex_symbol_dual` – (default: `None`) same as `latex_symbol` but for the dual basis

OUTPUT:

- a *VectorFrame* representing a basis on `self`

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()   # makes M parallelizable
sage: XM = M.vector_field_module()
sage: e = XM.basis('e'); e
Vector frame (M, (e_0,e_1))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)# makes M␣
↪parallelizable
>>> XM = M.vector_field_module()
>>> e = XM.basis('e'); e
Vector frame (M, (e_0,e_1))
```

See *VectorFrame* for more examples and documentation.

destination_map()

Return the differential map associated to `self`.

The differential map associated to this module is the map

$$\Phi : U \longrightarrow M$$

such that this module is the set $\mathfrak{X}(U, \Phi)$ of all vector fields of the type

$$v : U \longrightarrow TM$$

(where TM is the tangent bundle of M) such that

$$\forall p \in U, \ v(p) \in T_{\Phi(p)}M,$$

where $T_{\Phi(p)}M$ is the tangent space to M at the point $\Phi(p)$.

OUTPUT:

- a *DiffMap* representing the differential map Φ

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()  # makes M parallelizable
sage: XM = M.vector_field_module()
sage: XM.destination_map()
Identity map Id_M of the 3-dimensional differentiable manifold M
sage: U = Manifold(2, 'U')
sage: Y.<u,v> = U.chart()
sage: Phi = U.diff_map(M, {(Y,X): [u+v, u-v, u*v]}, name='Phi')
sage: XU = U.vector_field_module(dest_map=Phi)
sage: XU.destination_map()
Differentiable map Phi from the 2-dimensional differentiable
 manifold U to the 3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)#␣
↪makes M parallelizable
>>> XM = M.vector_field_module()
>>> XM.destination_map()
Identity map Id_M of the 3-dimensional differentiable manifold M
>>> U = Manifold(Integer(2), 'U')
>>> Y = U.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)
>>> Phi = U.diff_map(M, {(Y,X): [u+v, u-v, u*v]}, name='Phi')
>>> XU = U.vector_field_module(dest_map=Phi)
>>> XU.destination_map()
Differentiable map Phi from the 2-dimensional differentiable
 manifold U to the 3-dimensional differentiable manifold M
```

domain()

Return the domain of the vector fields in `self`.

If the module is $\mathfrak{X}(U, \Phi)$, returns the domain U of Φ.

OUTPUT:

- a *DifferentiableManifold* representing the domain of the vector fields that belong to this module

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()  # makes M parallelizable
sage: XM = M.vector_field_module()
sage: XM.domain()
3-dimensional differentiable manifold M
sage: U = Manifold(2, 'U')
sage: Y.<u,v> = U.chart()
sage: Phi = U.diff_map(M, {(Y,X): [u+v, u-v, u*v]}, name='Phi')
sage: XU = U.vector_field_module(dest_map=Phi)
sage: XU.domain()
2-dimensional differentiable manifold U
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)#␣
↪makes M parallelizable
>>> XM = M.vector_field_module()
>>> XM.domain()
3-dimensional differentiable manifold M
>>> U = Manifold(Integer(2), 'U')
>>> Y = U.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)
>>> Phi = U.diff_map(M, {(Y,X): [u+v, u-v, u*v]}, name='Phi')
>>> XU = U.vector_field_module(dest_map=Phi)
>>> XU.domain()
2-dimensional differentiable manifold U
```

dual_exterior_power(p)

Return the p-th exterior power of the dual of `self`.

If the vector field module `self` is $\mathfrak{X}(U,\Phi)$, the p-th exterior power of its dual is the set $\Omega^p(U,\Phi)$ of p-forms along U with values on $\Phi(U)$. It is a free module over $C^k(U)$, the ring (algebra) of differentiable scalar fields on U.

INPUT:

- `p` – nonnegative integer

OUTPUT:

- for $p = 0$, the base ring, i.e. $C^k(U)$
- for $p \geq 1$, a `DiffFormFreeModule` representing the module $\Omega^p(U,\Phi)$

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()  # makes M parallelizable
sage: XM = M.vector_field_module()
sage: XM.dual_exterior_power(2)
Free module Omega^2(M) of 2-forms on the 2-dimensional
 differentiable manifold M
sage: XM.dual_exterior_power(1)
Free module Omega^1(M) of 1-forms on the 2-dimensional differentiable␣
↪manifold M
sage: XM.dual_exterior_power(1) is XM.dual()
True
```

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```
sage: XM.dual_exterior_power(0)
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
sage: XM.dual_exterior_power(0) is M.scalar_field_algebra()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)# makes M␣
↪parallelizable
>>> XM = M.vector_field_module()
>>> XM.dual_exterior_power(Integer(2))
Free module Omega^2(M) of 2-forms on the 2-dimensional
 differentiable manifold M
>>> XM.dual_exterior_power(Integer(1))
Free module Omega^1(M) of 1-forms on the 2-dimensional differentiable␣
↪manifold M
>>> XM.dual_exterior_power(Integer(1)) is XM.dual()
True
>>> XM.dual_exterior_power(Integer(0))
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
>>> XM.dual_exterior_power(Integer(0)) is M.scalar_field_algebra()
True
```

See also:

DiffFormFreeModule for more examples and documentation.

exterior_power(*p*)

Return the p-th exterior power of `self`.

If the vector field module `self` is $\mathfrak{X}(U,\Phi)$, its p-th exterior power is the set $A^p(U,\Phi)$ of p-vector fields along U with values on $\Phi(U)$. It is a free module over $C^k(U)$, the ring (algebra) of differentiable scalar fields on U.

INPUT:

- `p` – nonnegative integer

OUTPUT:

- for $p = 0$, the base ring, i.e. $C^k(U)$
- for $p = 1$, the vector field free module `self`, since $A^1(U,\Phi) = \mathfrak{X}(U,\Phi)$
- for $p \geq 2$, instance of *MultivectorFreeModule* representing the module $A^p(U,\Phi)$

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()   # makes M parallelizable
sage: XM = M.vector_field_module()
sage: XM.exterior_power(2)
```

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```
Free module A^2(M) of 2-vector fields on the 2-dimensional
 differentiable manifold M
sage: XM.exterior_power(1)
Free module X(M) of vector fields on the 2-dimensional
 differentiable manifold M
sage: XM.exterior_power(1) is XM
True
sage: XM.exterior_power(0)
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
sage: XM.exterior_power(0) is M.scalar_field_algebra()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)# makes M␣
↪parallelizable
>>> XM = M.vector_field_module()
>>> XM.exterior_power(Integer(2))
Free module A^2(M) of 2-vector fields on the 2-dimensional
 differentiable manifold M
>>> XM.exterior_power(Integer(1))
Free module X(M) of vector fields on the 2-dimensional
 differentiable manifold M
>>> XM.exterior_power(Integer(1)) is XM
True
>>> XM.exterior_power(Integer(0))
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
>>> XM.exterior_power(Integer(0)) is M.scalar_field_algebra()
True
```

See also

`MultivectorFreeModule` for more examples and documentation.

general_linear_group()

Return the general linear group of `self`.

If the vector field module is $\mathfrak{X}(U,\Phi)$, the *general linear group* is the group $\mathrm{GL}(\mathfrak{X}(U,\Phi))$ of automorphisms of $\mathfrak{X}(U,\Phi)$. Note that an automorphism of $\mathfrak{X}(U,\Phi)$ can also be viewed as a *field* along U of automorphisms of the tangent spaces of $V = \Phi(U)$.

OUTPUT:

- a `AutomorphismFieldParalGroup` representing $\mathrm{GL}(\mathfrak{X}(U,\Phi))$

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()  # makes M parallelizable
sage: XM = M.vector_field_module()
```

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```
sage: XM.general_linear_group()
General linear group of the Free module X(M) of vector fields on
 the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)# makes M␣
↪parallelizable
>>> XM = M.vector_field_module()
>>> XM.general_linear_group()
General linear group of the Free module X(M) of vector fields on
 the 2-dimensional differentiable manifold M
```

See also:

AutomorphismFieldParalGroup for more examples and documentation.

metric(*name*, *signature=None*, *latex_name=None*)

Construct a pseudo-Riemannian metric (nondegenerate symmetric bilinear form) on the current vector field module.

A pseudo-Riemannian metric of the vector field module is actually a field of tangent-space non-degenerate symmetric bilinear forms along the manifold U on which the vector field module is defined.

INPUT:

- `name` – string; name given to the metric
- `signature` – integer (default: `None`); signature S of the metric: $S = n_+ - n_-$, where n_+ (resp. n_-) is the number of positive terms (resp. number of negative terms) in any diagonal writing of the metric components; if `signature` is not provided, S is set to the manifold's dimension (Riemannian signature)
- `latex_name` – (string; default: `None`) LaTeX symbol to denote the metric; if `None`, it is formed from `name`

OUTPUT:

- instance of *PseudoRiemannianMetricParal* representing the defined pseudo-Riemannian metric.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()  # makes M parallelizable
sage: XM = M.vector_field_module()
sage: XM.metric('g')
Riemannian metric g on the 2-dimensional differentiable manifold M
sage: XM.metric('g', signature=0)
Lorentzian metric g on the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)# makes M␣
↪parallelizable
>>> XM = M.vector_field_module()
```

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```
>>> XM.metric('g')
Riemannian metric g on the 2-dimensional differentiable manifold M
>>> XM.metric('g', signature=Integer(0))
Lorentzian metric g on the 2-dimensional differentiable manifold M
```

See also

PseudoRiemannianMetricParal for more documentation.

poisson_tensor(*name=None*, *latex_name=None*)

Construct a Poisson tensor on the current vector field module.

OUTPUT:

- instance of *PoissonTensorFieldParal*

EXAMPLES:

Standard Poisson tensor on $\mathbf{R}^2$:

```
sage: M.<q, p> = EuclideanSpace(2)
sage: poisson = M.vector_field_module().poisson_tensor('varpi')
sage: poisson.set_comp()[1,2] = -1
sage: poisson.display()
varpi = -e_q∧e_p
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(Integer(2), names=('q', 'p',)); (q, p,) = M._first_
↪ngens(2)
>>> poisson = M.vector_field_module().poisson_tensor('varpi')
>>> poisson.set_comp()[Integer(1),Integer(2)] = -Integer(1)
>>> poisson.display()
varpi = -e_q∧e_p
```

sym_bilinear_form(*name=None*, *latex_name=None*)

Construct a symmetric bilinear form on `self`.

A symmetric bilinear form on the vector field module is actually a field of tangent-space symmetric bilinear forms along the differentiable manifold U over which the vector field module is defined.

INPUT:

- `name` – string (default: `None`); name given to the symmetric bilinear form
- `latex_name` – string (default: `None`); LaTeX symbol to denote the symmetric bilinear form; if `None`, the LaTeX symbol is set to `name`

OUTPUT:

- a *TensorFieldParal* of tensor type $(0, 2)$ and symmetric

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()   # makes M parallelizable
sage: XM = M.vector_field_module()
```

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```
sage: XM.sym_bilinear_form(name='a')
Field of symmetric bilinear forms a on the 2-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)# makes M␣
↪parallelizable
>>> XM = M.vector_field_module()
>>> XM.sym_bilinear_form(name='a')
Field of symmetric bilinear forms a on the 2-dimensional
 differentiable manifold M
```

See also:

TensorFieldParal for more examples and documentation.

symplectic_form(*name=None*, *latex_name=None*)

Construct a symplectic form on the current vector field module.

OUTPUT:

- instance of *SymplecticFormParal*

EXAMPLES:

Standard symplectic form on $\mathbf{R}^2$:

```
sage: M.<q, p> = EuclideanSpace(2)
sage: omega = M.vector_field_module().symplectic_form('omega', r'\omega')
sage: omega.set_comp()[1,2] = -1
sage: omega.display()
omega = -dq∧dp
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(Integer(2), names=('q', 'p',)); (q, p,) = M._first_
↪ngens(2)
>>> omega = M.vector_field_module().symplectic_form('omega', r'\omega')
>>> omega.set_comp()[Integer(1),Integer(2)] = -Integer(1)
>>> omega.display()
omega = -dq∧dp
```

tensor_from_comp(*tensor_type*, *comp*, *name=None*, *latex_name=None*)

Construct a tensor on `self` from a set of components.

The tensor is actually a tensor field along the differentiable manifold U over which the vector field module is defined. The tensor symmetries are deduced from those of the components.

INPUT:

- `tensor_type` – pair (k, l) with k being the contravariant rank and l the covariant rank
- `comp` – `Components`; the tensor components in a given basis
- `name` – string (default: `None`); name given to the tensor

- latex_name – string (default: None); LaTeX symbol to denote the tensor; if None, the LaTeX symbol is set to name

OUTPUT:

- a *TensorFieldParal* representing the tensor defined on the vector field module with the provided characteristics

EXAMPLES:

A 2-dimensional set of components transformed into a type-$(1, 1)$ tensor field:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: XM = M.vector_field_module()
sage: from sage.tensor.modules.comp import Components
sage: comp = Components(M.scalar_field_algebra(), X.frame(), 2,
....:                   output_formatter=XM._output_formatter)
sage: comp[:] = [[1+x, -y], [x*y, 2-y^2]]
sage: t = XM.tensor_from_comp((1,1), comp, name='t'); t
Tensor field t of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: t.display()
t = (x + 1) ∂/∂x⊗dx - y ∂/∂x⊗dy + x*y ∂/∂y⊗dx + (-y^2 + 2) ∂/∂y⊗dy
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> XM = M.vector_field_module()
>>> from sage.tensor.modules.comp import Components
>>> comp = Components(M.scalar_field_algebra(), X.frame(), Integer(2),
...                   output_formatter=XM._output_formatter)
>>> comp[:] = [[Integer(1)+x, -y], [x*y, Integer(2)-y**Integer(2)]]
>>> t = XM.tensor_from_comp((Integer(1),Integer(1)), comp, name='t'); t
Tensor field t of type (1,1) on the 2-dimensional differentiable
 manifold M
>>> t.display()
t = (x + 1) ∂/∂x⊗dx - y ∂/∂x⊗dy + x*y ∂/∂y⊗dx + (-y^2 + 2) ∂/∂y⊗dy
```

The same set of components transformed into a type-$(0, 2)$ tensor field:

```
sage: t = XM.tensor_from_comp((0,2), comp, name='t'); t
Tensor field t of type (0,2) on the 2-dimensional differentiable
 manifold M
sage: t.display()
t = (x + 1) dx⊗dx - y dx⊗dy + x*y dy⊗dx + (-y^2 + 2) dy⊗dy
```

```
>>> from sage.all import *
>>> t = XM.tensor_from_comp((Integer(0),Integer(2)), comp, name='t'); t
Tensor field t of type (0,2) on the 2-dimensional differentiable
 manifold M
>>> t.display()
t = (x + 1) dx⊗dx - y dx⊗dy + x*y dy⊗dx + (-y^2 + 2) dy⊗dy
```

tensor_module(*k*, *l*, *sym*, *antisym*)

Return the free module of all tensors of type (k, l) defined on self.

INPUT:

- `k` – nonnegative integer; the contravariant rank, the tensor type being (k, l)
- `l` – nonnegative integer; the covariant rank, the tensor type being (k, l)

OUTPUT:

- a *TensorFieldFreeModule* representing the free module of type-(k, l) tensors on the vector field module

EXAMPLES:

A tensor field module on a 2-dimensional differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()  # makes M parallelizable
sage: XM = M.vector_field_module()
sage: XM.tensor_module(1,2)
Free module T^(1,2)(M) of type-(1,2) tensors fields on the
 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)# makes M␣
↪parallelizable
>>> XM = M.vector_field_module()
>>> XM.tensor_module(Integer(1),Integer(2))
Free module T^(1,2)(M) of type-(1,2) tensors fields on the
 2-dimensional differentiable manifold M
```

The special case of tensor fields of type (1,0):

```
sage: XM.tensor_module(1,0)
Free module X(M) of vector fields on the 2-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> XM.tensor_module(Integer(1),Integer(0))
Free module X(M) of vector fields on the 2-dimensional
 differentiable manifold M
```

The result is cached:

```
sage: XM.tensor_module(1,2) is XM.tensor_module(1,2)
True
sage: XM.tensor_module(1,0) is XM
True
```

```
>>> from sage.all import *
>>> XM.tensor_module(Integer(1),Integer(2)) is XM.tensor_module(Integer(1),
↪Integer(2))
True
>>> XM.tensor_module(Integer(1),Integer(0)) is XM
True
```

See also:

`TensorFieldFreeModule` for more examples and documentation.

class `sage.manifolds.differentiable.vectorfield_module.`**`VectorFieldModule`**(*domain: DifferentiableManifold*, *dest_map: DiffMap | None = None*)

Bases: `UniqueRepresentation`, `ReflexiveModule_base`

Module of vector fields along a differentiable manifold U with values on a differentiable manifold M, via a differentiable map $U \to M$.

Given a differentiable map

$$\Phi : U \longrightarrow M,$$

the *vector field module* $\mathfrak{X}(U, \Phi)$ is the set of all vector fields of the type

$$v : U \longrightarrow TM$$

(where TM is the tangent bundle of M) such that

$$\forall p \in U, \ v(p) \in T_{\Phi(p)}M,$$

where $T_{\Phi(p)}M$ is the tangent space to M at the point $\Phi(p)$.

The set $\mathfrak{X}(U, \Phi)$ is a module over $C^k(U)$, the ring (algebra) of differentiable scalar fields on U (see `DiffScalarFieldAlgebra`). Furthermore, it is a Lie algebroid under the Lie bracket (cf. Wikipedia article Lie_algebroid)

$$[X, Y] = X \circ Y - Y \circ X$$

over the scalarfields if Φ is the identity map. That is to say the Lie bracket is antisymmetric, bilinear over the base field, satisfies the Jacobi identity, and $[X, fY] = X(f)Y + f[X, Y]$.

The standard case of vector fields *on* a differentiable manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$; we then denote $\mathfrak{X}(M, \mathrm{Id}_M)$ by merely $\mathfrak{X}(M)$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note: If M is parallelizable, the class `VectorFieldFreeModule` should be used instead.

INPUT:

- `domain` – differentiable manifold U along which the vector fields are defined
- `dest_map` – (default: `None`) destination map $\Phi : U \to M$ (type: `DiffMap`); if `None`, it is assumed that $U = M$ and Φ is the identity map of M (case of vector fields *on* M)

EXAMPLES:

Module of vector fields on the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                intersection_name='W', restrictions1= x^2+y^2!=0,
....:                restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: XM = M.vector_field_module() ; XM
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> XM = M.vector_field_module() ; XM
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
```

$\mathfrak{X}(M)$ is a module over the algebra $C^k(M)$:

```
sage: XM.category()
Category of modules over Algebra of differentiable scalar fields on the
 2-dimensional differentiable manifold M
sage: XM.base_ring() is M.scalar_field_algebra()
True
```

```
>>> from sage.all import *
>>> XM.category()
Category of modules over Algebra of differentiable scalar fields on the
 2-dimensional differentiable manifold M
>>> XM.base_ring() is M.scalar_field_algebra()
True
```

$\mathfrak{X}(M)$ is not a free module:

```
sage: isinstance(XM, FiniteRankFreeModule)
False
```

```
>>> from sage.all import *
>>> isinstance(XM, FiniteRankFreeModule)
False
```

because $M = S^2$ is not parallelizable:

```
sage: M.is_manifestly_parallelizable()
False
```

```
>>> from sage.all import *
>>> M.is_manifestly_parallelizable()
False
```

On the contrary, the module of vector fields on U is a free module, since U is parallelizable (being a coordinate domain):

```
sage: XU = U.vector_field_module()
sage: isinstance(XU, FiniteRankFreeModule)
True
sage: U.is_manifestly_parallelizable()
True
```

```
>>> from sage.all import *
>>> XU = U.vector_field_module()
>>> isinstance(XU, FiniteRankFreeModule)
True
>>> U.is_manifestly_parallelizable()
True
```

The zero element of the module:

```
sage: z = XM.zero() ; z
Vector field zero on the 2-dimensional differentiable manifold M
sage: z.display(c_xy.frame())
zero = 0
sage: z.display(c_uv.frame())
zero = 0
```

```
>>> from sage.all import *
>>> z = XM.zero() ; z
Vector field zero on the 2-dimensional differentiable manifold M
>>> z.display(c_xy.frame())
zero = 0
>>> z.display(c_uv.frame())
zero = 0
```

The module $\mathfrak{X}(M)$ coerces to any module of vector fields defined on a subdomain of M, for instance $\mathfrak{X}(U)$:

```
sage: XU.has_coerce_map_from(XM)
True
sage: XU.coerce_map_from(XM)
Coercion map:
  From: Module X(M) of vector fields on the 2-dimensional
```

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```
  differentiable manifold M
 To:   Free module X(U) of vector fields on the Open subset U of the
  2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> XU.has_coerce_map_from(XM)
True
>>> XU.coerce_map_from(XM)
Coercion map:
  From: Module X(M) of vector fields on the 2-dimensional
   differentiable manifold M
  To:   Free module X(U) of vector fields on the Open subset U of the
   2-dimensional differentiable manifold M
```

The conversion map is actually the restriction of vector fields defined on M to U.

Element

alias of *VectorField*

alternating_contravariant_tensor(*degree*, *name=None*, *latex_name=None*)

Construct an alternating contravariant tensor on the vector field module `self`.

An alternating contravariant tensor on `self` is actually a multivector field along the differentiable manifold U over which `self` is defined.

INPUT:

- `degree` – degree of the alternating contravariant tensor (i.e. its tensor rank)
- `name` – (default: `None`) string; name given to the alternating contravariant tensor
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the alternating contravariant tensor; if none is provided, the LaTeX symbol is set to `name`

OUTPUT:

- instance of *MultivectorField*

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.alternating_contravariant_tensor(2, name='a')
2-vector field a on the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> XM = M.vector_field_module()
>>> XM.alternating_contravariant_tensor(Integer(2), name='a')
2-vector field a on the 2-dimensional differentiable
 manifold M
```

An alternating contravariant tensor of degree 1 is simply a vector field:

```
sage: XM.alternating_contravariant_tensor(1, name='a')
Vector field a on the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> XM.alternating_contravariant_tensor(Integer(1), name='a')
Vector field a on the 2-dimensional differentiable
 manifold M
```

See also

MultivectorField for more examples and documentation.

alternating_form(*degree*, *name=None*, *latex_name=None*)

Construct an alternating form on the vector field module `self`.

An alternating form on `self` is actually a differential form along the differentiable manifold U over which `self` is defined.

INPUT:

- `degree` – the degree of the alternating form (i.e. its tensor rank)
- `name` – (string; optional) name given to the alternating form
- `latex_name` – (string; optional) LaTeX symbol to denote the alternating form; if none is provided, the LaTeX symbol is set to `name`

OUTPUT:

- instance of *DiffForm*

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.alternating_form(2, name='a')
2-form a on the 2-dimensional differentiable manifold M
sage: XM.alternating_form(1, name='a')
1-form a on the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> XM = M.vector_field_module()
>>> XM.alternating_form(Integer(2), name='a')
2-form a on the 2-dimensional differentiable manifold M
>>> XM.alternating_form(Integer(1), name='a')
1-form a on the 2-dimensional differentiable manifold M
```

See also

DiffForm for more examples and documentation.

ambient_domain()

Return the manifold in which the vector fields of this module take their values.

If the module is $\mathfrak{X}(U, \Phi)$, returns the codomain M of Φ.

OUTPUT:

- instance of *DifferentiableManifold* representing the manifold in which the vector fields of this module take their values

EXAMPLES:

```
sage: M = Manifold(5, 'M')
sage: XM = M.vector_field_module()
sage: XM.ambient_domain()
5-dimensional differentiable manifold M
sage: U = Manifold(2, 'U')
sage: Phi = U.diff_map(M, name='Phi')
sage: XU = U.vector_field_module(dest_map=Phi)
sage: XU.ambient_domain()
5-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(5), 'M')
>>> XM = M.vector_field_module()
>>> XM.ambient_domain()
5-dimensional differentiable manifold M
>>> U = Manifold(Integer(2), 'U')
>>> Phi = U.diff_map(M, name='Phi')
>>> XU = U.vector_field_module(dest_map=Phi)
>>> XU.ambient_domain()
5-dimensional differentiable manifold M
```

automorphism(*name=None*, *latex_name=None*)

Construct an automorphism of the vector field module.

An automorphism of the vector field module is actually a field of tangent-space automorphisms along the differentiable manifold U over which the vector field module is defined.

INPUT:

- `name` – (string; optional) name given to the automorphism
- `latex_name` – (string; optional) LaTeX symbol to denote the automorphism; if none is provided, the LaTeX symbol is set to `name`

OUTPUT:

- instance of *AutomorphismField*

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.automorphism()
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
sage: XM.automorphism(name='a')
```

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```
Field of tangent-space automorphisms a on the 2-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> XM = M.vector_field_module()
>>> XM.automorphism()
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
>>> XM.automorphism(name='a')
Field of tangent-space automorphisms a on the 2-dimensional
 differentiable manifold M
```

See also

AutomorphismField for more examples and documentation.

destination_map()

Return the differential map associated to this module.

The differential map associated to this module is the map

$$\Phi : U \longrightarrow M$$

such that this module is the set $\mathfrak{X}(U, \Phi)$ of all vector fields of the type

$$v : U \longrightarrow TM$$

(where TM is the tangent bundle of M) such that

$$\forall p \in U, \ v(p) \in T_{\Phi(p)}M,$$

where $T_{\Phi(p)}M$ is the tangent space to M at the point $\Phi(p)$.

OUTPUT:

- instance of *DiffMap* representing the differential map Φ

EXAMPLES:

```
sage: M = Manifold(5, 'M')
sage: XM = M.vector_field_module()
sage: XM.destination_map()
Identity map Id_M of the 5-dimensional differentiable manifold M
sage: U = Manifold(2, 'U')
sage: Phi = U.diff_map(M, name='Phi')
sage: XU = U.vector_field_module(dest_map=Phi)
sage: XU.destination_map()
Differentiable map Phi from the 2-dimensional differentiable
 manifold U to the 5-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(5), 'M')
>>> XM = M.vector_field_module()
>>> XM.destination_map()
Identity map Id_M of the 5-dimensional differentiable manifold M
>>> U = Manifold(Integer(2), 'U')
>>> Phi = U.diff_map(M, name='Phi')
>>> XU = U.vector_field_module(dest_map=Phi)
>>> XU.destination_map()
Differentiable map Phi from the 2-dimensional differentiable
 manifold U to the 5-dimensional differentiable manifold M
```

domain()

Return the domain of the vector fields in this module.

If the module is $\mathfrak{X}(U,\Phi)$, returns the domain U of Φ.

OUTPUT:

- instance of *DifferentiableManifold* representing the domain of the vector fields that belong to this module

EXAMPLES:

```
sage: M = Manifold(5, 'M')
sage: XM = M.vector_field_module()
sage: XM.domain()
5-dimensional differentiable manifold M
sage: U = Manifold(2, 'U')
sage: Phi = U.diff_map(M, name='Phi')
sage: XU = U.vector_field_module(dest_map=Phi)
sage: XU.domain()
2-dimensional differentiable manifold U
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(5), 'M')
>>> XM = M.vector_field_module()
>>> XM.domain()
5-dimensional differentiable manifold M
>>> U = Manifold(Integer(2), 'U')
>>> Phi = U.diff_map(M, name='Phi')
>>> XU = U.vector_field_module(dest_map=Phi)
>>> XU.domain()
2-dimensional differentiable manifold U
```

dual()

Return the dual module.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.dual()
Module Omega^1(M) of 1-forms on the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> XM = M.vector_field_module()
>>> XM.dual()
Module Omega^1(M) of 1-forms on the 2-dimensional differentiable
 manifold M
```

dual_exterior_power(*p*)

Return the p-th exterior power of the dual of the vector field module.

If the vector field module is $\mathfrak{X}(U, \Phi)$, the p-th exterior power of its dual is the set $\Omega^p(U, \Phi)$ of p-forms along U with values on $\Phi(U)$. It is a module over $C^k(U)$, the ring (algebra) of differentiable scalar fields on U.

INPUT:

- `p` – nonnegative integer

OUTPUT:

- for $p = 0$, the base ring, i.e. $C^k(U)$
- for $p \geq 1$, instance of `DiffFormModule` representing the module $\Omega^p(U, \Phi)$

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.dual_exterior_power(2)
Module Omega^2(M) of 2-forms on the 2-dimensional differentiable
 manifold M
sage: XM.dual_exterior_power(1)
Module Omega^1(M) of 1-forms on the 2-dimensional differentiable
 manifold M
sage: XM.dual_exterior_power(1) is XM.dual()
True
sage: XM.dual_exterior_power(0)
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
sage: XM.dual_exterior_power(0) is M.scalar_field_algebra()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> XM = M.vector_field_module()
>>> XM.dual_exterior_power(Integer(2))
Module Omega^2(M) of 2-forms on the 2-dimensional differentiable
 manifold M
>>> XM.dual_exterior_power(Integer(1))
Module Omega^1(M) of 1-forms on the 2-dimensional differentiable
 manifold M
>>> XM.dual_exterior_power(Integer(1)) is XM.dual()
True
>>> XM.dual_exterior_power(Integer(0))
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
```

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```
>>> XM.dual_exterior_power(Integer(0)) is M.scalar_field_algebra()
True
```

See also:

`DiffFormModule` for more examples and documentation.

exterior_power(*p*)

Return the p-th exterior power of `self`.

If the vector field module `self` is $\mathfrak{X}(U,\Phi)$, its p-th exterior power is the set $A^p(U,\Phi)$ of p-vector fields along U with values on $\Phi(U)$. It is a module over $C^k(U)$, the ring (algebra) of differentiable scalar fields on U.

INPUT:

- `p` – nonnegative integer

OUTPUT:

- for $p = 0$, the base ring, i.e. $C^k(U)$
- for $p = 1$, the vector field module `self`, since $A^1(U,\Phi) = \mathfrak{X}(U,\Phi)$
- for $p \geq 2$, instance of `MultivectorModule` representing the module $A^p(U,\Phi)$

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.exterior_power(2)
Module A^2(M) of 2-vector fields on the 2-dimensional
 differentiable manifold M
sage: XM.exterior_power(1)
Module X(M) of vector fields on the 2-dimensional
 differentiable manifold M
sage: XM.exterior_power(1) is XM
True
sage: XM.exterior_power(0)
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
sage: XM.exterior_power(0) is M.scalar_field_algebra()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> XM = M.vector_field_module()
>>> XM.exterior_power(Integer(2))
Module A^2(M) of 2-vector fields on the 2-dimensional
 differentiable manifold M
>>> XM.exterior_power(Integer(1))
Module X(M) of vector fields on the 2-dimensional
 differentiable manifold M
>>> XM.exterior_power(Integer(1)) is XM
True
```

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```
>>> XM.exterior_power(Integer(0))
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
>>> XM.exterior_power(Integer(0)) is M.scalar_field_algebra()
True
```

See also:

`MultivectorModule` for more examples and documentation.

general_linear_group()

Return the general linear group of `self`.

If the vector field module is $\mathfrak{X}(U, \Phi)$, the *general linear group* is the group $\mathrm{GL}(\mathfrak{X}(U, \Phi))$ of automorphisms of $\mathfrak{X}(U, \Phi)$. Note that an automorphism of $\mathfrak{X}(U, \Phi)$ can also be viewed as a *field* along U of automorphisms of the tangent spaces of $M \supset \Phi(U)$.

OUTPUT:

- instance of class `AutomorphismFieldGroup` representing $\mathrm{GL}(\mathfrak{X}(U, \Phi))$

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.general_linear_group()
General linear group of the Module X(M) of vector fields on the
 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> XM = M.vector_field_module()
>>> XM.general_linear_group()
General linear group of the Module X(M) of vector fields on the
 2-dimensional differentiable manifold M
```

See also:

`AutomorphismFieldGroup` for more examples and documentation.

identity_map()

Construct the identity map on the vector field module.

The identity map on the vector field module is actually a field of tangent-space identity maps along the differentiable manifold U over which the vector field module is defined.

OUTPUT:

- instance of `AutomorphismField`

EXAMPLES:

Get the identity map on a vector field module:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: Id = XM.identity_map(); Id
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> XM = M.vector_field_module()
>>> Id = XM.identity_map(); Id
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold M
```

If the identity should be renamed, one has to create a copy:

```
sage: Id.set_name('1')
Traceback (most recent call last):
...
ValueError: the name of an immutable element cannot be changed
sage: one = Id.copy('1'); one
Field of tangent-space automorphisms 1 on the 2-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> Id.set_name('1')
Traceback (most recent call last):
...
ValueError: the name of an immutable element cannot be changed
>>> one = Id.copy('1'); one
Field of tangent-space automorphisms 1 on the 2-dimensional
 differentiable manifold M
```

linear_form(*name=None*, *latex_name=None*)

Construct a linear form on the vector field module.

A linear form on the vector field module is actually a field of linear forms (i.e. a 1-form) along the differentiable manifold U over which the vector field module is defined.

INPUT:

- `name` – (string; optional) name given to the linear form
- `latex_name` – (string; optional) LaTeX symbol to denote the linear form; if none is provided, the LaTeX symbol is set to `name`

OUTPUT:

- instance of `DiffForm`

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.linear_form()
1-form on the 2-dimensional differentiable manifold M
```

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```
sage: XM.linear_form(name='a')
1-form a on the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> XM = M.vector_field_module()
>>> XM.linear_form()
1-form on the 2-dimensional differentiable manifold M
>>> XM.linear_form(name='a')
1-form a on the 2-dimensional differentiable manifold M
```

See also:

`DiffForm` for more examples and documentation.

metric(*name*, *signature=None*, *latex_name=None*)

Construct a metric (symmetric bilinear form) on the current vector field module.

A metric of the vector field module is actually a field of tangent-space non-degenerate symmetric bilinear forms along the manifold U on which the vector field module is defined.

INPUT:

- `name` – string; name given to the metric
- `signature` – integer (default: `None`); signature S of the metric: $S = n_+ - n_-$, where n_+ (resp. n_-) is the number of positive terms (resp. number of negative terms) in any diagonal writing of the metric components; if `signature` is not provided, S is set to the manifold's dimension (Riemannian signature)
- `latex_name` – (string; default: `None`) LaTeX symbol to denote the metric; if `None`, it is formed from `name`

OUTPUT:

- instance of `PseudoRiemannianMetric` representing the defined pseudo-Riemannian metric.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.metric('g')
Riemannian metric g on the 2-dimensional differentiable manifold M
sage: XM.metric('g', signature=0)
Lorentzian metric g on the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> XM = M.vector_field_module()
>>> XM.metric('g')
Riemannian metric g on the 2-dimensional differentiable manifold M
>>> XM.metric('g', signature=Integer(0))
Lorentzian metric g on the 2-dimensional differentiable manifold M
```

> **See also:**
>
> *PseudoRiemannianMetric* for more documentation.

poisson_tensor(*name=None*, *latex_name=None*)

Construct a Poisson tensor on the current vector field module.

OUTPUT:

- instance of *PoissonTensorField*

EXAMPLES:

Poisson tensor on the 2-sphere:

```
sage: M = manifolds.Sphere(2, coordinates='stereographic')
sage: XM = M.vector_field_module()
sage: varpi = XM.poisson_tensor(name='varpi', latex_name=r'\varpi')
sage: varpi
2-vector field varpi on the 2-sphere S^2 of radius 1 smoothly embedded in the␣
↪Euclidean space E^3
```

```
>>> from sage.all import *
>>> M = manifolds.Sphere(Integer(2), coordinates='stereographic')
>>> XM = M.vector_field_module()
>>> varpi = XM.poisson_tensor(name='varpi', latex_name=r'\varpi')
>>> varpi
2-vector field varpi on the 2-sphere S^2 of radius 1 smoothly embedded in the␣
↪Euclidean space E^3
```

symplectic_form(*name=None*, *latex_name=None*)

Construct a symplectic form on the current vector field module.

OUTPUT:

- instance of *SymplecticForm*

EXAMPLES:

Symplectic form on the 2-sphere:

```
sage: M = manifolds.Sphere(2, coordinates='stereographic')
sage: XM = M.vector_field_module()
sage: omega = XM.symplectic_form(name='omega', latex_name=r'\omega')
sage: omega
Symplectic form omega on the 2-sphere S^2 of radius 1 smoothly
 embedded in the Euclidean space E^3
```

```
>>> from sage.all import *
>>> M = manifolds.Sphere(Integer(2), coordinates='stereographic')
>>> XM = M.vector_field_module()
>>> omega = XM.symplectic_form(name='omega', latex_name=r'\omega')
>>> omega
Symplectic form omega on the 2-sphere S^2 of radius 1 smoothly
 embedded in the Euclidean space E^3
```

tensor(**args*, ***kwds*)

Construct a tensor field on the domain of `self` or a tensor product of `self` with other modules.

If `args` consist of other parents, just delegate to `tensor_product()`.

Otherwise, construct a tensor (i.e., a tensor field on the domain of the vector field module) from the following input.

INPUT:

- `tensor_type` – pair (k,l) with k being the contravariant rank and l the covariant rank
- `name` – (string; default: `None`) name given to the tensor
- `latex_name` – (string; default: `None`) LaTeX symbol to denote the tensor; if none is provided, the LaTeX symbol is set to `name`
- `sym` – (default: `None`) a symmetry or a list of symmetries among the tensor arguments: each symmetry is described by a tuple containing the positions of the involved arguments, with the convention position=0 for the first argument; for instance:
  - `sym=(0,1)` for a symmetry between the 1st and 2nd arguments
  - `sym=[(0,2),(1,3,4)]` for a symmetry between the 1st and 3rd arguments and a symmetry between the 2nd, 4th and 5th arguments
- `antisym` – (default: `None`) antisymmetry or list of antisymmetries among the arguments, with the same convention as for `sym`
- `specific_type` – (default: `None`) specific subclass of *`TensorField`* for the output

OUTPUT:

- instance of *`TensorField`* representing the tensor defined on the vector field module with the provided characteristics

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.tensor((1,2), name='t')
Tensor field t of type (1,2) on the 2-dimensional differentiable
 manifold M
sage: XM.tensor((1,0), name='a')
Vector field a on the 2-dimensional differentiable manifold M
sage: XM.tensor((0,2), name='a', antisym=(0,1))
2-form a on the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> XM = M.vector_field_module()
>>> XM.tensor((Integer(1),Integer(2)), name='t')
Tensor field t of type (1,2) on the 2-dimensional differentiable
 manifold M
>>> XM.tensor((Integer(1),Integer(0)), name='a')
Vector field a on the 2-dimensional differentiable manifold M
>>> XM.tensor((Integer(0),Integer(2)), name='a', antisym=(Integer(0),
↪Integer(1)))
2-form a on the 2-dimensional differentiable manifold M
```

Delegation to `tensor_product()`:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.tensor(XM)
Module T^(2,0)(M) of type-(2,0) tensors fields on the 2-dimensional␣
↪differentiable manifold M
sage: XM.tensor(XM, XM.dual(), XM)
Module T^(3,1)(M) of type-(3,1) tensors fields on the 2-dimensional␣
↪differentiable manifold M
sage: XM.tensor(XM).tensor(XM.dual().tensor(XM.dual()))
Traceback (most recent call last):
...
AttributeError: 'TensorFieldModule_with_category' object has no attribute '_
↪basis_sym'...
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> XM = M.vector_field_module()
>>> XM.tensor(XM)
Module T^(2,0)(M) of type-(2,0) tensors fields on the 2-dimensional␣
↪differentiable manifold M
>>> XM.tensor(XM, XM.dual(), XM)
Module T^(3,1)(M) of type-(3,1) tensors fields on the 2-dimensional␣
↪differentiable manifold M
>>> XM.tensor(XM).tensor(XM.dual().tensor(XM.dual()))
Traceback (most recent call last):
...
AttributeError: 'TensorFieldModule_with_category' object has no attribute '_
↪basis_sym'...
```

See also

`TensorField` for more examples and documentation.

tensor_module(*k*, *l*, *sym*, *antisym*)

Return the module of type-(k, l) tensors on `self`.

INPUT:

- `k` – nonnegative integer; the contravariant rank, the tensor type being (k, l)
- `l` – nonnegative integer; the covariant rank, the tensor type being (k, l)

OUTPUT:

- instance of `TensorFieldModule` representing the module $T^{(k,l)}(U, \Phi)$ of type-(k, l) tensors on the vector field module

EXAMPLES:

A tensor field module on a 2-dimensional differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.tensor_module(1,2)
```

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```
Module T^(1,2)(M) of type-(1,2) tensors fields on the 2-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> XM = M.vector_field_module()
>>> XM.tensor_module(Integer(1),Integer(2))
Module T^(1,2)(M) of type-(1,2) tensors fields on the 2-dimensional
 differentiable manifold M
```

The special case of tensor fields of type (1,0):

```
sage: XM.tensor_module(1,0)
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> XM.tensor_module(Integer(1),Integer(0))
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
```

The result is cached:

```
sage: XM.tensor_module(1,2) is XM.tensor_module(1,2)
True
sage: XM.tensor_module(1,0) is XM
True
```

```
>>> from sage.all import *
>>> XM.tensor_module(Integer(1),Integer(2)) is XM.tensor_module(Integer(1),
↪Integer(2))
True
>>> XM.tensor_module(Integer(1),Integer(0)) is XM
True
```

See *TensorFieldModule* for more examples and documentation.

zero()

Return the zero of self.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()   # makes M parallelizable
sage: XM = M.vector_field_module()
sage: XM.zero()
Vector field zero on the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)# makes M␣
```

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```
↪parallelizable
>>> XM = M.vector_field_module()
>>> XM.zero()
Vector field zero on the 2-dimensional differentiable
 manifold M
```

2.7.2 Vector Fields

Given two differentiable manifolds U and M over the same topological field K and a differentiable map

$$\Phi: U \longrightarrow M,$$

we define a *vector field along U with values on M* to be a differentiable map

$$v: U \longrightarrow TM$$

(TM being the tangent bundle of M) such that

$$\forall p \in U, \ v(p) \in T_{\Phi(p)}M.$$

The standard case of vector fields *on* a differentiable manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Vector fields are implemented via two classes: `VectorFieldParal` and `VectorField`, depending respectively whether the manifold M is parallelizable or not, i.e. whether the bundle TM is trivial or not.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015) : initial version
- Marco Mancini (2015): parallelization of vector field plots
- Travis Scrimshaw (2016): review tweaks
- Eric Gourgoulhon (2017): vector fields inherit from multivector fields
- Eric Gourgoulhon (2018): dot and cross products, operators norm and curl

REFERENCES:

- [KN1963]
- [Lee2013]
- [ONe1983]
- [BG1988]

class `sage.manifolds.differentiable.vectorfield.`**`VectorField`**(*vector_field_module*, *name=None*, *latex_name=None*)

Bases: `MultivectorField`

Vector field along a differentiable manifold.

An instance of this class is a vector field along a differentiable manifold U with values on a differentiable manifold M, via a differentiable map $U \to M$. More precisely, given a differentiable map

$$\Phi: U \longrightarrow M,$$

a *vector field along U with values on M* is a differentiable map

$$v: U \longrightarrow TM$$

(TM being the tangent bundle of M) such that

$$\forall p \in U, \ v(p) \in T_{\Phi(p)}M.$$

The standard case of vector fields *on* a differentiable manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note

If M is parallelizable, then `VectorFieldParal` *must* be used instead.

INPUT:

- `vector_field_module` – module $\mathfrak{X}(U,\Phi)$ of vector fields along U with values on $M \supset \Phi(U)$
- `name` – (default: `None`) name given to the vector field
- `latex_name` – (default: `None`) LaTeX symbol to denote the vector field; if none is provided, the LaTeX symbol is set to `name`

EXAMPLES:

A vector field on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_tu.<t,u> = V.chart()
sage: transf = c_xy.transition_map(c_tu, (x+y, x-y), intersection_name='W',
....:                              restrictions1= x>0, restrictions2= t+u>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_tu.frame()
sage: c_tuW = c_tu.restrict(W) ; eVW = c_tuW.frame()
sage: v = M.vector_field(name='v') ; v
Vector field v on the 2-dimensional differentiable manifold M
sage: v.parent()
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_tu = V.
↪chart(names=('t', 'u',)); (t, u,) = c_tu._first_ngens(2)
>>> transf = c_xy.transition_map(c_tu, (x+y, x-y), intersection_name='W',
...                              restrictions1= x>Integer(0), restrictions2= t+u>
↪Integer(0))
>>> inv = transf.inverse()
>>> W = U.intersection(V)
>>> eU = c_xy.frame() ; eV = c_tu.frame()
>>> c_tuW = c_tu.restrict(W) ; eVW = c_tuW.frame()
>>> v = M.vector_field(name='v') ; v
Vector field v on the 2-dimensional differentiable manifold M
```

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```
>>> v.parent()
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
```

The vector field is first defined on the domain U by means of its components with respect to the frame `eU`:

```
sage: v[eU,:] = [-y, 1+x]
```

```
>>> from sage.all import *
>>> v[eU,:] = [-y, Integer(1)+x]
```

The components with respect to the frame `eV` are then deduced by continuation of the components with respect to the frame `eVW` on the domain $W = U \cap V$, expressed in terms on the coordinates covering V:

```
sage: v[eV,0] = v[eVW,0,c_tuW].expr()
sage: v[eV,1] = v[eVW,1,c_tuW].expr()
```

```
>>> from sage.all import *
>>> v[eV,Integer(0)] = v[eVW,Integer(0),c_tuW].expr()
>>> v[eV,Integer(1)] = v[eVW,Integer(1),c_tuW].expr()
```

At this stage, the vector field is fully defined on the whole manifold:

```
sage: v.display(eU)
v = -y ∂/∂x + (x + 1) ∂/∂y
sage: v.display(eV)
v = (u + 1) ∂/∂t + (-t - 1) ∂/∂u
```

```
>>> from sage.all import *
>>> v.display(eU)
v = -y ∂/∂x + (x + 1) ∂/∂y
>>> v.display(eV)
v = (u + 1) ∂/∂t + (-t - 1) ∂/∂u
```

The vector field acting on scalar fields:

```
sage: f = M.scalar_field({c_xy: (x+y)^2, c_tu: t^2}, name='f')
sage: s = v(f) ; s
Scalar field v(f) on the 2-dimensional differentiable manifold M
sage: s.display()
v(f): M → ℝ
on U: (x, y) ↦ 2*x^2 - 2*y^2 + 2*x + 2*y
on V: (t, u) ↦ 2*t*u + 2*t
```

```
>>> from sage.all import *
>>> f = M.scalar_field({c_xy: (x+y)**Integer(2), c_tu: t**Integer(2)}, name='f')
>>> s = v(f) ; s
Scalar field v(f) on the 2-dimensional differentiable manifold M
>>> s.display()
v(f): M → ℝ
on U: (x, y) ↦ 2*x^2 - 2*y^2 + 2*x + 2*y
on V: (t, u) ↦ 2*t*u + 2*t
```

Some checks:

```
sage: v(f) == f.differential()(v)
True
sage: v(f) == f.lie_der(v)
True
```

```
>>> from sage.all import *
>>> v(f) == f.differential()(v)
True
>>> v(f) == f.lie_der(v)
True
```

The result is defined on the intersection of the vector field's domain and the scalar field's one:

```
sage: s = v(f.restrict(U)) ; s
Scalar field v(f) on the Open subset U of the 2-dimensional
 differentiable manifold M
sage: s == v(f).restrict(U)
True
sage: s = v(f.restrict(W)) ; s
Scalar field v(f) on the Open subset W of the 2-dimensional
 differentiable manifold M
sage: s.display()
v(f): W → ℝ
   (x, y) ↦ 2*x^2 - 2*y^2 + 2*x + 2*y
   (t, u) ↦ 2*t*u + 2*t
sage: s = v.restrict(U)(f) ; s
Scalar field v(f) on the Open subset U of the 2-dimensional
 differentiable manifold M
sage: s.display()
v(f): U → ℝ
   (x, y) ↦ 2*x^2 - 2*y^2 + 2*x + 2*y
on W: (t, u) ↦ 2*t*u + 2*t
sage: s = v.restrict(U)(f.restrict(V)) ; s
Scalar field v(f) on the Open subset W of the 2-dimensional
 differentiable manifold M
sage: s.display()
v(f): W → ℝ
   (x, y) ↦ 2*x^2 - 2*y^2 + 2*x + 2*y
   (t, u) ↦ 2*t*u + 2*t
```

```
>>> from sage.all import *
>>> s = v(f.restrict(U)) ; s
Scalar field v(f) on the Open subset U of the 2-dimensional
 differentiable manifold M
>>> s == v(f).restrict(U)
True
>>> s = v(f.restrict(W)) ; s
Scalar field v(f) on the Open subset W of the 2-dimensional
 differentiable manifold M
>>> s.display()
v(f): W → ℝ
```

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```
   (x, y) ↦ 2*x^2 - 2*y^2 + 2*x + 2*y
   (t, u) ↦ 2*t*u + 2*t
>>> s = v.restrict(U)(f) ; s
Scalar field v(f) on the Open subset U of the 2-dimensional
 differentiable manifold M
>>> s.display()
v(f): U → ℝ
   (x, y) ↦ 2*x^2 - 2*y^2 + 2*x + 2*y
on W: (t, u) ↦ 2*t*u + 2*t
>>> s = v.restrict(U)(f.restrict(V)) ; s
Scalar field v(f) on the Open subset W of the 2-dimensional
 differentiable manifold M
>>> s.display()
v(f): W → ℝ
   (x, y) ↦ 2*x^2 - 2*y^2 + 2*x + 2*y
   (t, u) ↦ 2*t*u + 2*t
```

bracket(*other*)

Return the Lie bracket `[self, other]`.

INPUT:

- `other` – a *VectorField*

OUTPUT: the *VectorField* `[self, other]`

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: v = -X.frame()[0] + 2*X.frame()[1] - (x^2 - y)*X.frame()[2]
sage: w = (z + y) * X.frame()[1] - X.frame()[2]
sage: vw = v.bracket(w); vw
Vector field on the 3-dimensional differentiable manifold M
sage: vw.display()
(-x^2 + y + 2) ∂/∂y + (-y - z) ∂/∂z
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> v = -X.frame()[Integer(0)] + Integer(2)*X.frame()[Integer(1)] -␣
↪(x**Integer(2) - y)*X.frame()[Integer(2)]
>>> w = (z + y) * X.frame()[Integer(1)] - X.frame()[Integer(2)]
>>> vw = v.bracket(w); vw
Vector field on the 3-dimensional differentiable manifold M
>>> vw.display()
(-x^2 + y + 2) ∂/∂y + (-y - z) ∂/∂z
```

Some checks:

```
sage: vw == - w.bracket(v)
True
sage: f = M.scalar_field({X: x+y*z})
sage: vw(f) == v(w(f)) - w(v(f))
```

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```
True
sage: vw == w.lie_derivative(v)
True
```

```
>>> from sage.all import *
>>> vw == - w.bracket(v)
True
>>> f = M.scalar_field({X: x+y*z})
>>> vw(f) == v(w(f)) - w(v(f))
True
>>> vw == w.lie_derivative(v)
True
```

cross(*other*, *metric=None*)

Return the cross product of `self` with another vector field (with respect to a given metric), assuming that the domain of `self` is 3-dimensional.

If `self` is a vector field u on a 3-dimensional differentiable orientable manifold M and `other` is a vector field v on M, the *cross product* (also called *vector product*) *of* u *by* v with respect to a pseudo-Riemannian metric g on M is the vector field $w = u \times v$ defined by

$$w^i = \epsilon^i{}_{jk} u^j v^k = g^{il} \epsilon_{ljk} u^j v^k$$

where ϵ is the volume 3-form (Levi-Civita tensor) of g (cf. `volume_form()`)

Note

The method `cross_product` is meaningful only if for vector fields on a 3-dimensional manifold.

INPUT:

- `other` – a vector field, defined on the same domain as `self`
- `metric` – (default: `None`) the pseudo-Riemannian metric g involved in the definition of the cross product; if none is provided, the domain of `self` is supposed to be endowed with a default metric (i.e. is supposed to be pseudo-Riemannian manifold, see `PseudoRiemannianManifold`) and the latter is used to define the cross product

OUTPUT:

- instance of `VectorField` representing the cross product of `self` by `other`.

EXAMPLES:

Cross product in the Euclidean 3-space:

```
sage: M.<x,y,z> = EuclideanSpace()
sage: u = M.vector_field(-y, x, 0, name='u')
sage: v = M.vector_field(x, y, 0, name='v')
sage: w = u.cross_product(v); w
Vector field u x v on the Euclidean space E^3
sage: w.display()
u x v = (-x^2 - y^2) e_z
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y', 'z',)); (x, y, z,) = M._first_ngens(3)
>>> u = M.vector_field(-y, x, Integer(0), name='u')
>>> v = M.vector_field(x, y, Integer(0), name='v')
>>> w = u.cross_product(v); w
Vector field u x v on the Euclidean space E^3
>>> w.display()
u x v = (-x^2 - y^2) e_z
```

A shortcut alias of `cross_product` is `cross`:

```
sage: u.cross(v) == w
True
```

```
>>> from sage.all import *
>>> u.cross(v) == w
True
```

The cross product of a vector field with itself is zero:

```
sage: u.cross_product(u).display()
u x u = 0
```

```
>>> from sage.all import *
>>> u.cross_product(u).display()
u x u = 0
```

Cross product with respect to a metric that is not the default one:

```
sage: h = M.riemannian_metric('h')
sage: h[1,1], h[2,2], h[3,3] = 1/(1+y^2), 1/(1+z^2), 1/(1+x^2)
sage: w = u.cross_product(v, metric=h); w
Vector field on the Euclidean space E^3
sage: w.display()
-(x^2 + y^2)*sqrt(x^2 + 1)/(sqrt(y^2 + 1)*sqrt(z^2 + 1)) e_z
```

```
>>> from sage.all import *
>>> h = M.riemannian_metric('h')
>>> h[Integer(1),Integer(1)], h[Integer(2),Integer(2)], h[Integer(3),
↪Integer(3)] = Integer(1)/(Integer(1)+y**Integer(2)), Integer(1)/
↪(Integer(1)+z**Integer(2)), Integer(1)/(Integer(1)+x**Integer(2))
>>> w = u.cross_product(v, metric=h); w
Vector field on the Euclidean space E^3
>>> w.display()
-(x^2 + y^2)*sqrt(x^2 + 1)/(sqrt(y^2 + 1)*sqrt(z^2 + 1)) e_z
```

Cross product of two vector fields along a curve (arc of a helix):

```
sage: R.<t> = manifolds.RealLine()
sage: C = M.curve((cos(t), sin(t), t), (t, 0, 2*pi), name='C')
sage: u = C.tangent_vector_field()
sage: u.display()
C' = -sin(t) e_x + cos(t) e_y + e_z
```

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```
sage: I = C.domain(); I
Real interval (0, 2*pi)
sage: v = I.vector_field(-cos(t), sin(t), 0, dest_map=C)
sage: v.display()
-cos(t) e_x + sin(t) e_y
sage: w = u.cross_product(v); w
Vector field along the Real interval (0, 2*pi) with values on the
 Euclidean space E^3
sage: w.parent().destination_map()
Curve C in the Euclidean space E^3
sage: w.display()
-sin(t) e_x - cos(t) e_y + (2*cos(t)^2 - 1) e_z
```

```
>>> from sage.all import *
>>> R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1)
>>> C = M.curve((cos(t), sin(t), t), (t, Integer(0), Integer(2)*pi), name='C')
>>> u = C.tangent_vector_field()
>>> u.display()
C' = -sin(t) e_x + cos(t) e_y + e_z
>>> I = C.domain(); I
Real interval (0, 2*pi)
>>> v = I.vector_field(-cos(t), sin(t), Integer(0), dest_map=C)
>>> v.display()
-cos(t) e_x + sin(t) e_y
>>> w = u.cross_product(v); w
Vector field along the Real interval (0, 2*pi) with values on the
 Euclidean space E^3
>>> w.parent().destination_map()
Curve C in the Euclidean space E^3
>>> w.display()
-sin(t) e_x - cos(t) e_y + (2*cos(t)^2 - 1) e_z
```

Cross product between a vector field along the curve and a vector field on the ambient Euclidean space:

```
sage: e_x = M.cartesian_frame()[1]
sage: w = u.cross_product(e_x); w
Vector field C' x e_x along the Real interval (0, 2*pi) with values
 on the Euclidean space E^3
sage: w.display()
C' x e_x = e_y - cos(t) e_z
```

```
>>> from sage.all import *
>>> e_x = M.cartesian_frame()[Integer(1)]
>>> w = u.cross_product(e_x); w
Vector field C' x e_x along the Real interval (0, 2*pi) with values
 on the Euclidean space E^3
>>> w.display()
C' x e_x = e_y - cos(t) e_z
```

cross_product(*other*, *metric=None*)

Return the cross product of `self` with another vector field (with respect to a given metric), assuming that the domain of `self` is 3-dimensional.

If `self` is a vector field u on a 3-dimensional differentiable orientable manifold M and `other` is a vector field v on M, the *cross product* (also called *vector product*) *of* u *by* v with respect to a pseudo-Riemannian metric g on M is the vector field $w = u \times v$ defined by

$$w^i = \epsilon^i{}_{jk} u^j v^k = g^{il} \epsilon_{ljk} u^j v^k$$

where ϵ is the volume 3-form (Levi-Civita tensor) of g (cf. `volume_form()`)

Note

The method `cross_product` is meaningful only if for vector fields on a 3-dimensional manifold.

INPUT:

- `other` – a vector field, defined on the same domain as `self`
- `metric` – (default: `None`) the pseudo-Riemannian metric g involved in the definition of the cross product; if none is provided, the domain of `self` is supposed to be endowed with a default metric (i.e. is supposed to be pseudo-Riemannian manifold, see `PseudoRiemannianManifold`) and the latter is used to define the cross product

OUTPUT:

- instance of `VectorField` representing the cross product of `self` by `other`.

EXAMPLES:

Cross product in the Euclidean 3-space:

```
sage: M.<x,y,z> = EuclideanSpace()
sage: u = M.vector_field(-y, x, 0, name='u')
sage: v = M.vector_field(x, y, 0, name='v')
sage: w = u.cross_product(v); w
Vector field u x v on the Euclidean space E^3
sage: w.display()
u x v = (-x^2 - y^2) e_z
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y', 'z',)); (x, y, z,) = M._first_ngens(3)
>>> u = M.vector_field(-y, x, Integer(0), name='u')
>>> v = M.vector_field(x, y, Integer(0), name='v')
>>> w = u.cross_product(v); w
Vector field u x v on the Euclidean space E^3
>>> w.display()
u x v = (-x^2 - y^2) e_z
```

A shortcut alias of `cross_product` is `cross`:

```
sage: u.cross(v) == w
True
```

```
>>> from sage.all import *
>>> u.cross(v) == w
True
```

The cross product of a vector field with itself is zero:

```
sage: u.cross_product(u).display()
u x u = 0
```

```
>>> from sage.all import *
>>> u.cross_product(u).display()
u x u = 0
```

Cross product with respect to a metric that is not the default one:

```
sage: h = M.riemannian_metric('h')
sage: h[1,1], h[2,2], h[3,3] = 1/(1+y^2), 1/(1+z^2), 1/(1+x^2)
sage: w = u.cross_product(v, metric=h); w
Vector field on the Euclidean space E^3
sage: w.display()
-(x^2 + y^2)*sqrt(x^2 + 1)/(sqrt(y^2 + 1)*sqrt(z^2 + 1)) e_z
```

```
>>> from sage.all import *
>>> h = M.riemannian_metric('h')
>>> h[Integer(1),Integer(1)], h[Integer(2),Integer(2)], h[Integer(3),
↪Integer(3)] = Integer(1)/(Integer(1)+y**Integer(2)), Integer(1)/
↪(Integer(1)+z**Integer(2)), Integer(1)/(Integer(1)+x**Integer(2))
>>> w = u.cross_product(v, metric=h); w
Vector field on the Euclidean space E^3
>>> w.display()
-(x^2 + y^2)*sqrt(x^2 + 1)/(sqrt(y^2 + 1)*sqrt(z^2 + 1)) e_z
```

Cross product of two vector fields along a curve (arc of a helix):

```
sage: R.<t> = manifolds.RealLine()
sage: C = M.curve((cos(t), sin(t), t), (t, 0, 2*pi), name='C')
sage: u = C.tangent_vector_field()
sage: u.display()
C' = -sin(t) e_x + cos(t) e_y + e_z
sage: I = C.domain(); I
Real interval (0, 2*pi)
sage: v = I.vector_field(-cos(t), sin(t), 0, dest_map=C)
sage: v.display()
-cos(t) e_x + sin(t) e_y
sage: w = u.cross_product(v); w
Vector field along the Real interval (0, 2*pi) with values on the
 Euclidean space E^3
sage: w.parent().destination_map()
Curve C in the Euclidean space E^3
sage: w.display()
-sin(t) e_x - cos(t) e_y + (2*cos(t)^2 - 1) e_z
```

```
>>> from sage.all import *
>>> R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1)
>>> C = M.curve((cos(t), sin(t), t), (t, Integer(0), Integer(2)*pi), name='C')
>>> u = C.tangent_vector_field()
>>> u.display()
C' = -sin(t) e_x + cos(t) e_y + e_z
```

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```
>>> I = C.domain(); I
Real interval (0, 2*pi)
>>> v = I.vector_field(-cos(t), sin(t), Integer(0), dest_map=C)
>>> v.display()
-cos(t) e_x + sin(t) e_y
>>> w = u.cross_product(v); w
Vector field along the Real interval (0, 2*pi) with values on the
 Euclidean space E^3
>>> w.parent().destination_map()
Curve C in the Euclidean space E^3
>>> w.display()
-sin(t) e_x - cos(t) e_y + (2*cos(t)^2 - 1) e_z
```

Cross product between a vector field along the curve and a vector field on the ambient Euclidean space:

```
sage: e_x = M.cartesian_frame()[1]
sage: w = u.cross_product(e_x); w
Vector field C' x e_x along the Real interval (0, 2*pi) with values
 on the Euclidean space E^3
sage: w.display()
C' x e_x = e_y - cos(t) e_z
```

```
>>> from sage.all import *
>>> e_x = M.cartesian_frame()[Integer(1)]
>>> w = u.cross_product(e_x); w
Vector field C' x e_x along the Real interval (0, 2*pi) with values
 on the Euclidean space E^3
>>> w.display()
C' x e_x = e_y - cos(t) e_z
```

curl(*metric=None*)

Return the curl of `self` with respect to a given metric, assuming that the domain of `self` is 3-dimensional.

If `self` is a vector field v on a 3-dimensional differentiable orientable manifold M, the curl of v with respect to a metric g on M is the vector field defined by

$$\mathrm{curl}\, v = (*(\mathrm{d}v^\flat))^\sharp$$

where $v^\flat$ is the 1-form associated to v by the metric g (see *down()*), $*(\mathrm{d}v^\flat)$ is the Hodge dual with respect to g of the 2-form $\mathrm{d}v^\flat$ (exterior derivative of $v^\flat$) (see *hodge_dual()*) and $(*(\mathrm{d}v^\flat))^\sharp$ is corresponding vector field by g-duality (see *up()*).

An alternative expression of the curl is

$$(\mathrm{curl}\, v)^i = \epsilon^{ijk}\nabla_j v_k$$

where ∇ is the Levi-Civita connection of g (cf. *LeviCivitaConnection*) and ϵ the volume 3-form (Levi-Civita tensor) of g (cf. *volume_form()*)

> **Note**
>
> The method `curl` is meaningful only if `self` is a vector field on a 3-dimensional manifold.

INPUT:

- `metric` – (default: `None`) the pseudo-Riemannian metric g involved in the definition of the curl; if none is provided, the domain of `self` is supposed to be endowed with a default metric (i.e. is supposed to be pseudo-Riemannian manifold, see `PseudoRiemannianManifold`) and the latter is used to define the curl

OUTPUT: instance of `VectorField` representing the curl of `self`

EXAMPLES:

Curl of a vector field in the Euclidean 3-space:

```
sage: M.<x,y,z> = EuclideanSpace()
sage: v = M.vector_field(-y, x, 0, name='v')
sage: v.display()
v = -y e_x + x e_y
sage: s = v.curl(); s
Vector field curl(v) on the Euclidean space E^3
sage: s.display()
curl(v) = 2 e_z
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y', 'z',)); (x, y, z,) = M._first_ngens(3)
>>> v = M.vector_field(-y, x, Integer(0), name='v')
>>> v.display()
v = -y e_x + x e_y
>>> s = v.curl(); s
Vector field curl(v) on the Euclidean space E^3
>>> s.display()
curl(v) = 2 e_z
```

The function `curl()` from the `operators` module can be used instead of the method `curl()`:

```
sage: from sage.manifolds.operators import curl
sage: curl(v) == s
True
```

```
>>> from sage.all import *
>>> from sage.manifolds.operators import curl
>>> curl(v) == s
True
```

If one prefers the notation `rot` over `curl`, it suffices to do:

```
sage: from sage.manifolds.operators import curl as rot
sage: rot(v) == s
True
```

```
>>> from sage.all import *
>>> from sage.manifolds.operators import curl as rot
>>> rot(v) == s
True
```

The curl of a gradient vanishes identically:

```
sage: f = M.scalar_field(function('F')(x,y,z))
sage: gradf = f.gradient()
sage: gradf.display()
d(F)/dx e_x + d(F)/dy e_y + d(F)/dz e_z
sage: s = curl(gradf); s
Vector field on the Euclidean space E^3
sage: s.display()
0
```

```
>>> from sage.all import *
>>> f = M.scalar_field(function('F')(x,y,z))
>>> gradf = f.gradient()
>>> gradf.display()
d(F)/dx e_x + d(F)/dy e_y + d(F)/dz e_z
>>> s = curl(gradf); s
Vector field on the Euclidean space E^3
>>> s.display()
0
```

dot(*other*, *metric=None*)

Return the scalar product of `self` with another vector field (with respect to a given metric).

If `self` is the vector field u and other is the vector field v, the *scalar product of* u *by* v with respect to a given pseudo-Riemannian metric g is the scalar field s defined by

$$s = u \cdot v = g(u, v) = g_{ij} u^i v^j$$

INPUT:

- `other` – a vector field, defined on the same domain as `self`
- `metric` – (default: `None`) the pseudo-Riemannian metric g involved in the definition of the scalar product; if none is provided, the domain of `self` is supposed to be endowed with a default metric (i.e. is supposed to be pseudo-Riemannian manifold, see `PseudoRiemannianManifold`) and the latter is used to define the scalar product

OUTPUT:

- instance of `DiffScalarField` representing the scalar product of `self` by `other`.

EXAMPLES:

Scalar product in the Euclidean plane:

```
sage: M.<x,y> = EuclideanSpace()
sage: u = M.vector_field(x, y, name='u')
sage: v = M.vector_field(y, x, name='v')
sage: s = u.dot_product(v); s
Scalar field u.v on the Euclidean plane E^2
sage: s.display()
u.v: E^2 → ℝ
   (x, y) ↦ 2*x*y
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y',)); (x, y,) = M._first_ngens(2)
>>> u = M.vector_field(x, y, name='u')
```

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```
>>> v = M.vector_field(y, x, name='v')
>>> s = u.dot_product(v); s
Scalar field u.v on the Euclidean plane E^2
>>> s.display()
u.v: E^2 → ℝ
   (x, y) ↦ 2*x*y
```

A shortcut alias of `dot_product` is `dot`:

```
sage: u.dot(v) == s
True
```

```
>>> from sage.all import *
>>> u.dot(v) == s
True
```

A test of orthogonality:

```
sage: v[:] = -y, x
sage: u.dot_product(v) == 0
True
```

```
>>> from sage.all import *
>>> v[:] = -y, x
>>> u.dot_product(v) == Integer(0)
True
```

Scalar product with respect to a metric that is not the default one:

```
sage: h = M.riemannian_metric('h')
sage: h[1,1], h[2,2] = 1/(1+y^2), 1/(1+x^2)
sage: s = u.dot_product(v, metric=h); s
Scalar field h(u,v) on the Euclidean plane E^2
sage: s.display()
h(u,v): E^2 → ℝ
   (x, y) ↦ -(x^3*y - x*y^3)/((x^2 + 1)*y^2 + x^2 + 1)
```

```
>>> from sage.all import *
>>> h = M.riemannian_metric('h')
>>> h[Integer(1),Integer(1)], h[Integer(2),Integer(2)] = Integer(1)/
↪(Integer(1)+y**Integer(2)), Integer(1)/(Integer(1)+x**Integer(2))
>>> s = u.dot_product(v, metric=h); s
Scalar field h(u,v) on the Euclidean plane E^2
>>> s.display()
h(u,v): E^2 → ℝ
   (x, y) ↦ -(x^3*y - x*y^3)/((x^2 + 1)*y^2 + x^2 + 1)
```

Scalar product of two vector fields along a curve (a lemniscate of Gerono):

```
sage: R.<t> = manifolds.RealLine()
sage: C = M.curve([sin(t), sin(2*t)/2], (t, 0, 2*pi), name='C')
sage: u = C.tangent_vector_field(name='u')
```

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```
sage: u.display()
u = cos(t) e_x + (2*cos(t)^2 - 1) e_y
sage: I = C.domain(); I
Real interval (0, 2*pi)
sage: v = I.vector_field(cos(t), -1, dest_map=C, name='v')
sage: v.display()
v = cos(t) e_x - e_y
sage: s = u.dot_product(v); s
Scalar field u.v on the Real interval (0, 2*pi)
sage: s.display()
u.v: (0, 2*pi) → ℝ
   t ↦ sin(t)^2
```

```
>>> from sage.all import *
>>> R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1)
>>> C = M.curve([sin(t), sin(Integer(2)*t)/Integer(2)], (t, Integer(0),␣
↪Integer(2)*pi), name='C')
>>> u = C.tangent_vector_field(name='u')
>>> u.display()
u = cos(t) e_x + (2*cos(t)^2 - 1) e_y
>>> I = C.domain(); I
Real interval (0, 2*pi)
>>> v = I.vector_field(cos(t), -Integer(1), dest_map=C, name='v')
>>> v.display()
v = cos(t) e_x - e_y
>>> s = u.dot_product(v); s
Scalar field u.v on the Real interval (0, 2*pi)
>>> s.display()
u.v: (0, 2*pi) → ℝ
   t ↦ sin(t)^2
```

Scalar product between a vector field along the curve and a vector field on the ambient Euclidean plane:

```
sage: e_x = M.cartesian_frame()[1]
sage: s = u.dot_product(e_x); s
Scalar field u.e_x on the Real interval (0, 2*pi)
sage: s.display()
u.e_x: (0, 2*pi) → ℝ
   t ↦ cos(t)
```

```
>>> from sage.all import *
>>> e_x = M.cartesian_frame()[Integer(1)]
>>> s = u.dot_product(e_x); s
Scalar field u.e_x on the Real interval (0, 2*pi)
>>> s.display()
u.e_x: (0, 2*pi) → ℝ
   t ↦ cos(t)
```

dot_product(*other*, *metric=None*)

Return the scalar product of `self` with another vector field (with respect to a given metric).

If `self` is the vector field u and other is the vector field v, the *scalar product of* u *by* v with respect to a given pseudo-Riemannian metric g is the scalar field s defined by

$$s = u \cdot v = g(u, v) = g_{ij} u^i v^j$$

INPUT:

- `other` – a vector field, defined on the same domain as `self`
- `metric` – (default: `None`) the pseudo-Riemannian metric g involved in the definition of the scalar product; if none is provided, the domain of `self` is supposed to be endowed with a default metric (i.e. is supposed to be pseudo-Riemannian manifold, see *PseudoRiemannianManifold*) and the latter is used to define the scalar product

OUTPUT:

- instance of *DiffScalarField* representing the scalar product of `self` by `other`.

EXAMPLES:

Scalar product in the Euclidean plane:

```
sage: M.<x,y> = EuclideanSpace()
sage: u = M.vector_field(x, y, name='u')
sage: v = M.vector_field(y, x, name='v')
sage: s = u.dot_product(v); s
Scalar field u.v on the Euclidean plane E^2
sage: s.display()
u.v: E^2 → ℝ
   (x, y) ↦ 2*x*y
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y',)); (x, y,) = M._first_ngens(2)
>>> u = M.vector_field(x, y, name='u')
>>> v = M.vector_field(y, x, name='v')
>>> s = u.dot_product(v); s
Scalar field u.v on the Euclidean plane E^2
>>> s.display()
u.v: E^2 → ℝ
   (x, y) ↦ 2*x*y
```

A shortcut alias of `dot_product` is `dot`:

```
sage: u.dot(v) == s
True
```

```
>>> from sage.all import *
>>> u.dot(v) == s
True
```

A test of orthogonality:

```
sage: v[:] = -y, x
sage: u.dot_product(v) == 0
True
```

```
>>> from sage.all import *
>>> v[:] = -y, x
>>> u.dot_product(v) == Integer(0)
True
```

Scalar product with respect to a metric that is not the default one:

```
sage: h = M.riemannian_metric('h')
sage: h[1,1], h[2,2] = 1/(1+y^2), 1/(1+x^2)
sage: s = u.dot_product(v, metric=h); s
Scalar field h(u,v) on the Euclidean plane E^2
sage: s.display()
h(u,v): E^2 → ℝ
   (x, y) ↦ -(x^3*y - x*y^3)/((x^2 + 1)*y^2 + x^2 + 1)
```

```
>>> from sage.all import *
>>> h = M.riemannian_metric('h')
>>> h[Integer(1),Integer(1)], h[Integer(2),Integer(2)] = Integer(1)/
↪(Integer(1)+y**Integer(2)), Integer(1)/(Integer(1)+x**Integer(2))
>>> s = u.dot_product(v, metric=h); s
Scalar field h(u,v) on the Euclidean plane E^2
>>> s.display()
h(u,v): E^2 → ℝ
   (x, y) ↦ -(x^3*y - x*y^3)/((x^2 + 1)*y^2 + x^2 + 1)
```

Scalar product of two vector fields along a curve (a lemniscate of Gerono):

```
sage: R.<t> = manifolds.RealLine()
sage: C = M.curve([sin(t), sin(2*t)/2], (t, 0, 2*pi), name='C')
sage: u = C.tangent_vector_field(name='u')
sage: u.display()
u = cos(t) e_x + (2*cos(t)^2 - 1) e_y
sage: I = C.domain(); I
Real interval (0, 2*pi)
sage: v = I.vector_field(cos(t), -1, dest_map=C, name='v')
sage: v.display()
v = cos(t) e_x - e_y
sage: s = u.dot_product(v); s
Scalar field u.v on the Real interval (0, 2*pi)
sage: s.display()
u.v: (0, 2*pi) → ℝ
   t ↦ sin(t)^2
```

```
>>> from sage.all import *
>>> R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1)
>>> C = M.curve([sin(t), sin(Integer(2)*t)/Integer(2)], (t, Integer(0),␣
↪Integer(2)*pi), name='C')
>>> u = C.tangent_vector_field(name='u')
>>> u.display()
u = cos(t) e_x + (2*cos(t)^2 - 1) e_y
>>> I = C.domain(); I
Real interval (0, 2*pi)
>>> v = I.vector_field(cos(t), -Integer(1), dest_map=C, name='v')
```

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```
>>> v.display()
v = cos(t) e_x - e_y
>>> s = u.dot_product(v); s
Scalar field u.v on the Real interval (0, 2*pi)
>>> s.display()
u.v: (0, 2*pi) → ℝ
   t ↦ sin(t)^2
```

Scalar product between a vector field along the curve and a vector field on the ambient Euclidean plane:

```
sage: e_x = M.cartesian_frame()[1]
sage: s = u.dot_product(e_x); s
Scalar field u.e_x on the Real interval (0, 2*pi)
sage: s.display()
u.e_x: (0, 2*pi) → ℝ
   t ↦ cos(t)
```

```
>>> from sage.all import *
>>> e_x = M.cartesian_frame()[Integer(1)]
>>> s = u.dot_product(e_x); s
Scalar field u.e_x on the Real interval (0, 2*pi)
>>> s.display()
u.e_x: (0, 2*pi) → ℝ
   t ↦ cos(t)
```

norm(*metric=None*)

Return the norm of `self` (with respect to a given metric).

The *norm* of a vector field v with respect to a given pseudo-Riemannian metric g is the scalar field $\|v\|$ defined by

$$\|v\| = \sqrt{g(v, v)}$$

Note: If the metric g is not positive definite, it may be that $\|v\|$ takes imaginary values.

INPUT:

- `metric` – (default: `None`) the pseudo-Riemannian metric g involved in the definition of the norm; if none is provided, the domain of `self` is supposed to be endowed with a default metric (i.e. is supposed to be pseudo-Riemannian manifold, see `PseudoRiemannianManifold`) and the latter is used to define the norm

OUTPUT:

- instance of `DiffScalarField` representing the norm of `self`.

EXAMPLES:

Norm in the Euclidean plane:

```
sage: M.<x,y> = EuclideanSpace()
sage: v = M.vector_field(-y, x, name='v')
sage: s = v.norm(); s
Scalar field |v| on the Euclidean plane E^2
sage: s.display()
|v|: E^2 → ℝ
   (x, y) ↦ sqrt(x^2 + y^2)
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y',)); (x, y,) = M._first_ngens(2)
>>> v = M.vector_field(-y, x, name='v')
>>> s = v.norm(); s
Scalar field |v| on the Euclidean plane E^2
>>> s.display()
|v|: E^2 → ℝ
   (x, y) ↦ sqrt(x^2 + y^2)
```

The global function `norm()` can be used instead of the method `norm()`:

```
sage: norm(v) == s
True
```

```
>>> from sage.all import *
>>> norm(v) == s
True
```

Norm with respect to a metric that is not the default one:

```
sage: h = M.riemannian_metric('h')
sage: h[1,1], h[2,2] = 1/(1+y^2), 1/(1+x^2)
sage: s = v.norm(metric=h); s
Scalar field |v|_h on the Euclidean plane E^2
sage: s.display()
|v|_h: E^2 → ℝ
   (x, y) ↦ sqrt((2*x^2 + 1)*y^2 + x^2)/(sqrt(x^2 + 1)*sqrt(y^2 + 1))
```

```
>>> from sage.all import *
>>> h = M.riemannian_metric('h')
>>> h[Integer(1),Integer(1)], h[Integer(2),Integer(2)] = Integer(1)/
↪(Integer(1)+y**Integer(2)), Integer(1)/(Integer(1)+x**Integer(2))
>>> s = v.norm(metric=h); s
Scalar field |v|_h on the Euclidean plane E^2
>>> s.display()
|v|_h: E^2 → ℝ
   (x, y) ↦ sqrt((2*x^2 + 1)*y^2 + x^2)/(sqrt(x^2 + 1)*sqrt(y^2 + 1))
```

Norm of the tangent vector field to a curve (a lemniscate of Gerono):

```
sage: R.<t> = manifolds.RealLine()
sage: C = M.curve([sin(t), sin(2*t)/2], (t, 0, 2*pi), name='C')
sage: v = C.tangent_vector_field()
sage: v.display()
```

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```
C' = cos(t) e_x + (2*cos(t)^2 - 1) e_y
sage: s = v.norm(); s
Scalar field |C'| on the Real interval (0, 2*pi)
sage: s.display()
|C'|: (0, 2*pi) → ℝ
   t ↦ sqrt(4*cos(t)^4 - 3*cos(t)^2 + 1)
```

```
>>> from sage.all import *
>>> R = manifolds.RealLine(names=('t',)); (t,) = R._first_ngens(1)
>>> C = M.curve([sin(t), sin(Integer(2)*t)/Integer(2)], (t, Integer(0),␣
↪Integer(2)*pi), name='C')
>>> v = C.tangent_vector_field()
>>> v.display()
C' = cos(t) e_x + (2*cos(t)^2 - 1) e_y
>>> s = v.norm(); s
Scalar field |C'| on the Real interval (0, 2*pi)
>>> s.display()
|C'|: (0, 2*pi) → ℝ
   t ↦ sqrt(4*cos(t)^4 - 3*cos(t)^2 + 1)
```

plot(*chart=None*, *ambient_coords=None*, *mapping=None*, *chart_domain=None*, *fixed_coords=None*, *ranges=None*, *number_values=None*, *steps=None*, *parameters=None*, *label_axes=True*, *max_range=8*, *scale=1*, *color='blue'*, ***extra_options*)

Plot the vector field in a Cartesian graph based on the coordinates of some ambient chart.

The vector field is drawn in terms of two (2D graphics) or three (3D graphics) coordinates of a given chart, called hereafter the *ambient chart*. The vector field's base points p (or their images $\Phi(p)$ by some differentiable mapping Φ) must lie in the ambient chart's domain.

INPUT:

- `chart` – (default: `None`) the ambient chart (see above); if `None`, the default chart of the vector field's domain is used
- `ambient_coords` – (default: `None`) tuple containing the 2 or 3 coordinates of the ambient chart in terms of which the plot is performed; if `None`, all the coordinates of the ambient chart are considered
- `mapping` – *DiffMap* (default: `None`); differentiable map Φ providing the link between the vector field's domain and the ambient chart `chart`; if `None`, the identity map is assumed
- `chart_domain` – (default: `None`) chart on the vector field's domain to define the points at which vector arrows are to be plotted; if `None`, the default chart of the vector field's domain is used
- `fixed_coords` – (default: `None`) dictionary with keys the coordinates of `chart_domain` that are kept fixed and with values the value of these coordinates; if `None`, all the coordinates of `chart_domain` are used
- `ranges` – (default: `None`) dictionary with keys the coordinates of `chart_domain` to be used and values tuples `(x_min, x_max)` specifying the coordinate range for the plot; if `None`, the entire coordinate range declared during the construction of `chart_domain` is considered (with `-Infinity` replaced by `-max_range` and `+Infinity` by `max_range`)
- `number_values` – (default: `None`) either an integer or a dictionary with keys the coordinates of `chart_domain` to be used and values the number of values of the coordinate for sampling the part of the vector field's domain involved in the plot ; if `number_values` is a single integer, it represents the number of values for all coordinates; if `number_values` is `None`, it is set to 9 for a 2D plot and to 5 for a 3D plot

- `steps` – (default: `None`) dictionary with keys the coordinates of `chart_domain` to be used and values the step between each constant value of the coordinate; if `None`, the step is computed from the coordinate range (specified in `ranges`) and `number_values`; on the contrary, if the step is provided for some coordinate, the corresponding number of values is deduced from it and the coordinate range
- `parameters` – (default: `None`) dictionary giving the numerical values of the parameters that may appear in the coordinate expression of the vector field (see example below)
- `label_axes` – boolean (default: `True`); determining whether the labels of the coordinate axes of `chart` shall be added to the graph; can be set to `False` if the graph is 3D and must be superposed with another graph
- `color` – (default: `'blue'`) color of the arrows representing the vectors
- `max_range` – (default: 8) numerical value substituted to `+Infinity` if the latter is the upper bound of the range of a coordinate for which the plot is performed over the entire coordinate range (i.e. for which no specific plot range has been set in `ranges`); similarly `-max_range` is the numerical valued substituted for `-Infinity`
- `scale` – (default: 1) value by which the lengths of the arrows representing the vectors is multiplied
- `**extra_options` – extra options for the arrow plot, like `linestyle`, `width` or `arrowsize` (see `arrow2d()` and `arrow3d()` for details)

OUTPUT:

- a graphic object, either an instance of `Graphics` for a 2D plot (i.e. based on 2 coordinates of `chart`) or an instance of `Graphics3d` for a 3D plot (i.e. based on 3 coordinates of `chart`)

EXAMPLES:

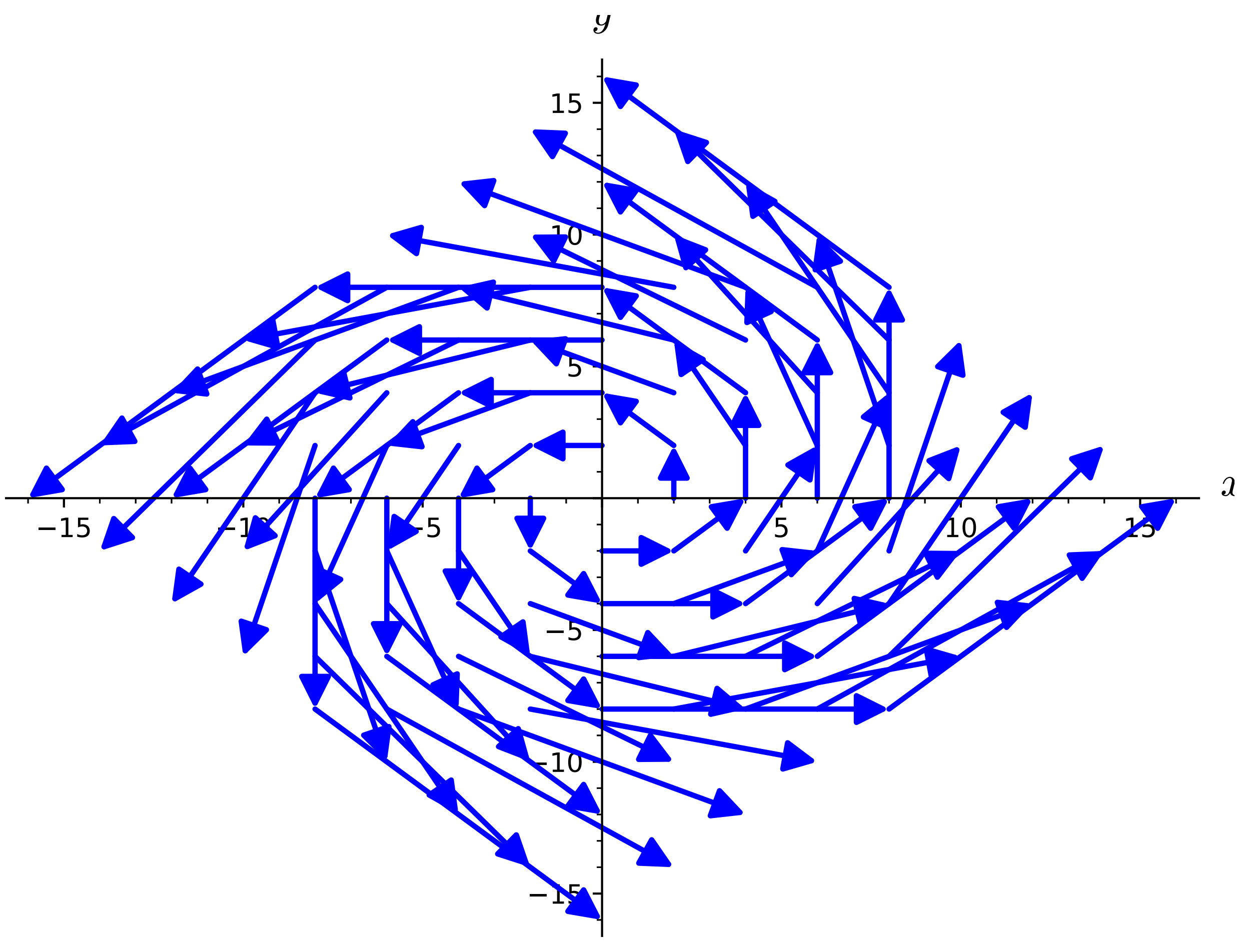
Plot of a vector field on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: v = M.vector_field(-y, x, name='v')
sage: v.display()
v = -y ∂/∂x + x ∂/∂y
sage: v.plot()                                                          #␣
↪needs sage.plot
Graphics object consisting of 80 graphics primitives
```

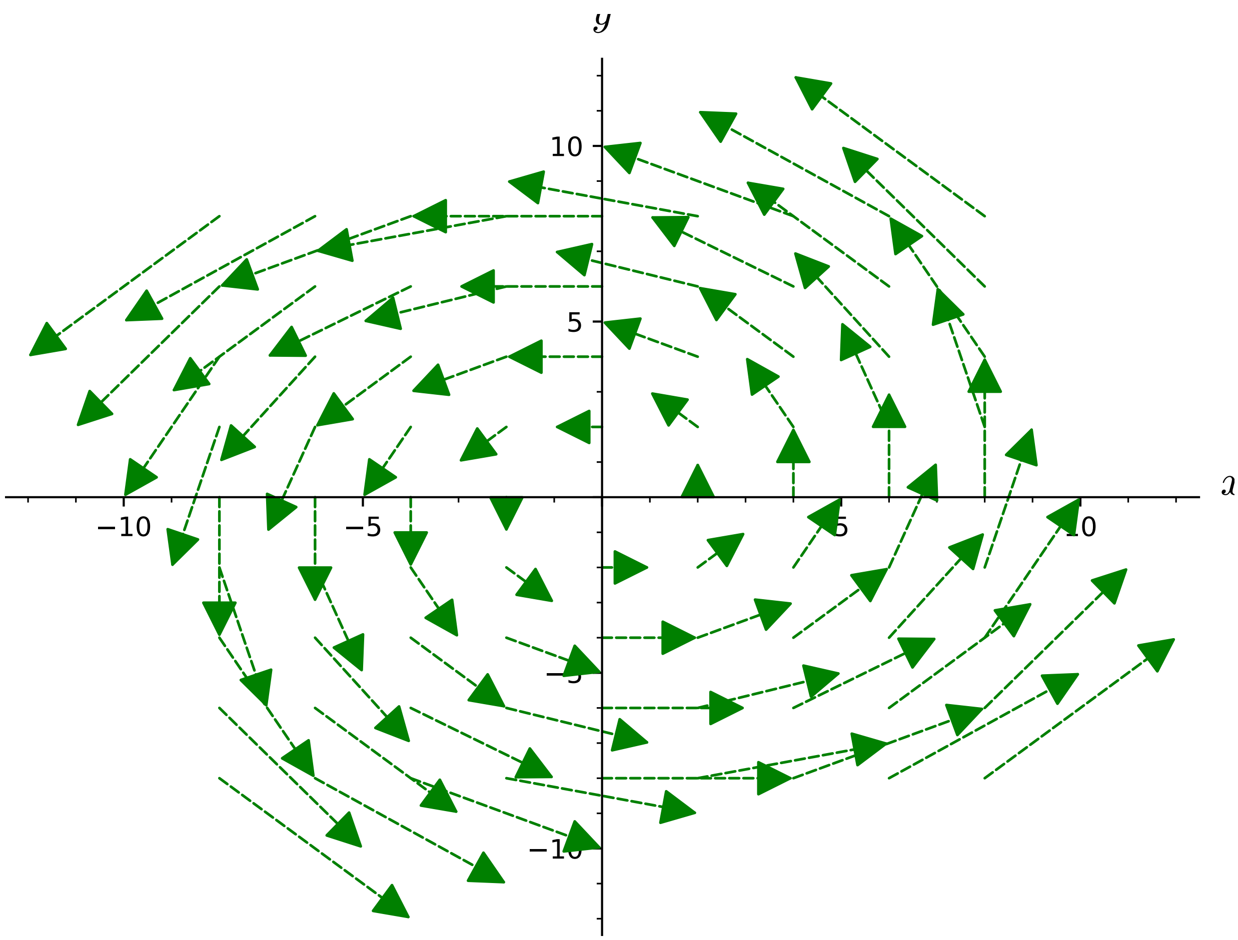
```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> v = M.vector_field(-y, x, name='v')
>>> v.display()
v = -y ∂/∂x + x ∂/∂y
>>> v.plot()                                                            #␣
↪needs sage.plot
Graphics object consisting of 80 graphics primitives
```

Plot with various options:

```
sage: v.plot(scale=0.5, color='green', linestyle='--', width=1,         #␣
↪needs sage.plot
....:        arrowsize=6)
Graphics object consisting of 80 graphics primitives
```


y
x
15
10
5
−5
−10
−15
−15
5
10

```
>>> from sage.all import *
>>> v.plot(scale=RealNumber('0.5'), color='green', linestyle='--',␣
↪width=Integer(1),             # needs sage.plot
...         arrowsize=Integer(6))
Graphics object consisting of 80 graphics primitives
```

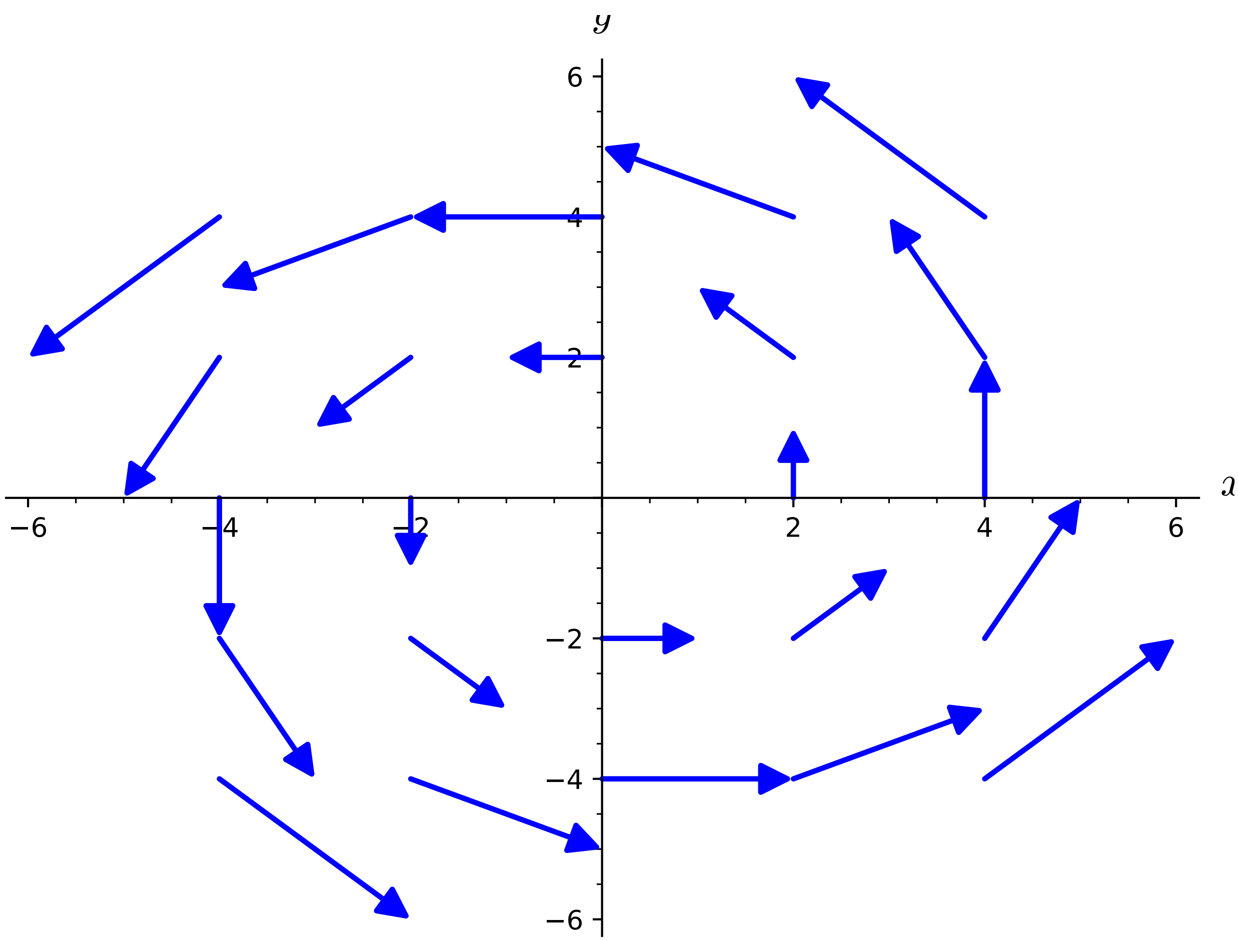



```
sage: v.plot(max_range=4, number_values=5, scale=0.5)                        #␣
↪needs sage.plot
Graphics object consisting of 24 graphics primitives
```

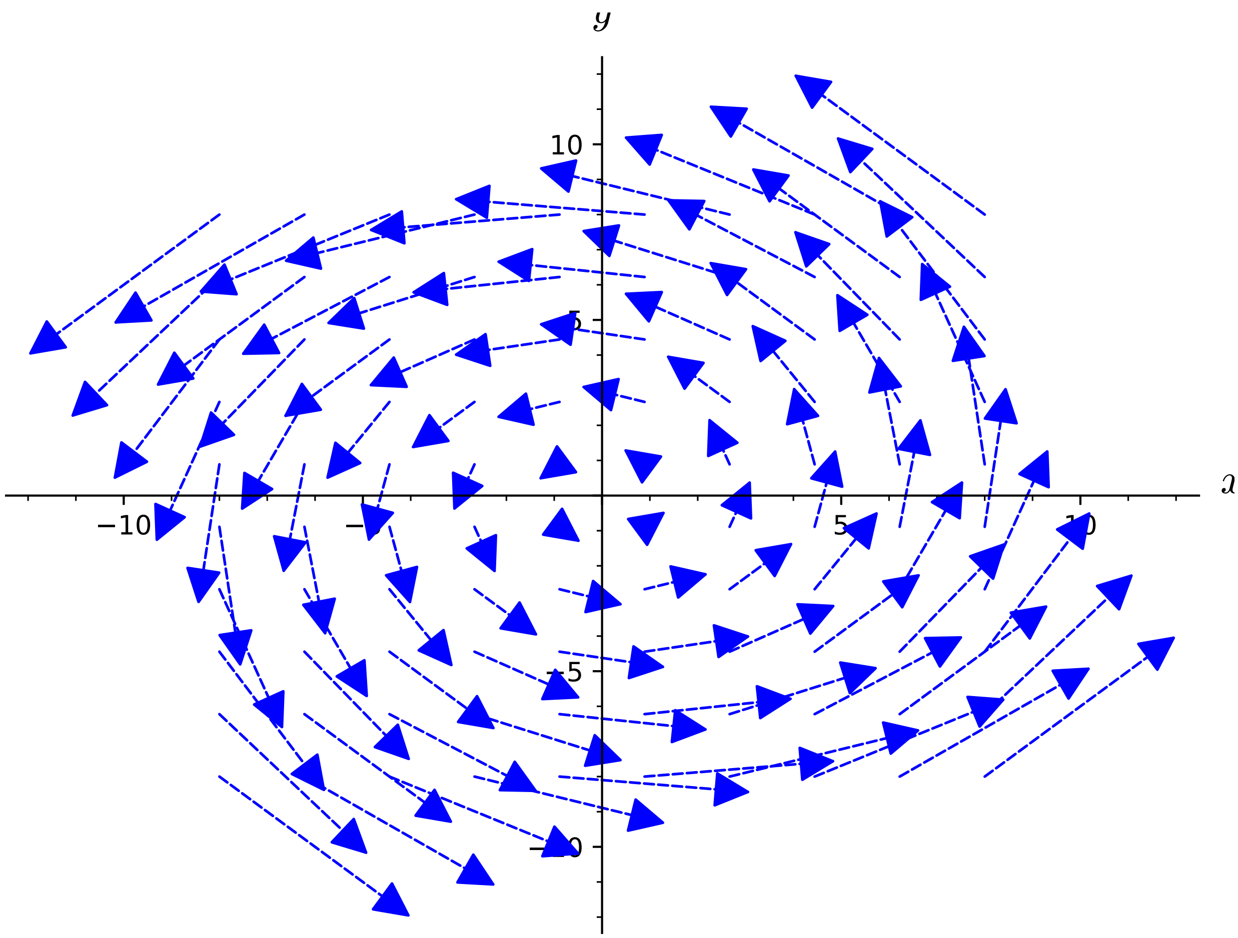
```
>>> from sage.all import *
>>> v.plot(max_range=Integer(4), number_values=Integer(5), scale=RealNumber(
↪'0.5'))                        # needs sage.plot
Graphics object consisting of 24 graphics primitives
```

Plot using parallel computation:

```
sage: Parallelism().set(nproc=2)
sage: v.plot(scale=0.5,  number_values=10, linestyle='--', width=1,           #␣
↪needs sage.plot
....:        arrowsize=6)
Graphics object consisting of 100 graphics primitives
```


y
6
4
2
−2
−4
−6
x
−6
−4
−2
2
4
6

```
>>> from sage.all import *
>>> Parallelism().set(nproc=Integer(2))
>>> v.plot(scale=RealNumber('0.5'),  number_values=Integer(10), linestyle='--
↪', width=Integer(1),          # needs sage.plot
...         arrowsize=Integer(6))
Graphics object consisting of 100 graphics primitives
```



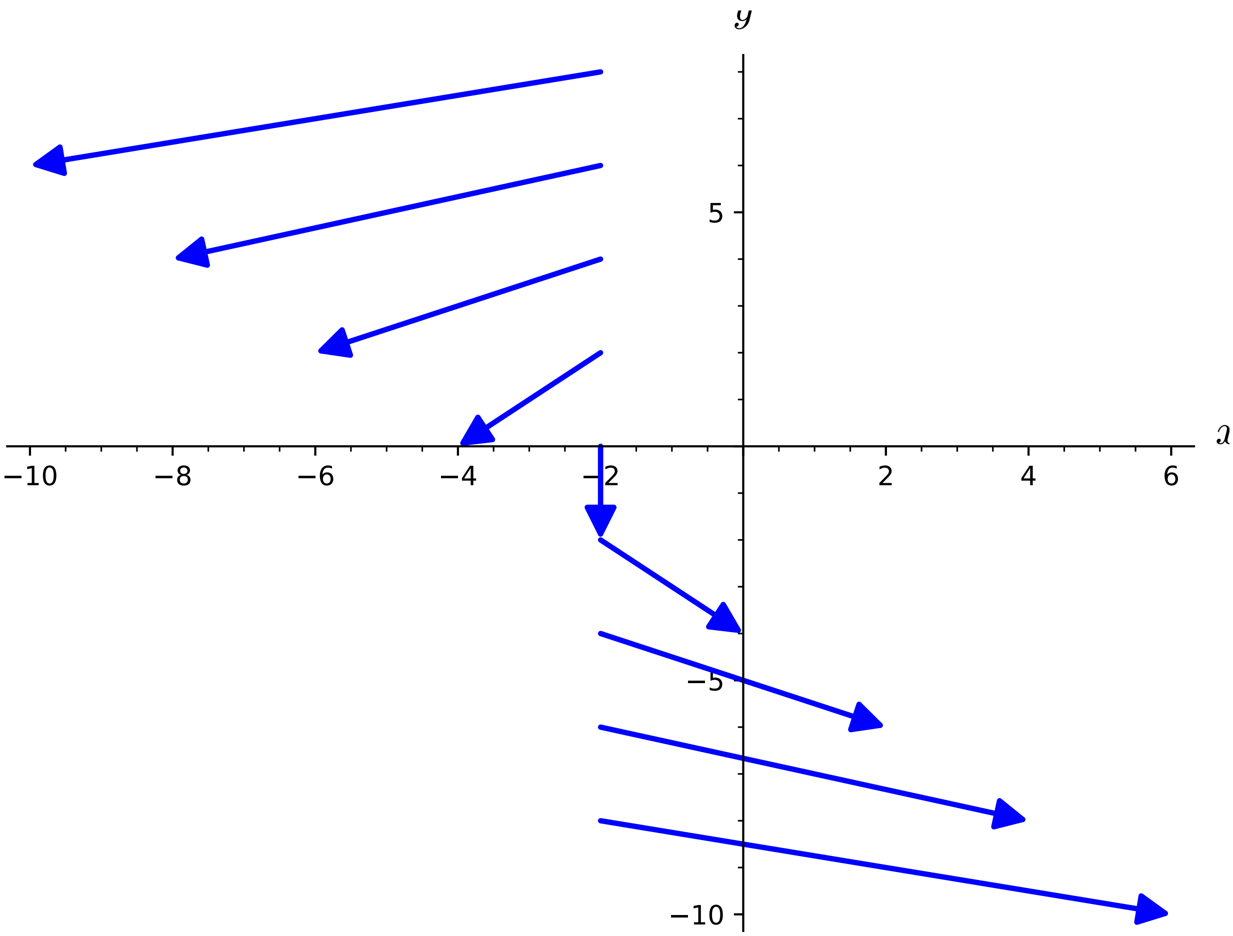

```
sage: Parallelism().set(nproc=1)   # switch off parallelization
```

```
>>> from sage.all import *
>>> Parallelism().set(nproc=Integer(1))   # switch off parallelization
```

Plots along a line of fixed coordinate:

```
sage: v.plot(fixed_coords={x: -2})                                          #␣
↪needs sage.plot
Graphics object consisting of 9 graphics primitives
```

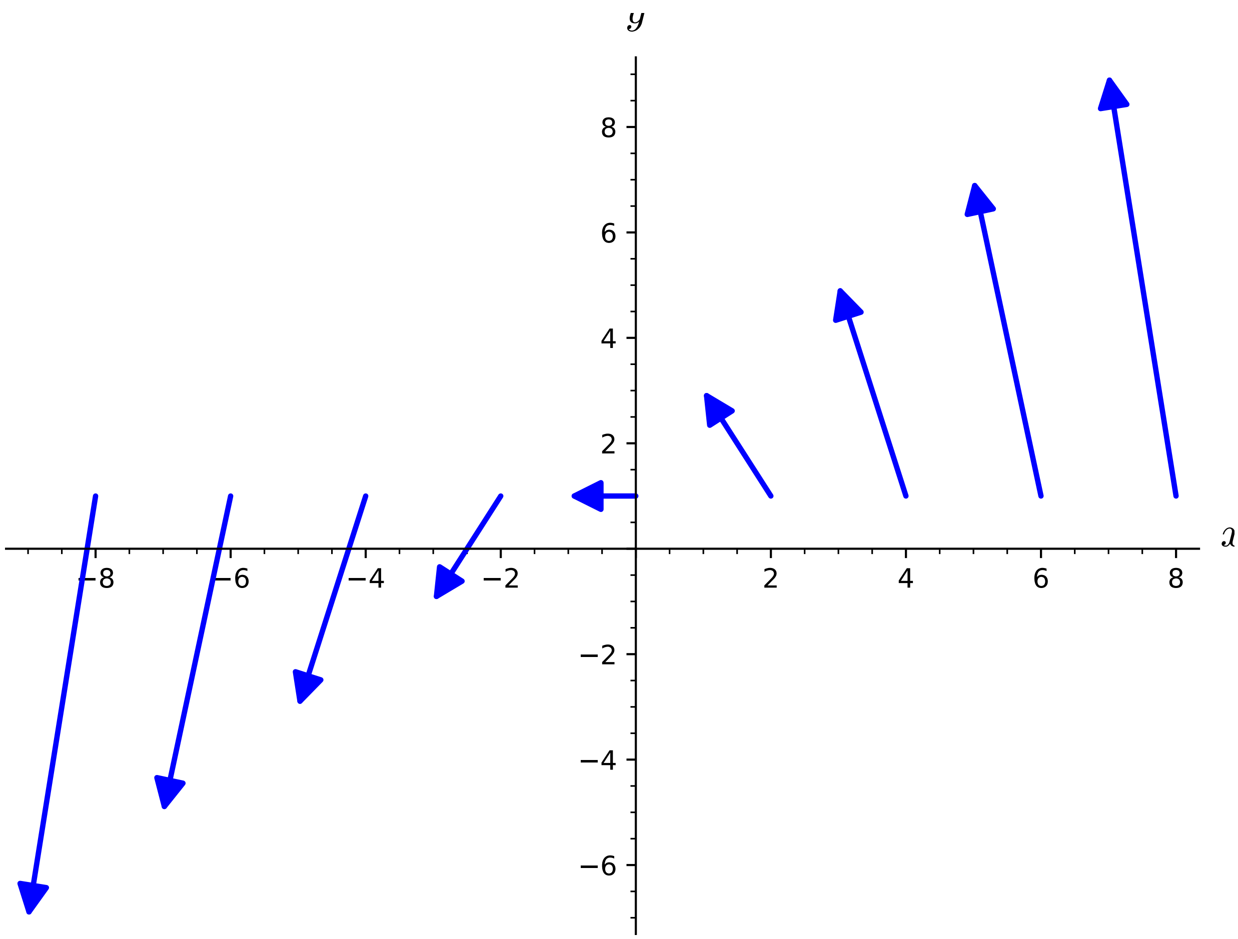
```
>>> from sage.all import *
>>> v.plot(fixed_coords={x: -Integer(2)})                                   ␣
↪     # needs sage.plot
Graphics object consisting of 9 graphics primitives
```



y
x
5
−5
−10
−10
−8
−6
−4
−2
2
4
6

```
sage: v.plot(fixed_coords={y: 1})                                       #␣
↪needs sage.plot
Graphics object consisting of 9 graphics primitives
```

```
>>> from sage.all import *
>>> v.plot(fixed_coords={y: Integer(1)})                                 ␣
↪     # needs sage.plot
Graphics object consisting of 9 graphics primitives
```



Let us now consider a vector field on a 4-dimensional manifold:

```
sage: M = Manifold(4, 'M')
sage: X.<t,x,y,z> = M.chart()
sage: v = M.vector_field((t/8)^2, -t*y/4, t*x/4, t*z/4, name='v')
sage: v.display()
v = 1/64*t^2 ∂/∂t - 1/4*t*y ∂/∂x + 1/4*t*x ∂/∂y + 1/4*t*z ∂/∂z
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> v = M.vector_field((t/Integer(8))**Integer(2), -t*y/Integer(4), t*x/
```

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```
↪Integer(4), t*z/Integer(4), name='v')
>>> v.display()
v = 1/64*t^2 ∂/∂t - 1/4*t*y ∂/∂x + 1/4*t*x ∂/∂y + 1/4*t*z ∂/∂z
```

We cannot make a 4D plot directly:

```
sage: v.plot()
Traceback (most recent call last):
...
ValueError: the number of ambient coordinates must be either 2 or 3, not 4
```

```
>>> from sage.all import *
>>> v.plot()
Traceback (most recent call last):
...
ValueError: the number of ambient coordinates must be either 2 or 3, not 4
```

Rather, we have to select some coordinates for the plot, via the argument `ambient_coords`. For instance, for a 3D plot:

```
sage: v.plot(ambient_coords=(x, y, z), fixed_coords={t: 1},          # long␣
↪time, needs sage.plot
....:        number_values=4)
Graphics3d Object
```

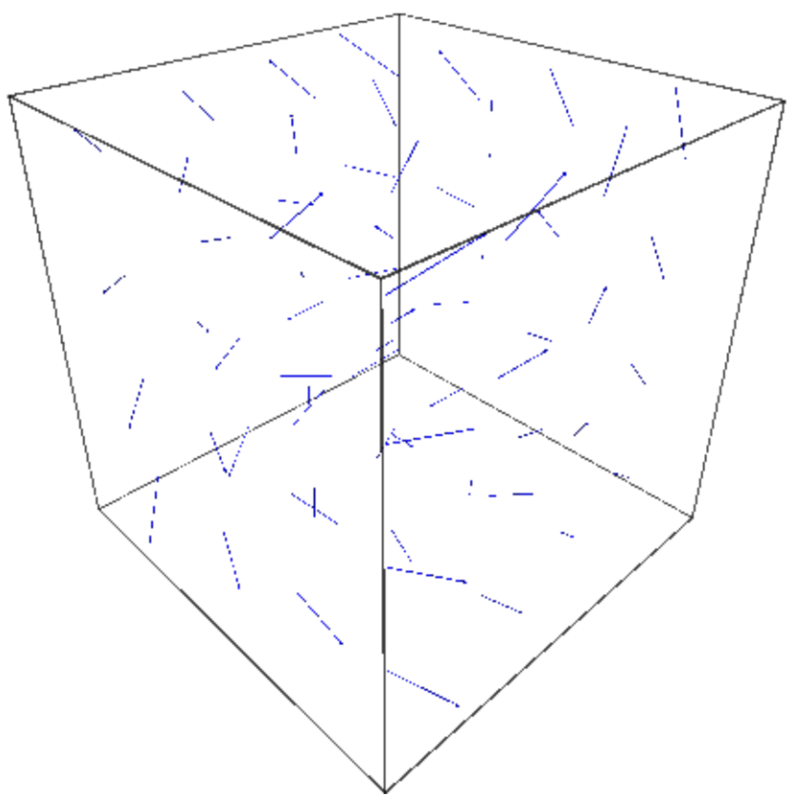
```
>>> from sage.all import *
>>> v.plot(ambient_coords=(x, y, z), fixed_coords={t: Integer(1)},          #␣
↪long time, needs sage.plot
...        number_values=Integer(4))
Graphics3d Object
```

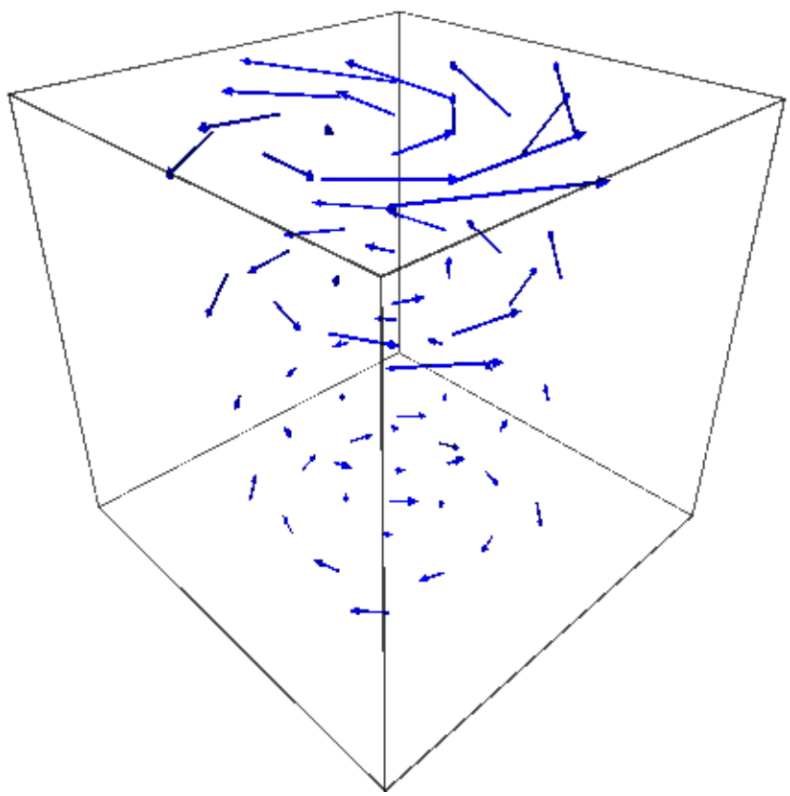
```
sage: v.plot(ambient_coords=(x, y, t), fixed_coords={z: 0},          # long␣
↪time, needs sage.plot
....:        ranges={x: (-2,2), y: (-2,2), t: (-1, 4)},
....:        number_values=4)
Graphics3d Object
```

```
>>> from sage.all import *
>>> v.plot(ambient_coords=(x, y, t), fixed_coords={z: Integer(0)},          #␣
↪long time, needs sage.plot
...        ranges={x: (-Integer(2),Integer(2)), y: (-Integer(2),Integer(2)),␣
↪t: (-Integer(1), Integer(4))},
...        number_values=Integer(4))
Graphics3d Object
```

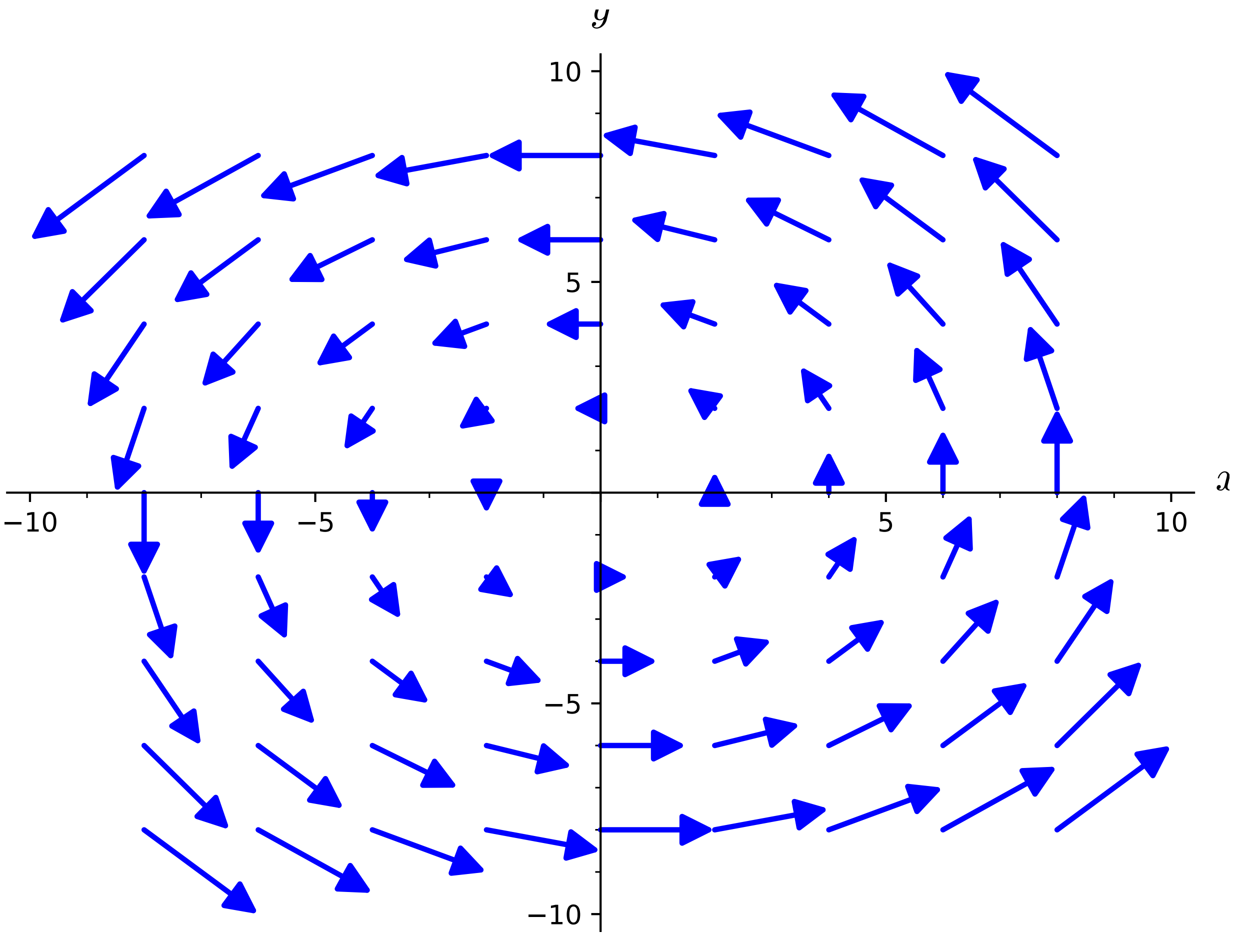
or, for a 2D plot:

```
sage: v.plot(ambient_coords=(x, y), fixed_coords={t: 1, z: 0})       # long␣
↪time, needs sage.plot
Graphics object consisting of 80 graphics primitives
```





```
>>> from sage.all import *
>>> v.plot(ambient_coords=(x, y), fixed_coords={t: Integer(1), z: Integer(0)}
↪)       # long time, needs sage.plot
Graphics object consisting of 80 graphics primitives
```




```
sage: v.plot(ambient_coords=(x, t), fixed_coords={y: 1, z: 0})       # long␣
↪time, needs sage.plot
Graphics object consisting of 72 graphics primitives
```

```
>>> from sage.all import *
>>> v.plot(ambient_coords=(x, t), fixed_coords={y: Integer(1), z: Integer(0)}
↪)       # long time, needs sage.plot
Graphics object consisting of 72 graphics primitives
```

An example of plot via a differential mapping: plot of a vector field tangent to a 2-sphere viewed in $\mathbf{R}^3$:

```
sage: S2 = Manifold(2, 'S^2')
sage: U = S2.open_subset('U') # the open set covered by spherical coord.
sage: XS.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: R3 = Manifold(3, 'R^3')
sage: X3.<x,y,z> = R3.chart()
sage: F = S2.diff_map(R3, {(XS, X3): [sin(th)*cos(ph),
```

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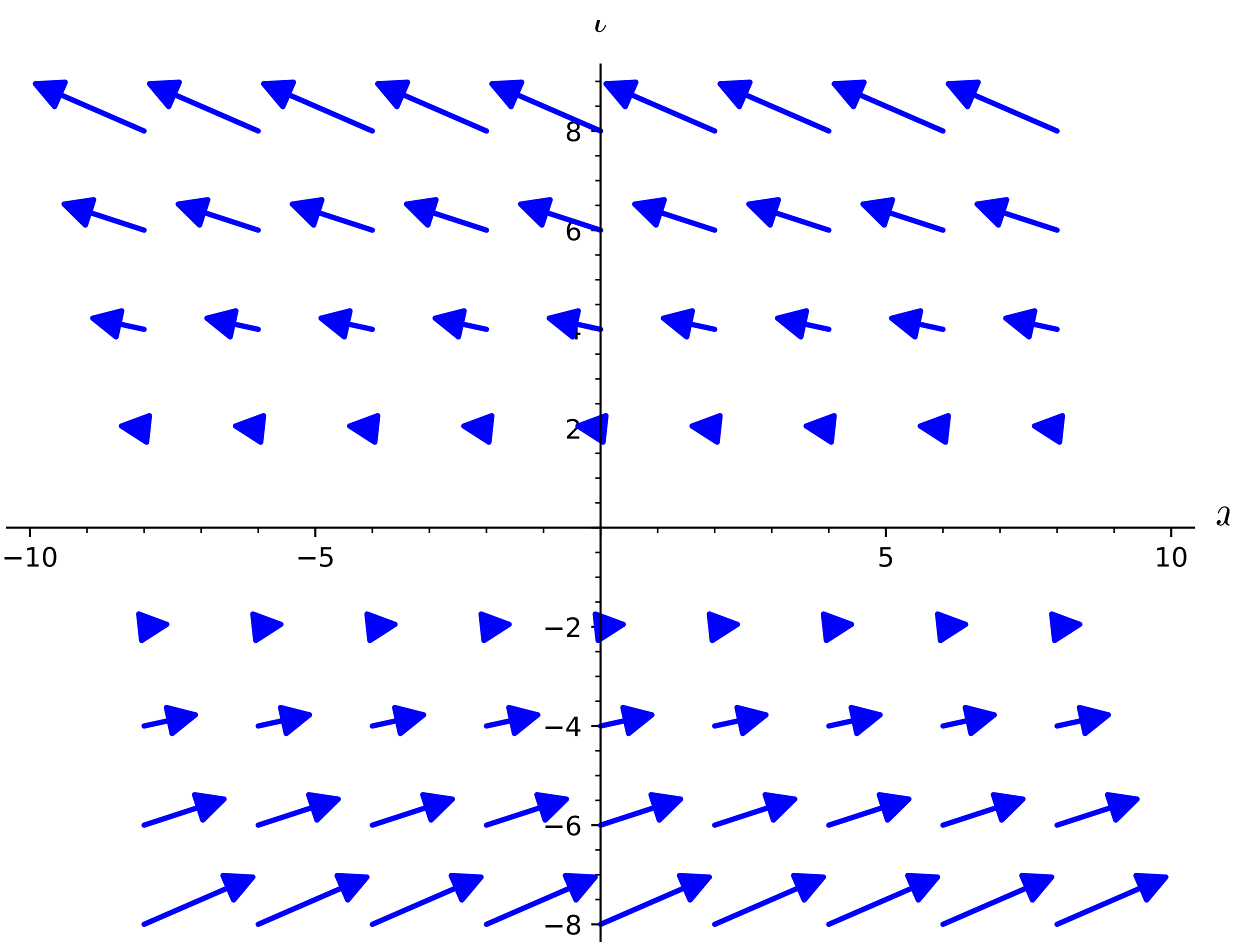
8
6
4
2
−10
−5
5
10
−2
−4
−6
−8

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```
....:                          sin(th)*sin(ph), cos(th)]}, name='F')
sage: F.display() # the standard embedding of S^2 into R^3
F: S^2 → R^3
on U: (th, ph) ↦ (x, y, z) = (cos(ph)*sin(th), sin(ph)*sin(th), cos(th))
sage: v = XS.frame()[1] ; v  # the coordinate vector ∂/∂phi
Vector field ∂/∂ph on the Open subset U of the 2-dimensional
 differentiable manifold S^2
sage: graph_v = v.plot(chart=X3, mapping=F, label_axes=False)            #␣
↪needs sage.plot
sage: graph_S2 = XS.plot(chart=X3, mapping=F, number_values=9)           #␣
↪needs sage.plot
sage: graph_v + graph_S2                                                 #␣
↪needs sage.plot
Graphics3d Object
```

```
>>> from sage.all import *
>>> S2 = Manifold(Integer(2), 'S^2')
>>> U = S2.open_subset('U') # the open set covered by spherical coord.
>>> XS = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi', names=('th', 'ph',));␣
↪(th, ph,) = XS._first_ngens(2)
>>> R3 = Manifold(Integer(3), 'R^3')
>>> X3 = R3.chart(names=('x', 'y', 'z',)); (x, y, z,) = X3._first_ngens(3)
>>> F = S2.diff_map(R3, {(XS, X3): [sin(th)*cos(ph),
...                          sin(th)*sin(ph), cos(th)]}, name='F')
>>> F.display() # the standard embedding of S^2 into R^3
F: S^2 → R^3
on U: (th, ph) ↦ (x, y, z) = (cos(ph)*sin(th), sin(ph)*sin(th), cos(th))
>>> v = XS.frame()[Integer(1)] ; v  # the coordinate vector ∂/∂phi
Vector field ∂/∂ph on the Open subset U of the 2-dimensional
 differentiable manifold S^2
>>> graph_v = v.plot(chart=X3, mapping=F, label_axes=False)            #␣
↪needs sage.plot
>>> graph_S2 = XS.plot(chart=X3, mapping=F, number_values=Integer(9))      ␣
↪     # needs sage.plot
>>> graph_v + graph_S2                                                 #␣
↪needs sage.plot
Graphics3d Object
```

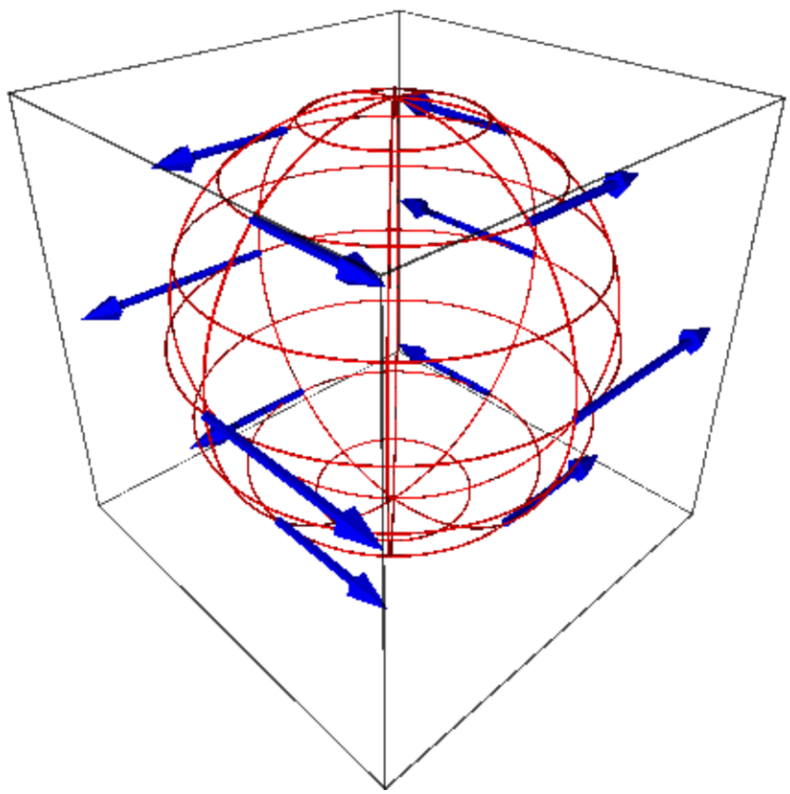
Note that the default values of some arguments of the method `plot` are stored in the dictionary `plot.options`:

```
sage: v.plot.options  # random (dictionary output)
{'color': 'blue', 'max_range': 8, 'scale': 1}
```

```
>>> from sage.all import *
>>> v.plot.options  # random (dictionary output)
{'color': 'blue', 'max_range': 8, 'scale': 1}
```

so that they can be adjusted by the user:

```
sage: v.plot.options['color'] = 'red'
```



```
>>> from sage.all import *
>>> v.plot.options['color'] = 'red'
```

From now on, all plots of vector fields will use red as the default color. To restore the original default options, it suffices to type:

```
sage: v.plot.reset()
```

```
>>> from sage.all import *
>>> v.plot.reset()
```

class sage.manifolds.differentiable.vectorfield.**VectorFieldParal**(*vector_field_module*, *name=None*, *latex_name=None*)

Bases: FiniteRankFreeModuleElement, *MultivectorFieldParal*, *VectorField*

Vector field along a differentiable manifold, with values on a parallelizable manifold.

An instance of this class is a vector field along a differentiable manifold U with values on a parallelizable manifold M, via a differentiable map $\Phi : U \to M$. More precisely, given a differentiable map

$$\Phi : U \longrightarrow M,$$

a *vector field along* U *with values on* M is a differentiable map

$$v : U \longrightarrow TM$$

(TM being the tangent bundle of M) such that

$$\forall p \in U, \ v(p) \in T_{\Phi(p)}M.$$

The standard case of vector fields *on* a differentiable manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note

If M is not parallelizable, then *VectorField* *must* be used instead.

INPUT:

- vector_field_module – free module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on $M \supset \Phi(U)$
- name – (default: None) name given to the vector field
- latex_name – (default: None) LaTeX symbol to denote the vector field; if none is provided, the LaTeX symbol is set to name

EXAMPLES:

A vector field on a parallelizable 3-dimensional manifold:

```
sage: M = Manifold(3, 'M')
sage: c_xyz.<x,y,z> = M.chart()
sage: v = M.vector_field(name='V') ; v
Vector field V on the 3-dimensional differentiable manifold M
sage: latex(v)
V
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_ngens(3)
>>> v = M.vector_field(name='V') ; v
Vector field V on the 3-dimensional differentiable manifold M
>>> latex(v)
V
```

Vector fields are considered as elements of a module over the ring (algebra) of scalar fields on M:

```
sage: v.parent()
Free module X(M) of vector fields on the 3-dimensional differentiable
 manifold M
sage: v.parent().base_ring()
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
sage: v.parent() is M.vector_field_module()
True
```

```
>>> from sage.all import *
>>> v.parent()
Free module X(M) of vector fields on the 3-dimensional differentiable
 manifold M
>>> v.parent().base_ring()
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
>>> v.parent() is M.vector_field_module()
True
```

A vector field is a tensor field of rank 1 and of type $(1, 0)$:

```
sage: v.tensor_rank()
1
sage: v.tensor_type()
(1, 0)
```

```
>>> from sage.all import *
>>> v.tensor_rank()
1
>>> v.tensor_type()
(1, 0)
```

Components of a vector field with respect to a given frame:

```
sage: e = M.vector_frame('e') ; M.set_default_frame(e)
sage: v[0], v[1], v[2] = (1+y, 4*x*z, 9)   # components on M's default frame (e)
sage: v.comp()
1-index components w.r.t. Vector frame (M, (e_0,e_1,e_2))
```

```
>>> from sage.all import *
>>> e = M.vector_frame('e') ; M.set_default_frame(e)
>>> v[Integer(0)], v[Integer(1)], v[Integer(2)] = (Integer(1)+y, Integer(4)*x*z,␣
```

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```
↪Integer(9))  # components on M's default frame (e)
>>> v.comp()
1-index components w.r.t. Vector frame (M, (e_0,e_1,e_2))
```

The totality of the components are accessed via the operator `[:]`:

```
sage: v[:] = (1+y, 4*x*z, 9)
sage: v[:]
[y + 1, 4*x*z, 9]
```

```
>>> from sage.all import *
>>> v[:] = (Integer(1)+y, Integer(4)*x*z, Integer(9))
>>> v[:]
[y + 1, 4*x*z, 9]
```

The components are also read on the expansion on the frame `e`, as provided by the method `display()`:

```
sage: v.display()  # expansion in the default frame
V = (y + 1) e_0 + 4*x*z e_1 + 9 e_2
```

```
>>> from sage.all import *
>>> v.display()  # expansion in the default frame
V = (y + 1) e_0 + 4*x*z e_1 + 9 e_2
```

A subset of the components can be accessed by using slice notation:

```
sage: v[1:] = (-2, -x*y)
sage: v[:]
[y + 1, -2, -x*y]
sage: v[:2]
[y + 1, -2]
```

```
>>> from sage.all import *
>>> v[Integer(1):] = (-Integer(2), -x*y)
>>> v[:]
[y + 1, -2, -x*y]
>>> v[:Integer(2)]
[y + 1, -2]
```

Components in another frame:

```
sage: f = M.vector_frame('f')
sage: for i in range(3):
....:     v.set_comp(f)[i] = (i+1)**3 * c_xyz[i]
sage: v.comp(f)[2]
27*z
sage: v[f, 2]  # equivalent to above
27*z
sage: v.display(f)
V = x f_0 + 8*y f_1 + 27*z f_2
```

```
>>> from sage.all import *
>>> f = M.vector_frame('f')
>>> for i in range(Integer(3)):
...     v.set_comp(f)[i] = (i+Integer(1))**Integer(3) * c_xyz[i]
>>> v.comp(f)[Integer(2)]
27*z
>>> v[f, Integer(2)]  # equivalent to above
27*z
>>> v.display(f)
V = x f_0 + 8*y f_1 + 27*z f_2
```

One can set the components at the vector definition:

```
sage: v = M.vector_field(1+y, 4*x*z, 9, name='V')
sage: v.display()
V = (y + 1) e_0 + 4*x*z e_1 + 9 e_2
```

```
>>> from sage.all import *
>>> v = M.vector_field(Integer(1)+y, Integer(4)*x*z, Integer(9), name='V')
>>> v.display()
V = (y + 1) e_0 + 4*x*z e_1 + 9 e_2
```

If the components regard a vector frame different from the default one, the vector frame has to be specified via the argument `frame`:

```
sage: v = M.vector_field(x, 8*y, 27*z, frame=f, name='V')
sage: v.display(f)
V = x f_0 + 8*y f_1 + 27*z f_2
```

```
>>> from sage.all import *
>>> v = M.vector_field(x, Integer(8)*y, Integer(27)*z, frame=f, name='V')
>>> v.display(f)
V = x f_0 + 8*y f_1 + 27*z f_2
```

For providing the components in various frames, one may use a dictionary:

```
sage: v = M.vector_field({e: [1+y, -2, -x*y], f: [x, 8*y, 27*z]},
....:                      name='V')
sage: v.display(e)
V = (y + 1) e_0 - 2 e_1 - x*y e_2
sage: v.display(f)
V = x f_0 + 8*y f_1 + 27*z f_2
```

```
>>> from sage.all import *
>>> v = M.vector_field({e: [Integer(1)+y, -Integer(2), -x*y], f: [x, Integer(8)*y,
↪ Integer(27)*z]},
...                      name='V')
>>> v.display(e)
V = (y + 1) e_0 - 2 e_1 - x*y e_2
>>> v.display(f)
V = x f_0 + 8*y f_1 + 27*z f_2
```

It is also possible to construct a vector field from a vector of symbolic expressions (or any other iterable):

```
sage: v = M.vector_field(vector([1+y, 4*x*z, 9]), name='V')
sage: v.display()
V = (y + 1) e_0 + 4*x*z e_1 + 9 e_2
```

```
>>> from sage.all import *
>>> v = M.vector_field(vector([Integer(1)+y, Integer(4)*x*z, Integer(9)]), name='V
↪')
>>> v.display()
V = (y + 1) e_0 + 4*x*z e_1 + 9 e_2
```

The range of the indices depends on the convention set for the manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: e = M.vector_frame('e') ; M.set_default_frame(e)
sage: v = M.vector_field(1+y, 4*x*z, 9, name='V')
sage: v[0]
Traceback (most recent call last):
...
IndexError: index out of range: 0 not in [1, 3]
sage: v[1]  # OK
y + 1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_ngens(3)
>>> e = M.vector_frame('e') ; M.set_default_frame(e)
>>> v = M.vector_field(Integer(1)+y, Integer(4)*x*z, Integer(9), name='V')
>>> v[Integer(0)]
Traceback (most recent call last):
...
IndexError: index out of range: 0 not in [1, 3]
>>> v[Integer(1)]  # OK
y + 1
```

A vector field acts on scalar fields (derivation along the vector field):

```
sage: M = Manifold(2, 'M')
sage: c_cart.<x,y> = M.chart()
sage: f = M.scalar_field(x*y^2, name='f')
sage: v = M.vector_field(-y, x, name='v')
sage: v.display()
v = -y ∂/∂x + x ∂/∂y
sage: v(f)
Scalar field v(f) on the 2-dimensional differentiable manifold M
sage: v(f).expr()
2*x^2*y - y^3
sage: latex(v(f))
v\left(f\right)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
```

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```
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)
>>> f = M.scalar_field(x*y**Integer(2), name='f')
>>> v = M.vector_field(-y, x, name='v')
>>> v.display()
v = -y ∂/∂x + x ∂/∂y
>>> v(f)
Scalar field v(f) on the 2-dimensional differentiable manifold M
>>> v(f).expr()
2*x^2*y - y^3
>>> latex(v(f))
v\left(f\right)
```

Example of a vector field associated with a non-trivial map Φ; a vector field along a curve in M:

```
sage: R = Manifold(1, 'R')
sage: T.<t> = R.chart()  # canonical chart on R
sage: Phi = R.diff_map(M, [cos(t), sin(t)], name='Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable manifold R
 to the 2-dimensional differentiable manifold M
sage: Phi.display()
Phi: R → M
   t ↦ (x, y) = (cos(t), sin(t))
sage: w = R.vector_field(-sin(t), cos(t), dest_map=Phi, name='w') ; w
Vector field w along the 1-dimensional differentiable manifold R with
 values on the 2-dimensional differentiable manifold M
sage: w.parent()
Free module X(R,Phi) of vector fields along the 1-dimensional
 differentiable manifold R mapped into the 2-dimensional differentiable
 manifold M
sage: w.display()
w = -sin(t) ∂/∂x + cos(t) ∂/∂y
```

```
>>> from sage.all import *
>>> R = Manifold(Integer(1), 'R')
>>> T = R.chart(names=('t',)); (t,) = T._first_ngens(1)# canonical chart on R
>>> Phi = R.diff_map(M, [cos(t), sin(t)], name='Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable manifold R
 to the 2-dimensional differentiable manifold M
>>> Phi.display()
Phi: R → M
   t ↦ (x, y) = (cos(t), sin(t))
>>> w = R.vector_field(-sin(t), cos(t), dest_map=Phi, name='w') ; w
Vector field w along the 1-dimensional differentiable manifold R with
 values on the 2-dimensional differentiable manifold M
>>> w.parent()
Free module X(R,Phi) of vector fields along the 1-dimensional
 differentiable manifold R mapped into the 2-dimensional differentiable
 manifold M
>>> w.display()
w = -sin(t) ∂/∂x + cos(t) ∂/∂y
```

Value at a given point:

```
sage: p = R((0,), name='p') ; p
Point p on the 1-dimensional differentiable manifold R
sage: w.at(p)
Tangent vector w at Point Phi(p) on the 2-dimensional differentiable
 manifold M
sage: w.at(p).display()
w = ∂/∂y
sage: w.at(p) == v.at(Phi(p))
True
```

```
>>> from sage.all import *
>>> p = R((Integer(0),), name='p') ; p
Point p on the 1-dimensional differentiable manifold R
>>> w.at(p)
Tangent vector w at Point Phi(p) on the 2-dimensional differentiable
 manifold M
>>> w.at(p).display()
w = ∂/∂y
>>> w.at(p) == v.at(Phi(p))
True
```

2.7.3 Vector Frames

The class `VectorFrame` implements vector frames on differentiable manifolds. By *vector frame*, it is meant a field e on some differentiable manifold U endowed with a differentiable map $\Phi : U \to M$ to a differentiable manifold M such that for each $p \in U$, $e(p)$ is a vector basis of the tangent space $T_{\Phi(p)}M$.

The standard case of a vector frame *on* U corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

A derived class of `VectorFrame` is `CoordFrame`; it regards the vector frames associated with a chart, i.e. the so-called *coordinate bases*.

The vector frame duals, i.e. the coframes, are implemented via the class `CoFrame`. The derived class `CoordCoFrame` is devoted to coframes deriving from a chart.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015): initial version
- Travis Scrimshaw (2016): review tweaks
- Eric Gourgoulhon (2018): some refactoring and more functionalities in the choice of symbols for vector frame elements (Issue #24792)

REFERENCES:

- [Lee2013]

EXAMPLES:

Introducing a chart on a manifold automatically endows it with a vector frame: the coordinate frame associated to the chart:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: M.frames()
[Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z))]
```

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```
sage: M.frames()[0] is X.frame()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> M.frames()
[Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z))]
>>> M.frames()[Integer(0)] is X.frame()
True
```

A new vector frame can be defined from a family of 3 linearly independent vector fields:

```
sage: e1 = M.vector_field(1, x, y)
sage: e2 = M.vector_field(z, -2, x*y)
sage: e3 = M.vector_field(1, 1, 0)
sage: e = M.vector_frame('e', (e1, e2, e3)); e
Vector frame (M, (e_0,e_1,e_2))
sage: latex(e)
\left(M, \left(e_{0},e_{1},e_{2}\right)\right)
```

```
>>> from sage.all import *
>>> e1 = M.vector_field(Integer(1), x, y)
>>> e2 = M.vector_field(z, -Integer(2), x*y)
>>> e3 = M.vector_field(Integer(1), Integer(1), Integer(0))
>>> e = M.vector_frame('e', (e1, e2, e3)); e
Vector frame (M, (e_0,e_1,e_2))
>>> latex(e)
\left(M, \left(e_{0},e_{1},e_{2}\right)\right)
```

The first frame defined on a manifold is its *default frame*; in the present case it is the coordinate frame associated to the chart `X`:

```
sage: M.default_frame()
Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z))
```

```
>>> from sage.all import *
>>> M.default_frame()
Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z))
```

The default frame can be changed via the method *`set_default_frame()`*:

```
sage: M.set_default_frame(e)
sage: M.default_frame()
Vector frame (M, (e_0,e_1,e_2))
```

```
>>> from sage.all import *
>>> M.set_default_frame(e)
>>> M.default_frame()
Vector frame (M, (e_0,e_1,e_2))
```

The elements of a vector frame are vector fields on the manifold:

```
sage: for vec in e:
....:     print(vec)
....:
Vector field e_0 on the 3-dimensional differentiable manifold M
Vector field e_1 on the 3-dimensional differentiable manifold M
Vector field e_2 on the 3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> for vec in e:
...     print(vec)
....:
Vector field e_0 on the 3-dimensional differentiable manifold M
Vector field e_1 on the 3-dimensional differentiable manifold M
Vector field e_2 on the 3-dimensional differentiable manifold M
```

Each element of a vector frame can be accessed by its index:

```
sage: e[0]
Vector field e_0 on the 3-dimensional differentiable manifold M
sage: e[0].display(X.frame())
e_0 = ∂/∂x + x ∂/∂y + y ∂/∂z
sage: X.frame()[1]
Vector field ∂/∂y on the 3-dimensional differentiable manifold M
sage: X.frame()[1].display(e)
∂/∂y = x/(x^2 - x + z + 2) e_0 - 1/(x^2 - x + z + 2) e_1
 - (x - z)/(x^2 - x + z + 2) e_2
```

```
>>> from sage.all import *
>>> e[Integer(0)]
Vector field e_0 on the 3-dimensional differentiable manifold M
>>> e[Integer(0)].display(X.frame())
e_0 = ∂/∂x + x ∂/∂y + y ∂/∂z
>>> X.frame()[Integer(1)]
Vector field ∂/∂y on the 3-dimensional differentiable manifold M
>>> X.frame()[Integer(1)].display(e)
∂/∂y = x/(x^2 - x + z + 2) e_0 - 1/(x^2 - x + z + 2) e_1
 - (x - z)/(x^2 - x + z + 2) e_2
```

The slice operator `:` can be used to access to more than one element:

```
sage: e[0:2]
(Vector field e_0 on the 3-dimensional differentiable manifold M,
 Vector field e_1 on the 3-dimensional differentiable manifold M)
sage: e[:]
(Vector field e_0 on the 3-dimensional differentiable manifold M,
 Vector field e_1 on the 3-dimensional differentiable manifold M,
 Vector field e_2 on the 3-dimensional differentiable manifold M)
```

```
>>> from sage.all import *
>>> e[Integer(0):Integer(2)]
(Vector field e_0 on the 3-dimensional differentiable manifold M,
 Vector field e_1 on the 3-dimensional differentiable manifold M)
```

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```
>>> e[:]
(Vector field e_0 on the 3-dimensional differentiable manifold M,
 Vector field e_1 on the 3-dimensional differentiable manifold M,
 Vector field e_2 on the 3-dimensional differentiable manifold M)
```

Vector frames can be constructed from scratch, without any connection to previously defined frames or vector fields (the connection can be performed later via the method *set_change_of_frame()*):

```
sage: f = M.vector_frame('f'); f
Vector frame (M, (f_0,f_1,f_2))
sage: M.frames()
[Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z)),
 Vector frame (M, (e_0,e_1,e_2)),
 Vector frame (M, (f_0,f_1,f_2))]
```

```
>>> from sage.all import *
>>> f = M.vector_frame('f'); f
Vector frame (M, (f_0,f_1,f_2))
>>> M.frames()
[Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z)),
 Vector frame (M, (e_0,e_1,e_2)),
 Vector frame (M, (f_0,f_1,f_2))]
```

The index range depends on the starting index defined on the manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: e = M.vector_frame('e')
sage: [e[i] for i in M.irange()]
[Vector field e_1 on the 3-dimensional differentiable manifold M,
 Vector field e_2 on the 3-dimensional differentiable manifold M,
 Vector field e_3 on the 3-dimensional differentiable manifold M]
sage: e[1], e[2], e[3]
(Vector field e_1 on the 3-dimensional differentiable manifold M,
 Vector field e_2 on the 3-dimensional differentiable manifold M,
 Vector field e_3 on the 3-dimensional differentiable manifold M)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> e = M.vector_frame('e')
>>> [e[i] for i in M.irange()]
[Vector field e_1 on the 3-dimensional differentiable manifold M,
 Vector field e_2 on the 3-dimensional differentiable manifold M,
 Vector field e_3 on the 3-dimensional differentiable manifold M]
>>> e[Integer(1)], e[Integer(2)], e[Integer(3)]
(Vector field e_1 on the 3-dimensional differentiable manifold M,
 Vector field e_2 on the 3-dimensional differentiable manifold M,
 Vector field e_3 on the 3-dimensional differentiable manifold M)
```

Let us check that the vector fields `e[i]` are the frame vectors from their components with respect to the frame e:

```
sage: e[1].comp(e)[:]
[1, 0, 0]
sage: e[2].comp(e)[:]
[0, 1, 0]
sage: e[3].comp(e)[:]
[0, 0, 1]
```

```
>>> from sage.all import *
>>> e[Integer(1)].comp(e)[:]
[1, 0, 0]
>>> e[Integer(2)].comp(e)[:]
[0, 1, 0]
>>> e[Integer(3)].comp(e)[:]
[0, 0, 1]
```

Defining a vector frame on a manifold automatically creates the dual coframe, which, by default, bares the same name (here e):

```
sage: M.coframes()
[Coordinate coframe (M, (dx,dy,dz)), Coframe (M, (e^1,e^2,e^3))]
sage: f = M.coframes()[1] ; f
Coframe (M, (e^1,e^2,e^3))
sage: f is e.coframe()
True
```

```
>>> from sage.all import *
>>> M.coframes()
[Coordinate coframe (M, (dx,dy,dz)), Coframe (M, (e^1,e^2,e^3))]
>>> f = M.coframes()[Integer(1)] ; f
Coframe (M, (e^1,e^2,e^3))
>>> f is e.coframe()
True
```

Each element of the coframe is a 1-form:

```
sage: f[1], f[2], f[3]
(1-form e^1 on the 3-dimensional differentiable manifold M,
1-form e^2 on the 3-dimensional differentiable manifold M,
1-form e^3 on the 3-dimensional differentiable manifold M)
sage: latex(f[1]), latex(f[2]), latex(f[3])
(e^{1}, e^{2}, e^{3})
```

```
>>> from sage.all import *
>>> f[Integer(1)], f[Integer(2)], f[Integer(3)]
(1-form e^1 on the 3-dimensional differentiable manifold M,
1-form e^2 on the 3-dimensional differentiable manifold M,
1-form e^3 on the 3-dimensional differentiable manifold M)
>>> latex(f[Integer(1)]), latex(f[Integer(2)]), latex(f[Integer(3)])
(e^{1}, e^{2}, e^{3})
```

Let us check that the coframe (e^i) is indeed the dual of the vector frame (e_i):

```
sage: f[1](e[1]) # the 1-form e^1 applied to the vector field e_1
Scalar field e^1(e_1) on the 3-dimensional differentiable manifold M
sage: f[1](e[1]).expr() # the explicit expression of e^1(e_1)
1
sage: f[1](e[1]).expr(), f[1](e[2]).expr(), f[1](e[3]).expr()
(1, 0, 0)
sage: f[2](e[1]).expr(), f[2](e[2]).expr(), f[2](e[3]).expr()
(0, 1, 0)
sage: f[3](e[1]).expr(), f[3](e[2]).expr(), f[3](e[3]).expr()
(0, 0, 1)
```

```
>>> from sage.all import *
>>> f[Integer(1)](e[Integer(1)]) # the 1-form e^1 applied to the vector field e_1
Scalar field e^1(e_1) on the 3-dimensional differentiable manifold M
>>> f[Integer(1)](e[Integer(1)]).expr() # the explicit expression of e^1(e_1)
1
>>> f[Integer(1)](e[Integer(1)]).expr(), f[Integer(1)](e[Integer(2)]).expr(),␣
↪f[Integer(1)](e[Integer(3)]).expr()
(1, 0, 0)
>>> f[Integer(2)](e[Integer(1)]).expr(), f[Integer(2)](e[Integer(2)]).expr(),␣
↪f[Integer(2)](e[Integer(3)]).expr()
(0, 1, 0)
>>> f[Integer(3)](e[Integer(1)]).expr(), f[Integer(3)](e[Integer(2)]).expr(),␣
↪f[Integer(3)](e[Integer(3)]).expr()
(0, 0, 1)
```

The coordinate frame associated to spherical coordinates of the sphere S^2:

```
sage: M = Manifold(2, 'S^2', start_index=1) # Part of S^2 covered by spherical coord.
sage: c_spher.<th,ph> = M.chart(r'th:[0,pi]:\theta ph:[0,2*pi):\phi')
sage: b = M.default_frame() ; b
Coordinate frame (S^2, (∂/∂th,∂/∂ph))
sage: b[1]
Vector field ∂/∂th on the 2-dimensional differentiable manifold S^2
sage: b[2]
Vector field ∂/∂ph on the 2-dimensional differentiable manifold S^2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', start_index=Integer(1)) # Part of S^2 covered by␣
↪spherical coord.
>>> c_spher = M.chart(r'th:[0,pi]:\theta ph:[0,2*pi):\phi', names=('th', 'ph',)); (th,
↪ ph,) = c_spher._first_ngens(2)
>>> b = M.default_frame() ; b
Coordinate frame (S^2, (∂/∂th,∂/∂ph))
>>> b[Integer(1)]
Vector field ∂/∂th on the 2-dimensional differentiable manifold S^2
>>> b[Integer(2)]
Vector field ∂/∂ph on the 2-dimensional differentiable manifold S^2
```

The orthonormal frame constructed from the coordinate frame:

```
sage: e = M.vector_frame('e', (b[1], b[2]/sin(th))); e
Vector frame (S^2, (e_1,e_2))
```

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```
sage: e[1].display()
e_1 = ∂/∂th
sage: e[2].display()
e_2 = 1/sin(th) ∂/∂ph
```

```
>>> from sage.all import *
>>> e = M.vector_frame('e', (b[Integer(1)], b[Integer(2)]/sin(th))); e
Vector frame (S^2, (e_1,e_2))
>>> e[Integer(1)].display()
e_1 = ∂/∂th
>>> e[Integer(2)].display()
e_2 = 1/sin(th) ∂/∂ph
```

The change-of-frame automorphisms and their matrices:

```
sage: M.change_of_frame(c_spher.frame(), e)
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold S^2
sage: M.change_of_frame(c_spher.frame(), e)[:]
[        1         0]
[        0 1/sin(th)]
sage: M.change_of_frame(e, c_spher.frame())
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold S^2
sage: M.change_of_frame(e, c_spher.frame())[:]
[      1       0]
[      0 sin(th)]
```

```
>>> from sage.all import *
>>> M.change_of_frame(c_spher.frame(), e)
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold S^2
>>> M.change_of_frame(c_spher.frame(), e)[:]
[        1         0]
[        0 1/sin(th)]
>>> M.change_of_frame(e, c_spher.frame())
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold S^2
>>> M.change_of_frame(e, c_spher.frame())[:]
[      1       0]
[      0 sin(th)]
```

class sage.manifolds.differentiable.vectorframe.**CoFrame**(*frame*, *symbol*, *latex_symbol=None*, *indices=None*, *latex_indices=None*)

Bases: FreeModuleCoBasis

Coframe on a differentiable manifold.

By *coframe*, it is meant a field f on some differentiable manifold U endowed with a differentiable map $\Phi : U \to M$ to a differentiable manifold M such that for each $p \in U$, $f(p)$ is a basis of the vector space $T^*_{\Phi(p)}M$ (the dual to the tangent space $T_{\Phi(p)}M$).

The standard case of a coframe *on* U corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

INPUT:

- `frame` – the vector frame dual to the coframe
- `symbol` – either a string, to be used as a common base for the symbols of the 1-forms constituting the coframe, or a tuple of strings, representing the individual symbols of the 1-forms
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the 1-forms constituting the coframe, or a tuple of strings, representing the individual LaTeX symbols of the 1-forms; if `None`, `symbol` is used in place of `latex_symbol`
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the 1-forms of the coframe; if `None`, the indices will be generated as integers within the range declared on the coframe's domain
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the 1-forms of the coframe; if `None`, `indices` is used instead

EXAMPLES:

Coframe on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: v = M.vector_frame('v')
sage: from sage.manifolds.differentiable.vectorframe import CoFrame
sage: e = CoFrame(v, 'e') ; e
Coframe (M, (e^1,e^2,e^3))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> v = M.vector_frame('v')
>>> from sage.manifolds.differentiable.vectorframe import CoFrame
>>> e = CoFrame(v, 'e') ; e
Coframe (M, (e^1,e^2,e^3))
```

Instead of importing CoFrame in the global namespace, the coframe can be obtained by means of the method `dual_basis()`; the symbol is then the same as that of the frame:

```
sage: a = v.dual_basis() ; a
Coframe (M, (v^1,v^2,v^3))
sage: a[1] == e[1]
True
sage: a[1] is e[1]
False
sage: e[1].display(v)
e^1 = v^1
```

```
>>> from sage.all import *
>>> a = v.dual_basis() ; a
Coframe (M, (v^1,v^2,v^3))
>>> a[Integer(1)] == e[Integer(1)]
True
>>> a[Integer(1)] is e[Integer(1)]
False
```

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```
>>> e[Integer(1)].display(v)
e^1 = v^1
```

The 1-forms composing the coframe are obtained via the operator `[]`:

```
sage: e[1], e[2], e[3]
(1-form e^1 on the 3-dimensional differentiable manifold M,
 1-form e^2 on the 3-dimensional differentiable manifold M,
 1-form e^3 on the 3-dimensional differentiable manifold M)
```

```
>>> from sage.all import *
>>> e[Integer(1)], e[Integer(2)], e[Integer(3)]
(1-form e^1 on the 3-dimensional differentiable manifold M,
 1-form e^2 on the 3-dimensional differentiable manifold M,
 1-form e^3 on the 3-dimensional differentiable manifold M)
```

Checking that e is the dual of v:

```
sage: e[1](v[1]).expr(), e[1](v[2]).expr(), e[1](v[3]).expr()
(1, 0, 0)
sage: e[2](v[1]).expr(), e[2](v[2]).expr(), e[2](v[3]).expr()
(0, 1, 0)
sage: e[3](v[1]).expr(), e[3](v[2]).expr(), e[3](v[3]).expr()
(0, 0, 1)
```

```
>>> from sage.all import *
>>> e[Integer(1)](v[Integer(1)]).expr(), e[Integer(1)](v[Integer(2)]).expr(),␣
↪e[Integer(1)](v[Integer(3)]).expr()
(1, 0, 0)
>>> e[Integer(2)](v[Integer(1)]).expr(), e[Integer(2)](v[Integer(2)]).expr(),␣
↪e[Integer(2)](v[Integer(3)]).expr()
(0, 1, 0)
>>> e[Integer(3)](v[Integer(1)]).expr(), e[Integer(3)](v[Integer(2)]).expr(),␣
↪e[Integer(3)](v[Integer(3)]).expr()
(0, 0, 1)
```

at(*point*)

Return the value of `self` at a given point on the manifold, this value being a basis of the dual of the tangent space at the point.

INPUT:

- `point` – *ManifoldPoint*; point p in the domain U of the coframe (denoted f hereafter)

OUTPUT:

- `FreeModuleCoBasis` representing the basis $f(p)$ of the vector space $T^*_{\Phi(p)}M$, dual to the tangent space $T_{\Phi(p)}M$, where $\Phi : U \to M$ is the differentiable map associated with f (possibly $\Phi = \mathrm{Id}_U$)

EXAMPLES:

Cobasis of a tangent space on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
```

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```
sage: p = M.point((-1,2), name='p')
sage: f = X.coframe() ; f
Coordinate coframe (M, (dx,dy))
sage: fp = f.at(p) ; fp
Dual basis (dx,dy) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
sage: type(fp)
<class 'sage.tensor.modules.free_module_basis.FreeModuleCoBasis_with_category
↪'>
sage: fp[0]
Linear form dx on the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
sage: fp[1]
Linear form dy on the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
sage: fp is X.frame().at(p).dual_basis()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> p = M.point((-Integer(1),Integer(2)), name='p')
>>> f = X.coframe() ; f
Coordinate coframe (M, (dx,dy))
>>> fp = f.at(p) ; fp
Dual basis (dx,dy) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
>>> type(fp)
<class 'sage.tensor.modules.free_module_basis.FreeModuleCoBasis_with_category
↪'>
>>> fp[Integer(0)]
Linear form dx on the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
>>> fp[Integer(1)]
Linear form dy on the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
>>> fp is X.frame().at(p).dual_basis()
True
```

set_name(*symbol*, *latex_symbol=None*, *indices=None*, *latex_indices=None*, *index_position='up'*, *include_domain=True*)

Set (or change) the text name and LaTeX name of `self`.

INPUT:

- `symbol` – either a string, to be used as a common base for the symbols of the 1-forms constituting the coframe, or a list/tuple of strings, representing the individual symbols of the 1-forms
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the 1-forms constituting the coframe, or a list/tuple of strings, representing the individual LaTeX symbols of the 1-forms; if `None`, `symbol` is used in place of `latex_symbol`
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the 1-forms of the coframe; if `None`, the indices will be generated as integers within the

range declared on `self`

- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the 1-forms; if `None`, `indices` is used instead
- `index_position` – (default: `'up'`) determines the position of the indices labelling the 1-forms of the coframe; can be either `'down'` or `'up'`
- `include_domain` – boolean (default: `True`); determining whether the name of the domain is included in the beginning of the coframe name

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: e = M.vector_frame('e').coframe(); e
Coframe (M, (e^0,e^1))
sage: e.set_name('f'); e
Coframe (M, (f^0,f^1))
sage: e.set_name('e', latex_symbol=r'\epsilon')
sage: latex(e)
\left(M, \left(\epsilon^{0},\epsilon^{1}\right)\right)
sage: e.set_name('e', include_domain=False); e
Coframe (e^0,e^1)
sage: e.set_name(['a', 'b'], latex_symbol=[r'\alpha', r'\beta']); e
Coframe (M, (a,b))
sage: latex(e)
\left(M, \left(\alpha,\beta\right)\right)
sage: e.set_name('e', indices=['x','y'],
....:            latex_indices=[r'\xi', r'\zeta']); e
Coframe (M, (e^x,e^y))
sage: latex(e)
\left(M, \left(e^{\xi},e^{\zeta}\right)\right)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> e = M.vector_frame('e').coframe(); e
Coframe (M, (e^0,e^1))
>>> e.set_name('f'); e
Coframe (M, (f^0,f^1))
>>> e.set_name('e', latex_symbol=r'\epsilon')
>>> latex(e)
\left(M, \left(\epsilon^{0},\epsilon^{1}\right)\right)
>>> e.set_name('e', include_domain=False); e
Coframe (e^0,e^1)
>>> e.set_name(['a', 'b'], latex_symbol=[r'\alpha', r'\beta']); e
Coframe (M, (a,b))
>>> latex(e)
\left(M, \left(\alpha,\beta\right)\right)
>>> e.set_name('e', indices=['x','y'],
...            latex_indices=[r'\xi', r'\zeta']); e
Coframe (M, (e^x,e^y))
>>> latex(e)
\left(M, \left(e^{\xi},e^{\zeta}\right)\right)
```

class `sage.manifolds.differentiable.vectorframe.`**`CoordCoFrame`**(*coord_frame*, *symbol*, *latex_symbol=None*, *indices=None*, *latex_indices=None*)

Bases: *CoFrame*

Coordinate coframe on a differentiable manifold.

By *coordinate coframe*, it is meant the n-tuple of the differentials of the coordinates of some chart on the manifold, with n being the manifold's dimension.

INPUT:

- `coord_frame` – coordinate frame dual to the coordinate coframe
- `symbol` – either a string, to be used as a common base for the symbols of the 1-forms constituting the coframe, or a tuple of strings, representing the individual symbols of the 1-forms
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the 1-forms constituting the coframe, or a tuple of strings, representing the individual LaTeX symbols of the 1-forms; if `None`, `symbol` is used in place of `latex_symbol`
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the 1-forms of the coframe; if `None`, the indices will be generated as integers within the range declared on the vector frame's domain
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the 1-forms of the coframe; if `None`, `indices` is used instead

EXAMPLES:

Coordinate coframe on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: M.frames()
[Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z))]
sage: M.coframes()
[Coordinate coframe (M, (dx,dy,dz))]
sage: dX = M.coframes()[0] ; dX
Coordinate coframe (M, (dx,dy,dz))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> M.frames()
[Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z))]
>>> M.coframes()
[Coordinate coframe (M, (dx,dy,dz))]
>>> dX = M.coframes()[Integer(0)] ; dX
Coordinate coframe (M, (dx,dy,dz))
```

The 1-forms composing the coframe are obtained via the operator `[]`:

```
sage: dX[1]
1-form dx on the 3-dimensional differentiable manifold M
sage: dX[2]
1-form dy on the 3-dimensional differentiable manifold M
sage: dX[3]
1-form dz on the 3-dimensional differentiable manifold M
sage: dX[1][:]
```

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```
[1, 0, 0]
sage: dX[2][:]
[0, 1, 0]
sage: dX[3][:]
[0, 0, 1]
```

```
>>> from sage.all import *
>>> dX[Integer(1)]
1-form dx on the 3-dimensional differentiable manifold M
>>> dX[Integer(2)]
1-form dy on the 3-dimensional differentiable manifold M
>>> dX[Integer(3)]
1-form dz on the 3-dimensional differentiable manifold M
>>> dX[Integer(1)][:]
[1, 0, 0]
>>> dX[Integer(2)][:]
[0, 1, 0]
>>> dX[Integer(3)][:]
[0, 0, 1]
```

The coframe is the dual of the coordinate frame:

```
sage: e = X.frame() ; e
Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z))
sage: dX[1](e[1]).expr(), dX[1](e[2]).expr(), dX[1](e[3]).expr()
(1, 0, 0)
sage: dX[2](e[1]).expr(), dX[2](e[2]).expr(), dX[2](e[3]).expr()
(0, 1, 0)
sage: dX[3](e[1]).expr(), dX[3](e[2]).expr(), dX[3](e[3]).expr()
(0, 0, 1)
```

```
>>> from sage.all import *
>>> e = X.frame() ; e
Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z))
>>> dX[Integer(1)](e[Integer(1)]).expr(), dX[Integer(1)](e[Integer(2)]).expr(),␣
↪dX[Integer(1)](e[Integer(3)]).expr()
(1, 0, 0)
>>> dX[Integer(2)](e[Integer(1)]).expr(), dX[Integer(2)](e[Integer(2)]).expr(),␣
↪dX[Integer(2)](e[Integer(3)]).expr()
(0, 1, 0)
>>> dX[Integer(3)](e[Integer(1)]).expr(), dX[Integer(3)](e[Integer(2)]).expr(),␣
↪dX[Integer(3)](e[Integer(3)]).expr()
(0, 0, 1)
```

Each 1-form of a coordinate coframe is closed:

```
sage: dX[1].exterior_derivative()
2-form ddx on the 3-dimensional differentiable manifold M
sage: dX[1].exterior_derivative() == 0
True
```

```
>>> from sage.all import *
>>> dX[Integer(1)].exterior_derivative()
2-form ddx on the 3-dimensional differentiable manifold M
>>> dX[Integer(1)].exterior_derivative() == Integer(0)
True
```

class sage.manifolds.differentiable.vectorframe.**CoordFrame**(*chart*)

Bases: *VectorFrame*

Coordinate frame on a differentiable manifold.

By *coordinate frame*, it is meant a vector frame on a differentiable manifold M that is associated to a coordinate chart on M.

INPUT:

- chart – the chart defining the coordinates

EXAMPLES:

The coordinate frame associated to spherical coordinates of the sphere S^2:

```
sage: M = Manifold(2, 'S^2', start_index=1)  # Part of S^2 covered by spherical␣
↪coord.
sage: M.chart(r'th:[0,pi]:\theta ph:[0,2*pi]:\phi')
Chart (S^2, (th, ph))
sage: b = M.default_frame()
sage: b
Coordinate frame (S^2, (∂/∂th,∂/∂ph))
sage: b[1]
Vector field ∂/∂th on the 2-dimensional differentiable manifold S^2
sage: b[2]
Vector field ∂/∂ph on the 2-dimensional differentiable manifold S^2
sage: latex(b)
\left(S^2, \left(\frac{\partial}{\partial {\theta} },\frac{\partial}{\partial {\
↪phi} }\right)\right)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', start_index=Integer(1))  # Part of S^2␣
↪covered by spherical coord.
>>> M.chart(r'th:[0,pi]:\theta ph:[0,2*pi]:\phi')
Chart (S^2, (th, ph))
>>> b = M.default_frame()
>>> b
Coordinate frame (S^2, (∂/∂th,∂/∂ph))
>>> b[Integer(1)]
Vector field ∂/∂th on the 2-dimensional differentiable manifold S^2
>>> b[Integer(2)]
Vector field ∂/∂ph on the 2-dimensional differentiable manifold S^2
>>> latex(b)
\left(S^2, \left(\frac{\partial}{\partial {\theta} },\frac{\partial}{\partial {\
↪phi} }\right)\right)
```

chart()

Return the chart defining this coordinate frame.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: e = X.frame()
sage: e.chart()
Chart (M, (x, y))
sage: U = M.open_subset('U', coord_def={X: x>0})
sage: e.restrict(U).chart()
Chart (U, (x, y))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> e = X.frame()
>>> e.chart()
Chart (M, (x, y))
>>> U = M.open_subset('U', coord_def={X: x>Integer(0)})
>>> e.restrict(U).chart()
Chart (U, (x, y))
```

structure_coeff()

Return the structure coefficients associated to `self`.

n being the manifold's dimension, the structure coefficients of the frame (e_i) are the n^3 scalar fields $C^k_{\ ij}$ defined by

$$[e_i, e_j] = C^k_{\ ij} e_k.$$

In the present case, since (e_i) is a coordinate frame, $C^k_{\ ij} = 0$.

OUTPUT:

- the structure coefficients $C^k_{\ ij}$, as a vanishing instance of `CompWithSym` with 3 indices ordered as (k, i, j)

EXAMPLES:

Structure coefficients of the coordinate frame associated to spherical coordinates in the Euclidean space $\mathbf{R}^3$:

```
sage: M = Manifold(3, 'R^3', r'\RR^3', start_index=1)  # Part of R^3 covered␣
↪by spherical coord.
sage: c_spher = M.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: b = M.default_frame() ; b
Coordinate frame (R^3, (∂/∂r,∂/∂th,∂/∂ph))
sage: c = b.structure_coeff() ; c
3-indices components w.r.t. Coordinate frame
 (R^3, (∂/∂r,∂/∂th,∂/∂ph)), with antisymmetry on the index
 positions (1, 2)
sage: c == 0
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'R^3', r'\RR^3', start_index=Integer(1))  # Part␣
↪of R^3 covered by spherical coord.
>>> c_spher = M.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi')
>>> b = M.default_frame() ; b
```

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```
Coordinate frame (R^3, (∂/∂r,∂/∂th,∂/∂ph))
>>> c = b.structure_coeff() ; c
3-indices components w.r.t. Coordinate frame
 (R^3, (∂/∂r,∂/∂th,∂/∂ph)), with antisymmetry on the index
 positions (1, 2)
>>> c == Integer(0)
True
```

class sage.manifolds.differentiable.vectorframe.**VectorFrame**(*vector_field_module*, *symbol*, *latex_symbol=None*, *from_frame=None*, *indices=None*, *latex_indices=None*, *symbol_dual=None*, *latex_symbol_dual=None*)

Bases: `FreeModuleBasis`

Vector frame on a differentiable manifold.

By *vector frame*, it is meant a field e on some differentiable manifold U endowed with a differentiable map $\Phi : U \to M$ to a differentiable manifold M such that for each $p \in U$, $e(p)$ is a vector basis of the tangent space $T_{\Phi(p)}M$.

The standard case of a vector frame *on* U corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

For each instantiation of a vector frame, a coframe is automatically created, as an instance of the class `CoFrame`. It is returned by the method `coframe()`.

INPUT:

- `vector_field_module` – free module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on $M \supset \Phi(U)$
- `symbol` – either a string, to be used as a common base for the symbols of the vector fields constituting the vector frame, or a tuple of strings, representing the individual symbols of the vector fields
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the vector fields constituting the vector frame, or a tuple of strings, representing the individual LaTeX symbols of the vector fields; if `None`, `symbol` is used in place of `latex_symbol`
- `from_frame` – (default: `None`) vector frame $\tilde{e}$ on the codomain M of the destination map Φ; the constructed frame e is then such that $\forall p \in U, e(p) = \tilde{e}(\Phi(p))$
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the vector fields of the frame; if `None`, the indices will be generated as integers within the range declared on the vector frame's domain
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the vector fields; if `None`, `indices` is used instead
- `symbol_dual` – (default: `None`) same as `symbol` but for the dual coframe; if `None`, `symbol` must be a string and is used for the common base of the symbols of the elements of the dual coframe
- `latex_symbol_dual` – (default: `None`) same as `latex_symbol` but for the dual coframe

EXAMPLES:

Defining a vector frame on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: e = M.vector_frame('e') ; e
Vector frame (M, (e_1,e_2,e_3))
sage: latex(e)
\left(M, \left(e_{1},e_{2},e_{3}\right)\right)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> e = M.vector_frame('e') ; e
Vector frame (M, (e_1,e_2,e_3))
>>> latex(e)
\left(M, \left(e_{1},e_{2},e_{3}\right)\right)
```

The individual elements of the vector frame are accessed via square brackets, with the possibility to invoke the slice operator ':' to get more than a single element:

```
sage: e[2]
Vector field e_2 on the 3-dimensional differentiable manifold M
sage: e[1:3]
(Vector field e_1 on the 3-dimensional differentiable manifold M,
 Vector field e_2 on the 3-dimensional differentiable manifold M)
sage: e[:]
(Vector field e_1 on the 3-dimensional differentiable manifold M,
 Vector field e_2 on the 3-dimensional differentiable manifold M,
 Vector field e_3 on the 3-dimensional differentiable manifold M)
```

```
>>> from sage.all import *
>>> e[Integer(2)]
Vector field e_2 on the 3-dimensional differentiable manifold M
>>> e[Integer(1):Integer(3)]
(Vector field e_1 on the 3-dimensional differentiable manifold M,
 Vector field e_2 on the 3-dimensional differentiable manifold M)
>>> e[:]
(Vector field e_1 on the 3-dimensional differentiable manifold M,
 Vector field e_2 on the 3-dimensional differentiable manifold M,
 Vector field e_3 on the 3-dimensional differentiable manifold M)
```

The LaTeX symbol can be specified:

```
sage: E = M.vector_frame('E', latex_symbol=r"\epsilon")
sage: latex(E)
\left(M, \left(\epsilon_{1},\epsilon_{2},\epsilon_{3}\right)\right)
```

```
>>> from sage.all import *
>>> E = M.vector_frame('E', latex_symbol=r"\epsilon")
>>> latex(E)
\left(M, \left(\epsilon_{1},\epsilon_{2},\epsilon_{3}\right)\right)
```

By default, the elements of the vector frame are labelled by integers within the range specified at the manifold declaration. It is however possible to fully customize the labels, via the argument `indices`:

```
sage: u = M.vector_frame('u', indices=('x', 'y', 'z')) ; u
Vector frame (M, (u_x,u_y,u_z))
sage: u[1]
Vector field u_x on the 3-dimensional differentiable manifold M
sage: u.coframe()
Coframe (M, (u^x,u^y,u^z))
```

```
>>> from sage.all import *
>>> u = M.vector_frame('u', indices=('x', 'y', 'z')) ; u
Vector frame (M, (u_x,u_y,u_z))
>>> u[Integer(1)]
Vector field u_x on the 3-dimensional differentiable manifold M
>>> u.coframe()
Coframe (M, (u^x,u^y,u^z))
```

The LaTeX format of the indices can be adjusted:

```
sage: v = M.vector_frame('v', indices=('a', 'b', 'c'),
....:                    latex_indices=(r'\alpha', r'\beta', r'\gamma'))
sage: v
Vector frame (M, (v_a,v_b,v_c))
sage: latex(v)
\left(M, \left(v_{\alpha},v_{\beta},v_{\gamma}\right)\right)
sage: latex(v.coframe())
\left(M, \left(v^{\alpha},v^{\beta},v^{\gamma}\right)\right)
```

```
>>> from sage.all import *
>>> v = M.vector_frame('v', indices=('a', 'b', 'c'),
...                    latex_indices=(r'\alpha', r'\beta', r'\gamma'))
>>> v
Vector frame (M, (v_a,v_b,v_c))
>>> latex(v)
\left(M, \left(v_{\alpha},v_{\beta},v_{\gamma}\right)\right)
>>> latex(v.coframe())
\left(M, \left(v^{\alpha},v^{\beta},v^{\gamma}\right)\right)
```

The symbol of each element of the vector frame can also be freely chosen, by providing a tuple of symbols as the first argument of `vector_frame`; it is then mandatory to specify as well some symbols for the dual coframe:

```
sage: h = M.vector_frame(('a', 'b', 'c'), symbol_dual=('A', 'B', 'C'))
sage: h
Vector frame (M, (a,b,c))
sage: h[1]
Vector field a on the 3-dimensional differentiable manifold M
sage: h.coframe()
Coframe (M, (A,B,C))
sage: h.coframe()[1]
1-form A on the 3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> h = M.vector_frame(('a', 'b', 'c'), symbol_dual=('A', 'B', 'C'))
>>> h
```

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```
Vector frame (M, (a,b,c))
>>> h[Integer(1)]
Vector field a on the 3-dimensional differentiable manifold M
>>> h.coframe()
Coframe (M, (A,B,C))
>>> h.coframe()[Integer(1)]
1-form A on the 3-dimensional differentiable manifold M
```

Example with a non-trivial map Φ (see above); a vector frame along a curve:

```
sage: U = Manifold(1, 'U')  # open interval (-1,1) as a 1-dimensional manifold
sage: T.<t> = U.chart('t:(-1,1)')  # canonical chart on U
sage: Phi = U.diff_map(M, [cos(t), sin(t), t], name='Phi',
....:                  latex_name=r'\Phi')
sage: Phi
Differentiable map Phi from the 1-dimensional differentiable manifold U
 to the 3-dimensional differentiable manifold M
sage: f = U.vector_frame('f', dest_map=Phi) ; f
Vector frame (U, (f_1,f_2,f_3)) with values on the 3-dimensional
 differentiable manifold M
sage: f.domain()
1-dimensional differentiable manifold U
sage: f.ambient_domain()
3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> U = Manifold(Integer(1), 'U')  # open interval (-1,1) as a 1-dimensional␣
↪manifold
>>> T = U.chart('t:(-1,1)', names=('t',)); (t,) = T._first_ngens(1)# canonical␣
↪chart on U
>>> Phi = U.diff_map(M, [cos(t), sin(t), t], name='Phi',
...                  latex_name=r'\Phi')
>>> Phi
Differentiable map Phi from the 1-dimensional differentiable manifold U
 to the 3-dimensional differentiable manifold M
>>> f = U.vector_frame('f', dest_map=Phi) ; f
Vector frame (U, (f_1,f_2,f_3)) with values on the 3-dimensional
 differentiable manifold M
>>> f.domain()
1-dimensional differentiable manifold U
>>> f.ambient_domain()
3-dimensional differentiable manifold M
```

The value of the vector frame at a given point is a basis of the corresponding tangent space:

```
sage: p = U((0,), name='p') ; p
Point p on the 1-dimensional differentiable manifold U
sage: f.at(p)
Basis (f_1,f_2,f_3) on the Tangent space at Point Phi(p) on the
 3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> p = U((Integer(0),), name='p') ; p
Point p on the 1-dimensional differentiable manifold U
>>> f.at(p)
Basis (f_1,f_2,f_3) on the Tangent space at Point Phi(p) on the
 3-dimensional differentiable manifold M
```

Vector frames are bases of free modules formed by vector fields:

```
sage: e.module()
Free module X(M) of vector fields on the 3-dimensional differentiable
 manifold M
sage: e.module().base_ring()
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
sage: e.module() is M.vector_field_module()
True
sage: e in M.vector_field_module().bases()
True
```

```
>>> from sage.all import *
>>> e.module()
Free module X(M) of vector fields on the 3-dimensional differentiable
 manifold M
>>> e.module().base_ring()
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
>>> e.module() is M.vector_field_module()
True
>>> e in M.vector_field_module().bases()
True
```

```
sage: f.module()
Free module X(U,Phi) of vector fields along the 1-dimensional
 differentiable manifold U mapped into the 3-dimensional differentiable
 manifold M
sage: f.module().base_ring()
Algebra of differentiable scalar fields on the 1-dimensional
 differentiable manifold U
sage: f.module() is U.vector_field_module(dest_map=Phi)
True
sage: f in U.vector_field_module(dest_map=Phi).bases()
True
```

```
>>> from sage.all import *
>>> f.module()
Free module X(U,Phi) of vector fields along the 1-dimensional
 differentiable manifold U mapped into the 3-dimensional differentiable
 manifold M
>>> f.module().base_ring()
Algebra of differentiable scalar fields on the 1-dimensional
 differentiable manifold U
```

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```
>>> f.module() is U.vector_field_module(dest_map=Phi)
True
>>> f in U.vector_field_module(dest_map=Phi).bases()
True
```

along(*mapping*)

Return the vector frame deduced from the current frame via a differentiable map, the codomain of which is included in the domain of of the current frame.

If e is the current vector frame, V its domain and if $\Phi : U \rightarrow V$ is a differentiable map from some differentiable manifold U to V, the returned object is a vector frame $\tilde{e}$ along U with values on V such that

$$\forall p \in U, \ \tilde{e}(p) = e(\Phi(p)).$$

INPUT:

- `mapping` – differentiable map $\Phi : U \rightarrow V$

OUTPUT: vector frame $\tilde{e}$ along U defined above

EXAMPLES:

Vector frame along a curve:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: R = Manifold(1, 'R')  # R as a 1-dimensional manifold
sage: T.<t> = R.chart()  # canonical chart on R
sage: Phi = R.diff_map(M, {(T,X): [cos(t), t]}, name='Phi',
....:                  latex_name=r'\Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable
 manifold R to the 2-dimensional differentiable manifold M
sage: e = X.frame() ; e
Coordinate frame (M, (∂/∂x,∂/∂y))
sage: te = e.along(Phi) ; te
Vector frame (R, (∂/∂x,∂/∂y)) with values on the 2-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> R = Manifold(Integer(1), 'R')  # R as a 1-dimensional manifold
>>> T = R.chart(names=('t',)); (t,) = T._first_ngens(1)# canonical chart on R
>>> Phi = R.diff_map(M, {(T,X): [cos(t), t]}, name='Phi',
...                  latex_name=r'\Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable
 manifold R to the 2-dimensional differentiable manifold M
>>> e = X.frame() ; e
Coordinate frame (M, (∂/∂x,∂/∂y))
>>> te = e.along(Phi) ; te
Vector frame (R, (∂/∂x,∂/∂y)) with values on the 2-dimensional
 differentiable manifold M
```

Check of the formula $\tilde{e}(p) = e(\Phi(p))$:

```
sage: p = R((pi,)) ; p
Point on the 1-dimensional differentiable manifold R
sage: te[0].at(p) == e[0].at(Phi(p))
True
sage: te[1].at(p) == e[1].at(Phi(p))
True
```

```
>>> from sage.all import *
>>> p = R((pi,)) ; p
Point on the 1-dimensional differentiable manifold R
>>> te[Integer(0)].at(p) == e[Integer(0)].at(Phi(p))
True
>>> te[Integer(1)].at(p) == e[Integer(1)].at(Phi(p))
True
```

The result is cached:

```
sage: te is e.along(Phi)
True
```

```
>>> from sage.all import *
>>> te is e.along(Phi)
True
```

ambient_domain()

Return the differentiable manifold in which `self` takes its values.

The ambient domain is the codomain M of the differentiable map $\Phi : U \to M$ associated with the frame.

OUTPUT:

- a *DifferentiableManifold* representing M

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: e = M.vector_frame('e')
sage: e.ambient_domain()
2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> e = M.vector_frame('e')
>>> e.ambient_domain()
2-dimensional differentiable manifold M
```

In the present case, since Φ is the identity map:

```
sage: e.ambient_domain() == e.domain()
True
```

```
>>> from sage.all import *
>>> e.ambient_domain() == e.domain()
True
```

An example with a non trivial map Φ:

```
sage: U = Manifold(1, 'U')
sage: T.<t> = U.chart()
sage: X.<x,y> = M.chart()
sage: Phi = U.diff_map(M, {(T,X): [cos(t), t]}, name='Phi',
....:                   latex_name=r'\Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable
 manifold U to the 2-dimensional differentiable manifold M
sage: f = U.vector_frame('f', dest_map=Phi); f
Vector frame (U, (f_0,f_1)) with values on the 2-dimensional
 differentiable manifold M
sage: f.ambient_domain()
2-dimensional differentiable manifold M
sage: f.domain()
1-dimensional differentiable manifold U
```

```
>>> from sage.all import *
>>> U = Manifold(Integer(1), 'U')
>>> T = U.chart(names=('t',)); (t,) = T._first_ngens(1)
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> Phi = U.diff_map(M, {(T,X): [cos(t), t]}, name='Phi',
...                  latex_name=r'\Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable
 manifold U to the 2-dimensional differentiable manifold M
>>> f = U.vector_frame('f', dest_map=Phi); f
Vector frame (U, (f_0,f_1)) with values on the 2-dimensional
 differentiable manifold M
>>> f.ambient_domain()
2-dimensional differentiable manifold M
>>> f.domain()
1-dimensional differentiable manifold U
```

at (*point*)

Return the value of `self` at a given point, this value being a basis of the tangent vector space at the point.

INPUT:

- `point` – *ManifoldPoint*; point p in the domain U of the vector frame (denoted e hereafter)

OUTPUT:

- `FreeModuleBasis` representing the basis $e(p)$ of the tangent vector space $T_{\Phi(p)}M$, where $\Phi : U \to M$ is the differentiable map associated with e (possibly $\Phi = \mathrm{Id}_U$)

EXAMPLES:

Basis of a tangent space to a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: p = M.point((-1,2), name='p')
sage: e = X.frame() ; e
Coordinate frame (M, (∂/∂x,∂/∂y))
sage: ep = e.at(p) ; ep
Basis (∂/∂x,∂/∂y) on the Tangent space at Point p on the
```

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```
 2-dimensional differentiable manifold M
sage: type(ep)
<class 'sage.tensor.modules.free_module_basis.FreeModuleBasis_with_category'>
sage: ep[0]
Tangent vector ∂/∂x at Point p on the 2-dimensional differentiable
 manifold M
sage: ep[1]
Tangent vector ∂/∂y at Point p on the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> p = M.point((-Integer(1),Integer(2)), name='p')
>>> e = X.frame() ; e
Coordinate frame (M, (∂/∂x,∂/∂y))
>>> ep = e.at(p) ; ep
Basis (∂/∂x,∂/∂y) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
>>> type(ep)
<class 'sage.tensor.modules.free_module_basis.FreeModuleBasis_with_category'>
>>> ep[Integer(0)]
Tangent vector ∂/∂x at Point p on the 2-dimensional differentiable
 manifold M
>>> ep[Integer(1)]
Tangent vector ∂/∂y at Point p on the 2-dimensional differentiable
 manifold M
```

Note that the symbols used to denote the vectors are same as those for the vector fields of the frame. At this stage, `ep` is the unique basis on the tangent space at `p`:

```
sage: Tp = M.tangent_space(p)
sage: Tp.bases()
[Basis (∂/∂x,∂/∂y) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M]
```

```
>>> from sage.all import *
>>> Tp = M.tangent_space(p)
>>> Tp.bases()
[Basis (∂/∂x,∂/∂y) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M]
```

Let us consider a vector frame that is a not a coordinate one:

```
sage: aut = M.automorphism_field()
sage: aut[:] = [[1+y^2, 0], [0, 2]]
sage: f = e.new_frame(aut, 'f') ; f
Vector frame (M, (f_0,f_1))
sage: fp = f.at(p) ; fp
Basis (f_0,f_1) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> aut = M.automorphism_field()
>>> aut[:] = [[Integer(1)+y**Integer(2), Integer(0)], [Integer(0),␣
↪Integer(2)]]
>>> f = e.new_frame(aut, 'f') ; f
Vector frame (M, (f_0,f_1))
>>> fp = f.at(p) ; fp
Basis (f_0,f_1) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
```

There are now two bases on the tangent space:

```
sage: Tp.bases()
[Basis (∂/∂x,∂/∂y) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M,
 Basis (f_0,f_1) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M]
```

```
>>> from sage.all import *
>>> Tp.bases()
[Basis (∂/∂x,∂/∂y) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M,
 Basis (f_0,f_1) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M]
```

Moreover, the changes of bases in the tangent space have been computed from the known relation between the frames `e` and `f` (field of automorphisms `aut` defined above):

```
sage: Tp.change_of_basis(ep, fp)
Automorphism of the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
sage: Tp.change_of_basis(ep, fp).display()
5 ∂/∂x⊗dx + 2 ∂/∂y⊗dy
sage: Tp.change_of_basis(fp, ep)
Automorphism of the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
sage: Tp.change_of_basis(fp, ep).display()
1/5 ∂/∂x⊗dx + 1/2 ∂/∂y⊗dy
```

```
>>> from sage.all import *
>>> Tp.change_of_basis(ep, fp)
Automorphism of the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
>>> Tp.change_of_basis(ep, fp).display()
5 ∂/∂x⊗dx + 2 ∂/∂y⊗dy
>>> Tp.change_of_basis(fp, ep)
Automorphism of the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
>>> Tp.change_of_basis(fp, ep).display()
1/5 ∂/∂x⊗dx + 1/2 ∂/∂y⊗dy
```

The dual bases:

```
sage: e.coframe()
Coordinate coframe (M, (dx,dy))
sage: ep.dual_basis()
Dual basis (dx,dy) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
sage: ep.dual_basis() is e.coframe().at(p)
True
sage: f.coframe()
Coframe (M, (f^0,f^1))
sage: fp.dual_basis()
Dual basis (f^0,f^1) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
sage: fp.dual_basis() is f.coframe().at(p)
True
```

```
>>> from sage.all import *
>>> e.coframe()
Coordinate coframe (M, (dx,dy))
>>> ep.dual_basis()
Dual basis (dx,dy) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
>>> ep.dual_basis() is e.coframe().at(p)
True
>>> f.coframe()
Coframe (M, (f^0,f^1))
>>> fp.dual_basis()
Dual basis (f^0,f^1) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
>>> fp.dual_basis() is f.coframe().at(p)
True
```

coframe()

Return the coframe of `self`.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: e = M.vector_frame('e')
sage: e.coframe()
Coframe (M, (e^0,e^1))
sage: X.<x,y> = M.chart()
sage: X.frame().coframe()
Coordinate coframe (M, (dx,dy))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> e = M.vector_frame('e')
>>> e.coframe()
Coframe (M, (e^0,e^1))
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> X.frame().coframe()
Coordinate coframe (M, (dx,dy))
```

destination_map()

Return the differential map associated to this vector frame.

Let e denote the vector frame; the differential map associated to it is the map $\Phi : U \to M$ such that for each $p \in U$, $e(p)$ is a vector basis of the tangent space $T_{\Phi(p)}M$.

OUTPUT:

- a *DiffMap* representing the differential map Φ

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: e = M.vector_frame('e')
sage: e.destination_map()
Identity map Id_M of the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> e = M.vector_frame('e')
>>> e.destination_map()
Identity map Id_M of the 2-dimensional differentiable manifold M
```

An example with a non trivial map Φ:

```
sage: U = Manifold(1, 'U')
sage: T.<t> = U.chart()
sage: X.<x,y> = M.chart()
sage: Phi = U.diff_map(M, {(T,X): [cos(t), t]}, name='Phi',
....:                     latex_name=r'\Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable
 manifold U to the 2-dimensional differentiable manifold M
sage: f = U.vector_frame('f', dest_map=Phi); f
Vector frame (U, (f_0,f_1)) with values on the 2-dimensional
 differentiable manifold M
sage: f.destination_map()
Differentiable map Phi from the 1-dimensional differentiable
 manifold U to the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> U = Manifold(Integer(1), 'U')
>>> T = U.chart(names=('t',)); (t,) = T._first_ngens(1)
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> Phi = U.diff_map(M, {(T,X): [cos(t), t]}, name='Phi',
...                     latex_name=r'\Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable
 manifold U to the 2-dimensional differentiable manifold M
>>> f = U.vector_frame('f', dest_map=Phi); f
Vector frame (U, (f_0,f_1)) with values on the 2-dimensional
 differentiable manifold M
>>> f.destination_map()
Differentiable map Phi from the 1-dimensional differentiable
 manifold U to the 2-dimensional differentiable manifold M
```

domain()

Return the domain on which `self` is defined.

OUTPUT:

- a *DifferentiableManifold*; representing the domain of the vector frame

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: e = M.vector_frame('e')
sage: e.domain()
2-dimensional differentiable manifold M
sage: U = M.open_subset('U')
sage: f = e.restrict(U)
sage: f.domain()
Open subset U of the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> e = M.vector_frame('e')
>>> e.domain()
2-dimensional differentiable manifold M
>>> U = M.open_subset('U')
>>> f = e.restrict(U)
>>> f.domain()
Open subset U of the 2-dimensional differentiable manifold M
```

new_frame(*change_of_frame*, *symbol*, *latex_symbol=None*, *indices=None*, *latex_indices=None*, *symbol_dual=None*, *latex_symbol_dual=None*)

Define a new vector frame from `self`.

The new vector frame is defined from a field of tangent-space automorphisms; its domain is the same as that of the current frame.

INPUT:

- `change_of_frame` – *AutomorphismFieldParal*; the field of tangent space automorphisms P that relates the current frame (e_i) to the new frame (n_i) according to $n_i = P(e_i)$
- `symbol` – either a string, to be used as a common base for the symbols of the vector fields constituting the vector frame, or a list/tuple of strings, representing the individual symbols of the vector fields
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the vector fields constituting the vector frame, or a list/tuple of strings, representing the individual LaTeX symbols of the vector fields; if `None`, `symbol` is used in place of `latex_symbol`
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the vector fields of the frame; if `None`, the indices will be generated as integers within the range declared on `self`
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the vector fields; if `None`, `indices` is used instead
- `symbol_dual` – (default: `None`) same as `symbol` but for the dual coframe; if `None`, `symbol` must be a string and is used for the common base of the symbols of the elements of the dual coframe
- `latex_symbol_dual` – (default: `None`) same as `latex_symbol` but for the dual coframe

OUTPUT:

- the new frame (n_i), as an instance of *VectorFrame*

EXAMPLES:

Frame resulting from a $\pi/3$-rotation in the Euclidean plane:

```
sage: M = Manifold(2, 'R^2')
sage: X.<x,y> = M.chart()
sage: e = M.vector_frame('e') ; M.set_default_frame(e)
sage: M._frame_changes
{}
sage: rot = M.automorphism_field()
sage: rot[:] = [[sqrt(3)/2, -1/2], [1/2, sqrt(3)/2]]
sage: n = e.new_frame(rot, 'n')
sage: n[0][:]
[1/2*sqrt(3), 1/2]
sage: n[1][:]
[-1/2, 1/2*sqrt(3)]
sage: a =  M.change_of_frame(e,n)
sage: a[:]
[1/2*sqrt(3)        -1/2]
[        1/2 1/2*sqrt(3)]
sage: a == rot
True
sage: a is rot
False
sage: a._components # random (dictionary output)
{Vector frame (R^2, (e_0,e_1)): 2-indices components w.r.t.
 Vector frame (R^2, (e_0,e_1)),
 Vector frame (R^2, (n_0,n_1)): 2-indices components w.r.t.
 Vector frame (R^2, (n_0,n_1))}
sage: a.comp(n)[:]
[1/2*sqrt(3)        -1/2]
[        1/2 1/2*sqrt(3)]
sage: a1 = M.change_of_frame(n,e)
sage: a1[:]
[1/2*sqrt(3)         1/2]
[       -1/2 1/2*sqrt(3)]
sage: a1 == rot.inverse()
True
sage: a1 is rot.inverse()
False
sage: e[0].comp(n)[:]
[1/2*sqrt(3), -1/2]
sage: e[1].comp(n)[:]
[1/2, 1/2*sqrt(3)]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> e = M.vector_frame('e') ; M.set_default_frame(e)
>>> M._frame_changes
{}
>>> rot = M.automorphism_field()
>>> rot[:] = [[sqrt(Integer(3))/Integer(2), -Integer(1)/Integer(2)],␣
↪[Integer(1)/Integer(2), sqrt(Integer(3))/Integer(2)]]
```

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```
>>> n = e.new_frame(rot, 'n')
>>> n[Integer(0)][:]
[1/2*sqrt(3), 1/2]
>>> n[Integer(1)][:]
[-1/2, 1/2*sqrt(3)]
>>> a =  M.change_of_frame(e,n)
>>> a[:]
[1/2*sqrt(3)        -1/2]
[        1/2 1/2*sqrt(3)]
>>> a == rot
True
>>> a is rot
False
>>> a._components # random (dictionary output)
{Vector frame (R^2, (e_0,e_1)): 2-indices components w.r.t.
 Vector frame (R^2, (e_0,e_1)),
 Vector frame (R^2, (n_0,n_1)): 2-indices components w.r.t.
 Vector frame (R^2, (n_0,n_1))}
>>> a.comp(n)[:]
[1/2*sqrt(3)        -1/2]
[        1/2 1/2*sqrt(3)]
>>> a1 = M.change_of_frame(n,e)
>>> a1[:]
[1/2*sqrt(3)         1/2]
[       -1/2 1/2*sqrt(3)]
>>> a1 == rot.inverse()
True
>>> a1 is rot.inverse()
False
>>> e[Integer(0)].comp(n)[:]
[1/2*sqrt(3), -1/2]
>>> e[Integer(1)].comp(n)[:]
[1/2, 1/2*sqrt(3)]
```

restrict(*subdomain*)

Return the restriction of `self` to some open subset of its domain.

If the restriction has not been defined yet, it is constructed here.

INPUT:

- `subdomain` – open subset V of the current frame domain U

OUTPUT: the restriction of the current frame to V as a `VectorFrame`

EXAMPLES:

Restriction of a frame defined on $\mathbf{R}^2$ to the unit disk:

```
sage: M = Manifold(2, 'R^2', start_index=1)
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: a = M.automorphism_field()
sage: a[:] = [[1-y^2,0], [1+x^2, 2]]
sage: e = c_cart.frame().new_frame(a, 'e') ; e
Vector frame (R^2, (e_1,e_2))
```

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```
sage: U = M.open_subset('U', coord_def={c_cart: x^2+y^2<1})
sage: e_U = e.restrict(U) ; e_U
Vector frame (U, (e_1,e_2))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2', start_index=Integer(1))
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#␣
↪Cartesian coordinates on R^2
>>> a = M.automorphism_field()
>>> a[:] = [[Integer(1)-y**Integer(2),Integer(0)], [Integer(1)+x**Integer(2),␣
↪Integer(2)]]
>>> e = c_cart.frame().new_frame(a, 'e') ; e
Vector frame (R^2, (e_1,e_2))
>>> U = M.open_subset('U', coord_def={c_cart: x**Integer(2)+y**Integer(2)
↪<Integer(1)})
>>> e_U = e.restrict(U) ; e_U
Vector frame (U, (e_1,e_2))
```

The vectors of the restriction have the same symbols as those of the original frame:

```
sage: e_U[1].display()
e_1 = (-y^2 + 1) ∂/∂x + (x^2 + 1) ∂/∂y
sage: e_U[2].display()
e_2 = 2 ∂/∂y
```

```
>>> from sage.all import *
>>> e_U[Integer(1)].display()
e_1 = (-y^2 + 1) ∂/∂x + (x^2 + 1) ∂/∂y
>>> e_U[Integer(2)].display()
e_2 = 2 ∂/∂y
```

They are actually the restrictions of the original frame vectors:

```
sage: e_U[1] is e[1].restrict(U)
True
sage: e_U[2] is e[2].restrict(U)
True
```

```
>>> from sage.all import *
>>> e_U[Integer(1)] is e[Integer(1)].restrict(U)
True
>>> e_U[Integer(2)] is e[Integer(2)].restrict(U)
True
```

set_name(*symbol*, *latex_symbol=None*, *indices=None*, *latex_indices=None*, *index_position='down'*, *include_domain=True*)

Set (or change) the text name and LaTeX name of `self`.

INPUT:

- `symbol` – either a string, to be used as a common base for the symbols of the vector fields constituting the vector frame, or a list/tuple of strings, representing the individual symbols of the vector fields

- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the vector fields constituting the vector frame, or a list/tuple of strings, representing the individual LaTeX symbols of the vector fields; if `None`, `symbol` is used in place of `latex_symbol`
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the vector fields of the frame; if `None`, the indices will be generated as integers within the range declared on `self`
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the vector fields; if `None`, `indices` is used instead
- `index_position` – (default: `'down'`) determines the position of the indices labelling the vector fields of the frame; can be either `'down'` or `'up'`
- `include_domain` – boolean (default: `True`); determining whether the name of the domain is included in the beginning of the vector frame name

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: e = M.vector_frame('e'); e
Vector frame (M, (e_0,e_1))
sage: e.set_name('f'); e
Vector frame (M, (f_0,f_1))
sage: e.set_name('e', include_domain=False); e
Vector frame (e_0,e_1)
sage: e.set_name(['a', 'b']); e
Vector frame (M, (a,b))
sage: e.set_name('e', indices=['x', 'y']); e
Vector frame (M, (e_x,e_y))
sage: e.set_name('e', latex_symbol=r'\epsilon')
sage: latex(e)
\left(M, \left(\epsilon_{0},\epsilon_{1}\right)\right)
sage: e.set_name('e', latex_symbol=[r'\alpha', r'\beta'])
sage: latex(e)
\left(M, \left(\alpha,\beta\right)\right)
sage: e.set_name('e', latex_symbol='E',
....:            latex_indices=[r'\alpha', r'\beta'])
sage: latex(e)
\left(M, \left(E_{\alpha},E_{\beta}\right)\right)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> e = M.vector_frame('e'); e
Vector frame (M, (e_0,e_1))
>>> e.set_name('f'); e
Vector frame (M, (f_0,f_1))
>>> e.set_name('e', include_domain=False); e
Vector frame (e_0,e_1)
>>> e.set_name(['a', 'b']); e
Vector frame (M, (a,b))
>>> e.set_name('e', indices=['x', 'y']); e
Vector frame (M, (e_x,e_y))
>>> e.set_name('e', latex_symbol=r'\epsilon')
>>> latex(e)
\left(M, \left(\epsilon_{0},\epsilon_{1}\right)\right)
```

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```
>>> e.set_name('e', latex_symbol=[r'\alpha', r'\beta'])
>>> latex(e)
\left(M, \left(\alpha,\beta\right)\right)
>>> e.set_name('e', latex_symbol='E',
...            latex_indices=[r'\alpha', r'\beta'])
>>> latex(e)
\left(M, \left(E_{\alpha},E_{\beta}\right)\right)
```

structure_coeff()

Evaluate the structure coefficients associated to `self`.

n being the manifold's dimension, the structure coefficients of the vector frame (e_i) are the n^3 scalar fields $C^k{}_{ij}$ defined by

$$[e_i, e_j] = C^k{}_{ij} e_k$$

OUTPUT:

- the structure coefficients $C^k{}_{ij}$, as an instance of `CompWithSym` with 3 indices ordered as (k, i, j).

EXAMPLES:

Structure coefficients of the orthonormal frame associated to spherical coordinates in the Euclidean space $\mathbf{R}^3$:

```
sage: M = Manifold(3, 'R^3', r'\RR^3', start_index=1)  # Part of R^3 covered␣
↪by spherical coordinates
sage: c_spher.<r,th,ph> = M.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\
↪phi')
sage: ch_frame = M.automorphism_field()
sage: ch_frame[1,1], ch_frame[2,2], ch_frame[3,3] = 1, 1/r, 1/(r*sin(th))
sage: M.frames()
[Coordinate frame (R^3, (∂/∂r,∂/∂th,∂/∂ph))]
sage: e = c_spher.frame().new_frame(ch_frame, 'e')
sage: e[1][:]  # components of e_1 in the manifold's default frame (∂/∂r, ∂/
↪∂th, ∂/∂th)
[1, 0, 0]
sage: e[2][:]
[0, 1/r, 0]
sage: e[3][:]
[0, 0, 1/(r*sin(th))]
sage: c = e.structure_coeff() ; c
3-indices components w.r.t. Vector frame (R^3, (e_1,e_2,e_3)), with
 antisymmetry on the index positions (1, 2)
sage: c[:]
[[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
 [[0, -1/r, 0], [1/r, 0, 0], [0, 0, 0]],
 [[0, 0, -1/r], [0, 0, -cos(th)/(r*sin(th))], [1/r, cos(th)/(r*sin(th)), 0]]]
sage: c[2,1,2]  # C^2_{12}
-1/r
sage: c[3,1,3]  # C^3_{13}
-1/r
sage: c[3,2,3]  # C^3_{23}
-cos(th)/(r*sin(th))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'R^3', r'\RR^3', start_index=Integer(1))  # Part␣
↪of R^3 covered by spherical coordinates
>>> c_spher = M.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi', names=(
↪'r', 'th', 'ph',)); (r, th, ph,) = c_spher._first_ngens(3)
>>> ch_frame = M.automorphism_field()
>>> ch_frame[Integer(1),Integer(1)], ch_frame[Integer(2),Integer(2)], ch_
↪frame[Integer(3),Integer(3)] = Integer(1), Integer(1)/r, Integer(1)/
↪(r*sin(th))
>>> M.frames()
[Coordinate frame (R^3, (∂/∂r,∂/∂th,∂/∂ph))]
>>> e = c_spher.frame().new_frame(ch_frame, 'e')
>>> e[Integer(1)][:]  # components of e_1 in the manifold's default frame (∂/
↪∂r, ∂/∂th, ∂/∂th)
[1, 0, 0]
>>> e[Integer(2)][:]
[0, 1/r, 0]
>>> e[Integer(3)][:]
[0, 0, 1/(r*sin(th))]
>>> c = e.structure_coeff() ; c
3-indices components w.r.t. Vector frame (R^3, (e_1,e_2,e_3)), with
 antisymmetry on the index positions (1, 2)
>>> c[:]
[[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
 [[0, -1/r, 0], [1/r, 0, 0], [0, 0, 0]],
 [[0, 0, -1/r], [0, 0, -cos(th)/(r*sin(th))], [1/r, cos(th)/(r*sin(th)), 0]]]
>>> c[Integer(2),Integer(1),Integer(2)]  # C^2_{12}
-1/r
>>> c[Integer(3),Integer(1),Integer(3)]  # C^3_{13}
-1/r
>>> c[Integer(3),Integer(2),Integer(3)]  # C^3_{23}
-cos(th)/(r*sin(th))
```

2.7.4 Group of Tangent-Space Automorphism Fields

Given a differentiable manifold U and a differentiable map $\Phi : U \to M$ to a differentiable manifold M (possibly $U = M$ and $\Phi = \mathrm{Id}_M$), the *group of tangent-space automorphism fields* associated with U and Φ is the general linear group $\mathrm{GL}(\mathfrak{X}(U, \Phi))$ of the module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on $M \supset \Phi(U)$ (see `VectorFieldModule`). Note that $\mathfrak{X}(U, \Phi)$ is a module over $C^k(U)$, the algebra of differentiable scalar fields on U. Elements of $\mathrm{GL}(\mathfrak{X}(U, \Phi))$ are fields along U of automorphisms of tangent spaces to M.

Two classes implement $\mathrm{GL}(\mathfrak{X}(U, \Phi))$ depending whether M is parallelizable or not: `AutomorphismFieldParalGroup` and `AutomorphismFieldGroup`.

AUTHORS:

- Eric Gourgoulhon (2015): initial version
- Travis Scrimshaw (2016): review tweaks
- Michael Jung (2019): improve treatment of the identity element

REFERENCES:

- Chap. 15 of [God1968]

class sage.manifolds.differentiable.automorphismfield_group.**AutomorphismFieldGroup**(*vector_field_module*)

Bases: UniqueRepresentation, Parent

General linear group of the module of vector fields along a differentiable manifold U with values on a differentiable manifold M.

Given a differentiable manifold U and a differentiable map $\Phi : U \to M$ to a differentiable manifold M (possibly $U = M$ and $\Phi = \mathrm{Id}_M$), the *group of tangent-space automorphism fields* associated with U and Φ is the general linear group $\mathrm{GL}(\mathfrak{X}(U, \Phi))$ of the module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on $M \supset \Phi(U)$ (see *VectorFieldModule*). Note that $\mathfrak{X}(U, \Phi)$ is a module over $C^k(U)$, the algebra of differentiable scalar fields on U. Elements of $\mathrm{GL}(\mathfrak{X}(U, \Phi))$ are fields along U of automorphisms of tangent spaces to M.

Note: If M is parallelizable, then *AutomorphismFieldParalGroup* *must* be used instead.

INPUT:

- `vector_field_module` – *VectorFieldModule*; module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on M

EXAMPLES:

Group of tangent-space automorphism fields of the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                intersection_name='W',
....:                restrictions1= x^2+y^2!=0, restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: G = M.automorphism_field_group() ; G
General linear group of the Module X(M) of vector fields on the
 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                intersection_name='W',
...                restrictions1= x**Integer(2)+y**Integer(2)!=Integer(0),␣
```

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```
↪restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> G = M.automorphism_field_group() ; G
General linear group of the Module X(M) of vector fields on the
 2-dimensional differentiable manifold M
```

G is the general linear group of the vector field module $\mathfrak{X}(M)$:

```
sage: XM = M.vector_field_module() ; XM
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
sage: G is XM.general_linear_group()
True
```

```
>>> from sage.all import *
>>> XM = M.vector_field_module() ; XM
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
>>> G is XM.general_linear_group()
True
```

G is a non-abelian group:

```
sage: G.category()
Category of groups
sage: G in Groups()
True
sage: G in CommutativeAdditiveGroups()
False
```

```
>>> from sage.all import *
>>> G.category()
Category of groups
>>> G in Groups()
True
>>> G in CommutativeAdditiveGroups()
False
```

The elements of G are tangent-space automorphisms:

```
sage: a = G.an_element(); a
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
sage: a.parent() is G
True
sage: a.restrict(U).display()
2 ∂/∂x⊗dx + 2 ∂/∂y⊗dy
sage: a.restrict(V).display()
2 ∂/∂u⊗du + 2 ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> a = G.an_element(); a
```

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```
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
>>> a.parent() is G
True
>>> a.restrict(U).display()
2 ∂/∂x⊗dx + 2 ∂/∂y⊗dy
>>> a.restrict(V).display()
2 ∂/∂u⊗du + 2 ∂/∂v⊗dv
```

The identity element of the group `G`:

```
sage: e = G.one() ; e
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold M
sage: eU = U.default_frame() ; eU
Coordinate frame (U, (∂/∂x,∂/∂y))
sage: eV = V.default_frame() ; eV
Coordinate frame (V, (∂/∂u,∂/∂v))
sage: e.display(eU)
Id = ∂/∂x⊗dx + ∂/∂y⊗dy
sage: e.display(eV)
Id = ∂/∂u⊗du + ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> e = G.one() ; e
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold M
>>> eU = U.default_frame() ; eU
Coordinate frame (U, (∂/∂x,∂/∂y))
>>> eV = V.default_frame() ; eV
Coordinate frame (V, (∂/∂u,∂/∂v))
>>> e.display(eU)
Id = ∂/∂x⊗dx + ∂/∂y⊗dy
>>> e.display(eV)
Id = ∂/∂u⊗du + ∂/∂v⊗dv
```

Element

alias of *AutomorphismField*

base_module()

Return the vector-field module of which `self` is the general linear group.

OUTPUT: *VectorFieldModule*

EXAMPLES:

Base module of the group of tangent-space automorphism fields of the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
```

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```
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                                 intersection_name='W', restrictions1= x^
↪2+y^2!=0,
....:                                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: G = M.automorphism_field_group()
sage: G.base_module()
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
sage: G.base_module() is M.vector_field_module()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                                 intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                                 restrictions2= u**Integer(2)+v**Integer(2)!
↪=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> G = M.automorphism_field_group()
>>> G.base_module()
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
>>> G.base_module() is M.vector_field_module()
True
```

one()

Return identity element of `self`.

The group identity element is the field of tangent-space identity maps.

OUTPUT:

- *AutomorphismField* representing the identity element

EXAMPLES:

Identity element of the group of tangent-space automorphism fields of the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
```

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```
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                                intersection_name='W', restrictions1= x^
↪2+y^2!=0,
....:                                restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: G = M.automorphism_field_group()
sage: G.one()
Field of tangent-space identity maps on the 2-dimensional differentiable␣
↪manifold M
sage: G.one().restrict(U)[:]
[1 0]
[0 1]
sage: G.one().restrict(V)[:]
[1 0]
[0 1]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                                intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                                restrictions2= u**Integer(2)+v**Integer(2)!
↪=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> G = M.automorphism_field_group()
>>> G.one()
Field of tangent-space identity maps on the 2-dimensional differentiable␣
↪manifold M
>>> G.one().restrict(U)[:]
[1 0]
[0 1]
>>> G.one().restrict(V)[:]
[1 0]
[0 1]
```

class sage.manifolds.differentiable.automorphismfield_group.**AutomorphismFieldParalGroup**(*vector_field_module*)

Bases: `FreeModuleLinearGroup`

General linear group of the module of vector fields along a differentiable manifold U with values on a parallelizable manifold M.

Given a differentiable manifold U and a differentiable map $\Phi : U \to M$ to a parallelizable manifold M (possibly $U = M$ and $\Phi = \mathrm{Id}_M$), the *group of tangent-space automorphism fields* associated with U and Φ is the general linear group $\mathrm{GL}(\mathfrak{X}(U,\Phi))$ of the module $\mathfrak{X}(U,\Phi)$ of vector fields along U with values on $M \supset \Phi(U)$ (see `VectorFieldFreeModule`). Note that $\mathfrak{X}(U,\Phi)$ is a free module over $C^k(U)$, the algebra of differentiable scalar fields on U. Elements of $\mathrm{GL}(\mathfrak{X}(U,\Phi))$ are fields along U of automorphisms of tangent spaces to M.

Note

If M is not parallelizable, the class `AutomorphismFieldGroup` must be used instead.

INPUT:

- `vector_field_module` – `VectorFieldFreeModule`; free module $\mathfrak{X}(U,\Phi)$ of vector fields along U with values on M

EXAMPLES:

Group of tangent-space automorphism fields of a 2-dimensional parallelizable manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: XM = M.vector_field_module() ; XM
Free module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
sage: G = M.automorphism_field_group(); G
General linear group of the Free module X(M) of vector fields on the
 2-dimensional differentiable manifold M
sage: latex(G)
\mathrm{GL}\left( \mathfrak{X}\left(M\right) \right)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> XM = M.vector_field_module() ; XM
Free module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
>>> G = M.automorphism_field_group(); G
General linear group of the Free module X(M) of vector fields on the
 2-dimensional differentiable manifold M
>>> latex(G)
\mathrm{GL}\left( \mathfrak{X}\left(M\right) \right)
```

`G` is nothing but the general linear group of the module $\mathfrak{X}(M)$:

```
sage: G is XM.general_linear_group()
True
```

```
>>> from sage.all import *
>>> G is XM.general_linear_group()
True
```

`G` is a group:

```
sage: G.category()
Category of groups
```

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```
sage: G in Groups()
True
```

```
>>> from sage.all import *
>>> G.category()
Category of groups
>>> G in Groups()
True
```

It is not an abelian group:

```
sage: G in CommutativeAdditiveGroups()
False
```

```
>>> from sage.all import *
>>> G in CommutativeAdditiveGroups()
False
```

The elements of `G` are tangent-space automorphisms:

```
sage: G.Element
<class 'sage.manifolds.differentiable.automorphismfield.AutomorphismFieldParal'>
sage: a = G.an_element() ; a
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
sage: a.parent() is G
True
```

```
>>> from sage.all import *
>>> G.Element
<class 'sage.manifolds.differentiable.automorphismfield.AutomorphismFieldParal'>
>>> a = G.an_element() ; a
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
>>> a.parent() is G
True
```

As automorphisms of $\mathfrak{X}(M)$, the elements of `G` map a vector field to a vector field:

```
sage: v = XM.an_element() ; v
Vector field on the 2-dimensional differentiable manifold M
sage: v.display()
2 ∂/∂x + 2 ∂/∂y
sage: a(v)
Vector field on the 2-dimensional differentiable manifold M
sage: a(v).display()
2 ∂/∂x - 2 ∂/∂y
```

```
>>> from sage.all import *
>>> v = XM.an_element() ; v
Vector field on the 2-dimensional differentiable manifold M
```

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```
>>> v.display()
2 ∂/∂x + 2 ∂/∂y
>>> a(v)
Vector field on the 2-dimensional differentiable manifold M
>>> a(v).display()
2 ∂/∂x - 2 ∂/∂y
```

Indeed the matrix of a with respect to the frame (∂_x, ∂_y) is:

```
sage: a[X.frame(),:]
[ 1  0]
[ 0 -1]
```

```
>>> from sage.all import *
>>> a[X.frame(),:]
[ 1  0]
[ 0 -1]
```

The elements of G can also be considered as tensor fields of type $(1, 1)$:

```
sage: a.tensor_type()
(1, 1)
sage: a.tensor_rank()
2
sage: a.domain()
2-dimensional differentiable manifold M
sage: a.display()
∂/∂x⊗dx - ∂/∂y⊗dy
```

```
>>> from sage.all import *
>>> a.tensor_type()
(1, 1)
>>> a.tensor_rank()
2
>>> a.domain()
2-dimensional differentiable manifold M
>>> a.display()
∂/∂x⊗dx - ∂/∂y⊗dy
```

The identity element of the group G is:

```
sage: id = G.one() ; id
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold M
sage: id*a == a
True
sage: a*id == a
True
sage: a*a^(-1) == id
True
sage: a^(-1)*a == id
True
```

```
>>> from sage.all import *
>>> id = G.one() ; id
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold M
>>> id*a == a
True
>>> a*id == a
True
>>> a*a**(-Integer(1)) == id
True
>>> a**(-Integer(1))*a == id
True
```

Construction of an element by providing its components with respect to the manifold's default frame (frame associated to the coordinates (x, y)):

```
sage: b = G([[1+x^2,0], [0,1+y^2]]) ; b
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
sage: b.display()
(x^2 + 1) ∂/∂x⊗dx + (y^2 + 1) ∂/∂y⊗dy
sage: (~b).display()  # the inverse automorphism
1/(x^2 + 1) ∂/∂x⊗dx + 1/(y^2 + 1) ∂/∂y⊗dy
```

```
>>> from sage.all import *
>>> b = G([[Integer(1)+x**Integer(2),Integer(0)], [Integer(0),
↪Integer(1)+y**Integer(2)]]) ; b
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
>>> b.display()
(x^2 + 1) ∂/∂x⊗dx + (y^2 + 1) ∂/∂y⊗dy
>>> (~b).display()  # the inverse automorphism
1/(x^2 + 1) ∂/∂x⊗dx + 1/(y^2 + 1) ∂/∂y⊗dy
```

We check the group law on these elements:

```
sage: (a*b)^(-1) == b^(-1) * a^(-1)
True
```

```
>>> from sage.all import *
>>> (a*b)**(-Integer(1)) == b**(-Integer(1)) * a**(-Integer(1))
True
```

Invertible tensor fields of type $(1, 1)$ can be converted to elements of `G`:

```
sage: t = M.tensor_field(1, 1, name='t')
sage: t[:] = [[1+exp(y), x*y], [0, 1+x^2]]
sage: t1 = G(t) ; t1
Field of tangent-space automorphisms t on the 2-dimensional
 differentiable manifold M
sage: t1 in G
True
sage: t1.display()
```

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```
t = (e^y + 1) ∂/∂x⊗dx + x*y ∂/∂x⊗dy + (x^2 + 1) ∂/∂y⊗dy
sage: t1^(-1)
Field of tangent-space automorphisms t^(-1) on the 2-dimensional
 differentiable manifold M
sage: (t1^(-1)).display()
t^(-1) = 1/(e^y + 1) ∂/∂x⊗dx - x*y/(x^2 + (x^2 + 1)*e^y + 1) ∂/∂x⊗dy
 + 1/(x^2 + 1) ∂/∂y⊗dy
```

```
>>> from sage.all import *
>>> t = M.tensor_field(Integer(1), Integer(1), name='t')
>>> t[:] = [[Integer(1)+exp(y), x*y], [Integer(0), Integer(1)+x**Integer(2)]]
>>> t1 = G(t) ; t1
Field of tangent-space automorphisms t on the 2-dimensional
 differentiable manifold M
>>> t1 in G
True
>>> t1.display()
t = (e^y + 1) ∂/∂x⊗dx + x*y ∂/∂x⊗dy + (x^2 + 1) ∂/∂y⊗dy
>>> t1**(-Integer(1))
Field of tangent-space automorphisms t^(-1) on the 2-dimensional
 differentiable manifold M
>>> (t1**(-Integer(1))).display()
t^(-1) = 1/(e^y + 1) ∂/∂x⊗dx - x*y/(x^2 + (x^2 + 1)*e^y + 1) ∂/∂x⊗dy
 + 1/(x^2 + 1) ∂/∂y⊗dy
```

Since any automorphism field can be considered as a tensor field of type-$(1,1)$ on `M`, there is a coercion map from `G` to the module $T^{(1,1)}(M)$ of type-$(1,1)$ tensor fields:

```
sage: T11 = M.tensor_field_module((1,1)) ; T11
Free module T^(1,1)(M) of type-(1,1) tensors fields on the
 2-dimensional differentiable manifold M
sage: T11.has_coerce_map_from(G)
True
```

```
>>> from sage.all import *
>>> T11 = M.tensor_field_module((Integer(1),Integer(1))) ; T11
Free module T^(1,1)(M) of type-(1,1) tensors fields on the
 2-dimensional differentiable manifold M
>>> T11.has_coerce_map_from(G)
True
```

An explicit call of this coercion map is:

```
sage: tt = T11(t1) ; tt
Tensor field t of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: tt == t
True
```

```
>>> from sage.all import *
>>> tt = T11(t1) ; tt
Tensor field t of type (1,1) on the 2-dimensional differentiable
```

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```
 manifold M
>>> tt == t
True
```

An implicit call of the coercion map is performed to subtract an element of G from an element of $T^{(1,1)}(M)$:

```
sage: s = t - t1 ; s
Tensor field t-t of type (1,1) on
 the 2-dimensional differentiable manifold M
sage: s.parent() is T11
True
sage: s.display()
t-t = 0
```

```
>>> from sage.all import *
>>> s = t - t1 ; s
Tensor field t-t of type (1,1) on
 the 2-dimensional differentiable manifold M
>>> s.parent() is T11
True
>>> s.display()
t-t = 0
```

as well as for the reverse operation:

```
sage: s = t1 - t ; s
Tensor field t-t of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: s.display()
t-t = 0
```

```
>>> from sage.all import *
>>> s = t1 - t ; s
Tensor field t-t of type (1,1) on the 2-dimensional differentiable
 manifold M
>>> s.display()
t-t = 0
```

Element

alias of *AutomorphismFieldParal*

2.7.5 Tangent-Space Automorphism Fields

The class *AutomorphismField* implements fields of automorphisms of tangent spaces to a generic (a priori not parallelizable) differentiable manifold, while the class *AutomorphismFieldParal* is devoted to fields of automorphisms of tangent spaces to a parallelizable manifold. The latter play the important role of transitions between vector frames sharing the same domain on a differentiable manifold.

AUTHORS:

- Eric Gourgoulhon (2015): initial version
- Travis Scrimshaw (2016): review tweaks

class sage.manifolds.differentiable.automorphismfield.**AutomorphismField**(*vector_field_module*, *name=None*, *latex_name=None*)

Bases: *TensorField*

Field of automorphisms of tangent spaces to a generic (a priori not parallelizable) differentiable manifold.

Given a differentiable manifold U and a differentiable map $\Phi : U \to M$ to a differentiable manifold M, a *field of tangent-space automorphisms along* U *with values on* $M \supset \Phi(U)$ is a differentiable map

$$a : U \longrightarrow T^{(1,1)}M,$$

with $T^{(1,1)}M$ being the tensor bundle of type $(1,1)$ over M, such that

$$\forall p \in U, \ a(p) \in \mathrm{Aut}(T_{\Phi(p)}M),$$

i.e. $a(p)$ is an automorphism of the tangent space to M at the point $\Phi(p)$.

The standard case of a field of tangent-space automorphisms *on* a manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of **R**).

Note

If M is parallelizable, then *AutomorphismFieldParal* *must* be used instead.

INPUT:

- vector_field_module – module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on M via the map Φ
- name – (default: None) name given to the field
- latex_name – (default: None) LaTeX symbol to denote the field; if none is provided, the LaTeX symbol is set to name
- is_identity – boolean (default: False); determines whether the constructed object is a field of identity automorphisms

EXAMPLES:

Field of tangent-space automorphisms on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
....:                              restrictions1= x>0, restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: a = M.automorphism_field(name='a') ; a
Field of tangent-space automorphisms a on the 2-dimensional
 differentiable manifold M
sage: a.parent()
General linear group of the Module X(M) of vector fields on the
 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv = V.
↪chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
...                               restrictions1= x>Integer(0), restrictions2= u+v>
↪Integer(0))
>>> inv = transf.inverse()
>>> a = M.automorphism_field(name='a') ; a
Field of tangent-space automorphisms a on the 2-dimensional
 differentiable manifold M
>>> a.parent()
General linear group of the Module X(M) of vector fields on the
 2-dimensional differentiable manifold M
```

We first define the components of a with respect to the coordinate frame on U:

```
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: a[eU,:] = [[1,x], [0,2]]
```

```
>>> from sage.all import *
>>> eU = c_xy.frame() ; eV = c_uv.frame()
>>> a[eU,:] = [[Integer(1),x], [Integer(0),Integer(2)]]
```

It is equivalent to pass the components while defining a:

```
sage: a = M.automorphism_field({eU: [[1,x], [0,2]]}, name='a')
```

```
>>> from sage.all import *
>>> a = M.automorphism_field({eU: [[Integer(1),x], [Integer(0),Integer(2)]]},␣
↪name='a')
```

We then set the components with respect to the coordinate frame on V by extending the expressions of the components in the corresponding subframe on $W = U \cap V$:

```
sage: W = U.intersection(V)
sage: a.add_comp_by_continuation(eV, W, c_uv)
```

```
>>> from sage.all import *
>>> W = U.intersection(V)
>>> a.add_comp_by_continuation(eV, W, c_uv)
```

At this stage, the automorphism field a is fully defined:

```
sage: a.display(eU)
a = ∂/∂x⊗dx + x ∂/∂x⊗dy + 2 ∂/∂y⊗dy
sage: a.display(eV)
a = (1/4*u + 1/4*v + 3/2) ∂/∂u⊗du + (-1/4*u - 1/4*v - 1/2) ∂/∂u⊗dv
 + (1/4*u + 1/4*v - 1/2) ∂/∂v⊗du + (-1/4*u - 1/4*v + 3/2) ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> a.display(eU)
a = ∂/∂x⊗dx + x ∂/∂x⊗dy + 2 ∂/∂y⊗dy
>>> a.display(eV)
a = (1/4*u + 1/4*v + 3/2) ∂/∂u⊗du + (-1/4*u - 1/4*v - 1/2) ∂/∂u⊗dv
 + (1/4*u + 1/4*v - 1/2) ∂/∂v⊗du + (-1/4*u - 1/4*v + 3/2) ∂/∂v⊗dv
```

In particular, we may ask for its inverse on the whole manifold M:

```
sage: ia = a.inverse() ; ia
Field of tangent-space automorphisms a^(-1) on the 2-dimensional
 differentiable manifold M
sage: ia.display(eU)
a^(-1) = ∂/∂x⊗dx - 1/2*x ∂/∂x⊗dy + 1/2 ∂/∂y⊗dy
sage: ia.display(eV)
a^(-1) = (-1/8*u - 1/8*v + 3/4) ∂/∂u⊗du + (1/8*u + 1/8*v + 1/4) ∂/∂u⊗dv
 + (-1/8*u - 1/8*v + 1/4) ∂/∂v⊗du + (1/8*u + 1/8*v + 3/4) ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> ia = a.inverse() ; ia
Field of tangent-space automorphisms a^(-1) on the 2-dimensional
 differentiable manifold M
>>> ia.display(eU)
a^(-1) = ∂/∂x⊗dx - 1/2*x ∂/∂x⊗dy + 1/2 ∂/∂y⊗dy
>>> ia.display(eV)
a^(-1) = (-1/8*u - 1/8*v + 3/4) ∂/∂u⊗du + (1/8*u + 1/8*v + 1/4) ∂/∂u⊗dv
 + (-1/8*u - 1/8*v + 1/4) ∂/∂v⊗du + (1/8*u + 1/8*v + 3/4) ∂/∂v⊗dv
```

Equivalently, one can use the power minus one to get the inverse:

```
sage: ia is a^(-1)
True
```

```
>>> from sage.all import *
>>> ia is a**(-Integer(1))
True
```

or the operator ~:

```
sage: ia is ~a
True
```

```
>>> from sage.all import *
>>> ia is ~a
True
```

add_comp(*basis=None*)

Return the components of `self` w.r.t. a given module basis for assignment, keeping the components w.r.t. other bases.

To delete the components w.r.t. other bases, use the method `set_comp()` instead.

INPUT:

- `basis` – (default: `None`) basis in which the components are defined; if none is provided, the components are assumed to refer to the module's default basis

> **Warning**
>
> If the automorphism field has already components in other bases, it is the user's responsibility to make sure that the components to be added are consistent with them.

OUTPUT:

- components in the given basis, as an instance of the class `Components`; if such components did not exist previously, they are created

EXAMPLES:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: e_uv = c_uv.frame()
sage: a= M.automorphism_field(name='a')
sage: a.add_comp(e_uv)
2-indices components w.r.t. Coordinate frame (V, (∂/∂u,∂/∂v))
sage: a.add_comp(e_uv)[0,0] = u+v
sage: a.add_comp(e_uv)[1,1] = u+v
sage: a.display(e_uv)
a = (u + v) ∂/∂u⊗du + (u + v) ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> e_uv = c_uv.frame()
>>> a= M.automorphism_field(name='a')
>>> a.add_comp(e_uv)
2-indices components w.r.t. Coordinate frame (V, (∂/∂u,∂/∂v))
>>> a.add_comp(e_uv)[Integer(0),Integer(0)] = u+v
>>> a.add_comp(e_uv)[Integer(1),Integer(1)] = u+v
>>> a.display(e_uv)
a = (u + v) ∂/∂u⊗du + (u + v) ∂/∂v⊗dv
```

Setting the components in a new frame:

```
sage: e = V.vector_frame('e')
sage: a.add_comp(e)
2-indices components w.r.t. Vector frame (V, (e_0,e_1))
```

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```
sage: a.add_comp(e)[0,1] = u*v
sage: a.add_comp(e)[1,0] = u*v
sage: a.display(e)
a = u*v e_0⊗e^1 + u*v e_1⊗e^0
```

```
>>> from sage.all import *
>>> e = V.vector_frame('e')
>>> a.add_comp(e)
2-indices components w.r.t. Vector frame (V, (e_0,e_1))
>>> a.add_comp(e)[Integer(0),Integer(1)] = u*v
>>> a.add_comp(e)[Integer(1),Integer(0)] = u*v
>>> a.display(e)
a = u*v e_0⊗e^1 + u*v e_1⊗e^0
```

The components with respect to `e_uv` are kept:

```
sage: a.display(e_uv)
a = (u + v) ∂/∂u⊗du + (u + v) ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> a.display(e_uv)
a = (u + v) ∂/∂u⊗du + (u + v) ∂/∂v⊗dv
```

Since the identity map is a special element, its components cannot be changed:

```
sage: id = M.tangent_identity_field()
sage: id.add_comp(e)[0,1] = u*v
Traceback (most recent call last):
...
ValueError: the components of an immutable element cannot be
 changed
```

```
>>> from sage.all import *
>>> id = M.tangent_identity_field()
>>> id.add_comp(e)[Integer(0),Integer(1)] = u*v
Traceback (most recent call last):
...
ValueError: the components of an immutable element cannot be
 changed
```

copy(*name=None*, *latex_name=None*)

Return an exact copy of the automorphism field `self`.

INPUT:

- `name` – (default: `None`) name given to the copy
- `latex_name` – (default: `None`) LaTeX symbol to denote the copy; if none is provided, the LaTeX symbol is set to `name`

Note

The name and the derived quantities are not copied.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                    intersection_name='W', restrictions1= x>0,
....:                    restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: Id = M.tangent_identity_field(); Id
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold M
sage: one = Id.copy('1'); one
Field of tangent-space automorphisms 1 on the 2-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
...                    intersection_name='W', restrictions1= x>Integer(0),
...                    restrictions2= u+v>Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> Id = M.tangent_identity_field(); Id
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold M
>>> one = Id.copy('1'); one
Field of tangent-space automorphisms 1 on the 2-dimensional
 differentiable manifold M
```

inverse()

Return the inverse automorphism of `self`.

EXAMPLES:

Inverse of a field of tangent-space automorphisms on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: W = U.intersection(V)
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y),
....:    intersection_name='W', restrictions1= x>0, restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: a = M.automorphism_field({eU: [[1,x], [0,2]]}, name='a')
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: ia = a.inverse() ; ia
Field of tangent-space automorphisms a^(-1) on the 2-dimensional
```

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```
 differentiable manifold M
sage: a[eU,:], ia[eU,:]
(
[1 x]  [      1 -1/2*x]
[0 2], [      0    1/2]
)
sage: a[eV,:], ia[eV,:]
(
[ 1/4*u + 1/4*v + 3/2 -1/4*u - 1/4*v - 1/2]
[ 1/4*u + 1/4*v - 1/2 -1/4*u - 1/4*v + 3/2],
[-1/8*u - 1/8*v + 3/4  1/8*u + 1/8*v + 1/4]
[-1/8*u - 1/8*v + 1/4  1/8*u + 1/8*v + 3/4]
)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> W = U.intersection(V)
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> transf = c_xy.transition_map(c_uv, (x+y, x-y),
...    intersection_name='W', restrictions1= x>Integer(0), restrictions2= u+v>
↪Integer(0))
>>> inv = transf.inverse()
>>> eU = c_xy.frame() ; eV = c_uv.frame()
>>> a = M.automorphism_field({eU: [[Integer(1),x], [Integer(0),Integer(2)]]},␣
↪name='a')
>>> a.add_comp_by_continuation(eV, W, c_uv)
>>> ia = a.inverse() ; ia
Field of tangent-space automorphisms a^(-1) on the 2-dimensional
 differentiable manifold M
>>> a[eU,:], ia[eU,:]
(
[1 x]  [      1 -1/2*x]
[0 2], [      0    1/2]
)
>>> a[eV,:], ia[eV,:]
(
[ 1/4*u + 1/4*v + 3/2 -1/4*u - 1/4*v - 1/2]
[ 1/4*u + 1/4*v - 1/2 -1/4*u - 1/4*v + 3/2],
[-1/8*u - 1/8*v + 3/4  1/8*u + 1/8*v + 1/4]
[-1/8*u - 1/8*v + 1/4  1/8*u + 1/8*v + 3/4]
)
```

Let us check that ia is indeed the inverse of a:

```
sage: s = a.contract(ia)
sage: s[eU,:], s[eV,:]
(
[1 0]  [1 0]
[0 1], [0 1]
```

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```
)
sage: s = ia.contract(a)
sage: s[eU,:], s[eV,:]
(
[1 0]  [1 0]
[0 1], [0 1]
)
```

```
>>> from sage.all import *
>>> s = a.contract(ia)
>>> s[eU,:], s[eV,:]
(
[1 0]  [1 0]
[0 1], [0 1]
)
>>> s = ia.contract(a)
>>> s[eU,:], s[eV,:]
(
[1 0]  [1 0]
[0 1], [0 1]
)
```

The result is cached:

```
sage: a.inverse() is ia
True
```

```
>>> from sage.all import *
>>> a.inverse() is ia
True
```

Instead of `inverse()`, one can use the power minus one to get the inverse:

```
sage: ia is a^(-1)
True
```

```
>>> from sage.all import *
>>> ia is a**(-Integer(1))
True
```

or the operator ~:

```
sage: ia is ~a
True
```

```
>>> from sage.all import *
>>> ia is ~a
True
```

restrict (*subdomain*, *dest_map=None*)

Return the restriction of `self` to some subdomain.

This is a redefinition of *sage.manifolds.differentiable.tensorfield.TensorField.restrict()* to take into account the identity map.

INPUT:

- `subdomain` – *DifferentiableManifold* open subset V of `self._domain`
- `dest_map` – (default: `None`) *DiffMap*; destination map $\Phi : V \to N$, where N is a subdomain of `self._codomain`; if `None`, the restriction of `self.base_module().destination_map()` to V is used

OUTPUT: a *AutomorphismField* representing the restriction

EXAMPLES:

Restrictions of an automorphism field on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: U = M.open_subset('U') # the complement of the North pole
sage: stereoN.<x,y> = U.chart()  # stereographic coordinates from the North␣
↪pole
sage: eN = stereoN.frame() # the associated vector frame
sage: V =  M.open_subset('V') # the complement of the South pole
sage: stereoS.<u,v> = V.chart()  # stereographic coordinates from the South␣
↪pole
sage: eS = stereoS.frame() # the associated vector frame
sage: transf = stereoN.transition_map(stereoS, (x/(x^2+y^2), y/(x^2+y^2)),
....:                                  intersection_name='W',
....:                                  restrictions1= x^2+y^2!=0,
....:                                  restrictions2= u^2+v^2!=0)
sage: inv = transf.inverse() # transformation from stereoS to stereoN
sage: W = U.intersection(V) # the complement of the North and South poles
sage: stereoN_W = W.atlas()[0]  # restriction of stereo. coord. from North␣
↪pole to W
sage: stereoS_W = W.atlas()[1]  # restriction of stereo. coord. from South␣
↪pole to W
sage: eN_W = stereoN_W.frame() ; eS_W = stereoS_W.frame()
sage: a = M.automorphism_field({eN: [[1, atan(x^2+y^2)], [0,3]]},
....:                          name='a')
sage: a.add_comp_by_continuation(eS, W, chart=stereoS); a
Field of tangent-space automorphisms a on the 2-dimensional
 differentiable manifold S^2
sage: a.restrict(U)
Field of tangent-space automorphisms a on the Open subset U of the
 2-dimensional differentiable manifold S^2
sage: a.restrict(U)[eN,:]
[                1 arctan(x^2 + y^2)]
[                0                 3]
sage: a.restrict(V)
Field of tangent-space automorphisms a on the Open subset V of the
 2-dimensional differentiable manifold S^2
sage: a.restrict(V)[eS,:]
[   (u^4 + 10*u^2*v^2 + v^4 + 2*(u^3*v - u*v^3)*arctan(1/(u^2 + v^2)))/(u^4 +␣
↪2*u^2*v^2 + v^4)  -(4*u^3*v - 4*u*v^3 + (u^4 - 2*u^2*v^2 + v^4)*arctan(1/(u^
↪2 + v^2)))/(u^4 + 2*u^2*v^2 + v^4)]
[                    4*(u^2*v^2*arctan(1/(u^2 + v^2)) - u^3*v + u*v^3)/(u^4 +␣
```

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```
↪2*u^2*v^2 + v^4) (3*u^4 - 2*u^2*v^2 + 3*v^4 - 2*(u^3*v - u*v^3)*arctan(1/(u^
↪2 + v^2)))/(u^4 + 2*u^2*v^2 + v^4)]
sage: a.restrict(W)
Field of tangent-space automorphisms a on the Open subset W of the
 2-dimensional differentiable manifold S^2
sage: a.restrict(W)[eN_W,:]
[                1 arctan(x^2 + y^2)]
[                0                 3]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', start_index=Integer(1))
>>> U = M.open_subset('U') # the complement of the North pole
>>> stereoN = U.chart(names=('x', 'y',)); (x, y,) = stereoN._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> eN = stereoN.frame() # the associated vector frame
>>> V =  M.open_subset('V') # the complement of the South pole
>>> stereoS = V.chart(names=('u', 'v',)); (u, v,) = stereoS._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> eS = stereoS.frame() # the associated vector frame
>>> transf = stereoN.transition_map(stereoS, (x/(x**Integer(2)+y**Integer(2)),
↪ y/(x**Integer(2)+y**Integer(2))),
...                                  intersection_name='W',
...                                  restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                                  restrictions2=␣
↪u**Integer(2)+v**Integer(2)!=Integer(0))
>>> inv = transf.inverse() # transformation from stereoS to stereoN
>>> W = U.intersection(V) # the complement of the North and South poles
>>> stereoN_W = W.atlas()[Integer(0)]  # restriction of stereo. coord. from␣
↪North pole to W
>>> stereoS_W = W.atlas()[Integer(1)]  # restriction of stereo. coord. from␣
↪South pole to W
>>> eN_W = stereoN_W.frame() ; eS_W = stereoS_W.frame()
>>> a = M.automorphism_field({eN: [[Integer(1),␣
↪atan(x**Integer(2)+y**Integer(2))], [Integer(0),Integer(3)]]},
...                          name='a')
>>> a.add_comp_by_continuation(eS, W, chart=stereoS); a
Field of tangent-space automorphisms a on the 2-dimensional
 differentiable manifold S^2
>>> a.restrict(U)
Field of tangent-space automorphisms a on the Open subset U of the
 2-dimensional differentiable manifold S^2
>>> a.restrict(U)[eN,:]
[                1 arctan(x^2 + y^2)]
[                0                 3]
>>> a.restrict(V)
Field of tangent-space automorphisms a on the Open subset V of the
 2-dimensional differentiable manifold S^2
>>> a.restrict(V)[eS,:]
[   (u^4 + 10*u^2*v^2 + v^4 + 2*(u^3*v - u*v^3)*arctan(1/(u^2 + v^2)))/(u^4 +␣
↪2*u^2*v^2 + v^4)  -(4*u^3*v - 4*u*v^3 + (u^4 - 2*u^2*v^2 + v^4)*arctan(1/(u^
↪2 + v^2)))/(u^4 + 2*u^2*v^2 + v^4)]
```

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```
[                    4*(u^2*v^2*arctan(1/(u^2 + v^2)) - u^3*v + u*v^3)/(u^4 +
↪2*u^2*v^2 + v^4) (3*u^4 - 2*u^2*v^2 + 3*v^4 - 2*(u^3*v - u*v^3)*arctan(1/(u^
↪2 + v^2)))/(u^4 + 2*u^2*v^2 + v^4)]
>>> a.restrict(W)
Field of tangent-space automorphisms a on the Open subset W of the
 2-dimensional differentiable manifold S^2
>>> a.restrict(W)[eN_W,:]
[                1 arctan(x^2 + y^2)]
[                0                 3]
```

Restrictions of the field of tangent-space identity maps:

```
sage: id = M.tangent_identity_field() ; id
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold S^2
sage: id.restrict(U)
Field of tangent-space identity maps on the Open subset U of the
 2-dimensional differentiable manifold S^2
sage: id.restrict(U)[eN,:]
[1 0]
[0 1]
sage: id.restrict(V)
Field of tangent-space identity maps on the Open subset V of the
 2-dimensional differentiable manifold S^2
sage: id.restrict(V)[eS,:]
[1 0]
[0 1]
sage: id.restrict(W)[eN_W,:]
[1 0]
[0 1]
sage: id.restrict(W)[eS_W,:]
[1 0]
[0 1]
```

```
>>> from sage.all import *
>>> id = M.tangent_identity_field() ; id
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold S^2
>>> id.restrict(U)
Field of tangent-space identity maps on the Open subset U of the
 2-dimensional differentiable manifold S^2
>>> id.restrict(U)[eN,:]
[1 0]
[0 1]
>>> id.restrict(V)
Field of tangent-space identity maps on the Open subset V of the
 2-dimensional differentiable manifold S^2
>>> id.restrict(V)[eS,:]
[1 0]
[0 1]
>>> id.restrict(W)[eN_W,:]
[1 0]
```

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```
[0 1]
>>> id.restrict(W)[eS_W,:]
[1 0]
[0 1]
```

set_comp(*basis=None*)

Return the components of `self` w.r.t. a given module basis for assignment.

The components with respect to other bases are deleted, in order to avoid any inconsistency. To keep them, use the method *add_comp()* instead.

INPUT:

- `basis` – (default: `None`) basis in which the components are defined; if none is provided, the components are assumed to refer to the module's default basis

OUTPUT:

- components in the given basis, as an instance of the class `Components`; if such components did not exist previously, they are created.

EXAMPLES:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: e_uv = c_uv.frame()
sage: a= M.automorphism_field(name='a')
sage: a.set_comp(e_uv)
2-indices components w.r.t. Coordinate frame (V, (∂/∂u,∂/∂v))
sage: a.set_comp(e_uv)[0,0] = u+v
sage: a.set_comp(e_uv)[1,1] = u+v
sage: a.display(e_uv)
a = (u + v) ∂/∂u⊗du + (u + v) ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> e_uv = c_uv.frame()
>>> a= M.automorphism_field(name='a')
>>> a.set_comp(e_uv)
2-indices components w.r.t. Coordinate frame (V, (∂/∂u,∂/∂v))
>>> a.set_comp(e_uv)[Integer(0),Integer(0)] = u+v
>>> a.set_comp(e_uv)[Integer(1),Integer(1)] = u+v
>>> a.display(e_uv)
a = (u + v) ∂/∂u⊗du + (u + v) ∂/∂v⊗dv
```

Setting the components in a new frame:

```
sage: e = V.vector_frame('e')
sage: a.set_comp(e)
2-indices components w.r.t. Vector frame (V, (e_0,e_1))
sage: a.set_comp(e)[0,1] = u*v
sage: a.set_comp(e)[1,0] = u*v
sage: a.display(e)
a = u*v e_0⊗e^1 + u*v e_1⊗e^0
```

```
>>> from sage.all import *
>>> e = V.vector_frame('e')
>>> a.set_comp(e)
2-indices components w.r.t. Vector frame (V, (e_0,e_1))
>>> a.set_comp(e)[Integer(0),Integer(1)] = u*v
>>> a.set_comp(e)[Integer(1),Integer(0)] = u*v
>>> a.display(e)
a = u*v e_0⊗e^1 + u*v e_1⊗e^0
```

Since the frames `e` and `e_uv` are defined on the same domain, the components w.r.t. `e_uv` have been erased:

```
sage: a.display(c_uv.frame())
Traceback (most recent call last):
...
ValueError: no basis could be found for computing the components
 in the Coordinate frame (V, (∂/∂u,∂/∂v))
```

```
>>> from sage.all import *
>>> a.display(c_uv.frame())
Traceback (most recent call last):
...
ValueError: no basis could be found for computing the components
 in the Coordinate frame (V, (∂/∂u,∂/∂v))
```

Since the identity map is an immutable element, its components cannot be changed:

```
sage: id = M.tangent_identity_field()
sage: id.add_comp(e)[0,1] = u*v
Traceback (most recent call last):
...
ValueError: the components of an immutable element cannot be
 changed
```

```
>>> from sage.all import *
>>> id = M.tangent_identity_field()
>>> id.add_comp(e)[Integer(0),Integer(1)] = u*v
Traceback (most recent call last):
...
ValueError: the components of an immutable element cannot be
 changed
```

class sage.manifolds.differentiable.automorphismfield.**AutomorphismFieldParal**(*vector_field_module*, *name=None*, *latex_name=None*)

Bases: `FreeModuleAutomorphism`, `TensorFieldParal`

Field of tangent-space automorphisms with values on a parallelizable manifold.

Given a differentiable manifold U and a differentiable map $\Phi : U \to M$ to a parallelizable manifold M, a *field of tangent-space automorphisms along* U *with values on* $M \supset \Phi(U)$ is a differentiable map

$$a : U \longrightarrow T^{(1,1)}M$$

($T^{(1,1)}M$ being the tensor bundle of type $(1,1)$ over M) such that

$$\forall p \in U, \ a(p) \in \mathrm{Aut}(T_{\Phi(p)}M)$$

i.e. $a(p)$ is an automorphism of the tangent space to M at the point $\Phi(p)$.

The standard case of a field of tangent-space automorphisms *on* a manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note

If M is not parallelizable, the class `AutomorphismField` *must* be used instead.

INPUT:

- `vector_field_module` – free module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on M via the map Φ
- `name` – (default: `None`) name given to the field
- `latex_name` – (default: `None`) LaTeX symbol to denote the field; if none is provided, the LaTeX symbol is set to `name`

EXAMPLES:

A $\pi/3$-rotation in the Euclidean 2-plane:

```
sage: M = Manifold(2, 'R^2')
sage: c_xy.<x,y> = M.chart()
sage: rot = M.automorphism_field([[sqrt(3)/2, -1/2], [1/2, sqrt(3)/2]],
....:                            name='R'); rot
Field of tangent-space automorphisms R on the 2-dimensional
 differentiable manifold R^2
sage: rot.parent()
General linear group of the Free module X(R^2) of vector fields on the
 2-dimensional differentiable manifold R^2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> rot = M.automorphism_field([[sqrt(Integer(3))/Integer(2), -Integer(1)/
→Integer(2)], [Integer(1)/Integer(2), sqrt(Integer(3))/Integer(2)]],
```

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```
...                               name='R'); rot
Field of tangent-space automorphisms R on the 2-dimensional
 differentiable manifold R^2
>>> rot.parent()
General linear group of the Free module X(R^2) of vector fields on the
 2-dimensional differentiable manifold R^2
```

The inverse automorphism is obtained via the method `inverse()`:

```
sage: inv = rot.inverse() ; inv
Field of tangent-space automorphisms R^(-1) on the 2-dimensional
 differentiable manifold R^2
sage: latex(inv)
R^{-1}
sage: inv[:]
[1/2*sqrt(3)         1/2]
[       -1/2 1/2*sqrt(3)]
sage: rot[:]
[1/2*sqrt(3)        -1/2]
[        1/2 1/2*sqrt(3)]
sage: inv[:] * rot[:]  # check
[1 0]
[0 1]
```

```
>>> from sage.all import *
>>> inv = rot.inverse() ; inv
Field of tangent-space automorphisms R^(-1) on the 2-dimensional
 differentiable manifold R^2
>>> latex(inv)
R^{-1}
>>> inv[:]
[1/2*sqrt(3)         1/2]
[       -1/2 1/2*sqrt(3)]
>>> rot[:]
[1/2*sqrt(3)        -1/2]
[        1/2 1/2*sqrt(3)]
>>> inv[:] * rot[:]  # check
[1 0]
[0 1]
```

Equivalently, one can use the power minus one to get the inverse:

```
sage: inv is rot^(-1)
True
```

```
>>> from sage.all import *
>>> inv is rot**(-Integer(1))
True
```

or the operator ~:

```
sage: inv is ~rot
True
```

```
>>> from sage.all import *
>>> inv is ~rot
True
```

at (*point*)

Value of `self` at a given point.

If the current field of tangent-space automorphisms is

$$a:\ U \longrightarrow T^{(1,1)}M$$

associated with the differentiable map

$$\Phi:\ U \longrightarrow M,$$

where U and M are two manifolds (possibly $U = M$ and $\Phi = \mathrm{Id}_M$), then for any point $p \in U$, $a(p)$ is an automorphism of the tangent space $T_{\Phi(p)}M$.

INPUT:

- `point` – *ManifoldPoint*; point p in the domain of the field of automorphisms a

OUTPUT:

- the automorphism $a(p)$ of the tangent vector space $T_{\Phi(p)}M$

EXAMPLES:

Automorphism at some point of a tangent space of a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: a = M.automorphism_field([[1+exp(y), x*y], [0, 1+x^2]],
....:                          name='a')
sage: a.display()
a = (e^y + 1) ∂/∂x⊗dx + x*y ∂/∂x⊗dy + (x^2 + 1) ∂/∂y⊗dy
sage: p = M.point((-2,3), name='p') ; p
Point p on the 2-dimensional differentiable manifold M
sage: ap = a.at(p) ; ap
Automorphism a of the Tangent space at Point p on the
 2-dimensional differentiable manifold M
sage: ap.display()
a = (e^3 + 1) ∂/∂x⊗dx - 6 ∂/∂x⊗dy + 5 ∂/∂y⊗dy
sage: ap.parent()
General linear group of the Tangent space at Point p on the
 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> a = M.automorphism_field([[Integer(1)+exp(y), x*y], [Integer(0),
↪Integer(1)+x**Integer(2)]],
...                          name='a')
```

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```
>>> a.display()
a = (e^y + 1) ∂/∂x⊗dx + x*y ∂/∂x⊗dy + (x^2 + 1) ∂/∂y⊗dy
>>> p = M.point((-Integer(2),Integer(3)), name='p') ; p
Point p on the 2-dimensional differentiable manifold M
>>> ap = a.at(p) ; ap
Automorphism a of the Tangent space at Point p on the
 2-dimensional differentiable manifold M
>>> ap.display()
a = (e^3 + 1) ∂/∂x⊗dx - 6 ∂/∂x⊗dy + 5 ∂/∂y⊗dy
>>> ap.parent()
General linear group of the Tangent space at Point p on the
 2-dimensional differentiable manifold M
```

The identity map of the tangent space at point `p`:

```
sage: id = M.tangent_identity_field() ; id
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold M
sage: idp = id.at(p) ; idp
Identity map of the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
sage: idp is M.tangent_space(p).identity_map()
True
sage: idp.display()
Id = ∂/∂x⊗dx + ∂/∂y⊗dy
sage: idp.parent()
General linear group of the Tangent space at Point p on the
 2-dimensional differentiable manifold M
sage: idp * ap == ap
True
```

```
>>> from sage.all import *
>>> id = M.tangent_identity_field() ; id
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold M
>>> idp = id.at(p) ; idp
Identity map of the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
>>> idp is M.tangent_space(p).identity_map()
True
>>> idp.display()
Id = ∂/∂x⊗dx + ∂/∂y⊗dy
>>> idp.parent()
General linear group of the Tangent space at Point p on the
 2-dimensional differentiable manifold M
>>> idp * ap == ap
True
```

inverse()

Return the inverse automorphism of `self`.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: a = M.automorphism_field([[0, 2], [-1, 0]], name='a')
sage: b = a.inverse(); b
Field of tangent-space automorphisms a^(-1) on the 2-dimensional
 differentiable manifold M
sage: b[:]
[  0  -1]
[1/2   0]
sage: a[:]
[ 0  2]
[-1  0]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> a = M.automorphism_field([[Integer(0), Integer(2)], [-Integer(1),
↪Integer(0)]], name='a')
>>> b = a.inverse(); b
Field of tangent-space automorphisms a^(-1) on the 2-dimensional
 differentiable manifold M
>>> b[:]
[  0  -1]
[1/2   0]
>>> a[:]
[ 0  2]
[-1  0]
```

The result is cached:

```
sage: a.inverse() is b
True
```

```
>>> from sage.all import *
>>> a.inverse() is b
True
```

Instead of `inverse()`, one can use the power minus one to get the inverse:

```
sage: b is a^(-1)
True
```

```
>>> from sage.all import *
>>> b is a**(-Integer(1))
True
```

or the operator ~:

```
sage: b is ~a
True
```

```
>>> from sage.all import *
>>> b is ~a
True
```

restrict (*subdomain*, *dest_map=None*)

Return the restriction of `self` to some subset of its domain.

If such restriction has not been defined yet, it is constructed here.

This is a redefinition of *sage.manifolds.differentiable.tensorfield_paral.TensorFieldParal.restrict()* to take into account the identity map.

INPUT:

- `subdomain` – *DifferentiableManifold*; open subset V of `self._domain`
- `dest_map` – (default: `None`) *DiffMap* destination map $\Phi : V \to N$, where N is a subset of `self._codomain`; if `None`, the restriction of `self.base_module().destination_map()` to V is used

OUTPUT: a *AutomorphismFieldParal* representing the restriction

EXAMPLES:

Restriction of an automorphism field defined on $\mathbf{R}^2$ to a disk:

```
sage: M = Manifold(2, 'R^2')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: D = M.open_subset('D') # the unit open disc
sage: c_cart_D = c_cart.restrict(D, x^2+y^2<1)
sage: a = M.automorphism_field([[1, x*y], [0, 3]], name='a'); a
Field of tangent-space automorphisms a on the 2-dimensional
 differentiable manifold R^2
sage: a.restrict(D)
Field of tangent-space automorphisms a on the Open subset D of the
 2-dimensional differentiable manifold R^2
sage: a.restrict(D)[:]
[  1 x*y]
[  0   3]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#
↪Cartesian coordinates on R^2
>>> D = M.open_subset('D') # the unit open disc
>>> c_cart_D = c_cart.restrict(D, x**Integer(2)+y**Integer(2)<Integer(1))
>>> a = M.automorphism_field([[Integer(1), x*y], [Integer(0), Integer(3)]],
↪name='a'); a
Field of tangent-space automorphisms a on the 2-dimensional
 differentiable manifold R^2
>>> a.restrict(D)
Field of tangent-space automorphisms a on the Open subset D of the
 2-dimensional differentiable manifold R^2
>>> a.restrict(D)[:]
[  1 x*y]
[  0   3]
```

Restriction to the disk of the field of tangent-space identity maps:

```
sage: id = M.tangent_identity_field() ; id
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold R^2
sage: id.restrict(D)
Field of tangent-space identity maps on the Open subset D of the
 2-dimensional differentiable manifold R^2
sage: id.restrict(D)[:]
[1 0]
[0 1]
sage: id.restrict(D) == D.tangent_identity_field()
True
```

```
>>> from sage.all import *
>>> id = M.tangent_identity_field() ; id
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold R^2
>>> id.restrict(D)
Field of tangent-space identity maps on the Open subset D of the
 2-dimensional differentiable manifold R^2
>>> id.restrict(D)[:]
[1 0]
[0 1]
>>> id.restrict(D) == D.tangent_identity_field()
True
```

2.8 Tensor Fields

2.8.1 Tensor Field Modules

The set of tensor fields along a differentiable manifold U with values on a differentiable manifold M via a differentiable map $\Phi : U \to M$ (possibly $U = M$ and $\Phi = \mathrm{Id}_M$) is a module over the algebra $C^k(U)$ of differentiable scalar fields on U. It is a free module if and only if M is parallelizable. Accordingly, two classes are devoted to tensor field modules:

- `TensorFieldModule` for tensor fields with values on a generic (in practice, not parallelizable) differentiable manifold M,
- `TensorFieldFreeModule` for tensor fields with values on a parallelizable manifold M.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2014-2015): initial version
- Travis Scrimshaw (2016): review tweaks

REFERENCES:

- [KN1963]
- [Lee2013]
- [ONe1983]

class `sage.manifolds.differentiable.tensorfield_module.`**`TensorFieldFreeModule`**(*vector_field_module*, *tensor_type*)

Bases: `TensorFreeModule`

Free module of tensor fields of a given type (k,l) along a differentiable manifold U with values on a parallelizable manifold M, via a differentiable map $U \to M$.

Given two nonnegative integers k and l and a differentiable map

$$\Phi : U \longrightarrow M,$$

the *tensor field module* $T^{(k,l)}(U,\Phi)$ is the set of all tensor fields of the type

$$t : U \longrightarrow T^{(k,l)}M$$

(where $T^{(k,l)}M$ is the tensor bundle of type (k,l) over M) such that

$$t(p) \in T^{(k,l)}(T_{\Phi(p)}M)$$

for all $p \in U$, i.e. $t(p)$ is a tensor of type (k,l) on the tangent vector space $T_{\Phi(p)}M$. Since M is parallelizable, the set $T^{(k,l)}(U,\Phi)$ is a free module over $C^k(U)$, the ring (algebra) of differentiable scalar fields on U (see `DiffScalarFieldAlgebra`).

The standard case of tensor fields *on* a differentiable manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$; we then denote $T^{(k,l)}(M,\mathrm{Id}_M)$ by merely $T^{(k,l)}(M)$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note

If M is not parallelizable, the class `TensorFieldModule` should be used instead, for $T^{(k,l)}(U,\Phi)$ is no longer a free module.

INPUT:

- `vector_field_module` – free module $\mathfrak{X}(U,\Phi)$ of vector fields along U associated with the map $\Phi : U \to M$
- `tensor_type` – pair (k,l) with k being the contravariant rank and l the covariant rank

EXAMPLES:

Module of type-$(2,0)$ tensor fields on $\mathbf{R}^3$:

```
sage: M = Manifold(3, 'R^3')
sage: c_xyz.<x,y,z> = M.chart()  # Cartesian coordinates
sage: T20 = M.tensor_field_module((2,0)) ; T20
Free module T^(2,0)(R^3) of type-(2,0) tensors fields on the
 3-dimensional differentiable manifold R^3
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'R^3')
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_ngens(3)#
↪Cartesian coordinates
>>> T20 = M.tensor_field_module((Integer(2),Integer(0))) ; T20
Free module T^(2,0)(R^3) of type-(2,0) tensors fields on the
 3-dimensional differentiable manifold R^3
```

$T^{(2,0)}(\mathbf{R}^3)$ is a module over the algebra $C^k(\mathbf{R}^3)$:

```
sage: T20.category()
Category of tensor products of finite dimensional modules over
 Algebra of differentiable scalar fields on the 3-dimensional differentiable␣
↪manifold R^3
sage: T20.base_ring() is M.scalar_field_algebra()
True
```

```
>>> from sage.all import *
>>> T20.category()
Category of tensor products of finite dimensional modules over
 Algebra of differentiable scalar fields on the 3-dimensional differentiable␣
↪manifold R^3
>>> T20.base_ring() is M.scalar_field_algebra()
True
```

$T^{(2,0)}(\mathbf{R}^3)$ is a free module:

```
sage: from sage.tensor.modules.finite_rank_free_module import␣
↪FiniteRankFreeModule_abstract
sage: isinstance(T20, FiniteRankFreeModule_abstract)
True
```

```
>>> from sage.all import *
>>> from sage.tensor.modules.finite_rank_free_module import FiniteRankFreeModule_
↪abstract
>>> isinstance(T20, FiniteRankFreeModule_abstract)
True
```

because $M = \mathbf{R}^3$ is parallelizable:

```
sage: M.is_manifestly_parallelizable()
True
```

```
>>> from sage.all import *
>>> M.is_manifestly_parallelizable()
True
```

The zero element:

```
sage: z = T20.zero() ; z
Tensor field zero of type (2,0) on the 3-dimensional differentiable
 manifold R^3
sage: z[:]
[0 0 0]
[0 0 0]
[0 0 0]
```

```
>>> from sage.all import *
>>> z = T20.zero() ; z
Tensor field zero of type (2,0) on the 3-dimensional differentiable
 manifold R^3
>>> z[:]
```

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```
[0 0 0]
[0 0 0]
[0 0 0]
```

A random element:

```
sage: t = T20.an_element() ; t
Tensor field of type (2,0) on the 3-dimensional differentiable
 manifold R^3
sage: t[:]
[2 0 0]
[0 0 0]
[0 0 0]
```

```
>>> from sage.all import *
>>> t = T20.an_element() ; t
Tensor field of type (2,0) on the 3-dimensional differentiable
 manifold R^3
>>> t[:]
[2 0 0]
[0 0 0]
[0 0 0]
```

The module $T^{(2,0)}(\mathbf{R}^3)$ coerces to any module of type-$(2,0)$ tensor fields defined on some subdomain of $\mathbf{R}^3$:

```
sage: U = M.open_subset('U', coord_def={c_xyz: x>0})
sage: T20U = U.tensor_field_module((2,0))
sage: T20U.has_coerce_map_from(T20)
True
sage: T20.has_coerce_map_from(T20U)  # the reverse is not true
False
sage: T20U.coerce_map_from(T20)
Coercion map:
  From: Free module T^(2,0)(R^3) of type-(2,0) tensors fields on the 3-
↪dimensional differentiable manifold R^3
  To:   Free module T^(2,0)(U) of type-(2,0) tensors fields on the Open subset U␣
↪of the 3-dimensional differentiable manifold R^3
```

```
>>> from sage.all import *
>>> U = M.open_subset('U', coord_def={c_xyz: x>Integer(0)})
>>> T20U = U.tensor_field_module((Integer(2),Integer(0)))
>>> T20U.has_coerce_map_from(T20)
True
>>> T20.has_coerce_map_from(T20U)  # the reverse is not true
False
>>> T20U.coerce_map_from(T20)
Coercion map:
  From: Free module T^(2,0)(R^3) of type-(2,0) tensors fields on the 3-
↪dimensional differentiable manifold R^3
  To:   Free module T^(2,0)(U) of type-(2,0) tensors fields on the Open subset U␣
↪of the 3-dimensional differentiable manifold R^3
```

The coercion map is actually the *restriction* of tensor fields defined on $\mathbf{R}^3$ to U.

There is also a coercion map from fields of tangent-space automorphisms to tensor fields of type $(1,1)$:

```
sage: T11 = M.tensor_field_module((1,1)) ; T11
Free module T^(1,1)(R^3) of type-(1,1) tensors fields on the
 3-dimensional differentiable manifold R^3
sage: GL = M.automorphism_field_group() ; GL
General linear group of the Free module X(R^3) of vector fields on the
 3-dimensional differentiable manifold R^3
sage: T11.has_coerce_map_from(GL)
True
```

```
>>> from sage.all import *
>>> T11 = M.tensor_field_module((Integer(1),Integer(1))) ; T11
Free module T^(1,1)(R^3) of type-(1,1) tensors fields on the
 3-dimensional differentiable manifold R^3
>>> GL = M.automorphism_field_group() ; GL
General linear group of the Free module X(R^3) of vector fields on the
 3-dimensional differentiable manifold R^3
>>> T11.has_coerce_map_from(GL)
True
```

An explicit call to this coercion map is:

```
sage: id = GL.one() ; id
Field of tangent-space identity maps on the 3-dimensional
 differentiable manifold R^3
sage: tid = T11(id) ; tid
Tensor field Id of type (1,1) on the 3-dimensional differentiable
 manifold R^3
sage: tid[:]
[1 0 0]
[0 1 0]
[0 0 1]
```

```
>>> from sage.all import *
>>> id = GL.one() ; id
Field of tangent-space identity maps on the 3-dimensional
 differentiable manifold R^3
>>> tid = T11(id) ; tid
Tensor field Id of type (1,1) on the 3-dimensional differentiable
 manifold R^3
>>> tid[:]
[1 0 0]
[0 1 0]
[0 0 1]
```

Element

alias of *TensorFieldParal*

class sage.manifolds.differentiable.tensorfield_module.**TensorFieldModule**(*vector_field_module*, *tensor_type*, *category=None*)

Bases: UniqueRepresentation, ReflexiveModule_tensor

Module of tensor fields of a given type (k, l) along a differentiable manifold U with values on a differentiable manifold M, via a differentiable map $U \to M$.

Given two nonnegative integers k and l and a differentiable map

$$\Phi : U \longrightarrow M,$$

the *tensor field module* $T^{(k,l)}(U, \Phi)$ is the set of all tensor fields of the type

$$t : U \longrightarrow T^{(k,l)}M$$

(where $T^{(k,l)}M$ is the tensor bundle of type (k, l) over M) such that

$$t(p) \in T^{(k,l)}(T_{\Phi(p)}M)$$

for all $p \in U$, i.e. $t(p)$ is a tensor of type (k, l) on the tangent vector space $T_{\Phi(p)}M$. The set $T^{(k,l)}(U, \Phi)$ is a module over $C^k(U)$, the ring (algebra) of differentiable scalar fields on U (see `DiffScalarFieldAlgebra`).

The standard case of tensor fields *on* a differentiable manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$; we then denote $T^{(k,l)}(M, \mathrm{Id}_M)$ by merely $T^{(k,l)}(M)$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note

If M is parallelizable, the class `TensorFieldFreeModule` should be used instead.

INPUT:

- `vector_field_module` – module $\mathfrak{X}(U, \Phi)$ of vector fields along U associated with the map $\Phi : U \to M$
- `tensor_type` – pair (k, l) with k being the contravariant rank and l the covariant rank

EXAMPLES:

Module of type-$(2, 0)$ tensor fields on the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: T20 = M.tensor_field_module((2,0)); T20
Module T^(2,0)(M) of type-(2,0) tensors fields on the 2-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
```

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```
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                    intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                    restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V)
>>> T20 = M.tensor_field_module((Integer(2),Integer(0))); T20
Module T^(2,0)(M) of type-(2,0) tensors fields on the 2-dimensional
 differentiable manifold M
```

$T^{(2,0)}(M)$ is a module over the algebra $C^k(M)$:

```
sage: T20.category()
Category of tensor products of modules over Algebra of differentiable scalar␣
↪fields
 on the 2-dimensional differentiable manifold M
sage: T20.base_ring() is M.scalar_field_algebra()
True
```

```
>>> from sage.all import *
>>> T20.category()
Category of tensor products of modules over Algebra of differentiable scalar␣
↪fields
 on the 2-dimensional differentiable manifold M
>>> T20.base_ring() is M.scalar_field_algebra()
True
```

$T^{(2,0)}(M)$ is not a free module:

```
sage: from sage.tensor.modules.finite_rank_free_module import␣
↪FiniteRankFreeModule_abstract
sage: isinstance(T20, FiniteRankFreeModule_abstract)
False
```

```
>>> from sage.all import *
>>> from sage.tensor.modules.finite_rank_free_module import FiniteRankFreeModule_
↪abstract
>>> isinstance(T20, FiniteRankFreeModule_abstract)
False
```

because $M = S^2$ is not parallelizable:

```
sage: M.is_manifestly_parallelizable()
False
```

```
>>> from sage.all import *
>>> M.is_manifestly_parallelizable()
```

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```
False
```

On the contrary, the module of type-$(2,0)$ tensor fields on U is a free module, since U is parallelizable (being a coordinate domain):

```
sage: T20U = U.tensor_field_module((2,0))
sage: isinstance(T20U, FiniteRankFreeModule_abstract)
True
sage: U.is_manifestly_parallelizable()
True
```

```
>>> from sage.all import *
>>> T20U = U.tensor_field_module((Integer(2),Integer(0)))
>>> isinstance(T20U, FiniteRankFreeModule_abstract)
True
>>> U.is_manifestly_parallelizable()
True
```

The zero element:

```
sage: z = T20.zero() ; z
Tensor field zero of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: z is T20(0)
True
sage: z[c_xy.frame(),:]
[0 0]
[0 0]
sage: z[c_uv.frame(),:]
[0 0]
[0 0]
```

```
>>> from sage.all import *
>>> z = T20.zero() ; z
Tensor field zero of type (2,0) on the 2-dimensional differentiable
 manifold M
>>> z is T20(Integer(0))
True
>>> z[c_xy.frame(),:]
[0 0]
[0 0]
>>> z[c_uv.frame(),:]
[0 0]
[0 0]
```

The module $T^{(2,0)}(M)$ coerces to any module of type-$(2,0)$ tensor fields defined on some subdomain of M, for instance $T^{(2,0)}(U)$:

```
sage: T20U.has_coerce_map_from(T20)
True
```

```
>>> from sage.all import *
>>> T20U.has_coerce_map_from(T20)
True
```

The reverse is not true:

```
sage: T20.has_coerce_map_from(T20U)
False
```

```
>>> from sage.all import *
>>> T20.has_coerce_map_from(T20U)
False
```

The coercion:

```
sage: T20U.coerce_map_from(T20)
Coercion map:
  From: Module T^(2,0)(M) of type-(2,0) tensors fields on the 2-dimensional␣
↪differentiable manifold M
  To:   Free module T^(2,0)(U) of type-(2,0) tensors fields on the Open subset U␣
↪of the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> T20U.coerce_map_from(T20)
Coercion map:
  From: Module T^(2,0)(M) of type-(2,0) tensors fields on the 2-dimensional␣
↪differentiable manifold M
  To:   Free module T^(2,0)(U) of type-(2,0) tensors fields on the Open subset U␣
↪of the 2-dimensional differentiable manifold M
```

The coercion map is actually the *restriction* of tensor fields defined on M to U:

```
sage: t = M.tensor_field(2,0, name='t')
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: t[eU,:] = [[2,0], [0,-3]]
sage: t.add_comp_by_continuation(eV, W, chart=c_uv)
sage: T20U(t)  # the conversion map in action
Tensor field t of type (2,0) on the Open subset U of the 2-dimensional
 differentiable manifold M
sage: T20U(t) is t.restrict(U)
True
```

```
>>> from sage.all import *
>>> t = M.tensor_field(Integer(2),Integer(0), name='t')
>>> eU = c_xy.frame() ; eV = c_uv.frame()
>>> t[eU,:] = [[Integer(2),Integer(0)], [Integer(0),-Integer(3)]]
>>> t.add_comp_by_continuation(eV, W, chart=c_uv)
>>> T20U(t)  # the conversion map in action
Tensor field t of type (2,0) on the Open subset U of the 2-dimensional
 differentiable manifold M
>>> T20U(t) is t.restrict(U)
True
```

There is also a coercion map from fields of tangent-space automorphisms to tensor fields of type-$(1,1)$:

```
sage: T11 = M.tensor_field_module((1,1)) ; T11
Module T^(1,1)(M) of type-(1,1) tensors fields on the 2-dimensional
 differentiable manifold M
sage: GL = M.automorphism_field_group() ; GL
General linear group of the Module X(M) of vector fields on the
 2-dimensional differentiable manifold M
sage: T11.has_coerce_map_from(GL)
True
```

```
>>> from sage.all import *
>>> T11 = M.tensor_field_module((Integer(1),Integer(1))) ; T11
Module T^(1,1)(M) of type-(1,1) tensors fields on the 2-dimensional
 differentiable manifold M
>>> GL = M.automorphism_field_group() ; GL
General linear group of the Module X(M) of vector fields on the
 2-dimensional differentiable manifold M
>>> T11.has_coerce_map_from(GL)
True
```

Explicit call to the coercion map:

```
sage: a = GL.one() ; a
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold M
sage: a.parent()
General linear group of the Module X(M) of vector fields on the
 2-dimensional differentiable manifold M
sage: ta = T11.coerce(a) ; ta
Tensor field Id of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: ta.parent()
Module T^(1,1)(M) of type-(1,1) tensors fields on the 2-dimensional
 differentiable manifold M
sage: ta[eU,:]  # ta on U
[1 0]
[0 1]
sage: ta[eV,:]  # ta on V
[1 0]
[0 1]
```

```
>>> from sage.all import *
>>> a = GL.one() ; a
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold M
>>> a.parent()
General linear group of the Module X(M) of vector fields on the
 2-dimensional differentiable manifold M
>>> ta = T11.coerce(a) ; ta
Tensor field Id of type (1,1) on the 2-dimensional differentiable
 manifold M
>>> ta.parent()
```

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```
Module T^(1,1)(M) of type-(1,1) tensors fields on the 2-dimensional
 differentiable manifold M
>>> ta[eU,:]  # ta on U
[1 0]
[0 1]
>>> ta[eV,:]  # ta on V
[1 0]
[0 1]
```

Element

alias of *TensorField*

base_module()

Return the vector field module on which `self` is constructed.

OUTPUT:

- a *VectorFieldModule* representing the module on which `self` is defined

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: T13 = M.tensor_field_module((1,3))
sage: T13.base_module()
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
sage: T13.base_module() is M.vector_field_module()
True
sage: T13.base_module().base_ring()
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> T13 = M.tensor_field_module((Integer(1),Integer(3)))
>>> T13.base_module()
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
>>> T13.base_module() is M.vector_field_module()
True
>>> T13.base_module().base_ring()
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
```

tensor_type()

Return the tensor type of `self`.

OUTPUT: pair (k, l) of nonnegative integers such that the tensor fields belonging to this module are of type (k, l)

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: T13 = M.tensor_field_module((1,3))
```

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```
sage: T13.tensor_type()
(1, 3)
sage: T20 = M.tensor_field_module((2,0))
sage: T20.tensor_type()
(2, 0)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> T13 = M.tensor_field_module((Integer(1),Integer(3)))
>>> T13.tensor_type()
(1, 3)
>>> T20 = M.tensor_field_module((Integer(2),Integer(0)))
>>> T20.tensor_type()
(2, 0)
```

zero()

Return the zero of `self`.

2.8.2 Tensor Fields

The class *TensorField* implements tensor fields on differentiable manifolds. The derived class *TensorFieldParal* is devoted to tensor fields with values on parallelizable manifolds.

Various derived classes of *TensorField* are devoted to specific tensor fields:

- *VectorField* for vector fields (rank-1 contravariant tensor fields)
- *AutomorphismField* for fields of tangent-space automorphisms
- *DiffForm* for differential forms (fully antisymmetric covariant tensor fields)
- *MultivectorField* for multivector fields (fully antisymmetric contravariant tensor fields)

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015) : initial version
- Travis Scrimshaw (2016): review tweaks
- Eric Gourgoulhon (2018): operators divergence, Laplacian and d'Alembertian; method *TensorField.along()*
- Florentin Jaffredo (2018) : series expansion with respect to a given parameter
- Michael Jung (2019): improve treatment of the zero element; add method *TensorField.copy_from()*
- Eric Gourgoulhon (2020): add method *TensorField.apply_map()*

REFERENCES:

- [KN1963]
- [Lee2013]
- [ONe1983]

class sage.manifolds.differentiable.tensorfield.**TensorField**(*vector_field_module: VectorFieldModule, tensor_type: TensorType, name: str | None = None, latex_name: str | None = None, sym=None, antisym=None, parent=None*)

Bases: `ModuleElementWithMutability`

Tensor field along a differentiable manifold.

An instance of this class is a tensor field along a differentiable manifold U with values on a differentiable manifold M, via a differentiable map $\Phi : U \to M$. More precisely, given two nonnegative integers k and l and a differentiable map

$$\Phi : U \longrightarrow M,$$

a *tensor field of type* (k, l) *along* U *with values on* M is a differentiable map

$$t : U \longrightarrow T^{(k,l)}M$$

(where $T^{(k,l)}M$ is the tensor bundle of type (k, l) over M) such that

$$\forall p \in U, \ t(p) \in T^{(k,l)}(T_q M)$$

i.e. $t(p)$ is a tensor of type (k, l) on the tangent space $T_q M$ at the point $q = \Phi(p)$, that is to say a multilinear map

$$t(p) : \underbrace{T_q^* M \times \cdots \times T_q^* M}_{k \text{ times}} \times \underbrace{T_q M \times \cdots \times T_q M}_{l \text{ times}} \longrightarrow K,$$

where $T_q^* M$ is the dual vector space to $T_q M$ and K is the topological field over which the manifold M is defined. The integer $k + l$ is called the *tensor rank*.

The standard case of a tensor field *on* a differentiable manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of **R**).

If M is parallelizable, the class *TensorFieldParal* should be used instead.

This is a Sage *element* class, the corresponding *parent* class being *TensorFieldModule*.

INPUT:

- `vector_field_module` – module $\mathfrak{X}(U, \Phi)$ of vector fields along U associated with the map $\Phi : U \to M$ (cf. *VectorFieldModule*)
- `tensor_type` – pair (k, l) with k being the contravariant rank and l the covariant rank
- `name` – (default: `None`) name given to the tensor field
- `latex_name` – (default: `None`) LaTeX symbol to denote the tensor field; if none is provided, the LaTeX symbol is set to `name`
- `sym` – (default: `None`) a symmetry or a list of symmetries among the tensor arguments: each symmetry is described by a tuple containing the positions of the involved arguments, with the convention `position = 0` for the first argument; for instance:
  - `sym = (0,1)` for a symmetry between the 1st and 2nd arguments
  - `sym = [(0,2), (1,3,4)]` for a symmetry between the 1st and 3rd arguments and a symmetry between the 2nd, 4th and 5th arguments.
- `antisym` – (default: `None`) antisymmetry or list of antisymmetries among the arguments, with the same convention as for `sym`
- `parent` – (default: `None`) some specific parent (e.g. exterior power for differential forms); if `None`, `vector_field_module.tensor_module(k,l)` is used

EXAMPLES:

Tensor field of type (0,2) on the sphere S^2:

```
sage: M = Manifold(2, 'S^2') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: t = M.tensor_field(0,2, name='t') ; t
Tensor field t of type (0,2) on the 2-dimensional differentiable
 manifold S^2
sage: t.parent()
Module T^(0,2)(S^2) of type-(0,2) tensors fields on the 2-dimensional
 differentiable manifold S^2
sage: t.parent().category()
Category of tensor products of modules over Algebra of differentiable scalar␣
↪fields
 on the 2-dimensional differentiable manifold S^2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                 intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                 restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V)
>>> t = M.tensor_field(Integer(0),Integer(2), name='t') ; t
Tensor field t of type (0,2) on the 2-dimensional differentiable
 manifold S^2
>>> t.parent()
Module T^(0,2)(S^2) of type-(0,2) tensors fields on the 2-dimensional
 differentiable manifold S^2
>>> t.parent().category()
Category of tensor products of modules over Algebra of differentiable scalar␣
↪fields
 on the 2-dimensional differentiable manifold S^2
```

The parent of t is not a free module, for the sphere S^2 is not parallelizable:

```
sage: isinstance(t.parent(), FiniteRankFreeModule)
False
```

```
>>> from sage.all import *
>>> isinstance(t.parent(), FiniteRankFreeModule)
False
```

To fully define t, we have to specify its components in some vector frames defined on subsets of S^2; let us start by the open subset U:

```
sage: eU = c_xy.frame()
sage: t[eU,:] = [[1,0], [-2,3]]
sage: t.display(eU)
t = dx⊗dx - 2 dy⊗dx + 3 dy⊗dy
```

```
>>> from sage.all import *
>>> eU = c_xy.frame()
>>> t[eU,:] = [[Integer(1),Integer(0)], [-Integer(2),Integer(3)]]
>>> t.display(eU)
t = dx⊗dx - 2 dy⊗dx + 3 dy⊗dy
```

To set the components of t on V consistently, we copy the expressions of the components in the common subset W:

```
sage: eV = c_uv.frame()
sage: eVW = eV.restrict(W)
sage: c_uvW = c_uv.restrict(W)
sage: t[eV,0,0] = t[eVW,0,0,c_uvW].expr()  # long time
sage: t[eV,0,1] = t[eVW,0,1,c_uvW].expr()  # long time
sage: t[eV,1,0] = t[eVW,1,0,c_uvW].expr()  # long time
sage: t[eV,1,1] = t[eVW,1,1,c_uvW].expr()  # long time
```

```
>>> from sage.all import *
>>> eV = c_uv.frame()
>>> eVW = eV.restrict(W)
>>> c_uvW = c_uv.restrict(W)
>>> t[eV,Integer(0),Integer(0)] = t[eVW,Integer(0),Integer(0),c_uvW].expr()  #␣
↪long time
>>> t[eV,Integer(0),Integer(1)] = t[eVW,Integer(0),Integer(1),c_uvW].expr()  #␣
↪long time
>>> t[eV,Integer(1),Integer(0)] = t[eVW,Integer(1),Integer(0),c_uvW].expr()  #␣
↪long time
>>> t[eV,Integer(1),Integer(1)] = t[eVW,Integer(1),Integer(1),c_uvW].expr()  #␣
↪long time
```

Actually, the above operation can be performed in a single line by means of the method *add_comp_by_continuation()*:

```
sage: t.add_comp_by_continuation(eV, W, chart=c_uv)  # long time
```

```
>>> from sage.all import *
>>> t.add_comp_by_continuation(eV, W, chart=c_uv)  # long time
```

At this stage, t is fully defined, having components in frames eU and eV and the union of the domains of eU and eV being the whole manifold:

```
sage: t.display(eV)  # long time
t = (u^4 - 4*u^3*v + 10*u^2*v^2 + 4*u*v^3 + v^4)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 +␣
↪4*u^2*v^6 + v^8) du⊗du
 - 4*(u^3*v + 2*u^2*v^2 - u*v^3)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 + 4*u^2*v^6 + v^8)␣
↪du⊗dv
 + 2*(u^4 - 2*u^3*v - 2*u^2*v^2 + 2*u*v^3 + v^4)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 +␣
↪4*u^2*v^6 + v^8) dv⊗du
 + (3*u^4 + 4*u^3*v - 2*u^2*v^2 - 4*u*v^3 + 3*v^4)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 +␣
↪4*u^2*v^6 + v^8) dv⊗dv
```

```
>>> from sage.all import *
>>> t.display(eV)  # long time
t = (u^4 - 4*u^3*v + 10*u^2*v^2 + 4*u*v^3 + v^4)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 +␣
↪4*u^2*v^6 + v^8) du⊗du
 - 4*(u^3*v + 2*u^2*v^2 - u*v^3)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 + 4*u^2*v^6 + v^8)␣
↪du⊗dv
 + 2*(u^4 - 2*u^3*v - 2*u^2*v^2 + 2*u*v^3 + v^4)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 +␣
↪4*u^2*v^6 + v^8) dv⊗du
 + (3*u^4 + 4*u^3*v - 2*u^2*v^2 - 4*u*v^3 + 3*v^4)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 +␣
↪4*u^2*v^6 + v^8) dv⊗dv
```

Let us consider two vector fields, a and b, on S^2:

```
sage: a = M.vector_field({eU: [-y, x]}, name='a')
sage: a.add_comp_by_continuation(eV, W, chart=c_uv)
sage: a.display(eV)
a = -v ∂/∂u + u ∂/∂v
sage: b = M.vector_field({eU: [y, -1]}, name='b')
sage: b.add_comp_by_continuation(eV, W, chart=c_uv)
sage: b.display(eV)
b = ((2*u + 1)*v^3 + (2*u^3 - u^2)*v)/(u^2 + v^2) ∂/∂u
 - (u^4 - v^4 + 2*u*v^2)/(u^2 + v^2) ∂/∂v
```

```
>>> from sage.all import *
>>> a = M.vector_field({eU: [-y, x]}, name='a')
>>> a.add_comp_by_continuation(eV, W, chart=c_uv)
>>> a.display(eV)
a = -v ∂/∂u + u ∂/∂v
>>> b = M.vector_field({eU: [y, -Integer(1)]}, name='b')
>>> b.add_comp_by_continuation(eV, W, chart=c_uv)
>>> b.display(eV)
b = ((2*u + 1)*v^3 + (2*u^3 - u^2)*v)/(u^2 + v^2) ∂/∂u
 - (u^4 - v^4 + 2*u*v^2)/(u^2 + v^2) ∂/∂v
```

As a tensor field of type $(0, 2)$, t acts on the pair (a, b), resulting in a scalar field:

```
sage: f = t(a,b); f
Scalar field t(a,b) on the 2-dimensional differentiable manifold S^2
sage: f.display()  # long time
t(a,b): S^2 → ℝ
on U: (x, y) ↦ -2*x*y - y^2 - 3*x
on V: (u, v) ↦ -(3*u^3 + (3*u + 1)*v^2 + 2*u*v)/(u^4 + 2*u^2*v^2 + v^4)
```

```
>>> from sage.all import *
>>> f = t(a,b); f
Scalar field t(a,b) on the 2-dimensional differentiable manifold S^2
>>> f.display()   # long time
t(a,b): S^2 → ℝ
on U: (x, y) ↦ -2*x*y - y^2 - 3*x
on V: (u, v) ↦ -(3*u^3 + (3*u + 1)*v^2 + 2*u*v)/(u^4 + 2*u^2*v^2 + v^4)
```

The vectors can be defined only on subsets of S^2, the domain of the result is then the common subset:

```
sage: # long time
sage: s = t(a.restrict(U), b) ; s
Scalar field t(a,b) on the Open subset U of the 2-dimensional
 differentiable manifold S^2
sage: s.display()
t(a,b): U → ℝ
   (x, y) ↦ -2*x*y - y^2 - 3*x
on W: (u, v) ↦ -(3*u^3 + (3*u + 1)*v^2 + 2*u*v)/(u^4 + 2*u^2*v^2 + v^4)
sage: s = t(a.restrict(U), b.restrict(W)) ; s
Scalar field t(a,b) on the Open subset W of the 2-dimensional
 differentiable manifold S^2
sage: s.display()
t(a,b): W → ℝ
   (x, y) ↦ -2*x*y - y^2 - 3*x
   (u, v) ↦ -(3*u^3 + (3*u + 1)*v^2 + 2*u*v)/(u^4 + 2*u^2*v^2 + v^4)
```

```
>>> from sage.all import *
>>> # long time
>>> s = t(a.restrict(U), b) ; s
Scalar field t(a,b) on the Open subset U of the 2-dimensional
 differentiable manifold S^2
>>> s.display()
t(a,b): U → ℝ
   (x, y) ↦ -2*x*y - y^2 - 3*x
on W: (u, v) ↦ -(3*u^3 + (3*u + 1)*v^2 + 2*u*v)/(u^4 + 2*u^2*v^2 + v^4)
>>> s = t(a.restrict(U), b.restrict(W)) ; s
Scalar field t(a,b) on the Open subset W of the 2-dimensional
 differentiable manifold S^2
>>> s.display()
t(a,b): W → ℝ
   (x, y) ↦ -2*x*y - y^2 - 3*x
   (u, v) ↦ -(3*u^3 + (3*u + 1)*v^2 + 2*u*v)/(u^4 + 2*u^2*v^2 + v^4)
```

The tensor itself can be defined only on some open subset of S^2, yielding a result whose domain is this subset:

```
sage: s = t.restrict(V)(a,b); s   # long time
Scalar field t(a,b) on the Open subset V of the 2-dimensional
 differentiable manifold S^2
sage: s.display()   # long time
t(a,b): V → ℝ
   (u, v) ↦ -(3*u^3 + (3*u + 1)*v^2 + 2*u*v)/(u^4 + 2*u^2*v^2 + v^4)
on W: (x, y) ↦ -2*x*y - y^2 - 3*x
```

```
>>> from sage.all import *
>>> s = t.restrict(V)(a,b); s  # long time
Scalar field t(a,b) on the Open subset V of the 2-dimensional
 differentiable manifold S^2
>>> s.display()  # long time
t(a,b): V → ℝ
   (u, v) ↦ -(3*u^3 + (3*u + 1)*v^2 + 2*u*v)/(u^4 + 2*u^2*v^2 + v^4)
on W: (x, y) ↦ -2*x*y - y^2 - 3*x
```

Tests regarding the multiplication by a scalar field:

```
sage: f = M.scalar_field({c_xy: 1/(1+x^2+y^2),
....:                     c_uv: (u^2 + v^2)/(u^2 + v^2 + 1)}, name='f')
sage: t.parent().base_ring() is f.parent()
True
sage: s = f*t; s  # long time
Tensor field f*t of type (0,2) on the 2-dimensional differentiable
 manifold S^2
sage: s[[0,0]] == f*t[[0,0]]  # long time
True
sage: s.restrict(U) == f.restrict(U) * t.restrict(U)  # long time
True
sage: s = f*t.restrict(U); s
Tensor field f*t of type (0,2) on the Open subset U of the 2-dimensional
 differentiable manifold S^2
sage: s.restrict(U) == f.restrict(U) * t.restrict(U)
True
```

```
>>> from sage.all import *
>>> f = M.scalar_field({c_xy: Integer(1)/(Integer(1)+x**Integer(2)+y**Integer(2)),
...                     c_uv: (u**Integer(2) + v**Integer(2))/(u**Integer(2) +␣
↪v**Integer(2) + Integer(1))}, name='f')
>>> t.parent().base_ring() is f.parent()
True
>>> s = f*t; s  # long time
Tensor field f*t of type (0,2) on the 2-dimensional differentiable
 manifold S^2
>>> s[[Integer(0),Integer(0)]] == f*t[[Integer(0),Integer(0)]]  # long time
True
>>> s.restrict(U) == f.restrict(U) * t.restrict(U)  # long time
True
>>> s = f*t.restrict(U); s
Tensor field f*t of type (0,2) on the Open subset U of the 2-dimensional
 differentiable manifold S^2
>>> s.restrict(U) == f.restrict(U) * t.restrict(U)
True
```

Same examples with SymPy as the symbolic engine

From now on, we ask that all symbolic calculus on manifold M are performed by SymPy:

```
sage: M.set_calculus_method('sympy')
```

```
>>> from sage.all import *
>>> M.set_calculus_method('sympy')
```

We define the tensor t as above:

```
sage: t = M.tensor_field(0, 2, {eU:  [[1,0], [-2,3]]}, name='t')
sage: t.display(eU)
t = dx⊗dx - 2 dy⊗dx + 3 dy⊗dy
sage: t.add_comp_by_continuation(eV, W, chart=c_uv)  # long time
sage: t.display(eV)  # long time
t = (u**4 - 4*u**3*v + 10*u**2*v**2 + 4*u*v**3 + v**4)/(u**8 +
 4*u**6*v**2 + 6*u**4*v**4 + 4*u**2*v**6 + v**8) du⊗du +
 4*u*v*(-u**2 - 2*u*v + v**2)/(u**8 + 4*u**6*v**2 + 6*u**4*v**4
 + 4*u**2*v**6 + v**8) du⊗dv + 2*(u**4 - 2*u**3*v - 2*u**2*v**2
 + 2*u*v**3 + v**4)/(u**8 + 4*u**6*v**2 + 6*u**4*v**4 +
 4*u**2*v**6 + v**8) dv⊗du + (3*u**4 + 4*u**3*v - 2*u**2*v**2 -
 4*u*v**3 + 3*v**4)/(u**8 + 4*u**6*v**2 + 6*u**4*v**4 +
 4*u**2*v**6 + v**8) dv⊗dv
```

```
>>> from sage.all import *
>>> t = M.tensor_field(Integer(0), Integer(2), {eU:  [[Integer(1),Integer(0)], [-
↪Integer(2),Integer(3)]]}, name='t')
>>> t.display(eU)
t = dx⊗dx - 2 dy⊗dx + 3 dy⊗dy
>>> t.add_comp_by_continuation(eV, W, chart=c_uv)  # long time
>>> t.display(eV)  # long time
t = (u**4 - 4*u**3*v + 10*u**2*v**2 + 4*u*v**3 + v**4)/(u**8 +
 4*u**6*v**2 + 6*u**4*v**4 + 4*u**2*v**6 + v**8) du⊗du +
 4*u*v*(-u**2 - 2*u*v + v**2)/(u**8 + 4*u**6*v**2 + 6*u**4*v**4
 + 4*u**2*v**6 + v**8) du⊗dv + 2*(u**4 - 2*u**3*v - 2*u**2*v**2
 + 2*u*v**3 + v**4)/(u**8 + 4*u**6*v**2 + 6*u**4*v**4 +
 4*u**2*v**6 + v**8) dv⊗du + (3*u**4 + 4*u**3*v - 2*u**2*v**2 -
 4*u*v**3 + 3*v**4)/(u**8 + 4*u**6*v**2 + 6*u**4*v**4 +
 4*u**2*v**6 + v**8) dv⊗dv
```

The default coordinate representations of tensor components are now SymPy objects:

```
sage: t[eV,1,1,c_uv].expr() # long time
(3*u**4 + 4*u**3*v - 2*u**2*v**2 - 4*u*v**3 + 3*v**4)/(u**8 +
 4*u**6*v**2 + 6*u**4*v**4 + 4*u**2*v**6 + v**8)
sage: type(t[eV,1,1,c_uv].expr()) # long time
<class 'sympy.core.mul.Mul'>
```

```
>>> from sage.all import *
>>> t[eV,Integer(1),Integer(1),c_uv].expr() # long time
(3*u**4 + 4*u**3*v - 2*u**2*v**2 - 4*u*v**3 + 3*v**4)/(u**8 +
 4*u**6*v**2 + 6*u**4*v**4 + 4*u**2*v**6 + v**8)
>>> type(t[eV,Integer(1),Integer(1),c_uv].expr()) # long time
<class 'sympy.core.mul.Mul'>
```

Let us consider two vector fields, a and b, on S^2:

```
sage: a = M.vector_field({eU: [-y, x]}, name='a')
sage: a.add_comp_by_continuation(eV, W, chart=c_uv)
sage: a.display(eV)
a = -v ∂/∂u + u ∂/∂v
sage: b = M.vector_field({eU: [y, -1]}, name='b')
sage: b.add_comp_by_continuation(eV, W, chart=c_uv)
sage: b.display(eV)
b = v*(2*u**3 - u**2 + 2*u*v**2 + v**2)/(u**2 + v**2) ∂/∂u
    + (-u**4 - 2*u*v**2 + v**4)/(u**2 + v**2) ∂/∂v
```

```
>>> from sage.all import *
>>> a = M.vector_field({eU: [-y, x]}, name='a')
>>> a.add_comp_by_continuation(eV, W, chart=c_uv)
>>> a.display(eV)
a = -v ∂/∂u + u ∂/∂v
>>> b = M.vector_field({eU: [y, -Integer(1)]}, name='b')
>>> b.add_comp_by_continuation(eV, W, chart=c_uv)
>>> b.display(eV)
b = v*(2*u**3 - u**2 + 2*u*v**2 + v**2)/(u**2 + v**2) ∂/∂u
    + (-u**4 - 2*u*v**2 + v**4)/(u**2 + v**2) ∂/∂v
```

As a tensor field of type $(0, 2)$, t acts on the pair (a, b), resulting in a scalar field:

```
sage: f = t(a,b)
sage: f.display()  # long time
t(a,b): S^2 → ℝ
on U: (x, y) ↦ -2*x*y - 3*x - y**2
on V: (u, v) ↦ (-3*u**3 - 3*u*v**2 - 2*u*v - v**2)/(u**4 + 2*u**2*v**2 + v**4)
```

```
>>> from sage.all import *
>>> f = t(a,b)
>>> f.display()  # long time
t(a,b): S^2 → ℝ
on U: (x, y) ↦ -2*x*y - 3*x - y**2
on V: (u, v) ↦ (-3*u**3 - 3*u*v**2 - 2*u*v - v**2)/(u**4 + 2*u**2*v**2 + v**4)
```

The vectors can be defined only on subsets of S^2, the domain of the result is then the common subset:

```
sage: s = t(a.restrict(U), b)
sage: s.display()  # long time
t(a,b): U → ℝ
   (x, y) ↦ -2*x*y - 3*x - y**2
on W: (u, v) ↦ (-3*u**3 - 3*u*v**2 - 2*u*v - v**2)/(u**4 + 2*u**2*v**2 + v**4)
sage: s = t(a.restrict(U), b.restrict(W))  # long time
sage: s.display()  # long time
t(a,b): W → ℝ
   (x, y) ↦ -2*x*y - 3*x - y**2
   (u, v) ↦ (-3*u**3 - 3*u*v**2 - 2*u*v - v**2)/(u**4 + 2*u**2*v**2 + v**4)
```

```
>>> from sage.all import *
>>> s = t(a.restrict(U), b)
>>> s.display() # long time
```

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```
t(a,b): U → ℝ
   (x, y) ↦ -2*x*y - 3*x - y**2
on W: (u, v) ↦ (-3*u**3 - 3*u*v**2 - 2*u*v - v**2)/(u**4 + 2*u**2*v**2 + v**4)
>>> s = t(a.restrict(U), b.restrict(W))  # long time
>>> s.display()  # long time
t(a,b): W → ℝ
   (x, y) ↦ -2*x*y - 3*x - y**2
   (u, v) ↦ (-3*u**3 - 3*u*v**2 - 2*u*v - v**2)/(u**4 + 2*u**2*v**2 + v**4)
```

The tensor itself can be defined only on some open subset of S^2, yielding a result whose domain is this subset:

```
sage: s = t.restrict(V)(a,b)  # long time
sage: s.display()  # long time
t(a,b): V → ℝ
   (u, v) ↦ (-3*u**3 - 3*u*v**2 - 2*u*v - v**2)/(u**4 + 2*u**2*v**2 + v**4)
on W: (x, y) ↦ -2*x*y - 3*x - y**2
```

```
>>> from sage.all import *
>>> s = t.restrict(V)(a,b)  # long time
>>> s.display()  # long time
t(a,b): V → ℝ
   (u, v) ↦ (-3*u**3 - 3*u*v**2 - 2*u*v - v**2)/(u**4 + 2*u**2*v**2 + v**4)
on W: (x, y) ↦ -2*x*y - 3*x - y**2
```

Tests regarding the multiplication by a scalar field:

```
sage: f = M.scalar_field({c_xy: 1/(1+x^2+y^2),
....:                     c_uv: (u^2 + v^2)/(u^2 + v^2 + 1)}, name='f')
sage: s = f*t # long time
sage: s[[0,0]] == f*t[[0,0]]  # long time
True
sage: s.restrict(U) == f.restrict(U) * t.restrict(U)  # long time
True
sage: s = f*t.restrict(U)
sage: s.restrict(U) == f.restrict(U) * t.restrict(U)
True
```

```
>>> from sage.all import *
>>> f = M.scalar_field({c_xy: Integer(1)/(Integer(1)+x**Integer(2)+y**Integer(2)),
...                     c_uv: (u**Integer(2) + v**Integer(2))/(u**Integer(2) +␣
↪v**Integer(2) + Integer(1))}, name='f')
>>> s = f*t # long time
>>> s[[Integer(0),Integer(0)]] == f*t[[Integer(0),Integer(0)]]  # long time
True
>>> s.restrict(U) == f.restrict(U) * t.restrict(U)  # long time
True
>>> s = f*t.restrict(U)
>>> s.restrict(U) == f.restrict(U) * t.restrict(U)
True
```

Notice that the zero tensor field is immutable, and therefore its components cannot be changed:

```
sage: zer = M.tensor_field_module((1, 1)).zero()
sage: zer.is_immutable()
True
sage: zer.set_comp()
Traceback (most recent call last):
...
ValueError: the components of an immutable element cannot be
 changed
```

```
>>> from sage.all import *
>>> zer = M.tensor_field_module((Integer(1), Integer(1))).zero()
>>> zer.is_immutable()
True
>>> zer.set_comp()
Traceback (most recent call last):
...
ValueError: the components of an immutable element cannot be
 changed
```

Other tensor fields can be declared immutable, too:

```
sage: t.is_immutable()
False
sage: t.set_immutable()
sage: t.is_immutable()
True
sage: t.set_comp()
Traceback (most recent call last):
...
ValueError: the components of an immutable element cannot be
 changed
sage: t.set_name('b')
Traceback (most recent call last):
...
ValueError: the name of an immutable element cannot be changed
```

```
>>> from sage.all import *
>>> t.is_immutable()
False
>>> t.set_immutable()
>>> t.is_immutable()
True
>>> t.set_comp()
Traceback (most recent call last):
...
ValueError: the components of an immutable element cannot be
 changed
>>> t.set_name('b')
Traceback (most recent call last):
...
ValueError: the name of an immutable element cannot be changed
```

add_comp(*basis=None*)

Return the components of `self` in a given vector frame for assignment.

The components with respect to other frames having the same domain as the provided vector frame are kept. To delete them, use the method `set_comp()` instead.

INPUT:

- `basis` – (default: `None`) vector frame in which the components are defined; if `None`, the components are assumed to refer to the tensor field domain's default frame

OUTPUT:

- components in the given frame, as a `Components`; if such components did not exist previously, they are created

EXAMPLES:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: e_uv = c_uv.frame()
sage: t = M.tensor_field(1, 2, name='t')
sage: t.add_comp(e_uv)
3-indices components w.r.t. Coordinate frame (V, (∂/∂u,∂/∂v))
sage: t.add_comp(e_uv)[1,0,1] = u+v
sage: t.display(e_uv)
t = (u + v) ∂/∂v⊗du⊗dv
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2) #␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2) #␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> e_uv = c_uv.frame()
>>> t = M.tensor_field(Integer(1), Integer(2), name='t')
>>> t.add_comp(e_uv)
3-indices components w.r.t. Coordinate frame (V, (∂/∂u,∂/∂v))
>>> t.add_comp(e_uv)[Integer(1),Integer(0),Integer(1)] = u+v
>>> t.display(e_uv)
t = (u + v) ∂/∂v⊗du⊗dv
```

Setting the components in a new frame:

```
sage: e = V.vector_frame('e')
sage: t.add_comp(e)
3-indices components w.r.t. Vector frame (V, (e_0,e_1))
sage: t.add_comp(e)[0,1,1] = u*v
sage: t.display(e)
t = u*v e_0⊗e^1⊗e^1
```

```
>>> from sage.all import *
>>> e = V.vector_frame('e')
>>> t.add_comp(e)
3-indices components w.r.t. Vector frame (V, (e_0,e_1))
>>> t.add_comp(e)[Integer(0),Integer(1),Integer(1)] = u*v
>>> t.display(e)
t = u*v e_0⊗e^1⊗e^1
```

The components with respect to `e_uv` are kept:

```
sage: t.display(e_uv)
t = (u + v) ∂/∂v⊗du⊗dv
```

```
>>> from sage.all import *
>>> t.display(e_uv)
t = (u + v) ∂/∂v⊗du⊗dv
```

Since zero is a special element, its components cannot be changed:

```
sage: z = M.tensor_field_module((1, 1)).zero()
sage: z.add_comp(e_uv)[1, 1] = u^2
Traceback (most recent call last):
...
ValueError: the components of an immutable element cannot be
 changed
```

```
>>> from sage.all import *
>>> z = M.tensor_field_module((Integer(1), Integer(1))).zero()
>>> z.add_comp(e_uv)[Integer(1), Integer(1)] = u**Integer(2)
Traceback (most recent call last):
...
ValueError: the components of an immutable element cannot be
 changed
```

add_comp_by_continuation(*frame*, *subdomain*, *chart=None*)

Set components with respect to a vector frame by continuation of the coordinate expression of the components in a subframe.

The continuation is performed by demanding that the components have the same coordinate expression as those on the restriction of the frame to a given subdomain.

INPUT:

- `frame` – vector frame e in which the components are to be set
- `subdomain` – open subset of e's domain in which the components are known or can be evaluated from other components
- `chart` – (default: `None`) coordinate chart on e's domain in which the extension of the expression of the components is to be performed; if `None`, the default's chart of e's domain is assumed

EXAMPLES:

Components of a vector field on the sphere S^2:

```
sage: M = Manifold(2, 'S^2', start_index=1)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', start_index=Integer(1))
```

The two open subsets covered by stereographic coordinates (North and South):

```
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart() # stereographic␣
↪coordinates
sage: transf = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:              intersection_name='W', restrictions1= x^2+y^2!=0,
....:              restrictions2= u^2+v^2!=0)
sage: inv = transf.inverse()
sage: W = U.intersection(V) # The complement of the two poles
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: a = M.vector_field({eU: [x, 2+y]}, name='a')
```

```
>>> from sage.all import *
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)# stereographic␣
↪coordinates
>>> transf = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...              intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...              restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> inv = transf.inverse()
>>> W = U.intersection(V) # The complement of the two poles
>>> eU = c_xy.frame() ; eV = c_uv.frame()
>>> a = M.vector_field({eU: [x, Integer(2)+y]}, name='a')
```

At this stage, the vector field has been defined only on the open subset `U` (through its components in the frame `eU`):

```
sage: a.display(eU)
a = x ∂/∂x + (y + 2) ∂/∂y
```

```
>>> from sage.all import *
>>> a.display(eU)
a = x ∂/∂x + (y + 2) ∂/∂y
```

The components with respect to the restriction of `eV` to the common subdomain `W`, in terms of the `(u,v)` coordinates, are obtained by a change-of-frame formula on `W`:

```
sage: a.display(eV.restrict(W), c_uv.restrict(W))
a = (-4*u*v - u) ∂/∂u + (2*u^2 - 2*v^2 - v) ∂/∂v
```

```
>>> from sage.all import *
>>> a.display(eV.restrict(W), c_uv.restrict(W))
a = (-4*u*v - u) ∂/∂u + (2*u^2 - 2*v^2 - v) ∂/∂v
```

The continuation consists in extending the definition of the vector field to the whole open subset `V` by demanding that the components in the frame eV have the same coordinate expression as the above one:

```
sage: a.add_comp_by_continuation(eV, W, chart=c_uv)
```

```
>>> from sage.all import *
>>> a.add_comp_by_continuation(eV, W, chart=c_uv)
```

We have then:

```
sage: a.display(eV)
a = (-4*u*v - u) ∂/∂u + (2*u^2 - 2*v^2 - v) ∂/∂v
```

```
>>> from sage.all import *
>>> a.display(eV)
a = (-4*u*v - u) ∂/∂u + (2*u^2 - 2*v^2 - v) ∂/∂v
```

and a is defined on the entire manifold S^2.

add_expr_from_subdomain(*frame*, *subdomain*)

Add an expression to an existing component from a subdomain.

INPUT:

- `frame` – vector frame e in which the components are to be set
- `subdomain` – open subset of e's domain in which the components have additional expressions

EXAMPLES:

We are going to consider a vector field in $\mathbf{R}^3$ along the 2-sphere:

```
sage: M = Manifold(3, 'M', structure='Riemannian')
sage: S = Manifold(2, 'S', structure='Riemannian')
sage: E.<X,Y,Z> = M.chart()
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='Riemannian')
>>> S = Manifold(Integer(2), 'S', structure='Riemannian')
>>> E = M.chart(names=('X', 'Y', 'Z',)); (X, Y, Z,) = E._first_ngens(3)
```

Let us define `S` in terms of stereographic charts:

```
sage: U = S.open_subset('U')
sage: V = S.open_subset('V')
sage: S.declare_union(U,V)
sage: stereoN.<x,y> = U.chart()
sage: stereoS.<xp,yp> = V.chart("xp:x' yp:y'")
sage: stereoN_to_S = stereoN.transition_map(stereoS,
....:                                (x/(x^2+y^2), y/(x^2+y^2)),
....:                                intersection_name='W',
....:                                restrictions1= x^2+y^2!=0,
....:                                restrictions2= xp^2+yp^2!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: W = U.intersection(V)
```

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```
sage: stereoN_W = stereoN.restrict(W)
sage: stereoS_W = stereoS.restrict(W)
```

```
>>> from sage.all import *
>>> U = S.open_subset('U')
>>> V = S.open_subset('V')
>>> S.declare_union(U,V)
>>> stereoN = U.chart(names=('x', 'y',)); (x, y,) = stereoN._first_ngens(2)
>>> stereoS = V.chart("xp:x' yp:y'", names=('xp', 'yp',)); (xp, yp,) =␣
↪stereoS._first_ngens(2)
>>> stereoN_to_S = stereoN.transition_map(stereoS,
...                                     (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                                     intersection_name='W',
...                                     restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                                     restrictions2=␣
↪xp**Integer(2)+yp**Integer(2)!=Integer(0))
>>> stereoS_to_N = stereoN_to_S.inverse()
>>> W = U.intersection(V)
>>> stereoN_W = stereoN.restrict(W)
>>> stereoS_W = stereoS.restrict(W)
```

The embedding of S^2 in $\mathbf{R}^3$:

```
sage: phi = S.diff_map(M, {(stereoN, E): [2*x/(1+x^2+y^2),
....:                                      2*y/(1+x^2+y^2),
....:                                      (x^2+y^2-1)/(1+x^2+y^2)],
....:                      (stereoS, E): [2*xp/(1+xp^2+yp^2),
....:                                      2*yp/(1+xp^2+yp^2),
....:                                (1-xp^2-yp^2)/(1+xp^2+yp^2)]},
....:                   name='Phi', latex_name=r'\Phi')
```

```
>>> from sage.all import *
>>> phi = S.diff_map(M, {(stereoN, E): [Integer(2)*x/
↪(Integer(1)+x**Integer(2)+y**Integer(2)),
...                                     Integer(2)*y/
↪(Integer(1)+x**Integer(2)+y**Integer(2)),
...                                     (x**Integer(2)+y**Integer(2)-
↪Integer(1))/(Integer(1)+x**Integer(2)+y**Integer(2))],
...                     (stereoS, E): [Integer(2)*xp/
↪(Integer(1)+xp**Integer(2)+yp**Integer(2)),
...                                     Integer(2)*yp/
↪(Integer(1)+xp**Integer(2)+yp**Integer(2)),
...                               (Integer(1)-xp**Integer(2)-yp**Integer(2))/
↪(Integer(1)+xp**Integer(2)+yp**Integer(2))]},
...                  name='Phi', latex_name=r'\Phi')
```

To define a vector field `v` along `S` taking its values in `M`, we first set the components on `U`:

```
sage: v = M.vector_field(name='v').along(phi)
sage: vU = v.restrict(U)
```

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```
sage: vU[:] = [x,y,x**2+y**2]
```

```
>>> from sage.all import *
>>> v = M.vector_field(name='v').along(phi)
>>> vU = v.restrict(U)
>>> vU[:] = [x,y,x**Integer(2)+y**Integer(2)]
```

But because `M` is parallelizable, these components can be extended to `S` itself:

```
sage: v.add_comp_by_continuation(E.frame().along(phi), U)
```

```
>>> from sage.all import *
>>> v.add_comp_by_continuation(E.frame().along(phi), U)
```

One can see that `v` is not yet fully defined: the components (scalar fields) do not have values on the whole manifold:

```
sage: sorted(v._components.values())[0]._comp[(0,)].display()
S → ℝ
on U: (x, y) ↦ x
on W: (xp, yp) ↦ xp/(xp^2 + yp^2)
```

```
>>> from sage.all import *
>>> sorted(v._components.values())[Integer(0)]._comp[(Integer(0),)].display()
S → ℝ
on U: (x, y) ↦ x
on W: (xp, yp) ↦ xp/(xp^2 + yp^2)
```

To fix that, we first extend the components from `W` to `V` using *add_comp_by_continuation()*:

```
sage: v.add_comp_by_continuation(E.frame().along(phi).restrict(V),
....:                            W, stereoS)
```

```
>>> from sage.all import *
>>> v.add_comp_by_continuation(E.frame().along(phi).restrict(V),
...                            W, stereoS)
```

Then, the expression on the subdomain `V` is added to the already known components on `S` by:

```
sage: v.add_expr_from_subdomain(E.frame().along(phi), V)
```

```
>>> from sage.all import *
>>> v.add_expr_from_subdomain(E.frame().along(phi), V)
```

The definition of `v` is now complete:

```
sage: sorted(v._components.values())[0]._comp[(2,)].display()
S → ℝ
on U: (x, y) ↦ x^2 + y^2
on V: (xp, yp) ↦ 1/(xp^2 + yp^2)
```

```
>>> from sage.all import *
>>> sorted(v._components.values())[Integer(0)]._comp[(Integer(2),)].display()
S → ℝ
on U: (x, y) ↦ x^2 + y^2
on V: (xp, yp) ↦ 1/(xp^2 + yp^2)
```

along(*mapping*)

Return the tensor field deduced from `self` via a differentiable map, the codomain of which is included in the domain of `self`.

More precisely, if `self` is a tensor field t on M and if $\Phi : U \to M$ is a differentiable map from some differentiable manifold U to M, the returned object is a tensor field $\tilde{t}$ along U with values on M such that

$$\forall p \in U, \ \tilde{t}(p) = t(\Phi(p)).$$

INPUT:

- `mapping` – differentiable map $\Phi : U \to M$

OUTPUT: tensor field $\tilde{t}$ along U defined above

EXAMPLES:

Let us consider the 2-dimensional sphere S^2:

```
sage: M = Manifold(2, 'S^2') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2) #␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2) #␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                 intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                 restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V)
```

and the following map from the open interval $(0, 5\pi/2)$ to S^2, the image of it being the great circle $x = 0$, $u = 0$, which goes through the North and South poles:

```
sage: I.<t> = manifolds.OpenInterval(0, 5*pi/2)
sage: J = I.open_interval(0, 3*pi/2)
sage: K = I.open_interval(pi, 5*pi/2)
sage: c_J = J.canonical_chart(); c_K = K.canonical_chart()
sage: Phi = I.diff_map(M, {(c_J, c_xy):
....:                      (0, sgn(pi-t)*sqrt((1+cos(t))/(1-cos(t)))),
....:                      (c_K, c_uv):
....:                      (0,  sgn(t-2*pi)*sqrt((1-cos(t))/(1+cos(t))))},
....:                  name='Phi')
```

```
>>> from sage.all import *
>>> I = manifolds.OpenInterval(Integer(0), Integer(5)*pi/Integer(2), names=('t
↪',)); (t,) = I._first_ngens(1)
>>> J = I.open_interval(Integer(0), Integer(3)*pi/Integer(2))
>>> K = I.open_interval(pi, Integer(5)*pi/Integer(2))
>>> c_J = J.canonical_chart(); c_K = K.canonical_chart()
>>> Phi = I.diff_map(M, {(c_J, c_xy):
...                      (Integer(0), sgn(pi-t)*sqrt((Integer(1)+cos(t))/
↪(Integer(1)-cos(t)))),
...                      (c_K, c_uv):
...                      (Integer(0),  sgn(t-Integer(2)*pi)*sqrt((Integer(1)-
↪cos(t))/(Integer(1)+cos(t))))},
...                  name='Phi')
```

Let us consider a vector field on S^2:

```
sage: eU = c_xy.frame(); eV = c_uv.frame()
sage: w = M.vector_field(name='w')
sage: w[eU,0] = 1
sage: w.add_comp_by_continuation(eV, W, chart=c_uv)
sage: w.display(eU)
w = ∂/∂x
sage: w.display(eV)
w = (-u^2 + v^2) ∂/∂u - 2*u*v ∂/∂v
```

```
>>> from sage.all import *
>>> eU = c_xy.frame(); eV = c_uv.frame()
>>> w = M.vector_field(name='w')
>>> w[eU,Integer(0)] = Integer(1)
>>> w.add_comp_by_continuation(eV, W, chart=c_uv)
>>> w.display(eU)
w = ∂/∂x
>>> w.display(eV)
w = (-u^2 + v^2) ∂/∂u - 2*u*v ∂/∂v
```

We have then:

```
sage: wa = w.along(Phi); wa
Vector field w along the Real interval (0, 5/2*pi) with values on
 the 2-dimensional differentiable manifold S^2
sage: wa.display(eU.along(Phi))
w = ∂/∂x
```

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```
sage: wa.display(eV.along(Phi))
w = -(cos(t) - 1)*sgn(-2*pi + t)^2/(cos(t) + 1) ∂/∂u
```

```
>>> from sage.all import *
>>> wa = w.along(Phi); wa
Vector field w along the Real interval (0, 5/2*pi) with values on
 the 2-dimensional differentiable manifold S^2
>>> wa.display(eU.along(Phi))
w = ∂/∂x
>>> wa.display(eV.along(Phi))
w = -(cos(t) - 1)*sgn(-2*pi + t)^2/(cos(t) + 1) ∂/∂u
```

Some tests:

```
sage: p = K.an_element()
sage: wa.at(p) == w.at(Phi(p))
True
sage: wa.at(J(4*pi/3)) == wa.at(K(4*pi/3))
True
sage: wa.at(I(4*pi/3)) == wa.at(K(4*pi/3))
True
sage: wa.at(K(7*pi/4)) == eU[0].at(Phi(I(7*pi/4))) # since eU[0]=∂/∂x
True
```

```
>>> from sage.all import *
>>> p = K.an_element()
>>> wa.at(p) == w.at(Phi(p))
True
>>> wa.at(J(Integer(4)*pi/Integer(3))) == wa.at(K(Integer(4)*pi/Integer(3)))
True
>>> wa.at(I(Integer(4)*pi/Integer(3))) == wa.at(K(Integer(4)*pi/Integer(3)))
True
>>> wa.at(K(Integer(7)*pi/Integer(4))) == eU[Integer(0)].
↪at(Phi(I(Integer(7)*pi/Integer(4)))) # since eU[0]=∂/∂x
True
```

antisymmetrize(**pos*)

Antisymmetrization over some arguments.

INPUT:

- `pos` – (default: `None`) list of argument positions involved in the antisymmetrization (with the convention `position=0` for the first argument); if `None`, the antisymmetrization is performed over all the arguments

OUTPUT:

- the antisymmetrized tensor field (instance of *TensorField*)

EXAMPLES:

Antisymmetrization of a type-$(0, 2)$ tensor field on a 2-dimensional non-parallelizable manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
```

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```
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
....:                              restrictions1= x>0, restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: a = M.tensor_field(0,2, {eU: [[1,x], [2,y]]}, name='a')
sage: a.add_comp_by_continuation(eV, W, chart=c_uv)
sage: a[eV,:]
[ 1/4*u + 3/4 -1/4*u + 3/4]
[ 1/4*v - 1/4 -1/4*v - 1/4]
sage: s = a.antisymmetrize() ; s
2-form on the 2-dimensional differentiable manifold M
sage: s[eU,:]
[         0  1/2*x - 1]
[-1/2*x + 1          0]
sage: s[eV,:]
[                    0 -1/8*u - 1/8*v + 1/2]
[ 1/8*u + 1/8*v - 1/2                    0]
sage: s == a.antisymmetrize(0,1)  # explicit positions
True
sage: s == a.antisymmetrize(1,0)  # the order of positions does not matter
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
...                              restrictions1= x>Integer(0), restrictions2=␣
↪u+v>Integer(0))
>>> inv = transf.inverse()
>>> W = U.intersection(V)
>>> eU = c_xy.frame() ; eV = c_uv.frame()
>>> a = M.tensor_field(Integer(0),Integer(2), {eU: [[Integer(1),x],␣
↪[Integer(2),y]]}, name='a')
>>> a.add_comp_by_continuation(eV, W, chart=c_uv)
>>> a[eV,:]
[ 1/4*u + 3/4 -1/4*u + 3/4]
[ 1/4*v - 1/4 -1/4*v - 1/4]
>>> s = a.antisymmetrize() ; s
2-form on the 2-dimensional differentiable manifold M
>>> s[eU,:]
[         0  1/2*x - 1]
[-1/2*x + 1          0]
>>> s[eV,:]
[                    0 -1/8*u - 1/8*v + 1/2]
[ 1/8*u + 1/8*v - 1/2                    0]
>>> s == a.antisymmetrize(Integer(0),Integer(1))  # explicit positions
True
```

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```
>>> s == a.antisymmetrize(Integer(1),Integer(0))  # the order of positions␣
↪does not matter
True
```

See also

For more details and examples, see `sage.tensor.modules.free_module_tensor.FreeModuleTensor.antisymmetrize()`.

apply_map(*fun*, *frame=None*, *chart=None*, *keep_other_components=False*)

Apply a function to the coordinate expressions of all components of `self` in a given vector frame.

This method allows operations like factorization, expansion, simplification or substitution to be performed on all components of `self` in a given vector frame (see examples below).

INPUT:

- `fun` – function to be applied to the coordinate expressions of the components
- `frame` – (default: `None`) vector frame defining the components on which the operation `fun` is to be performed; if `None`, the default frame of the domain of `self` is assumed
- `chart` – (default: `None`) coordinate chart; if specified, the operation `fun` is performed only on the coordinate expressions with respect to `chart` of the components w.r.t. `frame`; if `None`, the operation `fun` is performed on all available coordinate expressions
- `keep_other_components` – boolean (default: `False`); determine whether the components with respect to vector frames distinct from `frame` and having the same domain as `frame` are kept. If `fun` is non-destructive, `keep_other_components` can be set to `True`; otherwise, it is advised to set it to `False` (the default) in order to avoid any inconsistency between the various sets of components

EXAMPLES:

Factorizing all components in the default frame of a vector field:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: a, b = var('a b')
sage: v = M.vector_field(x^2 - y^2, a*(b^2 - b)*x)
sage: v.display()
(x^2 - y^2) ∂/∂x + (b^2 - b)*a*x ∂/∂y
sage: v.apply_map(factor)
sage: v.display()
(x + y)*(x - y) ∂/∂x + a*(b - 1)*b*x ∂/∂y
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> a, b = var('a b')
>>> v = M.vector_field(x**Integer(2) - y**Integer(2), a*(b**Integer(2) - b)*x)
>>> v.display()
(x^2 - y^2) ∂/∂x + (b^2 - b)*a*x ∂/∂y
>>> v.apply_map(factor)
>>> v.display()
(x + y)*(x - y) ∂/∂x + a*(b - 1)*b*x ∂/∂y
```

Performing a substitution in all components in the default frame:

```
sage: v.apply_map(lambda f: f.subs({a: 2}))
sage: v.display()
(x + y)*(x - y) ∂/∂x + 2*(b - 1)*b*x ∂/∂y
```

```
>>> from sage.all import *
>>> v.apply_map(lambda f: f.subs({a: Integer(2)}))
>>> v.display()
(x + y)*(x - y) ∂/∂x + 2*(b - 1)*b*x ∂/∂y
```

Specifying the vector frame via the argument `frame`:

```
sage: P.<p, q> = M.chart()
sage: X_to_P = X.transition_map(P, [x + 1, y - 1])
sage: P_to_X = X_to_P.inverse()
sage: v.display(P)
(p^2 - q^2 - 2*p - 2*q) ∂/∂p + (-2*b^2 + 2*(b^2 - b)*p + 2*b) ∂/∂q
sage: v.apply_map(lambda f: f.subs({b: pi}), frame=P.frame())
sage: v.display(P)
(p^2 - q^2 - 2*p - 2*q) ∂/∂p + (2*pi - 2*pi^2 - 2*(pi - pi^2)*p) ∂/∂q
```

```
>>> from sage.all import *
>>> P = M.chart(names=('p', 'q',)); (p, q,) = P._first_ngens(2)
>>> X_to_P = X.transition_map(P, [x + Integer(1), y - Integer(1)])
>>> P_to_X = X_to_P.inverse()
>>> v.display(P)
(p^2 - q^2 - 2*p - 2*q) ∂/∂p + (-2*b^2 + 2*(b^2 - b)*p + 2*b) ∂/∂q
>>> v.apply_map(lambda f: f.subs({b: pi}), frame=P.frame())
>>> v.display(P)
(p^2 - q^2 - 2*p - 2*q) ∂/∂p + (2*pi - 2*pi^2 - 2*(pi - pi^2)*p) ∂/∂q
```

Note that the required operation has been performed in all charts:

```
sage: v.display(P.frame(), P)
(p^2 - q^2 - 2*p - 2*q) ∂/∂p + (2*pi - 2*pi^2 - 2*(pi - pi^2)*p) ∂/∂q
sage: v.display(P.frame(), X)
(x + y)*(x - y) ∂/∂p + 2*pi*(pi - 1)*x ∂/∂q
```

```
>>> from sage.all import *
>>> v.display(P.frame(), P)
(p^2 - q^2 - 2*p - 2*q) ∂/∂p + (2*pi - 2*pi^2 - 2*(pi - pi^2)*p) ∂/∂q
>>> v.display(P.frame(), X)
(x + y)*(x - y) ∂/∂p + 2*pi*(pi - 1)*x ∂/∂q
```

By default, the components of `v` in frames distinct from the specified one have been deleted:

```
sage: X.frame() in v._components
False
```

```
>>> from sage.all import *
>>> X.frame() in v._components
False
```

When requested, they are recomputed by change-of-frame formulas, thereby enforcing the consistency between the representations in various vector frames. In particular, we can check that the substitution of `b` by `pi`, which was asked in `P.frame()`, is effective in `X.frame()` as well:

```
sage: v.display(X.frame(), X)
(x + y)*(x - y) ∂/∂x + 2*pi*(pi - 1)*x ∂/∂y
```

```
>>> from sage.all import *
>>> v.display(X.frame(), X)
(x + y)*(x - y) ∂/∂x + 2*pi*(pi - 1)*x ∂/∂y
```

When the requested operation does not change the value of the tensor field, one can use the keyword argument `keep_other_components=True`, in order to avoid the recomputation of the components in other frames:

```
sage: v.apply_map(factor, keep_other_components=True)
sage: v.display()
(x + y)*(x - y) ∂/∂x + 2*pi*(pi - 1)*x ∂/∂y
```

```
>>> from sage.all import *
>>> v.apply_map(factor, keep_other_components=True)
>>> v.display()
(x + y)*(x - y) ∂/∂x + 2*pi*(pi - 1)*x ∂/∂y
```

The components with respect to `P.frame()` have been kept:

```
sage: P.frame() in v._components
True
```

```
>>> from sage.all import *
>>> P.frame() in v._components
True
```

One can restrict the operation to expressions in a given chart, via the argument `chart`:

```
sage: v.display(X.frame(), P)
(p + q)*(p - q - 2) ∂/∂x + 2*pi*(pi - 1)*(p - 1) ∂/∂y
sage: v.apply_map(expand, chart=P)
sage: v.display(X.frame(), P)
(p^2 - q^2 - 2*p - 2*q) ∂/∂x + (2*pi + 2*pi^2*p - 2*pi^2 - 2*pi*p) ∂/∂y
sage: v.display(X.frame(), X)
(x + y)*(x - y) ∂/∂x + 2*pi*(pi - 1)*x ∂/∂y
```

```
>>> from sage.all import *
>>> v.display(X.frame(), P)
(p + q)*(p - q - 2) ∂/∂x + 2*pi*(pi - 1)*(p - 1) ∂/∂y
>>> v.apply_map(expand, chart=P)
>>> v.display(X.frame(), P)
(p^2 - q^2 - 2*p - 2*q) ∂/∂x + (2*pi + 2*pi^2*p - 2*pi^2 - 2*pi*p) ∂/∂y
>>> v.display(X.frame(), X)
(x + y)*(x - y) ∂/∂x + 2*pi*(pi - 1)*x ∂/∂y
```

at(*point*)

Value of `self` at a point of its domain.

If the current tensor field is

$$t: U \longrightarrow T^{(k,l)}M$$

associated with the differentiable map

$$\Phi: U \longrightarrow M,$$

where U and M are two manifolds (possibly $U = M$ and $\Phi = \mathrm{Id}_M$), then for any point $p \in U$, $t(p)$ is a tensor on the tangent space to M at the point $\Phi(p)$.

INPUT:

- `point` – *ManifoldPoint*; point p in the domain of the tensor field U

OUTPUT:

- `FreeModuleTensor` representing the tensor $t(p)$ on the tangent vector space $T_{\Phi(p)}M$

EXAMPLES:

Tensor on a tangent space of a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)    # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y),
....:                    intersection_name='W', restrictions1= x>0,
....:                    restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: a = M.tensor_field(1, 1, {eU: [[1+y,x], [0,x+y]]}, name='a')
sage: a.add_comp_by_continuation(eV, W, chart=c_uv)
sage: a.display(eU)
a = (y + 1) ∂/∂x⊗dx + x ∂/∂x⊗dy + (x + y) ∂/∂y⊗dy
sage: a.display(eV)
a = (u + 1/2) ∂/∂u⊗du + (-1/2*u - 1/2*v + 1/2) ∂/∂u⊗dv
 + 1/2 ∂/∂v⊗du + (1/2*u - 1/2*v + 1/2) ∂/∂v⊗dv
sage: p = M.point((2,3), chart=c_xy, name='p')
sage: ap = a.at(p) ; ap
Type-(1,1) tensor a on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
sage: ap.parent()
Free module of type-(1,1) tensors on the Tangent space at Point p
 on the 2-dimensional differentiable manifold M
sage: ap.display(eU.at(p))
a = 4 ∂/∂x⊗dx + 2 ∂/∂x⊗dy + 5 ∂/∂y⊗dy
sage: ap.display(eV.at(p))
a = 11/2 ∂/∂u⊗du - 3/2 ∂/∂u⊗dv + 1/2 ∂/∂v⊗du + 7/2 ∂/∂v⊗dv
sage: p.coord(c_uv) # to check the above expression
(5, -1)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
```

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```
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> transf = c_xy.transition_map(c_uv, (x+y, x-y),
...                    intersection_name='W', restrictions1= x>Integer(0),
...                    restrictions2= u+v>Integer(0))
>>> inv = transf.inverse()
>>> W = U.intersection(V)
>>> eU = c_xy.frame() ; eV = c_uv.frame()
>>> a = M.tensor_field(Integer(1), Integer(1), {eU: [[Integer(1)+y,x],␣
↪[Integer(0),x+y]]}, name='a')
>>> a.add_comp_by_continuation(eV, W, chart=c_uv)
>>> a.display(eU)
a = (y + 1) ∂/∂x⊗dx + x ∂/∂x⊗dy + (x + y) ∂/∂y⊗dy
>>> a.display(eV)
a = (u + 1/2) ∂/∂u⊗du + (-1/2*u - 1/2*v + 1/2) ∂/∂u⊗dv
 + 1/2 ∂/∂v⊗du + (1/2*u - 1/2*v + 1/2) ∂/∂v⊗dv
>>> p = M.point((Integer(2),Integer(3)), chart=c_xy, name='p')
>>> ap = a.at(p) ; ap
Type-(1,1) tensor a on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
>>> ap.parent()
Free module of type-(1,1) tensors on the Tangent space at Point p
 on the 2-dimensional differentiable manifold M
>>> ap.display(eU.at(p))
a = 4 ∂/∂x⊗dx + 2 ∂/∂x⊗dy + 5 ∂/∂y⊗dy
>>> ap.display(eV.at(p))
a = 11/2 ∂/∂u⊗du - 3/2 ∂/∂u⊗dv + 1/2 ∂/∂v⊗du + 7/2 ∂/∂v⊗dv
>>> p.coord(c_uv) # to check the above expression
(5, -1)
```

base_module()

Return the vector field module on which `self` acts as a tensor.

OUTPUT:

- instance of *VectorFieldModule*

EXAMPLES:

The module of vector fields on the 2-sphere as a "base module":

```
sage: M = Manifold(2, 'S^2')
sage: t = M.tensor_field(0,2)
sage: t.base_module()
Module X(S^2) of vector fields on the 2-dimensional differentiable
 manifold S^2
sage: t.base_module() is M.vector_field_module()
True
sage: XM = M.vector_field_module()
sage: XM.an_element().base_module() is XM
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2')
>>> t = M.tensor_field(Integer(0),Integer(2))
>>> t.base_module()
Module X(S^2) of vector fields on the 2-dimensional differentiable
 manifold S^2
>>> t.base_module() is M.vector_field_module()
True
>>> XM = M.vector_field_module()
>>> XM.an_element().base_module() is XM
True
```

comp(*basis=None*, *from_basis=None*)

Return the components in a given vector frame.

If the components are not known already, they are computed by the tensor change-of-basis formula from components in another vector frame.

INPUT:

- `basis` – (default: `None`) vector frame in which the components are required; if none is provided, the components are assumed to refer to the tensor field domain's default frame
- `from_basis` – (default: `None`) vector frame from which the required components are computed, via the tensor change-of-basis formula, if they are not known already in the basis `basis`

OUTPUT:

- components in the vector frame `basis`, as a `Components`

EXAMPLES:

Components of a type-$(1, 1)$ tensor field defined on two open subsets:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U')
sage: c_xy.<x, y> = U.chart()
sage: e = U.default_frame() ; e
Coordinate frame (U, (∂/∂x,∂/∂y))
sage: V = M.open_subset('V')
sage: c_uv.<u, v> = V.chart()
sage: f = V.default_frame() ; f
Coordinate frame (V, (∂/∂u,∂/∂v))
sage: M.declare_union(U,V)   # M is the union of U and V
sage: t = M.tensor_field(1,1, name='t')
sage: t[e,0,0] = - x + y^3
sage: t[e,0,1] = 2+x
sage: t[f,1,1] = - u*v
sage: t.comp(e)
2-indices components w.r.t. Coordinate frame (U, (∂/∂x,∂/∂y))
sage: t.comp(e)[:]
[y^3 - x    x + 2]
[      0        0]
sage: t.comp(f)
2-indices components w.r.t. Coordinate frame (V, (∂/∂u,∂/∂v))
sage: t.comp(f)[:]
```

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```
[    0    0]
[    0 -u*v]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U')
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> e = U.default_frame() ; e
Coordinate frame (U, (∂/∂x,∂/∂y))
>>> V = M.open_subset('V')
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> f = V.default_frame() ; f
Coordinate frame (V, (∂/∂u,∂/∂v))
>>> M.declare_union(U,V)   # M is the union of U and V
>>> t = M.tensor_field(Integer(1),Integer(1), name='t')
>>> t[e,Integer(0),Integer(0)] = - x + y**Integer(3)
>>> t[e,Integer(0),Integer(1)] = Integer(2)+x
>>> t[f,Integer(1),Integer(1)] = - u*v
>>> t.comp(e)
2-indices components w.r.t. Coordinate frame (U, (∂/∂x,∂/∂y))
>>> t.comp(e)[:]
[y^3 - x    x + 2]
[      0        0]
>>> t.comp(f)
2-indices components w.r.t. Coordinate frame (V, (∂/∂u,∂/∂v))
>>> t.comp(f)[:]
[    0    0]
[    0 -u*v]
```

Since e is M's default frame, the argument e can be omitted:

```
sage: e is M.default_frame()
True
sage: t.comp() is t.comp(e)
True
```

```
>>> from sage.all import *
>>> e is M.default_frame()
True
>>> t.comp() is t.comp(e)
True
```

Example of computation of the components via a change of frame:

```
sage: a = V.automorphism_field()
sage: a[:] = [[1+v, -u^2], [0, 1-u]]
sage: h = f.new_frame(a, 'h')
sage: t.comp(h)
2-indices components w.r.t. Vector frame (V, (h_0,h_1))
sage: t.comp(h)[:]
[             0 -u^3*v/(v + 1)]
[             0           -u*v]
```

```
>>> from sage.all import *
>>> a = V.automorphism_field()
>>> a[:] = [[Integer(1)+v, -u**Integer(2)], [Integer(0), Integer(1)-u]]
>>> h = f.new_frame(a, 'h')
>>> t.comp(h)
2-indices components w.r.t. Vector frame (V, (h_0,h_1))
>>> t.comp(h)[:]
[             0 -u^3*v/(v + 1)]
[             0           -u*v]
```

contract(**args*)

Contraction of `self` with another tensor field on one or more indices.

INPUT:

- `pos1` – positions of the indices in the current tensor field involved in the contraction; `pos1` must be a sequence of integers, with 0 standing for the first index position, 1 for the second one, etc.; if `pos1` is not provided, a single contraction on the last index position of the tensor field is assumed
- `other` – the tensor field to contract with
- `pos2` – positions of the indices in `other` involved in the contraction, with the same conventions as for `pos1`; if `pos2` is not provided, a single contraction on the first index position of `other` is assumed

OUTPUT:

- tensor field resulting from the contraction at the positions `pos1` and `pos2` of the tensor field with `other`

EXAMPLES:

Contractions of a type-$(1,1)$ tensor field with a type-$(2,0)$ one on a 2-dimensional non-parallelizable manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
....:                               restrictions1= x>0, restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: a = M.tensor_field(1, 1, {eU: [[1, x], [0, 2]]}, name='a')
sage: a.add_comp_by_continuation(eV, W, chart=c_uv)
sage: b = M.tensor_field(2, 0, {eU: [[y, -1], [x+y, 2]]}, name='b')
sage: b.add_comp_by_continuation(eV, W, chart=c_uv)
sage: s = a.contract(b) ; s   # contraction on last index of a and first one␣
↪of b
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
```

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```
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
...                              restrictions1= x>Integer(0), restrictions2=␣
↪u+v>Integer(0))
>>> inv = transf.inverse()
>>> W = U.intersection(V)
>>> eU = c_xy.frame() ; eV = c_uv.frame()
>>> a = M.tensor_field(Integer(1), Integer(1), {eU: [[Integer(1), x],␣
↪[Integer(0), Integer(2)]]}, name='a')
>>> a.add_comp_by_continuation(eV, W, chart=c_uv)
>>> b = M.tensor_field(Integer(2), Integer(0), {eU: [[y, -Integer(1)], [x+y,␣
↪Integer(2)]]}, name='b')
>>> b.add_comp_by_continuation(eV, W, chart=c_uv)
>>> s = a.contract(b) ; s   # contraction on last index of a and first one of␣
↪b
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
```

Check 1: components with respect to the manifold's default frame (`eU`):

```
sage: all(bool(s[i,j] == sum(a[i,k]*b[k,j] for k in M.irange()))
....:     for i in M.irange() for j in M.irange())
True
```

```
>>> from sage.all import *
>>> all(bool(s[i,j] == sum(a[i,k]*b[k,j] for k in M.irange()))
...     for i in M.irange() for j in M.irange())
True
```

Check 2: components with respect to the frame `eV`:

```
sage: all(bool(s[eV,i,j] == sum(a[eV,i,k]*b[eV,k,j]
....:                            for k in M.irange()))
....:      for i in M.irange() for j in M.irange())
True
```

```
>>> from sage.all import *
>>> all(bool(s[eV,i,j] == sum(a[eV,i,k]*b[eV,k,j]
...                            for k in M.irange()))
...      for i in M.irange() for j in M.irange())
True
```

Instead of the explicit call to the method `contract()`, one may use the index notation with Einstein convention (summation over repeated indices); it suffices to pass the indices as a string inside square brackets:

```
sage: a['^i_k']*b['^kj'] == s
True
```

```
>>> from sage.all import *
>>> a['^i_k']*b['^kj'] == s
True
```

Indices not involved in the contraction may be replaced by dots:

```
sage: a['^._k']*b['^k.'] == s
True
```

```
>>> from sage.all import *
>>> a['^._k']*b['^k.'] == s
True
```

LaTeX notation may be used:

```
sage: a['^{i}_{k}']*b['^{kj}'] == s
True
```

```
>>> from sage.all import *
>>> a['^{i}_{k}']*b['^{kj}'] == s
True
```

Contraction on the last index of `a` and last index of `b`:

```
sage: s = a.contract(b, 1) ; s
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: a['^i_k']*b['^jk'] == s
True
```

```
>>> from sage.all import *
>>> s = a.contract(b, Integer(1)) ; s
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
>>> a['^i_k']*b['^jk'] == s
True
```

Contraction on the first index of `b` and the last index of `a`:

```
sage: s = b.contract(0,a,1) ; s
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: b['^ki']*a['^j_k'] == s
True
```

```
>>> from sage.all import *
>>> s = b.contract(Integer(0),a,Integer(1)) ; s
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
>>> b['^ki']*a['^j_k'] == s
True
```

The domain of the result is the intersection of the domains of the two tensor fields:

```
sage: aU = a.restrict(U) ; bV = b.restrict(V)
sage: s = aU.contract(b) ; s
Tensor field of type (2,0) on the Open subset U of the
 2-dimensional differentiable manifold M
```

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```
sage: s = a.contract(bV) ; s
Tensor field of type (2,0) on the Open subset V of the
 2-dimensional differentiable manifold M
sage: s = aU.contract(bV) ; s
Tensor field of type (2,0) on the Open subset W of the
 2-dimensional differentiable manifold M
sage: s0 = a.contract(b)
sage: s == s0.restrict(W)
True
```

```
>>> from sage.all import *
>>> aU = a.restrict(U) ; bV = b.restrict(V)
>>> s = aU.contract(b) ; s
Tensor field of type (2,0) on the Open subset U of the
 2-dimensional differentiable manifold M
>>> s = a.contract(bV) ; s
Tensor field of type (2,0) on the Open subset V of the
 2-dimensional differentiable manifold M
>>> s = aU.contract(bV) ; s
Tensor field of type (2,0) on the Open subset W of the
 2-dimensional differentiable manifold M
>>> s0 = a.contract(b)
>>> s == s0.restrict(W)
True
```

The contraction can be performed on more than one index: `c` being a type-$(2,2)$ tensor, contracting the indices in positions 2 and 3 of `c` with respectively those in positions 0 and 1 of `b` is:

```
sage: c = a*a ; c
Tensor field of type (2,2) on the 2-dimensional differentiable
 manifold M
sage: s = c.contract(2,3, b, 0,1) ; s  # long time
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> c = a*a ; c
Tensor field of type (2,2) on the 2-dimensional differentiable
 manifold M
>>> s = c.contract(Integer(2),Integer(3), b, Integer(0),Integer(1)) ; s  #␣
↪long time
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
```

The same double contraction using index notation:

```
sage: s == c['^.._kl']*b['^kl']  # long time
True
```

```
>>> from sage.all import *
>>> s == c['^.._kl']*b['^kl']  # long time
True
```

The symmetries are either conserved or destroyed by the contraction:

```
sage: c = c.symmetrize(0,1).antisymmetrize(2,3)
sage: c.symmetries()
symmetry: (0, 1);  antisymmetry: (2, 3)
sage: s = b.contract(0, c, 2) ; s
Tensor field of type (3,1) on the 2-dimensional differentiable
 manifold M
sage: s.symmetries()
symmetry: (1, 2);  no antisymmetry
```

```
>>> from sage.all import *
>>> c = c.symmetrize(Integer(0),Integer(1)).antisymmetrize(Integer(2),
↪Integer(3))
>>> c.symmetries()
symmetry: (0, 1);  antisymmetry: (2, 3)
>>> s = b.contract(Integer(0), c, Integer(2)) ; s
Tensor field of type (3,1) on the 2-dimensional differentiable
 manifold M
>>> s.symmetries()
symmetry: (1, 2);  no antisymmetry
```

Case of a scalar field result:

```
sage: a = M.one_form({eU: [y, 1+x]}, name='a')
sage: a.add_comp_by_continuation(eV, W, chart=c_uv)
sage: b = M.vector_field({eU: [x, y^2]}, name='b')
sage: b.add_comp_by_continuation(eV, W, chart=c_uv)
sage: a.display(eU)
a = y dx + (x + 1) dy
sage: b.display(eU)
b = x ∂/∂x + y^2 ∂/∂y
sage: s = a.contract(b) ; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M → ℝ
on U: (x, y) ↦ (x + 1)*y^2 + x*y
on V: (u, v) ↦ 1/8*u^3 - 1/8*u*v^2 + 1/8*v^3 + 1/2*u^2 - 1/8*(u^2 + 4*u)*v
sage: s == a['_i']*b['^i'] # use of index notation
True
sage: s == b.contract(a)
True
```

```
>>> from sage.all import *
>>> a = M.one_form({eU: [y, Integer(1)+x]}, name='a')
>>> a.add_comp_by_continuation(eV, W, chart=c_uv)
>>> b = M.vector_field({eU: [x, y**Integer(2)]}, name='b')
>>> b.add_comp_by_continuation(eV, W, chart=c_uv)
>>> a.display(eU)
a = y dx + (x + 1) dy
>>> b.display(eU)
b = x ∂/∂x + y^2 ∂/∂y
>>> s = a.contract(b) ; s
```

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```
Scalar field on the 2-dimensional differentiable manifold M
>>> s.display()
M → ℝ
on U: (x, y) ↦ (x + 1)*y^2 + x*y
on V: (u, v) ↦ 1/8*u^3 - 1/8*u*v^2 + 1/8*v^3 + 1/2*u^2 - 1/8*(u^2 + 4*u)*v
>>> s == a['_i']*b['^i'] # use of index notation
True
>>> s == b.contract(a)
True
```

Case of a vanishing scalar field result:

```
sage: b = M.vector_field({eU: [1+x, -y]}, name='b')
sage: b.add_comp_by_continuation(eV, W, chart=c_uv)
sage: s = a.contract(b) ; s
Scalar field zero on the 2-dimensional differentiable manifold M
sage: s.display()
zero: M → ℝ
on U: (x, y) ↦ 0
on V: (u, v) ↦ 0
```

```
>>> from sage.all import *
>>> b = M.vector_field({eU: [Integer(1)+x, -y]}, name='b')
>>> b.add_comp_by_continuation(eV, W, chart=c_uv)
>>> s = a.contract(b) ; s
Scalar field zero on the 2-dimensional differentiable manifold M
>>> s.display()
zero: M → ℝ
on U: (x, y) ↦ 0
on V: (u, v) ↦ 0
```

copy(*name=None*, *latex_name=None*)

Return an exact copy of `self`.

INPUT:

- `name` – (default: `None`) name given to the copy
- `latex_name` – (default: `None`) LaTeX symbol to denote the copy; if none is provided, the LaTeX symbol is set to `name`

> **Note**
>
> The name and the derived quantities are not copied.

EXAMPLES:

Copy of a type-$(1, 1)$ tensor field on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
```

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```
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                         intersection_name='W', restrictions1= x>0,
....:                         restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
sage: t = M.tensor_field(1, 1, name='t')
sage: t[e_xy,:] = [[x+y, 0], [2, 1-y]]
sage: t.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
sage: s = t.copy(); s
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: s.display(e_xy)
(x + y) ∂/∂x⊗dx + 2 ∂/∂y⊗dx + (-y + 1) ∂/∂y⊗dy
sage: s == t
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
...                         intersection_name='W', restrictions1= x>Integer(0),
...                         restrictions2= u+v>Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> e_xy = c_xy.frame(); e_uv = c_uv.frame()
>>> t = M.tensor_field(Integer(1), Integer(1), name='t')
>>> t[e_xy,:] = [[x+y, Integer(0)], [Integer(2), Integer(1)-y]]
>>> t.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
>>> s = t.copy(); s
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
>>> s.display(e_xy)
(x + y) ∂/∂x⊗dx + 2 ∂/∂y⊗dx + (-y + 1) ∂/∂y⊗dy
>>> s == t
True
```

If the original tensor field is modified, the copy is not:

```
sage: t[e_xy,0,0] = -1
sage: t.display(e_xy)
t = -∂/∂x⊗dx + 2 ∂/∂y⊗dx + (-y + 1) ∂/∂y⊗dy
sage: s.display(e_xy)
(x + y) ∂/∂x⊗dx + 2 ∂/∂y⊗dx + (-y + 1) ∂/∂y⊗dy
sage: s == t
False
```

```
>>> from sage.all import *
>>> t[e_xy,Integer(0),Integer(0)] = -Integer(1)
>>> t.display(e_xy)
t = -∂/∂x⊗dx + 2 ∂/∂y⊗dx + (-y + 1) ∂/∂y⊗dy
```

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```
>>> s.display(e_xy)
(x + y) ∂/∂x⊗dx + 2 ∂/∂y⊗dx + (-y + 1) ∂/∂y⊗dy
>>> s == t
False
```

copy_from(*other*)

Make `self` a copy of `other`.

INPUT:

- `other` – other tensor field, in the same module as `self`

> **Note**
>
> While the derived quantities are not copied, the name is kept.

> **Warning**
>
> All previous defined components and restrictions will be deleted!

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                     intersection_name='W', restrictions1= x>0,
....:                     restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
sage: t = M.tensor_field(1, 1, name='t')
sage: t[e_xy,:] = [[x+y, 0], [2, 1-y]]
sage: t.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
sage: s = M.tensor_field(1, 1, name='s')
sage: s.copy_from(t)
sage: s.display(e_xy)
s = (x + y) ∂/∂x⊗dx + 2 ∂/∂y⊗dx + (-y + 1) ∂/∂y⊗dy
sage: s == t
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
...                     intersection_name='W', restrictions1= x>Integer(0),
...                     restrictions2= u+v>Integer(0))
```

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```
>>> uv_to_xy = xy_to_uv.inverse()
>>> e_xy = c_xy.frame(); e_uv = c_uv.frame()
>>> t = M.tensor_field(Integer(1), Integer(1), name='t')
>>> t[e_xy,:] = [[x+y, Integer(0)], [Integer(2), Integer(1)-y]]
>>> t.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
>>> s = M.tensor_field(Integer(1), Integer(1), name='s')
>>> s.copy_from(t)
>>> s.display(e_xy)
s = (x + y) ∂/∂x⊗dx + 2 ∂/∂y⊗dx + (-y + 1) ∂/∂y⊗dy
>>> s == t
True
```

While the original tensor field is modified, the copy is not:

```
sage: t[e_xy,0,0] = -1
sage: t.display(e_xy)
t = -∂/∂x⊗dx + 2 ∂/∂y⊗dx + (-y + 1) ∂/∂y⊗dy
sage: s.display(e_xy)
s = (x + y) ∂/∂x⊗dx + 2 ∂/∂y⊗dx + (-y + 1) ∂/∂y⊗dy
sage: s == t
False
```

```
>>> from sage.all import *
>>> t[e_xy,Integer(0),Integer(0)] = -Integer(1)
>>> t.display(e_xy)
t = -∂/∂x⊗dx + 2 ∂/∂y⊗dx + (-y + 1) ∂/∂y⊗dy
>>> s.display(e_xy)
s = (x + y) ∂/∂x⊗dx + 2 ∂/∂y⊗dx + (-y + 1) ∂/∂y⊗dy
>>> s == t
False
```

dalembertian(*metric=None*)

Return the d'Alembertian of `self` with respect to a given Lorentzian metric.

The *d'Alembertian* of a tensor field t with respect to a Lorentzian metric g is nothing but the Laplace-Beltrami operator of g applied to t (see `laplacian()`); if `self` a tensor field t of type (k, l), the d'Alembertian of t with respect to g is then the tensor field of type (k, l) defined by

$$(\Box t)^{a_1\ldots a_k}{}_{b_1\ldots b_k} = \nabla_i \nabla^i t^{a_1\ldots a_k}{}_{b_1\ldots b_k},$$

where ∇ is the Levi-Civita connection of g (cf. `LeviCivitaConnection`) and $\nabla^i := g^{ij}\nabla_j$.

Note

If the metric g is not Lorentzian, the name *d'Alembertian* is not appropriate and one should use `laplacian()` instead.

INPUT:

- `metric` – (default: `None`) the Lorentzian metric g involved in the definition of the d'Alembertian; if none is provided, the domain of `self` is supposed to be endowed with a default Lorentzian metric (i.e. is supposed to be Lorentzian manifold, see `PseudoRiemannianManifold`) and the latter is used to define the d'Alembertian

OUTPUT:

- instance of *TensorField* representing the d'Alembertian of `self`

EXAMPLES:

d'Alembertian of a vector field in Minkowski spacetime, representing the electric field of a simple plane electromagnetic wave:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1, 1, 1, 1
sage: e = M.vector_field(name='e')
sage: e[1] = cos(t-z)
sage: e.display()  # plane wave propagating in the z direction
e = cos(t - z) ∂/∂x
sage: De = e.dalembertian(); De # long time
Vector field Box(e) on the 4-dimensional Lorentzian manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1), Integer(1), Integer(1),
↪ Integer(1)
>>> e = M.vector_field(name='e')
>>> e[Integer(1)] = cos(t-z)
>>> e.display()  # plane wave propagating in the z direction
e = cos(t - z) ∂/∂x
>>> De = e.dalembertian(); De # long time
Vector field Box(e) on the 4-dimensional Lorentzian manifold M
```

The function *dalembertian()* from the *operators* module can be used instead of the method *dalembertian()*:

```
sage: from sage.manifolds.operators import dalembertian
sage: dalembertian(e) == De # long time
True
```

```
>>> from sage.all import *
>>> from sage.manifolds.operators import dalembertian
>>> dalembertian(e) == De # long time
True
```

We check that the electric field obeys the wave equation:

```
sage: De.display() # long time
Box(e) = 0
```

```
>>> from sage.all import *
>>> De.display() # long time
Box(e) = 0
```

disp(*frame=None*, *chart=None*)

Display the tensor field in terms of its expansion with respect to a given vector frame.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- `frame` – (default: `None`) vector frame with respect to which the tensor is expanded; if `frame` is `None` and `chart` is not `None`, the coordinate frame associated with `chart` is assumed; if both `frame` and `chart` are `None`, the default frame of the domain of definition of the tensor field is assumed
- `chart` – (default: `None`) chart with respect to which the components of the tensor field in the selected frame are expressed; if `None`, the default chart of the vector frame domain is assumed

EXAMPLES:

Display of a type-$(1, 1)$ tensor field on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                    intersection_name='W', restrictions1= x>0,
....:                    restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
sage: t = M.tensor_field(1,1, name='t')
sage: t[e_xy,:] = [[x, 1], [y, 0]]
sage: t.add_comp_by_continuation(e_uv, W, c_uv)
sage: t.display(e_xy)
t = x ∂/∂x⊗dx + ∂/∂x⊗dy + y ∂/∂y⊗dx
sage: t.display(e_uv)
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
...                    intersection_name='W', restrictions1= x>Integer(0),
...                    restrictions2= u+v>Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V)
>>> e_xy = c_xy.frame(); e_uv = c_uv.frame()
>>> t = M.tensor_field(Integer(1),Integer(1), name='t')
>>> t[e_xy,:] = [[x, Integer(1)], [y, Integer(0)]]
>>> t.add_comp_by_continuation(e_uv, W, c_uv)
>>> t.display(e_xy)
t = x ∂/∂x⊗dx + ∂/∂x⊗dy + y ∂/∂y⊗dx
>>> t.display(e_uv)
```

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```
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

Since `e_xy` is `M`'s default frame, the argument `e_xy` can be omitted:

```
sage: e_xy is M.default_frame()
True
sage: t.display()
t = x ∂/∂x⊗dx + ∂/∂x⊗dy + y ∂/∂y⊗dx
```

```
>>> from sage.all import *
>>> e_xy is M.default_frame()
True
>>> t.display()
t = x ∂/∂x⊗dx + ∂/∂x⊗dy + y ∂/∂y⊗dx
```

Similarly, since `e_uv` is `V`'s default frame, the argument `e_uv` can be omitted when considering the restriction of `t` to `V`:

```
sage: t.restrict(V).display()
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> t.restrict(V).display()
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

If the coordinate expression of the components are to be displayed in a chart distinct from the default one on the considered domain, then the chart has to be passed as the second argument of `display`. For instance, on $W = U \cap V$, two charts are available: `c_xy.restrict(W)` (the default one) and `c_uv.restrict(W)`. Accordingly, one can have two views of the expansion of `t` in the *same* vector frame `e_uv.restrict(W)`:

```
sage: t.display(e_uv.restrict(W))  # W's default chart assumed
t = (1/2*x + 1/2*y + 1/2) ∂/∂u⊗du + (1/2*x + 1/2*y - 1/2) ∂/∂u⊗dv
  + (1/2*x - 1/2*y + 1/2) ∂/∂v⊗du + (1/2*x - 1/2*y - 1/2) ∂/∂v⊗dv
sage: t.display(e_uv.restrict(W), c_uv.restrict(W))
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> t.display(e_uv.restrict(W))  # W's default chart assumed
t = (1/2*x + 1/2*y + 1/2) ∂/∂u⊗du + (1/2*x + 1/2*y - 1/2) ∂/∂u⊗dv
  + (1/2*x - 1/2*y + 1/2) ∂/∂v⊗du + (1/2*x - 1/2*y - 1/2) ∂/∂v⊗dv
>>> t.display(e_uv.restrict(W), c_uv.restrict(W))
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

As a shortcut, one can pass just a chart to `display`. It is then understood that the expansion is to be performed with respect to the coordinate frame associated with this chart. Therefore the above command can be abridged to:

```
sage: t.display(c_uv.restrict(W))
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> t.display(c_uv.restrict(W))
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

and one has:

```
sage: t.display(c_xy)
t = x ∂/∂x⊗dx + ∂/∂x⊗dy + y ∂/∂y⊗dx
sage: t.display(c_uv)
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
sage: t.display(c_xy.restrict(W))
t = x ∂/∂x⊗dx + ∂/∂x⊗dy + y ∂/∂y⊗dx
sage: t.restrict(W).display(c_uv.restrict(W))
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> t.display(c_xy)
t = x ∂/∂x⊗dx + ∂/∂x⊗dy + y ∂/∂y⊗dx
>>> t.display(c_uv)
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
>>> t.display(c_xy.restrict(W))
t = x ∂/∂x⊗dx + ∂/∂x⊗dy + y ∂/∂y⊗dx
>>> t.restrict(W).display(c_uv.restrict(W))
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

One can ask for the display with respect to a frame in which `t` has not been initialized yet (this will automatically trigger the use of the change-of-frame formula for tensors):

```
sage: a = V.automorphism_field()
sage: a[:] = [[1+v, -u^2], [0, 1-u]]
sage: f = e_uv.new_frame(a, 'f')
sage: [f[i].display() for i in M.irange()]
[f_0 = (v + 1) ∂/∂u, f_1 = -u^2 ∂/∂u + (-u + 1) ∂/∂v]
sage: t.display(f)
t = -1/2*(u^2*v + 1)/(u - 1) f_0⊗f^0
  - 1/2*(2*u^3 - 5*u^2 - (u^4 + u^3 - u^2)*v + 3*u - 1)/((u - 1)*v + u - 1) f_
↪0⊗f^1
  - 1/2*(v^2 + 2*v + 1)/(u - 1) f_1⊗f^0
  + 1/2*(u^2 + (u^2 + u - 1)*v - u + 1)/(u - 1) f_1⊗f^1
```

```
>>> from sage.all import *
>>> a = V.automorphism_field()
>>> a[:] = [[Integer(1)+v, -u**Integer(2)], [Integer(0), Integer(1)-u]]
```

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```
>>> f = e_uv.new_frame(a, 'f')
>>> [f[i].display() for i in M.irange()]
[f_0 = (v + 1) ∂/∂u, f_1 = -u^2 ∂/∂u + (-u + 1) ∂/∂v]
>>> t.display(f)
t = -1/2*(u^2*v + 1)/(u - 1) f_0⊗f^0
  - 1/2*(2*u^3 - 5*u^2 - (u^4 + u^3 - u^2)*v + 3*u - 1)/((u - 1)*v + u - 1) f_
↪0⊗f^1
  - 1/2*(v^2 + 2*v + 1)/(u - 1) f_1⊗f^0
  + 1/2*(u^2 + (u^2 + u - 1)*v - u + 1)/(u - 1) f_1⊗f^1
```

A shortcut of `display()` is `disp()`:

```
sage: t.disp(e_uv)
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> t.disp(e_uv)
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

display(*frame=None*, *chart=None*)

Display the tensor field in terms of its expansion with respect to a given vector frame.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- `frame` – (default: `None`) vector frame with respect to which the tensor is expanded; if `frame` is `None` and `chart` is not `None`, the coordinate frame associated with `chart` is assumed; if both `frame` and `chart` are `None`, the default frame of the domain of definition of the tensor field is assumed
- `chart` – (default: `None`) chart with respect to which the components of the tensor field in the selected frame are expressed; if `None`, the default chart of the vector frame domain is assumed

EXAMPLES:

Display of a type-$(1, 1)$ tensor field on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                    intersection_name='W', restrictions1= x>0,
....:                    restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
sage: t = M.tensor_field(1,1, name='t')
sage: t[e_xy,:] = [[x, 1], [y, 0]]
sage: t.add_comp_by_continuation(e_uv, W, c_uv)
sage: t.display(e_xy)
t = x ∂/∂x⊗dx + ∂/∂x⊗dy + y ∂/∂y⊗dx
sage: t.display(e_uv)
```

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```
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
...                    intersection_name='W', restrictions1= x>Integer(0),
...                    restrictions2= u+v>Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V)
>>> e_xy = c_xy.frame(); e_uv = c_uv.frame()
>>> t = M.tensor_field(Integer(1),Integer(1), name='t')
>>> t[e_xy,:] = [[x, Integer(1)], [y, Integer(0)]]
>>> t.add_comp_by_continuation(e_uv, W, c_uv)
>>> t.display(e_xy)
t = x ∂/∂x⊗dx + ∂/∂x⊗dy + y ∂/∂y⊗dx
>>> t.display(e_uv)
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

Since `e_xy` is `M`'s default frame, the argument `e_xy` can be omitted:

```
sage: e_xy is M.default_frame()
True
sage: t.display()
t = x ∂/∂x⊗dx + ∂/∂x⊗dy + y ∂/∂y⊗dx
```

```
>>> from sage.all import *
>>> e_xy is M.default_frame()
True
>>> t.display()
t = x ∂/∂x⊗dx + ∂/∂x⊗dy + y ∂/∂y⊗dx
```

Similarly, since `e_uv` is `V`'s default frame, the argument `e_uv` can be omitted when considering the restriction of `t` to `V`:

```
sage: t.restrict(V).display()
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> t.restrict(V).display()
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

If the coordinate expression of the components are to be displayed in a chart distinct from the default one on the considered domain, then the chart has to be passed as the second argument of `display`. For instance, on $W = U \cap V$, two charts are available: `c_xy.restrict(W)` (the default one) and `c_uv.restrict(W)`.

Accordingly, one can have two views of the expansion of `t` in the *same* vector frame `e_uv.restrict(W)`:

```
sage: t.display(e_uv.restrict(W))  # W's default chart assumed
t = (1/2*x + 1/2*y + 1/2) ∂/∂u⊗du + (1/2*x + 1/2*y - 1/2) ∂/∂u⊗dv
  + (1/2*x - 1/2*y + 1/2) ∂/∂v⊗du + (1/2*x - 1/2*y - 1/2) ∂/∂v⊗dv
sage: t.display(e_uv.restrict(W), c_uv.restrict(W))
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> t.display(e_uv.restrict(W))  # W's default chart assumed
t = (1/2*x + 1/2*y + 1/2) ∂/∂u⊗du + (1/2*x + 1/2*y - 1/2) ∂/∂u⊗dv
  + (1/2*x - 1/2*y + 1/2) ∂/∂v⊗du + (1/2*x - 1/2*y - 1/2) ∂/∂v⊗dv
>>> t.display(e_uv.restrict(W), c_uv.restrict(W))
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

As a shortcut, one can pass just a chart to `display`. It is then understood that the expansion is to be performed with respect to the coordinate frame associated with this chart. Therefore the above command can be abridged to:

```
sage: t.display(c_uv.restrict(W))
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> t.display(c_uv.restrict(W))
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

and one has:

```
sage: t.display(c_xy)
t = x ∂/∂x⊗dx + ∂/∂x⊗dy + y ∂/∂y⊗dx
sage: t.display(c_uv)
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
sage: t.display(c_xy.restrict(W))
t = x ∂/∂x⊗dx + ∂/∂x⊗dy + y ∂/∂y⊗dx
sage: t.restrict(W).display(c_uv.restrict(W))
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> t.display(c_xy)
t = x ∂/∂x⊗dx + ∂/∂x⊗dy + y ∂/∂y⊗dx
>>> t.display(c_uv)
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
>>> t.display(c_xy.restrict(W))
t = x ∂/∂x⊗dx + ∂/∂x⊗dy + y ∂/∂y⊗dx
>>> t.restrict(W).display(c_uv.restrict(W))
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
```

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```
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

One can ask for the display with respect to a frame in which `t` has not been initialized yet (this will automatically trigger the use of the change-of-frame formula for tensors):

```
sage: a = V.automorphism_field()
sage: a[:] = [[1+v, -u^2], [0, 1-u]]
sage: f = e_uv.new_frame(a, 'f')
sage: [f[i].display() for i in M.irange()]
[f_0 = (v + 1) ∂/∂u, f_1 = -u^2 ∂/∂u + (-u + 1) ∂/∂v]
sage: t.display(f)
t = -1/2*(u^2*v + 1)/(u - 1) f_0⊗f^0
  - 1/2*(2*u^3 - 5*u^2 - (u^4 + u^3 - u^2)*v + 3*u - 1)/((u - 1)*v + u - 1) f_
↪0⊗f^1
  - 1/2*(v^2 + 2*v + 1)/(u - 1) f_1⊗f^0
  + 1/2*(u^2 + (u^2 + u - 1)*v - u + 1)/(u - 1) f_1⊗f^1
```

```
>>> from sage.all import *
>>> a = V.automorphism_field()
>>> a[:] = [[Integer(1)+v, -u**Integer(2)], [Integer(0), Integer(1)-u]]
>>> f = e_uv.new_frame(a, 'f')
>>> [f[i].display() for i in M.irange()]
[f_0 = (v + 1) ∂/∂u, f_1 = -u^2 ∂/∂u + (-u + 1) ∂/∂v]
>>> t.display(f)
t = -1/2*(u^2*v + 1)/(u - 1) f_0⊗f^0
  - 1/2*(2*u^3 - 5*u^2 - (u^4 + u^3 - u^2)*v + 3*u - 1)/((u - 1)*v + u - 1) f_
↪0⊗f^1
  - 1/2*(v^2 + 2*v + 1)/(u - 1) f_1⊗f^0
  + 1/2*(u^2 + (u^2 + u - 1)*v - u + 1)/(u - 1) f_1⊗f^1
```

A shortcut of `display()` is `disp()`:

```
sage: t.disp(e_uv)
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> t.disp(e_uv)
t = (1/2*u + 1/2) ∂/∂u⊗du + (1/2*u - 1/2) ∂/∂u⊗dv
  + (1/2*v + 1/2) ∂/∂v⊗du + (1/2*v - 1/2) ∂/∂v⊗dv
```

display_comp(*frame=None*, *chart=None*, *coordinate_labels=True*, *only_nonzero=True*, *only_nonredundant=False*)

Display the tensor components with respect to a given frame, one per line.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- `frame` – (default: `None`) vector frame with respect to which the tensor field components are defined; if `None`, then
  - if `chart` is not `None`, the coordinate frame associated to `chart` is used
  - otherwise, the default basis of the vector field module on which the tensor field is defined is used

- `chart` – (default: `None`) chart specifying the coordinate expression of the components; if `None`, the default chart of the tensor field domain is used
- `coordinate_labels` – boolean (default: `True`); if `True`, coordinate symbols are used by default (instead of integers) as index labels whenever `frame` is a coordinate frame
- `only_nonzero` – boolean (default: `True`); if `True`, only nonzero components are displayed
- `only_nonredundant` – boolean (default: `False`); if `True`, only nonredundant components are displayed in case of symmetries

EXAMPLES:

Display of the components of a type-$(1,1)$ tensor field defined on two open subsets:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U')
sage: c_xy.<x, y> = U.chart()
sage: e = U.default_frame()
sage: V = M.open_subset('V')
sage: c_uv.<u, v> = V.chart()
sage: f = V.default_frame()
sage: M.declare_union(U,V)   # M is the union of U and V
sage: t = M.tensor_field(1,1, name='t')
sage: t[e,0,0] = - x + y^3
sage: t[e,0,1] = 2+x
sage: t[f,1,1] = - u*v
sage: t.display_comp(e)
t^x_x = y^3 - x
t^x_y = x + 2
sage: t.display_comp(f)
t^v_v = -u*v
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U')
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> e = U.default_frame()
>>> V = M.open_subset('V')
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> f = V.default_frame()
>>> M.declare_union(U,V)   # M is the union of U and V
>>> t = M.tensor_field(Integer(1),Integer(1), name='t')
>>> t[e,Integer(0),Integer(0)] = - x + y**Integer(3)
>>> t[e,Integer(0),Integer(1)] = Integer(2)+x
>>> t[f,Integer(1),Integer(1)] = - u*v
>>> t.display_comp(e)
t^x_x = y^3 - x
t^x_y = x + 2
>>> t.display_comp(f)
t^v_v = -u*v
```

Components in a chart frame:

```
sage: t.display_comp(chart=c_xy)
t^x_x = y^3 - x
```

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```
t^x_y = x + 2
sage: t.display_comp(chart=c_uv)
t^v_v = -u*v
```

```
>>> from sage.all import *
>>> t.display_comp(chart=c_xy)
t^x_x = y^3 - x
t^x_y = x + 2
>>> t.display_comp(chart=c_uv)
t^v_v = -u*v
```

See documentation of `sage.manifolds.differentiable.tensorfield_paral.TensorFieldParal.display_comp()` for more options.

div(*metric=None*)

Return the divergence of `self` (with respect to a given metric).

The divergence is taken on the *last* index: if `self` is a tensor field t of type $(k, 0)$ with $k \geq 1$, the divergence of t with respect to the metric g is the tensor field of type $(k-1, 0)$ defined by

$$(\operatorname{div} t)^{a_1 \ldots a_{k-1}} = \nabla_i t^{a_1 \ldots a_{k-1} i} = (\nabla t)^{a_1 \ldots a_{k-1} i}{}_i,$$

where ∇ is the Levi-Civita connection of g (cf. `LeviCivitaConnection`).

This definition is extended to tensor fields of type (k, l) with $k \geq 0$ and $l \geq 1$, by raising the last index with the metric g: $\operatorname{div} t$ is then the tensor field of type $(k, l-1)$ defined by

$$(\operatorname{div} t)^{a_1 \ldots a_k}{}_{b_1 \ldots b_{l-1}} = \nabla_i (g^{ij} t^{a_1 \ldots a_k}{}_{b_1 \ldots b_{l-1} j}) = (\nabla t^\sharp)^{a_1 \ldots a_k i}{}_{b_1 \ldots b_{l-1} i},$$

where $t^\sharp$ is the tensor field deduced from t by raising the last index with the metric g (see `up()`).

INPUT:

- `metric` – (default: `None`) the pseudo-Riemannian metric g involved in the definition of the divergence; if none is provided, the domain of `self` is supposed to be endowed with a default metric (i.e. is supposed to be pseudo-Riemannian manifold, see `PseudoRiemannianManifold`) and the latter is used to define the divergence.

OUTPUT:

- instance of either `DiffScalarField` if $(k, l) = (1, 0)$ (`self` is a vector field) or $(k, l) = (0, 1)$ (`self` is a 1-form) or of `TensorField` if $k + l \geq 2$ representing the divergence of `self` with respect to `metric`

EXAMPLES:

Divergence of a vector field in the Euclidean plane:

```
sage: M.<x,y> = EuclideanSpace()
sage: v = M.vector_field(x, y, name='v')
sage: s = v.divergence(); s
Scalar field div(v) on the Euclidean plane E^2
sage: s.display()
div(v): E^2 → ℝ
   (x, y) ↦ 2
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y',)); (x, y,) = M._first_ngens(2)
>>> v = M.vector_field(x, y, name='v')
>>> s = v.divergence(); s
Scalar field div(v) on the Euclidean plane E^2
>>> s.display()
div(v): E^2 → ℝ
   (x, y) ↦ 2
```

A shortcut alias of `divergence` is `div`:

```
sage: v.div() == s
True
```

```
>>> from sage.all import *
>>> v.div() == s
True
```

The function *div()* from the *operators* module can be used instead of the method *divergence()*:

```
sage: from sage.manifolds.operators import div
sage: div(v) == s
True
```

```
>>> from sage.all import *
>>> from sage.manifolds.operators import div
>>> div(v) == s
True
```

The divergence can be taken with respect to a metric tensor that is not the default one:

```
sage: h = M.lorentzian_metric('h')
sage: h[1,1], h[2,2] = -1, 1/(1+x^2+y^2)
sage: s = v.div(h); s
Scalar field div_h(v) on the Euclidean plane E^2
sage: s.display()
div_h(v): E^2 → ℝ
   (x, y) ↦ (x^2 + y^2 + 2)/(x^2 + y^2 + 1)
```

```
>>> from sage.all import *
>>> h = M.lorentzian_metric('h')
>>> h[Integer(1),Integer(1)], h[Integer(2),Integer(2)] = -Integer(1),␣
↪Integer(1)/(Integer(1)+x**Integer(2)+y**Integer(2))
>>> s = v.div(h); s
Scalar field div_h(v) on the Euclidean plane E^2
>>> s.display()
div_h(v): E^2 → ℝ
   (x, y) ↦ (x^2 + y^2 + 2)/(x^2 + y^2 + 1)
```

The standard formula

$$\mathrm{div}_h\, v = \frac{1}{\sqrt{|\det h|}} \frac{\partial}{\partial x^i} \left(\sqrt{|\det h|}\, v^i \right)$$

is checked as follows:

```
sage: sqrth = h.sqrt_abs_det().expr(); sqrth
1/sqrt(x^2 + y^2 + 1)
sage: s == 1/sqrth * sum( (sqrth*v[i]).diff(i) for i in M.irange())
True
```

```
>>> from sage.all import *
>>> sqrth = h.sqrt_abs_det().expr(); sqrth
1/sqrt(x^2 + y^2 + 1)
>>> s == Integer(1)/sqrth * sum( (sqrth*v[i]).diff(i) for i in M.irange())
True
```

A divergence-free vector:

```
sage: w = M.vector_field(-y, x, name='w')
sage: w.div().display()
div(w): E^2 → ℝ
   (x, y) ↦ 0
sage: w.div(h).display()
div_h(w): E^2 → ℝ
   (x, y) ↦ 0
```

```
>>> from sage.all import *
>>> w = M.vector_field(-y, x, name='w')
>>> w.div().display()
div(w): E^2 → ℝ
   (x, y) ↦ 0
>>> w.div(h).display()
div_h(w): E^2 → ℝ
   (x, y) ↦ 0
```

Divergence of a type-`(2,0)` tensor field:

```
sage: t = v*w; t
Tensor field v⊗w of type (2,0) on the Euclidean plane E^2
sage: s = t.div(); s
Vector field div(v⊗w) on the Euclidean plane E^2
sage: s.display()
div(v⊗w) = -y e_x + x e_y
```

```
>>> from sage.all import *
>>> t = v*w; t
Tensor field v⊗w of type (2,0) on the Euclidean plane E^2
>>> s = t.div(); s
Vector field div(v⊗w) on the Euclidean plane E^2
>>> s.display()
div(v⊗w) = -y e_x + x e_y
```

divergence(*metric=None*)

Return the divergence of `self` (with respect to a given metric).

The divergence is taken on the *last* index: if `self` is a tensor field t of type $(k,0)$ with $k \geq 1$, the divergence of t with respect to the metric g is the tensor field of type $(k-1,0)$ defined by

$$(\mathrm{div}\, t)^{a_1\ldots a_{k-1}} = \nabla_i t^{a_1\ldots a_{k-1}i} = (\nabla t)^{a_1\ldots a_{k-1}i}{}_i,$$

where ∇ is the Levi-Civita connection of g (cf. *LeviCivitaConnection*).

This definition is extended to tensor fields of type (k, l) with $k \geq 0$ and $l \geq 1$, by raising the last index with the metric g: $\operatorname{div} t$ is then the tensor field of type $(k, l-1)$ defined by

$$(\operatorname{div} t)^{a_1 \ldots a_k}{}_{b_1 \ldots b_{l-1}} = \nabla_i (g^{ij} t^{a_1 \ldots a_k}{}_{b_1 \ldots b_{l-1} j}) = (\nabla t^\sharp)^{a_1 \ldots a_k i}{}_{b_1 \ldots b_{l-1} i},$$

where $t^\sharp$ is the tensor field deduced from t by raising the last index with the metric g (see *up()*).

INPUT:

- `metric` – (default: `None`) the pseudo-Riemannian metric g involved in the definition of the divergence; if none is provided, the domain of `self` is supposed to be endowed with a default metric (i.e. is supposed to be pseudo-Riemannian manifold, see *PseudoRiemannianManifold*) and the latter is used to define the divergence.

OUTPUT:

- instance of either *DiffScalarField* if $(k, l) = (1, 0)$ (`self` is a vector field) or $(k, l) = (0, 1)$ (`self` is a 1-form) or of *TensorField* if $k + l \geq 2$ representing the divergence of `self` with respect to `metric`

EXAMPLES:

Divergence of a vector field in the Euclidean plane:

```
sage: M.<x,y> = EuclideanSpace()
sage: v = M.vector_field(x, y, name='v')
sage: s = v.divergence(); s
Scalar field div(v) on the Euclidean plane E^2
sage: s.display()
div(v): E^2 → ℝ
   (x, y) ↦ 2
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y',)); (x, y,) = M._first_ngens(2)
>>> v = M.vector_field(x, y, name='v')
>>> s = v.divergence(); s
Scalar field div(v) on the Euclidean plane E^2
>>> s.display()
div(v): E^2 → ℝ
   (x, y) ↦ 2
```

A shortcut alias of `divergence` is `div`:

```
sage: v.div() == s
True
```

```
>>> from sage.all import *
>>> v.div() == s
True
```

The function *div()* from the *operators* module can be used instead of the method *divergence()*:

```
sage: from sage.manifolds.operators import div
sage: div(v) == s
True
```

```
>>> from sage.all import *
>>> from sage.manifolds.operators import div
>>> div(v) == s
True
```

The divergence can be taken with respect to a metric tensor that is not the default one:

```
sage: h = M.lorentzian_metric('h')
sage: h[1,1], h[2,2] = -1, 1/(1+x^2+y^2)
sage: s = v.div(h); s
Scalar field div_h(v) on the Euclidean plane E^2
sage: s.display()
div_h(v): E^2 → ℝ
   (x, y) ↦ (x^2 + y^2 + 2)/(x^2 + y^2 + 1)
```

```
>>> from sage.all import *
>>> h = M.lorentzian_metric('h')
>>> h[Integer(1),Integer(1)], h[Integer(2),Integer(2)] = -Integer(1),␣
→Integer(1)/(Integer(1)+x**Integer(2)+y**Integer(2))
>>> s = v.div(h); s
Scalar field div_h(v) on the Euclidean plane E^2
>>> s.display()
div_h(v): E^2 → ℝ
   (x, y) ↦ (x^2 + y^2 + 2)/(x^2 + y^2 + 1)
```

The standard formula

$$\mathrm{div}_h\, v = \frac{1}{\sqrt{|\det h|}} \frac{\partial}{\partial x^i} \left(\sqrt{|\det h|}\, v^i \right)$$

is checked as follows:

```
sage: sqrth = h.sqrt_abs_det().expr(); sqrth
1/sqrt(x^2 + y^2 + 1)
sage: s == 1/sqrth * sum( (sqrth*v[i]).diff(i) for i in M.irange())
True
```

```
>>> from sage.all import *
>>> sqrth = h.sqrt_abs_det().expr(); sqrth
1/sqrt(x^2 + y^2 + 1)
>>> s == Integer(1)/sqrth * sum( (sqrth*v[i]).diff(i) for i in M.irange())
True
```

A divergence-free vector:

```
sage: w = M.vector_field(-y, x, name='w')
sage: w.div().display()
div(w): E^2 → ℝ
   (x, y) ↦ 0
sage: w.div(h).display()
div_h(w): E^2 → ℝ
   (x, y) ↦ 0
```

```
>>> from sage.all import *
>>> w = M.vector_field(-y, x, name='w')
>>> w.div().display()
div(w): E^2 → ℝ
   (x, y) ↦ 0
>>> w.div(h).display()
div_h(w): E^2 → ℝ
   (x, y) ↦ 0
```

Divergence of a type-`(2,0)` tensor field:

```
sage: t = v*w; t
Tensor field v⊗w of type (2,0) on the Euclidean plane E^2
sage: s = t.div(); s
Vector field div(v⊗w) on the Euclidean plane E^2
sage: s.display()
div(v⊗w) = -y e_x + x e_y
```

```
>>> from sage.all import *
>>> t = v*w; t
Tensor field v⊗w of type (2,0) on the Euclidean plane E^2
>>> s = t.div(); s
Vector field div(v⊗w) on the Euclidean plane E^2
>>> s.display()
div(v⊗w) = -y e_x + x e_y
```

domain()

Return the manifold on which `self` is defined.

OUTPUT:

- instance of class *DifferentiableManifold*

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: t = M.tensor_field(1,2)
sage: t.domain()
2-dimensional differentiable manifold M
sage: U = M.open_subset('U', coord_def={c_xy: x<0})
sage: h = t.restrict(U)
sage: h.domain()
Open subset U of the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> t = M.tensor_field(Integer(1),Integer(2))
>>> t.domain()
2-dimensional differentiable manifold M
>>> U = M.open_subset('U', coord_def={c_xy: x<Integer(0)})
>>> h = t.restrict(U)
```

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```
>>> h.domain()
Open subset U of the 2-dimensional differentiable manifold M
```

down (*non_degenerate_form*, *pos=None*)

Compute a dual of the tensor field by lowering some index with a given non-degenerate form (pseudo-Riemannian metric or symplectic form).

If T is the tensor field, (k, l) its type and p the position of a contravariant index (i.e. $0 \leq p < k$), this method called with `pos` $= p$ yields the tensor field $T^\flat$ of type $(k-1, l+1)$ whose components are

$$(T^\flat)^{a_1 \ldots a_{k-1}}{}_{b_1 \ldots b_{l+1}} = g_{i b_1} T^{a_1 \ldots a_p \, i \, a_{p+1} \ldots a_{k-1}}{}_{b_2 \ldots b_{l+1}},$$

g_{ab} being the components of the metric tensor or the symplectic form, respectively.

The reverse operation is `TensorField.up()`.

INPUT:

- `non_degenerate_form` – non-degenerate form g
- `pos` – (default: `None`) position of the index (with the convention `pos=0` for the first index); if `None`, the lowering is performed over all the contravariant indices, starting from the last one

OUTPUT:

- the tensor field $T^\flat$ resulting from the index lowering operation

EXAMPLES:

Lowering the index of a vector field results in a 1-form:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: c_xy.<x,y> = M.chart()
sage: g = M.metric('g')
sage: g[1,1], g[1,2], g[2,2] = 1+x, x*y, 1-y
sage: v = M.vector_field(-1, 2)
sage: w = v.down(g) ; w
1-form on the 2-dimensional differentiable manifold M
sage: w.display()
(2*x*y - x - 1) dx + (-(x + 2)*y + 2) dy
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', start_index=Integer(1))
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> g = M.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(1),Integer(2)], g[Integer(2),
↪Integer(2)] = Integer(1)+x, x*y, Integer(1)-y
>>> v = M.vector_field(-Integer(1), Integer(2))
>>> w = v.down(g) ; w
1-form on the 2-dimensional differentiable manifold M
>>> w.display()
(2*x*y - x - 1) dx + (-(x + 2)*y + 2) dy
```

Using the index notation instead of `down()`:

```
sage: w == g['_ab']*v['^b']
True
```

```
>>> from sage.all import *
>>> w == g['_ab']*v['^b']
True
```

The reverse operation:

```
sage: v1 = w.up(g) ; v1
Vector field on the 2-dimensional differentiable manifold M
sage: v1 == v
True
```

```
>>> from sage.all import *
>>> v1 = w.up(g) ; v1
Vector field on the 2-dimensional differentiable manifold M
>>> v1 == v
True
```

Lowering the indices of a tensor field of type (2,0):

```
sage: t = M.tensor_field(2, 0, [[1,2], [3,4]])
sage: td0 = t.down(g, 0) ; td0  # lowering the first index
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: td0 == g['_ac']*t['^cb'] # the same operation in index notation
True
sage: td0[:]
[  3*x*y + x + 1   (x - 3)*y + 3]
[4*x*y + 2*x + 2 2*(x - 2)*y + 4]
sage: tdd0 = td0.down(g) ; tdd0 # the two indices have been lowered, starting␣
↪from the first one
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
sage: tdd0 == g['_ac']*td0['^c_b'] # the same operation in index notation
True
sage: tdd0[:]
[      4*x^2*y^2 + x^2 + 5*(x^2 + x)*y + 2*x + 1 2*(x^2 - 2*x)*y^2 + (x^2 +␣
↪2*x - 3)*y + 3*x + 3]
[(3*x^2 - 4*x)*y^2 + (x^2 + 3*x - 2)*y + 2*x + 2           (x^2 - 5*x + 4)*y^
↪2 + (5*x - 8)*y + 4]
sage: td1 = t.down(g, 1) ; td1  # lowering the second index
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: td1 == g['_ac']*t['^bc'] # the same operation in index notation
True
sage: td1[:]
[  2*x*y + x + 1   (x - 2)*y + 2]
[4*x*y + 3*x + 3 (3*x - 4)*y + 4]
sage: tdd1 = td1.down(g) ; tdd1 # the two indices have been lowered, starting␣
↪from the second one
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
sage: tdd1 == g['_ac']*td1['^c_b'] # the same operation in index notation
True
```

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```
sage: tdd1[:]
[      4*x^2*y^2 + x^2 + 5*(x^2 + x)*y + 2*x + 1 (3*x^2 - 4*x)*y^2 + (x^2 +␣
↪3*x - 2)*y + 2*x + 2]
[2*(x^2 - 2*x)*y^2 + (x^2 + 2*x - 3)*y + 3*x + 3           (x^2 - 5*x + 4)*y^
↪2 + (5*x - 8)*y + 4]
sage: tdd1 == tdd0   # the order of index lowering is important
False
sage: tdd = t.down(g) ; tdd  # both indices are lowered, starting from the␣
↪last one
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
sage: tdd[:]
[      4*x^2*y^2 + x^2 + 5*(x^2 + x)*y + 2*x + 1 (3*x^2 - 4*x)*y^2 + (x^2 +␣
↪3*x - 2)*y + 2*x + 2]
[2*(x^2 - 2*x)*y^2 + (x^2 + 2*x - 3)*y + 3*x + 3           (x^2 - 5*x + 4)*y^
↪2 + (5*x - 8)*y + 4]
sage: tdd0 == tdd  # to get tdd0, indices have been lowered from the first␣
↪one, contrary to tdd
False
sage: tdd1 == tdd  # the same order for index lowering has been applied
True
sage: u0tdd = tdd.up(g, 0) ; u0tdd # the first index is raised again
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: uu0tdd = u0tdd.up(g) ; uu0tdd # the second index is then raised
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: u1tdd = tdd.up(g, 1) ; u1tdd  # raising operation, starting from the␣
↪last index
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: uu1tdd = u1tdd.up(g) ; uu1tdd
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: uutdd = tdd.up(g) ; uutdd  # both indices are raised, starting from the␣
↪first one
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: uutdd == t  # should be true
True
sage: uu0tdd == t # should be true
True
sage: uu1tdd == t # not true, because of the order of index raising to get␣
↪uu1tdd
False
```

```
>>> from sage.all import *
>>> t = M.tensor_field(Integer(2), Integer(0), [[Integer(1),Integer(2)],␣
↪[Integer(3),Integer(4)]])
>>> td0 = t.down(g, Integer(0)) ; td0  # lowering the first index
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
```

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```
>>> td0 == g['_ac']*t['^cb'] # the same operation in index notation
True
>>> td0[:]
[  3*x*y + x + 1   (x - 3)*y + 3]
[4*x*y + 2*x + 2 2*(x - 2)*y + 4]
>>> tdd0 = td0.down(g) ; tdd0 # the two indices have been lowered, starting␣
↪from the first one
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
>>> tdd0 == g['_ac']*td0['^c_b'] # the same operation in index notation
True
>>> tdd0[:]
[      4*x^2*y^2 + x^2 + 5*(x^2 + x)*y + 2*x + 1 2*(x^2 - 2*x)*y^2 + (x^2 +␣
↪2*x - 3)*y + 3*x + 3]
[(3*x^2 - 4*x)*y^2 + (x^2 + 3*x - 2)*y + 2*x + 2             (x^2 - 5*x + 4)*y^
↪2 + (5*x - 8)*y + 4]
>>> td1 = t.down(g, Integer(1)) ; td1  # lowering the second index
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
>>> td1 == g['_ac']*t['^bc'] # the same operation in index notation
True
>>> td1[:]
[  2*x*y + x + 1   (x - 2)*y + 2]
[4*x*y + 3*x + 3 (3*x - 4)*y + 4]
>>> tdd1 = td1.down(g) ; tdd1 # the two indices have been lowered, starting␣
↪from the second one
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
>>> tdd1 == g['_ac']*td1['^c_b'] # the same operation in index notation
True
>>> tdd1[:]
[      4*x^2*y^2 + x^2 + 5*(x^2 + x)*y + 2*x + 1 (3*x^2 - 4*x)*y^2 + (x^2 +␣
↪3*x - 2)*y + 2*x + 2]
[2*(x^2 - 2*x)*y^2 + (x^2 + 2*x - 3)*y + 3*x + 3             (x^2 - 5*x + 4)*y^
↪2 + (5*x - 8)*y + 4]
>>> tdd1 == tdd0   # the order of index lowering is important
False
>>> tdd = t.down(g) ; tdd  # both indices are lowered, starting from the last␣
↪one
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
>>> tdd[:]
[      4*x^2*y^2 + x^2 + 5*(x^2 + x)*y + 2*x + 1 (3*x^2 - 4*x)*y^2 + (x^2 +␣
↪3*x - 2)*y + 2*x + 2]
[2*(x^2 - 2*x)*y^2 + (x^2 + 2*x - 3)*y + 3*x + 3             (x^2 - 5*x + 4)*y^
↪2 + (5*x - 8)*y + 4]
>>> tdd0 == tdd  # to get tdd0, indices have been lowered from the first one,␣
↪contrary to tdd
False
>>> tdd1 == tdd  # the same order for index lowering has been applied
True
>>> u0tdd = tdd.up(g, Integer(0)) ; u0tdd # the first index is raised again
```

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```
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
>>> uu0tdd = u0tdd.up(g) ; uu0tdd # the second index is then raised
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
>>> u1tdd = tdd.up(g, Integer(1)) ; u1tdd  # raising operation, starting from␣
↪the last index
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
>>> uu1tdd = u1tdd.up(g) ; uu1tdd
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
>>> uutdd = tdd.up(g) ; uutdd  # both indices are raised, starting from the␣
↪first one
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
>>> uutdd == t  # should be true
True
>>> uu0tdd == t # should be true
True
>>> uu1tdd == t # not true, because of the order of index raising to get␣
↪uu1tdd
False
```

laplacian(*metric=None*)

Return the Laplacian of `self` with respect to a given metric (Laplace-Beltrami operator).

If `self` is a tensor field t of type (k,l), the Laplacian of t with respect to the metric g is the tensor field of type (k,l) defined by

$$(\Delta t)^{a_1\ldots a_k}{}_{b_1\ldots b_k} = \nabla_i \nabla^i t^{a_1\ldots a_k}{}_{b_1\ldots b_k},$$

where ∇ is the Levi-Civita connection of g (cf. *LeviCivitaConnection*) and $\nabla^i := g^{ij}\nabla_j$. The operator $\Delta = \nabla_i \nabla^i$ is called the *Laplace-Beltrami operator* of metric g.

INPUT:

- `metric` – (default: `None`) the pseudo-Riemannian metric g involved in the definition of the Laplacian; if none is provided, the domain of `self` is supposed to be endowed with a default metric (i.e. is supposed to be pseudo-Riemannian manifold, see *PseudoRiemannianManifold*) and the latter is used to define the Laplacian

OUTPUT:

- instance of *TensorField* representing the Laplacian of `self`

EXAMPLES:

Laplacian of a vector field in the Euclidean plane:

```
sage: M.<x,y> = EuclideanSpace()
sage: v = M.vector_field(x^3 + y^2, x*y, name='v')
sage: Dv = v.laplacian(); Dv
Vector field Delta(v) on the Euclidean plane E^2
sage: Dv.display()
Delta(v) = (6*x + 2) e_x
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y',)); (x, y,) = M._first_ngens(2)
>>> v = M.vector_field(x**Integer(3) + y**Integer(2), x*y, name='v')
>>> Dv = v.laplacian(); Dv
Vector field Delta(v) on the Euclidean plane E^2
>>> Dv.display()
Delta(v) = (6*x + 2) e_x
```

The function `laplacian()` from the `operators` module can be used instead of the method `laplacian()`:

```
sage: from sage.manifolds.operators import laplacian
sage: laplacian(v) == Dv
True
```

```
>>> from sage.all import *
>>> from sage.manifolds.operators import laplacian
>>> laplacian(v) == Dv
True
```

In the present case (Euclidean metric and Cartesian coordinates), the components of the Laplacian are the Laplacians of the components:

```
sage: all(Dv[[i]] == laplacian(v[[i]]) for i in M.irange())
True
```

```
>>> from sage.all import *
>>> all(Dv[[i]] == laplacian(v[[i]]) for i in M.irange())
True
```

The Laplacian can be taken with respect to a metric tensor that is not the default one:

```
sage: h = M.lorentzian_metric('h')
sage: h[1,1], h[2,2] = -1, 1+x^2
sage: Dv = v.laplacian(h); Dv
Vector field Delta_h(v) on the Euclidean plane E^2
sage: Dv.display()
Delta_h(v) = -(8*x^5 - 2*x^4 - x^2*y^2 + 15*x^3 - 4*x^2 + 6*x
 - 2)/(x^4 + 2*x^2 + 1) e_x - 3*x^3*y/(x^4 + 2*x^2 + 1) e_y
```

```
>>> from sage.all import *
>>> h = M.lorentzian_metric('h')
>>> h[Integer(1),Integer(1)], h[Integer(2),Integer(2)] = -Integer(1),␣
↪Integer(1)+x**Integer(2)
>>> Dv = v.laplacian(h); Dv
Vector field Delta_h(v) on the Euclidean plane E^2
>>> Dv.display()
Delta_h(v) = -(8*x^5 - 2*x^4 - x^2*y^2 + 15*x^3 - 4*x^2 + 6*x
 - 2)/(x^4 + 2*x^2 + 1) e_x - 3*x^3*y/(x^4 + 2*x^2 + 1) e_y
```

lie_der(*vector*)

Lie derivative of `self` with respect to a vector field.

INPUT:

- `vector` – vector field with respect to which the Lie derivative is to be taken

OUTPUT:

- the tensor field that is the Lie derivative of the current tensor field with respect to `vector`

EXAMPLES:

Lie derivative of a type-$(1, 1)$ tensor field along a vector field on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)    # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                    intersection_name='W', restrictions1= x>0,
....:                    restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
sage: t = M.tensor_field(1, 1, {e_xy: [[x, 1], [y, 0]]}, name='t')
sage: t.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
sage: w = M.vector_field({e_xy: [-y, x]}, name='w')
sage: w.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
sage: lt = t.lie_derivative(w); lt
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: lt.display(e_xy)
∂/∂x⊗dx - x ∂/∂x⊗dy + (-y - 1) ∂/∂y⊗dy
sage: lt.display(e_uv)
-1/2*u ∂/∂u⊗du + (1/2*u + 1) ∂/∂u⊗dv + (-1/2*v + 1) ∂/∂v⊗du + 1/2*v ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)    # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
...                    intersection_name='W', restrictions1= x>Integer(0),
...                    restrictions2= u+v>Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> e_xy = c_xy.frame(); e_uv = c_uv.frame()
>>> t = M.tensor_field(Integer(1), Integer(1), {e_xy: [[x, Integer(1)], [y,␣
↪Integer(0)]]}, name='t')
>>> t.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
>>> w = M.vector_field({e_xy: [-y, x]}, name='w')
>>> w.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
>>> lt = t.lie_derivative(w); lt
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
>>> lt.display(e_xy)
∂/∂x⊗dx - x ∂/∂x⊗dy + (-y - 1) ∂/∂y⊗dy
>>> lt.display(e_uv)
-1/2*u ∂/∂u⊗du + (1/2*u + 1) ∂/∂u⊗dv + (-1/2*v + 1) ∂/∂v⊗du + 1/2*v ∂/∂v⊗dv
```

The result is cached:

```
sage: t.lie_derivative(w) is lt
True
```

```
>>> from sage.all import *
>>> t.lie_derivative(w) is lt
True
```

An alias is `lie_der`:

```
sage: t.lie_der(w) is t.lie_derivative(w)
True
```

```
>>> from sage.all import *
>>> t.lie_der(w) is t.lie_derivative(w)
True
```

Lie derivative of a vector field:

```
sage: a = M.vector_field({e_xy: [1-x, x-y]}, name='a')
sage: a.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
sage: a.lie_der(w)
Vector field on the 2-dimensional differentiable manifold M
sage: a.lie_der(w).display(e_xy)
x ∂/∂x + (-y - 1) ∂/∂y
sage: a.lie_der(w).display(e_uv)
(v - 1) ∂/∂u + (u + 1) ∂/∂v
```

```
>>> from sage.all import *
>>> a = M.vector_field({e_xy: [Integer(1)-x, x-y]}, name='a')
>>> a.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
>>> a.lie_der(w)
Vector field on the 2-dimensional differentiable manifold M
>>> a.lie_der(w).display(e_xy)
x ∂/∂x + (-y - 1) ∂/∂y
>>> a.lie_der(w).display(e_uv)
(v - 1) ∂/∂u + (u + 1) ∂/∂v
```

The Lie derivative is antisymmetric:

```
sage: a.lie_der(w) == - w.lie_der(a)
True
```

```
>>> from sage.all import *
>>> a.lie_der(w) == - w.lie_der(a)
True
```

and it coincides with the commutator of the two vector fields:

```
sage: f = M.scalar_field({c_xy: 3*x-1, c_uv:  3/2*(u+v)-1})
sage: a.lie_der(w)(f) == w(a(f)) - a(w(f))  # long time
True
```

```
>>> from sage.all import *
>>> f = M.scalar_field({c_xy: Integer(3)*x-Integer(1), c_uv:  Integer(3)/
↪Integer(2)*(u+v)-Integer(1)})
>>> a.lie_der(w)(f) == w(a(f)) - a(w(f))   # long time
True
```

lie_derivative(*vector*)

Lie derivative of `self` with respect to a vector field.

INPUT:

- `vector` – vector field with respect to which the Lie derivative is to be taken

OUTPUT:

- the tensor field that is the Lie derivative of the current tensor field with respect to `vector`

EXAMPLES:

Lie derivative of a type-$(1,1)$ tensor field along a vector field on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                  intersection_name='W', restrictions1= x>0,
....:                  restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
sage: t = M.tensor_field(1, 1, {e_xy: [[x, 1], [y, 0]]}, name='t')
sage: t.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
sage: w = M.vector_field({e_xy: [-y, x]}, name='w')
sage: w.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
sage: lt = t.lie_derivative(w); lt
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: lt.display(e_xy)
∂/∂x⊗dx - x ∂/∂x⊗dy + (-y - 1) ∂/∂y⊗dy
sage: lt.display(e_uv)
-1/2*u ∂/∂u⊗du + (1/2*u + 1) ∂/∂u⊗dv + (-1/2*v + 1) ∂/∂v⊗du + 1/2*v ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
...                  intersection_name='W', restrictions1= x>Integer(0),
...                  restrictions2= u+v>Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> e_xy = c_xy.frame(); e_uv = c_uv.frame()
>>> t = M.tensor_field(Integer(1), Integer(1), {e_xy: [[x, Integer(1)], [y,␣
↪Integer(0)]]}, name='t')
```

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```
>>> t.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
>>> w = M.vector_field({e_xy: [-y, x]}, name='w')
>>> w.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
>>> lt = t.lie_derivative(w); lt
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
>>> lt.display(e_xy)
∂/∂x⊗dx - x ∂/∂x⊗dy + (-y - 1) ∂/∂y⊗dy
>>> lt.display(e_uv)
-1/2*u ∂/∂u⊗du + (1/2*u + 1) ∂/∂u⊗dv + (-1/2*v + 1) ∂/∂v⊗du + 1/2*v ∂/∂v⊗dv
```

The result is cached:

```
sage: t.lie_derivative(w) is lt
True
```

```
>>> from sage.all import *
>>> t.lie_derivative(w) is lt
True
```

An alias is `lie_der`:

```
sage: t.lie_der(w) is t.lie_derivative(w)
True
```

```
>>> from sage.all import *
>>> t.lie_der(w) is t.lie_derivative(w)
True
```

Lie derivative of a vector field:

```
sage: a = M.vector_field({e_xy: [1-x, x-y]}, name='a')
sage: a.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
sage: a.lie_der(w)
Vector field on the 2-dimensional differentiable manifold M
sage: a.lie_der(w).display(e_xy)
x ∂/∂x + (-y - 1) ∂/∂y
sage: a.lie_der(w).display(e_uv)
(v - 1) ∂/∂u + (u + 1) ∂/∂v
```

```
>>> from sage.all import *
>>> a = M.vector_field({e_xy: [Integer(1)-x, x-y]}, name='a')
>>> a.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
>>> a.lie_der(w)
Vector field on the 2-dimensional differentiable manifold M
>>> a.lie_der(w).display(e_xy)
x ∂/∂x + (-y - 1) ∂/∂y
>>> a.lie_der(w).display(e_uv)
(v - 1) ∂/∂u + (u + 1) ∂/∂v
```

The Lie derivative is antisymmetric:

```
sage: a.lie_der(w) == - w.lie_der(a)
True
```

```
>>> from sage.all import *
>>> a.lie_der(w) == - w.lie_der(a)
True
```

and it coincides with the commutator of the two vector fields:

```
sage: f = M.scalar_field({c_xy: 3*x-1, c_uv:  3/2*(u+v)-1})
sage: a.lie_der(w)(f) == w(a(f)) - a(w(f))  # long time
True
```

```
>>> from sage.all import *
>>> f = M.scalar_field({c_xy: Integer(3)*x-Integer(1), c_uv:  Integer(3)/
↪Integer(2)*(u+v)-Integer(1)})
>>> a.lie_der(w)(f) == w(a(f)) - a(w(f))  # long time
True
```

restrict (*subdomain*, *dest_map=None*)

Return the restriction of `self` to some subdomain.

If the restriction has not been defined yet, it is constructed here.

INPUT:

- `subdomain` – *DifferentiableManifold*; open subset U of the tensor field domain S
- `dest_map` – *DiffMap* (default: `None`); destination map $\Psi : U \to V$, where V is an open subset of the manifold M where the tensor field takes it values; if `None`, the restriction of Φ to U is used, Φ being the differentiable map $S \to M$ associated with the tensor field

OUTPUT: *TensorField* representing the restriction

EXAMPLES:

Restrictions of a vector field on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: U = M.open_subset('U') # the complement of the North pole
sage: stereoN.<x,y> = U.chart()  # stereographic coordinates from the North␣
↪pole
sage: eN = stereoN.frame() # the associated vector frame
sage: V =  M.open_subset('V') # the complement of the South pole
sage: stereoS.<u,v> = V.chart()  # stereographic coordinates from the South␣
↪pole
sage: eS = stereoS.frame() # the associated vector frame
sage: transf = stereoN.transition_map(stereoS, (x/(x^2+y^2), y/(x^2+y^2)),
....:                intersection_name='W', restrictions1= x^2+y^2!=0,
....:                restrictions2= u^2+v^2!=0)
sage: inv = transf.inverse() # transformation from stereoS to stereoN
sage: W = U.intersection(V) # the complement of the North and South poles
sage: stereoN_W = W.atlas()[0]  # restriction of stereographic coord. from␣
↪North pole to W
sage: stereoS_W = W.atlas()[1]  # restriction of stereographic coord. from␣
↪South pole to W
```

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```
sage: eN_W = stereoN_W.frame() ; eS_W = stereoS_W.frame()
sage: v = M.vector_field({eN: [1, 0]}, name='v')
sage: v.display()
v = ∂/∂x
sage: vU = v.restrict(U) ; vU
Vector field v on the Open subset U of the 2-dimensional
 differentiable manifold S^2
sage: vU.display()
v = ∂/∂x
sage: vU == eN[1]
True
sage: vW = v.restrict(W) ; vW
Vector field v on the Open subset W of the 2-dimensional
 differentiable manifold S^2
sage: vW.display()
v = ∂/∂x
sage: vW.display(eS_W, stereoS_W)
v = (-u^2 + v^2) ∂/∂u - 2*u*v ∂/∂v
sage: vW == eN_W[1]
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', start_index=Integer(1))
>>> U = M.open_subset('U') # the complement of the North pole
>>> stereoN = U.chart(names=('x', 'y',)); (x, y,) = stereoN._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> eN = stereoN.frame() # the associated vector frame
>>> V =  M.open_subset('V') # the complement of the South pole
>>> stereoS = V.chart(names=('u', 'v',)); (u, v,) = stereoS._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> eS = stereoS.frame() # the associated vector frame
>>> transf = stereoN.transition_map(stereoS, (x/(x**Integer(2)+y**Integer(2)),
↪ y/(x**Integer(2)+y**Integer(2))),
...                 intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                 restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> inv = transf.inverse() # transformation from stereoS to stereoN
>>> W = U.intersection(V) # the complement of the North and South poles
>>> stereoN_W = W.atlas()[Integer(0)]  # restriction of stereographic coord.␣
↪from North pole to W
>>> stereoS_W = W.atlas()[Integer(1)]  # restriction of stereographic coord.␣
↪from South pole to W
>>> eN_W = stereoN_W.frame() ; eS_W = stereoS_W.frame()
>>> v = M.vector_field({eN: [Integer(1), Integer(0)]}, name='v')
>>> v.display()
v = ∂/∂x
>>> vU = v.restrict(U) ; vU
Vector field v on the Open subset U of the 2-dimensional
 differentiable manifold S^2
>>> vU.display()
v = ∂/∂x
>>> vU == eN[Integer(1)]
```

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```
True
>>> vW = v.restrict(W) ; vW
Vector field v on the Open subset W of the 2-dimensional
 differentiable manifold S^2
>>> vW.display()
v = ∂/∂x
>>> vW.display(eS_W, stereoS_W)
v = (-u^2 + v^2) ∂/∂u - 2*u*v ∂/∂v
>>> vW == eN_W[Integer(1)]
True
```

At this stage, defining the restriction of `v` to the open subset `V` fully specifies `v`:

```
sage: v.restrict(V)[1] = vW[eS_W, 1, stereoS_W].expr()  # note that eS is the␣
↪default frame on V
sage: v.restrict(V)[2] = vW[eS_W, 2, stereoS_W].expr()
sage: v.display(eS, stereoS)
v = (-u^2 + v^2) ∂/∂u - 2*u*v ∂/∂v
sage: v.restrict(U).display()
v = ∂/∂x
sage: v.restrict(V).display()
v = (-u^2 + v^2) ∂/∂u - 2*u*v ∂/∂v
```

```
>>> from sage.all import *
>>> v.restrict(V)[Integer(1)] = vW[eS_W, Integer(1), stereoS_W].expr()  #␣
↪note that eS is the default frame on V
>>> v.restrict(V)[Integer(2)] = vW[eS_W, Integer(2), stereoS_W].expr()
>>> v.display(eS, stereoS)
v = (-u^2 + v^2) ∂/∂u - 2*u*v ∂/∂v
>>> v.restrict(U).display()
v = ∂/∂x
>>> v.restrict(V).display()
v = (-u^2 + v^2) ∂/∂u - 2*u*v ∂/∂v
```

The restriction of the vector field to its own domain is of course itself:

```
sage: v.restrict(M) is v
True
sage: vU.restrict(U) is vU
True
```

```
>>> from sage.all import *
>>> v.restrict(M) is v
True
>>> vU.restrict(U) is vU
True
```

set_calc_order(*symbol*, *order*, *truncate=False*)

Trigger a series expansion with respect to a small parameter in computations involving the tensor field.

This property is propagated by usual operations. The internal representation must be `SR` for this to take effect.

If the small parameter is ϵ and T is `self`, the power series expansion to order n is

$$T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \cdots + \epsilon^n T_n + O(\epsilon^{n+1}),$$

where $T_0, T_1, \ldots, T_n$ are $n+1$ tensor fields of the same tensor type as `self` and do not depend upon ϵ.

INPUT:

- `symbol` – symbolic variable (the "small parameter" ϵ) with respect to which the components of `self` are expanded in power series
- `order` – integer; the order n of the expansion, defined as the degree of the polynomial representing the truncated power series in `symbol`
- `truncate` – boolean (default: `False`); determines whether the components of `self` are replaced by their expansions to the given order

EXAMPLES:

Let us consider two vector fields depending on a small parameter h on a non-parallelizable manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
....:                              restrictions1= x>0, restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: a = M.vector_field()
sage: h = var('h', domain='real')
sage: a[eU,:] = (cos(h*x), -y)
sage: a.add_comp_by_continuation(eV, W, chart=c_uv)
sage: b = M.vector_field()
sage: b[eU,:] = (exp(h*x), exp(h*y))
sage: b.add_comp_by_continuation(eV, W, chart=c_uv)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
...                              restrictions1= x>Integer(0), restrictions2=␣
↪u+v>Integer(0))
>>> inv = transf.inverse()
>>> W = U.intersection(V)
>>> eU = c_xy.frame() ; eV = c_uv.frame()
>>> a = M.vector_field()
>>> h = var('h', domain='real')
>>> a[eU,:] = (cos(h*x), -y)
>>> a.add_comp_by_continuation(eV, W, chart=c_uv)
>>> b = M.vector_field()
>>> b[eU,:] = (exp(h*x), exp(h*y))
>>> b.add_comp_by_continuation(eV, W, chart=c_uv)
```

If we set the calculus order on one of the vector fields, any operation involving both of them is performed to that order:

```
sage: a.set_calc_order(h, 2)
sage: s = a + b
sage: s[eU,:]
[h*x + 2, 1/2*h^2*y^2 + h*y - y + 1]
sage: s[eV,:]
[1/8*(u^2 - 2*u*v + v^2)*h^2 + h*u - 1/2*u + 1/2*v + 3,
 -1/8*(u^2 - 2*u*v + v^2)*h^2 + h*v + 1/2*u - 1/2*v + 1]
```

```
>>> from sage.all import *
>>> a.set_calc_order(h, Integer(2))
>>> s = a + b
>>> s[eU,:]
[h*x + 2, 1/2*h^2*y^2 + h*y - y + 1]
>>> s[eV,:]
[1/8*(u^2 - 2*u*v + v^2)*h^2 + h*u - 1/2*u + 1/2*v + 3,
 -1/8*(u^2 - 2*u*v + v^2)*h^2 + h*v + 1/2*u - 1/2*v + 1]
```

Note that the components of `a` have not been affected by the above call to `set_calc_order`:

```
sage: a[eU,:]
[cos(h*x), -y]
sage: a[eV,:]
[cos(1/2*h*u)*cos(1/2*h*v) - sin(1/2*h*u)*sin(1/2*h*v) - 1/2*u + 1/2*v,
 cos(1/2*h*u)*cos(1/2*h*v) - sin(1/2*h*u)*sin(1/2*h*v) + 1/2*u - 1/2*v]
```

```
>>> from sage.all import *
>>> a[eU,:]
[cos(h*x), -y]
>>> a[eV,:]
[cos(1/2*h*u)*cos(1/2*h*v) - sin(1/2*h*u)*sin(1/2*h*v) - 1/2*u + 1/2*v,
 cos(1/2*h*u)*cos(1/2*h*v) - sin(1/2*h*u)*sin(1/2*h*v) + 1/2*u - 1/2*v]
```

To have `set_calc_order` act on them, set the optional argument `truncate` to `True`:

```
sage: a.set_calc_order(h, 2, truncate=True)
sage: a[eU,:]
[-1/2*h^2*x^2 + 1, -y]
sage: a[eV,:]
[-1/8*(u^2 + 2*u*v + v^2)*h^2 - 1/2*u + 1/2*v + 1,
 -1/8*(u^2 + 2*u*v + v^2)*h^2 + 1/2*u - 1/2*v + 1]
```

```
>>> from sage.all import *
>>> a.set_calc_order(h, Integer(2), truncate=True)
>>> a[eU,:]
[-1/2*h^2*x^2 + 1, -y]
>>> a[eV,:]
[-1/8*(u^2 + 2*u*v + v^2)*h^2 - 1/2*u + 1/2*v + 1,
 -1/8*(u^2 + 2*u*v + v^2)*h^2 + 1/2*u - 1/2*v + 1]
```

set_comp(*basis=None*)

Return the components of `self` in a given vector frame for assignment.

The components with respect to other frames having the same domain as the provided vector frame are deleted, in order to avoid any inconsistency. To keep them, use the method `add_comp()` instead.

INPUT:

- `basis` – (default: `None`) vector frame in which the components are defined; if none is provided, the components are assumed to refer to the tensor field domain's default frame

OUTPUT:

- components in the given frame, as a `Components`; if such components did not exist previously, they are created

EXAMPLES:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: e_uv = c_uv.frame()
sage: t = M.tensor_field(1, 2, name='t')
sage: t.set_comp(e_uv)
3-indices components w.r.t. Coordinate frame (V, (∂/∂u,∂/∂v))
sage: t.set_comp(e_uv)[1,0,1] = u+v
sage: t.display(e_uv)
t = (u + v) ∂/∂v⊗du⊗dv
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> e_uv = c_uv.frame()
>>> t = M.tensor_field(Integer(1), Integer(2), name='t')
>>> t.set_comp(e_uv)
3-indices components w.r.t. Coordinate frame (V, (∂/∂u,∂/∂v))
>>> t.set_comp(e_uv)[Integer(1),Integer(0),Integer(1)] = u+v
>>> t.display(e_uv)
t = (u + v) ∂/∂v⊗du⊗dv
```

Setting the components in a new frame (`e`):

```
sage: e = V.vector_frame('e')
sage: t.set_comp(e)
3-indices components w.r.t. Vector frame (V, (e_0,e_1))
sage: t.set_comp(e)[0,1,1] = u*v
sage: t.display(e)
t = u*v e_0⊗e^1⊗e^1
```

```
>>> from sage.all import *
>>> e = V.vector_frame('e')
>>> t.set_comp(e)
3-indices components w.r.t. Vector frame (V, (e_0,e_1))
>>> t.set_comp(e)[Integer(0),Integer(1),Integer(1)] = u*v
>>> t.display(e)
t = u*v e_0⊗e^1⊗e^1
```

Since the frames e and e_uv are defined on the same domain, the components w.r.t. e_uv have been erased:

```
sage: t.display(c_uv.frame())
Traceback (most recent call last):
...
ValueError: no basis could be found for computing the components
 in the Coordinate frame (V, (∂/∂u,∂/∂v))
```

```
>>> from sage.all import *
>>> t.display(c_uv.frame())
Traceback (most recent call last):
...
ValueError: no basis could be found for computing the components
 in the Coordinate frame (V, (∂/∂u,∂/∂v))
```

Since zero is an immutable, its components cannot be changed:

```
sage: z = M.tensor_field_module((1, 1)).zero()
sage: z.set_comp(e)[0,1] = u*v
Traceback (most recent call last):
...
ValueError: the components of an immutable element cannot be
 changed
```

```
>>> from sage.all import *
>>> z = M.tensor_field_module((Integer(1), Integer(1))).zero()
>>> z.set_comp(e)[Integer(0),Integer(1)] = u*v
Traceback (most recent call last):
...
ValueError: the components of an immutable element cannot be
 changed
```

set_immutable()

Set self and all restrictions of self immutable.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: U = M.open_subset('U', coord_def={X: x^2+y^2<1})
sage: a = M.tensor_field(1, 1, [[1+y,x], [0,x+y]], name='a')
sage: aU = a.restrict(U)
sage: a.set_immutable()
sage: aU.is_immutable()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> U = M.open_subset('U', coord_def={X: x**Integer(2)+y**Integer(2)
↪<Integer(1)})
>>> a = M.tensor_field(Integer(1), Integer(1), [[Integer(1)+y,x], [Integer(0),
↪x+y]], name='a')
>>> aU = a.restrict(U)
>>> a.set_immutable()
>>> aU.is_immutable()
True
```

set_name(*name=None*, *latex_name=None*)

Set (or change) the text name and LaTeX name of `self`.

INPUT:

- `name` – string (default: `None`); name given to the tensor field
- `latex_name` – string (default: `None`); LaTeX symbol to denote the tensor field; if `None` while `name` is provided, the LaTeX symbol is set to `name`

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: t = M.tensor_field(1, 3); t
Tensor field of type (1,3) on the 2-dimensional differentiable
 manifold M
sage: t.set_name(name='t')
sage: t
Tensor field t of type (1,3) on the 2-dimensional differentiable
 manifold M
sage: latex(t)
t
sage: t.set_name(latex_name=r'\tau')
sage: latex(t)
\tau
sage: t.set_name(name='a')
sage: t
Tensor field a of type (1,3) on the 2-dimensional differentiable
 manifold M
sage: latex(t)
a
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> t = M.tensor_field(Integer(1), Integer(3)); t
Tensor field of type (1,3) on the 2-dimensional differentiable
 manifold M
>>> t.set_name(name='t')
>>> t
Tensor field t of type (1,3) on the 2-dimensional differentiable
 manifold M
>>> latex(t)
```

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```
t
>>> t.set_name(latex_name=r'\tau')
>>> latex(t)
\tau
>>> t.set_name(name='a')
>>> t
Tensor field a of type (1,3) on the 2-dimensional differentiable
 manifold M
>>> latex(t)
a
```

set_restriction(*rst*)

Define a restriction of `self` to some subdomain.

INPUT:

- `rst` – *TensorField* of the same type and symmetries as the current tensor field `self`, defined on a subdomain of the domain of `self`

EXAMPLES:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: t = M.tensor_field(1, 2, name='t')
sage: s = U.tensor_field(1, 2)
sage: s[0,0,1] = x+y
sage: t.set_restriction(s)
sage: t.display(c_xy.frame())
t = (x + y) ∂/∂x⊗dx⊗dy
sage: t.restrict(U) == s
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> t = M.tensor_field(Integer(1), Integer(2), name='t')
>>> s = U.tensor_field(Integer(1), Integer(2))
>>> s[Integer(0),Integer(0),Integer(1)] = x+y
>>> t.set_restriction(s)
>>> t.display(c_xy.frame())
t = (x + y) ∂/∂x⊗dx⊗dy
>>> t.restrict(U) == s
True
```

If the restriction is defined on the very same domain, the tensor field becomes a copy of it (see `copy_from()`):

```
sage: v = M.tensor_field(1, 2, name='v')
sage: v.set_restriction(t)
sage: v.restrict(U) == t.restrict(U)
True
```

```
>>> from sage.all import *
>>> v = M.tensor_field(Integer(1), Integer(2), name='v')
>>> v.set_restriction(t)
>>> v.restrict(U) == t.restrict(U)
True
```

symmetries()

Print the list of symmetries and antisymmetries.

EXAMPLES:

```
sage: M = Manifold(2, 'S^2')
sage: t = M.tensor_field(1,2)
sage: t.symmetries()
no symmetry;  no antisymmetry
sage: t = M.tensor_field(1,2, sym=(1,2))
sage: t.symmetries()
symmetry: (1, 2);  no antisymmetry
sage: t = M.tensor_field(2,2, sym=(0,1), antisym=(2,3))
sage: t.symmetries()
symmetry: (0, 1);  antisymmetry: (2, 3)
sage: t = M.tensor_field(2,2, antisym=[(0,1),(2,3)])
sage: t.symmetries()
no symmetry;  antisymmetries: [(0, 1), (2, 3)]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2')
>>> t = M.tensor_field(Integer(1),Integer(2))
>>> t.symmetries()
no symmetry;  no antisymmetry
>>> t = M.tensor_field(Integer(1),Integer(2), sym=(Integer(1),Integer(2)))
>>> t.symmetries()
symmetry: (1, 2);  no antisymmetry
>>> t = M.tensor_field(Integer(2),Integer(2), sym=(Integer(0),Integer(1)),␣
↪antisym=(Integer(2),Integer(3)))
>>> t.symmetries()
symmetry: (0, 1);  antisymmetry: (2, 3)
>>> t = M.tensor_field(Integer(2),Integer(2), antisym=[(Integer(0),
↪Integer(1)),(Integer(2),Integer(3))])
>>> t.symmetries()
no symmetry;  antisymmetries: [(0, 1), (2, 3)]
```

symmetrize(**pos*)

Symmetrization over some arguments.

INPUT:

- `pos` – (default: `None`) list of argument positions involved in the symmetrization (with the convention `position=0` for the first argument); if `None`, the symmetrization is performed over all the arguments

OUTPUT:

- the symmetrized tensor field (instance of `TensorField`)

EXAMPLES:

Symmetrization of a type-$(0, 2)$ tensor field on a 2-dimensional non-parallelizable manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
....:                              restrictions1= x>0, restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: a = M.tensor_field(0,2, {eU: [[1,x], [2,y]]}, name='a')
sage: a.add_comp_by_continuation(eV, W, chart=c_uv)
sage: a[eV,:]
[ 1/4*u + 3/4 -1/4*u + 3/4]
[ 1/4*v - 1/4 -1/4*v - 1/4]
sage: s = a.symmetrize() ; s
Field of symmetric bilinear forms on the 2-dimensional
 differentiable manifold M
sage: s[eU,:]
[        1 1/2*x + 1]
[1/2*x + 1         y]
sage: s[eV,:]
[         1/4*u + 3/4 -1/8*u + 1/8*v + 1/4]
[-1/8*u + 1/8*v + 1/4          -1/4*v - 1/4]
sage: s == a.symmetrize(0,1)  # explicit positions
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
...                              restrictions1= x>Integer(0), restrictions2=␣
↪u+v>Integer(0))
>>> inv = transf.inverse()
>>> W = U.intersection(V)
>>> eU = c_xy.frame() ; eV = c_uv.frame()
>>> a = M.tensor_field(Integer(0),Integer(2), {eU: [[Integer(1),x],␣
↪[Integer(2),y]]}, name='a')
>>> a.add_comp_by_continuation(eV, W, chart=c_uv)
>>> a[eV,:]
[ 1/4*u + 3/4 -1/4*u + 3/4]
[ 1/4*v - 1/4 -1/4*v - 1/4]
```

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```
>>> s = a.symmetrize() ; s
Field of symmetric bilinear forms on the 2-dimensional
 differentiable manifold M
>>> s[eU,:]
[        1 1/2*x + 1]
[1/2*x + 1         y]
>>> s[eV,:]
[         1/4*u + 3/4 -1/8*u + 1/8*v + 1/4]
[-1/8*u + 1/8*v + 1/4         -1/4*v - 1/4]
>>> s == a.symmetrize(Integer(0),Integer(1))  # explicit positions
True
```

See also

For more details and examples, see `sage.tensor.modules.free_module_tensor.FreeModuleTensor.symmetrize()`.

tensor_rank()

Return the tensor rank of `self`.

OUTPUT:

- integer $k + l$, where k is the contravariant rank and l is the covariant rank

EXAMPLES:

```
sage: M = Manifold(2, 'S^2')
sage: t = M.tensor_field(1,2)
sage: t.tensor_rank()
3
sage: v = M.vector_field()
sage: v.tensor_rank()
1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2')
>>> t = M.tensor_field(Integer(1),Integer(2))
>>> t.tensor_rank()
3
>>> v = M.vector_field()
>>> v.tensor_rank()
1
```

tensor_type()

Return the tensor type of `self`.

OUTPUT:

- pair (k, l), where k is the contravariant rank and l is the covariant rank

EXAMPLES:

```
sage: M = Manifold(2, 'S^2')
sage: t = M.tensor_field(1,2)
sage: t.tensor_type()
(1, 2)
sage: v = M.vector_field()
sage: v.tensor_type()
(1, 0)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2')
>>> t = M.tensor_field(Integer(1),Integer(2))
>>> t.tensor_type()
(1, 2)
>>> v = M.vector_field()
>>> v.tensor_type()
(1, 0)
```

trace(*pos1=0*, *pos2=1*, *using=None*)

Trace (contraction) on two slots of the tensor field.

If a non-degenerate form is provided, the trace of a $(0, 2)$ tensor field is computed by first raising the last index.

INPUT:

- `pos1` – (default: 0) position of the first index for the contraction, with the convention `pos1=0` for the first slot
- `pos2` – (default: 1) position of the second index for the contraction, with the same convention as for `pos1`. The variance type of `pos2` must be opposite to that of `pos1`
- `using` – (default: `None`) a non-degenerate form

OUTPUT:

- tensor field resulting from the `(pos1, pos2)` contraction

EXAMPLES:

Trace of a type-$(1, 1)$ tensor field on a 2-dimensional non-parallelizable manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                    intersection_name='W', restrictions1= x>0,
....:                    restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
sage: W = U.intersection(V)
sage: a = M.tensor_field(1,1, name='a')
sage: a[e_xy,:] = [[1,x], [2,y]]
sage: a.add_comp_by_continuation(e_uv, W, chart=c_uv)
sage: s = a.trace() ; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
```

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```
M → ℝ
on U: (x, y) ↦ y + 1
on V: (u, v) ↦ 1/2*u - 1/2*v + 1
sage: s == a.trace(0,1) # explicit mention of the positions
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
...                    intersection_name='W', restrictions1= x>Integer(0),
...                    restrictions2= u+v>Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> e_xy = c_xy.frame(); e_uv = c_uv.frame()
>>> W = U.intersection(V)
>>> a = M.tensor_field(Integer(1),Integer(1), name='a')
>>> a[e_xy,:] = [[Integer(1),x], [Integer(2),y]]
>>> a.add_comp_by_continuation(e_uv, W, chart=c_uv)
>>> s = a.trace() ; s
Scalar field on the 2-dimensional differentiable manifold M
>>> s.display()
M → ℝ
on U: (x, y) ↦ y + 1
on V: (u, v) ↦ 1/2*u - 1/2*v + 1
>>> s == a.trace(Integer(0),Integer(1)) # explicit mention of the positions
True
```

The trace of a type-$(0, 2)$ tensor field using a metric:

```
sage: g = M.metric('g')
sage: g[0,0], g[0,1], g[1,1] = 1, 0, 1
sage: g.trace(using=g).display()
M → ℝ
on U: (x, y) ↦ 2
on W: (u, v) ↦ 2
```

```
>>> from sage.all import *
>>> g = M.metric('g')
>>> g[Integer(0),Integer(0)], g[Integer(0),Integer(1)], g[Integer(1),
↪Integer(1)] = Integer(1), Integer(0), Integer(1)
>>> g.trace(using=g).display()
M → ℝ
on U: (x, y) ↦ 2
on W: (u, v) ↦ 2
```

Instead of the explicit call to the method `trace()`, one may use the index notation with Einstein convention (summation over repeated indices); it suffices to pass the indices as a string inside square brackets:

```
sage: a['^i_i']
Scalar field on the 2-dimensional differentiable manifold M
sage: a['^i_i'] == s
True
```

```
>>> from sage.all import *
>>> a['^i_i']
Scalar field on the 2-dimensional differentiable manifold M
>>> a['^i_i'] == s
True
```

Any letter can be used to denote the repeated index:

```
sage: a['^b_b'] == s
True
```

```
>>> from sage.all import *
>>> a['^b_b'] == s
True
```

Trace of a type-$(1, 2)$ tensor field:

```
sage: b = M.tensor_field(1,2, name='b') ; b
Tensor field b of type (1,2) on the 2-dimensional differentiable
 manifold M
sage: b[e_xy,:] = [[[0,x+y], [y,0]], [[0,2], [3*x,-2]]]
sage: b.add_comp_by_continuation(e_uv, W, chart=c_uv)  # long time
sage: s = b.trace(0,1) ; s # contraction on first and second slots
1-form on the 2-dimensional differentiable manifold M
sage: s.display(e_xy)
3*x dx + (x + y - 2) dy
sage: s.display(e_uv)  # long time
(5/4*u + 3/4*v - 1) du + (1/4*u + 3/4*v + 1) dv
```

```
>>> from sage.all import *
>>> b = M.tensor_field(Integer(1),Integer(2), name='b') ; b
Tensor field b of type (1,2) on the 2-dimensional differentiable
 manifold M
>>> b[e_xy,:] = [[[Integer(0),x+y], [y,Integer(0)]], [[Integer(0),Integer(2)],
→ [Integer(3)*x,-Integer(2)]]]
>>> b.add_comp_by_continuation(e_uv, W, chart=c_uv)  # long time
>>> s = b.trace(Integer(0),Integer(1)) ; s # contraction on first and second␣
→slots
1-form on the 2-dimensional differentiable manifold M
>>> s.display(e_xy)
3*x dx + (x + y - 2) dy
>>> s.display(e_uv)  # long time
(5/4*u + 3/4*v - 1) du + (1/4*u + 3/4*v + 1) dv
```

Use of the index notation:

```
sage: b['^k_ki']
1-form on the 2-dimensional differentiable manifold M
```

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```
sage: b['^k_ki'] == s  # long time
True
```

```
>>> from sage.all import *
>>> b['^k_ki']
1-form on the 2-dimensional differentiable manifold M
>>> b['^k_ki'] == s  # long time
True
```

Indices not involved in the contraction may be replaced by dots:

```
sage: b['^k_k.'] == s  # long time
True
```

```
>>> from sage.all import *
>>> b['^k_k.'] == s  # long time
True
```

The symbol ^ may be omitted:

```
sage: b['k_k.'] == s  # long time
True
```

```
>>> from sage.all import *
>>> b['k_k.'] == s  # long time
True
```

LaTeX notations are allowed:

```
sage: b['^{k}_{ki}'] == s  # long time
True
```

```
>>> from sage.all import *
>>> b['^{k}_{ki}'] == s  # long time
True
```

Contraction on first and third slots:

```
sage: s = b.trace(0,2) ; s
1-form on the 2-dimensional differentiable manifold M
sage: s.display(e_xy)
2 dx + (y - 2) dy
sage: s.display(e_uv)  # long time
(1/4*u - 1/4*v) du + (-1/4*u + 1/4*v + 2) dv
```

```
>>> from sage.all import *
>>> s = b.trace(Integer(0),Integer(2)) ; s
1-form on the 2-dimensional differentiable manifold M
>>> s.display(e_xy)
2 dx + (y - 2) dy
>>> s.display(e_uv)  # long time
(1/4*u - 1/4*v) du + (-1/4*u + 1/4*v + 2) dv
```

Use of index notation:

```
sage: b['^k_.k'] == s  # long time
True
```

```
>>> from sage.all import *
>>> b['^k_.k'] == s  # long time
True
```

up(*non_degenerate_form*, *pos=None*)

Compute a dual of the tensor field by raising some index with the given tensor field (usually, a pseudo-Riemannian metric, a symplectic form or a Poisson tensor).

If T is the tensor field, (k,l) its type and p the position of a covariant index (i.e. $k \leq p < k+l$), this method called with `pos` $= p$ yields the tensor field $T^\sharp$ of type $(k+1, l-1)$ whose components are

$$(T^\sharp)^{a_1 \ldots a_{k+1}}{}_{b_1 \ldots b_{l-1}} = g^{a_{k+1} i} \, T^{a_1 \ldots a_k}{}_{b_1 \ldots b_{p-k} \, i \, b_{p-k+1} \ldots b_{l-1}},$$

g^{ab} being the components of the inverse metric or the Poisson tensor, respectively.

The reverse operation is `TensorField.down()`.

INPUT:

- `non_degenerate_form` – non-degenerate form g, or a Poisson tensor
- `pos` – (default: `None`) position of the index (with the convention `pos=0` for the first index); if `None`, the raising is performed over all the covariant indices, starting from the first one

OUTPUT:

- the tensor field $T^\sharp$ resulting from the index raising operation

EXAMPLES:

Raising the index of a 1-form results in a vector field:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: c_xy.<x,y> = M.chart()
sage: g = M.metric('g')
sage: g[1,1], g[1,2], g[2,2] = 1+x, x*y, 1-y
sage: w = M.one_form(-1, 2)
sage: v = w.up(g) ; v
Vector field on the 2-dimensional differentiable manifold M
sage: v.display()
((2*x - 1)*y + 1)/(x^2*y^2 + (x + 1)*y - x - 1) ∂/∂x
 - (x*y + 2*x + 2)/(x^2*y^2 + (x + 1)*y - x - 1) ∂/∂y
sage: ig = g.inverse(); ig[:]
[ (y - 1)/(x^2*y^2 + (x + 1)*y - x - 1)       x*y/(x^2*y^2 + (x + 1)*y - x -␣
↪1)]
[      x*y/(x^2*y^2 + (x + 1)*y - x - 1) -(x + 1)/(x^2*y^2 + (x + 1)*y - x -␣
↪1)]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', start_index=Integer(1))
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> g = M.metric('g')
```

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```
>>> g[Integer(1),Integer(1)], g[Integer(1),Integer(2)], g[Integer(2),
↪Integer(2)] = Integer(1)+x, x*y, Integer(1)-y
>>> w = M.one_form(-Integer(1), Integer(2))
>>> v = w.up(g) ; v
Vector field on the 2-dimensional differentiable manifold M
>>> v.display()
((2*x - 1)*y + 1)/(x^2*y^2 + (x + 1)*y - x - 1) ∂/∂x
 - (x*y + 2*x + 2)/(x^2*y^2 + (x + 1)*y - x - 1) ∂/∂y
>>> ig = g.inverse(); ig[:]
[ (y - 1)/(x^2*y^2 + (x + 1)*y - x - 1)      x*y/(x^2*y^2 + (x + 1)*y - x -
↪1)]
[     x*y/(x^2*y^2 + (x + 1)*y - x - 1) -(x + 1)/(x^2*y^2 + (x + 1)*y - x -
↪1)]
```

Using the index notation instead of *up()*:

```
sage: v == ig['^ab']*w['_b']
True
```

```
>>> from sage.all import *
>>> v == ig['^ab']*w['_b']
True
```

The reverse operation:

```
sage: w1 = v.down(g) ; w1
1-form on the 2-dimensional differentiable manifold M
sage: w1.display()
-dx + 2 dy
sage: w1 == w
True
```

```
>>> from sage.all import *
>>> w1 = v.down(g) ; w1
1-form on the 2-dimensional differentiable manifold M
>>> w1.display()
-dx + 2 dy
>>> w1 == w
True
```

The reverse operation in index notation:

```
sage: g['_ab']*v['^b'] == w
True
```

```
>>> from sage.all import *
>>> g['_ab']*v['^b'] == w
True
```

Raising the indices of a tensor field of type (0,2):

```
sage: t = M.tensor_field(0, 2, [[1,2], [3,4]])
sage: tu0 = t.up(g, 0) ; tu0  # raising the first index
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: tu0[:]
[  ((3*x + 1)*y - 1)/(x^2*y^2 + (x + 1)*y - x - 1) 2*((2*x + 1)*y - 1)/(x^2*y^
↪2 + (x + 1)*y - x - 1)]
[    (x*y - 3*x - 3)/(x^2*y^2 + (x + 1)*y - x - 1)   2*(x*y - 2*x - 2)/(x^2*y^
↪2 + (x + 1)*y - x - 1)]
sage: tu0 == ig['^ac']*t['_cb'] # the same operation in index notation
True
sage: tuu0 = tu0.up(g) ; tuu0 # the two indices have been raised, starting␣
↪from the first one
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: tuu0 == tu0['^a_c']*ig['^cb'] # the same operation in index notation
True
sage: tu1 = t.up(g, 1) ; tu1 # raising the second index
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: tu1 == ig['^ac']*t['_bc'] # the same operation in index notation
True
sage: tu1[:]
[((2*x + 1)*y - 1)/(x^2*y^2 + (x + 1)*y - x - 1) ((4*x + 3)*y - 3)/(x^2*y^2 +␣
↪(x + 1)*y - x - 1)]
[  (x*y - 2*x - 2)/(x^2*y^2 + (x + 1)*y - x - 1) (3*x*y - 4*x - 4)/(x^2*y^2 +␣
↪(x + 1)*y - x - 1)]
sage: tuu1 = tu1.up(g) ; tuu1 # the two indices have been raised, starting␣
↪from the second one
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: tuu1 == tu1['^a_c']*ig['^cb'] # the same operation in index notation
True
sage: tuu0 == tuu1 # the order of index raising is important
False
sage: tuu = t.up(g) ; tuu # both indices are raised, starting from the first␣
↪one
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: tuu0 == tuu # the same order for index raising has been applied
True
sage: tuu1 == tuu # to get tuu1, indices have been raised from the last one,␣
↪contrary to tuu
False
sage: d0tuu = tuu.down(g, 0) ; d0tuu # the first index is lowered again
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: dd0tuu = d0tuu.down(g) ; dd0tuu  # the second index is then lowered
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
sage: d1tuu = tuu.down(g, 1) ; d1tuu # lowering operation, starting from the␣
↪last index
Tensor field of type (1,1) on the 2-dimensional differentiable
```

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```
manifold M
sage: dd1tuu = d1tuu.down(g) ; dd1tuu
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
sage: ddtuu = tuu.down(g) ; ddtuu # both indices are lowered, starting from␣
↪the last one
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
sage: ddtuu == t # should be true
True
sage: dd0tuu == t # not true, because of the order of index lowering to get␣
↪dd0tuu
False
sage: dd1tuu == t # should be true
True
```

```
>>> from sage.all import *
>>> t = M.tensor_field(Integer(0), Integer(2), [[Integer(1),Integer(2)],␣
↪[Integer(3),Integer(4)]])
>>> tu0 = t.up(g, Integer(0)) ; tu0  # raising the first index
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
>>> tu0[:]
[  ((3*x + 1)*y - 1)/(x^2*y^2 + (x + 1)*y - x - 1) 2*((2*x + 1)*y - 1)/(x^2*y^
↪2 + (x + 1)*y - x - 1)]
[    (x*y - 3*x - 3)/(x^2*y^2 + (x + 1)*y - x - 1)    2*(x*y - 2*x - 2)/(x^2*y^
↪2 + (x + 1)*y - x - 1)]
>>> tu0 == ig['^ac']*t['_cb'] # the same operation in index notation
True
>>> tuu0 = tu0.up(g) ; tuu0 # the two indices have been raised, starting from␣
↪the first one
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
>>> tuu0 == tu0['^a_c']*ig['^cb'] # the same operation in index notation
True
>>> tu1 = t.up(g, Integer(1)) ; tu1 # raising the second index
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
>>> tu1 == ig['^ac']*t['_bc'] # the same operation in index notation
True
>>> tu1[:]
[((2*x + 1)*y - 1)/(x^2*y^2 + (x + 1)*y - x - 1) ((4*x + 3)*y - 3)/(x^2*y^2 +␣
↪(x + 1)*y - x - 1)]
[  (x*y - 2*x - 2)/(x^2*y^2 + (x + 1)*y - x - 1) (3*x*y - 4*x - 4)/(x^2*y^2 +␣
↪(x + 1)*y - x - 1)]
>>> tuu1 = tu1.up(g) ; tuu1 # the two indices have been raised, starting from␣
↪the second one
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
>>> tuu1 == tu1['^a_c']*ig['^cb'] # the same operation in index notation
True
>>> tuu0 == tuu1 # the order of index raising is important
```

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```
False
>>> tuu = t.up(g) ; tuu # both indices are raised, starting from the first one
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
>>> tuu0 == tuu # the same order for index raising has been applied
True
>>> tuu1 == tuu # to get tuu1, indices have been raised from the last one,␣
↪contrary to tuu
False
>>> d0tuu = tuu.down(g, Integer(0)) ; d0tuu # the first index is lowered again
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
>>> dd0tuu = d0tuu.down(g) ; dd0tuu  # the second index is then lowered
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
>>> d1tuu = tuu.down(g, Integer(1)) ; d1tuu # lowering operation, starting␣
↪from the last index
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
>>> dd1tuu = d1tuu.down(g) ; dd1tuu
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
>>> ddtuu = tuu.down(g) ; ddtuu # both indices are lowered, starting from the␣
↪last one
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
>>> ddtuu == t # should be true
True
>>> dd0tuu == t # not true, because of the order of index lowering to get␣
↪dd0tuu
False
>>> dd1tuu == t # should be true
True
```

2.8.3 Tensor Fields with Values on a Parallelizable Manifold

The class `TensorFieldParal` implements tensor fields along a differentiable manifolds with values on a parallelizable differentiable manifold. For non-parallelizable manifolds, see the class `TensorField`.

Various derived classes of `TensorFieldParal` are devoted to specific tensor fields:

- `VectorFieldParal` for vector fields (rank-1 contravariant tensor fields)
- `AutomorphismFieldParal` for fields of tangent-space automorphisms
- `DiffFormParal` for differential forms (fully antisymmetric covariant tensor fields)
- `MultivectorFieldParal` for multivector fields (fully antisymmetric contravariant tensor fields)

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015) : initial version
- Travis Scrimshaw (2016): review tweaks
- Eric Gourgoulhon (2018): method `TensorFieldParal.along()`

- Florentin Jaffredo (2018) : series expansion with respect to a given parameter

REFERENCES:

- [KN1963]
- [Lee2013]
- [ONe1983]

EXAMPLES:

A tensor field of type $(1, 1)$ on a 2-dimensional differentiable manifold:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: c_xy.<x,y> = M.chart()
sage: t = M.tensor_field(1, 1, name='T') ; t
Tensor field T of type (1,1) on the 2-dimensional differentiable manifold M
sage: t.tensor_type()
(1, 1)
sage: t.tensor_rank()
2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', start_index=Integer(1))
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> t = M.tensor_field(Integer(1), Integer(1), name='T') ; t
Tensor field T of type (1,1) on the 2-dimensional differentiable manifold M
>>> t.tensor_type()
(1, 1)
>>> t.tensor_rank()
2
```

Components with respect to the manifold's default frame are created by providing the relevant indices inside square brackets:

```
sage: t[1,1] = x^2
```

```
>>> from sage.all import *
>>> t[Integer(1),Integer(1)] = x**Integer(2)
```

Unset components are initialized to zero:

```
sage: t[:]  # list of components w.r.t. the manifold's default vector frame
[x^2   0]
[  0   0]
```

```
>>> from sage.all import *
>>> t[:]  # list of components w.r.t. the manifold's default vector frame
[x^2   0]
[  0   0]
```

It is also possible to initialize the components at the tensor field construction:

```
sage: t = M.tensor_field(1, 1, [[x^2, 0], [0, 0]], name='T')
sage: t[:]
```

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```
[x^2   0]
[  0   0]
```

```
>>> from sage.all import *
>>> t = M.tensor_field(Integer(1), Integer(1), [[x**Integer(2), Integer(0)],
↪[Integer(0), Integer(0)]], name='T')
>>> t[:]
[x^2   0]
[  0   0]
```

The full set of components with respect to a given vector frame is returned by the method `comp()`:

```
sage: t.comp(c_xy.frame())
2-indices components w.r.t. Coordinate frame (M, (∂/∂x,∂/∂y))
```

```
>>> from sage.all import *
>>> t.comp(c_xy.frame())
2-indices components w.r.t. Coordinate frame (M, (∂/∂x,∂/∂y))
```

If no vector frame is mentioned in the argument of `comp()`, it is assumed to be the manifold's default frame:

```
sage: M.default_frame()
Coordinate frame (M, (∂/∂x,∂/∂y))
sage: t.comp() is t.comp(c_xy.frame())
True
```

```
>>> from sage.all import *
>>> M.default_frame()
Coordinate frame (M, (∂/∂x,∂/∂y))
>>> t.comp() is t.comp(c_xy.frame())
True
```

Individual components with respect to the manifold's default frame are accessed by listing their indices inside double square brackets. They are `scalar fields` on the manifold:

```
sage: t[[1,1]]
Scalar field on the 2-dimensional differentiable manifold M
sage: t[[1,1]].display()
M → ℝ
(x, y) ↦ x^2
sage: t[[1,2]]
Scalar field zero on the 2-dimensional differentiable manifold M
sage: t[[1,2]].display()
zero: M → ℝ
   (x, y) ↦ 0
```

```
>>> from sage.all import *
>>> t[[Integer(1),Integer(1)]]
Scalar field on the 2-dimensional differentiable manifold M
>>> t[[Integer(1),Integer(1)]].display()
M → ℝ
```

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```
(x, y) ↦ x^2
>>> t[[Integer(1),Integer(2)]]
Scalar field zero on the 2-dimensional differentiable manifold M
>>> t[[Integer(1),Integer(2)]].display()
zero: M → ℝ
   (x, y) ↦ 0
```

A direct access to the coordinate expression of some component is obtained via the single square brackets:

```
sage: t[1,1]
x^2
sage: t[1,1] is t[[1,1]].coord_function() # the coordinate function
True
sage: t[1,1] is t[[1,1]].coord_function(c_xy)
True
sage: t[1,1].expr() is t[[1,1]].expr() # the symbolic expression
True
```

```
>>> from sage.all import *
>>> t[Integer(1),Integer(1)]
x^2
>>> t[Integer(1),Integer(1)] is t[[Integer(1),Integer(1)]].coord_function() # the␣
↪coordinate function
True
>>> t[Integer(1),Integer(1)] is t[[Integer(1),Integer(1)]].coord_function(c_xy)
True
>>> t[Integer(1),Integer(1)].expr() is t[[Integer(1),Integer(1)]].expr() # the␣
↪symbolic expression
True
```

Expressions in a chart different from the manifold's default one are obtained by specifying the chart as the last argument inside the single square brackets:

```
sage: c_uv.<u,v> = M.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, [x+y, x-y])
sage: uv_to_xy = xy_to_uv.inverse()
sage: t[1,1, c_uv]
1/4*u^2 + 1/2*u*v + 1/4*v^2
```

```
>>> from sage.all import *
>>> c_uv = M.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> xy_to_uv = c_xy.transition_map(c_uv, [x+y, x-y])
>>> uv_to_xy = xy_to_uv.inverse()
>>> t[Integer(1),Integer(1), c_uv]
1/4*u^2 + 1/2*u*v + 1/4*v^2
```

Note that `t[1,1, c_uv]` is the component of the tensor `t` with respect to the coordinate frame associated to the chart (x, y) expressed in terms of the coordinates (u, v). Indeed, `t[1,1, c_uv]` is a shortcut for `t.comp(c_xy.frame())[[1,1]].coord_function(c_uv)`:

```
sage: t[1,1, c_uv] is t.comp(c_xy.frame())[[1,1]].coord_function(c_uv)
True
```

```
>>> from sage.all import *
>>> t[Integer(1),Integer(1), c_uv] is t.comp(c_xy.frame())[[Integer(1),Integer(1)]].
↪coord_function(c_uv)
True
```

Similarly, `t[1,1]` is a shortcut for `t.comp(c_xy.frame())[[1,1]].coord_function(c_xy)`:

```
sage: t[1,1] is t.comp(c_xy.frame())[[1,1]].coord_function(c_xy)
True
sage: t[1,1] is t.comp()[[1,1]].coord_function()  # since c_xy.frame() and c_xy are␣
↪the manifold's default values
True
```

```
>>> from sage.all import *
>>> t[Integer(1),Integer(1)] is t.comp(c_xy.frame())[[Integer(1),Integer(1)]].coord_
↪function(c_xy)
True
>>> t[Integer(1),Integer(1)] is t.comp()[[Integer(1),Integer(1)]].coord_function()  #␣
↪since c_xy.frame() and c_xy are the manifold's default values
True
```

All the components can be set at once via `[:]`:

```
sage: t[:] = [[1, -x], [x*y, 2]]
sage: t[:]
[  1  -x]
[x*y   2]
```

```
>>> from sage.all import *
>>> t[:] = [[Integer(1), -x], [x*y, Integer(2)]]
>>> t[:]
[  1  -x]
[x*y   2]
```

To set the components in a vector frame different from the manifold's default one, the method *set_comp()* can be employed:

```
sage: e = M.vector_frame('e')
sage: t.set_comp(e)[1,1] = x+y
sage: t.set_comp(e)[2,1], t.set_comp(e)[2,2] = y, -3*x
```

```
>>> from sage.all import *
>>> e = M.vector_frame('e')
>>> t.set_comp(e)[Integer(1),Integer(1)] = x+y
>>> t.set_comp(e)[Integer(2),Integer(1)], t.set_comp(e)[Integer(2),Integer(2)] = y, -
↪Integer(3)*x
```

but, as a shortcut, one may simply specify the frame as the first argument of the square brackets:

```
sage: t[e,1,1] = x+y
sage: t[e,2,1], t[e,2,2] = y, -3*x
sage: t.comp(e)
2-indices components w.r.t. Vector frame (M, (e_1,e_2))
```

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```
sage: t.comp(e)[:]
[x + y     0]
[    y  -3*x]
sage: t[e,:]  # a shortcut of the above
[x + y     0]
[    y  -3*x]
```

```
>>> from sage.all import *
>>> t[e,Integer(1),Integer(1)] = x+y
>>> t[e,Integer(2),Integer(1)], t[e,Integer(2),Integer(2)] = y, -Integer(3)*x
>>> t.comp(e)
2-indices components w.r.t. Vector frame (M, (e_1,e_2))
>>> t.comp(e)[:]
[x + y     0]
[    y  -3*x]
>>> t[e,:]  # a shortcut of the above
[x + y     0]
[    y  -3*x]
```

All the components in some frame can be set at once, via the operator `[:]`:

```
sage: t[e,:] = [[x+y, 0], [y, -3*x]]
sage: t[e,:]  # same as above:
[x + y     0]
[    y  -3*x]
```

```
>>> from sage.all import *
>>> t[e,:] = [[x+y, Integer(0)], [y, -Integer(3)*x]]
>>> t[e,:]  # same as above:
[x + y     0]
[    y  -3*x]
```

Equivalently, one can initialize the components in `e` at the tensor field construction:

```
sage: t = M.tensor_field(1, 1, [[x+y, 0], [y, -3*x]], frame=e, name='T')
sage: t[e,:]  # same as above:
[x + y     0]
[    y  -3*x]
```

```
>>> from sage.all import *
>>> t = M.tensor_field(Integer(1), Integer(1), [[x+y, Integer(0)], [y, -
↪Integer(3)*x]], frame=e, name='T')
>>> t[e,:]  # same as above:
[x + y     0]
[    y  -3*x]
```

To avoid any inconsistency between the various components, the method *set_comp()* clears the components in other frames. To keep the other components, one must use the method *add_comp()*:

```
sage: t = M.tensor_field(1, 1, name='T')  # Let us restart
sage: t[:] = [[1, -x], [x*y, 2]]  # by first setting the components in the frame c_xy.
↪frame()
```

```
>>> from sage.all import *
>>> t = M.tensor_field(Integer(1), Integer(1), name='T')   # Let us restart
>>> t[:] = [[Integer(1), -x], [x*y, Integer(2)]]  # by first setting the components␣
↪in the frame c_xy.frame()
```

We now set the components in the frame e with add_comp:

```
sage: t.add_comp(e)[:] = [[x+y, 0], [y, -3*x]]
```

```
>>> from sage.all import *
>>> t.add_comp(e)[:] = [[x+y, Integer(0)], [y, -Integer(3)*x]]
```

The expansion of the tensor field in a given frame is obtained via the method `display`:

```
sage: t.display()  # expansion in the manifold's default frame
T = ∂/∂x⊗dx - x ∂/∂x⊗dy + x*y ∂/∂y⊗dx + 2 ∂/∂y⊗dy
sage: t.display(e)
T = (x + y) e_1⊗e^1 + y e_2⊗e^1 - 3*x e_2⊗e^2
```

```
>>> from sage.all import *
>>> t.display()  # expansion in the manifold's default frame
T = ∂/∂x⊗dx - x ∂/∂x⊗dy + x*y ∂/∂y⊗dx + 2 ∂/∂y⊗dy
>>> t.display(e)
T = (x + y) e_1⊗e^1 + y e_2⊗e^1 - 3*x e_2⊗e^2
```

See *display()* for more examples.

By definition, a tensor field acts as a multilinear map on 1-forms and vector fields; in the present case, `T` being of type $(1, 1)$, it acts on pairs (1-form, vector field):

```
sage: a = M.one_form(1, x, name='a')
sage: v = M.vector_field(y, 2, name='V')
sage: t(a,v)
Scalar field T(a,V) on the 2-dimensional differentiable manifold M
sage: t(a,v).display()
T(a,V): M → ℝ
   (x, y) ↦ x^2*y^2 + 2*x + y
   (u, v) ↦ 1/16*u^4 - 1/8*u^2*v^2 + 1/16*v^4 + 3/2*u + 1/2*v
sage: latex(t(a,v))
T\left(a,V\right)
```

```
>>> from sage.all import *
>>> a = M.one_form(Integer(1), x, name='a')
>>> v = M.vector_field(y, Integer(2), name='V')
>>> t(a,v)
Scalar field T(a,V) on the 2-dimensional differentiable manifold M
>>> t(a,v).display()
T(a,V): M → ℝ
   (x, y) ↦ x^2*y^2 + 2*x + y
   (u, v) ↦ 1/16*u^4 - 1/8*u^2*v^2 + 1/16*v^4 + 3/2*u + 1/2*v
>>> latex(t(a,v))
T\left(a,V\right)
```

Check by means of the component expression of `t(a,v)`:

```
sage: t(a,v).expr() - t[1,1]*a[1]*v[1] - t[1,2]*a[1]*v[2] \
....: - t[2,1]*a[2]*v[1] - t[2,2]*a[2]*v[2]
0
```

```
>>> from sage.all import *
>>> t(a,v).expr() - t[Integer(1),Integer(1)]*a[Integer(1)]*v[Integer(1)] -␣
↪t[Integer(1),Integer(2)]*a[Integer(1)]*v[Integer(2)] - t[Integer(2),
↪Integer(1)]*a[Integer(2)]*v[Integer(1)] - t[Integer(2),
↪Integer(2)]*a[Integer(2)]*v[Integer(2)]
0
```

A scalar field (rank-0 tensor field):

```
sage: f = M.scalar_field(x*y + 2, name='f') ; f
Scalar field f on the 2-dimensional differentiable manifold M
sage: f.tensor_type()
(0, 0)
```

```
>>> from sage.all import *
>>> f = M.scalar_field(x*y + Integer(2), name='f') ; f
Scalar field f on the 2-dimensional differentiable manifold M
>>> f.tensor_type()
(0, 0)
```

A scalar field acts on points on the manifold:

```
sage: p = M.point((1,2))
sage: f(p)
4
```

```
>>> from sage.all import *
>>> p = M.point((Integer(1),Integer(2)))
>>> f(p)
4
```

See *DiffScalarField* for more details on scalar fields.

A vector field (rank-1 contravariant tensor field):

```
sage: v = M.vector_field(-x, y, name='v') ; v
Vector field v on the 2-dimensional differentiable manifold M
sage: v.tensor_type()
(1, 0)
sage: v.display()
v = -x ∂/∂x + y ∂/∂y
```

```
>>> from sage.all import *
>>> v = M.vector_field(-x, y, name='v') ; v
Vector field v on the 2-dimensional differentiable manifold M
>>> v.tensor_type()
(1, 0)
>>> v.display()
v = -x ∂/∂x + y ∂/∂y
```

A field of symmetric bilinear forms:

```
sage: q = M.sym_bilin_form_field(name='Q') ; q
Field of symmetric bilinear forms Q on the 2-dimensional differentiable
 manifold M
sage: q.tensor_type()
(0, 2)
```

```
>>> from sage.all import *
>>> q = M.sym_bilin_form_field(name='Q') ; q
Field of symmetric bilinear forms Q on the 2-dimensional differentiable
 manifold M
>>> q.tensor_type()
(0, 2)
```

The components of a symmetric bilinear form are dealt by the subclass `CompFullySym` of the class `Components`, which takes into account the symmetry between the two indices:

```
sage: q[1,1], q[1,2], q[2,2] = (0, -x, y) # no need to set the component (2,1)
sage: type(q.comp())
<class 'sage.tensor.modules.comp.CompFullySym'>
sage: q[:] # note that the component (2,1) is equal to the component (1,2)
[ 0 -x]
[-x  y]
sage: q.display()
Q = -x dx⊗dy - x dy⊗dx + y dy⊗dy
```

```
>>> from sage.all import *
>>> q[Integer(1),Integer(1)], q[Integer(1),Integer(2)], q[Integer(2),Integer(2)] =␣
↪(Integer(0), -x, y) # no need to set the component (2,1)
>>> type(q.comp())
<class 'sage.tensor.modules.comp.CompFullySym'>
>>> q[:] # note that the component (2,1) is equal to the component (1,2)
[ 0 -x]
[-x  y]
>>> q.display()
Q = -x dx⊗dy - x dy⊗dx + y dy⊗dy
```

More generally, tensor symmetries or antisymmetries can be specified via the keywords `sym` and `antisym`. For instance a rank-4 covariant tensor symmetric with respect to its first two arguments (no. 0 and no. 1) and antisymmetric with respect to its last two ones (no. 2 and no. 3) is declared as follows:

```
sage: t = M.tensor_field(0, 4, name='T', sym=(0,1), antisym=(2,3))
sage: t[1,2,1,2] = 3
sage: t[2,1,1,2] # check of the symmetry with respect to the first 2 indices
3
sage: t[1,2,2,1] # check of the antisymmetry with respect to the last 2 indices
-3
```

```
>>> from sage.all import *
>>> t = M.tensor_field(Integer(0), Integer(4), name='T', sym=(Integer(0),Integer(1)),␣
↪antisym=(Integer(2),Integer(3)))
>>> t[Integer(1),Integer(2),Integer(1),Integer(2)] = Integer(3)
```

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```
>>> t[Integer(2),Integer(1),Integer(1),Integer(2)] # check of the symmetry with␣
↪respect to the first 2 indices
3
>>> t[Integer(1),Integer(2),Integer(2),Integer(1)] # check of the antisymmetry with␣
↪respect to the last 2 indices
-3
```

class sage.manifolds.differentiable.tensorfield_paral.**TensorFieldParal**(*vector_field_module*, *tensor_type*, *name=None*, *latex_name=None*, *sym=None*, *antisym=None*)

Bases: `FreeModuleTensor`, `TensorField`

Tensor field along a differentiable manifold, with values on a parallelizable manifold.

An instance of this class is a tensor field along a differentiable manifold U with values on a parallelizable manifold M, via a differentiable map $\Phi : U \to M$. More precisely, given two nonnegative integers k and l and a differentiable map

$$\Phi : U \longrightarrow M,$$

a *tensor field of type* (k, l) *along* U *with values on* M is a differentiable map

$$t : U \longrightarrow T^{(k,l)}M$$

(where $T^{(k,l)}M$ is the tensor bundle of type (k, l) over M) such that

$$t(p) \in T^{(k,l)}(T_q M)$$

for all $p \in U$, i.e. $t(p)$ is a tensor of type (k, l) on the tangent space $T_q M$ at the point $q = \Phi(p)$. That is to say a multilinear map

$$t(p) : \underbrace{T_q^*M \times \cdots \times T_q^*M}_{k \text{ times}} \times \underbrace{T_q M \times \cdots \times T_q M}_{l \text{ times}} \longrightarrow K,$$

where T_q^*M is the dual vector space to $T_q M$ and K is the topological field over which the manifold M is defined. The integer $k + l$ is called the *tensor rank*.

The standard case of a tensor field *on* a differentiable manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note

If M is not parallelizable, the class `TensorField` should be used instead.

INPUT:

- `vector_field_module` – free module $\mathfrak{X}(U, \Phi)$ of vector fields along U associated with the map $\Phi : U \to M$ (cf. `VectorFieldFreeModule`)
- `tensor_type` – pair (k, l) with k being the contravariant rank and l the covariant rank
- `name` – (default: `None`) name given to the tensor field

- latex_name – (default: None) LaTeX symbol to denote the tensor field; if none is provided, the LaTeX symbol is set to name
- sym – (default: None) a symmetry or a list of symmetries among the tensor arguments: each symmetry is described by a tuple containing the positions of the involved arguments, with the convention position=0 for the first argument; for instance:
  - sym=(0,1) for a symmetry between the 1st and 2nd arguments
  - sym=[(0,2),(1,3,4)] for a symmetry between the 1st and 3rd arguments and a symmetry between the 2nd, 4th and 5th arguments
- antisym – (default: None) antisymmetry or list of antisymmetries among the arguments, with the same convention as for sym

EXAMPLES:

A tensor field of type $(2,0)$ on a 3-dimensional parallelizable manifold:

```
sage: M = Manifold(3, 'M')
sage: c_xyz.<x,y,z> = M.chart()   # makes M parallelizable
sage: t = M.tensor_field(2, 0, name='T') ; t
Tensor field T of type (2,0) on the 3-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_ngens(3)#␣
↪makes M parallelizable
>>> t = M.tensor_field(Integer(2), Integer(0), name='T') ; t
Tensor field T of type (2,0) on the 3-dimensional differentiable
 manifold M
```

Tensor fields are considered as elements of a module over the ring $C^k(M)$ of scalar fields on M:

```
sage: t.parent()
Free module T^(2,0)(M) of type-(2,0) tensors fields on the
 3-dimensional differentiable manifold M
sage: t.parent().base_ring()
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> t.parent()
Free module T^(2,0)(M) of type-(2,0) tensors fields on the
 3-dimensional differentiable manifold M
>>> t.parent().base_ring()
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
```

The components with respect to the manifold's default frame are set or read by means of square brackets:

```
sage: e = M.vector_frame('e') ; M.set_default_frame(e)
sage: for i in M.irange():
....:     for j in M.irange():
....:         t[i,j] = (i+1)**(j+1)
```

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```
sage: [[ t[i,j] for j in M.irange()] for i in M.irange()]
[[1, 1, 1], [2, 4, 8], [3, 9, 27]]
```

```
>>> from sage.all import *
>>> e = M.vector_frame('e') ; M.set_default_frame(e)
>>> for i in M.irange():
...     for j in M.irange():
...         t[i,j] = (i+Integer(1))**(j+Integer(1))
>>> [[ t[i,j] for j in M.irange()] for i in M.irange()]
[[1, 1, 1], [2, 4, 8], [3, 9, 27]]
```

A shortcut for the above is using `[:]`:

```
sage: t[:]
[ 1  1  1]
[ 2  4  8]
[ 3  9 27]
```

```
>>> from sage.all import *
>>> t[:]
[ 1  1  1]
[ 2  4  8]
[ 3  9 27]
```

The components with respect to another frame are set via the method *set_comp()* and read via the method *comp()*; both return an instance of `Components`:

```
sage: f = M.vector_frame('f')  # a new frame defined on M, in addition to e
sage: t.set_comp(f)[0,0] = -3
sage: t.comp(f)
2-indices components w.r.t. Vector frame (M, (f_0,f_1,f_2))
sage: t.comp(f)[0,0]
-3
sage: t.comp(f)[:]  # the full list of components
[-3  0  0]
[ 0  0  0]
[ 0  0  0]
```

```
>>> from sage.all import *
>>> f = M.vector_frame('f')  # a new frame defined on M, in addition to e
>>> t.set_comp(f)[Integer(0),Integer(0)] = -Integer(3)
>>> t.comp(f)
2-indices components w.r.t. Vector frame (M, (f_0,f_1,f_2))
>>> t.comp(f)[Integer(0),Integer(0)]
-3
>>> t.comp(f)[:]  # the full list of components
[-3  0  0]
[ 0  0  0]
[ 0  0  0]
```

To avoid any inconsistency between the various components, the method *set_comp()* deletes the components in other frames. Accordingly, the components in the frame `e` have been deleted:

```
sage: t._components
{Vector frame (M, (f_0,f_1,f_2)): 2-indices components w.r.t. Vector
 frame (M, (f_0,f_1,f_2))}
```

```
>>> from sage.all import *
>>> t._components
{Vector frame (M, (f_0,f_1,f_2)): 2-indices components w.r.t. Vector
 frame (M, (f_0,f_1,f_2))}
```

To keep the other components, one must use the method *add_comp()*:

```
sage: t = M.tensor_field(2, 0, name='T')  # let us restart
sage: t[0,0] = 2                          # sets the components in the frame e
```

```
>>> from sage.all import *
>>> t = M.tensor_field(Integer(2), Integer(0), name='T')  # let us restart
>>> t[Integer(0),Integer(0)] = Integer(2)                 # sets the components␣
↪in the frame e
```

We now set the components in the frame f with add_comp:

```
sage: t.add_comp(f)[0,0] = -3
```

```
>>> from sage.all import *
>>> t.add_comp(f)[Integer(0),Integer(0)] = -Integer(3)
```

The components w.r.t. frame e have been kept:

```
sage: t._components  # random (dictionary output)
{Vector frame (M, (e_0,e_1,e_2)): 2-indices components w.r.t. Vector frame (M, (e_
↪0,e_1,e_2)),
 Vector frame (M, (f_0,f_1,f_2)): 2-indices components w.r.t. Vector frame (M, (f_
↪0,f_1,f_2))}
```

```
>>> from sage.all import *
>>> t._components  # random (dictionary output)
{Vector frame (M, (e_0,e_1,e_2)): 2-indices components w.r.t. Vector frame (M, (e_
↪0,e_1,e_2)),
 Vector frame (M, (f_0,f_1,f_2)): 2-indices components w.r.t. Vector frame (M, (f_
↪0,f_1,f_2))}
```

The basic properties of a tensor field are:

```
sage: t.domain()
3-dimensional differentiable manifold M
sage: t.tensor_type()
(2, 0)
```

```
>>> from sage.all import *
>>> t.domain()
3-dimensional differentiable manifold M
>>> t.tensor_type()
(2, 0)
```

Symmetries and antisymmetries are declared via the keywords `sym` and `antisym`. For instance, a rank-6 covariant tensor that is symmetric with respect to its 1st and 3rd arguments and antisymmetric with respect to the 2nd, 5th and 6th arguments is set up as follows:

```
sage: a = M.tensor_field(0, 6, name='T', sym=(0,2), antisym=(1,4,5))
sage: a[0,0,1,0,1,2] = 3
sage: a[1,0,0,0,1,2] # check of the symmetry
3
sage: a[0,1,1,0,0,2], a[0,1,1,0,2,0] # check of the antisymmetry
(-3, 3)
```

```
>>> from sage.all import *
>>> a = M.tensor_field(Integer(0), Integer(6), name='T', sym=(Integer(0),
↪Integer(2)), antisym=(Integer(1),Integer(4),Integer(5)))
>>> a[Integer(0),Integer(0),Integer(1),Integer(0),Integer(1),Integer(2)] =␣
↪Integer(3)
>>> a[Integer(1),Integer(0),Integer(0),Integer(0),Integer(1),Integer(2)] # check␣
↪of the symmetry
3
>>> a[Integer(0),Integer(1),Integer(1),Integer(0),Integer(0),Integer(2)],␣
↪a[Integer(0),Integer(1),Integer(1),Integer(0),Integer(2),Integer(0)] # check of␣
↪the antisymmetry
(-3, 3)
```

Multiple symmetries or antisymmetries are allowed; they must then be declared as a list. For instance, a rank-4 covariant tensor that is antisymmetric with respect to its 1st and 2nd arguments and with respect to its 3rd and 4th argument must be declared as:

```
sage: r = M.tensor_field(0, 4, name='T', antisym=[(0,1), (2,3)])
sage: r[0,1,2,0] = 3
sage: r[1,0,2,0] # first antisymmetry
-3
sage: r[0,1,0,2] # second antisymmetry
-3
sage: r[1,0,0,2] # both antisymmetries acting
3
```

```
>>> from sage.all import *
>>> r = M.tensor_field(Integer(0), Integer(4), name='T', antisym=[(Integer(0),
↪Integer(1)), (Integer(2),Integer(3))])
>>> r[Integer(0),Integer(1),Integer(2),Integer(0)] = Integer(3)
>>> r[Integer(1),Integer(0),Integer(2),Integer(0)] # first antisymmetry
-3
>>> r[Integer(0),Integer(1),Integer(0),Integer(2)] # second antisymmetry
-3
>>> r[Integer(1),Integer(0),Integer(0),Integer(2)] # both antisymmetries acting
3
```

Tensor fields of the same type can be added and subtracted:

```
sage: a = M.tensor_field(2, 0)
sage: a[0,0], a[0,1], a[0,2] = (1,2,3)
sage: b = M.tensor_field(2, 0)
```

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```
sage: b[0,0], b[1,1], b[2,2], b[0,2] = (4,5,6,7)
sage: s = a + 2*b ; s
Tensor field of type (2,0) on the 3-dimensional differentiable
 manifold M
sage: a[:], (2*b)[:], s[:]
(
[1 2 3]  [ 8  0 14]  [ 9  2 17]
[0 0 0]  [ 0 10  0]  [ 0 10  0]
[0 0 0], [ 0  0 12], [ 0  0 12]
)
sage: s = a - b ; s
Tensor field of type (2,0) on the 3-dimensional differentiable
 manifold M
sage: a[:], b[:], s[:]
(
[1 2 3]  [4 0 7]  [-3  2 -4]
[0 0 0]  [0 5 0]  [ 0 -5  0]
[0 0 0], [0 0 6], [ 0  0 -6]
)
```

```
>>> from sage.all import *
>>> a = M.tensor_field(Integer(2), Integer(0))
>>> a[Integer(0),Integer(0)], a[Integer(0),Integer(1)], a[Integer(0),Integer(2)]␣
↪= (Integer(1),Integer(2),Integer(3))
>>> b = M.tensor_field(Integer(2), Integer(0))
>>> b[Integer(0),Integer(0)], b[Integer(1),Integer(1)], b[Integer(2),Integer(2)],␣
↪b[Integer(0),Integer(2)] = (Integer(4),Integer(5),Integer(6),Integer(7))
>>> s = a + Integer(2)*b ; s
Tensor field of type (2,0) on the 3-dimensional differentiable
 manifold M
>>> a[:], (Integer(2)*b)[:], s[:]
(
[1 2 3]  [ 8  0 14]  [ 9  2 17]
[0 0 0]  [ 0 10  0]  [ 0 10  0]
[0 0 0], [ 0  0 12], [ 0  0 12]
)
>>> s = a - b ; s
Tensor field of type (2,0) on the 3-dimensional differentiable
 manifold M
>>> a[:], b[:], s[:]
(
[1 2 3]  [4 0 7]  [-3  2 -4]
[0 0 0]  [0 5 0]  [ 0 -5  0]
[0 0 0], [0 0 6], [ 0  0 -6]
)
```

Symmetries are preserved by the addition whenever it is possible:

```
sage: a = M.tensor_field(2, 0, sym=(0,1))
sage: a[0,0], a[0,1], a[0,2] = (1,2,3)
sage: s = a + b
sage: a[:], b[:], s[:]
```

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```
(
[1 2 3]  [4 0 7]  [ 5  2 10]
[2 0 0]  [0 5 0]  [ 2  5  0]
[3 0 0], [0 0 6], [ 3  0  6]
)
sage: a.symmetries()
symmetry: (0, 1);  no antisymmetry
sage: b.symmetries()
no symmetry;  no antisymmetry
sage: s.symmetries()
no symmetry;  no antisymmetry
```

```
>>> from sage.all import *
>>> a = M.tensor_field(Integer(2), Integer(0), sym=(Integer(0),Integer(1)))
>>> a[Integer(0),Integer(0)], a[Integer(0),Integer(1)], a[Integer(0),Integer(2)]␣
↪= (Integer(1),Integer(2),Integer(3))
>>> s = a + b
>>> a[:], b[:], s[:]
(
[1 2 3]  [4 0 7]  [ 5  2 10]
[2 0 0]  [0 5 0]  [ 2  5  0]
[3 0 0], [0 0 6], [ 3  0  6]
)
>>> a.symmetries()
symmetry: (0, 1);  no antisymmetry
>>> b.symmetries()
no symmetry;  no antisymmetry
>>> s.symmetries()
no symmetry;  no antisymmetry
```

Let us now make b symmetric:

```
sage: b = M.tensor_field(2, 0, sym=(0,1))
sage: b[0,0], b[1,1], b[2,2], b[0,2] = (4,5,6,7)
sage: s = a + b
sage: a[:], b[:], s[:]
(
[1 2 3]  [4 0 7]  [ 5  2 10]
[2 0 0]  [0 5 0]  [ 2  5  0]
[3 0 0], [7 0 6], [10  0  6]
)
sage: s.symmetries()  # s is symmetric because both a and b are
symmetry: (0, 1);  no antisymmetry
```

```
>>> from sage.all import *
>>> b = M.tensor_field(Integer(2), Integer(0), sym=(Integer(0),Integer(1)))
>>> b[Integer(0),Integer(0)], b[Integer(1),Integer(1)], b[Integer(2),Integer(2)],␣
↪b[Integer(0),Integer(2)] = (Integer(4),Integer(5),Integer(6),Integer(7))
>>> s = a + b
>>> a[:], b[:], s[:]
(
[1 2 3]  [4 0 7]  [ 5  2 10]
```

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```
[2 0 0]  [0 5 0]  [ 2  5  0]
[3 0 0], [7 0 6], [10  0  6]
)
>>> s.symmetries()  # s is symmetric because both a and b are
symmetry: (0, 1);  no antisymmetry
```

The tensor product is taken with the operator `*`:

```
sage: c = a*b ; c
Tensor field of type (4,0) on the 3-dimensional differentiable
 manifold M
sage: c.symmetries()  # since a and b are both symmetric, a*b has two symmetries:
symmetries: [(0, 1), (2, 3)];  no antisymmetry
```

```
>>> from sage.all import *
>>> c = a*b ; c
Tensor field of type (4,0) on the 3-dimensional differentiable
 manifold M
>>> c.symmetries()  # since a and b are both symmetric, a*b has two symmetries:
symmetries: [(0, 1), (2, 3)];  no antisymmetry
```

The tensor product of two fully contravariant tensors is not symmetric in general:

```
sage: a*b == b*a
False
```

```
>>> from sage.all import *
>>> a*b == b*a
False
```

The tensor product of a fully contravariant tensor by a fully covariant one is symmetric:

```
sage: d = M.diff_form(2)  # a fully covariant tensor field
sage: d[0,1], d[0,2], d[1,2] = (3, 2, 1)
sage: s = a*d ; s
Tensor field of type (2,2) on the 3-dimensional differentiable
 manifold M
sage: s.symmetries()
symmetry: (0, 1);  antisymmetry: (2, 3)
sage: s1 = d*a ; s1
Tensor field of type (2,2) on the 3-dimensional differentiable
 manifold M
sage: s1.symmetries()
symmetry: (0, 1);  antisymmetry: (2, 3)
sage: d*a == a*d
True
```

```
>>> from sage.all import *
>>> d = M.diff_form(Integer(2))  # a fully covariant tensor field
>>> d[Integer(0),Integer(1)], d[Integer(0),Integer(2)], d[Integer(1),Integer(2)]␣
↪= (Integer(3), Integer(2), Integer(1))
>>> s = a*d ; s
```

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```
Tensor field of type (2,2) on the 3-dimensional differentiable
 manifold M
>>> s.symmetries()
symmetry: (0, 1);  antisymmetry: (2, 3)
>>> s1 = d*a ; s1
Tensor field of type (2,2) on the 3-dimensional differentiable
 manifold M
>>> s1.symmetries()
symmetry: (0, 1);  antisymmetry: (2, 3)
>>> d*a == a*d
True
```

Example of tensor field associated with a non-trivial differentiable map Φ: tensor field along a curve in M:

```
sage: R = Manifold(1, 'R')  # R as a 1-dimensional manifold
sage: T.<t> = R.chart()  # canonical chart on R
sage: Phi = R.diff_map(M, [cos(t), sin(t), t], name='Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable manifold R
 to the 3-dimensional differentiable manifold M
sage: h = R.tensor_field(2, 0, name='h', dest_map=Phi) ; h
Tensor field h of type (2,0) along the 1-dimensional differentiable
 manifold R with values on the 3-dimensional differentiable manifold M
sage: h.parent()
Free module T^(2,0)(R,Phi) of type-(2,0) tensors fields along the
 1-dimensional differentiable manifold R mapped into the 3-dimensional
 differentiable manifold M
sage: h[0,0], h[0,1], h[2,0] = 1+t, t^2, sin(t)
sage: h.display()
h = (t + 1) ∂/∂x⊗∂/∂x + t^2 ∂/∂x⊗∂/∂y + sin(t) ∂/∂z⊗∂/∂x
```

```
>>> from sage.all import *
>>> R = Manifold(Integer(1), 'R')  # R as a 1-dimensional manifold
>>> T = R.chart(names=('t',)); (t,) = T._first_ngens(1)# canonical chart on R
>>> Phi = R.diff_map(M, [cos(t), sin(t), t], name='Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable manifold R
 to the 3-dimensional differentiable manifold M
>>> h = R.tensor_field(Integer(2), Integer(0), name='h', dest_map=Phi) ; h
Tensor field h of type (2,0) along the 1-dimensional differentiable
 manifold R with values on the 3-dimensional differentiable manifold M
>>> h.parent()
Free module T^(2,0)(R,Phi) of type-(2,0) tensors fields along the
 1-dimensional differentiable manifold R mapped into the 3-dimensional
 differentiable manifold M
>>> h[Integer(0),Integer(0)], h[Integer(0),Integer(1)], h[Integer(2),Integer(0)]
↪= Integer(1)+t, t**Integer(2), sin(t)
>>> h.display()
h = (t + 1) ∂/∂x⊗∂/∂x + t^2 ∂/∂x⊗∂/∂y + sin(t) ∂/∂z⊗∂/∂x
```

add_comp(*basis=None*)

Return the components of the tensor field in a given vector frame for assignment.

The components with respect to other frames on the same domain are kept. To delete them, use the method *set_comp()* instead.

INPUT:

- `basis` – (default: `None`) vector frame in which the components are defined; if none is provided, the components are assumed to refer to the tensor field domain's default frame

OUTPUT:

- components in the given frame, as an instance of the class `Components`; if such components did not exist previously, they are created

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: e_xy = X.frame()
sage: t = M.tensor_field(1,1, name='t')
sage: t.add_comp(e_xy)
2-indices components w.r.t. Coordinate frame (M, (∂/∂x,∂/∂y))
sage: t.add_comp(e_xy)[1,0] = 2
sage: t.display(e_xy)
t = 2 ∂/∂y⊗dx
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> e_xy = X.frame()
>>> t = M.tensor_field(Integer(1),Integer(1), name='t')
>>> t.add_comp(e_xy)
2-indices components w.r.t. Coordinate frame (M, (∂/∂x,∂/∂y))
>>> t.add_comp(e_xy)[Integer(1),Integer(0)] = Integer(2)
>>> t.display(e_xy)
t = 2 ∂/∂y⊗dx
```

Adding components with respect to a new frame (`e`):

```
sage: e = M.vector_frame('e')
sage: t.add_comp(e)
2-indices components w.r.t. Vector frame (M, (e_0,e_1))
sage: t.add_comp(e)[0,1] = x
sage: t.display(e)
t = x e_0⊗e^1
```

```
>>> from sage.all import *
>>> e = M.vector_frame('e')
>>> t.add_comp(e)
2-indices components w.r.t. Vector frame (M, (e_0,e_1))
>>> t.add_comp(e)[Integer(0),Integer(1)] = x
>>> t.display(e)
t = x e_0⊗e^1
```

The components with respect to the frame `e_xy` are kept:

```
sage: t.display(e_xy)
t = 2 ∂/∂y⊗dx
```

```
>>> from sage.all import *
>>> t.display(e_xy)
t = 2 ∂/∂y⊗dx
```

Adding components in a frame defined on a subdomain:

```
sage: U = M.open_subset('U', coord_def={X: x>0})
sage: f = U.vector_frame('f')
sage: t.add_comp(f)
2-indices components w.r.t. Vector frame (U, (f_0,f_1))
sage: t.add_comp(f)[0,1] = 1+y
sage: t.display(f)
t = (y + 1) f_0⊗f^1
```

```
>>> from sage.all import *
>>> U = M.open_subset('U', coord_def={X: x>Integer(0)})
>>> f = U.vector_frame('f')
>>> t.add_comp(f)
2-indices components w.r.t. Vector frame (U, (f_0,f_1))
>>> t.add_comp(f)[Integer(0),Integer(1)] = Integer(1)+y
>>> t.display(f)
t = (y + 1) f_0⊗f^1
```

The components previously defined are kept:

```
sage: t.display(e_xy)
t = 2 ∂/∂y⊗dx
sage: t.display(e)
t = x e_0⊗e^1
```

```
>>> from sage.all import *
>>> t.display(e_xy)
t = 2 ∂/∂y⊗dx
>>> t.display(e)
t = x e_0⊗e^1
```

along(*mapping*)

Return the tensor field deduced from `self` via a differentiable map, the codomain of which is included in the domain of `self`.

More precisely, if `self` is a tensor field t on M and if $\Phi : U \to M$ is a differentiable map from some differentiable manifold U to M, the returned object is a tensor field $\tilde{t}$ along U with values on M such that

$$\forall p \in U, \ \tilde{t}(p) = t(\Phi(p)).$$

INPUT:

- `mapping` – differentiable map $\Phi : U \to M$

OUTPUT: tensor field $\tilde{t}$ along U defined above

EXAMPLES:

Let us consider the map Φ between the interval $U = (0, 2\pi)$ and the Euclidean plane $M = \mathbf{R}^2$ defining the lemniscate of Gerono:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: t = var('t', domain='real')
sage: Phi = M.curve({X: [sin(t), sin(2*t)/2]}, (t, 0, 2*pi),
....:               name='Phi')
sage: U = Phi.domain(); U
Real interval (0, 2*pi)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> t = var('t', domain='real')
>>> Phi = M.curve({X: [sin(t), sin(Integer(2)*t)/Integer(2)]}, (t, Integer(0),
↪ Integer(2)*pi),
...               name='Phi')
>>> U = Phi.domain(); U
Real interval (0, 2*pi)
```

and a vector field on M:

```
sage: v = M.vector_field(-y , x, name='v')
```

```
>>> from sage.all import *
>>> v = M.vector_field(-y , x, name='v')
```

We have then:

```
sage: vU = v.along(Phi); vU
Vector field v along the Real interval (0, 2*pi) with values on
 the 2-dimensional differentiable manifold M
sage: vU.display()
v = -cos(t)*sin(t) ∂/∂x + sin(t) ∂/∂y
sage: vU.parent()
Free module X((0, 2*pi),Phi) of vector fields along the Real
 interval (0, 2*pi) mapped into the 2-dimensional differentiable
 manifold M
sage: vU.parent() is Phi.tangent_vector_field().parent()
True
```

```
>>> from sage.all import *
>>> vU = v.along(Phi); vU
Vector field v along the Real interval (0, 2*pi) with values on
 the 2-dimensional differentiable manifold M
>>> vU.display()
v = -cos(t)*sin(t) ∂/∂x + sin(t) ∂/∂y
>>> vU.parent()
Free module X((0, 2*pi),Phi) of vector fields along the Real
 interval (0, 2*pi) mapped into the 2-dimensional differentiable
 manifold M
>>> vU.parent() is Phi.tangent_vector_field().parent()
True
```

We check that the defining relation $\tilde{t}(p) = t(\Phi(p))$ holds:

```
sage: p = U(t)  # a generic point of U
sage: vU.at(p) == v.at(Phi(p))
True
```

```
>>> from sage.all import *
>>> p = U(t)  # a generic point of U
>>> vU.at(p) == v.at(Phi(p))
True
```

Case of a tensor field of type `(0,2)`:

```
sage: a = M.tensor_field(0, 2)
sage: a[0,0], a[0,1], a[1,1] = x+y, x*y, x^2-y^2
sage: aU = a.along(Phi); aU
Tensor field of type (0,2) along the Real interval (0, 2*pi) with
 values on the 2-dimensional differentiable manifold M
sage: aU.display()
(cos(t) + 1)*sin(t) dx⊗dx + cos(t)*sin(t)^2 dx⊗dy + sin(t)^4 dy⊗dy
sage: aU.parent()
Free module T^(0,2)((0, 2*pi),Phi) of type-(0,2) tensors fields
 along the Real interval (0, 2*pi) mapped into the 2-dimensional
 differentiable manifold M
sage: aU.at(p) == a.at(Phi(p))
True
```

```
>>> from sage.all import *
>>> a = M.tensor_field(Integer(0), Integer(2))
>>> a[Integer(0),Integer(0)], a[Integer(0),Integer(1)], a[Integer(1),
↪Integer(1)] = x+y, x*y, x**Integer(2)-y**Integer(2)
>>> aU = a.along(Phi); aU
Tensor field of type (0,2) along the Real interval (0, 2*pi) with
 values on the 2-dimensional differentiable manifold M
>>> aU.display()
(cos(t) + 1)*sin(t) dx⊗dx + cos(t)*sin(t)^2 dx⊗dy + sin(t)^4 dy⊗dy
>>> aU.parent()
Free module T^(0,2)((0, 2*pi),Phi) of type-(0,2) tensors fields
 along the Real interval (0, 2*pi) mapped into the 2-dimensional
 differentiable manifold M
>>> aU.at(p) == a.at(Phi(p))
True
```

at(*point*)

Value of `self` at a point of its domain.

If the current tensor field is

$$t: U \longrightarrow T^{(k,l)}M$$

associated with the differentiable map

$$\Phi: U \longrightarrow M,$$

where U and M are two manifolds (possibly $U = M$ and $\Phi = \mathrm{Id}_M$), then for any point $p \in U$, $t(p)$ is a tensor on the tangent space to M at the point $\Phi(p)$.

INPUT:

- `point` – `ManifoldPoint` point p in the domain of the tensor field U

OUTPUT:

- `FreeModuleTensor` representing the tensor $t(p)$ on the tangent vector space $T_{\Phi(p)}M$

EXAMPLES:

Vector in a tangent space of a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: p = M.point((-2,3), name='p')
sage: v = M.vector_field(y, x^2, name='v')
sage: v.display()
v = y ∂/∂x + x^2 ∂/∂y
sage: vp = v.at(p) ; vp
Tangent vector v at Point p on the 2-dimensional differentiable
 manifold M
sage: vp.parent()
Tangent space at Point p on the 2-dimensional differentiable
 manifold M
sage: vp.display()
v = 3 ∂/∂x + 4 ∂/∂y
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> p = M.point((-Integer(2),Integer(3)), name='p')
>>> v = M.vector_field(y, x**Integer(2), name='v')
>>> v.display()
v = y ∂/∂x + x^2 ∂/∂y
>>> vp = v.at(p) ; vp
Tangent vector v at Point p on the 2-dimensional differentiable
 manifold M
>>> vp.parent()
Tangent space at Point p on the 2-dimensional differentiable
 manifold M
>>> vp.display()
v = 3 ∂/∂x + 4 ∂/∂y
```

A 1-form gives birth to a linear form in the tangent space:

```
sage: w = M.one_form(-x, 1+y, name='w')
sage: w.display()
w = -x dx + (y + 1) dy
sage: wp = w.at(p) ; wp
Linear form w on the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
sage: wp.parent()
Dual of the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
sage: wp.display()
w = 2 dx + 4 dy
```

```
>>> from sage.all import *
>>> w = M.one_form(-x, Integer(1)+y, name='w')
>>> w.display()
w = -x dx + (y + 1) dy
>>> wp = w.at(p) ; wp
Linear form w on the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
>>> wp.parent()
Dual of the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
>>> wp.display()
w = 2 dx + 4 dy
```

A tensor field of type $(1, 1)$ yields a tensor of type $(1, 1)$ in the tangent space:

```
sage: t = M.tensor_field(1, 1, name='t')
sage: t[0,0], t[0,1], t[1,1] = 1+x, x*y, 1-y
sage: t.display()
t = (x + 1) ∂/∂x⊗dx + x*y ∂/∂x⊗dy + (-y + 1) ∂/∂y⊗dy
sage: tp = t.at(p) ; tp
Type-(1,1) tensor t on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
sage: tp.parent()
Free module of type-(1,1) tensors on the Tangent space at Point p
 on the 2-dimensional differentiable manifold M
sage: tp.display()
t = -∂/∂x⊗dx - 6 ∂/∂x⊗dy - 2 ∂/∂y⊗dy
```

```
>>> from sage.all import *
>>> t = M.tensor_field(Integer(1), Integer(1), name='t')
>>> t[Integer(0),Integer(0)], t[Integer(0),Integer(1)], t[Integer(1),
↪Integer(1)] = Integer(1)+x, x*y, Integer(1)-y
>>> t.display()
t = (x + 1) ∂/∂x⊗dx + x*y ∂/∂x⊗dy + (-y + 1) ∂/∂y⊗dy
>>> tp = t.at(p) ; tp
Type-(1,1) tensor t on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
>>> tp.parent()
Free module of type-(1,1) tensors on the Tangent space at Point p
 on the 2-dimensional differentiable manifold M
>>> tp.display()
t = -∂/∂x⊗dx - 6 ∂/∂x⊗dy - 2 ∂/∂y⊗dy
```

A 2-form yields an alternating form of degree 2 in the tangent space:

```
sage: a = M.diff_form(2, name='a')
sage: a[0,1] = x*y
sage: a.display()
a = x*y dx∧dy
sage: ap = a.at(p) ; ap
Alternating form a of degree 2 on the Tangent space at Point p on
 the 2-dimensional differentiable manifold M
sage: ap.parent()
```

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```
2nd exterior power of the dual of the Tangent space at Point p on
 the 2-dimensional differentiable manifold M
sage: ap.display()
a = -6 dx∧dy
```

```
>>> from sage.all import *
>>> a = M.diff_form(Integer(2), name='a')
>>> a[Integer(0),Integer(1)] = x*y
>>> a.display()
a = x*y dx∧dy
>>> ap = a.at(p) ; ap
Alternating form a of degree 2 on the Tangent space at Point p on
 the 2-dimensional differentiable manifold M
>>> ap.parent()
2nd exterior power of the dual of the Tangent space at Point p on
 the 2-dimensional differentiable manifold M
>>> ap.display()
a = -6 dx∧dy
```

Example with a non trivial map Φ:

```
sage: U = Manifold(1, 'U')  # (0,2*pi) as a 1-dimensional manifold
sage: T.<t> = U.chart(r't:(0,2*pi)')  # canonical chart on U
sage: Phi = U.diff_map(M, [cos(t), sin(t)], name='Phi',
....:                  latex_name=r'\Phi')
sage: v = U.vector_field(1+t, t^2, name='v', dest_map=Phi) ; v
Vector field v along the 1-dimensional differentiable manifold U
 with values on the 2-dimensional differentiable manifold M
sage: v.display()
v = (t + 1) ∂/∂x + t^2 ∂/∂y
sage: p = U((pi/6,))
sage: vp = v.at(p) ; vp
Tangent vector v at Point on the 2-dimensional differentiable
 manifold M
sage: vp.parent() is M.tangent_space(Phi(p))
True
sage: vp.display()
v = (1/6*pi + 1) ∂/∂x + 1/36*pi^2 ∂/∂y
```

```
>>> from sage.all import *
>>> U = Manifold(Integer(1), 'U')  # (0,2*pi) as a 1-dimensional manifold
>>> T = U.chart(r't:(0,2*pi)', names=('t',)); (t,) = T._first_ngens(1)#␣
↪canonical chart on U
>>> Phi = U.diff_map(M, [cos(t), sin(t)], name='Phi',
...                  latex_name=r'\Phi')
>>> v = U.vector_field(Integer(1)+t, t**Integer(2), name='v', dest_map=Phi) ;␣
↪v
Vector field v along the 1-dimensional differentiable manifold U
 with values on the 2-dimensional differentiable manifold M
>>> v.display()
v = (t + 1) ∂/∂x + t^2 ∂/∂y
>>> p = U((pi/Integer(6),))
```

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```
>>> vp = v.at(p) ; vp
Tangent vector v at Point on the 2-dimensional differentiable
 manifold M
>>> vp.parent() is M.tangent_space(Phi(p))
True
>>> vp.display()
v = (1/6*pi + 1) ∂/∂x + 1/36*pi^2 ∂/∂y
```

comp(*basis=None*, *from_basis=None*)

Return the components in a given vector frame.

If the components are not known already, they are computed by the tensor change-of-basis formula from components in another vector frame.

INPUT:

- `basis` – (default: `None`) vector frame in which the components are required; if none is provided, the components are assumed to refer to the tensor field domain's default frame
- `from_basis` – (default: `None`) vector frame from which the required components are computed, via the tensor change-of-basis formula, if they are not known already in the basis `basis`

OUTPUT:

- components in the vector frame `basis`, as an instance of the class `Components`

EXAMPLES:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: X.<x,y> = M.chart()
sage: t = M.tensor_field(1,2, name='t')
sage: t[1,2,1] = x*y
sage: t.comp(X.frame())
3-indices components w.r.t. Coordinate frame (M, (∂/∂x,∂/∂y))
sage: t.comp()  # the default frame is X.frame()
3-indices components w.r.t. Coordinate frame (M, (∂/∂x,∂/∂y))
sage: t.comp()[:]
[[[0, 0], [x*y, 0]], [[0, 0], [0, 0]]]
sage: e = M.vector_frame('e')
sage: t[e, 2,1,1] = x-3
sage: t.comp(e)
3-indices components w.r.t. Vector frame (M, (e_1,e_2))
sage: t.comp(e)[:]
[[[0, 0], [0, 0]], [[x - 3, 0], [0, 0]]]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> t = M.tensor_field(Integer(1),Integer(2), name='t')
>>> t[Integer(1),Integer(2),Integer(1)] = x*y
>>> t.comp(X.frame())
3-indices components w.r.t. Coordinate frame (M, (∂/∂x,∂/∂y))
>>> t.comp()  # the default frame is X.frame()
3-indices components w.r.t. Coordinate frame (M, (∂/∂x,∂/∂y))
>>> t.comp()[:]
```

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```
[[[0, 0], [x*y, 0]], [[0, 0], [0, 0]]]
>>> e = M.vector_frame('e')
>>> t[e, Integer(2),Integer(1),Integer(1)] = x-Integer(3)
>>> t.comp(e)
3-indices components w.r.t. Vector frame (M, (e_1,e_2))
>>> t.comp(e)[:]
[[[0, 0], [0, 0]], [[x - 3, 0], [0, 0]]]
```

contract(**args*)

Contraction with another tensor field, on one or more indices.

INPUT:

- `pos1` – positions of the indices in `self` involved in the contraction; `pos1` must be a sequence of integers, with 0 standing for the first index position, 1 for the second one, etc. If `pos1` is not provided, a single contraction on the last index position of `self` is assumed
- `other` – the tensor field to contract with
- `pos2` – positions of the indices in `other` involved in the contraction, with the same conventions as for `pos1`. If `pos2` is not provided, a single contraction on the first index position of `other` is assumed

OUTPUT:

- tensor field resulting from the contraction at the positions `pos1` and `pos2` of `self` with `other`

EXAMPLES:

Contraction of a tensor field of type $(2,0)$ with a tensor field of type $(1,1)$:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: a = M.tensor_field(2,0, [[1+x, 2], [y, -x^2]], name='a')
sage: b = M.tensor_field(1,1, [[-y, 1], [x, x+y]], name='b')
sage: s = a.contract(0, b, 1); s
Tensor field of type (2,0) on the 2-dimensional differentiable manifold M
sage: s.display()
-x*y ∂/∂x⊗∂/∂x + (x^2 + x*y + y^2 + x) ∂/∂x⊗∂/∂y
 + (-x^2 - 2*y) ∂/∂y⊗∂/∂x + (-x^3 - x^2*y + 2*x) ∂/∂y⊗∂/∂y
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> a = M.tensor_field(Integer(2),Integer(0), [[Integer(1)+x, Integer(2)], [y,
↪ -x**Integer(2)]], name='a')
>>> b = M.tensor_field(Integer(1),Integer(1), [[-y, Integer(1)], [x, x+y]],␣
↪name='b')
>>> s = a.contract(Integer(0), b, Integer(1)); s
Tensor field of type (2,0) on the 2-dimensional differentiable manifold M
>>> s.display()
-x*y ∂/∂x⊗∂/∂x + (x^2 + x*y + y^2 + x) ∂/∂x⊗∂/∂y
 + (-x^2 - 2*y) ∂/∂y⊗∂/∂x + (-x^3 - x^2*y + 2*x) ∂/∂y⊗∂/∂y
```

Check:

```
sage: all(s[ind] == sum(a[k, ind[0]]*b[ind[1], k] for k in [0..1])
....:     for ind in M.index_generator(2))
True
```

```
>>> from sage.all import *
>>> all(s[ind] == sum(a[k, ind[Integer(0)]]*b[ind[Integer(1)], k] for k in␣
↪(ellipsis_range(Integer(0),Ellipsis,Integer(1))))
...     for ind in M.index_generator(Integer(2)))
True
```

The same contraction with repeated index notation:

```
sage: s == a['^ki']*b['^j_k']
True
```

```
>>> from sage.all import *
>>> s == a['^ki']*b['^j_k']
True
```

Contraction on the second index of `a`:

```
sage: s = a.contract(1, b, 1); s
Tensor field of type (2,0) on the 2-dimensional differentiable manifold M
sage: s.display()
(-(x + 1)*y + 2) ∂/∂x⊗∂/∂x + (x^2 + 3*x + 2*y) ∂/∂x⊗∂/∂y
 + (-x^2 - y^2) ∂/∂y⊗∂/∂x + (-x^3 - (x^2 - x)*y) ∂/∂y⊗∂/∂y
```

```
>>> from sage.all import *
>>> s = a.contract(Integer(1), b, Integer(1)); s
Tensor field of type (2,0) on the 2-dimensional differentiable manifold M
>>> s.display()
(-(x + 1)*y + 2) ∂/∂x⊗∂/∂x + (x^2 + 3*x + 2*y) ∂/∂x⊗∂/∂y
 + (-x^2 - y^2) ∂/∂y⊗∂/∂x + (-x^3 - (x^2 - x)*y) ∂/∂y⊗∂/∂y
```

Check:

```
sage: all(s[ind] == sum(a[ind[0], k]*b[ind[1], k] for k in [0..1])
....:     for ind in M.index_generator(2))
True
```

```
>>> from sage.all import *
>>> all(s[ind] == sum(a[ind[Integer(0)], k]*b[ind[Integer(1)], k] for k in␣
↪(ellipsis_range(Integer(0),Ellipsis,Integer(1))))
...     for ind in M.index_generator(Integer(2)))
True
```

The same contraction with repeated index notation:

```
sage: s == a['^ik']*b['^j_k']
True
```

```
>>> from sage.all import *
>>> s == a['^ik']*b['^j_k']
True
```

See also:

`sage.manifolds.differentiable.tensorfield.TensorField.contract()` for more examples.

display_comp(*frame=None*, *chart=None*, *coordinate_labels=True*, *only_nonzero=True*, *only_nonredundant=False*)

Display the tensor components with respect to a given frame, one per line.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- `frame` – (default: `None`) vector frame with respect to which the tensor field components are defined; if `None`, then
  - if `chart` is not `None`, the coordinate frame associated to `chart` is used
  - otherwise, the default basis of the vector field module on which the tensor field is defined is used
- `chart` – (default: `None`) chart specifying the coordinate expression of the components; if `None`, the default chart of the tensor field domain is used
- `coordinate_labels` – boolean (default: `True`); if `True`, coordinate symbols are used by default (instead of integers) as index labels whenever `frame` is a coordinate frame
- `only_nonzero` – boolean (default: `True`); if `True`, only nonzero components are displayed
- `only_nonredundant` – boolean (default: `False`); if `True`, only nonredundant components are displayed in case of symmetries

EXAMPLES:

Display of the components of a type-$(2, 1)$ tensor field on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: t = M.tensor_field(2, 1, name='t', sym=(0,1))
sage: t[0,0,0], t[0,1,0], t[1,1,1] = x+y, x*y, -3
sage: t.display_comp()
t^xx_x = x + y
t^xy_x = x*y
t^yx_x = x*y
t^yy_y = -3
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> t = M.tensor_field(Integer(2), Integer(1), name='t', sym=(Integer(0),
↪Integer(1)))
>>> t[Integer(0),Integer(0),Integer(0)], t[Integer(0),Integer(1),Integer(0)],
↪t[Integer(1),Integer(1),Integer(1)] = x+y, x*y, -Integer(3)
```

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```
>>> t.display_comp()
t^xx_x = x + y
t^xy_x = x*y
t^yx_x = x*y
t^yy_y = -3
```

By default, only the non-vanishing components are displayed; to see all the components, the argument `only_nonzero` must be set to `False`:

```
sage: t.display_comp(only_nonzero=False)
t^xx_x = x + y
t^xx_y = 0
t^xy_x = x*y
t^xy_y = 0
t^yx_x = x*y
t^yx_y = 0
t^yy_x = 0
t^yy_y = -3
```

```
>>> from sage.all import *
>>> t.display_comp(only_nonzero=False)
t^xx_x = x + y
t^xx_y = 0
t^xy_x = x*y
t^xy_y = 0
t^yx_x = x*y
t^yx_y = 0
t^yy_x = 0
t^yy_y = -3
```

`t` being symmetric with respect to its first two indices, one may ask to skip the components that can be deduced by symmetry:

```
sage: t.display_comp(only_nonredundant=True)
t^xx_x = x + y
t^xy_x = x*y
t^yy_y = -3
```

```
>>> from sage.all import *
>>> t.display_comp(only_nonredundant=True)
t^xx_x = x + y
t^xy_x = x*y
t^yy_y = -3
```

Instead of coordinate labels, one may ask for integers:

```
sage: t.display_comp(coordinate_labels=False)
t^00_0 = x + y
t^01_0 = x*y
t^10_0 = x*y
t^11_1 = -3
```

```
>>> from sage.all import *
>>> t.display_comp(coordinate_labels=False)
t^00_0 = x + y
t^01_0 = x*y
t^10_0 = x*y
t^11_1 = -3
```

Display in a frame different from the default one (note that since `f` is not a coordinate frame, integer are used to label the indices):

```
sage: a = M.automorphism_field()
sage: a[:] = [[1+y^2, 0], [0, 2+x^2]]
sage: f = X.frame().new_frame(a, 'f')
sage: t.display_comp(frame=f)
t^00_0 = (x + y)/(y^2 + 1)
t^01_0 = x*y/(x^2 + 2)
t^10_0 = x*y/(x^2 + 2)
t^11_1 = -3/(x^2 + 2)
```

```
>>> from sage.all import *
>>> a = M.automorphism_field()
>>> a[:] = [[Integer(1)+y**Integer(2), Integer(0)], [Integer(0),␣
↪Integer(2)+x**Integer(2)]]
>>> f = X.frame().new_frame(a, 'f')
>>> t.display_comp(frame=f)
t^00_0 = (x + y)/(y^2 + 1)
t^01_0 = x*y/(x^2 + 2)
t^10_0 = x*y/(x^2 + 2)
t^11_1 = -3/(x^2 + 2)
```

Display with respect to a chart different from the default one:

```
sage: Y.<u,v> = M.chart()
sage: X_to_Y = X.transition_map(Y, [x+y, x-y])
sage: Y_to_X = X_to_Y.inverse()
sage: t.display_comp(chart=Y)
t^uu_u = 1/4*u^2 - 1/4*v^2 + 1/2*u - 3/2
t^uu_v = 1/4*u^2 - 1/4*v^2 + 1/2*u + 3/2
t^uv_u = 1/2*u + 3/2
t^uv_v = 1/2*u - 3/2
t^vu_u = 1/2*u + 3/2
t^vu_v = 1/2*u - 3/2
t^vv_u = -1/4*u^2 + 1/4*v^2 + 1/2*u - 3/2
t^vv_v = -1/4*u^2 + 1/4*v^2 + 1/2*u + 3/2
```

```
>>> from sage.all import *
>>> Y = M.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)
>>> X_to_Y = X.transition_map(Y, [x+y, x-y])
>>> Y_to_X = X_to_Y.inverse()
>>> t.display_comp(chart=Y)
t^uu_u = 1/4*u^2 - 1/4*v^2 + 1/2*u - 3/2
t^uu_v = 1/4*u^2 - 1/4*v^2 + 1/2*u + 3/2
t^uv_u = 1/2*u + 3/2
```

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```
t^uv_v = 1/2*u - 3/2
t^vu_u = 1/2*u + 3/2
t^vu_v = 1/2*u - 3/2
t^vv_u = -1/4*u^2 + 1/4*v^2 + 1/2*u - 3/2
t^vv_v = -1/4*u^2 + 1/4*v^2 + 1/2*u + 3/2
```

Note that the frame defining the components is the coordinate frame associated with chart `Y`, i.e. we have:

```
sage: str(t.display_comp(chart=Y)) == str(t.display_comp(frame=Y.frame(),␣
↪chart=Y))
True
```

```
>>> from sage.all import *
>>> str(t.display_comp(chart=Y)) == str(t.display_comp(frame=Y.frame(),␣
↪chart=Y))
True
```

Display of the components with respect to a specific frame, expressed in terms of a specific chart:

```
sage: t.display_comp(frame=f, chart=Y)
t^00_0 = 4*u/(u^2 - 2*u*v + v^2 + 4)
t^01_0 = (u^2 - v^2)/(u^2 + 2*u*v + v^2 + 8)
t^10_0 = (u^2 - v^2)/(u^2 + 2*u*v + v^2 + 8)
t^11_1 = -12/(u^2 + 2*u*v + v^2 + 8)
```

```
>>> from sage.all import *
>>> t.display_comp(frame=f, chart=Y)
t^00_0 = 4*u/(u^2 - 2*u*v + v^2 + 4)
t^01_0 = (u^2 - v^2)/(u^2 + 2*u*v + v^2 + 8)
t^10_0 = (u^2 - v^2)/(u^2 + 2*u*v + v^2 + 8)
t^11_1 = -12/(u^2 + 2*u*v + v^2 + 8)
```

lie_der(*vector*)

Compute the Lie derivative with respect to a vector field.

INPUT:

- `vector` – vector field with respect to which the Lie derivative is to be taken

OUTPUT:

- the tensor field that is the Lie derivative of `self` with respect to `vector`

EXAMPLES:

Lie derivative of a vector:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: c_xy.<x,y> = M.chart()
sage: v = M.vector_field(-y, x, name='v')
sage: w = M.vector_field(2*x+y, x*y)
sage: w.lie_derivative(v)
Vector field on the 2-dimensional differentiable manifold M
sage: w.lie_derivative(v).display()
((x - 2)*y + x) ∂/∂x + (x^2 - y^2 - 2*x - y) ∂/∂y
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', start_index=Integer(1))
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> v = M.vector_field(-y, x, name='v')
>>> w = M.vector_field(Integer(2)*x+y, x*y)
>>> w.lie_derivative(v)
Vector field on the 2-dimensional differentiable manifold M
>>> w.lie_derivative(v).display()
((x - 2)*y + x) ∂/∂x + (x^2 - y^2 - 2*x - y) ∂/∂y
```

The result is cached:

```
sage: w.lie_derivative(v) is w.lie_derivative(v)
True
```

```
>>> from sage.all import *
>>> w.lie_derivative(v) is w.lie_derivative(v)
True
```

An alias is `lie_der`:

```
sage: w.lie_der(v) is w.lie_derivative(v)
True
```

```
>>> from sage.all import *
>>> w.lie_der(v) is w.lie_derivative(v)
True
```

The Lie derivative is antisymmetric:

```
sage: w.lie_der(v) == -v.lie_der(w)
True
```

```
>>> from sage.all import *
>>> w.lie_der(v) == -v.lie_der(w)
True
```

For vectors, it coincides with the commutator:

```
sage: f = M.scalar_field(x^3 + x*y^2)
sage: w.lie_der(v)(f).display()
M → ℝ
(x, y) ↦ -(x + 2)*y^3 + 3*x^3 - x*y^2 + 5*(x^3 - 2*x^2)*y
sage: w.lie_der(v)(f) == v(w(f)) - w(v(f))  # rhs = commutator [v,w] acting␣
↪on f
True
```

```
>>> from sage.all import *
>>> f = M.scalar_field(x**Integer(3) + x*y**Integer(2))
>>> w.lie_der(v)(f).display()
M → ℝ
(x, y) ↦ -(x + 2)*y^3 + 3*x^3 - x*y^2 + 5*(x^3 - 2*x^2)*y
```

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```
>>> w.lie_der(v)(f) == v(w(f)) - w(v(f))  # rhs = commutator [v,w] acting on f
True
```

Lie derivative of a 1-form:

```
sage: om = M.one_form(y^2*sin(x), x^3*cos(y))
sage: om.lie_der(v)
1-form on the 2-dimensional differentiable manifold M
sage: om.lie_der(v).display()
(-y^3*cos(x) + x^3*cos(y) + 2*x*y*sin(x)) dx
 + (-x^4*sin(y) - 3*x^2*y*cos(y) - y^2*sin(x)) dy
```

```
>>> from sage.all import *
>>> om = M.one_form(y**Integer(2)*sin(x), x**Integer(3)*cos(y))
>>> om.lie_der(v)
1-form on the 2-dimensional differentiable manifold M
>>> om.lie_der(v).display()
(-y^3*cos(x) + x^3*cos(y) + 2*x*y*sin(x)) dx
 + (-x^4*sin(y) - 3*x^2*y*cos(y) - y^2*sin(x)) dy
```

Parallel computation:

```
sage: Parallelism().set('tensor', nproc=2)
sage: om.lie_der(v)
1-form on the 2-dimensional differentiable manifold M
sage: om.lie_der(v).display()
(-y^3*cos(x) + x^3*cos(y) + 2*x*y*sin(x)) dx
 + (-x^4*sin(y) - 3*x^2*y*cos(y) - y^2*sin(x)) dy
sage: Parallelism().set('tensor', nproc=1)  # switch off parallelization
```

```
>>> from sage.all import *
>>> Parallelism().set('tensor', nproc=Integer(2))
>>> om.lie_der(v)
1-form on the 2-dimensional differentiable manifold M
>>> om.lie_der(v).display()
(-y^3*cos(x) + x^3*cos(y) + 2*x*y*sin(x)) dx
 + (-x^4*sin(y) - 3*x^2*y*cos(y) - y^2*sin(x)) dy
>>> Parallelism().set('tensor', nproc=Integer(1))  # switch off␣
↪parallelization
```

Check of Cartan identity:

```
sage: om.lie_der(v) == (v.contract(0, om.exterior_derivative(), 0)
....:                   + om(v).exterior_derivative())
True
```

```
>>> from sage.all import *
>>> om.lie_der(v) == (v.contract(Integer(0), om.exterior_derivative(),␣
↪Integer(0))
...                   + om(v).exterior_derivative())
True
```

lie_derivative(*vector*)

Compute the Lie derivative with respect to a vector field.

INPUT:

- `vector` – vector field with respect to which the Lie derivative is to be taken

OUTPUT:

- the tensor field that is the Lie derivative of `self` with respect to `vector`

EXAMPLES:

Lie derivative of a vector:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: c_xy.<x,y> = M.chart()
sage: v = M.vector_field(-y, x, name='v')
sage: w = M.vector_field(2*x+y, x*y)
sage: w.lie_derivative(v)
Vector field on the 2-dimensional differentiable manifold M
sage: w.lie_derivative(v).display()
((x - 2)*y + x) ∂/∂x + (x^2 - y^2 - 2*x - y) ∂/∂y
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', start_index=Integer(1))
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> v = M.vector_field(-y, x, name='v')
>>> w = M.vector_field(Integer(2)*x+y, x*y)
>>> w.lie_derivative(v)
Vector field on the 2-dimensional differentiable manifold M
>>> w.lie_derivative(v).display()
((x - 2)*y + x) ∂/∂x + (x^2 - y^2 - 2*x - y) ∂/∂y
```

The result is cached:

```
sage: w.lie_derivative(v) is w.lie_derivative(v)
True
```

```
>>> from sage.all import *
>>> w.lie_derivative(v) is w.lie_derivative(v)
True
```

An alias is `lie_der`:

```
sage: w.lie_der(v) is w.lie_derivative(v)
True
```

```
>>> from sage.all import *
>>> w.lie_der(v) is w.lie_derivative(v)
True
```

The Lie derivative is antisymmetric:

```
sage: w.lie_der(v) == -v.lie_der(w)
True
```

```
>>> from sage.all import *
>>> w.lie_der(v) == -v.lie_der(w)
True
```

For vectors, it coincides with the commutator:

```
sage: f = M.scalar_field(x^3 + x*y^2)
sage: w.lie_der(v)(f).display()
M → ℝ
(x, y) ↦ -(x + 2)*y^3 + 3*x^3 - x*y^2 + 5*(x^3 - 2*x^2)*y
sage: w.lie_der(v)(f) == v(w(f)) - w(v(f))  # rhs = commutator [v,w] acting␣
↪on f
True
```

```
>>> from sage.all import *
>>> f = M.scalar_field(x**Integer(3) + x*y**Integer(2))
>>> w.lie_der(v)(f).display()
M → ℝ
(x, y) ↦ -(x + 2)*y^3 + 3*x^3 - x*y^2 + 5*(x^3 - 2*x^2)*y
>>> w.lie_der(v)(f) == v(w(f)) - w(v(f))  # rhs = commutator [v,w] acting on f
True
```

Lie derivative of a 1-form:

```
sage: om = M.one_form(y^2*sin(x), x^3*cos(y))
sage: om.lie_der(v)
1-form on the 2-dimensional differentiable manifold M
sage: om.lie_der(v).display()
(-y^3*cos(x) + x^3*cos(y) + 2*x*y*sin(x)) dx
 + (-x^4*sin(y) - 3*x^2*y*cos(y) - y^2*sin(x)) dy
```

```
>>> from sage.all import *
>>> om = M.one_form(y**Integer(2)*sin(x), x**Integer(3)*cos(y))
>>> om.lie_der(v)
1-form on the 2-dimensional differentiable manifold M
>>> om.lie_der(v).display()
(-y^3*cos(x) + x^3*cos(y) + 2*x*y*sin(x)) dx
 + (-x^4*sin(y) - 3*x^2*y*cos(y) - y^2*sin(x)) dy
```

Parallel computation:

```
sage: Parallelism().set('tensor', nproc=2)
sage: om.lie_der(v)
1-form on the 2-dimensional differentiable manifold M
sage: om.lie_der(v).display()
(-y^3*cos(x) + x^3*cos(y) + 2*x*y*sin(x)) dx
 + (-x^4*sin(y) - 3*x^2*y*cos(y) - y^2*sin(x)) dy
sage: Parallelism().set('tensor', nproc=1)  # switch off parallelization
```

```
>>> from sage.all import *
>>> Parallelism().set('tensor', nproc=Integer(2))
>>> om.lie_der(v)
```

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```
1-form on the 2-dimensional differentiable manifold M
>>> om.lie_der(v).display()
(-y^3*cos(x) + x^3*cos(y) + 2*x*y*sin(x)) dx
 + (-x^4*sin(y) - 3*x^2*y*cos(y) - y^2*sin(x)) dy
>>> Parallelism().set('tensor', nproc=Integer(1))  # switch off␣
↪parallelization
```

Check of Cartan identity:

```
sage: om.lie_der(v) == (v.contract(0, om.exterior_derivative(), 0)
....:                   + om(v).exterior_derivative())
True
```

```
>>> from sage.all import *
>>> om.lie_der(v) == (v.contract(Integer(0), om.exterior_derivative(),␣
↪Integer(0))
...                   + om(v).exterior_derivative())
True
```

restrict(*subdomain*, *dest_map=None*)

Return the restriction of `self` to some subdomain.

If the restriction has not been defined yet, it is constructed here.

INPUT:

- `subdomain` – *DifferentiableManifold*; open subset U of the tensor field domain S
- `dest_map` – *DiffMap* (default: `None`); destination map $\Psi : U \to V$, where V is an open subset of the manifold M where the tensor field takes it values; if `None`, the restriction of Φ to U is used, Φ being the differentiable map $S \to M$ associated with the tensor field

OUTPUT: instance of *TensorFieldParal* representing the restriction

EXAMPLES:

Restriction of a vector field defined on $\mathbf{R}^2$ to a disk:

```
sage: M = Manifold(2, 'R^2')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: v = M.vector_field(x+y, -1+x^2, name='v')
sage: D = M.open_subset('D') # the unit open disc
sage: c_cart_D = c_cart.restrict(D, x^2+y^2<1)
sage: v_D = v.restrict(D) ; v_D
Vector field v on the Open subset D of the 2-dimensional
 differentiable manifold R^2
sage: v_D.display()
v = (x + y) ∂/∂x + (x^2 - 1) ∂/∂y
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#␣
↪Cartesian coordinates on R^2
>>> v = M.vector_field(x+y, -Integer(1)+x**Integer(2), name='v')
>>> D = M.open_subset('D') # the unit open disc
```

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```
>>> c_cart_D = c_cart.restrict(D, x**Integer(2)+y**Integer(2)<Integer(1))
>>> v_D = v.restrict(D) ; v_D
Vector field v on the Open subset D of the 2-dimensional
 differentiable manifold R^2
>>> v_D.display()
v = (x + y) ∂/∂x + (x^2 - 1) ∂/∂y
```

The symbolic expressions of the components with respect to Cartesian coordinates are equal:

```
sage: bool( v_D[1].expr() == v[1].expr() )
True
```

```
>>> from sage.all import *
>>> bool( v_D[Integer(1)].expr() == v[Integer(1)].expr() )
True
```

but neither the chart functions representing the components (they are defined on different charts):

```
sage: v_D[1] == v[1]
False
```

```
>>> from sage.all import *
>>> v_D[Integer(1)] == v[Integer(1)]
False
```

nor the scalar fields representing the components (they are defined on different open subsets):

```
sage: v_D[[1]] == v[[1]]
False
```

```
>>> from sage.all import *
>>> v_D[[Integer(1)]] == v[[Integer(1)]]
False
```

The restriction of the vector field to its own domain is of course itself:

```
sage: v.restrict(M) is v
True
```

```
>>> from sage.all import *
>>> v.restrict(M) is v
True
```

series_expansion(*symbol*, *order*)

Expand the tensor field in power series with respect to a small parameter.

If the small parameter is ϵ and T is `self`, the power series expansion to order n is

$$T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \cdots + \epsilon^n T_n + O(\epsilon^{n+1}),$$

where $T_0, T_1, \ldots, T_n$ are $n+1$ tensor fields of the same tensor type as `self` and do not depend upon ϵ.

INPUT:

- `symbol` – symbolic variable (the "small parameter" ϵ) with respect to which the components of `self` are expanded in power series
- `order` – integer; the order n of the expansion, defined as the degree of the polynomial representing the truncated power series in `symbol`

OUTPUT:

- list of the tensor fields T_i (size `order+1`)

EXAMPLES:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: C.<t,x,y,z> = M.chart()
sage: e = var('e')
sage: g = M.metric()
sage: h1 = M.tensor_field(0,2,sym=(0,1))
sage: h2 = M.tensor_field(0,2,sym=(0,1))
sage: g[0, 0], g[1, 1], g[2, 2], g[3, 3] = -1, 1, 1, 1
sage: h1[0, 1], h1[1, 2], h1[2, 3] = 1, 1, 1
sage: h2[0, 2], h2[1, 3] = 1, 1
sage: g.set(g + e*h1 + e^2*h2)
sage: g_ser = g.series_expansion(e, 2); g_ser
[Field of symmetric bilinear forms on the 4-dimensional Lorentzian manifold M,
 Field of symmetric bilinear forms on the 4-dimensional Lorentzian manifold M,
 Field of symmetric bilinear forms on the 4-dimensional Lorentzian manifold M]
sage: g_ser[0][:]
[-1  0  0  0]
[ 0  1  0  0]
[ 0  0  1  0]
[ 0  0  0  1]
sage: g_ser[1][:]
[0 1 0 0]
[1 0 1 0]
[0 1 0 1]
[0 0 1 0]
sage: g_ser[2][:]
[0 0 1 0]
[0 0 0 1]
[1 0 0 0]
[0 1 0 0]
sage: all([g_ser[1] == h1, g_ser[2] == h2])
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> C = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = C._first_
↪ngens(4)
>>> e = var('e')
>>> g = M.metric()
>>> h1 = M.tensor_field(Integer(0),Integer(2),sym=(Integer(0),Integer(1)))
>>> h2 = M.tensor_field(Integer(0),Integer(2),sym=(Integer(0),Integer(1)))
>>> g[Integer(0), Integer(0)], g[Integer(1), Integer(1)], g[Integer(2),␣
↪Integer(2)], g[Integer(3), Integer(3)] = -Integer(1), Integer(1),␣
↪Integer(1), Integer(1)
```

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```
>>> h1[Integer(0), Integer(1)], h1[Integer(1), Integer(2)], h1[Integer(2),␣
↪Integer(3)] = Integer(1), Integer(1), Integer(1)
>>> h2[Integer(0), Integer(2)], h2[Integer(1), Integer(3)] = Integer(1),␣
↪Integer(1)
>>> g.set(g + e*h1 + e**Integer(2)*h2)
>>> g_ser = g.series_expansion(e, Integer(2)); g_ser
[Field of symmetric bilinear forms on the 4-dimensional Lorentzian manifold M,
 Field of symmetric bilinear forms on the 4-dimensional Lorentzian manifold M,
 Field of symmetric bilinear forms on the 4-dimensional Lorentzian manifold M]
>>> g_ser[Integer(0)][:]
[-1  0  0  0]
[ 0  1  0  0]
[ 0  0  1  0]
[ 0  0  0  1]
>>> g_ser[Integer(1)][:]
[0 1 0 0]
[1 0 1 0]
[0 1 0 1]
[0 0 1 0]
>>> g_ser[Integer(2)][:]
[0 0 1 0]
[0 0 0 1]
[1 0 0 0]
[0 1 0 0]
>>> all([g_ser[Integer(1)] == h1, g_ser[Integer(2)] == h2])
True
```

set_calc_order(*symbol*, *order*, *truncate=False*)

Trigger a power series expansion with respect to a small parameter in computations involving the tensor field.

This property is propagated by usual operations. The internal representation must be `SR` for this to take effect.

If the small parameter is ϵ and T is `self`, the power series expansion to order n is

$$T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \cdots + \epsilon^n T_n + O(\epsilon^{n+1}),$$

where $T_0, T_1, \ldots, T_n$ are $n + 1$ tensor fields of the same tensor type as `self` and do not depend upon ϵ.

INPUT:

- `symbol` – symbolic variable (the "small parameter" ϵ) with respect to which the components of `self` are expanded in power series
- `order` – integer; the order n of the expansion, defined as the degree of the polynomial representing the truncated power series in `symbol`
- `truncate` – boolean (default: `False`); determines whether the components of `self` are replaced by their expansions to the given order

EXAMPLES:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: C.<t,x,y,z> = M.chart()
sage: e = var('e')
sage: g = M.metric()
sage: h1 = M.tensor_field(0, 2, sym=(0,1))
```

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```
sage: h2 = M.tensor_field(0, 2, sym=(0,1))
sage: g[0, 0], g[1, 1], g[2, 2], g[3, 3] = -1, 1, 1, 1
sage: h1[0, 1], h1[1, 2], h1[2, 3] = 1, 1, 1
sage: h2[0, 2], h2[1, 3] = 1, 1
sage: g.set(g + e*h1 + e^2*h2)
sage: g.set_calc_order(e, 1)
sage: g[:]
[ -1   e e^2   0]
[  e   1   e e^2]
[e^2   e   1   e]
[  0 e^2   e   1]
sage: g.set_calc_order(e, 1, truncate=True)
sage: g[:]
[-1  e  0  0]
[ e  1  e  0]
[ 0  e  1  e]
[ 0  0  e  1]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> C = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = C._first_
↪ngens(4)
>>> e = var('e')
>>> g = M.metric()
>>> h1 = M.tensor_field(Integer(0), Integer(2), sym=(Integer(0),Integer(1)))
>>> h2 = M.tensor_field(Integer(0), Integer(2), sym=(Integer(0),Integer(1)))
>>> g[Integer(0), Integer(0)], g[Integer(1), Integer(1)], g[Integer(2),␣
↪Integer(2)], g[Integer(3), Integer(3)] = -Integer(1), Integer(1),␣
↪Integer(1), Integer(1)
>>> h1[Integer(0), Integer(1)], h1[Integer(1), Integer(2)], h1[Integer(2),␣
↪Integer(3)] = Integer(1), Integer(1), Integer(1)
>>> h2[Integer(0), Integer(2)], h2[Integer(1), Integer(3)] = Integer(1),␣
↪Integer(1)
>>> g.set(g + e*h1 + e**Integer(2)*h2)
>>> g.set_calc_order(e, Integer(1))
>>> g[:]
[ -1   e e^2   0]
[  e   1   e e^2]
[e^2   e   1   e]
[  0 e^2   e   1]
>>> g.set_calc_order(e, Integer(1), truncate=True)
>>> g[:]
[-1  e  0  0]
[ e  1  e  0]
[ 0  e  1  e]
[ 0  0  e  1]
```

set_comp(*basis=None*)

Return the components of the tensor field in a given vector frame for assignment.

The components with respect to other frames on the same domain are deleted, in order to avoid any inconsistency. To keep them, use the method `add_comp()` instead.

INPUT:

- basis – (default: None) vector frame in which the components are defined; if none is provided, the components are assumed to refer to the tensor field domain's default frame

OUTPUT:

- components in the given frame, as an instance of the class Components; if such components did not exist previously, they are created

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: e_xy = X.frame()
sage: t = M.tensor_field(1,1, name='t')
sage: t.set_comp(e_xy)
2-indices components w.r.t. Coordinate frame (M, (∂/∂x,∂/∂y))
sage: t.set_comp(e_xy)[1,0] = 2
sage: t.display(e_xy)
t = 2 ∂/∂y⊗dx
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> e_xy = X.frame()
>>> t = M.tensor_field(Integer(1),Integer(1), name='t')
>>> t.set_comp(e_xy)
2-indices components w.r.t. Coordinate frame (M, (∂/∂x,∂/∂y))
>>> t.set_comp(e_xy)[Integer(1),Integer(0)] = Integer(2)
>>> t.display(e_xy)
t = 2 ∂/∂y⊗dx
```

Setting components in a new frame (e):

```
sage: e = M.vector_frame('e')
sage: t.set_comp(e)
2-indices components w.r.t. Vector frame (M, (e_0,e_1))
sage: t.set_comp(e)[0,1] = x
sage: t.display(e)
t = x e_0⊗e^1
```

```
>>> from sage.all import *
>>> e = M.vector_frame('e')
>>> t.set_comp(e)
2-indices components w.r.t. Vector frame (M, (e_0,e_1))
>>> t.set_comp(e)[Integer(0),Integer(1)] = x
>>> t.display(e)
t = x e_0⊗e^1
```

The components with respect to the frame e_xy have be erased:

```
sage: t.display(e_xy)
Traceback (most recent call last):
...
```

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```
ValueError: no basis could be found for computing the components
 in the Coordinate frame (M, (∂/∂x,∂/∂y))
```

```
>>> from sage.all import *
>>> t.display(e_xy)
Traceback (most recent call last):
...
ValueError: no basis could be found for computing the components
 in the Coordinate frame (M, (∂/∂x,∂/∂y))
```

Setting components in a frame defined on a subdomain deletes previously defined components as well:

```
sage: U = M.open_subset('U', coord_def={X: x>0})
sage: f = U.vector_frame('f')
sage: t.set_comp(f)
2-indices components w.r.t. Vector frame (U, (f_0,f_1))
sage: t.set_comp(f)[0,1] = 1+y
sage: t.display(f)
t = (y + 1) f_0⊗f^1
sage: t.display(e)
Traceback (most recent call last):
...
ValueError: no basis could be found for computing the components
 in the Vector frame (M, (e_0,e_1))
```

```
>>> from sage.all import *
>>> U = M.open_subset('U', coord_def={X: x>Integer(0)})
>>> f = U.vector_frame('f')
>>> t.set_comp(f)
2-indices components w.r.t. Vector frame (U, (f_0,f_1))
>>> t.set_comp(f)[Integer(0),Integer(1)] = Integer(1)+y
>>> t.display(f)
t = (y + 1) f_0⊗f^1
>>> t.display(e)
Traceback (most recent call last):
...
ValueError: no basis could be found for computing the components
 in the Vector frame (M, (e_0,e_1))
```

truncate(*symbol*, *order*)

Return the tensor field truncated at a given order in the power series expansion with respect to some small parameter.

If the small parameter is ϵ and T is `self`, the power series expansion to order n is

$$T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \cdots + \epsilon^n T_n + O(\epsilon^{n+1}),$$

where $T_0, T_1, \ldots, T_n$ are $n+1$ tensor fields of the same tensor type as `self` and do not depend upon ϵ.

INPUT:

- `symbol` – symbolic variable (the "small parameter" ϵ) with respect to which the components of `self` are expanded in power series

- `order` – integer; the order n of the expansion, defined as the degree of the polynomial representing the truncated power series in `symbol`

OUTPUT:

- the tensor field $T_0 + \epsilon T_1 + \epsilon^2 T_2 + \cdots + \epsilon^n T_n$

EXAMPLES:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: C.<t,x,y,z> = M.chart()
sage: e = var('e')
sage: g = M.metric()
sage: h1 = M.tensor_field(0,2,sym=(0,1))
sage: h2 = M.tensor_field(0,2,sym=(0,1))
sage: g[0, 0], g[1, 1], g[2, 2], g[3, 3] = -1, 1, 1, 1
sage: h1[0, 1], h1[1, 2], h1[2, 3] = 1, 1, 1
sage: h2[0, 2], h2[1, 3] = 1, 1
sage: g.set(g + e*h1 + e^2*h2)
sage: g[:]
[ -1    e e^2    0]
[  e    1    e e^2]
[e^2    e    1    e]
[  0 e^2    e    1]
sage: g.truncate(e, 1)[:]
[-1  e  0  0]
[ e  1  e  0]
[ 0  e  1  e]
[ 0  0  e  1]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> C = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = C._first_
↪ngens(4)
>>> e = var('e')
>>> g = M.metric()
>>> h1 = M.tensor_field(Integer(0),Integer(2),sym=(Integer(0),Integer(1)))
>>> h2 = M.tensor_field(Integer(0),Integer(2),sym=(Integer(0),Integer(1)))
>>> g[Integer(0), Integer(0)], g[Integer(1), Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3), Integer(3)] = -Integer(1), Integer(1),
↪Integer(1), Integer(1)
>>> h1[Integer(0), Integer(1)], h1[Integer(1), Integer(2)], h1[Integer(2),
↪Integer(3)] = Integer(1), Integer(1), Integer(1)
>>> h2[Integer(0), Integer(2)], h2[Integer(1), Integer(3)] = Integer(1),
↪Integer(1)
>>> g.set(g + e*h1 + e**Integer(2)*h2)
>>> g[:]
[ -1    e e^2    0]
[  e    1    e e^2]
[e^2    e    1    e]
[  0 e^2    e    1]
>>> g.truncate(e, Integer(1))[:]
[-1  e  0  0]
[ e  1  e  0]
[ 0  e  1  e]
```

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```
[ 0  0  e  1]
```

2.9 Differential Forms

2.9.1 Differential Form Modules

The set $\Omega^p(U, \Phi)$ of p-forms along a differentiable manifold U with values on a differentiable manifold M via a differentiable map $\Phi : U \to M$ (possibly $U = M$ and $\Phi = \mathrm{Id}_M$) is a module over the algebra $C^k(U)$ of differentiable scalar fields on U. It is a free module if and only if M is parallelizable. Accordingly, two classes implement $\Omega^p(U, \Phi)$:

- *DiffFormModule* for differential forms with values on a generic (in practice, not parallelizable) differentiable manifold M
- *DiffFormFreeModule* for differential forms with values on a parallelizable manifold M (the subclass *VectorFieldDualFreeModule* implements the special case of differential 1-forms on a parallelizable manifold M)

AUTHORS:

- Eric Gourgoulhon (2015): initial version
- Travis Scrimshaw (2016): review tweaks
- Matthias Koeppe (2022): *VectorFieldDualFreeModule*

REFERENCES:

- [KN1963]
- [Lee2013]

class sage.manifolds.differentiable.diff_form_module.**DiffFormFreeModule**(*vector_field_module*, *degree*)

Bases: ExtPowerDualFreeModule

Free module of differential forms of a given degree p (p-forms) along a differentiable manifold U with values on a parallelizable manifold M.

Given a differentiable manifold U and a differentiable map $\Phi : U \to M$ to a parallelizable manifold M of dimension n, the set $\Omega^p(U, \Phi)$ of p-forms along U with values on M is a free module of rank $\binom{n}{p}$ over $C^k(U)$, the commutative algebra of differentiable scalar fields on U (see *DiffScalarFieldAlgebra*). The standard case of p-forms *on* a differentiable manifold M corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

> **Note**
>
> This class implements $\Omega^p(U, \Phi)$ in the case where M is parallelizable; $\Omega^p(U, \Phi)$ is then a *free* module. If M is not parallelizable, the class *DiffFormModule* must be used instead.

For the special case of 1-forms, use the class *VectorFieldDualFreeModule*.

INPUT:

- `vector_field_module` – free module $\mathfrak{X}(U, \Phi)$ of vector fields along U associated with the map $\Phi : U \to V$
- `degree` – positive integer; the degree p of the differential forms

EXAMPLES:

Free module of 2-forms on a parallelizable 3-dimensional manifold:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: XM = M.vector_field_module() ; XM
Free module X(M) of vector fields on the 3-dimensional differentiable
 manifold M
sage: A = M.diff_form_module(2) ; A
Free module Omega^2(M) of 2-forms on the 3-dimensional differentiable
 manifold M
sage: latex(A)
\Omega^{2}\left(M\right)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> XM = M.vector_field_module() ; XM
Free module X(M) of vector fields on the 3-dimensional differentiable
 manifold M
>>> A = M.diff_form_module(Integer(2)) ; A
Free module Omega^2(M) of 2-forms on the 3-dimensional differentiable
 manifold M
>>> latex(A)
\Omega^{2}\left(M\right)
```

A is nothing but the second exterior power of the dual of XM, i.e. we have $\Omega^2(M) = \Lambda^2(\mathfrak{X}(M)^*)$ (see ExtPowerDualFreeModule):

```
sage: A is XM.dual_exterior_power(2)
True
```

```
>>> from sage.all import *
>>> A is XM.dual_exterior_power(Integer(2))
True
```

$\Omega^2(M)$ is a module over the algebra $C^k(M)$ of (differentiable) scalar fields on M:

```
sage: A.category()
Category of finite dimensional modules over Algebra of differentiable
 scalar fields on the 3-dimensional differentiable manifold M
sage: CM = M.scalar_field_algebra() ; CM
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
sage: A in Modules(CM)
True
sage: A.base_ring()
Algebra of differentiable scalar fields on
 the 3-dimensional differentiable manifold M
sage: A.base_module()
Free module X(M) of vector fields on
 the 3-dimensional differentiable manifold M
sage: A.base_module() is XM
```

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```
True
sage: A.rank()
3
```

```
>>> from sage.all import *
>>> A.category()
Category of finite dimensional modules over Algebra of differentiable
 scalar fields on the 3-dimensional differentiable manifold M
>>> CM = M.scalar_field_algebra() ; CM
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
>>> A in Modules(CM)
True
>>> A.base_ring()
Algebra of differentiable scalar fields on
 the 3-dimensional differentiable manifold M
>>> A.base_module()
Free module X(M) of vector fields on
 the 3-dimensional differentiable manifold M
>>> A.base_module() is XM
True
>>> A.rank()
3
```

Elements can be constructed from A. In particular, `0` yields the zero element of A:

```
sage: A(0)
2-form zero on the 3-dimensional differentiable manifold M
sage: A(0) is A.zero()
True
```

```
>>> from sage.all import *
>>> A(Integer(0))
2-form zero on the 3-dimensional differentiable manifold M
>>> A(Integer(0)) is A.zero()
True
```

while nonzero elements are constructed by providing their components in a given vector frame:

```
sage: comp = [[0,3*x,-z],[-3*x,0,4],[z,-4,0]]
sage: a = A(comp, frame=X.frame(), name='a') ; a
2-form a on the 3-dimensional differentiable manifold M
sage: a.display()
a = 3*x dx∧dy - z dx∧dz + 4 dy∧dz
```

```
>>> from sage.all import *
>>> comp = [[Integer(0),Integer(3)*x,-z],[-Integer(3)*x,Integer(0),Integer(4)],[z,
↪-Integer(4),Integer(0)]]
>>> a = A(comp, frame=X.frame(), name='a') ; a
2-form a on the 3-dimensional differentiable manifold M
>>> a.display()
a = 3*x dx∧dy - z dx∧dz + 4 dy∧dz
```

An alternative is to construct the 2-form from an empty list of components and to set the nonzero nonredundant components afterwards:

```
sage: a = A([], name='a')
sage: a[0,1] = 3*x  # component in the manifold's default frame
sage: a[0,2] = -z
sage: a[1,2] = 4
sage: a.display()
a = 3*x dx∧dy - z dx∧dz + 4 dy∧dz
```

```
>>> from sage.all import *
>>> a = A([], name='a')
>>> a[Integer(0),Integer(1)] = Integer(3)*x  # component in the manifold's␣
↪default frame
>>> a[Integer(0),Integer(2)] = -z
>>> a[Integer(1),Integer(2)] = Integer(4)
>>> a.display()
a = 3*x dx∧dy - z dx∧dz + 4 dy∧dz
```

The module $\Omega^1(M)$ is nothing but the dual of $\mathfrak{X}(M)$ (the free module of vector fields on M):

```
sage: L1 = M.diff_form_module(1) ; L1
Free module Omega^1(M) of 1-forms on the 3-dimensional differentiable
 manifold M
sage: L1 is XM.dual()
True
```

```
>>> from sage.all import *
>>> L1 = M.diff_form_module(Integer(1)) ; L1
Free module Omega^1(M) of 1-forms on the 3-dimensional differentiable
 manifold M
>>> L1 is XM.dual()
True
```

Since any tensor field of type $(0, 1)$ is a 1-form, it is also equal to the set $T^{(0,1)}(M)$ of such tensors to $\Omega^1(M)$:

```
sage: T01 = M.tensor_field_module((0,1)) ; T01
Free module Omega^1(M) of 1-forms on the 3-dimensional differentiable manifold M
sage: L1 is T01
True
```

```
>>> from sage.all import *
>>> T01 = M.tensor_field_module((Integer(0),Integer(1))) ; T01
Free module Omega^1(M) of 1-forms on the 3-dimensional differentiable manifold M
>>> L1 is T01
True
```

For a degree $p \geq 2$, the coercion holds only in the direction $\Omega^p(M) \to T^{(0,p)}(M)$:

```
sage: T02 = M.tensor_field_module((0,2)); T02
Free module T^(0,2)(M) of type-(0,2) tensors fields on the
 3-dimensional differentiable manifold M
sage: T02.has_coerce_map_from(A)
```

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```
True
sage: A.has_coerce_map_from(T02)
False
```

```
>>> from sage.all import *
>>> T02 = M.tensor_field_module((Integer(0),Integer(2))); T02
Free module T^(0,2)(M) of type-(0,2) tensors fields on the
 3-dimensional differentiable manifold M
>>> T02.has_coerce_map_from(A)
True
>>> A.has_coerce_map_from(T02)
False
```

The coercion map $\Omega^2(M) \to T^{(0,2)}(M)$ in action:

```
sage: T02 = M.tensor_field_module((0,2)) ; T02
Free module T^(0,2)(M) of type-(0,2) tensors fields on the
 3-dimensional differentiable manifold M
sage: ta = T02(a) ; ta
Tensor field a of type (0,2) on the 3-dimensional differentiable
 manifold M
sage: ta.display()
a = 3*x dx⊗dy - z dx⊗dz - 3*x dy⊗dx + 4 dy⊗dz + z dz⊗dx - 4 dz⊗dy
sage: a.display()
a = 3*x dx∧dy - z dx∧dz + 4 dy∧dz
sage: ta.symmetries()  # the antisymmetry is preserved
no symmetry;  antisymmetry: (0, 1)
```

```
>>> from sage.all import *
>>> T02 = M.tensor_field_module((Integer(0),Integer(2))) ; T02
Free module T^(0,2)(M) of type-(0,2) tensors fields on the
 3-dimensional differentiable manifold M
>>> ta = T02(a) ; ta
Tensor field a of type (0,2) on the 3-dimensional differentiable
 manifold M
>>> ta.display()
a = 3*x dx⊗dy - z dx⊗dz - 3*x dy⊗dx + 4 dy⊗dz + z dz⊗dx - 4 dz⊗dy
>>> a.display()
a = 3*x dx∧dy - z dx∧dz + 4 dy∧dz
>>> ta.symmetries()  # the antisymmetry is preserved
no symmetry;  antisymmetry: (0, 1)
```

There is also coercion to subdomains, which is nothing but the restriction of the differential form to some subset of its domain:

```
sage: U = M.open_subset('U', coord_def={X: x^2+y^2<1})
sage: B = U.diff_form_module(2) ; B
Free module Omega^2(U) of 2-forms on the Open subset U of the
 3-dimensional differentiable manifold M
sage: B.has_coerce_map_from(A)
True
sage: a_U = B(a) ; a_U
```

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```
2-form a on the Open subset U of the 3-dimensional differentiable
 manifold M
sage: a_U.display()
a = 3*x dx∧dy - z dx∧dz + 4 dy∧dz
```

```
>>> from sage.all import *
>>> U = M.open_subset('U', coord_def={X: x**Integer(2)+y**Integer(2)<Integer(1)})
>>> B = U.diff_form_module(Integer(2)) ; B
Free module Omega^2(U) of 2-forms on the Open subset U of the
 3-dimensional differentiable manifold M
>>> B.has_coerce_map_from(A)
True
>>> a_U = B(a) ; a_U
2-form a on the Open subset U of the 3-dimensional differentiable
 manifold M
>>> a_U.display()
a = 3*x dx∧dy - z dx∧dz + 4 dy∧dz
```

Element

alias of *DiffFormParal*

class sage.manifolds.differentiable.diff_form_module.**DiffFormModule**(*vector_field_module*, *degree*)

Bases: UniqueRepresentation, Parent

Module of differential forms of a given degree p (p-forms) along a differentiable manifold U with values on a differentiable manifold M.

Given a differentiable manifold U and a differentiable map $\Phi : U \to M$ to a differentiable manifold M, the set $\Omega^p(U, \Phi)$ of p-forms along U with values on M is a module over $C^k(U)$, the commutative algebra of differentiable scalar fields on U (see *DiffScalarFieldAlgebra*). The standard case of p-forms *on* a differentiable manifold M corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note

This class implements $\Omega^p(U, \Phi)$ in the case where M is not assumed to be parallelizable; the module $\Omega^p(U, \Phi)$ is then not necessarily free. If M is parallelizable, the class *DiffFormFreeModule* must be used instead.

INPUT:

- `vector_field_module` – module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on M via the map $\Phi : U \to M$
- `degree` – positive integer; the degree p of the differential forms

EXAMPLES:

Module of 2-forms on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
```

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```
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y),
....:  intersection_name='W', restrictions1= x>0, restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: XM = M.vector_field_module() ; XM
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
sage: A = M.diff_form_module(2) ; A
Module Omega^2(M) of 2-forms on the 2-dimensional differentiable
 manifold M
sage: latex(A)
\Omega^{2}\left(M\right)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv = V.
↪chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> transf = c_xy.transition_map(c_uv, (x+y, x-y),
...  intersection_name='W', restrictions1= x>Integer(0), restrictions2= u+v>
↪Integer(0))
>>> inv = transf.inverse()
>>> W = U.intersection(V)
>>> eU = c_xy.frame() ; eV = c_uv.frame()
>>> XM = M.vector_field_module() ; XM
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
>>> A = M.diff_form_module(Integer(2)) ; A
Module Omega^2(M) of 2-forms on the 2-dimensional differentiable
 manifold M
>>> latex(A)
\Omega^{2}\left(M\right)
```

`A` is nothing but the second exterior power of the dual of `XM`, i.e. we have $\Omega^2(M) = \Lambda^2(\mathfrak{X}(M)^*)$:

```
sage: A is XM.dual_exterior_power(2)
True
```

```
>>> from sage.all import *
>>> A is XM.dual_exterior_power(Integer(2))
True
```

Modules of differential forms are unique:

```
sage: A is M.diff_form_module(2)
True
```

```
>>> from sage.all import *
>>> A is M.diff_form_module(Integer(2))
True
```

$\Omega^2(M)$ is a module over the algebra $C^k(M)$ of (differentiable) scalar fields on M:

```
sage: A.category()
Category of modules over Algebra of differentiable scalar fields on
 the 2-dimensional differentiable manifold M
sage: CM = M.scalar_field_algebra() ; CM
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
sage: A in Modules(CM)
True
sage: A.base_ring() is CM
True
sage: A.base_module()
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
sage: A.base_module() is XM
True
```

```
>>> from sage.all import *
>>> A.category()
Category of modules over Algebra of differentiable scalar fields on
 the 2-dimensional differentiable manifold M
>>> CM = M.scalar_field_algebra() ; CM
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
>>> A in Modules(CM)
True
>>> A.base_ring() is CM
True
>>> A.base_module()
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
>>> A.base_module() is XM
True
```

Elements can be constructed from `A()`. In particular, `0` yields the zero element of `A`:

```
sage: z = A(0) ; z
2-form zero on the 2-dimensional differentiable manifold M
sage: z.display(eU)
zero = 0
sage: z.display(eV)
zero = 0
sage: z is A.zero()
True
```

```
>>> from sage.all import *
>>> z = A(Integer(0)) ; z
2-form zero on the 2-dimensional differentiable manifold M
>>> z.display(eU)
zero = 0
>>> z.display(eV)
zero = 0
```

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```
>>> z is A.zero()
True
```

while nonzero elements are constructed by providing their components in a given vector frame:

```
sage: a = A([[0,3*x],[-3*x,0]], frame=eU, name='a') ; a
2-form a on the 2-dimensional differentiable manifold M
sage: a.add_comp_by_continuation(eV, W, c_uv) # finishes initializ. of a
sage: a.display(eU)
a = 3*x dx∧dy
sage: a.display(eV)
a = (-3/4*u - 3/4*v) du∧dv
```

```
>>> from sage.all import *
>>> a = A([[Integer(0),Integer(3)*x],[-Integer(3)*x,Integer(0)]], frame=eU, name=
↪'a') ; a
2-form a on the 2-dimensional differentiable manifold M
>>> a.add_comp_by_continuation(eV, W, c_uv) # finishes initializ. of a
>>> a.display(eU)
a = 3*x dx∧dy
>>> a.display(eV)
a = (-3/4*u - 3/4*v) du∧dv
```

An alternative is to construct the 2-form from an empty list of components and to set the nonzero nonredundant components afterwards:

```
sage: a = A([], name='a')
sage: a[eU,0,1] = 3*x
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: a.display(eU)
a = 3*x dx∧dy
sage: a.display(eV)
a = (-3/4*u - 3/4*v) du∧dv
```

```
>>> from sage.all import *
>>> a = A([], name='a')
>>> a[eU,Integer(0),Integer(1)] = Integer(3)*x
>>> a.add_comp_by_continuation(eV, W, c_uv)
>>> a.display(eU)
a = 3*x dx∧dy
>>> a.display(eV)
a = (-3/4*u - 3/4*v) du∧dv
```

The module $\Omega^1(M)$ is nothing but the dual of $\mathfrak{X}(M)$ (the module of vector fields on M):

```
sage: L1 = M.diff_form_module(1) ; L1
Module Omega^1(M) of 1-forms on the 2-dimensional differentiable
 manifold M
sage: L1 is XM.dual()
True
```

```
>>> from sage.all import *
>>> L1 = M.diff_form_module(Integer(1)) ; L1
Module Omega^1(M) of 1-forms on the 2-dimensional differentiable
 manifold M
>>> L1 is XM.dual()
True
```

Since any tensor field of type $(0, 1)$ is a 1-form, there is a coercion map from the set $T^{(0,1)}(M)$ of such tensors to $\Omega^1(M)$:

```
sage: T01 = M.tensor_field_module((0,1)) ; T01
Module T^(0,1)(M) of type-(0,1) tensors fields on the 2-dimensional
 differentiable manifold M
sage: L1.has_coerce_map_from(T01)
True
```

```
>>> from sage.all import *
>>> T01 = M.tensor_field_module((Integer(0),Integer(1))) ; T01
Module T^(0,1)(M) of type-(0,1) tensors fields on the 2-dimensional
 differentiable manifold M
>>> L1.has_coerce_map_from(T01)
True
```

There is also a coercion map in the reverse direction:

```
sage: T01.has_coerce_map_from(L1)
True
```

```
>>> from sage.all import *
>>> T01.has_coerce_map_from(L1)
True
```

For a degree $p \geq 2$, the coercion holds only in the direction $\Omega^p(M) \to T^{(0,p)}(M)$:

```
sage: T02 = M.tensor_field_module((0,2)) ; T02
Module T^(0,2)(M) of type-(0,2) tensors fields on the 2-dimensional
 differentiable manifold M
sage: T02.has_coerce_map_from(A)
True
sage: A.has_coerce_map_from(T02)
False
```

```
>>> from sage.all import *
>>> T02 = M.tensor_field_module((Integer(0),Integer(2))) ; T02
Module T^(0,2)(M) of type-(0,2) tensors fields on the 2-dimensional
 differentiable manifold M
>>> T02.has_coerce_map_from(A)
True
>>> A.has_coerce_map_from(T02)
False
```

The coercion map $T^{(0,1)}(M) \to \Omega^1(M)$ in action:

```
sage: b = T01([y,x], frame=eU, name='b') ; b
Tensor field b of type (0,1) on the 2-dimensional differentiable
 manifold M
sage: b.add_comp_by_continuation(eV, W, c_uv)
sage: b.display(eU)
b = y dx + x dy
sage: b.display(eV)
b = 1/2*u du - 1/2*v dv
sage: lb = L1(b) ; lb
1-form b on the 2-dimensional differentiable manifold M
sage: lb.display(eU)
b = y dx + x dy
sage: lb.display(eV)
b = 1/2*u du - 1/2*v dv
```

```
>>> from sage.all import *
>>> b = T01([y,x], frame=eU, name='b') ; b
Tensor field b of type (0,1) on the 2-dimensional differentiable
 manifold M
>>> b.add_comp_by_continuation(eV, W, c_uv)
>>> b.display(eU)
b = y dx + x dy
>>> b.display(eV)
b = 1/2*u du - 1/2*v dv
>>> lb = L1(b) ; lb
1-form b on the 2-dimensional differentiable manifold M
>>> lb.display(eU)
b = y dx + x dy
>>> lb.display(eV)
b = 1/2*u du - 1/2*v dv
```

The coercion map $\Omega^1(M) \to T^{(0,1)}(M)$ in action:

```
sage: tlb = T01(lb) ; tlb
Tensor field b of type (0,1) on the 2-dimensional differentiable
 manifold M
sage: tlb.display(eU)
b = y dx + x dy
sage: tlb.display(eV)
b = 1/2*u du - 1/2*v dv
sage: tlb == b
True
```

```
>>> from sage.all import *
>>> tlb = T01(lb) ; tlb
Tensor field b of type (0,1) on the 2-dimensional differentiable
 manifold M
>>> tlb.display(eU)
b = y dx + x dy
>>> tlb.display(eV)
b = 1/2*u du - 1/2*v dv
>>> tlb == b
True
```

The coercion map $\Omega^2(M) \to T^{(0,2)}(M)$ in action:

```
sage: ta = T02(a) ; ta
Tensor field a of type (0,2) on the 2-dimensional differentiable
 manifold M
sage: ta.display(eU)
a = 3*x dx⊗dy - 3*x dy⊗dx
sage: a.display(eU)
a = 3*x dx∧dy
sage: ta.display(eV)
a = (-3/4*u - 3/4*v) du⊗dv + (3/4*u + 3/4*v) dv⊗du
sage: a.display(eV)
a = (-3/4*u - 3/4*v) du∧dv
```

```
>>> from sage.all import *
>>> ta = T02(a) ; ta
Tensor field a of type (0,2) on the 2-dimensional differentiable
 manifold M
>>> ta.display(eU)
a = 3*x dx⊗dy - 3*x dy⊗dx
>>> a.display(eU)
a = 3*x dx∧dy
>>> ta.display(eV)
a = (-3/4*u - 3/4*v) du⊗dv + (3/4*u + 3/4*v) dv⊗du
>>> a.display(eV)
a = (-3/4*u - 3/4*v) du∧dv
```

There is also coercion to subdomains, which is nothing but the restriction of the differential form to some subset of its domain:

```
sage: L2U = U.diff_form_module(2) ; L2U
Free module Omega^2(U) of 2-forms on the Open subset U of the
 2-dimensional differentiable manifold M
sage: L2U.has_coerce_map_from(A)
True
sage: a_U = L2U(a) ; a_U
2-form a on the Open subset U of the 2-dimensional differentiable
 manifold M
sage: a_U.display(eU)
a = 3*x dx∧dy
```

```
>>> from sage.all import *
>>> L2U = U.diff_form_module(Integer(2)) ; L2U
Free module Omega^2(U) of 2-forms on the Open subset U of the
 2-dimensional differentiable manifold M
>>> L2U.has_coerce_map_from(A)
True
>>> a_U = L2U(a) ; a_U
2-form a on the Open subset U of the 2-dimensional differentiable
 manifold M
>>> a_U.display(eU)
a = 3*x dx∧dy
```

Element

alias of *DiffForm*

base_module()

Return the vector field module on which the differential form module `self` is constructed.

OUTPUT:

- a *VectorFieldModule* representing the module on which `self` is defined

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: A2 = M.diff_form_module(2) ; A2
Module Omega^2(M) of 2-forms on the 3-dimensional differentiable
 manifold M
sage: A2.base_module()
Module X(M) of vector fields on the 3-dimensional differentiable
 manifold M
sage: A2.base_module() is M.vector_field_module()
True
sage: U = M.open_subset('U')
sage: A2U = U.diff_form_module(2) ; A2U
Module Omega^2(U) of 2-forms on the Open subset U of the
 3-dimensional differentiable manifold M
sage: A2U.base_module()
Module X(U) of vector fields on the Open subset U of the
 3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> A2 = M.diff_form_module(Integer(2)) ; A2
Module Omega^2(M) of 2-forms on the 3-dimensional differentiable
 manifold M
>>> A2.base_module()
Module X(M) of vector fields on the 3-dimensional differentiable
 manifold M
>>> A2.base_module() is M.vector_field_module()
True
>>> U = M.open_subset('U')
>>> A2U = U.diff_form_module(Integer(2)) ; A2U
Module Omega^2(U) of 2-forms on the Open subset U of the
 3-dimensional differentiable manifold M
>>> A2U.base_module()
Module X(U) of vector fields on the Open subset U of the
 3-dimensional differentiable manifold M
```

degree()

Return the degree of the differential forms in `self`.

OUTPUT: integer p such that `self` is a set of p-forms

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: M.diff_form_module(1).degree()
```

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```
1
sage: M.diff_form_module(2).degree()
2
sage: M.diff_form_module(3).degree()
3
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> M.diff_form_module(Integer(1)).degree()
1
>>> M.diff_form_module(Integer(2)).degree()
2
>>> M.diff_form_module(Integer(3)).degree()
3
```

tensor(**others*)

Return the tensor product of `self` and `others`.

EXAMPLES:

```
sage: M = FiniteRankFreeModule(QQ, 2)
sage: M.tensor_product(M)
Free module of type-(2,0) tensors on the 2-dimensional vector space over the␣
↪Rational Field
sage: M.tensor_product(M.dual())
Free module of type-(1,1) tensors on the 2-dimensional vector space over the␣
↪Rational Field
sage: M.dual().tensor_product(M, M.dual())
Free module of type-(1,2) tensors on the 2-dimensional vector space over the␣
↪Rational Field
sage: M.tensor_product(M.tensor_module(1,2))
Free module of type-(2,2) tensors on the 2-dimensional vector space over the␣
↪Rational Field
sage: M.tensor_module(1,2).tensor_product(M)
Free module of type-(2,2) tensors on the 2-dimensional vector space over the␣
↪Rational Field
sage: M.tensor_module(1,1).tensor_product(M.tensor_module(1,2))
Free module of type-(2,3) tensors on the 2-dimensional vector space over the␣
↪Rational Field

sage: Sym2M = M.tensor_module(2, 0, sym=range(2)); Sym2M
Free module of fully symmetric type-(2,0) tensors on the 2-dimensional vector␣
↪space over the Rational Field
sage: Sym01x23M = Sym2M.tensor_product(Sym2M); Sym01x23M
Free module of type-(4,0) tensors on the 2-dimensional vector space over the␣
↪Rational Field,
 with symmetry on the index positions (0, 1), with symmetry on the index␣
↪positions (2, 3)
sage: Sym01x23M._index_maps
((0, 1), (2, 3))

sage: N = M.tensor_module(3, 3, sym=[1, 2], antisym=[3, 4]); N
```

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```
Free module of type-(3,3) tensors on the 2-dimensional vector space over the
↪Rational Field,
 with symmetry on the index positions (1, 2),
 with antisymmetry on the index positions (3, 4)
sage: NxN = N.tensor_product(N); NxN
Free module of type-(6,6) tensors on the 2-dimensional vector space over the
↪Rational Field,
 with symmetry on the index positions (1, 2), with symmetry on the index
↪positions (4, 5),
 with antisymmetry on the index positions (6, 7), with antisymmetry on the
↪index positions (9, 10)
sage: NxN._index_maps
((0, 1, 2, 6, 7, 8), (3, 4, 5, 9, 10, 11))
```

```
>>> from sage.all import *
>>> M = FiniteRankFreeModule(QQ, Integer(2))
>>> M.tensor_product(M)
Free module of type-(2,0) tensors on the 2-dimensional vector space over the
↪Rational Field
>>> M.tensor_product(M.dual())
Free module of type-(1,1) tensors on the 2-dimensional vector space over the
↪Rational Field
>>> M.dual().tensor_product(M, M.dual())
Free module of type-(1,2) tensors on the 2-dimensional vector space over the
↪Rational Field
>>> M.tensor_product(M.tensor_module(Integer(1),Integer(2)))
Free module of type-(2,2) tensors on the 2-dimensional vector space over the
↪Rational Field
>>> M.tensor_module(Integer(1),Integer(2)).tensor_product(M)
Free module of type-(2,2) tensors on the 2-dimensional vector space over the
↪Rational Field
>>> M.tensor_module(Integer(1),Integer(1)).tensor_product(M.tensor_
↪module(Integer(1),Integer(2)))
Free module of type-(2,3) tensors on the 2-dimensional vector space over the
↪Rational Field

>>> Sym2M = M.tensor_module(Integer(2), Integer(0), sym=range(Integer(2)));
↪Sym2M
Free module of fully symmetric type-(2,0) tensors on the 2-dimensional vector
↪space over the Rational Field
>>> Sym01x23M = Sym2M.tensor_product(Sym2M); Sym01x23M
Free module of type-(4,0) tensors on the 2-dimensional vector space over the
↪Rational Field,
 with symmetry on the index positions (0, 1), with symmetry on the index
↪positions (2, 3)
>>> Sym01x23M._index_maps
((0, 1), (2, 3))

>>> N = M.tensor_module(Integer(3), Integer(3), sym=[Integer(1), Integer(2)],
↪antisym=[Integer(3), Integer(4)]); N
Free module of type-(3,3) tensors on the 2-dimensional vector space over the
↪Rational Field,
```

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```
 with symmetry on the index positions (1, 2),
 with antisymmetry on the index positions (3, 4)
>>> NxN = N.tensor_product(N); NxN
Free module of type-(6,6) tensors on the 2-dimensional vector space over the␣
↪Rational Field,
 with symmetry on the index positions (1, 2), with symmetry on the index␣
↪positions (4, 5),
 with antisymmetry on the index positions (6, 7), with antisymmetry on the␣
↪index positions (9, 10)
>>> NxN._index_maps
((0, 1, 2, 6, 7, 8), (3, 4, 5, 9, 10, 11))
```

tensor_product (**others*)

Return the tensor product of `self` and `others`.

EXAMPLES:

```
sage: M = FiniteRankFreeModule(QQ, 2)
sage: M.tensor_product(M)
Free module of type-(2,0) tensors on the 2-dimensional vector space over the␣
↪Rational Field
sage: M.tensor_product(M.dual())
Free module of type-(1,1) tensors on the 2-dimensional vector space over the␣
↪Rational Field
sage: M.dual().tensor_product(M, M.dual())
Free module of type-(1,2) tensors on the 2-dimensional vector space over the␣
↪Rational Field
sage: M.tensor_product(M.tensor_module(1,2))
Free module of type-(2,2) tensors on the 2-dimensional vector space over the␣
↪Rational Field
sage: M.tensor_module(1,2).tensor_product(M)
Free module of type-(2,2) tensors on the 2-dimensional vector space over the␣
↪Rational Field
sage: M.tensor_module(1,1).tensor_product(M.tensor_module(1,2))
Free module of type-(2,3) tensors on the 2-dimensional vector space over the␣
↪Rational Field

sage: Sym2M = M.tensor_module(2, 0, sym=range(2)); Sym2M
Free module of fully symmetric type-(2,0) tensors on the 2-dimensional vector␣
↪space over the Rational Field
sage: Sym01x23M = Sym2M.tensor_product(Sym2M); Sym01x23M
Free module of type-(4,0) tensors on the 2-dimensional vector space over the␣
↪Rational Field,
 with symmetry on the index positions (0, 1), with symmetry on the index␣
↪positions (2, 3)
sage: Sym01x23M._index_maps
((0, 1), (2, 3))

sage: N = M.tensor_module(3, 3, sym=[1, 2], antisym=[3, 4]); N
Free module of type-(3,3) tensors on the 2-dimensional vector space over the␣
↪Rational Field,
 with symmetry on the index positions (1, 2),
```

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```
 with antisymmetry on the index positions (3, 4)
sage: NxN = N.tensor_product(N); NxN
Free module of type-(6,6) tensors on the 2-dimensional vector space over the␣
↪Rational Field,
 with symmetry on the index positions (1, 2), with symmetry on the index␣
↪positions (4, 5),
 with antisymmetry on the index positions (6, 7), with antisymmetry on the␣
↪index positions (9, 10)
sage: NxN._index_maps
((0, 1, 2, 6, 7, 8), (3, 4, 5, 9, 10, 11))
```

```
>>> from sage.all import *
>>> M = FiniteRankFreeModule(QQ, Integer(2))
>>> M.tensor_product(M)
Free module of type-(2,0) tensors on the 2-dimensional vector space over the␣
↪Rational Field
>>> M.tensor_product(M.dual())
Free module of type-(1,1) tensors on the 2-dimensional vector space over the␣
↪Rational Field
>>> M.dual().tensor_product(M, M.dual())
Free module of type-(1,2) tensors on the 2-dimensional vector space over the␣
↪Rational Field
>>> M.tensor_product(M.tensor_module(Integer(1),Integer(2)))
Free module of type-(2,2) tensors on the 2-dimensional vector space over the␣
↪Rational Field
>>> M.tensor_module(Integer(1),Integer(2)).tensor_product(M)
Free module of type-(2,2) tensors on the 2-dimensional vector space over the␣
↪Rational Field
>>> M.tensor_module(Integer(1),Integer(1)).tensor_product(M.tensor_
↪module(Integer(1),Integer(2)))
Free module of type-(2,3) tensors on the 2-dimensional vector space over the␣
↪Rational Field

>>> Sym2M = M.tensor_module(Integer(2), Integer(0), sym=range(Integer(2)));␣
↪Sym2M
Free module of fully symmetric type-(2,0) tensors on the 2-dimensional vector␣
↪space over the Rational Field
>>> Sym01x23M = Sym2M.tensor_product(Sym2M); Sym01x23M
Free module of type-(4,0) tensors on the 2-dimensional vector space over the␣
↪Rational Field,
 with symmetry on the index positions (0, 1), with symmetry on the index␣
↪positions (2, 3)
>>> Sym01x23M._index_maps
((0, 1), (2, 3))

>>> N = M.tensor_module(Integer(3), Integer(3), sym=[Integer(1), Integer(2)],␣
↪antisym=[Integer(3), Integer(4)]); N
Free module of type-(3,3) tensors on the 2-dimensional vector space over the␣
↪Rational Field,
 with symmetry on the index positions (1, 2),
 with antisymmetry on the index positions (3, 4)
>>> NxN = N.tensor_product(N); NxN
```

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```
Free module of type-(6,6) tensors on the 2-dimensional vector space over the␣
↪Rational Field,
 with symmetry on the index positions (1, 2), with symmetry on the index␣
↪positions (4, 5),
 with antisymmetry on the index positions (6, 7), with antisymmetry on the␣
↪index positions (9, 10)
>>> NxN._index_maps
((0, 1, 2, 6, 7, 8), (3, 4, 5, 9, 10, 11))
```

tensor_type()

Return the tensor type of `self` if `self` is a module of 1-forms.

In this case, the pair $(0, 1)$ is returned, indicating that the module is identified with the dual of the base module.

For differential forms of other degrees, an exception is raised.

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: M.diff_form_module(1).tensor_type()
(0, 1)
sage: M.diff_form_module(2).tensor_type()
Traceback (most recent call last):
...
NotImplementedError
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> M.diff_form_module(Integer(1)).tensor_type()
(0, 1)
>>> M.diff_form_module(Integer(2)).tensor_type()
Traceback (most recent call last):
...
NotImplementedError
```

zero()

Return the zero of `self`.

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: A2 = M.diff_form_module(2)
sage: A2.zero()
2-form zero on the 3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> A2 = M.diff_form_module(Integer(2))
>>> A2.zero()
2-form zero on the 3-dimensional differentiable manifold M
```

class sage.manifolds.differentiable.diff_form_module.**VectorFieldDualFreeModule**(*vector_field_module*)

Bases: `DiffFormFreeModule`

Free module of differential 1-forms along a differentiable manifold U with values on a parallelizable manifold M.

Given a differentiable manifold U and a differentiable map $\Phi : U \to M$ to a parallelizable manifold M of dimension n, the set $\Omega^1(U,\Phi)$ of 1-forms along U with values on M is a free module of rank n over $C^k(U)$, the commutative algebra of differentiable scalar fields on U (see `DiffScalarFieldAlgebra`). The standard case of 1-forms *on* a differentiable manifold M corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note

This class implements $\Omega^1(U,\Phi)$ in the case where M is parallelizable; $\Omega^1(U,\Phi)$ is then a *free* module. If M is not parallelizable, the class `DiffFormModule` must be used instead.

INPUT:

- `vector_field_module` – free module $\mathfrak{X}(U,\Phi)$ of vector fields along U associated with the map $\Phi : U \to V$

EXAMPLES:

Free module of 1-forms on a parallelizable 3-dimensional manifold:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: XM = M.vector_field_module() ; XM
Free module X(M) of vector fields on the 3-dimensional differentiable
 manifold M
sage: A = M.diff_form_module(1) ; A
Free module Omega^1(M) of 1-forms on the 3-dimensional differentiable manifold M
sage: latex(A)
\Omega^{1}\left(M\right)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> XM = M.vector_field_module() ; XM
Free module X(M) of vector fields on the 3-dimensional differentiable
 manifold M
>>> A = M.diff_form_module(Integer(1)) ; A
Free module Omega^1(M) of 1-forms on the 3-dimensional differentiable manifold M
>>> latex(A)
\Omega^{1}\left(M\right)
```

`A` is nothing but the dual of `XM` (the free module of vector fields on M) and thus also equal to the 1st exterior power of the dual, i.e. we have $\Omega^1(M) = \Lambda^1(\mathfrak{X}(M)^*) = \mathfrak{X}(M)^*$ (See `ExtPowerDualFreeModule`):

```
sage: A is XM.dual_exterior_power(1)
True
```

```
>>> from sage.all import *
>>> A is XM.dual_exterior_power(Integer(1))
True
```

$\Omega^1(M)$ is a module over the algebra $C^k(M)$ of (differentiable) scalar fields on M:

```
sage: A.category()
Category of finite dimensional modules over Algebra of differentiable
 scalar fields on the 3-dimensional differentiable manifold M
sage: CM = M.scalar_field_algebra() ; CM
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
sage: A in Modules(CM)
True
sage: A.base_ring()
Algebra of differentiable scalar fields on
 the 3-dimensional differentiable manifold M
sage: A.base_module()
Free module X(M) of vector fields on
 the 3-dimensional differentiable manifold M
sage: A.base_module() is XM
True
sage: A.rank()
3
```

```
>>> from sage.all import *
>>> A.category()
Category of finite dimensional modules over Algebra of differentiable
 scalar fields on the 3-dimensional differentiable manifold M
>>> CM = M.scalar_field_algebra() ; CM
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
>>> A in Modules(CM)
True
>>> A.base_ring()
Algebra of differentiable scalar fields on
 the 3-dimensional differentiable manifold M
>>> A.base_module()
Free module X(M) of vector fields on
 the 3-dimensional differentiable manifold M
>>> A.base_module() is XM
True
>>> A.rank()
3
```

Elements can be constructed from A. In particular, `0` yields the zero element of A:

```
sage: A(0)
1-form zero on the 3-dimensional differentiable manifold M
sage: A(0) is A.zero()
True
```

```
>>> from sage.all import *
>>> A(Integer(0))
1-form zero on the 3-dimensional differentiable manifold M
>>> A(Integer(0)) is A.zero()
True
```

while nonzero elements are constructed by providing their components in a given vector frame:

```
sage: comp = [3*x,-z,4]
sage: a = A(comp, frame=X.frame(), name='a') ; a
1-form a on the 3-dimensional differentiable manifold M
sage: a.display()
a = 3*x dx - z dy + 4 dz
```

```
>>> from sage.all import *
>>> comp = [Integer(3)*x,-z,Integer(4)]
>>> a = A(comp, frame=X.frame(), name='a') ; a
1-form a on the 3-dimensional differentiable manifold M
>>> a.display()
a = 3*x dx - z dy + 4 dz
```

An alternative is to construct the 1-form from an empty list of components and to set the nonzero nonredundant components afterwards:

```
sage: a = A([], name='a')
sage: a[0] = 3*x  # component in the manifold's default frame
sage: a[1] = -z
sage: a[2] = 4
sage: a.display()
a = 3*x dx - z dy + 4 dz
```

```
>>> from sage.all import *
>>> a = A([], name='a')
>>> a[Integer(0)] = Integer(3)*x  # component in the manifold's default frame
>>> a[Integer(1)] = -z
>>> a[Integer(2)] = Integer(4)
>>> a.display()
a = 3*x dx - z dy + 4 dz
```

Since any tensor field of type $(0, 1)$ is a 1-form, there is a coercion map from the set $T^{(0,1)}(M)$ of such tensors to $\Omega^1(M)$:

```
sage: T01 = M.tensor_field_module((0,1)) ; T01
Free module Omega^1(M) of 1-forms on the 3-dimensional differentiable manifold M
sage: A.has_coerce_map_from(T01)
True
```

```
>>> from sage.all import *
>>> T01 = M.tensor_field_module((Integer(0),Integer(1))) ; T01
Free module Omega^1(M) of 1-forms on the 3-dimensional differentiable manifold M
>>> A.has_coerce_map_from(T01)
True
```

There is also a coercion map in the reverse direction:

```
sage: T01.has_coerce_map_from(A)
True
```

```
>>> from sage.all import *
>>> T01.has_coerce_map_from(A)
True
```

The coercion map $T^{(0,1)}(M) \to \Omega^1(M)$ in action:

```
sage: b = T01([-x,2,3*y], name='b'); b
1-form b on the 3-dimensional differentiable manifold M
sage: b.display()
b = -x dx + 2 dy + 3*y dz
sage: lb = A(b) ; lb
1-form b on the 3-dimensional differentiable manifold M
sage: lb.display()
b = -x dx + 2 dy + 3*y dz
```

```
>>> from sage.all import *
>>> b = T01([-x,Integer(2),Integer(3)*y], name='b'); b
1-form b on the 3-dimensional differentiable manifold M
>>> b.display()
b = -x dx + 2 dy + 3*y dz
>>> lb = A(b) ; lb
1-form b on the 3-dimensional differentiable manifold M
>>> lb.display()
b = -x dx + 2 dy + 3*y dz
```

The coercion map $\Omega^1(M) \to T^{(0,1)}(M)$ in action:

```
sage: tlb = T01(lb); tlb
1-form b on the 3-dimensional differentiable manifold M
sage: tlb == b
True
```

```
>>> from sage.all import *
>>> tlb = T01(lb); tlb
1-form b on the 3-dimensional differentiable manifold M
>>> tlb == b
True
```

tensor_type()

Return the tensor type of `self`.

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: A = M.vector_field_module().dual(); A
Free module Omega^1(M) of 1-forms on the 3-dimensional differentiable
↪manifold M
sage: A.tensor_type()
(0, 1)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
```

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```
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> A = M.vector_field_module().dual(); A
Free module Omega^1(M) of 1-forms on the 3-dimensional differentiable␣
↪manifold M
>>> A.tensor_type()
(0, 1)
```

2.9.2 Differential Forms

Let U and M be two differentiable manifolds. Given a positive integer p and a differentiable map $\Phi : U \to M$, a *differential form of degree* p, or p*-form*, *along* U *with values on* M is a field along U of alternating multilinear forms of degree p in the tangent spaces to M. The standard case of a differential form *on* a differentiable manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Two classes implement differential forms, depending whether the manifold M is parallelizable:

- `DiffFormParal` when M is parallelizable
- `DiffForm` when M is not assumed parallelizable.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013, 2014): initial version
- Joris Vankerschaver (2010): developed a previous class, `DifferentialForm` (cf. Issue #24444), which inspired the storage of the nonzero components as a dictionary whose keys are the indices.
- Travis Scrimshaw (2016): review tweaks

REFERENCES:

- [KN1963]
- [Lee2013]

class `sage.manifolds.differentiable.diff_form.`**`DiffForm`**(*vector_field_module*, *degree*, *name=None*, *latex_name=None*)

Bases: `TensorField`

Differential form with values on a generic (i.e. a priori not parallelizable) differentiable manifold.

Given a differentiable manifold U, a differentiable map $\Phi : U \to M$ to a differentiable manifold M and a positive integer p, a *differential form of degree* p (or p*-form*) *along* U *with values on* $M \supset \Phi(U)$ is a differentiable map

$$a : U \longrightarrow T^{(0,p)}M$$

($T^{(0,p)}M$ being the tensor bundle of type $(0, p)$ over M) such that

$$\forall x \in U, \quad a(x) \in \Lambda^p(T^*_{\Phi(x)}M),$$

where $T^*_{\Phi(x)}M$ is the dual of the tangent space to M at $\Phi(x)$ and Λ^p stands for the exterior power of degree p (cf. `ExtPowerDualFreeModule`). In other words, $a(x)$ is an alternating multilinear form of degree p of the tangent vector space $T_{\Phi(x)}M$.

The standard case of a differential form *on* a manifold M corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note

If M is parallelizable, the class `DiffFormParal` must be used instead.

INPUT:

- `vector_field_module` – module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on M via the map Φ
- `degree` – the degree of the differential form (i.e. its tensor rank)
- `name` – (default: `None`) name given to the differential form
- `latex_name` – (default: `None`) LaTeX symbol to denote the differential form; if none is provided, the LaTeX symbol is set to `name`

EXAMPLES:

Differential form of degree 2 on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
....:                                restrictions1= x>0, restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: a = M.diff_form(2, name='a') ; a
2-form a on the 2-dimensional differentiable manifold M
sage: a.parent()
Module Omega^2(M) of 2-forms on the 2-dimensional differentiable
 manifold M
sage: a.degree()
2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv = V.
↪chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
...                                restrictions1= x>Integer(0), restrictions2=␣
↪u+v>Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V)
>>> eU = c_xy.frame() ; eV = c_uv.frame()
>>> a = M.diff_form(Integer(2), name='a') ; a
2-form a on the 2-dimensional differentiable manifold M
>>> a.parent()
Module Omega^2(M) of 2-forms on the 2-dimensional differentiable
 manifold M
>>> a.degree()
2
```

Setting the components of a:

```
sage: a[eU,0,1] = x*y^2 + 2*x
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: a.display(eU)
a = (x*y^2 + 2*x) dx∧dy
sage: a.display(eV)
a = (-1/16*u^3 + 1/16*u*v^2 - 1/16*v^3
 + 1/16*(u^2 - 8)*v - 1/2*u) du∧dv
```

```
>>> from sage.all import *
>>> a[eU,Integer(0),Integer(1)] = x*y**Integer(2) + Integer(2)*x
>>> a.add_comp_by_continuation(eV, W, c_uv)
>>> a.display(eU)
a = (x*y^2 + 2*x) dx∧dy
>>> a.display(eV)
a = (-1/16*u^3 + 1/16*u*v^2 - 1/16*v^3
 + 1/16*(u^2 - 8)*v - 1/2*u) du∧dv
```

A 1-form on M:

```
sage: a = M.one_form(name='a') ; a
1-form a on the 2-dimensional differentiable manifold M
sage: a.parent()
Module Omega^1(M) of 1-forms on the 2-dimensional differentiable
 manifold M
sage: a.degree()
1
```

```
>>> from sage.all import *
>>> a = M.one_form(name='a') ; a
1-form a on the 2-dimensional differentiable manifold M
>>> a.parent()
Module Omega^1(M) of 1-forms on the 2-dimensional differentiable
 manifold M
>>> a.degree()
1
```

Setting the components of the 1-form in a consistent way:

```
sage: a[eU,:] = [-y, x]
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: a.display(eU)
a = -y dx + x dy
sage: a.display(eV)
a = 1/2*v du - 1/2*u dv
```

```
>>> from sage.all import *
>>> a[eU,:] = [-y, x]
>>> a.add_comp_by_continuation(eV, W, c_uv)
>>> a.display(eU)
a = -y dx + x dy
>>> a.display(eV)
```

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```
a = 1/2*v du - 1/2*u dv
```

It is also possible to set the components at the 1-form definition, via a dictionary whose keys are the vector frames:

```
sage: a1 = M.one_form({eU: [-y, x], eV: [v/2, -u/2]}, name='a')
sage: a1 == a
True
```

```
>>> from sage.all import *
>>> a1 = M.one_form({eU: [-y, x], eV: [v/Integer(2), -u/Integer(2)]}, name='a')
>>> a1 == a
True
```

The exterior derivative of the 1-form is a 2-form:

```
sage: da = a.exterior_derivative() ; da
2-form da on the 2-dimensional differentiable manifold M
sage: da.display(eU)
da = 2 dx∧dy
sage: da.display(eV)
da = -du∧dv
```

```
>>> from sage.all import *
>>> da = a.exterior_derivative() ; da
2-form da on the 2-dimensional differentiable manifold M
>>> da.display(eU)
da = 2 dx∧dy
>>> da.display(eV)
da = -du∧dv
```

The exterior derivative can also be obtained by applying the function `diff` to a differentiable form:

```
sage: diff(a) is a.exterior_derivative()
True
```

```
>>> from sage.all import *
>>> diff(a) is a.exterior_derivative()
True
```

Another 1-form defined by its components in `eU`:

```
sage: b = M.one_form(1+x*y, x^2, frame=eU, name='b')
```

```
>>> from sage.all import *
>>> b = M.one_form(Integer(1)+x*y, x**Integer(2), frame=eU, name='b')
```

Since `eU` is the default vector frame on `M`, it can be omitted in the definition:

```
sage: b = M.one_form(1+x*y, x^2, name='b')
sage: b.add_comp_by_continuation(eV, W, c_uv)
```

```
>>> from sage.all import *
>>> b = M.one_form(Integer(1)+x*y, x**Integer(2), name='b')
>>> b.add_comp_by_continuation(eV, W, c_uv)
```

Adding two 1-forms results in another 1-form:

```
sage: s = a + b ; s
1-form a+b on the 2-dimensional differentiable manifold M
sage: s.display(eU)
a+b = ((x - 1)*y + 1) dx + (x^2 + x) dy
sage: s.display(eV)
a+b = (1/4*u^2 + 1/4*(u + 2)*v + 1/2) du
 + (-1/4*u*v - 1/4*v^2 - 1/2*u + 1/2) dv
```

```
>>> from sage.all import *
>>> s = a + b ; s
1-form a+b on the 2-dimensional differentiable manifold M
>>> s.display(eU)
a+b = ((x - 1)*y + 1) dx + (x^2 + x) dy
>>> s.display(eV)
a+b = (1/4*u^2 + 1/4*(u + 2)*v + 1/2) du
 + (-1/4*u*v - 1/4*v^2 - 1/2*u + 1/2) dv
```

The exterior product of two 1-forms is a 2-form:

```
sage: s = a.wedge(b) ; s
2-form a∧b on the 2-dimensional differentiable manifold M
sage: s.display(eU)
a∧b = (-2*x^2*y - x) dx∧dy
sage: s.display(eV)
a∧b = (1/8*u^3 - 1/8*u*v^2 - 1/8*v^3 + 1/8*(u^2 + 2)*v + 1/4*u) du∧dv
```

```
>>> from sage.all import *
>>> s = a.wedge(b) ; s
2-form a∧b on the 2-dimensional differentiable manifold M
>>> s.display(eU)
a∧b = (-2*x^2*y - x) dx∧dy
>>> s.display(eV)
a∧b = (1/8*u^3 - 1/8*u*v^2 - 1/8*v^3 + 1/8*(u^2 + 2)*v + 1/4*u) du∧dv
```

Multiplying a 1-form by a scalar field results in another 1-form:

```
sage: f = M.scalar_field({c_xy: (x+y)^2, c_uv: u^2}, name='f')
sage: s = f*a ; s
1-form f*a on the 2-dimensional differentiable manifold M
sage: s.display(eU)
f*a = (-x^2*y - 2*x*y^2 - y^3) dx + (x^3 + 2*x^2*y + x*y^2) dy
sage: s.display(eV)
f*a = 1/2*u^2*v du - 1/2*u^3 dv
```

```
>>> from sage.all import *
>>> f = M.scalar_field({c_xy: (x+y)**Integer(2), c_uv: u**Integer(2)}, name='f')
```

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```
>>> s = f*a ; s
1-form f*a on the 2-dimensional differentiable manifold M
>>> s.display(eU)
f*a = (-x^2*y - 2*x*y^2 - y^3) dx + (x^3 + 2*x^2*y + x*y^2) dy
>>> s.display(eV)
f*a = 1/2*u^2*v du - 1/2*u^3 dv
```

Examples with SymPy as the symbolic engine

From now on, we ask that all symbolic calculus on manifold M are performed by SymPy:

```
sage: M.set_calculus_method('sympy')
```

```
>>> from sage.all import *
>>> M.set_calculus_method('sympy')
```

We define a 2-form a as above:

```
sage: a = M.diff_form(2, name='a')
sage: a[eU,0,1] = x*y^2 + 2*x
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: a.display(eU)
a = (x*y**2 + 2*x) dx∧dy
sage: a.display(eV)
a = (-u**3/16 + u**2*v/16 + u*v**2/16 - u/2 - v**3/16 - v/2) du∧dv
```

```
>>> from sage.all import *
>>> a = M.diff_form(Integer(2), name='a')
>>> a[eU,Integer(0),Integer(1)] = x*y**Integer(2) + Integer(2)*x
>>> a.add_comp_by_continuation(eV, W, c_uv)
>>> a.display(eU)
a = (x*y**2 + 2*x) dx∧dy
>>> a.display(eV)
a = (-u**3/16 + u**2*v/16 + u*v**2/16 - u/2 - v**3/16 - v/2) du∧dv
```

A 1-form on `M`:

```
sage: a = M.one_form(-y, x, name='a')
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: a.display(eU)
a = -y dx + x dy
sage: a.display(eV)
a = v/2 du - u/2 dv
```

```
>>> from sage.all import *
>>> a = M.one_form(-y, x, name='a')
>>> a.add_comp_by_continuation(eV, W, c_uv)
>>> a.display(eU)
a = -y dx + x dy
>>> a.display(eV)
a = v/2 du - u/2 dv
```

The exterior derivative of a:

```
sage: da = a.exterior_derivative()
sage: da.display(eU)
da = 2 dx∧dy
sage: da.display(eV)
da = -du∧dv
```

```
>>> from sage.all import *
>>> da = a.exterior_derivative()
>>> da.display(eU)
da = 2 dx∧dy
>>> da.display(eV)
da = -du∧dv
```

Another 1-form:

```
sage: b = M.one_form(1+x*y, x^2, name='b')
sage: b.add_comp_by_continuation(eV, W, c_uv)
```

```
>>> from sage.all import *
>>> b = M.one_form(Integer(1)+x*y, x**Integer(2), name='b')
>>> b.add_comp_by_continuation(eV, W, c_uv)
```

Adding two 1-forms:

```
sage: s = a + b
sage: s.display(eU)
a+b = (x*y - y + 1) dx + x*(x + 1) dy
sage: s.display(eV)
a+b = (u**2/4 + u*v/4 + v/2 + 1/2) du + (-u*v/4 - u/2 - v**2/4 + 1/2) dv
```

```
>>> from sage.all import *
>>> s = a + b
>>> s.display(eU)
a+b = (x*y - y + 1) dx + x*(x + 1) dy
>>> s.display(eV)
a+b = (u**2/4 + u*v/4 + v/2 + 1/2) du + (-u*v/4 - u/2 - v**2/4 + 1/2) dv
```

The exterior product of two 1-forms:

```
sage: s = a.wedge(b)
sage: s.display(eU)
a∧b = x*(-2*x*y - 1) dx∧dy
sage: s.display(eV)
a∧b = (u**3/8 + u**2*v/8 - u*v**2/8 + u/4 - v**3/8 + v/4) du∧dv
```

```
>>> from sage.all import *
>>> s = a.wedge(b)
>>> s.display(eU)
a∧b = x*(-2*x*y - 1) dx∧dy
>>> s.display(eV)
a∧b = (u**3/8 + u**2*v/8 - u*v**2/8 + u/4 - v**3/8 + v/4) du∧dv
```

Multiplying a 1-form by a scalar field:

```
sage: f = M.scalar_field({c_xy: (x+y)^2, c_uv: u^2}, name='f')
sage: s = f*a
sage: s.display(eU)
f*a = y*(-x**2 - 2*x*y - y**2) dx + x*(x**2 + 2*x*y + y**2) dy
sage: s.display(eV)
f*a = u**2*v/2 du - u**3/2 dv
```

```
>>> from sage.all import *
>>> f = M.scalar_field({c_xy: (x+y)**Integer(2), c_uv: u**Integer(2)}, name='f')
>>> s = f*a
>>> s.display(eU)
f*a = y*(-x**2 - 2*x*y - y**2) dx + x*(x**2 + 2*x*y + y**2) dy
>>> s.display(eV)
f*a = u**2*v/2 du - u**3/2 dv
```

degree()

Return the degree of `self`.

OUTPUT: integer p such that the differential form is a p-form

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: a = M.diff_form(2); a
2-form on the 3-dimensional differentiable manifold M
sage: a.degree()
2
sage: b = M.diff_form(1); b
1-form on the 3-dimensional differentiable manifold M
sage: b.degree()
1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> a = M.diff_form(Integer(2)); a
2-form on the 3-dimensional differentiable manifold M
>>> a.degree()
2
>>> b = M.diff_form(Integer(1)); b
1-form on the 3-dimensional differentiable manifold M
>>> b.degree()
1
```

derivative()

Compute the exterior derivative of `self`.

OUTPUT:

- instance of *DiffForm* representing the exterior derivative of the differential form

EXAMPLES:

Exterior derivative of a 1-form on the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                intersection_name='W', restrictions1= x^2+y^2!=0,
....:                restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2) #␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2) #␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> e_xy = c_xy.frame(); e_uv = c_uv.frame()
```

The 1-form:

```
sage: a = M.one_form({e_xy: [-y^2, x^2]}, name='a')
sage: a.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
sage: a.display(e_xy)
a = -y^2 dx + x^2 dy
sage: a.display(e_uv)
a = -(2*u^3*v - u^2*v^2 + v^4)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 + 4*u^2*v^6 + v^
↪8) du
 + (u^4 - u^2*v^2 + 2*u*v^3)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 + 4*u^2*v^6 + v^8)␣
↪dv
```

```
>>> from sage.all import *
>>> a = M.one_form({e_xy: [-y**Integer(2), x**Integer(2)]}, name='a')
>>> a.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
>>> a.display(e_xy)
a = -y^2 dx + x^2 dy
>>> a.display(e_uv)
a = -(2*u^3*v - u^2*v^2 + v^4)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 + 4*u^2*v^6 + v^
↪8) du
 + (u^4 - u^2*v^2 + 2*u*v^3)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 + 4*u^2*v^6 + v^8)␣
↪dv
```

Its exterior derivative:

```
sage: da = a.exterior_derivative(); da
2-form da on the 2-dimensional differentiable manifold M
sage: da.display(e_xy)
da = (2*x + 2*y) dx∧dy
sage: da.display(e_uv)
da = -2*(u + v)/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6) du∧dv
```

```
>>> from sage.all import *
>>> da = a.exterior_derivative(); da
2-form da on the 2-dimensional differentiable manifold M
>>> da.display(e_xy)
da = (2*x + 2*y) dx∧dy
>>> da.display(e_uv)
da = -2*(u + v)/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6) du∧dv
```

The result is cached, i.e. is not recomputed unless `a` is changed:

```
sage: a.exterior_derivative() is da
True
```

```
>>> from sage.all import *
>>> a.exterior_derivative() is da
True
```

Instead of invoking the method `exterior_derivative()`, one may use the global function `diff`:

```
sage: diff(a) is a.exterior_derivative()
True
```

```
>>> from sage.all import *
>>> diff(a) is a.exterior_derivative()
True
```

Let us check Cartan's identity:

```
sage: v = M.vector_field({e_xy: [-y, x]}, name='v')
sage: v.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
sage: a.lie_der(v) == v.contract(diff(a)) + diff(a(v))  # long time
True
```

```
>>> from sage.all import *
>>> v = M.vector_field({e_xy: [-y, x]}, name='v')
>>> v.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
>>> a.lie_der(v) == v.contract(diff(a)) + diff(a(v))  # long time
True
```

exterior_derivative()

Compute the exterior derivative of `self`.

OUTPUT:

- instance of `DiffForm` representing the exterior derivative of the differential form

EXAMPLES:

Exterior derivative of a 1-form on the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                  intersection_name='W', restrictions1= x^2+y^2!=0,
....:                  restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                  intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                  restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> e_xy = c_xy.frame(); e_uv = c_uv.frame()
```

The 1-form:

```
sage: a = M.one_form({e_xy: [-y^2, x^2]}, name='a')
sage: a.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
sage: a.display(e_xy)
a = -y^2 dx + x^2 dy
sage: a.display(e_uv)
a = -(2*u^3*v - u^2*v^2 + v^4)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 + 4*u^2*v^6 + v^
↪8) du
 + (u^4 - u^2*v^2 + 2*u*v^3)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 + 4*u^2*v^6 + v^8)␣
↪dv
```

```
>>> from sage.all import *
>>> a = M.one_form({e_xy: [-y**Integer(2), x**Integer(2)]}, name='a')
>>> a.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
>>> a.display(e_xy)
a = -y^2 dx + x^2 dy
>>> a.display(e_uv)
a = -(2*u^3*v - u^2*v^2 + v^4)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 + 4*u^2*v^6 + v^
↪8) du
 + (u^4 - u^2*v^2 + 2*u*v^3)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 + 4*u^2*v^6 + v^8)␣
↪dv
```

Its exterior derivative:

```
sage: da = a.exterior_derivative(); da
2-form da on the 2-dimensional differentiable manifold M
sage: da.display(e_xy)
da = (2*x + 2*y) dx∧dy
sage: da.display(e_uv)
da = -2*(u + v)/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6) du∧dv
```

```
>>> from sage.all import *
>>> da = a.exterior_derivative(); da
2-form da on the 2-dimensional differentiable manifold M
>>> da.display(e_xy)
da = (2*x + 2*y) dx∧dy
>>> da.display(e_uv)
da = -2*(u + v)/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6) du∧dv
```

The result is cached, i.e. is not recomputed unless `a` is changed:

```
sage: a.exterior_derivative() is da
True
```

```
>>> from sage.all import *
>>> a.exterior_derivative() is da
True
```

Instead of invoking the method *exterior_derivative()*, one may use the global function `diff`:

```
sage: diff(a) is a.exterior_derivative()
True
```

```
>>> from sage.all import *
>>> diff(a) is a.exterior_derivative()
True
```

Let us check Cartan's identity:

```
sage: v = M.vector_field({e_xy: [-y, x]}, name='v')
sage: v.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
sage: a.lie_der(v) == v.contract(diff(a)) + diff(a(v))  # long time
True
```

```
>>> from sage.all import *
>>> v = M.vector_field({e_xy: [-y, x]}, name='v')
>>> v.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
>>> a.lie_der(v) == v.contract(diff(a)) + diff(a(v))  # long time
True
```

hodge_dual(*nondegenerate_tensor=None*, *minus_eigenvalues_convention=False*)

Compute the Hodge dual of the differential form with respect to some non-degenerate bilinear form (Riemannian metric or symplectic form).

If the differential form is a p-form A, its *Hodge dual* with respect to the non-degenerate form g is the $(n -$

p)-form $*A$ defined by

$$*A_{i_1\ldots i_{n-p}} = \frac{1}{p!} A^{k_1\ldots k_p} \epsilon_{k_1\ldots k_p\, i_1\ldots i_{n-p}}$$

where n is the manifold's dimension, ϵ is the volume n-form associated with g (see `volume_form()`) and the indices $k_1,\ldots,k_p$ are raised with g. If g is a pseudo-Riemannian metric, sometimes an additional multiplicative factor of $(-1)^s$ is introduced on the right-hand side, where s is the number of negative eigenvalues of g. This convention can be enforced by setting the option `minus_eigenvalues_convention`.

INPUT:

- `nondegenerate_tensor` – a non-degenerate bilinear form defined on the same manifold as the current differential form; must be an instance of `PseudoRiemannianMetric` or `SymplecticForm`. If none is provided, the ambient domain of `self` is supposed to be endowed with a default metric and this metric is then used.
- **`minus_eigenvalues_convention` – if $true$, a factor of $(-1)^s$ is**
  introduced with s being the number of negative eigenvalues of the `nondegenerate_tensor`.

OUTPUT:

- the $(n-p)$-form $*A$

EXAMPLES:

Hodge dual of a 1-form on the 2-sphere equipped with the standard metric: we first construct $\mathbb{S}^2$ and its metric g:

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart() # stereographic coord.␣
↪(North and South)
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V) # The complement of the two poles
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: g = M.metric('g')
sage: g[eU,1,1], g[eU,2,2] = 4/(1+x^2+y^2)^2, 4/(1+x^2+y^2)^2
sage: g[eV,1,1], g[eV,2,2] = 4/(1+u^2+v^2)^2, 4/(1+u^2+v^2)^2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', start_index=Integer(1))
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)# stereographic␣
↪coord. (North and South)
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                 intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                 restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
```

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```
>>> W = U.intersection(V) # The complement of the two poles
>>> eU = c_xy.frame() ; eV = c_uv.frame()
>>> g = M.metric('g')
>>> g[eU,Integer(1),Integer(1)], g[eU,Integer(2),Integer(2)] = Integer(4)/
↪(Integer(1)+x**Integer(2)+y**Integer(2))**Integer(2), Integer(4)/
↪(Integer(1)+x**Integer(2)+y**Integer(2))**Integer(2)
>>> g[eV,Integer(1),Integer(1)], g[eV,Integer(2),Integer(2)] = Integer(4)/
↪(Integer(1)+u**Integer(2)+v**Integer(2))**Integer(2), Integer(4)/
↪(Integer(1)+u**Integer(2)+v**Integer(2))**Integer(2)
```

We endow S^2 with the orientation defined by the stereographic frame from the North pole, i.e. `eU`; `eV` is then left-handed and in order to define an orientation on the whole manifold, we introduce a vector frame on `V` by swapping `eV`'s vectors:

```
sage: f = V.vector_frame('f', (eV[2], eV[1]))
sage: M.set_orientation([eU, f])
```

```
>>> from sage.all import *
>>> f = V.vector_frame('f', (eV[Integer(2)], eV[Integer(1)]))
>>> M.set_orientation([eU, f])
```

Then we construct the 1-form and take its Hodge dual w.r.t. g:

```
sage: a = M.one_form({eU: [-y, x]}, name='a')
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: a.display(eU)
a = -y dx + x dy
sage: a.display(eV)
a = -v/(u^4 + 2*u^2*v^2 + v^4) du + u/(u^4 + 2*u^2*v^2 + v^4) dv
sage: sa = a.hodge_dual(g); sa
1-form *a on the 2-dimensional differentiable manifold S^2
sage: sa.display(eU)
*a = -x dx - y dy
sage: sa.display(eV)
*a = u/(u^4 + 2*u^2*v^2 + v^4) du + v/(u^4 + 2*u^2*v^2 + v^4) dv
```

```
>>> from sage.all import *
>>> a = M.one_form({eU: [-y, x]}, name='a')
>>> a.add_comp_by_continuation(eV, W, c_uv)
>>> a.display(eU)
a = -y dx + x dy
>>> a.display(eV)
a = -v/(u^4 + 2*u^2*v^2 + v^4) du + u/(u^4 + 2*u^2*v^2 + v^4) dv
>>> sa = a.hodge_dual(g); sa
1-form *a on the 2-dimensional differentiable manifold S^2
>>> sa.display(eU)
*a = -x dx - y dy
>>> sa.display(eV)
*a = u/(u^4 + 2*u^2*v^2 + v^4) du + v/(u^4 + 2*u^2*v^2 + v^4) dv
```

Instead of calling the method *hodge_dual()* on the differential form, one can invoke the method *hodge_star()* of the metric:

```
sage: a.hodge_dual(g) == g.hodge_star(a)
True
```

```
>>> from sage.all import *
>>> a.hodge_dual(g) == g.hodge_star(a)
True
```

For a 1-form and a Riemannian metric in dimension 2, the Hodge dual applied twice is minus the identity:

```
sage: ssa = sa.hodge_dual(g); ssa
1-form **a on the 2-dimensional differentiable manifold S^2
sage: ssa == -a
True
```

```
>>> from sage.all import *
>>> ssa = sa.hodge_dual(g); ssa
1-form **a on the 2-dimensional differentiable manifold S^2
>>> ssa == -a
True
```

The Hodge dual of the metric volume 2-form is the constant scalar field 1 (considered as a 0-form):

```
sage: eps = g.volume_form(); eps
2-form eps_g on the 2-dimensional differentiable manifold S^2
sage: eps.display(eU)
eps_g = 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) dx∧dy
sage: eps.display(eV)
eps_g = -4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) du∧dv
sage: seps = eps.hodge_dual(g); seps
Scalar field *eps_g on the 2-dimensional differentiable manifold S^2
sage: seps.display()
*eps_g: S^2 → ℝ
on U: (x, y) ↦ 1
on V: (u, v) ↦ 1
```

```
>>> from sage.all import *
>>> eps = g.volume_form(); eps
2-form eps_g on the 2-dimensional differentiable manifold S^2
>>> eps.display(eU)
eps_g = 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) dx∧dy
>>> eps.display(eV)
eps_g = -4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) du∧dv
>>> seps = eps.hodge_dual(g); seps
Scalar field *eps_g on the 2-dimensional differentiable manifold S^2
>>> seps.display()
*eps_g: S^2 → ℝ
on U: (x, y) ↦ 1
on V: (u, v) ↦ 1
```

Hodge dual of a 1-form in the Euclidean space R^3:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
```

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```
sage: g = M.metric('g')  # the Euclidean metric
sage: g[1,1], g[2,2], g[3,3] = 1, 1, 1
sage: var('Ax Ay Az')
(Ax, Ay, Az)
sage: a = M.one_form(Ax, Ay, Az, name='A')
sage: sa = a.hodge_dual(g) ; sa
2-form *A on the 3-dimensional differentiable manifold M
sage: sa.display()
*A = Az dx∧dy - Ay dx∧dz + Ax dy∧dz
sage: ssa = sa.hodge_dual(g) ; ssa
1-form **A on the 3-dimensional differentiable manifold M
sage: ssa.display()
**A = Ax dx + Ay dy + Az dz
sage: ssa == a  # must hold for a Riemannian metric in dimension 3
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> g = M.metric('g')  # the Euclidean metric
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)], g[Integer(3),
↪Integer(3)] = Integer(1), Integer(1), Integer(1)
>>> var('Ax Ay Az')
(Ax, Ay, Az)
>>> a = M.one_form(Ax, Ay, Az, name='A')
>>> sa = a.hodge_dual(g) ; sa
2-form *A on the 3-dimensional differentiable manifold M
>>> sa.display()
*A = Az dx∧dy - Ay dx∧dz + Ax dy∧dz
>>> ssa = sa.hodge_dual(g) ; ssa
1-form **A on the 3-dimensional differentiable manifold M
>>> ssa.display()
**A = Ax dx + Ay dy + Az dz
>>> ssa == a  # must hold for a Riemannian metric in dimension 3
True
```

See the documentation of `hodge_star()` for more examples.

interior_product (*qvect*)

Interior product with a multivector field.

If `self` is a differential form A of degree p and B is a multivector field of degree $q \geq p$ on the same manifold, the interior product of A by B is the multivector field $\iota_A B$ of degree $q-p$ defined by

$$(\iota_A B)^{i_1 \dots i_{q-p}} = A_{k_1 \dots k_p} B^{k_1 \dots k_p i_1 \dots i_{q-p}}$$

Note

`A.interior_product(B)` yields the same result as `A.contract(0,..., p-1, B, 0,..., p-1)` (cf. `contract()`), but `interior_product` is more efficient, the alternating character of A being not used to reduce the computation in `contract()`

INPUT:

- `qvect` – multivector field B (instance of `MultivectorField`); the degree of B must be at least equal to the degree of `self`

OUTPUT:

- scalar field (case $p = q$) or `MultivectorField` (case $p < q$) representing the interior product $\iota_A B$, where A is `self`

See also

`interior_product()` for the interior product of a multivector field with a differential form

EXAMPLES:

Interior product of a 1-form with a 2-vector field on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1) # the sphere S^2
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: c_xy.<x,y> = U.chart() # stereographic coord. North
sage: c_uv.<u,v> = V.chart() # stereographic coord. South
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V) # The complement of the two poles
sage: e_xy = c_xy.frame() ; e_uv = c_uv.frame()
sage: a = M.one_form({e_xy: [y, x]}, name='a')
sage: a.add_comp_by_continuation(e_uv, W, c_uv)
sage: b = M.multivector_field(2, name='b')
sage: b[e_xy,1,2] = x*y
sage: b.add_comp_by_continuation(e_uv, W, c_uv)
sage: s = a.interior_product(b); s
Vector field i_a b on the 2-dimensional differentiable manifold S^2
sage: s.display(e_xy)
i_a b = -x^2*y ∂/∂x + x*y^2 ∂/∂y
sage: s.display(e_uv)
i_a b = (u^4*v - 3*u^2*v^3)/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6) ∂/∂u
 + (3*u^3*v^2 - u*v^4)/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6) ∂/∂v
sage: s == a.contract(b)
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', start_index=Integer(1)) # the sphere S^2
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coord. North
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coord. South
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
```

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```
...                 intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                 restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V) # The complement of the two poles
>>> e_xy = c_xy.frame() ; e_uv = c_uv.frame()
>>> a = M.one_form({e_xy: [y, x]}, name='a')
>>> a.add_comp_by_continuation(e_uv, W, c_uv)
>>> b = M.multivector_field(Integer(2), name='b')
>>> b[e_xy,Integer(1),Integer(2)] = x*y
>>> b.add_comp_by_continuation(e_uv, W, c_uv)
>>> s = a.interior_product(b); s
Vector field i_a b on the 2-dimensional differentiable manifold S^2
>>> s.display(e_xy)
i_a b = -x^2*y ∂/∂x + x*y^2 ∂/∂y
>>> s.display(e_uv)
i_a b = (u^4*v - 3*u^2*v^3)/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6) ∂/∂u
 + (3*u^3*v^2 - u*v^4)/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6) ∂/∂v
>>> s == a.contract(b)
True
```

Interior product of a 2-form with a 2-vector field:

```
sage: a = M.diff_form(2, name='a')
sage: a[e_xy,1,2] = 4/(x^2+y^2+1)^2   # the standard area 2-form
sage: a.add_comp_by_continuation(e_uv, W, c_uv)
sage: s = a.interior_product(b); s
Scalar field i_a b on the 2-dimensional differentiable manifold S^2
sage: s.display()
i_a b: S^2 → ℝ
on U: (x, y) ↦ 8*x*y/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1)
on V: (u, v) ↦ 8*u*v/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1)
```

```
>>> from sage.all import *
>>> a = M.diff_form(Integer(2), name='a')
>>> a[e_xy,Integer(1),Integer(2)] = Integer(4)/
↪(x**Integer(2)+y**Integer(2)+Integer(1))**Integer(2)   # the standard area␣
↪2-form
>>> a.add_comp_by_continuation(e_uv, W, c_uv)
>>> s = a.interior_product(b); s
Scalar field i_a b on the 2-dimensional differentiable manifold S^2
>>> s.display()
i_a b: S^2 → ℝ
on U: (x, y) ↦ 8*x*y/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1)
on V: (u, v) ↦ 8*u*v/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1)
```

Some checks:

```
sage: s == a.contract(0, 1, b, 0, 1)
True
sage: s.restrict(U) == 2 * a[[e_xy,1,2]] * b[[e_xy,1,2]]
True
```

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```
sage: s.restrict(V) == 2 * a[[e_uv,1,2]] * b[[e_uv,1,2]]
True
```

```
>>> from sage.all import *
>>> s == a.contract(Integer(0), Integer(1), b, Integer(0), Integer(1))
True
>>> s.restrict(U) == Integer(2) * a[[e_xy,Integer(1),Integer(2)]] * b[[e_xy,
↪Integer(1),Integer(2)]]
True
>>> s.restrict(V) == Integer(2) * a[[e_uv,Integer(1),Integer(2)]] * b[[e_uv,
↪Integer(1),Integer(2)]]
True
```

wedge(*other*)

Exterior product with another differential form.

INPUT:

- `other` – another differential form (on the same manifold)

OUTPUT:

- instance of *DiffForm* representing the exterior product `self ∧ other`

EXAMPLES:

Exterior product of two 1-forms on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1) # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart() # stereographic coord.␣
↪(North and South)
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V) # The complement of the two poles
sage: e_xy = c_xy.frame() ; e_uv = c_uv.frame()
sage: a = M.one_form({e_xy: [y, x]}, name='a')
sage: a.add_comp_by_continuation(e_uv, W, c_uv)
sage: b = M.one_form({e_xy: [x^2 + y^2, y]}, name='b')
sage: b.add_comp_by_continuation(e_uv, W, c_uv)
sage: c = a.wedge(b); c
2-form a∧b on the 2-dimensional differentiable manifold S^2
sage: c.display(e_xy)
a∧b = (-x^3 - (x - 1)*y^2) dx∧dy
sage: c.display(e_uv)
a∧b = -(v^2 - u)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 + 4*u^2*v^6 + v^8) du∧dv
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', start_index=Integer(1)) # the 2-
↪dimensional sphere S^2
>>> U = M.open_subset('U') ; V = M.open_subset('V')
```

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```
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)# stereographic␣
↪coord. (North and South)
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                    intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                    restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V) # The complement of the two poles
>>> e_xy = c_xy.frame() ; e_uv = c_uv.frame()
>>> a = M.one_form({e_xy: [y, x]}, name='a')
>>> a.add_comp_by_continuation(e_uv, W, c_uv)
>>> b = M.one_form({e_xy: [x**Integer(2) + y**Integer(2), y]}, name='b')
>>> b.add_comp_by_continuation(e_uv, W, c_uv)
>>> c = a.wedge(b); c
2-form a∧b on the 2-dimensional differentiable manifold S^2
>>> c.display(e_xy)
a∧b = (-x^3 - (x - 1)*y^2) dx∧dy
>>> c.display(e_uv)
a∧b = -(v^2 - u)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 + 4*u^2*v^6 + v^8) du∧dv
```

If one of the two operands is unnamed, the result is unnamed too:

```
sage: b1 = M.diff_form(1)  # no name set
sage: b1[e_xy,:] = x^2 + y^2, y
sage: b1.add_comp_by_continuation(e_uv, W, c_uv)
sage: c1 = a.wedge(b1); c1
2-form on the 2-dimensional differentiable manifold S^2
sage: c1.display(e_xy)
(-x^3 - (x - 1)*y^2) dx∧dy
```

```
>>> from sage.all import *
>>> b1 = M.diff_form(Integer(1))  # no name set
>>> b1[e_xy,:] = x**Integer(2) + y**Integer(2), y
>>> b1.add_comp_by_continuation(e_uv, W, c_uv)
>>> c1 = a.wedge(b1); c1
2-form on the 2-dimensional differentiable manifold S^2
>>> c1.display(e_xy)
(-x^3 - (x - 1)*y^2) dx∧dy
```

To give a name to the result, one shall use the method `set_name()`:

```
sage: c1.set_name('c');  c1
2-form c on the 2-dimensional differentiable manifold S^2
sage: c1.display(e_xy)
c = (-x^3 - (x - 1)*y^2) dx∧dy
```

```
>>> from sage.all import *
>>> c1.set_name('c');  c1
2-form c on the 2-dimensional differentiable manifold S^2
```

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```
>>> c1.display(e_xy)
c = (-x^3 - (x - 1)*y^2) dx∧dy
```

Wedging with scalar fields yields the multiplication from right:

```
sage: f = M.scalar_field(x, name='f')
sage: f.add_expr_by_continuation(c_uv, W)
sage: t = a.wedge(f)
sage: t.display()
f*a = x*y dx + x^2 dy
```

```
>>> from sage.all import *
>>> f = M.scalar_field(x, name='f')
>>> f.add_expr_by_continuation(c_uv, W)
>>> t = a.wedge(f)
>>> t.display()
f*a = x*y dx + x^2 dy
```

class sage.manifolds.differentiable.diff_form.**DiffFormParal**(*vector_field_module: VectorFieldModule*, *degree: int*, *name: str | None = None*, *latex_name: str | None = None*)

Bases: `FreeModuleAltForm`, `TensorFieldParal`, `DiffForm`

Differential form with values on a parallelizable manifold.

Given a differentiable manifold U, a differentiable map $\Phi : U \to M$ to a parallelizable manifold M and a positive integer p, a *differential form of degree* p (or p*-form*) *along* U *with values on* $M \supset \Phi(U)$ is a differentiable map

$$a : U \longrightarrow T^{(0,p)}M$$

($T^{(0,p)}M$ being the tensor bundle of type $(0, p)$ over M) such that

$$\forall x \in U, \quad a(x) \in \Lambda^p(T^*_{\Phi(x)}M),$$

where $T^*_{\Phi(x)}M$ is the dual of the tangent space to M at $\Phi(x)$ and Λ^p stands for the exterior power of degree p (cf. `ExtPowerDualFreeModule`). In other words, $a(x)$ is an alternating multilinear form of degree p of the tangent vector space $T_{\Phi(x)}M$.

The standard case of a differential form *on* a manifold M corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note

If M is not parallelizable, the class `DiffForm` must be used instead.

INPUT:

- `vector_field_module` – free module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on M via the map Φ
- `degree` – the degree of the differential form (i.e. its tensor rank)
- `name` – (default: `None`) name given to the differential form
- `latex_name` – (default: `None`) LaTeX symbol to denote the differential form; if none is provided, the LaTeX symbol is set to `name`

EXAMPLES:

A 2-form on a 4-dimensional manifold:

```
sage: M = Manifold(4, 'M')
sage: c_txyz.<t,x,y,z> = M.chart()
sage: a = M.diff_form(2, name='a') ; a
2-form a on the 4-dimensional differentiable manifold M
sage: a.parent()
Free module Omega^2(M) of 2-forms on the 4-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M')
>>> c_txyz = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = c_txyz._first_
↪ngens(4)
>>> a = M.diff_form(Integer(2), name='a') ; a
2-form a on the 4-dimensional differentiable manifold M
>>> a.parent()
Free module Omega^2(M) of 2-forms on the 4-dimensional differentiable
 manifold M
```

A differential form is a tensor field of purely covariant type:

```
sage: a.tensor_type()
(0, 2)
```

```
>>> from sage.all import *
>>> a.tensor_type()
(0, 2)
```

It is antisymmetric, its components being `CompFullyAntiSym`:

```
sage: a.symmetries()
no symmetry;  antisymmetry: (0, 1)
sage: a[0,1] = 2
sage: a[1,0]
-2
sage: a.comp()
Fully antisymmetric 2-indices components w.r.t. Coordinate frame (M, (∂/∂t,∂/∂x,∂/
↪∂y,∂/∂z))
sage: type(a.comp())
<class 'sage.tensor.modules.comp.CompFullyAntiSym'>
```

```
>>> from sage.all import *
>>> a.symmetries()
no symmetry;  antisymmetry: (0, 1)
>>> a[Integer(0),Integer(1)] = Integer(2)
>>> a[Integer(1),Integer(0)]
-2
>>> a.comp()
Fully antisymmetric 2-indices components w.r.t. Coordinate frame (M, (∂/∂t,∂/∂x,∂/
↪∂y,∂/∂z))
```

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```
>>> type(a.comp())
<class 'sage.tensor.modules.comp.CompFullyAntiSym'>
```

Setting a component with repeated indices to a nonzero value results in an error:

```
sage: a[1,1] = 3
Traceback (most recent call last):
...
ValueError: by antisymmetry, the component cannot have a nonzero value
 for the indices (1, 1)
sage: a[1,1] = 0  # OK, albeit useless
sage: a[1,2] = 3  # OK
```

```
>>> from sage.all import *
>>> a[Integer(1),Integer(1)] = Integer(3)
Traceback (most recent call last):
...
ValueError: by antisymmetry, the component cannot have a nonzero value
 for the indices (1, 1)
>>> a[Integer(1),Integer(1)] = Integer(0)  # OK, albeit useless
>>> a[Integer(1),Integer(2)] = Integer(3)  # OK
```

The expansion of a differential form with respect to a given coframe is displayed via the method `display()`:

```
sage: a.display() # expansion with respect to the default coframe (dt, dx, dy, dz)
a = 2 dt∧dx + 3 dx∧dy
sage: latex(a.display()) # output for the notebook
a = 2 \mathrm{d} t\wedge \mathrm{d} x
 + 3 \mathrm{d} x\wedge \mathrm{d} y
```

```
>>> from sage.all import *
>>> a.display() # expansion with respect to the default coframe (dt, dx, dy, dz)
a = 2 dt∧dx + 3 dx∧dy
>>> latex(a.display()) # output for the notebook
a = 2 \mathrm{d} t\wedge \mathrm{d} x
 + 3 \mathrm{d} x\wedge \mathrm{d} y
```

Differential forms can be added or subtracted:

```
sage: b = M.diff_form(2)
sage: b[0,1], b[0,2], b[0,3] = (1,2,3)
sage: s = a + b ; s
2-form on the 4-dimensional differentiable manifold M
sage: a[:], b[:], s[:]
(
[ 0  2  0  0]  [ 0  1  2  3]  [ 0  3  2  3]
[-2  0  3  0]  [-1  0  0  0]  [-3  0  3  0]
[ 0 -3  0  0]  [-2  0  0  0]  [-2 -3  0  0]
[ 0  0  0  0], [-3  0  0  0], [-3  0  0  0]
)
sage: s = a - b ; s
2-form on the 4-dimensional differentiable manifold M
```

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```
sage: s[:]
[ 0  1 -2 -3]
[-1  0  3  0]
[ 2 -3  0  0]
[ 3  0  0  0]
```

```
>>> from sage.all import *
>>> b = M.diff_form(Integer(2))
>>> b[Integer(0),Integer(1)], b[Integer(0),Integer(2)], b[Integer(0),Integer(3)]␣
↪= (Integer(1),Integer(2),Integer(3))
>>> s = a + b ; s
2-form on the 4-dimensional differentiable manifold M
>>> a[:], b[:], s[:]
(
[ 0  2  0  0]  [ 0  1  2  3]  [ 0  3  2  3]
[-2  0  3  0]  [-1  0  0  0]  [-3  0  3  0]
[ 0 -3  0  0]  [-2  0  0  0]  [-2 -3  0  0]
[ 0  0  0  0], [-3  0  0  0], [-3  0  0  0]
)
>>> s = a - b ; s
2-form on the 4-dimensional differentiable manifold M
>>> s[:]
[ 0  1 -2 -3]
[-1  0  3  0]
[ 2 -3  0  0]
[ 3  0  0  0]
```

An example of 3-form is the volume element on $\mathbf{R}^3$ in Cartesian coordinates:

```
sage: M = Manifold(3, 'R3', latex_name=r'\RR^3', start_index=1)
sage: c_cart.<x,y,z> = M.chart()
sage: eps = M.diff_form(3, name='epsilon', latex_name=r'\epsilon')
sage: eps[1,2,3] = 1  # the only independent component
sage: eps[:] # all the components are set from the previous line:
[[[0, 0, 0], [0, 0, 1], [0, -1, 0]], [[0, 0, -1], [0, 0, 0], [1, 0, 0]],
 [[0, 1, 0], [-1, 0, 0], [0, 0, 0]]]
sage: eps.display()
epsilon = dx∧dy∧dz
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'R3', latex_name=r'\RR^3', start_index=Integer(1))
>>> c_cart = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_cart._first_ngens(3)
>>> eps = M.diff_form(Integer(3), name='epsilon', latex_name=r'\epsilon')
>>> eps[Integer(1),Integer(2),Integer(3)] = Integer(1)  # the only independent␣
↪component
>>> eps[:] # all the components are set from the previous line:
[[[0, 0, 0], [0, 0, 1], [0, -1, 0]], [[0, 0, -1], [0, 0, 0], [1, 0, 0]],
 [[0, 1, 0], [-1, 0, 0], [0, 0, 0]]]
>>> eps.display()
epsilon = dx∧dy∧dz
```

Spherical components of the volume element from the tensorial change-of-frame formula:

```
sage: c_spher.<r,th,ph> = M.chart(r'r:[0,+oo) th:[0,pi]:\theta ph:[0,2*pi):\phi')
sage: spher_to_cart = c_spher.transition_map(c_cart,
....:                    [r*sin(th)*cos(ph), r*sin(th)*sin(ph), r*cos(th)])
sage: cart_to_spher = spher_to_cart.set_inverse(sqrt(x^2+y^2+z^2),
....:                                  atan2(sqrt(x^2+y^2),z), atan2(y, x))
Check of the inverse coordinate transformation:
  r == r  *passed*
  th == arctan2(r*sin(th), r*cos(th))  **failed**
  ph == arctan2(r*sin(ph)*sin(th), r*cos(ph)*sin(th))  **failed**
  x == x  *passed*
  y == y  *passed*
  z == z  *passed*
NB: a failed report can reflect a mere lack of simplification.
sage: eps.comp(c_spher.frame()) # computation of the components in the spherical␣
↪frame
Fully antisymmetric 3-indices components w.r.t. Coordinate frame
 (R3, (∂/∂r,∂/∂th,∂/∂ph))
sage: eps.comp(c_spher.frame())[1,2,3, c_spher]
r^2*sin(th)
sage: eps.display(c_spher.frame())
epsilon = sqrt(x^2 + y^2 + z^2)*sqrt(x^2 + y^2) dr∧dth∧dph
sage: eps.display(c_spher.frame(), c_spher)
epsilon = r^2*sin(th) dr∧dth∧dph
```

```
>>> from sage.all import *
>>> c_spher = M.chart(r'r:[0,+oo) th:[0,pi]:\theta ph:[0,2*pi):\phi', names=('r',
↪'th', 'ph',)); (r, th, ph,) = c_spher._first_ngens(3)
>>> spher_to_cart = c_spher.transition_map(c_cart,
...                    [r*sin(th)*cos(ph), r*sin(th)*sin(ph), r*cos(th)])
>>> cart_to_spher = spher_to_cart.set_
↪inverse(sqrt(x**Integer(2)+y**Integer(2)+z**Integer(2)),
...                                  atan2(sqrt(x**Integer(2)+y**Integer(2)),z),␣
↪atan2(y, x))
Check of the inverse coordinate transformation:
  r == r  *passed*
  th == arctan2(r*sin(th), r*cos(th))  **failed**
  ph == arctan2(r*sin(ph)*sin(th), r*cos(ph)*sin(th))  **failed**
  x == x  *passed*
  y == y  *passed*
  z == z  *passed*
NB: a failed report can reflect a mere lack of simplification.
>>> eps.comp(c_spher.frame()) # computation of the components in the spherical␣
↪frame
Fully antisymmetric 3-indices components w.r.t. Coordinate frame
 (R3, (∂/∂r,∂/∂th,∂/∂ph))
>>> eps.comp(c_spher.frame())[Integer(1),Integer(2),Integer(3), c_spher]
r^2*sin(th)
>>> eps.display(c_spher.frame())
epsilon = sqrt(x^2 + y^2 + z^2)*sqrt(x^2 + y^2) dr∧dth∧dph
>>> eps.display(c_spher.frame(), c_spher)
epsilon = r^2*sin(th) dr∧dth∧dph
```

As a shortcut of the above command, on can pass just the chart `c_spher` to `display`, the vector frame being then assumed to be the coordinate frame associated with the chart:

```
sage: eps.display(c_spher)
epsilon = r^2*sin(th) dr∧dth∧dph
```

```
>>> from sage.all import *
>>> eps.display(c_spher)
epsilon = r^2*sin(th) dr∧dth∧dph
```

The exterior product of two differential forms is performed via the method `wedge()`:

```
sage: a = M.one_form(x*y*z, -z*x, y*z, name='A')
sage: b = M.one_form(cos(z), sin(x), cos(y), name='B')
sage: ab = a.wedge(b) ; ab
2-form A∧B on the 3-dimensional differentiable manifold R3
sage: ab[:]
[                          0  x*y*z*sin(x) + x*z*cos(z)  x*y*z*cos(y) - y*z*cos(z)]
[-x*y*z*sin(x) - x*z*cos(z)                          0   -(x*cos(y) + y*sin(x))*z]
[-x*y*z*cos(y) + y*z*cos(z)    (x*cos(y) + y*sin(x))*z                          0]
sage: ab.display()
A∧B = (x*y*z*sin(x) + x*z*cos(z)) dx∧dy + (x*y*z*cos(y) - y*z*cos(z)) dx∧dz
 - (x*cos(y) + y*sin(x))*z dy∧dz
```

```
>>> from sage.all import *
>>> a = M.one_form(x*y*z, -z*x, y*z, name='A')
>>> b = M.one_form(cos(z), sin(x), cos(y), name='B')
>>> ab = a.wedge(b) ; ab
2-form A∧B on the 3-dimensional differentiable manifold R3
>>> ab[:]
[                          0  x*y*z*sin(x) + x*z*cos(z)  x*y*z*cos(y) - y*z*cos(z)]
[-x*y*z*sin(x) - x*z*cos(z)                          0   -(x*cos(y) + y*sin(x))*z]
[-x*y*z*cos(y) + y*z*cos(z)    (x*cos(y) + y*sin(x))*z                          0]
>>> ab.display()
A∧B = (x*y*z*sin(x) + x*z*cos(z)) dx∧dy + (x*y*z*cos(y) - y*z*cos(z)) dx∧dz
 - (x*cos(y) + y*sin(x))*z dy∧dz
```

Let us check the formula relating the exterior product to the tensor product for 1-forms:

```
sage: a.wedge(b) == a*b - b*a
True
```

```
>>> from sage.all import *
>>> a.wedge(b) == a*b - b*a
True
```

The tensor product of a 1-form and a 2-form is not a 3-form but a tensor field of type $(0, 3)$ with less symmetries:

```
sage: c = a*ab ; c
Tensor field A⊗(A∧B) of type (0,3) on the 3-dimensional differentiable
 manifold R3
sage: c.symmetries()  # the antisymmetry is only w.r.t. the last 2 arguments:
no symmetry;  antisymmetry: (1, 2)
sage: d = ab*a ; d
```

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```
Tensor field (A∧B)⊗A of type (0,3) on the 3-dimensional differentiable
 manifold R3
sage: d.symmetries()  # the antisymmetry is only w.r.t. the first 2 arguments:
no symmetry;  antisymmetry: (0, 1)
```

```
>>> from sage.all import *
>>> c = a*ab ; c
Tensor field A⊗(A∧B) of type (0,3) on the 3-dimensional differentiable
 manifold R3
>>> c.symmetries()  # the antisymmetry is only w.r.t. the last 2 arguments:
no symmetry;  antisymmetry: (1, 2)
>>> d = ab*a ; d
Tensor field (A∧B)⊗A of type (0,3) on the 3-dimensional differentiable
 manifold R3
>>> d.symmetries()  # the antisymmetry is only w.r.t. the first 2 arguments:
no symmetry;  antisymmetry: (0, 1)
```

The exterior derivative of a differential form is obtained by means of the method `exterior_derivative()`:

```
sage: da = a.exterior_derivative() ; da
2-form dA on the 3-dimensional differentiable manifold R3
sage: da.display()
dA = -(x + 1)*z dx∧dy - x*y dx∧dz + (x + z) dy∧dz
sage: db = b.exterior_derivative() ; db
2-form dB on the 3-dimensional differentiable manifold R3
sage: db.display()
dB = cos(x) dx∧dy + sin(z) dx∧dz - sin(y) dy∧dz
sage: dab = ab.exterior_derivative() ; dab
3-form d(A∧B) on the 3-dimensional differentiable manifold R3
```

```
>>> from sage.all import *
>>> da = a.exterior_derivative() ; da
2-form dA on the 3-dimensional differentiable manifold R3
>>> da.display()
dA = -(x + 1)*z dx∧dy - x*y dx∧dz + (x + z) dy∧dz
>>> db = b.exterior_derivative() ; db
2-form dB on the 3-dimensional differentiable manifold R3
>>> db.display()
dB = cos(x) dx∧dy + sin(z) dx∧dz - sin(y) dy∧dz
>>> dab = ab.exterior_derivative() ; dab
3-form d(A∧B) on the 3-dimensional differentiable manifold R3
```

or by applying the function `diff` to the differential form:

```
sage: diff(a) is a.exterior_derivative()
True
```

```
>>> from sage.all import *
>>> diff(a) is a.exterior_derivative()
True
```

As a 3-form over a 3-dimensional manifold, `d(A∧B)` is necessarily proportional to the volume 3-form:

```
sage: dab == dab[[1,2,3]]/eps[[1,2,3]]*eps
True
```

```
>>> from sage.all import *
>>> dab == dab[[Integer(1),Integer(2),Integer(3)]]/eps[[Integer(1),Integer(2),
↪Integer(3)]]*eps
True
```

We may also check that the classical anti-derivation formula is fulfilled:

```
sage: dab == da.wedge(b) - a.wedge(db)
True
```

```
>>> from sage.all import *
>>> dab == da.wedge(b) - a.wedge(db)
True
```

The Lie derivative of a 2-form is a 2-form:

```
sage: v = M.vector_field(y*z, -x*z, x*y, name='v')
sage: ab.lie_der(v)  # long time
2-form on the 3-dimensional differentiable manifold R3
```

```
>>> from sage.all import *
>>> v = M.vector_field(y*z, -x*z, x*y, name='v')
>>> ab.lie_der(v)  # long time
2-form on the 3-dimensional differentiable manifold R3
```

Let us check Cartan formula, which expresses the Lie derivative in terms of exterior derivatives:

```
sage: ab.lie_der(v) == (v.contract(ab.exterior_derivative())  # long time
....:                   + v.contract(ab).exterior_derivative())
True
```

```
>>> from sage.all import *
>>> ab.lie_der(v) == (v.contract(ab.exterior_derivative())  # long time
...                   + v.contract(ab).exterior_derivative())
True
```

A 1-form on a $\mathbf{R}^3$:

```
sage: om = M.one_form(name='omega', latex_name=r'\omega'); om
1-form omega on the 3-dimensional differentiable manifold R3
```

```
>>> from sage.all import *
>>> om = M.one_form(name='omega', latex_name=r'\omega'); om
1-form omega on the 3-dimensional differentiable manifold R3
```

A 1-form is of course a differential form:

```
sage: isinstance(om, sage.manifolds.differentiable.diff_form.DiffFormParal)
True
```

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```
sage: om.parent()
Free module Omega^1(R3) of 1-forms on the 3-dimensional differentiable
 manifold R3
sage: om.tensor_type()
(0, 1)
```

```
>>> from sage.all import *
>>> isinstance(om, sage.manifolds.differentiable.diff_form.DiffFormParal)
True
>>> om.parent()
Free module Omega^1(R3) of 1-forms on the 3-dimensional differentiable
 manifold R3
>>> om.tensor_type()
(0, 1)
```

Setting the components with respect to the manifold's default frame:

```
sage: om[:] = (2*z, x, x-y)
sage: om[:]
[2*z, x, x - y]
sage: om.display()
omega = 2*z dx + x dy + (x - y) dz
```

```
>>> from sage.all import *
>>> om[:] = (Integer(2)*z, x, x-y)
>>> om[:]
[2*z, x, x - y]
>>> om.display()
omega = 2*z dx + x dy + (x - y) dz
```

A 1-form acts on vector fields:

```
sage: v = M.vector_field(x, 2*y, 3*z, name='V')
sage: om(v)
Scalar field omega(V) on the 3-dimensional differentiable manifold R3
sage: om(v).display()
omega(V): R3 → ℝ
   (x, y, z) ↦ 2*x*y + (5*x - 3*y)*z
   (r, th, ph) ↦ 2*r^2*cos(ph)*sin(ph)*sin(th)^2 + r^2*(5*cos(ph)
                 - 3*sin(ph))*cos(th)*sin(th)
sage: latex(om(v))
\omega\left(V\right)
```

```
>>> from sage.all import *
>>> v = M.vector_field(x, Integer(2)*y, Integer(3)*z, name='V')
>>> om(v)
Scalar field omega(V) on the 3-dimensional differentiable manifold R3
>>> om(v).display()
omega(V): R3 → ℝ
   (x, y, z) ↦ 2*x*y + (5*x - 3*y)*z
   (r, th, ph) ↦ 2*r^2*cos(ph)*sin(ph)*sin(th)^2 + r^2*(5*cos(ph)
                 - 3*sin(ph))*cos(th)*sin(th)
```

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```
>>> latex(om(v))
\omega\left(V\right)
```

The tensor product of two 1-forms is a tensor field of type $(0, 2)$:

```
sage: a = M.one_form(1, 2, 3, name='A')
sage: b = M.one_form(6, 5, 4, name='B')
sage: c = a*b ; c
Tensor field A⊗B of type (0,2) on the 3-dimensional differentiable
 manifold R3
sage: c[:]
[ 6  5  4]
[12 10  8]
[18 15 12]
sage: c.symmetries()    # c has no symmetries:
no symmetry;  no antisymmetry
```

```
>>> from sage.all import *
>>> a = M.one_form(Integer(1), Integer(2), Integer(3), name='A')
>>> b = M.one_form(Integer(6), Integer(5), Integer(4), name='B')
>>> c = a*b ; c
Tensor field A⊗B of type (0,2) on the 3-dimensional differentiable
 manifold R3
>>> c[:]
[ 6  5  4]
[12 10  8]
[18 15 12]
>>> c.symmetries()    # c has no symmetries:
no symmetry;  no antisymmetry
```

derivative()

Compute the exterior derivative of `self`.

OUTPUT:

- a *DiffFormParal* representing the exterior derivative of the differential form

EXAMPLES:

Exterior derivative of a 1-form on a 4-dimensional manifold:

```
sage: M = Manifold(4, 'M')
sage: c_txyz.<t,x,y,z> = M.chart()
sage: a = M.one_form(t*x*y*z, z*y**2, x*z**2, x**2 + y**2, name='A')
sage: da = a.exterior_derivative() ; da
2-form dA on the 4-dimensional differentiable manifold M
sage: da.display()
dA = -t*y*z dt∧dx - t*x*z dt∧dy - t*x*y dt∧dz
 + (-2*y*z + z^2) dx∧dy + (-y^2 + 2*x) dx∧dz
 + (-2*x*z + 2*y) dy∧dz
sage: latex(da)
\mathrm{d}A
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M')
>>> c_txyz = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = c_txyz._
↪first_ngens(4)
>>> a = M.one_form(t*x*y*z, z*y**Integer(2), x*z**Integer(2), x**Integer(2) +␣
↪y**Integer(2), name='A')
>>> da = a.exterior_derivative() ; da
2-form dA on the 4-dimensional differentiable manifold M
>>> da.display()
dA = -t*y*z dt∧dx - t*x*z dt∧dy - t*x*y dt∧dz
 + (-2*y*z + z^2) dx∧dy + (-y^2 + 2*x) dx∧dz
 + (-2*x*z + 2*y) dy∧dz
>>> latex(da)
\mathrm{d}A
```

The result is cached, i.e. is not recomputed unless `a` is changed:

```
sage: a.exterior_derivative() is da
True
```

```
>>> from sage.all import *
>>> a.exterior_derivative() is da
True
```

Instead of invoking the method *exterior_derivative()*, one may use the global function `diff`:

```
sage: diff(a) is a.exterior_derivative()
True
```

```
>>> from sage.all import *
>>> diff(a) is a.exterior_derivative()
True
```

The exterior derivative is nilpotent:

```
sage: dda = da.exterior_derivative() ; dda
3-form ddA on the 4-dimensional differentiable manifold M
sage: dda.display()
ddA = 0
sage: dda == 0
True
```

```
>>> from sage.all import *
>>> dda = da.exterior_derivative() ; dda
3-form ddA on the 4-dimensional differentiable manifold M
>>> dda.display()
ddA = 0
>>> dda == Integer(0)
True
```

Let us check Cartan's identity:

```
sage: v = M.vector_field(-y, x, t, z, name='v')
sage: a.lie_der(v) == v.contract(diff(a)) + diff(a(v)) # long time
True
```

```
>>> from sage.all import *
>>> v = M.vector_field(-y, x, t, z, name='v')
>>> a.lie_der(v) == v.contract(diff(a)) + diff(a(v)) # long time
True
```

exterior_derivative()

Compute the exterior derivative of `self`.

OUTPUT:

- a *DiffFormParal* representing the exterior derivative of the differential form

EXAMPLES:

Exterior derivative of a 1-form on a 4-dimensional manifold:

```
sage: M = Manifold(4, 'M')
sage: c_txyz.<t,x,y,z> = M.chart()
sage: a = M.one_form(t*x*y*z, z*y**2, x*z**2, x**2 + y**2, name='A')
sage: da = a.exterior_derivative() ; da
2-form dA on the 4-dimensional differentiable manifold M
sage: da.display()
dA = -t*y*z dt∧dx - t*x*z dt∧dy - t*x*y dt∧dz
 + (-2*y*z + z^2) dx∧dy + (-y^2 + 2*x) dx∧dz
 + (-2*x*z + 2*y) dy∧dz
sage: latex(da)
\mathrm{d}A
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M')
>>> c_txyz = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = c_txyz._
↪first_ngens(4)
>>> a = M.one_form(t*x*y*z, z*y**Integer(2), x*z**Integer(2), x**Integer(2) +␣
↪y**Integer(2), name='A')
>>> da = a.exterior_derivative() ; da
2-form dA on the 4-dimensional differentiable manifold M
>>> da.display()
dA = -t*y*z dt∧dx - t*x*z dt∧dy - t*x*y dt∧dz
 + (-2*y*z + z^2) dx∧dy + (-y^2 + 2*x) dx∧dz
 + (-2*x*z + 2*y) dy∧dz
>>> latex(da)
\mathrm{d}A
```

The result is cached, i.e. is not recomputed unless `a` is changed:

```
sage: a.exterior_derivative() is da
True
```

```
>>> from sage.all import *
>>> a.exterior_derivative() is da
```

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```
True
```

Instead of invoking the method `exterior_derivative()`, one may use the global function `diff`:

```
sage: diff(a) is a.exterior_derivative()
True
```

```
>>> from sage.all import *
>>> diff(a) is a.exterior_derivative()
True
```

The exterior derivative is nilpotent:

```
sage: dda = da.exterior_derivative() ; dda
3-form ddA on the 4-dimensional differentiable manifold M
sage: dda.display()
ddA = 0
sage: dda == 0
True
```

```
>>> from sage.all import *
>>> dda = da.exterior_derivative() ; dda
3-form ddA on the 4-dimensional differentiable manifold M
>>> dda.display()
ddA = 0
>>> dda == Integer(0)
True
```

Let us check Cartan's identity:

```
sage: v = M.vector_field(-y, x, t, z, name='v')
sage: a.lie_der(v) == v.contract(diff(a)) + diff(a(v)) # long time
True
```

```
>>> from sage.all import *
>>> v = M.vector_field(-y, x, t, z, name='v')
>>> a.lie_der(v) == v.contract(diff(a)) + diff(a(v)) # long time
True
```

interior_product(*qvect*)

Interior product with a multivector field.

If `self` is a differential form A of degree p and B is a multivector field of degree $q \geq p$ on the same manifold, the interior product of A by B is the multivector field $\iota_A B$ of degree $q - p$ defined by

$$(\iota_A B)^{i_1 \ldots i_{q-p}} = A_{k_1 \ldots k_p} B^{k_1 \ldots k_p i_1 \ldots i_{q-p}}$$

Note

`A.interior_product(B)` yields the same result as `A.contract(0,..., p-1, B, 0,..., p-1)` (cf. `contract()`), but `interior_product` is more efficient, the alternating character of A being not used to reduce the computation in `contract()`

INPUT:

- `qvect` – multivector field B (instance of `MultivectorFieldParal`); the degree of B must be at least equal to the degree of `self`

OUTPUT:

- scalar field (case $p = q$) or `MultivectorFieldParal` (case $p < q$) representing the interior product $\iota_A B$, where A is `self`

See also

`interior_product()` for the interior product of a multivector field with a differential form

EXAMPLES:

Interior product of a 1-form with a 2-vector field on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: a = M.one_form(2, 1+x, y*z, name='a')
sage: b = M.multivector_field(2, name='b')
sage: b[1,2], b[1,3], b[2,3] = y^2, z+x, -z^2
sage: s = a.interior_product(b); s
Vector field i_a b on the 3-dimensional differentiable
 manifold M
sage: s.display()
i_a b = (-(x + 1)*y^2 - x*y*z - y*z^2) ∂/∂x
 + (y*z^3 + 2*y^2) ∂/∂y + (-(x + 1)*z^2 + 2*x + 2*z) ∂/∂z
sage: s == a.contract(b)
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> a = M.one_form(Integer(2), Integer(1)+x, y*z, name='a')
>>> b = M.multivector_field(Integer(2), name='b')
>>> b[Integer(1),Integer(2)], b[Integer(1),Integer(3)], b[Integer(2),
↪Integer(3)] = y**Integer(2), z+x, -z**Integer(2)
>>> s = a.interior_product(b); s
Vector field i_a b on the 3-dimensional differentiable
 manifold M
>>> s.display()
i_a b = (-(x + 1)*y^2 - x*y*z - y*z^2) ∂/∂x
 + (y*z^3 + 2*y^2) ∂/∂y + (-(x + 1)*z^2 + 2*x + 2*z) ∂/∂z
>>> s == a.contract(b)
True
```

Interior product of a 2-form with a 2-vector field:

```
sage: a = M.diff_form(2, name='a')
sage: a[1,2], a[1,3], a[2,3] = x*y, -3, z
sage: s = a.interior_product(b); s
Scalar field i_a b on the 3-dimensional differentiable manifold M
```

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```
sage: s.display()
i_a b: M → ℝ
   (x, y, z) ↦ 2*x*y^3 - 2*z^3 - 6*x - 6*z
sage: s == a.contract(0,1,b,0,1)
True
```

```
>>> from sage.all import *
>>> a = M.diff_form(Integer(2), name='a')
>>> a[Integer(1),Integer(2)], a[Integer(1),Integer(3)], a[Integer(2),
↪Integer(3)] = x*y, -Integer(3), z
>>> s = a.interior_product(b); s
Scalar field i_a b on the 3-dimensional differentiable manifold M
>>> s.display()
i_a b: M → ℝ
   (x, y, z) ↦ 2*x*y^3 - 2*z^3 - 6*x - 6*z
>>> s == a.contract(Integer(0),Integer(1),b,Integer(0),Integer(1))
True
```

wedge (*other*)

Exterior product of `self` with another differential form.

INPUT:

- `other` – another differential form

OUTPUT:

- instance of *DiffFormParal* representing the exterior product `self ∧ other`

EXAMPLES:

Exterior product of a 1-form and a 2-form on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: a = M.one_form(2, 1+x, y*z, name='a')
sage: b = M.diff_form(2, name='b')
sage: b[1,2], b[1,3], b[2,3] = y^2, z+x, z^2
sage: a.display()
a = 2 dx + (x + 1) dy + y*z dz
sage: b.display()
b = y^2 dx∧dy + (x + z) dx∧dz + z^2 dy∧dz
sage: s = a.wedge(b); s
3-form a∧b on the 3-dimensional differentiable manifold M
sage: s.display()
a∧b = (-x^2 + (y^3 - x - 1)*z + 2*z^2 - x) dx∧dy∧dz
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> a = M.one_form(Integer(2), Integer(1)+x, y*z, name='a')
>>> b = M.diff_form(Integer(2), name='b')
>>> b[Integer(1),Integer(2)], b[Integer(1),Integer(3)], b[Integer(2),
↪Integer(3)] = y**Integer(2), z+x, z**Integer(2)
```

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```
>>> a.display()
a = 2 dx + (x + 1) dy + y*z dz
>>> b.display()
b = y^2 dx∧dy + (x + z) dx∧dz + z^2 dy∧dz
>>> s = a.wedge(b); s
3-form a∧b on the 3-dimensional differentiable manifold M
>>> s.display()
a∧b = (-x^2 + (y^3 - x - 1)*z + 2*z^2 - x) dx∧dy∧dz
```

Check:

```
sage: s[1,2,3] == a[1]*b[2,3] + a[2]*b[3,1] + a[3]*b[1,2]
True
```

```
>>> from sage.all import *
>>> s[Integer(1),Integer(2),Integer(3)] == a[Integer(1)]*b[Integer(2),
↪Integer(3)] + a[Integer(2)]*b[Integer(3),Integer(1)] +␣
↪a[Integer(3)]*b[Integer(1),Integer(2)]
True
```

Wedging with scalar fields yields the multiplication from right:

```
sage: f = M.scalar_field(x, name='f')
sage: t = a.wedge(f)
sage: t.display()
f*a = 2*x dx + (x^2 + x) dy + x*y*z dz
```

```
>>> from sage.all import *
>>> f = M.scalar_field(x, name='f')
>>> t = a.wedge(f)
>>> t.display()
f*a = 2*x dx + (x^2 + x) dy + x*y*z dz
```

2.10 Mixed Differential Forms

2.10.1 Graded Algebra of Mixed Differential Forms

Let M and N be differentiable manifolds and $\varphi : M \to N$ a differentiable map. The space of *mixed differential forms along* φ, denoted by $\Omega^*(M,\varphi)$, is given by the direct sum $\bigoplus_{j=0}^{n} \Omega^j(M,\varphi)$ of differential form modules, where $n = \dim(N)$. With the wedge product, $\Omega^*(M,\varphi)$ inherits the structure of a graded algebra. See `MixedFormAlgebra` for details.

This algebra is endowed with a natural chain complex structure induced by the exterior derivative. The corresponding homology is called *de Rham cohomology*. See `DeRhamCohomologyRing` for details.

AUTHORS:

- Michael Jung (2019) : initial version

class sage.manifolds.differentiable.mixed_form_algebra.**MixedFormAlgebra**(*vector_field_module*)

Bases: `Parent`, `UniqueRepresentation`

An instance of this class represents the graded algebra of mixed forms. That is, if $\varphi : M \to N$ is a differentiable map between two differentiable manifolds M and N, the *graded algebra of mixed forms* $\Omega^*(M,\varphi)$ *along* φ is defined via the direct sum $\bigoplus_{j=0}^{n} \Omega^j(M,\varphi)$ consisting of differential form modules (cf. `DiffFormModule`), where n is the dimension of N. Hence, $\Omega^*(M,\varphi)$ is a module over $C^k(M)$ and a vector space over **R** or **C**. Furthermore notice, that

$$\Omega^*(M,\varphi) \cong C^k\left(\bigoplus_{j=0}^{n} \Lambda^j(\varphi^*T^*N)\right),$$

where C^k denotes the global section functor for differentiable sections of order k here.

The wedge product induces a multiplication on $\Omega^*(M,\varphi)$ and gives it the structure of a graded algebra since

$$\Omega^k(M,\varphi) \wedge \Omega^l(M,\varphi) \subset \Omega^{k+l}(M,\varphi).$$

Moreover, $\Omega^*(M,\varphi)$ inherits the structure of a chain complex, called *de Rham complex*, with the exterior derivative as boundary map, that is

$$0 \rightarrow \Omega^0(M,\varphi) \xrightarrow{\mathrm{d}_0} \Omega^1(M,\varphi) \xrightarrow{\mathrm{d}_1} \dots \xrightarrow{\mathrm{d}_{n-1}} \Omega^n(M,\varphi) \xrightarrow{\mathrm{d}_n} 0.$$

The induced cohomology is called *de Rham cohomology*, see `cohomology()` or `DeRhamCohomologyRing` respectively.

INPUT:

- `vector_field_module` – module $\mathfrak{X}(M,\varphi)$ of vector fields along M associated with the map $\varphi : M \to N$

EXAMPLES:

Graded algebra of mixed forms on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: Omega = M.mixed_form_algebra(); Omega
Graded algebra Omega^*(M) of mixed differential forms on the
 3-dimensional differentiable manifold M
sage: Omega.category()
Join of Category of graded algebras over Symbolic Ring and Category of
 chain complexes over Symbolic Ring
sage: Omega.base_ring()
Symbolic Ring
sage: Omega.vector_field_module()
Free module X(M) of vector fields on the 3-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> Omega = M.mixed_form_algebra(); Omega
Graded algebra Omega^*(M) of mixed differential forms on the
 3-dimensional differentiable manifold M
>>> Omega.category()
Join of Category of graded algebras over Symbolic Ring and Category of
 chain complexes over Symbolic Ring
>>> Omega.base_ring()
```

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```
Symbolic Ring
>>> Omega.vector_field_module()
Free module X(M) of vector fields on the 3-dimensional differentiable
 manifold M
```

Elements can be created from scratch:

```
sage: A = Omega(0); A
Mixed differential form zero on the 3-dimensional differentiable
 manifold M
sage: A is Omega.zero()
True
sage: B = Omega(1); B
Mixed differential form one on the 3-dimensional differentiable
 manifold M
sage: B is Omega.one()
True
sage: C = Omega([2,0,0,0]); C
Mixed differential form on the 3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> A = Omega(Integer(0)); A
Mixed differential form zero on the 3-dimensional differentiable
 manifold M
>>> A is Omega.zero()
True
>>> B = Omega(Integer(1)); B
Mixed differential form one on the 3-dimensional differentiable
 manifold M
>>> B is Omega.one()
True
>>> C = Omega([Integer(2),Integer(0),Integer(0),Integer(0)]); C
Mixed differential form on the 3-dimensional differentiable manifold M
```

There are some important coercions implemented:

```
sage: Omega0 = M.scalar_field_algebra(); Omega0
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
sage: Omega.has_coerce_map_from(Omega0)
True
sage: Omega2 = M.diff_form_module(2); Omega2
Free module Omega^2(M) of 2-forms on the 3-dimensional differentiable
 manifold M
sage: Omega.has_coerce_map_from(Omega2)
True
```

```
>>> from sage.all import *
>>> Omega0 = M.scalar_field_algebra(); Omega0
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
>>> Omega.has_coerce_map_from(Omega0)
```

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```
True
>>> Omega2 = M.diff_form_module(Integer(2)); Omega2
Free module Omega^2(M) of 2-forms on the 3-dimensional differentiable
 manifold M
>>> Omega.has_coerce_map_from(Omega2)
True
```

Restrictions induce coercions as well:

```
sage: U = M.open_subset('U'); U
Open subset U of the 3-dimensional differentiable manifold M
sage: OmegaU = U.mixed_form_algebra(); OmegaU
Graded algebra Omega^*(U) of mixed differential forms on the Open
 subset U of the 3-dimensional differentiable manifold M
sage: OmegaU.has_coerce_map_from(Omega)
True
```

```
>>> from sage.all import *
>>> U = M.open_subset('U'); U
Open subset U of the 3-dimensional differentiable manifold M
>>> OmegaU = U.mixed_form_algebra(); OmegaU
Graded algebra Omega^*(U) of mixed differential forms on the Open
 subset U of the 3-dimensional differentiable manifold M
>>> OmegaU.has_coerce_map_from(Omega)
True
```

Element

alias of `MixedForm`

cohomology(**args*, ***kwargs*)

Return the de Rham cohomology of the de Rham complex `self`.

The k-th de Rham cohomology is given by

$$H^k_{\mathrm{dR}}(M, \varphi) = \mathrm{ker}(\mathrm{d}_k)/\mathrm{im}(\mathrm{d}_{k-1})\ .$$

The corresponding ring is given by

$$H^*_{\mathrm{dR}}(M, \varphi) = \bigoplus_{k=0}^n H^k_{\mathrm{dR}}(M, \varphi),$$

endowed with the cup product as multiplication induced by the wedge product.

See also

See `DeRhamCohomologyRing` for details.

EXAMPLES:

```
sage: M = Manifold(3, 'M', latex_name=r'\mathcal{M}')
sage: A = M.mixed_form_algebra()
sage: A.cohomology()
```

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```
De Rham cohomology ring on the 3-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', latex_name=r'\mathcal{M}')
>>> A = M.mixed_form_algebra()
>>> A.cohomology()
De Rham cohomology ring on the 3-dimensional differentiable
 manifold M
```

differential (*degree=None*)

Return the differential of the de Rham complex `self` given by the exterior derivative.

INPUT:

- `degree` – (default: `None`) degree of the differential operator; if none is provided, the differential operator on `self` is returned.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: C = M.de_rham_complex()
sage: d = C.differential(); d
Generic endomorphism of Graded algebra Omega^*(M) of mixed
 differential forms on the 2-dimensional differentiable manifold M
sage: d0 = C.differential(0); d0
Generic morphism:
  From: Algebra of differentiable scalar fields on the
   2-dimensional differentiable manifold M
  To:   Free module Omega^1(M) of 1-forms on the 2-dimensional
   differentiable manifold M
sage: f = M.scalar_field(x, name='f'); f.display()
f: M → ℝ
   (x, y) ↦ x
sage: d0(f).display()
df = dx
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> C = M.de_rham_complex()
>>> d = C.differential(); d
Generic endomorphism of Graded algebra Omega^*(M) of mixed
 differential forms on the 2-dimensional differentiable manifold M
>>> d0 = C.differential(Integer(0)); d0
Generic morphism:
  From: Algebra of differentiable scalar fields on the
   2-dimensional differentiable manifold M
  To:   Free module Omega^1(M) of 1-forms on the 2-dimensional
   differentiable manifold M
>>> f = M.scalar_field(x, name='f'); f.display()
f: M → ℝ
```

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```
    (x, y) ↦ x
>>> d0(f).display()
df = dx
```

homology(**args*, ***kwargs*)

Return the de Rham cohomology of the de Rham complex `self`.

The k-th de Rham cohomology is given by

$$H^k_{\mathrm{dR}}(M, \varphi) = \ker(\mathrm{d}_k)/\mathrm{im}(\mathrm{d}_{k-1}) \ .$$

The corresponding ring is given by

$$H^*_{\mathrm{dR}}(M, \varphi) = \bigoplus_{k=0}^{n} H^k_{\mathrm{dR}}(M, \varphi),$$

endowed with the cup product as multiplication induced by the wedge product.

See also

See *DeRhamCohomologyRing* for details.

EXAMPLES:

```
sage: M = Manifold(3, 'M', latex_name=r'\mathcal{M}')
sage: A = M.mixed_form_algebra()
sage: A.cohomology()
De Rham cohomology ring on the 3-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', latex_name=r'\mathcal{M}')
>>> A = M.mixed_form_algebra()
>>> A.cohomology()
De Rham cohomology ring on the 3-dimensional differentiable
 manifold M
```

irange(*start=None*)

Single index generator.

INPUT:

- `start` – (default: `None`) initial value i_0 of the index between 0 and n, where n is the manifold's dimension; if none is provided, the value 0 is assumed

OUTPUT:

- an iterable index, starting from i_0 and ending at n, where n is the manifold's dimension

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: A = M.mixed_form_algebra()
sage: list(A.irange())
```

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```
[0, 1, 2, 3]
sage: list(A.irange(2))
[2, 3]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> A = M.mixed_form_algebra()
>>> list(A.irange())
[0, 1, 2, 3]
>>> list(A.irange(Integer(2)))
[2, 3]
```

lift_from_homology(*x*)

Lift a cohomology class to the algebra of mixed differential forms.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: C = M.de_rham_complex()
sage: H = C.cohomology()
sage: alpha = M.diff_form(1, [1,1], name='alpha')
sage: alpha.display()
alpha = dx + dy
sage: a = H(alpha); a
[alpha]
sage: C.lift_from_homology(a)
Mixed differential form alpha on the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> C = M.de_rham_complex()
>>> H = C.cohomology()
>>> alpha = M.diff_form(Integer(1), [Integer(1),Integer(1)], name='alpha')
>>> alpha.display()
alpha = dx + dy
>>> a = H(alpha); a
[alpha]
>>> C.lift_from_homology(a)
Mixed differential form alpha on the 2-dimensional differentiable
 manifold M
```

one()

Return the one of `self`.

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: A = M.mixed_form_algebra()
sage: A.one()
```

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```
Mixed differential form one on the 3-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> A = M.mixed_form_algebra()
>>> A.one()
Mixed differential form one on the 3-dimensional differentiable
 manifold M
```

vector_field_module()

Return the underlying vector field module.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: N = Manifold(3, 'N')
sage: Phi = M.diff_map(N, name='Phi'); Phi
Differentiable map Phi from the 2-dimensional differentiable
 manifold M to the 3-dimensional differentiable manifold N
sage: A = M.mixed_form_algebra(Phi); A
Graded algebra Omega^*(M,Phi) of mixed differential forms along the
 2-dimensional differentiable manifold M mapped into the
 3-dimensional differentiable manifold N via Phi
sage: A.vector_field_module()
Module X(M,Phi) of vector fields along the 2-dimensional
 differentiable manifold M mapped into the 3-dimensional
 differentiable manifold N
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> N = Manifold(Integer(3), 'N')
>>> Phi = M.diff_map(N, name='Phi'); Phi
Differentiable map Phi from the 2-dimensional differentiable
 manifold M to the 3-dimensional differentiable manifold N
>>> A = M.mixed_form_algebra(Phi); A
Graded algebra Omega^*(M,Phi) of mixed differential forms along the
 2-dimensional differentiable manifold M mapped into the
 3-dimensional differentiable manifold N via Phi
>>> A.vector_field_module()
Module X(M,Phi) of vector fields along the 2-dimensional
 differentiable manifold M mapped into the 3-dimensional
 differentiable manifold N
```

zero()

Return the zero of `self`.

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: A = M.mixed_form_algebra()
sage: A.zero()
```

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```
Mixed differential form zero on the 3-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> A = M.mixed_form_algebra()
>>> A.zero()
Mixed differential form zero on the 3-dimensional differentiable
 manifold M
```

2.10.2 Mixed Differential Forms

Let M and N be differentiable manifolds and $\varphi : M \longrightarrow N$ a differentiable map. A *mixed differential form along* φ is an element of the graded algebra represented by *MixedFormAlgebra*. Its homogeneous components consist of differential forms along φ. Mixed forms are useful to represent characteristic classes and perform computations of such.

AUTHORS:

- Michael Jung (2019) : initial version

class sage.manifolds.differentiable.mixed_form.**MixedForm**(*parent*, *name=None*, *latex_name=None*)

Bases: AlgebraElement, ModuleElementWithMutability

An instance of this class is a mixed form along some differentiable map $\varphi : M \to N$ between two differentiable manifolds M and N. More precisely, a mixed form a along $\varphi : M \to N$ can be considered as a differentiable map

$$a : M \longrightarrow \bigoplus_{k=0}^{n} T^{(0,k)}N,$$

where $T^{(0,k)}$ denotes the tensor bundle of type $(0, k)$, $\bigoplus$ the Whitney sum and n the dimension of N, such that

$$\forall x \in M, \quad a(x) \in \bigoplus_{k=0}^{n} \Lambda^k \left(T^*_{\varphi(x)}N\right),$$

where $\Lambda^k(T^*_{\varphi(x)}N)$ is the k-th exterior power of the dual of the tangent space $T_{\varphi(x)}N$. Thus, a mixed differential form a consists of homogeneous components a_i, $i = 0, 1, \dots, n$, where the i-th homogeneous component represents a differential form of degree i.

The standard case of a mixed form *on* M corresponds to $M = N$ with $\varphi = \mathrm{Id}_M$.

INPUT:

- `parent` – graded algebra of mixed forms represented by *MixedFormAlgebra* where the mixed form `self` shall belong to
- `comp` – (default: `None`) homogeneous components of the mixed form as a list; if none is provided, the components are set to innocent unnamed differential forms
- `name` – (default: `None`) name given to the mixed form
- `latex_name` – (default: `None`) LaTeX symbol to denote the mixed form; if none is provided, the LaTeX symbol is set to `name`

EXAMPLES:

Initialize a mixed form on a 2-dimensional parallelizable differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: e_xy = c_xy.frame()
sage: A = M.mixed_form(name='A'); A
Mixed differential form A on the 2-dimensional differentiable manifold M
sage: A.parent()
Graded algebra Omega^*(M) of mixed differential forms on the
 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> e_xy = c_xy.frame()
>>> A = M.mixed_form(name='A'); A
Mixed differential form A on the 2-dimensional differentiable manifold M
>>> A.parent()
Graded algebra Omega^*(M) of mixed differential forms on the
 2-dimensional differentiable manifold M
```

The default way to specify the i-th homogeneous component of a mixed form is by accessing it via `A[i]` or using *set_comp()*:

```
sage: A = M.mixed_form(name='A')
sage: A[0].set_expr(x) # scalar field
sage: A.set_comp(1)[0] = y*x
sage: A.set_comp(2)[0,1] = y^2*x
sage: A.display() # display names
A = A_0 + A_1 + A_2
sage: A.display_expansion() # display expansion in basis
A = x + x*y dx + x*y^2 dx∧dy
```

```
>>> from sage.all import *
>>> A = M.mixed_form(name='A')
>>> A[Integer(0)].set_expr(x) # scalar field
>>> A.set_comp(Integer(1))[Integer(0)] = y*x
>>> A.set_comp(Integer(2))[Integer(0),Integer(1)] = y**Integer(2)*x
>>> A.display() # display names
A = A_0 + A_1 + A_2
>>> A.display_expansion() # display expansion in basis
A = x + x*y dx + x*y^2 dx∧dy
```

Another way to define the homogeneous components is using predefined differential forms:

```
sage: f = M.scalar_field(x, name='f'); f
Scalar field f on the 2-dimensional differentiable manifold M
sage: omega = M.diff_form(1, name='omega'); omega
1-form omega on the 2-dimensional differentiable manifold M
sage: omega[e_xy,0] = y*x; omega.display()
omega = x*y dx
sage: eta = M.diff_form(2, name='eta'); eta
2-form eta on the 2-dimensional differentiable manifold M
sage: eta[e_xy,0,1] = y^2*x; eta.display()
eta = x*y^2 dx∧dy
```

```
>>> from sage.all import *
>>> f = M.scalar_field(x, name='f'); f
Scalar field f on the 2-dimensional differentiable manifold M
>>> omega = M.diff_form(Integer(1), name='omega'); omega
1-form omega on the 2-dimensional differentiable manifold M
>>> omega[e_xy,Integer(0)] = y*x; omega.display()
omega = x*y dx
>>> eta = M.diff_form(Integer(2), name='eta'); eta
2-form eta on the 2-dimensional differentiable manifold M
>>> eta[e_xy,Integer(0),Integer(1)] = y**Integer(2)*x; eta.display()
eta = x*y^2 dx∧dy
```

The components of a mixed form B can then be set as follows:

```
sage: B = M.mixed_form(name='B')
sage: B[:] = [f, omega, eta]; B.display() # display names
B = f + omega + eta
sage: B.display_expansion() # display in coordinates
B = x + x*y dx + x*y^2 dx∧dy
sage: B[0]
Scalar field f on the 2-dimensional differentiable manifold M
sage: B[1]
1-form omega on the 2-dimensional differentiable manifold M
sage: B[2]
2-form eta on the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> B = M.mixed_form(name='B')
>>> B[:] = [f, omega, eta]; B.display() # display names
B = f + omega + eta
>>> B.display_expansion() # display in coordinates
B = x + x*y dx + x*y^2 dx∧dy
>>> B[Integer(0)]
Scalar field f on the 2-dimensional differentiable manifold M
>>> B[Integer(1)]
1-form omega on the 2-dimensional differentiable manifold M
>>> B[Integer(2)]
2-form eta on the 2-dimensional differentiable manifold M
```

As we can see, the names are applied. However note that the differential forms are different instances:

```
sage: f is B[0]
False
```

```
>>> from sage.all import *
>>> f is B[Integer(0)]
False
```

Alternatively, the components can be determined from scratch:

```
sage: B = M.mixed_form([f, omega, eta], name='B')
sage: B.display()
B = f + omega + eta
```

```
>>> from sage.all import *
>>> B = M.mixed_form([f, omega, eta], name='B')
>>> B.display()
B = f + omega + eta
```

Mixed forms are elements of an algebra so they can be added, and multiplied via the wedge product:

```
sage: C = x*A; C
Mixed differential form x∧A on the 2-dimensional differentiable
 manifold M
sage: C.display_expansion()
x∧A = x^2 + x^2*y dx + x^2*y^2 dx∧dy
sage: D = A+C; D
Mixed differential form A+x∧A on the 2-dimensional differentiable
 manifold M
sage: D.display_expansion()
A+x∧A = x^2 + x + (x^2 + x)*y dx + (x^2 + x)*y^2 dx∧dy
sage: E = A*C; E
Mixed differential form A∧(x∧A) on the 2-dimensional differentiable
 manifold M
sage: E.display_expansion()
A∧(x∧A) = x^3 + 2*x^3*y dx + 2*x^3*y^2 dx∧dy
```

```
>>> from sage.all import *
>>> C = x*A; C
Mixed differential form x∧A on the 2-dimensional differentiable
 manifold M
>>> C.display_expansion()
x∧A = x^2 + x^2*y dx + x^2*y^2 dx∧dy
>>> D = A+C; D
Mixed differential form A+x∧A on the 2-dimensional differentiable
 manifold M
>>> D.display_expansion()
A+x∧A = x^2 + x + (x^2 + x)*y dx + (x^2 + x)*y^2 dx∧dy
>>> E = A*C; E
Mixed differential form A∧(x∧A) on the 2-dimensional differentiable
 manifold M
>>> E.display_expansion()
A∧(x∧A) = x^3 + 2*x^3*y dx + 2*x^3*y^2 dx∧dy
```

Coercions are fully implemented:

```
sage: F = omega*A
sage: F.display_expansion()
omega∧A = x^2*y dx
sage: G = omega+A
sage: G.display_expansion()
omega+A = x + 2*x*y dx + x*y^2 dx∧dy
```

```
>>> from sage.all import *
>>> F = omega*A
>>> F.display_expansion()
omega∧A = x^2*y dx
```

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```
>>> G = omega+A
>>> G.display_expansion()
omega+A = x + 2*x*y dx + x*y^2 dx∧dy
```

Moreover, it is possible to compute the exterior derivative of a mixed form:

```
sage: dA = A.exterior_derivative(); dA.display()
dA = zero + dA_0 + dA_1
sage: dA.display_expansion()
dA = dx - x dx∧dy
```

```
>>> from sage.all import *
>>> dA = A.exterior_derivative(); dA.display()
dA = zero + dA_0 + dA_1
>>> dA.display_expansion()
dA = dx - x dx∧dy
```

Initialize a mixed form on a 2-dimensional non-parallelizable differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y),
....:                  intersection_name='W', restrictions1= x>0,
....:                  restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame() # define frames
sage: A = M.mixed_form(name='A')
sage: A[0].set_expr(x, c_xy)
sage: A[0].display()
A_0: M → ℝ
on U: (x, y) ↦ x
on W: (u, v) ↦ 1/2*u + 1/2*v
sage: A[1][0] = y*x; A[1].display(e_xy)
A_1 = x*y dx
sage: A[2][e_uv,0,1] = u*v^2; A[2].display(e_uv)
A_2 = u*v^2 du∧dv
sage: A.add_comp_by_continuation(e_uv, W, c_uv)
sage: A.display_expansion(e_uv)
A = 1/2*u + 1/2*v + (1/8*u^2 - 1/8*v^2) du + (1/8*u^2 - 1/8*v^2) dv + u*v^2 du∧dv
sage: A.add_comp_by_continuation(e_xy, W, c_xy)
sage: A.display_expansion(e_xy)
A = x + x*y dx + (-2*x^3 + 2*x^2*y + 2*x*y^2 - 2*y^3) dx∧dy
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv = V.
→chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
```

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```
>>> transf = c_xy.transition_map(c_uv, (x+y, x-y),
...                      intersection_name='W', restrictions1= x>Integer(0),
...                      restrictions2= u+v>Integer(0))
>>> inv = transf.inverse()
>>> W = U.intersection(V)
>>> e_xy = c_xy.frame(); e_uv = c_uv.frame() # define frames
>>> A = M.mixed_form(name='A')
>>> A[Integer(0)].set_expr(x, c_xy)
>>> A[Integer(0)].display()
A_0: M → ℝ
on U: (x, y) ↦ x
on W: (u, v) ↦ 1/2*u + 1/2*v
>>> A[Integer(1)][Integer(0)] = y*x; A[Integer(1)].display(e_xy)
A_1 = x*y dx
>>> A[Integer(2)][e_uv,Integer(0),Integer(1)] = u*v**Integer(2); A[Integer(2)].
↪display(e_uv)
A_2 = u*v^2 du∧dv
>>> A.add_comp_by_continuation(e_uv, W, c_uv)
>>> A.display_expansion(e_uv)
A = 1/2*u + 1/2*v + (1/8*u^2 - 1/8*v^2) du + (1/8*u^2 - 1/8*v^2) dv + u*v^2 du∧dv
>>> A.add_comp_by_continuation(e_xy, W, c_xy)
>>> A.display_expansion(e_xy)
A = x + x*y dx + (-2*x^3 + 2*x^2*y + 2*x*y^2 - 2*y^3) dx∧dy
```

Since zero and one are special elements, their components cannot be changed:

```
sage: z = M.mixed_form_algebra().zero()
sage: z[0] = 1
Traceback (most recent call last):
...
ValueError: the components of an immutable element cannot be changed
sage: one = M.mixed_form_algebra().one()
sage: one[0] = 0
Traceback (most recent call last):
...
ValueError: the components of an immutable element cannot be changed
```

```
>>> from sage.all import *
>>> z = M.mixed_form_algebra().zero()
>>> z[Integer(0)] = Integer(1)
Traceback (most recent call last):
...
ValueError: the components of an immutable element cannot be changed
>>> one = M.mixed_form_algebra().one()
>>> one[Integer(0)] = Integer(0)
Traceback (most recent call last):
...
ValueError: the components of an immutable element cannot be changed
```

add_comp_by_continuation(*frame*, *subdomain*, *chart=None*)

Set components with respect to a vector frame by continuation of the coordinate expression of the components in a subframe.

The continuation is performed by demanding that the components have the same coordinate expression as those on the restriction of the frame to a given subdomain.

INPUT:

- `frame` – vector frame e in which the components are to be set
- `subdomain` – open subset of e's domain in which the components are known or can be evaluated from other components
- `chart` – (default: `None`) coordinate chart on e's domain in which the extension of the expression of the components is to be performed; if `None`, the default's chart of e's domain is assumed

EXAMPLES:

Mixed form defined by differential forms with components on different parts of the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
sage: F = M.mixed_form(name='F') # No predefined components, here
sage: F[0] = M.scalar_field(x, name='f')
sage: F[1] = M.diff_form(1, {e_xy: [x,0]}, name='omega')
sage: F[2].set_name(name='eta')
sage: F[2][e_uv,0,1] = u*v
sage: F.add_comp_by_continuation(e_uv, W, c_uv)
sage: F.add_comp_by_continuation(e_xy, W, c_xy) # Now, F is fully defined
sage: F.display_expansion(e_xy)
F = x + x dx - x*y/(x^8 + 4*x^6*y^2 + 6*x^4*y^4 + 4*x^2*y^6 + y^8) dx∧dy
sage: F.display_expansion(e_uv)
F = u/(u^2 + v^2) - (u^3 - u*v^2)/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6) du -␣
↪2*u^2*v/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6) dv + u*v du∧dv
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                 intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                 restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
```

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```
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V)
>>> e_xy = c_xy.frame(); e_uv = c_uv.frame()
>>> F = M.mixed_form(name='F') # No predefined components, here
>>> F[Integer(0)] = M.scalar_field(x, name='f')
>>> F[Integer(1)] = M.diff_form(Integer(1), {e_xy: [x,Integer(0)]}, name=
↪'omega')
>>> F[Integer(2)].set_name(name='eta')
>>> F[Integer(2)][e_uv,Integer(0),Integer(1)] = u*v
>>> F.add_comp_by_continuation(e_uv, W, c_uv)
>>> F.add_comp_by_continuation(e_xy, W, c_xy) # Now, F is fully defined
>>> F.display_expansion(e_xy)
F = x + x dx - x*y/(x^8 + 4*x^6*y^2 + 6*x^4*y^4 + 4*x^2*y^6 + y^8) dx∧dy
>>> F.display_expansion(e_uv)
F = u/(u^2 + v^2) - (u^3 - u*v^2)/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6) du -␣
↪2*u^2*v/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6) dv + u*v du∧dv
```

copy (*name=None*, *latex_name=None*)

Return an exact copy of `self`.

> **Note**
>
> The name and names of the components are not copied.

INPUT:

- `name` – (default: `None`) name given to the copy
- `latex_name` – (default: `None`) LaTeX symbol to denote the copy; if none is provided, the LaTeX symbol is set to `name`

EXAMPLES:

Initialize a 2-dimensional manifold and differential forms:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)    # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                   intersection_name='W', restrictions1= x>0,
....:                   restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
sage: f = M.scalar_field(x, name='f', chart=c_xy)
sage: f.add_expr_by_continuation(c_uv, W)
sage: f.display()
f: M → ℝ
on U: (x, y) ↦ x
on V: (u, v) ↦ 1/2*u + 1/2*v
sage: omega = M.diff_form(1, name='omega')
sage: omega[e_xy,0] = x
```

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```
sage: omega.add_comp_by_continuation(e_uv, W, c_uv)
sage: omega.display()
omega = x dx
sage: A = M.mixed_form([f, omega, 0], name='A'); A.display()
A = f + omega + zero
sage: A.display_expansion(e_uv)
A = 1/2*u + 1/2*v + (1/4*u + 1/4*v) du + (1/4*u + 1/4*v) dv
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)    # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
...                     intersection_name='W', restrictions1= x>Integer(0),
...                     restrictions2= u+v>Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V)
>>> e_xy = c_xy.frame(); e_uv = c_uv.frame()
>>> f = M.scalar_field(x, name='f', chart=c_xy)
>>> f.add_expr_by_continuation(c_uv, W)
>>> f.display()
f: M → ℝ
on U: (x, y) ↦ x
on V: (u, v) ↦ 1/2*u + 1/2*v
>>> omega = M.diff_form(Integer(1), name='omega')
>>> omega[e_xy,Integer(0)] = x
>>> omega.add_comp_by_continuation(e_uv, W, c_uv)
>>> omega.display()
omega = x dx
>>> A = M.mixed_form([f, omega, Integer(0)], name='A'); A.display()
A = f + omega + zero
>>> A.display_expansion(e_uv)
A = 1/2*u + 1/2*v + (1/4*u + 1/4*v) du + (1/4*u + 1/4*v) dv
```

An exact copy is made. The copy is an entirely new instance and has a different name, but has the very same values:

```
sage: B = A.copy(); B.display()
(unnamed scalar field) + (unnamed 1-form) + (unnamed 2-form)
sage: B.display_expansion(e_uv)
1/2*u + 1/2*v + (1/4*u + 1/4*v) du + (1/4*u + 1/4*v) dv
sage: A == B
True
sage: A is B
False
```

```
>>> from sage.all import *
>>> B = A.copy(); B.display()
(unnamed scalar field) + (unnamed 1-form) + (unnamed 2-form)
>>> B.display_expansion(e_uv)
```

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```
1/2*u + 1/2*v + (1/4*u + 1/4*v) du + (1/4*u + 1/4*v) dv
>>> A == B
True
>>> A is B
False
```

derivative()

Compute the exterior derivative of `self`.

More precisely, the *exterior derivative* on $\Omega^k(M,\varphi)$ is a linear map

$$\mathrm{d}_k : \Omega^k(M,\varphi) \to \Omega^{k+1}(M,\varphi),$$

where $\Omega^k(M,\varphi)$ denotes the space of differential forms of degree k along φ (see `exterior_derivative()` for further information). By linear extension, this induces a map on $\Omega^*(M,\varphi)$:

$$\mathrm{d} : \Omega^*(M,\varphi) \to \Omega^*(M,\varphi).$$

OUTPUT:

- a `MixedForm` representing the exterior derivative of the mixed form

EXAMPLES:

Exterior derivative of a mixed form on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: f = M.scalar_field(z^2, name='f')
sage: f.disp()
f: M → ℝ
    (x, y, z) ↦ z^2
sage: a = M.diff_form(2, 'a')
sage: a[1,2], a[1,3], a[2,3] = z+y^2, z+x, x^2
sage: a.disp()
a = (y^2 + z) dx∧dy + (x + z) dx∧dz + x^2 dy∧dz
sage: F = M.mixed_form([f, 0, a, 0], name='F'); F.display()
F = f + zero + a + zero
sage: dF = F.exterior_derivative()
sage: dF.display()
dF = zero + df + dzero + da
sage: dF = F.exterior_derivative()
sage: dF.display_expansion()
dF = 2*z dz + (2*x + 1) dx∧dy∧dz
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> f = M.scalar_field(z**Integer(2), name='f')
>>> f.disp()
f: M → ℝ
    (x, y, z) ↦ z^2
>>> a = M.diff_form(Integer(2), 'a')
```

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```
>>> a[Integer(1),Integer(2)], a[Integer(1),Integer(3)], a[Integer(2),
↪Integer(3)] = z+y**Integer(2), z+x, x**Integer(2)
>>> a.disp()
a = (y^2 + z) dx∧dy + (x + z) dx∧dz + x^2 dy∧dz
>>> F = M.mixed_form([f, Integer(0), a, Integer(0)], name='F'); F.display()
F = f + zero + a + zero
>>> dF = F.exterior_derivative()
>>> dF.display()
dF = zero + df + dzero + da
>>> dF = F.exterior_derivative()
>>> dF.display_expansion()
dF = 2*z dz + (2*x + 1) dx∧dy∧dz
```

Due to long calculation times, the result is cached:

```
sage: F.exterior_derivative() is dF
True
```

```
>>> from sage.all import *
>>> F.exterior_derivative() is dF
True
```

disp()

Display the homogeneous components of the mixed form.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: f = M.scalar_field(name='f')
sage: omega = M.diff_form(1, name='omega')
sage: eta = M.diff_form(2, name='eta')
sage: F = M.mixed_form([f, omega, eta], name='F'); F
Mixed differential form F on the 2-dimensional differentiable
 manifold M
sage: F.display() # display names of homogeneous components
F = f + omega + eta
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> f = M.scalar_field(name='f')
>>> omega = M.diff_form(Integer(1), name='omega')
>>> eta = M.diff_form(Integer(2), name='eta')
>>> F = M.mixed_form([f, omega, eta], name='F'); F
Mixed differential form F on the 2-dimensional differentiable
 manifold M
>>> F.display() # display names of homogeneous components
F = f + omega + eta
```

disp_exp(*frame=None*, *chart=None*, *from_chart=None*)

Display the expansion in a particular basis and chart of mixed forms.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- `frame` – (default: `None`) vector frame with respect to which the mixed form is expanded; if `None`, only the names of the components are displayed
- `chart` – (default: `None`) chart with respect to which the components of the mixed form in the selected frame are expressed; if `None`, the default chart of the vector frame domain is assumed

EXAMPLES:

Display the expansion of a mixed form on a 2-dimensional non-parallelizable differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x-y, x+y),
....:                   intersection_name='W', restrictions1= x>0,
....:                   restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame() # define frames
sage: omega = M.diff_form(1, name='omega')
sage: omega[e_xy,0] = x; omega.display(e_xy)
omega = x dx
sage: omega.add_comp_by_continuation(e_uv, W, c_uv) # continuation onto M
sage: eta = M.diff_form(2, name='eta')
sage: eta[e_uv,0,1] = u*v; eta.display(e_uv)
eta = u*v du∧dv
sage: eta.add_comp_by_continuation(e_xy, W, c_xy) # continuation onto M
sage: F = M.mixed_form([0, omega, eta], name='F'); F
Mixed differential form F on the 2-dimensional differentiable
 manifold M
sage: F.display() # display names of homogeneous components
F = zero + omega + eta
sage: F.display_expansion(e_uv)
F = (1/4*u + 1/4*v) du + (1/4*u + 1/4*v) dv + u*v du∧dv
sage: F.display_expansion(e_xy)
F = x dx + (2*x^2 - 2*y^2) dx∧dy
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> transf = c_xy.transition_map(c_uv, (x-y, x+y),
...                   intersection_name='W', restrictions1= x>Integer(0),
...                   restrictions2= u+v>Integer(0))
>>> inv = transf.inverse()
>>> W = U.intersection(V)
>>> e_xy = c_xy.frame(); e_uv = c_uv.frame() # define frames
>>> omega = M.diff_form(Integer(1), name='omega')
>>> omega[e_xy,Integer(0)] = x; omega.display(e_xy)
omega = x dx
```

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```
>>> omega.add_comp_by_continuation(e_uv, W, c_uv) # continuation onto M
>>> eta = M.diff_form(Integer(2), name='eta')
>>> eta[e_uv,Integer(0),Integer(1)] = u*v; eta.display(e_uv)
eta = u*v du∧dv
>>> eta.add_comp_by_continuation(e_xy, W, c_xy) # continuation onto M
>>> F = M.mixed_form([Integer(0), omega, eta], name='F'); F
Mixed differential form F on the 2-dimensional differentiable
 manifold M
>>> F.display() # display names of homogeneous components
F = zero + omega + eta
>>> F.display_expansion(e_uv)
F = (1/4*u + 1/4*v) du + (1/4*u + 1/4*v) dv + u*v du∧dv
>>> F.display_expansion(e_xy)
F = x dx + (2*x^2 - 2*y^2) dx∧dy
```

display()

Display the homogeneous components of the mixed form.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: f = M.scalar_field(name='f')
sage: omega = M.diff_form(1, name='omega')
sage: eta = M.diff_form(2, name='eta')
sage: F = M.mixed_form([f, omega, eta], name='F'); F
Mixed differential form F on the 2-dimensional differentiable
 manifold M
sage: F.display() # display names of homogeneous components
F = f + omega + eta
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> f = M.scalar_field(name='f')
>>> omega = M.diff_form(Integer(1), name='omega')
>>> eta = M.diff_form(Integer(2), name='eta')
>>> F = M.mixed_form([f, omega, eta], name='F'); F
Mixed differential form F on the 2-dimensional differentiable
 manifold M
>>> F.display() # display names of homogeneous components
F = f + omega + eta
```

display_exp(*frame=None*, *chart=None*, *from_chart=None*)

Display the expansion in a particular basis and chart of mixed forms.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- `frame` – (default: `None`) vector frame with respect to which the mixed form is expanded; if `None`, only the names of the components are displayed
- `chart` – (default: `None`) chart with respect to which the components of the mixed form in the selected frame are expressed; if `None`, the default chart of the vector frame domain is assumed

EXAMPLES:

Display the expansion of a mixed form on a 2-dimensional non-parallelizable differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x-y, x+y),
....:                    intersection_name='W', restrictions1= x>0,
....:                    restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame() # define frames
sage: omega = M.diff_form(1, name='omega')
sage: omega[e_xy,0] = x; omega.display(e_xy)
omega = x dx
sage: omega.add_comp_by_continuation(e_uv, W, c_uv) # continuation onto M
sage: eta = M.diff_form(2, name='eta')
sage: eta[e_uv,0,1] = u*v; eta.display(e_uv)
eta = u*v du∧dv
sage: eta.add_comp_by_continuation(e_xy, W, c_xy) # continuation onto M
sage: F = M.mixed_form([0, omega, eta], name='F'); F
Mixed differential form F on the 2-dimensional differentiable
 manifold M
sage: F.display() # display names of homogeneous components
F = zero + omega + eta
sage: F.display_expansion(e_uv)
F = (1/4*u + 1/4*v) du + (1/4*u + 1/4*v) dv + u*v du∧dv
sage: F.display_expansion(e_xy)
F = x dx + (2*x^2 - 2*y^2) dx∧dy
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> transf = c_xy.transition_map(c_uv, (x-y, x+y),
...                    intersection_name='W', restrictions1= x>Integer(0),
...                    restrictions2= u+v>Integer(0))
>>> inv = transf.inverse()
>>> W = U.intersection(V)
>>> e_xy = c_xy.frame(); e_uv = c_uv.frame() # define frames
>>> omega = M.diff_form(Integer(1), name='omega')
>>> omega[e_xy,Integer(0)] = x; omega.display(e_xy)
omega = x dx
>>> omega.add_comp_by_continuation(e_uv, W, c_uv) # continuation onto M
>>> eta = M.diff_form(Integer(2), name='eta')
>>> eta[e_uv,Integer(0),Integer(1)] = u*v; eta.display(e_uv)
eta = u*v du∧dv
>>> eta.add_comp_by_continuation(e_xy, W, c_xy) # continuation onto M
>>> F = M.mixed_form([Integer(0), omega, eta], name='F'); F
Mixed differential form F on the 2-dimensional differentiable
```

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```
 manifold M
>>> F.display() # display names of homogeneous components
F = zero + omega + eta
>>> F.display_expansion(e_uv)
F = (1/4*u + 1/4*v) du + (1/4*u + 1/4*v) dv + u*v du∧dv
>>> F.display_expansion(e_xy)
F = x dx + (2*x^2 - 2*y^2) dx∧dy
```

display_expansion(*frame=None*, *chart=None*, *from_chart=None*)

Display the expansion in a particular basis and chart of mixed forms.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- `frame` – (default: `None`) vector frame with respect to which the mixed form is expanded; if `None`, only the names of the components are displayed
- `chart` – (default: `None`) chart with respect to which the components of the mixed form in the selected frame are expressed; if `None`, the default chart of the vector frame domain is assumed

EXAMPLES:

Display the expansion of a mixed form on a 2-dimensional non-parallelizable differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x-y, x+y),
....:                    intersection_name='W', restrictions1= x>0,
....:                    restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame() # define frames
sage: omega = M.diff_form(1, name='omega')
sage: omega[e_xy,0] = x; omega.display(e_xy)
omega = x dx
sage: omega.add_comp_by_continuation(e_uv, W, c_uv) # continuation onto M
sage: eta = M.diff_form(2, name='eta')
sage: eta[e_uv,0,1] = u*v; eta.display(e_uv)
eta = u*v du∧dv
sage: eta.add_comp_by_continuation(e_xy, W, c_xy) # continuation onto M
sage: F = M.mixed_form([0, omega, eta], name='F'); F
Mixed differential form F on the 2-dimensional differentiable
 manifold M
sage: F.display() # display names of homogeneous components
F = zero + omega + eta
sage: F.display_expansion(e_uv)
F = (1/4*u + 1/4*v) du + (1/4*u + 1/4*v) dv + u*v du∧dv
sage: F.display_expansion(e_xy)
F = x dx + (2*x^2 - 2*y^2) dx∧dy
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
```

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```
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> transf = c_xy.transition_map(c_uv, (x-y, x+y),
...                   intersection_name='W', restrictions1= x>Integer(0),
...                   restrictions2= u+v>Integer(0))
>>> inv = transf.inverse()
>>> W = U.intersection(V)
>>> e_xy = c_xy.frame(); e_uv = c_uv.frame() # define frames
>>> omega = M.diff_form(Integer(1), name='omega')
>>> omega[e_xy,Integer(0)] = x; omega.display(e_xy)
omega = x dx
>>> omega.add_comp_by_continuation(e_uv, W, c_uv) # continuation onto M
>>> eta = M.diff_form(Integer(2), name='eta')
>>> eta[e_uv,Integer(0),Integer(1)] = u*v; eta.display(e_uv)
eta = u*v du∧dv
>>> eta.add_comp_by_continuation(e_xy, W, c_xy) # continuation onto M
>>> F = M.mixed_form([Integer(0), omega, eta], name='F'); F
Mixed differential form F on the 2-dimensional differentiable
 manifold M
>>> F.display() # display names of homogeneous components
F = zero + omega + eta
>>> F.display_expansion(e_uv)
F = (1/4*u + 1/4*v) du + (1/4*u + 1/4*v) dv + u*v du∧dv
>>> F.display_expansion(e_xy)
F = x dx + (2*x^2 - 2*y^2) dx∧dy
```

exterior_derivative()

Compute the exterior derivative of `self`.

More precisely, the *exterior derivative* on $\Omega^k(M,\varphi)$ is a linear map

$$\mathrm{d}_k : \Omega^k(M,\varphi) \to \Omega^{k+1}(M,\varphi),$$

where $\Omega^k(M,\varphi)$ denotes the space of differential forms of degree k along φ (see `exterior_derivative()` for further information). By linear extension, this induces a map on $\Omega^*(M,\varphi)$:

$$\mathrm{d} : \Omega^*(M,\varphi) \to \Omega^*(M,\varphi).$$

OUTPUT:

- a `MixedForm` representing the exterior derivative of the mixed form

EXAMPLES:

Exterior derivative of a mixed form on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: f = M.scalar_field(z^2, name='f')
sage: f.disp()
f: M → ℝ
    (x, y, z) ↦ z^2
```

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```
sage: a = M.diff_form(2, 'a')
sage: a[1,2], a[1,3], a[2,3] = z+y^2, z+x, x^2
sage: a.disp()
a = (y^2 + z) dx∧dy + (x + z) dx∧dz + x^2 dy∧dz
sage: F = M.mixed_form([f, 0, a, 0], name='F'); F.display()
F = f + zero + a + zero
sage: dF = F.exterior_derivative()
sage: dF.display()
dF = zero + df + dzero + da
sage: dF = F.exterior_derivative()
sage: dF.display_expansion()
dF = 2*z dz + (2*x + 1) dx∧dy∧dz
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> f = M.scalar_field(z**Integer(2), name='f')
>>> f.disp()
f: M → ℝ
    (x, y, z) ↦ z^2
>>> a = M.diff_form(Integer(2), 'a')
>>> a[Integer(1),Integer(2)], a[Integer(1),Integer(3)], a[Integer(2),
↪Integer(3)] = z+y**Integer(2), z+x, x**Integer(2)
>>> a.disp()
a = (y^2 + z) dx∧dy + (x + z) dx∧dz + x^2 dy∧dz
>>> F = M.mixed_form([f, Integer(0), a, Integer(0)], name='F'); F.display()
F = f + zero + a + zero
>>> dF = F.exterior_derivative()
>>> dF.display()
dF = zero + df + dzero + da
>>> dF = F.exterior_derivative()
>>> dF.display_expansion()
dF = 2*z dz + (2*x + 1) dx∧dy∧dz
```

Due to long calculation times, the result is cached:

```
sage: F.exterior_derivative() is dF
True
```

```
>>> from sage.all import *
>>> F.exterior_derivative() is dF
True
```

irange(*start=None*)

Single index generator.

INPUT:

- `start` – (default: `None`) initial value i_0 of the index between 0 and n, where n is the manifold's dimension; if none is provided, the value 0 is assumed

OUTPUT:

- an iterable index, starting from i_0 and ending at n, where n is the manifold's dimension

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: a = M.mixed_form(name='a')
sage: list(a.irange())
[0, 1, 2, 3]
sage: list(a.irange(2))
[2, 3]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> a = M.mixed_form(name='a')
>>> list(a.irange())
[0, 1, 2, 3]
>>> list(a.irange(Integer(2)))
[2, 3]
```

restrict(*subdomain*, *dest_map=None*)

Return the restriction of `self` to some subdomain.

INPUT:

- `subdomain` – *DifferentiableManifold*; open subset U of the domain of `self`
- `dest_map` – *DiffMap* (default: `None`); destination map $\Psi : U \to V$, where V is an open subset of the manifold N where the mixed form takes it values; if `None`, the restriction of Φ to U is used, Φ being the differentiable map $S \to M$ associated with the mixed form

OUTPUT: *MixedForm* representing the restriction

EXAMPLES:

Initialize the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                intersection_name='W', restrictions1= x^2+y^2!=0,
....:                restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#
```

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```
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                  intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                  restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V)
>>> e_xy = c_xy.frame(); e_uv = c_uv.frame()
```

And predefine some forms:

```
sage: f = M.scalar_field(x^2, name='f', chart=c_xy)
sage: f.add_expr_by_continuation(c_uv, W)
sage: omega = M.diff_form(1, name='omega')
sage: omega[e_xy,0] = y^2
sage: omega.add_comp_by_continuation(e_uv, W, c_uv)
sage: eta = M.diff_form(2, name='eta')
sage: eta[e_xy,0,1] = x^2*y^2
sage: eta.add_comp_by_continuation(e_uv, W, c_uv)
```

```
>>> from sage.all import *
>>> f = M.scalar_field(x**Integer(2), name='f', chart=c_xy)
>>> f.add_expr_by_continuation(c_uv, W)
>>> omega = M.diff_form(Integer(1), name='omega')
>>> omega[e_xy,Integer(0)] = y**Integer(2)
>>> omega.add_comp_by_continuation(e_uv, W, c_uv)
>>> eta = M.diff_form(Integer(2), name='eta')
>>> eta[e_xy,Integer(0),Integer(1)] = x**Integer(2)*y**Integer(2)
>>> eta.add_comp_by_continuation(e_uv, W, c_uv)
```

Now, a mixed form can be restricted to some subdomain:

```
sage: F = M.mixed_form([f, omega, eta], name='F')
sage: FV = F.restrict(V); FV
Mixed differential form F on the Open subset V of the 2-dimensional
 differentiable manifold M
sage: FV[:]
[Scalar field f on the Open subset V of the 2-dimensional
 differentiable manifold M,
 1-form omega on the Open subset V of the 2-dimensional
 differentiable manifold M,
 2-form eta on the Open subset V of the 2-dimensional
 differentiable manifold M]
sage: FV.display_expansion(e_uv)
F = u^2/(u^4 + 2*u^2*v^2 + v^4) - (u^2*v^2 - v^4)/(u^8 + 4*u^6*v^2 + 6*u^4*v^
↪4 + 4*u^2*v^6 + v^8) du - 2*u*v^3/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 + 4*u^2*v^6␣
↪+ v^8) dv - u^2*v^2/(u^12 + 6*u^10*v^2 + 15*u^8*v^4 + 20*u^6*v^6 + 15*u^4*v^
↪8 + 6*u^2*v^10 + v^12) du∧dv
```

```
>>> from sage.all import *
>>> F = M.mixed_form([f, omega, eta], name='F')
>>> FV = F.restrict(V); FV
Mixed differential form F on the Open subset V of the 2-dimensional
 differentiable manifold M
>>> FV[:]
[Scalar field f on the Open subset V of the 2-dimensional
 differentiable manifold M,
 1-form omega on the Open subset V of the 2-dimensional
 differentiable manifold M,
 2-form eta on the Open subset V of the 2-dimensional
 differentiable manifold M]
>>> FV.display_expansion(e_uv)
F = u^2/(u^4 + 2*u^2*v^2 + v^4) - (u^2*v^2 - v^4)/(u^8 + 4*u^6*v^2 + 6*u^4*v^
↪4 + 4*u^2*v^6 + v^8) du - 2*u*v^3/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 + 4*u^2*v^6␣
↪+ v^8) dv - u^2*v^2/(u^12 + 6*u^10*v^2 + 15*u^8*v^4 + 20*u^6*v^6 + 15*u^4*v^
↪8 + 6*u^2*v^10 + v^12) du∧dv
```

set_comp(*i*)

Return the i-th homogeneous component for assignment.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: A = M.mixed_form(name='A')
sage: A.set_comp(0).set_expr(x^2) # scalar field
sage: A.set_comp(1)[:] = [-y, x]
sage: A.set_comp(2)[0,1] = x-y
sage: A.display()
A = A_0 + A_1 + A_2
sage: A.display_expansion()
A = x^2 - y dx + x dy + (x - y) dx∧dy
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> A = M.mixed_form(name='A')
>>> A.set_comp(Integer(0)).set_expr(x**Integer(2)) # scalar field
>>> A.set_comp(Integer(1))[:] = [-y, x]
>>> A.set_comp(Integer(2))[Integer(0),Integer(1)] = x-y
>>> A.display()
A = A_0 + A_1 + A_2
>>> A.display_expansion()
A = x^2 - y dx + x dy + (x - y) dx∧dy
```

set_immutable()

Set `self` and homogeneous components of `self` immutable.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
```

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```
sage: f = M.scalar_field(x^2, name='f')
sage: A = M.mixed_form([f, 0, 0], name='A')
sage: A.set_immutable()
sage: A.is_immutable()
True
sage: A[0].is_immutable()
True
sage: f.is_immutable()
False
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> f = M.scalar_field(x**Integer(2), name='f')
>>> A = M.mixed_form([f, Integer(0), Integer(0)], name='A')
>>> A.set_immutable()
>>> A.is_immutable()
True
>>> A[Integer(0)].is_immutable()
True
>>> f.is_immutable()
False
```

set_name (*name=None*, *latex_name=None*, *apply_to_comp=True*)

Redefine the string and LaTeX representation of the object.

INPUT:

- `name` – (default: `None`) name given to the mixed form
- `latex_name` – (default: `None`) LaTeX symbol to denote the mixed form; if none is provided, the LaTeX symbol is set to `name`
- `apply_to_comp` – boolean (default: `True`); if `True` all homogeneous components will be renamed accordingly. If `False` only the mixed form will be renamed

EXAMPLES:

Rename a mixed form:

```
sage: M = Manifold(4, 'M')
sage: F = M.mixed_form(name='dummy', latex_name=r'\ugly'); F
Mixed differential form dummy on the 4-dimensional differentiable
 manifold M
sage: latex(F)
\ugly
sage: F.set_name(name='F', latex_name=r'\mathcal{F}'); F
Mixed differential form F on the 4-dimensional differentiable
 manifold M
sage: latex(F)
\mathcal{F}
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M')
```

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```
>>> F = M.mixed_form(name='dummy', latex_name=r'\ugly'); F
Mixed differential form dummy on the 4-dimensional differentiable
 manifold M
>>> latex(F)
\ugly
>>> F.set_name(name='F', latex_name=r'\mathcal{F}'); F
Mixed differential form F on the 4-dimensional differentiable
 manifold M
>>> latex(F)
\mathcal{F}
```

If not stated otherwise, all homogeneous components are renamed accordingly:

```
sage: F.display()
F = F_0 + F_1 + F_2 + F_3 + F_4
```

```
>>> from sage.all import *
>>> F.display()
F = F_0 + F_1 + F_2 + F_3 + F_4
```

Setting the argument `set_all` to `False` prevents the renaming in the homogeneous components:

```
sage: F.set_name(name='eta', latex_name=r'\eta', apply_to_comp=False)
sage: F.display()
eta = F_0 + F_1 + F_2 + F_3 + F_4
```

```
>>> from sage.all import *
>>> F.set_name(name='eta', latex_name=r'\eta', apply_to_comp=False)
>>> F.display()
eta = F_0 + F_1 + F_2 + F_3 + F_4
```

To rename a homogeneous component individually, we simply access the homogeneous component and use its `set_name()` method:

```
sage: F[0].set_name(name='g'); F.display()
eta = g + F_1 + F_2 + F_3 + F_4
```

```
>>> from sage.all import *
>>> F[Integer(0)].set_name(name='g'); F.display()
eta = g + F_1 + F_2 + F_3 + F_4
```

set_restriction(*rst*)

Set a (component-wise) restriction of `self` to some subdomain.

INPUT:

- `rst` – *MixedForm* of the same type as `self`, defined on a subdomain of the domain of `self`

EXAMPLES:

Initialize the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
```

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```
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coordinates from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coordinates from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                 intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                 restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V)
>>> e_xy = c_xy.frame(); e_uv = c_uv.frame()
```

And define some forms on the subset `U`:

```
sage: f = U.scalar_field(x, name='f', chart=c_xy)
sage: omega = U.diff_form(1, name='omega')
sage: omega[e_xy,0] = y
sage: AU = U.mixed_form([f, omega, 0], name='A'); AU
Mixed differential form A on the Open subset U of the 2-dimensional
 differentiable manifold M
sage: AU.display_expansion(e_xy)
A = x + y dx
```

```
>>> from sage.all import *
>>> f = U.scalar_field(x, name='f', chart=c_xy)
>>> omega = U.diff_form(Integer(1), name='omega')
>>> omega[e_xy,Integer(0)] = y
>>> AU = U.mixed_form([f, omega, Integer(0)], name='A'); AU
Mixed differential form A on the Open subset U of the 2-dimensional
 differentiable manifold M
>>> AU.display_expansion(e_xy)
A = x + y dx
```

A mixed form on `M` can be specified by some mixed form on a subset:

```
sage: A = M.mixed_form(name='A'); A
Mixed differential form A on the 2-dimensional differentiable
 manifold M
sage: A.set_restriction(AU)
sage: A.display_expansion(e_xy)
A = x + y dx
sage: A.add_comp_by_continuation(e_uv, W, c_uv)
sage: A.display_expansion(e_uv)
A = u/(u^2 + v^2) - (u^2*v - v^3)/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6) du -
↪2*u*v^2/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6) dv
sage: A.restrict(U) == AU
True
```

```
>>> from sage.all import *
>>> A = M.mixed_form(name='A'); A
Mixed differential form A on the 2-dimensional differentiable
 manifold M
>>> A.set_restriction(AU)
>>> A.display_expansion(e_xy)
A = x + y dx
>>> A.add_comp_by_continuation(e_uv, W, c_uv)
>>> A.display_expansion(e_uv)
A = u/(u^2 + v^2) - (u^2*v - v^3)/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6) du -
↪2*u*v^2/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6) dv
>>> A.restrict(U) == AU
True
```

wedge(*other*)

Wedge product on the graded algebra of mixed forms.

More precisely, the wedge product is a bilinear map

$$\wedge : \Omega^k(M,\varphi) \times \Omega^l(M,\varphi) \to \Omega^{k+l}(M,\varphi),$$

where $\Omega^k(M,\varphi)$ denotes the space of differential forms of degree k along φ. By bilinear extension, this induces a map

$$\wedge : \Omega^*(M,\varphi) \times \Omega^*(M,\varphi) \to \Omega^*(M,\varphi)``$$

and equips $\Omega^*(M,\varphi)$ with a multiplication such that it becomes a graded algebra.

INPUT:

- `other` – mixed form in the same algebra as `self`

OUTPUT:

- the mixed form resulting from the wedge product of `self` with `other`

EXAMPLES:

Initialize a mixed form on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M')
sage: c_xyz.<x,y,z> = M.chart()
sage: f = M.scalar_field(x, name='f')
```

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```
sage: f.display()
f: M → ℝ
   (x, y, z) ↦ x
sage: g = M.scalar_field(y, name='g')
sage: g.display()
g: M → ℝ
   (x, y, z) ↦ y
sage: omega = M.diff_form(1, name='omega')
sage: omega[0] = x
sage: omega.display()
omega = x dx
sage: eta = M.diff_form(1, name='eta')
sage: eta[1] = y
sage: eta.display()
eta = y dy
sage: mu = M.diff_form(2, name='mu')
sage: mu[0,2] = z
sage: mu.display()
mu = z dx∧dz
sage: A = M.mixed_form([f, omega, mu, 0], name='A')
sage: A.display_expansion()
A = x + x dx + z dx∧dz
sage: B = M.mixed_form([g, eta, mu, 0], name='B')
sage: B.display_expansion()
B = y + y dy + z dx∧dz
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> f = M.scalar_field(x, name='f')
>>> f.display()
f: M → ℝ
   (x, y, z) ↦ x
>>> g = M.scalar_field(y, name='g')
>>> g.display()
g: M → ℝ
   (x, y, z) ↦ y
>>> omega = M.diff_form(Integer(1), name='omega')
>>> omega[Integer(0)] = x
>>> omega.display()
omega = x dx
>>> eta = M.diff_form(Integer(1), name='eta')
>>> eta[Integer(1)] = y
>>> eta.display()
eta = y dy
>>> mu = M.diff_form(Integer(2), name='mu')
>>> mu[Integer(0),Integer(2)] = z
>>> mu.display()
mu = z dx∧dz
>>> A = M.mixed_form([f, omega, mu, Integer(0)], name='A')
>>> A.display_expansion()
```

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```
A = x + x dx + z dx∧dz
>>> B = M.mixed_form([g, eta, mu, Integer(0)], name='B')
>>> B.display_expansion()
B = y + y dy + z dx∧dz
```

The wedge product of A and B yields:

```
sage: C = A.wedge(B); C
Mixed differential form A∧B on the 3-dimensional differentiable
 manifold M
sage: C.display_expansion()
A∧B = x*y + x*y dx + x*y dy + x*y dx∧dy + (x + y)*z dx∧dz - y*z dx∧dy∧dz
sage: D = B.wedge(A); D # Don't even try, it's not commutative!
Mixed differential form B∧A on the 3-dimensional differentiable
 manifold M
sage: D.display_expansion() # I told you so!
B∧A = x*y + x*y dx + x*y dy - x*y dx∧dy + (x + y)*z dx∧dz - y*z dx∧dy∧dz
```

```
>>> from sage.all import *
>>> C = A.wedge(B); C
Mixed differential form A∧B on the 3-dimensional differentiable
 manifold M
>>> C.display_expansion()
A∧B = x*y + x*y dx + x*y dy + x*y dx∧dy + (x + y)*z dx∧dz - y*z dx∧dy∧dz
>>> D = B.wedge(A); D # Don't even try, it's not commutative!
Mixed differential form B∧A on the 3-dimensional differentiable
 manifold M
>>> D.display_expansion() # I told you so!
B∧A = x*y + x*y dx + x*y dy - x*y dx∧dy + (x + y)*z dx∧dz - y*z dx∧dy∧dz
```

Alternatively, the multiplication symbol can be used:

```
sage: A*B
Mixed differential form A∧B on the 3-dimensional differentiable
 manifold M
sage: A*B == C
True
```

```
>>> from sage.all import *
>>> A*B
Mixed differential form A∧B on the 3-dimensional differentiable
 manifold M
>>> A*B == C
True
```

Yet, the multiplication includes coercions:

```
sage: E = x*A; E.display_expansion()
x∧A = x^2 + x^2 dx + x*z dx∧dz
sage: F = A*eta; F.display_expansion()
A∧eta = x*y dy + x*y dx∧dy - y*z dx∧dy∧dz
```

```
>>> from sage.all import *
>>> E = x*A; E.display_expansion()
x∧A = x^2 + x^2 dx + x*z dx∧dz
>>> F = A*eta; F.display_expansion()
A∧eta = x*y dy + x*y dx∧dy - y*z dx∧dy∧dz
```

2.11 De Rham Cohomology

Let M and N be differentiable manifolds and $\varphi\colon M \to N$ be a differentiable map. Then the associated de Rham complex is given by

$$0 \rightarrow \Omega^0(M,\varphi) \xrightarrow{\mathrm{d}_0} \Omega^1(M,\varphi) \xrightarrow{\mathrm{d}_1} \dots \xrightarrow{\mathrm{d}_{n-1}} \Omega^n(M,\varphi) \xrightarrow{\mathrm{d}_n} 0,$$

where $\Omega^k(M,\varphi)$ is the module of differential forms of degree k, and d_k is the associated exterior derivative. Then the *k-th de Rham cohomology group* is given by

$$H^k_{\mathrm{dR}}(M,\varphi) = \mathrm{ker}(\mathrm{d}_k)/\mathrm{im}(\mathrm{d}_{k-1}) \; ,$$

and the corresponding ring is obtained by

$$H^*_{\mathrm{dR}}(M,\varphi) = \bigoplus^n_{k=0} H^k_{\mathrm{dR}}(M,\varphi).$$

The de Rham cohomology ring is implemented via `DeRhamCohomologyRing`. Its elements, the cohomology classes, are represented by `DeRhamCohomologyClass`.

AUTHORS:

- Michael Jung (2021) : initial version

class `sage.manifolds.differentiable.de_rham_cohomology.`**`DeRhamCohomologyClass`**(*parent*, *representative*)

Bases: `AlgebraElement`

Define a cohomology class in the de Rham cohomology ring.

INPUT:

- `parent` – de Rham cohomology ring represented by an instance of `DeRhamCohomologyRing`
- `representative` – a closed (mixed) differential form representing the cohomology class

Note

The current implementation only provides basic features. Comparison via exact forms are not supported at the time being.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: C = M.de_rham_complex()
sage: H = C.cohomology()
sage: omega = M.diff_form(1, [1,1], name='omega')
sage: u = H(omega); u
[omega]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> C = M.de_rham_complex()
>>> H = C.cohomology()
>>> omega = M.diff_form(Integer(1), [Integer(1),Integer(1)], name='omega')
>>> u = H(omega); u
[omega]
```

Cohomology classes can be lifted to the algebra of mixed differential forms:

```
sage: u.lift()
Mixed differential form omega on the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> u.lift()
Mixed differential form omega on the 2-dimensional differentiable
 manifold M
```

However, comparison of two cohomology classes is limited the time being:

```
sage: eta = M.diff_form(1, [1,1], name='eta')
sage: H(eta) == u
True
sage: H.one() == u
Traceback (most recent call last):
...
NotImplementedError: comparison via exact forms is currently not supported
```

```
>>> from sage.all import *
>>> eta = M.diff_form(Integer(1), [Integer(1),Integer(1)], name='eta')
>>> H(eta) == u
True
>>> H.one() == u
Traceback (most recent call last):
...
NotImplementedError: comparison via exact forms is currently not supported
```

cup (*other*)

Cup product of two cohomology classes.

INPUT:

- `other` – another cohomology class in the de Rham cohomology

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: C = M.de_rham_complex()
sage: H = C.cohomology()
sage: omega = M.diff_form(1, [1,1], name='omega')
sage: eta = M.diff_form(1, [1,-1], name='eta')
```

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```
sage: H(omega).cup(H(eta))
[omega∧eta]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> C = M.de_rham_complex()
>>> H = C.cohomology()
>>> omega = M.diff_form(Integer(1), [Integer(1),Integer(1)], name='omega')
>>> eta = M.diff_form(Integer(1), [Integer(1),-Integer(1)], name='eta')
>>> H(omega).cup(H(eta))
[omega∧eta]
```

lift()

Return a representative of `self` in the associated de Rham complex.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: C = M.de_rham_complex()
sage: H = C.cohomology()
sage: omega = M.diff_form(2, name='omega')
sage: omega[0,1] = x
sage: omega.display()
omega = x dx∧dy
sage: u = H(omega); u
[omega]
sage: u.representative()
Mixed differential form omega on the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> C = M.de_rham_complex()
>>> H = C.cohomology()
>>> omega = M.diff_form(Integer(2), name='omega')
>>> omega[Integer(0),Integer(1)] = x
>>> omega.display()
omega = x dx∧dy
>>> u = H(omega); u
[omega]
>>> u.representative()
Mixed differential form omega on the 2-dimensional differentiable
 manifold M
```

representative()

Return a representative of `self` in the associated de Rham complex.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: C = M.de_rham_complex()
sage: H = C.cohomology()
sage: omega = M.diff_form(2, name='omega')
sage: omega[0,1] = x
sage: omega.display()
omega = x dx∧dy
sage: u = H(omega); u
[omega]
sage: u.representative()
Mixed differential form omega on the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> C = M.de_rham_complex()
>>> H = C.cohomology()
>>> omega = M.diff_form(Integer(2), name='omega')
>>> omega[Integer(0),Integer(1)] = x
>>> omega.display()
omega = x dx∧dy
>>> u = H(omega); u
[omega]
>>> u.representative()
Mixed differential form omega on the 2-dimensional differentiable
 manifold M
```

class sage.manifolds.differentiable.de_rham_cohomology.**DeRhamCohomologyRing**(*de_rham_complex*)

Bases: `Parent`, `UniqueRepresentation`

The de Rham cohomology ring of a de Rham complex.

This ring is naturally endowed with a multiplication induced by the wedge product, called *cup product*, see `DeRhamCohomologyClass.cup()`.

> **Note**
>
> The current implementation only provides basic features. Comparison via exact forms are not supported at the time being.

INPUT:

- `de_rham_complex` – a de Rham complex, typically an instance of `MixedFormAlgebra`

EXAMPLES:

We define the de Rham cohomology ring on a parallelizable manifold M:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: C = M.de_rham_complex()
```

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```
sage: H = C.cohomology(); H
De Rham cohomology ring on the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> C = M.de_rham_complex()
>>> H = C.cohomology(); H
De Rham cohomology ring on the 2-dimensional differentiable manifold M
```

Its elements are induced by closed differential forms on M:

```
sage: beta = M.diff_form(1, [1,0], name='beta')
sage: beta.display()
beta = dx
sage: d1 = C.differential(1)
sage: d1(beta).display()
dbeta = 0
sage: b = H(beta); b
[beta]
```

```
>>> from sage.all import *
>>> beta = M.diff_form(Integer(1), [Integer(1),Integer(0)], name='beta')
>>> beta.display()
beta = dx
>>> d1 = C.differential(Integer(1))
>>> d1(beta).display()
dbeta = 0
>>> b = H(beta); b
[beta]
```

Cohomology classes can be lifted to the algebra of mixed differential forms:

```
sage: b.representative()
Mixed differential form beta on the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> b.representative()
Mixed differential form beta on the 2-dimensional differentiable
 manifold M
```

The ring admits a zero and unit element:

```
sage: H.zero()
[zero]
sage: H.one()
[one]
```

```
>>> from sage.all import *
>>> H.zero()
```

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```
[zero]
>>> H.one()
[one]
```

Element

alias of *DeRhamCohomologyClass*

one()

Return the one element of `self`.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: C = M.de_rham_complex()
sage: H = C.cohomology()
sage: H.one()
[one]
sage: H.one().representative()
Mixed differential form one on the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> C = M.de_rham_complex()
>>> H = C.cohomology()
>>> H.one()
[one]
>>> H.one().representative()
Mixed differential form one on the 2-dimensional differentiable
 manifold M
```

zero()

Return the zero element of `self`.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: C = M.de_rham_complex()
sage: H = C.cohomology()
sage: H.zero()
[zero]
sage: H.zero().representative()
Mixed differential form zero on the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> C = M.de_rham_complex()
>>> H = C.cohomology()
>>> H.zero()
[zero]
>>> H.zero().representative()
```

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```
Mixed differential form zero on the 2-dimensional differentiable
 manifold M
```

2.12 Alternating Multivector Fields

2.12.1 Multivector Field Modules

The set $A^p(U, \Phi)$ of p-vector fields along a differentiable manifold U with values on a differentiable manifold M via a differentiable map $\Phi : U \to M$ (possibly $U = M$ and $\Phi = \mathrm{Id}_M$) is a module over the algebra $C^k(U)$ of differentiable scalar fields on U. It is a free module if and only if M is parallelizable. Accordingly, two classes implement $A^p(U, \Phi)$:

- `MultivectorModule` for p-vector fields with values on a generic (in practice, not parallelizable) differentiable manifold M
- `MultivectorFreeModule` for p-vector fields with values on a parallelizable manifold M

AUTHORS:

- Eric Gourgoulhon (2017): initial version

REFERENCES:

- R. L. Bishop and S. L. Goldberg (1980) [BG1980]
- C.-M. Marle (1997) [Mar1997]

class `sage.manifolds.differentiable.multivector_module.`**`MultivectorFreeModule`**(*vector_field_module*, *degree*)

Bases: `ExtPowerFreeModule`

Free module of multivector fields of a given degree p (p-vector fields) along a differentiable manifold U with values on a parallelizable manifold M.

Given a differentiable manifold U and a differentiable map $\Phi : U \to M$ to a parallelizable manifold M of dimension n, the set $A^p(U, \Phi)$ of p-vector fields (i.e. alternating tensor fields of type $(p, 0)$) along U with values on M is a free module of rank $\binom{n}{p}$ over $C^k(U)$, the commutative algebra of differentiable scalar fields on U (see `DiffScalarFieldAlgebra`). The standard case of p-vector fields *on* a differentiable manifold M corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

> **Note**
>
> This class implements $A^p(U, \Phi)$ in the case where M is parallelizable; $A^p(U, \Phi)$ is then a *free* module. If M is not parallelizable, the class `MultivectorModule` must be used instead.

INPUT:

- `vector_field_module` – free module $\mathfrak{X}(U, \Phi)$ of vector fields along U associated with the map $\Phi : U \to V$
- `degree` – positive integer; the degree p of the multivector fields

EXAMPLES:

Free module of 2-vector fields on a parallelizable 3-dimensional manifold:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: XM = M.vector_field_module() ; XM
Free module X(M) of vector fields on the 3-dimensional
 differentiable manifold M
sage: A = M.multivector_module(2) ; A
Free module A^2(M) of 2-vector fields on the 3-dimensional
 differentiable manifold M
sage: latex(A)
A^{2}\left(M\right)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> XM = M.vector_field_module() ; XM
Free module X(M) of vector fields on the 3-dimensional
 differentiable manifold M
>>> A = M.multivector_module(Integer(2)) ; A
Free module A^2(M) of 2-vector fields on the 3-dimensional
 differentiable manifold M
>>> latex(A)
A^{2}\left(M\right)
```

`A` is nothing but the second exterior power of `XM`, i.e. we have $A^2(M) = \Lambda^2(\mathfrak{X}(M))$ (see `ExtPowerFreeModule`):

```
sage: A is XM.exterior_power(2)
True
```

```
>>> from sage.all import *
>>> A is XM.exterior_power(Integer(2))
True
```

$A^2(M)$ is a module over the algebra $C^k(M)$ of (differentiable) scalar fields on M:

```
sage: A.category()
Category of finite dimensional modules over Algebra of
 differentiable scalar fields on the 3-dimensional
 differentiable manifold M
sage: CM = M.scalar_field_algebra() ; CM
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
sage: A in Modules(CM)
True
sage: A.base_ring()
Algebra of differentiable scalar fields on
 the 3-dimensional differentiable manifold M
sage: A.base_module()
Free module X(M) of vector fields on
 the 3-dimensional differentiable manifold M
sage: A.base_module() is XM
True
```

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```
sage: A.rank()
3
```

```
>>> from sage.all import *
>>> A.category()
Category of finite dimensional modules over Algebra of
 differentiable scalar fields on the 3-dimensional
 differentiable manifold M
>>> CM = M.scalar_field_algebra() ; CM
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
>>> A in Modules(CM)
True
>>> A.base_ring()
Algebra of differentiable scalar fields on
 the 3-dimensional differentiable manifold M
>>> A.base_module()
Free module X(M) of vector fields on
 the 3-dimensional differentiable manifold M
>>> A.base_module() is XM
True
>>> A.rank()
3
```

Elements can be constructed from A. In particular, `0` yields the zero element of A:

```
sage: A(0)
2-vector field zero on the 3-dimensional differentiable
 manifold M
sage: A(0) is A.zero()
True
```

```
>>> from sage.all import *
>>> A(Integer(0))
2-vector field zero on the 3-dimensional differentiable
 manifold M
>>> A(Integer(0)) is A.zero()
True
```

while nonzero elements are constructed by providing their components in a given vector frame:

```
sage: comp = [[0,3*x,-z],[-3*x,0,4],[z,-4,0]]
sage: a = A(comp, frame=X.frame(), name='a') ; a
2-vector field a on the 3-dimensional differentiable manifold M
sage: a.display()
a = 3*x ∂/∂x∧∂/∂y - z ∂/∂x∧∂/∂z + 4 ∂/∂y∧∂/∂z
```

```
>>> from sage.all import *
>>> comp = [[Integer(0),Integer(3)*x,-z],[-Integer(3)*x,Integer(0),Integer(4)],[z,
↪-Integer(4),Integer(0)]]
>>> a = A(comp, frame=X.frame(), name='a') ; a
2-vector field a on the 3-dimensional differentiable manifold M
```

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```
>>> a.display()
a = 3*x ∂/∂x∧∂/∂y - z ∂/∂x∧∂/∂z + 4 ∂/∂y∧∂/∂z
```

An alternative is to construct the 2-vector field from an empty list of components and to set the nonzero nonredundant components afterwards:

```
sage: a = A([], name='a')
sage: a[0,1] = 3*x  # component in the manifold's default frame
sage: a[0,2] = -z
sage: a[1,2] = 4
sage: a.display()
a = 3*x ∂/∂x∧∂/∂y - z ∂/∂x∧∂/∂z + 4 ∂/∂y∧∂/∂z
```

```
>>> from sage.all import *
>>> a = A([], name='a')
>>> a[Integer(0),Integer(1)] = Integer(3)*x  # component in the manifold's␣
↪default frame
>>> a[Integer(0),Integer(2)] = -z
>>> a[Integer(1),Integer(2)] = Integer(4)
>>> a.display()
a = 3*x ∂/∂x∧∂/∂y - z ∂/∂x∧∂/∂z + 4 ∂/∂y∧∂/∂z
```

The module $A^1(M)$ is nothing but $\mathfrak{X}(M)$ (the free module of vector fields on M):

```
sage: A1 = M.multivector_module(1) ; A1
Free module X(M) of vector fields on the 3-dimensional
 differentiable manifold M
sage: A1 is XM
True
```

```
>>> from sage.all import *
>>> A1 = M.multivector_module(Integer(1)) ; A1
Free module X(M) of vector fields on the 3-dimensional
 differentiable manifold M
>>> A1 is XM
True
```

There is a coercion map $A^p(M) \to T^{(p,0)}(M)$:

```
sage: T20 = M.tensor_field_module((2,0)); T20
Free module T^(2,0)(M) of type-(2,0) tensors fields on the
 3-dimensional differentiable manifold M
sage: T20.has_coerce_map_from(A)
True
```

```
>>> from sage.all import *
>>> T20 = M.tensor_field_module((Integer(2),Integer(0))); T20
Free module T^(2,0)(M) of type-(2,0) tensors fields on the
 3-dimensional differentiable manifold M
>>> T20.has_coerce_map_from(A)
True
```

but of course not in the reverse direction, since not all contravariant tensor field is alternating:

```
sage: A.has_coerce_map_from(T20)
False
```

```
>>> from sage.all import *
>>> A.has_coerce_map_from(T20)
False
```

The coercion map $A^2(M) \to T^{(2,0)}(M)$ in action:

```
sage: T20 = M.tensor_field_module((2,0)) ; T20
Free module T^(2,0)(M) of type-(2,0) tensors fields on the
 3-dimensional differentiable manifold M
sage: ta = T20(a) ; ta
Tensor field a of type (2,0) on the 3-dimensional differentiable
 manifold M
sage: ta.display()
a = 3*x ∂/∂x⊗∂/∂y - z ∂/∂x⊗∂/∂z - 3*x ∂/∂y⊗∂/∂x + 4 ∂/∂y⊗∂/∂z
 + z ∂/∂z⊗∂/∂x - 4 ∂/∂z⊗∂/∂y
sage: a.display()
a = 3*x ∂/∂x∧∂/∂y - z ∂/∂x∧∂/∂z + 4 ∂/∂y∧∂/∂z
sage: ta.symmetries()  # the antisymmetry is preserved
no symmetry;  antisymmetry: (0, 1)
```

```
>>> from sage.all import *
>>> T20 = M.tensor_field_module((Integer(2),Integer(0))) ; T20
Free module T^(2,0)(M) of type-(2,0) tensors fields on the
 3-dimensional differentiable manifold M
>>> ta = T20(a) ; ta
Tensor field a of type (2,0) on the 3-dimensional differentiable
 manifold M
>>> ta.display()
a = 3*x ∂/∂x⊗∂/∂y - z ∂/∂x⊗∂/∂z - 3*x ∂/∂y⊗∂/∂x + 4 ∂/∂y⊗∂/∂z
 + z ∂/∂z⊗∂/∂x - 4 ∂/∂z⊗∂/∂y
>>> a.display()
a = 3*x ∂/∂x∧∂/∂y - z ∂/∂x∧∂/∂z + 4 ∂/∂y∧∂/∂z
>>> ta.symmetries()  # the antisymmetry is preserved
no symmetry;  antisymmetry: (0, 1)
```

There is also coercion to subdomains, which is nothing but the restriction of the multivector field to some subset of its domain:

```
sage: U = M.open_subset('U', coord_def={X: x^2+y^2<1})
sage: B = U.multivector_module(2) ; B
Free module A^2(U) of 2-vector fields on the Open subset U of the
 3-dimensional differentiable manifold M
sage: B.has_coerce_map_from(A)
True
sage: a_U = B(a) ; a_U
2-vector field a on the Open subset U of the 3-dimensional
 differentiable manifold M
sage: a_U.display()
a = 3*x ∂/∂x∧∂/∂y - z ∂/∂x∧∂/∂z + 4 ∂/∂y∧∂/∂z
```

```
>>> from sage.all import *
>>> U = M.open_subset('U', coord_def={X: x**Integer(2)+y**Integer(2)<Integer(1)})
>>> B = U.multivector_module(Integer(2)) ; B
Free module A^2(U) of 2-vector fields on the Open subset U of the
 3-dimensional differentiable manifold M
>>> B.has_coerce_map_from(A)
True
>>> a_U = B(a) ; a_U
2-vector field a on the Open subset U of the 3-dimensional
 differentiable manifold M
>>> a_U.display()
a = 3*x ∂/∂x∧∂/∂y - z ∂/∂x∧∂/∂z + 4 ∂/∂y∧∂/∂z
```

Element

alias of `MultivectorFieldParal`

class `sage.manifolds.differentiable.multivector_module.`**`MultivectorModule`**(*vector_field_module*, *degree*)

Bases: `UniqueRepresentation`, `Parent`

Module of multivector fields of a given degree p (p-vector fields) along a differentiable manifold U with values on a differentiable manifold M.

Given a differentiable manifold U and a differentiable map $\Phi : U \to M$ to a differentiable manifold M, the set $A^p(U, \Phi)$ of p-vector fields (i.e. alternating tensor fields of type $(p, 0)$) along U with values on M is a module over $C^k(U)$, the commutative algebra of differentiable scalar fields on U (see `DiffScalarFieldAlgebra`). The standard case of p-vector fields *on* a differentiable manifold M corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note

This class implements $A^p(U, \Phi)$ in the case where M is not assumed to be parallelizable; the module $A^p(U, \Phi)$ is then not necessarily free. If M is parallelizable, the class `MultivectorFreeModule` must be used instead.

INPUT:

- `vector_field_module` – module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on M via the map $\Phi : U \to M$
- `degree` – positive integer; the degree p of the multivector fields

EXAMPLES:

Module of 2-vector fields on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y),
....:  intersection_name='W', restrictions1= x>0,
....:  restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
```

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```
sage: XM = M.vector_field_module() ; XM
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
sage: A = M.multivector_module(2) ; A
Module A^2(M) of 2-vector fields on the 2-dimensional
 differentiable manifold M
sage: latex(A)
A^{2}\left(M\right)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)    # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv = V.
↪chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> transf = c_xy.transition_map(c_uv, (x+y, x-y),
...  intersection_name='W', restrictions1= x>Integer(0),
...  restrictions2= u+v>Integer(0))
>>> inv = transf.inverse()
>>> W = U.intersection(V)
>>> eU = c_xy.frame() ; eV = c_uv.frame()
>>> XM = M.vector_field_module() ; XM
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
>>> A = M.multivector_module(Integer(2)) ; A
Module A^2(M) of 2-vector fields on the 2-dimensional
 differentiable manifold M
>>> latex(A)
A^{2}\left(M\right)
```

`A` is nothing but the second exterior power of `XM`, i.e. we have $A^2(M) = \Lambda^2(\mathfrak{X}(M))$:

```
sage: A is XM.exterior_power(2)
True
```

```
>>> from sage.all import *
>>> A is XM.exterior_power(Integer(2))
True
```

Modules of multivector fields are unique:

```
sage: A is M.multivector_module(2)
True
```

```
>>> from sage.all import *
>>> A is M.multivector_module(Integer(2))
True
```

$A^2(M)$ is a module over the algebra $C^k(M)$ of (differentiable) scalar fields on M:

```
sage: A.category()
Category of modules over Algebra of differentiable scalar fields
```

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```
 on the 2-dimensional differentiable manifold M
sage: CM = M.scalar_field_algebra() ; CM
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
sage: A in Modules(CM)
True
sage: A.base_ring() is CM
True
sage: A.base_module()
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
sage: A.base_module() is XM
True
```

```
>>> from sage.all import *
>>> A.category()
Category of modules over Algebra of differentiable scalar fields
 on the 2-dimensional differentiable manifold M
>>> CM = M.scalar_field_algebra() ; CM
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
>>> A in Modules(CM)
True
>>> A.base_ring() is CM
True
>>> A.base_module()
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
>>> A.base_module() is XM
True
```

Elements can be constructed from `A()`. In particular, `0` yields the zero element of `A`:

```
sage: z = A(0) ; z
2-vector field zero on the 2-dimensional differentiable
 manifold M
sage: z.display(eU)
zero = 0
sage: z.display(eV)
zero = 0
sage: z is A.zero()
True
```

```
>>> from sage.all import *
>>> z = A(Integer(0)) ; z
2-vector field zero on the 2-dimensional differentiable
 manifold M
>>> z.display(eU)
zero = 0
>>> z.display(eV)
zero = 0
>>> z is A.zero()
```

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```
True
```

while nonzero elements are constructed by providing their components in a given vector frame:

```
sage: a = A([[0,3*x],[-3*x,0]], frame=eU, name='a') ; a
2-vector field a on the 2-dimensional differentiable manifold M
sage: a.add_comp_by_continuation(eV, W, c_uv) # finishes initializ. of a
sage: a.display(eU)
a = 3*x ∂/∂x∧∂/∂y
sage: a.display(eV)
a = (-3*u - 3*v) ∂/∂u∧∂/∂v
```

```
>>> from sage.all import *
>>> a = A([[Integer(0),Integer(3)*x],[-Integer(3)*x,Integer(0)]], frame=eU, name=
↪'a') ; a
2-vector field a on the 2-dimensional differentiable manifold M
>>> a.add_comp_by_continuation(eV, W, c_uv) # finishes initializ. of a
>>> a.display(eU)
a = 3*x ∂/∂x∧∂/∂y
>>> a.display(eV)
a = (-3*u - 3*v) ∂/∂u∧∂/∂v
```

An alternative is to construct the 2-vector field from an empty list of components and to set the nonzero nonredundant components afterwards:

```
sage: a = A([], name='a')
sage: a[eU,0,1] = 3*x
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: a.display(eU)
a = 3*x ∂/∂x∧∂/∂y
sage: a.display(eV)
a = (-3*u - 3*v) ∂/∂u∧∂/∂v
```

```
>>> from sage.all import *
>>> a = A([], name='a')
>>> a[eU,Integer(0),Integer(1)] = Integer(3)*x
>>> a.add_comp_by_continuation(eV, W, c_uv)
>>> a.display(eU)
a = 3*x ∂/∂x∧∂/∂y
>>> a.display(eV)
a = (-3*u - 3*v) ∂/∂u∧∂/∂v
```

The module $A^1(M)$ is nothing but the dual of $\mathfrak{X}(M)$ (the module of vector fields on M):

```
sage: A1 = M.multivector_module(1) ; A1
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
sage: A1 is XM
True
```

```
>>> from sage.all import *
>>> A1 = M.multivector_module(Integer(1)) ; A1
```

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```
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
>>> A1 is XM
True
```

There is a coercion map $A^p(M) \to T^{(p,0)}(M)$:

```
sage: T20 = M.tensor_field_module((2,0)) ; T20
Module T^(2,0)(M) of type-(2,0) tensors fields on the
 2-dimensional differentiable manifold M
sage: T20.has_coerce_map_from(A)
True
```

```
>>> from sage.all import *
>>> T20 = M.tensor_field_module((Integer(2),Integer(0))) ; T20
Module T^(2,0)(M) of type-(2,0) tensors fields on the
 2-dimensional differentiable manifold M
>>> T20.has_coerce_map_from(A)
True
```

but of course not in the reverse direction, since not all contravariant tensor field is alternating:

```
sage: A.has_coerce_map_from(T20)
False
```

```
>>> from sage.all import *
>>> A.has_coerce_map_from(T20)
False
```

The coercion map $A^2(M) \to T^{(2,0)}(M)$ in action:

```
sage: ta = T20(a) ; ta
Tensor field a of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: ta.display(eU)
a = 3*x ∂/∂x⊗∂/∂y - 3*x ∂/∂y⊗∂/∂x
sage: a.display(eU)
a = 3*x ∂/∂x∧∂/∂y
sage: ta.display(eV)
a = (-3*u - 3*v) ∂/∂u⊗∂/∂v + (3*u + 3*v) ∂/∂v⊗∂/∂u
sage: a.display(eV)
a = (-3*u - 3*v) ∂/∂u∧∂/∂v
```

```
>>> from sage.all import *
>>> ta = T20(a) ; ta
Tensor field a of type (2,0) on the 2-dimensional differentiable
 manifold M
>>> ta.display(eU)
a = 3*x ∂/∂x⊗∂/∂y - 3*x ∂/∂y⊗∂/∂x
>>> a.display(eU)
a = 3*x ∂/∂x∧∂/∂y
>>> ta.display(eV)
```

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```
a = (-3*u - 3*v) ∂/∂u⊗∂/∂v + (3*u + 3*v) ∂/∂v⊗∂/∂u
>>> a.display(eV)
a = (-3*u - 3*v) ∂/∂u∧∂/∂v
```

There is also coercion to subdomains, which is nothing but the restriction of the multivector field to some subset of its domain:

```
sage: A2U = U.multivector_module(2) ; A2U
Free module A^2(U) of 2-vector fields on the Open subset U of
 the 2-dimensional differentiable manifold M
sage: A2U.has_coerce_map_from(A)
True
sage: a_U = A2U(a) ; a_U
2-vector field a on the Open subset U of the 2-dimensional
 differentiable manifold M
sage: a_U.display(eU)
a = 3*x ∂/∂x∧∂/∂y
```

```
>>> from sage.all import *
>>> A2U = U.multivector_module(Integer(2)) ; A2U
Free module A^2(U) of 2-vector fields on the Open subset U of
 the 2-dimensional differentiable manifold M
>>> A2U.has_coerce_map_from(A)
True
>>> a_U = A2U(a) ; a_U
2-vector field a on the Open subset U of the 2-dimensional
 differentiable manifold M
>>> a_U.display(eU)
a = 3*x ∂/∂x∧∂/∂y
```

Element

alias of *MultivectorField*

base_module()

Return the vector field module on which the multivector field module `self` is constructed.

OUTPUT:

- a *VectorFieldModule* representing the module on which `self` is defined

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: A2 = M.multivector_module(2) ; A2
Module A^2(M) of 2-vector fields on the 3-dimensional
 differentiable manifold M
sage: A2.base_module()
Module X(M) of vector fields on the 3-dimensional
 differentiable manifold M
sage: A2.base_module() is M.vector_field_module()
True
sage: U = M.open_subset('U')
sage: A2U = U.multivector_module(2) ; A2U
Module A^2(U) of 2-vector fields on the Open subset U of the
```

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```
 3-dimensional differentiable manifold M
sage: A2U.base_module()
Module X(U) of vector fields on the Open subset U of the
 3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> A2 = M.multivector_module(Integer(2)) ; A2
Module A^2(M) of 2-vector fields on the 3-dimensional
 differentiable manifold M
>>> A2.base_module()
Module X(M) of vector fields on the 3-dimensional
 differentiable manifold M
>>> A2.base_module() is M.vector_field_module()
True
>>> U = M.open_subset('U')
>>> A2U = U.multivector_module(Integer(2)) ; A2U
Module A^2(U) of 2-vector fields on the Open subset U of the
 3-dimensional differentiable manifold M
>>> A2U.base_module()
Module X(U) of vector fields on the Open subset U of the
 3-dimensional differentiable manifold M
```

degree()

Return the degree of the multivector fields in `self`.

OUTPUT: integer p such that `self` is a set of p-vector fields

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: M.multivector_module(2).degree()
2
sage: M.multivector_module(3).degree()
3
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> M.multivector_module(Integer(2)).degree()
2
>>> M.multivector_module(Integer(3)).degree()
3
```

zero()

Return the zero of `self`.

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: A2 = M.multivector_module(2)
sage: A2.zero()
2-vector field zero on the 3-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> A2 = M.multivector_module(Integer(2))
>>> A2.zero()
2-vector field zero on the 3-dimensional differentiable
 manifold M
```

2.12.2 Multivector Fields

Let U and M be two differentiable manifolds. Given a positive integer p and a differentiable map $\Phi : U \to M$, a *multivector field of degree* p, or p*-vector field*, *along* U *with values on* M is a field along U of alternating contravariant tensors of rank p in the tangent spaces to M. The standard case of a multivector field *on* a differentiable manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of **R**).

Two classes implement multivector fields, depending whether the manifold M is parallelizable:

- `MultivectorFieldParal` when M is parallelizable
- `MultivectorField` when M is not assumed parallelizable.

AUTHORS:

- Eric Gourgoulhon (2017): initial version

REFERENCES:

- R. L. Bishop and S. L. Goldberg (1980) [BG1980]
- C.-M. Marle (1997) [Mar1997]

class `sage.manifolds.differentiable.multivectorfield.`**`MultivectorField`**(*vector_field_module*, *degree*, *name=None*, *latex_name=None*)

Bases: `TensorField`

Multivector field with values on a generic (i.e. a priori not parallelizable) differentiable manifold.

Given a differentiable manifold U, a differentiable map $\Phi : U \to M$ to a differentiable manifold M and a positive integer p, a *multivector field of degree* p (or p*-vector field*) *along* U *with values on* $M \supset \Phi(U)$ is a differentiable map

$$a : U \longrightarrow T^{(p,0)}M$$

($T^{(p,0)}M$ being the tensor bundle of type $(p, 0)$ over M) such that

$$\forall x \in U, \quad a(x) \in \Lambda^p(T_{\Phi(x)}M),$$

where $T_{\Phi(x)}M$ is the vector space tangent to M at $\Phi(x)$ and Λ^p stands for the exterior power of degree p (cf. `ExtPowerFreeModule`). In other words, $a(x)$ is an alternating contravariant tensor of degree p of the tangent vector space $T_{\Phi(x)}M$.

The standard case of a multivector field *on* a manifold M corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of **R**).

> **Note**
>
> If M is parallelizable, the class `MultivectorFieldParal` must be used instead.

INPUT:

- `vector_field_module` – module $\mathfrak{X}(U,\Phi)$ of vector fields along U with values on M via the map Φ
- `degree` – the degree of the multivector field (i.e. its tensor rank)
- `name` – (default: `None`) name given to the multivector field
- `latex_name` – (default: `None`) LaTeX symbol to denote the multivector field; if none is provided, the LaTeX symbol is set to `name`

EXAMPLES:

Multivector field of degree 2 on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                         intersection_name='W',
....:                         restrictions1= x>0, restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: a = M.multivector_field(2, name='a') ; a
2-vector field a on the 2-dimensional differentiable manifold M
sage: a.parent()
Module A^2(M) of 2-vector fields on the 2-dimensional differentiable
 manifold M
sage: a.degree()
2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv = V.
↪chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
...                         intersection_name='W',
...                         restrictions1= x>Integer(0), restrictions2= u+v>
↪Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V)
>>> eU = c_xy.frame() ; eV = c_uv.frame()
>>> a = M.multivector_field(Integer(2), name='a') ; a
2-vector field a on the 2-dimensional differentiable manifold M
>>> a.parent()
Module A^2(M) of 2-vector fields on the 2-dimensional differentiable
 manifold M
>>> a.degree()
2
```

Setting the components of `a`:

```
sage: a[eU,0,1] = x*y^2 + 2*x
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: a.display(eU)
a = (x*y^2 + 2*x) ∂/∂x∧∂/∂y
sage: a.display(eV)
a = (-1/4*u^3 + 1/4*u*v^2 - 1/4*v^3 + 1/4*(u^2 - 8)*v - 2*u) ∂/∂u∧∂/∂v
```

```
>>> from sage.all import *
>>> a[eU,Integer(0),Integer(1)] = x*y**Integer(2) + Integer(2)*x
>>> a.add_comp_by_continuation(eV, W, c_uv)
>>> a.display(eU)
a = (x*y^2 + 2*x) ∂/∂x∧∂/∂y
>>> a.display(eV)
a = (-1/4*u^3 + 1/4*u*v^2 - 1/4*v^3 + 1/4*(u^2 - 8)*v - 2*u) ∂/∂u∧∂/∂v
```

It is also possible to set the components while defining the 2-vector field definition, via a dictionary whose keys are the vector frames:

```
sage: a1 = M.multivector_field(2, {eU: [[0, x*y^2 + 2*x],
....:                                   [-x*y^2 - 2*x, 0]]}, name='a')
sage: a1.add_comp_by_continuation(eV, W, c_uv)
sage: a1 == a
True
```

```
>>> from sage.all import *
>>> a1 = M.multivector_field(Integer(2), {eU: [[Integer(0), x*y**Integer(2) +␣
↪Integer(2)*x],
...                                   [-x*y**Integer(2) - Integer(2)*x,␣
↪Integer(0)]]}, name='a')
>>> a1.add_comp_by_continuation(eV, W, c_uv)
>>> a1 == a
True
```

The exterior product of two vector fields is a 2-vector field:

```
sage: a = M.vector_field({eU: [-y, x]}, name='a')
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: b = M.vector_field({eU: [1+x*y, x^2]}, name='b')
sage: b.add_comp_by_continuation(eV, W, c_uv)
sage: s = a.wedge(b) ; s
2-vector field a∧b on the 2-dimensional differentiable manifold M
sage: s.display(eU)
a∧b = (-2*x^2*y - x) ∂/∂x∧∂/∂y
sage: s.display(eV)
a∧b = (1/2*u^3 - 1/2*u*v^2 - 1/2*v^3 + 1/2*(u^2 + 2)*v + u) ∂/∂u∧∂/∂v
```

```
>>> from sage.all import *
>>> a = M.vector_field({eU: [-y, x]}, name='a')
>>> a.add_comp_by_continuation(eV, W, c_uv)
>>> b = M.vector_field({eU: [Integer(1)+x*y, x**Integer(2)]}, name='b')
>>> b.add_comp_by_continuation(eV, W, c_uv)
>>> s = a.wedge(b) ; s
```

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```
2-vector field a∧b on the 2-dimensional differentiable manifold M
>>> s.display(eU)
a∧b = (-2*x^2*y - x) ∂/∂x∧∂/∂y
>>> s.display(eV)
a∧b = (1/2*u^3 - 1/2*u*v^2 - 1/2*v^3 + 1/2*(u^2 + 2)*v + u) ∂/∂u∧∂/∂v
```

Multiplying a 2-vector field by a scalar field results in another 2-vector field:

```
sage: f = M.scalar_field({c_xy: (x+y)^2, c_uv: u^2}, name='f')
sage: s = f*s ; s
2-vector field f*(a∧b) on the 2-dimensional differentiable manifold M
sage: s.display(eU)
f*(a∧b) = (-2*x^2*y^3 - x^3 - (4*x^3 + x)*y^2 - 2*(x^4 + x^2)*y) ∂/∂x∧∂/∂y
sage: s.display(eV)
f*(a∧b) = (1/2*u^5 - 1/2*u^3*v^2 - 1/2*u^2*v^3 + u^3 + 1/2*(u^4 + 2*u^2)*v)
  ∂/∂u∧∂/∂v
```

```
>>> from sage.all import *
>>> f = M.scalar_field({c_xy: (x+y)**Integer(2), c_uv: u**Integer(2)}, name='f')
>>> s = f*s ; s
2-vector field f*(a∧b) on the 2-dimensional differentiable manifold M
>>> s.display(eU)
f*(a∧b) = (-2*x^2*y^3 - x^3 - (4*x^3 + x)*y^2 - 2*(x^4 + x^2)*y) ∂/∂x∧∂/∂y
>>> s.display(eV)
f*(a∧b) = (1/2*u^5 - 1/2*u^3*v^2 - 1/2*u^2*v^3 + u^3 + 1/2*(u^4 + 2*u^2)*v)
  ∂/∂u∧∂/∂v
```

bracket(*other*)

Return the Schouten-Nijenhuis bracket of `self` with another multivector field.

The Schouten-Nijenhuis bracket extends the Lie bracket of vector fields (cf. `bracket()`) to multivector fields.

Denoting by $A^p(M)$ the $C^k(M)$-module of p-vector fields on the C^k-differentiable manifold M over the field K (cf. `MultivectorModule`), the *Schouten-Nijenhuis bracket* is a K-bilinear map

$$\begin{array}{ccc} A^p(M) \times A^q(M) & \longrightarrow & A^{p+q-1}(M) \\ (a,b) & \longmapsto & [a,b] \end{array}$$

which obeys the following properties:

- if $p=0$ and $q=0$, (i.e. a and b are two scalar fields), $[a,b]=0$
- if $p=0$ (i.e. a is a scalar field) and $q\geq 1$, $[a,b]=-\iota_{\mathrm{d}a}b$ (minus the interior product of the differential of a by b)
- if $p=1$ (i.e. a is a vector field), $[a,b]=\mathcal{L}_a b$ (the Lie derivative of b along a)
- $[a,b]=-(-1)^{(p-1)(q-1)}[b,a]$
- for any multivector field c and $(a,b)\in A^p(M)\times A^q(M)$, $[a,.]$ obeys the *graded Leibniz rule*

$$[a,b\wedge c]=[a,b]\wedge c+(-1)^{(p-1)q}b\wedge[a,c]$$

- for $(a,b,c)\in A^p(M)\times A^q(M)\times A^r(M)$, the *graded Jacobi identity* holds:

$$(-1)^{(p-1)(r-1)}[a,[b,c]]+(-1)^{(q-1)(p-1)}[b,[c,a]]+(-1)^{(r-1)(q-1)}[c,[a,b]]=0$$

Note

There are two definitions of the Schouten-Nijenhuis bracket in the literature, which differ from each other when p is even by an overall sign. The definition adopted here is that of [Mar1997], [Kos1985] and Wikipedia article Schouten-Nijenhuis_bracket. The other definition, adopted e.g. by [Nij1955], [Lic1977] and [Vai1994], is $[a, b]' = (-1)^{p+1}[a, b]$.

INPUT:

- `other` – a multivector field

OUTPUT:

- instance of `MultivectorField` (or of `DiffScalarField` if $p = 1$ and $q = 0$) representing the Schouten-Nijenhuis bracket $[a, b]$, where a is `self` and b is `other`

EXAMPLES:

Bracket of two vector fields on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1) # the sphere S^2
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: c_xy.<x,y> = U.chart() # stereographic coord. North
sage: c_uv.<u,v> = V.chart() # stereographic coord. South
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V) # The complement of the two poles
sage: e_xy = c_xy.frame() ; e_uv = c_uv.frame()
sage: a = M.vector_field({e_xy: [y, x]}, name='a')
sage: a.add_comp_by_continuation(e_uv, W, c_uv)
sage: b = M.vector_field({e_xy: [x*y, x-y]}, name='b')
sage: b.add_comp_by_continuation(e_uv, W, c_uv)
sage: s = a.bracket(b); s
Vector field [a,b] on the 2-dimensional differentiable manifold S^2
sage: s.display(e_xy)
[a,b] = (x^2 + y^2 - x + y) ∂/∂x + (-(x - 1)*y - x) ∂/∂y
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', start_index=Integer(1)) # the sphere S^2
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coord. North
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coord. South
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                 intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                 restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V) # The complement of the two poles
```

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```
>>> e_xy = c_xy.frame() ; e_uv = c_uv.frame()
>>> a = M.vector_field({e_xy: [y, x]}, name='a')
>>> a.add_comp_by_continuation(e_uv, W, c_uv)
>>> b = M.vector_field({e_xy: [x*y, x-y]}, name='b')
>>> b.add_comp_by_continuation(e_uv, W, c_uv)
>>> s = a.bracket(b); s
Vector field [a,b] on the 2-dimensional differentiable manifold S^2
>>> s.display(e_xy)
[a,b] = (x^2 + y^2 - x + y) ∂/∂x + (-(x - 1)*y - x) ∂/∂y
```

For two vector fields, the bracket coincides with the Lie derivative:

```
sage: s == b.lie_derivative(a)
True
```

```
>>> from sage.all import *
>>> s == b.lie_derivative(a)
True
```

Schouten-Nijenhuis bracket of a 2-vector field and a 1-vector field:

```
sage: c = a.wedge(b); c
2-vector field a∧b on the 2-dimensional differentiable
 manifold S^2
sage: s = c.bracket(a); s
2-vector field [a∧b,a] on the 2-dimensional differentiable
 manifold S^2
sage: s.display(e_xy)
[a∧b,a] = (x^3 + (2*x - 1)*y^2 - x^2 + 2*x*y) ∂/∂x∧∂/∂y
```

```
>>> from sage.all import *
>>> c = a.wedge(b); c
2-vector field a∧b on the 2-dimensional differentiable
 manifold S^2
>>> s = c.bracket(a); s
2-vector field [a∧b,a] on the 2-dimensional differentiable
 manifold S^2
>>> s.display(e_xy)
[a∧b,a] = (x^3 + (2*x - 1)*y^2 - x^2 + 2*x*y) ∂/∂x∧∂/∂y
```

Since a is a vector field, we have in this case:

```
sage: s == - c.lie_derivative(a)
True
```

```
>>> from sage.all import *
>>> s == - c.lie_derivative(a)
True
```

See also

MultivectorFieldParal.bracket() for more examples and check of standards identities involving the Schouten-Nijenhuis bracket

degree()

Return the degree of `self`.

OUTPUT: integer p such that `self` is a p-vector field

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: a = M.multivector_field(2); a
2-vector field on the 3-dimensional differentiable manifold M
sage: a.degree()
2
sage: b = M.vector_field(); b
Vector field on the 3-dimensional differentiable manifold M
sage: b.degree()
1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> a = M.multivector_field(Integer(2)); a
2-vector field on the 3-dimensional differentiable manifold M
>>> a.degree()
2
>>> b = M.vector_field(); b
Vector field on the 3-dimensional differentiable manifold M
>>> b.degree()
1
```

interior_product(*form*)

Interior product with a differential form.

If `self` is a multivector field A of degree p and B is a differential form of degree $q \geq p$ on the same manifold as A, the interior product of A by B is the differential form $\iota_A B$ of degree $q - p$ defined by

$$(\iota_A B)_{i_1 \dots i_{q-p}} = A^{k_1 \dots k_p} B_{k_1 \dots k_p i_1 \dots i_{q-p}}$$

Note

`A.interior_product(B)` yields the same result as `A.contract(0,..., p-1, B, 0,..., p-1)` (cf. *contract()*), but `interior_product` is more efficient, the alternating character of A being not used to reduce the computation in *contract()*

INPUT:

- `form` – differential form B (instance of *DiffForm*); the degree of B must be at least equal to the degree of `self`

OUTPUT:

- scalar field (case $p = q$) or *DiffForm* (case $p < q$) representing the interior product $\iota_A B$, where A is `self`

See also:

`interior_product()` for the interior product of a differential form with a multivector field

EXAMPLES:

Interior product of a vector field ($p = 1$) with a 2-form ($q = 2$) on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1) # the sphere S^2
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: c_xy.<x,y> = U.chart() # stereographic coord. North
sage: c_uv.<u,v> = V.chart() # stereographic coord. South
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V) # The complement of the two poles
sage: e_xy = c_xy.frame() ; e_uv = c_uv.frame()
sage: a = M.vector_field({e_xy: [-y, x]}, name='a')
sage: a.add_comp_by_continuation(e_uv, W, c_uv)
sage: b = M.diff_form(2, name='b')
sage: b[e_xy,1,2] = 4/(x^2+y^2+1)^2   # the standard area 2-form
sage: b.add_comp_by_continuation(e_uv, W, c_uv)
sage: b.display(e_xy)
b = 4/(x^2 + y^2 + 1)^2 dx∧dy
sage: b.display(e_uv)
b = -4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) du∧dv
sage: s = a.interior_product(b); s
1-form i_a b on the 2-dimensional differentiable manifold S^2
sage: s.display(e_xy)
i_a b = -4*x/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) dx
 - 4*y/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) dy
sage: s.display(e_uv)
 i_a b = 4*u/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) du
  + 4*v/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) dv
sage: s == a.contract(b)
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', start_index=Integer(1)) # the sphere S^2
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2) #␣
↪stereographic coord. North
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2) #␣
↪stereographic coord. South
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                 intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                 restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
```

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```
>>> W = U.intersection(V) # The complement of the two poles
>>> e_xy = c_xy.frame() ; e_uv = c_uv.frame()
>>> a = M.vector_field({e_xy: [-y, x]}, name='a')
>>> a.add_comp_by_continuation(e_uv, W, c_uv)
>>> b = M.diff_form(Integer(2), name='b')
>>> b[e_xy,Integer(1),Integer(2)] = Integer(4)/
↪(x**Integer(2)+y**Integer(2)+Integer(1))**Integer(2)    # the standard area␣
↪2-form
>>> b.add_comp_by_continuation(e_uv, W, c_uv)
>>> b.display(e_xy)
b = 4/(x^2 + y^2 + 1)^2 dx∧dy
>>> b.display(e_uv)
b = -4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) du∧dv
>>> s = a.interior_product(b); s
1-form i_a b on the 2-dimensional differentiable manifold S^2
>>> s.display(e_xy)
i_a b = -4*x/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) dx
 - 4*y/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) dy
>>> s.display(e_uv)
 i_a b = 4*u/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) du
  + 4*v/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) dv
>>> s == a.contract(b)
True
```

Example with $p = 2$ and $q = 2$:

```
sage: a = M.multivector_field(2, name='a')
sage: a[e_xy,1,2] = x*y
sage: a.add_comp_by_continuation(e_uv, W, c_uv)
sage: a.display(e_xy)
a = x*y ∂/∂x∧∂/∂y
sage: a.display(e_uv)
a = -u*v ∂/∂u∧∂/∂v
sage: s = a.interior_product(b); s
Scalar field i_a b on the 2-dimensional differentiable manifold S^2
sage: s.display()
i_a b: S^2 → ℝ
on U: (x, y) ↦ 8*x*y/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1)
on V: (u, v) ↦ 8*u*v/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1)
```

```
>>> from sage.all import *
>>> a = M.multivector_field(Integer(2), name='a')
>>> a[e_xy,Integer(1),Integer(2)] = x*y
>>> a.add_comp_by_continuation(e_uv, W, c_uv)
>>> a.display(e_xy)
a = x*y ∂/∂x∧∂/∂y
>>> a.display(e_uv)
a = -u*v ∂/∂u∧∂/∂v
>>> s = a.interior_product(b); s
Scalar field i_a b on the 2-dimensional differentiable manifold S^2
>>> s.display()
i_a b: S^2 → ℝ
```

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```
on U: (x, y) ↦ 8*x*y/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1)
on V: (u, v) ↦ 8*u*v/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1)
```

Some checks:

```
sage: s == a.contract(0, 1, b, 0, 1)
True
sage: s.restrict(U) == 2 * a[[e_xy,1,2]] * b[[e_xy,1,2]]
True
sage: s.restrict(V) == 2 * a[[e_uv,1,2]] * b[[e_uv,1,2]]
True
```

```
>>> from sage.all import *
>>> s == a.contract(Integer(0), Integer(1), b, Integer(0), Integer(1))
True
>>> s.restrict(U) == Integer(2) * a[[e_xy,Integer(1),Integer(2)]] * b[[e_xy,
↪Integer(1),Integer(2)]]
True
>>> s.restrict(V) == Integer(2) * a[[e_uv,Integer(1),Integer(2)]] * b[[e_uv,
↪Integer(1),Integer(2)]]
True
```

wedge(*other*)

Exterior product with another multivector field.

INPUT:

- `other` – another multivector field (on the same manifold)

OUTPUT:

- instance of *MultivectorField* representing the exterior product `self ∧ other`

EXAMPLES:

Exterior product of two vector fields on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1) # the sphere S^2
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: c_xy.<x,y> = U.chart() # stereographic coord. North
sage: c_uv.<u,v> = V.chart() # stereographic coord. South
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                  intersection_name='W', restrictions1= x^2+y^2!=0,
....:                  restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)  # The complement of the two poles
sage: e_xy = c_xy.frame() ; e_uv = c_uv.frame()
sage: a = M.vector_field({e_xy: [y, x]}, name='a')
sage: a.add_comp_by_continuation(e_uv, W, c_uv)
sage: b = M.vector_field({e_xy: [x^2 + y^2, y]}, name='b')
sage: b.add_comp_by_continuation(e_uv, W, c_uv)
sage: c = a.wedge(b); c
2-vector field a∧b on the 2-dimensional differentiable
 manifold S^2
```

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```
sage: c.display(e_xy)
a∧b = (-x^3 - (x - 1)*y^2) ∂/∂x∧∂/∂y
sage: c.display(e_uv)
a∧b = (-v^2 + u) ∂/∂u∧∂/∂v
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', start_index=Integer(1)) # the sphere S^2
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic coord. North
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic coord. South
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                 intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                 restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V) # The complement of the two poles
>>> e_xy = c_xy.frame() ; e_uv = c_uv.frame()
>>> a = M.vector_field({e_xy: [y, x]}, name='a')
>>> a.add_comp_by_continuation(e_uv, W, c_uv)
>>> b = M.vector_field({e_xy: [x**Integer(2) + y**Integer(2), y]}, name='b')
>>> b.add_comp_by_continuation(e_uv, W, c_uv)
>>> c = a.wedge(b); c
2-vector field a∧b on the 2-dimensional differentiable
 manifold S^2
>>> c.display(e_xy)
a∧b = (-x^3 - (x - 1)*y^2) ∂/∂x∧∂/∂y
>>> c.display(e_uv)
a∧b = (-v^2 + u) ∂/∂u∧∂/∂v
```

class sage.manifolds.differentiable.multivectorfield.**MultivectorFieldParal**(*vector_field_module*, *degree*, *name=None*, *latex_name=None*)

Bases: `AlternatingContrTensor`, *`TensorFieldParal`*

Multivector field with values on a parallelizable manifold.

Given a differentiable manifold U, a differentiable map $\Phi : U \to M$ to a parallelizable manifold M and a positive integer p, a *multivector field of degree* p (or p*-vector field*) *along* U *with values on* $M \supset \Phi(U)$ is a differentiable map

$$a : U \longrightarrow T^{(p,0)}M$$

($T^{(p,0)}M$ being the tensor bundle of type $(p,0)$ over M) such that

$$\forall x \in U, \quad a(x) \in \Lambda^p(T_{\Phi(x)}M),$$

where $T_{\Phi(x)}M$ is the vector space tangent to M at $\Phi(x)$ and Λ^p stands for the exterior power of degree p (cf. `ExtPowerFreeModule`). In other words, $a(x)$ is an alternating contravariant tensor of degree p of the tangent vector space $T_{\Phi(x)}M$.

The standard case of a multivector field *on* a manifold M corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note: If M is not parallelizable, the class `MultivectorField` must be used instead.

INPUT:

- `vector_field_module` – free module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on M via the map Φ
- `degree` – the degree of the multivector field (i.e. its tensor rank)
- `name` – (default: `None`) name given to the multivector field
- `latex_name` – (default: `None`) LaTeX symbol to denote the multivector field; if none is provided, the LaTeX symbol is set to `name`

EXAMPLES:

A 2-vector field on a 4-dimensional manifold:

```
sage: M = Manifold(4, 'M')
sage: c_txyz.<t,x,y,z> = M.chart()
sage: a = M.multivector_field(2, name='a') ; a
2-vector field a on the 4-dimensional differentiable manifold M
sage: a.parent()
Free module A^2(M) of 2-vector fields on the 4-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M')
>>> c_txyz = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = c_txyz._first_
↪ngens(4)
>>> a = M.multivector_field(Integer(2), name='a') ; a
2-vector field a on the 4-dimensional differentiable manifold M
>>> a.parent()
Free module A^2(M) of 2-vector fields on the 4-dimensional
 differentiable manifold M
```

A multivector field is a tensor field of purely contravariant type:

```
sage: a.tensor_type()
(2, 0)
```

```
>>> from sage.all import *
>>> a.tensor_type()
(2, 0)
```

It is antisymmetric, its components being `CompFullyAntiSym`:

```
sage: a.symmetries()
no symmetry;  antisymmetry: (0, 1)
sage: a[0,1] = 2*x
sage: a[1,0]
```

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```
-2*x
sage: a.comp()
Fully antisymmetric 2-indices components w.r.t. Coordinate frame
 (M, (∂/∂t,∂/∂x,∂/∂y,∂/∂z))
sage: type(a.comp())
<class 'sage.tensor.modules.comp.CompFullyAntiSym'>
```

```
>>> from sage.all import *
>>> a.symmetries()
no symmetry;  antisymmetry: (0, 1)
>>> a[Integer(0),Integer(1)] = Integer(2)*x
>>> a[Integer(1),Integer(0)]
-2*x
>>> a.comp()
Fully antisymmetric 2-indices components w.r.t. Coordinate frame
 (M, (∂/∂t,∂/∂x,∂/∂y,∂/∂z))
>>> type(a.comp())
<class 'sage.tensor.modules.comp.CompFullyAntiSym'>
```

Setting a component with repeated indices to a nonzero value results in an error:

```
sage: a[1,1] = 3
Traceback (most recent call last):
...
ValueError: by antisymmetry, the component cannot have a nonzero value
 for the indices (1, 1)
sage: a[1,1] = 0  # OK, albeit useless
sage: a[1,2] = 3  # OK
```

```
>>> from sage.all import *
>>> a[Integer(1),Integer(1)] = Integer(3)
Traceback (most recent call last):
...
ValueError: by antisymmetry, the component cannot have a nonzero value
 for the indices (1, 1)
>>> a[Integer(1),Integer(1)] = Integer(0)  # OK, albeit useless
>>> a[Integer(1),Integer(2)] = Integer(3)  # OK
```

The expansion of a multivector field with respect to a given frame is displayed via the method `display()`:

```
sage: a.display() # expansion w.r.t. the default frame
a = 2*x ∂/∂t∧∂/∂x + 3 ∂/∂x∧∂/∂y
sage: latex(a.display()) # output for the notebook
a = 2 \, x \frac{\partial}{\partial t }\wedge \frac{\partial}{\partial x }
 + 3 \frac{\partial}{\partial x }\wedge \frac{\partial}{\partial y }
```

```
>>> from sage.all import *
>>> a.display() # expansion w.r.t. the default frame
a = 2*x ∂/∂t∧∂/∂x + 3 ∂/∂x∧∂/∂y
>>> latex(a.display()) # output for the notebook
a = 2 \, x \frac{\partial}{\partial t }\wedge \frac{\partial}{\partial x }
 + 3 \frac{\partial}{\partial x }\wedge \frac{\partial}{\partial y }
```

Multivector fields can be added or subtracted:

```
sage: b = M.multivector_field(2)
sage: b[0,1], b[0,2], b[0,3] = y, 2, x+z
sage: s = a + b ; s
2-vector field on the 4-dimensional differentiable manifold M
sage: s.display()
(2*x + y) ∂/∂t∧∂/∂x + 2 ∂/∂t∧∂/∂y + (x + z) ∂/∂t∧∂/∂z + 3 ∂/∂x∧∂/∂y
sage: s = a - b ; s
2-vector field on the 4-dimensional differentiable manifold M
sage: s.display()
(2*x - y) ∂/∂t∧∂/∂x - 2 ∂/∂t∧∂/∂y + (-x - z) ∂/∂t∧∂/∂z + 3 ∂/∂x∧∂/∂y
```

```
>>> from sage.all import *
>>> b = M.multivector_field(Integer(2))
>>> b[Integer(0),Integer(1)], b[Integer(0),Integer(2)], b[Integer(0),Integer(3)]␣
↪= y, Integer(2), x+z
>>> s = a + b ; s
2-vector field on the 4-dimensional differentiable manifold M
>>> s.display()
(2*x + y) ∂/∂t∧∂/∂x + 2 ∂/∂t∧∂/∂y + (x + z) ∂/∂t∧∂/∂z + 3 ∂/∂x∧∂/∂y
>>> s = a - b ; s
2-vector field on the 4-dimensional differentiable manifold M
>>> s.display()
(2*x - y) ∂/∂t∧∂/∂x - 2 ∂/∂t∧∂/∂y + (-x - z) ∂/∂t∧∂/∂z + 3 ∂/∂x∧∂/∂y
```

An example of 3-vector field in $\mathbf{R}^3$ with Cartesian coordinates:

```
sage: M = Manifold(3, 'R3', latex_name=r'\RR^3', start_index=1)
sage: c_cart.<x,y,z> = M.chart()
sage: a = M.multivector_field(3, name='a')
sage: a[1,2,3] = x^2+y^2+z^2  # the only independent component
sage: a[:] # all the components are set from the previous line:
[[[0, 0, 0], [0, 0, x^2 + y^2 + z^2], [0, -x^2 - y^2 - z^2, 0]],
 [[0, 0, -x^2 - y^2 - z^2], [0, 0, 0], [x^2 + y^2 + z^2, 0, 0]],
 [[0, x^2 + y^2 + z^2, 0], [-x^2 - y^2 - z^2, 0, 0], [0, 0, 0]]]
sage: a.display()
a = (x^2 + y^2 + z^2) ∂/∂x∧∂/∂y∧∂/∂z
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'R3', latex_name=r'\RR^3', start_index=Integer(1))
>>> c_cart = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_cart._first_ngens(3)
>>> a = M.multivector_field(Integer(3), name='a')
>>> a[Integer(1),Integer(2),Integer(3)] =␣
↪x**Integer(2)+y**Integer(2)+z**Integer(2)  # the only independent component
>>> a[:] # all the components are set from the previous line:
[[[0, 0, 0], [0, 0, x^2 + y^2 + z^2], [0, -x^2 - y^2 - z^2, 0]],
 [[0, 0, -x^2 - y^2 - z^2], [0, 0, 0], [x^2 + y^2 + z^2, 0, 0]],
 [[0, x^2 + y^2 + z^2, 0], [-x^2 - y^2 - z^2, 0, 0], [0, 0, 0]]]
>>> a.display()
a = (x^2 + y^2 + z^2) ∂/∂x∧∂/∂y∧∂/∂z
```

Spherical components from the tensorial change-of-frame formula:

```
sage: c_spher.<r,th,ph> = M.chart(r'r:[0,+oo) th:[0,pi]:\theta ph:[0,2*pi):\phi')
sage: spher_to_cart = c_spher.transition_map(c_cart,
....:                 [r*sin(th)*cos(ph), r*sin(th)*sin(ph), r*cos(th)])
sage: cart_to_spher = spher_to_cart.set_inverse(sqrt(x^2+y^2+z^2),
....:                                 atan2(sqrt(x^2+y^2),z), atan2(y, x))
Check of the inverse coordinate transformation:
  r == r  *passed*
  th == arctan2(r*sin(th), r*cos(th))  **failed**
  ph == arctan2(r*sin(ph)*sin(th), r*cos(ph)*sin(th))  **failed**
  x == x  *passed*
  y == y  *passed*
  z == z  *passed*
NB: a failed report can reflect a mere lack of simplification.
sage: a.comp(c_spher.frame()) # computation of components w.r.t. spherical frame
Fully antisymmetric 3-indices components w.r.t. Coordinate frame
 (R3, (∂/∂r,∂/∂th,∂/∂ph))
sage: a.comp(c_spher.frame())[1,2,3, c_spher]
1/sin(th)
sage: a.display(c_spher.frame())
a = sqrt(x^2 + y^2 + z^2)/sqrt(x^2 + y^2) ∂/∂r∧∂/∂th∧∂/∂ph
sage: a.display(c_spher.frame(), c_spher)
a = 1/sin(th) ∂/∂r∧∂/∂th∧∂/∂ph
```

```
>>> from sage.all import *
>>> c_spher = M.chart(r'r:[0,+oo) th:[0,pi]:\theta ph:[0,2*pi):\phi', names=('r',
↪'th', 'ph',)); (r, th, ph,) = c_spher._first_ngens(3)
>>> spher_to_cart = c_spher.transition_map(c_cart,
...                 [r*sin(th)*cos(ph), r*sin(th)*sin(ph), r*cos(th)])
>>> cart_to_spher = spher_to_cart.set_
↪inverse(sqrt(x**Integer(2)+y**Integer(2)+z**Integer(2)),
...                               atan2(sqrt(x**Integer(2)+y**Integer(2)),z),␣
↪atan2(y, x))
Check of the inverse coordinate transformation:
  r == r  *passed*
  th == arctan2(r*sin(th), r*cos(th))  **failed**
  ph == arctan2(r*sin(ph)*sin(th), r*cos(ph)*sin(th))  **failed**
  x == x  *passed*
  y == y  *passed*
  z == z  *passed*
NB: a failed report can reflect a mere lack of simplification.
>>> a.comp(c_spher.frame()) # computation of components w.r.t. spherical frame
Fully antisymmetric 3-indices components w.r.t. Coordinate frame
 (R3, (∂/∂r,∂/∂th,∂/∂ph))
>>> a.comp(c_spher.frame())[Integer(1),Integer(2),Integer(3), c_spher]
1/sin(th)
>>> a.display(c_spher.frame())
a = sqrt(x^2 + y^2 + z^2)/sqrt(x^2 + y^2) ∂/∂r∧∂/∂th∧∂/∂ph
>>> a.display(c_spher.frame(), c_spher)
a = 1/sin(th) ∂/∂r∧∂/∂th∧∂/∂ph
```

As a shortcut of the above command, on can pass just the chart `c_spher` to `display`, the vector frame being then assumed to be the coordinate frame associated with the chart:

```
sage: a.display(c_spher)
a = 1/sin(th) ∂/∂r∧∂/∂th∧∂/∂ph
```

```
>>> from sage.all import *
>>> a.display(c_spher)
a = 1/sin(th) ∂/∂r∧∂/∂th∧∂/∂ph
```

The exterior product of two multivector fields is performed via the method `wedge()`:

```
sage: a = M.vector_field([x*y, -z*x, y], name='A')
sage: b = M.vector_field([y, z+y, x^2-z^2], name='B')
sage: ab = a.wedge(b) ; ab
2-vector field A∧B on the 3-dimensional differentiable manifold R3
sage: ab.display()
A∧B = (x*y^2 + 2*x*y*z) ∂/∂x∧∂/∂y + (x^3*y - x*y*z^2 - y^2) ∂/∂x∧∂/∂z
 + (x*z^3 - y^2 - (x^3 + y)*z) ∂/∂y∧∂/∂z
```

```
>>> from sage.all import *
>>> a = M.vector_field([x*y, -z*x, y], name='A')
>>> b = M.vector_field([y, z+y, x**Integer(2)-z**Integer(2)], name='B')
>>> ab = a.wedge(b) ; ab
2-vector field A∧B on the 3-dimensional differentiable manifold R3
>>> ab.display()
A∧B = (x*y^2 + 2*x*y*z) ∂/∂x∧∂/∂y + (x^3*y - x*y*z^2 - y^2) ∂/∂x∧∂/∂z
 + (x*z^3 - y^2 - (x^3 + y)*z) ∂/∂y∧∂/∂z
```

Let us check the formula relating the exterior product to the tensor product for vector fields:

```
sage: a.wedge(b) == a*b - b*a
True
```

```
>>> from sage.all import *
>>> a.wedge(b) == a*b - b*a
True
```

The tensor product of a vector field and a 2-vector field is not a 3-vector field but a tensor field of type $(3, 0)$ with less symmetries:

```
sage: c = a*ab ; c
Tensor field A⊗(A∧B) of type (3,0) on the 3-dimensional differentiable
 manifold R3
sage: c.symmetries()  # the antisymmetry is only w.r.t. the last 2 arguments:
no symmetry;  antisymmetry: (1, 2)
```

```
>>> from sage.all import *
>>> c = a*ab ; c
Tensor field A⊗(A∧B) of type (3,0) on the 3-dimensional differentiable
 manifold R3
>>> c.symmetries()  # the antisymmetry is only w.r.t. the last 2 arguments:
no symmetry;  antisymmetry: (1, 2)
```

The Lie derivative of a 2-vector field is a 2-vector field:

```
sage: ab.lie_der(a)
2-vector field on the 3-dimensional differentiable manifold R3
```

```
>>> from sage.all import *
>>> ab.lie_der(a)
2-vector field on the 3-dimensional differentiable manifold R3
```

bracket(*other*)

Return the Schouten-Nijenhuis bracket of `self` with another multivector field.

The Schouten-Nijenhuis bracket extends the Lie bracket of vector fields (cf. `bracket()`) to multivector fields.

Denoting by $A^p(M)$ the $C^k(M)$-module of p-vector fields on the C^k-differentiable manifold M over the field K (cf. `MultivectorModule`), the *Schouten-Nijenhuis bracket* is a K-bilinear map

$$\begin{array}{ccc} A^p(M) \times A^q(M) & \longrightarrow & A^{p+q-1}(M) \\ (a,b) & \longmapsto & [a,b] \end{array}$$

which obeys the following properties:

- if $p = 0$ and $q = 0$, (i.e. a and b are two scalar fields), $[a, b] = 0$
- if $p = 0$ (i.e. a is a scalar field) and $q \geq 1$, $[a, b] = -\iota_{\mathrm{d}a} b$ (minus the interior product of the differential of a by b)
- if $p = 1$ (i.e. a is a vector field), $[a, b] = \mathcal{L}_a b$ (the Lie derivative of b along a)
- $[a, b] = -(-1)^{(p-1)(q-1)}[b, a]$
- for any multivector field c and $(a, b) \in A^p(M) \times A^q(M)$, $[a, .]$ obeys the *graded Leibniz rule*

$$[a, b \wedge c] = [a, b] \wedge c + (-1)^{(p-1)q} b \wedge [a, c]$$

- for $(a, b, c) \in A^p(M) \times A^q(M) \times A^r(M)$, the *graded Jacobi identity* holds:

$$(-1)^{(p-1)(r-1)}[a, [b, c]] + (-1)^{(q-1)(p-1)}[b, [c, a]] + (-1)^{(r-1)(q-1)}[c, [a, b]] = 0$$

Note

There are two definitions of the Schouten-Nijenhuis bracket in the literature, which differ from each other when p is even by an overall sign. The definition adopted here is that of [Mar1997], [Kos1985] and Wikipedia article Schouten-Nijenhuis_bracket. The other definition, adopted e.g. by [Nij1955], [Lic1977] and [Vai1994], is $[a, b]' = (-1)^{p+1}[a, b]$.

INPUT:

- `other` – a multivector field

OUTPUT:

- instance of `MultivectorFieldParal` (or of `DiffScalarField` if $p = 1$ and $q = 0$) representing the Schouten-Nijenhuis bracket $[a, b]$, where a is `self` and b is `other`

EXAMPLES:

Let us consider two vector fields on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: a = M.vector_field([x*y+z, x+y-z, z-2*x+y], name='a')
sage: b = M.vector_field([y+2*z-x, x^2-y+z, z-x], name='b')
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> a = M.vector_field([x*y+z, x+y-z, z-Integer(2)*x+y], name='a')
>>> b = M.vector_field([y+Integer(2)*z-x, x**Integer(2)-y+z, z-x], name='b')
```

and form their Schouten-Nijenhuis bracket:

```
sage: s = a.bracket(b); s
Vector field [a,b] on the 3-dimensional differentiable manifold M
sage: s.display()
[a,b] = (-x^3 + (x + 3)*y - y^2 - (x + 2*y + 1)*z - 2*x) ∂/∂x
 + (2*x^2*y - x^2 + 2*x*z - 3*x) ∂/∂y
 + (-x^2 - (x - 4)*y - 3*x + 2*z) ∂/∂z
```

```
>>> from sage.all import *
>>> s = a.bracket(b); s
Vector field [a,b] on the 3-dimensional differentiable manifold M
>>> s.display()
[a,b] = (-x^3 + (x + 3)*y - y^2 - (x + 2*y + 1)*z - 2*x) ∂/∂x
 + (2*x^2*y - x^2 + 2*x*z - 3*x) ∂/∂y
 + (-x^2 - (x - 4)*y - 3*x + 2*z) ∂/∂z
```

Check that $[a, b]$ is actually the Lie bracket:

```
sage: f = M.scalar_field({X: x+y*z}, name='f')
sage: s(f) == a(b(f)) - b(a(f))
True
```

```
>>> from sage.all import *
>>> f = M.scalar_field({X: x+y*z}, name='f')
>>> s(f) == a(b(f)) - b(a(f))
True
```

Check that $[a, b]$ coincides with the Lie derivative of b along a:

```
sage: s == b.lie_derivative(a)
True
```

```
>>> from sage.all import *
>>> s == b.lie_derivative(a)
True
```

Schouten-Nijenhuis bracket for $p = 0$ and $q = 1$:

```
sage: s = f.bracket(a); s
Scalar field -i_df a on the 3-dimensional differentiable manifold M
```

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```
sage: s.display()
-i_df a: M → ℝ
   (x, y, z) ↦ x*y - y^2 - (x + 2*y + 1)*z + z^2
```

```
>>> from sage.all import *
>>> s = f.bracket(a); s
Scalar field -i_df a on the 3-dimensional differentiable manifold M
>>> s.display()
-i_df a: M → ℝ
   (x, y, z) ↦ x*y - y^2 - (x + 2*y + 1)*z + z^2
```

Check that $[f, a] = -\iota_{\mathrm{d}f} a = -\mathrm{d}f(a)$:

```
sage: s == - f.differential()(a)
True
```

```
>>> from sage.all import *
>>> s == - f.differential()(a)
True
```

Schouten-Nijenhuis bracket for $p = 0$ and $q = 2$:

```
sage: c = M.multivector_field(2, name='c')
sage: c[0,1], c[0,2], c[1,2] = x+z+1, x*y+z, x-y
sage: s = f.bracket(c); s
Vector field -i_df c on the 3-dimensional differentiable manifold M
sage: s.display()
-i_df c = (x*y^2 + (x + y + 1)*z + z^2) ∂/∂x
 + (x*y - y^2 - x - z - 1) ∂/∂y + (-x*y - (x - y + 1)*z) ∂/∂z
```

```
>>> from sage.all import *
>>> c = M.multivector_field(Integer(2), name='c')
>>> c[Integer(0),Integer(1)], c[Integer(0),Integer(2)], c[Integer(1),
↪Integer(2)] = x+z+Integer(1), x*y+z, x-y
>>> s = f.bracket(c); s
Vector field -i_df c on the 3-dimensional differentiable manifold M
>>> s.display()
-i_df c = (x*y^2 + (x + y + 1)*z + z^2) ∂/∂x
 + (x*y - y^2 - x - z - 1) ∂/∂y + (-x*y - (x - y + 1)*z) ∂/∂z
```

Check that $[f, c] = -\iota_{\mathrm{d}f} c$:

```
sage: s == - f.differential().interior_product(c)
True
```

```
>>> from sage.all import *
>>> s == - f.differential().interior_product(c)
True
```

Schouten-Nijenhuis bracket for $p = 1$ and $q = 2$:

```
sage: s = a.bracket(c); s
2-vector field [a,c] on the 3-dimensional differentiable manifold M
sage: s.display()
[a,c] = ((x - 1)*y - (y - 2)*z - 2*x - 1) ∂/∂x∧∂/∂y
 + ((x + 1)*y - (x + 1)*z - 3*x - 1) ∂/∂x∧∂/∂z
 + (-5*x + y - z - 2) ∂/∂y∧∂/∂z
```

```
>>> from sage.all import *
>>> s = a.bracket(c); s
2-vector field [a,c] on the 3-dimensional differentiable manifold M
>>> s.display()
[a,c] = ((x - 1)*y - (y - 2)*z - 2*x - 1) ∂/∂x∧∂/∂y
 + ((x + 1)*y - (x + 1)*z - 3*x - 1) ∂/∂x∧∂/∂z
 + (-5*x + y - z - 2) ∂/∂y∧∂/∂z
```

Again, since a is a vector field, the Schouten-Nijenhuis bracket coincides with the Lie derivative:

```
sage: s == c.lie_derivative(a)
True
```

```
>>> from sage.all import *
>>> s == c.lie_derivative(a)
True
```

Schouten-Nijenhuis bracket for $p = 2$ and $q = 2$:

```
sage: d = M.multivector_field(2, name='d')
sage: d[0,1], d[0,2], d[1,2] = x-y^2, x+z, z-x-1
sage: s = c.bracket(d); s
3-vector field [c,d] on the 3-dimensional differentiable manifold M
sage: s.display()
[c,d] = (-y^3 + (3*x + 1)*y - y^2 - x - z + 2) ∂/∂x∧∂/∂y∧∂/∂z
```

```
>>> from sage.all import *
>>> d = M.multivector_field(Integer(2), name='d')
>>> d[Integer(0),Integer(1)], d[Integer(0),Integer(2)], d[Integer(1),
↪Integer(2)] = x-y**Integer(2), x+z, z-x-Integer(1)
>>> s = c.bracket(d); s
3-vector field [c,d] on the 3-dimensional differentiable manifold M
>>> s.display()
[c,d] = (-y^3 + (3*x + 1)*y - y^2 - x - z + 2) ∂/∂x∧∂/∂y∧∂/∂z
```

Let us check the component formula (with respect to the manifold's default coordinate chart, i.e. `X`) for $p = q = 2$, taking into account the tensor antisymmetries:

```
sage: s[0,1,2] == - sum(c[i,0]*d[1,2].diff(i)
....:                  + c[i,1]*d[2,0].diff(i) + c[i,2]*d[0,1].diff(i)
....:                  + d[i,0]*c[1,2].diff(i) + d[i,1]*c[2,0].diff(i)
....:                  + d[i,2]*c[0,1].diff(i) for i in M.irange())
True
```

```
>>> from sage.all import *
>>> s[Integer(0),Integer(1),Integer(2)] == - sum(c[i,Integer(0)]*d[Integer(1),
↪Integer(2)].diff(i)
...                  + c[i,Integer(1)]*d[Integer(2),Integer(0)].diff(i) + c[i,
↪Integer(2)]*d[Integer(0),Integer(1)].diff(i)
...                  + d[i,Integer(0)]*c[Integer(1),Integer(2)].diff(i) + d[i,
↪Integer(1)]*c[Integer(2),Integer(0)].diff(i)
...                  + d[i,Integer(2)]*c[Integer(0),Integer(1)].diff(i) for i␣
↪in M.irange())
True
```

Schouten-Nijenhuis bracket for $p = 1$ and $q = 3$:

```
sage: e = M.multivector_field(3, name='e')
sage: e[0,1,2] = x+y*z+1
sage: s = a.bracket(e); s
3-vector field [a,e] on the 3-dimensional differentiable manifold M
sage: s.display()
[a,e] = (-(2*x + 1)*y + y^2 - (y^2 - x - 1)*z - z^2
 - 2*x - 2) ∂/∂x∧∂/∂y∧∂/∂z
```

```
>>> from sage.all import *
>>> e = M.multivector_field(Integer(3), name='e')
>>> e[Integer(0),Integer(1),Integer(2)] = x+y*z+Integer(1)
>>> s = a.bracket(e); s
3-vector field [a,e] on the 3-dimensional differentiable manifold M
>>> s.display()
[a,e] = (-(2*x + 1)*y + y^2 - (y^2 - x - 1)*z - z^2
 - 2*x - 2) ∂/∂x∧∂/∂y∧∂/∂z
```

Again, since $p = 1$, the bracket coincides with the Lie derivative:

```
sage: s == e.lie_derivative(a)
True
```

```
>>> from sage.all import *
>>> s == e.lie_derivative(a)
True
```

Schouten-Nijenhuis bracket for $p = 2$ and $q = 3$:

```
sage: s = c.bracket(e); s
4-vector field [c,e] on the 3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> s = c.bracket(e); s
4-vector field [c,e] on the 3-dimensional differentiable manifold M
```

Since on a 3-dimensional manifold, any 4-vector field is zero, we have:

```
sage: s.display()
[c,e] = 0
```

```
>>> from sage.all import *
>>> s.display()
[c,e] = 0
```

Let us check the graded commutation law $[a,b] = -(-1)^{(p-1)(q-1)}[b,a]$ for various values of p and q:

```
sage: f.bracket(a) == - a.bracket(f)   # p=0 and q=1
True
sage: f.bracket(c) == c.bracket(f)     # p=0 and q=2
True
sage: a.bracket(b) == - b.bracket(a)   # p=1 and q=1
True
sage: a.bracket(c) == - c.bracket(a)   # p=1 and q=2
True
sage: c.bracket(d) == d.bracket(c)     # p=2 and q=2
True
```

```
>>> from sage.all import *
>>> f.bracket(a) == - a.bracket(f)   # p=0 and q=1
True
>>> f.bracket(c) == c.bracket(f)     # p=0 and q=2
True
>>> a.bracket(b) == - b.bracket(a)   # p=1 and q=1
True
>>> a.bracket(c) == - c.bracket(a)   # p=1 and q=2
True
>>> c.bracket(d) == d.bracket(c)     # p=2 and q=2
True
```

Let us check the graded Leibniz rule for $p = 1$ and $q = 1$:

```
sage: a.bracket(b.wedge(c)) == a.bracket(b).wedge(c) + b.wedge(a.bracket(c))
↪# long time
True
```

```
>>> from sage.all import *
>>> a.bracket(b.wedge(c)) == a.bracket(b).wedge(c) + b.wedge(a.bracket(c))   #␣
↪long time
True
```

as well as for $p = 2$ and $q = 1$:

```
sage: c.bracket(a.wedge(b)) == c.bracket(a).wedge(b) - a.wedge(c.bracket(b))
↪# long time
True
```

```
>>> from sage.all import *
>>> c.bracket(a.wedge(b)) == c.bracket(a).wedge(b) - a.wedge(c.bracket(b))   #␣
↪long time
True
```

Finally let us check the graded Jacobi identity for $p = 1$, $q = 1$ and $r = 2$:

```
sage: # long time
sage: a_bc = a.bracket(b.bracket(c))
sage: b_ca = b.bracket(c.bracket(a))
sage: c_ab = c.bracket(a.bracket(b))
sage: a_bc + b_ca + c_ab == 0
True
```

```
>>> from sage.all import *
>>> # long time
>>> a_bc = a.bracket(b.bracket(c))
>>> b_ca = b.bracket(c.bracket(a))
>>> c_ab = c.bracket(a.bracket(b))
>>> a_bc + b_ca + c_ab == Integer(0)
True
```

as well as for $p = 1$, $q = 2$ and $r = 2$:

```
sage: # long time
sage: a_cd = a.bracket(c.bracket(d))
sage: c_da = c.bracket(d.bracket(a))
sage: d_ac = d.bracket(a.bracket(c))
sage: a_cd + c_da - d_ac == 0
True
```

```
>>> from sage.all import *
>>> # long time
>>> a_cd = a.bracket(c.bracket(d))
>>> c_da = c.bracket(d.bracket(a))
>>> d_ac = d.bracket(a.bracket(c))
>>> a_cd + c_da - d_ac == Integer(0)
True
```

interior_product(*form*)

Interior product with a differential form.

If `self` is a multivector field A of degree p and B is a differential form of degree $q \geq p$ on the same manifold as A, the interior product of A by B is the differential form $\iota_A B$ of degree $q - p$ defined by

$$(\iota_A B)_{i_1 \ldots i_{q-p}} = A^{k_1 \ldots k_p} B_{k_1 \ldots k_p i_1 \ldots i_{q-p}}$$

Note

`A.interior_product(B)` yields the same result as `A.contract(0,..., p-1, B, 0,..., p-1)` (cf. `contract()`), but `interior_product` is more efficient, the alternating character of A being not used to reduce the computation in `contract()`

INPUT:

- `form` – differential form B (instance of `DiffFormParal`); the degree of B must be at least equal to the degree of `self`

OUTPUT:

- scalar field (case $p = q$) or `DiffFormParal` (case $p < q$) representing the interior product $\iota_A B$, where A is `self`

See also

`interior_product()` for the interior product of a differential form with a multivector field

EXAMPLES:

Interior product with $p = 1$ and $q = 1$ on 4-dimensional manifold:

```
sage: M = Manifold(4, 'M')
sage: X.<t,x,y,z> = M.chart()
sage: a = M.vector_field([x, 1+t^2, x*z, y-3], name='a')
sage: b = M.one_form([-z^2, 2, x, x-y], name='b')
sage: s = a.interior_product(b); s
Scalar field i_a b on the 4-dimensional differentiable manifold M
sage: s.display()
i_a b: M → ℝ
   (t, x, y, z) ↦ x^2*z - x*z^2 + 2*t^2 + (x + 3)*y - y^2
    - 3*x + 2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> a = M.vector_field([x, Integer(1)+t**Integer(2), x*z, y-Integer(3)], name=
↪'a')
>>> b = M.one_form([-z**Integer(2), Integer(2), x, x-y], name='b')
>>> s = a.interior_product(b); s
Scalar field i_a b on the 4-dimensional differentiable manifold M
>>> s.display()
i_a b: M → ℝ
   (t, x, y, z) ↦ x^2*z - x*z^2 + 2*t^2 + (x + 3)*y - y^2
    - 3*x + 2
```

In this case, we have $\iota_a b = a^i b_i = a(b) = b(a)$:

```
sage: all([s == a.contract(b), s == a(b), s == b(a)])
True
```

```
>>> from sage.all import *
>>> all([s == a.contract(b), s == a(b), s == b(a)])
True
```

Case $p = 1$ and $q = 3$:

```
sage: c = M.diff_form(3, name='c')
sage: c[0,1,2], c[0,1,3] = x*y - z, -3*t
sage: c[0,2,3], c[1,2,3] = t+x, y
sage: s = a.interior_product(c); s
2-form i_a c on the 4-dimensional differentiable manifold M
sage: s.display()
```

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```
i_a c = (x^2*y*z - x*z^2 - 3*t*y + 9*t) dt∧dx
 + (-(t^2*x - t)*y + (t^2 + 1)*z - 3*t - 3*x) dt∧dy
 + (3*t^3 - (t*x + x^2)*z + 3*t) dt∧dz
 + ((x^2 - 3)*y + y^2 - x*z) dx∧dy
 + (-x*y*z - 3*t*x) dx∧dz + (t*x + x^2 + (t^2 + 1)*y) dy∧dz
sage: s == a.contract(c)
True
```

```
>>> from sage.all import *
>>> c = M.diff_form(Integer(3), name='c')
>>> c[Integer(0),Integer(1),Integer(2)], c[Integer(0),Integer(1),Integer(3)]␣
↪= x*y - z, -Integer(3)*t
>>> c[Integer(0),Integer(2),Integer(3)], c[Integer(1),Integer(2),Integer(3)]␣
↪= t+x, y
>>> s = a.interior_product(c); s
2-form i_a c on the 4-dimensional differentiable manifold M
>>> s.display()
i_a c = (x^2*y*z - x*z^2 - 3*t*y + 9*t) dt∧dx
 + (-(t^2*x - t)*y + (t^2 + 1)*z - 3*t - 3*x) dt∧dy
 + (3*t^3 - (t*x + x^2)*z + 3*t) dt∧dz
 + ((x^2 - 3)*y + y^2 - x*z) dx∧dy
 + (-x*y*z - 3*t*x) dx∧dz + (t*x + x^2 + (t^2 + 1)*y) dy∧dz
>>> s == a.contract(c)
True
```

Case $p = 2$ and $q = 3$:

```
sage: d = M.multivector_field(2, name='d')
sage: d[0,1], d[0,2], d[0,3] = t-x, 2*z, y-1
sage: d[1,2], d[1,3], d[2,3] = z, y+t, 4
sage: s = d.interior_product(c); s
1-form i_d c on the 4-dimensional differentiable manifold M
sage: s.display()
i_d c = (2*x*y*z - 6*t^2 - 6*t*y - 2*z^2 + 8*t + 8*x) dt
 + (-4*x*y*z + 2*(3*t + 4)*y + 4*z^2 - 6*t) dx
 + (2*((t - 1)*x - x^2 - 2*t)*y - 2*y^2 - 2*(t - x)*z + 2*t
 + 2*x) dy + (-6*t^2 + 6*t*x + 2*(2*t + 2*x + y)*z) dz
sage: s == d.contract(0, 1, c, 0, 1)
True
```

```
>>> from sage.all import *
>>> d = M.multivector_field(Integer(2), name='d')
>>> d[Integer(0),Integer(1)], d[Integer(0),Integer(2)], d[Integer(0),
↪Integer(3)] = t-x, Integer(2)*z, y-Integer(1)
>>> d[Integer(1),Integer(2)], d[Integer(1),Integer(3)], d[Integer(2),
↪Integer(3)] = z, y+t, Integer(4)
>>> s = d.interior_product(c); s
1-form i_d c on the 4-dimensional differentiable manifold M
>>> s.display()
i_d c = (2*x*y*z - 6*t^2 - 6*t*y - 2*z^2 + 8*t + 8*x) dt
 + (-4*x*y*z + 2*(3*t + 4)*y + 4*z^2 - 6*t) dx
 + (2*((t - 1)*x - x^2 - 2*t)*y - 2*y^2 - 2*(t - x)*z + 2*t
```

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```
 + 2*x) dy + (-6*t^2 + 6*t*x + 2*(2*t + 2*x + y)*z) dz
>>> s == d.contract(Integer(0), Integer(1), c, Integer(0), Integer(1))
True
```

wedge(*other*)

Exterior product of `self` with another multivector field.

INPUT:

- `other` – another multivector field

OUTPUT:

- instance of *MultivectorFieldParal* representing the exterior product `self ∧ other`

EXAMPLES:

Exterior product of a vector field and a 2-vector field on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: a = M.vector_field([2, 1+x, y*z], name='a')
sage: b = M.multivector_field(2, name='b')
sage: b[1,2], b[1,3], b[2,3] = y^2, z+x, z^2
sage: a.display()
a = 2 ∂/∂x + (x + 1) ∂/∂y + y*z ∂/∂z
sage: b.display()
b = y^2 ∂/∂x∧∂/∂y + (x + z) ∂/∂x∧∂/∂z + z^2 ∂/∂y∧∂/∂z
sage: s = a.wedge(b); s
3-vector field a∧b on the 3-dimensional differentiable manifold M
sage: s.display()
a∧b = (-x^2 + (y^3 - x - 1)*z + 2*z^2 - x) ∂/∂x∧∂/∂y∧∂/∂z
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> a = M.vector_field([Integer(2), Integer(1)+x, y*z], name='a')
>>> b = M.multivector_field(Integer(2), name='b')
>>> b[Integer(1),Integer(2)], b[Integer(1),Integer(3)], b[Integer(2),
↪Integer(3)] = y**Integer(2), z+x, z**Integer(2)
>>> a.display()
a = 2 ∂/∂x + (x + 1) ∂/∂y + y*z ∂/∂z
>>> b.display()
b = y^2 ∂/∂x∧∂/∂y + (x + z) ∂/∂x∧∂/∂z + z^2 ∂/∂y∧∂/∂z
>>> s = a.wedge(b); s
3-vector field a∧b on the 3-dimensional differentiable manifold M
>>> s.display()
a∧b = (-x^2 + (y^3 - x - 1)*z + 2*z^2 - x) ∂/∂x∧∂/∂y∧∂/∂z
```

Check:

```
sage: s[1,2,3] == a[1]*b[2,3] + a[2]*b[3,1] + a[3]*b[1,2]
True
```

```
>>> from sage.all import *
>>> s[Integer(1),Integer(2),Integer(3)] == a[Integer(1)]*b[Integer(2),
↪Integer(3)] + a[Integer(2)]*b[Integer(3),Integer(1)] +␣
↪a[Integer(3)]*b[Integer(1),Integer(2)]
True
```

Exterior product with a scalar field:

```
sage: f = M.scalar_field(x, name='f')
sage: s = b.wedge(f); s
2-vector field f*b on the 3-dimensional differentiable manifold M
sage: s.display()
f*b = x*y^2 ∂/∂x∧∂/∂y + (x^2 + x*z) ∂/∂x∧∂/∂z + x*z^2 ∂/∂y∧∂/∂z
sage: s == f*b
True
sage: s == f.wedge(b)
True
```

```
>>> from sage.all import *
>>> f = M.scalar_field(x, name='f')
>>> s = b.wedge(f); s
2-vector field f*b on the 3-dimensional differentiable manifold M
>>> s.display()
f*b = x*y^2 ∂/∂x∧∂/∂y + (x^2 + x*z) ∂/∂x∧∂/∂z + x*z^2 ∂/∂y∧∂/∂z
>>> s == f*b
True
>>> s == f.wedge(b)
True
```

2.13 Affine Connections

The class `AffineConnection` implements affine connections on smooth manifolds.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015) : initial version
- Marco Mancini (2015) : parallelization of some computations
- Florentin Jaffredo (2018) : series expansion with respect to a given parameter

REFERENCES:

- [Lee1997]
- [KN1963]
- [ONe1983]

class `sage.manifolds.differentiable.affine_connection.`**`AffineConnection`**(*domain*, *name*, *latex_name=None*)

Bases: `SageObject`

Affine connection on a smooth manifold.

Let M be a differentiable manifold of class C^∞ (smooth manifold) over a non-discrete topological field K (in most applications $K = \mathbf{R}$ or $K = \mathbf{C}$), let $C^\infty(M)$ be the algebra of smooth functions $M \to K$ (cf. `DiffS-`

calarFieldAlgebra) and let $\mathfrak{X}(M)$ be the $C^\infty(M)$-module of vector fields on M (cf. VectorFieldModule). An *affine connection* on M is an operator

$$\begin{array}{cccc} \nabla : & \mathfrak{X}(M) \times \mathfrak{X}(M) & \longrightarrow & \mathfrak{X}(M) \\ & (u, v) & \longmapsto & \nabla_u v \end{array}$$

that

- is K-bilinear, i.e. is bilinear when considering $\mathfrak{X}(M)$ as a vector space over K
- is $C^\infty(M)$-linear w.r.t. the first argument: $\forall f \in C^\infty(M),\ \nabla_{fu} v = f \nabla_u v$
- obeys Leibniz rule w.r.t. the second argument: $\forall f \in C^\infty(M),\ \nabla_u (fv) = \mathrm{d}f(u)\, v + f \nabla_u v$

The affine connection ∇ gives birth to the *covariant derivative operator* acting on tensor fields, denoted by the same symbol:

$$\begin{array}{cccc} \nabla : & T^{(k,l)}(M) & \longrightarrow & T^{(k,l+1)}(M) \\ & t & \longmapsto & \nabla t \end{array}$$

where $T^{(k,l)}(M)$ stands for the $C^\infty(M)$-module of tensor fields of type (k, l) on M (cf. TensorFieldModule), with the convention $T^{(0,0)}(M) := C^\infty(M)$. For a vector field v, the covariant derivative ∇v is a type-(1,1) tensor field such that

$$\forall u \in \mathfrak{X}(M),\ \nabla_u v = \nabla v(., u)$$

More generally for any tensor field $t \in T^{(k,l)}(M)$, we have

$$\forall u \in \mathfrak{X}(M),\ \nabla_u t = \nabla t(\ldots, u)$$

Note

The above convention means that, in terms of index notation, the "derivation index" in ∇t is the *last* one:

$$\nabla_c t^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} = (\nabla t)^{a_1 \ldots a_k}{}_{b_1 \ldots b_l c}$$

INPUT:

- `domain` – the manifold on which the connection is defined (must be an instance of class DifferentiableManifold)
- `name` – name given to the affine connection
- `latex_name` – (default: `None`) LaTeX symbol to denote the affine connection; if `None`, it is set to `name`

EXAMPLES:

Affine connection on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: nab = M.affine_connection('nabla', r'\nabla') ; nab
Affine connection nabla on the 3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_ngens(3)
>>> nab = M.affine_connection('nabla', r'\nabla') ; nab
Affine connection nabla on the 3-dimensional differentiable manifold M
```

A just-created connection has no connection coefficients:

```
sage: nab._coefficients
{}
```

```
>>> from sage.all import *
>>> nab._coefficients
{}
```

The connection coefficients relative to the manifold's default frame [here $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$], are created by providing the relevant indices inside square brackets:

```
sage: nab[1,1,2], nab[3,2,3] = x^2, y*z  # Gamma^1_{12} = x^2, Gamma^3_{23} = yz
sage: nab._coefficients
{Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z)): 3-indices components w.r.t.
 Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z))}
```

```
>>> from sage.all import *
>>> nab[Integer(1),Integer(1),Integer(2)], nab[Integer(3),Integer(2),Integer(3)]␣
↪= x**Integer(2), y*z  # Gamma^1_{12} = x^2, Gamma^3_{23} = yz
>>> nab._coefficients
{Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z)): 3-indices components w.r.t.
 Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z))}
```

If not the default one, the vector frame w.r.t. which the connection coefficients are defined can be specified as the first argument inside the square brackets; hence the above definition is equivalent to:

```
sage: nab[c_xyz.frame(), 1,1,2], nab[c_xyz.frame(),3,2,3] = x^2, y*z
sage: nab._coefficients
{Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z)): 3-indices components w.r.t.
 Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z))}
```

```
>>> from sage.all import *
>>> nab[c_xyz.frame(), Integer(1),Integer(1),Integer(2)], nab[c_xyz.frame(),
↪Integer(3),Integer(2),Integer(3)] = x**Integer(2), y*z
>>> nab._coefficients
{Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z)): 3-indices components w.r.t.
 Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z))}
```

Unset components are initialized to zero:

```
sage: nab[:]  # list of coefficients relative to the manifold's default vector␣
↪frame
[[[0, x^2, 0], [0, 0, 0], [0, 0, 0]],
 [[0, 0, 0], [0, 0, 0], [0, 0, 0]],
 [[0, 0, 0], [0, 0, y*z], [0, 0, 0]]]
```

```
>>> from sage.all import *
>>> nab[:]  # list of coefficients relative to the manifold's default vector frame
[[[0, x^2, 0], [0, 0, 0], [0, 0, 0]],
 [[0, 0, 0], [0, 0, 0], [0, 0, 0]],
 [[0, 0, 0], [0, 0, y*z], [0, 0, 0]]]
```

The treatment of connection coefficients in a given vector frame is similar to that of tensor components; see therefore the class `TensorField` for the documentation. In particular, the square brackets return the connection coefficients as instances of `ChartFunction`, while the double square brackets return a scalar field:

```
sage: nab[1,1,2]
x^2
sage: nab[1,1,2].display()
(x, y, z) ↦ x^2
sage: type(nab[1,1,2])
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_class'>
sage: nab[[1,1,2]]
Scalar field on the 3-dimensional differentiable manifold M
sage: nab[[1,1,2]].display()
M → ℝ
(x, y, z) ↦ x^2
sage: nab[[1,1,2]].coord_function() is nab[1,1,2]
True
```

```
>>> from sage.all import *
>>> nab[Integer(1),Integer(1),Integer(2)]
x^2
>>> nab[Integer(1),Integer(1),Integer(2)].display()
(x, y, z) ↦ x^2
>>> type(nab[Integer(1),Integer(1),Integer(2)])
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_class'>
>>> nab[[Integer(1),Integer(1),Integer(2)]]
Scalar field on the 3-dimensional differentiable manifold M
>>> nab[[Integer(1),Integer(1),Integer(2)]].display()
M → ℝ
(x, y, z) ↦ x^2
>>> nab[[Integer(1),Integer(1),Integer(2)]].coord_function() is nab[Integer(1),
↪Integer(1),Integer(2)]
True
```

Action on a scalar field:

```
sage: f = M.scalar_field(x^2 - y^2, name='f')
sage: Df = nab(f) ; Df
1-form df on the 3-dimensional differentiable manifold M
sage: Df[:]
[2*x, -2*y, 0]
```

```
>>> from sage.all import *
>>> f = M.scalar_field(x**Integer(2) - y**Integer(2), name='f')
>>> Df = nab(f) ; Df
1-form df on the 3-dimensional differentiable manifold M
>>> Df[:]
[2*x, -2*y, 0]
```

The action of an affine connection on a scalar field must coincide with the differential:

```
sage: Df == f.differential()
True
```

```
>>> from sage.all import *
>>> Df == f.differential()
True
```

A generic affine connection has some torsion:

```
sage: DDf = nab(Df) ; DDf
Tensor field nabla(df) of type (0,2) on the 3-dimensional
 differentiable manifold M
sage: DDf.antisymmetrize()[:] # nabla does not commute on scalar fields:
[   0 -x^3    0]
[ x^3    0    0]
[   0    0    0]
```

```
>>> from sage.all import *
>>> DDf = nab(Df) ; DDf
Tensor field nabla(df) of type (0,2) on the 3-dimensional
 differentiable manifold M
>>> DDf.antisymmetrize()[:] # nabla does not commute on scalar fields:
[   0 -x^3    0]
[ x^3    0    0]
[   0    0    0]
```

Let us check the standard formula

$$\nabla_j \nabla_i f - \nabla_i \nabla_j f = T^k_{\ \ ij} \nabla_k f,$$

where the $T^k_{\ \ ij}$'s are the components of the connection's torsion tensor:

```
sage: 2*DDf.antisymmetrize() == nab.torsion().contract(0,Df)
True
```

```
>>> from sage.all import *
>>> Integer(2)*DDf.antisymmetrize() == nab.torsion().contract(Integer(0),Df)
True
```

The connection acting on a vector field:

```
sage: v = M.vector_field(y*z, x*z, x*y, name='v')
sage: Dv = nab(v) ; Dv
Tensor field nabla(v) of type (1,1) on the 3-dimensional differentiable
 manifold M
sage: Dv[:]
[            0 (x^2*y + 1)*z             y]
[            z             0             x]
[            y             x       x*y*z^2]
```

```
>>> from sage.all import *
>>> v = M.vector_field(y*z, x*z, x*y, name='v')
>>> Dv = nab(v) ; Dv
Tensor field nabla(v) of type (1,1) on the 3-dimensional differentiable
 manifold M
>>> Dv[:]
```

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```
[              0  (x^2*y + 1)*z              y]
[              z              0              x]
[              y              x        x*y*z^2]
```

Another example: connection on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)    # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
....:                               restrictions1= x>0, restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: c_xyW = c_xy.restrict(W) ; c_uvW = c_uv.restrict(W)
sage: eUW = c_xyW.frame() ; eVW = c_uvW.frame()
sage: nab = M.affine_connection('nabla', r'\nabla')
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)    # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv = V.
↪chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
...                               restrictions1= x>Integer(0), restrictions2= u+v>
↪Integer(0))
>>> inv = transf.inverse()
>>> W = U.intersection(V)
>>> eU = c_xy.frame() ; eV = c_uv.frame()
>>> c_xyW = c_xy.restrict(W) ; c_uvW = c_uv.restrict(W)
>>> eUW = c_xyW.frame() ; eVW = c_uvW.frame()
>>> nab = M.affine_connection('nabla', r'\nabla')
```

The connection is first defined on the open subset U by means of its coefficients w.r.t. the frame eU (the manifold's default frame):

```
sage: nab[0,0,0], nab[1,0,1] = x, x*y
```

```
>>> from sage.all import *
>>> nab[Integer(0),Integer(0),Integer(0)], nab[Integer(1),Integer(0),Integer(1)]␣
↪= x, x*y
```

The coefficients w.r.t the frame eV are deduced by continuation of the coefficients w.r.t. the frame eVW on the open subset $W = U \cap V$:

```
sage: for i in M.irange():
....:     for j in M.irange():
....:         for k in M.irange():
....:             nab.add_coef(eV)[i,j,k] = nab.coef(eVW)[i,j,k,c_uvW].expr()
```

```
>>> from sage.all import *
>>> for i in M.irange():
...     for j in M.irange():
...         for k in M.irange():
...             nab.add_coef(eV)[i,j,k] = nab.coef(eVW)[i,j,k,c_uvW].expr()
```

At this stage, the connection is fully defined on all the manifold:

```
sage: nab.coef(eU)[:]
[[[x, 0], [0, 0]], [[0, x*y], [0, 0]]]
sage: nab.coef(eV)[:]
[[[1/16*u^2 - 1/16*v^2 + 1/8*u + 1/8*v, -1/16*u^2 + 1/16*v^2 + 1/8*u + 1/8*v],
  [1/16*u^2 - 1/16*v^2 + 1/8*u + 1/8*v, -1/16*u^2 + 1/16*v^2 + 1/8*u + 1/8*v]],
 [[-1/16*u^2 + 1/16*v^2 + 1/8*u + 1/8*v, 1/16*u^2 - 1/16*v^2 + 1/8*u + 1/8*v],
  [-1/16*u^2 + 1/16*v^2 + 1/8*u + 1/8*v, 1/16*u^2 - 1/16*v^2 + 1/8*u + 1/8*v]]]
```

```
>>> from sage.all import *
>>> nab.coef(eU)[:]
[[[x, 0], [0, 0]], [[0, x*y], [0, 0]]]
>>> nab.coef(eV)[:]
[[[1/16*u^2 - 1/16*v^2 + 1/8*u + 1/8*v, -1/16*u^2 + 1/16*v^2 + 1/8*u + 1/8*v],
  [1/16*u^2 - 1/16*v^2 + 1/8*u + 1/8*v, -1/16*u^2 + 1/16*v^2 + 1/8*u + 1/8*v]],
 [[-1/16*u^2 + 1/16*v^2 + 1/8*u + 1/8*v, 1/16*u^2 - 1/16*v^2 + 1/8*u + 1/8*v],
  [-1/16*u^2 + 1/16*v^2 + 1/8*u + 1/8*v, 1/16*u^2 - 1/16*v^2 + 1/8*u + 1/8*v]]]
```

We may let it act on a vector field defined globally on M:

```
sage: a = M.vector_field({eU: [-y,x]}, name='a')
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: a.display(eU)
a = -y ∂/∂x + x ∂/∂y
sage: a.display(eV)
a = v ∂/∂u - u ∂/∂v
sage: da = nab(a) ; da
Tensor field nabla(a) of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: da.display(eU)
nabla(a) = -x*y ∂/∂x⊗dx - ∂/∂x⊗dy + ∂/∂y⊗dx - x*y^2 ∂/∂y⊗dy
sage: da.display(eV)
nabla(a) = (-1/16*u^3 + 1/16*u^2*v + 1/16*(u + 2)*v^2 - 1/16*v^3 - 1/8*u^2) ∂/
↪∂u⊗du
 + (1/16*u^3 - 1/16*u^2*v - 1/16*(u - 2)*v^2 + 1/16*v^3 - 1/8*u^2 + 1) ∂/∂u⊗dv
 + (1/16*u^3 - 1/16*u^2*v - 1/16*(u - 2)*v^2 + 1/16*v^3 - 1/8*u^2 - 1) ∂/∂v⊗du
 + (-1/16*u^3 + 1/16*u^2*v + 1/16*(u + 2)*v^2 - 1/16*v^3 - 1/8*u^2) ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> a = M.vector_field({eU: [-y,x]}, name='a')
>>> a.add_comp_by_continuation(eV, W, c_uv)
>>> a.display(eU)
a = -y ∂/∂x + x ∂/∂y
>>> a.display(eV)
a = v ∂/∂u - u ∂/∂v
>>> da = nab(a) ; da
```

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```
Tensor field nabla(a) of type (1,1) on the 2-dimensional differentiable
 manifold M
>>> da.display(eU)
nabla(a) = -x*y ∂/∂x⊗dx - ∂/∂x⊗dy + ∂/∂y⊗dx - x*y^2 ∂/∂y⊗dy
>>> da.display(eV)
nabla(a) = (-1/16*u^3 + 1/16*u^2*v + 1/16*(u + 2)*v^2 - 1/16*v^3 - 1/8*u^2) ∂/
↪∂u⊗du
 + (1/16*u^3 - 1/16*u^2*v - 1/16*(u - 2)*v^2 + 1/16*v^3 - 1/8*u^2 + 1) ∂/∂u⊗dv
 + (1/16*u^3 - 1/16*u^2*v - 1/16*(u - 2)*v^2 + 1/16*v^3 - 1/8*u^2 - 1) ∂/∂v⊗du
 + (-1/16*u^3 + 1/16*u^2*v + 1/16*(u + 2)*v^2 - 1/16*v^3 - 1/8*u^2) ∂/∂v⊗dv
```

A few tests:

```
sage: nab(a.restrict(V)) == da.restrict(V)
True
sage: nab.restrict(V)(a) == da.restrict(V)
True
sage: nab.restrict(V)(a.restrict(U)) == da.restrict(W)
True
sage: nab.restrict(U)(a.restrict(V)) == da.restrict(W)  # long time
True
```

```
>>> from sage.all import *
>>> nab(a.restrict(V)) == da.restrict(V)
True
>>> nab.restrict(V)(a) == da.restrict(V)
True
>>> nab.restrict(V)(a.restrict(U)) == da.restrict(W)
True
>>> nab.restrict(U)(a.restrict(V)) == da.restrict(W)  # long time
True
```

Same examples with SymPy as the engine for symbolic calculus:

```
sage: M.set_calculus_method('sympy')
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[0,0,0], nab[1,0,1] = x, x*y
sage: for i in M.irange():
....:     for j in M.irange():
....:         for k in M.irange():
....:             nab.add_coef(eV)[i,j,k] = nab.coef(eVW)[i,j,k,c_uvW].expr()
```

```
>>> from sage.all import *
>>> M.set_calculus_method('sympy')
>>> nab = M.affine_connection('nabla', r'\nabla')
>>> nab[Integer(0),Integer(0),Integer(0)], nab[Integer(1),Integer(0),Integer(1)]␣
↪= x, x*y
>>> for i in M.irange():
...     for j in M.irange():
...         for k in M.irange():
...             nab.add_coef(eV)[i,j,k] = nab.coef(eVW)[i,j,k,c_uvW].expr()
```

At this stage, the connection is fully defined on all the manifold:

```
sage: nab.coef(eU)[:]
[[[x, 0], [0, 0]], [[0, x*y], [0, 0]]]
sage: nab.coef(eV)[:]
[[[u**2/16 + u/8 - v**2/16 + v/8, -u**2/16 + u/8 + v**2/16 + v/8],
 [u**2/16 + u/8 - v**2/16 + v/8, -u**2/16 + u/8 + v**2/16 + v/8]],
[[-u**2/16 + u/8 + v**2/16 + v/8, u**2/16 + u/8 - v**2/16 + v/8],
 [-u**2/16 + u/8 + v**2/16 + v/8, u**2/16 + u/8 - v**2/16 + v/8]]]
```

```
>>> from sage.all import *
>>> nab.coef(eU)[:]
[[[x, 0], [0, 0]], [[0, x*y], [0, 0]]]
>>> nab.coef(eV)[:]
[[[u**2/16 + u/8 - v**2/16 + v/8, -u**2/16 + u/8 + v**2/16 + v/8],
 [u**2/16 + u/8 - v**2/16 + v/8, -u**2/16 + u/8 + v**2/16 + v/8]],
[[-u**2/16 + u/8 + v**2/16 + v/8, u**2/16 + u/8 - v**2/16 + v/8],
 [-u**2/16 + u/8 + v**2/16 + v/8, u**2/16 + u/8 - v**2/16 + v/8]]]
```

We may let it act on a vector field defined globally on M:

```
sage: a = M.vector_field({eU: [-y,x]}, name='a')
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: a.display(eU)
a = -y ∂/∂x + x ∂/∂y
sage: a.display(eV)
a = v ∂/∂u - u ∂/∂v
sage: da = nab(a) ; da
Tensor field nabla(a) of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: da.display(eU)
nabla(a) = -x*y ∂/∂x⊗dx - ∂/∂x⊗dy + ∂/∂y⊗dx - x*y**2 ∂/∂y⊗dy
sage: da.display(eV)
nabla(a) = (-u**3/16 + u**2*v/16 - u**2/8 + u*v**2/16 - v**3/16 + v**2/8) ∂/∂u⊗du
 + (u**3/16 - u**2*v/16 - u**2/8 - u*v**2/16 + v**3/16 + v**2/8 + 1) ∂/∂u⊗dv
 + (u**3/16 - u**2*v/16 - u**2/8 - u*v**2/16 + v**3/16 + v**2/8 - 1) ∂/∂v⊗du
 + (-u**3/16 + u**2*v/16 - u**2/8 + u*v**2/16 - v**3/16 + v**2/8) ∂/∂v⊗dv
```

```
>>> from sage.all import *
>>> a = M.vector_field({eU: [-y,x]}, name='a')
>>> a.add_comp_by_continuation(eV, W, c_uv)
>>> a.display(eU)
a = -y ∂/∂x + x ∂/∂y
>>> a.display(eV)
a = v ∂/∂u - u ∂/∂v
>>> da = nab(a) ; da
Tensor field nabla(a) of type (1,1) on the 2-dimensional differentiable
 manifold M
>>> da.display(eU)
nabla(a) = -x*y ∂/∂x⊗dx - ∂/∂x⊗dy + ∂/∂y⊗dx - x*y**2 ∂/∂y⊗dy
>>> da.display(eV)
nabla(a) = (-u**3/16 + u**2*v/16 - u**2/8 + u*v**2/16 - v**3/16 + v**2/8) ∂/∂u⊗du
 + (u**3/16 - u**2*v/16 - u**2/8 - u*v**2/16 + v**3/16 + v**2/8 + 1) ∂/∂u⊗dv
 + (u**3/16 - u**2*v/16 - u**2/8 - u*v**2/16 + v**3/16 + v**2/8 - 1) ∂/∂v⊗du
 + (-u**3/16 + u**2*v/16 - u**2/8 + u*v**2/16 - v**3/16 + v**2/8) ∂/∂v⊗dv
```

To make affine connections hashable, they have to be set immutable before:

```
sage: nab.is_immutable()
False
sage: nab.set_immutable()
sage: nab.is_immutable()
True
```

```
>>> from sage.all import *
>>> nab.is_immutable()
False
>>> nab.set_immutable()
>>> nab.is_immutable()
True
```

Immutable connections cannot be changed anymore:

```
sage: nab.set_coef(eU)
Traceback (most recent call last):
...
ValueError: the coefficients of an immutable element cannot be
 changed
```

```
>>> from sage.all import *
>>> nab.set_coef(eU)
Traceback (most recent call last):
...
ValueError: the coefficients of an immutable element cannot be
 changed
```

However, they can now be used as keys for dictionaries:

```
sage: {nab: 1}[nab]
1
```

```
>>> from sage.all import *
>>> {nab: Integer(1)}[nab]
1
```

The immutability process cannot be made undone. If a connection is needed to be changed again, a copy has to be created:

```
sage: nab_copy = nab.copy('nablo'); nab_copy
Affine connection nablo on the 2-dimensional differentiable manifold M
sage: nab_copy is nab
False
sage: nab_copy == nab
True
sage: nab_copy.is_immutable()
False
```

```
>>> from sage.all import *
>>> nab_copy = nab.copy('nablo'); nab_copy
```

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```
Affine connection nablo on the 2-dimensional differentiable manifold M
>>> nab_copy is nab
False
>>> nab_copy == nab
True
>>> nab_copy.is_immutable()
False
```

add_coef(*frame=None*)

Return the connection coefficients in a given frame for assignment, keeping the coefficients in other frames.

See method `coef()` for details about the definition of the connection coefficients.

To delete the connection coefficients in other frames, use the method `set_coef()` instead.

INPUT:

- `frame` – (default: `None`) vector frame in which the connection coefficients are defined; if `None`, the default frame of the connection's domain is assumed.

> **Warning**
>
> If the connection has already coefficients in other frames, it is the user's responsibility to make sure that the coefficients to be added are consistent with them.

OUTPUT:

- connection coefficients in the given frame, as an instance of the class `Components`; if such connection coefficients did not exist previously, they are created. See method `coef()` for the storage convention of the connection coefficients.

EXAMPLES:

Setting the coefficients of an affine connection w.r.t. some coordinate frame:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: X.<x,y> = M.chart()
sage: nab = M.affine_connection('nabla', latex_name=r'\nabla')
sage: eX = X.frame(); eX
Coordinate frame (M, (∂/∂x,∂/∂y))
sage: nab.add_coef(eX)
3-indices components w.r.t. Coordinate frame (M, (∂/∂x,∂/∂y))
sage: nab.add_coef(eX)[1,2,1] = x*y
sage: nab.display(eX)
Gam^x_yx = x*y
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> nab = M.affine_connection('nabla', latex_name=r'\nabla')
>>> eX = X.frame(); eX
Coordinate frame (M, (∂/∂x,∂/∂y))
>>> nab.add_coef(eX)
3-indices components w.r.t. Coordinate frame (M, (∂/∂x,∂/∂y))
```

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```
>>> nab.add_coef(eX)[Integer(1),Integer(2),Integer(1)] = x*y
>>> nab.display(eX)
Gam^x_yx = x*y
```

Since `eX` is the manifold's default vector frame, its mention may be omitted:

```
sage: nab.add_coef()[1,2,1] = x*y
sage: nab.add_coef()
3-indices components w.r.t. Coordinate frame (M, (∂/∂x,∂/∂y))
sage: nab.add_coef()[1,2,1] = x*y
sage: nab.display()
Gam^x_yx = x*y
```

```
>>> from sage.all import *
>>> nab.add_coef()[Integer(1),Integer(2),Integer(1)] = x*y
>>> nab.add_coef()
3-indices components w.r.t. Coordinate frame (M, (∂/∂x,∂/∂y))
>>> nab.add_coef()[Integer(1),Integer(2),Integer(1)] = x*y
>>> nab.display()
Gam^x_yx = x*y
```

Adding connection coefficients w.r.t. to another vector frame:

```
sage: e = M.vector_frame('e')
sage: nab.add_coef(e)
3-indices components w.r.t. Vector frame (M, (e_1,e_2))
sage: nab.add_coef(e)[2,1,1] = x+y
sage: nab.add_coef(e)[2,1,2] = x-y
sage: nab.display(e)
Gam^2_11 = x + y
Gam^2_12 = x - y
```

```
>>> from sage.all import *
>>> e = M.vector_frame('e')
>>> nab.add_coef(e)
3-indices components w.r.t. Vector frame (M, (e_1,e_2))
>>> nab.add_coef(e)[Integer(2),Integer(1),Integer(1)] = x+y
>>> nab.add_coef(e)[Integer(2),Integer(1),Integer(2)] = x-y
>>> nab.display(e)
Gam^2_11 = x + y
Gam^2_12 = x - y
```

The coefficients w.r.t. the frame `eX` have been kept:

```
sage: nab.display(eX)
Gam^x_yx = x*y
```

```
>>> from sage.all import *
>>> nab.display(eX)
Gam^x_yx = x*y
```

To delete them, use the method `set_coef()` instead.

coef (*frame=None*)

Return the connection coefficients relative to the given frame.

n being the manifold's dimension, the connection coefficients relative to the vector frame (e_i) are the n^3 scalar fields $\Gamma^k_{\ ij}$ defined by

$$\nabla_{e_j} e_i = \Gamma^k_{\ ij} e_k$$

If the connection coefficients are not known already, they are computed from the above formula.

INPUT:

- `frame` – (default: `None`) vector frame relative to which the connection coefficients are required; if none is provided, the domain's default frame is assumed

OUTPUT:

- connection coefficients relative to the frame `frame`, as an instance of the class `Components` with 3 indices ordered as (k, i, j)

EXAMPLES:

Connection coefficient of an affine connection on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[1,1,2], nab[3,2,3] = x^2, y*z  # Gamma^1_{12} = x^2, Gamma^3_{23} =␣
↪yz
sage: nab.coef()
3-indices components w.r.t. Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z))
sage: type(nab.coef())
<class 'sage.tensor.modules.comp.Components'>
sage: M.default_frame()
Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z))
sage: nab.coef() is nab.coef(c_xyz.frame())
True
sage: nab.coef()[:]  # full list of coefficients:
[[[0, x^2, 0], [0, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, y*z], [0, 0, 0]]]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> nab = M.affine_connection('nabla', r'\nabla')
>>> nab[Integer(1),Integer(1),Integer(2)], nab[Integer(3),Integer(2),
↪Integer(3)] = x**Integer(2), y*z  # Gamma^1_{12} = x^2, Gamma^3_{23} = yz
>>> nab.coef()
3-indices components w.r.t. Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z))
>>> type(nab.coef())
<class 'sage.tensor.modules.comp.Components'>
>>> M.default_frame()
Coordinate frame (M, (∂/∂x,∂/∂y,∂/∂z))
>>> nab.coef() is nab.coef(c_xyz.frame())
True
```

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```
>>> nab.coef()[:]  # full list of coefficients:
[[[0, x^2, 0], [0, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, y*z], [0, 0, 0]]]
```

connection_form(*i*, *j*, *frame=None*)

Return the connection 1-form corresponding to the given index and vector frame.

The *connection 1-forms* with respect to the frame (e_i) are the n^2 1-forms $\omega^i{}_j$ defined by

$$\nabla_v e_j = \langle \omega^i{}_j, v \rangle \, e_i$$

for any vector v.

The components of $\omega^i{}_j$ in the coframe (e^i) dual to (e_i) are nothing but the connection coefficients $\Gamma^i{}_{jk}$ relative to the frame (e_i):

$$\omega^i{}_j = \Gamma^i{}_{jk} e^k$$

INPUT:

- `i`, `j` – indices identifying the 1-form $\omega^i{}_j$
- `frame` – (default: `None`) vector frame relative to which the connection 1-forms are defined; if `None`, the default frame of the connection's domain is assumed.

OUTPUT:

- the 1-form $\omega^i{}_j$, as an instance of `DiffForm`

EXAMPLES:

Connection 1-forms on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[1,1,1], nab[1,1,2], nab[1,1,3] = x*y*z, x^2, -y*z
sage: nab[1,2,3], nab[1,3,1], nab[1,3,2] = -x^3, y^2*z, y^2-x^2
sage: nab[2,1,1], nab[2,1,2], nab[2,2,1] = z^2, x*y*z^2, -x^2
sage: nab[2,3,1], nab[2,3,3], nab[3,1,2] = x^2+y^2+z^2, y^2-z^2, x*y+z^2
sage: nab[3,2,1], nab[3,2,2], nab[3,3,3] = x*y+z, z^3 -y^2, x*z^2 - z*y^2
sage: nab.connection_form(1,1)  # connection 1-form (i,j)=(1,1) w.r.t. M's␣
↪default frame
1-form nabla connection 1-form (1,1) on the 3-dimensional
 differentiable manifold M
sage: nab.connection_form(1,1)[:]
[x*y*z, x^2, -y*z]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> nab = M.affine_connection('nabla', r'\nabla')
>>> nab[Integer(1),Integer(1),Integer(1)], nab[Integer(1),Integer(1),
↪Integer(2)], nab[Integer(1),Integer(1),Integer(3)] = x*y*z, x**Integer(2), -
```

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```
↪y*z
>>> nab[Integer(1),Integer(2),Integer(3)], nab[Integer(1),Integer(3),
↪Integer(1)], nab[Integer(1),Integer(3),Integer(2)] = -x**Integer(3),␣
↪y**Integer(2)*z, y**Integer(2)-x**Integer(2)
>>> nab[Integer(2),Integer(1),Integer(1)], nab[Integer(2),Integer(1),
↪Integer(2)], nab[Integer(2),Integer(2),Integer(1)] = z**Integer(2),␣
↪x*y*z**Integer(2), -x**Integer(2)
>>> nab[Integer(2),Integer(3),Integer(1)], nab[Integer(2),Integer(3),
↪Integer(3)], nab[Integer(3),Integer(1),Integer(2)] =␣
↪x**Integer(2)+y**Integer(2)+z**Integer(2), y**Integer(2)-z**Integer(2),␣
↪x*y+z**Integer(2)
>>> nab[Integer(3),Integer(2),Integer(1)], nab[Integer(3),Integer(2),
↪Integer(2)], nab[Integer(3),Integer(3),Integer(3)] = x*y+z, z**Integer(3) -
↪y**Integer(2), x*z**Integer(2) - z*y**Integer(2)
>>> nab.connection_form(Integer(1),Integer(1))  # connection 1-form (i,j)=(1,
↪1) w.r.t. M's default frame
1-form nabla connection 1-form (1,1) on the 3-dimensional
 differentiable manifold M
>>> nab.connection_form(Integer(1),Integer(1))[:]
[x*y*z, x^2, -y*z]
```

The result is cached (until the connection is modified via `set_coef()` or `add_coef()`):

```
sage: nab.connection_form(1,1) is nab.connection_form(1,1)
True
```

```
>>> from sage.all import *
>>> nab.connection_form(Integer(1),Integer(1)) is nab.connection_
↪form(Integer(1),Integer(1))
True
```

Connection 1-forms w.r.t. a non-holonomic frame:

```
sage: ch_basis = M.automorphism_field()
sage: ch_basis[1,1], ch_basis[2,2], ch_basis[3,3] = y, z, x
sage: e = M.default_frame().new_frame(ch_basis, 'e')
sage: e[1][:], e[2][:], e[3][:]
([y, 0, 0], [0, z, 0], [0, 0, x])
sage: nab.connection_form(1,1,e)
1-form nabla connection 1-form (1,1) on the 3-dimensional
 differentiable manifold M
sage: nab.connection_form(1,1,e).comp(e)[:]
[x*y^2*z, (x^2*y + 1)*z/y, -x*y*z]
```

```
>>> from sage.all import *
>>> ch_basis = M.automorphism_field()
>>> ch_basis[Integer(1),Integer(1)], ch_basis[Integer(2),Integer(2)], ch_
↪basis[Integer(3),Integer(3)] = y, z, x
>>> e = M.default_frame().new_frame(ch_basis, 'e')
>>> e[Integer(1)][:], e[Integer(2)][:], e[Integer(3)][:]
([y, 0, 0], [0, z, 0], [0, 0, x])
>>> nab.connection_form(Integer(1),Integer(1),e)
```

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```
1-form nabla connection 1-form (1,1) on the 3-dimensional
 differentiable manifold M
>>> nab.connection_form(Integer(1),Integer(1),e).comp(e)[:]
[x*y^2*z, (x^2*y + 1)*z/y, -x*y*z]
```

Check of the formula $\omega^i{}_j = \Gamma^i{}_{jk} e^k$:

First on the manifold's default frame (∂/∂x, ∂/∂y, d:dz):

```
sage: dx = M.default_frame().coframe() ; dx
Coordinate coframe (M, (dx,dy,dz))
sage: check = []
sage: for i in M.irange():
....:     for j in M.irange():
....:         check.append( nab.connection_form(i,j) == \
....:               sum( nab[[i,j,k]]*dx[k] for k in M.irange() ) )
sage: check
[True, True, True, True, True, True, True, True, True]
```

```
>>> from sage.all import *
>>> dx = M.default_frame().coframe() ; dx
Coordinate coframe (M, (dx,dy,dz))
>>> check = []
>>> for i in M.irange():
...     for j in M.irange():
...         check.append( nab.connection_form(i,j) ==               sum(
↪nab[[i,j,k]]*dx[k] for k in M.irange() ) )
>>> check
[True, True, True, True, True, True, True, True, True]
```

Then on the frame e:

```
sage: ef = e.coframe() ; ef
Coframe (M, (e^1,e^2,e^3))
sage: check = []
sage: for i in M.irange():
....:     for j in M.irange():
....:         s = nab.connection_form(i,j,e).comp(c_xyz.frame(), from_basis=e)
....:         check.append( nab.connection_form(i,j,e) == sum( nab.coef(e)[[i,
↪j,k]]*ef[k] for k in M.irange() ) )
sage: check
[True, True, True, True, True, True, True, True, True]
```

```
>>> from sage.all import *
>>> ef = e.coframe() ; ef
Coframe (M, (e^1,e^2,e^3))
>>> check = []
>>> for i in M.irange():
...     for j in M.irange():
...         s = nab.connection_form(i,j,e).comp(c_xyz.frame(), from_basis=e)
...         check.append( nab.connection_form(i,j,e) == sum( nab.coef(e)[[i,j,
↪k]]*ef[k] for k in M.irange() ) )
```

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```
>>> check
[True, True, True, True, True, True, True, True, True]
```

Check of the formula $\nabla_v e_j = \langle \omega^i{}_j, v \rangle e_i$:

```
sage: v = M.vector_field()
sage: v[:] = (x*y, z^2-3*x, z+2*y)
sage: b = M.default_frame()
sage: for j in M.irange():  # check on M's default frame  # long time
....:     nab(b[j]).contract(v) == \
....:      sum( nab.connection_form(i,j)(v)*b[i] for i in M.irange())
True
True
True
sage: for j in M.irange():  # check on frame e  # long time
....:     nab(e[j]).contract(v) == \
....:      sum( nab.connection_form(i,j,e)(v)*e[i] for i in M.irange())
True
True
True
```

```
>>> from sage.all import *
>>> v = M.vector_field()
>>> v[:] = (x*y, z**Integer(2)-Integer(3)*x, z+Integer(2)*y)
>>> b = M.default_frame()
>>> for j in M.irange():  # check on M's default frame  # long time
...     nab(b[j]).contract(v) ==      sum( nab.connection_form(i,j)(v)*b[i]␣
↪for i in M.irange())
True
True
True
>>> for j in M.irange():  # check on frame e  # long time
...     nab(e[j]).contract(v) ==      sum( nab.connection_form(i,j,e)(v)*e[i]␣
↪for i in M.irange())
True
True
True
```

copy(*name*, *latex_name=None*)

Return an exact copy of `self`.

INPUT:

- `name` – name given to the copy
- `latex_name` – (default: `None`) LaTeX symbol to denote the copy; if none is provided, the LaTeX symbol is set to `name`

Note

The name and the derived quantities are not copied.

EXAMPLES:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: X.<x,y> = M.chart()
sage: nab = M.affine_connection('nabla', latex_name=r'\nabla')
sage: eX = X.frame()
sage: nab.set_coef(eX)[1,2,1] = x*y
sage: nab.set_coef(eX)[1,2,2] = x+y
sage: nab.display()
Gam^x_yx = x*y
Gam^x_yy = x + y
sage: nab_copy = nab.copy(name='nabla_1', latex_name=r'\nabla_1')
sage: nab is nab_copy
False
sage: nab == nab_copy
True
sage: nab_copy.display()
Gam^x_yx = x*y
Gam^x_yy = x + y
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> nab = M.affine_connection('nabla', latex_name=r'\nabla')
>>> eX = X.frame()
>>> nab.set_coef(eX)[Integer(1),Integer(2),Integer(1)] = x*y
>>> nab.set_coef(eX)[Integer(1),Integer(2),Integer(2)] = x+y
>>> nab.display()
Gam^x_yx = x*y
Gam^x_yy = x + y
>>> nab_copy = nab.copy(name='nabla_1', latex_name=r'\nabla_1')
>>> nab is nab_copy
False
>>> nab == nab_copy
True
>>> nab_copy.display()
Gam^x_yx = x*y
Gam^x_yy = x + y
```

curvature_form(*i*, *j*, *frame=None*)

Return the curvature 2-form corresponding to the given index and vector frame.

The *curvature 2-forms* with respect to the frame (e_i) are the n^2 2-forms $\Omega^i{}_j$ defined by

$$\Omega^i{}_j(u,v) = R(e^i, e_j, u, v)$$

where R is the connection's Riemann curvature tensor (cf. `riemann()`), (e^i) is the coframe dual to (e_i) and (u,v) is a generic pair of vectors.

INPUT:

- `i`, `j` – indices identifying the 2-form $\Omega^i{}_j$
- `frame` – (default: `None`) vector frame relative to which the curvature 2-forms are defined; if `None`, the default frame of the connection's domain is assumed.

OUTPUT:

- the 2-form $\Omega^i{}_j$, as an instance of `DiffForm`

EXAMPLES:

Curvature 2-forms on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[1,1,1], nab[1,1,2], nab[1,1,3] = x*y*z, x^2, -y*z
sage: nab[1,2,3], nab[1,3,1], nab[1,3,2] = -x^3, y^2*z, y^2-x^2
sage: nab[2,1,1], nab[2,1,2], nab[2,2,1] = z^2, x*y*z^2, -x^2
sage: nab[2,3,1], nab[2,3,3], nab[3,1,2] = x^2+y^2+z^2, y^2-z^2, x*y+z^2
sage: nab[3,2,1], nab[3,2,2], nab[3,3,3] = x*y+z, z^3 -y^2, x*z^2 - z*y^2
sage: nab.curvature_form(1,1)  # long time
2-form curvature (1,1) of connection nabla w.r.t. Coordinate frame
 (M, (∂/∂x,∂/∂y,∂/∂z)) on the 3-dimensional differentiable manifold M
sage: nab.curvature_form(1,1).display()  # long time (if above is skipped)
curvature (1,1) of connection nabla w.r.t. Coordinate frame
 (M, (∂/∂x,∂/∂y,∂/∂z)) = (y^2*z^3 + (x*y^3 - x)*z + 2*x) dx∧dy
  + (x^3*z^2 - x*y) dx∧dz + (x^4*y*z^2 - z) dy∧dz
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> nab = M.affine_connection('nabla', r'\nabla')
>>> nab[Integer(1),Integer(1),Integer(1)], nab[Integer(1),Integer(1),
↪Integer(2)], nab[Integer(1),Integer(1),Integer(3)] = x*y*z, x**Integer(2), -
↪y*z
>>> nab[Integer(1),Integer(2),Integer(3)], nab[Integer(1),Integer(3),
↪Integer(1)], nab[Integer(1),Integer(3),Integer(2)] = -x**Integer(3),␣
↪y**Integer(2)*z, y**Integer(2)-x**Integer(2)
>>> nab[Integer(2),Integer(1),Integer(1)], nab[Integer(2),Integer(1),
↪Integer(2)], nab[Integer(2),Integer(2),Integer(1)] = z**Integer(2),␣
↪x*y*z**Integer(2), -x**Integer(2)
>>> nab[Integer(2),Integer(3),Integer(1)], nab[Integer(2),Integer(3),
↪Integer(3)], nab[Integer(3),Integer(1),Integer(2)] =␣
↪x**Integer(2)+y**Integer(2)+z**Integer(2), y**Integer(2)-z**Integer(2),␣
↪x*y+z**Integer(2)
>>> nab[Integer(3),Integer(2),Integer(1)], nab[Integer(3),Integer(2),
↪Integer(2)], nab[Integer(3),Integer(3),Integer(3)] = x*y+z, z**Integer(3) -
↪y**Integer(2), x*z**Integer(2) - z*y**Integer(2)
>>> nab.curvature_form(Integer(1),Integer(1))  # long time
2-form curvature (1,1) of connection nabla w.r.t. Coordinate frame
 (M, (∂/∂x,∂/∂y,∂/∂z)) on the 3-dimensional differentiable manifold M
>>> nab.curvature_form(Integer(1),Integer(1)).display()  # long time (if␣
↪above is skipped)
curvature (1,1) of connection nabla w.r.t. Coordinate frame
 (M, (∂/∂x,∂/∂y,∂/∂z)) = (y^2*z^3 + (x*y^3 - x)*z + 2*x) dx∧dy
  + (x^3*z^2 - x*y) dx∧dz + (x^4*y*z^2 - z) dy∧dz
```

Curvature 2-forms w.r.t. a non-holonomic frame:

```
sage: ch_basis = M.automorphism_field()
sage: ch_basis[1,1], ch_basis[2,2], ch_basis[3,3] = y, z, x
sage: e = M.default_frame().new_frame(ch_basis, 'e')
sage: e[1].display(), e[2].display(), e[3].display()
(e_1 = y ∂/∂x, e_2 = z ∂/∂y, e_3 = x ∂/∂z)
sage: ef = e.coframe()
sage: ef[1].display(), ef[2].display(), ef[3].display()
(e^1 = 1/y dx, e^2 = 1/z dy, e^3 = 1/x dz)
sage: nab.curvature_form(1,1,e)  # long time
2-form curvature (1,1) of connection nabla w.r.t. Vector frame
 (M, (e_1,e_2,e_3)) on the 3-dimensional differentiable manifold M
sage: nab.curvature_form(1,1,e).display(e)  # long time (if above is skipped)
curvature (1,1) of connection nabla w.r.t. Vector frame
 (M, (e_1,e_2,e_3)) =
  (y^3*z^4 + 2*x*y*z + (x*y^4 - x*y)*z^2) e^1∧e^2
  + (x^4*y*z^2 - x^2*y^2) e^1∧e^3 + (x^5*y*z^3 - x*z^2) e^2∧e^3
```

```
>>> from sage.all import *
>>> ch_basis = M.automorphism_field()
>>> ch_basis[Integer(1),Integer(1)], ch_basis[Integer(2),Integer(2)], ch_
↪basis[Integer(3),Integer(3)] = y, z, x
>>> e = M.default_frame().new_frame(ch_basis, 'e')
>>> e[Integer(1)].display(), e[Integer(2)].display(), e[Integer(3)].display()
(e_1 = y ∂/∂x, e_2 = z ∂/∂y, e_3 = x ∂/∂z)
>>> ef = e.coframe()
>>> ef[Integer(1)].display(), ef[Integer(2)].display(), ef[Integer(3)].
↪display()
(e^1 = 1/y dx, e^2 = 1/z dy, e^3 = 1/x dz)
>>> nab.curvature_form(Integer(1),Integer(1),e)  # long time
2-form curvature (1,1) of connection nabla w.r.t. Vector frame
 (M, (e_1,e_2,e_3)) on the 3-dimensional differentiable manifold M
>>> nab.curvature_form(Integer(1),Integer(1),e).display(e)  # long time (if␣
↪above is skipped)
curvature (1,1) of connection nabla w.r.t. Vector frame
 (M, (e_1,e_2,e_3)) =
  (y^3*z^4 + 2*x*y*z + (x*y^4 - x*y)*z^2) e^1∧e^2
  + (x^4*y*z^2 - x^2*y^2) e^1∧e^3 + (x^5*y*z^3 - x*z^2) e^2∧e^3
```

Cartan's second structure equation is

$$\Omega^i{}_j = \mathrm{d}\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$$

where the $\omega^i{}_j$'s are the connection 1-forms (cf. `connection_form()`). Let us check it on the frame e:

```
sage: omega = nab.connection_form
sage: check = []
sage: for i in M.irange():  # long time
....:     for j in M.irange():
....:         check.append( nab.curvature_form(i,j,e) == \
....:                       omega(i,j,e).exterior_derivative() + \
....:         sum( omega(i,k,e).wedge(omega(k,j,e)) for k in M.irange()) )
sage: check  # long time
[True, True, True, True, True, True, True, True, True]
```

```
>>> from sage.all import *
>>> omega = nab.connection_form
>>> check = []
>>> for i in M.irange():  # long time
...     for j in M.irange():
...         check.append( nab.curvature_form(i,j,e) ==
↪omega(i,j,e).exterior_derivative() +         sum( omega(i,k,e).
↪wedge(omega(k,j,e)) for k in M.irange()) )
>>> check  # long time
[True, True, True, True, True, True, True, True, True]
```

del_other_coef(*frame=None*)

Delete all the coefficients but those corresponding to `frame`.

INPUT:

- `frame` – (default: `None`) vector frame, the connection coefficients w.r.t. which are to be kept; if `None`, the default frame of the connection's domain is assumed.

EXAMPLES:

We first create two sets of connection coefficients:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: X.<x,y> = M.chart()
sage: nab = M.affine_connection('nabla', latex_name=r'\nabla')
sage: eX = X.frame()
sage: nab.set_coef(eX)[1,2,1] = x*y
sage: e = M.vector_frame('e')
sage: nab.add_coef(e)[2,1,1] = x+y
sage: nab.display(eX)
Gam^x_yx = x*y
sage: nab.display(e)
Gam^2_11 = x + y
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> nab = M.affine_connection('nabla', latex_name=r'\nabla')
>>> eX = X.frame()
>>> nab.set_coef(eX)[Integer(1),Integer(2),Integer(1)] = x*y
>>> e = M.vector_frame('e')
>>> nab.add_coef(e)[Integer(2),Integer(1),Integer(1)] = x+y
>>> nab.display(eX)
Gam^x_yx = x*y
>>> nab.display(e)
Gam^2_11 = x + y
```

Let us delete the connection coefficients w.r.t. all frames except for frame `eX`:

```
sage: nab.del_other_coef(eX)
sage: nab.display(eX)
Gam^x_yx = x*y
```

```
>>> from sage.all import *
>>> nab.del_other_coef(eX)
>>> nab.display(eX)
Gam^x_yx = x*y
```

The connection coefficients w.r.t. frame `e` have indeed been deleted:

```
sage: nab.display(e)
Traceback (most recent call last):
...
ValueError: no common frame found for the computation
```

```
>>> from sage.all import *
>>> nab.display(e)
Traceback (most recent call last):
...
ValueError: no common frame found for the computation
```

display(*frame=None*, *chart=None*, *symbol=None*, *latex_symbol=None*, *index_labels=None*, *index_latex_labels=None*, *coordinate_labels=True*, *only_nonzero=True*, *only_nonredundant=False*)

Display all the connection coefficients w.r.t. to a given frame, one per line.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- `frame` – (default: `None`) vector frame relative to which the connection coefficients are defined; if `None`, the default frame of the connection's domain is used
- `chart` – (default: `None`) chart specifying the coordinate expression of the connection coefficients; if `None`, the default chart of the domain of `frame` is used
- `symbol` – (default: `None`) string specifying the symbol of the connection coefficients; if `None`, 'Gam' is used
- `latex_symbol` – (default: `None`) string specifying the LaTeX symbol for the components; if `None`, '\Gamma' is used
- `index_labels` – (default: `None`) list of strings representing the labels of each index; if `None`, integer labels are used, except if `frame` is a coordinate frame and `coordinate_symbols` is set to `True`, in which case the coordinate symbols are used
- `index_latex_labels` – (default: `None`) list of strings representing the LaTeX labels of each index; if `None`, integer labels are used, except if `frame` is a coordinate frame and `coordinate_symbols` is set to `True`, in which case the coordinate LaTeX symbols are used
- `coordinate_labels` – boolean (default: `True`); if `True`, coordinate symbols are used by default (instead of integers) as index labels whenever `frame` is a coordinate frame
- `only_nonzero` – boolean (default: `True`); if `True`, only nonzero connection coefficients are displayed
- `only_nonredundant` – boolean (default: `False`); if `True`, only nonredundant connection coefficients are displayed in case of symmetries

EXAMPLES:

Coefficients of a connection on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[1,1,2], nab[3,2,3] = x^2, y*z
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> nab = M.affine_connection('nabla', r'\nabla')
>>> nab[Integer(1),Integer(1),Integer(2)], nab[Integer(3),Integer(2),
↪Integer(3)] = x**Integer(2), y*z
```

By default, only the nonzero connection coefficients are displayed:

```
sage: nab.display()
Gam^x_xy = x^2
Gam^z_yz = y*z
sage: latex(nab.display())
\begin{array}{lcl} \Gamma_{ \phantom{\, x} \, x \, y }^{ \, x \phantom{\, x} \
↪phantom{\, y} }
& = & x^{2} \\
\Gamma_{ \phantom{\, z} \, y \, z }^{ \, z \phantom{\, y} \phantom{\, z} }
& = & y z \end{array}
```

```
>>> from sage.all import *
>>> nab.display()
Gam^x_xy = x^2
Gam^z_yz = y*z
>>> latex(nab.display())
\begin{array}{lcl} \Gamma_{ \phantom{\, x} \, x \, y }^{ \, x \phantom{\, x} \
↪phantom{\, y} }
& = & x^{2} \\
\Gamma_{ \phantom{\, z} \, y \, z }^{ \, z \phantom{\, y} \phantom{\, z} }
& = & y z \end{array}
```

By default, the displayed connection coefficients are those w.r.t. to the default frame of the connection's domain, so the above is equivalent to:

```
sage: nab.display(frame=M.default_frame())
Gam^x_xy = x^2
Gam^z_yz = y*z
```

```
>>> from sage.all import *
>>> nab.display(frame=M.default_frame())
Gam^x_xy = x^2
Gam^z_yz = y*z
```

Since the default frame is a coordinate frame, coordinate symbols are used to label the indices, but one may ask for integers instead:

```
sage: M.default_frame() is c_xyz.frame()
True
```

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```
sage: nab.display(coordinate_labels=False)
Gam^1_12 = x^2
Gam^3_23 = y*z
```

```
>>> from sage.all import *
>>> M.default_frame() is c_xyz.frame()
True
>>> nab.display(coordinate_labels=False)
Gam^1_12 = x^2
Gam^3_23 = y*z
```

The index labels can also be customized:

```
sage: nab.display(index_labels=['(1)', '(2)', '(3)'])
Gam^(1)_(1),(2) = x^2
Gam^(3)_(2),(3) = y*z
```

```
>>> from sage.all import *
>>> nab.display(index_labels=['(1)', '(2)', '(3)'])
Gam^(1)_(1),(2) = x^2
Gam^(3)_(2),(3) = y*z
```

The symbol ‘Gam’ can be changed:

```
sage: nab.display(symbol='C', latex_symbol='C')
C^x_xy = x^2
C^z_yz = y*z
sage: latex(nab.display(symbol='C', latex_symbol='C'))
\begin{array}{lcl} C_{ \phantom{\, x} \, x \, y }^{ \, x \phantom{\, x} \
↪phantom{\, y} }
& = & x^{2} \\
C_{ \phantom{\, z} \, y \, z }^{ \, z \phantom{\, y} \phantom{\, z} }
& = & y z \end{array}
```

```
>>> from sage.all import *
>>> nab.display(symbol='C', latex_symbol='C')
C^x_xy = x^2
C^z_yz = y*z
>>> latex(nab.display(symbol='C', latex_symbol='C'))
\begin{array}{lcl} C_{ \phantom{\, x} \, x \, y }^{ \, x \phantom{\, x} \
↪phantom{\, y} }
& = & x^{2} \\
C_{ \phantom{\, z} \, y \, z }^{ \, z \phantom{\, y} \phantom{\, z} }
& = & y z \end{array}
```

Display of Christoffel symbols, skipping the redundancy associated with the symmetry of the last two indices:

```
sage: M = Manifold(3, 'R^3', start_index=1)
sage: c_spher.<r,th,ph> = M.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\
↪phi')
sage: g = M.metric('g')
sage: g[1,1], g[2,2], g[3,3] = 1, r^2 , (r*sin(th))^2
```

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```
sage: g.display()
g = dr⊗dr + r^2 dth⊗dth + r^2*sin(th)^2 dph⊗dph
sage: g.connection().display(only_nonredundant=True)
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'R^3', start_index=Integer(1))
>>> c_spher = M.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi', names=(
↪'r', 'th', 'ph',)); (r, th, ph,) = c_spher._first_ngens(3)
>>> g = M.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)], g[Integer(3),
↪Integer(3)] = Integer(1), r**Integer(2) , (r*sin(th))**Integer(2)
>>> g.display()
g = dr⊗dr + r^2 dth⊗dth + r^2*sin(th)^2 dph⊗dph
>>> g.connection().display(only_nonredundant=True)
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
```

By default, the parameter `only_nonredundant` is set to `False`:

```
sage: g.connection().display()
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_th,r = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
Gam^ph_ph,r = 1/r
Gam^ph_ph,th = cos(th)/sin(th)
```

```
>>> from sage.all import *
>>> g.connection().display()
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_th,r = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
Gam^ph_ph,r = 1/r
Gam^ph_ph,th = cos(th)/sin(th)
```

domain()

Return the manifold subset on which the affine connection is defined.

OUTPUT:

- instance of class *DifferentiableManifold* representing the manifold on which `self` is defined.

EXAMPLES:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab.domain()
3-dimensional differentiable manifold M
sage: U = M.open_subset('U', coord_def={c_xyz: x>0})
sage: nabU = U.affine_connection('D')
sage: nabU.domain()
Open subset U of the 3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> nab = M.affine_connection('nabla', r'\nabla')
>>> nab.domain()
3-dimensional differentiable manifold M
>>> U = M.open_subset('U', coord_def={c_xyz: x>Integer(0)})
>>> nabU = U.affine_connection('D')
>>> nabU.domain()
Open subset U of the 3-dimensional differentiable manifold M
```

is_immutable()

Return `True` if this object is immutable, i.e. its coefficients cannot be chanced, and `False` if it is not.

To set an affine connection immutable, use *set_immutable()*.

EXAMPLES:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: X.<x,y> = M.chart()
sage: nab = M.affine_connection('nabla', latex_name=r'\nabla')
sage: nab.is_immutable()
False
sage: nab.set_immutable()
sage: nab.is_immutable()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> nab = M.affine_connection('nabla', latex_name=r'\nabla')
>>> nab.is_immutable()
False
>>> nab.set_immutable()
>>> nab.is_immutable()
True
```

is_mutable()

Return `True` if this object is mutable, i.e. its coefficients can be changed, and `False` if it is not.

EXAMPLES:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: X.<x,y> = M.chart()
sage: nab = M.affine_connection('nabla', latex_name=r'\nabla')
sage: nab.is_mutable()
True
sage: nab.set_immutable()
sage: nab.is_mutable()
False
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> nab = M.affine_connection('nabla', latex_name=r'\nabla')
>>> nab.is_mutable()
True
>>> nab.set_immutable()
>>> nab.is_mutable()
False
```

restrict(*subdomain*)

Return the restriction of the connection to some subdomain.

If such restriction has not been defined yet, it is constructed here.

INPUT:

- `subdomain` – open subset U of the connection's domain (must be an instance of `DifferentiableManifold`)

OUTPUT: instance of `AffineConnection` representing the restriction

EXAMPLES:

Restriction of a connection on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: c_xy.<x,y> = M.chart()
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[1,1,2], nab[2,1,1] = x^2, x+y
sage: nab[:]
[[[0, x^2], [0, 0]], [[x + y, 0], [0, 0]]]
sage: U = M.open_subset('U', coord_def={c_xy: x>0})
sage: nabU = nab.restrict(U) ; nabU
Affine connection nabla on the Open subset U of the 2-dimensional
 differentiable manifold M
sage: nabU.domain()
Open subset U of the 2-dimensional differentiable manifold M
sage: nabU[:]
[[[0, x^2], [0, 0]], [[x + y, 0], [0, 0]]]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', start_index=Integer(1))
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> nab = M.affine_connection('nabla', r'\nabla')
>>> nab[Integer(1),Integer(1),Integer(2)], nab[Integer(2),Integer(1),
↪Integer(1)] = x**Integer(2), x+y
>>> nab[:]
[[[0, x^2], [0, 0]], [[x + y, 0], [0, 0]]]
>>> U = M.open_subset('U', coord_def={c_xy: x>Integer(0)})
>>> nabU = nab.restrict(U) ; nabU
Affine connection nabla on the Open subset U of the 2-dimensional
 differentiable manifold M
>>> nabU.domain()
Open subset U of the 2-dimensional differentiable manifold M
>>> nabU[:]
[[[0, x^2], [0, 0]], [[x + y, 0], [0, 0]]]
```

The result is cached:

```
sage: nab.restrict(U) is nabU
True
```

```
>>> from sage.all import *
>>> nab.restrict(U) is nabU
True
```

until the connection is modified:

```
sage: nab[1,2,2] = -y
sage: nab.restrict(U) is nabU
False
sage: nab.restrict(U)[:]
[[[0, x^2], [0, -y]], [[x + y, 0], [0, 0]]]
```

```
>>> from sage.all import *
>>> nab[Integer(1),Integer(2),Integer(2)] = -y
>>> nab.restrict(U) is nabU
False
>>> nab.restrict(U)[:]
[[[0, x^2], [0, -y]], [[x + y, 0], [0, 0]]]
```

ricci()

Return the connection's Ricci tensor.

The *Ricci tensor* is the tensor field Ric of type (0,2) defined from the Riemann curvature tensor R by

$$Ric(u, v) = R(e^i, u, e_i, v)$$

for any vector fields u and v, (e_i) being any vector frame and (e^i) the dual coframe.

OUTPUT:

- the Ricci tensor Ric, as an instance of `TensorField`

EXAMPLES:

Ricci tensor of an affine connection on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: nab = M.affine_connection('nabla', r'\nabla') ; nab
Affine connection nabla on the 3-dimensional differentiable
 manifold M
sage: nab[1,1,2], nab[3,2,3] = x^2, y*z  # Gamma^1_{12} = x^2, Gamma^3_{23} =␣
↪yz
sage: r = nab.ricci() ; r
Tensor field of type (0,2) on the 3-dimensional differentiable
 manifold M
sage: r[:]
[  0 2*x   0]
[  0  -z   0]
[  0   0   0]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> nab = M.affine_connection('nabla', r'\nabla') ; nab
Affine connection nabla on the 3-dimensional differentiable
 manifold M
>>> nab[Integer(1),Integer(1),Integer(2)], nab[Integer(3),Integer(2),
↪Integer(3)] = x**Integer(2), y*z  # Gamma^1_{12} = x^2, Gamma^3_{23} = yz
>>> r = nab.ricci() ; r
Tensor field of type (0,2) on the 3-dimensional differentiable
 manifold M
>>> r[:]
[  0 2*x   0]
[  0  -z   0]
[  0   0   0]
```

The result is cached (until the connection is modified via `set_coef()` or `add_coef()`):

```
sage: nab.ricci() is r
True
```

```
>>> from sage.all import *
>>> nab.ricci() is r
True
```

riemann()

Return the connection's Riemann curvature tensor.

The *Riemann curvature tensor* is the tensor field R of type (1,3) defined by

$$R(\omega, w, u, v) = \left\langle \omega, \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w \right\rangle$$

for any 1-form ω and any vector fields u, v and w.

OUTPUT:

- the Riemann curvature tensor R, as an instance of `TensorField`

EXAMPLES:

Curvature of an affine connection on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: nab = M.affine_connection('nabla', r'\nabla') ; nab
Affine connection nabla on the 3-dimensional differentiable
 manifold M
sage: nab[1,1,2], nab[3,2,3] = x^2, y*z  # Gamma^1_{12} = x^2, Gamma^3_{23} =␣
↪yz
sage: r = nab.riemann() ; r
Tensor field of type (1,3) on the 3-dimensional differentiable
 manifold M
sage: r.parent()
Free module T^(1,3)(M) of type-(1,3) tensors fields on the
 3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> nab = M.affine_connection('nabla', r'\nabla') ; nab
Affine connection nabla on the 3-dimensional differentiable
 manifold M
>>> nab[Integer(1),Integer(1),Integer(2)], nab[Integer(3),Integer(2),
↪Integer(3)] = x**Integer(2), y*z  # Gamma^1_{12} = x^2, Gamma^3_{23} = yz
>>> r = nab.riemann() ; r
Tensor field of type (1,3) on the 3-dimensional differentiable
 manifold M
>>> r.parent()
Free module T^(1,3)(M) of type-(1,3) tensors fields on the
 3-dimensional differentiable manifold M
```

By construction, the Riemann tensor is antisymmetric with respect to its last two arguments (denoted u and v in the definition above), which are at positions 2 and 3 (the first argument being at position 0):

```
sage: r.symmetries()
no symmetry;  antisymmetry: (2, 3)
```

```
>>> from sage.all import *
>>> r.symmetries()
no symmetry;  antisymmetry: (2, 3)
```

The components:

```
sage: r[:]
[[[[0, 2*x, 0], [-2*x, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, 0], [0, 0, 0]]],
[[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, 0], [0, 0, 0]]],
[[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
```

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```
[[0, 0, 0], [0, 0, z], [0, -z, 0]],
[[0, 0, 0], [0, 0, 0], [0, 0, 0]]]]
```

```
>>> from sage.all import *
>>> r[:]
[[[[0, 2*x, 0], [-2*x, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, 0], [0, 0, 0]]],
[[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, 0], [0, 0, 0]]],
[[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, z], [0, -z, 0]],
[[0, 0, 0], [0, 0, 0], [0, 0, 0]]]]
```

The result is cached (until the connection is modified via `set_coef()` or `add_coef()`):

```
sage: nab.riemann() is r
True
```

```
>>> from sage.all import *
>>> nab.riemann() is r
True
```

Another example: Riemann curvature tensor of some connection on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
....:                              restrictions1= x>0, restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: c_xyW = c_xy.restrict(W) ; c_uvW = c_uv.restrict(W)
sage: eUW = c_xyW.frame() ; eVW = c_uvW.frame()
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[0,0,0], nab[0,1,0], nab[1,0,1] = x, x-y, x*y
sage: for i in M.irange():
....:     for j in M.irange():
....:         for k in M.irange():
....:             nab.add_coef(eV)[i,j,k] = nab.coef(eVW)[i,j,k,c_uvW].expr()
sage: r = nab.riemann() ; r     # long time
Tensor field of type (1,3) on the 2-dimensional differentiable
 manifold M
sage: r.parent()                # long time
Module T^(1,3)(M) of type-(1,3) tensors fields on the 2-dimensional
 differentiable manifold M
sage: r.display(eU)             # long time
(x^2*y - x*y^2) ∂/∂x⊗dx⊗dx⊗dy + (-x^2*y + x*y^2) ∂/∂x⊗dx⊗dy⊗dx + ∂/∂x⊗dy⊗dx⊗dy
```

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```
 - ∂/∂x⊗dy⊗dy⊗dx - (x^2 - 1)*y ∂/∂y⊗dx⊗dx⊗dy + (x^2 - 1)*y ∂/∂y⊗dx⊗dy⊗dx
 + (-x^2*y + x*y^2) ∂/∂y⊗dy⊗dx⊗dy + (x^2*y - x*y^2) ∂/∂y⊗dy⊗dy⊗dx
sage: r.display(eV)            # long time
(1/32*u^3 - 1/32*u*v^2 - 1/32*v^3 + 1/32*(u^2 + 4)*v - 1/8*u - 1/4) ∂/
↪∂u⊗du⊗du⊗dv
 + (-1/32*u^3 + 1/32*u*v^2 + 1/32*v^3 - 1/32*(u^2 + 4)*v + 1/8*u + 1/4) ∂/
↪∂u⊗du⊗dv⊗du
 + (1/32*u^3 - 1/32*u*v^2 + 3/32*v^3 - 1/32*(3*u^2 - 4)*v - 1/8*u + 1/4) ∂/
↪∂u⊗dv⊗du⊗dv
 + (-1/32*u^3 + 1/32*u*v^2 - 3/32*v^3 + 1/32*(3*u^2 - 4)*v + 1/8*u - 1/4) ∂/
↪∂u⊗dv⊗dv⊗du
 + (-1/32*u^3 + 1/32*u*v^2 + 5/32*v^3 - 1/32*(5*u^2 + 4)*v + 1/8*u - 1/4) ∂/
↪∂v⊗du⊗du⊗dv
 + (1/32*u^3 - 1/32*u*v^2 - 5/32*v^3 + 1/32*(5*u^2 + 4)*v - 1/8*u + 1/4) ∂/
↪∂v⊗du⊗dv⊗du
 + (-1/32*u^3 + 1/32*u*v^2 + 1/32*v^3 - 1/32*(u^2 + 4)*v + 1/8*u + 1/4) ∂/
↪∂v⊗dv⊗du⊗dv
 + (1/32*u^3 - 1/32*u*v^2 - 1/32*v^3 + 1/32*(u^2 + 4)*v - 1/8*u - 1/4) ∂/
↪∂v⊗dv⊗dv⊗du
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
...                              restrictions1= x>Integer(0), restrictions2=␣
↪u+v>Integer(0))
>>> inv = transf.inverse()
>>> W = U.intersection(V)
>>> eU = c_xy.frame() ; eV = c_uv.frame()
>>> c_xyW = c_xy.restrict(W) ; c_uvW = c_uv.restrict(W)
>>> eUW = c_xyW.frame() ; eVW = c_uvW.frame()
>>> nab = M.affine_connection('nabla', r'\nabla')
>>> nab[Integer(0),Integer(0),Integer(0)], nab[Integer(0),Integer(1),
↪Integer(0)], nab[Integer(1),Integer(0),Integer(1)] = x, x-y, x*y
>>> for i in M.irange():
...     for j in M.irange():
...         for k in M.irange():
...             nab.add_coef(eV)[i,j,k] = nab.coef(eVW)[i,j,k,c_uvW].expr()
>>> r = nab.riemann() ; r     # long time
Tensor field of type (1,3) on the 2-dimensional differentiable
 manifold M
>>> r.parent()                # long time
Module T^(1,3)(M) of type-(1,3) tensors fields on the 2-dimensional
 differentiable manifold M
>>> r.display(eU)             # long time
(x^2*y - x*y^2) ∂/∂x⊗dx⊗dx⊗dy + (-x^2*y + x*y^2) ∂/∂x⊗dx⊗dy⊗dx + ∂/∂x⊗dy⊗dx⊗dy
 - ∂/∂x⊗dy⊗dy⊗dx - (x^2 - 1)*y ∂/∂y⊗dx⊗dx⊗dy + (x^2 - 1)*y ∂/∂y⊗dx⊗dy⊗dx
 + (-x^2*y + x*y^2) ∂/∂y⊗dy⊗dx⊗dy + (x^2*y - x*y^2) ∂/∂y⊗dy⊗dy⊗dx
>>> r.display(eV)             # long time
```

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```
(1/32*u^3 - 1/32*u*v^2 - 1/32*v^3 + 1/32*(u^2 + 4)*v - 1/8*u - 1/4) ∂/
↪∂u⊗du⊗du⊗dv
 + (-1/32*u^3 + 1/32*u*v^2 + 1/32*v^3 - 1/32*(u^2 + 4)*v + 1/8*u + 1/4) ∂/
↪∂u⊗du⊗dv⊗du
 + (1/32*u^3 - 1/32*u*v^2 + 3/32*v^3 - 1/32*(3*u^2 - 4)*v - 1/8*u + 1/4) ∂/
↪∂u⊗dv⊗du⊗dv
 + (-1/32*u^3 + 1/32*u*v^2 - 3/32*v^3 + 1/32*(3*u^2 - 4)*v + 1/8*u - 1/4) ∂/
↪∂u⊗dv⊗dv⊗du
 + (-1/32*u^3 + 1/32*u*v^2 + 5/32*v^3 - 1/32*(5*u^2 + 4)*v + 1/8*u - 1/4) ∂/
↪∂v⊗du⊗du⊗dv
 + (1/32*u^3 - 1/32*u*v^2 - 5/32*v^3 + 1/32*(5*u^2 + 4)*v - 1/8*u + 1/4) ∂/
↪∂v⊗du⊗dv⊗du
 + (-1/32*u^3 + 1/32*u*v^2 + 1/32*v^3 - 1/32*(u^2 + 4)*v + 1/8*u + 1/4) ∂/
↪∂v⊗dv⊗du⊗dv
 + (1/32*u^3 - 1/32*u*v^2 - 1/32*v^3 + 1/32*(u^2 + 4)*v - 1/8*u - 1/4) ∂/
↪∂v⊗dv⊗dv⊗du
```

The same computation parallelized on 2 cores:

```
sage: Parallelism().set(nproc=2)
sage: r_backup = r              # long time
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[0,0,0], nab[0,1,0], nab[1,0,1] = x, x-y, x*y
sage: for i in M.irange():
....:     for j in M.irange():
....:         for k in M.irange():
....:             nab.add_coef(eV)[i,j,k] = nab.coef(eVW)[i,j,k,c_uvW].expr()
sage: r = nab.riemann() ; r     # long time
Tensor field of type (1,3) on the 2-dimensional differentiable
 manifold M
sage: r.parent()                # long time
Module T^(1,3)(M) of type-(1,3) tensors fields on the 2-dimensional
 differentiable manifold M
sage: r == r_backup             # long time
True
sage: Parallelism().set(nproc=1)  # switch off parallelization
```

```
>>> from sage.all import *
>>> Parallelism().set(nproc=Integer(2))
>>> r_backup = r              # long time
>>> nab = M.affine_connection('nabla', r'\nabla')
>>> nab[Integer(0),Integer(0),Integer(0)], nab[Integer(0),Integer(1),
↪Integer(0)], nab[Integer(1),Integer(0),Integer(1)] = x, x-y, x*y
>>> for i in M.irange():
...     for j in M.irange():
...         for k in M.irange():
...             nab.add_coef(eV)[i,j,k] = nab.coef(eVW)[i,j,k,c_uvW].expr()
>>> r = nab.riemann() ; r     # long time
Tensor field of type (1,3) on the 2-dimensional differentiable
 manifold M
>>> r.parent()                # long time
Module T^(1,3)(M) of type-(1,3) tensors fields on the 2-dimensional
```

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```
differentiable manifold M
>>> r == r_backup            # long time
True
>>> Parallelism().set(nproc=Integer(1))  # switch off parallelization
```

set_calc_order(*symbol*, *order*, *truncate=False*)

Trigger a series expansion with respect to a small parameter in computations involving `self`.

This property is propagated by usual operations. The internal representation must be `SR` for this to take effect.

INPUT:

- `symbol` – symbolic variable (the "small parameter" ϵ) with respect to which the connection coefficients are expanded in power series
- `order` – integer; the order n of the expansion, defined as the degree of the polynomial representing the truncated power series in `symbol`
- `truncate` – boolean (default: `False`); determines whether the connection coefficients are replaced by their expansions to the given order

EXAMPLES:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: C.<t,x,y,z> = M.chart()
sage: e = var('e')
sage: g = M.metric()
sage: h = M.tensor_field(0, 2, sym=(0,1))
sage: g[0, 0], g[1, 1], g[2, 2], g[3, 3] = -1, 1, 1, 1
sage: h[0, 1] = x
sage: g.set(g + e*h)
sage: g[:]
[ -1 e*x   0   0]
[e*x   1   0   0]
[  0   0   1   0]
[  0   0   0   1]
sage: nab = g.connection()
sage: nab[0, 1, 1]
-e/(e^2*x^2 + 1)
sage: nab.set_calc_order(e, 1, truncate=True)
sage: nab[0, 1, 1]
-e
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> C = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = C._first_
↪ngens(4)
>>> e = var('e')
>>> g = M.metric()
>>> h = M.tensor_field(Integer(0), Integer(2), sym=(Integer(0),Integer(1)))
>>> g[Integer(0), Integer(0)], g[Integer(1), Integer(1)], g[Integer(2),␣
↪Integer(2)], g[Integer(3), Integer(3)] = -Integer(1), Integer(1),␣
↪Integer(1), Integer(1)
>>> h[Integer(0), Integer(1)] = x
>>> g.set(g + e*h)
```

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```
>>> g[:]
[ -1 e*x   0   0]
[e*x   1   0   0]
[  0   0   1   0]
[  0   0   0   1]
>>> nab = g.connection()
>>> nab[Integer(0), Integer(1), Integer(1)]
-e/(e^2*x^2 + 1)
>>> nab.set_calc_order(e, Integer(1), truncate=True)
>>> nab[Integer(0), Integer(1), Integer(1)]
-e
```

set_coef(*frame=None*)

Return the connection coefficients in a given frame for assignment.

See method `coef()` for details about the definition of the connection coefficients.

The connection coefficients with respect to other frames are deleted, in order to avoid any inconsistency. To keep them, use the method `add_coef()` instead.

INPUT:

- `frame` – (default: `None`) vector frame in which the connection coefficients are defined; if `None`, the default frame of the connection's domain is assumed.

OUTPUT:

- connection coefficients in the given frame, as an instance of the class `Components`; if such connection coefficients did not exist previously, they are created. See method `coef()` for the storage convention of the connection coefficients.

EXAMPLES:

Setting the coefficients of an affine connection w.r.t. some coordinate frame:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: X.<x,y> = M.chart()
sage: nab = M.affine_connection('nabla', latex_name=r'\nabla')
sage: eX = X.frame(); eX
Coordinate frame (M, (∂/∂x,∂/∂y))
sage: nab.set_coef(eX)
3-indices components w.r.t. Coordinate frame (M, (∂/∂x,∂/∂y))
sage: nab.set_coef(eX)[1,2,1] = x*y
sage: nab.display(eX)
Gam^x_yx = x*y
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> nab = M.affine_connection('nabla', latex_name=r'\nabla')
>>> eX = X.frame(); eX
Coordinate frame (M, (∂/∂x,∂/∂y))
>>> nab.set_coef(eX)
3-indices components w.r.t. Coordinate frame (M, (∂/∂x,∂/∂y))
>>> nab.set_coef(eX)[Integer(1),Integer(2),Integer(1)] = x*y
```

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```
>>> nab.display(eX)
Gam^x_yx = x*y
```

Since `eX` is the manifold's default vector frame, its mention may be omitted:

```
sage: nab.set_coef()[1,2,1] = x*y
sage: nab.set_coef()
3-indices components w.r.t. Coordinate frame (M, (∂/∂x,∂/∂y))
sage: nab.set_coef()[1,2,1] = x*y
sage: nab.display()
Gam^x_yx = x*y
```

```
>>> from sage.all import *
>>> nab.set_coef()[Integer(1),Integer(2),Integer(1)] = x*y
>>> nab.set_coef()
3-indices components w.r.t. Coordinate frame (M, (∂/∂x,∂/∂y))
>>> nab.set_coef()[Integer(1),Integer(2),Integer(1)] = x*y
>>> nab.display()
Gam^x_yx = x*y
```

To set the coefficients in the default frame, one can even bypass the method `set_coef()` and call directly the operator `[]` on the connection object:

```
sage: nab[1,2,1] = x*y
sage: nab.display()
    Gam^x_yx = x*y
```

```
>>> from sage.all import *
>>> nab[Integer(1),Integer(2),Integer(1)] = x*y
>>> nab.display()
    Gam^x_yx = x*y
```

Setting the connection coefficients w.r.t. to another vector frame:

```
sage: e = M.vector_frame('e')
sage: nab.set_coef(e)
3-indices components w.r.t. Vector frame (M, (e_1,e_2))
sage: nab.set_coef(e)[2,1,1] = x+y
sage: nab.set_coef(e)[2,1,2] = x-y
sage: nab.display(e)
Gam^2_11 = x + y
Gam^2_12 = x - y
```

```
>>> from sage.all import *
>>> e = M.vector_frame('e')
>>> nab.set_coef(e)
3-indices components w.r.t. Vector frame (M, (e_1,e_2))
>>> nab.set_coef(e)[Integer(2),Integer(1),Integer(1)] = x+y
>>> nab.set_coef(e)[Integer(2),Integer(1),Integer(2)] = x-y
>>> nab.display(e)
Gam^2_11 = x + y
Gam^2_12 = x - y
```

The coefficients w.r.t. the frame `eX` have been deleted:

```
sage: nab.display(eX)
Traceback (most recent call last):
...
ValueError: no common frame found for the computation
```

```
>>> from sage.all import *
>>> nab.display(eX)
Traceback (most recent call last):
...
ValueError: no common frame found for the computation
```

To keep them, use the method `add_coef()` instead.

set_immutable()

Set `self` and all restrictions of `self` immutable.

EXAMPLES:

An affine connection can be set immutable:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: X.<x,y> = M.chart()
sage: U = M.open_subset('U', coord_def={X: x^2+y^2<1})
sage: nab = M.affine_connection('nabla', latex_name=r'\nabla')
sage: eX = X.frame()
sage: nab.set_coef(eX)[1,2,1] = x*y
sage: nab.is_immutable()
False
sage: nab.set_immutable()
sage: nab.is_immutable()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> U = M.open_subset('U', coord_def={X: x**Integer(2)+y**Integer(2)
↪<Integer(1)})
>>> nab = M.affine_connection('nabla', latex_name=r'\nabla')
>>> eX = X.frame()
>>> nab.set_coef(eX)[Integer(1),Integer(2),Integer(1)] = x*y
>>> nab.is_immutable()
False
>>> nab.set_immutable()
>>> nab.is_immutable()
True
```

The coefficients of immutable elements cannot be changed:

```
sage: nab.add_coef(eX)[2,1,1] = x+y
Traceback (most recent call last):
...
ValueError: the coefficients of an immutable element cannot
 be changed
```

```
>>> from sage.all import *
>>> nab.add_coef(eX)[Integer(2),Integer(1),Integer(1)] = x+y
Traceback (most recent call last):
...
ValueError: the coefficients of an immutable element cannot
 be changed
```

The restriction are set immutable as well:

```
sage: nabU = nab.restrict(U)
sage: nabU.is_immutable()
True
```

```
>>> from sage.all import *
>>> nabU = nab.restrict(U)
>>> nabU.is_immutable()
True
```

torsion()

Return the connection's torsion tensor.

The torsion tensor is the tensor field T of type (1,2) defined by

$$T(\omega, u, v) = \langle \omega, \nabla_u v - \nabla_v u - [u, v] \rangle$$

for any 1-form ω and any vector fields u and v.

OUTPUT:

- the torsion tensor T, as an instance of *TensorField*

EXAMPLES:

Torsion of an affine connection on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[1,1,2], nab[3,2,3] = x^2, y*z  # Gamma^1_{12} = x^2, Gamma^3_{23} =
↪yz
sage: t = nab.torsion() ; t
Tensor field of type (1,2) on the 3-dimensional differentiable
 manifold M
sage: t.symmetries()
no symmetry;  antisymmetry: (1, 2)
sage: t[:]
[[[0, -x^2, 0], [x^2, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, -y*z], [0, y*z, 0]]]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> nab = M.affine_connection('nabla', r'\nabla')
```

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```
>>> nab[Integer(1),Integer(1),Integer(2)], nab[Integer(3),Integer(2),
↪Integer(3)] = x**Integer(2), y*z  # Gamma^1_{12} = x^2, Gamma^3_{23} = yz
>>> t = nab.torsion() ; t
Tensor field of type (1,2) on the 3-dimensional differentiable
 manifold M
>>> t.symmetries()
no symmetry;  antisymmetry: (1, 2)
>>> t[:]
[[[0, -x^2, 0], [x^2, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, -y*z], [0, y*z, 0]]]
```

The torsion expresses the lack of commutativity of two successive derivatives of a scalar field:

```
sage: f = M.scalar_field(x*z^2 + y^2 - z^2, name='f')
sage: DDf = nab(nab(f)) ; DDf
Tensor field nabla(df) of type (0,2) on the 3-dimensional
 differentiable manifold M
sage: DDf.antisymmetrize()[:]  # two successive derivatives do not commute:
[             0   -1/2*x^2*z^2              0]
[   1/2*x^2*z^2              0 -(x - 1)*y*z^2]
[             0  (x - 1)*y*z^2              0]
sage: 2*DDf.antisymmetrize() == nab.torsion().contract(0,nab(f))
True
```

```
>>> from sage.all import *
>>> f = M.scalar_field(x*z**Integer(2) + y**Integer(2) - z**Integer(2), name=
↪'f')
>>> DDf = nab(nab(f)) ; DDf
Tensor field nabla(df) of type (0,2) on the 3-dimensional
 differentiable manifold M
>>> DDf.antisymmetrize()[:]  # two successive derivatives do not commute:
[             0   -1/2*x^2*z^2              0]
[   1/2*x^2*z^2              0 -(x - 1)*y*z^2]
[             0  (x - 1)*y*z^2              0]
>>> Integer(2)*DDf.antisymmetrize() == nab.torsion().contract(Integer(0),
↪nab(f))
True
```

The above identity is the standard formula

$$\nabla_j \nabla_i \, f - \nabla_i \nabla_j \, f = T^k_{\ \, ij} \nabla_k \, f,$$

where the $T^k_{\ \, ij}$'s are the components of the torsion tensor.

The result is cached:

```
sage: nab.torsion() is t
True
```

```
>>> from sage.all import *
>>> nab.torsion() is t
True
```

as long as the connection remains unchanged:

```
sage: nab[2,1,3] = 1+x     # changing the connection
sage: nab.torsion() is t   # a new computation of the torsion has been made
False
sage: (nab.torsion() - t).display()
(-x - 1) ∂/∂y⊗dx⊗dz + (x + 1) ∂/∂y⊗dz⊗dx
```

```
>>> from sage.all import *
>>> nab[Integer(2),Integer(1),Integer(3)] = Integer(1)+x     # changing the␣
↪connection
>>> nab.torsion() is t   # a new computation of the torsion has been made
False
>>> (nab.torsion() - t).display()
(-x - 1) ∂/∂y⊗dx⊗dz + (x + 1) ∂/∂y⊗dz⊗dx
```

Another example: torsion of some connection on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
....:                              restrictions1= x>0, restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: c_xyW = c_xy.restrict(W) ; c_uvW = c_uv.restrict(W)
sage: eUW = c_xyW.frame() ; eVW = c_uvW.frame()
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[0,0,0], nab[0,1,0], nab[1,0,1] = x, x-y, x*y
sage: for i in M.irange():
....:     for j in M.irange():
....:         for k in M.irange():
....:             nab.add_coef(eV)[i,j,k] = nab.coef(eVW)[i,j,k,c_uvW].expr()
sage: t = nab.torsion() ; t
Tensor field of type (1,2) on the 2-dimensional differentiable
 manifold M
sage: t.parent()
Module T^(1,2)(M) of type-(1,2) tensors fields on the 2-dimensional
 differentiable manifold M
sage: t[eU,:]
[[[0, x - y], [-x + y, 0]], [[0, -x*y], [x*y, 0]]]
sage: t[eV,:]
[[[0, 1/8*u^2 - 1/8*v^2 - 1/2*v], [-1/8*u^2 + 1/8*v^2 + 1/2*v, 0]],
 [[0, -1/8*u^2 + 1/8*v^2 - 1/2*v], [1/8*u^2 - 1/8*v^2 + 1/2*v, 0]]]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
```

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```
>>> transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
...                              restrictions1= x>Integer(0), restrictions2=␣
↪u+v>Integer(0))
>>> inv = transf.inverse()
>>> W = U.intersection(V)
>>> eU = c_xy.frame() ; eV = c_uv.frame()
>>> c_xyW = c_xy.restrict(W) ; c_uvW = c_uv.restrict(W)
>>> eUW = c_xyW.frame() ; eVW = c_uvW.frame()
>>> nab = M.affine_connection('nabla', r'\nabla')
>>> nab[Integer(0),Integer(0),Integer(0)], nab[Integer(0),Integer(1),
↪Integer(0)], nab[Integer(1),Integer(0),Integer(1)] = x, x-y, x*y
>>> for i in M.irange():
...     for j in M.irange():
...         for k in M.irange():
...             nab.add_coef(eV)[i,j,k] = nab.coef(eVW)[i,j,k,c_uvW].expr()
>>> t = nab.torsion() ; t
Tensor field of type (1,2) on the 2-dimensional differentiable
 manifold M
>>> t.parent()
Module T^(1,2)(M) of type-(1,2) tensors fields on the 2-dimensional
 differentiable manifold M
>>> t[eU,:]
[[[0, x - y], [-x + y, 0]], [[0, -x*y], [x*y, 0]]]
>>> t[eV,:]
[[[0, 1/8*u^2 - 1/8*v^2 - 1/2*v], [-1/8*u^2 + 1/8*v^2 + 1/2*v, 0]],
 [[0, -1/8*u^2 + 1/8*v^2 - 1/2*v], [1/8*u^2 - 1/8*v^2 + 1/2*v, 0]]]
```

Check of the torsion formula:

```
sage: f = M.scalar_field({c_xy: (x+y)^2, c_uv: u^2}, name='f')
sage: DDf = nab(nab(f)) ; DDf
Tensor field nabla(df) of type (0,2) on the 2-dimensional
 differentiable manifold M
sage: DDf.antisymmetrize().display(eU)
(-x^2*y - (x + 1)*y^2 + x^2) dx∧dy
sage: DDf.antisymmetrize().display(eV)
(1/8*u^3 - 1/8*u*v^2 - 1/2*u*v) du∧dv
sage: 2*DDf.antisymmetrize() == nab(f).contract(nab.torsion())
True
```

```
>>> from sage.all import *
>>> f = M.scalar_field({c_xy: (x+y)**Integer(2), c_uv: u**Integer(2)}, name='f
↪')
>>> DDf = nab(nab(f)) ; DDf
Tensor field nabla(df) of type (0,2) on the 2-dimensional
 differentiable manifold M
>>> DDf.antisymmetrize().display(eU)
(-x^2*y - (x + 1)*y^2 + x^2) dx∧dy
>>> DDf.antisymmetrize().display(eV)
(1/8*u^3 - 1/8*u*v^2 - 1/2*u*v) du∧dv
>>> Integer(2)*DDf.antisymmetrize() == nab(f).contract(nab.torsion())
True
```

torsion_form(*i*, *frame=None*)

Return the torsion 2-form corresponding to the given index and vector frame.

The *torsion 2-forms* with respect to the frame (e_i) are the n 2-forms θ^i defined by

$$\theta^i(u, v) = T(e^i, u, v)$$

where T is the connection's torsion tensor (cf. `torsion()`), (e^i) is the coframe dual to (e_i) and (u, v) is a generic pair of vectors.

INPUT:

- `i` – index identifying the 2-form θ^i
- `frame` – (default: `None`) vector frame relative to which the torsion 2-forms are defined; if `None`, the default frame of the connection's domain is assumed.

OUTPUT:

- the 2-form θ^i, as an instance of `DiffForm`

EXAMPLES:

Torsion 2-forms on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[1,1,1], nab[1,1,2], nab[1,1,3] = x*y*z, x^2, -y*z
sage: nab[1,2,3], nab[1,3,1], nab[1,3,2] = -x^3, y^2*z, y^2-x^2
sage: nab[2,1,1], nab[2,1,2], nab[2,2,1] = z^2, x*y*z^2, -x^2
sage: nab[2,3,1], nab[2,3,3], nab[3,1,2] = x^2+y^2+z^2, y^2-z^2, x*y+z^2
sage: nab[3,2,1], nab[3,2,2], nab[3,3,3] = x*y+z, z^3 -y^2, x*z^2 - z*y^2
sage: nab.torsion_form(1)
2-form torsion (1) of connection nabla w.r.t. Coordinate frame
 (M, (∂/∂x,∂/∂y,∂/∂z)) on the 3-dimensional differentiable manifold M
sage: nab.torsion_form(1)[:]
[               0             -x^2       (y^2 + y)*z]
[             x^2                0   x^3 - x^2 + y^2]
[    -(y^2 + y)*z -x^3 + x^2 - y^2                 0]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> nab = M.affine_connection('nabla', r'\nabla')
>>> nab[Integer(1),Integer(1),Integer(1)], nab[Integer(1),Integer(1),
↪Integer(2)], nab[Integer(1),Integer(1),Integer(3)] = x*y*z, x**Integer(2), -
↪y*z
>>> nab[Integer(1),Integer(2),Integer(3)], nab[Integer(1),Integer(3),
↪Integer(1)], nab[Integer(1),Integer(3),Integer(2)] = -x**Integer(3),␣
↪y**Integer(2)*z, y**Integer(2)-x**Integer(2)
>>> nab[Integer(2),Integer(1),Integer(1)], nab[Integer(2),Integer(1),
↪Integer(2)], nab[Integer(2),Integer(2),Integer(1)] = z**Integer(2),␣
↪x*y*z**Integer(2), -x**Integer(2)
>>> nab[Integer(2),Integer(3),Integer(1)], nab[Integer(2),Integer(3),
↪Integer(3)], nab[Integer(3),Integer(1),Integer(2)] =␣
```

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```
↪x**Integer(2)+y**Integer(2)+z**Integer(2), y**Integer(2)-z**Integer(2),␣
↪x*y+z**Integer(2)
>>> nab[Integer(3),Integer(2),Integer(1)], nab[Integer(3),Integer(2),
↪Integer(2)], nab[Integer(3),Integer(3),Integer(3)] = x*y+z, z**Integer(3) -
↪y**Integer(2), x*z**Integer(2) - z*y**Integer(2)
>>> nab.torsion_form(Integer(1))
2-form torsion (1) of connection nabla w.r.t. Coordinate frame
 (M, (∂/∂x,∂/∂y,∂/∂z)) on the 3-dimensional differentiable manifold M
>>> nab.torsion_form(Integer(1))[:]
[              0          -x^2      (y^2 + y)*z]
[            x^2             0  x^3 - x^2 + y^2]
[   -(y^2 + y)*z -x^3 + x^2 - y^2                0]
```

Torsion 2-forms w.r.t. a non-holonomic frame:

```
sage: ch_basis = M.automorphism_field()
sage: ch_basis[1,1], ch_basis[2,2], ch_basis[3,3] = y, z, x
sage: e = M.default_frame().new_frame(ch_basis, 'e')
sage: e[1][:], e[2][:], e[3][:]
([y, 0, 0], [0, z, 0], [0, 0, x])
sage: ef = e.coframe()
sage: ef[1][:], ef[2][:], ef[3][:]
([1/y, 0, 0], [0, 1/z, 0], [0, 0, 1/x])
sage: nab.torsion_form(1, e)  # long time
2-form torsion (1) of connection nabla w.r.t. Vector frame
 (M, (e_1,e_2,e_3)) on the 3-dimensional differentiable manifold M
sage: nab.torsion_form(1, e).comp(e)[:]  # long time
[                       0                    -x^2*z          (x*y^2 + x*y)*z]
[                   x^2*z                         0  (x^4 - x^3 + x*y^2)*z/y]
[        -(x*y^2 + x*y)*z -(x^4 - x^3 + x*y^2)*z/y                        0]
```

```
>>> from sage.all import *
>>> ch_basis = M.automorphism_field()
>>> ch_basis[Integer(1),Integer(1)], ch_basis[Integer(2),Integer(2)], ch_
↪basis[Integer(3),Integer(3)] = y, z, x
>>> e = M.default_frame().new_frame(ch_basis, 'e')
>>> e[Integer(1)][:], e[Integer(2)][:], e[Integer(3)][:]
([y, 0, 0], [0, z, 0], [0, 0, x])
>>> ef = e.coframe()
>>> ef[Integer(1)][:], ef[Integer(2)][:], ef[Integer(3)][:]
([1/y, 0, 0], [0, 1/z, 0], [0, 0, 1/x])
>>> nab.torsion_form(Integer(1), e)  # long time
2-form torsion (1) of connection nabla w.r.t. Vector frame
 (M, (e_1,e_2,e_3)) on the 3-dimensional differentiable manifold M
>>> nab.torsion_form(Integer(1), e).comp(e)[:]  # long time
[                       0                    -x^2*z          (x*y^2 + x*y)*z]
[                   x^2*z                         0  (x^4 - x^3 + x*y^2)*z/y]
[        -(x*y^2 + x*y)*z -(x^4 - x^3 + x*y^2)*z/y                        0]
```

Cartan's first structure equation is

$$\theta^i = \mathrm{d}e^i + \omega^i{}_j \wedge e^j$$

where the $\omega^i{}_j$'s are the connection 1-forms (cf. `connection_form()`). Let us check it on the frame e:

```
sage: for i in M.irange():  # long time
....:     nab.torsion_form(i, e) == ef[i].exterior_derivative() + \
....:      sum(nab.connection_form(i,j,e).wedge(ef[j]) for j in M.irange())
True
True
True
```

```
>>> from sage.all import *
>>> for i in M.irange():  # long time
...     nab.torsion_form(i, e) == ef[i].exterior_derivative() +      sum(nab.
↪connection_form(i,j,e).wedge(ef[j]) for j in M.irange())
True
True
True
```

2.14 Submanifolds of differentiable manifolds

Given two differentiable manifolds N and M, an *immersion* ϕ is a differentiable map $N \to M$ whose differential is everywhere injective. One then says that N is an *immersed submanifold* of M, via ϕ.

If in addition, ϕ is a differentiable embedding (i.e. ϕ is an immersion that is a homeomorphism onto its image), then N is called an *embedded submanifold* of M (or simply a *submanifold*).

ϕ can also depend on one or multiple parameters. As long as the differential of ϕ remains injective in these parameters, it represents a *foliation*. The *dimension* of the foliation is defined as the number of parameters.

AUTHORS:

- Florentin Jaffredo (2018): initial version
- Eric Gourgoulhon (2018-2019): add documentation
- Matthias Koeppe (2021): open subsets of submanifolds

REFERENCES:

- J. M. Lee: *Introduction to Smooth Manifolds* [Lee2013]

class sage.manifolds.differentiable.differentiable_submanifold.**DifferentiableSubmanifold**(*n*, *name*, *field*, *structure*, *ambient=None*, *base_manifold=None*, *diff_degree=+Infinity*, *latex_name=None*, *start_index=0*, *category=None*, *unique_tag=None*)

Bases: *DifferentiableManifold*, *TopologicalSubmanifold*

Submanifold of a differentiable manifold.

Given two differentiable manifolds N and M, an *immersion* ϕ is a differentiable map $N \to M$ whose differential is everywhere injective. One then says that N is an *immersed submanifold* of M, via ϕ.

If in addition, ϕ is a differentiable embedding (i.e. ϕ is an immersion that is a homeomorphism onto its image), then N is called an *embedded submanifold* of M (or simply a *submanifold*).

ϕ can also depend on one or multiple parameters. As long as the differential of ϕ remains injective in these parameters, it represents a *foliation*. The *dimension* of the foliation is defined as the number of parameters.

INPUT:

- `n` – positive integer; dimension of the submanifold
- `name` – string; name (symbol) given to the submanifold
- `field` – field K on which the sub manifold is defined; allowed values are
  - `'real'` or an object of type `RealField` (e.g., `RR`) for a manifold over **R**
  - `'complex'` or an object of type `ComplexField` (e.g., `CC`) for a manifold over **C**
  - an object in the category of topological fields (see `Fields` and `TopologicalSpaces`) for other types of manifolds
- `structure` – manifold structure (see *TopologicalStructure* or *RealTopologicalStructure*)
- `ambient` – (default: `None`) codomain M of the immersion ϕ; must be a differentiable manifold. If `None`, it is set to `self`
- `base_manifold` – (default: `None`) if not `None`, must be a differentiable manifold; the created object is then an open subset of `base_manifold`
- `diff_degree` – (default: `infinity`) degree of differentiability

- `latex_name` – (default: `None`) string; LaTeX symbol to denote the submanifold; if none are provided, it is set to `name`
- `start_index` – (default: 0) integer; lower value of the range of indices used for "indexed objects" on the submanifold, e.g., coordinates in a chart
- `category` – (default: `None`) to specify the category; if `None`, `Manifolds(field).Differentiable()` (or `Manifolds(field).Smooth()` if `diff_degree` = `infinity`) is assumed (see the category `Manifolds`)
- `unique_tag` – (default: `None`) tag used to force the construction of a new object when all the other arguments have been used previously (without `unique_tag`, the `UniqueRepresentation` behavior inherited from `ManifoldSubset` via `DifferentiableManifold` would return the previously constructed object corresponding to these arguments)

EXAMPLES:

Let N be a 2-dimensional submanifold of a 3-dimensional manifold M:

```
sage: M = Manifold(3, 'M')
sage: N = Manifold(2, 'N', ambient=M)
sage: N
2-dimensional differentiable submanifold N immersed in the
 3-dimensional differentiable manifold M
sage: CM.<x,y,z> = M.chart()
sage: CN.<u,v> = N.chart()
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> N = Manifold(Integer(2), 'N', ambient=M)
>>> N
2-dimensional differentiable submanifold N immersed in the
 3-dimensional differentiable manifold M
>>> CM = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = CM._first_ngens(3)
>>> CN = N.chart(names=('u', 'v',)); (u, v,) = CN._first_ngens(2)
```

Let us define a 1-dimensional foliation indexed by t:

```
sage: t = var('t')
sage: phi = N.continuous_map(M, {(CN,CM): [u, v, t+u^2+v^2]})
sage: phi.display()
N → M
   (u, v) ↦ (x, y, z) = (u, v, u^2 + v^2 + t)
```

```
>>> from sage.all import *
>>> t = var('t')
>>> phi = N.continuous_map(M, {(CN,CM): [u, v, t+u**Integer(2)+v**Integer(2)]})
>>> phi.display()
N → M
   (u, v) ↦ (x, y, z) = (u, v, u^2 + v^2 + t)
```

The foliation inverse maps are needed for computing the adapted chart on the ambient manifold:

```
sage: phi_inv = M.continuous_map(N, {(CM, CN): [x, y]})
sage: phi_inv.display()
M → N
```

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```
    (x, y, z) ↦ (u, v) = (x, y)
sage: phi_inv_t = M.scalar_field({CM: z-x^2-y^2})
sage: phi_inv_t.display()
M → ℝ
    (x, y, z) ↦ -x^2 - y^2 + z
```

```
>>> from sage.all import *
>>> phi_inv = M.continuous_map(N, {(CM, CN): [x, y]})
>>> phi_inv.display()
M → N
    (x, y, z) ↦ (u, v) = (x, y)
>>> phi_inv_t = M.scalar_field({CM: z-x**Integer(2)-y**Integer(2)})
>>> phi_inv_t.display()
M → ℝ
    (x, y, z) ↦ -x^2 - y^2 + z
```

ϕ can then be declared as an embedding $N \to M$:

```
sage: N.set_embedding(phi, inverse=phi_inv, var=t,
....:                 t_inverse={t: phi_inv_t})
```

```
>>> from sage.all import *
>>> N.set_embedding(phi, inverse=phi_inv, var=t,
...                 t_inverse={t: phi_inv_t})
```

The foliation can also be used to find new charts on the ambient manifold that are adapted to the foliation, ie in which the expression of the immersion is trivial. At the same time, the appropriate coordinate changes are computed:

```
sage: N.adapted_chart()
[Chart (M, (u_M, v_M, t_M))]
sage: M.atlas()
[Chart (M, (x, y, z)), Chart (M, (u_M, v_M, t_M))]
sage: len(M.coord_changes())
2
```

```
>>> from sage.all import *
>>> N.adapted_chart()
[Chart (M, (u_M, v_M, t_M))]
>>> M.atlas()
[Chart (M, (x, y, z)), Chart (M, (u_M, v_M, t_M))]
>>> len(M.coord_changes())
2
```

See also

manifold and *topological_submanifold*

open_subset(*name*, *latex_name=None*, *coord_def={}*, *supersets=None*)

Create an open subset of the manifold.

An open subset is a set that is (i) included in the manifold and (ii) open with respect to the manifold's topology.

It is a differentiable manifold by itself.

As `self` is a submanifold of its ambient manifold, the new open subset is also considered a submanifold of that. Hence the returned object is an instance of `DifferentiableSubmanifold`.

INPUT:

- `name` – name given to the open subset
- `latex_name` – (default: `None`) LaTeX symbol to denote the subset; if none is provided, it is set to `name`
- `coord_def` – (default: `{}`) definition of the subset in terms of coordinates; `coord_def` must a be dictionary with keys charts in the manifold's atlas and values the symbolic expressions formed by the coordinates to define the subset.
- `supersets` – (default: only `self`) list of sets that the new open subset is a subset of

OUTPUT: the open subset, as an instance of `DifferentiableSubmanifold`

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='differentiable')
sage: N = Manifold(2, 'N', ambient=M, structure='differentiable'); N
2-dimensional differentiable submanifold N immersed in the
 3-dimensional differentiable manifold M
sage: S = N.subset('S'); S
Subset S of the
 2-dimensional differentiable submanifold N immersed in the
  3-dimensional differentiable manifold M
sage: O = N.subset('O', is_open=True); O  # indirect doctest
Open subset O of the
 2-dimensional differentiable submanifold N immersed in the
  3-dimensional differentiable manifold M

sage: phi = N.diff_map(M)
sage: N.set_embedding(phi)
sage: N
2-dimensional differentiable submanifold N embedded in the
 3-dimensional differentiable manifold M
sage: S = N.subset('S'); S
Subset S of the
 2-dimensional differentiable submanifold N embedded in the
  3-dimensional differentiable manifold M
sage: O = N.subset('O', is_open=True); O  # indirect doctest
Open subset O of the
 2-dimensional differentiable submanifold N embedded in the
  3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='differentiable')
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='differentiable'); N
2-dimensional differentiable submanifold N immersed in the
 3-dimensional differentiable manifold M
>>> S = N.subset('S'); S
Subset S of the
 2-dimensional differentiable submanifold N immersed in the
  3-dimensional differentiable manifold M
```

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```
>>> O = N.subset('O', is_open=True); O  # indirect doctest
Open subset O of the
 2-dimensional differentiable submanifold N immersed in the
  3-dimensional differentiable manifold M

>>> phi = N.diff_map(M)
>>> N.set_embedding(phi)
>>> N
2-dimensional differentiable submanifold N embedded in the
 3-dimensional differentiable manifold M
>>> S = N.subset('S'); S
Subset S of the
 2-dimensional differentiable submanifold N embedded in the
  3-dimensional differentiable manifold M
>>> O = N.subset('O', is_open=True); O  # indirect doctest
Open subset O of the
 2-dimensional differentiable submanifold N embedded in the
  3-dimensional differentiable manifold M
```

2.15 Differentiable Vector Bundles

2.15.1 Differentiable Vector Bundles

Let K be a topological field. A C^k-differentiable *vector bundle* of rank n over the field K and over a C^k-differentiable manifold M (base space) is a C^k-differentiable manifold E (total space) together with a C^k differentiable and surjective map $\pi : E \to M$ such that for every point $x \in M$:

- the set $E_x = \pi^{-1}(x)$ has the vector space structure of K^n,
- there is a neighborhood $U \subset M$ of x and a C^k-diffeomorphism $\varphi : \pi^{-1}(x) \to U \times K^n$ such that $v \mapsto \varphi^{-1}(y, v)$ is a linear isomorphism for any $y \in U$.

An important case of a differentiable vector bundle over a differentiable manifold is the tensor bundle (see `TensorBundle`)

AUTHORS:

- Michael Jung (2019) : initial version

class `sage.manifolds.differentiable.vector_bundle.`**`DifferentiableVectorBundle`**(*rank*, *name*, *base_space*, *field='real'*, *latex_name=None*, *category=None*, *unique_tag=None*)

Bases: `TopologicalVectorBundle`

An instance of this class represents a differentiable vector bundle $E \to M$

INPUT:

- `rank` – positive integer; rank of the vector bundle
- `name` – string representation given to the total space

- `base_space` – the base space (differentiable manifold) M over which the vector bundle is defined
- `field` – field K which gives the fibers the structure of a vector space over K; allowed values are
  - `'real'` or an object of type `RealField` (e.g., `RR`) for a vector bundle over **R**
  - `'complex'` or an object of type `ComplexField` (e.g., `CC`) for a vector bundle over **C**
  - an object in the category of topological fields (see `Fields` and `TopologicalSpaces`) for other types of topological fields
- `latex_name` – (default: `None`) LaTeX representation given to the total space
- `category` – (default: `None`) to specify the category; if `None`, `VectorBundles(base_space, c_field).Differentiable()` is assumed (see the category `VectorBundles`)

EXAMPLES:

A differentiable vector bundle of rank 2 over a 3-dimensional differentiable manifold:

```
sage: M = Manifold(3, 'M')
sage: E = M.vector_bundle(2, 'E', field='complex'); E
Differentiable complex vector bundle E -> M of rank 2 over the base
 space 3-dimensional differentiable manifold M
sage: E.category()
Category of smooth vector bundles over Complex Field with 53 bits of
 precision with base space 3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> E = M.vector_bundle(Integer(2), 'E', field='complex'); E
Differentiable complex vector bundle E -> M of rank 2 over the base
 space 3-dimensional differentiable manifold M
>>> E.category()
Category of smooth vector bundles over Complex Field with 53 bits of
 precision with base space 3-dimensional differentiable manifold M
```

At this stage, the differentiable vector bundle has the same differentiability degree as the base manifold:

```
sage: M.diff_degree() == E.diff_degree()
True
```

```
>>> from sage.all import *
>>> M.diff_degree() == E.diff_degree()
True
```

bundle_connection(*name*, *latex_name=None*)

Return a bundle connection on `self`.

OUTPUT:

- a bundle connection on `self` as an instance of `BundleConnection`

EXAMPLES:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: E = M.vector_bundle(2, 'E')
```

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```
sage: e = E.local_frame('e') # standard frame for E
sage: nab = E.bundle_connection('nabla', latex_name=r'\nabla'); nab
Bundle connection nabla on the Differentiable real vector bundle
 E -> M of rank 2 over the base space 3-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e') # standard frame for E
>>> nab = E.bundle_connection('nabla', latex_name=r'\nabla'); nab
Bundle connection nabla on the Differentiable real vector bundle
 E -> M of rank 2 over the base space 3-dimensional differentiable
 manifold M
```

See also

Further examples can be found in `BundleConnection`.

characteristic_class(**args*, ***kwds*)

Deprecated: Use `characteristic_cohomology_class()` instead. See Issue #29581 for details.

characteristic_cohomology_class(**args*, ***kwargs*)

Return a characteristic cohomology class associated with the input data.

INPUT:

- `val` – the input data associated with the characteristic class using the Chern-Weil homomorphism; this argument can be either a symbolic expression, a polynomial or one of the following predefined classes:
  - `'Chern'` – total Chern class,
  - `'ChernChar'` – Chern character,
  - `'Todd'` – Todd class,
  - `'Pontryagin'` – total Pontryagin class,
  - `'Hirzebruch'` – Hirzebruch class,
  - `'AHat'` – $\hat{A}$ class,
  - `'Euler'` – Euler class.
- `base_ring` – (default: `QQ`) base ring over which the characteristic cohomology class ring shall be defined
- `name` – (default: `None`) string representation given to the characteristic cohomology class; if `None` the default algebra representation or predefined name is used
- `latex_name` – (default: `None`) LaTeX name given to the characteristic class; if `None` the value of `name` is used
- `class_type` – (default: `None`) class type of the characteristic cohomology class; the following options are possible:
  - `'multiplicative'` – returns a class of multiplicative type

- 'additive' – returns a class of additive type
- 'Pfaffian' – returns a class of Pfaffian type

This argument must be stated if `val` is a polynomial or symbolic expression.

EXAMPLES:

Pontryagin class on the Minkowski space:

```
sage: M = Manifold(4, 'M', structure='Lorentzian', start_index=1)
sage: X.<t,x,y,z> = M.chart()
sage: g = M.metric()
sage: g[1,1] = -1
sage: g[2,2] = 1
sage: g[3,3] = 1
sage: g[4,4] = 1
sage: g.display()
g = -dt⊗dt + dx⊗dx + dy⊗dy + dz⊗dz
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian', start_
↪index=Integer(1))
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> g = M.metric()
>>> g[Integer(1),Integer(1)] = -Integer(1)
>>> g[Integer(2),Integer(2)] = Integer(1)
>>> g[Integer(3),Integer(3)] = Integer(1)
>>> g[Integer(4),Integer(4)] = Integer(1)
>>> g.display()
g = -dt⊗dt + dx⊗dx + dy⊗dy + dz⊗dz
```

Let us introduce the corresponding Levi-Civita connection:

```
sage: nab = g.connection(); nab
Levi-Civita connection nabla_g associated with the Lorentzian
 metric g on the 4-dimensional Lorentzian manifold M
sage: nab.set_immutable()   # make nab immutable
```

```
>>> from sage.all import *
>>> nab = g.connection(); nab
Levi-Civita connection nabla_g associated with the Lorentzian
 metric g on the 4-dimensional Lorentzian manifold M
>>> nab.set_immutable()   # make nab immutable
```

Of course, ∇_g is flat:

```
sage: nab.display()
```

```
>>> from sage.all import *
>>> nab.display()
```

Let us check the total Pontryagin class which must be the one element in the corresponding cohomology ring in this case:

```
sage: TM = M.tangent_bundle(); TM
Tangent bundle TM over the 4-dimensional Lorentzian manifold M
sage: p = TM.characteristic_cohomology_class('Pontryagin'); p
Characteristic cohomology class p(TM) of the Tangent bundle TM over
 the 4-dimensional Lorentzian manifold M
sage: p_form = p.get_form(nab); p_form.display_expansion()
p(TM, nabla_g) = 1
```

```
>>> from sage.all import *
>>> TM = M.tangent_bundle(); TM
Tangent bundle TM over the 4-dimensional Lorentzian manifold M
>>> p = TM.characteristic_cohomology_class('Pontryagin'); p
Characteristic cohomology class p(TM) of the Tangent bundle TM over
 the 4-dimensional Lorentzian manifold M
>>> p_form = p.get_form(nab); p_form.display_expansion()
p(TM, nabla_g) = 1
```

See also

More examples can be found in `CharacteristicClass`.

characteristic_cohomology_class_ring(*base=Rational Field*)

Return the characteristic cohomology class ring of `self` over a given base.

INPUT:

- `base` – (default: `QQ`) base over which the ring should be constructed; typically that would be **Z**, **Q**, **R** or the symbolic ring

EXAMPLES:

```
sage: M = Manifold(4, 'M', start_index=1)
sage: R = M.tangent_bundle().characteristic_cohomology_class_ring()
sage: R
Algebra of characteristic cohomology classes of the Tangent bundle
 TM over the 4-dimensional differentiable manifold M
sage: p1 = R.gen(0); p1
Characteristic cohomology class (p_1)(TM) of the Tangent bundle TM
 over the 4-dimensional differentiable manifold M
sage: 1 + p1
Characteristic cohomology class (1 + p_1)(TM) of the Tangent bundle
 TM over the 4-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', start_index=Integer(1))
>>> R = M.tangent_bundle().characteristic_cohomology_class_ring()
>>> R
Algebra of characteristic cohomology classes of the Tangent bundle
 TM over the 4-dimensional differentiable manifold M
>>> p1 = R.gen(Integer(0)); p1
Characteristic cohomology class (p_1)(TM) of the Tangent bundle TM
 over the 4-dimensional differentiable manifold M
```

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```
>>> Integer(1) + p1
Characteristic cohomology class (1 + p_1)(TM) of the Tangent bundle
 TM over the 4-dimensional differentiable manifold M
```

diff_degree()

Return the vector bundle's degree of differentiability.

The degree of differentiability is the integer k (possibly $k = \infty$) such that the vector bundle is of class C^k over its base field. The degree always corresponds to the degree of differentiability of it's base space.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: E = M.vector_bundle(2, 'E')
sage: E.diff_degree()
+Infinity
sage: M = Manifold(2, 'M', structure='differentiable',
....:              diff_degree=3)
sage: E = M.vector_bundle(2, 'E')
sage: E.diff_degree()
3
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> E.diff_degree()
+Infinity
>>> M = Manifold(Integer(2), 'M', structure='differentiable',
...              diff_degree=Integer(3))
>>> E = M.vector_bundle(Integer(2), 'E')
>>> E.diff_degree()
3
```

total_space()

Return the total space of `self`.

Note

At this stage, the total space does not come with induced charts.

OUTPUT:

- the total space of `self` as an instance of *DifferentiableManifold*

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: E = M.vector_bundle(2, 'E')
sage: E.total_space()
6-dimensional differentiable manifold E
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> E = M.vector_bundle(Integer(2), 'E')
>>> E.total_space()
6-dimensional differentiable manifold E
```

class sage.manifolds.differentiable.vector_bundle.**TensorBundle**(*base_space*, *k*, *l*, *dest_map=None*)

Bases: *DifferentiableVectorBundle*

Tensor bundle over a differentiable manifold along a differentiable map.

An instance of this class represents the pullback tensor bundle $\Phi^* T^{(k,l)}N$ along a differentiable map (called *destination map*)

$$\Phi : M \longrightarrow N$$

between two differentiable manifolds M and N over the topological field K.

More precisely, $\Phi^* T^{(k,l)}N$ consists of all pairs $(p,t) \in M \times T^{(k,l)}N$ such that $t \in T_q^{(k,l)}N$ for $q = \Phi(p)$, namely

$$t : \underbrace{T_q^*N \times \cdots \times T_q^*N}_{k \text{ times}} \times \underbrace{T_qN \times \cdots \times T_qN}_{l \text{ times}} \longrightarrow K$$

(k is called the *contravariant* and l the *covariant* rank of the tensor bundle).

The trivializations are directly given by charts on the codomain (called *ambient domain*) of Φ. In particular, let (V,φ) be a chart of N with components $(x^1,\dots,x^n)$ such that $q = \Phi(p) \in V$. Then, the matrix entries of $t \in T_q^{(k,l)}N$ are given by

$$t^{a_1\dots a_k}{}_{b_1\dots b_l} = t\left(\left.\frac{\partial}{\partial x^{a_1}}\right|_q, \dots, \left.\frac{\partial}{\partial x^{a_k}}\right|_q, \left.\mathrm{d}x^{b_1}\right|_q, \dots, \left.\mathrm{d}x^{b_l}\right|_q\right) \in K$$

and a trivialization over $U = \Phi^{-1}(V) \subset M$ is obtained via

$$(p,t) \mapsto (p, t^{1\dots 1}{}_{1\dots 1}, \dots, t^{n\dots n}{}_{n\dots n}) \in U \times K^{n^{(k+l)}}.$$

The standard case of a tensor bundle over a differentiable manifold corresponds to $M = N$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in N (M is then an open interval of **R**).

INPUT:

- `base_space` – the base space (differentiable manifold) M over which the tensor bundle is defined
- `k` – the contravariant rank of the corresponding tensor bundle
- `l` – the covariant rank of the corresponding tensor bundle
- `dest_map` – (default: `None`) destination map $\Phi : M \to N$ (type: *DiffMap*); if `None`, it is assumed that $M = M$ and Φ is the identity map of M (case of the standard tensor bundle over M)

EXAMPLES:

Pullback tangent bundle of R^2 along a curve Φ:

```
sage: M = Manifold(2, 'M')
sage: c_cart.<x,y> = M.chart()
sage: R = Manifold(1, 'R')
```

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```
sage: T.<t> = R.chart()   # canonical chart on R
sage: Phi = R.diff_map(M, [cos(t), sin(t)], name='Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable manifold R
 to the 2-dimensional differentiable manifold M
sage: Phi.display()
Phi: R → M
   t ↦ (x, y) = (cos(t), sin(t))
sage: PhiTM = R.tangent_bundle(dest_map=Phi); PhiTM
Tangent bundle Phi^*TM over the 1-dimensional differentiable manifold R
 along the Differentiable map Phi from the 1-dimensional differentiable
 manifold R to the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)
>>> R = Manifold(Integer(1), 'R')
>>> T = R.chart(names=('t',)); (t,) = T._first_ngens(1)# canonical chart on R
>>> Phi = R.diff_map(M, [cos(t), sin(t)], name='Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable manifold R
 to the 2-dimensional differentiable manifold M
>>> Phi.display()
Phi: R → M
   t ↦ (x, y) = (cos(t), sin(t))
>>> PhiTM = R.tangent_bundle(dest_map=Phi); PhiTM
Tangent bundle Phi^*TM over the 1-dimensional differentiable manifold R
 along the Differentiable map Phi from the 1-dimensional differentiable
 manifold R to the 2-dimensional differentiable manifold M
```

The section module is the corresponding tensor field module:

```
sage: R_tensor_module = R.tensor_field_module((1,0), dest_map=Phi)
sage: R_tensor_module is PhiTM.section_module()
True
```

```
>>> from sage.all import *
>>> R_tensor_module = R.tensor_field_module((Integer(1),Integer(0)), dest_map=Phi)
>>> R_tensor_module is PhiTM.section_module()
True
```

ambient_domain()

Return the codomain of the destination map.

OUTPUT:

- a *DifferentiableManifold* representing the codomain of the destination map

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: c_cart.<x,y> = M.chart()
sage: e_cart = c_cart.frame() # standard basis
sage: R = Manifold(1, 'R')
sage: T.<t> = R.chart()   # canonical chart on R
```

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```
sage: Phi = R.diff_map(M, [cos(t), sin(t)], name='Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable
 manifold R to the 2-dimensional differentiable manifold M
sage: Phi.display()
Phi: R → M
   t ↦ (x, y) = (cos(t), sin(t))
sage: PhiT11 = R.tensor_bundle(1, 1, dest_map=Phi)
sage: PhiT11.ambient_domain()
2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)
>>> e_cart = c_cart.frame() # standard basis
>>> R = Manifold(Integer(1), 'R')
>>> T = R.chart(names=('t',)); (t,) = T._first_ngens(1)# canonical chart on R
>>> Phi = R.diff_map(M, [cos(t), sin(t)], name='Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable
 manifold R to the 2-dimensional differentiable manifold M
>>> Phi.display()
Phi: R → M
   t ↦ (x, y) = (cos(t), sin(t))
>>> PhiT11 = R.tensor_bundle(Integer(1), Integer(1), dest_map=Phi)
>>> PhiT11.ambient_domain()
2-dimensional differentiable manifold M
```

atlas()

Return the list of charts that have been defined on the codomain of the destination map.

Note

Since an atlas of charts gives rise to an atlas of trivializations, this method directly invokes *atlas()* of the class *TopologicalManifold*.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: Y.<u,v> = M.chart()
sage: TM = M.tangent_bundle()
sage: TM.atlas()
[Chart (M, (x, y)), Chart (M, (u, v))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> Y = M.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)
>>> TM = M.tangent_bundle()
>>> TM.atlas()
[Chart (M, (x, y)), Chart (M, (u, v))]
```

change_of_frame(*frame1*, *frame2*)

Return a change of vector frames defined on the base space of `self`.

See also:

For further details on frames on `self` see `local_frame()`.

Note: Since frames on `self` are directly induced by vector frames on the base space, this method directly invokes `change_of_frame()` of the class `DifferentiableManifold`.

INPUT:

- `frame1` – local frame 1
- `frame2` – local frame 2

OUTPUT:

- a `FreeModuleAutomorphism` representing, at each point, the vector space automorphism P that relates frame 1, (e_i) say, to frame 2, (f_i) say, according to $f_i = P(e_i)$

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: c_uv.<u,v> = M.chart()
sage: c_xy.transition_map(c_uv, (x+y, x-y))
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
sage: TM = M.tangent_bundle()
sage: TM.change_of_frame(c_xy.frame(), c_uv.frame())
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
sage: TM.change_of_frame(c_xy.frame(), c_uv.frame())[:]
[ 1/2  1/2]
[ 1/2 -1/2]
sage: TM.change_of_frame(c_uv.frame(), c_xy.frame())
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
sage: TM.change_of_frame(c_uv.frame(), c_xy.frame())[:]
[ 1  1]
[ 1 -1]
sage: TM.change_of_frame(c_uv.frame(), c_xy.frame()) == \
....:       M.change_of_frame(c_xy.frame(), c_uv.frame()).inverse()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> c_uv = M.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> c_xy.transition_map(c_uv, (x+y, x-y))
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
```

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```
>>> TM = M.tangent_bundle()
>>> TM.change_of_frame(c_xy.frame(), c_uv.frame())
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
>>> TM.change_of_frame(c_xy.frame(), c_uv.frame())[:]
[ 1/2  1/2]
[ 1/2 -1/2]
>>> TM.change_of_frame(c_uv.frame(), c_xy.frame())
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
>>> TM.change_of_frame(c_uv.frame(), c_xy.frame())[:]
[ 1  1]
[ 1 -1]
>>> TM.change_of_frame(c_uv.frame(), c_xy.frame()) ==       M.change_of_
↪frame(c_xy.frame(), c_uv.frame()).inverse()
True
```

changes_of_frame()

Return the changes of vector frames defined on the base space of `self` with respect to the destination map.

See also

For further details on frames on `self` see *local_frame()*.

OUTPUT:

- dictionary of automorphisms on the tangent bundle representing the changes of frames, the keys being the pair of frames

EXAMPLES:

Let us consider a first vector frame on a 2-dimensional differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: TM = M.tangent_bundle()
sage: e = X.frame(); e
Coordinate frame (M, (∂/∂x,∂/∂y))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> TM = M.tangent_bundle()
>>> e = X.frame(); e
Coordinate frame (M, (∂/∂x,∂/∂y))
```

At this stage, the dictionary of changes of frame is empty:

```
sage: TM.changes_of_frame()
{}
```

```
>>> from sage.all import *
>>> TM.changes_of_frame()
{}
```

We introduce a second frame on the manifold, relating it to frame `e` by a field of tangent space automorphisms:

```
sage: a = M.automorphism_field(name='a')
sage: a[:] = [[-y, x], [1, 2]]
sage: f = e.new_frame(a, 'f'); f
Vector frame (M, (f_0,f_1))
```

```
>>> from sage.all import *
>>> a = M.automorphism_field(name='a')
>>> a[:] = [[-y, x], [Integer(1), Integer(2)]]
>>> f = e.new_frame(a, 'f'); f
Vector frame (M, (f_0,f_1))
```

Then we have:

```
sage: TM.changes_of_frame()  # random (dictionary output)
{(Coordinate frame (M, (∂/∂x,∂/∂y)),
  Vector frame (M, (f_0,f_1))): Field of tangent-space
   automorphisms on the 2-dimensional differentiable manifold M,
 (Vector frame (M, (f_0,f_1)),
  Coordinate frame (M, (∂/∂x,∂/∂y))): Field of tangent-space
   automorphisms on the 2-dimensional differentiable manifold M}
```

```
>>> from sage.all import *
>>> TM.changes_of_frame()  # random (dictionary output)
{(Coordinate frame (M, (∂/∂x,∂/∂y)),
  Vector frame (M, (f_0,f_1))): Field of tangent-space
   automorphisms on the 2-dimensional differentiable manifold M,
 (Vector frame (M, (f_0,f_1)),
  Coordinate frame (M, (∂/∂x,∂/∂y))): Field of tangent-space
   automorphisms on the 2-dimensional differentiable manifold M}
```

Some checks:

```
sage: TM.changes_of_frame()[(e,f)] == a
True
sage: TM.changes_of_frame()[(f,e)] == a^(-1)
True
```

```
>>> from sage.all import *
>>> TM.changes_of_frame()[(e,f)] == a
True
>>> TM.changes_of_frame()[(f,e)] == a**(-Integer(1))
True
```

coframes()

Return the list of coframes defined on the base manifold of `self` with respect to the destination map.

See also:

For further details on frames on `self` see `local_frame()`.

OUTPUT: list of coframes defined on `self`

EXAMPLES:

Coframes on subsets of $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'R^2')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: TM = M.tangent_bundle()
sage: TM.coframes()
[Coordinate coframe (R^2, (dx,dy))]
sage: e = TM.vector_frame('e')
sage: M.coframes()
[Coordinate coframe (R^2, (dx,dy)), Coframe (R^2, (e^0,e^1))]
sage: U = M.open_subset('U', coord_def={c_cart: x^2+y^2<1})
sage: TU = U.tangent_bundle()
sage: TU.coframes()
[Coordinate coframe (U, (dx,dy))]
sage: e.restrict(U)
Vector frame (U, (e_0,e_1))
sage: TU.coframes()
[Coordinate coframe (U, (dx,dy)), Coframe (U, (e^0,e^1))]
sage: TM.coframes()
[Coordinate coframe (R^2, (dx,dy)),
 Coframe (R^2, (e^0,e^1)),
 Coordinate coframe (U, (dx,dy)),
 Coframe (U, (e^0,e^1))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#
↪Cartesian coordinates on R^2
>>> TM = M.tangent_bundle()
>>> TM.coframes()
[Coordinate coframe (R^2, (dx,dy))]
>>> e = TM.vector_frame('e')
>>> M.coframes()
[Coordinate coframe (R^2, (dx,dy)), Coframe (R^2, (e^0,e^1))]
>>> U = M.open_subset('U', coord_def={c_cart: x**Integer(2)+y**Integer(2)
↪<Integer(1)})
>>> TU = U.tangent_bundle()
>>> TU.coframes()
[Coordinate coframe (U, (dx,dy))]
>>> e.restrict(U)
Vector frame (U, (e_0,e_1))
>>> TU.coframes()
[Coordinate coframe (U, (dx,dy)), Coframe (U, (e^0,e^1))]
>>> TM.coframes()
[Coordinate coframe (R^2, (dx,dy)),
```

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```
Coframe (R^2, (e^0,e^1)),
Coordinate coframe (U, (dx,dy)),
Coframe (U, (e^0,e^1))]
```

default_frame()

Return the default vector frame defined on `self`.

By *vector frame*, it is meant a field on the manifold that provides, at each point p, a vector basis of the pulled back tangent space at p.

If the destination map is the identity map, the default frame is the the first one defined on the manifold, usually the coordinate frame, unless it is changed via *set_default_frame()*.

If the destination map is non-trivial, the default frame usually must be set via *set_default_frame()*.

OUTPUT:

- a *VectorFrame* representing the default vector frame

EXAMPLES:

The default vector frame is often the coordinate frame associated with the first chart defined on the manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: TM = M.tangent_bundle()
sage: TM.default_frame()
Coordinate frame (M, (∂/∂x,∂/∂y))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> TM = M.tangent_bundle()
>>> TM.default_frame()
Coordinate frame (M, (∂/∂x,∂/∂y))
```

destination_map()

Return the destination map.

OUTPUT:

- a `DifferentialMap` representing the destination map

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: c_cart.<x,y> = M.chart()
sage: e_cart = c_cart.frame() # standard basis
sage: R = Manifold(1, 'R')
sage: T.<t> = R.chart()  # canonical chart on R
sage: Phi = R.diff_map(M, [cos(t), sin(t)], name='Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable
 manifold R to the 2-dimensional differentiable manifold M
sage: Phi.display()
Phi: R → M
   t ↦ (x, y) = (cos(t), sin(t))
sage: PhiT11 = R.tensor_bundle(1, 1, dest_map=Phi)
```

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```
sage: PhiT11.destination_map()
Differentiable map Phi from the 1-dimensional differentiable
 manifold R to the 2-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)
>>> e_cart = c_cart.frame() # standard basis
>>> R = Manifold(Integer(1), 'R')
>>> T = R.chart(names=('t',)); (t,) = T._first_ngens(1)# canonical chart on R
>>> Phi = R.diff_map(M, [cos(t), sin(t)], name='Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable
 manifold R to the 2-dimensional differentiable manifold M
>>> Phi.display()
Phi: R → M
   t ↦ (x, y) = (cos(t), sin(t))
>>> PhiT11 = R.tensor_bundle(Integer(1), Integer(1), dest_map=Phi)
>>> PhiT11.destination_map()
Differentiable map Phi from the 1-dimensional differentiable
 manifold R to the 2-dimensional differentiable manifold M
```

fiber (*point*)

Return the tensor bundle fiber over a point.

INPUT:

- `point` – *ManifoldPoint*; point p of the base manifold of `self`

OUTPUT:

- an instance of `FiniteRankFreeModule` representing the tensor bundle fiber over p

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: p = M((0,2,1), name='p'); p
Point p on the 3-dimensional differentiable manifold M
sage: TM = M.tangent_bundle(); TM
Tangent bundle TM over the 3-dimensional differentiable manifold M
sage: TM.fiber(p)
Tangent space at Point p on the 3-dimensional differentiable
 manifold M
sage: TM.fiber(p) is M.tangent_space(p)
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> p = M((Integer(0),Integer(2),Integer(1)), name='p'); p
Point p on the 3-dimensional differentiable manifold M
>>> TM = M.tangent_bundle(); TM
Tangent bundle TM over the 3-dimensional differentiable manifold M
>>> TM.fiber(p)
```

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```
Tangent space at Point p on the 3-dimensional differentiable
 manifold M
>>> TM.fiber(p) is M.tangent_space(p)
True
```

```
sage: T11M = M.tensor_bundle(1,1); T11M
Tensor bundle T^(1,1)M over the 3-dimensional differentiable
 manifold M
sage: T11M.fiber(p)
Free module of type-(1,1) tensors on the Tangent space at Point p
 on the 3-dimensional differentiable manifold M
sage: T11M.fiber(p) is M.tangent_space(p).tensor_module(1,1)
True
```

```
>>> from sage.all import *
>>> T11M = M.tensor_bundle(Integer(1),Integer(1)); T11M
Tensor bundle T^(1,1)M over the 3-dimensional differentiable
 manifold M
>>> T11M.fiber(p)
Free module of type-(1,1) tensors on the Tangent space at Point p
 on the 3-dimensional differentiable manifold M
>>> T11M.fiber(p) is M.tangent_space(p).tensor_module(Integer(1),Integer(1))
True
```

frames()

Return the list of all vector frames defined on the base space of `self` with respect to the destination map.

See also

For further details on frames on `self` see *local_frame()*.

OUTPUT: list of local frames defined on `self`

EXAMPLES:

Vector frames on subsets of $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'R^2')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: TM = M.tangent_bundle()
sage: TM.frames()
[Coordinate frame (R^2, (∂/∂x,∂/∂y))]
sage: e = TM.vector_frame('e')
sage: TM.frames()
[Coordinate frame (R^2, (∂/∂x,∂/∂y)),
 Vector frame (R^2, (e_0,e_1))]
sage: U = M.open_subset('U', coord_def={c_cart: x^2+y^2<1})
sage: TU = U.tangent_bundle()
sage: TU.frames()
[Coordinate frame (U, (∂/∂x,∂/∂y))]
sage: TM.frames()
```

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```
[Coordinate frame (R^2, (∂/∂x,∂/∂y)),
 Vector frame (R^2, (e_0,e_1)),
 Coordinate frame (U, (∂/∂x,∂/∂y))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'R^2')
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#␣
↪Cartesian coordinates on R^2
>>> TM = M.tangent_bundle()
>>> TM.frames()
[Coordinate frame (R^2, (∂/∂x,∂/∂y))]
>>> e = TM.vector_frame('e')
>>> TM.frames()
[Coordinate frame (R^2, (∂/∂x,∂/∂y)),
 Vector frame (R^2, (e_0,e_1))]
>>> U = M.open_subset('U', coord_def={c_cart: x**Integer(2)+y**Integer(2)
↪<Integer(1)})
>>> TU = U.tangent_bundle()
>>> TU.frames()
[Coordinate frame (U, (∂/∂x,∂/∂y))]
>>> TM.frames()
[Coordinate frame (R^2, (∂/∂x,∂/∂y)),
 Vector frame (R^2, (e_0,e_1)),
 Coordinate frame (U, (∂/∂x,∂/∂y))]
```

List of vector frames of a tensor bundle of type $(1, 1)$ along a curve:

```
sage: M = Manifold(2, 'M')
sage: c_cart.<x,y> = M.chart()
sage: e_cart = c_cart.frame() # standard basis
sage: R = Manifold(1, 'R')
sage: T.<t> = R.chart()  # canonical chart on R
sage: Phi = R.diff_map(M, [cos(t), sin(t)], name='Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable
 manifold R to the 2-dimensional differentiable manifold M
sage: Phi.display()
Phi: R → M
   t ↦ (x, y) = (cos(t), sin(t))
sage: PhiT11 = R.tensor_bundle(1, 1, dest_map=Phi); PhiT11
Tensor bundle Phi^*T^(1,1)M over the 1-dimensional differentiable
 manifold R along the Differentiable map Phi from the 1-dimensional
 differentiable manifold R to the 2-dimensional differentiable
 manifold M
sage: f = PhiT11.local_frame(); f
Vector frame (R, (∂/∂x,∂/∂y)) with values on the 2-dimensional
 differentiable manifold M
sage: PhiT11.frames()
[Vector frame (R, (∂/∂x,∂/∂y)) with values on the 2-dimensional
 differentiable manifold M]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
```

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```
>>> c_cart = M.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)
>>> e_cart = c_cart.frame() # standard basis
>>> R = Manifold(Integer(1), 'R')
>>> T = R.chart(names=('t',)); (t,) = T._first_ngens(1)# canonical chart on R
>>> Phi = R.diff_map(M, [cos(t), sin(t)], name='Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable
 manifold R to the 2-dimensional differentiable manifold M
>>> Phi.display()
Phi: R → M
   t ↦ (x, y) = (cos(t), sin(t))
>>> PhiT11 = R.tensor_bundle(Integer(1), Integer(1), dest_map=Phi); PhiT11
Tensor bundle Phi^*T^(1,1)M over the 1-dimensional differentiable
 manifold R along the Differentiable map Phi from the 1-dimensional
 differentiable manifold R to the 2-dimensional differentiable
 manifold M
>>> f = PhiT11.local_frame(); f
Vector frame (R, (∂/∂x,∂/∂y)) with values on the 2-dimensional
 differentiable manifold M
>>> PhiT11.frames()
[Vector frame (R, (∂/∂x,∂/∂y)) with values on the 2-dimensional
 differentiable manifold M]
```

is_manifestly_trivial()

Return `True` if `self` is known to be a trivial and `False` otherwise.

If `False` is returned, either the tensor bundle is not trivial or no vector frame has been defined on it yet.

EXAMPLES:

A just created manifold has a priori no manifestly trivial tangent bundle:

```
sage: M = Manifold(2, 'M')
sage: TM = M.tangent_bundle()
sage: TM.is_manifestly_trivial()
False
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> TM = M.tangent_bundle()
>>> TM.is_manifestly_trivial()
False
```

Defining a vector frame on it makes it trivial:

```
sage: e = TM.vector_frame('e')
sage: TM.is_manifestly_trivial()
True
```

```
>>> from sage.all import *
>>> e = TM.vector_frame('e')
>>> TM.is_manifestly_trivial()
True
```

Defining a coordinate chart on the whole manifold also makes it trivial:

```
sage: N = Manifold(4, 'N')
sage: X.<t,x,y,z> = N.chart()
sage: TN = N.tangent_bundle()
sage: TN.is_manifestly_trivial()
True
```

```
>>> from sage.all import *
>>> N = Manifold(Integer(4), 'N')
>>> X = N.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> TN = N.tangent_bundle()
>>> TN.is_manifestly_trivial()
True
```

The situation is not so clear anymore when a destination map to a non-parallelizable manifold is stated:

```
sage: M = Manifold(2, 'S^2') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereo coord from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereo coord from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2),
....:                                        y/(x^2+y^2)),
....:                                 intersection_name='W',
....:                                 restrictions1= x^2+y^2!=0,
....:                                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: Phi = U.diff_map(M, {(c_xy, c_xy): [x, y]},
....:                  name='Phi') # inclusion map
sage: PhiTU = U.tangent_bundle(dest_map=Phi); PhiTU
Tangent bundle Phi^*TS^2 over the Open subset U of the
 2-dimensional differentiable manifold S^2 along the
 Differentiable map Phi from the Open subset U of the
 2-dimensional differentiable manifold S^2 to the 2-dimensional
 differentiable manifold S^2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2') # the 2-dimensional sphere S^2
>>> U = M.open_subset('U') # complement of the North pole
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)# stereo␣
↪coord from the North pole
>>> V = M.open_subset('V') # complement of the South pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)# stereo␣
↪coord from the South pole
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)),
...                                        y/(x**Integer(2)+y**Integer(2))),
...                                 intersection_name='W',
...                                 restrictions1= x**Integer(2)+y**Integer(2)!
↪=Integer(0),
```

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```
...                               restrictions2= u**Integer(2)+v**Integer(2)!
↪=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V)
>>> Phi = U.diff_map(M, {(c_xy, c_xy): [x, y]},
...                  name='Phi') # inclusion map
>>> PhiTU = U.tangent_bundle(dest_map=Phi); PhiTU
Tangent bundle Phi^*TS^2 over the Open subset U of the
 2-dimensional differentiable manifold S^2 along the
 Differentiable map Phi from the Open subset U of the
 2-dimensional differentiable manifold S^2 to the 2-dimensional
 differentiable manifold S^2
```

A priori, the pullback tangent bundle is not trivial:

```
sage: PhiTU.is_manifestly_trivial()
False
```

```
>>> from sage.all import *
>>> PhiTU.is_manifestly_trivial()
False
```

But certainly, this bundle must be trivial since U is parallelizable. To ensure this, we need to define a local frame on U with values in Φ^*TS^2:

```
sage: PhiTU.local_frame('e', from_frame=c_xy.frame())
Vector frame (U, (e_0,e_1)) with values on the 2-dimensional
 differentiable manifold S^2
sage: PhiTU.is_manifestly_trivial()
True
```

```
>>> from sage.all import *
>>> PhiTU.local_frame('e', from_frame=c_xy.frame())
Vector frame (U, (e_0,e_1)) with values on the 2-dimensional
 differentiable manifold S^2
>>> PhiTU.is_manifestly_trivial()
True
```

local_frame(**args*, ***kwargs*)

Define a vector frame on `domain`, possibly with values in the tangent bundle of the ambient domain.

If the basis specified by the given symbol already exists, it is simply returned. If no argument is provided the vector field module's default frame is returned.

Notice, that a vector frame automatically induces a local frame on the tensor bundle `self`. More precisely, if $e : U \to \Phi^*TN$ is a vector frame on $U \subset M$ with values in Φ^*TN along the destination map

$$\Phi : M \longrightarrow N$$

then the map

$$p \mapsto \Big(\underbrace{e^*(p), \dots, e^*(p)}_{k \text{ times}}, \underbrace{e(p), \dots, e(p)}_{l \text{ times}} \Big) \in T_q^{(k,l)} N,$$

with $q = \Phi(p)$, defines a basis at each point $p \in U$ and therefore gives rise to a local frame on $\Phi^* T^{(k,l)} N$ on the domain U.

See also:

VectorFrame for complete documentation.

INPUT:

- `symbol` – (default: `None`) either a string, to be used as a common base for the symbols of the vector fields constituting the vector frame, or a list/tuple of strings, representing the individual symbols of the vector fields; can be `None` only if `from_frame` is not `None` (see below)
- `vector_fields` – tuple or list of n linearly independent vector fields on `domain` (n being the dimension of `domain`) defining the vector frame; can be omitted if the vector frame is created from scratch or if `from_frame` is not `None`
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the vector fields constituting the vector frame, or a list/tuple of strings, representing the individual LaTeX symbols of the vector fields; if `None`, `symbol` is used in place of `latex_symbol`
- `from_frame` – (default: `None`) vector frame $\tilde{e}$ on the codomain N of the destination map Φ; the returned frame e is then such that for all $p \in U$, we have $e(p) = \tilde{e}(\Phi(p))$
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the vector fields of the frame; if `None`, the indices will be generated as integers within the range declared on `self`
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the vector fields; if `None`, `indices` is used instead
- `symbol_dual` – (default: `None`) same as `symbol` but for the dual coframe; if `None`, `symbol` must be a string and is used for the common base of the symbols of the elements of the dual coframe
- `latex_symbol_dual` – (default: `None`) same as `latex_symbol` but for the dual coframe
- `domain` – (default: `None`) domain on which the local frame is defined; if `None` is provided, the base space of `self` is assumed

OUTPUT:

- the vector frame corresponding to the above specifications; this is an instance of *VectorFrame*.

EXAMPLES:

Defining a local frame for the tangent bundle of a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M')
sage: TM = M.tangent_bundle()
sage: e = TM.local_frame('e'); e
Vector frame (M, (e_0,e_1,e_2))
sage: e[0]
Vector field e_0 on the 3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> TM = M.tangent_bundle()
>>> e = TM.local_frame('e'); e
Vector frame (M, (e_0,e_1,e_2))
```

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```
>>> e[Integer(0)]
Vector field e_0 on the 3-dimensional differentiable manifold M
```

Specifying the domain of the vector frame:

```
sage: U = M.open_subset('U')
sage: f = TM.local_frame('f', domain=U); f
Vector frame (U, (f_0,f_1,f_2))
sage: f[0]
Vector field f_0 on the Open subset U of the 3-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> U = M.open_subset('U')
>>> f = TM.local_frame('f', domain=U); f
Vector frame (U, (f_0,f_1,f_2))
>>> f[Integer(0)]
Vector field f_0 on the Open subset U of the 3-dimensional
 differentiable manifold M
```

See also

For more options, in particular for the choice of symbols and indices, see `VectorFrame`.

orientation()

Get the preferred orientation of `self` if available.

See `orientation()` for details regarding orientations on vector bundles.

The tensor bundle $\Phi^* T^{(k,l)} N$ of a manifold is orientable if the manifold $\Phi(M)$ is orientable. The converse does not necessarily hold true. The usual case corresponds to Φ being the identity map, where the tensor bundle $T^{(k,l)} M$ is orientable if and only if the manifold M is orientable.

Note

Notice that the orientation of a general tensor bundle $\Phi^* T^{(k,l)} N$ is canonically induced by the orientation of the tensor bundle $\Phi^* T^{(1,0)} N$ as each local frame there induces the frames on $\Phi^* T^{(k,l)} N$ in a canonical way.

If no preferred orientation has been set before, and if the ambient space already admits a preferred orientation, the corresponding orientation is returned and henceforth fixed for the tensor bundle.

EXAMPLES:

In the trivial case, i.e. if the destination map is the identity and the tangent bundle is covered by one frame, the orientation is easily obtained:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: T11 = M.tensor_bundle(1, 1)
sage: T11.orientation()
[Coordinate frame (M, (∂/∂x,∂/∂y))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> T11 = M.tensor_bundle(Integer(1), Integer(1))
>>> T11.orientation()
[Coordinate frame (M, (∂/∂x,∂/∂y))]
```

The same holds true if the ambient domain admits a trivial orientation:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: R = Manifold(1, 'R')
sage: c_t.<t> = R.chart()
sage: Phi = R.diff_map(M, name='Phi')
sage: PhiT22 = R.tensor_bundle(2, 2, dest_map=Phi); PhiT22
Tensor bundle Phi^*T^(2,2)M over the 1-dimensional differentiable
 manifold R along the Differentiable map Phi from the 1-dimensional
 differentiable manifold R to the 2-dimensional differentiable
 manifold M
sage: PhiT22.local_frame()  # initialize frame
Vector frame (R, (∂/∂x,∂/∂y)) with values on the 2-dimensional
 differentiable manifold M
sage: PhiT22.orientation()
[Vector frame (R, (∂/∂x,∂/∂y)) with values on the 2-dimensional
 differentiable manifold M]
sage: PhiT22.local_frame() is PhiT22.orientation()[0]
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> R = Manifold(Integer(1), 'R')
>>> c_t = R.chart(names=('t',)); (t,) = c_t._first_ngens(1)
>>> Phi = R.diff_map(M, name='Phi')
>>> PhiT22 = R.tensor_bundle(Integer(2), Integer(2), dest_map=Phi); PhiT22
Tensor bundle Phi^*T^(2,2)M over the 1-dimensional differentiable
 manifold R along the Differentiable map Phi from the 1-dimensional
 differentiable manifold R to the 2-dimensional differentiable
 manifold M
>>> PhiT22.local_frame()  # initialize frame
Vector frame (R, (∂/∂x,∂/∂y)) with values on the 2-dimensional
 differentiable manifold M
>>> PhiT22.orientation()
[Vector frame (R, (∂/∂x,∂/∂y)) with values on the 2-dimensional
 differentiable manifold M]
>>> PhiT22.local_frame() is PhiT22.orientation()[Integer(0)]
True
```

In the non-trivial case, however, the orientation must be set manually by the user:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U'); V = M.open_subset('V')
sage: M.declare_union(U, V)
```

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```
sage: c_xy.<x,y> = U.chart(); c_uv.<u,v> = V.chart()
sage: T11 = M.tensor_bundle(1, 1); T11
Tensor bundle T^(1,1)M over the 2-dimensional differentiable
 manifold M
sage: T11.orientation()
[]
sage: T11.set_orientation([c_xy.frame(), c_uv.frame()])
sage: T11.orientation()
[Coordinate frame (U, (∂/∂x,∂/∂y)),
 Coordinate frame (V, (∂/∂u,∂/∂v))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U'); V = M.open_subset('V')
>>> M.declare_union(U, V)
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> T11 = M.tensor_bundle(Integer(1), Integer(1)); T11
Tensor bundle T^(1,1)M over the 2-dimensional differentiable
 manifold M
>>> T11.orientation()
[]
>>> T11.set_orientation([c_xy.frame(), c_uv.frame()])
>>> T11.orientation()
[Coordinate frame (U, (∂/∂x,∂/∂y)),
 Coordinate frame (V, (∂/∂u,∂/∂v))]
```

If the destination map is the identity, the orientation is automatically set for the manifold, too:

```
sage: M.orientation()
[Coordinate frame (U, (∂/∂x,∂/∂y)),
 Coordinate frame (V, (∂/∂u,∂/∂v))]
```

```
>>> from sage.all import *
>>> M.orientation()
[Coordinate frame (U, (∂/∂x,∂/∂y)),
 Coordinate frame (V, (∂/∂u,∂/∂v))]
```

Conversely, if one sets an orientation on the manifold, the orientation on its tensor bundles is set accordingly:

```
sage: c_tz.<t,z> = U.chart()
sage: M.set_orientation([c_tz, c_uv])
sage: T11.orientation()
[Coordinate frame (U, (∂/∂t,∂/∂z)),
 Coordinate frame (V, (∂/∂u,∂/∂v))]
```

```
>>> from sage.all import *
>>> c_tz = U.chart(names=('t', 'z',)); (t, z,) = c_tz._first_ngens(2)
>>> M.set_orientation([c_tz, c_uv])
>>> T11.orientation()
[Coordinate frame (U, (∂/∂t,∂/∂z)),
 Coordinate frame (V, (∂/∂u,∂/∂v))]
```

section(**args*, ***kwargs*)

Return a section of `self` on `domain`, namely a tensor field on the subset `domain` of the base space.

> **Note**
>
> This method directly invokes *tensor_field()* of the class *DifferentiableManifold*.

INPUT:

- `comp` – (optional) either the components of the tensor field with respect to the vector frame specified by the argument `frame` or a dictionary of components, the keys of which are vector frames or pairs `(f, c)` where `f` is a vector frame and `c` the chart in which the components are expressed
- `frame` – (default: `None`; unused if `comp` is not given or is a dictionary) vector frame in which the components are given; if `None`, the default vector frame of `self` is assumed
- `chart` – (default: `None`; unused if `comp` is not given or is a dictionary) coordinate chart in which the components are expressed; if `None`, the default chart on the domain of `frame` is assumed
- `domain` – (default: `None`) domain of the section; if `None`, `self.base_space()` is assumed
- `name` – (default: `None`) name given to the tensor field
- `latex_name` – (default: `None`) LaTeX symbol to denote the tensor field; if `None`, the LaTeX symbol is set to `name`
- `sym` – (default: `None`) a symmetry or a list of symmetries among the tensor arguments: each symmetry is described by a tuple containing the positions of the involved arguments, with the convention `position=0` for the first argument; for instance:
  - `sym = (0,1)` for a symmetry between the 1st and 2nd arguments
  - `sym = [(0,2), (1,3,4)]` for a symmetry between the 1st and 3rd arguments and a symmetry between the 2nd, 4th and 5th arguments
- `antisym` – (default: `None`) antisymmetry or list of antisymmetries among the arguments, with the same convention as for `sym`

OUTPUT:

- a *TensorField* (or if N is parallelizable, a *TensorFieldParal*) representing the defined tensor field on the domain $U \subset M$

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y),
....:                              intersection_name='W',
....:                              restrictions1= x>0,
....:                              restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: T11M = M.tensor_bundle(1, 1); T11M
Tensor bundle T^(1,1)M over the 2-dimensional differentiable
 manifold M
```

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```
sage: t = T11M.section({eU: [[1, x], [0, 2]]}, name='t'); t
Tensor field t of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: t.display()
t = ∂/∂x⊗dx + x ∂/∂x⊗dy + 2 ∂/∂y⊗dy
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> transf = c_xy.transition_map(c_uv, (x+y, x-y),
...                               intersection_name='W',
...                               restrictions1= x>Integer(0),
...                               restrictions2= u+v>Integer(0))
>>> inv = transf.inverse()
>>> W = U.intersection(V)
>>> eU = c_xy.frame() ; eV = c_uv.frame()
>>> T11M = M.tensor_bundle(Integer(1), Integer(1)); T11M
Tensor bundle T^(1,1)M over the 2-dimensional differentiable
 manifold M
>>> t = T11M.section({eU: [[Integer(1), x], [Integer(0), Integer(2)]]}, name=
↪'t'); t
Tensor field t of type (1,1) on the 2-dimensional differentiable
 manifold M
>>> t.display()
t = ∂/∂x⊗dx + x ∂/∂x⊗dy + 2 ∂/∂y⊗dy
```

An example of use with the arguments `comp` and `domain`:

```
sage: TM = M.tangent_bundle()
sage: w = TM.section([-y, x], domain=U); w
Vector field on the Open subset U of the 2-dimensional
 differentiable manifold M
sage: w.display()
-y ∂/∂x + x ∂/∂y
```

```
>>> from sage.all import *
>>> TM = M.tangent_bundle()
>>> w = TM.section([-y, x], domain=U); w
Vector field on the Open subset U of the 2-dimensional
 differentiable manifold M
>>> w.display()
-y ∂/∂x + x ∂/∂y
```

section_module(*domain=None*)

Return the section module on `domain`, namely the corresponding tensor field module, of `self` on `domain`.

Note

This method directly invokes *tensor_field_module()* of the class *DifferentiableManifold*.

INPUT:

- `domain` – (default: `None`) the domain of the corresponding section module; if `None`, the base space is assumed

OUTPUT:

- a *TensorFieldModule* (or if N is parallelizable, a *TensorFieldFreeModule*) representing the module $\mathcal{T}^{(k,l)}(U,\Phi)$ of type-(k,l) tensor fields on the domain $U \subset M$ taking values on $\Phi(U) \subset N$

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: U = M.open_subset('U')
sage: TM = M.tangent_bundle()
sage: TUM = TM.section_module(domain=U); TUM
Module X(U) of vector fields on the Open subset U of the
 2-dimensional differentiable manifold M
sage: TUM is U.tensor_field_module((1,0))
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> U = M.open_subset('U')
>>> TM = M.tangent_bundle()
>>> TUM = TM.section_module(domain=U); TUM
Module X(U) of vector fields on the Open subset U of the
 2-dimensional differentiable manifold M
>>> TUM is U.tensor_field_module((Integer(1),Integer(0)))
True
```

set_change_of_frame(*frame1*, *frame2*, *change_of_frame*, *compute_inverse=True*)

Relate two vector frames by an automorphism.

This updates the internal dictionary `self._frame_changes` of the base space M.

See also

For further details on frames on `self` see *local_frame()*.

Note

Since frames on `self` are directly induced by vector frames on the base space, this method directly invokes *set_change_of_frame()* of the class *DifferentiableManifold*.

INPUT:

- `frame1` – frame 1, denoted (e_i) below
- `frame2` – frame 2, denoted (f_i) below

- `change_of_frame` – instance of class `FreeModuleAutomorphism` describing the automorphism P that relates the basis (e_i) to the basis (f_i) according to $f_i = P(e_i)$
- `compute_inverse` – boolean (default: `True`); if set to `True`, the inverse automorphism is computed and the change from basis (f_i) to (e_i) is set to it in the internal dictionary `self._frame_changes`

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: e = M.vector_frame('e')
sage: f = M.vector_frame('f')
sage: a = M.automorphism_field()
sage: a[e,:] = [[1,2],[0,3]]
sage: TM = M.tangent_bundle()
sage: TM.set_change_of_frame(e, f, a)
sage: f[0].display(e)
f_0 = e_0
sage: f[1].display(e)
f_1 = 2 e_0 + 3 e_1
sage: e[0].display(f)
e_0 = f_0
sage: e[1].display(f)
e_1 = -2/3 f_0 + 1/3 f_1
sage: TM.change_of_frame(e,f)[e,:]
[1 2]
[0 3]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> e = M.vector_frame('e')
>>> f = M.vector_frame('f')
>>> a = M.automorphism_field()
>>> a[e,:] = [[Integer(1),Integer(2)],[Integer(0),Integer(3)]]
>>> TM = M.tangent_bundle()
>>> TM.set_change_of_frame(e, f, a)
>>> f[Integer(0)].display(e)
f_0 = e_0
>>> f[Integer(1)].display(e)
f_1 = 2 e_0 + 3 e_1
>>> e[Integer(0)].display(f)
e_0 = f_0
>>> e[Integer(1)].display(f)
e_1 = -2/3 f_0 + 1/3 f_1
>>> TM.change_of_frame(e,f)[e,:]
[1 2]
[0 3]
```

set_default_frame(*frame*)

Changing the default vector frame on `self`.

Note: If the destination map is the identity, the default frame of the base manifold gets changed here as well.

INPUT:

- `frame` – *VectorFrame* a vector frame defined on the base manifold

EXAMPLES:

Changing the default frame on the tangent bundle of a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: TM = M.tangent_bundle()
sage: e = TM.vector_frame('e')
sage: TM.default_frame()
Coordinate frame (M, (∂/∂x,∂/∂y))
sage: TM.set_default_frame(e)
sage: TM.default_frame()
Vector frame (M, (e_0,e_1))
sage: M.default_frame()
Vector frame (M, (e_0,e_1))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> TM = M.tangent_bundle()
>>> e = TM.vector_frame('e')
>>> TM.default_frame()
Coordinate frame (M, (∂/∂x,∂/∂y))
>>> TM.set_default_frame(e)
>>> TM.default_frame()
Vector frame (M, (e_0,e_1))
>>> M.default_frame()
Vector frame (M, (e_0,e_1))
```

set_orientation(*orientation*)

Set the preferred orientation of `self`.

INPUT:

- `orientation` – a vector frame or a list of vector frames, covering the base space of `self`

Note: If the destination map is the identity, the preferred orientation of the base manifold gets changed here as well.

Warning: It is the user's responsibility that the orientation set here is indeed an orientation. There is no check going on in the background. See `orientation()` for the definition of an orientation.

EXAMPLES:

Set an orientation on a tensor bundle:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: T11 = M.tensor_bundle(1, 1)
sage: e = T11.local_frame('e'); e
Vector frame (M, (e_0,e_1))
sage: T11.set_orientation(e)
sage: T11.orientation()
[Vector frame (M, (e_0,e_1))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> T11 = M.tensor_bundle(Integer(1), Integer(1))
>>> e = T11.local_frame('e'); e
Vector frame (M, (e_0,e_1))
>>> T11.set_orientation(e)
>>> T11.orientation()
[Vector frame (M, (e_0,e_1))]
```

Set an orientation in the non-trivial case:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U'); V = M.open_subset('V')
sage: M.declare_union(U, V)
sage: c_xy.<x,y> = U.chart(); c_uv.<u,v> = V.chart()
sage: T12 = M.tensor_bundle(1, 2)
sage: e = T12.local_frame('e', domain=U)
sage: f = T12.local_frame('f', domain=V)
sage: T12.set_orientation([e, f])
sage: T12.orientation()
[Vector frame (U, (e_0,e_1)), Vector frame (V, (f_0,f_1))]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U'); V = M.open_subset('V')
>>> M.declare_union(U, V)
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> T12 = M.tensor_bundle(Integer(1), Integer(2))
>>> e = T12.local_frame('e', domain=U)
>>> f = T12.local_frame('f', domain=V)
>>> T12.set_orientation([e, f])
>>> T12.orientation()
[Vector frame (U, (e_0,e_1)), Vector frame (V, (f_0,f_1))]
```

transition(*chart1*, *chart2*)

Return the change of trivializations in terms of a coordinate change between two differentiable charts defined on the codomain of the destination map.

The differentiable chart must have been defined previously, for instance by the method `transition_map()`.

Note: Since a chart gives direct rise to a trivialization, this method is nothing but an invocation of *coord_change()* of the class *TopologicalManifold*.

INPUT:

- chart1 – chart 1
- chart2 – chart 2

OUTPUT:

- instance of *CoordChange* representing the transition map from chart 1 to chart 2

EXAMPLES:

Change of coordinates on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: c_uv.<u,v> = M.chart()
sage: c_xy.transition_map(c_uv, (x+y, x-y)) # defines coord. change
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
sage: TM = M.tangent_bundle()
sage: TM.transition(c_xy, c_uv) # returns the coord. change above
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> c_uv = M.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> c_xy.transition_map(c_uv, (x+y, x-y)) # defines coord. change
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
>>> TM = M.tangent_bundle()
>>> TM.transition(c_xy, c_uv) # returns the coord. change above
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
```

transitions()

Return the transition maps between trivialization maps in terms of coordinate changes defined via charts on the codomain of the destination map.

Note: Since a chart gives direct rise to a trivialization, this method is nothing but an invocation of *coord_changes()* of the class *TopologicalManifold*.

EXAMPLES:

Various changes of coordinates on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: c_uv.<u,v> = M.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, [x+y, x-y])
```

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```
sage: TM = M.tangent_bundle()
sage: TM.transitions()
{(Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y))
   to Chart (M, (u, v))}
sage: uv_to_xy = xy_to_uv.inverse()
sage: TM.transitions()  # random (dictionary output)
{(Chart (M, (u, v)),
  Chart (M, (x, y))): Change of coordinates from Chart (M, (u, v))
  to Chart (M, (x, y)),
 (Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y))
   to Chart (M, (u, v))}
sage: c_rs.<r,s> = M.chart()
sage: uv_to_rs = c_uv.transition_map(c_rs, [-u+2*v, 3*u-v])
sage: TM.transitions()  # random (dictionary output)
{(Chart (M, (u, v)),
  Chart (M, (r, s))): Change of coordinates from Chart (M, (u, v))
   to Chart (M, (r, s)),
 (Chart (M, (u, v)),
  Chart (M, (x, y))): Change of coordinates from Chart (M, (u, v))
  to Chart (M, (x, y)),
 (Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y))
   to Chart (M, (u, v))}
sage: xy_to_rs = uv_to_rs * xy_to_uv
sage: TM.transitions()  # random (dictionary output)
{(Chart (M, (u, v)),
  Chart (M, (r, s))): Change of coordinates from Chart (M, (u, v))
   to Chart (M, (r, s)),
 (Chart (M, (u, v)),
  Chart (M, (x, y))): Change of coordinates from Chart (M, (u, v))
  to Chart (M, (x, y)),
 (Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y))
   to Chart (M, (u, v)),
 (Chart (M, (x, y)),
  Chart (M, (r, s))): Change of coordinates from Chart (M, (x, y))
   to Chart (M, (r, s))}
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> c_uv = M.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> xy_to_uv = c_xy.transition_map(c_uv, [x+y, x-y])
>>> TM = M.tangent_bundle()
>>> TM.transitions()
{(Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y))
   to Chart (M, (u, v))}
>>> uv_to_xy = xy_to_uv.inverse()
>>> TM.transitions()  # random (dictionary output)
```

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```
{(Chart (M, (u, v)),
  Chart (M, (x, y))): Change of coordinates from Chart (M, (u, v))
  to Chart (M, (x, y)),
 (Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y))
   to Chart (M, (u, v))}
>>> c_rs = M.chart(names=('r', 's',)); (r, s,) = c_rs._first_ngens(2)
>>> uv_to_rs = c_uv.transition_map(c_rs, [-u+Integer(2)*v, Integer(3)*u-v])
>>> TM.transitions()  # random (dictionary output)
{(Chart (M, (u, v)),
  Chart (M, (r, s))): Change of coordinates from Chart (M, (u, v))
   to Chart (M, (r, s)),
 (Chart (M, (u, v)),
  Chart (M, (x, y))): Change of coordinates from Chart (M, (u, v))
  to Chart (M, (x, y)),
 (Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y))
   to Chart (M, (u, v))}
>>> xy_to_rs = uv_to_rs * xy_to_uv
>>> TM.transitions()  # random (dictionary output)
{(Chart (M, (u, v)),
  Chart (M, (r, s))): Change of coordinates from Chart (M, (u, v))
   to Chart (M, (r, s)),
 (Chart (M, (u, v)),
  Chart (M, (x, y))): Change of coordinates from Chart (M, (u, v))
  to Chart (M, (x, y)),
 (Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y))
   to Chart (M, (u, v)),
 (Chart (M, (x, y)),
  Chart (M, (r, s))): Change of coordinates from Chart (M, (x, y))
   to Chart (M, (r, s))}
```

trivialization(*coordinates=''*, *names=None*, *calc_method=None*)

Return a trivialization of `self` in terms of a chart on the codomain of the destination map.

> **Note**
>
> Since a chart gives direct rise to a trivialization, this method is nothing but an invocation of *chart()* of the class *TopologicalManifold*.

INPUT:

- `coordinates` – (default: `''` (empty string)) string defining the coordinate symbols, ranges and possible periodicities, see below
- `names` – (default: `None`) unused argument, except if `coordinates` is not provided; it must then be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)
- `calc_method` – (default: `None`) string defining the calculus method to be used on this chart; must be one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)

– 'sympy': SymPy
– None: the current calculus method defined on the manifold is used (cf. *set_calculus_method()*)

The coordinates declared in the string coordinates are separated by ' ' (whitespace) and each coordinate has at most four fields, separated by a colon (':'):

1. The coordinate symbol (a letter or a few letters).
2. (optional, only for manifolds over **R**) The interval I defining the coordinate range: if not provided, the coordinate is assumed to span all **R**; otherwise I must be provided in the form (a,b) (or equivalently]a, b[) The bounds a and b can be +/-Infinity, Inf, infinity, inf or oo. For *singular* coordinates, non-open intervals such as [a,b] and (a,b] (or equivalently]a,b]) are allowed. Note that the interval declaration must not contain any space character.
3. (optional) Indicator of the periodic character of the coordinate, either as period=T, where T is the period, or, for manifolds over **R** only, as the keyword periodic (the value of the period is then deduced from the interval I declared in field 2; see the example below)
4. (optional) The LaTeX spelling of the coordinate; if not provided the coordinate symbol given in the first field will be used.

The order of fields 2 to 4 does not matter and each of them can be omitted. If it contains any LaTeX expression, the string coordinates must be declared with the prefix 'r' (for "raw") to allow for a proper treatment of the backslash character (see examples below). If no interval range, no period and no LaTeX spelling is to be set for any coordinate, the argument coordinates can be omitted when the shortcut operator <,> is used to declare the trivialization.

OUTPUT:

- the created chart, as an instance of *Chart* or one of its subclasses, like *RealDiffChart* for differentiable manifolds over **R**.

EXAMPLES:

Chart on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: TM = M.tangent_bundle()
sage: X = TM.trivialization('x y'); X
Chart (M, (x, y))
sage: X[0]
x
sage: X[1]
y
sage: X[:]
(x, y)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> TM = M.tangent_bundle()
>>> X = TM.trivialization('x y'); X
Chart (M, (x, y))
>>> X[Integer(0)]
x
>>> X[Integer(1)]
y
```

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```
>>> X[:]
(x, y)
```

vector_frame(**args*, ***kwargs*)

Define a vector frame on `domain`, possibly with values in the tangent bundle of the ambient domain.

If the basis specified by the given symbol already exists, it is simply returned. If no argument is provided the vector field module's default frame is returned.

Notice, that a vector frame automatically induces a local frame on the tensor bundle `self`. More precisely, if $e : U \to \Phi^* TN$ is a vector frame on $U \subset M$ with values in $\Phi^* TN$ along the destination map

$$\Phi : M \longrightarrow N$$

then the map

$$p \mapsto \Big(\underbrace{e^*(p), \ldots, e^*(p)}_{k \text{ times}}, \underbrace{e(p), \ldots, e(p)}_{l \text{ times}} \Big) \in T_q^{(k,l)} N,$$

with $q = \Phi(p)$, defines a basis at each point $p \in U$ and therefore gives rise to a local frame on $\Phi^* T^{(k,l)} N$ on the domain U.

See also

`VectorFrame` for complete documentation.

INPUT:

- `symbol` – (default: `None`) either a string, to be used as a common base for the symbols of the vector fields constituting the vector frame, or a list/tuple of strings, representing the individual symbols of the vector fields; can be `None` only if `from_frame` is not `None` (see below)
- `vector_fields` – tuple or list of n linearly independent vector fields on `domain` (n being the dimension of `domain`) defining the vector frame; can be omitted if the vector frame is created from scratch or if `from_frame` is not `None`
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the vector fields constituting the vector frame, or a list/tuple of strings, representing the individual LaTeX symbols of the vector fields; if `None`, `symbol` is used in place of `latex_symbol`
- `from_frame` – (default: `None`) vector frame $\tilde{e}$ on the codomain N of the destination map Φ; the returned frame e is then such that for all $p \in U$, we have $e(p) = \tilde{e}(\Phi(p))$
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the vector fields of the frame; if `None`, the indices will be generated as integers within the range declared on `self`
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the vector fields; if `None`, `indices` is used instead
- `symbol_dual` – (default: `None`) same as `symbol` but for the dual coframe; if `None`, `symbol` must be a string and is used for the common base of the symbols of the elements of the dual coframe
- `latex_symbol_dual` – (default: `None`) same as `latex_symbol` but for the dual coframe
- `domain` – (default: `None`) domain on which the local frame is defined; if `None` is provided, the base space of `self` is assumed

OUTPUT:

- the vector frame corresponding to the above specifications; this is an instance of `VectorFrame`.

EXAMPLES:

Defining a local frame for the tangent bundle of a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M')
sage: TM = M.tangent_bundle()
sage: e = TM.local_frame('e'); e
Vector frame (M, (e_0,e_1,e_2))
sage: e[0]
Vector field e_0 on the 3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> TM = M.tangent_bundle()
>>> e = TM.local_frame('e'); e
Vector frame (M, (e_0,e_1,e_2))
>>> e[Integer(0)]
Vector field e_0 on the 3-dimensional differentiable manifold M
```

Specifying the domain of the vector frame:

```
sage: U = M.open_subset('U')
sage: f = TM.local_frame('f', domain=U); f
Vector frame (U, (f_0,f_1,f_2))
sage: f[0]
Vector field f_0 on the Open subset U of the 3-dimensional
 differentiable manifold M
```

```
>>> from sage.all import *
>>> U = M.open_subset('U')
>>> f = TM.local_frame('f', domain=U); f
Vector frame (U, (f_0,f_1,f_2))
>>> f[Integer(0)]
Vector field f_0 on the Open subset U of the 3-dimensional
 differentiable manifold M
```

See also:

For more options, in particular for the choice of symbols and indices, see `VectorFrame`.

2.15.2 Bundle Connections

Let $E \to M$ be a smooth vector bundle of rank n over a smooth manifold M and over a non-discrete topological field K (typically $K = \mathbf{R}$ or $K = \mathbf{C}$). A *bundle connection* on this vector bundle is a K-linear map

$$\nabla : C^\infty(M; E) \to C^\infty(M; E \otimes T^*M)$$

such that the Leibniz rule applies for each scalar field $f \in C^\infty(M)$ and section $s \in C^\infty(M; E)$:

$$\nabla(f\, s) = f \cdot \nabla s + s \otimes \mathrm{d}f.$$

If e is a local frame on E, we have

$$\nabla e_i = \sum_{j=1}^{n} e_j \otimes \omega^j_i,$$

and the corresponding $n \times n$-matrix $(\omega^j_i)_{i,j}$ consisting of one forms is called *connection matrix of* ∇ *with respect to* e.

AUTHORS:

- Michael Jung (2019) : initial version

class `sage.manifolds.differentiable.bundle_connection.`**`BundleConnection`**(*vbundle*, *name*, *latex_name=None*)

Bases: `SageObject`, `Mutability`

An instance of this class represents a bundle connection ∇ on a smooth vector bundle $E \to M$.

INPUT:

- `vbundle` – the vector bundle on which the connection is defined (must be an instance of class `DifferentiableVectorBundle`)
- `name` – name given to the bundle connection
- `latex_name` – (default: `None`) LaTeX symbol to denote the bundle connection; if `None`, it is set to `name`

EXAMPLES:

Define a bundle connection on a rank 2 vector bundle over some 3-dimensional smooth manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e') # standard frame for E
sage: nab = E.bundle_connection('nabla'); nab
Bundle connection nabla on the Differentiable real vector bundle E -> M
 of rank 2 over the base space 3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e') # standard frame for E
>>> nab = E.bundle_connection('nabla'); nab
Bundle connection nabla on the Differentiable real vector bundle E -> M
 of rank 2 over the base space 3-dimensional differentiable manifold M
```

First, let us initialize all connection 1-forms w.r.t. the frame `e` to zero:

```
sage: nab[e, :] = [[0, 0], [0, 0]]
```

```
>>> from sage.all import *
>>> nab[e, :] = [[Integer(0), Integer(0)], [Integer(0), Integer(0)]]
```

This line can be shortened by the following:

```
sage: nab[e, :] = 0  # initialize to zero
```

```
>>> from sage.all import *
>>> nab[e, :] = Integer(0)  # initialize to zero
```

The connection 1-forms are now initialized being differential 1-forms:

```
sage: nab[e, 1, 1].parent()
Free module Omega^1(M) of 1-forms on the 3-dimensional differentiable
 manifold M
sage: nab[e, 1, 1].display()
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = 0
```

```
>>> from sage.all import *
>>> nab[e, Integer(1), Integer(1)].parent()
Free module Omega^1(M) of 1-forms on the 3-dimensional differentiable
 manifold M
>>> nab[e, Integer(1), Integer(1)].display()
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = 0
```

Now, we want to specify some nonzero entries:

```
sage: nab[e, 1, 2][:] = [x*z, y*z, z^2]
sage: nab[e, 2, 1][:] = [x, x^2, x^3]
sage: nab[e, 1, 1][:] = [x+z, y-z, x*y*z]
sage: nab.display()
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = (x + z) dx + (y - z) dy + x*y*z dz
connection (1,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x*z dx + y*z dy + z^2 dz
connection (2,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x dx + x^2 dy + x^3 dz
```

```
>>> from sage.all import *
>>> nab[e, Integer(1), Integer(2)][:] = [x*z, y*z, z**Integer(2)]
>>> nab[e, Integer(2), Integer(1)][:] = [x, x**Integer(2), x**Integer(3)]
>>> nab[e, Integer(1), Integer(1)][:] = [x+z, y-z, x*y*z]
>>> nab.display()
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = (x + z) dx + (y - z) dy + x*y*z dz
connection (1,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x*z dx + y*z dy + z^2 dz
connection (2,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x dx + x^2 dy + x^3 dz
```

Notice, when we omit the frame, the default frame of the vector bundle is assumed (in this case `e`):

```
sage: nab[2, 2].display()
connection (2,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = 0
```

```
>>> from sage.all import *
>>> nab[Integer(2), Integer(2)].display()
```

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```
connection (2,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = 0
```

The same holds for the assignment:

```
sage: nab[1, 2] = 0
sage: nab[e, 1, 2].display()
connection (1,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = 0
```

```
>>> from sage.all import *
>>> nab[Integer(1), Integer(2)] = Integer(0)
>>> nab[e, Integer(1), Integer(2)].display()
connection (1,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = 0
```

Keep noticed that item assignments for bundle connections only copy the right-hand-side and never create a binding to the original instance:

```
sage: omega = M.one_form('omega')
sage: omega[:] = [x*z, y*z, z^2]
sage: nab[1, 2] = omega
sage: nab[1, 2] == omega
True
sage: nab[1, 2] is omega
False
```

```
>>> from sage.all import *
>>> omega = M.one_form('omega')
>>> omega[:] = [x*z, y*z, z**Integer(2)]
>>> nab[Integer(1), Integer(2)] = omega
>>> nab[Integer(1), Integer(2)] == omega
True
>>> nab[Integer(1), Integer(2)] is omega
False
```

Hence, this is therefore equivalent to:

```
sage: nab[2, 2].copy_from(omega)
```

```
>>> from sage.all import *
>>> nab[Integer(2), Integer(2)].copy_from(omega)
```

Preferably, we use `set_connection_form()` to specify the connection 1-forms:

```
sage: nab[:] = 0  # re-initialize to zero
sage: nab.set_connection_form(1, 2)[:] = [x*z, y*z, z^2]
sage: nab.set_connection_form(2, 1)[:] = [x, x^2, x^3]
sage: nab[1, 2].display()
connection (1,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x*z dx + y*z dy + z^2 dz
sage: nab[2, 1].display()
```

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```
connection (2,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x dx + x^2 dy + x^3 dz
```

```
>>> from sage.all import *
>>> nab[:] = Integer(0)  # re-initialize to zero
>>> nab.set_connection_form(Integer(1), Integer(2))[:] = [x*z, y*z, z**Integer(2)]
>>> nab.set_connection_form(Integer(2), Integer(1))[:] = [x, x**Integer(2),␣
↪x**Integer(3)]
>>> nab[Integer(1), Integer(2)].display()
connection (1,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x*z dx + y*z dy + z^2 dz
>>> nab[Integer(2), Integer(1)].display()
connection (2,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x dx + x^2 dy + x^3 dz
```

Note

Notice that item assignments and *set_connection_form()* delete the connection 1-forms w.r.t. other frames for consistency reasons. To avoid this behavior, *add_connection_form()* must be used instead.

In conclusion, the connection 1-forms of a bundle connection are mutable until the connection itself is set immutable:

```
sage: nab.set_immutable()
sage: nab[1, 2] = omega
Traceback (most recent call last):
...
ValueError: object is immutable; please change a copy instead
```

```
>>> from sage.all import *
>>> nab.set_immutable()
>>> nab[Integer(1), Integer(2)] = omega
Traceback (most recent call last):
...
ValueError: object is immutable; please change a copy instead
```

By definition, a bundle connection acts on vector fields and sections:

```
sage: v = M.vector_field((x^2,y^2,z^2), name='v'); v.display()
v = x^2 ∂/∂x + y^2 ∂/∂y + z^2 ∂/∂z
sage: s = E.section((x-y^2, -z), name='s'); s.display()
s = (-y^2 + x) e_1 - z e_2
sage: nab_vs = nab(v, s); nab_vs
Section nabla_v(s) on the 3-dimensional differentiable manifold M with
 values in the real vector bundle E of rank 2
sage: nab_vs.display()
nabla_v(s) = (-x^3*z^3 - 2*y^3 + x^2 - (x^2*y^2 + x^3)*z) e_1 +
 (-(y^2 - x)*z^4 - (x^3*y^2 + y^5 - x^4 - x*y^3)*z - z^2) e_2
```

```
>>> from sage.all import *
>>> v = M.vector_field((x**Integer(2),y**Integer(2),z**Integer(2)), name='v'); v.
↪display()
v = x^2 ∂/∂x + y^2 ∂/∂y + z^2 ∂/∂z
>>> s = E.section((x-y**Integer(2), -z), name='s'); s.display()
s = (-y^2 + x) e_1 - z e_2
>>> nab_vs = nab(v, s); nab_vs
Section nabla_v(s) on the 3-dimensional differentiable manifold M with
 values in the real vector bundle E of rank 2
>>> nab_vs.display()
nabla_v(s) = (-x^3*z^3 - 2*y^3 + x^2 - (x^2*y^2 + x^3)*z) e_1 +
 (-(y^2 - x)*z^4 - (x^3*y^2 + y^5 - x^4 - x*y^3)*z - z^2) e_2
```

The bundle connection action certainly obeys the defining formula for the connection 1-forms:

```
sage: vframe = X.frame()
sage: all(nab(vframe[k], e[i]) == sum(nab[e, i, j](vframe[k])*e[j]
....:                                  for j in E.irange())
....:     for i in E.irange() for k in M.irange())
True
```

```
>>> from sage.all import *
>>> vframe = X.frame()
>>> all(nab(vframe[k], e[i]) == sum(nab[e, i, j](vframe[k])*e[j]
...                                  for j in E.irange())
...     for i in E.irange() for k in M.irange())
True
```

The connection 1-forms are computed automatically for different frames:

```
sage: f = E.local_frame('f', ((1+x^2)*e[1], e[1]-e[2]))
sage: nab.display(frame=f)
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (f_1,f_2)) = ((x^3 + x)*z + 2*x)/(x^2 + 1) dx + y*z dy + z^2 dz
connection (1,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (f_1,f_2)) = -(x^3 + x)*z dx - (x^2 + 1)*y*z dy -
  (x^2 + 1)*z^2 dz
connection (2,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (f_1,f_2)) = (x*z - x)/(x^2 + 1) dx -
  (x^2 - y*z)/(x^2 + 1) dy - (x^3 - z^2)/(x^2 + 1) dz
connection (2,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (f_1,f_2)) = -x*z dx - y*z dy - z^2 dz
```

```
>>> from sage.all import *
>>> f = E.local_frame('f', ((Integer(1)+x**Integer(2))*e[Integer(1)],␣
↪e[Integer(1)]-e[Integer(2)]))
>>> nab.display(frame=f)
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (f_1,f_2)) = ((x^3 + x)*z + 2*x)/(x^2 + 1) dx + y*z dy + z^2 dz
connection (1,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (f_1,f_2)) = -(x^3 + x)*z dx - (x^2 + 1)*y*z dy -
  (x^2 + 1)*z^2 dz
connection (2,1) of bundle connection nabla w.r.t. Local frame
```

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```
 (E|_M, (f_1,f_2)) = (x*z - x)/(x^2 + 1) dx -
  (x^2 - y*z)/(x^2 + 1) dy - (x^3 - z^2)/(x^2 + 1) dz
connection (2,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (f_1,f_2)) = -x*z dx - y*z dy - z^2 dz
```

The new connection 1-forms obey the defining formula, too:

```
sage: all(nab(vframe[k], f[i]) == sum(nab[f, i, j](vframe[k])*f[j]
....:                                    for j in E.irange())
....:     for i in E.irange() for k in M.irange())
True
```

```
>>> from sage.all import *
>>> all(nab(vframe[k], f[i]) == sum(nab[f, i, j](vframe[k])*f[j]
...                                    for j in E.irange())
...     for i in E.irange() for k in M.irange())
True
```

After the connection has been specified, the curvature 2-forms can be derived:

```
sage: Omega = nab.curvature_form
sage: for i in E.irange():
....:     for j in E.irange():
....:         print(Omega(i ,j, e).display())
curvature (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = -(x^3 - x*y)*z dx∧dy + (-x^4*z + x*z^2) dx∧dz +
  (-x^3*y*z + x^2*z^2) dy∧dz
curvature (1,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = -x dx∧dz - y dy∧dz
curvature (2,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = 2*x dx∧dy + 3*x^2 dx∧dz
curvature (2,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = (x^3 - x*y)*z dx∧dy + (x^4*z - x*z^2) dx∧dz +
  (x^3*y*z - x^2*z^2) dy∧dz
```

```
>>> from sage.all import *
>>> Omega = nab.curvature_form
>>> for i in E.irange():
...     for j in E.irange():
...         print(Omega(i ,j, e).display())
curvature (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = -(x^3 - x*y)*z dx∧dy + (-x^4*z + x*z^2) dx∧dz +
  (-x^3*y*z + x^2*z^2) dy∧dz
curvature (1,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = -x dx∧dz - y dy∧dz
curvature (2,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = 2*x dx∧dy + 3*x^2 dx∧dz
curvature (2,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = (x^3 - x*y)*z dx∧dy + (x^4*z - x*z^2) dx∧dz +
  (x^3*y*z - x^2*z^2) dy∧dz
```

The derived forms certainly obey the structure equations, see `curvature_form()` for details:

```
sage: omega = nab.connection_form
sage: check = []
sage: for i in E.irange():  # long time
....:     for j in E.irange():
....:         check.append(Omega(i,j,e) == \
....:                       omega(i,j,e).exterior_derivative() + \
....:         sum(omega(k,j,e).wedge(omega(i,k,e))
....:             for k in E.irange()))
sage: check  # long time
[True, True, True, True]
```

```
>>> from sage.all import *
>>> omega = nab.connection_form
>>> check = []
>>> for i in E.irange():  # long time
...     for j in E.irange():
...         check.append(Omega(i,j,e) ==                       omega(i,j,e).
↪exterior_derivative() +         sum(omega(k,j,e).wedge(omega(i,k,e))
...             for k in E.irange()))
>>> check  # long time
[True, True, True, True]
```

add_connection_form(*i*, *j*, *frame=None*)

Return the connection form ω^j_i in a given frame for assignment.

See method *connection_forms()* for details about the definition of the connection forms.

To delete the connection forms in other frames, use the method *set_connection_form()* instead.

INPUT:

- `i`, `j` – indices identifying the 1-form ω^j_i
- `frame` – (default: `None`) local frame in which the connection 1-form is defined; if `None`, the default frame of the vector bundle is assumed.

> **Warning**
>
> If the connection has already forms in other frames, it is the user's responsibility to make sure that the 1-forms to be added are consistent with them.

OUTPUT:

- connection 1-form ω^j_i in the given frame, as an instance of the class *DiffForm*; if such connection 1-form did not exist previously, it is created. See method *connection_forms()* for the storage convention of the connection 1-forms.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e') # standard frame for E
sage: nab = E.bundle_connection('nabla', latex_name=r'\nabla')
sage: nab.add_connection_form(0, 1, frame=e)[:] = [x^2, x]
```

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```
sage: nab[e, 0, 1].display()
connection (0,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_0,e_1)) = x^2 dx + x dy
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e') # standard frame for E
>>> nab = E.bundle_connection('nabla', latex_name=r'\nabla')
>>> nab.add_connection_form(Integer(0), Integer(1), frame=e)[:] =␣
↪[x**Integer(2), x]
>>> nab[e, Integer(0), Integer(1)].display()
connection (0,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_0,e_1)) = x^2 dx + x dy
```

Since e is the vector bundle's default local frame, its mention may be omitted:

```
sage: nab.add_connection_form(1, 0)[:] = [y^2, y]
sage: nab[1, 0].display()
connection (1,0) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_0,e_1)) = y^2 dx + y dy
```

```
>>> from sage.all import *
>>> nab.add_connection_form(Integer(1), Integer(0))[:] = [y**Integer(2), y]
>>> nab[Integer(1), Integer(0)].display()
connection (1,0) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_0,e_1)) = y^2 dx + y dy
```

Adding connection 1-forms w.r.t. to another local frame:

```
sage: f = E.local_frame('f')
sage: nab.add_connection_form(1, 1, frame=f)[:] = [x, y]
sage: nab[f, 1, 1].display()
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (f_0,f_1)) = x dx + y dy
```

```
>>> from sage.all import *
>>> f = E.local_frame('f')
>>> nab.add_connection_form(Integer(1), Integer(1), frame=f)[:] = [x, y]
>>> nab[f, Integer(1), Integer(1)].display()
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (f_0,f_1)) = x dx + y dy
```

The forms w.r.t. the frame e have been kept:

```
sage: nab[e, 0, 1].display()
connection (0,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_0,e_1)) = x^2 dx + x dy
```

```
>>> from sage.all import *
>>> nab[e, Integer(0), Integer(1)].display()
```

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```
connection (0,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_0,e_1)) = x^2 dx + x dy
```

To delete them, use the method `set_connection_form()` instead.

connection_form(*i*, *j*, *frame=None*)

Return the connection 1-form corresponding to the given index and local frame.

See also

Consult `connection_forms()` for detailed information.

INPUT:

- `i, j` – indices identifying the 1-form ω_i^j
- `frame` – (default: `None`) local frame relative to which the connection 1-forms are defined; if `None`, the default frame of the vector bundle's corresponding section module is assumed.

OUTPUT:

- the 1-form ω_i^j, as an instance of `DiffForm`

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e') # standard frame for E
sage: nab = E.bundle_connection('nabla', latex_name=r'\nabla')
sage: nab.set_connection_form(0, 1)[:] = [x^2, x]
sage: nab.set_connection_form(1, 0)[:] = [y^2, y]
sage: nab.connection_form(0, 1).display()
connection (0,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_0,e_1)) = x^2 dx + x dy
sage: nab.connection_form(1, 0).display()
connection (1,0) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_0,e_1)) = y^2 dx + y dy
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e') # standard frame for E
>>> nab = E.bundle_connection('nabla', latex_name=r'\nabla')
>>> nab.set_connection_form(Integer(0), Integer(1))[:] = [x**Integer(2), x]
>>> nab.set_connection_form(Integer(1), Integer(0))[:] = [y**Integer(2), y]
>>> nab.connection_form(Integer(0), Integer(1)).display()
connection (0,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_0,e_1)) = x^2 dx + x dy
>>> nab.connection_form(Integer(1), Integer(0)).display()
connection (1,0) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_0,e_1)) = y^2 dx + y dy
```

`connection_forms`(*frame=None*)

Return the connection forms relative to the given frame.

If e is a local frame on E, we have

$$\nabla e_i = \sum_{j=1}^{n} e_j \otimes \omega^j_i,$$

and the corresponding $n \times n$-matrix $(\omega^j_i)_{i,j}$ consisting of one forms is called *connection matrix of* ∇ *with respect to* e.

If the connection coefficients are not known already, they are computed from the above formula.

INPUT:

- `frame` – (default: `None`) local frame relative to which the connection forms are required; if none is provided, the vector bundle's default frame is assumed

OUTPUT:

- connection forms relative to the frame `frame`, as a dictionary with tuples (i, j) as key and one forms as instances of `diff_form` as value representing the matrix entries.

EXAMPLES:

Connection forms of a bundle connection on a rank 2 vector bundle over a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e')
sage: nab = E.bundle_connection('nabla', r'\nabla')
sage: nab[:] = 0  # initialize curvature forms
sage: forms = nab.connection_forms()
sage: [forms[k] for k in sorted(forms)]
[1-form connection (1,1) of bundle connection nabla w.r.t. Local
 frame (E|_M, (e_1,e_2)) on the 3-dimensional differentiable
 manifold M,
1-form connection (1,2) of bundle connection nabla w.r.t. Local
 frame (E|_M, (e_1,e_2)) on the 3-dimensional differentiable
 manifold M,
1-form connection (2,1) of bundle connection nabla w.r.t. Local
 frame (E|_M, (e_1,e_2)) on the 3-dimensional differentiable
 manifold M,
1-form connection (2,2) of bundle connection nabla w.r.t. Local
 frame (E|_M, (e_1,e_2)) on the 3-dimensional differentiable
 manifold M]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e')
>>> nab = E.bundle_connection('nabla', r'\nabla')
>>> nab[:] = Integer(0)  # initialize curvature forms
>>> forms = nab.connection_forms()
```

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```
>>> [forms[k] for k in sorted(forms)]
[1-form connection (1,1) of bundle connection nabla w.r.t. Local
 frame (E|_M, (e_1,e_2)) on the 3-dimensional differentiable
 manifold M,
1-form connection (1,2) of bundle connection nabla w.r.t. Local
 frame (E|_M, (e_1,e_2)) on the 3-dimensional differentiable
 manifold M,
1-form connection (2,1) of bundle connection nabla w.r.t. Local
 frame (E|_M, (e_1,e_2)) on the 3-dimensional differentiable
 manifold M,
1-form connection (2,2) of bundle connection nabla w.r.t. Local
 frame (E|_M, (e_1,e_2)) on the 3-dimensional differentiable
 manifold M]
```

copy (*name*, *latex_name=None*)

Return an exact copy of `self`.

INPUT:

- `name` – name given to the copy
- `latex_name` – (default: `None`) LaTeX symbol to denote the copy; if none is provided, the LaTeX symbol is set to `name`

Note

The name and the derived quantities are not copied.

EXAMPLES:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e')
sage: nab = E.bundle_connection('nabla')
sage: nab.set_connection_form(1, 1)[:] = [x^2, x-z, y^3]
sage: nab.set_connection_form(1, 2)[:] = [1, x, z*y^3]
sage: nab.set_connection_form(2, 1)[:] = [1, 2, 3]
sage: nab.set_connection_form(2, 2)[:] = [0, 0, 0]
sage: nab.display()
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x^2 dx + (x - z) dy + y^3 dz
connection (1,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = dx + x dy + y^3*z dz
connection (2,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = dx + 2 dy + 3 dz
sage: nab_copy = nab.copy('nablo'); nab_copy
Bundle connection nablo on the Differentiable real vector bundle
 E -> M of rank 2 over the base space 3-dimensional differentiable
 manifold M
sage: nab is nab_copy
False
```

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```
sage: nab == nab_copy
True
sage: nab_copy.display()
connection (1,1) of bundle connection nablo w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x^2 dx + (x - z) dy + y^3 dz
connection (1,2) of bundle connection nablo w.r.t. Local frame
 (E|_M, (e_1,e_2)) = dx + x dy + y^3*z dz
connection (2,1) of bundle connection nablo w.r.t. Local frame
 (E|_M, (e_1,e_2)) = dx + 2 dy + 3 dz
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e')
>>> nab = E.bundle_connection('nabla')
>>> nab.set_connection_form(Integer(1), Integer(1))[:] = [x**Integer(2), x-z,
↪y**Integer(3)]
>>> nab.set_connection_form(Integer(1), Integer(2))[:] = [Integer(1), x,
↪z*y**Integer(3)]
>>> nab.set_connection_form(Integer(2), Integer(1))[:] = [Integer(1),
↪Integer(2), Integer(3)]
>>> nab.set_connection_form(Integer(2), Integer(2))[:] = [Integer(0),
↪Integer(0), Integer(0)]
>>> nab.display()
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x^2 dx + (x - z) dy + y^3 dz
connection (1,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = dx + x dy + y^3*z dz
connection (2,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = dx + 2 dy + 3 dz
>>> nab_copy = nab.copy('nablo'); nab_copy
Bundle connection nablo on the Differentiable real vector bundle
 E -> M of rank 2 over the base space 3-dimensional differentiable
 manifold M
>>> nab is nab_copy
False
>>> nab == nab_copy
True
>>> nab_copy.display()
connection (1,1) of bundle connection nablo w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x^2 dx + (x - z) dy + y^3 dz
connection (1,2) of bundle connection nablo w.r.t. Local frame
 (E|_M, (e_1,e_2)) = dx + x dy + y^3*z dz
connection (2,1) of bundle connection nablo w.r.t. Local frame
 (E|_M, (e_1,e_2)) = dx + 2 dy + 3 dz
```

curvature_form(*i*, *j*, *frame=None*)

Return the curvature 2-form corresponding to the given index and local frame.

The *curvature 2-forms* with respect to the frame e are the 2-forms Ω^j_i given by the formula

$$\Omega^j_i = \mathrm{d}\omega^j_i + \sum_{k=1}^{n} \omega^j_k \wedge \omega^k_i$$

INPUT:

- `i, j` – indices identifying the 2-form Ω^j_i
- `frame` – (default: `None`) local frame relative to which the curvature 2-forms are defined; if `None`, the default frame of the vector bundle is assumed.

OUTPUT:

- the 2-form Ω^j_i, as an instance of `DiffForm`

EXAMPLES:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: X.<x,y> = M.chart()
sage: E = M.vector_bundle(1, 'E')
sage: nab = E.bundle_connection('nabla', latex_name=r'\nabla')
sage: e = E.local_frame('e')
sage: nab.set_connection_form(1, 1)[:] = [x^2, x]
sage: curv = nab.curvature_form(1, 1); curv
2-form curvature (1,1) of bundle connection nabla w.r.t. Local
 frame (E|_M, (e_1)) on the 2-dimensional differentiable manifold M
sage: curv.display()
curvature (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1)) = dx∧dy
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> E = M.vector_bundle(Integer(1), 'E')
>>> nab = E.bundle_connection('nabla', latex_name=r'\nabla')
>>> e = E.local_frame('e')
>>> nab.set_connection_form(Integer(1), Integer(1))[:] = [x**Integer(2), x]
>>> curv = nab.curvature_form(Integer(1), Integer(1)); curv
2-form curvature (1,1) of bundle connection nabla w.r.t. Local
 frame (E|_M, (e_1)) on the 2-dimensional differentiable manifold M
>>> curv.display()
curvature (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1)) = dx∧dy
```

del_other_forms(*frame=None*)

Delete all the connection forms but those corresponding to `frame`.

INPUT:

- `frame` – (default: `None`) local frame, the connection forms w.r.t. which are to be kept; if `None`, the default frame of the vector bundle is assumed.

EXAMPLES:

We first create two sets of connection forms:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: X.<x,y> = M.chart()
sage: E = M.vector_bundle(2, 'E')
sage: nab = E.bundle_connection('nabla', latex_name=r'\nabla')
sage: e = E.local_frame('e')
sage: nab.set_connection_form(1, 1, frame=e)[:] = [x^2, x]
sage: f = E.local_frame('f')
sage: nab.add_connection_form(1, 1, frame=f)[:] = [y^2, y]
sage: nab[e, 1, 1].display()
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x^2 dx + x dy
sage: nab[f, 1, 1].display()
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (f_1,f_2)) = y^2 dx + y dy
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> nab = E.bundle_connection('nabla', latex_name=r'\nabla')
>>> e = E.local_frame('e')
>>> nab.set_connection_form(Integer(1), Integer(1), frame=e)[:] =␣
↪[x**Integer(2), x]
>>> f = E.local_frame('f')
>>> nab.add_connection_form(Integer(1), Integer(1), frame=f)[:] =␣
↪[y**Integer(2), y]
>>> nab[e, Integer(1), Integer(1)].display()
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x^2 dx + x dy
>>> nab[f, Integer(1), Integer(1)].display()
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (f_1,f_2)) = y^2 dx + y dy
```

Let us delete the connection forms w.r.t. all frames except for frame `e`:

```
sage: nab.del_other_forms(e)
sage: nab[e, 1, 1].display()
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x^2 dx + x dy
```

```
>>> from sage.all import *
>>> nab.del_other_forms(e)
>>> nab[e, Integer(1), Integer(1)].display()
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x^2 dx + x dy
```

The connection forms w.r.t. frame `e` have indeed been deleted:

```
sage: nab[f, :]
Traceback (most recent call last):
...
ValueError: no basis could be found for computing the components in
 the Local frame (E|_M, (e_1,e_2))
```

```
>>> from sage.all import *
>>> nab[f, :]
Traceback (most recent call last):
...
ValueError: no basis could be found for computing the components in
 the Local frame (E|_M, (e_1,e_2))
```

display (*frame=None*, *vector_frame=None*, *chart=None*, *only_nonzero=True*)

Display all the connection 1-forms w.r.t. to a given local frame, one per line.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- `frame` – (default: `None`) local frame of the vector bundle relative to which the connection 1-forms are defined; if `None`, the default frame of the bundle is used
- `vector_frame` – (default: `None`) vector frame of the manifold relative to which the connection 1-forms should be displayed; if `None`, the default frame of the local frame's domain is used
- `chart` – (default: `None`) chart specifying the coordinate expression of the connection 1-forms; if `None`, the default chart of the domain of `frame` is used
- `only_nonzero` – boolean (default: `True`); if `True`, only nonzero connection coefficients are displayed

EXAMPLES:

Set connection 1-forms:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e') # standard frame for E
sage: nab = E.bundle_connection('nabla', latex_name=r'\nabla'); nab
Bundle connection nabla on the Differentiable real vector bundle
 E -> M of rank 2 over the base space 3-dimensional differentiable
 manifold M
sage: nab[:] = 0
sage: nab[1, 1][:] = [x, y, z]
sage: nab[2, 2][:] = [x^2, y^2, z^2]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e') # standard frame for E
>>> nab = E.bundle_connection('nabla', latex_name=r'\nabla'); nab
Bundle connection nabla on the Differentiable real vector bundle
 E -> M of rank 2 over the base space 3-dimensional differentiable
 manifold M
>>> nab[:] = Integer(0)
>>> nab[Integer(1), Integer(1)][:] = [x, y, z]
>>> nab[Integer(2), Integer(2)][:] = [x**Integer(2), y**Integer(2),␣
↪z**Integer(2)]
```

By default, only the nonzero connection coefficients are displayed:

```
sage: nab.display()
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x dx + y dy + z dz
connection (2,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x^2 dx + y^2 dy + z^2 dz
sage: latex(nab.display())
\begin{array}{lcl} \omega^1_{\ \, 1} = x \mathrm{d} x +
 y \mathrm{d} y + z \mathrm{d} z \\ \omega^2_{\ \, 2} = x^{2}
 \mathrm{d} x + y^{2} \mathrm{d} y + z^{2} \mathrm{d} z \end{array}
```

```
>>> from sage.all import *
>>> nab.display()
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x dx + y dy + z dz
connection (2,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x^2 dx + y^2 dy + z^2 dz
>>> latex(nab.display())
\begin{array}{lcl} \omega^1_{\ \, 1} = x \mathrm{d} x +
 y \mathrm{d} y + z \mathrm{d} z \\ \omega^2_{\ \, 2} = x^{2}
 \mathrm{d} x + y^{2} \mathrm{d} y + z^{2} \mathrm{d} z \end{array}
```

By default, the displayed connection 1-forms are those w.r.t. the default frame of the vector bundle. The aforementioned is therefore equivalent to:

```
sage: nab.display(frame=E.default_frame())
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x dx + y dy + z dz
connection (2,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x^2 dx + y^2 dy + z^2 dz
```

```
>>> from sage.all import *
>>> nab.display(frame=E.default_frame())
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x dx + y dy + z dz
connection (2,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x^2 dx + y^2 dy + z^2 dz
```

Moreover, the connection 1-forms are displayed w.r.t. the default vector frame on the local frame's domain, i.e.:

```
sage: domain = e.domain()
sage: nab.display(vector_frame=domain.default_frame())
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x dx + y dy + z dz
connection (2,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x^2 dx + y^2 dy + z^2 dz
```

```
>>> from sage.all import *
>>> domain = e.domain()
>>> nab.display(vector_frame=domain.default_frame())
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x dx + y dy + z dz
```

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```
connection (2,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x^2 dx + y^2 dy + z^2 dz
```

By default, the parameter `only_nonzero` is set to `True`. Otherwise, the connection 1-forms being zero are shown as well:

```
sage: nab.display(only_nonzero=False)
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x dx + y dy + z dz
connection (1,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = 0
connection (2,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = 0
connection (2,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x^2 dx + y^2 dy + z^2 dz
```

```
>>> from sage.all import *
>>> nab.display(only_nonzero=False)
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x dx + y dy + z dz
connection (1,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = 0
connection (2,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = 0
connection (2,2) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_1,e_2)) = x^2 dx + y^2 dy + z^2 dz
```

set_connection_form(*i*, *j*, *frame=None*)

Return the connection form ω^j_i in a given frame for assignment.

See method *connection_forms()* for details about the definition of the connection forms.

The connection forms with respect to other frames are deleted, in order to avoid any inconsistency. To keep them, use the method *add_connection_form()* instead.

INPUT:

- `i`, `j` – indices identifying the 1-form ω^j_i
- `frame` – (default: `None`) local frame in which the connection 1-form is defined; if `None`, the default frame of the vector bundle is assumed.

OUTPUT:

- connection 1-form ω^j_i in the given frame, as an instance of the class *DiffForm*; if such connection 1-form did not exist previously, it is created. See method *connection_forms()* for the storage convention of the connection 1-forms.

EXAMPLES:

Setting the connection forms of a bundle connection w.r.t. some local frame:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e') # standard frame for E
```

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```
sage: nab = E.bundle_connection('nabla', latex_name=r'\nabla')
sage: nab.set_connection_form(0, 1)[:] = [x^2, x]
sage: nab[0, 1].display()
connection (0,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_0,e_1)) = x^2 dx + x dy
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e') # standard frame for E
>>> nab = E.bundle_connection('nabla', latex_name=r'\nabla')
>>> nab.set_connection_form(Integer(0), Integer(1))[:] = [x**Integer(2), x]
>>> nab[Integer(0), Integer(1)].display()
connection (0,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_0,e_1)) = x^2 dx + x dy
```

Since `e` is the vector bundle's default local frame, its mention may be omitted:

```
sage: nab.set_connection_form(1, 0)[:] = [y^2, y]
sage: nab[1, 0].display()
connection (1,0) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_0,e_1)) = y^2 dx + y dy
```

```
>>> from sage.all import *
>>> nab.set_connection_form(Integer(1), Integer(0))[:] = [y**Integer(2), y]
>>> nab[Integer(1), Integer(0)].display()
connection (1,0) of bundle connection nabla w.r.t. Local frame
 (E|_M, (e_0,e_1)) = y^2 dx + y dy
```

Setting connection 1-forms w.r.t. to another local frame:

```
sage: f = E.local_frame('f')
sage: nab.set_connection_form(1, 1, frame=f)[:] = [x, y]
sage: nab[f, 1, 1].display()
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (f_0,f_1)) = x dx + y dy
```

```
>>> from sage.all import *
>>> f = E.local_frame('f')
>>> nab.set_connection_form(Integer(1), Integer(1), frame=f)[:] = [x, y]
>>> nab[f, Integer(1), Integer(1)].display()
connection (1,1) of bundle connection nabla w.r.t. Local frame
 (E|_M, (f_0,f_1)) = x dx + y dy
```

The forms w.r.t. the frame `e` have been deleted:

```
sage: nab[e, 0, 1].display()
Traceback (most recent call last):
...
ValueError: no basis could be found for computing the components in
 the Local frame (E|_M, (f_0,f_1))
```

```
>>> from sage.all import *
>>> nab[e, Integer(0), Integer(1)].display()
Traceback (most recent call last):
...
ValueError: no basis could be found for computing the components in
 the Local frame (E|_M, (f_0,f_1))
```

To keep them, use the method `add_connection_form()` instead.

set_immutable()

Set `self` and all restrictions of `self` immutable.

EXAMPLES:

An affine connection can be set immutable:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: E = M.vector_bundle(2, 'E')
sage: e = E.local_frame('e')
sage: nab = E.bundle_connection('nabla')
sage: nab.set_connection_form(1, 1)[:] = [x^2, x-z, y^3]
sage: nab.set_connection_form(1, 2)[:] = [1, x, z*y^3]
sage: nab.set_connection_form(2, 1)[:] = [1, 2, 3]
sage: nab.set_connection_form(2, 2)[:] = [0, 0, 0]
sage: nab.is_immutable()
False
sage: nab.set_immutable()
sage: nab.is_immutable()
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> e = E.local_frame('e')
>>> nab = E.bundle_connection('nabla')
>>> nab.set_connection_form(Integer(1), Integer(1))[:] = [x**Integer(2), x-z,
↪y**Integer(3)]
>>> nab.set_connection_form(Integer(1), Integer(2))[:] = [Integer(1), x,
↪z*y**Integer(3)]
>>> nab.set_connection_form(Integer(2), Integer(1))[:] = [Integer(1),
↪Integer(2), Integer(3)]
>>> nab.set_connection_form(Integer(2), Integer(2))[:] = [Integer(0),
↪Integer(0), Integer(0)]
>>> nab.is_immutable()
False
>>> nab.set_immutable()
>>> nab.is_immutable()
True
```

The coefficients of immutable elements cannot be changed:

```
sage: f = E.local_frame('f')
sage: nab.add_connection_form(1, 1, frame=f)[:] = [x, y, z]
Traceback (most recent call last):
...
ValueError: object is immutable; please change a copy instead
```

```
>>> from sage.all import *
>>> f = E.local_frame('f')
>>> nab.add_connection_form(Integer(1), Integer(1), frame=f)[:] = [x, y, z]
Traceback (most recent call last):
...
ValueError: object is immutable; please change a copy instead
```

vector_bundle()

Return the vector bundle on which the bundle connection is defined.

OUTPUT:

- instance of class *DifferentiableVectorBundle* representing the vector bundle on which `self` is defined.

EXAMPLES:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: E = M.vector_bundle(2, 'E')
sage: nab = E.bundle_connection('nabla', r'\nabla')
sage: nab.vector_bundle()
Differentiable real vector bundle E -> M of rank 2 over the base
 space 3-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> c_xyz = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_xyz._first_
↪ngens(3)
>>> E = M.vector_bundle(Integer(2), 'E')
>>> nab = E.bundle_connection('nabla', r'\nabla')
>>> nab.vector_bundle()
Differentiable real vector bundle E -> M of rank 2 over the base
 space 3-dimensional differentiable manifold M
```

2.15.3 Characteristic cohomology classes

A *characteristic class* κ is a natural transformation that associates with each vector bundle $E \to M$ a cohomology class $\kappa(E) \in H^*(M; R)$ such that for any continuous map $f\colon N \to M$ from another topological manifold N, the *naturality condition* is satisfied:

$$f^*\kappa(E) = \kappa(f^*E) \in H^*(N; R)$$

The cohomology class $\kappa(E)$ is called *characteristic cohomology class*. Roughly speaking, characteristic cohomology classes measure the non-triviality of vector bundles.

One way to obtain and compute characteristic classes in the de Rham cohomology with coefficients in the ring **C** is via the so-called *Chern-Weil theory* using the curvature of a differentiable vector bundle.

For that let ∇ be a connection on E, e a local frame on E and Ω be the corresponding curvature matrix (see: `curvature_form()`).

Namely, if $P : \mathrm{Mat}_{n \times n}(\mathbf{C}) \to \mathbf{C}$ is an invariant polynomial, the object

$$[P(\Omega)] \in H^{2*}_{\mathrm{dR}}(M, \mathbf{C})$$

is well-defined, independent of the choice of ∇ (the proof can be found in [Roe1988] pp. 31) and fulfills the naturality condition. This is the foundation of the Chern-Weil theory and the following definitions.

Note

This documentation is rich of examples, but sparse in explanations. Please consult the references for more details.

AUTHORS:

- Michael Jung (2019) : initial version
- Michael Jung (2021) : new algorithm; complete refactoring

REFERENCES:

- [Mil1974]
- [Roe1988]

Contents

We consider the following three types of classes:

- *Additive Classes*
- *Multiplicative Classes*
- *Pfaffian Classes*

Additive Classes

In the **complex** case, let f be a holomorphic function around zero. Then we call

$$\left[\mathrm{tr}\left(f\left(\frac{\Omega}{2\pi i}\right)\right)\right] \in H^{2*}_{\mathrm{dR}}(M, \mathbf{C})$$

the *additive characteristic class associated to* f of the complex vector bundle E.

Important and predefined additive classes are:

- *Chern Character* with $f(x) = \exp(x)$

In the **real** case, let g be a holomorphic function around zero with $g(0) = 0$. Then we call

$$\left[\mathrm{tr}\left(\frac{1}{2}g\left(-\frac{\Omega^2}{4\pi^2}\right)\right)\right] \in H^{4*}_{\mathrm{dR}}(M, \mathbf{C})$$

the *additive characteristic class associated to* g of the **real** vector bundle E.

EXAMPLES:

Consider the **Chern character** on some 2-dimensional spacetime:

```
sage: M = Manifold(2, 'M', structure='Lorentzian')
sage: X.<t,x> = M.chart()
sage: E = M.vector_bundle(1, 'E', field='complex'); E
Differentiable complex vector bundle E -> M of rank 1 over the base space
 2-dimensional Lorentzian manifold M
sage: e = E.local_frame('e')
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x',)); (t, x,) = X._first_ngens(2)
>>> E = M.vector_bundle(Integer(1), 'E', field='complex'); E
Differentiable complex vector bundle E -> M of rank 1 over the base space
 2-dimensional Lorentzian manifold M
>>> e = E.local_frame('e')
```

Let us define the connection ∇^E in terms of an electro-magnetic potential $A(t)$:

```
sage: nab = E.bundle_connection('nabla^E', latex_name=r'\nabla^E')
sage: omega = M.one_form(name='omega')
sage: A = function('A')
sage: nab.set_connection_form(0, 0)[1] = I*A(t)
sage: nab[0, 0].display()
connection (0,0) of bundle connection nabla^E w.r.t. Local frame
 (E|_M, (e_0)) = I*A(t) dx
sage: nab.set_immutable()
```

```
>>> from sage.all import *
>>> nab = E.bundle_connection('nabla^E', latex_name=r'\nabla^E')
>>> omega = M.one_form(name='omega')
>>> A = function('A')
>>> nab.set_connection_form(Integer(0), Integer(0))[Integer(1)] = I*A(t)
>>> nab[Integer(0), Integer(0)].display()
connection (0,0) of bundle connection nabla^E w.r.t. Local frame
 (E|_M, (e_0)) = I*A(t) dx
>>> nab.set_immutable()
```

The Chern character is then given by:

```
sage: ch = E.characteristic_cohomology_class('ChernChar'); ch
Characteristic cohomology class ch(E) of the Differentiable complex vector
 bundle E -> M of rank 1 over the base space 2-dimensional Lorentzian
 manifold M
```

```
>>> from sage.all import *
>>> ch = E.characteristic_cohomology_class('ChernChar'); ch
Characteristic cohomology class ch(E) of the Differentiable complex vector
 bundle E -> M of rank 1 over the base space 2-dimensional Lorentzian
 manifold M
```

The corresponding characteristic form w.r.t. the bundle connection can be obtained via `get_form()`:

```
sage: ch_form = ch.get_form(nab); ch_form.display_expansion()
ch(E, nabla^E) = 1 + 1/2*d(A)/dt/pi dt∧dx
```

```
>>> from sage.all import *
>>> ch_form = ch.get_form(nab); ch_form.display_expansion()
ch(E, nabla^E) = 1 + 1/2*d(A)/dt/pi dt∧dx
```

Multiplicative Classes

In the **complex** case, let f be a holomorphic function around zero. Then we call

$$\left[\det\left(f\left(\frac{\Omega}{2\pi i}\right)\right)\right] \in H^{2*}_{\mathrm{dR}}(M, \mathbf{C})$$

the *multiplicative characteristic class associated to* f of the complex vector bundle E.

Important and predefined multiplicative classes on complex vector bundles are:

- *Chern class* with $f(x) = 1 + x$
- *Todd class* with $f(x) = \frac{x}{1-\exp(-x)}$

In the **real** case, let g be a holomorphic function around zero with $g(0) = 1$. Then we call

$$\left[\det\left(\sqrt{g\left(-\frac{\Omega^2}{4\pi^2}\right)}\right)\right] \in H^{4*}_{\mathrm{dR}}(M, \mathbf{C})$$

the *multiplicative characteristic class associated to* g on the **real** vector bundle E.

Important and predefined multiplicative classes on real vector bundles are:

- *Pontryagin class* with $g(x) = 1 + x$
- $\hat{A}$ *class* with $g(x) = \frac{\sqrt{x}/2}{\sinh(\sqrt{x}/2)}$
- *Hirzebruch class* with $g(x) = \frac{\sqrt{x}}{\tanh(\sqrt{x})}$

EXAMPLES:

We consider the **Chern class** of the tautological line bundle γ^1 over $\mathbf{CP}^1$:

```
sage: M = Manifold(2, 'CP^1', start_index=1)
sage: U = M.open_subset('U')
sage: c_cart.<x,y> = U.chart() # homogeneous coordinates in real terms
sage: c_comp.<z, zbar> = U.chart(r'z:z zbar:\bar{z}') # complexification
sage: cart_to_comp = c_cart.transition_map(c_comp, (x+I*y, x-I*y))
sage: comp_to_cart = cart_to_comp.inverse()
sage: E = M.vector_bundle(1, 'gamma^1', field='complex')
sage: e = E.local_frame('e', domain=U)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'CP^1', start_index=Integer(1))
>>> U = M.open_subset('U')
>>> c_cart = U.chart(names=('x', 'y',)); (x, y,) = c_cart._first_ngens(2)#␣
↪homogeneous coordinates in real terms
>>> c_comp = U.chart(r'z:z zbar:\bar{z}', names=('z', 'zbar',)); (z, zbar,) = c_comp._
↪first_ngens(2)# complexification
>>> cart_to_comp = c_cart.transition_map(c_comp, (x+I*y, x-I*y))
>>> comp_to_cart = cart_to_comp.inverse()
>>> E = M.vector_bundle(Integer(1), 'gamma^1', field='complex')
>>> e = E.local_frame('e', domain=U)
```

To apply the Chern-Weil approach, we need a bundle connection in terms of a connection one form. To achieve this, we take the connection induced from the hermitian metric on the trivial bundle $\mathbf{C}^2 \times \mathbf{CP}^1 \supset \gamma^1$. In this the frame e corresponds to the section $[z:1] \mapsto (z,1)$ and its magnitude-squared is given by $1+|z|^2$:

```
sage: nab = E.bundle_connection('nabla')
sage: omega = U.one_form(name='omega')
sage: omega[c_comp.frame(),1,c_comp] = zbar/(1+z*zbar)
sage: nab[e, 1, 1] = omega
sage: nab.set_immutable()
```

```
>>> from sage.all import *
>>> nab = E.bundle_connection('nabla')
>>> omega = U.one_form(name='omega')
>>> omega[c_comp.frame(),Integer(1),c_comp] = zbar/(Integer(1)+z*zbar)
>>> nab[e, Integer(1), Integer(1)] = omega
>>> nab.set_immutable()
```

Now, the Chern class can be constructed:

```
sage: c = E.characteristic_cohomology_class('Chern'); c
Characteristic cohomology class c(gamma^1) of the Differentiable complex
 vector bundle gamma^1 -> CP^1 of rank 1 over the base space 2-dimensional
 differentiable manifold CP^1
sage: c_form = c.get_form(nab)
sage: c_form.display_expansion(c_comp.frame(), chart=c_comp)
c(gamma^1, nabla) = 1 + 1/2*I/(pi + pi*z^2*zbar^2 + 2*pi*z*zbar) dz∧dzbar
```

```
>>> from sage.all import *
>>> c = E.characteristic_cohomology_class('Chern'); c
Characteristic cohomology class c(gamma^1) of the Differentiable complex
 vector bundle gamma^1 -> CP^1 of rank 1 over the base space 2-dimensional
 differentiable manifold CP^1
>>> c_form = c.get_form(nab)
>>> c_form.display_expansion(c_comp.frame(), chart=c_comp)
c(gamma^1, nabla) = 1 + 1/2*I/(pi + pi*z^2*zbar^2 + 2*pi*z*zbar) dz∧dzbar
```

Since U and $\mathbf{CP}^1$ differ only by a point and therefore a null set, it is enough to integrate the top form over the domain U:

```
sage: integrate(integrate(c_form[2][[1,2]].expr(c_cart), x, -infinity, infinity).full_
↪simplify(),
....:           y, -infinity, infinity)
1
```

```
>>> from sage.all import *
>>> integrate(integrate(c_form[Integer(2)][[Integer(1),Integer(2)]].expr(c_cart), x, -
↪infinity, infinity).full_simplify(),
...           y, -infinity, infinity)
1
```

The result shows that $c_1(\gamma^1)$ generates the second integer cohomology of $\mathbf{CP}^1$.

Pfaffian Classes

Usually, there is no such thing as "Pfaffian classes" in literature. However, using the matrix' Pfaffian and inspired by the aforementioned definitions, such classes can be defined as follows.

Let E be a real vector bundle of rank $2n$ and f an odd real function being analytic at zero. Furthermore, let Ω be skew-symmetric, which certainly will be true if ∇ is metric and e is orthonormal. Then we call

$$\left[\mathrm{Pf}\left(f\left(\frac{\Omega}{2\pi}\right)\right)\right] \in H^{2n*}(M, \mathbf{R})$$

the *Pfaffian class associated to* f.

The most important Pfaffian class is the *Euler class* which is simply given by $f(x) = x$.

EXAMPLES:

We consider the **Euler class** of S^2:

```
sage: M.<x,y> = manifolds.Sphere(2, coordinates='stereographic')
sage: TM = M.tangent_bundle()
sage: e_class = TM.characteristic_cohomology_class('Euler'); e_class
Characteristic cohomology class e(TS^2) of the Tangent bundle TS^2 over the
 2-sphere S^2 of radius 1 smoothly embedded in the Euclidean space E^3
```

```
>>> from sage.all import *
>>> M = manifolds.Sphere(Integer(2), coordinates='stereographic', names=('x', 'y',));
↪(x, y,) = M._first_ngens(2)
>>> TM = M.tangent_bundle()
>>> e_class = TM.characteristic_cohomology_class('Euler'); e_class
Characteristic cohomology class e(TS^2) of the Tangent bundle TS^2 over the
 2-sphere S^2 of radius 1 smoothly embedded in the Euclidean space E^3
```

To compute a particular representative of the Euler class, we need to determine a connection, which is in this case given by the standard metric:

```
sage: g = M.metric('g') # standard metric on S2, long time
sage: nab = g.connection() # long time
sage: nab.set_immutable() # long time
```

```
>>> from sage.all import *
>>> g = M.metric('g') # standard metric on S2, long time
>>> nab = g.connection() # long time
>>> nab.set_immutable() # long time
```

Now the representative of the Euler class with respect to the connection ∇_g induced by the standard metric can be computed:

```
sage: e_class_form = e_class.get_form(nab) # long time
sage: e_class_form.display_expansion() # long time
e(TS^2, nabla_g) = 2/(pi + pi*x^4 + pi*y^4 + 2*pi*x^2 + 2*(pi + pi*x^2)*y^2) dx∧dy
```

```
>>> from sage.all import *
>>> e_class_form = e_class.get_form(nab) # long time
>>> e_class_form.display_expansion() # long time
e(TS^2, nabla_g) = 2/(pi + pi*x^4 + pi*y^4 + 2*pi*x^2 + 2*(pi + pi*x^2)*y^2) dx∧dy
```

Let us check whether this form represents the Euler class correctly:

```
sage: # long time
sage: expr = e_class_form[2][[1,2]].expr()
sage: expr = integrate(expr, x, -infinity, infinity)
sage: expr = expr.simplify_full()
sage: integrate(expr, y, -infinity, infinity)
2
```

```
>>> from sage.all import *
>>> # long time
>>> expr = e_class_form[Integer(2)][[Integer(1),Integer(2)]].expr()
>>> expr = integrate(expr, x, -infinity, infinity)
>>> expr = expr.simplify_full()
>>> integrate(expr, y, -infinity, infinity)
2
```

As we can see, the integral coincides with the Euler characteristic of S^2 so that our form actually represents the Euler class appropriately.

class sage.manifolds.differentiable.characteristic_cohomology_class.**Algorithm_generic**

Bases: SageObject

Abstract algorithm class to compute the characteristic forms of the generators.

EXAMPLES:

```
sage: from sage.manifolds.differentiable.characteristic_cohomology_class import
↪Algorithm_generic
sage: algorithm = Algorithm_generic()
sage: algorithm.get
Cached version of <function Algorithm_generic.get at 0x...>
sage: algorithm.get_local
Traceback (most recent call last):
...
NotImplementedError: <abstract method get_local at 0x...>
sage: algorithm.get_gen_pow
Cached version of <function Algorithm_generic.get_gen_pow at 0x...>
```

```
>>> from sage.all import *
>>> from sage.manifolds.differentiable.characteristic_cohomology_class import
↪Algorithm_generic
>>> algorithm = Algorithm_generic()
>>> algorithm.get
Cached version of <function Algorithm_generic.get at 0x...>
>>> algorithm.get_local
Traceback (most recent call last):
...
NotImplementedError: <abstract method get_local at 0x...>
>>> algorithm.get_gen_pow
Cached version of <function Algorithm_generic.get_gen_pow at 0x...>
```

get(*nab*)

Return the global characteristic forms of the generators w.r.t. a given connection.

The result is cached.

OUTPUT:

- a list containing the generator's global characteristic forms as instances of `sage.manifolds.differentiable.diff_form.DiffForm`

EXAMPLES:

```
sage: M = manifolds.EuclideanSpace(4)
sage: TM = M.tangent_bundle()
sage: g = M.metric()
sage: nab = g.connection()
sage: nab.set_immutable()
```

```
>>> from sage.all import *
>>> M = manifolds.EuclideanSpace(Integer(4))
>>> TM = M.tangent_bundle()
>>> g = M.metric()
>>> nab = g.connection()
>>> nab.set_immutable()
```

```
sage: p = TM.characteristic_cohomology_class('Pontryagin')
sage: p_form = p.get_form(nab); p_form # long time
Mixed differential form p(TE^4, nabla_g) on the 4-dimensional
 Euclidean space E^4
sage: p_form.display_expansion() # long time
p(TE^4, nabla_g) = 1
```

```
>>> from sage.all import *
>>> p = TM.characteristic_cohomology_class('Pontryagin')
>>> p_form = p.get_form(nab); p_form # long time
Mixed differential form p(TE^4, nabla_g) on the 4-dimensional
 Euclidean space E^4
>>> p_form.display_expansion() # long time
p(TE^4, nabla_g) = 1
```

get_gen_pow(*nab*, *i*, *n*)

Return the n-th power of the i-th generator's characteristic form w.r.t `nab`.

The result is cached.

EXAMPLES:

```
sage: M = manifolds.EuclideanSpace(8)
sage: TM = M.tangent_bundle()
sage: g = M.metric()
sage: nab = g.connection()
sage: nab.set_immutable()
```

```
>>> from sage.all import *
>>> M = manifolds.EuclideanSpace(Integer(8))
>>> TM = M.tangent_bundle()
>>> g = M.metric()
>>> nab = g.connection()
>>> nab.set_immutable()
```

```
sage: A = TM.characteristic_cohomology_class('AHat')
sage: A_form = A.get_form(nab); A_form  # long time
Mixed differential form A^(TE^8, nabla_g) on the 8-dimensional
 Euclidean space E^8
sage: A_form.display_expansion()  # long time
A^(TE^8, nabla_g) = 1
```

```
>>> from sage.all import *
>>> A = TM.characteristic_cohomology_class('AHat')
>>> A_form = A.get_form(nab); A_form  # long time
Mixed differential form A^(TE^8, nabla_g) on the 8-dimensional
 Euclidean space E^8
>>> A_form.display_expansion()  # long time
A^(TE^8, nabla_g) = 1
```

get_local(*cmat*)

Abstract method to get the local forms of the generators w.r.t. a given curvature form matrix `cmat`.

OUTPUT: list containing the generator's local characteristic forms

ALGORITHM:

The inherited class determines the algorithm.

EXAMPLES:

4-dimensional Euclidean space:

```
sage: M = manifolds.EuclideanSpace(4)
sage: TM = M.tangent_bundle()
sage: g = M.metric()
sage: nab = g.connection()
sage: e = M.frames()[0]  # select standard frame
sage: cmat = [ [nab.curvature_form(i, j, e)  # long time
....:            for j in TM.irange()]
....:           for i in TM.irange()]
```

```
>>> from sage.all import *
>>> M = manifolds.EuclideanSpace(Integer(4))
>>> TM = M.tangent_bundle()
>>> g = M.metric()
>>> nab = g.connection()
>>> e = M.frames()[Integer(0)]  # select standard frame
>>> cmat = [ [nab.curvature_form(i, j, e)  # long time
...            for j in TM.irange()]
...           for i in TM.irange()]
```

Import the algorithm:

```
sage: from sage.manifolds.differentiable.characteristic_cohomology_class␣
↪import PontryaginAlgorithm
sage: algorithm = PontryaginAlgorithm()
sage: [p1] = algorithm.get_local(cmat) # long time
sage: p1.display() # long time
0
```

```
>>> from sage.all import *
>>> from sage.manifolds.differentiable.characteristic_cohomology_class import␣
↪PontryaginAlgorithm
>>> algorithm = PontryaginAlgorithm()
>>> [p1] = algorithm.get_local(cmat) # long time
>>> p1.display() # long time
0
```

A concrete implementation is given by a `sage.misc.fast_methods.Singleton`:

```
sage: algorithm is PontryaginAlgorithm()
True
```

```
>>> from sage.all import *
>>> algorithm is PontryaginAlgorithm()
True
```

class `sage.manifolds.differentiable.characteristic_cohomology_class.`**`CharacteristicCohomologyClassRin`**

Bases: `FiniteGCAlgebra`

Characteristic cohomology class ring.

Let $E \to M$ be a real or complex vector bundle of rank k and R be a torsion-free subring of $\mathbf{C}$.

Let BG be the classifying space of the group G. As for vector bundles, we consider

- $G = O(k)$ if E is real,
- $G = SO(k)$ if E is real and oriented,
- $G = U(k)$ if E is complex.

The cohomology ring $H^*(BG; R)$ can be explicitly expressed for the aforementioned cases:

$$H^*(BG; R) \cong \begin{cases} R[c_1, \ldots c_k] & \text{if } G = U(k), \\ R[p_1, \ldots p_{\lfloor \frac{k}{2} \rfloor}] & \text{if } G = O(k), \\ R[p_1, \ldots p_k, e]/(p_k - e^2) & \text{if } G = SO(2k), \\ R[p_1, \ldots p_k, e] & \text{if } G = SO(2k+1). \end{cases}$$

The Chern-Weil homomorphism relates the generators in the de Rham cohomology as follows. If Ω is a curvature form matrix on E, for the Chern classes we get

$$\left[\det\left(1 + \frac{t\Omega}{2\pi i}\right)\right] = 1 + \sum_{n=1}^{k} c_n(E) t^n,$$

for the Pontryagin classes we have

$$\left[\det\left(1 - \frac{t\Omega}{2\pi}\right)\right] = 1 + \sum_{n=1}^{\lfloor \frac{k}{2} \rfloor} p_n(E) t^n,$$

and for the Euler class we obtain

$$\left[\mathrm{Pf}\left(\frac{\Omega}{2\pi}\right)\right] = e(E).$$

Consequently, the cohomology ring $H^*(BG; R)$ is mapped (not necessarily injectively) to a subring of $H^*_{\mathrm{dR}}(M, \mathbf{C})$ via the Chern-Weil homomorphism. This implementation attempts to represent this subring.

Note

Some generators might have torsion in $H^*(BG; R)$ giving a zero element in the de Rham cohomology. Those generators are still considered in the implementation. Generators whose degree exceed the dimension of the base space, however, are ignored.

INPUT:

- `base` – base ring
- `vbundle` – vector bundle

EXAMPLES:

Characteristic cohomology class ring over the tangent bundle of an 8-dimensional manifold:

```
sage: M = Manifold(8, 'M')
sage: TM = M.tangent_bundle()
sage: CR = TM.characteristic_cohomology_class_ring(); CR
Algebra of characteristic cohomology classes of the Tangent bundle TM
 over the 8-dimensional differentiable manifold M
sage: CR.gens()
(Characteristic cohomology class (p_1)(TM) of the Tangent bundle TM over
 the 8-dimensional differentiable manifold M,
 Characteristic cohomology class (p_2)(TM) of the Tangent bundle TM
 over the 8-dimensional differentiable manifold M)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(8), 'M')
>>> TM = M.tangent_bundle()
>>> CR = TM.characteristic_cohomology_class_ring(); CR
Algebra of characteristic cohomology classes of the Tangent bundle TM
 over the 8-dimensional differentiable manifold M
>>> CR.gens()
(Characteristic cohomology class (p_1)(TM) of the Tangent bundle TM over
 the 8-dimensional differentiable manifold M,
 Characteristic cohomology class (p_2)(TM) of the Tangent bundle TM
 over the 8-dimensional differentiable manifold M)
```

The default base ring is $\mathbf{Q}$:

```
sage: CR.base_ring()
Rational Field
```

```
>>> from sage.all import *
>>> CR.base_ring()
Rational Field
```

Characteristic cohomology class ring over a complex vector bundle:

```
sage: M = Manifold(4, 'M')
sage: E = M.vector_bundle(2, 'E', field='complex')
sage: CR_E = E.characteristic_cohomology_class_ring(); CR_E
Algebra of characteristic cohomology classes of the Differentiable
 complex vector bundle E -> M of rank 2 over the base space
 4-dimensional differentiable manifold M
sage: CR_E.gens()
(Characteristic cohomology class (c_1)(E) of the Differentiable complex
 vector bundle E -> M of rank 2 over the base space 4-dimensional
 differentiable manifold M,
 Characteristic cohomology class (c_2)(E) of the Differentiable
 complex vector bundle E -> M of rank 2 over the base space
 4-dimensional differentiable manifold M)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M')
>>> E = M.vector_bundle(Integer(2), 'E', field='complex')
>>> CR_E = E.characteristic_cohomology_class_ring(); CR_E
Algebra of characteristic cohomology classes of the Differentiable
 complex vector bundle E -> M of rank 2 over the base space
 4-dimensional differentiable manifold M
>>> CR_E.gens()
(Characteristic cohomology class (c_1)(E) of the Differentiable complex
 vector bundle E -> M of rank 2 over the base space 4-dimensional
 differentiable manifold M,
 Characteristic cohomology class (c_2)(E) of the Differentiable
 complex vector bundle E -> M of rank 2 over the base space
 4-dimensional differentiable manifold M)
```

Characteristic cohomology class ring over an oriented manifold:

```
sage: S2 = manifolds.Sphere(2, coordinates='stereographic')
sage: TS2 = S2.tangent_bundle()
sage: S2.has_orientation()
True
sage: CR = TS2.characteristic_cohomology_class_ring()
sage: CR.gens()
(Characteristic cohomology class (e)(TS^2) of the Tangent bundle TS^2
 over the 2-sphere S^2 of radius 1 smoothly embedded in the Euclidean
 space E^3,)
```

```
>>> from sage.all import *
>>> S2 = manifolds.Sphere(Integer(2), coordinates='stereographic')
>>> TS2 = S2.tangent_bundle()
>>> S2.has_orientation()
True
>>> CR = TS2.characteristic_cohomology_class_ring()
>>> CR.gens()
(Characteristic cohomology class (e)(TS^2) of the Tangent bundle TS^2
 over the 2-sphere S^2 of radius 1 smoothly embedded in the Euclidean
 space E^3,)
```

Element

alias of `CharacteristicCohomologyClassRingElement`

class sage.manifolds.differentiable.characteristic_cohomology_class.**CharacteristicCohomologyClassRing**

Bases: `IndexedFreeModuleElement`

Characteristic cohomology class.

Let $E \to M$ be a real/complex vector bundle of rank k. A characteristic cohomology class of E is generated by either

- Chern classes if E is complex,
- Pontryagin classes if E is real,
- Pontryagin classes and the Euler class if E is real and orientable,

via the Chern-Weil homomorphism.

For details about the ring structure, see `CharacteristicCohomologyClassRing`.

To construct a characteristic cohomology class, please use `CharacteristicCohomologyClass()`.

EXAMPLES:

```
sage: M = Manifold(12, 'M')
sage: TM = M.tangent_bundle()
sage: CR = TM.characteristic_cohomology_class_ring()
sage: p1, p2, p3 = CR.gens()
sage: p1*p2+p3
Characteristic cohomology class (p_1⌣p_2 + p_3)(TM) of the Tangent
 bundle TM over the 12-dimensional differentiable manifold M
sage: A = TM.characteristic_cohomology_class('AHat'); A
Characteristic cohomology class A^(TM) of the Tangent bundle TM over
 the 12-dimensional differentiable manifold M
sage: A == 1 - p1/24 + (7*p1^2-4*p2)/5760 + (44*p1*p2-31*p1^3-16*p3)/967680
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(12), 'M')
>>> TM = M.tangent_bundle()
>>> CR = TM.characteristic_cohomology_class_ring()
>>> p1, p2, p3 = CR.gens()
>>> p1*p2+p3
Characteristic cohomology class (p_1⌣p_2 + p_3)(TM) of the Tangent
 bundle TM over the 12-dimensional differentiable manifold M
>>> A = TM.characteristic_cohomology_class('AHat'); A
Characteristic cohomology class A^(TM) of the Tangent bundle TM over
 the 12-dimensional differentiable manifold M
>>> A == Integer(1) - p1/Integer(24) + (Integer(7)*p1**Integer(2)-Integer(4)*p2)/
↪Integer(5760) + (Integer(44)*p1*p2-Integer(31)*p1**Integer(3)-Integer(16)*p3)/
↪Integer(967680)
True
```

get_form(*nab*)

Return the characteristic form of `self`.

INPUT:

- `nab` – get the characteristic form w.r.t. to the connection `nab`

OUTPUT:

- an instance of *sage.manifolds.differentiable.mixed_form.MixedForm*

EXAMPLES:

Trivial characteristic form on Euclidean space:

```
sage: M = manifolds.EuclideanSpace(4)
sage: TM = M.tangent_bundle()
sage: one = TM.characteristic_cohomology_class_ring().one()
sage: g = M.metric()
sage: nab = g.connection()
sage: nab.set_immutable()
sage: one.get_form(nab)
Mixed differential form one on the 4-dimensional Euclidean space E^4
```

```
>>> from sage.all import *
>>> M = manifolds.EuclideanSpace(Integer(4))
>>> TM = M.tangent_bundle()
>>> one = TM.characteristic_cohomology_class_ring().one()
>>> g = M.metric()
>>> nab = g.connection()
>>> nab.set_immutable()
>>> one.get_form(nab)
Mixed differential form one on the 4-dimensional Euclidean space E^4
```

Pontryagin form on the 4-sphere:

```
sage: M = manifolds.Sphere(4)
sage: TM = M.tangent_bundle()
sage: p = TM.characteristic_cohomology_class('Pontryagin'); p
Characteristic cohomology class p(TS^4) of the Tangent bundle TS^4
 over the 4-sphere S^4 of radius 1 smoothly embedded in the
 5-dimensional Euclidean space E^5
sage: g = M.metric() # long time
sage: nab = g.connection() # long time
sage: nab.set_immutable() # long time
sage: p_form = p.get_form(nab); p_form # long time
Mixed differential form p(TS^4, nabla_g) on the 4-sphere S^4 of
 radius 1 smoothly embedded in the 5-dimensional Euclidean space E^5
sage: p_form.display_expansion() # long time
p(TS^4, nabla_g) = 1
```

```
>>> from sage.all import *
>>> M = manifolds.Sphere(Integer(4))
>>> TM = M.tangent_bundle()
>>> p = TM.characteristic_cohomology_class('Pontryagin'); p
Characteristic cohomology class p(TS^4) of the Tangent bundle TS^4
```

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```
 over the 4-sphere S^4 of radius 1 smoothly embedded in the
 5-dimensional Euclidean space E^5
>>> g = M.metric() # long time
>>> nab = g.connection() # long time
>>> nab.set_immutable() # long time
>>> p_form = p.get_form(nab); p_form # long time
Mixed differential form p(TS^4, nabla_g) on the 4-sphere S^4 of
 radius 1 smoothly embedded in the 5-dimensional Euclidean space E^5
>>> p_form.display_expansion() # long time
p(TS^4, nabla_g) = 1
```

representative(*nab=None*)

Return any representative of `self`.

INPUT:

- `nab` – (default: `None`) if stated, return the representative w.r.t. to the connection `nab`; otherwise an arbitrary already computed representative will be chosen.

OUTPUT:

- an instance of *sage.manifolds.differentiable.mixed_form.MixedForm*

EXAMPLES:

Define the 4-dimensional Euclidean space:

```
sage: M = manifolds.EuclideanSpace(4)
sage: TM = M.tangent_bundle()
sage: one = TM.characteristic_cohomology_class_ring().one()
```

```
>>> from sage.all import *
>>> M = manifolds.EuclideanSpace(Integer(4))
>>> TM = M.tangent_bundle()
>>> one = TM.characteristic_cohomology_class_ring().one()
```

No characteristic form has been computed so far, thus we get an error:

```
sage: one.representative()
Traceback (most recent call last):
...
AttributeError: cannot pick a representative
```

```
>>> from sage.all import *
>>> one.representative()
Traceback (most recent call last):
...
AttributeError: cannot pick a representative
```

Get a characteristic form:

```
sage: g = M.metric()
sage: nab = g.connection()
sage: nab.set_immutable()
```

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```
sage: one.get_form(nab)
Mixed differential form one on the 4-dimensional Euclidean space E^4
```

```
>>> from sage.all import *
>>> g = M.metric()
>>> nab = g.connection()
>>> nab.set_immutable()
>>> one.get_form(nab)
Mixed differential form one on the 4-dimensional Euclidean space E^4
```

Now, the result is cached and *representative* returns a form:

```
sage: one.representative()
Mixed differential form one on the 4-dimensional Euclidean space E^4
```

```
>>> from sage.all import *
>>> one.representative()
Mixed differential form one on the 4-dimensional Euclidean space E^4
```

Alternatively, the option `nab` can be used to return the characteristic form w.r.t. a fixed connection:

```
sage: one.representative(nab)
Mixed differential form one on the 4-dimensional Euclidean space E^4
```

```
>>> from sage.all import *
>>> one.representative(nab)
Mixed differential form one on the 4-dimensional Euclidean space E^4
```

See also

CharacteristicCohomologyClassRingElement.get_form()

set_name(*name=None*, *latex_name=None*)

Set (or change) the text name and LaTeX name of the characteristic class.

INPUT:

- `name` – (string; default: `None`) name given to the characteristic cohomology class
- `latex_name` – (string; default: `None`) LaTeX symbol to denote the characteristic cohomology class; if `None` while `name` is provided, the LaTeX symbol is set to `name`

EXAMPLES:

```
sage: M = Manifold(8, 'M')
sage: TM = M.tangent_bundle()
sage: x = var('x')
sage: k = TM.characteristic_cohomology_class(1+x^2,
....:                                        class_type='multiplicative'); k
Characteristic cohomology class (1 + p_1^2 - 2*p_2)(TM) of the
 Tangent bundle TM over the 8-dimensional differentiable manifold M
sage: k.set_name(name='k', latex_name=r'\kappa')
```

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```
sage: k
Characteristic cohomology class k(TM) of the Tangent bundle TM over
 the 8-dimensional differentiable manifold M
sage: latex(k)
\kappa\left(TM\right)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(8), 'M')
>>> TM = M.tangent_bundle()
>>> x = var('x')
>>> k = TM.characteristic_cohomology_class(Integer(1)+x**Integer(2),
...                                    class_type='multiplicative'); k
Characteristic cohomology class (1 + p_1^2 - 2*p_2)(TM) of the
 Tangent bundle TM over the 8-dimensional differentiable manifold M
>>> k.set_name(name='k', latex_name=r'\kappa')
>>> k
Characteristic cohomology class k(TM) of the Tangent bundle TM over
 the 8-dimensional differentiable manifold M
>>> latex(k)
\kappa\left(TM\right)
```

class sage.manifolds.differentiable.characteristic_cohomology_class.**ChernAlgorithm**

Bases: `Singleton`, *`Algorithm_generic`*

Algorithm class to generate Chern forms.

EXAMPLES:

Define a complex line bundle over a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='Lorentzian')
sage: X.<t,x> = M.chart()
sage: E = M.vector_bundle(1, 'E', field='complex'); E
Differentiable complex vector bundle E -> M of rank 1 over the base space
 2-dimensional Lorentzian manifold M
sage: e = E.local_frame('e')
sage: nab = E.bundle_connection('nabla^E', latex_name=r'\nabla^E')
sage: omega = M.one_form(name='omega')
sage: A = function('A')
sage: nab.set_connection_form(0, 0)[1] = I*A(t)
sage: nab[0, 0].display()
connection (0,0) of bundle connection nabla^E w.r.t. Local frame
 (E|_M, (e_0)) = I*A(t) dx
sage: nab.set_immutable()
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x',)); (t, x,) = X._first_ngens(2)
>>> E = M.vector_bundle(Integer(1), 'E', field='complex'); E
Differentiable complex vector bundle E -> M of rank 1 over the base space
 2-dimensional Lorentzian manifold M
>>> e = E.local_frame('e')
>>> nab = E.bundle_connection('nabla^E', latex_name=r'\nabla^E')
```

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```
>>> omega = M.one_form(name='omega')
>>> A = function('A')
>>> nab.set_connection_form(Integer(0), Integer(0))[Integer(1)] = I*A(t)
>>> nab[Integer(0), Integer(0)].display()
connection (0,0) of bundle connection nabla^E w.r.t. Local frame
 (E|_M, (e_0)) = I*A(t) dx
>>> nab.set_immutable()
```

Import the algorithm and apply `nab` to it:

```
sage: from sage.manifolds.differentiable.characteristic_cohomology_class import␣
↪ChernAlgorithm
sage: algorithm = ChernAlgorithm()
sage: algorithm.get(nab)
[2-form on the 2-dimensional Lorentzian manifold M]
sage: algorithm.get(nab)[0].display()
1/2*d(A)/dt/pi dt∧dx
```

```
>>> from sage.all import *
>>> from sage.manifolds.differentiable.characteristic_cohomology_class import␣
↪ChernAlgorithm
>>> algorithm = ChernAlgorithm()
>>> algorithm.get(nab)
[2-form on the 2-dimensional Lorentzian manifold M]
>>> algorithm.get(nab)[Integer(0)].display()
1/2*d(A)/dt/pi dt∧dx
```

Check some equalities:

```
sage: cmat = [[nab.curvature_form(0, 0, e)]]
sage: algorithm.get(nab)[0] == algorithm.get_local(cmat)[0]  # bundle trivial
True
sage: algorithm.get_gen_pow(nab, 0, 1) == algorithm.get(nab)[0]
True
```

```
>>> from sage.all import *
>>> cmat = [[nab.curvature_form(Integer(0), Integer(0), e)]]
>>> algorithm.get(nab)[Integer(0)] == algorithm.get_local(cmat)[Integer(0)]  #␣
↪bundle trivial
True
>>> algorithm.get_gen_pow(nab, Integer(0), Integer(1)) == algorithm.
↪get(nab)[Integer(0)]
True
```

get_local(*cmat*)

Return the local Chern forms w.r.t. a given curvature form matrix.

OUTPUT:

- a list containing the local characteristic Chern forms as instances of *sage.manifolds.differentiable.diff_form.DiffForm*

ALGORITHM:

```
The algorithm is based on the Faddeev-LeVerrier algorithm for the
characteristic polynomial.
```

EXAMPLES:

Define a complex line bundle over a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='Lorentzian')
sage: X.<t,x> = M.chart()
sage: E = M.vector_bundle(1, 'E', field='complex'); E
Differentiable complex vector bundle E -> M of rank 1 over the base
 space 2-dimensional Lorentzian manifold M
sage: e = E.local_frame('e')
sage: nab = E.bundle_connection('nabla^E', latex_name=r'\nabla^E')
sage: omega = M.one_form(name='omega')
sage: A = function('A')
sage: nab.set_connection_form(0, 0)[1] = I*A(t)
sage: nab[0, 0].display()
connection (0,0) of bundle connection nabla^E w.r.t. Local frame
 (E|_M, (e_0)) = I*A(t) dx
sage: cmat = [[nab.curvature_form(i, j, e) for j in E.irange()]
....:          for i in E.irange()]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x',)); (t, x,) = X._first_ngens(2)
>>> E = M.vector_bundle(Integer(1), 'E', field='complex'); E
Differentiable complex vector bundle E -> M of rank 1 over the base
 space 2-dimensional Lorentzian manifold M
>>> e = E.local_frame('e')
>>> nab = E.bundle_connection('nabla^E', latex_name=r'\nabla^E')
>>> omega = M.one_form(name='omega')
>>> A = function('A')
>>> nab.set_connection_form(Integer(0), Integer(0))[Integer(1)] = I*A(t)
>>> nab[Integer(0), Integer(0)].display()
connection (0,0) of bundle connection nabla^E w.r.t. Local frame
 (E|_M, (e_0)) = I*A(t) dx
>>> cmat = [[nab.curvature_form(i, j, e) for j in E.irange()]
...          for i in E.irange()]
```

Import the algorithm and apply `cmat` to it:

```
sage: from sage.manifolds.differentiable.characteristic_cohomology_class␣
↪import ChernAlgorithm
sage: algorithm = ChernAlgorithm()
sage: algorithm.get_local(cmat)
[2-form on the 2-dimensional Lorentzian manifold M]
```

```
>>> from sage.all import *
>>> from sage.manifolds.differentiable.characteristic_cohomology_class import␣
↪ChernAlgorithm
>>> algorithm = ChernAlgorithm()
>>> algorithm.get_local(cmat)
```

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```
[2-form on the 2-dimensional Lorentzian manifold M]
```

class sage.manifolds.differentiable.characteristic_cohomology_class.**EulerAlgorithm**

Bases: Singleton, *Algorithm_generic*

Algorithm class to generate Euler forms.

EXAMPLES:

Consider the 2-dimensional Euclidean space:

```
sage: M = manifolds.EuclideanSpace(2)
sage: g = M.metric()
sage: nab = g.connection()
sage: nab.set_immutable()
```

```
>>> from sage.all import *
>>> M = manifolds.EuclideanSpace(Integer(2))
>>> g = M.metric()
>>> nab = g.connection()
>>> nab.set_immutable()
```

Import the algorithm and apply nab to it:

```
sage: from sage.manifolds.differentiable.characteristic_cohomology_class import
↪EulerAlgorithm
sage: algorithm = EulerAlgorithm()
sage: algorithm.get(nab)
[2-form on the Euclidean plane E^2]
sage: algorithm.get(nab)[0].display()
0
```

```
>>> from sage.all import *
>>> from sage.manifolds.differentiable.characteristic_cohomology_class import
↪EulerAlgorithm
>>> algorithm = EulerAlgorithm()
>>> algorithm.get(nab)
[2-form on the Euclidean plane E^2]
>>> algorithm.get(nab)[Integer(0)].display()
0
```

get(*nab*)

Return the global characteristic forms of the generators w.r.t. a given connection.

INPUT:

- nab – a metric connection ∇

OUTPUT: list containing the global characteristic Euler form

ALGORITHM:

Assume that ∇ is compatible with the metric g, and let $(s_1, \dots, s_n)$ be any oriented frame. Denote by $G_s = (g(s_i, s_j))_{ij}$ the metric tensor and let Ω_s be the curvature form matrix of ∇ w.r.t. s. Then, we get:

$$(G_s \cdot \Omega_s)_{ij} = g(R(.,.)s_i, s_j)\,,$$

where R is the Riemannian curvature tensor w.r.t. ∇.

The characteristic Euler form is now obtained by the expression

$$\frac{1}{\sqrt{|\det(G_s)|}} \operatorname{Pf}\left(G_s \cdot \frac{\Omega_s}{2\pi}\right).$$

EXAMPLES:

Consider the 2-sphere:

```
sage: M.<x,y> = manifolds.Sphere(2, coordinates='stereographic')
sage: g = M.metric() # long time
sage: nab = g.connection() # long time
sage: nab.set_immutable() # long time
```

```
>>> from sage.all import *
>>> M = manifolds.Sphere(Integer(2), coordinates='stereographic', names=('x',
↪'y',)); (x, y,) = M._first_ngens(2)
>>> g = M.metric() # long time
>>> nab = g.connection() # long time
>>> nab.set_immutable() # long time
```

Import the algorithm and apply `nab` to it:

```
sage: from sage.manifolds.differentiable.characteristic_cohomology_class␣
↪import EulerAlgorithm
sage: algorithm = EulerAlgorithm()
sage: algorithm.get(nab) # long time
[2-form on the 2-sphere S^2 of radius 1 smoothly embedded in the
 Euclidean space E^3]
sage: algorithm.get(nab)[0].display() # long time
2/(pi + pi*x^4 + pi*y^4 + 2*pi*x^2 + 2*(pi + pi*x^2)*y^2) dx∧dy
```

```
>>> from sage.all import *
>>> from sage.manifolds.differentiable.characteristic_cohomology_class import␣
↪EulerAlgorithm
>>> algorithm = EulerAlgorithm()
>>> algorithm.get(nab) # long time
[2-form on the 2-sphere S^2 of radius 1 smoothly embedded in the
 Euclidean space E^3]
>>> algorithm.get(nab)[Integer(0)].display() # long time
2/(pi + pi*x^4 + pi*y^4 + 2*pi*x^2 + 2*(pi + pi*x^2)*y^2) dx∧dy
```

REFERENCES:

- [Che1944]
- [Baer2020]

get_local(*cmat*)

Return the normalized Pfaffian w.r.t. a given curvature form matrix.

The normalization is given by the factor $\left(\frac{1}{2\pi}\right)^{\frac{k}{2}}$, where k is the dimension of the curvature matrix.

OUTPUT: list containing the normalized Pfaffian of a given curvature form

Note

In general, the output does *not* represent the local characteristic Euler form. The result is only guaranteed to be the local Euler form if `cmat` is given w.r.t. an orthonormal oriented frame. See `get()` for details.

ALGORITHM:

```
The algorithm is based on the Bär-Faddeev-LeVerrier algorithm for
the Pfaffian.
```

EXAMPLES:

Consider the 2-sphere:

```
sage: M.<th,phi> = manifolds.Sphere(2)  # use spherical coordinates
sage: TM = M.tangent_bundle()
sage: g = M.metric()
sage: nab = g.connection()
sage: e = M.frames()[0]  # select frame (opposite orientation!)
sage: cmat = [[nab.curvature_form(i, j, e) for j in TM.irange()]
....:         for i in TM.irange()]
```

```
>>> from sage.all import *
>>> M = manifolds.Sphere(Integer(2), names=('th', 'phi',)); (th, phi,) = M._
↪first_ngens(2)# use spherical coordinates
>>> TM = M.tangent_bundle()
>>> g = M.metric()
>>> nab = g.connection()
>>> e = M.frames()[Integer(0)]  # select frame (opposite orientation!)
>>> cmat = [[nab.curvature_form(i, j, e) for j in TM.irange()]
...         for i in TM.irange()]
```

We need some preprocessing because the frame is not orthonormal:

```
sage: gcmat = [[sum(g[[e, i, j]] * nab.curvature_form(j, k, e)
....:               for j in TM.irange())
....:           for k in TM.irange()] for i in TM.irange()]
```

```
>>> from sage.all import *
>>> gcmat = [[sum(g[[e, i, j]] * nab.curvature_form(j, k, e)
...               for j in TM.irange())
...           for k in TM.irange()] for i in TM.irange()]
```

Now, `gcmat` is guaranteed to be skew-symmetric and can be applied to `get_local()`. Then, the result must be normalized:

```
sage: from sage.manifolds.differentiable.characteristic_cohomology_class␣
↪import EulerAlgorithm
sage: algorithm = EulerAlgorithm()
sage: euler = -algorithm.get_local(gcmat)[0] / sqrt(g.det(frame=e))
sage: euler.display()
1/2*sin(th)/pi dth∧dphi
```

```
>>> from sage.all import *
>>> from sage.manifolds.differentiable.characteristic_cohomology_class import␣
↪EulerAlgorithm
>>> algorithm = EulerAlgorithm()
>>> euler = -algorithm.get_local(gcmat)[Integer(0)] / sqrt(g.det(frame=e))
>>> euler.display()
1/2*sin(th)/pi dth∧dphi
```

class sage.manifolds.differentiable.characteristic_cohomology_class.**PontryaginAlgorithm**

Bases: Singleton, *Algorithm_generic*

Algorithm class to generate Pontryagin forms.

EXAMPLES:

5-dimensional Euclidean space:

```
sage: M = manifolds.EuclideanSpace(5)
sage: g = M.metric()
sage: nab = g.connection()
sage: nab.set_immutable()
```

```
>>> from sage.all import *
>>> M = manifolds.EuclideanSpace(Integer(5))
>>> g = M.metric()
>>> nab = g.connection()
>>> nab.set_immutable()
```

Import the algorithm:

```
sage: from sage.manifolds.differentiable.characteristic_cohomology_class import␣
↪PontryaginAlgorithm
sage: algorithm = PontryaginAlgorithm()
sage: [p1] = algorithm.get(nab) # long time
sage: p1.display() # long time
0
```

```
>>> from sage.all import *
>>> from sage.manifolds.differentiable.characteristic_cohomology_class import␣
↪PontryaginAlgorithm
>>> algorithm = PontryaginAlgorithm()
>>> [p1] = algorithm.get(nab) # long time
>>> p1.display() # long time
0
```

get_local(*cmat*)

Return the local Pontryagin forms w.r.t. a given curvature form matrix.

OUTPUT: list containing the local characteristic Pontryagin forms

ALGORITHM:

```
The algorithm is based on the Faddeev-LeVerrier algorithm for the
characteristic polynomial.
```

EXAMPLES:

5-dimensional Euclidean space:

```
sage: M = manifolds.EuclideanSpace(5)
sage: TM = M.tangent_bundle()
sage: g = M.metric()
sage: nab = g.connection()
sage: e = M.frames()[0]  # select standard frame
sage: cmat = [ [nab.curvature_form(i, j, e) # long time
....:           for j in TM.irange()]
....:         for i in TM.irange()]
```

```
>>> from sage.all import *
>>> M = manifolds.EuclideanSpace(Integer(5))
>>> TM = M.tangent_bundle()
>>> g = M.metric()
>>> nab = g.connection()
>>> e = M.frames()[Integer(0)]  # select standard frame
>>> cmat = [ [nab.curvature_form(i, j, e) # long time
...           for j in TM.irange()]
...         for i in TM.irange()]
```

Import the algorithm:

```
sage: from sage.manifolds.differentiable.characteristic_cohomology_class␣
↪import PontryaginAlgorithm
sage: algorithm = PontryaginAlgorithm()
sage: [p1] = algorithm.get_local(cmat) # long time
sage: p1.display() # long time
0
```

```
>>> from sage.all import *
>>> from sage.manifolds.differentiable.characteristic_cohomology_class import␣
↪PontryaginAlgorithm
>>> algorithm = PontryaginAlgorithm()
>>> [p1] = algorithm.get_local(cmat) # long time
>>> p1.display() # long time
0
```

class sage.manifolds.differentiable.characteristic_cohomology_class.**PontryaginEulerAlgorithm**

Bases: `Singleton`, *`Algorithm_generic`*

Algorithm class to generate Euler and Pontryagin forms.

EXAMPLES:

6-dimensional Euclidean space:

```
sage: M = manifolds.EuclideanSpace(6)
sage: g = M.metric()
sage: nab = g.connection()
sage: nab.set_immutable()
```

```
>>> from sage.all import *
>>> M = manifolds.EuclideanSpace(Integer(6))
>>> g = M.metric()
>>> nab = g.connection()
>>> nab.set_immutable()
```

Import the algorithm:

```
sage: from sage.manifolds.differentiable.characteristic_cohomology_class import␣
↪PontryaginEulerAlgorithm
sage: algorithm = PontryaginEulerAlgorithm()
sage: e_form, p1_form = algorithm.get(nab)  # long time
sage: e_form.display()  # long time
0
sage: p1_form.display()  # long time
0
```

```
>>> from sage.all import *
>>> from sage.manifolds.differentiable.characteristic_cohomology_class import␣
↪PontryaginEulerAlgorithm
>>> algorithm = PontryaginEulerAlgorithm()
>>> e_form, p1_form = algorithm.get(nab)  # long time
>>> e_form.display()  # long time
0
>>> p1_form.display()  # long time
0
```

get(*nab*)

Return the global characteristic forms of the generators w.r.t. a given connection.

OUTPUT:

- a list containing the global Euler form in the first entry, and the global Pontryagin forms in the remaining entries.

EXAMPLES:

4-dimensional Euclidean space:

```
sage: M = manifolds.EuclideanSpace(4)
sage: g = M.metric()
sage: nab = g.connection()
sage: nab.set_immutable()
```

```
>>> from sage.all import *
>>> M = manifolds.EuclideanSpace(Integer(4))
>>> g = M.metric()
>>> nab = g.connection()
>>> nab.set_immutable()
```

Import the algorithm:

```
sage: from sage.manifolds.differentiable.characteristic_cohomology_class␣
↪import PontryaginEulerAlgorithm
```

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```
sage: algorithm = PontryaginEulerAlgorithm()
sage: algorithm.get(nab) # long time
[4-form on the 4-dimensional Euclidean space E^4,
 4-form on the 4-dimensional Euclidean space E^4]
```

```
>>> from sage.all import *
>>> from sage.manifolds.differentiable.characteristic_cohomology_class import␣
↪PontryaginEulerAlgorithm
>>> algorithm = PontryaginEulerAlgorithm()
>>> algorithm.get(nab) # long time
[4-form on the 4-dimensional Euclidean space E^4,
 4-form on the 4-dimensional Euclidean space E^4]
```

get_gen_pow(*nab*, *i*, *n*)

Return the n-th power of the i-th generator w.r.t `nab`.

The result is cached.

EXAMPLES:

4-dimensional Euclidean space:

```
sage: M = manifolds.EuclideanSpace(4)
sage: TM = M.tangent_bundle()
sage: g = M.metric()
sage: nab = g.connection()
sage: nab.set_immutable()
```

```
>>> from sage.all import *
>>> M = manifolds.EuclideanSpace(Integer(4))
>>> TM = M.tangent_bundle()
>>> g = M.metric()
>>> nab = g.connection()
>>> nab.set_immutable()
```

Import the algorithm:

```
sage: from sage.manifolds.differentiable.characteristic_cohomology_class␣
↪import PontryaginEulerAlgorithm
sage: algorithm = PontryaginEulerAlgorithm()
sage: e = algorithm.get_gen_pow(nab, 0, 1)  # Euler, long time
sage: e.display() # long time
0
sage: p1_pow2 = algorithm.get_gen_pow(nab, 1, 2)  # 1st Pontryagin squared,␣
↪long time
sage: p1_pow2 # long time
8-form zero on the 4-dimensional Euclidean space E^4
```

```
>>> from sage.all import *
>>> from sage.manifolds.differentiable.characteristic_cohomology_class import␣
↪PontryaginEulerAlgorithm
>>> algorithm = PontryaginEulerAlgorithm()
>>> e = algorithm.get_gen_pow(nab, Integer(0), Integer(1))  # Euler, long time
```

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```
>>> e.display() # long time
0
>>> p1_pow2 = algorithm.get_gen_pow(nab, Integer(1), Integer(2))  # 1st␣
↪Pontryagin squared, long time
>>> p1_pow2 # long time
8-form zero on the 4-dimensional Euclidean space E^4
```

get_local(*cmat*)

Return the local Euler and Pontryagin forms w.r.t. a given curvature form matrix.

> **Note**
>
> Similar as for `EulerAlgorithm`, the first entry only represents the Euler form if the curvature form matrix is chosen w.r.t. an orthonormal oriented frame. See `EulerAlgorithm.get_local()` for details.

OUTPUT:

- a list containing the local Euler form in the first entry, and the local Pontryagin forms in the remaining entries.

EXAMPLES:

6-dimensional Euclidean space:

```
sage: M = manifolds.EuclideanSpace(6)
sage: TM = M.tangent_bundle()
sage: g = M.metric()
sage: nab = g.connection()
sage: e = M.frames()[0]  # select the standard frame
sage: cmat = [ [nab.curvature_form(i, j, e)  # long time
....:           for j in TM.irange()]
....:          for i in TM.irange() ]
```

```
>>> from sage.all import *
>>> M = manifolds.EuclideanSpace(Integer(6))
>>> TM = M.tangent_bundle()
>>> g = M.metric()
>>> nab = g.connection()
>>> e = M.frames()[Integer(0)]  # select the standard frame
>>> cmat = [ [nab.curvature_form(i, j, e)  # long time
...           for j in TM.irange()]
...          for i in TM.irange() ]
```

Import the algorithm:

```
sage: from sage.manifolds.differentiable.characteristic_cohomology_class␣
↪import PontryaginEulerAlgorithm
sage: algorithm = PontryaginEulerAlgorithm()
sage: e, p1 = algorithm.get_local(cmat)  # long time
sage: e.display()  # long time
0
sage: p1.display()  # long time
0
```

```
>>> from sage.all import *
>>> from sage.manifolds.differentiable.characteristic_cohomology_class import␣
↪PontryaginEulerAlgorithm
>>> algorithm = PontryaginEulerAlgorithm()
>>> e, p1 = algorithm.get_local(cmat)  # long time
>>> e.display()  # long time
0
>>> p1.display()  # long time
0
```

sage.manifolds.differentiable.characteristic_cohomology_class.**additive_sequence**(*q*, *k*, *n=None*)

Turn the polynomial q into its additive sequence.

Let q be a polynomial and $x_1, \ldots x_n$ indeterminates. The *additive sequence of* q is then given by the polynomials Q_j

$$\sum_{j=0}^{n} Q_j(\sigma_1, \ldots, \sigma_j) z^j = \sum_{i=1}^{k} q(z\, x_i),$$

where σ_i is the i-th elementary symmetric polynomial in the indeterminates x_i.

INPUT:

- q – polynomial to turn into its additive sequence
- k – maximal index k of the sum
- n – (default: None) the highest order of the sequence n; if None, the order of q is assumed

OUTPUT: a symmetric polynomial representing the additive sequence

EXAMPLES:

```
sage: P.<x> = PolynomialRing(QQ)
sage: from sage.manifolds.differentiable.characteristic_cohomology_class import␣
↪additive_sequence
sage: f = 1 + x - x^2
sage: sym = additive_sequence(f, 2); sym
2*e[] + e[1] - e[1, 1] + 2*e[2]
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, names=('x',)); (x,) = P._first_ngens(1)
>>> from sage.manifolds.differentiable.characteristic_cohomology_class import␣
↪additive_sequence
>>> f = Integer(1) + x - x**Integer(2)
>>> sym = additive_sequence(f, Integer(2)); sym
2*e[] + e[1] - e[1, 1] + 2*e[2]
```

The maximal order of the result can be stated with n:

```
sage: sym_1 = additive_sequence(f, 2, 1); sym_1
2*e[] + e[1]
```

```
>>> from sage.all import *
>>> sym_1 = additive_sequence(f, Integer(2), Integer(1)); sym_1
2*e[] + e[1]
```

`sage.manifolds.differentiable.characteristic_cohomology_class.`**`fast_wedge_power`**`(`*form*, *n*`)`

Return the wedge product power of $form$ using a square-and-wedge algorithm.

INPUT:

- `form` – a differential form
- `n` – nonnegative integer

EXAMPLES:

```
sage: M = Manifold(4, 'M')
sage: X.<x,y,z,t> = M.chart()
sage: omega = M.diff_form(2, name='omega')
sage: omega[0,1] = t*y^2 + 2*x
sage: omega[0,3] = z - 2*y
sage: from sage.manifolds.differentiable.characteristic_cohomology_class import␣
↪fast_wedge_power
sage: fast_wedge_power(omega, 0)
Scalar field 1 on the 4-dimensional differentiable manifold M
sage: fast_wedge_power(omega, 1)
2-form omega on the 4-dimensional differentiable manifold M
sage: fast_wedge_power(omega, 2)
4-form omega∧omega on the 4-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M')
>>> X = M.chart(names=('x', 'y', 'z', 't',)); (x, y, z, t,) = X._first_ngens(4)
>>> omega = M.diff_form(Integer(2), name='omega')
>>> omega[Integer(0),Integer(1)] = t*y**Integer(2) + Integer(2)*x
>>> omega[Integer(0),Integer(3)] = z - Integer(2)*y
>>> from sage.manifolds.differentiable.characteristic_cohomology_class import␣
↪fast_wedge_power
>>> fast_wedge_power(omega, Integer(0))
Scalar field 1 on the 4-dimensional differentiable manifold M
>>> fast_wedge_power(omega, Integer(1))
2-form omega on the 4-dimensional differentiable manifold M
>>> fast_wedge_power(omega, Integer(2))
4-form omega∧omega on the 4-dimensional differentiable manifold M
```

`sage.manifolds.differentiable.characteristic_cohomology_class.`**`multiplicative_sequence`**`(`*q*, *n=None*`)`

Turn the polynomial `q` into its multiplicative sequence.

Let q be a polynomial and $x_1, \ldots x_n$ indeterminates. The *multiplicative sequence of* q is then given by the polynomials K_j

$$\sum_{j=0}^{n} K_j(\sigma_1, \ldots, \sigma_j) z^j = \prod_{i=1}^{n} q(z\, x_i),$$

where σ_i is the i-th elementary symmetric polynomial in the indeterminates x_i.

INPUT:

- q – polynomial to turn into its multiplicative sequence
- n – (default: None) the highest order n of the sequence; if None, the order of q is assumed

OUTPUT: a symmetric polynomial representing the multiplicative sequence

EXAMPLES:

```
sage: P.<x> = PolynomialRing(QQ)
sage: from sage.manifolds.differentiable.characteristic_cohomology_class import␣
↪multiplicative_sequence
sage: f = 1 + x - x^2
sage: sym = multiplicative_sequence(f); sym
e[] + e[1] - e[1, 1] + 3*e[2]
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, names=('x',)); (x,) = P._first_ngens(1)
>>> from sage.manifolds.differentiable.characteristic_cohomology_class import␣
↪multiplicative_sequence
>>> f = Integer(1) + x - x**Integer(2)
>>> sym = multiplicative_sequence(f); sym
e[] + e[1] - e[1, 1] + 3*e[2]
```

The maximal order of the result can be stated with n:

```
sage: sym_5 = multiplicative_sequence(f, n=5); sym_5
e[] + e[1] - e[1, 1] + 3*e[2] - e[2, 1] + e[2, 2] + 4*e[3] - 3*e[3, 1]
 + e[3, 2] + 7*e[4] - 4*e[4, 1] + 11*e[5]
```

```
>>> from sage.all import *
>>> sym_5 = multiplicative_sequence(f, n=Integer(5)); sym_5
e[] + e[1] - e[1, 1] + 3*e[2] - e[2, 1] + e[2, 2] + 4*e[3] - 3*e[3, 1]
 + e[3, 2] + 7*e[4] - 4*e[4, 1] + 11*e[5]
```

CHAPTER

THREE

PSEUDO-RIEMANNIAN MANIFOLDS

3.1 Pseudo-Riemannian Manifolds

A *pseudo-Riemannian manifold* is a pair (M, g) where M is a real differentiable manifold M (see `DifferentiableManifold`) and g is a field of non-degenerate symmetric bilinear forms on M, which is called the *metric tensor*, or simply the *metric* (see `PseudoRiemannianMetric`).

Two important subcases are

- *Riemannian manifold*: the metric g is positive definite, i.e. its signature is $n = \dim M$;
- *Lorentzian manifold*: the metric g has signature $n - 2$ (positive convention) or $2 - n$ (negative convention).

On a pseudo-Riemannian manifold, one may use various standard `operators` acting on scalar and tensor fields, like `grad()` or `div()`.

All pseudo-Riemannian manifolds, whatever the metric signature, are implemented via the class `PseudoRiemannianManifold`.

Example 1: the sphere as a Riemannian manifold of dimension 2

We start by declaring S^2 as a 2-dimensional Riemannian manifold:

```
sage: M = Manifold(2, 'S^2', structure='Riemannian')
sage: M
2-dimensional Riemannian manifold S^2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', structure='Riemannian')
>>> M
2-dimensional Riemannian manifold S^2
```

We then cover S^2 by two stereographic charts, from the North pole and from the South pole respectively:

```
sage: U = M.open_subset('U')
sage: stereoN.<x,y> = U.chart()
sage: V = M.open_subset('V')
sage: stereoS.<u,v> = V.chart()
sage: M.declare_union(U,V)
sage: stereoN_to_S = stereoN.transition_map(stereoS,
....:                 [x/(x^2+y^2), y/(x^2+y^2)], intersection_name='W',
....:                 restrictions1= x^2+y^2!=0, restrictions2= u^2+v^2!=0)
sage: W = U.intersection(V)
sage: stereoN_to_S
```

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```
Change of coordinates from Chart (W, (x, y)) to Chart (W, (u, v))
sage: stereoN_to_S.display()
u = x/(x^2 + y^2)
v = y/(x^2 + y^2)
sage: stereoN_to_S.inverse().display()
x = u/(u^2 + v^2)
y = v/(u^2 + v^2)
```

```
>>> from sage.all import *
>>> U = M.open_subset('U')
>>> stereoN = U.chart(names=('x', 'y',)); (x, y,) = stereoN._first_ngens(2)
>>> V = M.open_subset('V')
>>> stereoS = V.chart(names=('u', 'v',)); (u, v,) = stereoS._first_ngens(2)
>>> M.declare_union(U,V)
>>> stereoN_to_S = stereoN.transition_map(stereoS,
...                 [x/(x**Integer(2)+y**Integer(2)), y/(x**Integer(2)+y**Integer(2))],
↪ intersection_name='W',
...                 restrictions1= x**Integer(2)+y**Integer(2)!=Integer(0),␣
↪restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> W = U.intersection(V)
>>> stereoN_to_S
Change of coordinates from Chart (W, (x, y)) to Chart (W, (u, v))
>>> stereoN_to_S.display()
u = x/(x^2 + y^2)
v = y/(x^2 + y^2)
>>> stereoN_to_S.inverse().display()
x = u/(u^2 + v^2)
y = v/(u^2 + v^2)
```

We get the metric defining the Riemannian structure by:

```
sage: g = M.metric()
sage: g
Riemannian metric g on the 2-dimensional Riemannian manifold S^2
```

```
>>> from sage.all import *
>>> g = M.metric()
>>> g
Riemannian metric g on the 2-dimensional Riemannian manifold S^2
```

At this stage, the metric g is defined as a Python object but there remains to initialize it by setting its components with respect to the vector frames associated with the stereographic coordinates. Let us begin with the frame of chart `stereoN`:

```
sage: eU = stereoN.frame()
sage: g[eU, 0, 0] = 4/(1 + x^2 + y^2)^2
sage: g[eU, 1, 1] = 4/(1 + x^2 + y^2)^2
```

```
>>> from sage.all import *
>>> eU = stereoN.frame()
>>> g[eU, Integer(0), Integer(0)] = Integer(4)/(Integer(1) + x**Integer(2) +␣
↪y**Integer(2))**Integer(2)
```

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```
>>> g[eU, Integer(1), Integer(1)] = Integer(4)/(Integer(1) + x**Integer(2) +
↪y**Integer(2))**Integer(2)
```

The metric components in the frame of chart `stereoS` are obtained by continuation of the expressions found in $W = U \cap V$ from the known change-of-coordinate formulas:

```
sage: eV = stereoS.frame()
sage: g.add_comp_by_continuation(eV, W)
```

```
>>> from sage.all import *
>>> eV = stereoS.frame()
>>> g.add_comp_by_continuation(eV, W)
```

At this stage, the metric g is well defined in all S^2:

```
sage: g.display(eU)
g = 4/(x^2 + y^2 + 1)^2 dx⊗dx + 4/(x^2 + y^2 + 1)^2 dy⊗dy
sage: g.display(eV)
g = 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) du⊗du
 + 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) dv⊗dv
```

```
>>> from sage.all import *
>>> g.display(eU)
g = 4/(x^2 + y^2 + 1)^2 dx⊗dx + 4/(x^2 + y^2 + 1)^2 dy⊗dy
>>> g.display(eV)
g = 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) du⊗du
 + 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) dv⊗dv
```

The expression in frame `eV` can be given a shape similar to that in frame `eU`, by factorizing the components:

```
sage: g[eV, 0, 0].factor()
4/(u^2 + v^2 + 1)^2
sage: g[eV, 1, 1].factor()
4/(u^2 + v^2 + 1)^2
sage: g.display(eV)
g = 4/(u^2 + v^2 + 1)^2 du⊗du + 4/(u^2 + v^2 + 1)^2 dv⊗dv
```

```
>>> from sage.all import *
>>> g[eV, Integer(0), Integer(0)].factor()
4/(u^2 + v^2 + 1)^2
>>> g[eV, Integer(1), Integer(1)].factor()
4/(u^2 + v^2 + 1)^2
>>> g.display(eV)
g = 4/(u^2 + v^2 + 1)^2 du⊗du + 4/(u^2 + v^2 + 1)^2 dv⊗dv
```

Let us consider a scalar field f on S^2:

```
sage: f = M.scalar_field({stereoN: 1/(1+x^2+y^2)}, name='f')
sage: f.add_expr_by_continuation(stereoS, W)
sage: f.display()
f: S^2 → R
```

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```
on U: (x, y) ↦ 1/(x^2 + y^2 + 1)
on V: (u, v) ↦ (u^2 + v^2)/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> f = M.scalar_field({stereoN: Integer(1)/(Integer(1)+x**Integer(2)+y**Integer(2))},
↪ name='f')
>>> f.add_expr_by_continuation(stereoS, W)
>>> f.display()
f: S^2 → ℝ
on U: (x, y) ↦ 1/(x^2 + y^2 + 1)
on V: (u, v) ↦ (u^2 + v^2)/(u^2 + v^2 + 1)
```

The gradient of f (with respect to the metric g) is:

```
sage: gradf = f.gradient()
sage: gradf
Vector field grad(f) on the 2-dimensional Riemannian manifold S^2
sage: gradf.display(eU)
grad(f) = -1/2*x ∂/∂x - 1/2*y ∂/∂y
sage: gradf.display(eV)
grad(f) = 1/2*u ∂/∂u + 1/2*v ∂/∂v
```

```
>>> from sage.all import *
>>> gradf = f.gradient()
>>> gradf
Vector field grad(f) on the 2-dimensional Riemannian manifold S^2
>>> gradf.display(eU)
grad(f) = -1/2*x ∂/∂x - 1/2*y ∂/∂y
>>> gradf.display(eV)
grad(f) = 1/2*u ∂/∂u + 1/2*v ∂/∂v
```

It is possible to write `grad(f)` instead of `f.gradient()`, by importing the standard differential operators of vector calculus:

```
sage: from sage.manifolds.operators import *
sage: grad(f) == gradf
True
```

```
>>> from sage.all import *
>>> from sage.manifolds.operators import *
>>> grad(f) == gradf
True
```

The Laplacian of f (with respect to the metric g) is obtained either as `f.laplacian()` or, thanks to the above import, as `laplacian(f)`:

```
sage: Df = laplacian(f)
sage: Df
Scalar field Delta(f) on the 2-dimensional Riemannian manifold S^2
sage: Df.display()
Delta(f): S^2 → ℝ
```

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```
on U: (x, y) ↦ (x^2 + y^2 - 1)/(x^2 + y^2 + 1)
on V: (u, v) ↦ -(u^2 + v^2 - 1)/(u^2 + v^2 + 1)
```

```
>>> from sage.all import *
>>> Df = laplacian(f)
>>> Df
Scalar field Delta(f) on the 2-dimensional Riemannian manifold S^2
>>> Df.display()
Delta(f): S^2 → ℝ
on U: (x, y) ↦ (x^2 + y^2 - 1)/(x^2 + y^2 + 1)
on V: (u, v) ↦ -(u^2 + v^2 - 1)/(u^2 + v^2 + 1)
```

Let us check the standard formula $\Delta f = \mathrm{div}(\mathrm{grad}\, f)$:

```
sage: Df == div(gradf)
True
```

```
>>> from sage.all import *
>>> Df == div(gradf)
True
```

Since each open subset of S^2 inherits the structure of a Riemannian manifold, we can get the metric on it via the method `metric()`:

```
sage: gU = U.metric()
sage: gU
Riemannian metric g on the Open subset U of the 2-dimensional Riemannian
 manifold S^2
sage: gU.display()
g = 4/(x^2 + y^2 + 1)^2 dx⊗dx + 4/(x^2 + y^2 + 1)^2 dy⊗dy
```

```
>>> from sage.all import *
>>> gU = U.metric()
>>> gU
Riemannian metric g on the Open subset U of the 2-dimensional Riemannian
 manifold S^2
>>> gU.display()
g = 4/(x^2 + y^2 + 1)^2 dx⊗dx + 4/(x^2 + y^2 + 1)^2 dy⊗dy
```

Of course, `gU` is nothing but the restriction of g to U:

```
sage: gU is g.restrict(U)
True
```

```
>>> from sage.all import *
>>> gU is g.restrict(U)
True
```

Example 2: Minkowski spacetime as a Lorentzian manifold of dimension 4

We start by declaring a 4-dimensional Lorentzian manifold M:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: M
4-dimensional Lorentzian manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> M
4-dimensional Lorentzian manifold M
```

We define Minkowskian coordinates on M:

```
sage: X.<t,x,y,z> = M.chart()
```

```
>>> from sage.all import *
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_ngens(4)
```

We construct the metric tensor by:

```
sage: g = M.metric()
sage: g
Lorentzian metric g on the 4-dimensional Lorentzian manifold M
```

```
>>> from sage.all import *
>>> g = M.metric()
>>> g
Lorentzian metric g on the 4-dimensional Lorentzian manifold M
```

and initialize it to the Minkowskian value:

```
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1, 1, 1, 1
sage: g.display()
g = -dt⊗dt + dx⊗dx + dy⊗dy + dz⊗dz
sage: g[:]
[-1  0  0  0]
[ 0  1  0  0]
[ 0  0  1  0]
[ 0  0  0  1]
```

```
>>> from sage.all import *
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),Integer(2)],␣
↪g[Integer(3),Integer(3)] = -Integer(1), Integer(1), Integer(1), Integer(1)
>>> g.display()
g = -dt⊗dt + dx⊗dx + dy⊗dy + dz⊗dz
>>> g[:]
[-1  0  0  0]
[ 0  1  0  0]
[ 0  0  1  0]
[ 0  0  0  1]
```

We may check that the metric is flat, i.e. has a vanishing Riemann curvature tensor:

```
sage: g.riemann().display()
Riem(g) = 0
```

```
>>> from sage.all import *
>>> g.riemann().display()
Riem(g) = 0
```

A vector field on M:

```
sage: u = M.vector_field(name='u')
sage: u[0] = cosh(t)
sage: u[1] = sinh(t)
sage: u.display()
u = cosh(t) ∂/∂t + sinh(t) ∂/∂x
```

```
>>> from sage.all import *
>>> u = M.vector_field(name='u')
>>> u[Integer(0)] = cosh(t)
>>> u[Integer(1)] = sinh(t)
>>> u.display()
u = cosh(t) ∂/∂t + sinh(t) ∂/∂x
```

The scalar square of u is:

```
sage: s = u.dot(u); s
Scalar field u.u on the 4-dimensional Lorentzian manifold M
```

```
>>> from sage.all import *
>>> s = u.dot(u); s
Scalar field u.u on the 4-dimensional Lorentzian manifold M
```

Scalar products are taken with respect to the metric tensor:

```
sage: u.dot(u) == g(u,u)
True
```

```
>>> from sage.all import *
>>> u.dot(u) == g(u,u)
True
```

u is a unit timelike vector, i.e. its scalar square is identically -1:

```
sage: s.display()
u.u: M → ℝ
   (t, x, y, z) ↦ -1
sage: s.expr()
-1
```

```
>>> from sage.all import *
>>> s.display()
u.u: M → ℝ
   (t, x, y, z) ↦ -1
```

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```
>>> s.expr()
-1
```

Let us consider a unit spacelike vector:

```
sage: v = M.vector_field(name='v')
sage: v[0] = sinh(t)
sage: v[1] = cosh(t)
sage: v.display()
v = sinh(t) ∂/∂t + cosh(t) ∂/∂x
sage: v.dot(v).display()
v.v: M → ℝ
   (t, x, y, z) ↦ 1
sage: v.dot(v).expr()
1
```

```
>>> from sage.all import *
>>> v = M.vector_field(name='v')
>>> v[Integer(0)] = sinh(t)
>>> v[Integer(1)] = cosh(t)
>>> v.display()
v = sinh(t) ∂/∂t + cosh(t) ∂/∂x
>>> v.dot(v).display()
v.v: M → ℝ
   (t, x, y, z) ↦ 1
>>> v.dot(v).expr()
1
```

u and v are orthogonal vectors with respect to Minkowski metric:

```
sage: u.dot(v).display()
    u.v: M → ℝ
   (t, x, y, z) ↦ 0
sage: u.dot(v).expr()
0
```

```
>>> from sage.all import *
>>> u.dot(v).display()
    u.v: M → ℝ
   (t, x, y, z) ↦ 0
>>> u.dot(v).expr()
0
```

The divergence of u is:

```
sage: s = u.div(); s
Scalar field div(u) on the 4-dimensional Lorentzian manifold M
sage: s.display()
div(u): M → ℝ
   (t, x, y, z) ↦ sinh(t)
```

```
>>> from sage.all import *
>>> s = u.div(); s
Scalar field div(u) on the 4-dimensional Lorentzian manifold M
>>> s.display()
div(u): M → ℝ
   (t, x, y, z) ↦ sinh(t)
```

while its d'Alembertian is:

```
sage: Du = u.dalembertian(); Du
Vector field Box(u) on the 4-dimensional Lorentzian manifold M
sage: Du.display()
Box(u) = -cosh(t) ∂/∂t - sinh(t) ∂/∂x
```

```
>>> from sage.all import *
>>> Du = u.dalembertian(); Du
Vector field Box(u) on the 4-dimensional Lorentzian manifold M
>>> Du.display()
Box(u) = -cosh(t) ∂/∂t - sinh(t) ∂/∂x
```

AUTHORS:

- Eric Gourgoulhon (2018): initial version

REFERENCES:

- B. O'Neill : *Semi-Riemannian Geometry* [ONe1983]
- J. M. Lee : *Riemannian Manifolds* [Lee1997]

class sage.manifolds.differentiable.pseudo_riemannian.**PseudoRiemannianManifold**(*n*, *name*, *metric_name=None*, *signature=None*, *base_manifold=None*, *diff_degree=+Infinity*, *latex_name=None*, *metric_latex_name=None*, *start_index=0*, *category=None*, *unique_tag=None*)

Bases: `DifferentiableManifold`

PseudoRiemannian manifold.

A *pseudo-Riemannian manifold* is a pair (M, g) where M is a real differentiable manifold M (see `DifferentiableManifold`) and g is a field of non-degenerate symmetric bilinear forms on M, which is called the *metric tensor*, or simply the *metric* (see `PseudoRiemannianMetric`).

Two important subcases are

- *Riemannian manifold*: the metric g is positive definite, i.e. its signature is $n = \dim M$;
- *Lorentzian manifold*: the metric g has signature $n - 2$ (positive convention) or $2 - n$ (negative convention).

INPUT:

- `n` – positive integer; dimension of the manifold
- `name` – string; name (symbol) given to the manifold
- `metric_name` – (default: `None`) string; name (symbol) given to the metric; if `None`, `'g'` is used
- `signature` – (default: `None`) signature S of the metric as a single integer: $S = n_+ - n_-$, where n_+ (resp. n_-) is the number of positive terms (resp. number of negative terms) in any diagonal writing of the metric components; if `signature` is not provided, S is set to the manifold's dimension (Riemannian signature)
- `base_manifold` – (default: `None`) if not `None`, must be a differentiable manifold; the created object is then an open subset of `base_manifold`
- `diff_degree` – (default: `infinity`) degree k of differentiability
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the manifold; if none is provided, it is set to `name`
- `metric_latex_name` – (default: `None`) string; LaTeX symbol to denote the metric; if none is provided, it is set to `metric_name`
- `start_index` – (default: 0) integer; lower value of the range of indices used for "indexed objects" on the manifold, e.g. coordinates in a chart
- `category` – (default: `None`) to specify the category; if `None`, `Manifolds(RR).Differentiable()` (or `Manifolds(RR).Smooth()` if `diff_degree` = `infinity`) is assumed (see the category `Manifolds`)
- `unique_tag` – (default: `None`) tag used to force the construction of a new object when all the other arguments have been used previously (without `unique_tag`, the `UniqueRepresentation` behavior inherited from `ManifoldSubset`, via `DifferentiableManifold` and `TopologicalManifold`, would return the previously constructed object corresponding to these arguments).

EXAMPLES:

Pseudo-Riemannian manifolds are constructed via the generic function `Manifold()`, using the keyword `structure`:

```
sage: M = Manifold(4, 'M', structure='pseudo-Riemannian', signature=0)
sage: M
4-dimensional pseudo-Riemannian manifold M
sage: M.category()
Category of smooth manifolds over Real Field with 53 bits of precision
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='pseudo-Riemannian',␣
↪signature=Integer(0))
>>> M
4-dimensional pseudo-Riemannian manifold M
>>> M.category()
Category of smooth manifolds over Real Field with 53 bits of precision
```

The metric associated with `M` is:

```
sage: M.metric()
Pseudo-Riemannian metric g on the 4-dimensional pseudo-Riemannian
 manifold M
sage: M.metric().signature()
0
sage: M.metric().tensor_type()
(0, 2)
```

```
>>> from sage.all import *
>>> M.metric()
Pseudo-Riemannian metric g on the 4-dimensional pseudo-Riemannian
 manifold M
>>> M.metric().signature()
0
>>> M.metric().tensor_type()
(0, 2)
```

Its value has to be initialized either by setting its components in various vector frames (see the above examples regarding the 2-sphere and Minkowski spacetime) or by making it equal to a given field of symmetric bilinear forms (see the method *set()* of the metric class). Both methods are also covered in the documentation of method *metric()* below.

The metric object belongs to the class *PseudoRiemannianMetric*:

```
sage: isinstance(M.metric(), sage.manifolds.differentiable.metric.
....:                        PseudoRiemannianMetric)
True
```

```
>>> from sage.all import *
>>> isinstance(M.metric(), sage.manifolds.differentiable.metric.
...                        PseudoRiemannianMetric)
True
```

See the documentation of this class for all operations available on metrics.

The default name of the metric is `g`; it can be customized:

```
sage: M = Manifold(4, 'M', structure='pseudo-Riemannian',
....:              metric_name='gam', metric_latex_name=r'\gamma')
sage: M.metric()
Riemannian metric gam on the 4-dimensional Riemannian manifold M
sage: latex(M.metric())
\gamma
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='pseudo-Riemannian',
...              metric_name='gam', metric_latex_name=r'\gamma')
>>> M.metric()
Riemannian metric gam on the 4-dimensional Riemannian manifold M
>>> latex(M.metric())
\gamma
```

A Riemannian manifold is constructed by the proper setting of the keyword `structure`:

```
sage: M = Manifold(4, 'M', structure='Riemannian'); M
4-dimensional Riemannian manifold M
sage: M.metric()
Riemannian metric g on the 4-dimensional Riemannian manifold M
sage: M.metric().signature()
4
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Riemannian'); M
4-dimensional Riemannian manifold M
>>> M.metric()
Riemannian metric g on the 4-dimensional Riemannian manifold M
>>> M.metric().signature()
4
```

Similarly, a Lorentzian manifold is obtained by:

```
sage: M = Manifold(4, 'M', structure='Lorentzian'); M
4-dimensional Lorentzian manifold M
sage: M.metric()
Lorentzian metric g on the 4-dimensional Lorentzian manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian'); M
4-dimensional Lorentzian manifold M
>>> M.metric()
Lorentzian metric g on the 4-dimensional Lorentzian manifold M
```

The default Lorentzian signature is taken to be positive:

```
sage: M.metric().signature()
2
```

```
>>> from sage.all import *
>>> M.metric().signature()
2
```

but one can opt for the negative convention via the keyword `signature`:

```
sage: M = Manifold(4, 'M', structure='Lorentzian', signature='negative')
sage: M.metric()
Lorentzian metric g on the 4-dimensional Lorentzian manifold M
sage: M.metric().signature()
-2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian', signature='negative')
>>> M.metric()
Lorentzian metric g on the 4-dimensional Lorentzian manifold M
>>> M.metric().signature()
-2
```

metric(*name=None*, *signature=None*, *latex_name=None*, *dest_map=None*)

Return the metric giving the pseudo-Riemannian structure to the manifold, or define a new metric tensor on the manifold.

INPUT:

- `name` – (default: `None`) name given to the metric; if `name` is `None` or matches the name of the metric defining the pseudo-Riemannian structure of `self`, the latter metric is returned
- `signature` – (default: `None`; ignored if `name` is `None`) signature S of the metric as a single integer: $S = n_+ - n_-$, where n_+ (resp. n_-) is the number of positive terms (resp. number of negative terms) in any diagonal writing of the metric components; if `signature` is not provided, S is set to the manifold's dimension (Riemannian signature)
- `latex_name` – (default: `None`; ignored if `name` is `None`) LaTeX symbol to denote the metric; if `None`, it is formed from `name`
- `dest_map` – (default: `None`; ignored if `name` is `None`) instance of class `DiffMap` representing the destination map $\Phi : U \to M$, where U is the current manifold; if `None`, the identity map is assumed (case of a metric tensor field *on* U)

OUTPUT:

- instance of `PseudoRiemannianMetric`

EXAMPLES:

Metric of a 3-dimensional Riemannian manifold:

```
sage: M = Manifold(3, 'M', structure='Riemannian', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: g = M.metric(); g
Riemannian metric g on the 3-dimensional Riemannian manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='Riemannian', start_
↪index=Integer(1))
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> g = M.metric(); g
Riemannian metric g on the 3-dimensional Riemannian manifold M
```

The metric remains to be initialized, for instance by setting its components in the coordinate frame associated to the chart `X`:

```
sage: g[1,1], g[2,2], g[3,3] = 1, 1, 1
sage: g.display()
g = dx⊗dx + dy⊗dy + dz⊗dz
```

```
>>> from sage.all import *
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)], g[Integer(3),
↪Integer(3)] = Integer(1), Integer(1), Integer(1)
>>> g.display()
g = dx⊗dx + dy⊗dy + dz⊗dz
```

Alternatively, the metric can be initialized from a given field of nondegenerate symmetric bilinear forms; we may create the former object by:

```
sage: X.coframe()
Coordinate coframe (M, (dx,dy,dz))
sage: dx, dy, dz = X.coframe()[1], X.coframe()[2], X.coframe()[3]
sage: b = dx*dx + dy*dy + dz*dz
sage: b
Field of symmetric bilinear forms dx⊗dx+dy⊗dy+dz⊗dz on the
 3-dimensional Riemannian manifold M
```

```
>>> from sage.all import *
>>> X.coframe()
Coordinate coframe (M, (dx,dy,dz))
>>> dx, dy, dz = X.coframe()[Integer(1)], X.coframe()[Integer(2)], X.
↪coframe()[Integer(3)]
>>> b = dx*dx + dy*dy + dz*dz
>>> b
Field of symmetric bilinear forms dx⊗dx+dy⊗dy+dz⊗dz on the
 3-dimensional Riemannian manifold M
```

We then use the metric method `set()` to make `g` being equal to `b` as a symmetric tensor field of type `(0,2)`:

```
sage: g.set(b)
sage: g.display()
g = dx⊗dx + dy⊗dy + dz⊗dz
```

```
>>> from sage.all import *
>>> g.set(b)
>>> g.display()
g = dx⊗dx + dy⊗dy + dz⊗dz
```

Another metric can be defined on `M` by specifying a metric name distinct from that chosen at the creation of the manifold (which is `g` by default, but can be changed thanks to the keyword `metric_name` in `Manifold()`):

```
sage: h = M.metric('h'); h
Riemannian metric h on the 3-dimensional Riemannian manifold M
sage: h[1,1], h[2,2], h[3,3] = 1+y^2, 1+z^2, 1+x^2
sage: h.display()
h = (y^2 + 1) dx⊗dx + (z^2 + 1) dy⊗dy + (x^2 + 1) dz⊗dz
```

```
>>> from sage.all import *
>>> h = M.metric('h'); h
Riemannian metric h on the 3-dimensional Riemannian manifold M
>>> h[Integer(1),Integer(1)], h[Integer(2),Integer(2)], h[Integer(3),
↪Integer(3)] = Integer(1)+y**Integer(2), Integer(1)+z**Integer(2),␣
↪Integer(1)+x**Integer(2)
>>> h.display()
h = (y^2 + 1) dx⊗dx + (z^2 + 1) dy⊗dy + (x^2 + 1) dz⊗dz
```

The metric tensor `h` is distinct from the metric entering in the definition of the Riemannian manifold `M`:

```
sage: h is M.metric()
False
```

```
>>> from sage.all import *
>>> h is M.metric()
False
```

while we have of course:

```
sage: g is M.metric()
True
```

```
>>> from sage.all import *
>>> g is M.metric()
True
```

Providing the same name as the manifold's default metric returns the latter:

```
sage: M.metric('g') is M.metric()
True
```

```
>>> from sage.all import *
>>> M.metric('g') is M.metric()
True
```

In the present case (`M` is diffeomorphic to $\mathbf{R}^3$), we can even create a Lorentzian metric on `M`:

```
sage: h = M.metric('h', signature=1); h
Lorentzian metric h on the 3-dimensional Riemannian manifold M
```

```
>>> from sage.all import *
>>> h = M.metric('h', signature=Integer(1)); h
Lorentzian metric h on the 3-dimensional Riemannian manifold M
```

open_subset(*name*, *latex_name=None*, *coord_def={}*, *supersets=None*)

Create an open subset of `self`.

An open subset is a set that is (i) included in the manifold and (ii) open with respect to the manifold's topology. It is a differentiable manifold by itself. Moreover, equipped with the restriction of the manifold metric to itself, it is a pseudo-Riemannian manifold. Hence the returned object is an instance of *PseudoRiemannianManifold*.

INPUT:

- `name` – name given to the open subset
- `latex_name` – (default: `None`) LaTeX symbol to denote the subset; if none is provided, it is set to `name`
- `coord_def` – (default: `{}`) definition of the subset in terms of coordinates; `coord_def` must a be dictionary with keys charts in the manifold's atlas and values the symbolic expressions formed by the coordinates to define the subset.
- `supersets` – (default: only `self`) list of sets that the new open subset is a subset of

OUTPUT:

- instance of *PseudoRiemannianManifold* representing the created open subset

EXAMPLES:

Open subset of a 2-dimensional Riemannian manifold:

```
sage: M = Manifold(2, 'M', structure='Riemannian')
sage: X.<x,y> = M.chart()
sage: U = M.open_subset('U', coord_def={X: x>0}); U
Open subset U of the 2-dimensional Riemannian manifold M
sage: type(U)
<class 'sage.manifolds.differentiable.pseudo_riemannian.
↪PseudoRiemannianManifold_with_category'>
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='Riemannian')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> U = M.open_subset('U', coord_def={X: x>Integer(0)}); U
Open subset U of the 2-dimensional Riemannian manifold M
>>> type(U)
<class 'sage.manifolds.differentiable.pseudo_riemannian.
↪PseudoRiemannianManifold_with_category'>
```

We initialize the metric of `M`:

```
sage: g = M.metric()
sage: g[0,0], g[1,1] = 1, 1
```

```
>>> from sage.all import *
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)] = Integer(1),␣
↪Integer(1)
```

Then the metric on `U` is determined as the restriction of `g` to `U`:

```
sage: gU = U.metric(); gU
Riemannian metric g on the Open subset U of the 2-dimensional Riemannian␣
↪manifold M
sage: gU.display()
g = dx⊗dx + dy⊗dy
sage: gU is g.restrict(U)
True
```

```
>>> from sage.all import *
>>> gU = U.metric(); gU
Riemannian metric g on the Open subset U of the 2-dimensional Riemannian␣
↪manifold M
>>> gU.display()
g = dx⊗dx + dy⊗dy
>>> gU is g.restrict(U)
True
```

volume_form(*contra=0*)

Volume form (Levi-Civita tensor) ϵ associated with `self`.

This assumes that `self` is an orientable manifold, with a preferred orientation; see *orientation()* for details.

The volume form ϵ is a n-form (n being the manifold's dimension) such that, for any vector frame (e_i) that is orthonormal with respect to the metric of the pseudo-Riemannian manifold `self`,

$$\epsilon(e_1, \ldots, e_n) = \pm 1$$

There are only two such n-forms, which are opposite of each other. The volume form ϵ is selected as the one that returns $+1$ for any right-handed vector frame with respect to the chosen orientation of `self`.

INPUT:

- `contra` – (default: 0) number of contravariant indices of the returned tensor

OUTPUT:

- if `contra = 0` (default value): the volume n-form ϵ, as an instance of `DiffForm`
- if `contra = k`, with $1 \leq k \leq n$, the tensor field of type (k,n-k) formed from ϵ by raising the first k indices with the metric (see method `up()`); the output is then an instance of `TensorField`, with the appropriate antisymmetries, or of the subclass `MultivectorField` if $k = n$

EXAMPLES:

Volume form of the Euclidean 3-space:

```
sage: M = Manifold(3, 'M', structure='Riemannian', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: g = M.metric()
sage: g[1,1], g[2,2], g[3,3] = 1, 1, 1
sage: eps = M.volume_form(); eps
3-form eps_g on the 3-dimensional Riemannian manifold M
sage: eps.display()
eps_g = dx∧dy∧dz
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='Riemannian', start_
↪index=Integer(1))
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> g = M.metric()
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)], g[Integer(3),
↪Integer(3)] = Integer(1), Integer(1), Integer(1)
>>> eps = M.volume_form(); eps
3-form eps_g on the 3-dimensional Riemannian manifold M
>>> eps.display()
eps_g = dx∧dy∧dz
```

Raising the first index:

```
sage: eps1 = M.volume_form(1); eps1
Tensor field of type (1,2) on the 3-dimensional Riemannian
 manifold M
sage: eps1.display()
∂/∂x⊗dy⊗dz - ∂/∂x⊗dz⊗dy - ∂/∂y⊗dx⊗dz + ∂/∂y⊗dz⊗dx + ∂/∂z⊗dx⊗dy
 - ∂/∂z⊗dy⊗dx
sage: eps1.symmetries()
no symmetry; antisymmetry: (1, 2)
```

```
>>> from sage.all import *
>>> eps1 = M.volume_form(Integer(1)); eps1
```

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```
Tensor field of type (1,2) on the 3-dimensional Riemannian
 manifold M
>>> eps1.display()
∂/∂x⊗dy⊗dz - ∂/∂x⊗dz⊗dy - ∂/∂y⊗dx⊗dz + ∂/∂y⊗dz⊗dx + ∂/∂z⊗dx⊗dy
 - ∂/∂z⊗dy⊗dx
>>> eps1.symmetries()
no symmetry; antisymmetry: (1, 2)
```

Raising the first and second indices:

```
sage: eps2 = M.volume_form(2); eps2
Tensor field of type (2,1) on the 3-dimensional Riemannian
 manifold M
sage: eps2.display()
∂/∂x⊗∂/∂y⊗dz - ∂/∂x⊗∂/∂z⊗dy - ∂/∂y⊗∂/∂x⊗dz + ∂/∂y⊗∂/∂z⊗dx
 + ∂/∂z⊗∂/∂x⊗dy - ∂/∂z⊗∂/∂y⊗dx
sage: eps2.symmetries()
no symmetry; antisymmetry: (0, 1)
```

```
>>> from sage.all import *
>>> eps2 = M.volume_form(Integer(2)); eps2
Tensor field of type (2,1) on the 3-dimensional Riemannian
 manifold M
>>> eps2.display()
∂/∂x⊗∂/∂y⊗dz - ∂/∂x⊗∂/∂z⊗dy - ∂/∂y⊗∂/∂x⊗dz + ∂/∂y⊗∂/∂z⊗dx
 + ∂/∂z⊗∂/∂x⊗dy - ∂/∂z⊗∂/∂y⊗dx
>>> eps2.symmetries()
no symmetry; antisymmetry: (0, 1)
```

Fully contravariant version:

```
sage: eps3 = M.volume_form(3); eps3
3-vector field on the 3-dimensional Riemannian manifold M
sage: eps3.display()
∂/∂x∧∂/∂y∧∂/∂z
```

```
>>> from sage.all import *
>>> eps3 = M.volume_form(Integer(3)); eps3
3-vector field on the 3-dimensional Riemannian manifold M
>>> eps3.display()
∂/∂x∧∂/∂y∧∂/∂z
```

3.2 Euclidean Spaces and Vector Calculus

3.2.1 Euclidean Spaces

A *Euclidean space of dimension* n is an affine space E, whose associated vector space is a n-dimensional vector space over **R** and is equipped with a positive definite symmetric bilinear form, called the *scalar product* or *dot product* [Ber1987]. A Euclidean space of dimension n can also be viewed as a Riemannian manifold that is diffeomorphic to $\mathbf{R}^n$ and that has a flat metric g. The Euclidean scalar product is then that defined by the Riemannian metric g.

The current implementation of Euclidean spaces is based on the second point of view. This allows for the introduction of various coordinate systems in addition to the usual the Cartesian systems. Standard curvilinear systems (planar, spherical and cylindrical coordinates) are predefined for 2-dimensional and 3-dimensional Euclidean spaces, along with the corresponding transition maps between them. Another benefit of such an implementation is the direct use of methods for vector calculus already implemented at the level of Riemannian manifolds (see, e.g., the methods `cross_product()` and `curl()`, as well as the module `operators`).

Euclidean spaces are implemented via the following classes:

- `EuclideanSpace` for generic values n,
- `EuclideanPlane` for $n = 2$,
- `Euclidean3dimSpace` for $n = 3$.

The user interface is provided by `EuclideanSpace`.

Example 1: the Euclidean plane

We start by declaring the Euclidean plane `E`, with `(x, y)` as Cartesian coordinates:

```
sage: E.<x,y> = EuclideanSpace()
sage: E
Euclidean plane E^2
sage: dim(E)
2
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(names=('x', 'y',)); (x, y,) = E._first_ngens(2)
>>> E
Euclidean plane E^2
>>> dim(E)
2
```

`E` is automatically endowed with the chart of Cartesian coordinates:

```
sage: E.atlas()
[Chart (E^2, (x, y))]
sage: cartesian = E.default_chart(); cartesian
Chart (E^2, (x, y))
```

```
>>> from sage.all import *
>>> E.atlas()
[Chart (E^2, (x, y))]
>>> cartesian = E.default_chart(); cartesian
Chart (E^2, (x, y))
```

Thanks to the use of `<x,y>` when declaring `E`, the coordinates (x, y) have been injected in the global namespace, i.e. the Python variables `x` and `y` have been created and are available to form symbolic expressions:

```
sage: y
y
sage: type(y)
<class 'sage.symbolic.expression.Expression'>
sage: assumptions()
[x is real, y is real]
```

```
>>> from sage.all import *
>>> y
y
>>> type(y)
<class 'sage.symbolic.expression.Expression'>
>>> assumptions()
[x is real, y is real]
```

The metric tensor of `E` is predefined:

```
sage: g = E.metric(); g
Riemannian metric g on the Euclidean plane E^2
sage: g.display()
g = dx⊗dx + dy⊗dy
sage: g[:]
[1 0]
[0 1]
```

```
>>> from sage.all import *
>>> g = E.metric(); g
Riemannian metric g on the Euclidean plane E^2
>>> g.display()
g = dx⊗dx + dy⊗dy
>>> g[:]
[1 0]
[0 1]
```

It is a *flat* metric, i.e. it has a vanishing Riemann tensor:

```
sage: g.riemann()
Tensor field Riem(g) of type (1,3) on the Euclidean plane E^2
sage: g.riemann().display()
Riem(g) = 0
```

```
>>> from sage.all import *
>>> g.riemann()
Tensor field Riem(g) of type (1,3) on the Euclidean plane E^2
>>> g.riemann().display()
Riem(g) = 0
```

Polar coordinates (r, ϕ) are introduced by:

```
sage: polar.<r,ph> = E.polar_coordinates()
sage: polar
Chart (E^2, (r, ph))
```

```
>>> from sage.all import *
>>> polar = E.polar_coordinates(names=('r', 'ph',)); (r, ph,) = polar._first_ngens(2)
>>> polar
Chart (E^2, (r, ph))
```

`E` is now endowed with two coordinate charts:

```
sage: E.atlas()
[Chart (E^2, (x, y)), Chart (E^2, (r, ph))]
```

```
>>> from sage.all import *
>>> E.atlas()
[Chart (E^2, (x, y)), Chart (E^2, (r, ph))]
```

The ranges of the coordinates introduced so far are:

```
sage: cartesian.coord_range()
x: (-oo, +oo); y: (-oo, +oo)
sage: polar.coord_range()
r: (0, +oo); ph: [0, 2*pi] (periodic)
```

```
>>> from sage.all import *
>>> cartesian.coord_range()
x: (-oo, +oo); y: (-oo, +oo)
>>> polar.coord_range()
r: (0, +oo); ph: [0, 2*pi] (periodic)
```

The transition map from polar coordinates to Cartesian ones is:

```
sage: E.coord_change(polar, cartesian).display()
x = r*cos(ph)
y = r*sin(ph)
```

```
>>> from sage.all import *
>>> E.coord_change(polar, cartesian).display()
x = r*cos(ph)
y = r*sin(ph)
```

while the reverse one is:

```
sage: E.coord_change(cartesian, polar).display()
r = sqrt(x^2 + y^2)
ph = arctan2(y, x)
```

```
>>> from sage.all import *
>>> E.coord_change(cartesian, polar).display()
r = sqrt(x^2 + y^2)
ph = arctan2(y, x)
```

A point of `E` is constructed from its coordinates (by default in the Cartesian chart):

```
sage: p = E((-1,1), name='p'); p
Point p on the Euclidean plane E^2
sage: p.parent()
Euclidean plane E^2
```

```
>>> from sage.all import *
>>> p = E((-Integer(1),Integer(1)), name='p'); p
Point p on the Euclidean plane E^2
```

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```
>>> p.parent()
Euclidean plane E^2
```

The coordinates of a point are obtained by letting the corresponding chart act on it:

```
sage: cartesian(p)
(-1, 1)
sage: polar(p)
(sqrt(2), 3/4*pi)
```

```
>>> from sage.all import *
>>> cartesian(p)
(-1, 1)
>>> polar(p)
(sqrt(2), 3/4*pi)
```

At this stage, `E` is endowed with three vector frames:

```
sage: E.frames()
[Coordinate frame (E^2, (e_x,e_y)),
 Coordinate frame (E^2, (∂/∂r,∂/∂ph)),
 Vector frame (E^2, (e_r,e_ph))]
```

```
>>> from sage.all import *
>>> E.frames()
[Coordinate frame (E^2, (e_x,e_y)),
 Coordinate frame (E^2, (∂/∂r,∂/∂ph)),
 Vector frame (E^2, (e_r,e_ph))]
```

The third one is the standard orthonormal frame associated with polar coordinates, as we can check from the metric components in it:

```
sage: polar_frame = E.polar_frame(); polar_frame
Vector frame (E^2, (e_r,e_ph))
sage: g[polar_frame,:]
[1 0]
[0 1]
```

```
>>> from sage.all import *
>>> polar_frame = E.polar_frame(); polar_frame
Vector frame (E^2, (e_r,e_ph))
>>> g[polar_frame,:]
[1 0]
[0 1]
```

The expression of the metric tensor in terms of polar coordinates is:

```
sage: g.display(polar)
g = dr⊗dr + r^2 dph⊗dph
```

```
>>> from sage.all import *
>>> g.display(polar)
g = dr⊗dr + r^2 dph⊗dph
```

A vector field on E:

```
sage: v = E.vector_field(-y, x, name='v'); v
Vector field v on the Euclidean plane E^2
sage: v.display()
v = -y e_x + x e_y
sage: v[:]
[-y, x]
```

```
>>> from sage.all import *
>>> v = E.vector_field(-y, x, name='v'); v
Vector field v on the Euclidean plane E^2
>>> v.display()
v = -y e_x + x e_y
>>> v[:]
[-y, x]
```

By default, the components of `v`, as returned by `display` or the bracket operator, refer to the Cartesian frame on `E`; to get the components with respect to the orthonormal polar frame, one has to specify it explicitly, generally along with the polar chart for the coordinate expression of the components:

```
sage: v.display(polar_frame, polar)
v = r e_ph
sage: v[polar_frame,:,polar]
[0, r]
```

```
>>> from sage.all import *
>>> v.display(polar_frame, polar)
v = r e_ph
>>> v[polar_frame,:,polar]
[0, r]
```

Note that the default frame for the display of vector fields can be changed thanks to the method `set_default_frame()`; in the same vein, the default coordinates can be changed via the method `set_default_chart()`:

```
sage: E.set_default_frame(polar_frame)
sage: E.set_default_chart(polar)
sage: v.display()
v = r e_ph
sage: v[:]
[0, r]
sage: E.set_default_frame(E.cartesian_frame())   # revert to Cartesian frame
sage: E.set_default_chart(cartesian)             # and chart
```

```
>>> from sage.all import *
>>> E.set_default_frame(polar_frame)
>>> E.set_default_chart(polar)
>>> v.display()
```

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```
v = r e_ph
>>> v[:]
[0, r]
>>> E.set_default_frame(E.cartesian_frame())  # revert to Cartesian frame
>>> E.set_default_chart(cartesian)            # and chart
```

When defining a vector field from components relative to a vector frame different from the default one, the vector frame has to be specified explicitly:

```
sage: v = E.vector_field(1, 0, frame=polar_frame)
sage: v.display(polar_frame)
e_r
sage: v.display()
x/sqrt(x^2 + y^2) e_x + y/sqrt(x^2 + y^2) e_y
```

```
>>> from sage.all import *
>>> v = E.vector_field(Integer(1), Integer(0), frame=polar_frame)
>>> v.display(polar_frame)
e_r
>>> v.display()
x/sqrt(x^2 + y^2) e_x + y/sqrt(x^2 + y^2) e_y
```

The argument `chart` must be used to specify in which coordinate chart the components are expressed:

```
sage: v = E.vector_field(0, r, frame=polar_frame, chart=polar)
sage: v.display(polar_frame, polar)
r e_ph
sage: v.display()
-y e_x + x e_y
```

```
>>> from sage.all import *
>>> v = E.vector_field(Integer(0), r, frame=polar_frame, chart=polar)
>>> v.display(polar_frame, polar)
r e_ph
>>> v.display()
-y e_x + x e_y
```

It is also possible to pass the components as a dictionary, with a pair (vector frame, chart) as a key:

```
sage: v = E.vector_field({(polar_frame, polar): (0, r)})
sage: v.display(polar_frame, polar)
r e_ph
```

```
>>> from sage.all import *
>>> v = E.vector_field({(polar_frame, polar): (Integer(0), r)})
>>> v.display(polar_frame, polar)
r e_ph
```

The key can be reduced to the vector frame if the chart is the default one:

```
sage: v = E.vector_field({polar_frame: (0, 1)})
sage: v.display(polar_frame)
```

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```
e_ph
```

```
>>> from sage.all import *
>>> v = E.vector_field({polar_frame: (Integer(0), Integer(1))})
>>> v.display(polar_frame)
e_ph
```

Finally, it is possible to construct the vector field without initializing any component:

```
sage: v = E.vector_field(); v
Vector field on the Euclidean plane E^2
```

```
>>> from sage.all import *
>>> v = E.vector_field(); v
Vector field on the Euclidean plane E^2
```

The components can then by set in a second stage, via the square bracket operator, the unset components being assumed to be zero:

```
sage: v[1] = -y
sage: v.display()  # v[2] is zero
-y e_x
sage: v[2] = x
sage: v.display()
-y e_x + x e_y
```

```
>>> from sage.all import *
>>> v[Integer(1)] = -y
>>> v.display()  # v[2] is zero
-y e_x
>>> v[Integer(2)] = x
>>> v.display()
-y e_x + x e_y
```

The above is equivalent to:

```
sage: v[:] = -y, x
sage: v.display()
-y e_x + x e_y
```

```
>>> from sage.all import *
>>> v[:] = -y, x
>>> v.display()
-y e_x + x e_y
```

The square bracket operator can also be used to set components in a vector frame that is not the default one:

```
sage: v = E.vector_field(name='v')
sage: v[polar_frame, 2, polar] = r
sage: v.display(polar_frame, polar)
v = r e_ph
```

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```
sage: v.display()
v = -y e_x + x e_y
```

```
>>> from sage.all import *
>>> v = E.vector_field(name='v')
>>> v[polar_frame, Integer(2), polar] = r
>>> v.display(polar_frame, polar)
v = r e_ph
>>> v.display()
v = -y e_x + x e_y
```

The value of the vector field `v` at point `p`:

```
sage: vp = v.at(p); vp
Vector v at Point p on the Euclidean plane E^2
sage: vp.display()
v = -e_x - e_y
sage: vp.display(polar_frame.at(p))
v = sqrt(2) e_ph
```

```
>>> from sage.all import *
>>> vp = v.at(p); vp
Vector v at Point p on the Euclidean plane E^2
>>> vp.display()
v = -e_x - e_y
>>> vp.display(polar_frame.at(p))
v = sqrt(2) e_ph
```

A scalar field on `E`:

```
sage: f = E.scalar_field(x*y, name='f'); f
Scalar field f on the Euclidean plane E^2
sage: f.display()
f: E^2 → R
   (x, y) ↦ x*y
   (r, ph) ↦ r^2*cos(ph)*sin(ph)
```

```
>>> from sage.all import *
>>> f = E.scalar_field(x*y, name='f'); f
Scalar field f on the Euclidean plane E^2
>>> f.display()
f: E^2 → R
   (x, y) ↦ x*y
   (r, ph) ↦ r^2*cos(ph)*sin(ph)
```

The value of `f` at point `p`:

```
sage: f(p)
-1
```

```
>>> from sage.all import *
>>> f(p)
-1
```

The gradient of `f`:

```
sage: from sage.manifolds.operators import * # to get grad, div, etc.
sage: w = grad(f); w
Vector field grad(f) on the Euclidean plane E^2
sage: w.display()
grad(f) = y e_x + x e_y
sage: w.display(polar_frame, polar)
grad(f) = 2*r*cos(ph)*sin(ph) e_r + (2*cos(ph)^2 - 1)*r e_ph
```

```
>>> from sage.all import *
>>> from sage.manifolds.operators import * # to get grad, div, etc.
>>> w = grad(f); w
Vector field grad(f) on the Euclidean plane E^2
>>> w.display()
grad(f) = y e_x + x e_y
>>> w.display(polar_frame, polar)
grad(f) = 2*r*cos(ph)*sin(ph) e_r + (2*cos(ph)^2 - 1)*r e_ph
```

The dot product of two vector fields:

```
sage: s = v.dot(w); s
Scalar field v.grad(f) on the Euclidean plane E^2
sage: s.display()
v.grad(f): E^2 → ℝ
   (x, y) ↦ x^2 - y^2
   (r, ph) ↦ (2*cos(ph)^2 - 1)*r^2
sage: s.expr()
x^2 - y^2
```

```
>>> from sage.all import *
>>> s = v.dot(w); s
Scalar field v.grad(f) on the Euclidean plane E^2
>>> s.display()
v.grad(f): E^2 → ℝ
   (x, y) ↦ x^2 - y^2
   (r, ph) ↦ (2*cos(ph)^2 - 1)*r^2
>>> s.expr()
x^2 - y^2
```

The norm is related to the dot product by the standard formula:

```
sage: norm(v)^2 == v.dot(v)
True
```

```
>>> from sage.all import *
>>> norm(v)**Integer(2) == v.dot(v)
True
```

The divergence of the vector field v:

```
sage: s = div(v); s
Scalar field div(v) on the Euclidean plane E^2
sage: s.display()
div(v): E^2 → ℝ
   (x, y) ↦ 0
   (r, ph) ↦ 0
```

```
>>> from sage.all import *
>>> s = div(v); s
Scalar field div(v) on the Euclidean plane E^2
>>> s.display()
div(v): E^2 → ℝ
   (x, y) ↦ 0
   (r, ph) ↦ 0
```

Example 2: Vector calculus in the Euclidean 3-space

We start by declaring the 3-dimensional Euclidean space E, with (x,y,z) as Cartesian coordinates:

```
sage: E.<x,y,z> = EuclideanSpace()
sage: E
Euclidean space E^3
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(names=('x', 'y', 'z',)); (x, y, z,) = E._first_ngens(3)
>>> E
Euclidean space E^3
```

A simple vector field on E:

```
sage: v = E.vector_field(-y, x, 0, name='v')
sage: v.display()
v = -y e_x + x e_y
sage: v[:]
[-y, x, 0]
```

```
>>> from sage.all import *
>>> v = E.vector_field(-y, x, Integer(0), name='v')
>>> v.display()
v = -y e_x + x e_y
>>> v[:]
[-y, x, 0]
```

The Euclidean norm of v:

```
sage: s = norm(v); s
Scalar field |v| on the Euclidean space E^3
sage: s.display()
|v|: E^3 → ℝ
   (x, y, z) ↦ sqrt(x^2 + y^2)
```

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```
sage: s.expr()
sqrt(x^2 + y^2)
```

```
>>> from sage.all import *
>>> s = norm(v); s
Scalar field |v| on the Euclidean space E^3
>>> s.display()
|v|: E^3 → ℝ
   (x, y, z) ↦ sqrt(x^2 + y^2)
>>> s.expr()
sqrt(x^2 + y^2)
```

The divergence of v is zero:

```
sage: from sage.manifolds.operators import *
sage: div(v)
Scalar field div(v) on the Euclidean space E^3
sage: div(v).display()
div(v): E^3 → ℝ
   (x, y, z) ↦ 0
```

```
>>> from sage.all import *
>>> from sage.manifolds.operators import *
>>> div(v)
Scalar field div(v) on the Euclidean space E^3
>>> div(v).display()
div(v): E^3 → ℝ
   (x, y, z) ↦ 0
```

while its curl is a constant vector field along e_z:

```
sage: w = curl(v); w
Vector field curl(v) on the Euclidean space E^3
sage: w.display()
curl(v) = 2 e_z
```

```
>>> from sage.all import *
>>> w = curl(v); w
Vector field curl(v) on the Euclidean space E^3
>>> w.display()
curl(v) = 2 e_z
```

The gradient of a scalar field:

```
sage: f = E.scalar_field(sin(x*y*z), name='f')
sage: u = grad(f); u
Vector field grad(f) on the Euclidean space E^3
sage: u.display()
grad(f) = y*z*cos(x*y*z) e_x + x*z*cos(x*y*z) e_y + x*y*cos(x*y*z) e_z
```

```
>>> from sage.all import *
>>> f = E.scalar_field(sin(x*y*z), name='f')
```

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```
>>> u = grad(f); u
Vector field grad(f) on the Euclidean space E^3
>>> u.display()
grad(f) = y*z*cos(x*y*z) e_x + x*z*cos(x*y*z) e_y + x*y*cos(x*y*z) e_z
```

The curl of a gradient is zero:

```
sage: curl(u).display()
curl(grad(f)) = 0
```

```
>>> from sage.all import *
>>> curl(u).display()
curl(grad(f)) = 0
```

The dot product of two vector fields:

```
sage: s = u.dot(v); s
Scalar field grad(f).v on the Euclidean space E^3
sage: s.expr()
(x^2 - y^2)*z*cos(x*y*z)
```

```
>>> from sage.all import *
>>> s = u.dot(v); s
Scalar field grad(f).v on the Euclidean space E^3
>>> s.expr()
(x^2 - y^2)*z*cos(x*y*z)
```

The cross product of two vector fields:

```
sage: a = u.cross(v); a
Vector field grad(f) x v on the Euclidean space E^3
sage: a.display()
grad(f) x v = -x^2*y*cos(x*y*z) e_x - x*y^2*cos(x*y*z) e_y
 + 2*x*y*z*cos(x*y*z) e_z
```

```
>>> from sage.all import *
>>> a = u.cross(v); a
Vector field grad(f) x v on the Euclidean space E^3
>>> a.display()
grad(f) x v = -x^2*y*cos(x*y*z) e_x - x*y^2*cos(x*y*z) e_y
 + 2*x*y*z*cos(x*y*z) e_z
```

The scalar triple product of three vector fields:

```
sage: triple_product = E.scalar_triple_product()
sage: s = triple_product(u, v, w); s
Scalar field epsilon(grad(f),v,curl(v)) on the Euclidean space E^3
sage: s.expr()
4*x*y*z*cos(x*y*z)
```

```
>>> from sage.all import *
>>> triple_product = E.scalar_triple_product()
```

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```
>>> s = triple_product(u, v, w); s
Scalar field epsilon(grad(f),v,curl(v)) on the Euclidean space E^3
>>> s.expr()
4*x*y*z*cos(x*y*z)
```

Let us check that the scalar triple product of u, v and w is $u \cdot (v \times w)$:

```
sage: s == u.dot(v.cross(w))
True
```

```
>>> from sage.all import *
>>> s == u.dot(v.cross(w))
True
```

AUTHORS:

- Eric Gourgoulhon (2018): initial version

REFERENCES:

- M. Berger: *Geometry I* [Ber1987]

class sage.manifolds.differentiable.examples.euclidean.**Euclidean3dimSpace**(*name=None*, *latex_name=None*, *coordinates='Cartesian'*, *symbols=None*, *metric_name='g'*, *metric_latex_name=None*, *start_index=1*, *base_manifold=None*, *category=None*, *unique_tag=None*)

Bases: `EuclideanSpace`

3-dimensional Euclidean space.

A *3-dimensional Euclidean space* is an affine space E, whose associated vector space is a 3-dimensional vector space over $\mathbf{R}$ and is equipped with a positive definite symmetric bilinear form, called the *scalar product* or *dot product*.

The class `Euclidean3dimSpace` inherits from `PseudoRiemannianManifold` (via `EuclideanSpace`) since a 3-dimensional Euclidean space can be viewed as a Riemannian manifold that is diffeomorphic to $\mathbf{R}^3$ and that has a flat metric g. The Euclidean scalar product is the one defined by the Riemannian metric g.

INPUT:

- `name` – (default: `None`) string; name (symbol) given to the Euclidean 3-space; if `None`, the name will be set to `'E^3'`
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the Euclidean 3-space; if `None`, it is set to `'\mathbb{E}^{3}'` if `name` is `None` and to `name` otherwise
- `coordinates` – (default: `'Cartesian'`) string describing the type of coordinates to be initialized at the Euclidean 3-space creation; allowed values are `'Cartesian'` (see `cartesian_coordinates()`),

'spherical' (see *spherical_coordinates()*) and 'cylindrical' (see *cylindrical_coordinates()*)

- symbols – (default: None) string defining the coordinate text symbols and LaTeX symbols, with the same conventions as the argument coordinates in *RealDiffChart*, namely symbols is a string of coordinate fields separated by a blank space, where each field contains the coordinate's text symbol and possibly the coordinate's LaTeX symbol (when the latter is different from the text symbol), both symbols being separated by a colon (:); if None, the symbols will be automatically generated according to the value of coordinates
- metric_name – (default: 'g') string; name (symbol) given to the Euclidean metric tensor
- metric_latex_name – (default: None) string; LaTeX symbol to denote the Euclidean metric tensor; if none is provided, it is set to metric_name
- start_index – (default: 1) integer; lower value of the range of indices used for "indexed objects" in the Euclidean 3-space, e.g. coordinates of a chart
- base_manifold – (default: None) if not None, must be a Euclidean 3-space; the created object is then an open subset of base_manifold
- category – (default: None) to specify the category; if None, Manifolds(RR).Smooth() & MetricSpaces().Complete() is assumed
- names – (default: None) unused argument, except if symbols is not provided; it must then be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator <,> is used)
- init_coord_methods – (default: None) dictionary of methods to initialize the various type of coordinates, with each key being a string describing the type of coordinates; to be used by derived classes only
- unique_tag – (default: None) tag used to force the construction of a new object when all the other arguments have been used previously (without unique_tag, the UniqueRepresentation behavior inherited from *PseudoRiemannianManifold* would return the previously constructed object corresponding to these arguments)

EXAMPLES:

A 3-dimensional Euclidean space:

```
sage: E = EuclideanSpace(3); E
Euclidean space E^3
sage: latex(E)
\mathbb{E}^{3}
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(Integer(3)); E
Euclidean space E^3
>>> latex(E)
\mathbb{E}^{3}
```

E belongs to the class *Euclidean3dimSpace* (actually to a dynamically generated subclass of it via SageMath's category framework):

```
sage: type(E)
<class 'sage.manifolds.differentiable.examples.euclidean.Euclidean3dimSpace_with_
→category'>
```

```
>>> from sage.all import *
>>> type(E)
```

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```
<class 'sage.manifolds.differentiable.examples.euclidean.Euclidean3dimSpace_with_
↪category'>
```

E is both a real smooth manifold of dimension 3 and a complete metric space:

```
sage: E.category()
Join of Category of smooth manifolds over Real Field with 53 bits of
 precision and Category of connected manifolds over Real Field with
 53 bits of precision and Category of complete metric spaces
sage: dim(E)
3
```

```
>>> from sage.all import *
>>> E.category()
Join of Category of smooth manifolds over Real Field with 53 bits of
 precision and Category of connected manifolds over Real Field with
 53 bits of precision and Category of complete metric spaces
>>> dim(E)
3
```

It is endowed with a default coordinate chart, which is that of Cartesian coordinates (x, y, z):

```
sage: E.atlas()
[Chart (E^3, (x, y, z))]
sage: E.default_chart()
Chart (E^3, (x, y, z))
sage: cartesian = E.cartesian_coordinates()
sage: cartesian is E.default_chart()
True
```

```
>>> from sage.all import *
>>> E.atlas()
[Chart (E^3, (x, y, z))]
>>> E.default_chart()
Chart (E^3, (x, y, z))
>>> cartesian = E.cartesian_coordinates()
>>> cartesian is E.default_chart()
True
```

A point of E:

```
sage: p = E((3,-2,1)); p
Point on the Euclidean space E^3
sage: cartesian(p)
(3, -2, 1)
sage: p in E
True
sage: p.parent() is E
True
```

```
>>> from sage.all import *
>>> p = E((Integer(3),-Integer(2),Integer(1))); p
```

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```
Point on the Euclidean space E^3
>>> cartesian(p)
(3, -2, 1)
>>> p in E
True
>>> p.parent() is E
True
```

`E` is endowed with a default metric tensor, which defines the Euclidean scalar product:

```
sage: g = E.metric(); g
Riemannian metric g on the Euclidean space E^3
sage: g.display()
g = dx⊗dx + dy⊗dy + dz⊗dz
```

```
>>> from sage.all import *
>>> g = E.metric(); g
Riemannian metric g on the Euclidean space E^3
>>> g.display()
g = dx⊗dx + dy⊗dy + dz⊗dz
```

Curvilinear coordinates can be introduced on `E`: see *spherical_coordinates()* and *cylindrical_coordinates()*.

See also:

Example 2: Vector calculus in the Euclidean 3-space

cartesian_coordinates(*symbols=None*, *names=None*)

Return the chart of Cartesian coordinates, possibly creating it if it does not already exist.

INPUT:

- `symbols` – (default: `None`) string defining the coordinate text symbols and LaTeX symbols, with the same conventions as the argument `coordinates` in *RealDiffChart*; this is used only if the Cartesian chart has not been already defined; if `None` the symbols are generated as (x, y, z).
- `names` – (default: `None`) unused argument, except if `symbols` is not provided; it must be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)

OUTPUT:

- the chart of Cartesian coordinates, as an instance of *RealDiffChart*

EXAMPLES:

```
sage: E = EuclideanSpace(3)
sage: E.cartesian_coordinates()
Chart (E^3, (x, y, z))
sage: E.cartesian_coordinates().coord_range()
x: (-oo, +oo); y: (-oo, +oo); z: (-oo, +oo)
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(Integer(3))
```

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```
>>> E.cartesian_coordinates()
Chart (E^3, (x, y, z))
>>> E.cartesian_coordinates().coord_range()
x: (-oo, +oo); y: (-oo, +oo); z: (-oo, +oo)
```

An example where the Cartesian coordinates have not been previously created:

```
sage: E = EuclideanSpace(3, coordinates='spherical')
sage: E.atlas()  # only spherical coordinates have been initialized
[Chart (E^3, (r, th, ph))]
sage: E.cartesian_coordinates(symbols='X Y Z')
Chart (E^3, (X, Y, Z))
sage: E.atlas()  # the Cartesian chart has been added to the atlas
[Chart (E^3, (r, th, ph)), Chart (E^3, (X, Y, Z))]
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(Integer(3), coordinates='spherical')
>>> E.atlas()  # only spherical coordinates have been initialized
[Chart (E^3, (r, th, ph))]
>>> E.cartesian_coordinates(symbols='X Y Z')
Chart (E^3, (X, Y, Z))
>>> E.atlas()  # the Cartesian chart has been added to the atlas
[Chart (E^3, (r, th, ph)), Chart (E^3, (X, Y, Z))]
```

The coordinate variables are returned by the square bracket operator:

```
sage: E.cartesian_coordinates()[1]
X
sage: E.cartesian_coordinates()[3]
Z
sage: E.cartesian_coordinates()[:]
(X, Y, Z)
```

```
>>> from sage.all import *
>>> E.cartesian_coordinates()[Integer(1)]
X
>>> E.cartesian_coordinates()[Integer(3)]
Z
>>> E.cartesian_coordinates()[:]
(X, Y, Z)
```

It is also possible to use the operator `<,>` to set symbolic variable containing the coordinates:

```
sage: E = EuclideanSpace(3, coordinates='spherical')
sage: cartesian.<u,v,w> = E.cartesian_coordinates()
sage: cartesian
Chart (E^3, (u, v, w))
sage: u, v, w
(u, v, w)
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(Integer(3), coordinates='spherical')
```

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```
>>> cartesian = E.cartesian_coordinates(names=('u', 'v', 'w',)); (u, v, w,) =␣
↪cartesian._first_ngens(3)
>>> cartesian
Chart (E^3, (u, v, w))
>>> u, v, w
(u, v, w)
```

The command `cartesian.<u,v,w> = E.cartesian_coordinates()` is actually a shortcut for:

```
sage: cartesian = E.cartesian_coordinates(symbols='u v w')
sage: u, v, w = cartesian[:]
```

```
>>> from sage.all import *
>>> cartesian = E.cartesian_coordinates(symbols='u v w')
>>> u, v, w = cartesian[:]
```

cylindrical_coordinates(*symbols=None*, *names=None*)

Return the chart of cylindrical coordinates, possibly creating it if it does not already exist.

INPUT:

- `symbols` – (default: `None`) string defining the coordinate text symbols and LaTeX symbols, with the same conventions as the argument `coordinates` in *RealDiffChart*; this is used only if the cylindrical chart has not been already defined; if `None` the symbols are generated as (ρ, ϕ, z).
- `names` – (default: `None`) unused argument, except if `symbols` is not provided; it must be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)

OUTPUT:

- the chart of cylindrical coordinates, as an instance of *RealDiffChart*

EXAMPLES:

```
sage: E = EuclideanSpace(3)
sage: E.cylindrical_coordinates()
Chart (E^3, (rh, ph, z))
sage: latex(_)
\left(\mathbb{E}^{3},({\rho}, {\phi}, z)\right)
sage: E.cylindrical_coordinates().coord_range()
rh: (0, +oo); ph: [0, 2*pi] (periodic); z: (-oo, +oo)
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(Integer(3))
>>> E.cylindrical_coordinates()
Chart (E^3, (rh, ph, z))
>>> latex(_)
\left(\mathbb{E}^{3},({\rho}, {\phi}, z)\right)
>>> E.cylindrical_coordinates().coord_range()
rh: (0, +oo); ph: [0, 2*pi] (periodic); z: (-oo, +oo)
```

The relation to Cartesian coordinates is:

```
sage: E.coord_change(E.cylindrical_coordinates(),
....:                E.cartesian_coordinates()).display()
```

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```
x = rh*cos(ph)
y = rh*sin(ph)
z = z
sage: E.coord_change(E.cartesian_coordinates(),
....:                E.cylindrical_coordinates()).display()
rh = sqrt(x^2 + y^2)
ph = arctan2(y, x)
z = z
```

```
>>> from sage.all import *
>>> E.coord_change(E.cylindrical_coordinates(),
...                E.cartesian_coordinates()).display()
x = rh*cos(ph)
y = rh*sin(ph)
z = z
>>> E.coord_change(E.cartesian_coordinates(),
...                E.cylindrical_coordinates()).display()
rh = sqrt(x^2 + y^2)
ph = arctan2(y, x)
z = z
```

The coordinate variables are returned by the square bracket operator:

```
sage: E.cylindrical_coordinates()[1]
rh
sage: E.cylindrical_coordinates()[3]
z
sage: E.cylindrical_coordinates()[:]
(rh, ph, z)
```

```
>>> from sage.all import *
>>> E.cylindrical_coordinates()[Integer(1)]
rh
>>> E.cylindrical_coordinates()[Integer(3)]
z
>>> E.cylindrical_coordinates()[:]
(rh, ph, z)
```

They can also be obtained via the operator <,>:

```
sage: cylindrical.<rh,ph,z> = E.cylindrical_coordinates()
sage: cylindrical
Chart (E^3, (rh, ph, z))
sage: rh, ph, z
(rh, ph, z)
```

```
>>> from sage.all import *
>>> cylindrical = E.cylindrical_coordinates(names=('rh', 'ph', 'z',)); (rh,␣
→ph, z,) = cylindrical._first_ngens(3)
>>> cylindrical
Chart (E^3, (rh, ph, z))
```

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```
>>> rh, ph, z
(rh, ph, z)
```

Actually, `cylindrical.<rh,ph,z> = E.cylindrical_coordinates()` is a shortcut for:

```
sage: cylindrical = E.cylindrical_coordinates()
sage: rh, ph, z = cylindrical[:]
```

```
>>> from sage.all import *
>>> cylindrical = E.cylindrical_coordinates()
>>> rh, ph, z = cylindrical[:]
```

The coordinate symbols can be customized:

```
sage: E = EuclideanSpace(3)
sage: E.cylindrical_coordinates(symbols=r"R Phi:\Phi Z")
Chart (E^3, (R, Phi, Z))
sage: latex(E.cylindrical_coordinates())
\left(\mathbb{E}^{3},(R, {\Phi}, Z)\right)
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(Integer(3))
>>> E.cylindrical_coordinates(symbols=r"R Phi:\Phi Z")
Chart (E^3, (R, Phi, Z))
>>> latex(E.cylindrical_coordinates())
\left(\mathbb{E}^{3},(R, {\Phi}, Z)\right)
```

Note that if the cylindrical coordinates have been already initialized, the argument `symbols` has no effect:

```
sage: E.cylindrical_coordinates(symbols=r"rh:\rho ph:\phi z")
Chart (E^3, (R, Phi, Z))
```

```
>>> from sage.all import *
>>> E.cylindrical_coordinates(symbols=r"rh:\rho ph:\phi z")
Chart (E^3, (R, Phi, Z))
```

cylindrical_frame()

Return the orthonormal vector frame associated with cylindrical coordinates.

OUTPUT: *VectorFrame*

EXAMPLES:

```
sage: E = EuclideanSpace(3)
sage: E.cylindrical_frame()
Vector frame (E^3, (e_rh,e_ph,e_z))
sage: E.cylindrical_frame()[1]
Vector field e_rh on the Euclidean space E^3
sage: E.cylindrical_frame()[:]
(Vector field e_rh on the Euclidean space E^3,
 Vector field e_ph on the Euclidean space E^3,
 Vector field e_z on the Euclidean space E^3)
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(Integer(3))
>>> E.cylindrical_frame()
Vector frame (E^3, (e_rh,e_ph,e_z))
>>> E.cylindrical_frame()[Integer(1)]
Vector field e_rh on the Euclidean space E^3
>>> E.cylindrical_frame()[:]
(Vector field e_rh on the Euclidean space E^3,
 Vector field e_ph on the Euclidean space E^3,
 Vector field e_z on the Euclidean space E^3)
```

The cylindrical frame expressed in terms of the Cartesian one:

```
sage: for e in E.cylindrical_frame():
....:     e.display(E.cartesian_frame(), E.cylindrical_coordinates())
e_rh = cos(ph) e_x + sin(ph) e_y
e_ph = -sin(ph) e_x + cos(ph) e_y
e_z = e_z
```

```
>>> from sage.all import *
>>> for e in E.cylindrical_frame():
...     e.display(E.cartesian_frame(), E.cylindrical_coordinates())
e_rh = cos(ph) e_x + sin(ph) e_y
e_ph = -sin(ph) e_x + cos(ph) e_y
e_z = e_z
```

The orthonormal frame (e_r, e_ϕ, e_z) expressed in terms of the coordinate frame $\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial z}\right)$:

```
sage: for e in E.cylindrical_frame():
....:     e.display(E.cylindrical_coordinates())
e_rh = ∂/∂rh
e_ph = 1/rh ∂/∂ph
e_z = ∂/∂z
```

```
>>> from sage.all import *
>>> for e in E.cylindrical_frame():
...     e.display(E.cylindrical_coordinates())
e_rh = ∂/∂rh
e_ph = 1/rh ∂/∂ph
e_z = ∂/∂z
```

scalar_triple_product(*name=None*, *latex_name=None*)

Return the scalar triple product operator, as a 3-form.

The *scalar triple product* (also called *mixed product*) of three vector fields u, v and w defined on a Euclidean space E is the scalar field

$$\epsilon(u, v, w) = u \cdot (v \times w).$$

The scalar triple product operator ϵ is a *3-form*, i.e. a field of fully antisymmetric trilinear forms; it is also called the *volume form* of E or the *Levi-Civita tensor* of E.

INPUT:

- `name` – (default: `None`) string; name given to the scalar triple product operator; if `None`, `'epsilon'` is used
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the scalar triple product; if `None`, it is set to `r'\epsilon'` if `name` is `None` and to `name` otherwise.

OUTPUT:

- the scalar triple product operator ϵ, as an instance of `DiffFormParal`

EXAMPLES:

```
sage: E.<x,y,z> = EuclideanSpace()
sage: triple_product = E.scalar_triple_product()
sage: triple_product
3-form epsilon on the Euclidean space E^3
sage: latex(triple_product)
\epsilon
sage: u = E.vector_field(x, y, z, name='u')
sage: v = E.vector_field(-y, x, 0, name='v')
sage: w = E.vector_field(y*z, x*z, x*y, name='w')
sage: s = triple_product(u, v, w); s
Scalar field epsilon(u,v,w) on the Euclidean space E^3
sage: s.display()
epsilon(u,v,w): E^3 → ℝ
   (x, y, z) ↦ x^3*y + x*y^3 - 2*x*y*z^2
sage: s.expr()
x^3*y + x*y^3 - 2*x*y*z^2
sage: latex(s)
\epsilon\left(u,v,w\right)
sage: s == - triple_product(w, v, u)
True
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(names=('x', 'y', 'z',)); (x, y, z,) = E._first_ngens(3)
>>> triple_product = E.scalar_triple_product()
>>> triple_product
3-form epsilon on the Euclidean space E^3
>>> latex(triple_product)
\epsilon
>>> u = E.vector_field(x, y, z, name='u')
>>> v = E.vector_field(-y, x, Integer(0), name='v')
>>> w = E.vector_field(y*z, x*z, x*y, name='w')
>>> s = triple_product(u, v, w); s
Scalar field epsilon(u,v,w) on the Euclidean space E^3
>>> s.display()
epsilon(u,v,w): E^3 → ℝ
   (x, y, z) ↦ x^3*y + x*y^3 - 2*x*y*z^2
>>> s.expr()
x^3*y + x*y^3 - 2*x*y*z^2
>>> latex(s)
\epsilon\left(u,v,w\right)
>>> s == - triple_product(w, v, u)
True
```

Check of the identity $\epsilon(u, v, w) = u \cdot (v \times w)$:

```
sage: s == u.dot(v.cross(w))
True
```

```
>>> from sage.all import *
>>> s == u.dot(v.cross(w))
True
```

Customizing the name:

```
sage: E.scalar_triple_product(name='S')
3-form S on the Euclidean space E^3
sage: latex(_)
S
sage: E.scalar_triple_product(name='Omega', latex_name=r'\Omega')
3-form Omega on the Euclidean space E^3
sage: latex(_)
\Omega
```

```
>>> from sage.all import *
>>> E.scalar_triple_product(name='S')
3-form S on the Euclidean space E^3
>>> latex(_)
S
>>> E.scalar_triple_product(name='Omega', latex_name=r'\Omega')
3-form Omega on the Euclidean space E^3
>>> latex(_)
\Omega
```

spherical_coordinates(*symbols=None*, *names=None*)

Return the chart of spherical coordinates, possibly creating it if it does not already exist.

INPUT:

- `symbols` – (default: `None`) string defining the coordinate text symbols and LaTeX symbols, with the same conventions as the argument `coordinates` in *RealDiffChart*; this is used only if the spherical chart has not been already defined; if `None` the symbols are generated as (r, θ, ϕ).
- `names` – (default: `None`) unused argument, except if `symbols` is not provided; it must be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)

OUTPUT:

- the chart of spherical coordinates, as an instance of *RealDiffChart*

EXAMPLES:

```
sage: E = EuclideanSpace(3)
sage: E.spherical_coordinates()
Chart (E^3, (r, th, ph))
sage: latex(_)
\left(\mathbb{E}^{3},(r, {\theta}, {\phi})\right)
sage: E.spherical_coordinates().coord_range()
r: (0, +oo); th: (0, pi); ph: [0, 2*pi] (periodic)
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(Integer(3))
>>> E.spherical_coordinates()
Chart (E^3, (r, th, ph))
>>> latex(_)
\left(\mathbb{E}^{3},(r, {\theta}, {\phi})\right)
>>> E.spherical_coordinates().coord_range()
r: (0, +oo); th: (0, pi); ph: [0, 2*pi] (periodic)
```

The relation to Cartesian coordinates is:

```
sage: E.coord_change(E.spherical_coordinates(),
....:                E.cartesian_coordinates()).display()
x = r*cos(ph)*sin(th)
y = r*sin(ph)*sin(th)
z = r*cos(th)
sage: E.coord_change(E.cartesian_coordinates(),
....:                E.spherical_coordinates()).display()
r = sqrt(x^2 + y^2 + z^2)
th = arctan2(sqrt(x^2 + y^2), z)
ph = arctan2(y, x)
```

```
>>> from sage.all import *
>>> E.coord_change(E.spherical_coordinates(),
...                E.cartesian_coordinates()).display()
x = r*cos(ph)*sin(th)
y = r*sin(ph)*sin(th)
z = r*cos(th)
>>> E.coord_change(E.cartesian_coordinates(),
...                E.spherical_coordinates()).display()
r = sqrt(x^2 + y^2 + z^2)
th = arctan2(sqrt(x^2 + y^2), z)
ph = arctan2(y, x)
```

The coordinate variables are returned by the square bracket operator:

```
sage: E.spherical_coordinates()[1]
r
sage: E.spherical_coordinates()[3]
ph
sage: E.spherical_coordinates()[:]
(r, th, ph)
```

```
>>> from sage.all import *
>>> E.spherical_coordinates()[Integer(1)]
r
>>> E.spherical_coordinates()[Integer(3)]
ph
>>> E.spherical_coordinates()[:]
(r, th, ph)
```

They can also be obtained via the operator `<,>`:

```
sage: spherical.<r,th,ph> = E.spherical_coordinates()
sage: spherical
Chart (E^3, (r, th, ph))
sage: r, th, ph
(r, th, ph)
```

```
>>> from sage.all import *
>>> spherical = E.spherical_coordinates(names=('r', 'th', 'ph',)); (r, th, ph,
↪) = spherical._first_ngens(3)
>>> spherical
Chart (E^3, (r, th, ph))
>>> r, th, ph
(r, th, ph)
```

Actually, `spherical.<r,th,ph> = E.spherical_coordinates()` is a shortcut for:

```
sage: spherical = E.spherical_coordinates()
sage: r, th, ph = spherical[:]
```

```
>>> from sage.all import *
>>> spherical = E.spherical_coordinates()
>>> r, th, ph = spherical[:]
```

The coordinate symbols can be customized:

```
sage: E = EuclideanSpace(3)
sage: E.spherical_coordinates(symbols=r"R T:\Theta F:\Phi")
Chart (E^3, (R, T, F))
sage: latex(E.spherical_coordinates())
\left(\mathbb{E}^{3},(R, {\Theta}, {\Phi})\right)
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(Integer(3))
>>> E.spherical_coordinates(symbols=r"R T:\Theta F:\Phi")
Chart (E^3, (R, T, F))
>>> latex(E.spherical_coordinates())
\left(\mathbb{E}^{3},(R, {\Theta}, {\Phi})\right)
```

Note that if the spherical coordinates have been already initialized, the argument `symbols` has no effect:

```
sage: E.spherical_coordinates(symbols=r"r th:\theta ph:\phi")
Chart (E^3, (R, T, F))
```

```
>>> from sage.all import *
>>> E.spherical_coordinates(symbols=r"r th:\theta ph:\phi")
Chart (E^3, (R, T, F))
```

spherical_frame()

Return the orthonormal vector frame associated with spherical coordinates.

OUTPUT: *VectorFrame*

EXAMPLES:

```
sage: E = EuclideanSpace(3)
sage: E.spherical_frame()
Vector frame (E^3, (e_r,e_th,e_ph))
sage: E.spherical_frame()[1]
Vector field e_r on the Euclidean space E^3
sage: E.spherical_frame()[:]
(Vector field e_r on the Euclidean space E^3,
 Vector field e_th on the Euclidean space E^3,
 Vector field e_ph on the Euclidean space E^3)
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(Integer(3))
>>> E.spherical_frame()
Vector frame (E^3, (e_r,e_th,e_ph))
>>> E.spherical_frame()[Integer(1)]
Vector field e_r on the Euclidean space E^3
>>> E.spherical_frame()[:]
(Vector field e_r on the Euclidean space E^3,
 Vector field e_th on the Euclidean space E^3,
 Vector field e_ph on the Euclidean space E^3)
```

The spherical frame expressed in terms of the Cartesian one:

```
sage: for e in E.spherical_frame():
....:     e.display(E.cartesian_frame(), E.spherical_coordinates())
e_r = cos(ph)*sin(th) e_x + sin(ph)*sin(th) e_y + cos(th) e_z
e_th = cos(ph)*cos(th) e_x + cos(th)*sin(ph) e_y - sin(th) e_z
e_ph = -sin(ph) e_x + cos(ph) e_y
```

```
>>> from sage.all import *
>>> for e in E.spherical_frame():
...     e.display(E.cartesian_frame(), E.spherical_coordinates())
e_r = cos(ph)*sin(th) e_x + sin(ph)*sin(th) e_y + cos(th) e_z
e_th = cos(ph)*cos(th) e_x + cos(th)*sin(ph) e_y - sin(th) e_z
e_ph = -sin(ph) e_x + cos(ph) e_y
```

The orthonormal frame (e_r, e_θ, e_ϕ) expressed in terms of the coordinate frame $\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\right)$:

```
sage: for e in E.spherical_frame():
....:     e.display(E.spherical_coordinates())
e_r = ∂/∂r
e_th = 1/r ∂/∂th
e_ph = 1/(r*sin(th)) ∂/∂ph
```

```
>>> from sage.all import *
>>> for e in E.spherical_frame():
...     e.display(E.spherical_coordinates())
e_r = ∂/∂r
e_th = 1/r ∂/∂th
e_ph = 1/(r*sin(th)) ∂/∂ph
```

class sage.manifolds.differentiable.examples.euclidean.**EuclideanPlane**(*name=None*, *latex_name=None*, *coordinates='Cartesian'*, *symbols=None*, *metric_name='g'*, *metric_latex_name=None*, *start_index=1*, *base_manifold=None*, *category=None*, *unique_tag=None*)

Bases: *EuclideanSpace*

Euclidean plane.

A *Euclidean plane* is an affine space E, whose associated vector space is a 2-dimensional vector space over **R** and is equipped with a positive definite symmetric bilinear form, called the *scalar product* or *dot product*.

The class *EuclideanPlane* inherits from *PseudoRiemannianManifold* (via *EuclideanSpace*) since a Euclidean plane can be viewed as a Riemannian manifold that is diffeomorphic to $\mathbf{R}^2$ and that has a flat metric g. The Euclidean scalar product is the one defined by the Riemannian metric g.

INPUT:

- `name` – (default: `None`) string; name (symbol) given to the Euclidean plane; if `None`, the name will be set to `'E^2'`
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the Euclidean plane; if `None`, it is set to `'\mathbb{E}^{2}'` if `name` is `None` and to `name` otherwise
- `coordinates` – (default: `'Cartesian'`) string describing the type of coordinates to be initialized at the Euclidean plane creation; allowed values are `'Cartesian'` (see *cartesian_coordinates()*) and `'polar'` (see *polar_coordinates()*)
- `symbols` – (default: `None`) string defining the coordinate text symbols and LaTeX symbols, with the same conventions as the argument `coordinates` in *RealDiffChart*, namely `symbols` is a string of coordinate fields separated by a blank space, where each field contains the coordinate's text symbol and possibly the coordinate's LaTeX symbol (when the latter is different from the text symbol), both symbols being separated by a colon (`:`); if `None`, the symbols will be automatically generated according to the value of `coordinates`
- `metric_name` – (default: `'g'`) string; name (symbol) given to the Euclidean metric tensor
- `metric_latex_name` – (default: `None`) string; LaTeX symbol to denote the Euclidean metric tensor; if none is provided, it is set to `metric_name`
- `start_index` – (default: 1) integer; lower value of the range of indices used for "indexed objects" in the Euclidean plane, e.g. coordinates of a chart
- `base_manifold` – (default: `None`) if not `None`, must be an Euclidean plane; the created object is then an open subset of `base_manifold`
- `category` – (default: `None`) to specify the category; if `None`, `Manifolds(RR).Smooth() & MetricSpaces().Complete()` is assumed
- `names` – (default: `None`) unused argument, except if `symbols` is not provided; it must then be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)
- `init_coord_methods` – (default: `None`) dictionary of methods to initialize the various type of coordinates, with each key being a string describing the type of coordinates; to be used by derived classes only
- `unique_tag` – (default: `None`) tag used to force the construction of a new object when all the other arguments have been used previously (without `unique_tag`, the `UniqueRepresentation` behavior inherited

from `PseudoRiemannianManifold` would return the previously constructed object corresponding to these arguments)

EXAMPLES:

One creates a Euclidean plane `E` with:

```
sage: E.<x,y> = EuclideanSpace(); E
Euclidean plane E^2
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(names=('x', 'y',)); (x, y,) = E._first_ngens(2); E
Euclidean plane E^2
```

`E` is both a real smooth manifold of dimension 2 and a complete metric space:

```
sage: E.category()
Join of Category of smooth manifolds over Real Field with 53 bits of
 precision and Category of connected manifolds over Real Field with
 53 bits of precision and Category of complete metric spaces
sage: dim(E)
2
```

```
>>> from sage.all import *
>>> E.category()
Join of Category of smooth manifolds over Real Field with 53 bits of
 precision and Category of connected manifolds over Real Field with
 53 bits of precision and Category of complete metric spaces
>>> dim(E)
2
```

It is endowed with a default coordinate chart, which is that of Cartesian coordinates (x, y):

```
sage: E.atlas()
[Chart (E^2, (x, y))]
sage: E.default_chart()
Chart (E^2, (x, y))
sage: cartesian = E.cartesian_coordinates()
sage: cartesian is E.default_chart()
True
```

```
>>> from sage.all import *
>>> E.atlas()
[Chart (E^2, (x, y))]
>>> E.default_chart()
Chart (E^2, (x, y))
>>> cartesian = E.cartesian_coordinates()
>>> cartesian is E.default_chart()
True
```

A point of `E`:

```
sage: p = E((3,-2)); p
Point on the Euclidean plane E^2
```

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```
sage: cartesian(p)
(3, -2)
sage: p in E
True
sage: p.parent() is E
True
```

```
>>> from sage.all import *
>>> p = E((Integer(3),-Integer(2))); p
Point on the Euclidean plane E^2
>>> cartesian(p)
(3, -2)
>>> p in E
True
>>> p.parent() is E
True
```

`E` is endowed with a default metric tensor, which defines the Euclidean scalar product:

```
sage: g = E.metric(); g
Riemannian metric g on the Euclidean plane E^2
sage: g.display()
g = dx⊗dx + dy⊗dy
```

```
>>> from sage.all import *
>>> g = E.metric(); g
Riemannian metric g on the Euclidean plane E^2
>>> g.display()
g = dx⊗dx + dy⊗dy
```

Curvilinear coordinates can be introduced on `E`: see `polar_coordinates()`.

See also

Example 1: the Euclidean plane

cartesian_coordinates(*symbols=None*, *names=None*)

Return the chart of Cartesian coordinates, possibly creating it if it does not already exist.

INPUT:

- `symbols` – (default: `None`) string defining the coordinate text symbols and LaTeX symbols, with the same conventions as the argument `coordinates` in `RealDiffChart`; this is used only if the Cartesian chart has not been already defined; if `None` the symbols are generated as (x, y).
- `names` – (default: `None`) unused argument, except if `symbols` is not provided; it must be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)

OUTPUT:

- the chart of Cartesian coordinates, as an instance of `RealDiffChart`

EXAMPLES:

```
sage: E = EuclideanSpace(2)
sage: E.cartesian_coordinates()
Chart (E^2, (x, y))
sage: E.cartesian_coordinates().coord_range()
x: (-oo, +oo); y: (-oo, +oo)
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(Integer(2))
>>> E.cartesian_coordinates()
Chart (E^2, (x, y))
>>> E.cartesian_coordinates().coord_range()
x: (-oo, +oo); y: (-oo, +oo)
```

An example where the Cartesian coordinates have not been previously created:

```
sage: E = EuclideanSpace(2, coordinates='polar')
sage: E.atlas()  # only polar coordinates have been initialized
[Chart (E^2, (r, ph))]
sage: E.cartesian_coordinates(symbols='X Y')
Chart (E^2, (X, Y))
sage: E.atlas()  # the Cartesian chart has been added to the atlas
[Chart (E^2, (r, ph)), Chart (E^2, (X, Y))]
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(Integer(2), coordinates='polar')
>>> E.atlas()  # only polar coordinates have been initialized
[Chart (E^2, (r, ph))]
>>> E.cartesian_coordinates(symbols='X Y')
Chart (E^2, (X, Y))
>>> E.atlas()  # the Cartesian chart has been added to the atlas
[Chart (E^2, (r, ph)), Chart (E^2, (X, Y))]
```

Note that if the Cartesian coordinates have been already initialized, the argument `symbols` has no effect:

```
sage: E.cartesian_coordinates(symbols='x y')
Chart (E^2, (X, Y))
```

```
>>> from sage.all import *
>>> E.cartesian_coordinates(symbols='x y')
Chart (E^2, (X, Y))
```

The coordinate variables are returned by the square bracket operator:

```
sage: E.cartesian_coordinates()[1]
X
sage: E.cartesian_coordinates()[2]
Y
sage: E.cartesian_coordinates()[:]
(X, Y)
```

```
>>> from sage.all import *
>>> E.cartesian_coordinates()[Integer(1)]
```

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```
X
>>> E.cartesian_coordinates()[Integer(2)]
Y
>>> E.cartesian_coordinates()[:]
(X, Y)
```

It is also possible to use the operator `<,>` to set symbolic variable containing the coordinates:

```
sage: E = EuclideanSpace(2, coordinates='polar')
sage: cartesian.<u,v> = E.cartesian_coordinates()
sage: cartesian
Chart (E^2, (u, v))
sage: u,v
(u, v)
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(Integer(2), coordinates='polar')
>>> cartesian = E.cartesian_coordinates(names=('u', 'v',)); (u, v,) =␣
↪cartesian._first_ngens(2)
>>> cartesian
Chart (E^2, (u, v))
>>> u,v
(u, v)
```

The command `cartesian.<u,v> = E.cartesian_coordinates()` is actually a shortcut for:

```
sage: cartesian = E.cartesian_coordinates(symbols='u v')
sage: u, v = cartesian[:]
```

```
>>> from sage.all import *
>>> cartesian = E.cartesian_coordinates(symbols='u v')
>>> u, v = cartesian[:]
```

polar_coordinates(*symbols=None*, *names=None*)

Return the chart of polar coordinates, possibly creating it if it does not already exist.

INPUT:

- `symbols` – (default: `None`) string defining the coordinate text symbols and LaTeX symbols, with the same conventions as the argument `coordinates` in *RealDiffChart*; this is used only if the polar chart has not been already defined; if `None` the symbols are generated as (r, ϕ).
- `names` – (default: `None`) unused argument, except if `symbols` is not provided; it must be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)

OUTPUT:

- the chart of polar coordinates, as an instance of *RealDiffChart*

EXAMPLES:

```
sage: E = EuclideanSpace(2)
sage: E.polar_coordinates()
Chart (E^2, (r, ph))
sage: latex(_)
```

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```
\left(\mathbb{E}^{2},(r, {\phi})\right)
sage: E.polar_coordinates().coord_range()
r: (0, +oo); ph: [0, 2*pi] (periodic)
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(Integer(2))
>>> E.polar_coordinates()
Chart (E^2, (r, ph))
>>> latex(_)
\left(\mathbb{E}^{2},(r, {\phi})\right)
>>> E.polar_coordinates().coord_range()
r: (0, +oo); ph: [0, 2*pi] (periodic)
```

The relation to Cartesian coordinates is:

```
sage: E.coord_change(E.polar_coordinates(),
....:                E.cartesian_coordinates()).display()
x = r*cos(ph)
y = r*sin(ph)
sage: E.coord_change(E.cartesian_coordinates(),
....:                E.polar_coordinates()).display()
r = sqrt(x^2 + y^2)
ph = arctan2(y, x)
```

```
>>> from sage.all import *
>>> E.coord_change(E.polar_coordinates(),
...                E.cartesian_coordinates()).display()
x = r*cos(ph)
y = r*sin(ph)
>>> E.coord_change(E.cartesian_coordinates(),
...                E.polar_coordinates()).display()
r = sqrt(x^2 + y^2)
ph = arctan2(y, x)
```

The coordinate variables are returned by the square bracket operator:

```
sage: E.polar_coordinates()[1]
r
sage: E.polar_coordinates()[2]
ph
sage: E.polar_coordinates()[:]
(r, ph)
```

```
>>> from sage.all import *
>>> E.polar_coordinates()[Integer(1)]
r
>>> E.polar_coordinates()[Integer(2)]
ph
>>> E.polar_coordinates()[:]
(r, ph)
```

They can also be obtained via the operator `<,>`:

```
sage: polar.<r,ph> = E.polar_coordinates(); polar
Chart (E^2, (r, ph))
sage: r, ph
(r, ph)
```

```
>>> from sage.all import *
>>> polar = E.polar_coordinates(names=('r', 'ph',)); (r, ph,) = polar._first_
↪ngens(2); polar
Chart (E^2, (r, ph))
>>> r, ph
(r, ph)
```

Actually, `polar.<r,ph> = E.polar_coordinates()` is a shortcut for:

```
sage: polar = E.polar_coordinates()
sage: r, ph = polar[:]
```

```
>>> from sage.all import *
>>> polar = E.polar_coordinates()
>>> r, ph = polar[:]
```

The coordinate symbols can be customized:

```
sage: E = EuclideanSpace(2)
sage: E.polar_coordinates(symbols=r"r th:\theta")
Chart (E^2, (r, th))
sage: latex(E.polar_coordinates())
\left(\mathbb{E}^{2},(r, {\theta})\right)
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(Integer(2))
>>> E.polar_coordinates(symbols=r"r th:\theta")
Chart (E^2, (r, th))
>>> latex(E.polar_coordinates())
\left(\mathbb{E}^{2},(r, {\theta})\right)
```

Note that if the polar coordinates have been already initialized, the argument `symbols` has no effect:

```
sage: E.polar_coordinates(symbols=r"R Th:\Theta")
Chart (E^2, (r, th))
```

```
>>> from sage.all import *
>>> E.polar_coordinates(symbols=r"R Th:\Theta")
Chart (E^2, (r, th))
```

polar_frame()

Return the orthonormal vector frame associated with polar coordinates.

OUTPUT:

- instance of *VectorFrame*

EXAMPLES:

```
sage: E = EuclideanSpace(2)
sage: E.polar_frame()
Vector frame (E^2, (e_r,e_ph))
sage: E.polar_frame()[1]
Vector field e_r on the Euclidean plane E^2
sage: E.polar_frame()[:]
(Vector field e_r on the Euclidean plane E^2,
 Vector field e_ph on the Euclidean plane E^2)
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(Integer(2))
>>> E.polar_frame()
Vector frame (E^2, (e_r,e_ph))
>>> E.polar_frame()[Integer(1)]
Vector field e_r on the Euclidean plane E^2
>>> E.polar_frame()[:]
(Vector field e_r on the Euclidean plane E^2,
 Vector field e_ph on the Euclidean plane E^2)
```

The orthonormal polar frame expressed in terms of the Cartesian one:

```
sage: for e in E.polar_frame():
....:     e.display(E.cartesian_frame(), E.polar_coordinates())
e_r = cos(ph) e_x + sin(ph) e_y
e_ph = -sin(ph) e_x + cos(ph) e_y
```

```
>>> from sage.all import *
>>> for e in E.polar_frame():
...     e.display(E.cartesian_frame(), E.polar_coordinates())
e_r = cos(ph) e_x + sin(ph) e_y
e_ph = -sin(ph) e_x + cos(ph) e_y
```

The orthonormal frame (e_r, e_ϕ) expressed in terms of the coordinate frame $\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \phi}\right)$:

```
sage: for e in E.polar_frame():
....:     e.display(E.polar_coordinates())
e_r = ∂/∂r
e_ph = 1/r ∂/∂ph
```

```
>>> from sage.all import *
>>> for e in E.polar_frame():
...     e.display(E.polar_coordinates())
e_r = ∂/∂r
e_ph = 1/r ∂/∂ph
```

class sage.manifolds.differentiable.examples.euclidean.**EuclideanSpace**(*n*, *name=None*, *latex_name=None*, *coordinates='Cartesian'*, *symbols=None*, *metric_name='g'*, *metric_latex_name=None*, *start_index=1*, *base_manifold=None*, *category=None*, *init_coord_methods=None*, *unique_tag=None*)

Bases: *PseudoRiemannianManifold*

Euclidean space.

A *Euclidean space of dimension* n is an affine space E, whose associated vector space is a n-dimensional vector space over $\mathbf{R}$ and is equipped with a positive definite symmetric bilinear form, called the *scalar product* or *dot product*.

Euclidean space of dimension n can be viewed as a Riemannian manifold that is diffeomorphic to $\mathbf{R}^n$ and that has a flat metric g. The Euclidean scalar product is the one defined by the Riemannian metric g.

INPUT:

- `n` – positive integer; dimension of the space over the real field
- `name` – (default: `None`) string; name (symbol) given to the Euclidean space; if `None`, the name will be set to `'E^n'`
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the space; if `None`, it is set to `'\mathbb{E}^{n}'` if `name` is `None` and to `name` otherwise
- `coordinates` – (default: `'Cartesian'`) string describing the type of coordinates to be initialized at the Euclidean space creation; allowed values are
  - `'Cartesian'` (canonical coordinates on $\mathbf{R}^n$)
  - `'polar'` for `n=2` only (see *polar_coordinates()*)
  - `'spherical'` for `n=3` only (see *spherical_coordinates()*)
  - `'cylindrical'` for `n=3` only (see *cylindrical_coordinates()*)
- `symbols` – (default: `None`) string defining the coordinate text symbols and LaTeX symbols, with the same conventions as the argument `coordinates` in *RealDiffChart*, namely `symbols` is a string of coordinate fields separated by a blank space, where each field contains the coordinate's text symbol and possibly the coordinate's LaTeX symbol (when the latter is different from the text symbol), both symbols being separated by a colon (`:`); if `None`, the symbols will be automatically generated according to the value of `coordinates`
- `metric_name` – (default: `'g'`) string; name (symbol) given to the Euclidean metric tensor
- `metric_latex_name` – (default: `None`) string; LaTeX symbol to denote the Euclidean metric tensor; if none is provided, it is set to `metric_name`
- `start_index` – (default: 1) integer; lower value of the range of indices used for "indexed objects" in the Euclidean space, e.g. coordinates of a chart
- `names` – (default: `None`) unused argument, except if `symbols` is not provided; it must then be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)

If `names` is specified, then `n` does not have to be specified.

EXAMPLES:

Constructing a 2-dimensional Euclidean space:

```
sage: E = EuclideanSpace(2); E
Euclidean plane E^2
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(Integer(2)); E
Euclidean plane E^2
```

Each call to `EuclideanSpace` creates a different object:

```
sage: E1 = EuclideanSpace(2)
sage: E1 is E
False
sage: E1 == E
False
```

```
>>> from sage.all import *
>>> E1 = EuclideanSpace(Integer(2))
>>> E1 is E
False
>>> E1 == E
False
```

The LaTeX symbol of the Euclidean space is by default $\mathbb{E}^n$, where n is the dimension:

```
sage: latex(E)
\mathbb{E}^{2}
```

```
>>> from sage.all import *
>>> latex(E)
\mathbb{E}^{2}
```

But both the name and LaTeX names of the Euclidean space can be customized:

```
sage: F = EuclideanSpace(2, name='F', latex_name=r'\mathcal{F}'); F
Euclidean plane F
sage: latex(F)
\mathcal{F}
```

```
>>> from sage.all import *
>>> F = EuclideanSpace(Integer(2), name='F', latex_name=r'\mathcal{F}'); F
Euclidean plane F
>>> latex(F)
\mathcal{F}
```

By default, a Euclidean space is created with a single coordinate chart: that of Cartesian coordinates:

```
sage: E.atlas()
[Chart (E^2, (x, y))]
sage: E.cartesian_coordinates()
Chart (E^2, (x, y))
```

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```
sage: E.default_chart() is E.cartesian_coordinates()
True
```

```
>>> from sage.all import *
>>> E.atlas()
[Chart (E^2, (x, y))]
>>> E.cartesian_coordinates()
Chart (E^2, (x, y))
>>> E.default_chart() is E.cartesian_coordinates()
True
```

The coordinate variables can be initialized, as the Python variables `x` and `y`, by:

```
sage: x, y = E.cartesian_coordinates()[:]
```

```
>>> from sage.all import *
>>> x, y = E.cartesian_coordinates()[:]
```

However, it is possible to both construct the Euclidean space and initialize the coordinate variables in a single stage, thanks to SageMath operator `<,>`:

```
sage: E.<x,y> = EuclideanSpace()
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(names=('x', 'y',)); (x, y,) = E._first_ngens(2)
```

Note that providing the dimension as an argument of `EuclideanSpace` is not necessary in that case, since it can be deduced from the number of coordinates within `<,>`. Besides, the coordinate symbols can be customized:

```
sage: E.<X,Y> = EuclideanSpace()
sage: E.cartesian_coordinates()
Chart (E^2, (X, Y))
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(names=('X', 'Y',)); (X, Y,) = E._first_ngens(2)
>>> E.cartesian_coordinates()
Chart (E^2, (X, Y))
```

By default, the LaTeX symbols of the coordinates coincide with the text ones:

```
sage: latex(X+Y)
X + Y
```

```
>>> from sage.all import *
>>> latex(X+Y)
X + Y
```

However, it is possible to customize them, via the argument `symbols`, which must be a string, usually prefixed by `r` (for *raw* string, in order to allow for the backslash character of LaTeX expressions). This string contains the coordinate fields separated by a blank space; each field contains the coordinate's text symbol and possibly the coordinate's LaTeX symbol (when the latter is different from the text symbol), both symbols being separated by a colon (`:`):

```
sage: E.<xi,ze> = EuclideanSpace(symbols=r"xi:\xi ze:\zeta")
sage: E.cartesian_coordinates()
Chart (E^2, (xi, ze))
sage: latex(xi+ze)
{\xi} + {\zeta}
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(symbols=r"xi:\xi ze:\zeta", names=('xi', 'ze',)); (xi, ze,
↪) = E._first_ngens(2)
>>> E.cartesian_coordinates()
Chart (E^2, (xi, ze))
>>> latex(xi+ze)
{\xi} + {\zeta}
```

Thanks to the argument `coordinates`, a Euclidean space can be constructed with curvilinear coordinates initialized instead of the Cartesian ones:

```
sage: E.<r,ph> = EuclideanSpace(coordinates='polar')
sage: E.atlas()   # no Cartesian coordinates have been constructed
[Chart (E^2, (r, ph))]
sage: polar = E.polar_coordinates(); polar
Chart (E^2, (r, ph))
sage: E.default_chart() is polar
True
sage: latex(r+ph)
{\phi} + r
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(coordinates='polar', names=('r', 'ph',)); (r, ph,) = E._
↪first_ngens(2)
>>> E.atlas()   # no Cartesian coordinates have been constructed
[Chart (E^2, (r, ph))]
>>> polar = E.polar_coordinates(); polar
Chart (E^2, (r, ph))
>>> E.default_chart() is polar
True
>>> latex(r+ph)
{\phi} + r
```

The Cartesian coordinates, along with the transition maps to and from the curvilinear coordinates, can be constructed at any time by:

```
sage: cartesian.<x,y> = E.cartesian_coordinates()
sage: E.atlas()  # both polar and Cartesian coordinates now exist
[Chart (E^2, (r, ph)), Chart (E^2, (x, y))]
```

```
>>> from sage.all import *
>>> cartesian = E.cartesian_coordinates(names=('x', 'y',)); (x, y,) = cartesian._
↪first_ngens(2)
>>> E.atlas()  # both polar and Cartesian coordinates now exist
[Chart (E^2, (r, ph)), Chart (E^2, (x, y))]
```

The transition maps have been initialized by the command `E.cartesian_coordinates()`:

```
sage: E.coord_change(polar, cartesian).display()
x = r*cos(ph)
y = r*sin(ph)
sage: E.coord_change(cartesian, polar).display()
r = sqrt(x^2 + y^2)
ph = arctan2(y, x)
```

```
>>> from sage.all import *
>>> E.coord_change(polar, cartesian).display()
x = r*cos(ph)
y = r*sin(ph)
>>> E.coord_change(cartesian, polar).display()
r = sqrt(x^2 + y^2)
ph = arctan2(y, x)
```

The default name of the Euclidean metric tensor is g:

```
sage: E.metric()
Riemannian metric g on the Euclidean plane E^2
sage: latex(_)
g
```

```
>>> from sage.all import *
>>> E.metric()
Riemannian metric g on the Euclidean plane E^2
>>> latex(_)
g
```

But this can be customized:

```
sage: E = EuclideanSpace(2, metric_name='h')
sage: E.metric()
Riemannian metric h on the Euclidean plane E^2
sage: latex(_)
h
sage: E = EuclideanSpace(2, metric_latex_name=r'\mathbf{g}')
sage: E.metric()
Riemannian metric g on the Euclidean plane E^2
sage: latex(_)
\mathbf{g}
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(Integer(2), metric_name='h')
>>> E.metric()
Riemannian metric h on the Euclidean plane E^2
>>> latex(_)
h
>>> E = EuclideanSpace(Integer(2), metric_latex_name=r'\mathbf{g}')
>>> E.metric()
Riemannian metric g on the Euclidean plane E^2
>>> latex(_)
\mathbf{g}
```

A 4-dimensional Euclidean space:

```
sage: E = EuclideanSpace(4); E
4-dimensional Euclidean space E^4
sage: latex(E)
\mathbb{E}^{4}
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(Integer(4)); E
4-dimensional Euclidean space E^4
>>> latex(E)
\mathbb{E}^{4}
```

`E` is both a real smooth manifold of dimension 4 and a complete metric space:

```
sage: E.category()
Join of Category of smooth manifolds over Real Field with 53 bits of
 precision and Category of connected manifolds over Real Field with
 53 bits of precision and Category of complete metric spaces
sage: dim(E)
4
```

```
>>> from sage.all import *
>>> E.category()
Join of Category of smooth manifolds over Real Field with 53 bits of
 precision and Category of connected manifolds over Real Field with
 53 bits of precision and Category of complete metric spaces
>>> dim(E)
4
```

It is endowed with a default coordinate chart, which is that of Cartesian coordinates (x_1, x_2, x_3, x_4):

```
sage: E.atlas()
[Chart (E^4, (x1, x2, x3, x4))]
sage: E.default_chart()
Chart (E^4, (x1, x2, x3, x4))
sage: E.default_chart() is E.cartesian_coordinates()
True
```

```
>>> from sage.all import *
>>> E.atlas()
[Chart (E^4, (x1, x2, x3, x4))]
>>> E.default_chart()
Chart (E^4, (x1, x2, x3, x4))
>>> E.default_chart() is E.cartesian_coordinates()
True
```

`E` is also endowed with a default metric tensor, which defines the Euclidean scalar product:

```
sage: g = E.metric(); g
Riemannian metric g on the 4-dimensional Euclidean space E^4
sage: g.display()
g = dx1⊗dx1 + dx2⊗dx2 + dx3⊗dx3 + dx4⊗dx4
```

```
>>> from sage.all import *
>>> g = E.metric(); g
Riemannian metric g on the 4-dimensional Euclidean space E^4
>>> g.display()
g = dx1⊗dx1 + dx2⊗dx2 + dx3⊗dx3 + dx4⊗dx4
```

cartesian_coordinates(*symbols=None*, *names=None*)

Return the chart of Cartesian coordinates, possibly creating it if it does not already exist.

INPUT:

- `symbols` – (default: `None`) string defining the coordinate text symbols and LaTeX symbols, with the same conventions as the argument `coordinates` in *RealDiffChart*; this is used only if the Cartesian chart has not been already defined; if `None` the symbols are generated as $(x_1, \ldots, x_n)$.
- `names` – (default: `None`) unused argument, except if `symbols` is not provided; it must be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)

OUTPUT:

- the chart of Cartesian coordinates, as an instance of *RealDiffChart*

EXAMPLES:

```
sage: E = EuclideanSpace(4)
sage: X = E.cartesian_coordinates(); X
Chart (E^4, (x1, x2, x3, x4))
sage: X.coord_range()
x1: (-oo, +oo); x2: (-oo, +oo); x3: (-oo, +oo); x4: (-oo, +oo)
sage: X[2]
x2
sage: X[:]
(x1, x2, x3, x4)
sage: latex(X[:])
\left({x_{1}}, {x_{2}}, {x_{3}}, {x_{4}}\right)
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(Integer(4))
>>> X = E.cartesian_coordinates(); X
Chart (E^4, (x1, x2, x3, x4))
>>> X.coord_range()
x1: (-oo, +oo); x2: (-oo, +oo); x3: (-oo, +oo); x4: (-oo, +oo)
>>> X[Integer(2)]
x2
>>> X[:]
(x1, x2, x3, x4)
>>> latex(X[:])
\left({x_{1}}, {x_{2}}, {x_{3}}, {x_{4}}\right)
```

cartesian_frame()

Return the orthonormal vector frame associated with Cartesian coordinates.

OUTPUT: *CoordFrame*

EXAMPLES:

```
sage: E = EuclideanSpace(2)
sage: E.cartesian_frame()
Coordinate frame (E^2, (e_x,e_y))
sage: E.cartesian_frame()[1]
Vector field e_x on the Euclidean plane E^2
sage: E.cartesian_frame()[:]
(Vector field e_x on the Euclidean plane E^2,
 Vector field e_y on the Euclidean plane E^2)
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(Integer(2))
>>> E.cartesian_frame()
Coordinate frame (E^2, (e_x,e_y))
>>> E.cartesian_frame()[Integer(1)]
Vector field e_x on the Euclidean plane E^2
>>> E.cartesian_frame()[:]
(Vector field e_x on the Euclidean plane E^2,
 Vector field e_y on the Euclidean plane E^2)
```

For Cartesian coordinates, the orthonormal frame coincides with the coordinate frame:

```
sage: E.cartesian_frame() is E.cartesian_coordinates().frame()
True
```

```
>>> from sage.all import *
>>> E.cartesian_frame() is E.cartesian_coordinates().frame()
True
```

dist (p, q)

Euclidean distance between two points.

INPUT:

- `p` – an element of `self`
- `q` – an element of `self`

OUTPUT:

- the Euclidean distance $d(p, q)$

EXAMPLES:

```
sage: E.<x,y> = EuclideanSpace()
sage: p = E((1,0))
sage: q = E((0,2))
sage: E.dist(p, q)
sqrt(5)
sage: p.dist(q)  # indirect doctest
sqrt(5)
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(names=('x', 'y',)); (x, y,) = E._first_ngens(2)
>>> p = E((Integer(1),Integer(0)))
>>> q = E((Integer(0),Integer(2)))
```

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```
>>> E.dist(p, q)
sqrt(5)
>>> p.dist(q)   # indirect doctest
sqrt(5)
```

sphere(*radius=1*, *center=None*, *name=None*, *latex_name=None*, *coordinates='spherical'*, *names=None*)

Return an $(n-1)$-sphere smoothly embedded in `self`.

INPUT:

- `radius` – (default: `1`) the radius greater than 1 of the sphere
- `center` – (default: `None`) point on `self` representing the barycenter of the sphere
- `name` – (default: `None`) string; name (symbol) given to the sphere; if `None`, the name will be generated according to the input
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the sphere; if `None`, the symbol will be generated according to the input
- `coordinates` – (default: `'spherical'`) string describing the type of coordinates to be initialized at the sphere's creation; allowed values are
  - `'spherical'` spherical coordinates (see `spherical_coordinates()`)
  - `'stereographic'` stereographic coordinates given by the stereographic projection (see `stereographic_coordinates()`)
- `names` – (default: `None`) must be a tuple containing the coordinate symbols (this guarantees the shortcut operator `<,>` to function); if `None`, the usual conventions are used (see examples in `Sphere` for details)

EXAMPLES:

Define a 2-sphere with radius 2 centered at $(1,2,3)$ in Cartesian coordinates:

```
sage: E3 = EuclideanSpace(3)
sage: c = E3.point((1,2,3), name='c'); c
Point c on the Euclidean space E^3
sage: S2_2 = E3.sphere(radius=2, center=c); S2_2
2-sphere S^2_2(c) of radius 2 smoothly embedded in the Euclidean
 space E^3 centered at the Point c
```

```
>>> from sage.all import *
>>> E3 = EuclideanSpace(Integer(3))
>>> c = E3.point((Integer(1),Integer(2),Integer(3)), name='c'); c
Point c on the Euclidean space E^3
>>> S2_2 = E3.sphere(radius=Integer(2), center=c); S2_2
2-sphere S^2_2(c) of radius 2 smoothly embedded in the Euclidean
 space E^3 centered at the Point c
```

The ambient space is precisely our previously defined Euclidean space:

```
sage: S2_2.ambient() is E3
True
```

```
>>> from sage.all import *
>>> S2_2.ambient() is E3
True
```

The embedding into Euclidean space:

```
sage: S2_2.embedding().display()
iota: S^2_2(c) → E^3
on A: (theta, phi) ↦ (x, y, z) = (2*cos(phi)*sin(theta) + 1,
                                  2*sin(phi)*sin(theta) + 2,
                                  2*cos(theta) + 3)
```

```
>>> from sage.all import *
>>> S2_2.embedding().display()
iota: S^2_2(c) → E^3
on A: (theta, phi) ↦ (x, y, z) = (2*cos(phi)*sin(theta) + 1,
                                  2*sin(phi)*sin(theta) + 2,
                                  2*cos(theta) + 3)
```

See `Sphere` for more examples.

3.2.2 Spheres smoothly embedded in Euclidean Space

Let E^{n+1} be a Euclidean space of dimension $n+1$ and $c \in E^{n+1}$. An n-sphere with radius r and centered at c, usually denoted by $\mathbb{S}^n_r(c)$, smoothly embedded in the Euclidean space E^{n+1} is an n-dimensional smooth manifold together with a smooth embedding

$$\iota\colon \mathbb{S}^n_r \to E^{n+1}$$

whose image consists of all points having the same Euclidean distance to the fixed point c. If we choose Cartesian coordinates $(x_1, \ldots, x_{n+1})$ on E^{n+1} with $x(c) = 0$ then the above translates to

$$\iota(\mathbb{S}^n_r(c)) = \left\{ p \in E^{n+1} : \|x(p)\| = r \right\}.$$

This corresponds to the standard n-sphere of radius r centered at c.

AUTHORS:

- Michael Jung (2020): initial version

REFERENCES:

- M. Berger: *Geometry I&II* [Ber1987], [Ber1987a]
- J. Lee: *Introduction to Smooth Manifolds* [Lee2013]

EXAMPLES:

We start by defining a 2-sphere of unspecified radius r:

```
sage: r = var('r')
sage: S2_r = manifolds.Sphere(2, radius=r); S2_r
2-sphere S^2_r of radius r smoothly embedded in the Euclidean space E^3
```

```
>>> from sage.all import *
>>> r = var('r')
>>> S2_r = manifolds.Sphere(Integer(2), radius=r); S2_r
2-sphere S^2_r of radius r smoothly embedded in the Euclidean space E^3
```

The embedding ι is constructed from scratch and can be returned by the following command:

```
sage: i = S2_r.embedding(); i
Differentiable map iota from the 2-sphere S^2_r of radius r smoothly
 embedded in the Euclidean space E^3 to the Euclidean space E^3
sage: i.display()
iota: S^2_r → E^3
on A: (theta, phi) ↦ (x, y, z) = (r*cos(phi)*sin(theta),
                                    r*sin(phi)*sin(theta),
                                    r*cos(theta))
```

```
>>> from sage.all import *
>>> i = S2_r.embedding(); i
Differentiable map iota from the 2-sphere S^2_r of radius r smoothly
 embedded in the Euclidean space E^3 to the Euclidean space E^3
>>> i.display()
iota: S^2_r → E^3
on A: (theta, phi) ↦ (x, y, z) = (r*cos(phi)*sin(theta),
                                    r*sin(phi)*sin(theta),
                                    r*cos(theta))
```

As a submanifold of a Riemannian manifold, namely the Euclidean space, the 2-sphere admits an induced metric:

```
sage: g = S2_r.induced_metric()
sage: g.display()
g = r^2 dtheta⊗dtheta + r^2*sin(theta)^2 dphi⊗dphi
```

```
>>> from sage.all import *
>>> g = S2_r.induced_metric()
>>> g.display()
g = r^2 dtheta⊗dtheta + r^2*sin(theta)^2 dphi⊗dphi
```

The induced metric is also known as the *first fundamental form* (see `first_fundamental_form()`):

```
sage: g is S2_r.first_fundamental_form()
True
```

```
>>> from sage.all import *
>>> g is S2_r.first_fundamental_form()
True
```

The *second fundamental form* encodes the extrinsic curvature of the 2-sphere as hypersurface of Euclidean space (see `second_fundamental_form()`):

```
sage: K = S2_r.second_fundamental_form(); K
Field of symmetric bilinear forms K on the 2-sphere S^2_r of radius r
 smoothly embedded in the Euclidean space E^3
sage: K.display()
K = r dtheta⊗dtheta + r*sin(theta)^2 dphi⊗dphi
```

```
>>> from sage.all import *
>>> K = S2_r.second_fundamental_form(); K
Field of symmetric bilinear forms K on the 2-sphere S^2_r of radius r
```

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```
 smoothly embedded in the Euclidean space E^3
>>> K.display()
K = r dtheta⊗dtheta + r*sin(theta)^2 dphi⊗dphi
```

One quantity that can be derived from the second fundamental form is the Gaussian curvature:

```
sage: K = S2_r.gauss_curvature()
sage: K.display()
S^2_r → ℝ
on A: (theta, phi) ↦ r^(-2)
```

```
>>> from sage.all import *
>>> K = S2_r.gauss_curvature()
>>> K.display()
S^2_r → ℝ
on A: (theta, phi) ↦ r^(-2)
```

As we have seen, spherical coordinates are initialized by default. To initialize stereographic coordinates retrospectively, we can use the following command:

```
sage: S2_r.stereographic_coordinates()
Chart (S^2_r-{NP}, (y1, y2))
```

```
>>> from sage.all import *
>>> S2_r.stereographic_coordinates()
Chart (S^2_r-{NP}, (y1, y2))
```

To get all charts corresponding to stereographic coordinates, we can use the `coordinate_charts()`:

```
sage: stereoN, stereoS = S2_r.coordinate_charts('stereographic')
sage: stereoN, stereoS
(Chart (S^2_r-{NP}, (y1, y2)), Chart (S^2_r-{SP}, (yp1, yp2)))
```

```
>>> from sage.all import *
>>> stereoN, stereoS = S2_r.coordinate_charts('stereographic')
>>> stereoN, stereoS
(Chart (S^2_r-{NP}, (y1, y2)), Chart (S^2_r-{SP}, (yp1, yp2)))
```

See also

See `stereographic_coordinates()` and `spherical_coordinates()` for details.

Note

Notice that the derived quantities such as the embedding as well as the first and second fundamental forms must be computed from scratch again when new coordinates have been initialized. That makes the usage of previously declared objects obsolete.

Consider now a 1-sphere with barycenter $(1, 0)$ in Cartesian coordinates:

```
sage: E2 = EuclideanSpace(2)
sage: c = E2.point((1,0), name='c')
sage: S1c.<chi> = E2.sphere(center=c); S1c
1-sphere S^1(c) of radius 1 smoothly embedded in the Euclidean plane
 E^2 centered at the Point c
sage: S1c.spherical_coordinates()
Chart (A, (chi,))
```

```
>>> from sage.all import *
>>> E2 = EuclideanSpace(Integer(2))
>>> c = E2.point((Integer(1),Integer(0)), name='c')
>>> S1c = E2.sphere(center=c, names=('chi',)); (chi,) = S1c._first_ngens(1); S1c
1-sphere S^1(c) of radius 1 smoothly embedded in the Euclidean plane
 E^2 centered at the Point c
>>> S1c.spherical_coordinates()
Chart (A, (chi,))
```

Get stereographic coordinates:

```
sage: stereoN, stereoS = S1c.coordinate_charts('stereographic')
sage: stereoN, stereoS
(Chart (S^1(c)-{NP}, (y1,)), Chart (S^1(c)-{SP}, (yp1,)))
```

```
>>> from sage.all import *
>>> stereoN, stereoS = S1c.coordinate_charts('stereographic')
>>> stereoN, stereoS
(Chart (S^1(c)-{NP}, (y1,)), Chart (S^1(c)-{SP}, (yp1,)))
```

The embedding takes now the following form in all coordinates:

```
sage: S1c.embedding().display()
iota: S^1(c) → E^2
on A: chi ↦ (x, y) = (cos(chi) + 1, sin(chi))
on S^1(c)-{NP}: y1 ↦ (x, y) = (2*y1/(y1^2 + 1) + 1, (y1^2 - 1)/(y1^2 + 1))
on S^1(c)-{SP}: yp1 ↦ (x, y) = (2*yp1/(yp1^2 + 1) + 1, -(yp1^2 - 1)/(yp1^2 + 1))
```

```
>>> from sage.all import *
>>> S1c.embedding().display()
iota: S^1(c) → E^2
on A: chi ↦ (x, y) = (cos(chi) + 1, sin(chi))
on S^1(c)-{NP}: y1 ↦ (x, y) = (2*y1/(y1^2 + 1) + 1, (y1^2 - 1)/(y1^2 + 1))
on S^1(c)-{SP}: yp1 ↦ (x, y) = (2*yp1/(yp1^2 + 1) + 1, -(yp1^2 - 1)/(yp1^2 + 1))
```

Since the sphere is a hypersurface, we can get a normal vector field by using `normal`:

```
sage: n = S1c.normal(); n
Vector field n along the 1-sphere S^1(c) of radius 1 smoothly embedded in
 the Euclidean plane E^2 centered at the Point c with values on the
 Euclidean plane E^2
sage: n.display()
n = -cos(chi) e_x - sin(chi) e_y
```

```
>>> from sage.all import *
>>> n = S1c.normal(); n
Vector field n along the 1-sphere S^1(c) of radius 1 smoothly embedded in
 the Euclidean plane E^2 centered at the Point c with values on the
 Euclidean plane E^2
>>> n.display()
n = -cos(chi) e_x - sin(chi) e_y
```

Notice that this is just *one* normal field with arbitrary direction, in this particular case n points inwards whereas $-n$ points outwards. However, the vector field n is indeed non-vanishing and hence the sphere admits an orientation (as all spheres do):

```
sage: orient = S1c.orientation(); orient
[Coordinate frame (S^1(c)-{SP}, (∂/∂yp1)), Vector frame (S^1(c)-{NP}, (f_1))]
sage: f = orient[1]
sage: f[1].display()
f_1 = -∂/∂y1
```

```
>>> from sage.all import *
>>> orient = S1c.orientation(); orient
[Coordinate frame (S^1(c)-{SP}, (∂/∂yp1)), Vector frame (S^1(c)-{NP}, (f_1))]
>>> f = orient[Integer(1)]
>>> f[Integer(1)].display()
f_1 = -∂/∂y1
```

Notice that the orientation is chosen is such a way that $(\iota_*(f_1), -n)$ is oriented in the ambient Euclidean space, i.e. the last entry is the normal vector field pointing outwards. Henceforth, the manifold admits a volume form:

```
sage: g = S1c.induced_metric()
sage: g.display()
g = dchi⊗dchi
sage: eps = g.volume_form()
sage: eps.display()
eps_g = -dchi
```

```
>>> from sage.all import *
>>> g = S1c.induced_metric()
>>> g.display()
g = dchi⊗dchi
>>> eps = g.volume_form()
>>> eps.display()
eps_g = -dchi
```

class `sage.manifolds.differentiable.examples.sphere.`**`Sphere`**(*n*, *radius=1*, *ambient_space=None*, *center=None*, *name=None*, *latex_name=None*, *coordinates='spherical'*, *names=None*, *category=None*, *init_coord_methods=None*, *unique_tag=None*)

Bases: `PseudoRiemannianSubmanifold`

Sphere smoothly embedded in Euclidean Space.

An n-sphere of radius $r`smoothlyembeddedinaEuclideanspace`E^{n+1}$ is a smooth n-dimensional manifold smoothly embedded into E^{n+1}, such that the embedding constitutes a standard n-sphere of radius r in that Euclidean space (possibly shifted by a point).

- `n` – positive integer representing dimension of the sphere
- `radius` – (default: `1`) positive number that states the radius of the sphere
- `name` – (default: `None`) string; name (symbol) given to the sphere; if `None`, the name will be set according to the input (see convention above)
- `ambient_space` – (default: `None`) Euclidean space in which the sphere should be embedded; if `None`, a new instance of Euclidean space is created
- `center` – (default: `None`) the barycenter of the sphere as point of the ambient Euclidean space; if `None` the barycenter is set to the origin of the ambient space's standard Cartesian coordinates
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the space; if `None`, it will be set according to the input (see convention above)
- `coordinates` – (default: `'spherical'`) string describing the type of coordinates to be initialized at the sphere's creation; allowed values are
  - `'spherical'` spherical coordinates (see `spherical_coordinates()`)
  - `'stereographic'` stereographic coordinates given by the stereographic projection (see `stereographic_coordinates()`)
- `names` – (default: `None`) must be a tuple containing the coordinate symbols (this guarantees the shortcut operator `<,>` to function); if `None`, the usual conventions are used (see examples below for details)
- `unique_tag` – (default: `None`) tag used to force the construction of a new object when all the other arguments have been used previously (without `unique_tag`, the `UniqueRepresentation` behavior inherited from `PseudoRiemannianManifold` would return the previously constructed object corresponding to these arguments)

EXAMPLES:

A 2-sphere embedded in Euclidean space:

```
sage: S2 = manifolds.Sphere(2); S2
2-sphere S^2 of radius 1 smoothly embedded in the Euclidean space E^3
sage: latex(S2)
\mathbb{S}^{2}
```

```
>>> from sage.all import *
>>> S2 = manifolds.Sphere(Integer(2)); S2
2-sphere S^2 of radius 1 smoothly embedded in the Euclidean space E^3
>>> latex(S2)
\mathbb{S}^{2}
```

The ambient Euclidean space is constructed incidentally:

```
sage: S2.ambient()
Euclidean space E^3
```

```
>>> from sage.all import *
>>> S2.ambient()
Euclidean space E^3
```

Another call creates another sphere and hence another Euclidean space:

```
sage: S2 is manifolds.Sphere(2)
False
sage: S2.ambient() is manifolds.Sphere(2).ambient()
False
```

```
>>> from sage.all import *
>>> S2 is manifolds.Sphere(Integer(2))
False
>>> S2.ambient() is manifolds.Sphere(Integer(2)).ambient()
False
```

By default, the barycenter is set to the coordinate origin of the standard Cartesian coordinates in the ambient Euclidean space:

```
sage: c = S2.center(); c
Point on the Euclidean space E^3
sage: c.coord()
(0, 0, 0)
```

```
>>> from sage.all import *
>>> c = S2.center(); c
Point on the Euclidean space E^3
>>> c.coord()
(0, 0, 0)
```

Each n-sphere is a compact manifold and a complete metric space:

```
sage: S2.category()
Join of Category of compact topological spaces and Category of smooth
 manifolds over Real Field with 53 bits of precision and Category of
 connected manifolds over Real Field with 53 bits of precision and
 Category of complete metric spaces
```

```
>>> from sage.all import *
>>> S2.category()
Join of Category of compact topological spaces and Category of smooth
 manifolds over Real Field with 53 bits of precision and Category of
 connected manifolds over Real Field with 53 bits of precision and
 Category of complete metric spaces
```

If not stated otherwise, each n-sphere is automatically endowed with spherical coordinates:

```
sage: S2.atlas()
[Chart (A, (theta, phi))]
sage: S2.default_chart()
Chart (A, (theta, phi))
sage: spher = S2.spherical_coordinates()
sage: spher is S2.default_chart()
True
```

```
>>> from sage.all import *
>>> S2.atlas()
[Chart (A, (theta, phi))]
>>> S2.default_chart()
Chart (A, (theta, phi))
>>> spher = S2.spherical_coordinates()
>>> spher is S2.default_chart()
True
```

Notice that the spherical coordinates do not cover the whole sphere. To cover the entire sphere with charts, use stereographic coordinates instead:

```
sage: stereoN, stereoS = S2.coordinate_charts('stereographic')
sage: stereoN, stereoS
(Chart (S^2-{NP}, (y1, y2)), Chart (S^2-{SP}, (yp1, yp2)))
sage: list(S2.open_covers())
[Set {S^2} of open subsets of the 2-sphere S^2 of radius 1 smoothly embedded in
↪the Euclidean space E^3,
 Set {S^2-{NP}, S^2-{SP}} of open subsets of the 2-sphere S^2 of radius 1
↪smoothly embedded in the Euclidean space E^3]
```

```
>>> from sage.all import *
>>> stereoN, stereoS = S2.coordinate_charts('stereographic')
>>> stereoN, stereoS
(Chart (S^2-{NP}, (y1, y2)), Chart (S^2-{SP}, (yp1, yp2)))
>>> list(S2.open_covers())
[Set {S^2} of open subsets of the 2-sphere S^2 of radius 1 smoothly embedded in
↪the Euclidean space E^3,
 Set {S^2-{NP}, S^2-{SP}} of open subsets of the 2-sphere S^2 of radius 1
↪smoothly embedded in the Euclidean space E^3]
```

Note: Keep in mind that the initialization process of stereographic coordinates and their transition maps is computational complex in higher dimensions. Henceforth, high computation times are expected with increasing dimension.

center()

Return the barycenter of `self` in the ambient Euclidean space.

EXAMPLES:

2-sphere embedded in Euclidean space centered at $(1, 2, 3)$ in Cartesian coordinates:

```
sage: E3 = EuclideanSpace(3)
sage: c = E3.point((1,2,3), name='c')
sage: S2c = manifolds.Sphere(2, ambient_space=E3, center=c); S2c
2-sphere S^2(c) of radius 1 smoothly embedded in the Euclidean space
 E^3 centered at the Point c
sage: S2c.center()
Point c on the Euclidean space E^3
```

```
>>> from sage.all import *
>>> E3 = EuclideanSpace(Integer(3))
>>> c = E3.point((Integer(1),Integer(2),Integer(3)), name='c')
>>> S2c = manifolds.Sphere(Integer(2), ambient_space=E3, center=c); S2c
2-sphere S^2(c) of radius 1 smoothly embedded in the Euclidean space
 E^3 centered at the Point c
>>> S2c.center()
Point c on the Euclidean space E^3
```

We can see that the embedding is shifted accordingly:

```
sage: S2c.embedding().display()
iota: S^2(c) → E^3
on A: (theta, phi) ↦ (x, y, z) = (cos(phi)*sin(theta) + 1,
                                   sin(phi)*sin(theta) + 2,
                                   cos(theta) + 3)
```

```
>>> from sage.all import *
>>> S2c.embedding().display()
iota: S^2(c) → E^3
on A: (theta, phi) ↦ (x, y, z) = (cos(phi)*sin(theta) + 1,
                                   sin(phi)*sin(theta) + 2,
                                   cos(theta) + 3)
```

coordinate_charts(*coord_name*, *names=None*)

Return a list of all charts belonging to the coordinates `coord_name`.

INPUT:

- `coord_name` – string describing the type of coordinates
- `names` – (default: `None`) must be a tuple containing the coordinate symbols for the first chart in the list; if `None`, the standard convention is used

EXAMPLES:

Spherical coordinates on S^1:

```
sage: S1 = manifolds.Sphere(1)
sage: S1.coordinate_charts('spherical')
[Chart (A, (phi,))]
```

```
>>> from sage.all import *
>>> S1 = manifolds.Sphere(Integer(1))
>>> S1.coordinate_charts('spherical')
[Chart (A, (phi,))]
```

Stereographic coordinates on S^1:

```
sage: stereo_charts = S1.coordinate_charts('stereographic', names=['a'])
sage: stereo_charts
[Chart (S^1-{NP}, (a,)), Chart (S^1-{SP}, (ap,))]
```

```
>>> from sage.all import *
>>> stereo_charts = S1.coordinate_charts('stereographic', names=['a'])
```

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```
>>> stereo_charts
[Chart (S^1-{NP}, (a,)), Chart (S^1-{SP}, (ap,))]
```

dist (p, q)

Return the great circle distance between the points `p` and `q` on `self`.

INPUT:

- `p` – an element of `self`
- `q` – an element of `self`

OUTPUT:

- the great circle distance $d(p, q)$ on `self`

The great circle distance $d(p, q)$ of the points $p, q \in \mathbb{S}^n_r(c)$ is the length of the shortest great circle segment on $\mathbb{S}^n_r(c)$ that joins p and q. If we choose Cartesian coordinates $(x_1, \ldots, x_{n+1})$ of the ambient Euclidean space such that the center lies in the coordinate origin, i.e. $x(c) = 0$, the great circle distance can be expressed in terms of the following formula:

$$d(p, q) = r \arccos\left(\frac{x(\iota(p)) \cdot x(\iota(q))}{r^2}\right).$$

EXAMPLES:

Define a 2-sphere with unspecified radius:

```
sage: r = var('r')
sage: S2_r = manifolds.Sphere(2, radius=r); S2_r
2-sphere S^2_r of radius r smoothly embedded in the Euclidean space E^3
```

```
>>> from sage.all import *
>>> r = var('r')
>>> S2_r = manifolds.Sphere(Integer(2), radius=r); S2_r
2-sphere S^2_r of radius r smoothly embedded in the Euclidean space E^3
```

Given two antipodal points in spherical coordinates:

```
sage: p = S2_r.point((pi/2, pi/2), name='p'); p
Point p on the 2-sphere S^2_r of radius r smoothly embedded in the
 Euclidean space E^3
sage: q = S2_r.point((pi/2, -pi/2), name='q'); q
Point q on the 2-sphere S^2_r of radius r smoothly embedded in the
 Euclidean space E^3
```

```
>>> from sage.all import *
>>> p = S2_r.point((pi/Integer(2), pi/Integer(2)), name='p'); p
Point p on the 2-sphere S^2_r of radius r smoothly embedded in the
 Euclidean space E^3
>>> q = S2_r.point((pi/Integer(2), -pi/Integer(2)), name='q'); q
Point q on the 2-sphere S^2_r of radius r smoothly embedded in the
 Euclidean space E^3
```

The distance is determined as the length of the half great circle:

```
sage: S2_r.dist(p, q)
pi*r
```

```
>>> from sage.all import *
>>> S2_r.dist(p, q)
pi*r
```

minimal_triangulation()

Return the minimal triangulation of `self` as a simplicial complex.

EXAMPLES:

Minimal triangulation of the 2-sphere:

```
sage: S2 = manifolds.Sphere(2)
sage: S = S2.minimal_triangulation(); S
Minimal triangulation of the 2-sphere
```

```
>>> from sage.all import *
>>> S2 = manifolds.Sphere(Integer(2))
>>> S = S2.minimal_triangulation(); S
Minimal triangulation of the 2-sphere
```

The Euler characteristic of a 2-sphere:

```
sage: S.euler_characteristic()
2
```

```
>>> from sage.all import *
>>> S.euler_characteristic()
2
```

radius()

Return the radius of `self`.

EXAMPLES:

3-sphere with radius 3:

```
sage: S3_2 = manifolds.Sphere(3, radius=2); S3_2
3-sphere S^3_2 of radius 2 smoothly embedded in the 4-dimensional
 Euclidean space E^4
sage: S3_2.radius()
2
```

```
>>> from sage.all import *
>>> S3_2 = manifolds.Sphere(Integer(3), radius=Integer(2)); S3_2
3-sphere S^3_2 of radius 2 smoothly embedded in the 4-dimensional
 Euclidean space E^4
>>> S3_2.radius()
2
```

2-sphere with unspecified radius:

```
sage: r = var('r')
sage: S2_r = manifolds.Sphere(3, radius=r); S2_r
3-sphere S^3_r of radius r smoothly embedded in the 4-dimensional
 Euclidean space E^4
sage: S2_r.radius()
r
```

```
>>> from sage.all import *
>>> r = var('r')
>>> S2_r = manifolds.Sphere(Integer(3), radius=r); S2_r
3-sphere S^3_r of radius r smoothly embedded in the 4-dimensional
 Euclidean space E^4
>>> S2_r.radius()
r
```

spherical_coordinates(*names=None*)

Return the spherical coordinates of `self`.

INPUT:

- `names` – (default: `None`) must be a tuple containing the coordinate symbols (this guarantees the usage of the shortcut operator `<,>`)

OUTPUT:

- the chart of spherical coordinates, as an instance of *RealDiffChart*

Let $\mathbb{S}^n_r(c)$ be an n-sphere of radius r smoothly embedded in the Euclidean space E^{n+1} centered at $c \in E^{n+1}$. We say that $(\varphi_1, \ldots, \varphi_n)$ define *spherical coordinates* on the open subset $A \subset \mathbb{S}^n_r(c)$ for the Cartesian coordinates $(x_1, \ldots, x_{n+1})$ on E^{n+1} (not necessarily centered at c) if

$$\begin{aligned} x_1 \circ \iota|_A &= r \cos(\varphi_n) \sin(\varphi_{n-1}) \cdots \sin(\varphi_1) + x_1(c), \\ x_1 \circ \iota|_A &= r \sin(\varphi_n) \sin(\varphi_{n-1}) \cdots \sin(\varphi_1) + x_1(c), \\ x_2 \circ \iota|_A &= r \cos(\varphi_{n-1}) \sin(\varphi_{n-2}) \cdots \sin(\varphi_1) + x_2(c), \\ x_3 \circ \iota|_A &= r \cos(\varphi_{n-2}) \sin(\varphi_{n-3}) \cdots \sin(\varphi_1) + x_3(c), \\ &\vdots \\ x_{n+1} \circ \iota|_A &= r \cos(\varphi_1) + x_{n+1}(c), \end{aligned}$$

where φ_i has range $(0, \pi)$ for $i = 1, \ldots, n-1$ and φ_n lies in $(-\pi, \pi)$. Notice that the above expressions together with the ranges of the φ_i fully determine the open set A.

Note

Notice that our convention slightly differs from the one given on the Wikipedia article N-sphere#Spherical_coordinates. The definition above ensures that the conventions for the most common cases $n = 1$ and $n = 2$ are maintained.

EXAMPLES:

The spherical coordinates on a 2-sphere follow the common conventions:

```
sage: S2 = manifolds.Sphere(2)
sage: spher = S2.spherical_coordinates(); spher
Chart (A, (theta, phi))
```

```
>>> from sage.all import *
>>> S2 = manifolds.Sphere(Integer(2))
>>> spher = S2.spherical_coordinates(); spher
Chart (A, (theta, phi))
```

The coordinate range of spherical coordinates:

```
sage: spher.coord_range()
theta: (0, pi); phi: [-pi, pi] (periodic)
```

```
>>> from sage.all import *
>>> spher.coord_range()
theta: (0, pi); phi: [-pi, pi] (periodic)
```

Spherical coordinates do not cover the 2-sphere entirely:

```
sage: A = spher.domain(); A
Open subset A of the 2-sphere S^2 of radius 1 smoothly embedded in
 the Euclidean space E^3
```

```
>>> from sage.all import *
>>> A = spher.domain(); A
Open subset A of the 2-sphere S^2 of radius 1 smoothly embedded in
 the Euclidean space E^3
```

The embedding of a 2-sphere in Euclidean space via spherical coordinates:

```
sage: S2.embedding().display()
iota: S^2 → E^3
 on A: (theta, phi) ↦ (x, y, z) =
                         (cos(phi)*sin(theta),
                          sin(phi)*sin(theta),
                          cos(theta))
```

```
>>> from sage.all import *
>>> S2.embedding().display()
iota: S^2 → E^3
 on A: (theta, phi) ↦ (x, y, z) =
                         (cos(phi)*sin(theta),
                          sin(phi)*sin(theta),
                          cos(theta))
```

Now, consider spherical coordinates on a 3-sphere:

```
sage: S3 = manifolds.Sphere(3)
sage: spher = S3.spherical_coordinates(); spher
Chart (A, (chi, theta, phi))
sage: S3.embedding().display()
iota: S^3 → E^4
on A: (chi, theta, phi) ↦ (x1, x2, x3, x4) =
                             (cos(phi)*sin(chi)*sin(theta),
                              sin(chi)*sin(phi)*sin(theta),
```

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```
                                          cos(theta)*sin(chi),
                                          cos(chi))
```

```
>>> from sage.all import *
>>> S3 = manifolds.Sphere(Integer(3))
>>> spher = S3.spherical_coordinates(); spher
Chart (A, (chi, theta, phi))
>>> S3.embedding().display()
iota: S^3 → E^4
on A: (chi, theta, phi) ↦ (x1, x2, x3, x4) =
                                         (cos(phi)*sin(chi)*sin(theta),
                                          sin(chi)*sin(phi)*sin(theta),
                                          cos(theta)*sin(chi),
                                          cos(chi))
```

By convention, the last coordinate is periodic:

```
sage: spher.coord_range()
chi: (0, pi); theta: (0, pi); phi: [-pi, pi] (periodic)
```

```
>>> from sage.all import *
>>> spher.coord_range()
chi: (0, pi); theta: (0, pi); phi: [-pi, pi] (periodic)
```

stereographic_coordinates(*pole='north'*, *names=None*)

Return stereographic coordinates given by the stereographic projection of `self` w.r.t. to a given pole.

INPUT:

- `pole` – (default: `'north'`) the pole determining the stereographic projection; possible options are `'north'` and `'south'`
- `names` – (default: `None`) must be a tuple containing the coordinate symbols (this guarantees the usage of the shortcut operator `<,>`)

OUTPUT:

- the chart of stereographic coordinates w.r.t. to the given pole, as an instance of *RealDiffChart*

Let $\mathbb{S}^n_r(c)$ be an n-sphere of radius r smoothly embedded in the Euclidean space E^{n+1} centered at $c \in E^{n+1}$. We denote the north pole of $\mathbb{S}^n_r(c)$ by NP and the south pole by SP. These poles are uniquely determined by the requirement

$$x(\iota(\mathrm{NP})) = (0, \ldots, 0, r) + x(c),$$
$$x(\iota(\mathrm{SP})) = (0, \ldots, 0, -r) + x(c).$$

The coordinates $(y_1, \ldots, y_n)$ ($(y'_1, \ldots, y'_n)$ respectively) define *stereographic coordinates* on $\mathbb{S}^n_r(c)$ for the Cartesian coordinates $(x_1, \ldots, x_{n+1})$ on E^{n+1} if they arise from the stereographic projection from $\iota(\mathrm{NP})$ ($\iota(\mathrm{SP})$) to the hypersurface $x_n = x_n(c)$. In concrete formulas, this means:

$$x \circ \iota|_{\mathbb{S}^n_r(c)\setminus\{\mathrm{NP}\}} = \left(\frac{2y_1 r^2}{r^2 + \sum_{i=1}^n y_i^2}, \ldots, \frac{2y_n r^2}{r^2 + \sum_{i=1}^n y_i^2}, \frac{r\sum_{i=1}^n y_i^2 - r^3}{r^2 + \sum_{i=1}^n y_i^2} \right) + x(c),$$
$$x \circ \iota|_{\mathbb{S}^n_r(c)\setminus\{\mathrm{SP}\}} = \left(\frac{2y'_1 r^2}{r^2 + \sum_{i=1}^n y_i'^2}, \ldots, \frac{2y'_n r^2}{r^2 + \sum_{i=1}^n y_i'^2}, \frac{r^3 - r\sum_{i=1}^n y_i'^2}{r^2 + \sum_{i=1}^n y_i'^2} \right) + x(c).$$

EXAMPLES:

Initialize a 1-sphere centered at $(1, 0)$ in the Euclidean plane using the shortcut operator:

```
sage: E2 = EuclideanSpace(2)
sage: c = E2.point((1,0), name='c')
sage: S1.<a> = E2.sphere(center=c, coordinates='stereographic'); S1
1-sphere S^1(c) of radius 1 smoothly embedded in the Euclidean plane
 E^2 centered at the Point c
```

```
>>> from sage.all import *
>>> E2 = EuclideanSpace(Integer(2))
>>> c = E2.point((Integer(1),Integer(0)), name='c')
>>> S1 = E2.sphere(center=c, coordinates='stereographic', names=('a',)); (a,)
↪= S1._first_ngens(1); S1
1-sphere S^1(c) of radius 1 smoothly embedded in the Euclidean plane
 E^2 centered at the Point c
```

By default, the shortcut variables belong to the stereographic projection from the north pole:

```
sage: S1.coordinate_charts('stereographic')
[Chart (S^1(c)-{NP}, (a,)), Chart (S^1(c)-{SP}, (ap,))]
sage: S1.embedding().display()
iota: S^1(c) → E^2
on S^1(c)-{NP}: a ↦ (x, y) = (2*a/(a^2 + 1) + 1, (a^2 - 1)/(a^2 + 1))
on S^1(c)-{SP}: ap ↦ (x, y) = (2*ap/(ap^2 + 1) + 1, -(ap^2 - 1)/(ap^2 + 1))
```

```
>>> from sage.all import *
>>> S1.coordinate_charts('stereographic')
[Chart (S^1(c)-{NP}, (a,)), Chart (S^1(c)-{SP}, (ap,))]
>>> S1.embedding().display()
iota: S^1(c) → E^2
on S^1(c)-{NP}: a ↦ (x, y) = (2*a/(a^2 + 1) + 1, (a^2 - 1)/(a^2 + 1))
on S^1(c)-{SP}: ap ↦ (x, y) = (2*ap/(ap^2 + 1) + 1, -(ap^2 - 1)/(ap^2 + 1))
```

Initialize a 2-sphere from scratch:

```
sage: S2 = manifolds.Sphere(2)
sage: S2.atlas()
[Chart (A, (theta, phi))]
```

```
>>> from sage.all import *
>>> S2 = manifolds.Sphere(Integer(2))
>>> S2.atlas()
[Chart (A, (theta, phi))]
```

In the previous block, the stereographic coordinates have not been initialized. This happens subsequently with the invocation of `stereographic_coordinates`:

```
sage: stereoS.<u,v> = S2.stereographic_coordinates(pole='south')
sage: S2.coordinate_charts('stereographic')
[Chart (S^2-{NP}, (up, vp)), Chart (S^2-{SP}, (u, v))]
```

```
>>> from sage.all import *
>>> stereoS = S2.stereographic_coordinates(pole='south', names=('u', 'v',));␣
```

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```
↪(u, v,) = stereoS._first_ngens(2)
>>> S2.coordinate_charts('stereographic')
[Chart (S^2-{NP}, (up, vp)), Chart (S^2-{SP}, (u, v))]
```

If not specified by the user, the default coordinate names are given by $(y_1, \ldots, y_n)$ and $(y'_1, \ldots, y'_n)$ respectively:

```
sage: S3 = manifolds.Sphere(3, coordinates='stereographic')
sage: S3.stereographic_coordinates(pole='north')
Chart (S^3-{NP}, (y1, y2, y3))
sage: S3.stereographic_coordinates(pole='south')
Chart (S^3-{SP}, (yp1, yp2, yp3))
```

```
>>> from sage.all import *
>>> S3 = manifolds.Sphere(Integer(3), coordinates='stereographic')
>>> S3.stereographic_coordinates(pole='north')
Chart (S^3-{NP}, (y1, y2, y3))
>>> S3.stereographic_coordinates(pole='south')
Chart (S^3-{SP}, (yp1, yp2, yp3))
```

3.2.3 Operators for vector calculus

This module defines the following operators for scalar, vector and tensor fields on any pseudo-Riemannian manifold (see `pseudo_riemannian`), and in particular on Euclidean spaces (see `euclidean`):

- `grad()`: gradient of a scalar field
- `div()`: divergence of a vector field, and more generally of a tensor field
- `curl()`: curl of a vector field (3-dimensional case only)
- `laplacian()`: Laplace-Beltrami operator acting on a scalar field, a vector field, or more generally a tensor field
- `dalembertian()`: d'Alembert operator acting on a scalar field, a vector field, or more generally a tensor field, on a Lorentzian manifold

All these operators are implemented as functions that call the appropriate method on their argument. The purpose is to allow one to use standard mathematical notations, e.g. to write `curl(v)` instead of `v.curl()`.

Note that the `norm()` operator is defined in the module `functional`.

See also:

Examples 1 and 2 in `euclidean` for examples involving these operators in the Euclidean plane and in the Euclidean 3-space.

AUTHORS:

- Eric Gourgoulhon (2018): initial version

`sage.manifolds.operators.`**`curl`**`(`*vector*`)`

Curl operator.

The *curl* of a vector field v on an orientable pseudo-Riemannian manifold (M, g) of dimension 3 is the vector field defined by

$$\operatorname{curl} v = (*(\mathrm{d}v^\flat))^\sharp$$

where $v^\flat$ is the 1-form associated to v by the metric g (see `down()`), $*(\mathrm{d}v^\flat)$ is the Hodge dual with respect to g of the 2-form $\mathrm{d}v^\flat$ (exterior derivative of $v^\flat$) (see `hodge_dual()`) and $(*(\mathrm{d}v^\flat))^\sharp$ is corresponding vector field by g-duality (see `up()`).

An alternative expression of the curl is

$$(\operatorname{curl} v)^i = \epsilon^{ijk} \nabla_j v_k$$

where ∇ is the Levi-Civita connection of g (cf. `LeviCivitaConnection`) and ϵ the volume 3-form (Levi-Civita tensor) of g (cf. `volume_form()`)

INPUT:

- `vector` – vector field on an orientable 3-dimensional pseudo-Riemannian manifold, as an instance of `VectorField`

OUTPUT:

- instance of `VectorField` representing the curl of `vector`

EXAMPLES:

Curl of a vector field in the Euclidean 3-space:

```
sage: E.<x,y,z> = EuclideanSpace()
sage: v = E.vector_field(sin(y), sin(x), 0, name='v')
sage: v.display()
v = sin(y) e_x + sin(x) e_y
sage: from sage.manifolds.operators import curl
sage: s = curl(v); s
Vector field curl(v) on the Euclidean space E^3
sage: s.display()
curl(v) = (cos(x) - cos(y)) e_z
sage: s[:]
[0, 0, cos(x) - cos(y)]
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(names=('x', 'y', 'z',)); (x, y, z,) = E._first_ngens(3)
>>> v = E.vector_field(sin(y), sin(x), Integer(0), name='v')
>>> v.display()
v = sin(y) e_x + sin(x) e_y
>>> from sage.manifolds.operators import curl
>>> s = curl(v); s
Vector field curl(v) on the Euclidean space E^3
>>> s.display()
curl(v) = (cos(x) - cos(y)) e_z
>>> s[:]
[0, 0, cos(x) - cos(y)]
```

See the method `curl()` of `VectorField` for more details and examples.

`sage.manifolds.operators.`**`dalembertian`**`(`*field*`)`

d'Alembert operator.

The *d'Alembert operator* or *d'Alembertian* on a Lorentzian manifold (M, g) is nothing but the Laplace-Beltrami operator:

$$\Box = \nabla_i \nabla^i = g^{ij} \nabla_i \nabla_j$$

where ∇ is the Levi-Civita connection of the metric g (cf. `LeviCivitaConnection`) and $\nabla^i := g^{ij}\nabla_j$

INPUT:

- `field` – a scalar field f (instance of `DiffScalarField`) or a tensor field f (instance of `TensorField`) on a pseudo-Riemannian manifold

OUTPUT:

- $\Box f$, as an instance of `DiffScalarField` or of `TensorField`

EXAMPLES:

d'Alembertian of a scalar field in the 2-dimensional Minkowski spacetime:

```
sage: M = Manifold(2, 'M', structure='Lorentzian')
sage: X.<t,x> = M.chart()
sage: g = M.metric()
sage: g[0,0], g[1,1] = -1, 1
sage: f = M.scalar_field((x-t)^3 + (x+t)^2, name='f')
sage: from sage.manifolds.operators import dalembertian
sage: Df = dalembertian(f); Df
Scalar field Box(f) on the 2-dimensional Lorentzian manifold M
sage: Df.display()
Box(f): M → ℝ
   (t, x) ↦ 0
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x',)); (t, x,) = X._first_ngens(2)
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)] = -Integer(1), Integer(1)
>>> f = M.scalar_field((x-t)**Integer(3) + (x+t)**Integer(2), name='f')
>>> from sage.manifolds.operators import dalembertian
>>> Df = dalembertian(f); Df
Scalar field Box(f) on the 2-dimensional Lorentzian manifold M
>>> Df.display()
Box(f): M → ℝ
   (t, x) ↦ 0
```

See the method `dalembertian()` of `DiffScalarField` and the method `dalembertian()` of `TensorField` for more details and examples.

`sage.manifolds.operators.`**`div`**`(`*tensor*`)`

Divergence operator.

Let t be a tensor field of type $(k, 0)$ with $k \geq 1$ on a pseudo-Riemannian manifold (M, g). The *divergence* of t is the tensor field of type $(k-1, 0)$ defined by

$$(\operatorname{div} t)^{a_1 \ldots a_{k-1}} = \nabla_i t^{a_1 \ldots a_{k-1} i} = (\nabla t)^{a_1 \ldots a_{k-1} i}{}_i$$

where ∇ is the Levi-Civita connection of g (cf. `LeviCivitaConnection`).

Note that the divergence is taken on the *last* index of the tensor field. This definition is extended to tensor fields of type (k,l) with $k \geq 0$ and $l \geq 1$, by raising the last index with the metric g: $\mathrm{div}\, t$ is then the tensor field of type $(k, l-1)$ defined by

$$(\mathrm{div}\, t)^{a_1\ldots a_k}{}_{b_1\ldots b_{l-1}} = \nabla_i (g^{ij} t^{a_1\ldots a_k}{}_{b_1\ldots b_{l-1} j}) = (\nabla t^\sharp)^{a_1\ldots a_k i}{}_{b_1\ldots b_{l-1} i}$$

where $t^\sharp$ is the tensor field deduced from t by raising the last index with the metric g (see `up()`).

INPUT:

- `tensor` – tensor field t on a pseudo-Riemannian manifold (M,g), as an instance of `TensorField` (possibly via one of its derived classes, like `VectorField`)

OUTPUT:

- the divergence of `tensor` as an instance of either `DiffScalarField` if $(k,l) = (1,0)$ (`tensor` is a vector field) or $(k,l) = (0,1)$ (`tensor` is a 1-form) or of `TensorField` if $k + l \geq 2$

EXAMPLES:

Divergence of a vector field in the Euclidean plane:

```
sage: E.<x,y> = EuclideanSpace()
sage: v = E.vector_field(cos(x*y), sin(x*y), name='v')
sage: v.display()
v = cos(x*y) e_x + sin(x*y) e_y
sage: from sage.manifolds.operators import div
sage: s = div(v); s
Scalar field div(v) on the Euclidean plane E^2
sage: s.display()
div(v): E^2 → ℝ
   (x, y) ↦ x*cos(x*y) - y*sin(x*y)
sage: s.expr()
x*cos(x*y) - y*sin(x*y)
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(names=('x', 'y',)); (x, y,) = E._first_ngens(2)
>>> v = E.vector_field(cos(x*y), sin(x*y), name='v')
>>> v.display()
v = cos(x*y) e_x + sin(x*y) e_y
>>> from sage.manifolds.operators import div
>>> s = div(v); s
Scalar field div(v) on the Euclidean plane E^2
>>> s.display()
div(v): E^2 → ℝ
   (x, y) ↦ x*cos(x*y) - y*sin(x*y)
>>> s.expr()
x*cos(x*y) - y*sin(x*y)
```

See the method `divergence()` of `TensorField` for more details and examples.

`sage.manifolds.operators.`**`grad`**`(`*`scalar`*`)`

Gradient operator.

The *gradient* of a scalar field f on a pseudo-Riemannian manifold (M,g) is the vector field $\mathrm{grad}\, f$ whose components in any coordinate frame are

$$(\mathrm{grad}\, f)^i = g^{ij} \frac{\partial F}{\partial x^j}$$

where the x^j's are the coordinates with respect to which the frame is defined and F is the chart function representing f in these coordinates: $f(p) = F(x^1(p), \ldots, x^n(p))$ for any point p in the chart domain. In other words, the gradient of f is the vector field that is the g-dual of the differential of f.

INPUT:

- `scalar` – scalar field f, as an instance of `DiffScalarField`

OUTPUT:

- instance of `VectorField` representing grad f

EXAMPLES:

Gradient of a scalar field in the Euclidean plane:

```
sage: E.<x,y> = EuclideanSpace()
sage: f = E.scalar_field(sin(x*y), name='f')
sage: from sage.manifolds.operators import grad
sage: grad(f)
Vector field grad(f) on the Euclidean plane E^2
sage: grad(f).display()
grad(f) = y*cos(x*y) e_x + x*cos(x*y) e_y
sage: grad(f)[:]
[y*cos(x*y), x*cos(x*y)]
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(names=('x', 'y',)); (x, y,) = E._first_ngens(2)
>>> f = E.scalar_field(sin(x*y), name='f')
>>> from sage.manifolds.operators import grad
>>> grad(f)
Vector field grad(f) on the Euclidean plane E^2
>>> grad(f).display()
grad(f) = y*cos(x*y) e_x + x*cos(x*y) e_y
>>> grad(f)[:]
[y*cos(x*y), x*cos(x*y)]
```

See the method `gradient()` of `DiffScalarField` for more details and examples.

`sage.manifolds.operators.`**`laplacian`**`(`*field*`)`

Laplace-Beltrami operator.

The *Laplace-Beltrami operator* on a pseudo-Riemannian manifold (M, g) is the operator

$$\Delta = \nabla_i \nabla^i = g^{ij} \nabla_i \nabla_j$$

where ∇ is the Levi-Civita connection of the metric g (cf. `LeviCivitaConnection`) and $\nabla^i := g^{ij} \nabla_j$

INPUT:

- `field` – a scalar field f (instance of `DiffScalarField`) or a tensor field f (instance of `TensorField`) on a pseudo-Riemannian manifold

OUTPUT:

- Δf, as an instance of `DiffScalarField` or of `TensorField`

EXAMPLES:

Laplacian of a scalar field on the Euclidean plane:

```
sage: E.<x,y> = EuclideanSpace()
sage: f = E.scalar_field(sin(x*y), name='f')
sage: from sage.manifolds.operators import laplacian
sage: Df = laplacian(f); Df
Scalar field Delta(f) on the Euclidean plane E^2
sage: Df.display()
Delta(f): E^2 → ℝ
   (x, y) ↦ -(x^2 + y^2)*sin(x*y)
sage: Df.expr()
-(x^2 + y^2)*sin(x*y)
```

```
>>> from sage.all import *
>>> E = EuclideanSpace(names=('x', 'y',)); (x, y,) = E._first_ngens(2)
>>> f = E.scalar_field(sin(x*y), name='f')
>>> from sage.manifolds.operators import laplacian
>>> Df = laplacian(f); Df
Scalar field Delta(f) on the Euclidean plane E^2
>>> Df.display()
Delta(f): E^2 → ℝ
   (x, y) ↦ -(x^2 + y^2)*sin(x*y)
>>> Df.expr()
-(x^2 + y^2)*sin(x*y)
```

The Laplacian of a scalar field is the divergence of its gradient:

```
sage: from sage.manifolds.operators import div, grad
sage: Df == div(grad(f))
True
```

```
>>> from sage.all import *
>>> from sage.manifolds.operators import div, grad
>>> Df == div(grad(f))
True
```

See the method `laplacian()` of `DiffScalarField` and the method `laplacian()` of `TensorField` for more details and examples.

3.3 Pseudo-Riemannian Metrics and Degenerate Metrics

The class `PseudoRiemannianMetric` implements pseudo-Riemannian metrics on differentiable manifolds over **R**. The derived class `PseudoRiemannianMetricParal` is devoted to metrics with values on a parallelizable manifold.

The class `DegenerateMetric` implements degenerate (or null or lightlike) metrics on differentiable manifolds over **R**. The derived class `DegenerateMetricParal` is devoted to metrics with values on a parallelizable manifold.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015) : initial version
- Pablo Angulo (2016) : Schouten, Cotton and Cotton-York tensors
- Florentin Jaffredo (2018) : series expansion for the inverse metric
- Hans Fotsing Tetsing (2019) : degenerate metrics
- Marius Gerbershagen (2022) : compute volume forms with contravariant indices only as needed

REFERENCES:

- [KN1963]
- [Lee1997]
- [ONe1983]
- [DB1996]
- [DS2010]

class sage.manifolds.differentiable.metric.**DegenerateMetric**(*vector_field_module*, *name*, *signature=None*, *latex_name=None*)

Bases: *TensorField*

Degenerate (or null or lightlike) metric with values on an open subset of a differentiable manifold.

An instance of this class is a field of degenerate symmetric bilinear forms (metric field) along a differentiable manifold U with values on a differentiable manifold M over $\mathbf{R}$, via a differentiable mapping $\Phi : U \to M$. The standard case of a degenerate metric field *on* a manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

A *degenerate metric* g is a field on U, such that at each point $p \in U$, $g(p)$ is a bilinear map of the type:

$$g(p) : T_q M \times T_q M \longrightarrow \mathbf{R}$$

where $T_q M$ stands for the tangent space to the manifold M at the point $q = \Phi(p)$, such that $g(p)$ is symmetric: $\forall (u,v) \in T_q M \times T_q M,\ g(p)(v,u) = g(p)(u,v)$ and degenerate: $\exists v \in T_q M;\ \ g(p)(u,v) = 0\ \ \forall u \in T_q M$.

Note

If M is parallelizable, the class *DegenerateMetricParal* should be used instead.

INPUT:

- `vector_field_module` – module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on $\Phi(U) \subset M$
- `name` – name given to the metric
- `signature` – (default: `None`) signature S of the metric as a tuple: $S = (n_+, n_-, n_0)$, where n_+ (resp. n_-, resp. n_0) is the number of positive terms (resp. negative terms, resp. zero tems) in any diagonal writing of the metric components; if `signature` is not provided, S is set to $(ndim - 1, 0, 1)$, being $ndim$ the manifold's dimension
- `latex_name` – (default: `None`) LaTeX symbol to denote the metric; if `None`, it is formed from `name`

EXAMPLES:

Lightlike cone:

```
sage: M = Manifold(3, 'M'); X.<x,y,z> = M.chart()
sage: g = M.metric('g', signature=(2,0,1)); g
degenerate metric g on the 3-dimensional differentiable manifold M
sage: det(g)
Scalar field zero on the 3-dimensional differentiable manifold M
sage: g.parent()
Free module T^(0,2)(M) of type-(0,2) tensors fields on the
3-dimensional differentiable manifold M
sage: g[0,0], g[0,1], g[0,2] = (y^2 + z^2)/(x^2 + y^2 + z^2), \
```

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```
....: - x*y/(x^2 + y^2 + z^2), - x*z/(x^2 + y^2 + z^2)
sage: g[1,1], g[1,2], g[2,2] = (x^2 + z^2)/(x^2 + y^2 + z^2), \
....: - y*z/(x^2 + y^2 + z^2), (x^2 + y^2)/(x^2 + y^2 + z^2)
sage: g.disp()
g = (y^2 + z^2)/(x^2 + y^2 + z^2) dx⊗dx - x*y/(x^2 + y^2 + z^2) dx⊗dy
- x*z/(x^2 + y^2 + z^2) dx⊗dz - x*y/(x^2 + y^2 + z^2) dy⊗dx
+ (x^2 + z^2)/(x^2 + y^2 + z^2) dy⊗dy - y*z/(x^2 + y^2 + z^2) dy⊗dz
- x*z/(x^2 + y^2 + z^2) dz⊗dx - y*z/(x^2 + y^2 + z^2) dz⊗dy
+ (x^2 + y^2)/(x^2 + y^2 + z^2) dz⊗dz
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M'); X = M.chart(names=('x', 'y', 'z',)); (x, y, z,
↪) = X._first_ngens(3)
>>> g = M.metric('g', signature=(Integer(2),Integer(0),Integer(1))); g
degenerate metric g on the 3-dimensional differentiable manifold M
>>> det(g)
Scalar field zero on the 3-dimensional differentiable manifold M
>>> g.parent()
Free module T^(0,2)(M) of type-(0,2) tensors fields on the
3-dimensional differentiable manifold M
>>> g[Integer(0),Integer(0)], g[Integer(0),Integer(1)], g[Integer(0),Integer(2)]␣
↪= (y**Integer(2) + z**Integer(2))/(x**Integer(2) + y**Integer(2) +␣
↪z**Integer(2)), - x*y/(x**Integer(2) + y**Integer(2) + z**Integer(2)), - x*z/
↪(x**Integer(2) + y**Integer(2) + z**Integer(2))
>>> g[Integer(1),Integer(1)], g[Integer(1),Integer(2)], g[Integer(2),Integer(2)]␣
↪= (x**Integer(2) + z**Integer(2))/(x**Integer(2) + y**Integer(2) +␣
↪z**Integer(2)), - y*z/(x**Integer(2) + y**Integer(2) + z**Integer(2)),␣
↪(x**Integer(2) + y**Integer(2))/(x**Integer(2) + y**Integer(2) + z**Integer(2))
>>> g.disp()
g = (y^2 + z^2)/(x^2 + y^2 + z^2) dx⊗dx - x*y/(x^2 + y^2 + z^2) dx⊗dy
- x*z/(x^2 + y^2 + z^2) dx⊗dz - x*y/(x^2 + y^2 + z^2) dy⊗dx
+ (x^2 + z^2)/(x^2 + y^2 + z^2) dy⊗dy - y*z/(x^2 + y^2 + z^2) dy⊗dz
- x*z/(x^2 + y^2 + z^2) dz⊗dx - y*z/(x^2 + y^2 + z^2) dz⊗dy
+ (x^2 + y^2)/(x^2 + y^2 + z^2) dz⊗dz
```

The position vector is a lightlike vector field:

```
sage: v = M.vector_field()
sage: v[0], v[1], v[2] = x , y, z
sage: g(v, v).disp()
M → ℝ
(x, y, z) ↦ 0
```

```
>>> from sage.all import *
>>> v = M.vector_field()
>>> v[Integer(0)], v[Integer(1)], v[Integer(2)] = x , y, z
>>> g(v, v).disp()
M → ℝ
(x, y, z) ↦ 0
```

det()

Determinant of a degenerate metric is always ‘0’.

EXAMPLES:

```
sage: S = Manifold(2, 'S')
sage: g = S.metric('g', signature=([0,1,1]))
sage: g.determinant()
Scalar field zero on the 2-dimensional differentiable manifold S
```

```
>>> from sage.all import *
>>> S = Manifold(Integer(2), 'S')
>>> g = S.metric('g', signature=([Integer(0),Integer(1),Integer(1)]))
>>> g.determinant()
Scalar field zero on the 2-dimensional differentiable manifold S
```

determinant()

Determinant of a degenerate metric is always '0'.

EXAMPLES:

```
sage: S = Manifold(2, 'S')
sage: g = S.metric('g', signature=([0,1,1]))
sage: g.determinant()
Scalar field zero on the 2-dimensional differentiable manifold S
```

```
>>> from sage.all import *
>>> S = Manifold(Integer(2), 'S')
>>> g = S.metric('g', signature=([Integer(0),Integer(1),Integer(1)]))
>>> g.determinant()
Scalar field zero on the 2-dimensional differentiable manifold S
```

restrict(*subdomain*, *dest_map=None*)

Return the restriction of the metric to some subdomain.

If the restriction has not been defined yet, it is constructed here.

INPUT:

- `subdomain` – open subset U of the metric's domain (must be an instance of *DifferentiableManifold*)
- `dest_map` – (default: `None`) destination map $\Phi : U \to V$, where V is a subdomain of `self._codomain` (type: *DiffMap*) If `None`, the restriction of `self._vmodule._dest_map` to U is used.

OUTPUT:

- instance of *DegenerateMetric* representing the restriction.

EXAMPLES:

```
sage: M = Manifold(5, 'M')
sage: g = M.metric('g', signature=(3,1,1))
sage: U = M.open_subset('U')
sage: g.restrict(U)
degenerate metric g on the Open subset U of the 5-dimensional
differentiable manifold M
sage: g.restrict(U).signature()
(3, 1, 1)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(5), 'M')
>>> g = M.metric('g', signature=(Integer(3),Integer(1),Integer(1)))
>>> U = M.open_subset('U')
>>> g.restrict(U)
degenerate metric g on the Open subset U of the 5-dimensional
differentiable manifold M
>>> g.restrict(U).signature()
(3, 1, 1)
```

See the top documentation of `DegenerateMetric` for more examples.

set (*symbiform*)

Define the metric from a field of symmetric bilinear forms.

INPUT:

- `symbiform` – instance of `TensorField` representing a field of symmetric bilinear forms

EXAMPLES:

Metric defined from a field of symmetric bilinear forms on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
....:                               restrictions1= x>0, restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: h = M.sym_bilin_form_field(name='h')
sage: h[eU,0,0], h[eU,0,1], h[eU,1,1] = 1+x, x*y, 1-y
sage: h.add_comp_by_continuation(eV, W, c_uv)
sage: h.display(eU)
h = (x + 1) dx⊗dx + x*y dx⊗dy + x*y dy⊗dx + (-y + 1) dy⊗dy
sage: h.display(eV)
h = (1/8*u^2 - 1/8*v^2 + 1/4*v + 1/2) du⊗du + 1/4*u du⊗dv
 + 1/4*u dv⊗du + (-1/8*u^2 + 1/8*v^2 + 1/4*v + 1/2) dv⊗dv
sage: g = M.metric('g')
sage: g.set(h)
sage: g.display(eU)
g = (x + 1) dx⊗dx + x*y dx⊗dy + x*y dy⊗dx + (-y + 1) dy⊗dy
sage: g.display(eV)
g = (1/8*u^2 - 1/8*v^2 + 1/4*v + 1/2) du⊗du + 1/4*u du⊗dv
 + 1/4*u dv⊗du + (-1/8*u^2 + 1/8*v^2 + 1/4*v + 1/2) dv⊗dv
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
```

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```
...                                  restrictions1= x>Integer(0), restrictions2=
↪u+v>Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V)
>>> eU = c_xy.frame() ; eV = c_uv.frame()
>>> h = M.sym_bilin_form_field(name='h')
>>> h[eU,Integer(0),Integer(0)], h[eU,Integer(0),Integer(1)], h[eU,Integer(1),
↪Integer(1)] = Integer(1)+x, x*y, Integer(1)-y
>>> h.add_comp_by_continuation(eV, W, c_uv)
>>> h.display(eU)
h = (x + 1) dx⊗dx + x*y dx⊗dy + x*y dy⊗dx + (-y + 1) dy⊗dy
>>> h.display(eV)
h = (1/8*u^2 - 1/8*v^2 + 1/4*v + 1/2) du⊗du + 1/4*u du⊗dv
 + 1/4*u dv⊗du + (-1/8*u^2 + 1/8*v^2 + 1/4*v + 1/2) dv⊗dv
>>> g = M.metric('g')
>>> g.set(h)
>>> g.display(eU)
g = (x + 1) dx⊗dx + x*y dx⊗dy + x*y dy⊗dx + (-y + 1) dy⊗dy
>>> g.display(eV)
g = (1/8*u^2 - 1/8*v^2 + 1/4*v + 1/2) du⊗du + 1/4*u du⊗dv
 + 1/4*u dv⊗du + (-1/8*u^2 + 1/8*v^2 + 1/4*v + 1/2) dv⊗dv
```

signature()

Signature of the metric.

OUTPUT:

- signature of a degenerate metric is defined as the tuple (n_+, n_-, n_0), where n_+ (resp. n_-, resp. n_0) is the number of positive terms (resp. negative terms, resp. zero terms) eigenvalues

EXAMPLES:

Signatures on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M')
sage: g = M.metric('g', signature=(1,1,1))
sage: g.signature()
(1, 1, 1)
sage: M = Manifold(3, 'M', structure='degenerate_metric')
sage: g = M.metric()
sage: g.signature()
(0, 2, 1)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M')
>>> g = M.metric('g', signature=(Integer(1),Integer(1),Integer(1)))
>>> g.signature()
(1, 1, 1)
>>> M = Manifold(Integer(3), 'M', structure='degenerate_metric')
>>> g = M.metric()
>>> g.signature()
(0, 2, 1)
```

class sage.manifolds.differentiable.metric.**DegenerateMetricParal**(*vector_field_module*, *name*, *signature=None*, *latex_name=None*)

Bases: `DegenerateMetric`, `TensorFieldParal`

Degenerate (or null or lightlike) metric with values on an open subset of a differentiable manifold.

An instance of this class is a field of degenerate symmetric bilinear forms (metric field) along a differentiable manifold U with values on a differentiable manifold M over $\mathbf{R}$, via a differentiable mapping $\Phi : U \to M$. The standard case of a degenerate metric field *on* a manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

A *degenerate metric* g is a field on U, such that at each point $p \in U$, $g(p)$ is a bilinear map of the type:

$$g(p) : T_qM \times T_qM \longrightarrow \mathbf{R}$$

where T_qM stands for the tangent space to the manifold M at the point $q = \Phi(p)$, such that $g(p)$ is symmetric: $\forall (u, v) \in T_qM \times T_qM, \; g(p)(v, u) = g(p)(u, v)$ and degenerate: $\exists v \in T_qM; \;\; g(p)(u, v) = 0 \;\; \forall u \in T_qM$.

Note: If M is not parallelizable, the class `DegenerateMetric` should be used instead.

INPUT:

- `vector_field_module` – module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on $\Phi(U) \subset M$
- `name` – name given to the metric
- `signature` – (default: `None`) signature S of the metric as a tuple: $S = (n_+, n_-, n_0)$, where n_+ (resp. n_-, resp. n_0) is the number of positive terms (resp. negative terms, resp. zero tems) in any diagonal writing of the metric components; if `signature` is not provided, S is set to $(ndim - 1, 0, 1)$, being $ndim$ the manifold's dimension
- `latex_name` – (default: `None`) LaTeX symbol to denote the metric; if `None`, it is formed from `name`

EXAMPLES:

Lightlike cone:

```
sage: M = Manifold(3, 'M'); X.<x,y,z> = M.chart()
sage: g = M.metric('g', signature=(2,0,1)); g
degenerate metric g on the 3-dimensional differentiable manifold M
sage: det(g)
Scalar field zero on the 3-dimensional differentiable manifold M
sage: g.parent()
Free module T^(0,2)(M) of type-(0,2) tensors fields on the
3-dimensional differentiable manifold M
sage: g[0,0], g[0,1], g[0,2] = (y^2 + z^2)/(x^2 + y^2 + z^2), \
....: - x*y/(x^2 + y^2 + z^2), - x*z/(x^2 + y^2 + z^2)
sage: g[1,1], g[1,2], g[2,2] = (x^2 + z^2)/(x^2 + y^2 + z^2), \
....: - y*z/(x^2 + y^2 + z^2), (x^2 + y^2)/(x^2 + y^2 + z^2)
sage: g.disp()
g = (y^2 + z^2)/(x^2 + y^2 + z^2) dx⊗dx - x*y/(x^2 + y^2 + z^2) dx⊗dy
- x*z/(x^2 + y^2 + z^2) dx⊗dz - x*y/(x^2 + y^2 + z^2) dy⊗dx
+ (x^2 + z^2)/(x^2 + y^2 + z^2) dy⊗dy - y*z/(x^2 + y^2 + z^2) dy⊗dz
- x*z/(x^2 + y^2 + z^2) dz⊗dx - y*z/(x^2 + y^2 + z^2) dz⊗dy
+ (x^2 + y^2)/(x^2 + y^2 + z^2) dz⊗dz
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M'); X = M.chart(names=('x', 'y', 'z',)); (x, y, z,
↪) = X._first_ngens(3)
>>> g = M.metric('g', signature=(Integer(2),Integer(0),Integer(1))); g
degenerate metric g on the 3-dimensional differentiable manifold M
>>> det(g)
Scalar field zero on the 3-dimensional differentiable manifold M
>>> g.parent()
Free module T^(0,2)(M) of type-(0,2) tensors fields on the
3-dimensional differentiable manifold M
>>> g[Integer(0),Integer(0)], g[Integer(0),Integer(1)], g[Integer(0),Integer(2)]␣
↪= (y**Integer(2) + z**Integer(2))/(x**Integer(2) + y**Integer(2) +␣
↪z**Integer(2)), - x*y/(x**Integer(2) + y**Integer(2) + z**Integer(2)), - x*z/
↪(x**Integer(2) + y**Integer(2) + z**Integer(2))
>>> g[Integer(1),Integer(1)], g[Integer(1),Integer(2)], g[Integer(2),Integer(2)]␣
↪= (x**Integer(2) + z**Integer(2))/(x**Integer(2) + y**Integer(2) +␣
↪z**Integer(2)), - y*z/(x**Integer(2) + y**Integer(2) + z**Integer(2)),␣
↪(x**Integer(2) + y**Integer(2))/(x**Integer(2) + y**Integer(2) + z**Integer(2))
>>> g.disp()
g = (y^2 + z^2)/(x^2 + y^2 + z^2) dx⊗dx - x*y/(x^2 + y^2 + z^2) dx⊗dy
- x*z/(x^2 + y^2 + z^2) dx⊗dz - x*y/(x^2 + y^2 + z^2) dy⊗dx
+ (x^2 + z^2)/(x^2 + y^2 + z^2) dy⊗dy - y*z/(x^2 + y^2 + z^2) dy⊗dz
- x*z/(x^2 + y^2 + z^2) dz⊗dx - y*z/(x^2 + y^2 + z^2) dz⊗dy
+ (x^2 + y^2)/(x^2 + y^2 + z^2) dz⊗dz
```

The position vector is a lightlike vector field:

```
sage: v = M.vector_field()
sage: v[0], v[1], v[2] = x , y, z
sage: g(v, v).disp()
M → ℝ
(x, y, z) ↦ 0
```

```
>>> from sage.all import *
>>> v = M.vector_field()
>>> v[Integer(0)], v[Integer(1)], v[Integer(2)] = x , y, z
>>> g(v, v).disp()
M → ℝ
(x, y, z) ↦ 0
```

restrict(*subdomain*, *dest_map=None*)

Return the restriction of the metric to some subdomain.

If the restriction has not been defined yet, it is constructed here.

INPUT:

- subdomain – open subset U of the metric's domain (must be an instance of *DifferentiableManifold*)
- dest_map – (default: None) destination map $\Phi : U \to V$, where V is a subdomain of self._codomain (type: *DiffMap*) If None, the restriction of self._vmodule._dest_map to U is used.

OUTPUT:

- instance of *DegenerateMetric* representing the restriction.

EXAMPLES:

```
sage: M = Manifold(5, 'M')
sage: g = M.metric('g', signature=(3,1,1))
sage: U = M.open_subset('U')
sage: g.restrict(U)
degenerate metric g on the Open subset U of the 5-dimensional differentiable␣
↪manifold M
sage: g.restrict(U).signature()
(3, 1, 1)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(5), 'M')
>>> g = M.metric('g', signature=(Integer(3),Integer(1),Integer(1)))
>>> U = M.open_subset('U')
>>> g.restrict(U)
degenerate metric g on the Open subset U of the 5-dimensional differentiable␣
↪manifold M
>>> g.restrict(U).signature()
(3, 1, 1)
```

See the top documentation of *DegenerateMetric* for more examples.

set (*symbiform*)

Define the metric from a field of symmetric bilinear forms.

INPUT:

- `symbiform` – instance of *TensorField* representing a field of symmetric bilinear forms

EXAMPLES:

Metric defined from a field of symmetric bilinear forms on a parallelizable 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1);
sage: X.<x,y,z> = M.chart()
sage: dx, dy = X.coframe()[1], X.coframe()[2]
sage: b = dx*dx + dy*dy
sage: g = M.metric('g', signature=(1,1,1)); g
degenerate metric g on the 3-dimensional differentiable manifold M
sage: g.set(b)
sage: g.display()
g = dx⊗dx + dy⊗dy
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1));
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> dx, dy = X.coframe()[Integer(1)], X.coframe()[Integer(2)]
>>> b = dx*dx + dy*dy
>>> g = M.metric('g', signature=(Integer(1),Integer(1),Integer(1))); g
degenerate metric g on the 3-dimensional differentiable manifold M
>>> g.set(b)
>>> g.display()
g = dx⊗dx + dy⊗dy
```

class sage.manifolds.differentiable.metric.**PseudoRiemannianMetric**(*vector_field_module*, *name*, *signature=None*, *latex_name=None*)

Bases: `TensorField`

Pseudo-Riemannian metric with values on an open subset of a differentiable manifold.

An instance of this class is a field of nondegenerate symmetric bilinear forms (metric field) along a differentiable manifold U with values on a differentiable manifold M over $\mathbf{R}$, via a differentiable mapping $\Phi : U \to M$. The standard case of a metric field *on* a manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

A *metric* g is a field on U, such that at each point $p \in U$, $g(p)$ is a bilinear map of the type:

$$g(p) : T_q M \times T_q M \longrightarrow \mathbf{R}$$

where $T_q M$ stands for the tangent space to the manifold M at the point $q = \Phi(p)$, such that $g(p)$ is symmetric: $\forall (u, v) \in T_q M \times T_q M, \ g(p)(v, u) = g(p)(u, v)$ and nondegenerate: $(\forall v \in T_q M, \ \ g(p)(u, v) = 0) \Longrightarrow u = 0$.

Note: If M is parallelizable, the class `PseudoRiemannianMetricParal` should be used instead.

INPUT:

- `vector_field_module` – module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on $\Phi(U) \subset M$
- `name` – name given to the metric
- `signature` – (default: `None`) signature S of the metric as a single integer: $S = n_+ - n_-$, where n_+ (resp. n_-) is the number of positive terms (resp. number of negative terms) in any diagonal writing of the metric components; if `signature` is `None`, S is set to the dimension of manifold M (Riemannian signature)
- `latex_name` – (default: `None`) LaTeX symbol to denote the metric; if `None`, it is formed from `name`

EXAMPLES:

Let us construct the standard metric on the sphere S^2, described in terms of stereographic coordinates, from the North pole (open subset U) and from the South pole (open subset V):

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart() # stereographic coord
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: g = M.metric('g') ; g
Riemannian metric g on the 2-dimensional differentiable manifold S^2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', start_index=Integer(1))
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # S^2 is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv = V.
```

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```
↪chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)# stereographic coord
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                     intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                     restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> eU = c_xy.frame() ; eV = c_uv.frame()
>>> g = M.metric('g') ; g
Riemannian metric g on the 2-dimensional differentiable manifold S^2
```

The metric is considered as a tensor field of type (0,2) on S^2:

```
sage: g.parent()
Module T^(0,2)(S^2) of type-(0,2) tensors fields on the 2-dimensional
 differentiable manifold S^2
```

```
>>> from sage.all import *
>>> g.parent()
Module T^(0,2)(S^2) of type-(0,2) tensors fields on the 2-dimensional
 differentiable manifold S^2
```

We define g by its components on domain U:

```
sage: g[eU,1,1], g[eU,2,2] = 4/(1+x^2+y^2)^2, 4/(1+x^2+y^2)^2
sage: g.display(eU)
g = 4/(x^2 + y^2 + 1)^2 dx⊗dx + 4/(x^2 + y^2 + 1)^2 dy⊗dy
```

```
>>> from sage.all import *
>>> g[eU,Integer(1),Integer(1)], g[eU,Integer(2),Integer(2)] = Integer(4)/
↪(Integer(1)+x**Integer(2)+y**Integer(2))**Integer(2), Integer(4)/
↪(Integer(1)+x**Integer(2)+y**Integer(2))**Integer(2)
>>> g.display(eU)
g = 4/(x^2 + y^2 + 1)^2 dx⊗dx + 4/(x^2 + y^2 + 1)^2 dy⊗dy
```

A matrix view of the components:

```
sage: g[eU,:]
[4/(x^2 + y^2 + 1)^2                   0]
[                  0 4/(x^2 + y^2 + 1)^2]
```

```
>>> from sage.all import *
>>> g[eU,:]
[4/(x^2 + y^2 + 1)^2                   0]
[                  0 4/(x^2 + y^2 + 1)^2]
```

The components of g on domain V expressed in terms of coordinates (u, v) are obtained by applying (i) the tensor transformation law on $W = U \cap V$ and (ii) some analytical continuation:

```
sage: W = U.intersection(V)
sage: g.add_comp_by_continuation(eV, W, chart=c_uv)
sage: g.apply_map(factor, frame=eV, keep_other_components=True) # for a nicer␣
```

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```
↪display
sage: g.display(eV)
g = 4/(u^2 + v^2 + 1)^2 du⊗du + 4/(u^2 + v^2 + 1)^2 dv⊗dv
```

```
>>> from sage.all import *
>>> W = U.intersection(V)
>>> g.add_comp_by_continuation(eV, W, chart=c_uv)
>>> g.apply_map(factor, frame=eV, keep_other_components=True) # for a nicer␣
↪display
>>> g.display(eV)
g = 4/(u^2 + v^2 + 1)^2 du⊗du + 4/(u^2 + v^2 + 1)^2 dv⊗dv
```

At this stage, the metric is fully defined on the whole sphere. Its restriction to some subdomain is itself a metric (by default, it bears the same symbol):

```
sage: g.restrict(U)
Riemannian metric g on the Open subset U of the 2-dimensional
 differentiable manifold S^2
sage: g.restrict(U).parent()
Free module T^(0,2)(U) of type-(0,2) tensors fields on the Open subset
 U of the 2-dimensional differentiable manifold S^2
```

```
>>> from sage.all import *
>>> g.restrict(U)
Riemannian metric g on the Open subset U of the 2-dimensional
 differentiable manifold S^2
>>> g.restrict(U).parent()
Free module T^(0,2)(U) of type-(0,2) tensors fields on the Open subset
 U of the 2-dimensional differentiable manifold S^2
```

The parent of $g|_U$ is a free module because is U is a parallelizable domain, contrary to S^2. Actually, g and $g|_U$ have different Python type:

```
sage: type(g)
<class 'sage.manifolds.differentiable.metric.PseudoRiemannianMetric'>
sage: type(g.restrict(U))
<class 'sage.manifolds.differentiable.metric.PseudoRiemannianMetricParal'>
```

```
>>> from sage.all import *
>>> type(g)
<class 'sage.manifolds.differentiable.metric.PseudoRiemannianMetric'>
>>> type(g.restrict(U))
<class 'sage.manifolds.differentiable.metric.PseudoRiemannianMetricParal'>
```

As a field of bilinear forms, the metric acts on pairs of vector fields, yielding a scalar field:

```
sage: a = M.vector_field({eU: [x, 2+y]}, name='a')
sage: a.add_comp_by_continuation(eV, W, chart=c_uv)
sage: b = M.vector_field({eU: [-y, x]}, name='b')
sage: b.add_comp_by_continuation(eV, W, chart=c_uv)
sage: s = g(a,b) ; s
Scalar field g(a,b) on the 2-dimensional differentiable manifold S^2
```

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```
sage: s.display()
g(a,b): S^2 → ℝ
on U: (x, y) ↦ 8*x/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1)
on V: (u, v) ↦ 8*(u^3 + u*v^2)/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1)
```

```
>>> from sage.all import *
>>> a = M.vector_field({eU: [x, Integer(2)+y]}, name='a')
>>> a.add_comp_by_continuation(eV, W, chart=c_uv)
>>> b = M.vector_field({eU: [-y, x]}, name='b')
>>> b.add_comp_by_continuation(eV, W, chart=c_uv)
>>> s = g(a,b) ; s
Scalar field g(a,b) on the 2-dimensional differentiable manifold S^2
>>> s.display()
g(a,b): S^2 → ℝ
on U: (x, y) ↦ 8*x/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1)
on V: (u, v) ↦ 8*(u^3 + u*v^2)/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1)
```

The inverse metric is:

```
sage: ginv = g.inverse() ; ginv
Tensor field inv_g of type (2,0) on the 2-dimensional differentiable
 manifold S^2
sage: ginv.parent()
Module T^(2,0)(S^2) of type-(2,0) tensors fields on the 2-dimensional
 differentiable manifold S^2
sage: latex(ginv)
g^{-1}
sage: ginv.display(eU)
inv_g = (1/4*x^4 + 1/4*y^4 + 1/2*(x^2 + 1)*y^2 + 1/2*x^2 + 1/4) ∂/∂x⊗∂/∂x
 + (1/4*x^4 + 1/4*y^4 + 1/2*(x^2 + 1)*y^2 + 1/2*x^2 + 1/4) ∂/∂y⊗∂/∂y
sage: ginv.display(eV)
inv_g = (1/4*u^4 + 1/4*v^4 + 1/2*(u^2 + 1)*v^2 + 1/2*u^2 + 1/4) ∂/∂u⊗∂/∂u
 + (1/4*u^4 + 1/4*v^4 + 1/2*(u^2 + 1)*v^2 + 1/2*u^2 + 1/4) ∂/∂v⊗∂/∂v
```

```
>>> from sage.all import *
>>> ginv = g.inverse() ; ginv
Tensor field inv_g of type (2,0) on the 2-dimensional differentiable
 manifold S^2
>>> ginv.parent()
Module T^(2,0)(S^2) of type-(2,0) tensors fields on the 2-dimensional
 differentiable manifold S^2
>>> latex(ginv)
g^{-1}
>>> ginv.display(eU)
inv_g = (1/4*x^4 + 1/4*y^4 + 1/2*(x^2 + 1)*y^2 + 1/2*x^2 + 1/4) ∂/∂x⊗∂/∂x
 + (1/4*x^4 + 1/4*y^4 + 1/2*(x^2 + 1)*y^2 + 1/2*x^2 + 1/4) ∂/∂y⊗∂/∂y
>>> ginv.display(eV)
inv_g = (1/4*u^4 + 1/4*v^4 + 1/2*(u^2 + 1)*v^2 + 1/2*u^2 + 1/4) ∂/∂u⊗∂/∂u
 + (1/4*u^4 + 1/4*v^4 + 1/2*(u^2 + 1)*v^2 + 1/2*u^2 + 1/4) ∂/∂v⊗∂/∂v
```

We have:

```
sage: ginv.restrict(U) is g.restrict(U).inverse()
True
sage: ginv.restrict(V) is g.restrict(V).inverse()
True
sage: ginv.restrict(W) is g.restrict(W).inverse()
True
```

```
>>> from sage.all import *
>>> ginv.restrict(U) is g.restrict(U).inverse()
True
>>> ginv.restrict(V) is g.restrict(V).inverse()
True
>>> ginv.restrict(W) is g.restrict(W).inverse()
True
```

To get the volume form (Levi-Civita tensor) associated with g, we have first to define an orientation on S^2. The standard orientation is that in which `eV` is right-handed; indeed, once supplemented by the outward unit normal, `eV` give birth to a right-handed frame with respect to the standard orientation of the ambient Euclidean space E^3. With such an orientation, `eU` is then left-handed and in order to define an orientation on the whole of S^2, we introduce a vector frame on U by swapping `eU`'s vectors:

```
sage: f = U.vector_frame('f', (eU[2], eU[1]))
sage: M.set_orientation([eV, f])
```

```
>>> from sage.all import *
>>> f = U.vector_frame('f', (eU[Integer(2)], eU[Integer(1)]))
>>> M.set_orientation([eV, f])
```

We have then, factorizing the components for a nicer display:

```
sage: eps = g.volume_form() ; eps
2-form eps_g on the 2-dimensional differentiable manifold S^2
sage: eps.apply_map(factor, frame=eU, keep_other_components=True)
sage: eps.apply_map(factor, frame=eV, keep_other_components=True)
sage: eps.display(eU)
eps_g = -4/(x^2 + y^2 + 1)^2 dx∧dy
sage: eps.display(eV)
eps_g = 4/(u^2 + v^2 + 1)^2 du∧dv
```

```
>>> from sage.all import *
>>> eps = g.volume_form() ; eps
2-form eps_g on the 2-dimensional differentiable manifold S^2
>>> eps.apply_map(factor, frame=eU, keep_other_components=True)
>>> eps.apply_map(factor, frame=eV, keep_other_components=True)
>>> eps.display(eU)
eps_g = -4/(x^2 + y^2 + 1)^2 dx∧dy
>>> eps.display(eV)
eps_g = 4/(u^2 + v^2 + 1)^2 du∧dv
```

The unique non-trivial component of the volume form is, up to a sign depending of the chosen orientation, nothing but the square root of the determinant of g in the corresponding frame:

```
sage: eps[[eU,1,2]] == -g.sqrt_abs_det(eU)
True
sage: eps[[eV,1,2]] == g.sqrt_abs_det(eV)
True
```

```
>>> from sage.all import *
>>> eps[[eU,Integer(1),Integer(2)]] == -g.sqrt_abs_det(eU)
True
>>> eps[[eV,Integer(1),Integer(2)]] == g.sqrt_abs_det(eV)
True
```

The Levi-Civita connection associated with the metric g:

```
sage: nabla = g.connection() ; nabla
Levi-Civita connection nabla_g associated with the Riemannian metric g
 on the 2-dimensional differentiable manifold S^2
sage: latex(nabla)
\nabla_{g}
```

```
>>> from sage.all import *
>>> nabla = g.connection() ; nabla
Levi-Civita connection nabla_g associated with the Riemannian metric g
 on the 2-dimensional differentiable manifold S^2
>>> latex(nabla)
\nabla_{g}
```

The Christoffel symbols ${\Gamma^i}_{jk}$ associated with some coordinates:

```
sage: g.christoffel_symbols(c_xy)
3-indices components w.r.t. Coordinate frame (U, (∂/∂x,∂/∂y)), with
 symmetry on the index positions (1, 2)
sage: g.christoffel_symbols(c_xy)[:]
[[[-2*x/(x^2 + y^2 + 1), -2*y/(x^2 + y^2 + 1)],
  [-2*y/(x^2 + y^2 + 1), 2*x/(x^2 + y^2 + 1)]],
 [[2*y/(x^2 + y^2 + 1), -2*x/(x^2 + y^2 + 1)],
  [-2*x/(x^2 + y^2 + 1), -2*y/(x^2 + y^2 + 1)]]]
sage: g.christoffel_symbols(c_uv)[:]
[[[-2*u/(u^2 + v^2 + 1), -2*v/(u^2 + v^2 + 1)],
  [-2*v/(u^2 + v^2 + 1), 2*u/(u^2 + v^2 + 1)]],
 [[2*v/(u^2 + v^2 + 1), -2*u/(u^2 + v^2 + 1)],
  [-2*u/(u^2 + v^2 + 1), -2*v/(u^2 + v^2 + 1)]]]
```

```
>>> from sage.all import *
>>> g.christoffel_symbols(c_xy)
3-indices components w.r.t. Coordinate frame (U, (∂/∂x,∂/∂y)), with
 symmetry on the index positions (1, 2)
>>> g.christoffel_symbols(c_xy)[:]
[[[-2*x/(x^2 + y^2 + 1), -2*y/(x^2 + y^2 + 1)],
  [-2*y/(x^2 + y^2 + 1), 2*x/(x^2 + y^2 + 1)]],
 [[2*y/(x^2 + y^2 + 1), -2*x/(x^2 + y^2 + 1)],
  [-2*x/(x^2 + y^2 + 1), -2*y/(x^2 + y^2 + 1)]]]
>>> g.christoffel_symbols(c_uv)[:]
```

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```
[[[-2*u/(u^2 + v^2 + 1), -2*v/(u^2 + v^2 + 1)],
  [-2*v/(u^2 + v^2 + 1), 2*u/(u^2 + v^2 + 1)]],
 [[2*v/(u^2 + v^2 + 1), -2*u/(u^2 + v^2 + 1)],
  [-2*u/(u^2 + v^2 + 1), -2*v/(u^2 + v^2 + 1)]]]
```

The Christoffel symbols are nothing but the connection coefficients w.r.t. the coordinate frame:

```
sage: g.christoffel_symbols(c_xy) is nabla.coef(c_xy.frame())
True
sage: g.christoffel_symbols(c_uv) is nabla.coef(c_uv.frame())
True
```

```
>>> from sage.all import *
>>> g.christoffel_symbols(c_xy) is nabla.coef(c_xy.frame())
True
>>> g.christoffel_symbols(c_uv) is nabla.coef(c_uv.frame())
True
```

Test that ∇ is the connection compatible with g:

```
sage: t = nabla(g) ; t
Tensor field nabla_g(g) of type (0,3) on the 2-dimensional
 differentiable manifold S^2
sage: t.display(eU)
nabla_g(g) = 0
sage: t.display(eV)
nabla_g(g) = 0
sage: t == 0
True
```

```
>>> from sage.all import *
>>> t = nabla(g) ; t
Tensor field nabla_g(g) of type (0,3) on the 2-dimensional
 differentiable manifold S^2
>>> t.display(eU)
nabla_g(g) = 0
>>> t.display(eV)
nabla_g(g) = 0
>>> t == Integer(0)
True
```

The Riemann curvature tensor of g:

```
sage: riem = g.riemann() ; riem
Tensor field Riem(g) of type (1,3) on the 2-dimensional differentiable
 manifold S^2
sage: riem.display(eU)
Riem(g) = 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) ∂/∂x⊗dy⊗dx⊗dy
 - 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) ∂/∂x⊗dy⊗dy⊗dx
 - 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) ∂/∂y⊗dx⊗dx⊗dy
 + 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) ∂/∂y⊗dx⊗dy⊗dx
sage: riem.display(eV)
```

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```
Riem(g) = 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) ∂/∂u⊗dv⊗du⊗dv
 - 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) ∂/∂u⊗dv⊗dv⊗du
 - 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) ∂/∂v⊗du⊗du⊗dv
 + 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) ∂/∂v⊗du⊗dv⊗du
```

```
>>> from sage.all import *
>>> riem = g.riemann() ; riem
Tensor field Riem(g) of type (1,3) on the 2-dimensional differentiable
 manifold S^2
>>> riem.display(eU)
Riem(g) = 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) ∂/∂x⊗dy⊗dx⊗dy
 - 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) ∂/∂x⊗dy⊗dy⊗dx
 - 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) ∂/∂y⊗dx⊗dx⊗dy
 + 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) ∂/∂y⊗dx⊗dy⊗dx
>>> riem.display(eV)
Riem(g) = 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) ∂/∂u⊗dv⊗du⊗dv
 - 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) ∂/∂u⊗dv⊗dv⊗du
 - 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) ∂/∂v⊗du⊗du⊗dv
 + 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) ∂/∂v⊗du⊗dv⊗du
```

The Ricci tensor of g:

```
sage: ric = g.ricci() ; ric
Field of symmetric bilinear forms Ric(g) on the 2-dimensional
 differentiable manifold S^2
sage: ric.display(eU)
Ric(g) = 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) dx⊗dx
 + 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) dy⊗dy
sage: ric.display(eV)
Ric(g) = 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) du⊗du
 + 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) dv⊗dv
sage: ric == g
True
```

```
>>> from sage.all import *
>>> ric = g.ricci() ; ric
Field of symmetric bilinear forms Ric(g) on the 2-dimensional
 differentiable manifold S^2
>>> ric.display(eU)
Ric(g) = 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) dx⊗dx
 + 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) dy⊗dy
>>> ric.display(eV)
Ric(g) = 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) du⊗du
 + 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) dv⊗dv
>>> ric == g
True
```

The Ricci scalar of g:

```
sage: r = g.ricci_scalar() ; r
Scalar field r(g) on the 2-dimensional differentiable manifold S^2
sage: r.display()
```

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```
r(g): S^2 → ℝ
on U: (x, y) ↦ 2
on V: (u, v) ↦ 2
```

```
>>> from sage.all import *
>>> r = g.ricci_scalar() ; r
Scalar field r(g) on the 2-dimensional differentiable manifold S^2
>>> r.display()
r(g): S^2 → ℝ
on U: (x, y) ↦ 2
on V: (u, v) ↦ 2
```

In dimension 2, the Riemann tensor can be expressed entirely in terms of the Ricci scalar r:

$$R^i{}_{jlk} = \frac{r}{2}\left(\delta^i{}_k g_{jl} - \delta^i{}_l g_{jk}\right)$$

This formula can be checked here, with the r.h.s. rewritten as $-rg_{j[k}\delta^i{}_{l]}$:

```
sage: delta = M.tangent_identity_field()
sage: riem == - r*(g*delta).antisymmetrize(2,3)
True
```

```
>>> from sage.all import *
>>> delta = M.tangent_identity_field()
>>> riem == - r*(g*delta).antisymmetrize(Integer(2),Integer(3))
True
```

christoffel_symbols(*chart=None*)

Christoffel symbols of `self` with respect to a chart.

INPUT:

- `chart` – (default: `None`) chart with respect to which the Christoffel symbols are required; if none is provided, the default chart of the metric's domain is assumed.

OUTPUT:

- the set of Christoffel symbols in the given chart, as an instance of `CompWithSym`

EXAMPLES:

Christoffel symbols of the flat metric on $\mathbf{R}^3$ with respect to spherical coordinates:

```
sage: M = Manifold(3, 'R3', r'\RR^3', start_index=1)
sage: U = M.open_subset('U') # the complement of the half-plane (y=0, x>=0)
sage: X.<r,th,ph> = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: g = U.metric('g')
sage: g[1,1], g[2,2], g[3,3] = 1, r^2, r^2*sin(th)^2
sage: g.display()  # the standard flat metric expressed in spherical␣
↪coordinates
g = dr⊗dr + r^2 dth⊗dth + r^2*sin(th)^2 dph⊗dph
sage: Gam = g.christoffel_symbols() ; Gam
3-indices components w.r.t. Coordinate frame (U, (∂/∂r,∂/∂th,∂/∂ph)),
 with symmetry on the index positions (1, 2)
sage: type(Gam)
```

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```
<class 'sage.tensor.modules.comp.CompWithSym'>
sage: Gam[:]
[[[0, 0, 0], [0, -r, 0], [0, 0, -r*sin(th)^2]],
[[0, 1/r, 0], [1/r, 0, 0], [0, 0, -cos(th)*sin(th)]],
[[0, 0, 1/r], [0, 0, cos(th)/sin(th)], [1/r, cos(th)/sin(th), 0]]]
sage: Gam[1,2,2]
-r
sage: Gam[2,1,2]
1/r
sage: Gam[3,1,3]
1/r
sage: Gam[3,2,3]
cos(th)/sin(th)
sage: Gam[2,3,3]
-cos(th)*sin(th)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'R3', r'\RR^3', start_index=Integer(1))
>>> U = M.open_subset('U') # the complement of the half-plane (y=0, x>=0)
>>> X = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi', names=('r',
↪'th', 'ph',)); (r, th, ph,) = X._first_ngens(3)
>>> g = U.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)], g[Integer(3),
↪Integer(3)] = Integer(1), r**Integer(2), r**Integer(2)*sin(th)**Integer(2)
>>> g.display()  # the standard flat metric expressed in spherical coordinates
g = dr⊗dr + r^2 dth⊗dth + r^2*sin(th)^2 dph⊗dph
>>> Gam = g.christoffel_symbols() ; Gam
3-indices components w.r.t. Coordinate frame (U, (∂/∂r,∂/∂th,∂/∂ph)),
 with symmetry on the index positions (1, 2)
>>> type(Gam)
<class 'sage.tensor.modules.comp.CompWithSym'>
>>> Gam[:]
[[[0, 0, 0], [0, -r, 0], [0, 0, -r*sin(th)^2]],
[[0, 1/r, 0], [1/r, 0, 0], [0, 0, -cos(th)*sin(th)]],
[[0, 0, 1/r], [0, 0, cos(th)/sin(th)], [1/r, cos(th)/sin(th), 0]]]
>>> Gam[Integer(1),Integer(2),Integer(2)]
-r
>>> Gam[Integer(2),Integer(1),Integer(2)]
1/r
>>> Gam[Integer(3),Integer(1),Integer(3)]
1/r
>>> Gam[Integer(3),Integer(2),Integer(3)]
cos(th)/sin(th)
>>> Gam[Integer(2),Integer(3),Integer(3)]
-cos(th)*sin(th)
```

Note that a better display of the Christoffel symbols is provided by the method *christoffel_symbols_display()*:

```
sage: g.christoffel_symbols_display()
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
```

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```
Gam^th_r,th = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
```

```
>>> from sage.all import *
>>> g.christoffel_symbols_display()
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
```

christoffel_symbols_display(*chart=None*, *symbol=None*, *latex_symbol=None*, *index_labels=None*, *index_latex_labels=None*, *coordinate_labels=True*, *only_nonzero=True*, *only_nonredundant=True*)

Display the Christoffel symbols w.r.t. to a given chart, one per line.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- `chart` – (default: `None`) chart with respect to which the Christoffel symbols are defined; if none is provided, the default chart of the metric's domain is assumed.
- `symbol` – (default: `None`) string specifying the symbol of the connection coefficients; if `None`, 'Gam' is used
- `latex_symbol` – (default: `None`) string specifying the LaTeX symbol for the components; if `None`, '\Gamma' is used
- `index_labels` – (default: `None`) list of strings representing the labels of each index; if `None`, coordinate symbols are used except if `coordinate_symbols` is set to `False`, in which case integer labels are used
- `index_latex_labels` – (default: `None`) list of strings representing the LaTeX labels of each index; if `None`, coordinate LaTeX symbols are used, except if `coordinate_symbols` is set to `False`, in which case integer labels are used
- `coordinate_labels` – boolean (default: `True`); if `True`, coordinate symbols are used by default (instead of integers)
- `only_nonzero` – boolean (default: `True`); if `True`, only nonzero connection coefficients are displayed
- `only_nonredundant` – boolean (default: `True`); if `True`, only nonredundant (w.r.t. the symmetry of the last two indices) connection coefficients are displayed

EXAMPLES:

Christoffel symbols of the flat metric on $\mathbf{R}^3$ with respect to spherical coordinates:

```
sage: M = Manifold(3, 'R3', r'\RR^3', start_index=1)
sage: U = M.open_subset('U') # the complement of the half-plane (y=0, x>=0)
sage: X.<r,th,ph> = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: g = U.metric('g')
sage: g[1,1], g[2,2], g[3,3] = 1, r^2, r^2*sin(th)^2
```

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```
sage: g.display()  # the standard flat metric expressed in spherical␣
↪coordinates
g = dr⊗dr + r^2 dth⊗dth + r^2*sin(th)^2 dph⊗dph
sage: g.christoffel_symbols_display()
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'R3', r'\RR^3', start_index=Integer(1))
>>> U = M.open_subset('U') # the complement of the half-plane (y=0, x>=0)
>>> X = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi', names=('r',
↪'th', 'ph',)); (r, th, ph,) = X._first_ngens(3)
>>> g = U.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)], g[Integer(3),
↪Integer(3)] = Integer(1), r**Integer(2), r**Integer(2)*sin(th)**Integer(2)
>>> g.display()  # the standard flat metric expressed in spherical coordinates
g = dr⊗dr + r^2 dth⊗dth + r^2*sin(th)^2 dph⊗dph
>>> g.christoffel_symbols_display()
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
```

To list all nonzero Christoffel symbols, including those that can be deduced by symmetry, use `only_nonredundant=False`:

```
sage: g.christoffel_symbols_display(only_nonredundant=False)
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_th,r = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
Gam^ph_ph,r = 1/r
Gam^ph_ph,th = cos(th)/sin(th)
```

```
>>> from sage.all import *
>>> g.christoffel_symbols_display(only_nonredundant=False)
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_th,r = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
```

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```
Gam^ph_ph,r = 1/r
Gam^ph_ph,th = cos(th)/sin(th)
```

Listing all Christoffel symbols (except those that can be deduced by symmetry), including the vanishing one:

```
sage: g.christoffel_symbols_display(only_nonzero=False)
Gam^r_r,r = 0
Gam^r_r,th = 0
Gam^r_r,ph = 0
Gam^r_th,th = -r
Gam^r_th,ph = 0
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,r = 0
Gam^th_r,th = 1/r
Gam^th_r,ph = 0
Gam^th_th,th = 0
Gam^th_th,ph = 0
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,r = 0
Gam^ph_r,th = 0
Gam^ph_r,ph = 1/r
Gam^ph_th,th = 0
Gam^ph_th,ph = cos(th)/sin(th)
Gam^ph_ph,ph = 0
```

```
>>> from sage.all import *
>>> g.christoffel_symbols_display(only_nonzero=False)
Gam^r_r,r = 0
Gam^r_r,th = 0
Gam^r_r,ph = 0
Gam^r_th,th = -r
Gam^r_th,ph = 0
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,r = 0
Gam^th_r,th = 1/r
Gam^th_r,ph = 0
Gam^th_th,th = 0
Gam^th_th,ph = 0
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,r = 0
Gam^ph_r,th = 0
Gam^ph_r,ph = 1/r
Gam^ph_th,th = 0
Gam^ph_th,ph = cos(th)/sin(th)
Gam^ph_ph,ph = 0
```

Using integer labels:

```
sage: g.christoffel_symbols_display(coordinate_labels=False)
Gam^1_22 = -r
Gam^1_33 = -r*sin(th)^2
Gam^2_12 = 1/r
```

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```
Gam^2_33 = -cos(th)*sin(th)
Gam^3_13 = 1/r
Gam^3_23 = cos(th)/sin(th)
```

```
>>> from sage.all import *
>>> g.christoffel_symbols_display(coordinate_labels=False)
Gam^1_22 = -r
Gam^1_33 = -r*sin(th)^2
Gam^2_12 = 1/r
Gam^2_33 = -cos(th)*sin(th)
Gam^3_13 = 1/r
Gam^3_23 = cos(th)/sin(th)
```

connection(*name=None*, *latex_name=None*, *init_coef=True*)

Return the unique torsion-free affine connection compatible with `self`.

This is the so-called Levi-Civita connection.

INPUT:

- `name` – (default: `None`) name given to the Levi-Civita connection; if `None`, it is formed from the metric name
- `latex_name` – (default: `None`) LaTeX symbol to denote the Levi-Civita connection; if `None`, it is set to `name`, or if the latter is None as well, it formed from the symbol ∇ and the metric symbol
- `init_coef` – boolean (default: `True`); determines whether the connection coefficients are initialized, as Christoffel symbols in the top charts of the domain of `self` (i.e. disregarding the subcharts)

OUTPUT:

- the Levi-Civita connection, as an instance of *LeviCivitaConnection*

EXAMPLES:

Levi-Civita connection associated with the Euclidean metric on $\mathbf{R}^3$:

```
sage: M = Manifold(3, 'R^3', start_index=1)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'R^3', start_index=Integer(1))
```

Let us use spherical coordinates on $\mathbf{R}^3$:

```
sage: U = M.open_subset('U') # the complement of the half-plane (y=0, x>=0)
sage: c_spher.<r,th,ph> = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\
↪phi')
sage: g = U.metric('g')
sage: g[1,1], g[2,2], g[3,3] = 1, r^2 , (r*sin(th))^2  # the Euclidean metric
sage: g.connection()
Levi-Civita connection nabla_g associated with the Riemannian
 metric g on the Open subset U of the 3-dimensional differentiable
 manifold R^3
sage: g.connection().display()  # Nonzero connection coefficients
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
```

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```
Gam^th_r,th = 1/r
Gam^th_th,r = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
Gam^ph_ph,r = 1/r
Gam^ph_ph,th = cos(th)/sin(th)
```

```
>>> from sage.all import *
>>> U = M.open_subset('U') # the complement of the half-plane (y=0, x>=0)
>>> c_spher = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi', names=(
↪'r', 'th', 'ph',)); (r, th, ph,) = c_spher._first_ngens(3)
>>> g = U.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)], g[Integer(3),
↪Integer(3)] = Integer(1), r**Integer(2) , (r*sin(th))**Integer(2)  # the␣
↪Euclidean metric
>>> g.connection()
Levi-Civita connection nabla_g associated with the Riemannian
 metric g on the Open subset U of the 3-dimensional differentiable
 manifold R^3
>>> g.connection().display()  # Nonzero connection coefficients
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_th,r = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
Gam^ph_ph,r = 1/r
Gam^ph_ph,th = cos(th)/sin(th)
```

Test of compatibility with the metric:

```
sage: Dg = g.connection()(g) ; Dg
Tensor field nabla_g(g) of type (0,3) on the Open subset U of the
 3-dimensional differentiable manifold R^3
sage: Dg == 0
True
sage: Dig = g.connection()(g.inverse()) ; Dig
Tensor field nabla_g(inv_g) of type (2,1) on the Open subset U of
 the 3-dimensional differentiable manifold R^3
sage: Dig == 0
True
```

```
>>> from sage.all import *
>>> Dg = g.connection()(g) ; Dg
Tensor field nabla_g(g) of type (0,3) on the Open subset U of the
 3-dimensional differentiable manifold R^3
>>> Dg == Integer(0)
True
>>> Dig = g.connection()(g.inverse()) ; Dig
Tensor field nabla_g(inv_g) of type (2,1) on the Open subset U of
```

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```
 the 3-dimensional differentiable manifold R^3
>>> Dig == Integer(0)
True
```

cotton(*name=None*, *latex_name=None*)

Return the Cotton conformal tensor associated with the metric. The tensor has type (0,3) and is defined in terms of the Schouten tensor S (see *schouten()*):

$$C_{ijk} = (n-2)\left(\nabla_k S_{ij} - \nabla_j S_{ik}\right)$$

INPUT:

- `name` – (default: `None`) name given to the Cotton conformal tensor; if `None`, it is set to "Cot(g)", where "g" is the metric's name
- `latex_name` – (default: `None`) LaTeX symbol to denote the Cotton conformal tensor; if `None`, it is set to "\mathrm{Cot}(g)", where "g" is the metric's name

OUTPUT:

- the Cotton conformal tensor Cot, as an instance of *TensorField*

EXAMPLES:

Checking that the Cotton tensor identically vanishes on a conformally flat 3-dimensional manifold, for instance the hyperbolic space H^3:

```
sage: M = Manifold(3, 'H^3', start_index=1)
sage: U = M.open_subset('U') # the complement of the half-plane (y=0, x>=0)
sage: X.<rh,th,ph> = U.chart(r'rh:(0,+oo):\rho th:(0,pi):\theta  ph:(0,2*pi):\
↪phi')
sage: g = U.metric('g')
sage: b = var('b')
sage: g[1,1], g[2,2], g[3,3] = b^2, (b*sinh(rh))^2, (b*sinh(rh)*sin(th))^2
sage: g.display()  # standard metric on H^3:
g = b^2 drh⊗drh + b^2*sinh(rh)^2 dth⊗dth
 + b^2*sin(th)^2*sinh(rh)^2 dph⊗dph
sage: Cot = g.cotton() ; Cot # long time
Tensor field Cot(g) of type (0,3) on the Open subset U of the
 3-dimensional differentiable manifold H^3
sage: Cot == 0 # long time
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'H^3', start_index=Integer(1))
>>> U = M.open_subset('U') # the complement of the half-plane (y=0, x>=0)
>>> X = U.chart(r'rh:(0,+oo):\rho th:(0,pi):\theta  ph:(0,2*pi):\phi', names=(
↪'rh', 'th', 'ph',)); (rh, th, ph,) = X._first_ngens(3)
>>> g = U.metric('g')
>>> b = var('b')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)], g[Integer(3),
↪Integer(3)] = b**Integer(2), (b*sinh(rh))**Integer(2),␣
↪(b*sinh(rh)*sin(th))**Integer(2)
>>> g.display()  # standard metric on H^3:
g = b^2 drh⊗drh + b^2*sinh(rh)^2 dth⊗dth
```

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```
 + b^2*sin(th)^2*sinh(rh)^2 dph⊗dph
>>> Cot = g.cotton() ; Cot # long time
Tensor field Cot(g) of type (0,3) on the Open subset U of the
 3-dimensional differentiable manifold H^3
>>> Cot == Integer(0) # long time
True
```

cotton_york(*name=None*, *latex_name=None*)

Return the Cotton-York conformal tensor associated with the metric. The tensor has type (0,2) and is only defined for manifolds of dimension 3. It is defined in terms of the Cotton tensor C (see `cotton()`) or the Schouten tensor S (see `schouten()`):

$$CY_{ij} = \frac{1}{2}\epsilon^{kl}{}_{i}C_{jlk} = \epsilon^{kl}{}_{i}\nabla_k S_{lj}$$

INPUT:

- `name` – (default: `None`) name given to the Cotton-York tensor; if `None`, it is set to "CY(g)", where "g" is the metric's name
- `latex_name` – (default: `None`) LaTeX symbol to denote the Cotton-York tensor; if `None`, it is set to "\mathrm{CY}(g)", where "g" is the metric's name

OUTPUT:

- the Cotton-York conformal tensor CY, as an instance of `TensorField`

EXAMPLES:

Compute the determinant of the Cotton-York tensor for the Heisenberg group with the left invariant metric:

```
sage: M = Manifold(3, 'Nil', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: g = M.riemannian_metric('g')
sage: g[1,1], g[2,2], g[2,3], g[3,3] = 1, 1+x^2, -x, 1
sage: g.display()
g = dx⊗dx + (x^2 + 1) dy⊗dy - x dy⊗dz - x dz⊗dy + dz⊗dz
sage: CY = g.cotton_york() ; CY # long time
Tensor field CY(g) of type (0,2) on the 3-dimensional
 differentiable manifold Nil
sage: CY.display()  # long time
CY(g) = 1/2 dx⊗dx + (-x^2 + 1/2) dy⊗dy + x dy⊗dz + x dz⊗dy - dz⊗dz
sage: det(CY[:]) # long time
-1/4
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'Nil', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> g = M.riemannian_metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)], g[Integer(2),
↪Integer(3)], g[Integer(3),Integer(3)] = Integer(1),␣
↪Integer(1)+x**Integer(2), -x, Integer(1)
>>> g.display()
g = dx⊗dx + (x^2 + 1) dy⊗dy - x dy⊗dz - x dz⊗dy + dz⊗dz
>>> CY = g.cotton_york() ; CY # long time
Tensor field CY(g) of type (0,2) on the 3-dimensional
```

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```
differentiable manifold Nil
>>> CY.display()  # long time
CY(g) = 1/2 dx⊗dx + (-x^2 + 1/2) dy⊗dy + x dy⊗dz + x dz⊗dy - dz⊗dz
>>> det(CY[:]) # long time
-1/4
```

det(*frame=None*)

Determinant of the metric components in the specified frame.

INPUT:

- `frame` – (default: `None`) vector frame with respect to which the components g_{ij} of the metric are defined; if `None`, the default frame of the metric's domain is used. If a chart is provided instead of a frame, the associated coordinate frame is used

OUTPUT:

- the determinant $\det(g_{ij})$, as an instance of `DiffScalarField`

EXAMPLES:

Metric determinant on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: X.<x,y> = M.chart()
sage: g = M.metric('g')
sage: g[1,1], g[1, 2], g[2, 2] = 1+x, x*y , 1-y
sage: g[:]
[ x + 1    x*y]
[   x*y -y + 1]
sage: s = g.determinant()  # determinant in M's default frame
sage: s.expr()
-x^2*y^2 - (x + 1)*y + x + 1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> g = M.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(1), Integer(2)], g[Integer(2),
↪Integer(2)] = Integer(1)+x, x*y , Integer(1)-y
>>> g[:]
[ x + 1    x*y]
[   x*y -y + 1]
>>> s = g.determinant()  # determinant in M's default frame
>>> s.expr()
-x^2*y^2 - (x + 1)*y + x + 1
```

A shortcut is `det()`:

```
sage: g.det() == g.determinant()
True
```

```
>>> from sage.all import *
>>> g.det() == g.determinant()
True
```

The notation `det(g)` can be used:

```
sage: det(g) == g.determinant()
True
```

```
>>> from sage.all import *
>>> det(g) == g.determinant()
True
```

Determinant in a frame different from the default's one:

```
sage: Y.<u,v> = M.chart()
sage: ch_X_Y = X.transition_map(Y, [x+y, x-y])
sage: ch_X_Y.inverse()
Change of coordinates from Chart (M, (u, v)) to Chart (M, (x, y))
sage: g.comp(Y.frame())[:, Y]
[ 1/8*u^2 - 1/8*v^2 + 1/4*v + 1/2                            1/4*u]
[                           1/4*u -1/8*u^2 + 1/8*v^2 + 1/4*v + 1/2]
sage: g.determinant(Y.frame()).expr()
-1/4*x^2*y^2 - 1/4*(x + 1)*y + 1/4*x + 1/4
sage: g.determinant(Y.frame()).expr(Y)
-1/64*u^4 - 1/64*v^4 + 1/32*(u^2 + 2)*v^2 - 1/16*u^2 + 1/4*v + 1/4
```

```
>>> from sage.all import *
>>> Y = M.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)
>>> ch_X_Y = X.transition_map(Y, [x+y, x-y])
>>> ch_X_Y.inverse()
Change of coordinates from Chart (M, (u, v)) to Chart (M, (x, y))
>>> g.comp(Y.frame())[:, Y]
[ 1/8*u^2 - 1/8*v^2 + 1/4*v + 1/2                            1/4*u]
[                           1/4*u -1/8*u^2 + 1/8*v^2 + 1/4*v + 1/2]
>>> g.determinant(Y.frame()).expr()
-1/4*x^2*y^2 - 1/4*(x + 1)*y + 1/4*x + 1/4
>>> g.determinant(Y.frame()).expr(Y)
-1/64*u^4 - 1/64*v^4 + 1/32*(u^2 + 2)*v^2 - 1/16*u^2 + 1/4*v + 1/4
```

A chart can be passed instead of a frame:

```
sage: g.determinant(X) is g.determinant(X.frame())
True
sage: g.determinant(Y) is g.determinant(Y.frame())
True
```

```
>>> from sage.all import *
>>> g.determinant(X) is g.determinant(X.frame())
True
>>> g.determinant(Y) is g.determinant(Y.frame())
True
```

The metric determinant depends on the frame:

```
sage: g.determinant(X.frame()) == g.determinant(Y.frame())
False
```

```
>>> from sage.all import *
>>> g.determinant(X.frame()) == g.determinant(Y.frame())
False
```

Using SymPy as symbolic engine:

```
sage: M.set_calculus_method('sympy')
sage: g = M.metric('g')
sage: g[1,1], g[1, 2], g[2, 2] = 1+x, x*y , 1-y
sage: s = g.determinant()  # determinant in M's default frame
sage: s.expr()
-x**2*y**2 + x - y*(x + 1) + 1
```

```
>>> from sage.all import *
>>> M.set_calculus_method('sympy')
>>> g = M.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(1), Integer(2)], g[Integer(2),␣
↪Integer(2)] = Integer(1)+x, x*y , Integer(1)-y
>>> s = g.determinant()  # determinant in M's default frame
>>> s.expr()
-x**2*y**2 + x - y*(x + 1) + 1
```

determinant(*frame=None*)

Determinant of the metric components in the specified frame.

INPUT:

- `frame` – (default: `None`) vector frame with respect to which the components g_{ij} of the metric are defined; if `None`, the default frame of the metric's domain is used. If a chart is provided instead of a frame, the associated coordinate frame is used

OUTPUT:

- the determinant $\det(g_{ij})$, as an instance of *DiffScalarField*

EXAMPLES:

Metric determinant on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: X.<x,y> = M.chart()
sage: g = M.metric('g')
sage: g[1,1], g[1, 2], g[2, 2] = 1+x, x*y , 1-y
sage: g[:]
[ x + 1    x*y]
[   x*y -y + 1]
sage: s = g.determinant()  # determinant in M's default frame
sage: s.expr()
-x^2*y^2 - (x + 1)*y + x + 1
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> g = M.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(1), Integer(2)], g[Integer(2),␣
```

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```
↪Integer(2)] = Integer(1)+x, x*y , Integer(1)-y
>>> g[:]
[ x + 1     x*y]
[   x*y -y + 1]
>>> s = g.determinant()  # determinant in M's default frame
>>> s.expr()
-x^2*y^2 - (x + 1)*y + x + 1
```

A shortcut is `det()`:

```
sage: g.det() == g.determinant()
True
```

```
>>> from sage.all import *
>>> g.det() == g.determinant()
True
```

The notation `det(g)` can be used:

```
sage: det(g) == g.determinant()
True
```

```
>>> from sage.all import *
>>> det(g) == g.determinant()
True
```

Determinant in a frame different from the default's one:

```
sage: Y.<u,v> = M.chart()
sage: ch_X_Y = X.transition_map(Y, [x+y, x-y])
sage: ch_X_Y.inverse()
Change of coordinates from Chart (M, (u, v)) to Chart (M, (x, y))
sage: g.comp(Y.frame())[:, Y]
[ 1/8*u^2 - 1/8*v^2 + 1/4*v + 1/2                            1/4*u]
[                           1/4*u -1/8*u^2 + 1/8*v^2 + 1/4*v + 1/2]
sage: g.determinant(Y.frame()).expr()
-1/4*x^2*y^2 - 1/4*(x + 1)*y + 1/4*x + 1/4
sage: g.determinant(Y.frame()).expr(Y)
-1/64*u^4 - 1/64*v^4 + 1/32*(u^2 + 2)*v^2 - 1/16*u^2 + 1/4*v + 1/4
```

```
>>> from sage.all import *
>>> Y = M.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)
>>> ch_X_Y = X.transition_map(Y, [x+y, x-y])
>>> ch_X_Y.inverse()
Change of coordinates from Chart (M, (u, v)) to Chart (M, (x, y))
>>> g.comp(Y.frame())[:, Y]
[ 1/8*u^2 - 1/8*v^2 + 1/4*v + 1/2                            1/4*u]
[                           1/4*u -1/8*u^2 + 1/8*v^2 + 1/4*v + 1/2]
>>> g.determinant(Y.frame()).expr()
-1/4*x^2*y^2 - 1/4*(x + 1)*y + 1/4*x + 1/4
>>> g.determinant(Y.frame()).expr(Y)
-1/64*u^4 - 1/64*v^4 + 1/32*(u^2 + 2)*v^2 - 1/16*u^2 + 1/4*v + 1/4
```

A chart can be passed instead of a frame:

```
sage: g.determinant(X) is g.determinant(X.frame())
True
sage: g.determinant(Y) is g.determinant(Y.frame())
True
```

```
>>> from sage.all import *
>>> g.determinant(X) is g.determinant(X.frame())
True
>>> g.determinant(Y) is g.determinant(Y.frame())
True
```

The metric determinant depends on the frame:

```
sage: g.determinant(X.frame()) == g.determinant(Y.frame())
False
```

```
>>> from sage.all import *
>>> g.determinant(X.frame()) == g.determinant(Y.frame())
False
```

Using SymPy as symbolic engine:

```
sage: M.set_calculus_method('sympy')
sage: g = M.metric('g')
sage: g[1,1], g[1, 2], g[2, 2] = 1+x, x*y , 1-y
sage: s = g.determinant()  # determinant in M's default frame
sage: s.expr()
-x**2*y**2 + x - y*(x + 1) + 1
```

```
>>> from sage.all import *
>>> M.set_calculus_method('sympy')
>>> g = M.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(1), Integer(2)], g[Integer(2),
↪Integer(2)] = Integer(1)+x, x*y , Integer(1)-y
>>> s = g.determinant()  # determinant in M's default frame
>>> s.expr()
-x**2*y**2 + x - y*(x + 1) + 1
```

hodge_star(*pform*)

Compute the Hodge dual of a differential form with respect to the metric.

If the differential form is a p-form A, its *Hodge dual* with respect to the metric g is the $(n-p)$-form $*A$ defined by

$$*A_{i_1\ldots i_{n-p}} = \frac{1}{p!} A_{k_1\ldots k_p} \epsilon^{k_1\ldots k_p}{}_{i_1\ldots i_{n-p}}$$

where n is the manifold's dimension, ϵ is the volume n-form associated with g (see `volume_form()`) and the indices $k_1,\ldots,k_p$ are raised with g.

Notice that the hodge star dual requires an orientable manifold with a preferred orientation, see `orientation()` for details.

INPUT:

- `pform` – a p-form A; must be an instance of `DiffScalarField` for $p = 0$ and of `DiffForm` or `DiffFormParal` for $p \geq 1$.

OUTPUT:

- the $(n - p)$-form $*A$

EXAMPLES:

Hodge dual of a 1-form in the Euclidean space R^3:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: g = M.metric('g')
sage: g[1,1], g[2,2], g[3,3] = 1, 1, 1
sage: var('Ax Ay Az')
(Ax, Ay, Az)
sage: a = M.one_form(Ax, Ay, Az, name='A')
sage: sa = g.hodge_star(a) ; sa
2-form *A on the 3-dimensional differentiable manifold M
sage: sa.display()
*A = Az dx∧dy - Ay dx∧dz + Ax dy∧dz
sage: ssa = g.hodge_star(sa) ; ssa
1-form **A on the 3-dimensional differentiable manifold M
sage: ssa.display()
**A = Ax dx + Ay dy + Az dz
sage: ssa == a  # must hold for a Riemannian metric in dimension 3
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> g = M.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)], g[Integer(3),
↪Integer(3)] = Integer(1), Integer(1), Integer(1)
>>> var('Ax Ay Az')
(Ax, Ay, Az)
>>> a = M.one_form(Ax, Ay, Az, name='A')
>>> sa = g.hodge_star(a) ; sa
2-form *A on the 3-dimensional differentiable manifold M
>>> sa.display()
*A = Az dx∧dy - Ay dx∧dz + Ax dy∧dz
>>> ssa = g.hodge_star(sa) ; ssa
1-form **A on the 3-dimensional differentiable manifold M
>>> ssa.display()
**A = Ax dx + Ay dy + Az dz
>>> ssa == a  # must hold for a Riemannian metric in dimension 3
True
```

Hodge dual of a 0-form (scalar field) in R^3:

```
sage: f = M.scalar_field(function('F')(x,y,z), name='f')
sage: sf = g.hodge_star(f) ; sf
3-form *f on the 3-dimensional differentiable manifold M
sage: sf.display()
```

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```
*f = F(x, y, z) dx∧dy∧dz
sage: ssf = g.hodge_star(sf) ; ssf
Scalar field **f on the 3-dimensional differentiable manifold M
sage: ssf.display()
**f: M → ℝ
   (x, y, z) ↦ F(x, y, z)
sage: ssf == f # must hold for a Riemannian metric
True
```

```
>>> from sage.all import *
>>> f = M.scalar_field(function('F')(x,y,z), name='f')
>>> sf = g.hodge_star(f) ; sf
3-form *f on the 3-dimensional differentiable manifold M
>>> sf.display()
*f = F(x, y, z) dx∧dy∧dz
>>> ssf = g.hodge_star(sf) ; ssf
Scalar field **f on the 3-dimensional differentiable manifold M
>>> ssf.display()
**f: M → ℝ
   (x, y, z) ↦ F(x, y, z)
>>> ssf == f # must hold for a Riemannian metric
True
```

Hodge dual of a 0-form in Minkowski spacetime:

```
sage: M = Manifold(4, 'M')
sage: X.<t,x,y,z> = M.chart()
sage: g = M.lorentzian_metric('g')
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1, 1, 1, 1
sage: g.display()  # Minkowski metric
g = -dt⊗dt + dx⊗dx + dy⊗dy + dz⊗dz
sage: var('f0')
f0
sage: f = M.scalar_field(f0, name='f')
sage: sf = g.hodge_star(f) ; sf
4-form *f on the 4-dimensional differentiable manifold M
sage: sf.display()
*f = f0 dt∧dx∧dy∧dz
sage: ssf = g.hodge_star(sf) ; ssf
Scalar field **f on the 4-dimensional differentiable manifold M
sage: ssf.display()
**f: M → ℝ
   (t, x, y, z) ↦ -f0
sage: ssf == -f  # must hold for a Lorentzian metric
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> g = M.lorentzian_metric('g')
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
```

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```
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1), Integer(1), Integer(1),
↪ Integer(1)
>>> g.display()  # Minkowski metric
g = -dt⊗dt + dx⊗dx + dy⊗dy + dz⊗dz
>>> var('f0')
f0
>>> f = M.scalar_field(f0, name='f')
>>> sf = g.hodge_star(f) ; sf
4-form *f on the 4-dimensional differentiable manifold M
>>> sf.display()
*f = f0 dt∧dx∧dy∧dz
>>> ssf = g.hodge_star(sf) ; ssf
Scalar field **f on the 4-dimensional differentiable manifold M
>>> ssf.display()
**f: M → ℝ
   (t, x, y, z) ↦ -f0
>>> ssf == -f  # must hold for a Lorentzian metric
True
```

Hodge dual of a 1-form in Minkowski spacetime:

```
sage: var('At Ax Ay Az')
(At, Ax, Ay, Az)
sage: a = M.one_form(At, Ax, Ay, Az, name='A')
sage: a.display()
A = At dt + Ax dx + Ay dy + Az dz
sage: sa = g.hodge_star(a) ; sa
3-form *A on the 4-dimensional differentiable manifold M
sage: sa.display()
*A = -Az dt∧dx∧dy + Ay dt∧dx∧dz - Ax dt∧dy∧dz - At dx∧dy∧dz
sage: ssa = g.hodge_star(sa) ; ssa
1-form **A on the 4-dimensional differentiable manifold M
sage: ssa.display()
**A = At dt + Ax dx + Ay dy + Az dz
sage: ssa == a  # must hold for a Lorentzian metric in dimension 4
True
```

```
>>> from sage.all import *
>>> var('At Ax Ay Az')
(At, Ax, Ay, Az)
>>> a = M.one_form(At, Ax, Ay, Az, name='A')
>>> a.display()
A = At dt + Ax dx + Ay dy + Az dz
>>> sa = g.hodge_star(a) ; sa
3-form *A on the 4-dimensional differentiable manifold M
>>> sa.display()
*A = -Az dt∧dx∧dy + Ay dt∧dx∧dz - Ax dt∧dy∧dz - At dx∧dy∧dz
>>> ssa = g.hodge_star(sa) ; ssa
1-form **A on the 4-dimensional differentiable manifold M
>>> ssa.display()
**A = At dt + Ax dx + Ay dy + Az dz
>>> ssa == a  # must hold for a Lorentzian metric in dimension 4
```

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```
True
```

Hodge dual of a 2-form in Minkowski spacetime:

```
sage: F = M.diff_form(2, name='F')
sage: var('Ex Ey Ez Bx By Bz')
(Ex, Ey, Ez, Bx, By, Bz)
sage: F[0,1], F[0,2], F[0,3] = -Ex, -Ey, -Ez
sage: F[1,2], F[1,3], F[2,3] = Bz, -By, Bx
sage: F[:]
[  0 -Ex -Ey -Ez]
[ Ex   0  Bz -By]
[ Ey -Bz   0  Bx]
[ Ez  By -Bx   0]
sage: sF = g.hodge_star(F) ; sF
2-form *F on the 4-dimensional differentiable manifold M
sage: sF[:]
[  0  Bx  By  Bz]
[-Bx   0  Ez -Ey]
[-By -Ez   0  Ex]
[-Bz  Ey -Ex   0]
sage: ssF = g.hodge_star(sF) ; ssF
2-form **F on the 4-dimensional differentiable manifold M
sage: ssF[:]
[  0  Ex  Ey  Ez]
[-Ex   0 -Bz  By]
[-Ey  Bz   0 -Bx]
[-Ez -By  Bx   0]
sage: ssF.display()
**F = Ex dt∧dx + Ey dt∧dy + Ez dt∧dz - Bz dx∧dy + By dx∧dz
 - Bx dy∧dz
sage: F.display()
F = -Ex dt∧dx - Ey dt∧dy - Ez dt∧dz + Bz dx∧dy - By dx∧dz
 + Bx dy∧dz
sage: ssF == -F  # must hold for a Lorentzian metric in dimension 4
True
```

```
>>> from sage.all import *
>>> F = M.diff_form(Integer(2), name='F')
>>> var('Ex Ey Ez Bx By Bz')
(Ex, Ey, Ez, Bx, By, Bz)
>>> F[Integer(0),Integer(1)], F[Integer(0),Integer(2)], F[Integer(0),
↪Integer(3)] = -Ex, -Ey, -Ez
>>> F[Integer(1),Integer(2)], F[Integer(1),Integer(3)], F[Integer(2),
↪Integer(3)] = Bz, -By, Bx
>>> F[:]
[  0 -Ex -Ey -Ez]
[ Ex   0  Bz -By]
[ Ey -Bz   0  Bx]
[ Ez  By -Bx   0]
>>> sF = g.hodge_star(F) ; sF
2-form *F on the 4-dimensional differentiable manifold M
```

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```
>>> sF[:]
[  0  Bx  By  Bz]
[-Bx   0  Ez -Ey]
[-By -Ez   0  Ex]
[-Bz  Ey -Ex   0]
>>> ssF = g.hodge_star(sF) ; ssF
2-form **F on the 4-dimensional differentiable manifold M
>>> ssF[:]
[  0  Ex  Ey  Ez]
[-Ex   0 -Bz  By]
[-Ey  Bz   0 -Bx]
[-Ez -By  Bx   0]
>>> ssF.display()
**F = Ex dt∧dx + Ey dt∧dy + Ez dt∧dz - Bz dx∧dy + By dx∧dz
 - Bx dy∧dz
>>> F.display()
F = -Ex dt∧dx - Ey dt∧dy - Ez dt∧dz + Bz dx∧dy - By dx∧dz
 + Bx dy∧dz
>>> ssF == -F  # must hold for a Lorentzian metric in dimension 4
True
```

Test of the standard identity

$$*(A \wedge B) = \epsilon(A^\sharp, B^\sharp, ., .)$$

where A and B are any 1-forms and $A^\sharp$ and $B^\sharp$ the vectors associated to them by the metric g (index raising):

```
sage: var('Bt Bx By Bz')
(Bt, Bx, By, Bz)
sage: b = M.one_form(Bt, Bx, By, Bz, name='B')
sage: b.display()
B = Bt dt + Bx dx + By dy + Bz dz
sage: epsilon = g.volume_form()
sage: g.hodge_star(a.wedge(b)) == epsilon.contract(0,a.up(g)).contract(0,b.
↪up(g))
True
```

```
>>> from sage.all import *
>>> var('Bt Bx By Bz')
(Bt, Bx, By, Bz)
>>> b = M.one_form(Bt, Bx, By, Bz, name='B')
>>> b.display()
B = Bt dt + Bx dx + By dy + Bz dz
>>> epsilon = g.volume_form()
>>> g.hodge_star(a.wedge(b)) == epsilon.contract(Integer(0),a.up(g)).
↪contract(Integer(0),b.up(g))
True
```

inverse(*expansion_symbol=None*, *order=1*)

Return the inverse metric.

INPUT:

- `expansion_symbol` – (default: `None`) symbolic variable; if specified, the inverse will be expanded in power series with respect to this variable (around its zero value)
- `order` – integer (default: 1); the order of the expansion if `expansion_symbol` is not `None`; the *order* is defined as the degree of the polynomial representing the truncated power series in `expansion_symbol`; currently only first order inverse is supported

If `expansion_symbol` is set, then the zeroth order metric must be invertible. Moreover, subsequent calls to this method will return a cached value, even when called with the default value (to enable computation of derived quantities). To reset, use `_del_derived()`.

OUTPUT:

- instance of *TensorField* with `tensor_type` = (2,0) representing the inverse metric

EXAMPLES:

Inverse of the standard metric on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)  # S^2 is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart() # stereographic coord.
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)  # the complement of the two poles
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: g = M.metric('g')
sage: g[eU,1,1], g[eU,2,2] = 4/(1+x^2+y^2)^2, 4/(1+x^2+y^2)^2
sage: g.add_comp_by_continuation(eV, W, c_uv)
sage: ginv = g.inverse(); ginv
Tensor field inv_g of type (2,0) on the 2-dimensional differentiable manifold␣
↪S^2
sage: ginv.display(eU)
inv_g = (1/4*x^4 + 1/4*y^4 + 1/2*(x^2 + 1)*y^2 + 1/2*x^2 + 1/4) ∂/∂x⊗∂/∂x
 + (1/4*x^4 + 1/4*y^4 + 1/2*(x^2 + 1)*y^2 + 1/2*x^2 + 1/4) ∂/∂y⊗∂/∂y
sage: ginv.display(eV)
inv_g = (1/4*u^4 + 1/4*v^4 + 1/2*(u^2 + 1)*v^2 + 1/2*u^2 + 1/4) ∂/∂u⊗∂/∂u
 + (1/4*u^4 + 1/4*v^4 + 1/2*(u^2 + 1)*v^2 + 1/2*u^2 + 1/4) ∂/∂v⊗∂/∂v
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', start_index=Integer(1))
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)  # S^2 is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)# stereographic␣
↪coord.
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                 intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                 restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V)  # the complement of the two poles
```

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```
>>> eU = c_xy.frame() ; eV = c_uv.frame()
>>> g = M.metric('g')
>>> g[eU,Integer(1),Integer(1)], g[eU,Integer(2),Integer(2)] = Integer(4)/
↪(Integer(1)+x**Integer(2)+y**Integer(2))**Integer(2), Integer(4)/
↪(Integer(1)+x**Integer(2)+y**Integer(2))**Integer(2)
>>> g.add_comp_by_continuation(eV, W, c_uv)
>>> ginv = g.inverse(); ginv
Tensor field inv_g of type (2,0) on the 2-dimensional differentiable manifold␣
↪S^2
>>> ginv.display(eU)
inv_g = (1/4*x^4 + 1/4*y^4 + 1/2*(x^2 + 1)*y^2 + 1/2*x^2 + 1/4) ∂/∂x⊗∂/∂x
 + (1/4*x^4 + 1/4*y^4 + 1/2*(x^2 + 1)*y^2 + 1/2*x^2 + 1/4) ∂/∂y⊗∂/∂y
>>> ginv.display(eV)
inv_g = (1/4*u^4 + 1/4*v^4 + 1/2*(u^2 + 1)*v^2 + 1/2*u^2 + 1/4) ∂/∂u⊗∂/∂u
 + (1/4*u^4 + 1/4*v^4 + 1/2*(u^2 + 1)*v^2 + 1/2*u^2 + 1/4) ∂/∂v⊗∂/∂v
```

Let us check that `ginv` is indeed the inverse of `g`:

```
sage: s = g.contract(ginv); s  # contraction of last index of g with first␣
↪index of ginv
Tensor field of type (1,1) on the 2-dimensional differentiable manifold S^2
sage: s == M.tangent_identity_field()
True
```

```
>>> from sage.all import *
>>> s = g.contract(ginv); s  # contraction of last index of g with first␣
↪index of ginv
Tensor field of type (1,1) on the 2-dimensional differentiable manifold S^2
>>> s == M.tangent_identity_field()
True
```

restrict(*subdomain*, *dest_map=None*)

Return the restriction of the metric to some subdomain.

If the restriction has not been defined yet, it is constructed here.

INPUT:

- `subdomain` – open subset U of the metric's domain (must be an instance of *DifferentiableManifold*)
- `dest_map` – (default: `None`) destination map $\Phi : U \to V$, where V is a subdomain of `self._codomain` (type: *DiffMap*) If `None`, the restriction of `self._vmodule._dest_map` to U is used.

OUTPUT:

- instance of *PseudoRiemannianMetric* representing the restriction.

EXAMPLES:

```
sage: M = Manifold(5, 'M')
sage: g = M.metric('g', signature=3)
sage: U = M.open_subset('U')
sage: g.restrict(U)
Lorentzian metric g on the Open subset U of the
```

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```
 5-dimensional differentiable manifold M
sage: g.restrict(U).signature()
3
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(5), 'M')
>>> g = M.metric('g', signature=Integer(3))
>>> U = M.open_subset('U')
>>> g.restrict(U)
Lorentzian metric g on the Open subset U of the
 5-dimensional differentiable manifold M
>>> g.restrict(U).signature()
3
```

See the top documentation of `PseudoRiemannianMetric` for more examples.

ricci(*name=None*, *latex_name=None*)

Return the Ricci tensor associated with the metric.

This method is actually a shortcut for `self.connection().ricci()`

The Ricci tensor is the tensor field Ric of type (0,2) defined from the Riemann curvature tensor R by

$$Ric(u, v) = R(e^i, u, e_i, v)$$

for any vector fields u and v, (e_i) being any vector frame and (e^i) the dual coframe.

INPUT:

- `name` – (default: `None`) name given to the Ricci tensor; if none, it is set to "Ric(g)", where "g" is the metric's name
- `latex_name` – (default: `None`) LaTeX symbol to denote the Ricci tensor; if none, it is set to "\mathrm{Ric}(g)", where "g" is the metric's name

OUTPUT:

- the Ricci tensor Ric, as an instance of `TensorField` of tensor type (0,2) and symmetric

EXAMPLES:

Ricci tensor of the standard metric on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: U = M.open_subset('U') # the complement of a meridian (domain of␣
↪spherical coordinates)
sage: c_spher.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: a = var('a') # the sphere radius
sage: g = U.metric('g')
sage: g[1,1], g[2,2] = a^2, a^2*sin(th)^2
sage: g.display() # standard metric on the 2-sphere of radius a:
g = a^2 dth⊗dth + a^2*sin(th)^2 dph⊗dph
sage: g.ricci()
Field of symmetric bilinear forms Ric(g) on the Open subset U of
 the 2-dimensional differentiable manifold S^2
sage: g.ricci()[:]
[        1         0]
```

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```
[          0 sin(th)^2]
sage: g.ricci() == a^(-2) * g
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', start_index=Integer(1))
>>> U = M.open_subset('U') # the complement of a meridian (domain of␣
↪spherical coordinates)
>>> c_spher = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi', names=('th', 'ph',
↪)); (th, ph,) = c_spher._first_ngens(2)
>>> a = var('a') # the sphere radius
>>> g = U.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)] = a**Integer(2),␣
↪a**Integer(2)*sin(th)**Integer(2)
>>> g.display() # standard metric on the 2-sphere of radius a:
g = a^2 dth⊗dth + a^2*sin(th)^2 dph⊗dph
>>> g.ricci()
Field of symmetric bilinear forms Ric(g) on the Open subset U of
 the 2-dimensional differentiable manifold S^2
>>> g.ricci()[:]
[          1          0]
[          0 sin(th)^2]
>>> g.ricci() == a**(-Integer(2)) * g
True
```

ricci_scalar(*name=None*, *latex_name=None*)

Return the Ricci scalar associated with the metric.

The Ricci scalar is the scalar field r defined from the Ricci tensor Ric and the metric tensor g by

$$r = g^{ij} Ric_{ij}$$

INPUT:

- `name` – (default: `None`) name given to the Ricci scalar; if none, it is set to "r(g)", where "g" is the metric's name
- `latex_name` – (default: `None`) LaTeX symbol to denote the Ricci scalar; if none, it is set to "\mathrm{r}(g)", where "g" is the metric's name

OUTPUT:

- the Ricci scalar r, as an instance of `DiffScalarField`

EXAMPLES:

Ricci scalar of the standard metric on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: U = M.open_subset('U') # the complement of a meridian (domain of␣
↪spherical coordinates)
sage: c_spher.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: a = var('a') # the sphere radius
sage: g = U.metric('g')
sage: g[1,1], g[2,2] = a^2, a^2*sin(th)^2
```

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```
sage: g.display() # standard metric on the 2-sphere of radius a:
g = a^2 dth⊗dth + a^2*sin(th)^2 dph⊗dph
sage: g.ricci_scalar()
Scalar field r(g) on the Open subset U of the 2-dimensional
 differentiable manifold S^2
sage: g.ricci_scalar().display() # The Ricci scalar is constant:
r(g): U → ℝ
   (th, ph) ↦ 2/a^2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', start_index=Integer(1))
>>> U = M.open_subset('U') # the complement of a meridian (domain of␣
↪spherical coordinates)
>>> c_spher = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi', names=('th', 'ph',
↪)); (th, ph,) = c_spher._first_ngens(2)
>>> a = var('a') # the sphere radius
>>> g = U.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)] = a**Integer(2),␣
↪a**Integer(2)*sin(th)**Integer(2)
>>> g.display() # standard metric on the 2-sphere of radius a:
g = a^2 dth⊗dth + a^2*sin(th)^2 dph⊗dph
>>> g.ricci_scalar()
Scalar field r(g) on the Open subset U of the 2-dimensional
 differentiable manifold S^2
>>> g.ricci_scalar().display() # The Ricci scalar is constant:
r(g): U → ℝ
   (th, ph) ↦ 2/a^2
```

riemann(*name=None*, *latex_name=None*)

Return the Riemann curvature tensor associated with the metric.

This method is actually a shortcut for `self.connection().riemann()`

The Riemann curvature tensor is the tensor field R of type (1,3) defined by

$$R(\omega, u, v, w) = \left\langle \omega, \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w \right\rangle$$

for any 1-form ω and any vector fields u, v and w.

INPUT:

- `name` – (default: `None`) name given to the Riemann tensor; if none, it is set to "Riem(g)", where "g" is the metric's name
- `latex_name` – (default: `None`) LaTeX symbol to denote the Riemann tensor; if none, it is set to "\mathrm{Riem}(g)", where "g" is the metric's name

OUTPUT:

- the Riemann curvature tensor R, as an instance of *TensorField*

EXAMPLES:

Riemann tensor of the standard metric on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: U = M.open_subset('U') # the complement of a meridian (domain of␣
↪spherical coordinates)
sage: c_spher.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: a = var('a') # the sphere radius
sage: g = U.metric('g')
sage: g[1,1], g[2,2] = a^2, a^2*sin(th)^2
sage: g.display() # standard metric on the 2-sphere of radius a:
g = a^2 dth⊗dth + a^2*sin(th)^2 dph⊗dph
sage: g.riemann()
Tensor field Riem(g) of type (1,3) on the Open subset U of the
 2-dimensional differentiable manifold S^2
sage: g.riemann()[:]
[[[[0, 0], [0, 0]], [[0, sin(th)^2], [-sin(th)^2, 0]]],
 [[[0, -1], [1, 0]], [[0, 0], [0, 0]]]]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', start_index=Integer(1))
>>> U = M.open_subset('U') # the complement of a meridian (domain of␣
↪spherical coordinates)
>>> c_spher = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi', names=('th', 'ph',
↪)); (th, ph,) = c_spher._first_ngens(2)
>>> a = var('a') # the sphere radius
>>> g = U.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)] = a**Integer(2),␣
↪a**Integer(2)*sin(th)**Integer(2)
>>> g.display() # standard metric on the 2-sphere of radius a:
g = a^2 dth⊗dth + a^2*sin(th)^2 dph⊗dph
>>> g.riemann()
Tensor field Riem(g) of type (1,3) on the Open subset U of the
 2-dimensional differentiable manifold S^2
>>> g.riemann()[:]
[[[[0, 0], [0, 0]], [[0, sin(th)^2], [-sin(th)^2, 0]]],
 [[[0, -1], [1, 0]], [[0, 0], [0, 0]]]]
```

In dimension 2, the Riemann tensor can be expressed entirely in terms of the Ricci scalar r:

$$R^i_{\ jlk} = \frac{r}{2} \left(\delta^i_{\ k} g_{jl} - \delta^i_{\ l} g_{jk} \right)$$

This formula can be checked here, with the r.h.s. rewritten as $-r g_{j[k} \delta^i_{\ l]}$:

```
sage: g.riemann() == \
....:  -g.ricci_scalar()*(g*U.tangent_identity_field()).antisymmetrize(2,3)
True
```

```
>>> from sage.all import *
>>> g.riemann() ==  -g.ricci_scalar()*(g*U.tangent_identity_field()).
↪antisymmetrize(Integer(2),Integer(3))
True
```

Using SymPy as symbolic engine:

```
sage: M.set_calculus_method('sympy')
sage: g = U.metric('g')
sage: g[1,1], g[2,2] = a**2, a**2*sin(th)**2
sage: g.riemann()[:]
[[[[0, 0], [0, 0]],
  [[0, sin(2*th)/(2*tan(th)) - cos(2*th)],
   [-sin(2*th)/(2*tan(th)) + cos(2*th), 0]]],
 [[[0, -1], [1, 0]], [[0, 0], [0, 0]]]]
```

```
>>> from sage.all import *
>>> M.set_calculus_method('sympy')
>>> g = U.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)] = a**Integer(2),␣
↪a**Integer(2)*sin(th)**Integer(2)
>>> g.riemann()[:]
[[[[0, 0], [0, 0]],
  [[0, sin(2*th)/(2*tan(th)) - cos(2*th)],
   [-sin(2*th)/(2*tan(th)) + cos(2*th), 0]]],
 [[[0, -1], [1, 0]], [[0, 0], [0, 0]]]]
```

schouten(*name=None*, *latex_name=None*)

Return the Schouten tensor associated with the metric.

The Schouten tensor is the tensor field Sc of type (0,2) defined from the Ricci curvature tensor Ric (see `ricci()`) and the scalar curvature r (see `ricci_scalar()`) and the metric g by

$$Sc(u,v) = \frac{1}{n-2}\left(Ric(u,v) + \frac{r}{2(n-1)}g(u,v)\right)$$

for any vector fields u and v.

INPUT:

- `name` – (default: `None`) name given to the Schouten tensor; if none, it is set to "Schouten(g)", where "g" is the metric's name
- `latex_name` – (default: `None`) LaTeX symbol to denote the Schouten tensor; if none, it is set to "\mathrm{Schouten}(g)", where "g" is the metric's name

OUTPUT:

- the Schouten tensor Sc, as an instance of `TensorField` of tensor type (0,2) and symmetric

EXAMPLES:

Schouten tensor of the left invariant metric of Heisenberg's Nil group:

```
sage: M = Manifold(3, 'Nil', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: g = M.riemannian_metric('g')
sage: g[1,1], g[2,2], g[2,3], g[3,3] = 1, 1+x^2, -x, 1
sage: g.display()
g = dx⊗dx + (x^2 + 1) dy⊗dy - x dy⊗dz - x dz⊗dy + dz⊗dz
sage: g.schouten()
Field of symmetric bilinear forms Schouten(g) on the 3-dimensional
 differentiable manifold Nil
sage: g.schouten().display()
```

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```
Schouten(g) = -3/8 dx⊗dx + (5/8*x^2 - 3/8) dy⊗dy - 5/8*x dy⊗dz
 - 5/8*x dz⊗dy + 5/8 dz⊗dz
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'Nil', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> g = M.riemannian_metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)], g[Integer(2),
↪Integer(3)], g[Integer(3),Integer(3)] = Integer(1),␣
↪Integer(1)+x**Integer(2), -x, Integer(1)
>>> g.display()
g = dx⊗dx + (x^2 + 1) dy⊗dy - x dy⊗dz - x dz⊗dy + dz⊗dz
>>> g.schouten()
Field of symmetric bilinear forms Schouten(g) on the 3-dimensional
 differentiable manifold Nil
>>> g.schouten().display()
Schouten(g) = -3/8 dx⊗dx + (5/8*x^2 - 3/8) dy⊗dy - 5/8*x dy⊗dz
 - 5/8*x dz⊗dy + 5/8 dz⊗dz
```

set (*symbiform*)

Define the metric from a field of symmetric bilinear forms.

INPUT:

- `symbiform` – instance of *TensorField* representing a field of symmetric bilinear forms

EXAMPLES:

Metric defined from a field of symmetric bilinear forms on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
....:                               restrictions1= x>0, restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: h = M.sym_bilin_form_field(name='h')
sage: h[eU,0,0], h[eU,0,1], h[eU,1,1] = 1+x, x*y, 1-y
sage: h.add_comp_by_continuation(eV, W, c_uv)
sage: h.display(eU)
h = (x + 1) dx⊗dx + x*y dx⊗dy + x*y dy⊗dx + (-y + 1) dy⊗dy
sage: h.display(eV)
h = (1/8*u^2 - 1/8*v^2 + 1/4*v + 1/2) du⊗du + 1/4*u du⊗dv
 + 1/4*u dv⊗du + (-1/8*u^2 + 1/8*v^2 + 1/4*v + 1/2) dv⊗dv
sage: g = M.metric('g')
sage: g.set(h)
sage: g.display(eU)
g = (x + 1) dx⊗dx + x*y dx⊗dy + x*y dy⊗dx + (-y + 1) dy⊗dy
sage: g.display(eV)
g = (1/8*u^2 - 1/8*v^2 + 1/4*v + 1/2) du⊗du + 1/4*u du⊗dv
 + 1/4*u dv⊗du + (-1/8*u^2 + 1/8*v^2 + 1/4*v + 1/2) dv⊗dv
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> U = M.open_subset('U') ; V = M.open_subset('V')
>>> M.declare_union(U,V)   # M is the union of U and V
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2); c_uv =␣
↪V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)
>>> xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
...                                restrictions1= x>Integer(0), restrictions2=␣
↪u+v>Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
>>> W = U.intersection(V)
>>> eU = c_xy.frame() ; eV = c_uv.frame()
>>> h = M.sym_bilin_form_field(name='h')
>>> h[eU,Integer(0),Integer(0)], h[eU,Integer(0),Integer(1)], h[eU,Integer(1),
↪Integer(1)] = Integer(1)+x, x*y, Integer(1)-y
>>> h.add_comp_by_continuation(eV, W, c_uv)
>>> h.display(eU)
h = (x + 1) dx⊗dx + x*y dx⊗dy + x*y dy⊗dx + (-y + 1) dy⊗dy
>>> h.display(eV)
h = (1/8*u^2 - 1/8*v^2 + 1/4*v + 1/2) du⊗du + 1/4*u du⊗dv
 + 1/4*u dv⊗du + (-1/8*u^2 + 1/8*v^2 + 1/4*v + 1/2) dv⊗dv
>>> g = M.metric('g')
>>> g.set(h)
>>> g.display(eU)
g = (x + 1) dx⊗dx + x*y dx⊗dy + x*y dy⊗dx + (-y + 1) dy⊗dy
>>> g.display(eV)
g = (1/8*u^2 - 1/8*v^2 + 1/4*v + 1/2) du⊗du + 1/4*u du⊗dv
 + 1/4*u dv⊗du + (-1/8*u^2 + 1/8*v^2 + 1/4*v + 1/2) dv⊗dv
```

signature()

Signature of the metric.

OUTPUT:

- signature S of the metric, defined as the integer $S = n_+ - n_-$, where n_+ (resp. n_-) is the number of positive terms (resp. number of negative terms) in any diagonal writing of the metric components

EXAMPLES:

Signatures on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: g = M.metric('g') # if not specified, the signature is Riemannian
sage: g.signature()
2
sage: h = M.metric('h', signature=0)
sage: h.signature()
0
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> g = M.metric('g') # if not specified, the signature is Riemannian
>>> g.signature()
2
```

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```
>>> h = M.metric('h', signature=Integer(0))
>>> h.signature()
0
```

sqrt_abs_det(*frame=None*)

Square root of the absolute value of the determinant of the metric components in the specified frame.

INPUT:

- `frame` – (default: `None`) vector frame with respect to which the components g_{ij} of `self` are defined; if `None`, the domain's default frame is used. If a chart is provided, the associated coordinate frame is used

OUTPUT:

- $\sqrt{|\det(g_{ij})|}$, as an instance of `DiffScalarField`

EXAMPLES:

Standard metric in the Euclidean space $\mathbf{R}^3$ with spherical coordinates:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: U = M.open_subset('U') # the complement of the half-plane (y=0, x>=0)
sage: c_spher.<r,th,ph> = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\
↪phi')
sage: g = U.metric('g')
sage: g[1,1], g[2,2], g[3,3] = 1, r^2, (r*sin(th))^2
sage: g.display()
g = dr⊗dr + r^2 dth⊗dth + r^2*sin(th)^2 dph⊗dph
sage: g.sqrt_abs_det().expr()
r^2*sin(th)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> U = M.open_subset('U') # the complement of the half-plane (y=0, x>=0)
>>> c_spher = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi', names=(
↪'r', 'th', 'ph',)); (r, th, ph,) = c_spher._first_ngens(3)
>>> g = U.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)], g[Integer(3),
↪Integer(3)] = Integer(1), r**Integer(2), (r*sin(th))**Integer(2)
>>> g.display()
g = dr⊗dr + r^2 dth⊗dth + r^2*sin(th)^2 dph⊗dph
>>> g.sqrt_abs_det().expr()
r^2*sin(th)
```

Metric determinant on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: X.<x,y> = M.chart()
sage: g = M.metric('g')
sage: g[1,1], g[1, 2], g[2, 2] = 1+x, x*y , 1-y
sage: g[:]
[ x + 1    x*y]
[   x*y -y + 1]
sage: s = g.sqrt_abs_det() ; s
Scalar field on the 2-dimensional differentiable manifold M
```

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```
sage: s.expr()
sqrt(-x^2*y^2 - (x + 1)*y + x + 1)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', start_index=Integer(1))
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> g = M.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(1), Integer(2)], g[Integer(2),
↪Integer(2)] = Integer(1)+x, x*y , Integer(1)-y
>>> g[:]
[ x + 1    x*y]
[   x*y -y + 1]
>>> s = g.sqrt_abs_det() ; s
Scalar field on the 2-dimensional differentiable manifold M
>>> s.expr()
sqrt(-x^2*y^2 - (x + 1)*y + x + 1)
```

Determinant in a frame different from the default's one:

```
sage: Y.<u,v> = M.chart()
sage: ch_X_Y = X.transition_map(Y, [x+y, x-y])
sage: ch_X_Y.inverse()
Change of coordinates from Chart (M, (u, v)) to Chart (M, (x, y))
sage: g[Y.frame(),:,Y]
[ 1/8*u^2 - 1/8*v^2 + 1/4*v + 1/2                            1/4*u]
[                           1/4*u -1/8*u^2 + 1/8*v^2 + 1/4*v + 1/2]
sage: g.sqrt_abs_det(Y.frame()).expr()
1/2*sqrt(-x^2*y^2 - (x + 1)*y + x + 1)
sage: g.sqrt_abs_det(Y.frame()).expr(Y)
1/8*sqrt(-u^4 - v^4 + 2*(u^2 + 2)*v^2 - 4*u^2 + 16*v + 16)
```

```
>>> from sage.all import *
>>> Y = M.chart(names=('u', 'v',)); (u, v,) = Y._first_ngens(2)
>>> ch_X_Y = X.transition_map(Y, [x+y, x-y])
>>> ch_X_Y.inverse()
Change of coordinates from Chart (M, (u, v)) to Chart (M, (x, y))
>>> g[Y.frame(),:,Y]
[ 1/8*u^2 - 1/8*v^2 + 1/4*v + 1/2                            1/4*u]
[                           1/4*u -1/8*u^2 + 1/8*v^2 + 1/4*v + 1/2]
>>> g.sqrt_abs_det(Y.frame()).expr()
1/2*sqrt(-x^2*y^2 - (x + 1)*y + x + 1)
>>> g.sqrt_abs_det(Y.frame()).expr(Y)
1/8*sqrt(-u^4 - v^4 + 2*(u^2 + 2)*v^2 - 4*u^2 + 16*v + 16)
```

A chart can be passed instead of a frame:

```
sage: g.sqrt_abs_det(Y) is g.sqrt_abs_det(Y.frame())
True
```

```
>>> from sage.all import *
>>> g.sqrt_abs_det(Y) is g.sqrt_abs_det(Y.frame())
True
```

The metric determinant depends on the frame:

```
sage: g.sqrt_abs_det(X.frame()) == g.sqrt_abs_det(Y.frame())
False
```

```
>>> from sage.all import *
>>> g.sqrt_abs_det(X.frame()) == g.sqrt_abs_det(Y.frame())
False
```

Using SymPy as symbolic engine:

```
sage: M.set_calculus_method('sympy')
sage: g = M.metric('g')
sage: g[1,1], g[1, 2], g[2, 2] = 1+x, x*y , 1-y
sage: g.sqrt_abs_det().expr()
sqrt(-x**2*y**2 - x*y + x - y + 1)
sage: g.sqrt_abs_det(Y.frame()).expr()
sqrt(-x**2*y**2 - x*y + x - y + 1)/2
sage: g.sqrt_abs_det(Y.frame()).expr(Y)
sqrt(-u**4 + 2*u**2*v**2 - 4*u**2 - v**4 + 4*v**2 + 16*v + 16)/8
```

```
>>> from sage.all import *
>>> M.set_calculus_method('sympy')
>>> g = M.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(1), Integer(2)], g[Integer(2),␣
↪Integer(2)] = Integer(1)+x, x*y , Integer(1)-y
>>> g.sqrt_abs_det().expr()
sqrt(-x**2*y**2 - x*y + x - y + 1)
>>> g.sqrt_abs_det(Y.frame()).expr()
sqrt(-x**2*y**2 - x*y + x - y + 1)/2
>>> g.sqrt_abs_det(Y.frame()).expr(Y)
sqrt(-u**4 + 2*u**2*v**2 - 4*u**2 - v**4 + 4*v**2 + 16*v + 16)/8
```

volume_form(*contra=0*)

Volume form (Levi-Civita tensor) ϵ associated with the metric.

The volume form ϵ is an n-form (n being the manifold's dimension) such that for any oriented vector basis (e_i) which is orthonormal with respect to the metric, the condition

$$\epsilon(e_1, \dots, e_n) = 1$$

holds.

Notice that a volume form requires an orientable manifold with a preferred orientation, see `orientation()` for details.

INPUT:

- `contra` – (default: 0) number of contravariant indices of the returned tensor

OUTPUT:

- if `contra = 0` (default value): the volume n-form ϵ, as an instance of `DiffForm`
- if `contra = k`, with $1 \leq k \leq n$, the tensor field of type (k,n-k) formed from ϵ by raising the first k indices with the metric (see method `up()`); the output is then an instance of `TensorField`, with the appropriate antisymmetries, or of the subclass `MultivectorField` if $k = n$

EXAMPLES:

Volume form on $\mathbf{R}^3$ with spherical coordinates, using the standard orientation, which is predefined:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: U = M.open_subset('U') # the complement of the half-plane (y=0, x>=0)
sage: c_spher.<r,th,ph> = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\
↪phi')
sage: g = U.metric('g')
sage: g[1,1], g[2,2], g[3,3] = 1, r^2, (r*sin(th))^2
sage: g.display()
g = dr⊗dr + r^2 dth⊗dth + r^2*sin(th)^2 dph⊗dph
sage: eps = g.volume_form() ; eps
3-form eps_g on the Open subset U of the 3-dimensional
 differentiable manifold M
sage: eps.display()
eps_g = r^2*sin(th) dr∧dth∧dph
sage: eps[[1,2,3]] == g.sqrt_abs_det()
True
sage: latex(eps)
\epsilon_{g}
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', start_index=Integer(1))
>>> U = M.open_subset('U') # the complement of the half-plane (y=0, x>=0)
>>> c_spher = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi', names=(
↪'r', 'th', 'ph',)); (r, th, ph,) = c_spher._first_ngens(3)
>>> g = U.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)], g[Integer(3),
↪Integer(3)] = Integer(1), r**Integer(2), (r*sin(th))**Integer(2)
>>> g.display()
g = dr⊗dr + r^2 dth⊗dth + r^2*sin(th)^2 dph⊗dph
>>> eps = g.volume_form() ; eps
3-form eps_g on the Open subset U of the 3-dimensional
 differentiable manifold M
>>> eps.display()
eps_g = r^2*sin(th) dr∧dth∧dph
>>> eps[[Integer(1),Integer(2),Integer(3)]] == g.sqrt_abs_det()
True
>>> latex(eps)
\epsilon_{g}
```

The tensor field of components ${\epsilon^i}_{jk}$ (`contra=1`):

```
sage: eps1 = g.volume_form(1) ; eps1
Tensor field of type (1,2) on the Open subset U of the
 3-dimensional differentiable manifold M
sage: eps1.symmetries()
no symmetry;  antisymmetry: (1, 2)
sage: eps1[:]
[[[0, 0, 0], [0, 0, r^2*sin(th)], [0, -r^2*sin(th), 0]],
 [[0, 0, -sin(th)], [0, 0, 0], [sin(th), 0, 0]],
 [[0, 1/sin(th), 0], [-1/sin(th), 0, 0], [0, 0, 0]]]
```

```
>>> from sage.all import *
>>> eps1 = g.volume_form(Integer(1)) ; eps1
Tensor field of type (1,2) on the Open subset U of the
 3-dimensional differentiable manifold M
>>> eps1.symmetries()
no symmetry;  antisymmetry: (1, 2)
>>> eps1[:]
[[[0, 0, 0], [0, 0, r^2*sin(th)], [0, -r^2*sin(th), 0]],
 [[0, 0, -sin(th)], [0, 0, 0], [sin(th), 0, 0]],
 [[0, 1/sin(th), 0], [-1/sin(th), 0, 0], [0, 0, 0]]]
```

The tensor field of components $\epsilon^{ij}{}_k$ (`contra=2`):

```
sage: eps2 = g.volume_form(2) ; eps2
Tensor field of type (2,1) on the Open subset U of the
 3-dimensional differentiable manifold M
sage: eps2.symmetries()
no symmetry;  antisymmetry: (0, 1)
sage: eps2[:]
[[[0, 0, 0], [0, 0, sin(th)], [0, -1/sin(th), 0]],
 [[0, 0, -sin(th)], [0, 0, 0], [1/(r^2*sin(th)), 0, 0]],
 [[0, 1/sin(th), 0], [-1/(r^2*sin(th)), 0, 0], [0, 0, 0]]]
```

```
>>> from sage.all import *
>>> eps2 = g.volume_form(Integer(2)) ; eps2
Tensor field of type (2,1) on the Open subset U of the
 3-dimensional differentiable manifold M
>>> eps2.symmetries()
no symmetry;  antisymmetry: (0, 1)
>>> eps2[:]
[[[0, 0, 0], [0, 0, sin(th)], [0, -1/sin(th), 0]],
 [[0, 0, -sin(th)], [0, 0, 0], [1/(r^2*sin(th)), 0, 0]],
 [[0, 1/sin(th), 0], [-1/(r^2*sin(th)), 0, 0], [0, 0, 0]]]
```

The tensor field of components ϵ^{ijk} (`contra=3`):

```
sage: eps3 = g.volume_form(3) ; eps3
3-vector field on the Open subset U of the 3-dimensional
 differentiable manifold M
sage: eps3.tensor_type()
(3, 0)
sage: eps3.symmetries()
no symmetry;  antisymmetry: (0, 1, 2)
sage: eps3[:]
[[[0, 0, 0], [0, 0, 1/(r^2*sin(th))], [0, -1/(r^2*sin(th)), 0]],
 [[0, 0, -1/(r^2*sin(th))], [0, 0, 0], [1/(r^2*sin(th)), 0, 0]],
 [[0, 1/(r^2*sin(th)), 0], [-1/(r^2*sin(th)), 0, 0], [0, 0, 0]]]
sage: eps3[1,2,3]
1/(r^2*sin(th))
sage: eps3[[1,2,3]] * g.sqrt_abs_det() == 1
True
```

```
>>> from sage.all import *
>>> eps3 = g.volume_form(Integer(3)) ; eps3
3-vector field on the Open subset U of the 3-dimensional
 differentiable manifold M
>>> eps3.tensor_type()
(3, 0)
>>> eps3.symmetries()
no symmetry;  antisymmetry: (0, 1, 2)
>>> eps3[:]
[[[0, 0, 0], [0, 0, 1/(r^2*sin(th))], [0, -1/(r^2*sin(th)), 0]],
 [[0, 0, -1/(r^2*sin(th))], [0, 0, 0], [1/(r^2*sin(th)), 0, 0]],
 [[0, 1/(r^2*sin(th)), 0], [-1/(r^2*sin(th)), 0, 0], [0, 0, 0]]]
>>> eps3[Integer(1),Integer(2),Integer(3)]
1/(r^2*sin(th))
>>> eps3[[Integer(1),Integer(2),Integer(3)]] * g.sqrt_abs_det() == Integer(1)
True
```

If the manifold has no predefined orientation, an orientation must be set before invoking `volume_form()`. For instance let consider the 2-sphere described by the stereographic charts from the North and South pole:

```
sage: M = Manifold(2, 'M', structure='Riemannian')
sage: U = M.open_subset('U'); V = M.open_subset('V')
sage: M.declare_union(U, V)
sage: c_xy.<x,y> = U.chart()  # stereographic chart from the North pole
sage: c_uv.<u,v> = V.chart()  # stereographic chart from the South pole
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                intersection_name='W', restrictions1= x^2+y^2!=0,
....:                restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: eU = c_xy.frame(); eV = c_uv.frame()
sage: g = M.metric()
sage: g[eU,0,0], g[eU,1,1] = 4/(1+x^2+y^2)^2, 4/(1+x^2+y^2)^2
sage: g.add_comp_by_continuation(eV, U.intersection(V), chart=c_uv)
sage: eps = g.volume_form()
Traceback (most recent call last):
...
ValueError: 2-dimensional Riemannian manifold M must admit an
 orientation
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='Riemannian')
>>> U = M.open_subset('U'); V = M.open_subset('V')
>>> M.declare_union(U, V)
>>> c_xy = U.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)#␣
↪stereographic chart from the North pole
>>> c_uv = V.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)#␣
↪stereographic chart from the South pole
>>> xy_to_uv = c_xy.transition_map(c_uv, (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                intersection_name='W', restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                restrictions2= u**Integer(2)+v**Integer(2)!=Integer(0))
>>> uv_to_xy = xy_to_uv.inverse()
```

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```
>>> eU = c_xy.frame(); eV = c_uv.frame()
>>> g = M.metric()
>>> g[eU,Integer(0),Integer(0)], g[eU,Integer(1),Integer(1)] = Integer(4)/
↪(Integer(1)+x**Integer(2)+y**Integer(2))**Integer(2), Integer(4)/
↪(Integer(1)+x**Integer(2)+y**Integer(2))**Integer(2)
>>> g.add_comp_by_continuation(eV, U.intersection(V), chart=c_uv)
>>> eps = g.volume_form()
Traceback (most recent call last):
...
ValueError: 2-dimensional Riemannian manifold M must admit an
 orientation
```

Let us define the orientation of `M` such that `eU` is right-handed; `eV` is then left-handed and in order to define an orientation on the whole of `M`, we introduce a vector frame on `V` by swapping `eV`'s vectors:

```
sage: f = V.vector_frame('f', (eV[1], eV[0]))
sage: M.set_orientation([eU, f])
```

```
>>> from sage.all import *
>>> f = V.vector_frame('f', (eV[Integer(1)], eV[Integer(0)]))
>>> M.set_orientation([eU, f])
```

We have then, factorizing the components for a nicer display:

```
sage: eps = g.volume_form()
sage: eps.apply_map(factor, frame=eU, keep_other_components=True)
sage: eps.apply_map(factor, frame=eV, keep_other_components=True)
sage: eps.display(eU)
eps_g = 4/(x^2 + y^2 + 1)^2 dx∧dy
sage: eps.display(eV)
eps_g = -4/(u^2 + v^2 + 1)^2 du∧dv
```

```
>>> from sage.all import *
>>> eps = g.volume_form()
>>> eps.apply_map(factor, frame=eU, keep_other_components=True)
>>> eps.apply_map(factor, frame=eV, keep_other_components=True)
>>> eps.display(eU)
eps_g = 4/(x^2 + y^2 + 1)^2 dx∧dy
>>> eps.display(eV)
eps_g = -4/(u^2 + v^2 + 1)^2 du∧dv
```

Note the minus sign in the above expression, reflecting the fact that `eV` is left-handed with respect to the chosen orientation.

weyl(*name=None*, *latex_name=None*)

Return the Weyl conformal tensor associated with the metric.

The Weyl conformal tensor is the tensor field C of type (1,3) defined as the trace-free part of the Riemann curvature tensor R

INPUT:

- `name` – (default: `None`) name given to the Weyl conformal tensor; if `None`, it is set to "C(g)", where "g" is the metric's name

- `latex_name` – (default: `None`) LaTeX symbol to denote the Weyl conformal tensor; if `None`, it is set to "\mathrm{C}(g)", where "g" is the metric's name

OUTPUT:

- the Weyl conformal tensor C, as an instance of `TensorField`

EXAMPLES:

Checking that the Weyl tensor identically vanishes on a 3-dimensional manifold, for instance the hyperbolic space H^3:

```
sage: M = Manifold(3, 'H^3', start_index=1)
sage: U = M.open_subset('U') # the complement of the half-plane (y=0, x>=0)
sage: X.<rh,th,ph> = U.chart(r'rh:(0,+oo):\rho th:(0,pi):\theta  ph:(0,2*pi):\
↪phi')
sage: g = U.metric('g')
sage: b = var('b')
sage: g[1,1], g[2,2], g[3,3] = b^2, (b*sinh(rh))^2, (b*sinh(rh)*sin(th))^2
sage: g.display()  # standard metric on H^3:
g = b^2 drh⊗drh + b^2*sinh(rh)^2 dth⊗dth
 + b^2*sin(th)^2*sinh(rh)^2 dph⊗dph
sage: C = g.weyl() ; C
Tensor field C(g) of type (1,3) on the Open subset U of the
 3-dimensional differentiable manifold H^3
sage: C == 0
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'H^3', start_index=Integer(1))
>>> U = M.open_subset('U') # the complement of the half-plane (y=0, x>=0)
>>> X = U.chart(r'rh:(0,+oo):\rho th:(0,pi):\theta  ph:(0,2*pi):\phi', names=(
↪'rh', 'th', 'ph',)); (rh, th, ph,) = X._first_ngens(3)
>>> g = U.metric('g')
>>> b = var('b')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)], g[Integer(3),
↪Integer(3)] = b**Integer(2), (b*sinh(rh))**Integer(2),␣
↪(b*sinh(rh)*sin(th))**Integer(2)
>>> g.display()  # standard metric on H^3:
g = b^2 drh⊗drh + b^2*sinh(rh)^2 dth⊗dth
 + b^2*sin(th)^2*sinh(rh)^2 dph⊗dph
>>> C = g.weyl() ; C
Tensor field C(g) of type (1,3) on the Open subset U of the
 3-dimensional differentiable manifold H^3
>>> C == Integer(0)
True
```

class `sage.manifolds.differentiable.metric.`**`PseudoRiemannianMetricParal`**(*vector_field_module*, *name*, *signature=None*, *latex_name=None*)

Bases: `PseudoRiemannianMetric`, `TensorFieldParal`

Pseudo-Riemannian metric with values on a parallelizable manifold.

An instance of this class is a field of nondegenerate symmetric bilinear forms (metric field) along a differentiable manifold U with values in a parallelizable manifold M over $\mathbf{R}$, via a differentiable mapping $\Phi : U \rightarrow M$. The standard case of a metric field *on* a manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

A *metric* g is a field on U, such that at each point $p \in U$, $g(p)$ is a bilinear map of the type:

$$g(p) : T_qM \times T_qM \longrightarrow \mathbf{R}$$

where T_qM stands for the tangent space to manifold M at the point $q = \Phi(p)$, such that $g(p)$ is symmetric: $\forall (u, v) \in T_qM \times T_qM,\ g(p)(v, u) = g(p)(u, v)$ and nondegenerate: $(\forall v \in T_qM,\ \ g(p)(u, v) = 0) \Longrightarrow u = 0$.

Note

If M is not parallelizable, the class *PseudoRiemannianMetric* should be used instead.

INPUT:

- `vector_field_module` – free module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on $\Phi(U) \subset M$
- `name` – name given to the metric
- `signature` – (default: `None`) signature S of the metric as a single integer: $S = n_+ - n_-$, where n_+ (resp. n_-) is the number of positive terms (resp. number of negative terms) in any diagonal writing of the metric components; if `signature` is `None`, S is set to the dimension of manifold M (Riemannian signature)
- `latex_name` – (default: `None`) LaTeX symbol to denote the metric; if `None`, it is formed from `name`

EXAMPLES:

Metric on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: c_xy.<x,y> = M.chart()
sage: g = M.metric('g') ; g
Riemannian metric g on the 2-dimensional differentiable manifold M
sage: latex(g)
g
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', start_index=Integer(1))
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> g = M.metric('g') ; g
Riemannian metric g on the 2-dimensional differentiable manifold M
>>> latex(g)
g
```

A metric is a special kind of tensor field and therefore inheritates all the properties from class *TensorField*:

```
sage: g.parent()
Free module T^(0,2)(M) of type-(0,2) tensors fields on the
 2-dimensional differentiable manifold M
sage: g.tensor_type()
(0, 2)
sage: g.symmetries()   # g is symmetric:
symmetry: (0, 1);  no antisymmetry
```

```
>>> from sage.all import *
>>> g.parent()
Free module T^(0,2)(M) of type-(0,2) tensors fields on the
 2-dimensional differentiable manifold M
>>> g.tensor_type()
(0, 2)
>>> g.symmetries()  # g is symmetric:
symmetry: (0, 1);  no antisymmetry
```

Setting the metric components in the manifold's default frame:

```
sage: g[1,1], g[1,2], g[2,2] = 1+x, x*y, 1-x
sage: g[:]
[ x + 1    x*y]
[   x*y -x + 1]
sage: g.display()
g = (x + 1) dx⊗dx + x*y dx⊗dy + x*y dy⊗dx + (-x + 1) dy⊗dy
```

```
>>> from sage.all import *
>>> g[Integer(1),Integer(1)], g[Integer(1),Integer(2)], g[Integer(2),Integer(2)]␣
↪= Integer(1)+x, x*y, Integer(1)-x
>>> g[:]
[ x + 1    x*y]
[   x*y -x + 1]
>>> g.display()
g = (x + 1) dx⊗dx + x*y dx⊗dy + x*y dy⊗dx + (-x + 1) dy⊗dy
```

Metric components in a frame different from the manifold's default one:

```
sage: c_uv.<u,v> = M.chart()  # new chart on M
sage: xy_to_uv = c_xy.transition_map(c_uv, [x+y, x-y]) ; xy_to_uv
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
sage: uv_to_xy = xy_to_uv.inverse() ; uv_to_xy
Change of coordinates from Chart (M, (u, v)) to Chart (M, (x, y))
sage: M.atlas()
[Chart (M, (x, y)), Chart (M, (u, v))]
sage: M.frames()
[Coordinate frame (M, (∂/∂x,∂/∂y)), Coordinate frame (M, (∂/∂u,∂/∂v))]
sage: g[c_uv.frame(),:]  # metric components in frame c_uv.frame() expressed in M
↪'s default chart (x,y)
[ 1/2*x*y + 1/2          1/2*x]
[         1/2*x -1/2*x*y + 1/2]
sage: g.display(c_uv.frame())
g = (1/2*x*y + 1/2) du⊗du + 1/2*x du⊗dv + 1/2*x dv⊗du
 + (-1/2*x*y + 1/2) dv⊗dv
sage: g[c_uv.frame(),:,c_uv]   # metric components in frame c_uv.frame()␣
↪expressed in chart (u,v)
[ 1/8*u^2 - 1/8*v^2 + 1/2            1/4*u + 1/4*v]
[           1/4*u + 1/4*v -1/8*u^2 + 1/8*v^2 + 1/2]
sage: g.display(c_uv.frame(), c_uv)
g = (1/8*u^2 - 1/8*v^2 + 1/2) du⊗du + (1/4*u + 1/4*v) du⊗dv
 + (1/4*u + 1/4*v) dv⊗du + (-1/8*u^2 + 1/8*v^2 + 1/2) dv⊗dv
```

```
>>> from sage.all import *
>>> c_uv = M.chart(names=('u', 'v',)); (u, v,) = c_uv._first_ngens(2)# new chart␣
↪on M
>>> xy_to_uv = c_xy.transition_map(c_uv, [x+y, x-y]) ; xy_to_uv
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
>>> uv_to_xy = xy_to_uv.inverse() ; uv_to_xy
Change of coordinates from Chart (M, (u, v)) to Chart (M, (x, y))
>>> M.atlas()
[Chart (M, (x, y)), Chart (M, (u, v))]
>>> M.frames()
[Coordinate frame (M, (∂/∂x,∂/∂y)), Coordinate frame (M, (∂/∂u,∂/∂v))]
>>> g[c_uv.frame(),:]  # metric components in frame c_uv.frame() expressed in M's␣
↪default chart (x,y)
[ 1/2*x*y + 1/2          1/2*x]
[         1/2*x -1/2*x*y + 1/2]
>>> g.display(c_uv.frame())
g = (1/2*x*y + 1/2) du⊗du + 1/2*x du⊗dv + 1/2*x dv⊗du
 + (-1/2*x*y + 1/2) dv⊗dv
>>> g[c_uv.frame(),:,c_uv]   # metric components in frame c_uv.frame() expressed␣
↪in chart (u,v)
[ 1/8*u^2 - 1/8*v^2 + 1/2             1/4*u + 1/4*v]
[           1/4*u + 1/4*v -1/8*u^2 + 1/8*v^2 + 1/2]
>>> g.display(c_uv.frame(), c_uv)
g = (1/8*u^2 - 1/8*v^2 + 1/2) du⊗du + (1/4*u + 1/4*v) du⊗dv
 + (1/4*u + 1/4*v) dv⊗du + (-1/8*u^2 + 1/8*v^2 + 1/2) dv⊗dv
```

As a shortcut of the above command, on can pass just the chart `c_uv` to `display`, the vector frame being then assumed to be the coordinate frame associated with the chart:

```
sage: g.display(c_uv)
g = (1/8*u^2 - 1/8*v^2 + 1/2) du⊗du + (1/4*u + 1/4*v) du⊗dv
 + (1/4*u + 1/4*v) dv⊗du + (-1/8*u^2 + 1/8*v^2 + 1/2) dv⊗dv
```

```
>>> from sage.all import *
>>> g.display(c_uv)
g = (1/8*u^2 - 1/8*v^2 + 1/2) du⊗du + (1/4*u + 1/4*v) du⊗dv
 + (1/4*u + 1/4*v) dv⊗du + (-1/8*u^2 + 1/8*v^2 + 1/2) dv⊗dv
```

The inverse metric is obtained via *inverse()*:

```
sage: ig = g.inverse() ; ig
Tensor field inv_g of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: ig[:]
[ (x - 1)/(x^2*y^2 + x^2 - 1)      x*y/(x^2*y^2 + x^2 - 1)]
[     x*y/(x^2*y^2 + x^2 - 1) -(x + 1)/(x^2*y^2 + x^2 - 1)]
sage: ig.display()
inv_g = (x - 1)/(x^2*y^2 + x^2 - 1) ∂/∂x⊗∂/∂x
 + x*y/(x^2*y^2 + x^2 - 1) ∂/∂x⊗∂/∂y + x*y/(x^2*y^2 + x^2 - 1) ∂/∂y⊗∂/∂x
 - (x + 1)/(x^2*y^2 + x^2 - 1) ∂/∂y⊗∂/∂y
```

```
>>> from sage.all import *
>>> ig = g.inverse() ; ig
```

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```
Tensor field inv_g of type (2,0) on the 2-dimensional differentiable
 manifold M
>>> ig[:]
[ (x - 1)/(x^2*y^2 + x^2 - 1)       x*y/(x^2*y^2 + x^2 - 1)]
[     x*y/(x^2*y^2 + x^2 - 1) -(x + 1)/(x^2*y^2 + x^2 - 1)]
>>> ig.display()
inv_g = (x - 1)/(x^2*y^2 + x^2 - 1) ∂/∂x⊗∂/∂x
 + x*y/(x^2*y^2 + x^2 - 1) ∂/∂x⊗∂/∂y + x*y/(x^2*y^2 + x^2 - 1) ∂/∂y⊗∂/∂x
 - (x + 1)/(x^2*y^2 + x^2 - 1) ∂/∂y⊗∂/∂y
```

inverse(*expansion_symbol=None*, *order=1*)

Return the inverse metric.

INPUT:

- `expansion_symbol` – (default: `None`) symbolic variable; if specified, the inverse will be expanded in power series with respect to this variable (around its zero value)
- `order` – integer (default: 1); the order of the expansion if `expansion_symbol` is not `None`; the *order* is defined as the degree of the polynomial representing the truncated power series in `expansion_symbol`; currently only first order inverse is supported

If `expansion_symbol` is set, then the zeroth order metric must be invertible. Moreover, subsequent calls to this method will return a cached value, even when called with the default value (to enable computation of derived quantities). To reset, use `_del_derived()`.

OUTPUT:

- instance of *TensorFieldParal* with `tensor_type` = (2,0) representing the inverse metric

EXAMPLES:

Inverse metric on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: c_xy.<x,y> = M.chart()
sage: g = M.metric('g')
sage: g[1,1], g[1,2], g[2,2] = 1+x, x*y, 1-x
sage: g[:]  # components in the manifold's default frame
[ x + 1    x*y]
[   x*y -x + 1]
sage: ig = g.inverse() ; ig
Tensor field inv_g of type (2,0) on the 2-dimensional
  differentiable manifold M
sage: ig[:]
[ (x - 1)/(x^2*y^2 + x^2 - 1)       x*y/(x^2*y^2 + x^2 - 1)]
[     x*y/(x^2*y^2 + x^2 - 1) -(x + 1)/(x^2*y^2 + x^2 - 1)]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', start_index=Integer(1))
>>> c_xy = M.chart(names=('x', 'y',)); (x, y,) = c_xy._first_ngens(2)
>>> g = M.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(1),Integer(2)], g[Integer(2),
↪Integer(2)] = Integer(1)+x, x*y, Integer(1)-x
>>> g[:]  # components in the manifold's default frame
[ x + 1    x*y]
```

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```
[   x*y -x + 1]
>>> ig = g.inverse() ; ig
Tensor field inv_g of type (2,0) on the 2-dimensional
  differentiable manifold M
>>> ig[:]
[ (x - 1)/(x^2*y^2 + x^2 - 1)       x*y/(x^2*y^2 + x^2 - 1)]
[     x*y/(x^2*y^2 + x^2 - 1) -(x + 1)/(x^2*y^2 + x^2 - 1)]
```

If the metric is modified, the inverse metric is automatically updated:

```
sage: g[1,2] = 0 ; g[:]
[ x + 1      0]
[     0 -x + 1]
sage: g.inverse()[:]
[ 1/(x + 1)          0]
[         0 -1/(x - 1)]
```

```
>>> from sage.all import *
>>> g[Integer(1),Integer(2)] = Integer(0) ; g[:]
[ x + 1      0]
[     0 -x + 1]
>>> g.inverse()[:]
[ 1/(x + 1)          0]
[         0 -1/(x - 1)]
```

Using SymPy as symbolic engine:

```
sage: M.set_calculus_method('sympy')
sage: g[1,1], g[1,2], g[2,2] = 1+x, x*y, 1-x
sage: g[:]  # components in the manifold's default frame
[x + 1   x*y]
[  x*y 1 - x]
sage: g.inverse()[:]
[ (x - 1)/(x**2*y**2 + x**2 - 1)       x*y/(x**2*y**2 + x**2 - 1)]
[     x*y/(x**2*y**2 + x**2 - 1) -(x + 1)/(x**2*y**2 + x**2 - 1)]
```

```
>>> from sage.all import *
>>> M.set_calculus_method('sympy')
>>> g[Integer(1),Integer(1)], g[Integer(1),Integer(2)], g[Integer(2),
↪Integer(2)] = Integer(1)+x, x*y, Integer(1)-x
>>> g[:]  # components in the manifold's default frame
[x + 1   x*y]
[  x*y 1 - x]
>>> g.inverse()[:]
[ (x - 1)/(x**2*y**2 + x**2 - 1)       x*y/(x**2*y**2 + x**2 - 1)]
[     x*y/(x**2*y**2 + x**2 - 1) -(x + 1)/(x**2*y**2 + x**2 - 1)]
```

Demonstration of the series expansion capabilities:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: C.<t,x,y,z> = M.chart()
sage: e = var('e')
```

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```
sage: g = M.metric()
sage: h = M.tensor_field(0, 2, sym=(0,1))
sage: g[0, 0], g[1, 1], g[2, 2], g[3, 3] = -1, 1, 1, 1
sage: h[0, 1], h[1, 2], h[2, 3] = 1, 1, 1
sage: g.set(g + e*h)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> C = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = C._first_
↪ngens(4)
>>> e = var('e')
>>> g = M.metric()
>>> h = M.tensor_field(Integer(0), Integer(2), sym=(Integer(0),Integer(1)))
>>> g[Integer(0), Integer(0)], g[Integer(1), Integer(1)], g[Integer(2),␣
↪Integer(2)], g[Integer(3), Integer(3)] = -Integer(1), Integer(1),␣
↪Integer(1), Integer(1)
>>> h[Integer(0), Integer(1)], h[Integer(1), Integer(2)], h[Integer(2),␣
↪Integer(3)] = Integer(1), Integer(1), Integer(1)
>>> g.set(g + e*h)
```

If e is a small parameter, g is a tridiagonal approximation of the Minkowski metric:

```
sage: g[:]
[-1  e  0  0]
[ e  1  e  0]
[ 0  e  1  e]
[ 0  0  e  1]
```

```
>>> from sage.all import *
>>> g[:]
[-1  e  0  0]
[ e  1  e  0]
[ 0  e  1  e]
[ 0  0  e  1]
```

The inverse, truncated to first order in e, is:

```
sage: g.inverse(expansion_symbol=e)[:]
[-1  e  0  0]
[ e  1 -e  0]
[ 0 -e  1 -e]
[ 0  0 -e  1]
```

```
>>> from sage.all import *
>>> g.inverse(expansion_symbol=e)[:]
[-1  e  0  0]
[ e  1 -e  0]
[ 0 -e  1 -e]
[ 0  0 -e  1]
```

If inverse() is called subsequently, the result will be the same. This allows for all computations to be made to first order:

```
sage: g.inverse()[:]
[-1  e  0  0]
[ e  1 -e  0]
[ 0 -e  1 -e]
[ 0  0 -e  1]
```

```
>>> from sage.all import *
>>> g.inverse()[:]
[-1  e  0  0]
[ e  1 -e  0]
[ 0 -e  1 -e]
[ 0  0 -e  1]
```

restrict(*subdomain*, *dest_map=None*)

Return the restriction of the metric to some subdomain.

If the restriction has not been defined yet, it is constructed here.

INPUT:

- `subdomain` – open subset U of `self._domain` (must be an instance of *DifferentiableManifold*)
- `dest_map` – (default: `None`) destination map $\Phi : U \to V$, where V is a subdomain of `self._codomain` (type: *DiffMap*) If `None`, the restriction of `self._vmodule._dest_map` to U is used.

OUTPUT:

- instance of *PseudoRiemannianMetricParal* representing the restriction.

EXAMPLES:

Restriction of a Lorentzian metric on $\mathbf{R}^2$ to the upper half plane:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: g = M.lorentzian_metric('g')
sage: g[0,0], g[1,1] = -1, 1
sage: U = M.open_subset('U', coord_def={X: y>0})
sage: gU = g.restrict(U); gU
Lorentzian metric g on the Open subset U of the 2-dimensional
 differentiable manifold M
sage: gU.signature()
0
sage: gU.display()
g = -dx⊗dx + dy⊗dy
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> g = M.lorentzian_metric('g')
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)] = -Integer(1),␣
↪Integer(1)
>>> U = M.open_subset('U', coord_def={X: y>Integer(0)})
>>> gU = g.restrict(U); gU
Lorentzian metric g on the Open subset U of the 2-dimensional
 differentiable manifold M
```

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```
>>> gU.signature()
0
>>> gU.display()
g = -dx⊗dx + dy⊗dy
```

ricci_scalar(*name=None*, *latex_name=None*)

Return the metric's Ricci scalar.

The Ricci scalar is the scalar field r defined from the Ricci tensor Ric and the metric tensor g by

$$r = g^{ij} Ric_{ij}$$

INPUT:

- `name` – (default: `None`) name given to the Ricci scalar; if none, it is set to "r(g)", where "g" is the metric's name
- `latex_name` – (default: `None`) LaTeX symbol to denote the Ricci scalar; if none, it is set to "\mathrm{r}(g)", where "g" is the metric's name

OUTPUT:

- the Ricci scalar r, as an instance of `DiffScalarField`

EXAMPLES:

Ricci scalar of the standard metric on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: U = M.open_subset('U') # the complement of a meridian (domain of␣
↪spherical coordinates)
sage: c_spher.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: a = var('a') # the sphere radius
sage: g = U.metric('g')
sage: g[1,1], g[2,2] = a^2, a^2*sin(th)^2
sage: g.display() # standard metric on the 2-sphere of radius a:
g = a^2 dth⊗dth + a^2*sin(th)^2 dph⊗dph
sage: g.ricci_scalar()
Scalar field r(g) on the Open subset U of the 2-dimensional
 differentiable manifold S^2
sage: g.ricci_scalar().display() # The Ricci scalar is constant:
r(g): U → ℝ
    (th, ph) ↦ 2/a^2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', start_index=Integer(1))
>>> U = M.open_subset('U') # the complement of a meridian (domain of␣
↪spherical coordinates)
>>> c_spher = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi', names=('th', 'ph',
↪)); (th, ph,) = c_spher._first_ngens(2)
>>> a = var('a') # the sphere radius
>>> g = U.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)] = a**Integer(2),␣
↪a**Integer(2)*sin(th)**Integer(2)
>>> g.display() # standard metric on the 2-sphere of radius a:
```

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```
g = a^2 dth⊗dth + a^2*sin(th)^2 dph⊗dph
>>> g.ricci_scalar()
Scalar field r(g) on the Open subset U of the 2-dimensional
 differentiable manifold S^2
>>> g.ricci_scalar().display() # The Ricci scalar is constant:
r(g): U → ℝ
   (th, ph) ↦ 2/a^2
```

set (*symbiform*)

Define the metric from a field of symmetric bilinear forms.

INPUT:

- `symbiform` – instance of *TensorFieldParal* representing a field of symmetric bilinear forms

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: s = M.sym_bilin_form_field(name='s')
sage: s[0,0], s[0,1], s[1,1] = 1+x^2, x*y, 1+y^2
sage: g = M.metric('g')
sage: g.set(s)
sage: g.display()
g = (x^2 + 1) dx⊗dx + x*y dx⊗dy + x*y dy⊗dx + (y^2 + 1) dy⊗dy
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> s = M.sym_bilin_form_field(name='s')
>>> s[Integer(0),Integer(0)], s[Integer(0),Integer(1)], s[Integer(1),
↪Integer(1)] = Integer(1)+x**Integer(2), x*y, Integer(1)+y**Integer(2)
>>> g = M.metric('g')
>>> g.set(s)
>>> g.display()
g = (x^2 + 1) dx⊗dx + x*y dx⊗dy + x*y dy⊗dx + (y^2 + 1) dy⊗dy
```

3.4 Levi-Civita Connections

The class *LeviCivitaConnection* implements the Levi-Civita connection associated with some pseudo-Riemannian metric on a smooth manifold.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015) : initial version
- Marco Mancini (2015) : parallelization of some computations
- Marius Gerbershagen (2022) : use the first Bianchi identity in the computation of the Riemann tensor

REFERENCES:

- [KN1963]
- [Lee1997]
- [ONe1983]

class sage.manifolds.differentiable.levi_civita_connection.**LeviCivitaConnection**(*metric*, *name*, *latex_name=None*, *init_coef=True*)

Bases: `AffineConnection`

Levi-Civita connection on a pseudo-Riemannian manifold.

Let M be a differentiable manifold of class C^∞ (smooth manifold) over $\mathbf{R}$ endowed with a pseudo-Riemannian metric g. Let $C^\infty(M)$ be the algebra of smooth functions $M \to \mathbf{R}$ (cf. `DiffScalarFieldAlgebra`) and let $\mathfrak{X}(M)$ be the $C^\infty(M)$-module of vector fields on M (cf. `VectorFieldModule`). The *Levi-Civita connection associated with* g is the unique operator

$$\begin{array}{cccc} \nabla : & \mathfrak{X}(M) \times \mathfrak{X}(M) & \longrightarrow & \mathfrak{X}(M) \\ & (u, v) & \longmapsto & \nabla_u v \end{array}$$

that

- is $\mathbf{R}$-bilinear, i.e. is bilinear when considering $\mathfrak{X}(M)$ as a vector space over $\mathbf{R}$
- is $C^\infty(M)$-linear w.r.t. the first argument: $\forall f \in C^\infty(M), \ \nabla_{fu} v = f \nabla_u v$
- obeys Leibniz rule w.r.t. the second argument: $\forall f \in C^\infty(M), \ \nabla_u (f v) = \mathrm{d}f(u) \, v + f \nabla_u v$
- is torsion-free
- is compatible with g: $\forall (u, v, w) \in \mathfrak{X}(M)^3, \ u(g(v, w)) = g(\nabla_u v, w) + g(v, \nabla_u w)$

The Levi-Civita connection ∇ gives birth to the *covariant derivative operator* acting on tensor fields, denoted by the same symbol:

$$\begin{array}{cccc} \nabla : & T^{(k,l)}(M) & \longrightarrow & T^{(k,l+1)}(M) \\ & t & \longmapsto & \nabla t \end{array}$$

where $T^{(k,l)}(M)$ stands for the $C^\infty(M)$-module of tensor fields of type (k, l) on M (cf. `TensorFieldModule`), with the convention $T^{(0,0)}(M) := C^\infty(M)$. For a vector field v, the covariant derivative ∇v is a type-(1,1) tensor field such that

$$\forall u \in \mathfrak{X}(M), \ \nabla_u v = \nabla v(., u)$$

More generally for any tensor field $t \in T^{(k,l)}(M)$, we have

$$\forall u \in \mathfrak{X}(M), \ \nabla_u t = \nabla t(\ldots, u)$$

Note

The above convention means that, in terms of index notation, the "derivation index" in ∇t is the *last* one:

$$\nabla_c t^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} = (\nabla t)^{a_1 \ldots a_k}{}_{b_1 \ldots b_l c}$$

INPUT:

- `metric` – the metric g defining the Levi-Civita connection, as an instance of class `PseudoRiemannianMetric`
- `name` – name given to the connection
- `latex_name` – (default: `None`) LaTeX symbol to denote the connection

- `init_coef` – boolean (default: `True`); determines whether the Christoffel symbols are initialized (in the top charts on the domain, i.e. disregarding the subcharts)

EXAMPLES:

Levi-Civita connection associated with the Euclidean metric on $\mathbf{R}^3$ expressed in spherical coordinates:

```
sage: forget() # for doctests only
sage: M = Manifold(3, 'R^3', start_index=1)
sage: c_spher.<r,th,ph> = M.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: g = M.metric('g')
sage: g[1,1], g[2,2], g[3,3] = 1, r^2 , (r*sin(th))^2
sage: g.display()
g = dr⊗dr + r^2 dth⊗dth + r^2*sin(th)^2 dph⊗dph
sage: nab = g.connection(name='nabla', latex_name=r'\nabla') ; nab
Levi-Civita connection nabla associated with the Riemannian metric g on
 the 3-dimensional differentiable manifold R^3
```

```
>>> from sage.all import *
>>> forget() # for doctests only
>>> M = Manifold(Integer(3), 'R^3', start_index=Integer(1))
>>> c_spher = M.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi', names=('r',
↪'th', 'ph',)); (r, th, ph,) = c_spher._first_ngens(3)
>>> g = M.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)], g[Integer(3),Integer(3)]␣
↪= Integer(1), r**Integer(2) , (r*sin(th))**Integer(2)
>>> g.display()
g = dr⊗dr + r^2 dth⊗dth + r^2*sin(th)^2 dph⊗dph
>>> nab = g.connection(name='nabla', latex_name=r'\nabla') ; nab
Levi-Civita connection nabla associated with the Riemannian metric g on
 the 3-dimensional differentiable manifold R^3
```

Let us check that the connection is compatible with the metric:

```
sage: Dg = nab(g) ; Dg
Tensor field nabla(g) of type (0,3) on the 3-dimensional
 differentiable manifold R^3
sage: Dg == 0
True
```

```
>>> from sage.all import *
>>> Dg = nab(g) ; Dg
Tensor field nabla(g) of type (0,3) on the 3-dimensional
 differentiable manifold R^3
>>> Dg == Integer(0)
True
```

and that it is torsionless:

```
sage: nab.torsion() == 0
True
```

```
>>> from sage.all import *
>>> nab.torsion() == Integer(0)
```

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```
True
```

As a check, let us enforce the computation of the torsion:

```
sage: sage.manifolds.differentiable.affine_connection.AffineConnection.
↪torsion(nab) == 0
True
```

```
>>> from sage.all import *
>>> sage.manifolds.differentiable.affine_connection.AffineConnection.torsion(nab)␣
↪== Integer(0)
True
```

The connection coefficients in the manifold's default frame are Christoffel symbols, since the default frame is a coordinate frame:

```
sage: M.default_frame()
Coordinate frame (R^3, (∂/∂r,∂/∂th,∂/∂ph))
sage: nab.coef()
3-indices components w.r.t. Coordinate frame (R^3, (∂/∂r,∂/∂th,∂/∂ph)),
 with symmetry on the index positions (1, 2)
```

```
>>> from sage.all import *
>>> M.default_frame()
Coordinate frame (R^3, (∂/∂r,∂/∂th,∂/∂ph))
>>> nab.coef()
3-indices components w.r.t. Coordinate frame (R^3, (∂/∂r,∂/∂th,∂/∂ph)),
 with symmetry on the index positions (1, 2)
```

We note that the Christoffel symbols are symmetric with respect to their last two indices (positions (1,2)); their expression is:

```
sage: nab.coef()[:]  # display as a array
[[[0, 0, 0], [0, -r, 0], [0, 0, -r*sin(th)^2]],
 [[0, 1/r, 0], [1/r, 0, 0], [0, 0, -cos(th)*sin(th)]],
 [[0, 0, 1/r], [0, 0, cos(th)/sin(th)], [1/r, cos(th)/sin(th), 0]]]
sage: nab.display()  # display only the non-vanishing symbols
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_th,r = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
Gam^ph_ph,r = 1/r
Gam^ph_ph,th = cos(th)/sin(th)
sage: nab.display(only_nonredundant=True)  # skip redundancy due to symmetry
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
```

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```
Gam^ph_th,ph = cos(th)/sin(th)
```

```
>>> from sage.all import *
>>> nab.coef()[:]  # display as a array
[[[0, 0, 0], [0, -r, 0], [0, 0, -r*sin(th)^2]],
 [[0, 1/r, 0], [1/r, 0, 0], [0, 0, -cos(th)*sin(th)]],
 [[0, 0, 1/r], [0, 0, cos(th)/sin(th)], [1/r, cos(th)/sin(th), 0]]]
>>> nab.display()  # display only the non-vanishing symbols
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_th,r = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
Gam^ph_ph,r = 1/r
Gam^ph_ph,th = cos(th)/sin(th)
>>> nab.display(only_nonredundant=True)  # skip redundancy due to symmetry
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
```

The same display can be obtained via the function *christoffel_symbols_display()* acting on the metric:

```
sage: g.christoffel_symbols_display(chart=c_spher)
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
```

```
>>> from sage.all import *
>>> g.christoffel_symbols_display(chart=c_spher)
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
```

coef(*frame=None*)

Return the connection coefficients relative to the given frame.

n being the manifold's dimension, the connection coefficients relative to the vector frame (e_i) are the n^3 scalar fields $\Gamma^k_{\ ij}$ defined by

$$\nabla_{e_j} e_i = \Gamma^k_{\ ij} e_k$$

If the connection coefficients are not known already, they are computed

- as Christoffel symbols if the frame (e_i) is a coordinate frame
- from the above formula otherwise

INPUT:

- `frame` – (default: `None`) vector frame relative to which the connection coefficients are required; if none is provided, the domain's default frame is assumed

OUTPUT:

- connection coefficients relative to the frame `frame`, as an instance of the class `Components` with 3 indices ordered as (k, i, j); for Christoffel symbols, an instance of the subclass `CompWithSym` is returned.

EXAMPLES:

Christoffel symbols of the Levi-Civita connection associated to the Euclidean metric on $\mathbf{R}^3$ expressed in spherical coordinates:

```
sage: M = Manifold(3, 'R^3', start_index=1)
sage: c_spher.<r,th,ph> = M.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\
↪phi')
sage: g = M.metric('g')
sage: g[1,1], g[2,2], g[3,3] = 1, r^2 , (r*sin(th))^2
sage: g.display()
g = dr⊗dr + r^2 dth⊗dth + r^2*sin(th)^2 dph⊗dph
sage: nab = g.connection()
sage: gam = nab.coef() ; gam
3-indices components w.r.t. Coordinate frame (R^3, (∂/∂r,∂/∂th,∂/∂ph)),
 with symmetry on the index positions (1, 2)
sage: gam[:]
[[[0, 0, 0], [0, -r, 0], [0, 0, -r*sin(th)^2]],
[[0, 1/r, 0], [1/r, 0, 0], [0, 0, -cos(th)*sin(th)]],
[[0, 0, 1/r], [0, 0, cos(th)/sin(th)], [1/r, cos(th)/sin(th), 0]]]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'R^3', start_index=Integer(1))
>>> c_spher = M.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi', names=(
↪'r', 'th', 'ph',)); (r, th, ph,) = c_spher._first_ngens(3)
>>> g = M.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)], g[Integer(3),
↪Integer(3)] = Integer(1), r**Integer(2) , (r*sin(th))**Integer(2)
>>> g.display()
g = dr⊗dr + r^2 dth⊗dth + r^2*sin(th)^2 dph⊗dph
>>> nab = g.connection()
>>> gam = nab.coef() ; gam
3-indices components w.r.t. Coordinate frame (R^3, (∂/∂r,∂/∂th,∂/∂ph)),
 with symmetry on the index positions (1, 2)
>>> gam[:]
[[[0, 0, 0], [0, -r, 0], [0, 0, -r*sin(th)^2]],
[[0, 1/r, 0], [1/r, 0, 0], [0, 0, -cos(th)*sin(th)]],
[[0, 0, 1/r], [0, 0, cos(th)/sin(th)], [1/r, cos(th)/sin(th), 0]]]
```

The only nonzero Christoffel symbols:

```
sage: gam[1,2,2], gam[1,3,3]
(-r, -r*sin(th)^2)
```

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```
sage: gam[2,1,2], gam[2,3,3]
(1/r, -cos(th)*sin(th))
sage: gam[3,1,3], gam[3,2,3]
(1/r, cos(th)/sin(th))
```

```
>>> from sage.all import *
>>> gam[Integer(1),Integer(2),Integer(2)], gam[Integer(1),Integer(3),
↪Integer(3)]
(-r, -r*sin(th)^2)
>>> gam[Integer(2),Integer(1),Integer(2)], gam[Integer(2),Integer(3),
↪Integer(3)]
(1/r, -cos(th)*sin(th))
>>> gam[Integer(3),Integer(1),Integer(3)], gam[Integer(3),Integer(2),
↪Integer(3)]
(1/r, cos(th)/sin(th))
```

Connection coefficients of the same connection with respect to the orthonormal frame associated to spherical coordinates:

```
sage: ch_basis = M.automorphism_field()
sage: ch_basis[1,1], ch_basis[2,2], ch_basis[3,3] = 1, 1/r, 1/(r*sin(th))
sage: e = c_spher.frame().new_frame(ch_basis, 'e')
sage: gam_e = nab.coef(e) ; gam_e
3-indices components w.r.t. Vector frame (R^3, (e_1,e_2,e_3))
sage: gam_e[:]
[[[0, 0, 0], [0, -1/r, 0], [0, 0, -1/r]],
[[0, 1/r, 0], [0, 0, 0], [0, 0, -cos(th)/(r*sin(th))]],
[[0, 0, 1/r], [0, 0, cos(th)/(r*sin(th))], [0, 0, 0]]]
```

```
>>> from sage.all import *
>>> ch_basis = M.automorphism_field()
>>> ch_basis[Integer(1),Integer(1)], ch_basis[Integer(2),Integer(2)], ch_
↪basis[Integer(3),Integer(3)] = Integer(1), Integer(1)/r, Integer(1)/
↪(r*sin(th))
>>> e = c_spher.frame().new_frame(ch_basis, 'e')
>>> gam_e = nab.coef(e) ; gam_e
3-indices components w.r.t. Vector frame (R^3, (e_1,e_2,e_3))
>>> gam_e[:]
[[[0, 0, 0], [0, -1/r, 0], [0, 0, -1/r]],
[[0, 1/r, 0], [0, 0, 0], [0, 0, -cos(th)/(r*sin(th))]],
[[0, 0, 1/r], [0, 0, cos(th)/(r*sin(th))], [0, 0, 0]]]
```

The only nonzero connection coefficients:

```
sage: gam_e[1,2,2], gam_e[2,1,2]
(-1/r, 1/r)
sage: gam_e[1,3,3], gam_e[3,1,3]
(-1/r, 1/r)
sage: gam_e[2,3,3], gam_e[3,2,3]
(-cos(th)/(r*sin(th)), cos(th)/(r*sin(th)))
```

```
>>> from sage.all import *
>>> gam_e[Integer(1),Integer(2),Integer(2)], gam_e[Integer(2),Integer(1),
↪Integer(2)]
(-1/r, 1/r)
>>> gam_e[Integer(1),Integer(3),Integer(3)], gam_e[Integer(3),Integer(1),
↪Integer(3)]
(-1/r, 1/r)
>>> gam_e[Integer(2),Integer(3),Integer(3)], gam_e[Integer(3),Integer(2),
↪Integer(3)]
(-cos(th)/(r*sin(th)), cos(th)/(r*sin(th)))
```

restrict(*subdomain*)

Return the restriction of the connection to some subdomain.

If such restriction has not been defined yet, it is constructed here.

INPUT:

- subdomain – open subset U of the connection's domain (must be an instance of *DifferentiableManifold*)

OUTPUT:

- instance of *LeviCivitaConnection* representing the restriction.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: g = M.metric('g')
sage: g[0,0], g[1,1] = 1+y^2, 1+x^2
sage: nab = g.connection()
sage: nab[:]
[[[0, y/(y^2 + 1)], [y/(y^2 + 1), -x/(y^2 + 1)]],
 [[-y/(x^2 + 1), x/(x^2 + 1)], [x/(x^2 + 1), 0]]]
sage: U = M.open_subset('U', coord_def={X: x>0})
sage: nabU = nab.restrict(U); nabU
Levi-Civita connection nabla_g associated with the Riemannian
 metric g on the Open subset U of the 2-dimensional differentiable
 manifold M
sage: nabU[:]
[[[0, y/(y^2 + 1)], [y/(y^2 + 1), -x/(y^2 + 1)]],
 [[-y/(x^2 + 1), x/(x^2 + 1)], [x/(x^2 + 1), 0]]]
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> g = M.metric('g')
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)] =␣
↪Integer(1)+y**Integer(2), Integer(1)+x**Integer(2)
>>> nab = g.connection()
>>> nab[:]
[[[0, y/(y^2 + 1)], [y/(y^2 + 1), -x/(y^2 + 1)]],
 [[-y/(x^2 + 1), x/(x^2 + 1)], [x/(x^2 + 1), 0]]]
>>> U = M.open_subset('U', coord_def={X: x>Integer(0)})
```

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```
>>> nabU = nab.restrict(U); nabU
Levi-Civita connection nabla_g associated with the Riemannian
 metric g on the Open subset U of the 2-dimensional differentiable
 manifold M
>>> nabU[:]
[[[0, y/(y^2 + 1)], [y/(y^2 + 1), -x/(y^2 + 1)]],
 [[-y/(x^2 + 1), x/(x^2 + 1)], [x/(x^2 + 1), 0]]]
```

Let us check that the restriction is the connection compatible with the restriction of the metric:

```
sage: nabU(g.restrict(U)).display()
nabla_g(g) = 0
```

```
>>> from sage.all import *
>>> nabU(g.restrict(U)).display()
nabla_g(g) = 0
```

ricci(*name=None*, *latex_name=None*)

Return the connection's Ricci tensor.

This method redefines `sage.manifolds.differentiable.affine_connection.AffineConnection.ricci()` to take into account the symmetry of the Ricci tensor for a Levi-Civita connection.

The Ricci tensor is the tensor field Ric of type (0,2) defined from the Riemann curvature tensor R by

$$Ric(u, v) = R(e^i, u, e_i, v)$$

for any vector fields u and v, (e_i) being any vector frame and (e^i) the dual coframe.

INPUT:

- `name` – (default: `None`) name given to the Ricci tensor; if none, it is set to "Ric(g)", where "g" is the metric's name
- `latex_name` – (default: `None`) LaTeX symbol to denote the Ricci tensor; if none, it is set to "\mathrm{Ric}(g)", where "g" is the metric's name

OUTPUT:

- the Ricci tensor Ric, as an instance of `TensorField` of tensor type (0,2) and symmetric

EXAMPLES:

Ricci tensor of the standard connection on the 2-dimensional sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: c_spher.<th,ph> = M.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: g = M.metric('g')
sage: g[1,1], g[2,2] = 1, sin(th)^2
sage: g.display() # standard metric on S^2:
g = dth⊗dth + sin(th)^2 dph⊗dph
sage: nab = g.connection() ; nab
Levi-Civita connection nabla_g associated with the Riemannian
 metric g on the 2-dimensional differentiable manifold S^2
sage: ric = nab.ricci() ; ric
Field of symmetric bilinear forms Ric(g) on the 2-dimensional
```

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```
 differentiable manifold S^2
sage: ric.display()
Ric(g) = dth⊗dth + sin(th)^2 dph⊗dph
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'S^2', start_index=Integer(1))
>>> c_spher = M.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi', names=('th', 'ph',
↪)); (th, ph,) = c_spher._first_ngens(2)
>>> g = M.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)] = Integer(1),␣
↪sin(th)**Integer(2)
>>> g.display() # standard metric on S^2:
g = dth⊗dth + sin(th)^2 dph⊗dph
>>> nab = g.connection() ; nab
Levi-Civita connection nabla_g associated with the Riemannian
 metric g on the 2-dimensional differentiable manifold S^2
>>> ric = nab.ricci() ; ric
Field of symmetric bilinear forms Ric(g) on the 2-dimensional
 differentiable manifold S^2
>>> ric.display()
Ric(g) = dth⊗dth + sin(th)^2 dph⊗dph
```

Checking that the Ricci tensor of the Levi-Civita connection associated to Schwarzschild metric is identically zero (as a solution of the Einstein equation):

```
sage: M = Manifold(4, 'M')
sage: c_BL.<t,r,th,ph> = M.chart(r't r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\
↪phi') # Schwarzschild-Droste coordinates
sage: g = M.lorentzian_metric('g')
sage: m = var('m')  # mass in Schwarzschild metric
sage: g[0,0], g[1,1] = -(1-2*m/r), 1/(1-2*m/r)
sage: g[2,2], g[3,3] = r^2, (r*sin(th))^2
sage: g.display()
g = (2*m/r - 1) dt⊗dt - 1/(2*m/r - 1) dr⊗dr + r^2 dth⊗dth
 + r^2*sin(th)^2 dph⊗dph
sage: nab = g.connection() ; nab
Levi-Civita connection nabla_g associated with the Lorentzian
 metric g on the 4-dimensional differentiable manifold M
sage: ric = nab.ricci() ; ric
Field of symmetric bilinear forms Ric(g) on the 4-dimensional
 differentiable manifold M
sage: ric == 0
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M')
>>> c_BL = M.chart(r't r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi', names=('t
↪', 'r', 'th', 'ph',)); (t, r, th, ph,) = c_BL._first_ngens(4)#␣
↪Schwarzschild-Droste coordinates
>>> g = M.lorentzian_metric('g')
>>> m = var('m')  # mass in Schwarzschild metric
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)] = -(Integer(1)-
```

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```
↪Integer(2)*m/r), Integer(1)/(Integer(1)-Integer(2)*m/r)
>>> g[Integer(2),Integer(2)], g[Integer(3),Integer(3)] = r**Integer(2),␣
↪(r*sin(th))**Integer(2)
>>> g.display()
g = (2*m/r - 1) dt⊗dt - 1/(2*m/r - 1) dr⊗dr + r^2 dth⊗dth
 + r^2*sin(th)^2 dph⊗dph
>>> nab = g.connection() ; nab
Levi-Civita connection nabla_g associated with the Lorentzian
 metric g on the 4-dimensional differentiable manifold M
>>> ric = nab.ricci() ; ric
Field of symmetric bilinear forms Ric(g) on the 4-dimensional
 differentiable manifold M
>>> ric == Integer(0)
True
```

riemann(*name=None*, *latex_name=None*)

Return the Riemann curvature tensor of the connection.

This method redefines `sage.manifolds.differentiable.affine_connection.AffineConnection.riemann()` to take into account the symmetry of the Riemann tensor for a Levi-Civita connection.

The Riemann curvature tensor is the tensor field R of type (1,3) defined by

$$R(\omega, w, u, v) = \left\langle \omega, \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w \right\rangle$$

for any 1-form ω and any vector fields u, v and w.

INPUT:

- `name` – (default: `None`) name given to the Riemann tensor; if none, it is set to "Riem(g)", where "g" is the metric's name
- `latex_name` – (default: `None`) LaTeX symbol to denote the Riemann tensor; if none, it is set to "\mathrm{Riem}(g)", where "g" is the metric's name

OUTPUT:

- the Riemann curvature tensor R, as an instance of `TensorField`

EXAMPLES:

Riemann tensor of the Levi-Civita connection associated with the metric of the hyperbolic plane (Poincaré disk model):

```
sage: M = Manifold(2, 'M', start_index=1)
sage: X.<x,y> = M.chart('x:(-1,1) y:(-1,1)', coord_restrictions=lambda x,y: x^
↪2+y^2<1)
....:   # Cartesian coord. on the Poincaré disk
sage: g = M.metric('g')
sage: g[1,1], g[2,2] = 4/(1-x^2-y^2)^2, 4/(1-x^2-y^2)^2
sage: nab = g.connection()
sage: riem = nab.riemann(); riem
Tensor field Riem(g) of type (1,3) on the 2-dimensional
 differentiable manifold M
sage: riem.display_comp()
Riem(g)^x_yxy = -4/(x^4 + y^4 + 2*(x^2 - 1)*y^2 - 2*x^2 + 1)
```

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```
Riem(g)^x_yyx = 4/(x^4 + y^4 + 2*(x^2 - 1)*y^2 - 2*x^2 + 1)
Riem(g)^y_xxy = 4/(x^4 + y^4 + 2*(x^2 - 1)*y^2 - 2*x^2 + 1)
Riem(g)^y_xyx = -4/(x^4 + y^4 + 2*(x^2 - 1)*y^2 - 2*x^2 + 1)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', start_index=Integer(1))
>>> X = M.chart('x:(-1,1) y:(-1,1)', coord_restrictions=lambda x,y:␣
↪x**Integer(2)+y**Integer(2)<Integer(1), names=('x', 'y',)); (x, y,) = X._
↪first_ngens(2)
...    # Cartesian coord. on the Poincaré disk
>>> g = M.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)] = Integer(4)/
↪(Integer(1)-x**Integer(2)-y**Integer(2))**Integer(2), Integer(4)/
↪(Integer(1)-x**Integer(2)-y**Integer(2))**Integer(2)
>>> nab = g.connection()
>>> riem = nab.riemann(); riem
Tensor field Riem(g) of type (1,3) on the 2-dimensional
 differentiable manifold M
>>> riem.display_comp()
Riem(g)^x_yxy = -4/(x^4 + y^4 + 2*(x^2 - 1)*y^2 - 2*x^2 + 1)
Riem(g)^x_yyx = 4/(x^4 + y^4 + 2*(x^2 - 1)*y^2 - 2*x^2 + 1)
Riem(g)^y_xxy = 4/(x^4 + y^4 + 2*(x^2 - 1)*y^2 - 2*x^2 + 1)
Riem(g)^y_xyx = -4/(x^4 + y^4 + 2*(x^2 - 1)*y^2 - 2*x^2 + 1)
```

The same computation parallelized on 2 cores:

```
sage: Parallelism().set(nproc=2)
sage: riem_backup = riem
sage: g = M.metric('g')
sage: g[1,1], g[2,2] = 4/(1-x^2-y^2)^2, 4/(1-x^2-y^2)^2
sage: nab = g.connection()
sage: riem = nab.riemann(); riem
Tensor field Riem(g) of type (1,3) on the 2-dimensional
 differentiable manifold M
sage: riem == riem_backup
True
sage: Parallelism().set(nproc=1)  # switch off parallelization
```

```
>>> from sage.all import *
>>> Parallelism().set(nproc=Integer(2))
>>> riem_backup = riem
>>> g = M.metric('g')
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)] = Integer(4)/
↪(Integer(1)-x**Integer(2)-y**Integer(2))**Integer(2), Integer(4)/
↪(Integer(1)-x**Integer(2)-y**Integer(2))**Integer(2)
>>> nab = g.connection()
>>> riem = nab.riemann(); riem
Tensor field Riem(g) of type (1,3) on the 2-dimensional
 differentiable manifold M
>>> riem == riem_backup
True
>>> Parallelism().set(nproc=Integer(1))  # switch off parallelization
```

torsion()

Return the connection's torsion tensor (identically zero for a Levi-Civita connection).

See *sage.manifolds.differentiable.affine_connection.AffineConnection.torsion()* for the general definition of the torsion tensor.

OUTPUT:

- the torsion tensor T, as a vanishing instance of *TensorField*

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: g = M.metric('g')
sage: g[0,0], g[1,1] = 1+y^2, 1+x^2
sage: nab = g.connection()
sage: t = nab.torsion(); t
Tensor field of type (1,2) on the 2-dimensional differentiable
 manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> g = M.metric('g')
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)] =␣
↪Integer(1)+y**Integer(2), Integer(1)+x**Integer(2)
>>> nab = g.connection()
>>> t = nab.torsion(); t
Tensor field of type (1,2) on the 2-dimensional differentiable
 manifold M
```

The torsion of a Levi-Civita connection is always zero:

```
sage: t.display()
0
```

```
>>> from sage.all import *
>>> t.display()
0
```

3.5 Pseudo-Riemannian submanifolds

An *embedded (resp. immersed) submanifold of a pseudo-Riemannian manifold* (M, g) is an embedded (resp. immersed) submanifold N of M as a differentiable manifold (see *differentiable_submanifold*) such that pull back of the metric tensor g via the embedding (resp. immersion) endows N with the structure of a pseudo-Riemannian manifold.

The following example shows how to compute the various quantities related to the intrinsic and extrinsic geometries of a hyperbolic slicing of the 3-dimensional Minkowski space.

We start by declaring the ambient manifold M and the submanifold N:

```
sage: M = Manifold(3, 'M', structure='Lorentzian')
sage: N = Manifold(2, 'N', ambient=M, structure='Riemannian', start_index=1)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='Lorentzian')
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='Riemannian', start_
↪index=Integer(1))
```

The considered slices being spacelike hypersurfaces, they are Riemannian manifolds.

Let us introduce the Minkowskian coordinates (w, x, y) on M and the polar coordinates (ρ, θ) on the submanifold N:

```
sage: E.<w,x,y> = M.chart()
sage: C.<rh,th> = N.chart(r'rh:(0,+oo):\rho th:(0,2*pi):\theta')
```

```
>>> from sage.all import *
>>> E = M.chart(names=('w', 'x', 'y',)); (w, x, y,) = E._first_ngens(3)
>>> C = N.chart(r'rh:(0,+oo):\rho th:(0,2*pi):\theta', names=('rh', 'th',)); (rh, th,
↪) = C._first_ngens(2)
```

Let b be the hyperbola semi-major axis and t the parameter of the foliation:

```
sage: b = var('b', domain='real')
sage: assume(b>0)
sage: t = var('t', domain='real')
```

```
>>> from sage.all import *
>>> b = var('b', domain='real')
>>> assume(b>Integer(0))
>>> t = var('t', domain='real')
```

One can then define the embedding ϕ_t:

```
sage: phi = N.diff_map(M, {(C,E): [b*cosh(rh)+t,
....:                              b*sinh(rh)*cos(th),
....:                              b*sinh(rh)*sin(th)]})
sage: phi.display()
N → M
   (rh, th) ↦ (w, x, y) = (b*cosh(rh) + t, b*cos(th)*sinh(rh),
                            b*sin(th)*sinh(rh))
```

```
>>> from sage.all import *
>>> phi = N.diff_map(M, {(C,E): [b*cosh(rh)+t,
...                              b*sinh(rh)*cos(th),
...                              b*sinh(rh)*sin(th)]})
>>> phi.display()
N → M
   (rh, th) ↦ (w, x, y) = (b*cosh(rh) + t, b*cos(th)*sinh(rh),
                            b*sin(th)*sinh(rh))
```

as well as its inverse (when considered as a diffeomorphism onto its image):

```
sage: phi_inv = M.diff_map(N, {(E,C): [log(sqrt(x^2+y^2+b^2)/b+
....:                                  sqrt((x^2+y^2+b^2)/b^2-1)),
....:                                  atan2(y,x)]})
sage: phi_inv.display()
```

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```
M → N
   (w, x, y) ↦ (rh, th) = (log(sqrt((b^2 + x^2 + y^2)/b^2 - 1)
                                  + sqrt(b^2 + x^2 + y^2)/b), arctan2(y, x))
```

```
>>> from sage.all import *
>>> phi_inv = M.diff_map(N, {(E,C):␣
↪[log(sqrt(x**Integer(2)+y**Integer(2)+b**Integer(2))/b+
...                                  sqrt((x**Integer(2)+y**Integer(2)+b**Integer(2))/
↪b**Integer(2)-Integer(1))),
...                                  atan2(y,x)]})
>>> phi_inv.display()
M → N
   (w, x, y) ↦ (rh, th) = (log(sqrt((b^2 + x^2 + y^2)/b^2 - 1)
                                  + sqrt(b^2 + x^2 + y^2)/b), arctan2(y, x))
```

and the partial inverse expressing the foliation parameter t as a scalar field on M:

```
sage: phi_inv_t = M.scalar_field({E: w-sqrt(x^2+y^2+b^2)})
sage: phi_inv_t.display()
M → ℝ
(w, x, y) ↦ w - sqrt(b^2 + x^2 + y^2)
```

```
>>> from sage.all import *
>>> phi_inv_t = M.scalar_field({E: w-sqrt(x**Integer(2)+y**Integer(2)+b**Integer(2))})
>>> phi_inv_t.display()
M → ℝ
(w, x, y) ↦ w - sqrt(b^2 + x^2 + y^2)
```

One can check that the inverse is correct with:

```
sage: (phi*phi_inv).display()
M → M
   (w, x, y) ↦ ((b^2 + x^2 + y^2 + sqrt(b^2 + x^2 + y^2)*(t + sqrt(x^2 +
 y^2)) + sqrt(x^2 + y^2)*t)/(sqrt(b^2 + x^2 + y^2) + sqrt(x^2 + y^2)), x, y)
```

```
>>> from sage.all import *
>>> (phi*phi_inv).display()
M → M
   (w, x, y) ↦ ((b^2 + x^2 + y^2 + sqrt(b^2 + x^2 + y^2)*(t + sqrt(x^2 +
 y^2)) + sqrt(x^2 + y^2)*t)/(sqrt(b^2 + x^2 + y^2) + sqrt(x^2 + y^2)), x, y)
```

The first item of the 3-uple in the right-hand does not appear as w because t has not been replaced by its value provided by `phi_inv_t`. Once this is done, we do get w:

```
sage: (phi*phi_inv).expr()[0].subs({t: phi_inv_t.expr()}).simplify_full()
w
```

```
>>> from sage.all import *
>>> (phi*phi_inv).expr()[Integer(0)].subs({t: phi_inv_t.expr()}).simplify_full()
w
```

The embedding can then be declared:

```
sage: N.set_embedding(phi, inverse=phi_inv, var=t,
....:                 t_inverse = {t: phi_inv_t})
```

```
>>> from sage.all import *
>>> N.set_embedding(phi, inverse=phi_inv, var=t,
...                 t_inverse = {t: phi_inv_t})
```

This line does not perform any calculation yet. It just check the coherence of the arguments, but not the inverse, the user is trusted on this point.

Finally, we initialize the metric of M to be that of Minkowski space:

```
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2] = -1, 1, 1
sage: g.display()
g = -dw⊗dw + dx⊗dx + dy⊗dy
```

```
>>> from sage.all import *
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),Integer(2)] = -
↪Integer(1), Integer(1), Integer(1)
>>> g.display()
g = -dw⊗dw + dx⊗dx + dy⊗dy
```

With this, the declaration the ambient manifold and its foliation parametrized by t is finished, and calculations can be performed.

The first step is always to find a chart adapted to the foliation. This is done by the method "adapted_chart":

```
sage: T = N.adapted_chart(); T
[Chart (M, (rh_M, th_M, t_M))]
```

```
>>> from sage.all import *
>>> T = N.adapted_chart(); T
[Chart (M, (rh_M, th_M, t_M))]
```

`T` contains a new chart defined on M. By default, the coordinate names are constructed from the names of the submanifold coordinates and the foliation parameter indexed by the name of the ambient manifold. By this can be customized, see *adapted_chart()*.

One can check that the adapted chart has been added to M's atlas, along with some coordinates changes:

```
sage: M.atlas()
[Chart (M, (w, x, y)), Chart (M, (rh_M, th_M, t_M))]
sage: len(M.coord_changes())
2
```

```
>>> from sage.all import *
>>> M.atlas()
[Chart (M, (w, x, y)), Chart (M, (rh_M, th_M, t_M))]
>>> len(M.coord_changes())
2
```

Let us compute the induced metric (or first fundamental form):

```
sage: # long time
sage: gamma = N.induced_metric()
sage: gamma.display()
gamma = b^2 drh⊗drh + b^2*sinh(rh)^2 dth⊗dth
sage: gamma[:]
[           b^2              0]
[             0 b^2*sinh(rh)^2]
sage: gamma[1,1]
b^2
```

```
>>> from sage.all import *
>>> # long time
>>> gamma = N.induced_metric()
>>> gamma.display()
gamma = b^2 drh⊗drh + b^2*sinh(rh)^2 dth⊗dth
>>> gamma[:]
[           b^2              0]
[             0 b^2*sinh(rh)^2]
>>> gamma[Integer(1),Integer(1)]
b^2
```

the normal vector:

```
sage: N.normal().display()  # long time
n = sqrt(b^2 + x^2 + y^2)/b ∂/∂w + x/b ∂/∂x + y/b ∂/∂y
```

```
>>> from sage.all import *
>>> N.normal().display()  # long time
n = sqrt(b^2 + x^2 + y^2)/b ∂/∂w + x/b ∂/∂x + y/b ∂/∂y
```

Check that the hypersurface is indeed spacelike, i.e. that its normal is timelike:

```
sage: N.ambient_metric()(N.normal(), N.normal()).display()  # long time
g(n,n): M → ℝ
   (w, x, y) ↦ -1
   (rh_M, th_M, t_M) ↦ -1
```

```
>>> from sage.all import *
>>> N.ambient_metric()(N.normal(), N.normal()).display()  # long time
g(n,n): M → ℝ
   (w, x, y) ↦ -1
   (rh_M, th_M, t_M) ↦ -1
```

The lapse function is:

```
sage: N.lapse().display()  # long time
N: M → ℝ
   (w, x, y) ↦ sqrt(b^2 + x^2 + y^2)/b
   (rh_M, th_M, t_M) ↦ cosh(rh_M)
```

```
>>> from sage.all import *
>>> N.lapse().display()  # long time
```

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```
N: M → ℝ
   (w, x, y) ↦ sqrt(b^2 + x^2 + y^2)/b
   (rh_M, th_M, t_M) ↦ cosh(rh_M)
```

while the shift vector is:

```
sage: N.shift().display()  # long time
beta = -(x^2 + y^2)/b^2 ∂/∂w - sqrt(b^2 + x^2 + y^2)*x/b^2 ∂/∂x
 - sqrt(b^2 + x^2 + y^2)*y/b^2 ∂/∂y
```

```
>>> from sage.all import *
>>> N.shift().display()  # long time
beta = -(x^2 + y^2)/b^2 ∂/∂w - sqrt(b^2 + x^2 + y^2)*x/b^2 ∂/∂x
 - sqrt(b^2 + x^2 + y^2)*y/b^2 ∂/∂y
```

The extrinsic curvature (or second fundamental form) as a tensor field on the ambient manifold:

```
sage: N.ambient_extrinsic_curvature()[:] # long time
[                                 -(x^2 + y^2)/b^3 (b^2*x + x^3 + x*y^2)/(sqrt(b^2 +␣
↪x^2 + y^2)*b^3) (y^3 + (b^2 + x^2)*y)/(sqrt(b^2 + x^2 + y^2)*b^3)]
[                      sqrt(b^2 + x^2 + y^2)*x/b^3                                  -
↪(b^2 + x^2)/b^3                                    -x*y/b^3]
[                      sqrt(b^2 + x^2 + y^2)*y/b^3                                  ␣
↪      -x*y/b^3                                 -(b^2 + y^2)/b^3]
```

```
>>> from sage.all import *
>>> N.ambient_extrinsic_curvature()[:] # long time
[                                 -(x^2 + y^2)/b^3 (b^2*x + x^3 + x*y^2)/(sqrt(b^2 +␣
↪x^2 + y^2)*b^3) (y^3 + (b^2 + x^2)*y)/(sqrt(b^2 + x^2 + y^2)*b^3)]
[                      sqrt(b^2 + x^2 + y^2)*x/b^3                                  -
↪(b^2 + x^2)/b^3                                    -x*y/b^3]
[                      sqrt(b^2 + x^2 + y^2)*y/b^3                                  ␣
↪      -x*y/b^3                                 -(b^2 + y^2)/b^3]
```

The extrinsic curvature as a tensor field on the submanifold:

```
sage: N.extrinsic_curvature()[:] # long time
[           -b              0]
[            0 -b*sinh(rh)^2]
```

```
>>> from sage.all import *
>>> N.extrinsic_curvature()[:] # long time
[           -b              0]
[            0 -b*sinh(rh)^2]
```

AUTHORS:

- Florentin Jaffredo (2018): initial version
- Eric Gourgoulhon (2018-2019): add documentation
- Matthias Koeppe (2021): open subsets of submanifolds

REFERENCES:

- B. O'Neill : *Semi-Riemannian Geometry* [ONe1983]
- J. M. Lee : *Riemannian Manifolds* [Lee1997]

class sage.manifolds.differentiable.pseudo_riemannian_submanifold.**PseudoRiemannianSubmanifold**(*n*, *name*, *ambient=None*, *metric_name*, *signature=None*, *base_manifold=None*, *diff_degree=+Infinity*, *latex_name*, *metric_latex_name*, *start_index=0*, *category=None*, *unique_t*

Bases: *PseudoRiemannianManifold*, *DifferentiableSubmanifold*

Pseudo-Riemannian submanifold.

An *embedded (resp. immersed) submanifold of a pseudo-Riemannian manifold* (M, g) is an embedded (resp. immersed) submanifold N of M as a differentiable manifold such that pull back of the metric tensor g via the embedding (resp. immersion) endows N with the structure of a pseudo-Riemannian manifold.

INPUT:

- n – positive integer; dimension of the submanifold
- name – string; name (symbol) given to the submanifold
- ambient – (default: None) pseudo-Riemannian manifold M in which the submanifold is embedded (or immersed). If None, it is set to self
- metric_name – (default: None) string; name (symbol) given to the metric; if None, 'gamma' is used
- signature – (default: None) signature S of the metric as a single integer: $S = n_+ - n_-$, where n_+ (resp. n_-) is the number of positive terms (resp. number of negative terms) in any diagonal writing of the metric components; if signature is not provided, S is set to the submanifold's dimension (Riemannian signature)
- base_manifold – (default: None) if not None, must be a differentiable manifold; the created object is then an open subset of base_manifold
- diff_degree – (default: infinity) degree of differentiability

- `latex_name` – (default: `None`) string; LaTeX symbol to denote the submanifold; if none is provided, it is set to `name`
- `metric_latex_name` – (default: `None`) string; LaTeX symbol to denote the metric; if none is provided, it is set to `metric_name` if the latter is not `None` and to `r'\gamma'` otherwise
- `start_index` – (default: 0) integer; lower value of the range of indices used for "indexed objects" on the submanifold, e.g. coordinates in a chart
- `category` – (default: `None`) to specify the category; if `None`, `Manifolds(RR).Differentiable()` (or `Manifolds(RR).Smooth()` if `diff_degree = infinity`) is assumed (see the category `Manifolds`)
- `unique_tag` – (default: `None`) tag used to force the construction of a new object when all the other arguments have been used previously (without `unique_tag`, the `UniqueRepresentation` behavior inherited from `ManifoldSubset`, via `DifferentiableManifold` and `TopologicalManifold`, would return the previously constructed object corresponding to these arguments).

EXAMPLES:

Let N be a 2-dimensional submanifold of a 3-dimensional Riemannian manifold M:

```
sage: M = Manifold(3, 'M', structure ='Riemannian')
sage: N = Manifold(2, 'N', ambient=M, structure='Riemannian')
sage: N
2-dimensional Riemannian submanifold N immersed in the 3-dimensional
 Riemannian manifold M
sage: CM.<x,y,z> = M.chart()
sage: CN.<u,v> = N.chart()
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure ='Riemannian')
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='Riemannian')
>>> N
2-dimensional Riemannian submanifold N immersed in the 3-dimensional
 Riemannian manifold M
>>> CM = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = CM._first_ngens(3)
>>> CN = N.chart(names=('u', 'v',)); (u, v,) = CN._first_ngens(2)
```

Let us define a 1-dimension foliation indexed by t. The inverse map is needed in order to compute the adapted chart in the ambient manifold:

```
sage: t = var('t')
sage: phi = N.diff_map(M, {(CN,CM):[u, v, t+u^2+v^2]}); phi
Differentiable map from the 2-dimensional Riemannian submanifold N
 immersed in the 3-dimensional Riemannian manifold M to the
 3-dimensional Riemannian manifold M
sage: phi_inv = M.diff_map(N,{(CM, CN): [x,y]})
sage: phi_inv_t = M.scalar_field({CM: z-x^2-y^2})
```

```
>>> from sage.all import *
>>> t = var('t')
>>> phi = N.diff_map(M, {(CN,CM):[u, v, t+u**Integer(2)+v**Integer(2)]}); phi
Differentiable map from the 2-dimensional Riemannian submanifold N
 immersed in the 3-dimensional Riemannian manifold M to the
 3-dimensional Riemannian manifold M
```

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```
>>> phi_inv = M.diff_map(N,{(CM, CN): [x,y]})
>>> phi_inv_t = M.scalar_field({CM: z-x**Integer(2)-y**Integer(2)})
```

ϕ can then be declared as an embedding $N \to M$:

```
sage: N.set_embedding(phi, inverse=phi_inv, var=t,
....:                 t_inverse={t: phi_inv_t})
```

```
>>> from sage.all import *
>>> N.set_embedding(phi, inverse=phi_inv, var=t,
...                 t_inverse={t: phi_inv_t})
```

The foliation can also be used to find new charts on the ambient manifold that are adapted to the foliation, ie in which the expression of the immersion is trivial. At the same time, the appropriate coordinate changes are computed:

```
sage: N.adapted_chart()
[Chart (M, (u_M, v_M, t_M))]
sage: len(M.coord_changes())
2
```

```
>>> from sage.all import *
>>> N.adapted_chart()
[Chart (M, (u_M, v_M, t_M))]
>>> len(M.coord_changes())
2
```

See also

manifold and *differentiable_submanifold*

ambient_extrinsic_curvature()

Return the second fundamental form of the submanifold as a tensor field on the ambient manifold.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- (0,2) tensor field on the ambient manifold equal to the second fundamental form once orthogonally projected onto the submanifold

EXAMPLES:

A unit circle embedded in the Euclidean plane:

```
sage: M.<X,Y> = EuclideanSpace()
sage: N = Manifold(1, 'N', ambient=M, structure='Riemannian')
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
sage: N.declare_union(U,V)
sage: stereoN.<x> = U.chart()
sage: stereoS.<y> = V.chart()
sage: stereoN_to_S = stereoN.transition_map(stereoS, (4/x),
....:                 intersection_name='W',
```

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```
....:                     restrictions1=x!=0, restrictions2=y!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M,
....:          {(stereoN, E): [1/sqrt(1+x^2/4), x/2/sqrt(1+x^2/4)],
....:           (stereoS, E): [1/sqrt(1+4/y^2), 2/y/sqrt(1+4/y^2)]})
sage: N.set_embedding(phi)
sage: N.ambient_second_fundamental_form()  # long time
Field of symmetric bilinear forms K along the 1-dimensional
 Riemannian submanifold N embedded in the Euclidean plane E^2 with
 values on the Euclidean plane E^2
sage: N.ambient_second_fundamental_form()[:] # long time
[-x^2/(x^2 + 4)  2*x/(x^2 + 4)]
[ 2*x/(x^2 + 4)   -4/(x^2 + 4)]
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('X', 'Y',)); (X, Y,) = M._first_ngens(2)
>>> N = Manifold(Integer(1), 'N', ambient=M, structure='Riemannian')
>>> U = N.open_subset('U')
>>> V = N.open_subset('V')
>>> N.declare_union(U,V)
>>> stereoN = U.chart(names=('x',)); (x,) = stereoN._first_ngens(1)
>>> stereoS = V.chart(names=('y',)); (y,) = stereoS._first_ngens(1)
>>> stereoN_to_S = stereoN.transition_map(stereoS, (Integer(4)/x),
...                     intersection_name='W',
...                     restrictions1=x!=Integer(0), restrictions2=y!
↪=Integer(0))
>>> stereoS_to_N = stereoN_to_S.inverse()
>>> E = M.cartesian_coordinates()
>>> phi = N.diff_map(M,
...          {(stereoN, E): [Integer(1)/sqrt(Integer(1)+x**Integer(2)/
↪Integer(4)), x/Integer(2)/sqrt(Integer(1)+x**Integer(2)/Integer(4))],
...           (stereoS, E): [Integer(1)/sqrt(Integer(1)+Integer(4)/
↪y**Integer(2)), Integer(2)/y/sqrt(Integer(1)+Integer(4)/y**Integer(2))]})
>>> N.set_embedding(phi)
>>> N.ambient_second_fundamental_form()  # long time
Field of symmetric bilinear forms K along the 1-dimensional
 Riemannian submanifold N embedded in the Euclidean plane E^2 with
 values on the Euclidean plane E^2
>>> N.ambient_second_fundamental_form()[:] # long time
[-x^2/(x^2 + 4)  2*x/(x^2 + 4)]
[ 2*x/(x^2 + 4)   -4/(x^2 + 4)]
```

An alias is `ambient_extrinsic_curvature`:

```
sage: N.ambient_extrinsic_curvature()[:]  # long time
[-x^2/(x^2 + 4)  2*x/(x^2 + 4)]
[ 2*x/(x^2 + 4)   -4/(x^2 + 4)]
```

```
>>> from sage.all import *
>>> N.ambient_extrinsic_curvature()[:]  # long time
[-x^2/(x^2 + 4)  2*x/(x^2 + 4)]
```

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```
[ 2*x/(x^2 + 4)   -4/(x^2 + 4)]
```

ambient_first_fundamental_form()

Return the first fundamental form of the submanifold as a tensor of the ambient manifold.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- (0,2) tensor field on the ambient manifold describing the induced metric before projection on the submanifold

EXAMPLES:

A unit circle embedded in the Euclidean plane:

```
sage: M.<X,Y> = EuclideanSpace()
sage: N = Manifold(1, 'N', ambient=M, structure='Riemannian')
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
sage: N.declare_union(U,V)
sage: stereoN.<x> = U.chart()
sage: stereoS.<y> = V.chart()
sage: stereoN_to_S = stereoN.transition_map(stereoS, (4/x),
....:                    intersection_name='W',
....:                    restrictions1=x!=0, restrictions2=y!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M,
....:          {(stereoN, E): [1/sqrt(1+x^2/4), x/2/sqrt(1+x^2/4)],
....:           (stereoS, E): [1/sqrt(1+4/y^2), 2/y/sqrt(1+4/y^2)]})
sage: N.set_embedding(phi)
sage: N.ambient_first_fundamental_form()
Tensor field gamma of type (0,2) along the 1-dimensional Riemannian
 submanifold N embedded in the Euclidean plane E^2 with values on
 the Euclidean plane E^2
sage: N.ambient_first_fundamental_form()[:]
[ x^2/(x^2 + 4) -2*x/(x^2 + 4)]
[-2*x/(x^2 + 4)    4/(x^2 + 4)]
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('X', 'Y',)); (X, Y,) = M._first_ngens(2)
>>> N = Manifold(Integer(1), 'N', ambient=M, structure='Riemannian')
>>> U = N.open_subset('U')
>>> V = N.open_subset('V')
>>> N.declare_union(U,V)
>>> stereoN = U.chart(names=('x',)); (x,) = stereoN._first_ngens(1)
>>> stereoS = V.chart(names=('y',)); (y,) = stereoS._first_ngens(1)
>>> stereoN_to_S = stereoN.transition_map(stereoS, (Integer(4)/x),
...                    intersection_name='W',
...                    restrictions1=x!=Integer(0), restrictions2=y!
↪=Integer(0))
>>> stereoS_to_N = stereoN_to_S.inverse()
>>> E = M.cartesian_coordinates()
```

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```
>>> phi = N.diff_map(M,
...           {(stereoN, E): [Integer(1)/sqrt(Integer(1)+x**Integer(2)/
↪Integer(4)), x/Integer(2)/sqrt(Integer(1)+x**Integer(2)/Integer(4))],
...            (stereoS, E): [Integer(1)/sqrt(Integer(1)+Integer(4)/
↪y**Integer(2)), Integer(2)/y/sqrt(Integer(1)+Integer(4)/y**Integer(2))]})
>>> N.set_embedding(phi)
>>> N.ambient_first_fundamental_form()
Tensor field gamma of type (0,2) along the 1-dimensional Riemannian
 submanifold N embedded in the Euclidean plane E^2 with values on
 the Euclidean plane E^2
>>> N.ambient_first_fundamental_form()[:]
[ x^2/(x^2 + 4) -2*x/(x^2 + 4)]
[-2*x/(x^2 + 4)     4/(x^2 + 4)]
```

An alias is `ambient_induced_metric`:

```
sage: N.ambient_induced_metric()[:]
[ x^2/(x^2 + 4) -2*x/(x^2 + 4)]
[-2*x/(x^2 + 4)     4/(x^2 + 4)]
```

```
>>> from sage.all import *
>>> N.ambient_induced_metric()[:]
[ x^2/(x^2 + 4) -2*x/(x^2 + 4)]
[-2*x/(x^2 + 4)     4/(x^2 + 4)]
```

ambient_induced_metric()

Return the first fundamental form of the submanifold as a tensor of the ambient manifold.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- (0,2) tensor field on the ambient manifold describing the induced metric before projection on the submanifold

EXAMPLES:

A unit circle embedded in the Euclidean plane:

```
sage: M.<X,Y> = EuclideanSpace()
sage: N = Manifold(1, 'N', ambient=M, structure='Riemannian')
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
sage: N.declare_union(U,V)
sage: stereoN.<x> = U.chart()
sage: stereoS.<y> = V.chart()
sage: stereoN_to_S = stereoN.transition_map(stereoS, (4/x),
....:                  intersection_name='W',
....:                  restrictions1=x!=0, restrictions2=y!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M,
....:         {(stereoN, E): [1/sqrt(1+x^2/4), x/2/sqrt(1+x^2/4)],
....:          (stereoS, E): [1/sqrt(1+4/y^2), 2/y/sqrt(1+4/y^2)]})
```

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```
sage: N.set_embedding(phi)
sage: N.ambient_first_fundamental_form()
Tensor field gamma of type (0,2) along the 1-dimensional Riemannian
 submanifold N embedded in the Euclidean plane E^2 with values on
 the Euclidean plane E^2
sage: N.ambient_first_fundamental_form()[:]
[ x^2/(x^2 + 4) -2*x/(x^2 + 4)]
[-2*x/(x^2 + 4)     4/(x^2 + 4)]
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('X', 'Y',)); (X, Y,) = M._first_ngens(2)
>>> N = Manifold(Integer(1), 'N', ambient=M, structure='Riemannian')
>>> U = N.open_subset('U')
>>> V = N.open_subset('V')
>>> N.declare_union(U,V)
>>> stereoN = U.chart(names=('x',)); (x,) = stereoN._first_ngens(1)
>>> stereoS = V.chart(names=('y',)); (y,) = stereoS._first_ngens(1)
>>> stereoN_to_S = stereoN.transition_map(stereoS, (Integer(4)/x),
...                   intersection_name='W',
...                   restrictions1=x!=Integer(0), restrictions2=y!
↪=Integer(0))
>>> stereoS_to_N = stereoN_to_S.inverse()
>>> E = M.cartesian_coordinates()
>>> phi = N.diff_map(M,
...         {(stereoN, E): [Integer(1)/sqrt(Integer(1)+x**Integer(2)/
↪Integer(4)), x/Integer(2)/sqrt(Integer(1)+x**Integer(2)/Integer(4))],
...          (stereoS, E): [Integer(1)/sqrt(Integer(1)+Integer(4)/
↪y**Integer(2)), Integer(2)/y/sqrt(Integer(1)+Integer(4)/y**Integer(2))]})
>>> N.set_embedding(phi)
>>> N.ambient_first_fundamental_form()
Tensor field gamma of type (0,2) along the 1-dimensional Riemannian
 submanifold N embedded in the Euclidean plane E^2 with values on
 the Euclidean plane E^2
>>> N.ambient_first_fundamental_form()[:]
[ x^2/(x^2 + 4) -2*x/(x^2 + 4)]
[-2*x/(x^2 + 4)     4/(x^2 + 4)]
```

An alias is `ambient_induced_metric`:

```
sage: N.ambient_induced_metric()[:]
[ x^2/(x^2 + 4) -2*x/(x^2 + 4)]
[-2*x/(x^2 + 4)     4/(x^2 + 4)]
```

```
>>> from sage.all import *
>>> N.ambient_induced_metric()[:]
[ x^2/(x^2 + 4) -2*x/(x^2 + 4)]
[-2*x/(x^2 + 4)     4/(x^2 + 4)]
```

ambient_metric()

Return the metric of the ambient manifold.

OUTPUT: the metric of the ambient manifold

EXAMPLES:

```
sage: M.<x,y,z> = EuclideanSpace()
sage: N = Manifold(2, 'N', ambient=M, structure='Riemannian')
sage: N.ambient_metric()
Riemannian metric g on the Euclidean space E^3
sage: N.ambient_metric().display()
g = dx⊗dx + dy⊗dy + dz⊗dz
sage: N.ambient_metric() is M.metric()
True
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y', 'z',)); (x, y, z,) = M._first_ngens(3)
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='Riemannian')
>>> N.ambient_metric()
Riemannian metric g on the Euclidean space E^3
>>> N.ambient_metric().display()
g = dx⊗dx + dy⊗dy + dz⊗dz
>>> N.ambient_metric() is M.metric()
True
```

ambient_second_fundamental_form()

Return the second fundamental form of the submanifold as a tensor field on the ambient manifold.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- (0,2) tensor field on the ambient manifold equal to the second fundamental form once orthogonally projected onto the submanifold

EXAMPLES:

A unit circle embedded in the Euclidean plane:

```
sage: M.<X,Y> = EuclideanSpace()
sage: N = Manifold(1, 'N', ambient=M, structure='Riemannian')
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
sage: N.declare_union(U,V)
sage: stereoN.<x> = U.chart()
sage: stereoS.<y> = V.chart()
sage: stereoN_to_S = stereoN.transition_map(stereoS, (4/x),
....:                     intersection_name='W',
....:                     restrictions1=x!=0, restrictions2=y!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M,
....:         {(stereoN, E): [1/sqrt(1+x^2/4), x/2/sqrt(1+x^2/4)],
....:          (stereoS, E): [1/sqrt(1+4/y^2), 2/y/sqrt(1+4/y^2)]})
sage: N.set_embedding(phi)
sage: N.ambient_second_fundamental_form()  # long time
Field of symmetric bilinear forms K along the 1-dimensional
 Riemannian submanifold N embedded in the Euclidean plane E^2 with
 values on the Euclidean plane E^2
sage: N.ambient_second_fundamental_form()[:] # long time
```

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```
[-x^2/(x^2 + 4)  2*x/(x^2 + 4)]
[ 2*x/(x^2 + 4)   -4/(x^2 + 4)]
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('X', 'Y',)); (X, Y,) = M._first_ngens(2)
>>> N = Manifold(Integer(1), 'N', ambient=M, structure='Riemannian')
>>> U = N.open_subset('U')
>>> V = N.open_subset('V')
>>> N.declare_union(U,V)
>>> stereoN = U.chart(names=('x',)); (x,) = stereoN._first_ngens(1)
>>> stereoS = V.chart(names=('y',)); (y,) = stereoS._first_ngens(1)
>>> stereoN_to_S = stereoN.transition_map(stereoS, (Integer(4)/x),
...                   intersection_name='W',
...                   restrictions1=x!=Integer(0), restrictions2=y!
↪=Integer(0))
>>> stereoS_to_N = stereoN_to_S.inverse()
>>> E = M.cartesian_coordinates()
>>> phi = N.diff_map(M,
...         {(stereoN, E): [Integer(1)/sqrt(Integer(1)+x**Integer(2)/
↪Integer(4)), x/Integer(2)/sqrt(Integer(1)+x**Integer(2)/Integer(4))],
...          (stereoS, E): [Integer(1)/sqrt(Integer(1)+Integer(4)/
↪y**Integer(2)), Integer(2)/y/sqrt(Integer(1)+Integer(4)/y**Integer(2))]})
>>> N.set_embedding(phi)
>>> N.ambient_second_fundamental_form()  # long time
Field of symmetric bilinear forms K along the 1-dimensional
 Riemannian submanifold N embedded in the Euclidean plane E^2 with
 values on the Euclidean plane E^2
>>> N.ambient_second_fundamental_form()[:] # long time
[-x^2/(x^2 + 4)  2*x/(x^2 + 4)]
[ 2*x/(x^2 + 4)   -4/(x^2 + 4)]
```

An alias is `ambient_extrinsic_curvature`:

```
sage: N.ambient_extrinsic_curvature()[:]  # long time
[-x^2/(x^2 + 4)  2*x/(x^2 + 4)]
[ 2*x/(x^2 + 4)   -4/(x^2 + 4)]
```

```
>>> from sage.all import *
>>> N.ambient_extrinsic_curvature()[:]  # long time
[-x^2/(x^2 + 4)  2*x/(x^2 + 4)]
[ 2*x/(x^2 + 4)   -4/(x^2 + 4)]
```

clear_cache()

Reset all the cached functions and the derived quantities.

Use this function if you modified the immersion (or embedding) of the submanifold. Note that when calling a calculus function after clearing, new Python objects will be created.

EXAMPLES:

```
sage: M.<x,y,z> = EuclideanSpace()
sage: N = Manifold(2, 'N', ambient=M, structure='Riemannian')
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
```

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```
sage: r = var('r', domain='real') # foliation parameter
sage: assume(r>0)
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M, {(C,E): [r*sin(th)*cos(ph),
....:                              r*sin(th)*sin(ph),
....:                              r*cos(th)]})
sage: phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
sage: phi_inv_r = M.scalar_field({E: sqrt(x^2+y^2+z^2)})
sage: N.set_embedding(phi, inverse=phi_inv, var=r,
....:                 t_inverse={r: phi_inv_r})
sage: T = N.adapted_chart()
sage: n = N.normal()
sage: n is N.normal()
True
sage: N.clear_cache()
sage: n is N.normal()
False
sage: n == N.normal()
True
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y', 'z',)); (x, y, z,) = M._first_ngens(3)
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='Riemannian')
>>> C = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi', names=('th', 'ph',));
↪(th, ph,) = C._first_ngens(2)
>>> r = var('r', domain='real') # foliation parameter
>>> assume(r>Integer(0))
>>> E = M.cartesian_coordinates()
>>> phi = N.diff_map(M, {(C,E): [r*sin(th)*cos(ph),
...                              r*sin(th)*sin(ph),
...                              r*cos(th)]})
>>> phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
>>> phi_inv_r = M.scalar_field({E:
↪sqrt(x**Integer(2)+y**Integer(2)+z**Integer(2))})
>>> N.set_embedding(phi, inverse=phi_inv, var=r,
...                 t_inverse={r: phi_inv_r})
>>> T = N.adapted_chart()
>>> n = N.normal()
>>> n is N.normal()
True
>>> N.clear_cache()
>>> n is N.normal()
False
>>> n == N.normal()
True
```

difft()

Return the differential of the scalar field on the ambient manifold representing the first parameter of the foliation associated to `self`.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT: 1-form field on the ambient manifold

EXAMPLES:

Foliation of the Euclidean 3-space by 2-spheres parametrized by their radii:

```
sage: M.<x,y,z> = EuclideanSpace()
sage: N = Manifold(2, 'N', ambient=M, structure='Riemannian')
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
sage: r = var('r', domain='real')
sage: assume(r>0)
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M, {(C,E): [r*sin(th)*cos(ph),
....:                              r*sin(th)*sin(ph),
....:                              r*cos(th)]})
sage: phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
sage: phi_inv_r = M.scalar_field({E: sqrt(x^2+y^2+z^2)})
sage: N.set_embedding(phi, inverse=phi_inv, var=r,
....:                 t_inverse={r: phi_inv_r})
sage: N.difft()
1-form dr on the Euclidean space E^3
sage: N.difft().display()
dr = x/sqrt(x^2 + y^2 + z^2) dx + y/sqrt(x^2 + y^2 + z^2) dy +
 z/sqrt(x^2 + y^2 + z^2) dz
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y', 'z',)); (x, y, z,) = M._first_ngens(3)
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='Riemannian')
>>> C = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi', names=('th', 'ph',));
↪(th, ph,) = C._first_ngens(2)
>>> r = var('r', domain='real')
>>> assume(r>Integer(0))
>>> E = M.cartesian_coordinates()
>>> phi = N.diff_map(M, {(C,E): [r*sin(th)*cos(ph),
...                              r*sin(th)*sin(ph),
...                              r*cos(th)]})
>>> phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
>>> phi_inv_r = M.scalar_field({E:
↪sqrt(x**Integer(2)+y**Integer(2)+z**Integer(2))})
>>> N.set_embedding(phi, inverse=phi_inv, var=r,
...                 t_inverse={r: phi_inv_r})
>>> N.difft()
1-form dr on the Euclidean space E^3
>>> N.difft().display()
dr = x/sqrt(x^2 + y^2 + z^2) dx + y/sqrt(x^2 + y^2 + z^2) dy +
 z/sqrt(x^2 + y^2 + z^2) dz
```

extrinsic_curvature()

Return the second fundamental form of the submanifold.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- the second fundamental form, as a symmetric tensor field of type (0,2) on the submanifold

EXAMPLES:

A unit circle embedded in the Euclidean plane:

```
sage: M.<X,Y> = EuclideanSpace()
sage: N = Manifold(1, 'N', ambient=M, structure='Riemannian')
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
sage: N.declare_union(U,V)
sage: stereoN.<x> = U.chart()
sage: stereoS.<y> = V.chart()
sage: stereoN_to_S = stereoN.transition_map(stereoS, (4/x),
....:                 intersection_name='W',
....:                 restrictions1=x!=0, restrictions2=y!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M,
....:         {(stereoN, E): [1/sqrt(1+x^2/4), x/2/sqrt(1+x^2/4)],
....:          (stereoS, E): [1/sqrt(1+4/y^2), 2/y/sqrt(1+4/y^2)]})
sage: N.set_embedding(phi)
sage: N.second_fundamental_form()  # long time
Field of symmetric bilinear forms K on the 1-dimensional Riemannian
 submanifold N embedded in the Euclidean plane E^2
sage: N.second_fundamental_form().display()  # long time
K = -4/(x^4 + 8*x^2 + 16) dx⊗dx
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('X', 'Y',)); (X, Y,) = M._first_ngens(2)
>>> N = Manifold(Integer(1), 'N', ambient=M, structure='Riemannian')
>>> U = N.open_subset('U')
>>> V = N.open_subset('V')
>>> N.declare_union(U,V)
>>> stereoN = U.chart(names=('x',)); (x,) = stereoN._first_ngens(1)
>>> stereoS = V.chart(names=('y',)); (y,) = stereoS._first_ngens(1)
>>> stereoN_to_S = stereoN.transition_map(stereoS, (Integer(4)/x),
...                 intersection_name='W',
...                 restrictions1=x!=Integer(0), restrictions2=y!
↪=Integer(0))
>>> stereoS_to_N = stereoN_to_S.inverse()
>>> E = M.cartesian_coordinates()
>>> phi = N.diff_map(M,
...         {(stereoN, E): [Integer(1)/sqrt(Integer(1)+x**Integer(2)/
↪Integer(4)), x/Integer(2)/sqrt(Integer(1)+x**Integer(2)/Integer(4))],
...          (stereoS, E): [Integer(1)/sqrt(Integer(1)+Integer(4)/
↪y**Integer(2)), Integer(2)/y/sqrt(Integer(1)+Integer(4)/y**Integer(2))]})
>>> N.set_embedding(phi)
>>> N.second_fundamental_form()  # long time
Field of symmetric bilinear forms K on the 1-dimensional Riemannian
 submanifold N embedded in the Euclidean plane E^2
>>> N.second_fundamental_form().display()  # long time
K = -4/(x^4 + 8*x^2 + 16) dx⊗dx
```

An alias is `extrinsic_curvature`:

```
sage: N.extrinsic_curvature().display()  # long time
K = -4/(x^4 + 8*x^2 + 16) dx⊗dx
```

```
>>> from sage.all import *
>>> N.extrinsic_curvature().display()  # long time
K = -4/(x^4 + 8*x^2 + 16) dx⊗dx
```

An example with a non-Euclidean ambient metric:

```
sage: M = Manifold(2, 'M', structure='Riemannian')
sage: N = Manifold(1, 'N', ambient=M, structure='Riemannian',
....:              start_index=1)
sage: CM.<x,y> = M.chart()
sage: CN.<u> = N.chart()
sage: g = M.metric()
sage: g[0, 0], g[1, 1] = 1, 1/(1 + y^2)^2
sage: phi = N.diff_map(M, (u, u))
sage: N.set_embedding(phi)
sage: N.second_fundamental_form()
Field of symmetric bilinear forms K on the 1-dimensional Riemannian
 submanifold N embedded in the 2-dimensional Riemannian manifold M
sage: N.second_fundamental_form().display()
K = 2*sqrt(u^4 + 2*u^2 + 2)*u/(u^6 + 3*u^4 + 4*u^2 + 2) du⊗du
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='Riemannian')
>>> N = Manifold(Integer(1), 'N', ambient=M, structure='Riemannian',
...              start_index=Integer(1))
>>> CM = M.chart(names=('x', 'y',)); (x, y,) = CM._first_ngens(2)
>>> CN = N.chart(names=('u',)); (u,) = CN._first_ngens(1)
>>> g = M.metric()
>>> g[Integer(0), Integer(0)], g[Integer(1), Integer(1)] = Integer(1),␣
↪Integer(1)/(Integer(1) + y**Integer(2))**Integer(2)
>>> phi = N.diff_map(M, (u, u))
>>> N.set_embedding(phi)
>>> N.second_fundamental_form()
Field of symmetric bilinear forms K on the 1-dimensional Riemannian
 submanifold N embedded in the 2-dimensional Riemannian manifold M
>>> N.second_fundamental_form().display()
K = 2*sqrt(u^4 + 2*u^2 + 2)*u/(u^6 + 3*u^4 + 4*u^2 + 2) du⊗du
```

first_fundamental_form()

Return the first fundamental form of the submanifold.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- the first fundamental form, as an instance of *PseudoRiemannianMetric*

EXAMPLES:

A sphere embedded in Euclidean space:

```
sage: M.<x,y,z> = EuclideanSpace()
sage: N = Manifold(2, 'N', ambient=M, structure='Riemannian')
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
sage: r = var('r', domain='real')
```

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```
sage: assume(r>0)
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M, {(C,E): [r*sin(th)*cos(ph),
....:                              r*sin(th)*sin(ph),
....:                              r*cos(th)]})
sage: N.set_embedding(phi)
sage: N.first_fundamental_form()  # long time
Riemannian metric gamma on the 2-dimensional Riemannian
 submanifold N embedded in the Euclidean space E^3
sage: N.first_fundamental_form()[:]  # long time
[          r^2                0]
[            0 r^2*sin(th)^2]
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y', 'z',)); (x, y, z,) = M._first_ngens(3)
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='Riemannian')
>>> C = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi', names=('th', 'ph',));
↪(th, ph,) = C._first_ngens(2)
>>> r = var('r', domain='real')
>>> assume(r>Integer(0))
>>> E = M.cartesian_coordinates()
>>> phi = N.diff_map(M, {(C,E): [r*sin(th)*cos(ph),
...                              r*sin(th)*sin(ph),
...                              r*cos(th)]})
>>> N.set_embedding(phi)
>>> N.first_fundamental_form()  # long time
Riemannian metric gamma on the 2-dimensional Riemannian
 submanifold N embedded in the Euclidean space E^3
>>> N.first_fundamental_form()[:]  # long time
[          r^2                0]
[            0 r^2*sin(th)^2]
```

An alias is `induced_metric`:

```
sage: N.induced_metric()[:]  # long time
[          r^2                0]
[            0 r^2*sin(th)^2]
```

```
>>> from sage.all import *
>>> N.induced_metric()[:]  # long time
[          r^2                0]
[            0 r^2*sin(th)^2]
```

By default, the first fundamental form is named `gamma`, but this can be customized by means of the argument `metric_name` when declaring the submanifold:

```
sage: P = Manifold(1, 'P', ambient=M, structure='Riemannian',
....:              metric_name='g')
sage: CP.<t> = P.chart()
sage: F = P.diff_map(M, [t, 2*t, 3*t])
sage: P.set_embedding(F)
sage: P.induced_metric()
```

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```
Riemannian metric g on the 1-dimensional Riemannian submanifold P
 embedded in the Euclidean space E^3
sage: P.induced_metric().display()
g = 14 dt⊗dt
```

```
>>> from sage.all import *
>>> P = Manifold(Integer(1), 'P', ambient=M, structure='Riemannian',
...              metric_name='g')
>>> CP = P.chart(names=('t',)); (t,) = CP._first_ngens(1)
>>> F = P.diff_map(M, [t, Integer(2)*t, Integer(3)*t])
>>> P.set_embedding(F)
>>> P.induced_metric()
Riemannian metric g on the 1-dimensional Riemannian submanifold P
 embedded in the Euclidean space E^3
>>> P.induced_metric().display()
g = 14 dt⊗dt
```

gauss_curvature()

Return the Gauss curvature of the submanifold.

The *Gauss curvature* is the product or the principal curvatures, or equivalently the determinant of the projection operator.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT: the Gauss curvature as a scalar field on the submanifold

EXAMPLES:

A unit circle embedded in the Euclidean plane:

```
sage: M.<X,Y> = EuclideanSpace()
sage: N = Manifold(1, 'N', ambient=M, structure='Riemannian')
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
sage: N.declare_union(U,V)
sage: stereoN.<x> = U.chart()
sage: stereoS.<y> = V.chart()
sage: stereoN_to_S = stereoN.transition_map(stereoS, (4/x),
....:                   intersection_name='W',
....:                   restrictions1=x!=0, restrictions2=y!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M,
....:         {(stereoN, E): [1/sqrt(1+x^2/4), x/2/sqrt(1+x^2/4)],
....:          (stereoS, E): [1/sqrt(1+4/y^2), 2/y/sqrt(1+4/y^2)]})
sage: N.set_embedding(phi)
sage: N.gauss_curvature()  # long time
Scalar field on the 1-dimensional Riemannian submanifold N embedded
 in the Euclidean plane E^2
sage: N.gauss_curvature().display()  # long time
N → ℝ
on U: x ↦ -1
on V: y ↦ -1
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('X', 'Y',)); (X, Y,) = M._first_ngens(2)
>>> N = Manifold(Integer(1), 'N', ambient=M, structure='Riemannian')
>>> U = N.open_subset('U')
>>> V = N.open_subset('V')
>>> N.declare_union(U,V)
>>> stereoN = U.chart(names=('x',)); (x,) = stereoN._first_ngens(1)
>>> stereoS = V.chart(names=('y',)); (y,) = stereoS._first_ngens(1)
>>> stereoN_to_S = stereoN.transition_map(stereoS, (Integer(4)/x),
...                   intersection_name='W',
...                   restrictions1=x!=Integer(0), restrictions2=y!
↪=Integer(0))
>>> stereoS_to_N = stereoN_to_S.inverse()
>>> E = M.cartesian_coordinates()
>>> phi = N.diff_map(M,
...         {(stereoN, E): [Integer(1)/sqrt(Integer(1)+x**Integer(2)/
↪Integer(4)), x/Integer(2)/sqrt(Integer(1)+x**Integer(2)/Integer(4))],
...          (stereoS, E): [Integer(1)/sqrt(Integer(1)+Integer(4)/
↪y**Integer(2)), Integer(2)/y/sqrt(Integer(1)+Integer(4)/y**Integer(2))]})
>>> N.set_embedding(phi)
>>> N.gauss_curvature()  # long time
Scalar field on the 1-dimensional Riemannian submanifold N embedded
 in the Euclidean plane E^2
>>> N.gauss_curvature().display()  # long time
N → ℝ
on U: x ↦ -1
on V: y ↦ -1
```

gradt()

Return the gradient of the scalar field on the ambient manifold representing the first parameter of the foliation associated to `self`.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT: vector field on the ambient manifold

EXAMPLES:

Foliation of the Euclidean 3-space by 2-spheres parametrized by their radii:

```
sage: M.<x,y,z> = EuclideanSpace()
sage: N = Manifold(2, 'N', ambient=M, structure='Riemannian')
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
sage: r = var('r', domain='real')
sage: assume(r>0)
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M, {(C,E): [r*sin(th)*cos(ph),
....:                              r*sin(th)*sin(ph),
....:                              r*cos(th)]})
sage: phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
sage: phi_inv_r = M.scalar_field({E: sqrt(x^2+y^2+z^2)})
sage: N.set_embedding(phi, inverse=phi_inv, var=r,
....:                 t_inverse={r: phi_inv_r})
sage: N.gradt()
Vector field grad(r) on the Euclidean space E^3
```

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```
sage: N.gradt().display()
grad(r) = x/sqrt(x^2 + y^2 + z^2) e_x + y/sqrt(x^2 + y^2 + z^2) e_y
 + z/sqrt(x^2 + y^2 + z^2) e_z
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y', 'z',)); (x, y, z,) = M._first_ngens(3)
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='Riemannian')
>>> C = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi', names=('th', 'ph',));␣
↪(th, ph,) = C._first_ngens(2)
>>> r = var('r', domain='real')
>>> assume(r>Integer(0))
>>> E = M.cartesian_coordinates()
>>> phi = N.diff_map(M, {(C,E): [r*sin(th)*cos(ph),
...                              r*sin(th)*sin(ph),
...                              r*cos(th)]})
>>> phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
>>> phi_inv_r = M.scalar_field({E:␣
↪sqrt(x**Integer(2)+y**Integer(2)+z**Integer(2))})
>>> N.set_embedding(phi, inverse=phi_inv, var=r,
...                 t_inverse={r: phi_inv_r})
>>> N.gradt()
Vector field grad(r) on the Euclidean space E^3
>>> N.gradt().display()
grad(r) = x/sqrt(x^2 + y^2 + z^2) e_x + y/sqrt(x^2 + y^2 + z^2) e_y
 + z/sqrt(x^2 + y^2 + z^2) e_z
```

induced_metric()

Return the first fundamental form of the submanifold.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- the first fundamental form, as an instance of *PseudoRiemannianMetric*

EXAMPLES:

A sphere embedded in Euclidean space:

```
sage: M.<x,y,z> = EuclideanSpace()
sage: N = Manifold(2, 'N', ambient=M, structure='Riemannian')
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
sage: r = var('r', domain='real')
sage: assume(r>0)
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M, {(C,E): [r*sin(th)*cos(ph),
....:                              r*sin(th)*sin(ph),
....:                              r*cos(th)]})
sage: N.set_embedding(phi)
sage: N.first_fundamental_form()  # long time
Riemannian metric gamma on the 2-dimensional Riemannian
 submanifold N embedded in the Euclidean space E^3
sage: N.first_fundamental_form()[:]  # long time
[          r^2               0]
```

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```
[              0 r^2*sin(th)^2]
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y', 'z',)); (x, y, z,) = M._first_ngens(3)
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='Riemannian')
>>> C = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi', names=('th', 'ph',));
↪(th, ph,) = C._first_ngens(2)
>>> r = var('r', domain='real')
>>> assume(r>Integer(0))
>>> E = M.cartesian_coordinates()
>>> phi = N.diff_map(M, {(C,E): [r*sin(th)*cos(ph),
...                              r*sin(th)*sin(ph),
...                              r*cos(th)]})
>>> N.set_embedding(phi)
>>> N.first_fundamental_form()   # long time
Riemannian metric gamma on the 2-dimensional Riemannian
 submanifold N embedded in the Euclidean space E^3
>>> N.first_fundamental_form()[:]   # long time
[           r^2               0]
[             0 r^2*sin(th)^2]
```

An alias is `induced_metric`:

```
sage: N.induced_metric()[:]   # long time
[           r^2               0]
[             0 r^2*sin(th)^2]
```

```
>>> from sage.all import *
>>> N.induced_metric()[:]   # long time
[           r^2               0]
[             0 r^2*sin(th)^2]
```

By default, the first fundamental form is named `gamma`, but this can be customized by means of the argument `metric_name` when declaring the submanifold:

```
sage: P = Manifold(1, 'P', ambient=M, structure='Riemannian',
....:              metric_name='g')
sage: CP.<t> = P.chart()
sage: F = P.diff_map(M, [t, 2*t, 3*t])
sage: P.set_embedding(F)
sage: P.induced_metric()
Riemannian metric g on the 1-dimensional Riemannian submanifold P
 embedded in the Euclidean space E^3
sage: P.induced_metric().display()
g = 14 dt⊗dt
```

```
>>> from sage.all import *
>>> P = Manifold(Integer(1), 'P', ambient=M, structure='Riemannian',
...              metric_name='g')
>>> CP = P.chart(names=('t',)); (t,) = CP._first_ngens(1)
>>> F = P.diff_map(M, [t, Integer(2)*t, Integer(3)*t])
>>> P.set_embedding(F)
```

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```
>>> P.induced_metric()
Riemannian metric g on the 1-dimensional Riemannian submanifold P
 embedded in the Euclidean space E^3
>>> P.induced_metric().display()
g = 14 dt⊗dt
```

lapse()

Return the lapse function of the foliation.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT: the lapse function, as a scalar field on the ambient manifold

EXAMPLES:

Foliation of the Euclidean 3-space by 2-spheres parametrized by their radii:

```
sage: M.<x,y,z> = EuclideanSpace()
sage: N = Manifold(2, 'N', ambient=M, structure='Riemannian')
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
sage: r = var('r', domain='real') # foliation parameter
sage: assume(r>0)
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M, {(C,E): [r*sin(th)*cos(ph),
....:                              r*sin(th)*sin(ph),
....:                              r*cos(th)]})
sage: phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
sage: phi_inv_r = M.scalar_field({E: sqrt(x^2+y^2+z^2)})
sage: N.set_embedding(phi, inverse=phi_inv, var=r,
....:                 t_inverse={r: phi_inv_r})
sage: T = N.adapted_chart()
sage: N.lapse()
Scalar field N on the Euclidean space E^3
sage: N.lapse().display()
N: E^3 → ℝ
   (x, y, z) ↦ 1
   (th_E3, ph_E3, r_E3) ↦ 1
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y', 'z',)); (x, y, z,) = M._first_ngens(3)
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='Riemannian')
>>> C = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi', names=('th', 'ph',));␣
↪(th, ph,) = C._first_ngens(2)
>>> r = var('r', domain='real') # foliation parameter
>>> assume(r>Integer(0))
>>> E = M.cartesian_coordinates()
>>> phi = N.diff_map(M, {(C,E): [r*sin(th)*cos(ph),
...                              r*sin(th)*sin(ph),
...                              r*cos(th)]})
>>> phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
>>> phi_inv_r = M.scalar_field({E:␣
↪sqrt(x**Integer(2)+y**Integer(2)+z**Integer(2))})
>>> N.set_embedding(phi, inverse=phi_inv, var=r,
...                 t_inverse={r: phi_inv_r})
```

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```
>>> T = N.adapted_chart()
>>> N.lapse()
Scalar field N on the Euclidean space E^3
>>> N.lapse().display()
N: E^3 → ℝ
   (x, y, z) ↦ 1
   (th_E3, ph_E3, r_E3) ↦ 1
```

mean_curvature()

Return the mean curvature of the submanifold.

The *mean curvature* is the arithmetic mean of the principal curvatures, or equivalently the trace of the projection operator.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT: the mean curvature, as a scalar field on the submanifold

EXAMPLES:

A unit circle embedded in the Euclidean plane:

```
sage: M.<X,Y> = EuclideanSpace()
sage: N = Manifold(1, 'N', ambient=M, structure='Riemannian')
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
sage: N.declare_union(U,V)
sage: stereoN.<x> = U.chart()
sage: stereoS.<y> = V.chart()
sage: stereoN_to_S = stereoN.transition_map(stereoS, (4/x),
....:                   intersection_name='W',
....:                   restrictions1=x!=0, restrictions2=y!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M,
....:          {(stereoN, E): [1/sqrt(1+x^2/4), x/2/sqrt(1+x^2/4)],
....:           (stereoS, E): [1/sqrt(1+4/y^2), 2/y/sqrt(1+4/y^2)]})
sage: N.set_embedding(phi)
sage: N.mean_curvature()  # long time
Scalar field on the 1-dimensional Riemannian submanifold N
 embedded in the Euclidean plane E^2
sage: N.mean_curvature().display()  # long time
N → ℝ
on U: x ↦ -1
on V: y ↦ -1
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('X', 'Y',)); (X, Y,) = M._first_ngens(2)
>>> N = Manifold(Integer(1), 'N', ambient=M, structure='Riemannian')
>>> U = N.open_subset('U')
>>> V = N.open_subset('V')
>>> N.declare_union(U,V)
>>> stereoN = U.chart(names=('x',)); (x,) = stereoN._first_ngens(1)
>>> stereoS = V.chart(names=('y',)); (y,) = stereoS._first_ngens(1)
```

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```
>>> stereoN_to_S = stereoN.transition_map(stereoS, (Integer(4)/x),
...                     intersection_name='W',
...                     restrictions1=x!=Integer(0), restrictions2=y!
↪=Integer(0))
>>> stereoS_to_N = stereoN_to_S.inverse()
>>> E = M.cartesian_coordinates()
>>> phi = N.diff_map(M,
...          {(stereoN, E): [Integer(1)/sqrt(Integer(1)+x**Integer(2)/
↪Integer(4)), x/Integer(2)/sqrt(Integer(1)+x**Integer(2)/Integer(4))],
...           (stereoS, E): [Integer(1)/sqrt(Integer(1)+Integer(4)/
↪y**Integer(2)), Integer(2)/y/sqrt(Integer(1)+Integer(4)/y**Integer(2))]})
>>> N.set_embedding(phi)
>>> N.mean_curvature()  # long time
Scalar field on the 1-dimensional Riemannian submanifold N
 embedded in the Euclidean plane E^2
>>> N.mean_curvature().display()  # long time
N → ℝ
on U: x ↦ -1
on V: y ↦ -1
```

metric(*name=None*, *signature=None*, *latex_name=None*, *dest_map=None*)

Return the induced metric (first fundamental form) or define a new metric tensor on the submanifold.

A new (uninitialized) metric is returned only if the argument `name` is provided and differs from the metric name declared at the construction of the submanifold; otherwise, the first fundamental form is returned.

INPUT:

- `name` – (default: `None`) name given to the metric; if `name` is `None` or equals the metric name declared when constructing the submanifold, the first fundamental form is returned (see *first_fundamental_form()*)
- `signature` – (default: `None`; ignored if `name` is `None`) signature S of the metric as a single integer: $S = n_+ - n_-$, where n_+ (resp. n_-) is the number of positive terms (resp. number of negative terms) in any diagonal writing of the metric components; if `signature` is not provided, S is set to the submanifold's dimension (Riemannian signature)
- `latex_name` – (default: `None`; ignored if `name` is `None`) LaTeX symbol to denote the metric; if `None`, it is formed from `name`
- `dest_map` – (default: `None`; ignored if `name` is `None`) instance of class *DiffMap* representing the destination map $\Phi : U \to M$, where U is the current submanifold; if `None`, the identity map is assumed (case of a metric tensor field *on* U)

OUTPUT:

- instance of *PseudoRiemannianMetric*

EXAMPLES:

Induced metric on a straight line of the Euclidean plane:

```
sage: M.<x,y> = EuclideanSpace()
sage: N = Manifold(1, 'N', ambient=M, structure='Riemannian')
sage: CN.<t> = N.chart()
sage: F = N.diff_map(M, [t, 2*t])
sage: N.set_embedding(F)
```

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```
sage: N.metric()
Riemannian metric gamma on the 1-dimensional Riemannian
 submanifold N embedded in the Euclidean plane E^2
sage: N.metric().display()
gamma = 5 dt⊗dt
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y',)); (x, y,) = M._first_ngens(2)
>>> N = Manifold(Integer(1), 'N', ambient=M, structure='Riemannian')
>>> CN = N.chart(names=('t',)); (t,) = CN._first_ngens(1)
>>> F = N.diff_map(M, [t, Integer(2)*t])
>>> N.set_embedding(F)
>>> N.metric()
Riemannian metric gamma on the 1-dimensional Riemannian
 submanifold N embedded in the Euclidean plane E^2
>>> N.metric().display()
gamma = 5 dt⊗dt
```

Setting the argument `name` to that declared while constructing the submanifold (default: `'gamma'`) yields the same result:

```
sage: N.metric(name='gamma') is N.metric()
True
```

```
>>> from sage.all import *
>>> N.metric(name='gamma') is N.metric()
True
```

while using a different name allows one to define a new metric on the submanifold:

```
sage: h = N.metric(name='h'); h
Riemannian metric h on the 1-dimensional Riemannian submanifold N
 embedded in the Euclidean plane E^2
sage: h[0, 0] = 1  # initialization
sage: h.display()
h = dt⊗dt
```

```
>>> from sage.all import *
>>> h = N.metric(name='h'); h
Riemannian metric h on the 1-dimensional Riemannian submanifold N
 embedded in the Euclidean plane E^2
>>> h[Integer(0), Integer(0)] = Integer(1)  # initialization
>>> h.display()
h = dt⊗dt
```

mixed_projection(*tensor*, *indices=0*)

Return de n+1 decomposition of a tensor on the submanifold and the normal vector.

The n+1 decomposition of a tensor of rank k can be obtained by contracting each index either with the normal vector or the projection operator of the submanifold (see `projector()`).

INPUT:

- `tensor` – any tensor field, eventually along the submanifold if no foliation is provided

- `indices` – (default: `0`) list of integers containing the indices on which the projection is made on the normal vector. By default, all projections are made on the submanifold. If an integer n is provided, the n first contractions are made with the normal vector, all the other ones with the orthogonal projection operator.

OUTPUT: tensor field of rank k-`len(indices)`

EXAMPLES:

Foliation of the Euclidean 3-space by 2-spheres parametrized by their radii:

```
sage: M.<x,y,z> = EuclideanSpace()
sage: N = Manifold(2, 'N', ambient=M, structure='Riemannian')
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
sage: r = var('r', domain='real') # foliation parameter
sage: assume(r>0)
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M, {(C,E): [r*sin(th)*cos(ph),
....:                             r*sin(th)*sin(ph),
....:                             r*cos(th)]})
sage: phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
sage: phi_inv_r = M.scalar_field({E: sqrt(x^2+y^2+z^2)})
sage: N.set_embedding(phi, inverse=phi_inv, var=r,
....:                 t_inverse={r: phi_inv_r})
sage: T = N.adapted_chart()
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y', 'z',)); (x, y, z,) = M._first_ngens(3)
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='Riemannian')
>>> C = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi', names=('th', 'ph',));
↪(th, ph,) = C._first_ngens(2)
>>> r = var('r', domain='real') # foliation parameter
>>> assume(r>Integer(0))
>>> E = M.cartesian_coordinates()
>>> phi = N.diff_map(M, {(C,E): [r*sin(th)*cos(ph),
...                             r*sin(th)*sin(ph),
...                             r*cos(th)]})
>>> phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
>>> phi_inv_r = M.scalar_field({E:
↪sqrt(x**Integer(2)+y**Integer(2)+z**Integer(2))})
>>> N.set_embedding(phi, inverse=phi_inv, var=r,
...                 t_inverse={r: phi_inv_r})
>>> T = N.adapted_chart()
```

If `indices` is not specified, the mixed projection of the ambient metric coincides with the first fundamental form:

```
sage: g = M.metric()
sage: gpp = N.mixed_projection(g); gpp  # long time
Tensor field of type (0,2) on the Euclidean space E^3
sage: gpp == N.ambient_first_fundamental_form()  # long time
True
```

```
>>> from sage.all import *
>>> g = M.metric()
```

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```
>>> gpp = N.mixed_projection(g); gpp  # long time
Tensor field of type (0,2) on the Euclidean space E^3
>>> gpp == N.ambient_first_fundamental_form()  # long time
True
```

The other non-redundant projections are:

```
sage: gnp = N.mixed_projection(g, [0]); gnp  # long time
1-form on the Euclidean space E^3
```

```
>>> from sage.all import *
>>> gnp = N.mixed_projection(g, [Integer(0)]); gnp  # long time
1-form on the Euclidean space E^3
```

and:

```
sage: gnn = N.mixed_projection(g, [0,1]); gnn
Scalar field on the Euclidean space E^3
```

```
>>> from sage.all import *
>>> gnn = N.mixed_projection(g, [Integer(0),Integer(1)]); gnn
Scalar field on the Euclidean space E^3
```

which is constant and equal to 1 (the norm of the unit normal vector):

```
sage: gnn.display()
E^3 → ℝ
(x, y, z) ↦ 1
(th_E3, ph_E3, r_E3) ↦ 1
```

```
>>> from sage.all import *
>>> gnn.display()
E^3 → ℝ
(x, y, z) ↦ 1
(th_E3, ph_E3, r_E3) ↦ 1
```

normal()

Return a normal unit vector to the submanifold.

If a foliation is defined, it is used to compute the gradient of the foliation parameter and then the normal vector. If not, the normal vector is computed using the following formula:

$$n = \vec{*}(\mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n-1})$$

where the star stands for the Hodge dual operator and the wedge for the exterior product.

This formula does not always define a proper vector field when multiple charts overlap, because of the arbitrariness of the direction of the normal vector. To avoid this problem, the method `normal()` considers the graph defined by the atlas of the submanifold and the changes of coordinates, and only calculate the normal vector once by connected component. The expression is then propagate by restriction, continuation, or change of coordinates using a breadth-first exploration of the graph.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- vector field on the ambient manifold (case of a foliation) or along the submanifold with values in the ambient manifold (case of a single submanifold)

EXAMPLES:

Foliation of the Euclidean 3-space by 2-spheres parametrized by their radii:

```
sage: M.<x,y,z> = EuclideanSpace()
sage: N = Manifold(2, 'N', ambient=M, structure='Riemannian')
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
sage: r = var('r', domain='real') # foliation parameter
sage: assume(r>0)
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M, {(C,E): [r*sin(th)*cos(ph),
....:                              r*sin(th)*sin(ph),
....:                              r*cos(th)]})
sage: phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
sage: phi_inv_r = M.scalar_field({E: sqrt(x^2+y^2+z^2)})
sage: N.set_embedding(phi, inverse=phi_inv, var=r,
....:                 t_inverse={r: phi_inv_r})
sage: T = N.adapted_chart()
sage: N.normal()  # long time
Vector field n on the Euclidean space E^3
sage: N.normal().display()  # long time
n = x/sqrt(x^2 + y^2 + z^2) e_x + y/sqrt(x^2 + y^2 + z^2) e_y
 + z/sqrt(x^2 + y^2 + z^2) e_z
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y', 'z',)); (x, y, z,) = M._first_ngens(3)
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='Riemannian')
>>> C = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi', names=('th', 'ph',));
↪(th, ph,) = C._first_ngens(2)
>>> r = var('r', domain='real') # foliation parameter
>>> assume(r>Integer(0))
>>> E = M.cartesian_coordinates()
>>> phi = N.diff_map(M, {(C,E): [r*sin(th)*cos(ph),
...                              r*sin(th)*sin(ph),
...                              r*cos(th)]})
>>> phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
>>> phi_inv_r = M.scalar_field({E:
↪sqrt(x**Integer(2)+y**Integer(2)+z**Integer(2))})
>>> N.set_embedding(phi, inverse=phi_inv, var=r,
...                 t_inverse={r: phi_inv_r})
>>> T = N.adapted_chart()
>>> N.normal()  # long time
Vector field n on the Euclidean space E^3
>>> N.normal().display()  # long time
n = x/sqrt(x^2 + y^2 + z^2) e_x + y/sqrt(x^2 + y^2 + z^2) e_y
 + z/sqrt(x^2 + y^2 + z^2) e_z
```

Or in spherical coordinates:

```
sage: N.normal().display(T[0].frame(),T[0])  # long time
n = ∂/∂r_E3
```

```
>>> from sage.all import *
>>> N.normal().display(T[Integer(0)].frame(),T[Integer(0)])   # long time
n = ∂/∂r_E3
```

Let us now consider a sphere of constant radius, i.e. not assumed to be part of a foliation, in stereographic coordinates:

```
sage: M.<X,Y,Z> = EuclideanSpace()
sage: N = Manifold(2, 'N', ambient=M, structure='Riemannian')
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
sage: N.declare_union(U, V)
sage: stereoN.<x,y> = U.chart()
sage: stereoS.<xp,yp> = V.chart("xp:x' yp:y'")
sage: stereoN_to_S = stereoN.transition_map(stereoS,
....:                                 (x/(x^2+y^2), y/(x^2+y^2)),
....:                                 intersection_name='W',
....:                                 restrictions1= x^2+y^2!=0,
....:                                 restrictions2= xp^2+yp^2!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: W = U.intersection(V)
sage: stereoN_W = stereoN.restrict(W)
sage: stereoS_W = stereoS.restrict(W)
sage: A = W.open_subset('A', coord_def={stereoN_W: (y!=0, x<0),
....:                                   stereoS_W: (yp!=0, xp<0)})
sage: spher.<the,phi> = A.chart(r'the:(0,pi):\theta phi:(0,2*pi):\phi')
sage: stereoN_A = stereoN_W.restrict(A)
sage: spher_to_stereoN = spher.transition_map(stereoN_A,
....:                               (sin(the)*cos(phi)/(1-cos(the)),
....:                                sin(the)*sin(phi)/(1-cos(the))))
sage: spher_to_stereoN.set_inverse(2*atan(1/sqrt(x^2+y^2)),
....:                               atan2(-y,-x)+pi)
Check of the inverse coordinate transformation:
  the == 2*arctan(sqrt(-cos(the) + 1)/sqrt(cos(the) + 1))  **failed**
  phi == pi + arctan2(sin(phi)*sin(the)/(cos(the) - 1),
                      cos(phi)*sin(the)/(cos(the) - 1))  **failed**
  x == x  *passed*
  y == y  *passed*
NB: a failed report can reflect a mere lack of simplification.
sage: stereoN_to_S_A = stereoN_to_S.restrict(A)
sage: spher_to_stereoS = stereoN_to_S_A * spher_to_stereoN
sage: stereoS_to_N_A = stereoN_to_S.inverse().restrict(A)
sage: stereoS_to_spher = spher_to_stereoN.inverse() * stereoS_to_N_A
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M, {(stereoN, E): [2*x/(1+x^2+y^2),
....:                                      2*y/(1+x^2+y^2),
....:                                      (x^2+y^2-1)/(1+x^2+y^2)],
....:                      (stereoS, E): [2*xp/(1+xp^2+yp^2),
....:                                     2*yp/(1+xp^2+yp^2),
....:                                    (1-xp^2-yp^2)/(1+xp^2+yp^2)]},
....:                    name='Phi', latex_name=r'\Phi')
sage: N.set_embedding(phi)
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('X', 'Y', 'Z',)); (X, Y, Z,) = M._first_ngens(3)
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='Riemannian')
>>> U = N.open_subset('U')
>>> V = N.open_subset('V')
>>> N.declare_union(U, V)
>>> stereoN = U.chart(names=('x', 'y',)); (x, y,) = stereoN._first_ngens(2)
>>> stereoS = V.chart("xp:x' yp:y'", names=('xp', 'yp',)); (xp, yp,) =␣
↪stereoS._first_ngens(2)
>>> stereoN_to_S = stereoN.transition_map(stereoS,
...                                     (x/(x**Integer(2)+y**Integer(2)), y/
↪(x**Integer(2)+y**Integer(2))),
...                                     intersection_name='W',
...                                     restrictions1=␣
↪x**Integer(2)+y**Integer(2)!=Integer(0),
...                                     restrictions2=␣
↪xp**Integer(2)+yp**Integer(2)!=Integer(0))
>>> stereoS_to_N = stereoN_to_S.inverse()
>>> W = U.intersection(V)
>>> stereoN_W = stereoN.restrict(W)
>>> stereoS_W = stereoS.restrict(W)
>>> A = W.open_subset('A', coord_def={stereoN_W: (y!=Integer(0), x
↪<Integer(0)),
...                                       stereoS_W: (yp!=Integer(0), xp
↪<Integer(0))})
>>> spher = A.chart(r'the:(0,pi):\theta phi:(0,2*pi):\phi', names=('the', 'phi
↪',)); (the, phi,) = spher._first_ngens(2)
>>> stereoN_A = stereoN_W.restrict(A)
>>> spher_to_stereoN = spher.transition_map(stereoN_A,
...                                 (sin(the)*cos(phi)/(Integer(1)-cos(the)),
...                                  sin(the)*sin(phi)/(Integer(1)-cos(the))))
>>> spher_to_stereoN.set_inverse(Integer(2)*atan(Integer(1)/
↪sqrt(x**Integer(2)+y**Integer(2))),
...                                 atan2(-y,-x)+pi)
Check of the inverse coordinate transformation:
  the == 2*arctan(sqrt(-cos(the) + 1)/sqrt(cos(the) + 1))  **failed**
  phi == pi + arctan2(sin(phi)*sin(the)/(cos(the) - 1),
                      cos(phi)*sin(the)/(cos(the) - 1))  **failed**
  x == x  *passed*
  y == y  *passed*
NB: a failed report can reflect a mere lack of simplification.
>>> stereoN_to_S_A = stereoN_to_S.restrict(A)
>>> spher_to_stereoS = stereoN_to_S_A * spher_to_stereoN
>>> stereoS_to_N_A = stereoN_to_S.inverse().restrict(A)
>>> stereoS_to_spher = spher_to_stereoN.inverse() * stereoS_to_N_A
>>> E = M.cartesian_coordinates()
>>> phi = N.diff_map(M, {(stereoN, E): [Integer(2)*x/
↪(Integer(1)+x**Integer(2)+y**Integer(2)),
...                                        Integer(2)*y/
↪(Integer(1)+x**Integer(2)+y**Integer(2)),
...                                        (x**Integer(2)+y**Integer(2)-
↪Integer(1))/(Integer(1)+x**Integer(2)+y**Integer(2))],
...                       (stereoS, E): [Integer(2)*xp/
```

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```
↪(Integer(1)+xp**Integer(2)+yp**Integer(2)),
...                                     Integer(2)*yp/
↪(Integer(1)+xp**Integer(2)+yp**Integer(2)),
...                                    (Integer(1)-xp**Integer(2)-
↪yp**Integer(2))/(Integer(1)+xp**Integer(2)+yp**Integer(2))]},
...                     name='Phi', latex_name=r'\Phi')
>>> N.set_embedding(phi)
```

The method `normal()` now returns a tensor field along `N`:

```
sage: n = N.normal()  # long time
sage: n  # long time
Vector field n along the 2-dimensional Riemannian submanifold N
 embedded in the Euclidean space E^3 with values on the Euclidean
 space E^3
```

```
>>> from sage.all import *
>>> n = N.normal()  # long time
>>> n  # long time
Vector field n along the 2-dimensional Riemannian submanifold N
 embedded in the Euclidean space E^3 with values on the Euclidean
 space E^3
```

Let us check that the choice of orientation is coherent on the two top frames:

```
sage: n.restrict(V).display(format_spec=spher)  # long time
n = -cos(phi)*sin(the) e_X - sin(phi)*sin(the) e_Y - cos(the) e_Z
sage: n.restrict(U).display(format_spec=spher)  # long time
n = -cos(phi)*sin(the) e_X - sin(phi)*sin(the) e_Y - cos(the) e_Z
```

```
>>> from sage.all import *
>>> n.restrict(V).display(format_spec=spher)  # long time
n = -cos(phi)*sin(the) e_X - sin(phi)*sin(the) e_Y - cos(the) e_Z
>>> n.restrict(U).display(format_spec=spher)  # long time
n = -cos(phi)*sin(the) e_X - sin(phi)*sin(the) e_Y - cos(the) e_Z
```

open_subset(*name*, *latex_name=None*, *coord_def={}*, *supersets=None*)

Create an open subset of `self`.

An open subset is a set that is (i) included in the manifold and (ii) open with respect to the manifold's topology. It is a differentiable manifold by itself. Moreover, equipped with the restriction of the manifold metric to itself, it is a pseudo-Riemannian manifold.

As `self` is a submanifold of its ambient manifold, the new open subset is also considered a submanifold of that. Hence the returned object is an instance of *PseudoRiemannianSubmanifold*.

INPUT:

- `name` – name given to the open subset
- `latex_name` – (default: `None`) LaTeX symbol to denote the subset; if none is provided, it is set to `name`
- `coord_def` – (default: `{}`) definition of the subset in terms of coordinates; `coord_def` must a be dictionary with keys charts in the manifold's atlas and values the symbolic expressions formed by the coordinates to define the subset.

- `supersets` – (default: only `self`) list of sets that the new open subset is a subset of

OUTPUT:

- instance of `PseudoRiemannianSubmanifold` representing the created open subset

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='Riemannian')
sage: N = Manifold(2, 'N', ambient=M, structure='Riemannian'); N
2-dimensional Riemannian submanifold N immersed in the
 3-dimensional Riemannian manifold M
sage: S = N.subset('S'); S
Subset S of the
 2-dimensional Riemannian submanifold N immersed in the
  3-dimensional Riemannian manifold M
sage: O = N.subset('O', is_open=True); O  # indirect doctest
Open subset O of the
 2-dimensional Riemannian submanifold N immersed in the
  3-dimensional Riemannian manifold M

sage: phi = N.diff_map(M)
sage: N.set_embedding(phi)
sage: N
2-dimensional Riemannian submanifold N embedded in the
 3-dimensional Riemannian manifold M
sage: S = N.subset('S'); S
Subset S of the
 2-dimensional Riemannian submanifold N embedded in the
  3-dimensional Riemannian manifold M
sage: O = N.subset('O', is_open=True); O  # indirect doctest
Open subset O of the
 2-dimensional Riemannian submanifold N embedded in the
  3-dimensional Riemannian manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='Riemannian')
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='Riemannian'); N
2-dimensional Riemannian submanifold N immersed in the
 3-dimensional Riemannian manifold M
>>> S = N.subset('S'); S
Subset S of the
 2-dimensional Riemannian submanifold N immersed in the
  3-dimensional Riemannian manifold M
>>> O = N.subset('O', is_open=True); O  # indirect doctest
Open subset O of the
 2-dimensional Riemannian submanifold N immersed in the
  3-dimensional Riemannian manifold M

>>> phi = N.diff_map(M)
>>> N.set_embedding(phi)
>>> N
2-dimensional Riemannian submanifold N embedded in the
 3-dimensional Riemannian manifold M
>>> S = N.subset('S'); S
```

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```
Subset S of the
 2-dimensional Riemannian submanifold N embedded in the
  3-dimensional Riemannian manifold M
>>> O = N.subset('O', is_open=True); O  # indirect doctest
Open subset O of the
 2-dimensional Riemannian submanifold N embedded in the
  3-dimensional Riemannian manifold M
```

principal_curvatures(*chart*)

Return the principal curvatures of the submanifold.

The *principal curvatures* are the eigenvalues of the projection operator. The resulting scalar fields are named `k_i` with the index `i` ranging from 0 to the submanifold dimension minus one.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

INPUT:

- `chart` – chart in which the principal curvatures are to be computed

OUTPUT:

- the principal curvatures, as a list of scalar fields on the submanifold

EXAMPLES:

A unit circle embedded in the Euclidean plane:

```
sage: M.<X,Y> = EuclideanSpace()
sage: N = Manifold(1, 'N', ambient=M, structure='Riemannian')
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
sage: N.declare_union(U,V)
sage: stereoN.<x> = U.chart()
sage: stereoS.<y> = V.chart()
sage: stereoN_to_S = stereoN.transition_map(stereoS, (4/x),
....:                 intersection_name='W',
....:                 restrictions1=x!=0, restrictions2=y!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M,
....:         {(stereoN, E): [1/sqrt(1+x^2/4), x/2/sqrt(1+x^2/4)],
....:          (stereoS, E): [1/sqrt(1+4/y^2), 2/y/sqrt(1+4/y^2)]})
sage: N.set_embedding(phi)
sage: N.principal_curvatures(stereoN)  # long time
[Scalar field k_0 on the 1-dimensional Riemannian submanifold N
 embedded in the Euclidean plane E^2]
sage: N.principal_curvatures(stereoN)[0].display()  # long time
k_0: N → ℝ
on U: x ↦ -1
on W: y ↦ -1
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('X', 'Y',)); (X, Y,) = M._first_ngens(2)
>>> N = Manifold(Integer(1), 'N', ambient=M, structure='Riemannian')
>>> U = N.open_subset('U')
```

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```
>>> V = N.open_subset('V')
>>> N.declare_union(U,V)
>>> stereoN = U.chart(names=('x',)); (x,) = stereoN._first_ngens(1)
>>> stereoS = V.chart(names=('y',)); (y,) = stereoS._first_ngens(1)
>>> stereoN_to_S = stereoN.transition_map(stereoS, (Integer(4)/x),
...                    intersection_name='W',
...                    restrictions1=x!=Integer(0), restrictions2=y!
↪=Integer(0))
>>> stereoS_to_N = stereoN_to_S.inverse()
>>> E = M.cartesian_coordinates()
>>> phi = N.diff_map(M,
...          {(stereoN, E): [Integer(1)/sqrt(Integer(1)+x**Integer(2)/
↪Integer(4)), x/Integer(2)/sqrt(Integer(1)+x**Integer(2)/Integer(4))],
...           (stereoS, E): [Integer(1)/sqrt(Integer(1)+Integer(4)/
↪y**Integer(2)), Integer(2)/y/sqrt(Integer(1)+Integer(4)/y**Integer(2))]})
>>> N.set_embedding(phi)
>>> N.principal_curvatures(stereoN)  # long time
[Scalar field k_0 on the 1-dimensional Riemannian submanifold N
 embedded in the Euclidean plane E^2]
>>> N.principal_curvatures(stereoN)[Integer(0)].display()  # long time
k_0: N → ℝ
on U: x ↦ -1
on W: y ↦ -1
```

principal_directions(*chart*)

Return the principal directions of the submanifold.

The *principal directions* are the eigenvectors of the projection operator. The result is formatted as a list of pairs (eigenvector, eigenvalue).

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

INPUT:

- `chart` – chart in which the principal directions are to be computed

OUTPUT:

- list of pairs (vector field, scalar field) representing the principal directions and the associated principal curvatures

EXAMPLES:

A unit circle embedded in the Euclidean plane:

```
sage: M.<X,Y> = EuclideanSpace()
sage: N = Manifold(1, 'N', ambient=M, structure='Riemannian')
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
sage: N.declare_union(U,V)
sage: stereoN.<x> = U.chart()
sage: stereoS.<y> = V.chart()
sage: stereoN_to_S = stereoN.transition_map(stereoS, (4/x),
....:                   intersection_name='W',
....:                   restrictions1=x!=0, restrictions2=y!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
```

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```
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M,
....:          {(stereoN, E): [1/sqrt(1+x^2/4), x/2/sqrt(1+x^2/4)],
....:           (stereoS, E): [1/sqrt(1+4/y^2), 2/y/sqrt(1+4/y^2)]})
sage: N.set_embedding(phi)
sage: N.principal_directions(stereoN)  # long time
[(Vector field e_0 on the 1-dimensional Riemannian submanifold N
  embedded in the Euclidean plane E^2, -1)]
sage: N.principal_directions(stereoN)[0][0].display()  # long time
e_0 = ∂/∂x
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('X', 'Y',)); (X, Y,) = M._first_ngens(2)
>>> N = Manifold(Integer(1), 'N', ambient=M, structure='Riemannian')
>>> U = N.open_subset('U')
>>> V = N.open_subset('V')
>>> N.declare_union(U,V)
>>> stereoN = U.chart(names=('x',)); (x,) = stereoN._first_ngens(1)
>>> stereoS = V.chart(names=('y',)); (y,) = stereoS._first_ngens(1)
>>> stereoN_to_S = stereoN.transition_map(stereoS, (Integer(4)/x),
...                   intersection_name='W',
...                   restrictions1=x!=Integer(0), restrictions2=y!
↪=Integer(0))
>>> stereoS_to_N = stereoN_to_S.inverse()
>>> E = M.cartesian_coordinates()
>>> phi = N.diff_map(M,
...          {(stereoN, E): [Integer(1)/sqrt(Integer(1)+x**Integer(2)/
↪Integer(4)), x/Integer(2)/sqrt(Integer(1)+x**Integer(2)/Integer(4))],
...           (stereoS, E): [Integer(1)/sqrt(Integer(1)+Integer(4)/
↪y**Integer(2)), Integer(2)/y/sqrt(Integer(1)+Integer(4)/y**Integer(2))]})
>>> N.set_embedding(phi)
>>> N.principal_directions(stereoN)  # long time
[(Vector field e_0 on the 1-dimensional Riemannian submanifold N
  embedded in the Euclidean plane E^2, -1)]
>>> N.principal_directions(stereoN)[Integer(0)][Integer(0)].display()  # long␣
↪time
e_0 = ∂/∂x
```

project (*tensor*)

Return the orthogonal projection of a tensor field onto the submanifold.

INPUT:

- `tensor` – any tensor field to be projected onto the submanifold. If no foliation is provided, must be a tensor field along the submanifold.

OUTPUT:

- orthogonal projection of `tensor` onto the submanifold, as a tensor field of the *ambient* manifold

EXAMPLES:

Foliation of the Euclidean 3-space by 2-spheres parametrized by their radii:

```
sage: M.<x,y,z> = EuclideanSpace()
sage: N = Manifold(2, 'N', ambient=M, structure='Riemannian')
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
sage: r = var('r', domain='real') # foliation parameter
sage: assume(r>0)
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M, {(C,E): [r*sin(th)*cos(ph),
....:                              r*sin(th)*sin(ph),
....:                              r*cos(th)]})
sage: phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
sage: phi_inv_r = M.scalar_field({E: sqrt(x^2+y^2+z^2)})
sage: N.set_embedding(phi, inverse=phi_inv, var=r,
....:                 t_inverse={r: phi_inv_r})
sage: T = N.adapted_chart()
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y', 'z',)); (x, y, z,) = M._first_ngens(3)
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='Riemannian')
>>> C = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi', names=('th', 'ph',));
↪(th, ph,) = C._first_ngens(2)
>>> r = var('r', domain='real') # foliation parameter
>>> assume(r>Integer(0))
>>> E = M.cartesian_coordinates()
>>> phi = N.diff_map(M, {(C,E): [r*sin(th)*cos(ph),
...                              r*sin(th)*sin(ph),
...                              r*cos(th)]})
>>> phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
>>> phi_inv_r = M.scalar_field({E:
↪sqrt(x**Integer(2)+y**Integer(2)+z**Integer(2))})
>>> N.set_embedding(phi, inverse=phi_inv, var=r,
...                 t_inverse={r: phi_inv_r})
>>> T = N.adapted_chart()
```

Let us perform the projection of the ambient metric and check that it is equal to the first fundamental form:

```
sage: pg = N.project(M.metric()); pg  # long time
Tensor field of type (0,2) on the Euclidean space E^3
sage: pg == N.ambient_first_fundamental_form()  # long time
True
```

```
>>> from sage.all import *
>>> pg = N.project(M.metric()); pg  # long time
Tensor field of type (0,2) on the Euclidean space E^3
>>> pg == N.ambient_first_fundamental_form()  # long time
True
```

Note that the output of `project()` is not cached.

projector()

Return the orthogonal projector onto the submanifold.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- the orthogonal projector onto the submanifold, as tensor field of type (1,1) on the ambient manifold

EXAMPLES:

Foliation of the Euclidean 3-space by 2-spheres parametrized by their radii:

```
sage: M.<x,y,z> = EuclideanSpace()
sage: N = Manifold(2, 'N', ambient=M, structure='Riemannian')
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
sage: r = var('r', domain='real') # foliation parameter
sage: assume(r>0)
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M, {(C,E): [r*sin(th)*cos(ph),
....:                             r*sin(th)*sin(ph),
....:                             r*cos(th)]})
sage: phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
sage: phi_inv_r = M.scalar_field({E: sqrt(x^2+y^2+z^2)})
sage: N.set_embedding(phi, inverse=phi_inv, var=r,
....:                 t_inverse={r: phi_inv_r})
sage: T = N.adapted_chart()
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y', 'z',)); (x, y, z,) = M._first_ngens(3)
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='Riemannian')
>>> C = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi', names=('th', 'ph',));
↪(th, ph,) = C._first_ngens(2)
>>> r = var('r', domain='real') # foliation parameter
>>> assume(r>Integer(0))
>>> E = M.cartesian_coordinates()
>>> phi = N.diff_map(M, {(C,E): [r*sin(th)*cos(ph),
...                             r*sin(th)*sin(ph),
...                             r*cos(th)]})
>>> phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
>>> phi_inv_r = M.scalar_field({E:
↪sqrt(x**Integer(2)+y**Integer(2)+z**Integer(2))})
>>> N.set_embedding(phi, inverse=phi_inv, var=r,
...                 t_inverse={r: phi_inv_r})
>>> T = N.adapted_chart()
```

The orthogonal projector onto `N` is a type-(1,1) tensor field on `M`:

```
sage: N.projector()  # long time
Tensor field gamma of type (1,1) on the Euclidean space E^3
```

```
>>> from sage.all import *
>>> N.projector()  # long time
Tensor field gamma of type (1,1) on the Euclidean space E^3
```

Check that the orthogonal projector applied to the normal vector is zero:

```
sage: N.projector().contract(N.normal()).display()  # long time
0
```

```
>>> from sage.all import *
>>> N.projector().contract(N.normal()).display()   # long time
0
```

second_fundamental_form()

Return the second fundamental form of the submanifold.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- the second fundamental form, as a symmetric tensor field of type (0,2) on the submanifold

EXAMPLES:

A unit circle embedded in the Euclidean plane:

```
sage: M.<X,Y> = EuclideanSpace()
sage: N = Manifold(1, 'N', ambient=M, structure='Riemannian')
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
sage: N.declare_union(U,V)
sage: stereoN.<x> = U.chart()
sage: stereoS.<y> = V.chart()
sage: stereoN_to_S = stereoN.transition_map(stereoS, (4/x),
....:                     intersection_name='W',
....:                     restrictions1=x!=0, restrictions2=y!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M,
....:          {(stereoN, E): [1/sqrt(1+x^2/4), x/2/sqrt(1+x^2/4)],
....:           (stereoS, E): [1/sqrt(1+4/y^2), 2/y/sqrt(1+4/y^2)]})
sage: N.set_embedding(phi)
sage: N.second_fundamental_form()  # long time
Field of symmetric bilinear forms K on the 1-dimensional Riemannian
 submanifold N embedded in the Euclidean plane E^2
sage: N.second_fundamental_form().display()  # long time
K = -4/(x^4 + 8*x^2 + 16) dx⊗dx
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('X', 'Y',)); (X, Y,) = M._first_ngens(2)
>>> N = Manifold(Integer(1), 'N', ambient=M, structure='Riemannian')
>>> U = N.open_subset('U')
>>> V = N.open_subset('V')
>>> N.declare_union(U,V)
>>> stereoN = U.chart(names=('x',)); (x,) = stereoN._first_ngens(1)
>>> stereoS = V.chart(names=('y',)); (y,) = stereoS._first_ngens(1)
>>> stereoN_to_S = stereoN.transition_map(stereoS, (Integer(4)/x),
...                     intersection_name='W',
...                     restrictions1=x!=Integer(0), restrictions2=y!
↪=Integer(0))
>>> stereoS_to_N = stereoN_to_S.inverse()
>>> E = M.cartesian_coordinates()
>>> phi = N.diff_map(M,
...          {(stereoN, E): [Integer(1)/sqrt(Integer(1)+x**Integer(2)/
```

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```
↪Integer(4)), x/Integer(2)/sqrt(Integer(1)+x**Integer(2)/Integer(4))],
...            (stereoS, E): [Integer(1)/sqrt(Integer(1)+Integer(4)/
↪y**Integer(2)), Integer(2)/y/sqrt(Integer(1)+Integer(4)/y**Integer(2))]})
>>> N.set_embedding(phi)
>>> N.second_fundamental_form()  # long time
Field of symmetric bilinear forms K on the 1-dimensional Riemannian
 submanifold N embedded in the Euclidean plane E^2
>>> N.second_fundamental_form().display()  # long time
K = -4/(x^4 + 8*x^2 + 16) dx⊗dx
```

An alias is `extrinsic_curvature`:

```
sage: N.extrinsic_curvature().display()  # long time
K = -4/(x^4 + 8*x^2 + 16) dx⊗dx
```

```
>>> from sage.all import *
>>> N.extrinsic_curvature().display()  # long time
K = -4/(x^4 + 8*x^2 + 16) dx⊗dx
```

An example with a non-Euclidean ambient metric:

```
sage: M = Manifold(2, 'M', structure='Riemannian')
sage: N = Manifold(1, 'N', ambient=M, structure='Riemannian',
....:              start_index=1)
sage: CM.<x,y> = M.chart()
sage: CN.<u> = N.chart()
sage: g = M.metric()
sage: g[0, 0], g[1, 1] = 1, 1/(1 + y^2)^2
sage: phi = N.diff_map(M, (u, u))
sage: N.set_embedding(phi)
sage: N.second_fundamental_form()
Field of symmetric bilinear forms K on the 1-dimensional Riemannian
 submanifold N embedded in the 2-dimensional Riemannian manifold M
sage: N.second_fundamental_form().display()
K = 2*sqrt(u^4 + 2*u^2 + 2)*u/(u^6 + 3*u^4 + 4*u^2 + 2) du⊗du
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='Riemannian')
>>> N = Manifold(Integer(1), 'N', ambient=M, structure='Riemannian',
...              start_index=Integer(1))
>>> CM = M.chart(names=('x', 'y',)); (x, y,) = CM._first_ngens(2)
>>> CN = N.chart(names=('u',)); (u,) = CN._first_ngens(1)
>>> g = M.metric()
>>> g[Integer(0), Integer(0)], g[Integer(1), Integer(1)] = Integer(1),␣
↪Integer(1)/(Integer(1) + y**Integer(2))**Integer(2)
>>> phi = N.diff_map(M, (u, u))
>>> N.set_embedding(phi)
>>> N.second_fundamental_form()
Field of symmetric bilinear forms K on the 1-dimensional Riemannian
 submanifold N embedded in the 2-dimensional Riemannian manifold M
>>> N.second_fundamental_form().display()
K = 2*sqrt(u^4 + 2*u^2 + 2)*u/(u^6 + 3*u^4 + 4*u^2 + 2) du⊗du
```

shape_operator()

Return the shape operator of the submanifold.

The shape operator is equal to the second fundamental form with one of the indices upped.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- the shape operator, as a tensor field of type (1,1) on the submanifold

EXAMPLES:

A unit circle embedded in the Euclidean plane:

```
sage: M.<X,Y> = EuclideanSpace()
sage: N = Manifold(1, 'N', ambient=M, structure='Riemannian')
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
sage: N.declare_union(U,V)
sage: stereoN.<x> = U.chart()
sage: stereoS.<y> = V.chart()
sage: stereoN_to_S = stereoN.transition_map(stereoS, (4/x),
....:                   intersection_name='W',
....:                   restrictions1=x!=0, restrictions2=y!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M,
....:         {(stereoN, E): [1/sqrt(1+x^2/4), x/2/sqrt(1+x^2/4)],
....:          (stereoS, E): [1/sqrt(1+4/y^2), 2/y/sqrt(1+4/y^2)]})
sage: N.set_embedding(phi)
sage: N.shape_operator()  # long time
Tensor field of type (1,1) on the 1-dimensional Riemannian
 submanifold N embedded in the Euclidean plane E^2
sage: N.shape_operator().display()  # long time
-∂/∂x⊗dx
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('X', 'Y',)); (X, Y,) = M._first_ngens(2)
>>> N = Manifold(Integer(1), 'N', ambient=M, structure='Riemannian')
>>> U = N.open_subset('U')
>>> V = N.open_subset('V')
>>> N.declare_union(U,V)
>>> stereoN = U.chart(names=('x',)); (x,) = stereoN._first_ngens(1)
>>> stereoS = V.chart(names=('y',)); (y,) = stereoS._first_ngens(1)
>>> stereoN_to_S = stereoN.transition_map(stereoS, (Integer(4)/x),
...                   intersection_name='W',
...                   restrictions1=x!=Integer(0), restrictions2=y!
↪=Integer(0))
>>> stereoS_to_N = stereoN_to_S.inverse()
>>> E = M.cartesian_coordinates()
>>> phi = N.diff_map(M,
...         {(stereoN, E): [Integer(1)/sqrt(Integer(1)+x**Integer(2)/
↪Integer(4)), x/Integer(2)/sqrt(Integer(1)+x**Integer(2)/Integer(4))],
...          (stereoS, E): [Integer(1)/sqrt(Integer(1)+Integer(4)/
↪y**Integer(2)), Integer(2)/y/sqrt(Integer(1)+Integer(4)/y**Integer(2))]})
```

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```
>>> N.set_embedding(phi)
>>> N.shape_operator()  # long time
Tensor field of type (1,1) on the 1-dimensional Riemannian
 submanifold N embedded in the Euclidean plane E^2
>>> N.shape_operator().display()  # long time
-∂/∂x⊗dx
```

shift()

Return the shift vector associated with the first adapted chart of the foliation.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT: shift vector field on the ambient manifold

EXAMPLES:

Foliation of the Euclidean 3-space by 2-spheres parametrized by their radii:

```
sage: M.<x,y,z> = EuclideanSpace()
sage: N = Manifold(2, 'N', ambient=M, structure='Riemannian')
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
sage: r = var('r', domain='real') # foliation parameter
sage: assume(r>0)
sage: E = M.cartesian_coordinates()
sage: phi = N.diff_map(M, {(C,E): [r*sin(th)*cos(ph),
....:                             r*sin(th)*sin(ph),
....:                             r*cos(th)]})
sage: phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
sage: phi_inv_r = M.scalar_field({E: sqrt(x^2+y^2+z^2)})
sage: N.set_embedding(phi, inverse=phi_inv, var=r,
....:                 t_inverse={r: phi_inv_r})
sage: T = N.adapted_chart()
sage: N.shift()  # long time
Vector field beta on the Euclidean space E^3
sage: N.shift().display()  # long time
beta = 0
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(names=('x', 'y', 'z',)); (x, y, z,) = M._first_ngens(3)
>>> N = Manifold(Integer(2), 'N', ambient=M, structure='Riemannian')
>>> C = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi', names=('th', 'ph',));
↪(th, ph,) = C._first_ngens(2)
>>> r = var('r', domain='real') # foliation parameter
>>> assume(r>Integer(0))
>>> E = M.cartesian_coordinates()
>>> phi = N.diff_map(M, {(C,E): [r*sin(th)*cos(ph),
...                             r*sin(th)*sin(ph),
...                             r*cos(th)]})
>>> phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
>>> phi_inv_r = M.scalar_field({E:
↪sqrt(x**Integer(2)+y**Integer(2)+z**Integer(2))})
>>> N.set_embedding(phi, inverse=phi_inv, var=r,
...                 t_inverse={r: phi_inv_r})
>>> T = N.adapted_chart()
```

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```
>>> N.shift()  # long time
Vector field beta on the Euclidean space E^3
>>> N.shift().display()  # long time
beta = 0
```

3.6 Degenerate Metric Manifolds

3.6.1 Degenerate manifolds

class sage.manifolds.differentiable.degenerate.**DegenerateManifold**(*n*, *name*, *metric_name=None*, *signature=None*, *base_manifold=None*, *diff_degree=+Infinity*, *latex_name=None*, *metric_latex_name=None*, *start_index=0*, *category=None*, *unique_tag=None*)

Bases: *DifferentiableManifold*

Degenerate Manifolds

A *degenerate manifold* (or a *null manifold*) is a pair (M, g) where M is a real differentiable manifold (see *DifferentiableManifold*) and g is a field of degenerate symmetric bilinear forms on M (see *DegenerateMetric*).

INPUT:

- `n` – positive integer; dimension of the manifold
- `name` – string; name (symbol) given to the manifold
- `metric_name` – (default: `None`) string; name (symbol) given to the metric; if `None`, `'g'` is used
- `signature` – (default: `None`) signature S of the metric as a tuple: $S = (n_+, n_-, n_0)$, where n_+ (resp. n_-, resp. n_0) is the number of positive terms (resp. negative terms, resp. zero tems) in any diagonal writing of the metric components; if `signature` is not provided, S is set to $(ndim - 1, 0, 1)$, being $ndim$ the manifold's dimension
- `ambient` – (default: `None`) if not `None`, must be a differentiable manifold; the created object is then an open subset of `ambient`
- `diff_degree` – (default: `infinity`) degree k of differentiability
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the manifold; if none is provided, it is set to `name`
- `metric_latex_name` – (default: `None`) string; LaTeX symbol to denote the metric; if none is provided, it is set to `metric_name`
- `start_index` – (default: 0) integer; lower value of the range of indices used for "indexed objects" on the manifold, e.g. coordinates in a chart
- `category` – (default: `None`) to specify the category; if `None`, `Manifolds(RR).Differentiable()` (or `Manifolds(RR).Smooth()` if `diff_degree` = `infinity`) is assumed (see the category `Manifolds`)
- `unique_tag` – (default: `None`) tag used to force the construction of a new object when all the other arguments have been used previously (without `unique_tag`, the `UniqueRepresentation` behavior inherited

from `ManifoldSubset`, via `DifferentiableManifold` and `TopologicalManifold`, would return the previously constructed object corresponding to these arguments).

EXAMPLES:

A degenerate manifold is constructed via the generic function `Manifold()`, using the keyword `structure`:

```
sage: M = Manifold(3, 'M', structure='degenerate_metric')
sage: M
3-dimensional degenerate_metric manifold M
sage: M.parent()
<class 'sage.manifolds.differentiable.degenerate.DegenerateManifold_with_category
↪'>
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='degenerate_metric')
>>> M
3-dimensional degenerate_metric manifold M
>>> M.parent()
<class 'sage.manifolds.differentiable.degenerate.DegenerateManifold_with_category
↪'>
```

The metric associated with `M` is:

```
sage: g = M.metric()
sage: g
degenerate metric g on the 3-dimensional degenerate_metric manifold M
sage: g.signature()
(0, 2, 1)
```

```
>>> from sage.all import *
>>> g = M.metric()
>>> g
degenerate metric g on the 3-dimensional degenerate_metric manifold M
>>> g.signature()
(0, 2, 1)
```

Its value has to be initialized either by setting its components in various vector frames (see the above examples regarding the 2-sphere and Minkowski spacetime) or by making it equal to a given field of symmetric bilinear forms (see the method `set()` of the metric class). Both methods are also covered in the documentation of method `metric()` below.

REFERENCES:

- [DB1996]
- [DS2010]

metric(*name=None*, *signature=None*, *latex_name=None*, *dest_map=None*)

Return the metric giving the null manifold structure to the manifold, or define a new metric tensor on the manifold.

INPUT:

- `name` – (default: `None`) name given to the metric; if `name` is `None` or matches the name of the metric defining the null manifold structure of `self`, the latter metric is returned

- `signature` – (default: `None`; ignored if `name` is `None`) signature S of the metric as a tuple: $S = (n_+, n_-, n_0)$, where n_+ (resp. n_-, resp. n_0) is the number of positive terms (resp. negative terms, resp. zero tems) in any diagonal writing of the metric components; if `signature` is not provided, S is set to $(ndim - 1, 0, 1)$, being $ndim$ the manifold's dimension
- `latex_name` – (default: `None`; ignored if `name` is `None`) LaTeX symbol to denote the metric; if `None`, it is formed from `name`
- `dest_map` – (default: `None`; ignored if `name` is `None`) instance of class `DiffMap` representing the destination map $\Phi : U \rightarrow M$, where U is the current manifold; if `None`, the identity map is assumed (case of a metric tensor field *on* U)

OUTPUT:

- instance of `DegenerateMetric`

EXAMPLES:

Metric of a 3-dimensional degenerate manifold:

```
sage: M = Manifold(3, 'M', structure='degenerate_metric', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: g = M.metric(); g
degenerate metric g on the 3-dimensional degenerate_metric manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='degenerate_metric', start_
↪index=Integer(1))
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> g = M.metric(); g
degenerate metric g on the 3-dimensional degenerate_metric manifold M
```

The metric remains to be initialized, for instance by setting its components in the coordinate frame associated to the chart `X`:

```
sage: g[1,1], g[2,2] = -1, 1
sage: g.display()
g = -dx⊗dx + dy⊗dy
sage: g[:]
[-1  0  0]
[ 0  1  0]
[ 0  0  0]
```

```
>>> from sage.all import *
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)] = -Integer(1),␣
↪Integer(1)
>>> g.display()
g = -dx⊗dx + dy⊗dy
>>> g[:]
[-1  0  0]
[ 0  1  0]
[ 0  0  0]
```

Alternatively, the metric can be initialized from a given field of degenerate symmetric bilinear forms; we may create the former object by:

```
sage: X.coframe()
Coordinate coframe (M, (dx,dy,dz))
sage: dx, dy = X.coframe()[1], X.coframe()[2]
sage: b = dx*dx + dy*dy
sage: b
Field of symmetric bilinear forms dx⊗dx+dy⊗dy on the 3-dimensional
degenerate_metric manifold M
```

```
>>> from sage.all import *
>>> X.coframe()
Coordinate coframe (M, (dx,dy,dz))
>>> dx, dy = X.coframe()[Integer(1)], X.coframe()[Integer(2)]
>>> b = dx*dx + dy*dy
>>> b
Field of symmetric bilinear forms dx⊗dx+dy⊗dy on the 3-dimensional
degenerate_metric manifold M
```

We then use the metric method *set()* to make `g` being equal to `b` as a symmetric tensor field of type `(0,2)`:

```
sage: g.set(b)
sage: g.display()
g = dx⊗dx + dy⊗dy
```

```
>>> from sage.all import *
>>> g.set(b)
>>> g.display()
g = dx⊗dx + dy⊗dy
```

Another metric can be defined on `M` by specifying a metric name distinct from that chosen at the creation of the manifold (which is `g` by default, but can be changed thanks to the keyword `metric_name` in *Manifold()*):

```
sage: h = M.metric('h'); h
degenerate metric h on the 3-dimensional degenerate_metric manifold M
sage: h[1,1], h[2,2], h[3,3] = 1+y^2, 1+z^2, 1+x^2
sage: h.display()
h = (y^2 + 1) dx⊗dx + (z^2 + 1) dy⊗dy + (x^2 + 1) dz⊗dz
```

```
>>> from sage.all import *
>>> h = M.metric('h'); h
degenerate metric h on the 3-dimensional degenerate_metric manifold M
>>> h[Integer(1),Integer(1)], h[Integer(2),Integer(2)], h[Integer(3),
↪Integer(3)] = Integer(1)+y**Integer(2), Integer(1)+z**Integer(2),␣
↪Integer(1)+x**Integer(2)
>>> h.display()
h = (y^2 + 1) dx⊗dx + (z^2 + 1) dy⊗dy + (x^2 + 1) dz⊗dz
```

The metric tensor `h` is distinct from the metric entering in the definition of the degenerate manifold `M`:

```
sage: h is M.metric()
False
```

```
>>> from sage.all import *
>>> h is M.metric()
False
```

while we have of course:

```
sage: g is M.metric()
True
```

```
>>> from sage.all import *
>>> g is M.metric()
True
```

Providing the same name as the manifold's default metric returns the latter:

```
sage: M.metric('g') is M.metric()
True
```

```
>>> from sage.all import *
>>> M.metric('g') is M.metric()
True
```

open_subset(*name*, *latex_name=None*, *coord_def={}*)

Create an open subset of `self`.

An open subset is a set that is (i) included in the manifold and (ii) open with respect to the manifold's topology. It is a differentiable manifold by itself. Moreover, equipped with the restriction of the manifold metric to itself, it is a null manifold. Hence the returned object is an instance of `DegenerateManifold`.

INPUT:

- `name` – name given to the open subset
- `latex_name` – (default: `None`) LaTeX symbol to denote the subset; if none is provided, it is set to `name`
- `coord_def` – (default: `{}`) definition of the subset in terms of coordinates; `coord_def` must a be dictionary with keys charts in the manifold's atlas and values the symbolic expressions formed by the coordinates to define the subset.

OUTPUT:

- instance of `DegenerateManifold` representing the created open subset

EXAMPLES:

Open subset of a 3-dimensional degenerate manifold:

```
sage: M = Manifold(3, 'M', structure='degenerate_metric', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: U = M.open_subset('U', coord_def={X: [x>0, y>0]}); U
Open subset U of the 3-dimensional degenerate_metric manifold M
sage: type(U)
<class 'sage.manifolds.differentiable.degenerate.DegenerateManifold_with_
↪category'>
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='degenerate_metric', start_
↪index=Integer(1))
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> U = M.open_subset('U', coord_def={X: [x>Integer(0), y>Integer(0)]}); U
Open subset U of the 3-dimensional degenerate_metric manifold M
>>> type(U)
<class 'sage.manifolds.differentiable.degenerate.DegenerateManifold_with_
↪category'>
```

We initialize the metric of M:

```
sage: g = M.metric()
sage: g[1,1], g[2,2] = -1, 1
```

```
>>> from sage.all import *
>>> g = M.metric()
>>> g[Integer(1),Integer(1)], g[Integer(2),Integer(2)] = -Integer(1),
↪Integer(1)
```

Then the metric on U is determined as the restriction of g to U:

```
sage: gU = U.metric(); gU
degenerate metric g on the Open subset U of the 3-dimensional
degenerate_metric manifold M
sage: gU.display()
g = -dx⊗dx + dy⊗dy
sage: gU is g.restrict(U)
True
```

```
>>> from sage.all import *
>>> gU = U.metric(); gU
degenerate metric g on the Open subset U of the 3-dimensional
degenerate_metric manifold M
>>> gU.display()
g = -dx⊗dx + dy⊗dy
>>> gU is g.restrict(U)
True
```

class sage.manifolds.differentiable.degenerate.**TangentTensor**(*tensor*, *embedding*, *screen=None*)

Bases: *TensorFieldParal*

Let S be a lightlike submanifold embedded in a pseudo-Riemannian manifold (M,g) with Phi the embedding map. Let T1 be a tensor on M along S or not. TangentTensor(T1, Phi) returns the restriction T2 of T1 along S that in addition can be applied only on vector fields tangent to S, when T1 has a covariant part.

INPUT:

- tensor – a tensor field on the ambient manifold
- embedding – the embedding map Phi

EXAMPLES:

Section of the lightcone of the Minkowski space with a hyperplane passing through the origin:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: S = Manifold(2, 'S', ambient=M, structure='degenerate_metric')
sage: X_S.<u,v> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [sqrt(u^2+v^2), u, v, 0]},
....:                  name='Phi', latex_name=r'\Phi')
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x, y]}, name='Phi_inv',
....:                      latex_name=r'\Phi^{-1}')
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1, 1, 1, 1
sage: V = M.vector_field(0,0,0,1)
sage: S.set_transverse(rigging=t, normal=V)
sage: xi = M.vector_field(sqrt(x^2+y^2+z^2), x, y, 0)
sage: U = M.vector_field(0, -y, x, 0)
sage: Sc = S.screen('Sc', U, xi);
sage: T1 = M.tensor_field(1,1).along(Phi); T1[0,0] = 1
sage: V1 = M.vector_field().along(Phi); V1[0] = 1; V1[1]=1
sage: T1(V1).display()
∂/∂t
sage: from sage.manifolds.differentiable.degenerate_submanifold import
↪TangentTensor
sage: T2 = TangentTensor(T1, Phi)
sage: T2
Tensor field of type (1,1) along the 2-dimensional degenerate
 submanifold S embedded in 4-dimensional differentiable manifold M
 with values on the 4-dimensional Lorentzian manifold M
sage: V2 = S.projection(V1)
sage: T2(V2).display()
u/sqrt(u^2 + v^2) ∂/∂t
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> S = Manifold(Integer(2), 'S', ambient=M, structure='degenerate_metric
↪')
>>> X_S = S.chart(names=('u', 'v',)); (u, v,) = X_S._first_ngens(2)
>>> Phi = S.diff_map(M, {(X_S, X): [sqrt(u**Integer(2)+v**Integer(2)), u,
↪v, Integer(0)]},
...                  name='Phi', latex_name=r'\Phi')
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x, y]}, name='Phi_inv',
...                      latex_name=r'\Phi^{-1}')
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1), Integer(1),
↪Integer(1), Integer(1)
>>> V = M.vector_field(Integer(0),Integer(0),Integer(0),Integer(1))
>>> S.set_transverse(rigging=t, normal=V)
>>> xi = M.vector_field(sqrt(x**Integer(2)+y**Integer(2)+z**Integer(2)),
↪x, y, Integer(0))
```

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```
>>> U = M.vector_field(Integer(0), -y, x, Integer(0))
>>> Sc = S.screen('Sc', U, xi);
>>> T1 = M.tensor_field(Integer(1),Integer(1)).along(Phi); T1[Integer(0),
↪Integer(0)] = Integer(1)
>>> V1 = M.vector_field().along(Phi); V1[Integer(0)] = Integer(1);␣
↪V1[Integer(1)]=Integer(1)
>>> T1(V1).display()
∂/∂t
>>> from sage.manifolds.differentiable.degenerate_submanifold import␣
↪TangentTensor
>>> T2 = TangentTensor(T1, Phi)
>>> T2
Tensor field of type (1,1) along the 2-dimensional degenerate
 submanifold S embedded in 4-dimensional differentiable manifold M
 with values on the 4-dimensional Lorentzian manifold M
>>> V2 = S.projection(V1)
>>> T2(V2).display()
u/sqrt(u^2 + v^2) ∂/∂t
```

Of course $T1$ and $T2$ give the same output on vector fields tangent to S:

```
sage: T1(xi.along(Phi)).display()
sqrt(u^2 + v^2) ∂/∂t
sage: T2(xi.along(Phi)).display()
sqrt(u^2 + v^2) ∂/∂t
```

```
>>> from sage.all import *
>>> T1(xi.along(Phi)).display()
sqrt(u^2 + v^2) ∂/∂t
>>> T2(xi.along(Phi)).display()
sqrt(u^2 + v^2) ∂/∂t
```

extension()

Return initial tensor

EXAMPLES:

Section of the lightcone of the Minkowski space with a hyperplane passing through the origin:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: S = Manifold(2, 'S', ambient=M, structure='degenerate_metric')
sage: X_S.<u,v> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [sqrt(u^2+v^2), u, v, 0]},
....:                 name='Phi', latex_name=r'\Phi')
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x, y]}, name='Phi_inv',
....:                         latex_name=r'\Phi^{-1}')
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1,1,1,1
sage: V = M.vector_field(); V[3] = 1
sage: S.set_transverse(rigging=t, normal=V)
sage: xi = M.vector_field(); xi[0] = sqrt(x^2+y^2+z^2); xi[1] = x; xi[2] = y
```

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```
sage: U = M.vector_field(); U[1] = -y; U[2] = x
sage: Sc = S.screen('Sc', U, xi);
sage: T1 = M.tensor_field(1,1).along(Phi); T1[0,0] = 1
sage: from sage.manifolds.differentiable.degenerate_submanifold import␣
↪TangentTensor
sage: T2 = TangentTensor(T1, Phi); T3 = T2.extension()
sage: T3 is T2
False
sage: T3 is T1
True
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> S = Manifold(Integer(2), 'S', ambient=M, structure='degenerate_metric')
>>> X_S = S.chart(names=('u', 'v',)); (u, v,) = X_S._first_ngens(2)
>>> Phi = S.diff_map(M, {(X_S, X): [sqrt(u**Integer(2)+v**Integer(2)), u, v,␣
↪Integer(0)]},
...                 name='Phi', latex_name=r'\Phi')
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x, y]}, name='Phi_inv',
...                         latex_name=r'\Phi^{-1}')
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1),Integer(1),Integer(1),
↪Integer(1)
>>> V = M.vector_field(); V[Integer(3)] = Integer(1)
>>> S.set_transverse(rigging=t, normal=V)
>>> xi = M.vector_field(); xi[Integer(0)] =␣
↪sqrt(x**Integer(2)+y**Integer(2)+z**Integer(2)); xi[Integer(1)] = x;␣
↪xi[Integer(2)] = y
>>> U = M.vector_field(); U[Integer(1)] = -y; U[Integer(2)] = x
>>> Sc = S.screen('Sc', U, xi);
>>> T1 = M.tensor_field(Integer(1),Integer(1)).along(Phi); T1[Integer(0),
↪Integer(0)] = Integer(1)
>>> from sage.manifolds.differentiable.degenerate_submanifold import␣
↪TangentTensor
>>> T2 = TangentTensor(T1, Phi); T3 = T2.extension()
>>> T3 is T2
False
>>> T3 is T1
True
```

3.6.2 Degenerate submanifolds

An *embedded (resp. immersed) degenerate submanifold of a proper pseudo-Riemannian manifold* (M, g) is an embedded (resp. immersed) submanifold H of M as a differentiable manifold such that pull back of the metric tensor g via the embedding (resp. immersion) endows H with the structure of a degenerate manifold.

Degenerate submanifolds are study in many fields of mathematics and physics, for instance in Differential Geometry (especially in geometry of lightlike submanifold) and in General Relativity. In geometry of lightlike submanifolds, according to the dimension r of the radical distribution (see below for definition of radical distribution), degenerate submanifolds have been classified into 4 subgroups: r-lightlike submanifolds, Coisotropic submanifolds, Isotropic submanifolds and Totally lightlike submanifolds. (See the book of Krishan L. Duggal and Aurel Bejancu [DS2010].)

In the present module, you can define any of the 4 types but most of the methods are implemented only for degenerate hypersurfaces who belong to r-lightlike submanifolds. However, they might be generalized to 1-lightlike submanifolds. In the literature there is a new approach (the rigging technique) for studying 1-lightlike submanifolds but here we use the method of Krishan L. Duggal and Aurel Bejancu based on the screen distribution.

Let H be a lightlike hypersurface of a pseudo-Riemannian manifold (M, g). Then the normal bundle $TH^\perp$ intersect the tangent bundle TH. The radical distribution is defined as $Rad(TH) = TH \cap TH^\perp$. In case of hypersurfaces, and more generally 1-lightlike submanifolds, this is a rank 1 distribution. A screen distribution $S(TH)$ is a complementary of $Rad(TH)$ in TH.

Giving a screen distribution $S(TH)$ and a null vector field ξ locally defined and spanning $Rad(TH)$, there exists a unique transversal null vector field 'N' locally defined and such that $g(N, \xi) = 1$. From a transverse vector 'v', the null rigging 'N' is giving by the formula

$$N = \frac{1}{g(\xi, v)}\left(v - \frac{g(v, v)}{2g(\xi, v)}\xi\right)$$

Tensors on the ambient manifold M are projected on H along N to obtain induced objects. For instance, induced connection is the linear connection defined on H through the Levi-Civitta connection of g along N.

To work on a degenerate submanifold, after defining H as an instance of *DifferentiableManifold*, with the keyword `structure='degenerate_metric'`, you have to set a transvervector v and screen distribution together with the radical distribution.

An example of degenerate submanifold from General Relativity is the horizon of the Schwarzschild black hole. Allow us to recall that Schwarzschild black hole is the first non-trivial solution of Einstein's equations. It describes the metric inside a star of radius $R = 2m$, being m the inertial mass of the star. It can be seen as an open ball in a Lorentzian manifold structure on $\mathbf{R}^4$:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X_M.<t, r, th, ph> = \
....: M.chart(r"t r:(0,oo) th:(0,pi):\theta ph:(0,2*pi):\phi")
sage: var('m'); assume(m>0)
m
sage: g = M.metric()
sage: g[0,0], g[0,1], g[1,1], g[2,2], g[3,3] = \
....: -1+2*m/r, 2*m/r, 1+2*m/r, r^2, r^2*sin(th)^2
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X_M = M.chart(r"t r:(0,oo) th:(0,pi):\theta ph:(0,2*pi):\phi", names=('t', 'r',
↪'th', 'ph',)); (t, r, th, ph,) = X_M._first_ngens(4)
>>> var('m'); assume(m>Integer(0))
m
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(0),Integer(1)], g[Integer(1),Integer(1)],␣
↪g[Integer(2),Integer(2)], g[Integer(3),Integer(3)] = -Integer(1)+Integer(2)*m/r,␣
↪Integer(2)*m/r, Integer(1)+Integer(2)*m/r, r**Integer(2),␣
↪r**Integer(2)*sin(th)**Integer(2)
```

Let us define the horizon as a degenerate hypersurface:

```
sage: H = Manifold(3, 'H', ambient=M, structure='degenerate_metric')
sage: H
```

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```
degenerate hypersurface H embedded in 4-dimensional differentiable
manifold M
```

```
>>> from sage.all import *
>>> H = Manifold(Integer(3), 'H', ambient=M, structure='degenerate_metric')
>>> H
degenerate hypersurface H embedded in 4-dimensional differentiable
manifold M
```

A 2-dimensional degenerate submanifold of a Lorentzian manifold:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: S = Manifold(2, 'S', ambient=M, structure='degenerate_metric')
sage: S
2-dimensional degenerate submanifold S embedded in 4-dimensional
differentiable manifold M
sage: X_S.<u,v> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [u, u, u, v]},
....:         name='Phi', latex_name=r'\Phi');
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x,y]}, name='Phi_inv',
....:           latex_name=r'\Phi^{-1}');
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1,1,1,1
sage: S.set_transverse(rigging=[x,y])
sage: xi = M.vector_field(); xi[0] = 1; xi[1] = 1
sage: V = M.vector_field(); V[3] = 1
sage: Sc = S.screen('Sc', V, xi)

sage: S.default_screen()
screen distribution Sc along the 2-dimensional degenerate submanifold
S embedded in 4-dimensional differentiable manifold M mapped into
the 4-dimensional Lorentzian manifold M

sage: S.ambient_metric()
Lorentzian metric g on the 4-dimensional Lorentzian manifold M

sage: S.induced_metric()
degenerate metric gamma on the 2-dimensional degenerate submanifold S
embedded in 4-dimensional differentiable manifold M

sage: S.first_fundamental_form()
Field of symmetric bilinear forms g|S along the 2-dimensional
degenerate submanifold S embedded in 4-dimensional differentiable manifold M
with values on the 4-dimensional Lorentzian manifold M

sage: S.adapted_frame()
Vector frame (S, (vv_0,vv_1,vv_2,vv_3)) with values on the 4-dimensional Lorentzian
↪manifold M

sage: S.projection(V)
```

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```
Tensor field of type (1,0) along the 2-dimensional degenerate submanifold S
embedded in 4-dimensional differentiable manifold M
with values on the 4-dimensional Lorentzian manifold M

sage: S.weingarten_map()  # long time
Tensor field nabla_g(xi)|X(S) of type (1,1) along the 2-dimensional
degenerate submanifold S embedded in 4-dimensional differentiable manifold M
with values on the 4-dimensional Lorentzian manifold M

sage: SO = S.shape_operator()  # long time
sage: SO.display()             # long time
A^* = 0

sage: S.is_tangent(xi.along(Phi))
True
sage: v = M.vector_field(); v[1] = 1
sage: S.is_tangent(v.along(Phi))
False
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_ngens(4)
>>> S = Manifold(Integer(2), 'S', ambient=M, structure='degenerate_metric')
>>> S
2-dimensional degenerate submanifold S embedded in 4-dimensional
differentiable manifold M
>>> X_S = S.chart(names=('u', 'v',)); (u, v,) = X_S._first_ngens(2)
>>> Phi = S.diff_map(M, {(X_S, X): [u, u, u, v]},
...         name='Phi', latex_name=r'\Phi');
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x,y]}, name='Phi_inv',
...           latex_name=r'\Phi^{-1}');
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),Integer(2)],
↪g[Integer(3),Integer(3)] = -Integer(1),Integer(1),Integer(1),Integer(1)
>>> S.set_transverse(rigging=[x,y])
>>> xi = M.vector_field(); xi[Integer(0)] = Integer(1); xi[Integer(1)] = Integer(1)
>>> V = M.vector_field(); V[Integer(3)] = Integer(1)
>>> Sc = S.screen('Sc', V, xi)

>>> S.default_screen()
screen distribution Sc along the 2-dimensional degenerate submanifold
S embedded in 4-dimensional differentiable manifold M mapped into
the 4-dimensional Lorentzian manifold M

>>> S.ambient_metric()
Lorentzian metric g on the 4-dimensional Lorentzian manifold M

>>> S.induced_metric()
degenerate metric gamma on the 2-dimensional degenerate submanifold S
embedded in 4-dimensional differentiable manifold M
```

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```
>>> S.first_fundamental_form()
Field of symmetric bilinear forms g|S along the 2-dimensional
degenerate submanifold S embedded in 4-dimensional differentiable manifold M
with values on the 4-dimensional Lorentzian manifold M

>>> S.adapted_frame()
Vector frame (S, (vv_0,vv_1,vv_2,vv_3)) with values on the 4-dimensional Lorentzian␣
↪manifold M

>>> S.projection(V)
Tensor field of type (1,0) along the 2-dimensional degenerate submanifold S
embedded in 4-dimensional differentiable manifold M
with values on the 4-dimensional Lorentzian manifold M

>>> S.weingarten_map()  # long time
Tensor field nabla_g(xi)|X(S) of type (1,1) along the 2-dimensional
degenerate submanifold S embedded in 4-dimensional differentiable manifold M
with values on the 4-dimensional Lorentzian manifold M

>>> SO = S.shape_operator()  # long time
>>> SO.display()             # long time
A^* = 0

>>> S.is_tangent(xi.along(Phi))
True
>>> v = M.vector_field(); v[Integer(1)] = Integer(1)
>>> S.is_tangent(v.along(Phi))
False
```

AUTHORS:

- Hans Fotsing Tetsing (2019) : initial version

REFERENCES:

- [DB1996]
- [DS2010]
- [FNO2019]

class sage.manifolds.differentiable.degenerate_submanifold.**DegenerateSubmanifold**(*n*, *name*, *ambient=None*, *metric_name=None*, *signature=None*, *base_manifold=None*, *diff_degree=+Infinity*, *latex_name=None*, *metric_latex_name=None*, *start_index=0*, *category=None*, *unique_tag=None*)

Bases: *DegenerateManifold*, *DifferentiableSubmanifold*

Degenerate submanifolds.

An *embedded (resp. immersed) degenerate submanifold of a proper pseudo-Riemannian manifold* (M, g) is an embedded (resp. immersed) submanifold H of M as a differentiable manifold such that pull back of the metric tensor g via the embedding (resp. immersion) endows H with the structure of a degenerate manifold.

INPUT:

- `n` – positive integer; dimension of the manifold
- `name` – string; name (symbol) given to the manifold
- `ambient` – (default: `None`) pseudo-Riemannian manifold M in which the submanifold is embedded (or immersed). If `None`, it is set to `self`
- `metric_name` – (default: `None`) string; name (symbol) given to the metric; if `None`, `'g'` is used
- `signature` – (default: `None`) signature S of the metric as a tuple: $S = (n_+, n_-, n_0)$, where n_+ (resp. n_-, resp. n_0) is the number of positive terms (resp. negative terms, resp. zero tems) in any diagonal writing of the metric components; if `signature` is not provided, S is set to $(ndim - 1, 0, 1)$, being $ndim$ the manifold's dimension
- `base_manifold` – (default: `None`) if not `None`, must be a topological manifold; the created object is then an open subset of `base_manifold`
- `diff_degree` – (default: `infinity`) degree of differentiability
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the manifold; if none are provided, it is set to `name`
- `metric_latex_name` – (default: `None`) string; LaTeX symbol to denote the metric; if none is provided, it is set to `metric_name`

- start_index – (default: 0) integer; lower value of the range of indices used for "indexed objects" on the manifold, e.g., coordinates in a chart - category – (default: None) to specify the category; if None, Manifolds(field) is assumed (see the category Manifolds)
- unique_tag – (default: None) tag used to force the construction of a new object when all the other arguments have been used previously (without unique_tag, the UniqueRepresentation behavior inherited from *ManifoldSubset* would return the previously constructed object corresponding to these arguments)

See also:

manifold and *differentiable_submanifold*

adapted_frame(*screen=None*)

Return a frame $(e_1, \ldots, e_p, \xi_1, \ldots, \xi_r, v_1, \ldots, v_q, N_1, \ldots, N_n)$ of the ambient manifold along the submanifold, being e_i vector fields spanning the giving screen distribution, ξ_i vector fields spanning radical distribution, v_i normal transversal vector fields, N_i rigging vector fields corresponding to the giving screen.

INPUT:

- screen – (default: None) an instance of *Screen*. if None default screen is used.

OUTPUT: a frame on the ambient manifold along the submanifold

EXAMPLES:

A degenerate hyperplane the 4-dimensional Minkowski space $\mathbf{R}^4_1$:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: S = Manifold(3, 'S', ambient=M, structure='degenerate_metric')
sage: X_S.<u,v,w> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
....:         name='Phi', latex_name=r'\Phi');
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
....:           latex_name=r'\Phi^{-1}');
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1,1,1,1
sage: S.set_transverse(rigging=x)
sage: xi = M.vector_field(); xi[0] = 1; xi[1] = 1
sage: U = M.vector_field(); U[2] = 1; V = M.vector_field(); V[3] = 1
sage: Sc = S.screen('Sc', (U,V), xi);  # long time
sage: T = S.adapted_frame(); T         # long time
Vector frame (S, (vv_0,vv_1,vv_2,vv_3)) with values on the 4-dimensional
Lorentzian manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> S = Manifold(Integer(3), 'S', ambient=M, structure='degenerate_metric')
>>> X_S = S.chart(names=('u', 'v', 'w',)); (u, v, w,) = X_S._first_ngens(3)
>>> Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
...         name='Phi', latex_name=r'\Phi');
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
```

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```
...             latex_name=r'\Phi^{-1}');
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1),Integer(1),Integer(1),
↪Integer(1)
>>> S.set_transverse(rigging=x)
>>> xi = M.vector_field(); xi[Integer(0)] = Integer(1); xi[Integer(1)] =␣
↪Integer(1)
>>> U = M.vector_field(); U[Integer(2)] = Integer(1); V = M.vector_field();␣
↪V[Integer(3)] = Integer(1)
>>> Sc = S.screen('Sc', (U,V), xi);  # long time
>>> T = S.adapted_frame(); T         # long time
Vector frame (S, (vv_0,vv_1,vv_2,vv_3)) with values on the 4-dimensional
Lorentzian manifold M
```

ambient_metric()

Return the metric of the ambient manifold. The submanifold has to be embedded

OUTPUT: the metric of the ambient manifold

EXAMPLES:

The lightcone of the 3D Minkowski space:

```
sage: M = Manifold(3, 'M', structure='Lorentzian')
sage: X.<t,x,y> = M.chart()
sage: S = Manifold(2, 'S', ambient=M, structure='degenerate_metric')
sage: X_S.<u,v> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [sqrt(u^2+v^2), u, v]},
....:            name='Phi', latex_name=r'\Phi');
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x, y]}, name='Phi_inv',
....:                latex_name=r'\Phi^{-1}');
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: S.ambient_metric()
Lorentzian metric g on the 3-dimensional Lorentzian manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y',)); (t, x, y,) = X._first_ngens(3)
>>> S = Manifold(Integer(2), 'S', ambient=M, structure='degenerate_metric')
>>> X_S = S.chart(names=('u', 'v',)); (u, v,) = X_S._first_ngens(2)
>>> Phi = S.diff_map(M, {(X_S, X): [sqrt(u**Integer(2)+v**Integer(2)), u, v]},
...              name='Phi', latex_name=r'\Phi');
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x, y]}, name='Phi_inv',
...                  latex_name=r'\Phi^{-1}');
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> S.ambient_metric()
Lorentzian metric g on the 3-dimensional Lorentzian manifold M
```

default_screen()

Return the default screen distribution.

OUTPUT:

- an instance of `Screen`

EXAMPLES:

A degenerate hyperplane the 4-dimensional Minkowski space $\mathbf{R}^4_1$:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: S = Manifold(3, 'S', ambient=M, structure='degenerate_metric')
sage: X_S.<u,v,w> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
....:         name='Phi', latex_name=r'\Phi');
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
....:           latex_name=r'\Phi^{-1}');
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1,1,1,1
sage: S.set_transverse(rigging=x)
sage: xi = M.vector_field(); xi[0] = 1; xi[1] = 1
sage: U = M.vector_field(); U[2] = 1; V = M.vector_field(); V[3] = 1
sage: Sc = S.screen('Sc', (U,V), xi)  # long time
sage: S.default_screen()              # long time
screen distribution Sc along the degenerate hypersurface S embedded
in 4-dimensional differentiable manifold M mapped into the 4-dimensional
Lorentzian manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> S = Manifold(Integer(3), 'S', ambient=M, structure='degenerate_metric')
>>> X_S = S.chart(names=('u', 'v', 'w',)); (u, v, w,) = X_S._first_ngens(3)
>>> Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
...         name='Phi', latex_name=r'\Phi');
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
...           latex_name=r'\Phi^{-1}');
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1),Integer(1),Integer(1),
↪Integer(1)
>>> S.set_transverse(rigging=x)
>>> xi = M.vector_field(); xi[Integer(0)] = Integer(1); xi[Integer(1)] =␣
↪Integer(1)
>>> U = M.vector_field(); U[Integer(2)] = Integer(1); V = M.vector_field();␣
↪V[Integer(3)] = Integer(1)
>>> Sc = S.screen('Sc', (U,V), xi)  # long time
>>> S.default_screen()              # long time
screen distribution Sc along the degenerate hypersurface S embedded
in 4-dimensional differentiable manifold M mapped into the 4-dimensional
Lorentzian manifold M
```

extrinsic_curvature(*screen=None*)

This method is implemented only for null hypersurfaces. The method returns a tensor B of type $(0,2)$ instance of `TangentTensor` such that for two vector fields U, V on the ambient manifold along the null hypersurface, one has:

$$\nabla_U V = D(U,V) + B(U,V)N$$

being ∇ the ambient connection, D the induced connection and N the chosen rigging.

INPUT:

- `screen` – (default: `None`) an instance of *Screen*. If `None`, the default screen is used

OUTPUT:

- an instance of *TangentTensor*

EXAMPLES:

A degenerate hyperplane the 4-dimensional Minkowski space $\mathbf{R}^4_1$:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: S = Manifold(3, 'S', ambient=M, structure='degenerate_metric')
sage: X_S.<u,v,w> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
....:         name='Phi', latex_name=r'\Phi');
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
....:           latex_name=r'\Phi^{-1}');
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1,1,1,1
sage: S.set_transverse(rigging=x)
sage: xi = M.vector_field(); xi[0] = 1; xi[1] = 1
sage: U = M.vector_field(); U[2] = 1; V = M.vector_field(); V[3] = 1
sage: Sc = S.screen('Sc', (U,V), xi);   # long time
sage: B = S.second_fundamental_form();  # long time
sage: B.display()                       # long time
B = 0
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> S = Manifold(Integer(3), 'S', ambient=M, structure='degenerate_metric')
>>> X_S = S.chart(names=('u', 'v', 'w',)); (u, v, w,) = X_S._first_ngens(3)
>>> Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
...         name='Phi', latex_name=r'\Phi');
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
...           latex_name=r'\Phi^{-1}');
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1),Integer(1),Integer(1),
↪Integer(1)
>>> S.set_transverse(rigging=x)
>>> xi = M.vector_field(); xi[Integer(0)] = Integer(1); xi[Integer(1)] =
↪Integer(1)
>>> U = M.vector_field(); U[Integer(2)] = Integer(1); V = M.vector_field();
```

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```
↪V[Integer(3)] = Integer(1)
>>> Sc = S.screen('Sc', (U,V), xi);    # long time
>>> B = S.second_fundamental_form();   # long time
>>> B.display()                        # long time
B = 0
```

first_fundamental_form()

Return the restriction of the ambient metric on vector field along the submanifold and tangent to it. It is difference from induced metric who gives the pullback of the ambient metric on the submanifold.

OUTPUT:

- the first fundamental form, as an instance of *TangentTensor*

EXAMPLES:

A degenerate hyperplane the 4-dimensional Minkowski space $\mathbf{R}^4_1$:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: S = Manifold(3, 'S', ambient=M, structure='degenerate_metric')
sage: X_S.<u,v,w> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
....:         name='Phi', latex_name=r'\Phi');
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
....:           latex_name=r'\Phi^{-1}');
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1,1,1,1
sage: S.set_transverse(rigging=x)
sage: xi = M.vector_field(); xi[0] = 1; xi[1] = 1
sage: U = M.vector_field(); U[2] = 1; V = M.vector_field(); V[3] = 1
sage: Sc = S.screen('Sc', (U,V), xi);  # long time
sage: h = S.first_fundamental_form()   # long time
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> S = Manifold(Integer(3), 'S', ambient=M, structure='degenerate_metric')
>>> X_S = S.chart(names=('u', 'v', 'w',)); (u, v, w,) = X_S._first_ngens(3)
>>> Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
...         name='Phi', latex_name=r'\Phi');
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
...           latex_name=r'\Phi^{-1}');
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1),Integer(1),Integer(1),
↪Integer(1)
>>> S.set_transverse(rigging=x)
>>> xi = M.vector_field(); xi[Integer(0)] = Integer(1); xi[Integer(1)] =␣
↪Integer(1)
>>> U = M.vector_field(); U[Integer(2)] = Integer(1); V = M.vector_field();␣
```

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```
↪V[Integer(3)] = Integer(1)
>>> Sc = S.screen('Sc', (U,V), xi);  # long time
>>> h = S.first_fundamental_form()   # long time
```

gauss_curvature(*screen=None*)

Gauss curvature is the product of all eigenfunctions of the shape operator.

INPUT:

- `screen` – (default: `None`) an instance of `Screen`. If `None` the default screen is used.

OUTPUT: a scalar function on `self`

EXAMPLES:

A degenerate hyperplane the 4-dimensional Minkowski space $\mathbf{R}^4_1$:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: S = Manifold(3, 'S', ambient=M, structure='degenerate_metric')
sage: X_S.<u,v,w> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
....:         name='Phi', latex_name=r'\Phi');
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
....:           latex_name=r'\Phi^{-1}');
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1,1,1,1
sage: S.set_transverse(rigging=x)
sage: xi = M.vector_field(); xi[0] = 1; xi[1] = 1
sage: U = M.vector_field(); U[2] = 1; V = M.vector_field(); V[3] = 1
sage: Sc = S.screen('Sc', (U,V), xi);  # long time
sage: K = S.gauss_curvature();         # long time
sage: K.display()                      # long time
S → ℝ
(u, v, w) ↦ 0
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> S = Manifold(Integer(3), 'S', ambient=M, structure='degenerate_metric')
>>> X_S = S.chart(names=('u', 'v', 'w',)); (u, v, w,) = X_S._first_ngens(3)
>>> Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
...         name='Phi', latex_name=r'\Phi');
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
...           latex_name=r'\Phi^{-1}');
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1),Integer(1),Integer(1),
↪Integer(1)
>>> S.set_transverse(rigging=x)
>>> xi = M.vector_field(); xi[Integer(0)] = Integer(1); xi[Integer(1)] =␣
```

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```
↪Integer(1)
>>> U = M.vector_field(); U[Integer(2)] = Integer(1); V = M.vector_field();␣
↪V[Integer(3)] = Integer(1)
>>> Sc = S.screen('Sc', (U,V), xi);  # long time
>>> K = S.gauss_curvature();         # long time
>>> K.display()                      # long time
S → ℝ
(u, v, w) ↦ 0
```

induced_metric()

Return the pullback of the ambient metric.

OUTPUT:

- induced metric, as an instance of *DegenerateMetric*

EXAMPLES:

Section of the lightcone of the Minkowski space with a hyperplane passing through the origin:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: S = Manifold(2, 'S', ambient=M, structure='degenerate_metric')
sage: X_S.<u,v> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [sqrt(u^2+v^2), u, v, 0]},
....:                  name='Phi', latex_name=r'\Phi');
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x, y]}, name='Phi_inv',
....:               latex_name=r'\Phi^{-1}');
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1,1,1,1
sage: h = S.induced_metric(); h  # long time
degenerate metric gamma on the 2-dimensional degenerate
submanifold S embedded in 4-dimensional differentiable manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> S = Manifold(Integer(2), 'S', ambient=M, structure='degenerate_metric')
>>> X_S = S.chart(names=('u', 'v',)); (u, v,) = X_S._first_ngens(2)
>>> Phi = S.diff_map(M, {(X_S, X): [sqrt(u**Integer(2)+v**Integer(2)), u, v,␣
↪Integer(0)]},
...                name='Phi', latex_name=r'\Phi');
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x, y]}, name='Phi_inv',
...              latex_name=r'\Phi^{-1}');
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1),Integer(1),Integer(1),
↪Integer(1)
>>> h = S.induced_metric(); h  # long time
degenerate metric gamma on the 2-dimensional degenerate
submanifold S embedded in 4-dimensional differentiable manifold M
```

is_tangent(*v*)

Determine whether a vector field on the ambient manifold along `self` is tangent to `self` or not.

INPUT:

- `v` – field on the ambient manifold along `self`

OUTPUT:

- `True` if `v` is everywhere tangent to `self` or `False` if not

EXAMPLES:

A degenerate hyperplane the 4-dimensional Minkowski space $\mathbf{R}^4_1$:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: S = Manifold(3, 'S', ambient=M, structure='degenerate_metric')
sage: X_S.<u,v,w> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
....:         name='Phi', latex_name=r'\Phi');
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
....:           latex_name=r'\Phi^{-1}');
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1,1,1,1
sage: v = M.vector_field(); v[1] = 1
sage: S.set_transverse(rigging=v)
sage: xi = M.vector_field(); xi[0] = 1; xi[1] = 1
sage: U = M.vector_field(); U[2] = 1; V = M.vector_field(); V[3] = 1
sage: Sc = S.screen('Sc', (U,V), xi);  # long time
sage: S.is_tangent(xi.along(Phi))      # long time
True
sage: S.is_tangent(v.along(Phi))       # long time
False
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> S = Manifold(Integer(3), 'S', ambient=M, structure='degenerate_metric')
>>> X_S = S.chart(names=('u', 'v', 'w',)); (u, v, w,) = X_S._first_ngens(3)
>>> Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
...         name='Phi', latex_name=r'\Phi');
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
...           latex_name=r'\Phi^{-1}');
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1),Integer(1),Integer(1),
↪Integer(1)
>>> v = M.vector_field(); v[Integer(1)] = Integer(1)
>>> S.set_transverse(rigging=v)
>>> xi = M.vector_field(); xi[Integer(0)] = Integer(1); xi[Integer(1)] =␣
↪Integer(1)
>>> U = M.vector_field(); U[Integer(2)] = Integer(1); V = M.vector_field();␣
```

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```
↪V[Integer(3)] = Integer(1)
>>> Sc = S.screen('Sc', (U,V), xi);  # long time
>>> S.is_tangent(xi.along(Phi))       # long time
True
>>> S.is_tangent(v.along(Phi))        # long time
False
```

list_of_screens()

Return the default screen distribution.

OUTPUT:

- an instance of *Screen*

EXAMPLES:

A degenerate hyperplane the 4-dimensional Minkowski space $\mathbf{R}^4_1$:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: S = Manifold(3, 'S', ambient=M, structure='degenerate_metric')
sage: X_S.<u,v,w> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
....:                  name='Phi', latex_name=r'\Phi')
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
....:                      latex_name=r'\Phi^{-1}')
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1,1,1,1
sage: S.set_transverse(rigging=x)
sage: xi = M.vector_field(); xi[0] = 1; xi[1] = 1
sage: U = M.vector_field(); U[2] = 1; V = M.vector_field(); V[3] = 1
sage: Sc = S.screen('Sc', (U,V), xi)  # long time
sage: S.list_of_screens()             # long time
{'Sc': screen distribution Sc along the degenerate hypersurface S
embedded in 4-dimensional differentiable manifold M mapped into the
4-dimensional Lorentzian manifold M}
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> S = Manifold(Integer(3), 'S', ambient=M, structure='degenerate_metric')
>>> X_S = S.chart(names=('u', 'v', 'w',)); (u, v, w,) = X_S._first_ngens(3)
>>> Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
...                  name='Phi', latex_name=r'\Phi')
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
...                      latex_name=r'\Phi^{-1}')
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1),Integer(1),Integer(1),
↪Integer(1)
>>> S.set_transverse(rigging=x)
```

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```
>>> xi = M.vector_field(); xi[Integer(0)] = Integer(1); xi[Integer(1)] =␣
↪Integer(1)
>>> U = M.vector_field(); U[Integer(2)] = Integer(1); V = M.vector_field();␣
↪V[Integer(3)] = Integer(1)
>>> Sc = S.screen('Sc', (U,V), xi)  # long time
>>> S.list_of_screens()             # long time
{'Sc': screen distribution Sc along the degenerate hypersurface S
embedded in 4-dimensional differentiable manifold M mapped into the
4-dimensional Lorentzian manifold M}
```

mean_curvature(*screen=None*)

Mean curvature is the sum of principal curvatures. This method is implemented only for hypersurfaces.

INPUT:

- `screen` – (default: `None`) an instance of *Screen*. If `None` the default screen is used.

OUTPUT: the mean curvature, as a scalar field on the submanifold

EXAMPLES:

A degenerate hyperplane the 4-dimensional Minkowski space $\mathbf{R}^4_1$:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: S = Manifold(3, 'S', ambient=M, structure='degenerate_metric')
sage: X_S.<u,v,w> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
....:         name='Phi', latex_name=r'\Phi');
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
....:           latex_name=r'\Phi^{-1}');
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1,1,1,1
sage: S.set_transverse(rigging=x)
sage: xi = M.vector_field(); xi[0] = 1; xi[1] = 1
sage: U = M.vector_field(); U[2] = 1; V = M.vector_field(); V[3] = 1
sage: Sc = S.screen('Sc', (U,V), xi);  # long time
sage: m = S.mean_curvature(); m        # long time
Scalar field on the degenerate hypersurface S embedded in 4-dimensional
differentiable manifold M
sage: m.display()                      # long time
S → ℝ
(u, v, w) ↦ 0
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> S = Manifold(Integer(3), 'S', ambient=M, structure='degenerate_metric')
>>> X_S = S.chart(names=('u', 'v', 'w',)); (u, v, w,) = X_S._first_ngens(3)
>>> Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
...         name='Phi', latex_name=r'\Phi');
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
```

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```
...             latex_name=r'\Phi^{-1}');
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1),Integer(1),Integer(1),
↪Integer(1)
>>> S.set_transverse(rigging=x)
>>> xi = M.vector_field(); xi[Integer(0)] = Integer(1); xi[Integer(1)] =␣
↪Integer(1)
>>> U = M.vector_field(); U[Integer(2)] = Integer(1); V = M.vector_field();␣
↪V[Integer(3)] = Integer(1)
>>> Sc = S.screen('Sc', (U,V), xi);  # long time
>>> m = S.mean_curvature(); m        # long time
Scalar field on the degenerate hypersurface S embedded in 4-dimensional
differentiable manifold M
>>> m.display()                      # long time
S → ℝ
(u, v, w) ↦ 0
```

principal_directions(*screen=None*)

Principal directions are eigenvectors of the shape operator. This method is implemented only for hypersurfaces.

INPUT:

- `screen` – (default: `None`) an instance of *Screen*. If `None` default screen is used.

OUTPUT:

- list of pairs (vector field, scalar field) representing the principal directions and the associated principal curvatures

EXAMPLES:

A degenerate hyperplane the 4-dimensional Minkowski space $\mathbf{R}^4_1$:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: S = Manifold(3, 'S', ambient=M, structure='degenerate_metric')
sage: X_S.<u,v,w> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
....:         name='Phi', latex_name=r'\Phi');
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
....:           latex_name=r'\Phi^{-1}');
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1,1,1,1
sage: S.set_transverse(rigging=x)
sage: xi = M.vector_field(); xi[0] = 1; xi[1] = 1
sage: U = M.vector_field(); U[2] = 1; V = M.vector_field(); V[3] = 1
sage: Sc = S.screen('Sc', (U,V), xi); T = S.adapted_frame()  # long time
sage: PD = S.principal_directions()                          # long time
sage: PD[2][0].display(T)                                    # long time
e_2 = xi
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> S = Manifold(Integer(3), 'S', ambient=M, structure='degenerate_metric')
>>> X_S = S.chart(names=('u', 'v', 'w',)); (u, v, w,) = X_S._first_ngens(3)
>>> Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
...         name='Phi', latex_name=r'\Phi');
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
...           latex_name=r'\Phi^{-1}');
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1),Integer(1),Integer(1),
↪Integer(1)
>>> S.set_transverse(rigging=x)
>>> xi = M.vector_field(); xi[Integer(0)] = Integer(1); xi[Integer(1)] =␣
↪Integer(1)
>>> U = M.vector_field(); U[Integer(2)] = Integer(1); V = M.vector_field();␣
↪V[Integer(3)] = Integer(1)
>>> Sc = S.screen('Sc', (U,V), xi); T = S.adapted_frame()   # long time
>>> PD = S.principal_directions()                           # long time
>>> PD[Integer(2)][Integer(0)].display(T)
↪# long time
e_2 = xi
```

projection(*tensor*, *screen=None*)

For a given tensor T of type $(r, 1)$ on the ambient manifold, this method returns the tensor T' of type $(r, 1)$ such that for r vector fields $v_1, \dots, v_r$, $T'(v_1, \dots, v_r)$ is the projection of $T(v_1, \dots, v_r)$ on `self` along the bundle spanned by the transversal vector fields provided by *set_transverse()*.

INPUT:

- `tensor` – a tensor of type $(r, 1)$ on the ambient manifold

OUTPUT:

- a tensor of type $(r, 1)$ on the ambient manifold along `self`

EXAMPLES:

A degenerate hyperplane the 4-dimensional Minkowski space $\mathbf{R}^4_1$:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: S = Manifold(3, 'S', ambient=M, structure='degenerate_metric')
sage: X_S.<u,v,w> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
....:         name='Phi', latex_name=r'\Phi');
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
....:           latex_name=r'\Phi^{-1}');
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1,1,1,1
sage: S.set_transverse(rigging=x)
sage: xi = M.vector_field(); xi[0] = 1; xi[1] = 1
```

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```
sage: U = M.vector_field(); U[2] = 1; V = M.vector_field(); V[3] = 1
sage: Sc = S.screen('Sc', (U,V), xi);  # long time
sage: U1 = S.projection(U)             # long time
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> S = Manifold(Integer(3), 'S', ambient=M, structure='degenerate_metric')
>>> X_S = S.chart(names=('u', 'v', 'w',)); (u, v, w,) = X_S._first_ngens(3)
>>> Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
...         name='Phi', latex_name=r'\Phi');
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
...           latex_name=r'\Phi^{-1}');
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1),Integer(1),Integer(1),
↪Integer(1)
>>> S.set_transverse(rigging=x)
>>> xi = M.vector_field(); xi[Integer(0)] = Integer(1); xi[Integer(1)] =␣
↪Integer(1)
>>> U = M.vector_field(); U[Integer(2)] = Integer(1); V = M.vector_field();␣
↪V[Integer(3)] = Integer(1)
>>> Sc = S.screen('Sc', (U,V), xi);  # long time
>>> U1 = S.projection(U)             # long time
```

screen (*name*, *screen*, *rad*, *latex_name=None*)

For setting a screen distribution and vector fields of the radical distribution that will be used for computations

INPUT:

- `name` – string (default: `None`); name given to the screen
- `latex_name` – string (default: `None`); LaTeX symbol to denote the screen; if `None`, the LaTeX symbol is set to `name`
- `screen` – list or tuple of vector fields of the ambient manifold or chart function; of the ambient manifold in the latter case, the corresponding gradient vector field with respect to the ambient metric is calculated; the vectors must be linearly independent, tangent to the submanifold but not normal
- `rad` – – list or tuple of vector fields of the ambient manifold or chart function; of the ambient manifold in the latter case, the corresponding gradient vector field with respect to the ambient metric is calculated; the vectors must be linearly independent, tangent and normal to the submanifold

EXAMPLES:

A degenerate hyperplane the 4-dimensional Minkowski space $\mathbf{R}^4_1$:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: S = Manifold(3, 'S', ambient=M, structure='degenerate_metric')
sage: X_S.<u,v,w> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
....:         name='Phi', latex_name=r'\Phi');
```

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```
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
....:               latex_name=r'\Phi^{-1}');
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1,1,1,1
sage: S.set_transverse(rigging=x)
sage: xi = M.vector_field(); xi[0] = 1; xi[1] = 1
sage: U = M.vector_field(); U[2] = 1; V = M.vector_field(); V[3] = 1
sage: Sc = S.screen('Sc', (U,V), xi); Sc  # long time
screen distribution Sc along the degenerate hypersurface S embedded
in 4-dimensional differentiable manifold M mapped into the 4-dimensional
Lorentzian manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> S = Manifold(Integer(3), 'S', ambient=M, structure='degenerate_metric')
>>> X_S = S.chart(names=('u', 'v', 'w',)); (u, v, w,) = X_S._first_ngens(3)
>>> Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
...         name='Phi', latex_name=r'\Phi');
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
...             latex_name=r'\Phi^{-1}');
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1),Integer(1),Integer(1),
↪Integer(1)
>>> S.set_transverse(rigging=x)
>>> xi = M.vector_field(); xi[Integer(0)] = Integer(1); xi[Integer(1)] =␣
↪Integer(1)
>>> U = M.vector_field(); U[Integer(2)] = Integer(1); V = M.vector_field();␣
↪V[Integer(3)] = Integer(1)
>>> Sc = S.screen('Sc', (U,V), xi); Sc  # long time
screen distribution Sc along the degenerate hypersurface S embedded
in 4-dimensional differentiable manifold M mapped into the 4-dimensional
Lorentzian manifold M
```

screen_projection (*tensor*, *screen=None*)

For a given tensor T of type $(r,1)$ on the ambient manifold, this method returns the tensor T' of type $(r,1)$ such that for r vector fields $v_1,\ldots,v_r$, $T'(v_1,\ldots,v_r)$ is the projection of $T(v_1,\ldots,v_r)$ on the bundle spanned by `screen` along the bundle spanned by the transversal plus the radical vector fields provided.

INPUT:

- `tensor` – a tensor of type $(r,1)$ on the ambient manifold

OUTPUT:

- a tensor of type $(r,1)$ on the ambient manifold

EXAMPLES:

A degenerate hyperplane the 4-dimensional Minkowski space $\mathbf{R}^4_1$:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: S = Manifold(3, 'S', ambient=M, structure='degenerate_metric')
sage: X_S.<u,v,w> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
....:         name='Phi', latex_name=r'\Phi');
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
....:           latex_name=r'\Phi^{-1}');
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1,1,1,1
sage: S.set_transverse(rigging=x)
sage: xi = M.vector_field(); xi[0] = 1; xi[1] = 1
sage: U = M.vector_field(); U[2] = 1; V = M.vector_field(); V[3] = 1
sage: Sc = S.screen('Sc', (U,V), xi);  # long time
sage: U1 = S.screen_projection(U);     # long time
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> S = Manifold(Integer(3), 'S', ambient=M, structure='degenerate_metric')
>>> X_S = S.chart(names=('u', 'v', 'w',)); (u, v, w,) = X_S._first_ngens(3)
>>> Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
...         name='Phi', latex_name=r'\Phi');
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
...           latex_name=r'\Phi^{-1}');
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1),Integer(1),Integer(1),
↪Integer(1)
>>> S.set_transverse(rigging=x)
>>> xi = M.vector_field(); xi[Integer(0)] = Integer(1); xi[Integer(1)] =␣
↪Integer(1)
>>> U = M.vector_field(); U[Integer(2)] = Integer(1); V = M.vector_field();␣
↪V[Integer(3)] = Integer(1)
>>> Sc = S.screen('Sc', (U,V), xi);  # long time
>>> U1 = S.screen_projection(U);     # long time
```

second_fundamental_form(*screen=None*)

This method is implemented only for null hypersurfaces. The method returns a tensor B of type $(0,2)$ instance of *TangentTensor* such that for two vector fields U, V on the ambient manifold along the null hypersurface, one has:

$$\nabla_U V = D(U,V) + B(U,V)N$$

being ∇ the ambient connection, D the induced connection and N the chosen rigging.

INPUT:

- `screen` – (default: `None`) an instance of *Screen*. If `None`, the default screen is used

OUTPUT:

- an instance of *TangentTensor*

EXAMPLES:

A degenerate hyperplane the 4-dimensional Minkowski space $\mathbf{R}^4_1$:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: S = Manifold(3, 'S', ambient=M, structure='degenerate_metric')
sage: X_S.<u,v,w> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
....:         name='Phi', latex_name=r'\Phi');
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
....:           latex_name=r'\Phi^{-1}');
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1,1,1,1
sage: S.set_transverse(rigging=x)
sage: xi = M.vector_field(); xi[0] = 1; xi[1] = 1
sage: U = M.vector_field(); U[2] = 1; V = M.vector_field(); V[3] = 1
sage: Sc = S.screen('Sc', (U,V), xi);   # long time
sage: B = S.second_fundamental_form();  # long time
sage: B.display()                        # long time
B = 0
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> S = Manifold(Integer(3), 'S', ambient=M, structure='degenerate_metric')
>>> X_S = S.chart(names=('u', 'v', 'w',)); (u, v, w,) = X_S._first_ngens(3)
>>> Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
...         name='Phi', latex_name=r'\Phi');
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
...           latex_name=r'\Phi^{-1}');
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1),Integer(1),Integer(1),
↪Integer(1)
>>> S.set_transverse(rigging=x)
>>> xi = M.vector_field(); xi[Integer(0)] = Integer(1); xi[Integer(1)] =␣
↪Integer(1)
>>> U = M.vector_field(); U[Integer(2)] = Integer(1); V = M.vector_field();␣
↪V[Integer(3)] = Integer(1)
>>> Sc = S.screen('Sc', (U,V), xi);   # long time
>>> B = S.second_fundamental_form();  # long time
>>> B.display()                        # long time
B = 0
```

set_transverse(*rigging=None*, *normal=None*)

For setting a transversal distribution of the degenerate submanifold.

According to the type of the submanifold among the 4 possible types, one must enter a list of normal transversal vector fields and/or a list of transversal and not normal vector fields spanning a transverse distribution.

INPUT:

- `rigging` – list or tuple (default: `None`); list of vector fields of the ambient manifold or chart function; of the ambient manifold in the latter case, the corresponding gradient vector field with respect to the ambient metric is calculated; the vectors must be linearly independent, transversal to the submanifold but not normal
- `normal` – list or tuple (default: `None`); list of vector fields of the ambient manifold or chart function; of the ambient manifold in the latter case, the corresponding gradient vector field with respect to the ambient metric is calculated; the vectors must be linearly independent, transversal and normal to the submanifold

EXAMPLES:

The lightcone of the 3-dimensional Minkowski space $\mathbf{R}^3_1$:

```
sage: M = Manifold(3, 'M', structure='Lorentzian')
sage: X.<t,x,y> = M.chart()
sage: S = Manifold(2, 'S', ambient=M, structure='degenerate_metric')
sage: X_S.<u,v> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [sqrt(u^2+v^2), u, v]},
....:                  name='Phi', latex_name=r'\Phi')
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x, y]}, name='Phi_inv',
....:                          latex_name=r'\Phi^{-1}')
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2] = -1,1,1
sage: S.set_transverse(rigging=t)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y',)); (t, x, y,) = X._first_ngens(3)
>>> S = Manifold(Integer(2), 'S', ambient=M, structure='degenerate_metric')
>>> X_S = S.chart(names=('u', 'v',)); (u, v,) = X_S._first_ngens(2)
>>> Phi = S.diff_map(M, {(X_S, X): [sqrt(u**Integer(2)+v**Integer(2)), u, v]},
...                  name='Phi', latex_name=r'\Phi')
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x, y]}, name='Phi_inv',
...                          latex_name=r'\Phi^{-1}')
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)] = -Integer(1),Integer(1),Integer(1)
>>> S.set_transverse(rigging=t)
```

shape_operator(*screen=None*)

This method is implemented only for hypersurfaces. *shape operator* is the projection of the Weingarten map on the screen distribution along the radical distribution.

INPUT:

- `screen` – (default: `None`) an instance of *Screen*. If `None` the default screen is used.

OUTPUT:

- tensor of type $(1, 1)$ instance of *TangentTensor*

EXAMPLES:

A degenerate hyperplane the 4-dimensional Minkowski space $\mathbf{R}^4_1$:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: S = Manifold(3, 'S', ambient=M, structure='degenerate_metric')
sage: X_S.<u,v,w> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
....:         name='Phi', latex_name=r'\Phi');
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
....:           latex_name=r'\Phi^{-1}');
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1,1,1,1
sage: v = M.vector_field(); v[1] = 1
sage: S.set_transverse(rigging=v)
sage: xi = M.vector_field(); xi[0] = 1; xi[1] = 1
sage: U = M.vector_field(); U[2] = 1; V = M.vector_field(); V[3] = 1
sage: Sc = S.screen('Sc', (U,V), xi);  # long time
sage: SO = S.shape_operator();         # long time
sage: SO.display()                     # long time
A^* = 0
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> S = Manifold(Integer(3), 'S', ambient=M, structure='degenerate_metric')
>>> X_S = S.chart(names=('u', 'v', 'w',)); (u, v, w,) = X_S._first_ngens(3)
>>> Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
...         name='Phi', latex_name=r'\Phi');
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
...           latex_name=r'\Phi^{-1}');
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1),Integer(1),Integer(1),
↪Integer(1)
>>> v = M.vector_field(); v[Integer(1)] = Integer(1)
>>> S.set_transverse(rigging=v)
>>> xi = M.vector_field(); xi[Integer(0)] = Integer(1); xi[Integer(1)] =␣
↪Integer(1)
>>> U = M.vector_field(); U[Integer(2)] = Integer(1); V = M.vector_field();␣
↪V[Integer(3)] = Integer(1)
>>> Sc = S.screen('Sc', (U,V), xi);  # long time
>>> SO = S.shape_operator();         # long time
>>> SO.display()                     # long time
A^* = 0
```

weingarten_map(*screen=None*)

This method is implemented only for hypersurfaces. *Weigarten map* is the 1-form W defined for a vector field U tangent to `self` by

$$W(U) = \nabla_U \xi$$

being ∇ the Levi-Civita connection of the ambient manifold and ξ the chosen vector field spanning the radical distribution.

INPUT:

- `screen` – (default: `None`) an instance of `Screen`. If `None` the default screen is used.

OUTPUT:

- tensor of type $(1, 1)$ instance of `TangentTensor`

EXAMPLES:

A degenerate hyperplane the 4-dimensional Minkowski space $\mathbf{R}^4_1$:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: S = Manifold(3, 'S', ambient=M, structure='degenerate_metric')
sage: X_S.<u,v,w> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
....:         name='Phi', latex_name=r'\Phi');
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
....:           latex_name=r'\Phi^{-1}');
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1,1,1,1
sage: v = M.vector_field(); v[1] = 1
sage: S.set_transverse(rigging=v)
sage: xi = M.vector_field(); xi[0] = 1; xi[1] = 1
sage: U = M.vector_field(); U[2] = 1; V = M.vector_field(); V[3] = 1
sage: Sc = S.screen('Sc', (U,V), xi);  # long time
sage: W = S.weingarten_map();          # long time
sage: W.display()                      # long time
nabla_g(xi)|X(S) = 0
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> S = Manifold(Integer(3), 'S', ambient=M, structure='degenerate_metric')
>>> X_S = S.chart(names=('u', 'v', 'w',)); (u, v, w,) = X_S._first_ngens(3)
>>> Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
...         name='Phi', latex_name=r'\Phi');
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
...           latex_name=r'\Phi^{-1}');
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1),Integer(1),Integer(1),
↪Integer(1)
>>> v = M.vector_field(); v[Integer(1)] = Integer(1)
>>> S.set_transverse(rigging=v)
>>> xi = M.vector_field(); xi[Integer(0)] = Integer(1); xi[Integer(1)] =
↪Integer(1)
>>> U = M.vector_field(); U[Integer(2)] = Integer(1); V = M.vector_field();
↪V[Integer(3)] = Integer(1)
>>> Sc = S.screen('Sc', (U,V), xi);  # long time
>>> W = S.weingarten_map();          # long time
```

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```
>>> W.display()                              # long time
nabla_g(xi)|X(S) = 0
```

class sage.manifolds.differentiable.degenerate_submanifold.**Screen**(*submanifold*, *name*, *screen*, *rad*, *latex_name=None*)

Bases: *VectorFieldModule*

Let H be a lightlike submanifold embedded in a pseudo-Riemannian manifold (M, g) with Φ the embedding map. A screen distribution is a complementary $S(TH)$ of the radical distribution $Rad(TM) = TH \cap TH^{\perp}$ in TH. One then has

$$TH = S(TH) \oplus_{orth} Rad(TH)$$

INPUT:

- submanifold – a lightlike submanifold, as an instance of *DegenerateSubmanifold*
- name – name given to the screen distribution
- screen – vector fields of the ambient manifold which span the screen distribution
- rad – vector fields of the ambient manifold which span the radical distribution
- latex_name – (default: None) LaTeX symbol to denote the screen distribution; if None, it is formed from name

EXAMPLES:

The horizon of the Schwarzschild black hole:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X_M.<t, r, th, ph> = \
....: M.chart(r"t r:(0,oo) th:(0,pi):\theta ph:(0,2*pi):\phi")
sage: var('m'); assume(m>0)
m
sage: g = M.metric()
sage: g[0,0], g[0,1], g[1,1], g[2,2], g[3,3] = \
....: -1+2*m/r, 2*m/r, 1+2*m/r, r^2, r^2*sin(th)^2
sage: H = Manifold(3, 'H', ambient=M, structure='degenerate_metric')
sage: X_H.<ht,hth,hph> = \
....: H.chart(r"ht:(-oo,oo):t hth:(0,pi):\theta hph:(0,2*pi):\phi")
sage: Phi = H.diff_map(M, {(X_H, X_M): [ht, 2*m,hth, hph]}, \
....: name='Phi', latex_name=r'\Phi')
sage: Phi_inv = M.diff_map(H, {(X_M, X_H): [t,th, ph]}, \
....: name='Phi_inv', latex_name=r'\Phi^{-1}')
sage: H.set_immersion(Phi, inverse=Phi_inv); H.declare_embedding()
sage: xi = M.vector_field(-1, 0, 0, 0)
sage: v = M.vector_field(r, -r, 0, 0)
sage: e1 = M.vector_field(0, 0, 1, 0)
sage: e2 = M.vector_field(0, 0, 0, 1)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X_M = M.chart(r"t r:(0,oo) th:(0,pi):\theta ph:(0,2*pi):\phi", names=('t', 'r
↪', 'th', 'ph',)); (t, r, th, ph,) = X_M._first_ngens(4)
>>> var('m'); assume(m>Integer(0))
```

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```
m
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(0),Integer(1)], g[Integer(1),Integer(1)],␣
↪g[Integer(2),Integer(2)], g[Integer(3),Integer(3)] = -Integer(1)+Integer(2)*m/r,
↪ Integer(2)*m/r, Integer(1)+Integer(2)*m/r, r**Integer(2),␣
↪r**Integer(2)*sin(th)**Integer(2)
>>> H = Manifold(Integer(3), 'H', ambient=M, structure='degenerate_metric')
>>> X_H = H.chart(r"ht:(-oo,oo):t hth:(0,pi):\theta hph:(0,2*pi):\phi", names=('ht
↪', 'hth', 'hph',)); (ht, hth, hph,) = X_H._first_ngens(3)
>>> Phi = H.diff_map(M, {(X_H, X_M): [ht, Integer(2)*m,hth, hph]}, name='Phi',␣
↪latex_name=r'\Phi')
>>> Phi_inv = M.diff_map(H, {(X_M, X_H): [t,th, ph]}, name='Phi_inv', latex_name=r
↪'\Phi^{-1}')
>>> H.set_immersion(Phi, inverse=Phi_inv); H.declare_embedding()
>>> xi = M.vector_field(-Integer(1), Integer(0), Integer(0), Integer(0))
>>> v = M.vector_field(r, -r, Integer(0), Integer(0))
>>> e1 = M.vector_field(Integer(0), Integer(0), Integer(1), Integer(0))
>>> e2 = M.vector_field(Integer(0), Integer(0), Integer(0), Integer(1))
```

A screen distribution for the Schwarzschild black hole horizon:

```
sage: H.set_transverse(rigging=v)
sage: S = H.screen('S', [e1, e2], (xi)); S  # long time
screen distribution S along the degenerate hypersurface H embedded
in 4-dimensional differentiable manifold M mapped into the
4-dimensional Lorentzian manifold M
```

```
>>> from sage.all import *
>>> H.set_transverse(rigging=v)
>>> S = H.screen('S', [e1, e2], (xi)); S  # long time
screen distribution S along the degenerate hypersurface H embedded
in 4-dimensional differentiable manifold M mapped into the
4-dimensional Lorentzian manifold M
```

The corresponding normal tangent null vector field and null transversal vector field:

```
sage: xi = S.normal_tangent_vector(); xi.display()  # long time
xi = -∂/∂t
sage: N = S.rigging(); N.display()  # long time
N = ∂/∂t - ∂/∂r
```

```
>>> from sage.all import *
>>> xi = S.normal_tangent_vector(); xi.display()  # long time
xi = -∂/∂t
>>> N = S.rigging(); N.display()  # long time
N = ∂/∂t - ∂/∂r
```

Those vector fields are normalized by $g(\xi, N) = 1$:

```
sage: g.along(Phi)(xi, N).display()  # long time
g(xi,N): H → ℝ
(ht, hth, hph) ↦ 1
```

```
>>> from sage.all import *
>>> g.along(Phi)(xi, N).display()  # long time
g(xi,N): H → ℝ
(ht, hth, hph) ↦ 1
```

normal_tangent_vector()

Return either a list Rad of vector fields spanning the radical distribution or (in case of a hypersurface) a normal tangent null vector field spanning the radical distribution.

OUTPUT:

- either a list of vector fields or a single vector field in case of a hypersurface

EXAMPLES:

A degenerate hyperplane the 4-dimensional Minkowski space $\mathbf{R}^4_1$:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: S = Manifold(3, 'S', ambient=M, structure='degenerate_metric')
sage: X_S.<u,v,w> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
....:                 name='Phi', latex_name=r'\Phi');
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
....:                     latex_name=r'\Phi^{-1}');
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1,1,1,1
sage: v = M.vector_field(); v[1] = 1
sage: S.set_transverse(rigging=v)
sage: xi = M.vector_field(); xi[0] = 1; xi[1] = 1
sage: U = M.vector_field(); U[2] = 1; V = M.vector_field(); V[3] = 1
sage: Sc = S.screen('Sc', (U,V), xi);                      # long time
sage: Rad = Sc.normal_tangent_vector(); Rad.display()  # long time
xi = ∂/∂t + ∂/∂x
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> S = Manifold(Integer(3), 'S', ambient=M, structure='degenerate_metric')
>>> X_S = S.chart(names=('u', 'v', 'w',)); (u, v, w,) = X_S._first_ngens(3)
>>> Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
...                 name='Phi', latex_name=r'\Phi');
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
...                     latex_name=r'\Phi^{-1}');
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1),Integer(1),Integer(1),
↪Integer(1)
>>> v = M.vector_field(); v[Integer(1)] = Integer(1)
>>> S.set_transverse(rigging=v)
>>> xi = M.vector_field(); xi[Integer(0)] = Integer(1); xi[Integer(1)] =␣
```

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```
↪Integer(1)
>>> U = M.vector_field(); U[Integer(2)] = Integer(1); V = M.vector_field();␣
↪V[Integer(3)] = Integer(1)
>>> Sc = S.screen('Sc', (U,V), xi);                 # long time
>>> Rad = Sc.normal_tangent_vector(); Rad.display()  # long time
xi = ∂/∂t + ∂/∂x
```

rigging()

Return either a list Rad of vector fields spanning the complementary of the normal distribution $TH^{\perp}$ in the transverse bundle or (when H is a null hypersurface) the null transversal vector field defined in [DB1996].

OUTPUT:

- either a list made by vector fields or a vector field in case of hypersurface

EXAMPLES:

A degenerate hyperplane the 4-dimensional Minkowski space $\mathbf{R}^4_1$:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: S = Manifold(3, 'S', ambient=M, structure='degenerate_metric')
sage: X_S.<u,v,w> = S.chart()
sage: Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
....:         name='Phi', latex_name=r'\Phi');
sage: Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
....:           latex_name=r'\Phi^{-1}');
sage: S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1,1,1,1
sage: v = M.vector_field(); v[1] = 1
sage: S.set_transverse(rigging=v)
sage: xi = M.vector_field(); xi[0] = 1; xi[1] = 1
sage: U = M.vector_field(); U[2] = 1; V = M.vector_field(); V[3] = 1
sage: Sc = S.screen('Sc', (U,V), xi);     # long time
sage: rig = Sc.rigging(); rig.display()   # long time
N = -1/2 ∂/∂t + 1/2 ∂/∂x
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(4), 'M', structure='Lorentzian')
>>> X = M.chart(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = X._first_
↪ngens(4)
>>> S = Manifold(Integer(3), 'S', ambient=M, structure='degenerate_metric')
>>> X_S = S.chart(names=('u', 'v', 'w',)); (u, v, w,) = X_S._first_ngens(3)
>>> Phi = S.diff_map(M, {(X_S, X): [u, u, v, w]},
...         name='Phi', latex_name=r'\Phi');
>>> Phi_inv = M.diff_map(S, {(X, X_S): [x,y, z]}, name='Phi_inv',
...           latex_name=r'\Phi^{-1}');
>>> S.set_immersion(Phi, inverse=Phi_inv); S.declare_embedding()
>>> g = M.metric()
>>> g[Integer(0),Integer(0)], g[Integer(1),Integer(1)], g[Integer(2),
↪Integer(2)], g[Integer(3),Integer(3)] = -Integer(1),Integer(1),Integer(1),
↪Integer(1)
>>> v = M.vector_field(); v[Integer(1)] = Integer(1)
```

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```
>>> S.set_transverse(rigging=v)
>>> xi = M.vector_field(); xi[Integer(0)] = Integer(1); xi[Integer(1)] =␣
↪Integer(1)
>>> U = M.vector_field(); U[Integer(2)] = Integer(1); V = M.vector_field();␣
↪V[Integer(3)] = Integer(1)
>>> Sc = S.screen('Sc', (U,V), xi);     # long time
>>> rig = Sc.rigging(); rig.display()  # long time
N = -1/2 ∂/∂t + 1/2 ∂/∂x
```

CHAPTER

FOUR

POISSON MANIFOLDS

4.1 Poisson tensors

AUTHORS:

- Tobias Diez (2021): initial version

class sage.manifolds.differentiable.poisson_tensor.**PoissonTensorField**(*manifold: DifferentiableManifold | VectorFieldModule, name: str | None = 'varpi', latex_name: str | None = '\\varpi'*)

Bases: *MultivectorField*

A Poisson bivector field ϖ on a differentiable manifold.

That is, at each point $m \in M$, ϖ_m is a bilinear map of the type:

$$\varpi_m : T_m^*M \times T_m^*M \to \mathbf{R}$$

where T_m^*M stands for the cotangent space to the manifold M at the point m, such that ϖ_m is skew-symmetric and the Schouten-Nijenhuis bracket (cf. *bracket()*) of ϖ with itself vanishes.

INPUT:

- manifold – module $\mathfrak{X}(M)$ of vector fields on the manifold M, or the manifold M itself
- name – (default: varpi) name given to the Poisson tensor
- latex_name – (default: \\varpi) LaTeX symbol to denote the Poisson tensor; if None, it is formed from name

EXAMPLES:

A Poisson tensor on the 2-sphere:

```
sage: M.<x,y> = manifolds.Sphere(2, coordinates='stereographic')
sage: stereoN = M.stereographic_coordinates(pole='north')
sage: stereoS = M.stereographic_coordinates(pole='south')
sage: varpi = M.poisson_tensor(name='varpi', latex_name=r'\varpi')
sage: varpi
2-vector field varpi on the 2-sphere S^2 of radius 1 smoothly embedded
 in the Euclidean space E^3
```

```
>>> from sage.all import *
>>> M = manifolds.Sphere(Integer(2), coordinates='stereographic', names=('x', 'y',
↪)); (x, y,) = M._first_ngens(2)
>>> stereoN = M.stereographic_coordinates(pole='north')
>>> stereoS = M.stereographic_coordinates(pole='south')
>>> varpi = M.poisson_tensor(name='varpi', latex_name=r'\varpi')
>>> varpi
2-vector field varpi on the 2-sphere S^2 of radius 1 smoothly embedded
 in the Euclidean space E^3
```

`varpi` is initialized by providing its single nonvanishing component w.r.t. the vector frame associated to `stereoN`, which is the default frame on `M`:

```
sage: varpi[1, 2] = 1
```

```
>>> from sage.all import *
>>> varpi[Integer(1), Integer(2)] = Integer(1)
```

The components w.r.t. the vector frame associated to `stereoS` are obtained thanks to the method *add_comp_by_continuation()*:

```
sage: varpi.add_comp_by_continuation(stereoS.frame(),
....:                   stereoS.domain().intersection(stereoN.domain()))
sage: varpi.display()
varpi = ∂/∂x∧∂/∂y
sage: varpi.display(stereoS)
varpi = (-xp^4 - 2*xp^2*yp^2 - yp^4) ∂/∂xp∧∂/∂yp
```

```
>>> from sage.all import *
>>> varpi.add_comp_by_continuation(stereoS.frame(),
...                   stereoS.domain().intersection(stereoN.domain()))
>>> varpi.display()
varpi = ∂/∂x∧∂/∂y
>>> varpi.display(stereoS)
varpi = (-xp^4 - 2*xp^2*yp^2 - yp^4) ∂/∂xp∧∂/∂yp
```

The Schouten-Nijenhuis bracket of a Poisson tensor with itself vanishes (this is trivial here, since `M` is 2-dimensional):

```
sage: varpi.bracket(varpi).display()
[varpi,varpi] = 0
```

```
>>> from sage.all import *
>>> varpi.bracket(varpi).display()
[varpi,varpi] = 0
```

hamiltonian_vector_field(*function*)

Return the Hamiltonian vector field X_f generated by the given function $f : M \to \mathbf{R}$.

The Hamiltonian vector field is defined by

$$X_f = -\varpi^\sharp(df),$$

where $\varpi^\sharp : T^*M \to TM$ is given by $\beta(\varpi^\sharp(\alpha)) = \varpi(\alpha, \beta)$.

INPUT:

- `function` – the function generating the Hamiltonian vector field

EXAMPLES:

```
sage: M.<q, p> = EuclideanSpace(2)
sage: poisson = M.poisson_tensor('varpi')
sage: poisson.set_comp()[1,2] = -1
sage: f = M.scalar_field(function('f')(q, p), name='f')
sage: Xf = poisson.hamiltonian_vector_field(f)
sage: Xf.display()
Xf = d(f)/dp e_q - d(f)/dq e_p
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(Integer(2), names=('q', 'p',)); (q, p,) = M._first_
↪ngens(2)
>>> poisson = M.poisson_tensor('varpi')
>>> poisson.set_comp()[Integer(1),Integer(2)] = -Integer(1)
>>> f = M.scalar_field(function('f')(q, p), name='f')
>>> Xf = poisson.hamiltonian_vector_field(f)
>>> Xf.display()
Xf = d(f)/dp e_q - d(f)/dq e_p
```

poisson_bracket(f, g)

Return the Poisson bracket

$$\{f, g\} = \varpi(df, dg)$$

of the given functions.

INPUT:

- `f` – first function
- `g` – second function

EXAMPLES:

```
sage: M.<q, p> = EuclideanSpace(2)
sage: poisson = M.poisson_tensor('varpi')
sage: poisson.set_comp()[1,2] = -1
sage: f = M.scalar_field(function('f')(q, p), name='f')
sage: g = M.scalar_field(function('g')(q, p), name='g')
sage: poisson.poisson_bracket(f, g).display()
poisson(f, g): E^2 → ℝ
   (q, p) ↦ d(f)/dp*d(g)/dq - d(f)/dq*d(g)/dp
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(Integer(2), names=('q', 'p',)); (q, p,) = M._first_
↪ngens(2)
>>> poisson = M.poisson_tensor('varpi')
>>> poisson.set_comp()[Integer(1),Integer(2)] = -Integer(1)
>>> f = M.scalar_field(function('f')(q, p), name='f')
>>> g = M.scalar_field(function('g')(q, p), name='g')
>>> poisson.poisson_bracket(f, g).display()
```

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```
poisson(f, g): E^2 → ℝ
   (q, p) ↦ d(f)/dp*d(g)/dq - d(f)/dq*d(g)/dp
```

sharp(*form*)

Return the image of the given differential form under the map $\varpi^\sharp : T^*M \to TM$ defined by

$$\beta(\varpi^\sharp(\alpha)) = \varpi(\alpha, \beta).$$

for all $\alpha, \beta \in T^*_m M$.

In indices, $\alpha^i = \varpi^{ij}\alpha_j$.

INPUT:

- `form` – the differential form to calculate its sharp of

EXAMPLES:

```
sage: M.<q, p> = EuclideanSpace(2)
sage: poisson = M.poisson_tensor('varpi')
sage: poisson.set_comp()[1,2] = -1
sage: a = M.one_form(1, 0, name='a')
sage: poisson.sharp(a).display()
a_sharp = e_p
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(Integer(2), names=('q', 'p',)); (q, p,) = M._first_
↪ngens(2)
>>> poisson = M.poisson_tensor('varpi')
>>> poisson.set_comp()[Integer(1),Integer(2)] = -Integer(1)
>>> a = M.one_form(Integer(1), Integer(0), name='a')
>>> poisson.sharp(a).display()
a_sharp = e_p
```

class sage.manifolds.differentiable.poisson_tensor.**PoissonTensorFieldParal**(*manifold: DifferentiableManifold | VectorFieldModule*, *name: str | None = None*, *latex_name: str | None = None*)

Bases: *PoissonTensorField*, *MultivectorFieldParal*

A Poisson bivector field ϖ on a parallelizable manifold.

INPUT:

- `manifold` – module $\mathfrak{X}(M)$ of vector fields on the manifold M, or the manifold M itself
- `name` – (default: `varpi`) name given to the Poisson tensor
- `latex_name` – (default: `\\varpi`) LaTeX symbol to denote the Poisson tensor; if `None`, it is formed from `name`

EXAMPLES:

Standard Poisson tensor on $\mathbf{R}^2$:

```
sage: M.<q, p> = EuclideanSpace(2)
sage: varpi = M.poisson_tensor(name='varpi', latex_name=r'\varpi')
sage: varpi[1,2] = -1
sage: varpi
2-vector field varpi on the Euclidean plane E^2
sage: varpi.display()
varpi = -e_q∧e_p
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(Integer(2), names=('q', 'p',)); (q, p,) = M._first_ngens(2)
>>> varpi = M.poisson_tensor(name='varpi', latex_name=r'\varpi')
>>> varpi[Integer(1),Integer(2)] = -Integer(1)
>>> varpi
2-vector field varpi on the Euclidean plane E^2
>>> varpi.display()
varpi = -e_q∧e_p
```

4.2 Symplectic structures

The class *SymplecticForm* implements symplectic structures on differentiable manifolds over $\mathbf{R}$. The derived class *SymplecticFormParal* is devoted to symplectic forms on a parallelizable manifold.

AUTHORS:

- Tobias Diez (2021) : initial version

REFERENCES:

- [AM1990]
- [RS2012]

class sage.manifolds.differentiable.symplectic_form.**SymplecticForm**(*manifold: DifferentiableManifold | VectorFieldModule, name: str | None = None, latex_name: str | None = None*)

Bases: *DiffForm*

A symplectic form on a differentiable manifold.

An instance of this class is a closed nondegenerate differential 2-form ω on a differentiable manifold M over $\mathbf{R}$.

In particular, at each point $m \in M$, ω_m is a bilinear map of the type:

$$\omega_m : T_m M \times T_m M \to \mathbf{R},$$

where $T_m M$ stands for the tangent space to the manifold M at the point m, such that ω_m is skew-symmetric: $\forall u, v \in T_m M, \ \omega_m(v, u) = -\omega_m(u, v)$ and nondegenerate: $(\forall v \in T_m M, \ \ \omega_m(u, v) = 0) \Longrightarrow u = 0$.

Note

If M is parallelizable, the class *SymplecticFormParal* should be used instead.

INPUT:

- `manifold` – module $\mathfrak{X}(M)$ of vector fields on the manifold M, or the manifold M itself
- `name` – (default: `omega`) name given to the symplectic form
- `latex_name` – (default: `None`) LaTeX symbol to denote the symplectic form; if `None`, it is formed from `name`

EXAMPLES:

A symplectic form on the 2-sphere:

```
sage: M.<x,y> = manifolds.Sphere(2, coordinates='stereographic')
sage: stereoN = M.stereographic_coordinates(pole='north')
sage: stereoS = M.stereographic_coordinates(pole='south')
sage: omega = M.symplectic_form(name='omega', latex_name=r'\omega')
sage: omega
Symplectic form omega on the 2-sphere S^2 of radius 1 smoothly embedded
 in the Euclidean space E^3
```

```
>>> from sage.all import *
>>> M = manifolds.Sphere(Integer(2), coordinates='stereographic', names=('x', 'y',
↪)); (x, y,) = M._first_ngens(2)
>>> stereoN = M.stereographic_coordinates(pole='north')
>>> stereoS = M.stereographic_coordinates(pole='south')
>>> omega = M.symplectic_form(name='omega', latex_name=r'\omega')
>>> omega
Symplectic form omega on the 2-sphere S^2 of radius 1 smoothly embedded
 in the Euclidean space E^3
```

`omega` is initialized by providing its single nonvanishing component w.r.t. the vector frame associated to `stereoN`, which is the default frame on `M`:

```
sage: omega[1, 2] = 1/(1 + x^2 + y^2)^2
```

```
>>> from sage.all import *
>>> omega[Integer(1), Integer(2)] = Integer(1)/(Integer(1) + x**Integer(2) +
↪y**Integer(2))**Integer(2)
```

The components w.r.t. the vector frame associated to `stereoS` are obtained thanks to the method *add_comp_by_continuation()*:

```
sage: omega.add_comp_by_continuation(stereoS.frame(),
....:                    stereoS.domain().intersection(stereoN.domain()))
sage: omega.display()
omega = (x^2 + y^2 + 1)^(-2) dx∧dy
sage: omega.display(stereoS)
omega = -1/(xp^4 + yp^4 + 2*(xp^2 + 1)*yp^2 + 2*xp^2 + 1) dxp∧dyp
```

```
>>> from sage.all import *
>>> omega.add_comp_by_continuation(stereoS.frame(),
...                  stereoS.domain().intersection(stereoN.domain()))
>>> omega.display()
omega = (x^2 + y^2 + 1)^(-2) dx∧dy
>>> omega.display(stereoS)
omega = -1/(xp^4 + yp^4 + 2*(xp^2 + 1)*yp^2 + 2*xp^2 + 1) dxp∧dyp
```

`omega` is an exact 2-form (this is trivial here, since `M` is 2-dimensional):

```
sage: diff(omega).display()
domega = 0
```

```
>>> from sage.all import *
>>> diff(omega).display()
domega = 0
```

flat(*vector_field*)

Return the image of the given differential form under the map $\omega^\flat : TM \to T^*M$ defined by

$$< \omega^\flat(X), Y >= \omega_m(X, Y)$$

for all $X, Y \in T_m M$.

In indices, $X_i = \omega_{ji} X^j$.

INPUT:

- `vector_field` – the vector field to calculate its flat of

EXAMPLES:

```
sage: M = manifolds.StandardSymplecticSpace(2)
sage: omega = M.symplectic_form()
sage: X = M.vector_field_module().an_element()
sage: X.set_name('X')
sage: X.display()
X = 2 e_q + 2 e_p
sage: omega.flat(X).display()
X_flat = 2 dq - 2 dp
```

```
>>> from sage.all import *
>>> M = manifolds.StandardSymplecticSpace(Integer(2))
>>> omega = M.symplectic_form()
>>> X = M.vector_field_module().an_element()
>>> X.set_name('X')
>>> X.display()
X = 2 e_q + 2 e_p
>>> omega.flat(X).display()
X_flat = 2 dq - 2 dp
```

hamiltonian_vector_field(*function*)

The Hamiltonian vector field X_f generated by a function $f : M \to \mathbf{R}$.

The Hamiltonian vector field is defined by

$$X_f \lrcorner \omega + df = 0.$$

INPUT:

- `function` – the function generating the Hamiltonian vector field

EXAMPLES:

```
sage: M = manifolds.StandardSymplecticSpace(2)
sage: omega = M.symplectic_form()
sage: f = M.scalar_field({ chart: function('f')(*chart[:]) for chart in M.
↪atlas() }, name='f')
sage: f.display()
f: R2 → ℝ
   (q, p) ↦ f(q, p)
sage: Xf = omega.hamiltonian_vector_field(f)
sage: Xf.display()
Xf = d(f)/dp e_q - d(f)/dq e_p
```

```
>>> from sage.all import *
>>> M = manifolds.StandardSymplecticSpace(Integer(2))
>>> omega = M.symplectic_form()
>>> f = M.scalar_field({ chart: function('f')(*chart[:]) for chart in M.
↪atlas() }, name='f')
>>> f.display()
f: R2 → ℝ
   (q, p) ↦ f(q, p)
>>> Xf = omega.hamiltonian_vector_field(f)
>>> Xf.display()
Xf = d(f)/dp e_q - d(f)/dq e_p
```

hodge_star(*pform*)

Compute the Hodge dual of a differential form with respect to the symplectic form.

See *hodge_dual()* for the definition and more details.

INPUT:

- `pform` – a p-form A; must be an instance of *DiffScalarField* for $p = 0$ and of *DiffForm* or *DiffFormParal* for $p \geq 1$.

OUTPUT:

- the $(n - p)$-form $*A$

EXAMPLES:

Hodge dual of any form on the symplectic vector space R^2:

```
sage: M = manifolds.StandardSymplecticSpace(2)
sage: omega = M.symplectic_form()
sage: a = M.one_form(1, 0, name='a')
sage: omega.hodge_star(a).display()
*a = dq
sage: b = M.one_form(0, 1, name='b')
sage: omega.hodge_star(b).display()
*b = dp
sage: f = M.scalar_field(1, name='f')
sage: omega.hodge_star(f).display()
*f = -dq∧dp
```

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```
sage: omega.hodge_star(omega).display()
*omega: R2 → ℝ
   (q, p) ↦ 1
```

```
>>> from sage.all import *
>>> M = manifolds.StandardSymplecticSpace(Integer(2))
>>> omega = M.symplectic_form()
>>> a = M.one_form(Integer(1), Integer(0), name='a')
>>> omega.hodge_star(a).display()
*a = dq
>>> b = M.one_form(Integer(0), Integer(1), name='b')
>>> omega.hodge_star(b).display()
*b = dp
>>> f = M.scalar_field(Integer(1), name='f')
>>> omega.hodge_star(f).display()
*f = -dq∧dp
>>> omega.hodge_star(omega).display()
*omega: R2 → ℝ
   (q, p) ↦ 1
```

on_forms(*first*, *second*)

Return the contraction of the two forms with respect to the symplectic form.

The symplectic form ω gives rise to a bilinear form, also denoted by ω on the space of 1-forms by

$$\omega(\alpha, \beta) = \omega(\alpha^\sharp, \beta^\sharp),$$

where $\alpha^\sharp$ is the dual of α with respect to ω, see `up()`. This bilinear form induces a bilinear form on the space of all forms determined by its value on decomposable elements as:

$$\omega(\alpha_1 \wedge \ldots \wedge \alpha_p, \beta_1 \wedge \ldots \wedge \beta_p) = det(\omega(\alpha_i, \beta_j)).$$

INPUT:

- `first` – a p-form α
- `second` – a p-form β

OUTPUT:

- the scalar field $\omega(\alpha, \beta)$

EXAMPLES:

sage: M = manifolds.StandardSymplecticSpace(2) sage: omega = M.symplectic_form() sage: a = M.one_form(1, 0, name='a') sage: b = M.one_form(0, 1, name='b') sage: omega.on_forms(a, b).display() R2 → ℝ (q, p) ↦ -1

poisson(*expansion_symbol=None*, *order=1*)

Return the Poisson tensor associated with the symplectic form.

INPUT:

- `expansion_symbol` – (default: `None`) symbolic variable; if specified, the inverse will be expanded in power series with respect to this variable (around its zero value)
- `order` – integer (default: 1); the order of the expansion if `expansion_symbol` is not `None`; the *order* is defined as the degree of the polynomial representing the truncated power series in `expansion_symbol`; currently only first order inverse is supported

If `expansion_symbol` is set, then the zeroth order symplectic form must be invertible. Moreover, subsequent calls to this method will return a cached value, even when called with the default value (to enable computation of derived quantities). To reset, use `_del_derived()`.

OUTPUT:

- the Poisson tensor, as an instance of `PoissonTensorField()`

EXAMPLES:

Poisson tensor of 2-dimensional symplectic vector space:

```
sage: M = manifolds.StandardSymplecticSpace(2)
sage: omega = M.symplectic_form()
sage: poisson = omega.poisson(); poisson
2-vector field poisson_omega on the Standard symplectic space R2
sage: poisson.display()
poisson_omega = -e_q∧e_p
```

```
>>> from sage.all import *
>>> M = manifolds.StandardSymplecticSpace(Integer(2))
>>> omega = M.symplectic_form()
>>> poisson = omega.poisson(); poisson
2-vector field poisson_omega on the Standard symplectic space R2
>>> poisson.display()
poisson_omega = -e_q∧e_p
```

poisson_bracket (f, g)

Return the Poisson bracket

$$\{f, g\} = \omega(X_f, X_g)$$

of the given functions.

INPUT:

- `f` – function inserted in the first slot
- `g` – function inserted in the second slot

EXAMPLES:

```
sage: M.<q, p> = EuclideanSpace(2)
sage: poisson = M.poisson_tensor('varpi')
sage: poisson.set_comp()[1,2] = -1
sage: f = M.scalar_field({ chart: function('f')(*chart[:]) for chart in M.
↪atlas() }, name='f')
sage: g = M.scalar_field({ chart: function('g')(*chart[:]) for chart in M.
↪atlas() }, name='g')
sage: poisson.poisson_bracket(f, g).display()
poisson(f, g): E^2 → ℝ
   (q, p) ↦ d(f)/dp*d(g)/dq - d(f)/dq*d(g)/dp
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(Integer(2), names=('q', 'p',)); (q, p,) = M._first_
↪ngens(2)
>>> poisson = M.poisson_tensor('varpi')
```

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```
>>> poisson.set_comp()[Integer(1),Integer(2)] = -Integer(1)
>>> f = M.scalar_field({ chart: function('f')(*chart[:]) for chart in M.
↪atlas() }, name='f')
>>> g = M.scalar_field({ chart: function('g')(*chart[:]) for chart in M.
↪atlas() }, name='g')
>>> poisson.poisson_bracket(f, g).display()
poisson(f, g): E^2 → R
   (q, p) ↦ d(f)/dp*d(g)/dq - d(f)/dq*d(g)/dp
```

restrict(*subdomain*, *dest_map=None*)

Return the restriction of the symplectic form to some subdomain.

If the restriction has not been defined yet, it is constructed here.

INPUT:

- `subdomain` – open subset U of the symplectic form's domain
- `dest_map` – (default: `None`) smooth destination map $\Phi : U \to V$, where V is a subdomain of the symplectic form's domain If `None`, the restriction of the initial vector field module is used.

OUTPUT: the restricted symplectic form

EXAMPLES:

```
sage: M = Manifold(6, 'M')
sage: omega = M.symplectic_form()
sage: U = M.open_subset('U')
sage: omega.restrict(U)
2-form omega on the Open subset U of the 6-dimensional differentiable␣
↪manifold M
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(6), 'M')
>>> omega = M.symplectic_form()
>>> U = M.open_subset('U')
>>> omega.restrict(U)
2-form omega on the Open subset U of the 6-dimensional differentiable␣
↪manifold M
```

sharp(*form*)

Return the image of the given differential form under the map $\omega^\sharp : T^*M \to TM$ defined by

$$\omega(\omega^\sharp(\alpha), X) = \alpha(X)$$

for all $X \in T_m M$ and $\alpha \in T^*_m M$. The sharp map is inverse to the flat map.

In indices, $\alpha^i = \varpi^{ij}\alpha_j$, where ϖ is the Poisson tensor associated with the symplectic form.

INPUT:

- `form` – the differential form to calculate its sharp of

EXAMPLES:

```
sage: M = manifolds.StandardSymplecticSpace(2)
sage: omega = M.symplectic_form()
sage: X = M.vector_field_module().an_element()
sage: alpha = omega.flat(X)
sage: alpha.set_name('alpha')
sage: alpha.display()
alpha = 2 dq - 2 dp
sage: omega.sharp(alpha).display()
alpha_sharp = 2 e_q + 2 e_p
```

```
>>> from sage.all import *
>>> M = manifolds.StandardSymplecticSpace(Integer(2))
>>> omega = M.symplectic_form()
>>> X = M.vector_field_module().an_element()
>>> alpha = omega.flat(X)
>>> alpha.set_name('alpha')
>>> alpha.display()
alpha = 2 dq - 2 dp
>>> omega.sharp(alpha).display()
alpha_sharp = 2 e_q + 2 e_p
```

volume_form(*contra=0*)

Liouville volume form $\frac{1}{n!}\omega^n$ associated with the symplectic form ω, where $2n$ is the dimension of the manifold.

INPUT:

- `contra` – (default: 0) number of contravariant indices of the returned tensor

OUTPUT:

- if `contra = 0`: volume form associated with the symplectic form
- if `contra = k`, with $1 \leq k \leq n$, the tensor field of type (k,n-k) formed from ϵ by raising the first k indices with the symplectic form (see method `up()`)

EXAMPLES:

Volume form on $\mathbf{R}^4$:

```
sage: M = manifolds.StandardSymplecticSpace(4)
sage: omega = M.symplectic_form()
sage: vol = omega.volume_form() ; vol
4-form mu_omega on the Standard symplectic space R4
sage: vol.display()
mu_omega = dq1∧dp1∧dq2∧dp2
```

```
>>> from sage.all import *
>>> M = manifolds.StandardSymplecticSpace(Integer(4))
>>> omega = M.symplectic_form()
>>> vol = omega.volume_form() ; vol
4-form mu_omega on the Standard symplectic space R4
>>> vol.display()
mu_omega = dq1∧dp1∧dq2∧dp2
```

static wrap(*form*, *name=None*, *latex_name=None*)

Define the symplectic form from a differential form.

INPUT:

- `form` – differential 2-form

EXAMPLES:

Volume form on the sphere as a symplectic form:

```
sage: from sage.manifolds.differentiable.symplectic_form import SymplecticForm
sage: M = manifolds.Sphere(2, coordinates='stereographic')
sage: vol_form = M.induced_metric().volume_form()                      # long␣
↪time
sage: omega = SymplecticForm.wrap(vol_form, 'omega', r'\omega')        # long␣
↪time
sage: omega.display()                                                  # long␣
↪time
omega = -4/(y1^4 + y2^4 + 2*(y1^2 + 1)*y2^2 + 2*y1^2 + 1) dy1∧dy2
```

```
>>> from sage.all import *
>>> from sage.manifolds.differentiable.symplectic_form import SymplecticForm
>>> M = manifolds.Sphere(Integer(2), coordinates='stereographic')
>>> vol_form = M.induced_metric().volume_form()                     # long time
>>> omega = SymplecticForm.wrap(vol_form, 'omega', r'\omega')       # long time
>>> omega.display()                                                 # long time
omega = -4/(y1^4 + y2^4 + 2*(y1^2 + 1)*y2^2 + 2*y1^2 + 1) dy1∧dy2
```

class sage.manifolds.differentiable.symplectic_form.**SymplecticFormParal**(*manifold: VectorFieldModule | DifferentiableManifold*, *name: str | None*, *latex_name: str | None = None*)

Bases: *SymplecticForm*, *DiffFormParal*

A symplectic form on a parallelizable manifold.

> **Note**
>
> If M is not parallelizable, the class *SymplecticForm* should be used instead.

INPUT:

- `manifold` – module $\mathfrak{X}(M)$ of vector fields on the manifold M, or the manifold M itself
- `name` – (default: `omega`) name given to the symplectic form
- `latex_name` – (default: `None`) LaTeX symbol to denote the symplectic form; if `None`, it is formed from `name`

EXAMPLES:

Standard symplectic form on $\mathbf{R}^2$:

```
sage: M.<q, p> = EuclideanSpace(name='R2', latex_name=r"\mathbb{R}^2")
sage: omega = M.symplectic_form(name='omega', latex_name=r'\omega')
sage: omega
Symplectic form omega on the Euclidean plane R2
sage: omega.set_comp()[1,2] = -1
sage: omega.display()
omega = -dq∧dp
```

```
>>> from sage.all import *
>>> M = EuclideanSpace(name='R2', latex_name=r"\mathbb{R}^2", names=('q', 'p',));
↪(q, p,) = M._first_ngens(2)
>>> omega = M.symplectic_form(name='omega', latex_name=r'\omega')
>>> omega
Symplectic form omega on the Euclidean plane R2
>>> omega.set_comp()[Integer(1),Integer(2)] = -Integer(1)
>>> omega.display()
omega = -dq∧dp
```

poisson(*expansion_symbol=None*, *order=1*)

Return the Poisson tensor associated with the symplectic form.

INPUT:

- `expansion_symbol` – (default: `None`) symbolic variable; if specified, the inverse will be expanded in power series with respect to this variable (around its zero value)
- `order` – integer (default: 1); the order of the expansion if `expansion_symbol` is not `None`; the *order* is defined as the degree of the polynomial representing the truncated power series in `expansion_symbol`; currently only first order inverse is supported

If `expansion_symbol` is set, then the zeroth order symplectic form must be invertible. Moreover, subsequent calls to this method will return a cached value, even when called with the default value (to enable computation of derived quantities). To reset, use `_del_derived()`.

OUTPUT:

- the Poisson tensor, , as an instance of *PoissonTensorFieldParal()*

EXAMPLES:

Poisson tensor of 2-dimensional symplectic vector space:

```
sage: from sage.manifolds.differentiable.symplectic_form import
↪SymplecticFormParal
sage: M.<q, p> = EuclideanSpace(2, "R2", r"\mathbb{R}^2", symbols=r"q:q p:p")
sage: omega = SymplecticFormParal(M, 'omega', r'\omega')
sage: omega[1,2] = -1
sage: poisson = omega.poisson(); poisson
2-vector field poisson_omega on the Euclidean plane R2
sage: poisson.display()
poisson_omega = -e_q∧e_p
```

```
>>> from sage.all import *
>>> from sage.manifolds.differentiable.symplectic_form import
↪SymplecticFormParal
>>> M = EuclideanSpace(Integer(2), "R2", r"\mathbb{R}^2", symbols=r"q:q p:p",
```

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```
↪names=('q', 'p',)); (q, p,) = M._first_ngens(2)
>>> omega = SymplecticFormParal(M, 'omega', r'\omega')
>>> omega[Integer(1),Integer(2)] = -Integer(1)
>>> poisson = omega.poisson(); poisson
2-vector field poisson_omega on the Euclidean plane R2
>>> poisson.display()
poisson_omega = -e_q∧e_p
```

restrict(*subdomain*, *dest_map=None*)

Return the restriction of the symplectic form to some subdomain.

If the restriction has not been defined yet, it is constructed here.

INPUT:

- `subdomain` – open subset U of the symplectic form's domain
- `dest_map` – (default: `None`) smooth destination map $\Phi : U \to V$, where V is a subdomain of the symplectic form's domain. If `None`, the restriction of the initial vector field module is used.

OUTPUT: the restricted symplectic form

EXAMPLES:

Restriction of the standard symplectic form on $\mathbf{R}^2$ to the upper half plane:

```
sage: from sage.manifolds.differentiable.symplectic_form import␣
↪SymplecticFormParal
sage: M = EuclideanSpace(2, "R2", r"\mathbb{R}^2", symbols=r"q:q p:p")
sage: X.<q, p> = M.chart()
sage: omega = SymplecticFormParal(M, 'omega', r'\omega')
sage: omega[1,2] = -1
sage: U = M.open_subset('U', coord_def={X: q>0})
sage: omegaU = omega.restrict(U); omegaU
Symplectic form omega on the Open subset U of the Euclidean plane R2
sage: omegaU.display()
omega = -dq∧dp
```

```
>>> from sage.all import *
>>> from sage.manifolds.differentiable.symplectic_form import␣
↪SymplecticFormParal
>>> M = EuclideanSpace(Integer(2), "R2", r"\mathbb{R}^2", symbols=r"q:q p:p")
>>> X = M.chart(names=('q', 'p',)); (q, p,) = X._first_ngens(2)
>>> omega = SymplecticFormParal(M, 'omega', r'\omega')
>>> omega[Integer(1),Integer(2)] = -Integer(1)
>>> U = M.open_subset('U', coord_def={X: q>Integer(0)})
>>> omegaU = omega.restrict(U); omegaU
Symplectic form omega on the Open subset U of the Euclidean plane R2
>>> omegaU.display()
omega = -dq∧dp
```

4.3 Symplectic vector spaces

AUTHORS:

- Tobias Diez (2021): initial version

class sage.manifolds.differentiable.examples.symplectic_space.**StandardSymplecticSpace**(*dimension: int*, *name: str | None = None*, *latex_name: str | None = None*, *coordinates: str = 'Cartesian'*, *symbols: str | None = None*, *symplectic_name: str | None = 'omega'*, *symplectic_latex_name: str | None = None*, *start_index: int = 1*, *base_manifold: StandardSym-

Bases: *EuclideanSpace*

The vector space $\mathbf{R}^{2n}$ equipped with its standard symplectic form.

symplectic_form()

Return the symplectic form.

EXAMPLES:

Standard symplectic form on $\mathbf{R}^2$:

```
sage: M.<q, p> = manifolds.StandardSymplecticSpace(2, symplectic_name='omega')
sage: omega = M.symplectic_form()
sage: omega.display()
omega = -dq∧dp
```

```
>>> from sage.all import *
>>> M = manifolds.StandardSymplecticSpace(Integer(2), symplectic_name='omega',
↪ names=('q', 'p',)); (q, p,) = M._first_ngens(2)
>>> omega = M.symplectic_form()
>>> omega.display()
omega = -dq∧dp
```

CHAPTER

FIVE

UTILITIES FOR CALCULUS

This module defines helper functions which are used for simplifications and display of symbolic expressions.

AUTHORS:

- Michal Bejger (2015) : class *ExpressionNice*
- Eric Gourgoulhon (2015, 2017) : simplification functions
- Travis Scrimshaw (2016): review tweaks
- Marius Gerbershagen (2022) : skip simplification of expressions with a single number or symbolic variable

class sage.manifolds.utilities.**ExpressionNice**(*ex*)

Bases: Expression

Subclass of Expression for a "human-friendly" display of partial derivatives and the possibility to shorten the display by skipping the arguments of symbolic functions.

INPUT:

- ex – symbolic expression

EXAMPLES:

An expression formed with callable symbolic expressions:

```
sage: var('x y z')
(x, y, z)
sage: f = function('f')(x, y)
sage: g = f.diff(y).diff(x)
sage: h = function('h')(y, z)
sage: k = h.diff(z)
sage: fun = x*g + y*(k-z)^2
```

```
>>> from sage.all import *
>>> var('x y z')
(x, y, z)
>>> f = function('f')(x, y)
>>> g = f.diff(y).diff(x)
>>> h = function('h')(y, z)
>>> k = h.diff(z)
>>> fun = x*g + y*(k-z)**Integer(2)
```

The standard Pynac display of partial derivatives:

```
sage: fun
y*(z - diff(h(y, z), z))^2 + x*diff(f(x, y), x, y)
sage: latex(fun)
y {\left(z - \frac{\partial}{\partial z}h\left(y, z\right)\right)}^{2} + x \frac{\
↪partial^{2}}{\partial x\partial y}f\left(x, y\right)
```

```
>>> from sage.all import *
>>> fun
y*(z - diff(h(y, z), z))^2 + x*diff(f(x, y), x, y)
>>> latex(fun)
y {\left(z - \frac{\partial}{\partial z}h\left(y, z\right)\right)}^{2} + x \frac{\
↪partial^{2}}{\partial x\partial y}f\left(x, y\right)
```

With *ExpressionNice*, the Pynac notation `D[...]` is replaced by textbook-like notation:

```
sage: from sage.manifolds.utilities import ExpressionNice
sage: ExpressionNice(fun)
y*(z - d(h)/dz)^2 + x*d^2(f)/dxdy
sage: latex(ExpressionNice(fun))
y {\left(z - \frac{\partial\,h}{\partial z}\right)}^{2}
 + x \frac{\partial^2\,f}{\partial x\partial y}
```

```
>>> from sage.all import *
>>> from sage.manifolds.utilities import ExpressionNice
>>> ExpressionNice(fun)
y*(z - d(h)/dz)^2 + x*d^2(f)/dxdy
>>> latex(ExpressionNice(fun))
y {\left(z - \frac{\partial\,h}{\partial z}\right)}^{2}
 + x \frac{\partial^2\,f}{\partial x\partial y}
```

An example when function variables are themselves functions:

```
sage: f = function('f')(x, y)
sage: g = function('g')(x, f)   # the second variable is the function f
sage: fun = (g.diff(x))*x - x^2*f.diff(x,y)
sage: fun
-x^2*diff(f(x, y), x, y) + (diff(f(x, y), x)*D[1](g)(x, f(x, y)) + D[0](g)(x, f(x,
↪ y)))*x
sage: ExpressionNice(fun)
-x^2*d^2(f)/dxdy + (d(f)/dx*d(g)/d(f(x, y)) + d(g)/dx)*x
sage: latex(ExpressionNice(fun))
-x^{2} \frac{\partial^2\,f}{\partial x\partial y}
 + {\left(\frac{\partial\,f}{\partial x}
   \frac{\partial\,g}{\partial \left( f\left(x, y\right) \right)}
 + \frac{\partial\,g}{\partial x}\right)} x
```

```
>>> from sage.all import *
>>> f = function('f')(x, y)
>>> g = function('g')(x, f)   # the second variable is the function f
>>> fun = (g.diff(x))*x - x**Integer(2)*f.diff(x,y)
>>> fun
-x^2*diff(f(x, y), x, y) + (diff(f(x, y), x)*D[1](g)(x, f(x, y)) + D[0](g)(x, f(x,
```

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```
↪ y)))*x
>>> ExpressionNice(fun)
-x^2*d^2(f)/dxdy + (d(f)/dx*d(g)/d(f(x, y)) + d(g)/dx)*x
>>> latex(ExpressionNice(fun))
-x^{2} \frac{\partial^2\,f}{\partial x\partial y}
 + {\left(\frac{\partial\,f}{\partial x}
   \frac{\partial\,g}{\partial \left( f\left(x, y\right) \right)}
 + \frac{\partial\,g}{\partial x}\right)} x
```

Note that `D[1](g)(x, f(x,y))` is rendered as `d(g)/d(f(x, y))`.

An example with multiple differentiations:

```
sage: fun = f.diff(x,x,y,y,x)*x
sage: fun
x*diff(f(x, y), x, x, x, y, y)
sage: ExpressionNice(fun)
x*d^5(f)/dx^3dy^2
sage: latex(ExpressionNice(fun))
x \frac{\partial^5\,f}{\partial x ^ 3\partial y ^ 2}
```

```
>>> from sage.all import *
>>> fun = f.diff(x,x,y,y,x)*x
>>> fun
x*diff(f(x, y), x, x, x, y, y)
>>> ExpressionNice(fun)
x*d^5(f)/dx^3dy^2
>>> latex(ExpressionNice(fun))
x \frac{\partial^5\,f}{\partial x ^ 3\partial y ^ 2}
```

Parentheses are added around powers of partial derivatives to avoid any confusion:

```
sage: fun = f.diff(y)^2
sage: fun
diff(f(x, y), y)^2
sage: ExpressionNice(fun)
(d(f)/dy)^2
sage: latex(ExpressionNice(fun))
\left(\frac{\partial\,f}{\partial y}\right)^{2}
```

```
>>> from sage.all import *
>>> fun = f.diff(y)**Integer(2)
>>> fun
diff(f(x, y), y)^2
>>> ExpressionNice(fun)
(d(f)/dy)^2
>>> latex(ExpressionNice(fun))
\left(\frac{\partial\,f}{\partial y}\right)^{2}
```

The explicit mention of function arguments can be omitted for the sake of brevity:

```
sage: fun = fun*f
sage: ExpressionNice(fun)
```

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```
f(x, y)*(d(f)/dy)^2
sage: Manifold.options.omit_function_arguments=True
sage: ExpressionNice(fun)
f*(d(f)/dy)^2
sage: latex(ExpressionNice(fun))
f \left(\frac{\partial\,f}{\partial y}\right)^{2}
sage: Manifold.options._reset()
sage: ExpressionNice(fun)
f(x, y)*(d(f)/dy)^2
sage: latex(ExpressionNice(fun))
f\left(x, y\right) \left(\frac{\partial\,f}{\partial y}\right)^{2}
```

```
>>> from sage.all import *
>>> fun = fun*f
>>> ExpressionNice(fun)
f(x, y)*(d(f)/dy)^2
>>> Manifold.options.omit_function_arguments=True
>>> ExpressionNice(fun)
f*(d(f)/dy)^2
>>> latex(ExpressionNice(fun))
f \left(\frac{\partial\,f}{\partial y}\right)^{2}
>>> Manifold.options._reset()
>>> ExpressionNice(fun)
f(x, y)*(d(f)/dy)^2
>>> latex(ExpressionNice(fun))
f\left(x, y\right) \left(\frac{\partial\,f}{\partial y}\right)^{2}
```

class sage.manifolds.utilities.**SimplifyAbsTrig**(*ex*)

Bases: ExpressionTreeWalker

Class for simplifying absolute values of cosines or sines (in the real domain), by walking the expression tree.

The end user interface is the function *simplify_abs_trig()*.

INPUT:

- ex – a symbolic expression

EXAMPLES:

Let us consider the following symbolic expression with some assumption on the range of the variable x:

```
sage: assume(pi/2<x, x<pi)
sage: a = abs(cos(x)) + abs(sin(x))
```

```
>>> from sage.all import *
>>> assume(pi/Integer(2)<x, x<pi)
>>> a = abs(cos(x)) + abs(sin(x))
```

The method simplify_full() is ineffective on such an expression:

```
sage: a.simplify_full()
abs(cos(x)) + abs(sin(x))
```

```
>>> from sage.all import *
>>> a.simplify_full()
abs(cos(x)) + abs(sin(x))
```

We construct a `SimplifyAbsTrig` object `s` from the symbolic expression `a`:

```
sage: from sage.manifolds.utilities import SimplifyAbsTrig
sage: s = SimplifyAbsTrig(a)
```

```
>>> from sage.all import *
>>> from sage.manifolds.utilities import SimplifyAbsTrig
>>> s = SimplifyAbsTrig(a)
```

We use the `__call__` method to walk the expression tree and produce a correctly simplified expression, given that $x \in (\pi/2, \pi)$:

```
sage: s()
-cos(x) + sin(x)
```

```
>>> from sage.all import *
>>> s()
-cos(x) + sin(x)
```

Calling the simplifier `s` with an expression actually simplifies this expression:

```
sage: s(a)  # same as s() since s is built from a
-cos(x) + sin(x)
sage: s(abs(cos(x/2)) + abs(sin(x/2)))  #  pi/4 < x/2 < pi/2
cos(1/2*x) + sin(1/2*x)
sage: s(abs(cos(2*x)) + abs(sin(2*x)))  #  pi < 2 x < 2*pi
abs(cos(2*x)) - sin(2*x)
sage: s(abs(sin(2+abs(cos(x)))))  # nested abs(sin_or_cos(...))
sin(-cos(x) + 2)
```

```
>>> from sage.all import *
>>> s(a)  # same as s() since s is built from a
-cos(x) + sin(x)
>>> s(abs(cos(x/Integer(2))) + abs(sin(x/Integer(2))))  #  pi/4 < x/2 < pi/2
cos(1/2*x) + sin(1/2*x)
>>> s(abs(cos(Integer(2)*x)) + abs(sin(Integer(2)*x)))  #  pi < 2 x < 2*pi
abs(cos(2*x)) - sin(2*x)
>>> s(abs(sin(Integer(2)+abs(cos(x)))))  # nested abs(sin_or_cos(...))
sin(-cos(x) + 2)
```

See also

simplify_abs_trig() for more examples with *SimplifyAbsTrig* at work.

composition(*ex*, *operator*)

This is the only method of the base class `ExpressionTreeWalker` that is reimplemented, since it manages the composition of `abs` with `cos` or `sin`.

INPUT:

- ex – a symbolic expression
- operator – an operator

OUTPUT:

- a symbolic expression, equivalent to ex with abs(cos(...)) and abs(sin(...)) simplified, according to the range of their argument.

EXAMPLES:

```
sage: from sage.manifolds.utilities import SimplifyAbsTrig
sage: assume(-pi/2 < x, x<0)
sage: a = abs(sin(x))
sage: s = SimplifyAbsTrig(a)
sage: a.operator()
abs
sage: s.composition(a, a.operator())
sin(-x)
```

```
>>> from sage.all import *
>>> from sage.manifolds.utilities import SimplifyAbsTrig
>>> assume(-pi/Integer(2) < x, x<Integer(0))
>>> a = abs(sin(x))
>>> s = SimplifyAbsTrig(a)
>>> a.operator()
abs
>>> s.composition(a, a.operator())
sin(-x)
```

```
sage: a = exp(function('f')(x))  # no abs(sin_or_cos(...))
sage: a.operator()
exp
sage: s.composition(a, a.operator())
e^f(x)
```

```
>>> from sage.all import *
>>> a = exp(function('f')(x))  # no abs(sin_or_cos(...))
>>> a.operator()
exp
>>> s.composition(a, a.operator())
e^f(x)
```

```
sage: forget()  # no longer any assumption on x
sage: a = abs(cos(sin(x)))  # simplifiable since -1 <= sin(x) <= 1
sage: s.composition(a, a.operator())
cos(sin(x))
sage: a = abs(sin(cos(x)))  # not simplifiable
sage: s.composition(a, a.operator())
abs(sin(cos(x)))
```

```
>>> from sage.all import *
>>> forget()  # no longer any assumption on x
```

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```
>>> a = abs(cos(sin(x)))   # simplifiable since -1 <= sin(x) <= 1
>>> s.composition(a, a.operator())
cos(sin(x))
>>> a = abs(sin(cos(x)))   # not simplifiable
>>> s.composition(a, a.operator())
abs(sin(cos(x)))
```

class sage.manifolds.utilities.**SimplifySqrtReal**(*ex*)

Bases: `ExpressionTreeWalker`

Class for simplifying square roots in the real domain, by walking the expression tree.

The end user interface is the function *simplify_sqrt_real()*.

INPUT:

- `ex` – a symbolic expression

EXAMPLES:

Let us consider the square root of an exact square under some assumption:

```
sage: assume(x<1)
sage: a = sqrt(x^2-2*x+1)
```

```
>>> from sage.all import *
>>> assume(x<Integer(1))
>>> a = sqrt(x**Integer(2)-Integer(2)*x+Integer(1))
```

The method `simplify_full()` is ineffective on such an expression:

```
sage: a.simplify_full()
sqrt(x^2 - 2*x + 1)
```

```
>>> from sage.all import *
>>> a.simplify_full()
sqrt(x^2 - 2*x + 1)
```

and the more aggressive method `canonicalize_radical()` yields a wrong result, given that $x < 1$:

```
sage: a.canonicalize_radical()   # wrong output!
x - 1
```

```
>>> from sage.all import *
>>> a.canonicalize_radical()   # wrong output!
x - 1
```

We construct a *SimplifySqrtReal* object `s` from the symbolic expression `a`:

```
sage: from sage.manifolds.utilities import SimplifySqrtReal
sage: s = SimplifySqrtReal(a)
```

```
>>> from sage.all import *
>>> from sage.manifolds.utilities import SimplifySqrtReal
>>> s = SimplifySqrtReal(a)
```

We use the `__call__` method to walk the expression tree and produce a correctly simplified expression:

```
sage: s()
-x + 1
```

```
>>> from sage.all import *
>>> s()
-x + 1
```

Calling the simplifier `s` with an expression actually simplifies this expression:

```
sage: s(a)  # same as s() since s is built from a
-x + 1
sage: s(sqrt(x^2))
abs(x)
sage: s(sqrt(1+sqrt(x^2-2*x+1)))  # nested sqrt's
sqrt(-x + 2)
```

```
>>> from sage.all import *
>>> s(a)  # same as s() since s is built from a
-x + 1
>>> s(sqrt(x**Integer(2)))
abs(x)
>>> s(sqrt(Integer(1)+sqrt(x**Integer(2)-Integer(2)*x+Integer(1))))  # nested sqrt
↪'s
sqrt(-x + 2)
```

Another example where both `simplify_full()` and `canonicalize_radical()` fail:

```
sage: b = sqrt((x-1)/(x-2))*sqrt(1-x)
sage: b.simplify_full()  # does not simplify
sqrt(-x + 1)*sqrt((x - 1)/(x - 2))
sage: b.canonicalize_radical()  # wrong output, given that x<1
(I*x - I)/sqrt(x - 2)
sage: SimplifySqrtReal(b)()  # OK, given that x<1
-(x - 1)/sqrt(-x + 2)
```

```
>>> from sage.all import *
>>> b = sqrt((x-Integer(1))/(x-Integer(2)))*sqrt(Integer(1)-x)
>>> b.simplify_full()  # does not simplify
sqrt(-x + 1)*sqrt((x - 1)/(x - 2))
>>> b.canonicalize_radical()  # wrong output, given that x<1
(I*x - I)/sqrt(x - 2)
>>> SimplifySqrtReal(b)()  # OK, given that x<1
-(x - 1)/sqrt(-x + 2)
```

See also:

`simplify_sqrt_real()` for more examples with *`SimplifySqrtReal`* at work.

`arithmetic`(*ex*, *operator*)

This is the only method of the base class `ExpressionTreeWalker` that is reimplemented, since square roots are considered as arithmetic operations with `operator = pow` and `ex.operands()[1] = 1/2` or `-1/2`.

INPUT:

- `ex` – a symbolic expression
- `operator` – an arithmetic operator

OUTPUT:

- a symbolic expression, equivalent to `ex` with square roots simplified

EXAMPLES:

```
sage: from sage.manifolds.utilities import SimplifySqrtReal
sage: a = sqrt(x^2+2*x+1)
sage: s = SimplifySqrtReal(a)
sage: a.operator()
<built-in function pow>
sage: s.arithmetic(a, a.operator())
abs(x + 1)
```

```
>>> from sage.all import *
>>> from sage.manifolds.utilities import SimplifySqrtReal
>>> a = sqrt(x**Integer(2)+Integer(2)*x+Integer(1))
>>> s = SimplifySqrtReal(a)
>>> a.operator()
<built-in function pow>
>>> s.arithmetic(a, a.operator())
abs(x + 1)
```

```
sage: a = x + 1  # no square root
sage: s.arithmetic(a, a.operator())
x + 1
```

```
>>> from sage.all import *
>>> a = x + Integer(1)  # no square root
>>> s.arithmetic(a, a.operator())
x + 1
```

```
sage: a = x + 1 + sqrt(function('f')(x)^2)
sage: s.arithmetic(a, a.operator())
x + abs(f(x)) + 1
```

```
>>> from sage.all import *
>>> a = x + Integer(1) + sqrt(function('f')(x)**Integer(2))
>>> s.arithmetic(a, a.operator())
x + abs(f(x)) + 1
```

`sage.manifolds.utilities.`**`exterior_derivative`**`(`*form*`)`

Exterior derivative of a differential form.

INPUT:

- `form` – a differential form; this must an instance of either
  - *`DiffScalarField`* for a 0-form (scalar field)

   – `DiffFormParal` for a p-form ($p \geq 1$) on a parallelizable manifold
   – `DiffForm` for a a p-form ($p \geq 1$) on a non-parallelizable manifold

OUTPUT:

- the $(p+1)$-form that is the exterior derivative of `form`

EXAMPLES:

Exterior derivative of a scalar field (0-form):

```
sage: from sage.manifolds.utilities import exterior_derivative
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: f = M.scalar_field({X: x+y^2+z^3}, name='f')
sage: df = exterior_derivative(f); df
1-form df on the 3-dimensional differentiable manifold M
sage: df.display()
df = dx + 2*y dy + 3*z^2 dz
```

```
>>> from sage.all import *
>>> from sage.manifolds.utilities import exterior_derivative
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> f = M.scalar_field({X: x+y**Integer(2)+z**Integer(3)}, name='f')
>>> df = exterior_derivative(f); df
1-form df on the 3-dimensional differentiable manifold M
>>> df.display()
df = dx + 2*y dy + 3*z^2 dz
```

An alias is `xder`:

```
sage: from sage.manifolds.utilities import xder
sage: df == xder(f)
True
```

```
>>> from sage.all import *
>>> from sage.manifolds.utilities import xder
>>> df == xder(f)
True
```

Exterior derivative of a 1-form:

```
sage: a = M.one_form(name='a')
sage: a[:] = [x+y*z, x-y*z, x*y*z]
sage: da = xder(a); da
2-form da on the 3-dimensional differentiable manifold M
sage: da.display()
da = (-z + 1) dx∧dy + (y*z - y) dx∧dz + (x*z + y) dy∧dz
sage: dda = xder(da); dda
3-form dda on the 3-dimensional differentiable manifold M
sage: dda.display()
dda = 0
```

```
>>> from sage.all import *
>>> a = M.one_form(name='a')
>>> a[:] = [x+y*z, x-y*z, x*y*z]
>>> da = xder(a); da
2-form da on the 3-dimensional differentiable manifold M
>>> da.display()
da = (-z + 1) dx∧dy + (y*z - y) dx∧dz + (x*z + y) dy∧dz
>>> dda = xder(da); dda
3-form dda on the 3-dimensional differentiable manifold M
>>> dda.display()
dda = 0
```

See also:

sage.manifolds.differentiable.diff_form.DiffFormParal.exterior_derivative or *sage.manifolds.differentiable.diff_form.DiffForm.exterior_derivative* for more examples.

sage.manifolds.utilities.**set_axes_labels**(*graph*, *xlabel*, *ylabel*, *zlabel*, ***kwds*)

Set axes labels for a 3D graphics object `graph`.

This is a workaround for the lack of axes labels in 3D plots. This sets the labels as `text3d()` objects at locations determined from the bounding box of the graphic object `graph`.

INPUT:

- `graph` – `Graphics3d`; a 3D graphic object
- `xlabel` – string for the x-axis label
- `ylabel` – string for the y-axis label
- `zlabel` – string for the z-axis label
- `**kwds` – options (e.g. color) for text3d

OUTPUT: the 3D graphic object with text3d labels added

EXAMPLES:

```
sage: # needs sage.plot
sage: g = sphere()
sage: g.all
[Graphics3d Object]
sage: from sage.manifolds.utilities import set_axes_labels
sage: ga = set_axes_labels(g, 'X', 'Y', 'Z', color='red')
sage: ga.all  # the 3D frame has now axes labels
[Graphics3d Object, Graphics3d Object,
 Graphics3d Object, Graphics3d Object]
```

```
>>> from sage.all import *
>>> # needs sage.plot
>>> g = sphere()
>>> g.all
[Graphics3d Object]
```

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```
>>> from sage.manifolds.utilities import set_axes_labels
>>> ga = set_axes_labels(g, 'X', 'Y', 'Z', color='red')
>>> ga.all  # the 3D frame has now axes labels
[Graphics3d Object, Graphics3d Object,
 Graphics3d Object, Graphics3d Object]
```

sage.manifolds.utilities.**simplify_abs_trig**(*expr*)

Simplify `abs(sin(...))` and `abs(cos(...))` in symbolic expressions.

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: X.<x,y,z> = M.chart(r'x y:(0,pi) z:(-pi/3,0)')
sage: X.coord_range()
x: (-oo, +oo); y: (0, pi); z: (-1/3*pi, 0)
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(3), 'M', structure='topological')
>>> X = M.chart(r'x y:(0,pi) z:(-pi/3,0)', names=('x', 'y', 'z',)); (x, y, z,) =
↪X._first_ngens(3)
>>> X.coord_range()
x: (-oo, +oo); y: (0, pi); z: (-1/3*pi, 0)
```

Since x spans all **R**, no simplification of `abs(sin(x))` occurs, while `abs(sin(y))` and `abs(sin(3*z))` are correctly simplified, given that $y \in (0, \pi)$ and $z \in (-\pi/3, 0)$:

```
sage: from sage.manifolds.utilities import simplify_abs_trig
sage: simplify_abs_trig( abs(sin(x)) + abs(sin(y)) + abs(sin(3*z)) )
abs(sin(x)) + sin(y) + sin(-3*z)
```

```
>>> from sage.all import *
>>> from sage.manifolds.utilities import simplify_abs_trig
>>> simplify_abs_trig( abs(sin(x)) + abs(sin(y)) + abs(sin(Integer(3)*z)) )
abs(sin(x)) + sin(y) + sin(-3*z)
```

Note that neither `simplify_trig()` nor `simplify_full()` works in this case:

```
sage: s = abs(sin(x)) + abs(sin(y)) + abs(sin(3*z))
sage: s.simplify_trig()
abs(4*cos(-z)^2 - 1)*abs(sin(-z)) + abs(sin(x)) + abs(sin(y))
sage: s.simplify_full()
abs(4*cos(-z)^2 - 1)*abs(sin(-z)) + abs(sin(x)) + abs(sin(y))
```

```
>>> from sage.all import *
>>> s = abs(sin(x)) + abs(sin(y)) + abs(sin(Integer(3)*z))
>>> s.simplify_trig()
abs(4*cos(-z)^2 - 1)*abs(sin(-z)) + abs(sin(x)) + abs(sin(y))
>>> s.simplify_full()
abs(4*cos(-z)^2 - 1)*abs(sin(-z)) + abs(sin(x)) + abs(sin(y))
```

despite the following assumptions hold:

```
sage: assumptions()
[x is real, y is real, y > 0, y < pi, z is real, z > -1/3*pi, z < 0]
```

```
>>> from sage.all import *
>>> assumptions()
[x is real, y is real, y > 0, y < pi, z is real, z > -1/3*pi, z < 0]
```

Additional checks are:

```
sage: simplify_abs_trig( abs(sin(y/2)) )  # shall simplify
sin(1/2*y)
sage: simplify_abs_trig( abs(sin(2*y)) )  # must not simplify
abs(sin(2*y))
sage: simplify_abs_trig( abs(sin(z/2)) )  # shall simplify
sin(-1/2*z)
sage: simplify_abs_trig( abs(sin(4*z)) )  # must not simplify
abs(sin(-4*z))
```

```
>>> from sage.all import *
>>> simplify_abs_trig( abs(sin(y/Integer(2))) )  # shall simplify
sin(1/2*y)
>>> simplify_abs_trig( abs(sin(Integer(2)*y)) )  # must not simplify
abs(sin(2*y))
>>> simplify_abs_trig( abs(sin(z/Integer(2))) )  # shall simplify
sin(-1/2*z)
>>> simplify_abs_trig( abs(sin(Integer(4)*z)) )  # must not simplify
abs(sin(-4*z))
```

Simplification of `abs(cos(...))`:

```
sage: forget()
sage: M = Manifold(3, 'M', structure='topological')
sage: X.<x,y,z> = M.chart(r'x y:(0,pi/2) z:(pi/4,3*pi/4)')
sage: X.coord_range()
x: (-oo, +oo); y: (0, 1/2*pi); z: (1/4*pi, 3/4*pi)
sage: simplify_abs_trig( abs(cos(x)) + abs(cos(y)) + abs(cos(2*z)) )
abs(cos(x)) + cos(y) - cos(2*z)
```

```
>>> from sage.all import *
>>> forget()
>>> M = Manifold(Integer(3), 'M', structure='topological')
>>> X = M.chart(r'x y:(0,pi/2) z:(pi/4,3*pi/4)', names=('x', 'y', 'z',)); (x, y,␣
↪z,) = X._first_ngens(3)
>>> X.coord_range()
x: (-oo, +oo); y: (0, 1/2*pi); z: (1/4*pi, 3/4*pi)
>>> simplify_abs_trig( abs(cos(x)) + abs(cos(y)) + abs(cos(Integer(2)*z)) )
abs(cos(x)) + cos(y) - cos(2*z)
```

Additional tests:

```
sage: simplify_abs_trig(abs(cos(y-pi/2)))  # shall simplify
cos(-1/2*pi + y)
```

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```
sage: simplify_abs_trig(abs(cos(y+pi/2)))  # shall simplify
-cos(1/2*pi + y)
sage: simplify_abs_trig(abs(cos(y-pi)))  # shall simplify
-cos(-pi + y)
sage: simplify_abs_trig(abs(cos(2*y)))  # must not simplify
abs(cos(2*y))
sage: simplify_abs_trig(abs(cos(y/2)) * abs(sin(z)))  # shall simplify
cos(1/2*y)*sin(z)
```

```
>>> from sage.all import *
>>> simplify_abs_trig(abs(cos(y-pi/Integer(2))))  # shall simplify
cos(-1/2*pi + y)
>>> simplify_abs_trig(abs(cos(y+pi/Integer(2))))  # shall simplify
-cos(1/2*pi + y)
>>> simplify_abs_trig(abs(cos(y-pi)))  # shall simplify
-cos(-pi + y)
>>> simplify_abs_trig(abs(cos(Integer(2)*y)))  # must not simplify
abs(cos(2*y))
>>> simplify_abs_trig(abs(cos(y/Integer(2))) * abs(sin(z)))  # shall simplify
cos(1/2*y)*sin(z)
```

sage.manifolds.utilities.**simplify_chain_generic**(*expr*)

Apply a chain of simplifications to a symbolic expression.

This is the simplification chain used in calculus involving coordinate functions on manifolds over fields different from **R**, as implemented in `ChartFunction`.

The chain is formed by the following functions, called successively:

1. `simplify_factorial()`
2. `simplify_rectform()`
3. `simplify_trig()`
4. `simplify_rational()`
5. `expand_sum()`

NB: for the time being, this is identical to `simplify_full()`.

EXAMPLES:

We consider variables that are coordinates of a chart on a complex manifold:

```
sage: M = Manifold(2, 'M', structure='topological', field='complex')
sage: X.<x,y> = M.chart()
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological', field='complex')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
```

Then neither `x` nor `y` is assumed to be real:

```
sage: assumptions()
[]
```

```
>>> from sage.all import *
>>> assumptions()
[]
```

Accordingly, `simplify_chain_generic` does not simplify `sqrt(x^2)` to `abs(x)`:

```
sage: from sage.manifolds.utilities import simplify_chain_generic
sage: s = sqrt(x^2)
sage: simplify_chain_generic(s)
sqrt(x^2)
```

```
>>> from sage.all import *
>>> from sage.manifolds.utilities import simplify_chain_generic
>>> s = sqrt(x**Integer(2))
>>> simplify_chain_generic(s)
sqrt(x^2)
```

This contrasts with the behavior of *simplify_chain_real()*.

Other simplifications:

```
sage: s = (x+y)^2 - x^2 -2*x*y - y^2
sage: simplify_chain_generic(s)
0
sage: s = (x^2 - 2*x + 1) / (x^2 -1)
sage: simplify_chain_generic(s)
(x - 1)/(x + 1)
sage: s = cos(2*x) - 2*cos(x)^2 + 1
sage: simplify_chain_generic(s)
0
```

```
>>> from sage.all import *
>>> s = (x+y)**Integer(2) - x**Integer(2) -Integer(2)*x*y - y**Integer(2)
>>> simplify_chain_generic(s)
0
>>> s = (x**Integer(2) - Integer(2)*x + Integer(1)) / (x**Integer(2) -Integer(1))
>>> simplify_chain_generic(s)
(x - 1)/(x + 1)
>>> s = cos(Integer(2)*x) - Integer(2)*cos(x)**Integer(2) + Integer(1)
>>> simplify_chain_generic(s)
0
```

sage.manifolds.utilities.**simplify_chain_generic_sympy**(*expr*)

Apply a chain of simplifications to a sympy expression.

This is the simplification chain used in calculus involving coordinate functions on manifolds over fields different from **R**, as implemented in *ChartFunction*.

The chain is formed by the following functions, called successively:

1. `combsimp()`
2. `trigsimp()`
3. `expand()`
4. `simplify()`

EXAMPLES:

We consider variables that are coordinates of a chart on a complex manifold:

```
sage: forget()   # for doctest only
sage: M = Manifold(2, 'M', structure='topological', field='complex', calc_method=
↪'sympy')
sage: X.<x,y> = M.chart()
```

```
>>> from sage.all import *
>>> forget()   # for doctest only
>>> M = Manifold(Integer(2), 'M', structure='topological', field='complex', calc_
↪method='sympy')
>>> X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)
```

Then neither `x` nor `y` is assumed to be real:

```
sage: assumptions()
[]
```

```
>>> from sage.all import *
>>> assumptions()
[]
```

Accordingly, `simplify_chain_generic_sympy` does not simplify `sqrt(x^2)` to `abs(x)`:

```
sage: from sage.manifolds.utilities import simplify_chain_generic_sympy
sage: s = (sqrt(x^2))._sympy_()
sage: simplify_chain_generic_sympy(s)
sqrt(x**2)
```

```
>>> from sage.all import *
>>> from sage.manifolds.utilities import simplify_chain_generic_sympy
>>> s = (sqrt(x**Integer(2)))._sympy_()
>>> simplify_chain_generic_sympy(s)
sqrt(x**2)
```

This contrasts with the behavior of *simplify_chain_real_sympy()*.

Other simplifications:

```
sage: s = ((x+y)^2 - x^2 -2*x*y - y^2)._sympy_()
sage: simplify_chain_generic_sympy(s)
0
sage: s = ((x^2 - 2*x + 1) / (x^2 -1))._sympy_()
sage: simplify_chain_generic_sympy(s)
(x - 1)/(x + 1)
sage: s = (cos(2*x) - 2*cos(x)^2 + 1)._sympy_()
sage: simplify_chain_generic_sympy(s)
0
```

```
>>> from sage.all import *
>>> s = ((x+y)**Integer(2) - x**Integer(2) -Integer(2)*x*y - y**Integer(2))._
↪sympy_()
```

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```
>>> simplify_chain_generic_sympy(s)
0
>>> s = ((x**Integer(2) - Integer(2)*x + Integer(1)) / (x**Integer(2) -
↪Integer(1)))._sympy_()
>>> simplify_chain_generic_sympy(s)
(x - 1)/(x + 1)
>>> s = (cos(Integer(2)*x) - Integer(2)*cos(x)**Integer(2) + Integer(1))._sympy_()
>>> simplify_chain_generic_sympy(s)
0
```

sage.manifolds.utilities.**simplify_chain_real**(*expr*)

Apply a chain of simplifications to a symbolic expression, assuming the real domain.

This is the simplification chain used in calculus involving coordinate functions on real manifolds, as implemented in *ChartFunction*.

The chain is formed by the following functions, called successively:

1. simplify_factorial()
2. simplify_trig()
3. simplify_rational()
4. *simplify_sqrt_real()*
5. *simplify_abs_trig()*
6. canonicalize_radical()
7. simplify_log()
8. simplify_rational()
9. simplify_trig()

EXAMPLES:

We consider variables that are coordinates of a chart on a real manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart('x:(0,1) y')
```

```
>>> from sage.all import *
>>> M = Manifold(Integer(2), 'M', structure='topological')
>>> X = M.chart('x:(0,1) y', names=('x', 'y',)); (x, y,) = X._first_ngens(2)
```

The following assumptions then hold:

```
sage: assumptions()
[x is real, x > 0, x < 1, y is real]
```

```
>>> from sage.all import *
>>> assumptions()
[x is real, x > 0, x < 1, y is real]
```

and we have:

```
sage: from sage.manifolds.utilities import simplify_chain_real
sage: s = sqrt(y^2)
sage: simplify_chain_real(s)
abs(y)
```

```
>>> from sage.all import *
>>> from sage.manifolds.utilities import simplify_chain_real
>>> s = sqrt(y**Integer(2))
>>> simplify_chain_real(s)
abs(y)
```

The above result is correct since `y` is real. It is obtained by `simplify_real()` as well:

```
sage: s.simplify_real()
abs(y)
sage: s.simplify_full()
abs(y)
```

```
>>> from sage.all import *
>>> s.simplify_real()
abs(y)
>>> s.simplify_full()
abs(y)
```

Furthermore, we have:

```
sage: s = sqrt(x^2-2*x+1)
sage: simplify_chain_real(s)
-x + 1
```

```
>>> from sage.all import *
>>> s = sqrt(x**Integer(2)-Integer(2)*x+Integer(1))
>>> simplify_chain_real(s)
-x + 1
```

which is correct since $x \in (0, 1)$. On this example, neither `simplify_real()` nor `simplify_full()`, nor `canonicalize_radical()` give satisfactory results:

```
sage: s.simplify_real()  # unsimplified output
sqrt(x^2 - 2*x + 1)
sage: s.simplify_full()  # unsimplified output
sqrt(x^2 - 2*x + 1)
sage: s.canonicalize_radical()  # wrong output since x in (0,1)
x - 1
```

```
>>> from sage.all import *
>>> s.simplify_real()  # unsimplified output
sqrt(x^2 - 2*x + 1)
>>> s.simplify_full()  # unsimplified output
sqrt(x^2 - 2*x + 1)
>>> s.canonicalize_radical()  # wrong output since x in (0,1)
x - 1
```

Other simplifications:

```
sage: s = abs(sin(pi*x))
sage: simplify_chain_real(s)  # correct output since x in (0,1)
sin(pi*x)
sage: s.simplify_real()  # unsimplified output
abs(sin(pi*x))
sage: s.simplify_full()  # unsimplified output
abs(sin(pi*x))
```

```
>>> from sage.all import *
>>> s = abs(sin(pi*x))
>>> simplify_chain_real(s)  # correct output since x in (0,1)
sin(pi*x)
>>> s.simplify_real()  # unsimplified output
abs(sin(pi*x))
>>> s.simplify_full()  # unsimplified output
abs(sin(pi*x))
```

```
sage: s = cos(y)^2 + sin(y)^2
sage: simplify_chain_real(s)
1
sage: s.simplify_real()  # unsimplified output
cos(y)^2 + sin(y)^2
sage: s.simplify_full()  # OK
1
```

```
>>> from sage.all import *
>>> s = cos(y)**Integer(2) + sin(y)**Integer(2)
>>> simplify_chain_real(s)
1
>>> s.simplify_real()  # unsimplified output
cos(y)^2 + sin(y)^2
>>> s.simplify_full()  # OK
1
```

sage.manifolds.utilities.**simplify_chain_real_sympy**(*expr*)

Apply a chain of simplifications to a sympy expression, assuming the real domain.

This is the simplification chain used in calculus involving coordinate functions on real manifolds, as implemented in *ChartFunction*.

The chain is formed by the following functions, called successively:

1. combsimp()
2. trigsimp()
3. *simplify_sqrt_real()*
4. *simplify_abs_trig()*
5. expand()
6. simplify()

EXAMPLES:

We consider variables that are coordinates of a chart on a real manifold:

```
sage: forget()   # for doctest only
sage: M = Manifold(2, 'M', structure='topological',calc_method='sympy')
sage: X.<x,y> = M.chart('x:(0,1) y')
```

```
>>> from sage.all import *
>>> forget()   # for doctest only
>>> M = Manifold(Integer(2), 'M', structure='topological',calc_method='sympy')
>>> X = M.chart('x:(0,1) y', names=('x', 'y',)); (x, y,) = X._first_ngens(2)
```

The following assumptions then hold:

```
sage: assumptions()
[x is real, x > 0, x < 1, y is real]
```

```
>>> from sage.all import *
>>> assumptions()
[x is real, x > 0, x < 1, y is real]
```

and we have:

```
sage: from sage.manifolds.utilities import simplify_chain_real_sympy
sage: s = (sqrt(y^2))._sympy_()
sage: simplify_chain_real_sympy(s)
Abs(y)
```

```
>>> from sage.all import *
>>> from sage.manifolds.utilities import simplify_chain_real_sympy
>>> s = (sqrt(y**Integer(2)))._sympy_()
>>> simplify_chain_real_sympy(s)
Abs(y)
```

Furthermore, we have:

```
sage: s = (sqrt(x^2-2*x+1))._sympy_()
sage: simplify_chain_real_sympy(s)
1 - x
```

```
>>> from sage.all import *
>>> s = (sqrt(x**Integer(2)-Integer(2)*x+Integer(1)))._sympy_()
>>> simplify_chain_real_sympy(s)
1 - x
```

Other simplifications:

```
sage: s = (abs(sin(pi*x)))._sympy_()
sage: simplify_chain_real_sympy(s)   # correct output since x in (0,1)
sin(pi*x)
```

```
>>> from sage.all import *
>>> s = (abs(sin(pi*x)))._sympy_()
```

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```
>>> simplify_chain_real_sympy(s)  # correct output since x in (0,1)
sin(pi*x)
```

```
sage: s = (cos(y)^2 + sin(y)^2)._sympy_()
sage: simplify_chain_real_sympy(s)
1
```

```
>>> from sage.all import *
>>> s = (cos(y)**Integer(2) + sin(y)**Integer(2))._sympy_()
>>> simplify_chain_real_sympy(s)
1
```

sage.manifolds.utilities.**simplify_sqrt_real**(*expr*)

Simplify `sqrt` in symbolic expressions in the real domain.

EXAMPLES:

Simplifications of basic expressions:

```
sage: from sage.manifolds.utilities import simplify_sqrt_real
sage: simplify_sqrt_real( sqrt(x^2) )
abs(x)
sage: assume(x<0)
sage: simplify_sqrt_real( sqrt(x^2) )
-x
sage: simplify_sqrt_real( sqrt(x^2-2*x+1) )
-x + 1
sage: simplify_sqrt_real( sqrt(x^2) + sqrt(x^2-2*x+1) )
-2*x + 1
```

```
>>> from sage.all import *
>>> from sage.manifolds.utilities import simplify_sqrt_real
>>> simplify_sqrt_real( sqrt(x**Integer(2)) )
abs(x)
>>> assume(x<Integer(0))
>>> simplify_sqrt_real( sqrt(x**Integer(2)) )
-x
>>> simplify_sqrt_real( sqrt(x**Integer(2)-Integer(2)*x+Integer(1)) )
-x + 1
>>> simplify_sqrt_real( sqrt(x**Integer(2)) + sqrt(x**Integer(2)-
↪Integer(2)*x+Integer(1)) )
-2*x + 1
```

This improves over `canonicalize_radical()`, which yields incorrect results when `x < 0`:

```
sage: forget()  # removes the assumption x<0
sage: sqrt(x^2).canonicalize_radical()
x
sage: assume(x<0)
sage: sqrt(x^2).canonicalize_radical()
-x
sage: sqrt(x^2-2*x+1).canonicalize_radical() # wrong output
```

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```
x - 1
sage: ( sqrt(x^2) + sqrt(x^2-2*x+1) ).canonicalize_radical() # wrong output
-1
```

```
>>> from sage.all import *
>>> forget()  # removes the assumption x<0
>>> sqrt(x**Integer(2)).canonicalize_radical()
x
>>> assume(x<Integer(0))
>>> sqrt(x**Integer(2)).canonicalize_radical()
-x
>>> sqrt(x**Integer(2)-Integer(2)*x+Integer(1)).canonicalize_radical() # wrong␣
↪output
x - 1
>>> ( sqrt(x**Integer(2)) + sqrt(x**Integer(2)-Integer(2)*x+Integer(1)) ).
↪canonicalize_radical() # wrong output
-1
```

Simplification of nested `sqrt`'s:

```
sage: forget()  # removes the assumption x<0
sage: simplify_sqrt_real( sqrt(1 + sqrt(x^2)) )
sqrt(abs(x) + 1)
sage: assume(x<0)
sage: simplify_sqrt_real( sqrt(1 + sqrt(x^2)) )
sqrt(-x + 1)
sage: simplify_sqrt_real( sqrt(x^2 + sqrt(4*x^2) + 1) )
-x + 1
```

```
>>> from sage.all import *
>>> forget()  # removes the assumption x<0
>>> simplify_sqrt_real( sqrt(Integer(1) + sqrt(x**Integer(2))) )
sqrt(abs(x) + 1)
>>> assume(x<Integer(0))
>>> simplify_sqrt_real( sqrt(Integer(1) + sqrt(x**Integer(2))) )
sqrt(-x + 1)
>>> simplify_sqrt_real( sqrt(x**Integer(2) + sqrt(Integer(4)*x**Integer(2)) +␣
↪Integer(1)) )
-x + 1
```

Again, `canonicalize_radical()` fails on the last one:

```
sage: (sqrt(x^2 + sqrt(4*x^2) + 1)).canonicalize_radical()
x - 1
```

```
>>> from sage.all import *
>>> (sqrt(x**Integer(2) + sqrt(Integer(4)*x**Integer(2)) + Integer(1))).
↪canonicalize_radical()
x - 1
```

`sage.manifolds.utilities.`**`xder`**`(`*form*`)`

Exterior derivative of a differential form.

INPUT:

- `form` – a differential form; this must an instance of either
  - *DiffScalarField* for a 0-form (scalar field)
  - *DiffFormParal* for a p-form ($p \geq 1$) on a parallelizable manifold
  - *DiffForm* for a a p-form ($p \geq 1$) on a non-parallelizable manifold

OUTPUT:

- the $(p + 1)$-form that is the exterior derivative of `form`

EXAMPLES:

Exterior derivative of a scalar field (0-form):

```
sage: from sage.manifolds.utilities import exterior_derivative
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: f = M.scalar_field({X: x+y^2+z^3}, name='f')
sage: df = exterior_derivative(f); df
1-form df on the 3-dimensional differentiable manifold M
sage: df.display()
df = dx + 2*y dy + 3*z^2 dz
```

```
>>> from sage.all import *
>>> from sage.manifolds.utilities import exterior_derivative
>>> M = Manifold(Integer(3), 'M')
>>> X = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = X._first_ngens(3)
>>> f = M.scalar_field({X: x+y**Integer(2)+z**Integer(3)}, name='f')
>>> df = exterior_derivative(f); df
1-form df on the 3-dimensional differentiable manifold M
>>> df.display()
df = dx + 2*y dy + 3*z^2 dz
```

An alias is `xder`:

```
sage: from sage.manifolds.utilities import xder
sage: df == xder(f)
True
```

```
>>> from sage.all import *
>>> from sage.manifolds.utilities import xder
>>> df == xder(f)
True
```

Exterior derivative of a 1-form:

```
sage: a = M.one_form(name='a')
sage: a[:] = [x+y*z, x-y*z, x*y*z]
sage: da = xder(a); da
2-form da on the 3-dimensional differentiable manifold M
sage: da.display()
da = (-z + 1) dx∧dy + (y*z - y) dx∧dz + (x*z + y) dy∧dz
sage: dda = xder(da); dda
```

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```
3-form dda on the 3-dimensional differentiable manifold M
sage: dda.display()
dda = 0
```

```
>>> from sage.all import *
>>> a = M.one_form(name='a')
>>> a[:] = [x+y*z, x-y*z, x*y*z]
>>> da = xder(a); da
2-form da on the 3-dimensional differentiable manifold M
>>> da.display()
da = (-z + 1) dx∧dy + (y*z - y) dx∧dz + (x*z + y) dy∧dz
>>> dda = xder(da); dda
3-form dda on the 3-dimensional differentiable manifold M
>>> dda.display()
dda = 0
```

See also

sage.manifolds.differentiable.diff_form.DiffFormParal.exterior_derivative or *sage.manifolds.differentiable.diff_form.DiffForm.exterior_derivative* for more examples.

CHAPTER

SIX

MANIFOLDS CATALOG

A catalog of manifolds to rapidly create various simple manifolds.

The current entries to the catalog are obtained by typing `manifolds.<tab>`, where `<tab>` indicates pressing the `Tab` key. They are:

- *`EuclideanSpace`*: Euclidean space
- *`RealLine`*: real line
- *`OpenInterval`*: open interval on the real line
- *`Sphere`*: sphere embedded in Euclidean space
- *`Torus()`*: torus embedded in Euclidean space
- *`Minkowski()`*: 4-dimensional Minkowski space
- *`Kerr()`*: Kerr spacetime
- *`RealProjectiveSpace()`*: n-dimensional real projective space

AUTHORS:

- Florentin Jaffredo (2018) : initial version
- Trevor K. Karn (2022) : projective space

`sage.manifolds.catalog.`**`Kerr`**`(`*m=1*, *a=0*, *coordinates='BL'*, *names=None*`)`

Generate a Kerr spacetime.

A Kerr spacetime is a 4 dimensional manifold describing a rotating black hole. Two coordinate systems are implemented: Boyer-Lindquist and Kerr (3+1 version).

The shortcut operator `.<,>` can be used to specify the coordinates.

INPUT:

- `m` – (default: `1`) mass of the black hole in natural units ($c = 1$, $G = 1$)
- `a` – (default: `0`) angular momentum in natural units; if set to `0`, the resulting spacetime corresponds to a Schwarzschild black hole
- `coordinates` – (default: `'BL'`) either `'BL'` for Boyer-Lindquist coordinates or `'Kerr'` for Kerr coordinates (3+1 version)
- `names` – (default: `None`) name of the coordinates, automatically set by the shortcut operator

OUTPUT: Lorentzian manifold

EXAMPLES:

```
sage: m, a = var('m, a')
sage: K = manifolds.Kerr(m, a)
sage: K
4-dimensional Lorentzian manifold M
sage: K.atlas()
[Chart (M, (t, r, th, ph))]
```

```
>>> from sage.all import *
>>> m, a = var('m, a')
>>> K = manifolds.Kerr(m, a)
>>> K
4-dimensional Lorentzian manifold M
>>> K.atlas()
[Chart (M, (t, r, th, ph))]
```

The Kerr metric in Boyer-Lindquist coordinates (cf. Wikipedia article Kerr_metric):

```
sage: K.metric().display()
g = (2*m*r/(a^2*cos(th)^2 + r^2) - 1) dt⊗dt
 - 2*a*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2) dt⊗dph
 + (a^2*cos(th)^2 + r^2)/(a^2 - 2*m*r + r^2) dr⊗dr
 + (a^2*cos(th)^2 + r^2) dth⊗dth
 - 2*a*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2) dph⊗dt
 + (2*a^2*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2) + a^2 + r^2)*sin(th)^2 dph⊗dph
```

```
>>> from sage.all import *
>>> K.metric().display()
g = (2*m*r/(a^2*cos(th)^2 + r^2) - 1) dt⊗dt
 - 2*a*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2) dt⊗dph
 + (a^2*cos(th)^2 + r^2)/(a^2 - 2*m*r + r^2) dr⊗dr
 + (a^2*cos(th)^2 + r^2) dth⊗dth
 - 2*a*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2) dph⊗dt
 + (2*a^2*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2) + a^2 + r^2)*sin(th)^2 dph⊗dph
```

The Schwarzschild spacetime with the mass parameter set to 1:

```
sage: K.<t, r, th, ph> = manifolds.Kerr()
sage: K
4-dimensional Lorentzian manifold M
sage: K.metric().display()
g = (2/r - 1) dt⊗dt + r^2/(r^2 - 2*r) dr⊗dr
 + r^2 dth⊗dth + r^2*sin(th)^2 dph⊗dph
sage: K.default_chart().coord_range()
t: (-oo, +oo); r: (0, +oo); th: (0, pi); ph: [-pi, pi] (periodic)
```

```
>>> from sage.all import *
>>> K = manifolds.Kerr(names=('t', 'r', 'th', 'ph',)); (t, r, th, ph,) = K._first_
↪ngens(4)
>>> K
4-dimensional Lorentzian manifold M
>>> K.metric().display()
g = (2/r - 1) dt⊗dt + r^2/(r^2 - 2*r) dr⊗dr
```

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```
 + r^2 dth⊗dth + r^2*sin(th)^2 dph⊗dph
>>> K.default_chart().coord_range()
t: (-oo, +oo); r: (0, +oo); th: (0, pi); ph: [-pi, pi] (periodic)
```

The Kerr spacetime in Kerr coordinates:

```
sage: m, a = var('m, a')
sage: K.<t, r, th, ph> = manifolds.Kerr(m, a, coordinates='Kerr')
sage: K
4-dimensional Lorentzian manifold M
sage: K.atlas()
[Chart (M, (t, r, th, ph))]
sage: K.metric().display()
g = (2*m*r/(a^2*cos(th)^2 + r^2) - 1) dt⊗dt
 + 2*m*r/(a^2*cos(th)^2 + r^2) dt⊗dr
 - 2*a*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2) dt⊗dph
 + 2*m*r/(a^2*cos(th)^2 + r^2) dr⊗dt
 + (2*m*r/(a^2*cos(th)^2 + r^2) + 1) dr⊗dr
 - a*(2*m*r/(a^2*cos(th)^2 + r^2) + 1)*sin(th)^2 dr⊗dph
 + (a^2*cos(th)^2 + r^2) dth⊗dth
 - 2*a*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2) dph⊗dt
 - a*(2*m*r/(a^2*cos(th)^2 + r^2) + 1)*sin(th)^2 dph⊗dr
 + (2*a^2*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2)
 + a^2 + r^2)*sin(th)^2 dph⊗dph
sage: K.default_chart().coord_range()
t: (-oo, +oo); r: (0, +oo); th: (0, pi); ph: [-pi, pi] (periodic)
```

```
>>> from sage.all import *
>>> m, a = var('m, a')
>>> K = manifolds.Kerr(m, a, coordinates='Kerr', names=('t', 'r', 'th', 'ph',));␣
↪(t, r, th, ph,) = K._first_ngens(4)
>>> K
4-dimensional Lorentzian manifold M
>>> K.atlas()
[Chart (M, (t, r, th, ph))]
>>> K.metric().display()
g = (2*m*r/(a^2*cos(th)^2 + r^2) - 1) dt⊗dt
 + 2*m*r/(a^2*cos(th)^2 + r^2) dt⊗dr
 - 2*a*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2) dt⊗dph
 + 2*m*r/(a^2*cos(th)^2 + r^2) dr⊗dt
 + (2*m*r/(a^2*cos(th)^2 + r^2) + 1) dr⊗dr
 - a*(2*m*r/(a^2*cos(th)^2 + r^2) + 1)*sin(th)^2 dr⊗dph
 + (a^2*cos(th)^2 + r^2) dth⊗dth
 - 2*a*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2) dph⊗dt
 - a*(2*m*r/(a^2*cos(th)^2 + r^2) + 1)*sin(th)^2 dph⊗dr
 + (2*a^2*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2)
 + a^2 + r^2)*sin(th)^2 dph⊗dph
>>> K.default_chart().coord_range()
t: (-oo, +oo); r: (0, +oo); th: (0, pi); ph: [-pi, pi] (periodic)
```

`sage.manifolds.catalog.`**`Minkowski`**(*positive_spacelike=True*, *names=None*)

Generate a Minkowski space of dimension 4.

By default the signature is set to $(- + ++)$, but can be changed to $(+ - --)$ by setting the optional argument `positive_spacelike` to `False`. The shortcut operator `.<,>` can be used to specify the coordinates.

INPUT:

- `positive_spacelike` – boolean (default: `True`); if `False`, then the spacelike vectors yield a negative sign (i.e., the signature is $(+ - --)$)
- `names` – (default: `None`) name of the coordinates, automatically set by the shortcut operator

OUTPUT:

- Lorentzian manifold of dimension 4 with (flat) Minkowskian metric

EXAMPLES:

```
sage: M.<t, x, y, z> = manifolds.Minkowski()
sage: M.metric()[:]
[-1  0  0  0]
[ 0  1  0  0]
[ 0  0  1  0]
[ 0  0  0  1]

sage: M.<t, x, y, z> = manifolds.Minkowski(False)
sage: M.metric()[:]
[ 1  0  0  0]
[ 0 -1  0  0]
[ 0  0 -1  0]
[ 0  0  0 -1]
```

```
>>> from sage.all import *
>>> M = manifolds.Minkowski(names=('t', 'x', 'y', 'z',)); (t, x, y, z,) = M._
↪first_ngens(4)
>>> M.metric()[:]
[-1  0  0  0]
[ 0  1  0  0]
[ 0  0  1  0]
[ 0  0  0  1]

>>> M = manifolds.Minkowski(False, names=('t', 'x', 'y', 'z',)); (t, x, y, z,) =␣
↪M._first_ngens(4)
>>> M.metric()[:]
[ 1  0  0  0]
[ 0 -1  0  0]
[ 0  0 -1  0]
[ 0  0  0 -1]
```

`sage.manifolds.catalog.`**`RealProjectiveSpace`**`(`*dim=2*`)`

Generate projective space of dimension `dim` over the reals.

This is the topological space of lines through the origin in $\mathbf{R}^{d+1}$. The standard atlas consists of $d+2$ charts, which sends the set $U_i = \{[x_1, x_2, \ldots, x_{d+1}] : x_i \neq 0\}$ to k^d by dividing by x_i and omitting the i-th coordinate $x_i/x_i = 1$.

INPUT:

- `dim` – (default: `2`) the dimension of projective space

OUTPUT: `P` – the projective space $\mathbf{RP}^d$ where $d =$ `dim`

EXAMPLES:

```
sage: RP2 = manifolds.RealProjectiveSpace(); RP2
2-dimensional topological manifold RP2
sage: latex(RP2)
\mathbb{RP}^{2}

sage: C0, C1, C2 = RP2.top_charts()
sage: p = RP2.point((2,0), chart = C0)
sage: q = RP2.point((0,3), chart = C0)
sage: p in C0.domain()
True
sage: p in C1.domain()
True
sage: C1(p)
(1/2, 0)
sage: p in C2.domain()
False
sage: q in C0.domain()
True
sage: q in C1.domain()
False
sage: q in C2.domain()
True
sage: C2(q)
(1/3, 0)

sage: r = RP2.point((2,3))
sage: r in C0.domain() and r in C1.domain() and r in C2.domain()
True
sage: C0(r)
(2, 3)
sage: C1(r)
(1/2, 3/2)
sage: C2(r)
(1/3, 2/3)

sage: p = RP2.point((2,3), chart = C1)
sage: p in C0.domain() and p in C1.domain() and p in C2.domain()
True
sage: C0(p)
(1/2, 3/2)
sage: C2(p)
(2/3, 1/3)

sage: RP1 = manifolds.RealProjectiveSpace(1); RP1
1-dimensional topological manifold RP1
sage: C0, C1 = RP1.top_charts()
sage: p, q = RP1.point((2,)), RP1.point((0,))
sage: p in C0.domain()
True
sage: p in C1.domain()
```

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```
True
sage: q in C0.domain()
True
sage: q in C1.domain()
False
sage: C1(p)
(1/2,)

sage: p, q = RP1.point((3,), chart = C1), RP1.point((0,), chart = C1)
sage: p in C0.domain()
True
sage: q in C0.domain()
False
sage: C0(p)
(1/3,)
```

```
>>> from sage.all import *
>>> RP2 = manifolds.RealProjectiveSpace(); RP2
2-dimensional topological manifold RP2
>>> latex(RP2)
\mathbb{RP}^{2}

>>> C0, C1, C2 = RP2.top_charts()
>>> p = RP2.point((Integer(2),Integer(0)), chart = C0)
>>> q = RP2.point((Integer(0),Integer(3)), chart = C0)
>>> p in C0.domain()
True
>>> p in C1.domain()
True
>>> C1(p)
(1/2, 0)
>>> p in C2.domain()
False
>>> q in C0.domain()
True
>>> q in C1.domain()
False
>>> q in C2.domain()
True
>>> C2(q)
(1/3, 0)

>>> r = RP2.point((Integer(2),Integer(3)))
>>> r in C0.domain() and r in C1.domain() and r in C2.domain()
True
>>> C0(r)
(2, 3)
>>> C1(r)
(1/2, 3/2)
>>> C2(r)
(1/3, 2/3)
```

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```
>>> p = RP2.point((Integer(2),Integer(3)), chart = C1)
>>> p in C0.domain() and p in C1.domain() and p in C2.domain()
True
>>> C0(p)
(1/2, 3/2)
>>> C2(p)
(2/3, 1/3)

>>> RP1 = manifolds.RealProjectiveSpace(Integer(1)); RP1
1-dimensional topological manifold RP1
>>> C0, C1 = RP1.top_charts()
>>> p, q = RP1.point((Integer(2),)), RP1.point((Integer(0),))
>>> p in C0.domain()
True
>>> p in C1.domain()
True
>>> q in C0.domain()
True
>>> q in C1.domain()
False
>>> C1(p)
(1/2,)

>>> p, q = RP1.point((Integer(3),), chart = C1), RP1.point((Integer(0),), chart =␣
↪C1)
>>> p in C0.domain()
True
>>> q in C0.domain()
False
>>> C0(p)
(1/3,)
```

`sage.manifolds.catalog.`**`Torus`**`(`*R=2*, *r=1*, *names=None*`)`

Generate a 2-dimensional torus embedded in Euclidean space.

The shortcut operator `.<,>` can be used to specify the coordinates.

INPUT:

- `R` – (default: `2`) distance form the center to the center of the tube
- `r` – (default: `1`) radius of the tube
- `names` – (default: `None`) name of the coordinates, automatically set by the shortcut operator

OUTPUT: Riemannian manifold

EXAMPLES:

```
sage: T.<theta, phi> = manifolds.Torus(3, 1)
sage: T
2-dimensional Riemannian submanifold T embedded in the Euclidean
 space E^3
sage: T.atlas()
[Chart (T, (theta, phi))]
```

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```
sage: T.embedding().display()
T → E^3
   (theta, phi) ↦ (X, Y, Z) = ((cos(theta) + 3)*cos(phi),
                               (cos(theta) + 3)*sin(phi),
                               sin(theta))
sage: T.metric().display()
gamma = dtheta⊗dtheta + (cos(theta)^2 + 6*cos(theta) + 9) dphi⊗dphi
```

```
>>> from sage.all import *
>>> T = manifolds.Torus(Integer(3), Integer(1), names=('theta', 'phi',)); (theta,␣
↪phi,) = T._first_ngens(2)
>>> T
2-dimensional Riemannian submanifold T embedded in the Euclidean
 space E^3
>>> T.atlas()
[Chart (T, (theta, phi))]
>>> T.embedding().display()
T → E^3
   (theta, phi) ↦ (X, Y, Z) = ((cos(theta) + 3)*cos(phi),
                               (cos(theta) + 3)*sin(phi),
                               sin(theta))
>>> T.metric().display()
gamma = dtheta⊗dtheta + (cos(theta)^2 + 6*cos(theta) + 9) dphi⊗dphi
```

CHAPTER SEVEN

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PYTHON MODULE INDEX

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