Modules

Release 9.6

The Sage Development Team

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In this tutorial, we show how to construct and manipulate free modules and vector spaces and their elements.

Sage currently provides two implementations of free modules: `FreeModule` and `CombinatorialFreeModule`. The distinction between the two is mostly an accident in history. The latter allows for the basis to be indexed by any kind of objects, instead of just 0, 1, 2, .... They also differ by feature set and efficiency. Eventually, both implementations will be merged under the name `FreeModule`. In the mean time, we focus here on `CombinatorialFreeModule`. We recommend to start by browsing its documentation:

```
sage: CombinatorialFreeModule # not tested
```

### 1.1 Construction, arithmetic, and basic usage

We begin with a minimal example:

```
sage: G = Zmod(5)
sage: F = CombinatorialFreeModule(ZZ, G)
sage: F.an_element()
```

$F$ is the free module over the ring integers $\mathbb{Z}$ whose canonical basis is indexed by the set of integers modulo 5.

We can use any set, finite or not, to index the basis, as long as its elements are immutable. Here are some $\mathbb{Z}$-free modules; what is the indexing set for the basis in each example below?

```
sage: F = CombinatorialFreeModule(ZZ, CC); F.an_element()
B[1.00000000000000*I]
sage: F = CombinatorialFreeModule(ZZ, Partitions(NonNegativeIntegers(), max_part=3)); F.an_element()
sage: F = CombinatorialFreeModule(ZZ, ["spam", "eggs", '42']); F.an_element()
3*B['42'] + 2*B['eggs'] + 2*B['spam']
```

Note that we use ‘42’ (and not the number 42) in order to ensure that all objects are comparable in a deterministic way, which allows the elements to be printed in a predictable manner. It is not mandatory that indices have such a stable ordering, but if they do not, then the elements may be displayed in some random order.

Lists are not hashable, and thus cannot be used to index the basis; instead one can use tuples:

```
sage: F = CombinatorialFreeModule(ZZ, ([1],[2],[3])); F.an_element()
Traceback (most recent call last):
```
...TypeError: unhashable type: 'list'

```
sage: F = CombinatorialFreeModule(ZZ, ((1,), (2,), (3,))); F.an_element()
2*B[(1,)] + 2*B[(2,)] + 3*B[(3,)]
```

The name of the basis can be customized:

```
sage: F = CombinatorialFreeModule(ZZ, Zmod(5), prefix='a'); F.an_element()
2*a[0] + 2*a[1] + 3*a[2]
```

Let us do some arithmetic with elements of $\mathcal{A}$:

```
sage: f = F.an_element(); f
2*a[0] + 2*a[1] + 3*a[2]
sage: 2*f
4*a[0] + 4*a[1] + 6*a[2]
sage: 2*f - f
2*a[0] + 2*a[1] + 3*a[2]
```

Inputing elements as they are output does not work by default:

```
sage: a[0] + 3*a[1]
Traceback (most recent call last):
  ...NameError: name 'a' is not defined
```

To enable this, we must first get the canonical basis for the module:

```
sage: a = F.basis(); a
Lazy family (Term map from Ring of integers modulo 5 to Free module generated by Ring of␣˓→integers modulo 5 over Integer Ring(i))_{i in Ring of integers modulo 5}
```

This gadget models the family $(B_i)_{i \in \mathbb{Z}_5}$. In particular, one can run through its elements:

```
sage: list(a)
[a[0], a[1], a[2], a[3], a[4]]
```

recover its indexing set:

```
sage: a.keys()
Ring of integers modulo 5
```

or construct an element from the corresponding index:

```
sage: a[2]
a[2]
```

So now we can do:

```
sage: a[0] + 3*a[1]
a[0] + 3*a[1]
```
which enables copy-pasting outputs as long as the prefix matches the name of the basis:

```
True
```

Be careful that the input is currently not checked:

```
sage: a['is'] + a['this'] + a['a'] + a['bug']
```

```
a['a'] + a['bug'] + a['is'] + a['this']
```

### 1.2 Manipulating free module elements

The elements of our module come with many methods for exploring and manipulating them:

```
sage: f.<tab>  # not tested
```

Some definitions:

- A *monomial* is an element of the basis $B_i$;
- A *term* is an element of the basis multiplied by a non zero *coefficient*: $cB_i$;
- The support of that term is $i$.
- The corresponding *item* is the tuple $(i, c)$.
- The *support* of an element $f$ is the collection of indices $i$ such that $B_i$ appears in $f$ with non zero coefficient.
- The *monomials*, *terms*, *items*, and *coefficients* of an element $f$ are defined accordingly.
- *Leading/trailing* refers to the greatest/least index. Elements are printed starting with the least index (for lexicographic order by default).

Let us investigate those definitions on our example:

```
sage: f
2*a[0] + 2*a[1] + 3*a[2]
sage: f.leading_term()
3*a[2]
sage: f.leading_monomial()
a[2]
sage: f.leading_support()
2
sage: f.leading_coefficient()
3
sage: f.leading_item()
(2, 3)
sage: f.support()
[0, 1, 2]
sage: f.monomials()
[a[0], a[1], a[2]]
sage: f.coefficients()
[2, 2, 3]
```

We can iterate through the items of an element:
sage: for index, coeff in f:
    ....:    print("The coefficient of a_{%s} is %s"%(index, coeff))
The coefficient of a_{0} is 2
The coefficient of a_{1} is 2
The coefficient of a_{2} is 3

This element can be thought of as a dictionary index→coefficient:

sage: f[0], f[1], f[2]
(2, 2, 3)

This dictionary can be accessed explicitly with the monomial_coefficients method:

sage: f.monomial_coefficients()
{0: 2, 1: 2, 2: 3}

The map methods are useful to transform elements:

sage: f
2*a[0] + 2*a[1] + 3*a[2]
sage: f.map_support(lambda i: i+1)
sage: f.map_coefficients(lambda c: c-3)
-a[0] - a[1]
sage: f.map_item(lambda i,c: (i+1,c-3))

Note: this last function should be called map_items!

## 1.3 Manipulating free modules

The free module itself (A in our example) has several utility methods for constructing elements:

sage: F.zero()
0
sage: F.term(1)
a[1]
sage: F.sum_of_monomials(i for i in Zmod(5) if i > 2)
sage: F.sum_of_terms((i+1,i) for i in Zmod(5) if i > 2)
4*a[0] + 3*a[4]
sage: F.sum(ZZ(i)*a[i+1] for i in Zmod(5) if i > 2)  # Note coeff is not (currently) implicitly coerced
4*a[0] + 3*a[4]

It is safer to use F.sum() than to use sum(): in case the input is an empty iterable, it makes sure the zero of A is returned, and not a plain 0:

sage: F.sum([]), parent(F.sum([]))
(0, Free module generated by Ring of integers modulo 5 over Integer Ring)
sage: sum([]), parent(sum([]))
(0, <... 'int'>)
Todo: Introduce echelon forms, submodules, quotients in the finite dimensional case

1.4 Review

In this tutorial we have seen how to construct vector spaces and free modules with a basis indexed by any kind of objects.

To learn how to endow such free modules with additional structure, define morphisms, or implement modules with several distinguished basis, see the Implementing Algebraic Structures thematic tutorial.
ABSTRACT BASE CLASS FOR MODULES

AUTHORS:

- William Stein: initial version

EXAMPLES:

A minimal example of a module:

```python
sage: from sage.structure.richcmp import richcmp
sage: class MyElement(sage.structure.element.ModuleElement):
    ....:    def __init__(self, parent, x):
    ....:        self.x = x
    ....:        sage.structure.element.ModuleElement.__init__(self, parent=parent)
    ....:    def _lmul_(self, c):
    ....:        return self.parent()(c*self.x)
    ....:    def _add_(self, other):
    ....:        return self.parent()(self.x + other.x)
    ....:    def _richcmp_(self, other, op):
    ....:        return richcmp(self.x, other.x, op)
    ....:    def __hash__(self):
    ....:        return hash(self.x)
    ....:    def __repr__(self):
    ....:        return repr(self.x)

sage: class MyModule(sage.modules.module.Module):
    ....:    Element = MyElement
    ....:    def _element_constructor_(self, x):
    ....:        if isinstance(x, MyElement): x = x.x
    ....:        return self.element_class(self, self.base_ring()(x))
    ....:    def __eq__(self, other):
    ....:        if not isinstance(other, MyModule): return False
    ....:        return self.base_ring() == other.base_ring()
    ....:    def __hash__(self):
    ....:        return hash(self.base_ring())

sage: M = MyModule(QQ)
sage: M(1)
1
sage: import __main__
```

(continues on next page)
class sage.modules.module.Module

    Bases: sage.structure.parent.Parent

    Generic module class.

    INPUT:

    - `base` – a ring. The base ring of the module.
    - `category` – a category (default: None), the category for this module. If None, then this is set to the category of modules/vector spaces over base.
    - `names` – names of generators

    EXAMPLES:

    ```python
    sage: from sage.modules.module import Module
    sage: M = Module(ZZ)
    sage: M.base_ring()
    Integer Ring
    sage: M.category()
    Category of modules over Integer Ring
    ```

    Normally the category is set to the category of modules over base. If base is a field, then the category is the category of vector spaces over base:

    ```python
    sage: M_QQ = Module(QQ)
    sage: M_QQ.category()
    Category of vector spaces over Rational Field
    ```

    The `category` parameter can be used to set a more specific category:

    ```python
    sage: N = Module(ZZ, category=FiniteDimensionalModulesWithBasis(ZZ))
    sage: N.category()
    Category of finite dimensional modules with basis over Integer Ring
    ```

    `base_extend(R)`

    Return the base extension of self to R.

    This is the same as `self.change_ring(R)` except that a `TypeError` is raised if there is no canonical coerce map from the base ring of self to R.

    INPUT:

    - `R` – ring

    EXAMPLES:

    ```python
    sage: V = ZZ^7
    sage: V.base_extend(QQ)
    Vector space of dimension 7 over Rational Field
    ```

    `change_ring(R)`

    Return the base change of self to R.
EXAMPLES:

```python
sage: sage.modular.modform.space.ModularFormsSpace(Gamma0(11), 2,
     ...DirichletGroup(1)[0], QQ).change_ring(GF(7))
Traceback (most recent call last):
... 
NotImplementedError: the method change_ring() has not yet been implemented
```

```python
domain
```

Return the endomorphism ring of this module in its category.

EXAMPLES:

```python
sage: from sage.modules.module import Module
sage: M = Module(ZZ)
sage: M.endomorphism_ring()
Set of Morphisms from <sage.modules.module.Module object at ...> to <sage.
     ...modules.module.Module object at ...> in Category of modules over Integer Ring
```

```python
gens
```

Return True if x is a module, False otherwise.

INPUT:

* x – anything.

EXAMPLES:

```python
sage: from sage.modules.module import is_Module
sage: M = FreeModule(RationalField(),30)
sage: is_Module(M)
True
sage: is_Module(10)
False
```

```python
sage: from sage.modules.module import is_VectorSpace
sage: M = FreeModule(RationalField(),30)
sage: is_VectorSpace(M)
True
sage: M = FreeModule(IntegerRing(),30)
sage: is_Module(M)
True
sage: is_VectorSpace(M)
False
```

```python
endomorphism_ring()
```

Return True if x is a vector space, False otherwise.

INPUT:

* x – anything.

EXAMPLES:
Chapter 2. Abstract base class for modules
FREE MODULES

Sage supports computation with free modules over an arbitrary commutative ring. Nontrivial functionality is available
over \( \mathbb{Z} \), fields, and some principal ideal domains (e.g. \( \mathbb{Q}[x] \) and rings of integers of number fields). All free modules
over an integral domain are equipped with an embedding in an ambient vector space and an inner product, which you
can specify and change.

Create the free module of rank \( n \) over an arbitrary commutative ring \( R \) using the command \texttt{FreeModule}(R, n). Equiv-
alanently, \( R^n \) also creates that free module.

The following example illustrates the creation of both a vector space and a free module over the integers and a submodule
of it. Use the functions \texttt{FreeModule}, \texttt{span} and member functions of free modules to create free modules. \textit{Do not use
the \texttt{FreeModule}_xxx constructors directly.}

EXAMPLES:

```python
sage: V = VectorSpace(QQ,3)
sage: W = V.subspace([[1,2,7], [1,1,0]])
sage: W
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1 0 -7]
[ 0 1 7]
sage: C = VectorSpaces(FiniteField(7))
sage: C
Category of vector spaces over Finite Field of size 7
sage: C(W)
Vector space of degree 3 and dimension 2 over Finite Field of size 7
Basis matrix:
[1 0 0]
[0 1 0]
sage: M = ZZ^3
sage: C = VectorSpaces(FiniteField(7))
sage: C(M)
Vector space of dimension 3 over Finite Field of size 7
sage: W = M.submodule([[1,2,7], [8,8,0]])
sage: C(W)
Vector space of degree 3 and dimension 2 over Finite Field of size 7
Basis matrix:
[1 0 0]
[0 1 0]
```

We illustrate the exponent notation for creation of free modules.
### Modules, Release 9.6

**sage:** $\mathbb{Z}^4$

Ambient free module of rank 4 over the principal ideal domain Integer Ring

**sage:** $\mathbb{Q}^2$

Vector space of dimension 2 over Rational Field

**sage:** $\mathbb{R}^3$

Vector space of dimension 3 over Real Field with 53 bits of precision

Base ring:

```python
sage: R.<x,y> = QQ[]
sage: M = FreeModule(R,2)
sage: M.base_ring()
Multivariate Polynomial Ring in x, y over Rational Field
```

**sage:** VectorSpace(QQ, 10).base_ring()

Rational Field

Enumeration of $\mathbb{Z}^n$ happens in order of increasing 1-norm primarily and increasing $\infty$-norm secondarily:

```python
sage: print([(0, 0, 0),
          (1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, -1, 0), (0, 0, 1), (0, 0, -1),
          (1, 1, 0), (-1, 1, 0), (1, -1, 0), (-1, -1, 0), (1, 0, 1), (-1, 0, 1), (1, 0, -1), (-1, 0, -1),
          (0, 1, 1), (0, -1, 1), (0, 1, -1), (0, -1, -1),
          (2, 0, 0), (-2, 0, 0), (0, 2, 0), (0, -2, 0), (0, 0, 2), (0, 0, -2),
          (1, 1, 1), (-1, 1, 1), (1, -1, 1), (-1, -1, 1), (1, 1, -1), ...
])
```

For other infinite enumerated base rings (i.e., rings which are objects of the category `InfiniteEnumeratedSets`), a free module of rank $r$ is enumerated by applying `FreeModule_ambient.linear_combination_of_basis()` to all vectors in $\mathbb{Z}^r$, enumerated in the way shown above.

**AUTHORS:**

- David Kohel (2007, 2008)
- Niles Johnson (2010-08): (trac ticket #3893) `random_element()` should pass on *args and **kwds.
- Simon King (2010-12): trac ticket #8800 : Fixing a bug in denominator().
- Simon King (2010-12), Peter Bruin (June 2014): trac ticket #10513 : New coercion model and category framework.

**class** `sage.modules.free_module.ComplexDoubleVectorSpace_class(n)`

Bases: `sage.modules.free_module.FreeModule_ambient_field`

`coordinates(v)`

**class** `sage.modules.free_module.EchelonMatrixKey(obj)`

Bases: `object`

A total ordering on free modules for sorting.

This class orders modules by their ambient spaces, then by dimension, then in order by their echelon matrices. If a function returns a list of free modules, this can be used to sort the output and thus render it deterministic.

**INPUT:**

- `obj` – a free module
EXAMPLES:

```python
sage: V = span([[1,2,3], [5,6,7], [8,9,10]], QQ)
sage: W = span([[5,6,7], [8,9,10]], QQ)
sage: X = span([[5,6,7]], ZZ).scale(1/11)
sage: Y = CC^3
sage: Z = ZZ^2
sage: modules = [V,W,X,Y,Z]
sage: modulesSorted = [Z,X,V,W,Y]
sage: from sage.modules.free_module import EchelonMatrixKey
sage: modules.sort(key=EchelonMatrixKey)
sage: modules == modulesSorted
True
```

```python
def FreeModule(base_ring, rank_or_basis_keys, sparse, inner_product_matrix, with_basis=None, rank=False, basis_keys=None, **args):
    create a free module over the given commutative base_ring
    FreeModule can be called with the following positional arguments:
    • FreeModule(base_ring, rank, ...)
    • FreeModule(base_ring, basis_keys, ...)
    INPUT:
    • base_ring - a commutative ring
    • rank - a nonnegative integer
    • basis_keys - a finite or enumerated family of arbitrary objects
    • sparse - bool; (default False)
    • inner_product_matrix - the inner product matrix (default None)
    • with_basis - either "standard" (the default), in which case a free module with the standard basis as the
distinguished basis is created; or None, in which case a free module without distinguished basis is created.
    • further options may be accepted by various implementation classes
    OUTPUT: a free module
    This factory function creates instances of various specialized classes depending on the input. Not all combina-
tions of options are implemented.
    • If the parameter basis_keys is provided, it must be a finite or enumerated family of objects, and an instance
of CombinatorialFreeModule is created.
    EXAMPLES:
```

```python
sage: CombinatorialFreeModule(QQ, ['a','b','c'])
Free module generated by {'a', 'b', 'c'} over Rational Field
```

It has a distinguished standard basis that is indexed by the provided basis_keys. See the documentation
of CombinatorialFreeModule for more examples and details, including its UniqueRepresentation
semantics.

• If the parameter with_basis is set to None, then a free module of the given rank without distinguished
basis is created. It is represented by an instance of FiniteRankFreeModule.
```
See the documentation of `FiniteRankFreeModule` for more options, examples, and details.

- If `rank` is provided and the option `with_basis` is left at its default value, "standard", then a free ambient module with distinguished standard basis indexed by `range(rank)` is created. There is only one dense and one sparse free ambient module of given `rank` over `base_ring`.

**EXAMPLES:**

```python
sage: FreeModule(Integers(8), 10)
Ambient free module of rank 10 over Ring of integers modulo 8
```

The remainder of this documentation discusses this case of free ambient modules.

**EXAMPLES:**

First we illustrate creating free modules over various base fields. The base field affects the free module that is created. For example, free modules over a field are vector spaces, and free modules over a principal ideal domain are special in that more functionality is available for them than for completely general free modules.

```python
sage: FreeModule(QQ,10)
Vector space of dimension 10 over Rational Field
sage: FreeModule(ZZ,10)
Ambient free module of rank 10 over the principal ideal domain Integer Ring
sage: FreeModule(FiniteField(5),10)
Vector space of dimension 10 over Finite Field of size 5
sage: FreeModule(Integers(7),10)
Ambient free module of rank 10 over Ring of integers modulo 7
sage: FreeModule(PolynomialRing(QQ,'x'),5)
Ambient free module of rank 5 over the principal ideal domain Univariate Polynomial Ring in x over Rational Field
sage: FreeModule(PolynomialRing(ZZ,'x'),5)
Ambient free module of rank 5 over the integral domain Univariate Polynomial Ring in x over Integer Ring
```

Of course we can make rank 0 free modules:

```python
sage: FreeModule(RealField(100),0)
Vector space of dimension 0 over Real Field with 100 bits of precision
```

Next we create a free module with sparse representation of elements. Functionality with sparse modules is identical to dense modules, but they may use less memory and arithmetic may be faster (or slower!).

```python
sage: M = FreeModule(ZZ,200,sparse=True)
sage: M.is_sparse()
True
sage: type(M.0)
<class 'sage.modules.free_module_element.FreeModuleElement_generic_sparse'>
```

The default is dense.

```python
sage: M = ZZ^200
sage: type(M.0)
<class 'sage.modules.vector_integer_dense.Vector_integer_dense'>
```
Note that matrices associated in some way to sparse free modules are sparse by default:

```python
sage: M = FreeModule(Integers(8), 2)
sage: A = M.basis_matrix()
sage: A.is_sparse()
False
sage: Ms = FreeModule(Integers(8), 2, sparse=True)
sage: M == Ms  # as mathematical objects they are equal
True
sage: Ms.basis_matrix().is_sparse()
True
```

We can also specify an inner product matrix, which is used when computing inner products of elements.

```python
sage: A = MatrixSpace(ZZ,2)([[1,0],[0,-1]])
sage: M = FreeModule(ZZ,2,inner_product_matrix=A)
sage: v, w = M.gens()
sage: v.inner_product(w)
0
sage: v.inner_product(v)
1
sage: w.inner_product(w)
-1
sage: (v+2*w).inner_product(w)
-2
```

You can also specify the inner product matrix by giving anything that coerces to an appropriate matrix. This is only useful if the inner product matrix takes values in the base ring.

```python
sage: FreeModule(ZZ,2,inner_product_matrix=1).inner_product_matrix()
[1 0]
[0 1]
sage: FreeModule(ZZ,2,inner_product_matrix=[1,2,3,4]).inner_product_matrix()
[1 2]
[3 4]
sage: FreeModule(ZZ,2,inner_product_matrix=[[1,2],[3,4]]).inner_product_matrix()
[1 2]
[3 4]
```

**Todo:** Refactor modules such that it only counts what category the base ring belongs to, but not what is its Python class.

**EXAMPLES:**

```python
sage: FreeModule(QQ, ['a', 'b', 'c'])
Free module generated by {'a', 'b', 'c'} over Rational Field
sage: _.category()
Category of finite dimensional vector spaces with basis over Rational Field
sage: FreeModule(QQ, 3, with_basis=None)
3-dimensional vector space over the Rational Field
sage: _.category()
Category of finite dimensional vector spaces over Rational Field
```
class sage.modules.free_module.FreeModuleFactory
Bases: sage.structure.factory.UniqueFactory

Factory class for the finite-dimensional free modules with standard basis

create_key(base_ring, rank, sparse=False, inner_product_matrix=None)

Todo: Replace the above by TestSuite(...).run(), once _test_pickling() will test unique representation and not only equality.

create_object(version, key)

class sage.modules.free_module.FreeModule_ambient(base_ring, rank, sparse=False, coordinate_ring=None)
Bases: sage.modules.free_module.FreeModule_generic

Ambient free module over a commutative ring.

ambient_module()
Return self, since self is ambient.

EXAMPLES:

sage: A = QQ^5; A.ambient_module()
Vector space of dimension 5 over Rational Field
sage: A = ZZ^5; A.ambient_module()
Ambient free module of rank 5 over the principal ideal domain Integer Ring

basis()
Return a basis for this ambient free module.

OUTPUT:

* Sequence - an immutable sequence with universe this ambient free module

EXAMPLES:

sage: A = ZZ^3; B = A.basis(); B
[ (1, 0, 0),
(0, 1, 0),
(0, 0, 1)
]
sage: B.universe()
Ambient free module of rank 3 over the principal ideal domain Integer Ring

change_ring(R)
Return the ambient free module over R of the same rank as self.

EXAMPLES:

sage: A = ZZ^3; A.change_ring(QQ)
Vector space of dimension 3 over Rational Field
sage: A = ZZ^3; A.change_ring(GF(5))
Vector space of dimension 3 over Finite Field of size 5

For ambient modules any change of rings is defined.
sage: A = GF(5)**3; A.change_ring(QQ)
Vector space of dimension 3 over Rational Field

coordinate_vector(v, check=True)
Write $v$ in terms of the standard basis for self and return the resulting coefficients in a vector over the fraction field of the base ring.

Returns a vector $c$ such that if $B$ is the basis for self, then

$$\sum c_i B_i = v.$$  

If $v$ is not in self, raise an \texttt{ArithmeticError} exception.

EXAMPLES:

sage: V = Integers(16)^3
sage: v = V.coordinate_vector([1,5,9]); v
(1, 5, 9)
sage: v.parent()
Ambient free module of rank 3 over Ring of integers modulo 16

echelon_coordinate_vector(v, check=True)
Same as \texttt{self.coordinate_vector(v)}, since self is an ambient free module.

INPUT:

- v - vector
- check - bool (default: True); if True, also verify that v is really in self.

OUTPUT: list

EXAMPLES:

sage: V = QQ^4
sage: v = V([-1/2,1/2,-1/2,1/2])
(1, -1)
sage: V.coordinate_vector(v)
(-1/2, 1/2, -1/2, 1/2)
sage: V.echelon_coordinate_vector(v)
(-1/2, 1/2)

echelon_coordinates(v, check=True)
Returns the coordinate vector of $v$ in terms of the echelon basis for self.

EXAMPLES:

sage: U = VectorSpace(QQ,3)
sage: [ U.coordinates(v) for v in U.basis() ]
[[1, 0, 0], [0, 1, 0], [0, 0, 1]]
sage: [ U.echelon_coordinates(v) for v in U.basis() ]
\begin{verbatim}
[[1, 0, 0], [0, 1, 0], [0, 0, 1]]
sage: V = U.submodule([[1,1,0],[0,1,1]])
sage: V
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1 0 -1]
[ 0 1 1]
sage: [ V.coordinates(v) for v in V.basis() ]
[[1, 0], [0, 1]]
sage: [ V.echelon_coordinates(v) for v in V.basis() ]
[[1, 0], [0, 1]]

W = U.submodule_with_basis([[1,1,0],[0,1,1]])
sage: W
Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[1 1 0]
[0 1 1]
sage: [ W.coordinates(v) for v in W.basis() ]
[[1, 0], [0, 1]]
sage: [ W.echelon_coordinates(v) for v in W.basis() ]
[[1, 1], [0, 1]]
\end{verbatim}

echelonized_basis()

Return a basis for this ambient free module in echelon form.

EXAMPLES:

\begin{verbatim}
sage: A = ZZ^3; A.echelonized_basis()
[ (1, 0, 0),
 (0, 1, 0),
 (0, 0, 1)]
\end{verbatim}

echelonized_basis_matrix()

The echelonized basis matrix of self.

EXAMPLES:

\begin{verbatim}
sage: V = ZZ^4
sage: W = V.submodule([ V.gen(i)-V.gen(0) for i in range(1,4) ])
sage: W.echelonized_basis_matrix()
[ 1 0 0 -1]
[ 0 1 0 -1]
[ 0 0 1 -1]
sage: U = V.submodule_with_basis([ V.gen(i)-V.gen(0) for i in range(1,4) ])
sage: U.echelonized_basis_matrix()
[ 1 1 0 0]
[ 0 1 0 1]
\end{verbatim}

gen($i=0$)
Return the $i$-th generator for self.
Here $i$ is between 0 and rank - 1, inclusive.

**INPUT:**
- $i$ – an integer (default 0)

**OUTPUT:** $i$-th basis vector for self.

**EXAMPLES:**

```python
sage: n = 5
sage: V = QQ^n
sage: B = [V.gen(i) for i in range(n)]
sage: B
[(1, 0, 0, 0, 0),
 (0, 1, 0, 0, 0),
 (0, 0, 1, 0, 0),
 (0, 0, 0, 1, 0),
 (0, 0, 0, 0, 1)]
sage: V.gens() == tuple(B)
True
```

is_ambient()
Return True since this module is an ambient module.

**EXAMPLES:**

```python
sage: A = QQ^5; A.is_ambient()
True
sage: A = (QQ^5).span([[1,2,3,4,5]]); A.is_ambient()
False
```

linear_combination_of_basis(v)
Return the linear combination of the basis for self obtained from the elements of the list $v$.

**INPUT:**
- $v$ - list

**EXAMPLES:**

```python
sage: V = span([[1,2,3], [4,5,6]], ZZ)
sage: V
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[1 2 3]
[0 3 6]
```
sage: V.linear_combination_of_basis([1,1])
(1, 5, 9)

This should raise an error if the resulting element is not in self:

sage: W = span([[2,4]], ZZ)
sage: W.linear_combination_of_basis([1/2])
Traceback (most recent call last):
...  
TypeError: element [1, 2] is not in free module

**random_element**(prob=1.0, *args, **kwds)

Returns a random element of self.

INPUT:

* **prob** - float. Each coefficient will be set to zero with probability \(1 - \text{prob}\). Otherwise coefficients will be chosen randomly from base ring (and may be zero).

  * **args, **kwds - passed on to random_element function of base ring.

EXAMPLES:

sage: M = FreeModule(ZZ, 3)
sage: M.random_element().parent() is M
True

Passes extra positional or keyword arguments through:

sage: all(i in range(5, 10) for i in M.random_element(1.0, 5, 10))
True

sage: M = FreeModule(ZZ, 16)
sage: M.random_element().parent() is M
True

sage: def add_sample(**kwds):
.....:   global total, zeros
.....:   v = M.random_element(**kwds)
.....:   total += M.rank()
.....:   zeros += sum(i == 0 for i in v)

sage: total = 0
sage: zeros = 0
sage: add_sample()
sage: expected = 1/5
sage: expected = 1/5 * 3/10 + 7/10
sage: while abs(zeros/total - expected) > 0.01:
.....:   add_sample()  

add_sample(prob=0.3)

sage: total = 0
sage: zeros = 0
sage: add_sample(prob=0.7)

sage: while abs(zeros/total - expected) > 0.01:
    ....:     add_sample(prob=0.7)

class sage.modules.free_module.FreeModule_ambient_domain(base_ring, rank, sparse=False, coordinate_ring=None)

Bases: sage.modules.free_module.FreeModule_ambient

Ambient free module over an integral domain.

ambient_vector_space()

Return the ambient vector space, which is this free module tensored with its fraction field.

EXAMPLES:

sage: M = ZZ^3
sage: V = M.ambient_vector_space(); V
Vector space of dimension 3 over Rational Field

If an inner product on the module is specified, then this is preserved on the ambient vector space.

sage: N = FreeModule(ZZ,4,inner_product_matrix=1)
sage: U = N.ambient_vector_space()
sage: U
Ambient quadratic space of dimension 4 over Rational Field
Inner product matrix:
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
sage: P = N.submodule_with_basis(
[[1,-1,0,0],[0,1,-1,0],[0,0,1,-1]])
sage: P.gram_matrix()
[ 2 -1  0]
[-1  2 -1]
[ 0  -1  2]
sage: U == N.ambient_vector_space()
True
sage: U == V
False

coordinate_vector(v, check=True)

Write $v$ in terms of the standard basis for self and return the resulting coefficients in a vector over the fraction field of the base ring.

INPUT:

* $v$ – vector

* check – bool (default: True); if True, also verify that $v$ is really in self.

OUTPUT: list
Returns a vector \( c \) such that if \( B \) is the basis for self, then

\[
\sum c_i B_i = v.
\]

If \( v \) is not in self, raise an \texttt{ArithmeticError} exception.

**EXAMPLES:**

```python
sage: V = ZZ^3
sage: v = V.coordinate_vector([1,5,9]); v
(1, 5, 9)
sage: v.parent()
Vector space of dimension 3 over Rational Field
```

\texttt{vector\_space}(\texttt{base\_field}=None)

Returns the vector space obtained from \texttt{self} by tensoring with the fraction field of the base ring and extending to the field.

**EXAMPLES:**

```python
sage: M = ZZ^3; M.vector_space()
Vector space of dimension 3 over Rational Field
```

\texttt{class} \texttt{sage.modules.free_module.\texttt{FreeModule\_ambient\_field}(\texttt{base\_field}, \texttt{dimension}, \texttt{sparse}=False)}

Bases: \texttt{sage.modules.free_module.\texttt{FreeModule\_generic\_field}}, \texttt{sage.modules.free_module.\texttt{FreeModule\_ambient\_pid}}

\texttt{ambient\_vector\_space}()

Returns \texttt{self} as the ambient vector space.

**EXAMPLES:**

```python
sage: M = QQ^3
sage: M.ambient_vector_space()
Vector space of dimension 3 over Rational Field
```

\texttt{base\_field}()

Returns the base field of this vector space.

**EXAMPLES:**

```python
sage: M = QQ^3
sage: M.base_field()
Rational Field
```

\texttt{class} \texttt{sage.modules.free_module.\texttt{FreeModule\_ambient\_pid}(\texttt{base\_ring}, \texttt{rank}, \texttt{sparse}=False, \texttt{coordinate\_ring}=None)}

Bases: \texttt{sage.modules.free_module.\texttt{FreeModule\_generic\_pid}}, \texttt{sage.modules.free_module.\texttt{FreeModule\_ambient\_domain}}

Ambient free module over a principal ideal domain.

\texttt{class} \texttt{sage.modules.free_module.\texttt{FreeModule\_generic}(\texttt{base\_ring}, \texttt{rank}, \texttt{degree}, \texttt{sparse}=False, \texttt{coordinate\_ring}=None, \texttt{category}=None)}

Bases: \texttt{sage.modules.module.Module}

Base class for all free modules.

\texttt{ambient\_module}()

Return the ambient module associated to this module.
EXAMPLES:

```python
sage: R.<x,y> = QQ[]
sage: M = FreeModule(R,2)
sage: M.ambient_module()
Ambient free module of rank 2 over the integral domain Multivariate Polynomial Ring in x, y over Rational Field
```

```python
sage: V = FreeModule(QQ, 4).span([[1,2,3,4], [1,0,0,0]]); V
Vector space of degree 4 and dimension 2 over Rational Field
Basis matrix:
[ 1 0 0 0]
[ 0 1 3/2 2]
sage: V.ambient_module()
Vector space of dimension 4 over Rational Field
```

`are_linearly_dependent(vecs)`

Return True if the vectors `vecs` are linearly dependent and False otherwise.

EXAMPLES:

```python
sage: M = QQ^3
sage: vecs = [M([1,2,3]), M([4,5,6])]
sage: M.are_linearly_dependent(vecs)
False
sage: vecs.append(M([3,3,3]))
sage: M.are_linearly_dependent(vecs)
True
```

```python
sage: R.<x> = QQ[]
sage: M = FreeModule(R, 2)
sage: vecs = [M([x^2+1, x+1]), M([x+2, 2*x+1])]
sage: M.are_linearly_dependent(vecs)
False
sage: vecs.append(M([-2*x+1, -2*x^2+1]))
sage: M.are_linearly_dependent(vecs)
True
```

`base_field()`

Return the base field, which is the fraction field of the base ring of this module.

EXAMPLES:

```python
sage: FreeModule(GF(3), 2).base_field()
Finite Field of size 3
sage: FreeModule(ZZ, 2).base_field()
Rational Field
sage: FreeModule(PolynomialRing(GF(7), 'x'), 2).base_field()
Fraction Field of Univariate Polynomial Ring in x over Finite Field of size 7
```

`basis()`

Return the basis of this module.

EXAMPLES:
\begin{verbatim}
sage: FreeModule(Integers(12),3).basis()
[ (1, 0, 0),
  (0, 1, 0),
  (0, 0, 1)
]

basis_matrix(ring=None)
Return the matrix whose rows are the basis for this free module.

INPUT:
- \texttt{ring} – (default: self.coordinate_ring()) a ring over which the matrix is defined

EXAMPLES:

\begin{verbatim}
sage: FreeModule(Integers(12),3).basis_matrix()
[1 0 0]
[0 1 0]
[0 0 1]
sage: M = FreeModule(GF(7),3).span([[2,3,4],[1,1,1]]); M
Vector space of degree 3 and dimension 2 over Finite Field of size 7
Basis matrix:
[1 0 6]
[0 1 2]
sage: M.basis_matrix()
[1 0 6]
[0 1 2]
sage: M = FreeModule(GF(7),3).span_of_basis([[2,3,4],[1,1,1]])
sage: M.basis_matrix()
[2 3 4]
[1 1 1]
sage: M = FreeModule(QQ,2).span_of_basis([[1,-1],[1,0]]); M
Vector space of degree 2 and dimension 2 over Rational Field
User basis matrix:
[ 1 -1]
[ 1 0]
sage: M.basis_matrix()
[ 1 -1]
[ 1 0]
\end{verbatim}
\end{verbatim}

cardinality()
Return the cardinality of the free module.

OUTPUT:
Either an integer or \texttt{+Infinity}.

EXAMPLES:

\begin{verbatim}
sage: k.<a> = FiniteField(9)
sage: V = VectorSpace(k,3)
\end{verbatim}
\end{verbatim}
sage: V.cardinality()
729
sage: W = V.span([[1,2,1],[0,1,1]])
sage: W.cardinality()
81
sage: R = IntegerModRing(12)
sage: M = FreeModule(R,2)
sage: M.cardinality()
144
sage: (QQ^3).cardinality() +Infinity

**codimension()**

Return the codimension of this free module, which is the dimension of the ambient space minus the dimension of this free module.

**EXAMPLES:**

```sage
sage: M = Matrix(3, 4, range(12))
sage: V = M.left_kernel(); V
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[ 1 -2 1]
sage: V.dimension()
1
sage: V.codimension()
2
```

The codimension of an ambient space is always zero:

```sage
sage: (QQ^10).codimension()
0
```

**construction()**

The construction functor and base ring for self.

**EXAMPLES:**

```sage
sage: R = PolynomialRing(QQ,3,'x')
sage: V = R^5
sage: V.construction()
(VectorFunctor, Multivariate Polynomial Ring in x0, x1, x2 over Rational Field)
```

**coordinate_module(V)**

Suppose V is a submodule of self (or a module commensurable with self), and that self is a free module over $R$ of rank $n$. Let $\phi$ be the map from self to $R^n$ that sends the basis vectors of self in order to the standard basis of $R^n$. This function returns the image $\phi(V)$.

**Warning:** If there is no integer $d$ such that $dV$ is a submodule of self, then this function will give total nonsense.

**EXAMPLES:**

We illustrate this function with some Z-submodules of Q^3:
\begin{verbatim}
sage: V = (ZZ^3).span([[1/2,3,5], [0,1,-3]])
sage: W = (ZZ^3).span([[1/2,4,2]])
sage: V.coordinate_module(W)
Free module of degree 2 and rank 1 over Integer Ring
User basis matrix:
[1 4]
sage: V.0 + 4*V.1
(1/2, 4, 2)
\end{verbatim}

In this example, the coordinate module isn’t even in \(\mathbb{Z}^3\):

\begin{verbatim}
sage: W = (ZZ^3).span([[1/4,2,1]])
sage: V.coordinate_module(W)
Free module of degree 2 and rank 1 over Integer Ring
User basis matrix:
[1/2 2]
\end{verbatim}

The following more elaborate example illustrates using this function to write a submodule in terms of integral cuspidal modular symbols:

\begin{verbatim}
sage: M = ModularSymbols(54)
sage: S = M.cuspidal_subspace()
sage: K = S.integral_structure(); K
Free module of degree 19 and rank 8 over Integer Ring
Echelon basis matrix:
[ 0 1 0 0 -1 0 0 0 0 0 0 0 0 0 0 0 0 0 0]
...
sage: L = M[0].integral_structure(); L
Free module of degree 19 and rank 2 over Integer Ring
Echelon basis matrix:
[ 0 1 1 0 -2 1 -1 -2 2 0 0 0 0 0 0 0 0 0 0]
[ 0 0 3 0 -3 2 -1 2 -1 -4 2 -1 -2 1 2 0 0 -1 1]
sage: K.coordinate_module(L)
Free module of degree 8 and rank 2 over Integer Ring
User basis matrix:
[ 1 1 1 -1 1 -1 0 0]
[ 0 3 2 -1 2 -1 -1 -2]
sage: K.coordinate_module(L).basis_matrix() * K.basis_matrix()
[ 0 1 1 0 -2 1 -1 -2 2 0 0 0 0 0 0 0 0 0 0]
[ 0 0 3 0 -3 2 -1 2 -1 -4 2 -1 -2 1 2 0 0 -1 1]
\end{verbatim}

c\text{coordinate}_{\text{ring}}()

Return the ring over which the entries of the vectors are defined.

This is the same as \text{base}_{\text{ring}}() unless an explicit basis was given over the fraction field.

EXAMPLES:

\begin{verbatim}
sage: M = ZZ^2
sage: M.coordinate_ring()
Integer Ring
\end{verbatim}

\begin{verbatim}
sage: M = (ZZ^2) * (1/2)
sage: M.base_ring()
\end{verbatim}

(continues on next page)
Integer Ring

`sage: M.coordinate_ring()`

Rational Field

`sage: R.<x> = QQ[]
sage: L = R^2
sage: L.coordinate_ring()`

Univariate Polynomial Ring in x over Rational Field

`sage: L.span([(x,0), (1,x)]).coordinate_ring()`

Univariate Polynomial Ring in x over Rational Field

`sage: L.span([(x,0), (1,1/x)]).coordinate_ring()`

Fraction Field of Univariate Polynomial Ring in x over Rational Field

`sage: L.span([]).coordinate_ring()`

Univariate Polynomial Ring in x over Rational Field

```
coordinate_vector(v, check=True)
```

Return the vector whose coefficients give \( v \) as a linear combination of the basis for self.

**INPUT:**

- \( v \) – vector
- \( \text{check} \) – bool (default: True); if True, also verify that \( v \) is really in self.

**OUTPUT:** list

**EXAMPLES:**

```
sage: M = FreeModule(ZZ, 2); M0,M1=M.gens()
sage: W = M.submodule([M0 + M1, M0 - 2*M1])
sage: W.coordinate_vector(2*M0 - M1)
(2, -1)
```

```
coordinates(v, check=True)
```

Write \( v \) in terms of the basis for self.

**INPUT:**

- \( v \) – vector
- \( \text{check} \) – bool (default: True); if True, also verify that \( v \) is really in self.

**OUTPUT:** list

Returns a list \( c \) such that if \( B \) is the basis for self, then

\[
\sum c_i B_i = v.
\]

If \( v \) is not in self, raise an `ArithmeticError` exception.

**EXAMPLES:**

```
sage: M = FreeModule(ZZ, 2); M0,M1=M.gens()
sage: W = M.submodule([M0 + M1, M0 - 2*M1])
sage: W.coordinates(2*M0-M1)
[2, -1]
```
degree()
Return the degree of this free module. This is the dimension of the ambient vector space in which it is
embedded.

EXAMPLES:

```
sage: M = FreeModule(ZZ, 10)
sage: W = M.submodule([M.gen(0), 2*M.gen(3) - M.gen(0), M.gen(0) + M.gen(3)])
sage: W.degree()
10
sage: W.rank()
2
```

dense_module()
Return corresponding dense module.

EXAMPLES:

We first illustrate conversion with ambient spaces:

```
sage: M = FreeModule(QQ,3)
sage: S = FreeModule(QQ,3, sparse=True)
sage: M.sparse_module()
Sparse vector space of dimension 3 over Rational Field
sage: S.dense_module()
Vector space of dimension 3 over Rational Field
sage: M.sparse_module() == S
True
sage: S.dense_module() == M
True
sage: M.dense_module() == M
True
sage: S.sparse_module() == S
True
```

Next we create a subspace:

```
sage: M = FreeModule(QQ,3, sparse=True)
sage: V = M.span([[1,2,3]])
Sparse vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[1 2 3]
sage: V.sparse_module()
Sparse vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[1 2 3]
```

dimension()
Return the dimension of this free module.

EXAMPLES:

```
sage: M = FreeModule(FiniteField(19), 100)
sage: W = M.submodule([M.gen(50)])
sage: W.dimension()
1
```
**direct_sum**(other)
Return the direct sum of self and other as a free module.

EXAMPLES:

```
sage: V = (ZZ^3).span([[1/2,3,5], [0,1,-3]]); V
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[1/2  0  14]
[ 0  1 -3]
sage: W = (ZZ^3).span([[1/2,4,2]]); W
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[1/2  4  2]
sage: V.direct_sum(W)
Free module of degree 6 and rank 3 over Integer Ring
Echelon basis matrix:
[1/2  0  14  0  0  0]
[ 0  1 -3  0  0  0]
[ 0  0  0  1/2  4  2]
```

**discriminant()**
Return the discriminant of this free module.

EXAMPLES:

```
sage: M = FreeModule(ZZ, 3)
sage: M.discriminant()
1
sage: W = M.span([[1,2,3]])
sage: W.discriminant()
14
sage: W2 = M.span([[1,2,3], [1,1,1]])
sage: W2.discriminant()
6
```

**echelonized_basis_matrix()**
The echelonized basis matrix (not implemented for this module).

This example works because M is an ambient module. Submodule creation should exist for generic modules.

EXAMPLES:

```
sage: R = IntegerModRing(12)
sage: S.<x,y> = R[]
sage: M = FreeModule(S,3)
sage: M.echelonized_basis_matrix()
[1 0 0]
[0 1 0]
[0 0 1]
```

**free_module()**
Return this free module. (This is used by the FreeModule functor, and simply returns self.)

EXAMPLES:
```
sage: M = FreeModule(ZZ, 3)
sage: M.free_module()
Ambient free module of rank 3 over the principal ideal domain Integer Ring
```

**gen**(i=0)

Return the \(i\)-th generator for \(self\).

Here \(i\) is between 0 and rank - 1, inclusive.

**INPUT:**

- \(i\) – an integer (default 0)

**OUTPUT:** \(i\)-th basis vector for \(self\).

**EXAMPLES:**

```
sage: n = 5
sage: V = QQ^n
sage: B = [V.gen(i) for i in range(n)]
sage: B
[(1, 0, 0, 0, 0),
 (0, 1, 0, 0, 0),
 (0, 0, 1, 0, 0),
 (0, 0, 0, 1, 0),
 (0, 0, 0, 0, 1)]
sage: V.gens() == tuple(B)
True
```

**gens**()

Return a tuple of basis elements of \(self\).

**EXAMPLES:**

```
sage: FreeModule(Integers(12),3).gens()
((1, 0, 0), (0, 1, 0), (0, 0, 1))
```

**gram_matrix**()

Return the gram matrix associated to this free module, defined to be \(G = B \cdot A \cdot B.transpose()\), where \(A\) is the inner product matrix (induced from the ambient space), and \(B\) the basis matrix.

**EXAMPLES:**

```
sage: V = VectorSpace(QQ,4)
sage: u = V([1/2,1/2,1/2,1/2])
sage: v = V([0,1,1,0])
sage: w = V([0,0,1,1])
sage: M = span([u,v,w], ZZ)
sage: M.inner_product_matrix() == V.inner_product_matrix()
True
sage: L = M.submodule_with_basis([u,v,w])
sage: L.inner_product_matrix() == M.inner_product_matrix()
True
sage: L.gram_matrix()
[1 1 1]
[1 2 1]
[1 1 2]
```
**has_user_basis()**

Return True if the basis of this free module is specified by the user, as opposed to being the default echelon form.

EXAMPLES:

```
sage: V = QQ^3
sage: W = V.subspace([[2,'1/2', 1]])
sage: W.has_user_basis()
False
sage: W = V.subspace_with_basis([[2,'1/2',1]])
sage: W.has_user_basis()
True
```

**hom**(im_gens, codomain=None, **kwds)

Override the hom method to handle the case of morphisms given by left-multiplication of a matrix and the codomain is not given.

EXAMPLES:

```
sage: W = ZZ^2; W.hom(matrix(1, [1, 2]), side="right")
Free module morphism defined as left-multiplication by the matrix
[1 2]
Domain: Ambient free module of rank 2 over the principal ideal domain Integer
→ Ring
Codomain: Ambient free module of rank 1 over the principal ideal domain Integer
→ Ring
sage: V = QQ^2; V.hom(identity_matrix(2), side="right")
Vector space morphism represented as left-multiplication by the matrix:
[1 0]
[0 1]
Domain: Vector space of dimension 2 over Rational Field
Codomain: Vector space of dimension 2 over Rational Field
```

**inner_product_matrix()**

Return the default identity inner product matrix associated to this module.

By definition this is the inner product matrix of the ambient space, hence may be of degree greater than the rank of the module.

TODO: Differentiate the image ring of the inner product from the base ring of the module and/or ambient space. E.g. On an integral module over ZZ the inner product pairing could naturally take values in ZZ, QQ, RR, or CC.

EXAMPLES:

```
sage: M = FreeModule(ZZ, 3)
sage: M.inner_product_matrix()
[1 0 0]
[0 1 0]
[0 0 1]
```

**is_ambient()**

Returns False since this is not an ambient free module.

EXAMPLES:
is_dense()
Return True if the underlying representation of this module uses dense vectors, and False otherwise.

EXAMPLES:

```
sage: FreeModule(ZZ, 2).is_dense()
True
sage: FreeModule(ZZ, 2, sparse=True).is_dense()
False
```

is_finite()
Returns True if the underlying set of this free module is finite.

EXAMPLES:

```
sage: FreeModule(ZZ, 2).is_finite()
False
sage: FreeModule(Integers(8), 2).is_finite()
True
sage: FreeModule(ZZ, 0).is_finite()
True
```

is_full()
Return True if the rank of this module equals its degree.

EXAMPLES:

```
sage: FreeModule(ZZ, 2).is_full()
True
sage: M = FreeModule(ZZ, 2).span([[1,2]])
sage: M.is_full()
False
```

is_sparse()
Return True if the underlying representation of this module uses sparse vectors, and False otherwise.

EXAMPLES:
is_submodule(other)
Return True if self is a submodule of other.

EXAMPLES:

\begin{verbatim}
    sage: M = FreeModule(ZZ, 3)
    sage: V = M.ambient_vector_space()
    sage: X = V.span([[1/2,1/2,0],[1/2,0,1/2]], ZZ)
    sage: Y = V.span([[1,1,1]], ZZ)
    sage: N = X + Y
    sage: M.is_submodule(X)
    False
    sage: M.is_submodule(Y)
    False
    sage: Y.is_submodule(M)
    True
    sage: N.is_submodule(M)
    False
    sage: M.is_submodule(N)
    True

    sage: M = FreeModule(ZZ, 2)
    sage: M.is_submodule(M)
    True
    sage: N = M.scale(2)
    sage: N.is_submodule(M)
    True
    sage: M.is_submodule(N)
    False
    sage: N = M.scale(1/2)
    sage: N.is_submodule(M)
    False
    sage: M.is_submodule(N)
    True
\end{verbatim}

Since \texttt{basis()} is not implemented in general, submodule testing does not work for all PID's. However, trivial cases are already used (and useful) for coercion, e.g.:

\begin{verbatim}
    sage: QQ(1/2) * vector(ZZ['x','y'],[1,2,3,4])
    (1/2, 1, 3/2, 2)
    sage: vector(ZZ['x','y'],[1,2,3,4]) * QQ(1/2)
    (1/2, 1, 3/2, 2)
\end{verbatim}

matrix()
Return the basis matrix of this module, which is the matrix whose rows are a basis for this module.

EXAMPLES:
sage: M = FreeModule(ZZ, 2)
sage: M.matrix()
\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]
sage: M.submodule([M.gen(0) + M.gen(1), M.gen(0) - 2*M.gen(1)]).matrix()
\[
\begin{bmatrix}
1 & 1 \\
0 & 3 \\
\end{bmatrix}
\]

**ngens()**

Returns the number of basis elements of this free module.

**EXAMPLES:**

```sage
sage: FreeModule(ZZ, 2).ngens()
2
sage: FreeModule(ZZ, 0).ngens()
0
sage: FreeModule(ZZ, 2).span([[1,1]]).ngens()
1
```

**nonembedded_free_module()**

Returns an ambient free module that is isomorphic to this free module.

Thus if this free module is of rank \(n\) over a ring \(R\), then this function returns \(R^n\), as an ambient free module.

**EXAMPLES:**

```sage
sage: FreeModule(ZZ, 2).span([[1,1]]).nonembedded_free_module()
Ambient free module of rank 1 over the principal ideal domain Integer Ring
```

**random_element**(prob=1.0, *args, **kwds)

Returns a random element of self.

**INPUT:**

– **prob** - float. Each coefficient will be set to zero with probability \(1-\text{prob}\). Otherwise coefficients will be chosen randomly from base ring (and may be zero).

– **args, **kwds - passed on to `random_element()` function of base ring.

**EXAMPLES:**

```sage
M = FreeModule(ZZ, 2).span([[1, 1]])
sage: v = M.random_element()
sage: v.parent() is M
True
sage: v in M
True
```

Small entries are likely:

```sage
sage: for i in [-2, -1, 0, 1, 2]:
    ....:     while vector([i, i]) != M.random_element():
    ....:         pass
```

Large entries appear as well:
sage: while abs(M.random_element()[0]) < 100:
....:   pass

Passes extra positional or keyword arguments through:
sage: all(i in range(5, 10) for i in M.random_element(1.0, 5, 10))
True

**rank()**
Return the rank of this free module.

**EXAMPLES:**
sage: FreeModule(Integers(6), 1000000).rank()
1000000
sage: FreeModule(ZZ, 2).span([\[1,1\], \[2,2\], \[3,4\]]).rank()
2

**some_elements()**
Return some elements of this free module.

See TestSuite for a typical use case.

**OUTPUT:**
An iterator.

**EXAMPLES:**
sage: F = FreeModule(ZZ, 2)
sage: tuple(F.some_elements())
((1, 0),
 (1, 1),
 (0, 1),
 (-1, 2),
 (-2, 3),
 ...
 (-49, 50))
sage: F = FreeModule(QQ, 3)
sage: tuple(F.some_elements())
((1, 0, 0),
 (1/2, 1/2, 1/2),
 (1/2, -1/2, 2),
 (-2, 0, 1),
 (-1, 42, 2/3),
 (-2/3, 3/2, -3/2),
 (4/5, -4/5, 5/4),
 ...
 (46/103823, -46/103823, 103823/46))
sage: F = FreeModule(SR, 2)
sage: tuple(F.some_elements())
((1, 0), (some_variable, some_variable))

**sparse_module()**
Return the corresponding sparse module with the same defining data.
EXAMPLES:

We first illustrate conversion with ambient spaces:

```
sage: M = FreeModule(Integers(8),3)
sage: S = FreeModule(Integers(8),3, sparse=True)
sage: M.sparse_module()
Ambient sparse free module of rank 3 over Ring of integers modulo 8
sage: S.dense_module()
Ambient free module of rank 3 over Ring of integers modulo 8
sage: M.sparse_module() is S
True
sage: S.dense_module() is M
True
sage: M.dense_module() is M
True
sage: S.sparse_module() is S
True
```

Next we convert a subspace:

```
sage: M = FreeModule(QQ,3)
sage: V = M.span([ [1,2,3] ]); V
Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[1 2 3]
sage: V.sparse_module()
Sparse vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[1 2 3]
```

`uses_ambient_inner_product`()

Return True if the inner product on this module is the one induced by the ambient inner product.

EXAMPLES:

```
sage: M = FreeModule(ZZ, 2)
sage: W = M.submodule([[1,2]])
sage: W.uses_ambient_inner_product()
True
sage: W.inner_product_matrix()
[1 0]
[0 1]
sage: W.gram_matrix()
[5]
```

`zero()`

Returns the zero vector in this free module.

EXAMPLES:

```
sage: M = FreeModule(ZZ, 2)
sage: M.zero()
(0, 0)
```
zero_vector()

Returns the zero vector in this free module.

EXAMPLES:

```python
sage: M = FreeModule(ZZ, 2)
sage: M.zero_vector()
(0, 0)
sage: N(0)
(0, 0)
sage: M.span([[1,1]]).zero_vector()
(0, 0)
sage: M.zero_submodule().zero_vector()
(0, 0)
```

class sage.modules.free_module.FreeModule_generic_field(base_field, dimension, degree, sparse=False)

Bases: `sage.modules.free_module.FreeModule_generic_pid`

Base class for all free modules over fields.

complement()

Return the complement of self in the ambient vector space().

EXAMPLES:

```python
sage: V = QQ^3
sage: V.complement()
Vector space of degree 3 and dimension 0 over Rational Field
Basis matrix: []

sage: V == V.complement().complement()
True
sage: W = V.span([[1, 0, 1]])
sage: X = W.complement(); X
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1 0 -1]
[ 0 1 0]

sage: X.complement() == W
True
sage: X + W == V
True
```

Even though we construct a subspace of a subspace, the orthogonal complement is still done in the ambient vector space $\mathbb{Q}^3$: 
```python
sage: V = QQ^3
sage: W = V.subspace_with_basis([\[1,0,1\],\[-1,1,0\]])
sage: X = W.subspace_with_basis([\[1,0,1\]])

sage: X.complement()
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 0
\end{bmatrix}
\]

All these complements are only done with respect to the inner product in the usual basis. Over finite fields, this means we can get complements which are only isomorphic to a vector space decomposition complement.

```
sage: W1.intersection(W2)
Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[0 1 0]
sage: W2.intersection(W1)
Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[0 1 0]
sage: V.intersection(W1)
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[1 0 0]
[0 1 0]
sage: W1.intersection(V)
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[1 0 0]
[0 1 0]

**is_subspace**(other)

True if this vector space is a subspace of other.

**EXAMPLES:**

```
sage: V = VectorSpace(QQ,3)
sage: W = V.subspace([V.gen(0), V.gen(0) + V.gen(1)])
sage: W2 = V.subspace([V.gen(1)])
sage: W.is_subspace(V)
True
sage: W2.is_subspace(V)
True
sage: W.is_subspace(W2)
False
sage: W2.is_subspace(W)
True
```

**linear_dependence**(vectors, zeros='left', check=True)

Returns a list of vectors giving relations of linear dependence for the input list of vectors. Can be used to check linear independence of a set of vectors.

**INPUT:**

- vectors – A list of vectors, all from the same vector space.
- zeros – default: 'left' - 'left' or 'right' as a general preference for where zeros are located in the returned coefficients
- check – default: True - if True each item in the list vectors is checked for membership in self. Set to False if you can be certain the vectors come from the vector space.

**OUTPUT:**
Returns a list of vectors. The scalar entries of each vector provide the coefficients for a linear combination of the input vectors that will equal the zero vector in self. Furthermore, the returned list is linearly independent in the vector space over the same base field with degree equal to the length of the list vectors.

The linear independence of vectors is equivalent to the returned list being empty, so this provides a test — see the examples below.

The returned vectors are always independent, and with zeros set to 'left' they have 1’s in their first non-zero entries and a qualitative disposition to having zeros in the low-index entries. With zeros set to 'right' the situation is reversed with a qualitative disposition for zeros in the high-index entries.

If the vectors in vectors are made the rows of a matrix \( V \) and the returned vectors are made the rows of a matrix \( R \), then the matrix product \( RV \) is a zero matrix of the proper size. And \( R \) is a matrix of full rank. This routine uses kernels of matrices to compute these relations of linear dependence, but handles all the conversions between sets of vectors and matrices. If speed is important, consider working with the appropriate matrices and kernels instead.

EXAMPLS:

We begin with two linearly independent vectors, and add three non-trivial linear combinations to the set. We illustrate both types of output and check a selected relation of linear dependence.

```
sage: v1 = vector(QQ, [2, 1, -4, 3])
sage: v2 = vector(QQ, [1, 5, 2, -2])
sage: V = QQ^4
sage: V.linear_dependence([v1,v2])
[
]
sage: v3 = v1 + v2
sage: v4 = 3*v1 - 4*v2
sage: v5 = -v1 + 2*v2
sage: L = [v1, v2, v3, v4, v5]
sage: relations = V.linear_dependence(L, zeros='left')
sage: relations
[(1, 0, 0, -1, -2),
(0, 1, 0, -1/2, -3/2),
(0, 0, 1, -3/2, -7/2)]
sage: v2 + (-1/2)*v4 + (-3/2)*v5
(0, 0, 0, 0)
sage: relations = V.linear_dependence(L, zeros='right')
sage: relations
[
(-1, -1, 1, 0, 0),
(-3, 4, 0, 1, 0),
(1, -2, 0, 0, 1)]
sage: z = sum([relations[2][i]*L[i] for i in range(len(L))])
sage: z == zero_vector(QQ, 4)
True
```

A linearly independent set returns an empty list, a result that can be tested.
sage: v1 = vector(QQ, [0,1,-3])
sage: v2 = vector(QQ, [4,1,0])
sage: V = QQ^3
sage: relations = V.linear_dependence([v1, v2]); relations
[]
sage: relations == []
True

Exact results result from exact fields. We start with three linearly independent vectors and add in two linear combinations to make a linearly dependent set of five vectors.

sage: F = FiniteField(17)
sage: v1 = vector(F, [1, 2, 3, 4, 5])
sage: v2 = vector(F, [2, 4, 8, 16, 15])
sage: v3 = vector(F, [1, 0, 0, 0, 1])
sage: (F^5).linear_dependence([v1, v2, v3]) == []
True
sage: L = [v1, v2, v3, 2*v1+v2, 3*v2+6*v3]
sage: (F^5).linear_dependence(L)
[(1, 0, 16, 8, 3),
 (0, 1, 2, 0, 11)]
sage: v1 + 16*v3 + 8*(2*v1+v2) + 3*(3*v2+6*v3)
(0, 0, 0, 0, 0)
sage: v2 + 2*v3 + 11*(3*v2+6*v3)
(0, 0, 0, 0, 0)
sage: (F^5).linear_dependence(L, zeros='right')
[(15, 16, 0, 1, 0),
 (0, 14, 11, 0, 1)]

**quotient**(sub, check=True)

Return the quotient of self by the given subspace sub.

INPUT:

- sub - a submodule of self, or something that can be turned into one via self.submodule(sub).
- check - (default: True) whether or not to check that sub is a submodule.

EXAMPLES:

sage: A = QQ^3; V = A.span([[1,2,3], [4,5,6]])
sage: Q = V.quotient([V.0 + V.1]); Q
Vector space quotient V/W of dimension 1 over Rational Field where
V: Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1 0 -1]
[ 0 1 2]
W: Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
We illustrate that the base rings must be the same:

```
sage: (QQ^2)/(ZZ^2)
Traceback (most recent call last):
...
ValueError: base rings must be the same
```

`quotient_abstract(sub, check=True, **kwds)`

Return an ambient free module isomorphic to the quotient space of `self` modulo `sub`, together with maps from `self` to the quotient, and a lifting map in the other direction.

Use `self.quotient(sub)` to obtain the quotient module as an object equipped with natural maps in both directions, and a canonical coercion.

**INPUT:**
- `sub` – a submodule of `self` or something that can be turned into one via `self.submodule(sub)`
- `check` – (default: `True`) whether or not to check that `sub` is a submodule
- further named arguments, that are currently ignored.

**OUTPUT:**
- `U` – the quotient as an abstract ambient free module
- `pi` – projection map to the quotient
- `lift` – lifting map back from quotient

**EXAMPLES:**

```
sage: V = GF(19)^3
sage: W = V.span_of_basis([ [1,2,3], [1,0,1] ])
sage: U,pi,lift = V.quotient_abstract(W)
sage: pi(V.2)
(18)
sage: pi(V.0)
(1)
sage: pi(V.0 + V.2)
(0)
```

Another example involving a quotient of one subspace by another:

```
sage: A = matrix(QQ,4,4,[[0,1,0,0, 0,0,1,0, 0,0,0,1, 0,0,0,0]])
sage: V = (A^3).kernel()
sage: W = A.kernel()
sage: U, pi, lift = V.quotient_abstract(W)
sage: [pi(v) == 0 for v in W.gens()]
[True]
sage: [pi(lift(b)) == b for b in U.basis()]
[True, True]
```
**scale**(other)

Return the product of self by the number other, which is the module spanned by other times each basis vector. Since self is a vector space this product equals self if other is nonzero, and is the zero vector space if other is 0.

EXAMPLES:

```python
sage: V = QQ^4
sage: V.scale(5)
Vector space of dimension 4 over Rational Field
sage: V.scale(0)
Vector space of degree 4 and dimension 0 over Rational Field
Basis matrix:
[]
```

```python
sage: W = V.span([[1,1,1,1]])
```

```python
sage: W.scale(2)
Vector space of degree 4 and dimension 1 over Rational Field
Basis matrix:
[1 1 1 1]
sage: W.scale(0)
Vector space of degree 4 and dimension 0 over Rational Field
Basis matrix:
[]
```

```python
sage: V = QQ^4; V
Vector space of dimension 4 over Rational Field
sage: V.scale(3)
Vector space of dimension 4 over Rational Field
sage: V.scale(0)
Vector space of degree 4 and dimension 0 over Rational Field
Basis matrix:
[]
```

**span**(gens, base_ring=None, check=True, already_echelonized=False)

Return the K-span of the given list of gens, where K is the base field of self or the user-specified base_ring. Note that this span is a subspace of the ambient vector space, but need not be a subspace of self.

INPUT:

- **gens** - list of vectors
- **check** - bool (default: True): whether or not to coerce entries of gens into base field
- **already_echelonized** - bool (default: False): set this if you know the gens are already in echelon form

EXAMPLES:

```python
sage: V = VectorSpace(GF(7), 3)
sage: W = V.subspace([[2,3,4]]); W
Vector space of degree 3 and dimension 1 over Finite Field of size 7
Basis matrix:
[1 5 2]
sage: W.span([[1,1,1]])
Vector space of degree 3 and dimension 1 over Finite Field of size 7
```
span_of_basis(basis, base_ring=None, check=True, already_echelonized=False)
Return the free $K$-module with the given basis, where $K$ is the base field of self or user specified base_ring.

Note that this span is a subspace of the ambient vector space, but need not be a subspace of self.

INPUT:

• basis - list of vectors
• check - bool (default: True): whether or not to coerce entries of gens into base field
• already_echelonized - bool (default: False): set this if you know the gens are already in echelon form

EXAMPLES:

sage: V = VectorSpace(GF(7), 3)
sage: W = V.span_of_basis([[2,3,4]]); W
Vector space of degree 3 and dimension 1 over Finite Field of size 7
Basis matrix:
[1 5 2]
sage: W.span_of_basis([[2,2,2], [3,3,0]])
Vector space of degree 3 and dimension 2 over Finite Field of size 7
User basis matrix:
[2 2 2]
[3 3 0]

The basis vectors must be linearly independent or a ValueError exception is raised:

sage: W.span_of_basis([[2,2,2], [3,3,3]])
Traceback (most recent call last):
...
ValueError: The given basis vectors must be linearly independent.

subspace(gens, check=True, already_echelonized=False)
Return the subspace of self spanned by the elements of gens.

INPUT:

• gens - list of vectors
• check - bool (default: True) verify that gens are all in self.
• already_echelonized - bool (default: False) set to True if you know the gens are in Echelon form.

EXAMPLES:

First we create a 1-dimensional vector subspace of an ambient 3-dimensional space over the finite field of order 7:

sage: V = VectorSpace(GF(7), 3)
sage: W = V.subspace([[2,3,4]]); W
Vector space of degree 3 and dimension 1 over Finite Field of size 7
Basis matrix:
[1 5 2]
Next we create an invalid subspace, but it’s allowed since check=False. This is just equivalent to computing the span of the element:

```python
sage: W.subspace([\[1,1,0\]], check=False)
Vector space of degree 3 and dimension 1 over Finite Field of size 7
Basis matrix:
[1 1 0]
```

With check=True (the default) the mistake is correctly detected and reported with an ArithmeticError exception:

```python
sage: W.subspace([\[1,1,0\]], check=True)
Traceback (most recent call last):
...
ArithmeticError: Argument gens (= \[[1, 1, 0]\]) does not generate a submodule of...
```

**subspace_with_basis** *(gens, check=True, already_echelonized=False)*

Same as `self.submodule_with_basis(...)`.

**EXAMPLES:**

We create a subspace with a user-defined basis.

```python
sage: V = VectorSpace(GF(7), 3)
sage: W = V.subspace_with_basis([[2,2,2], [1,2,3]]); W
Vector space of degree 3 and dimension 2 over Finite Field of size 7
User basis matrix:
[2 2 2]
[1 2 3]
```

We then create a subspace of the subspace with user-defined basis.

```python
sage: W1 = W.subspace_with_basis([[3,4,5]]); W1
Vector space of degree 3 and dimension 1 over Finite Field of size 7
User basis matrix:
[3 4 5]
```

Notice how the basis for the same subspace is different if we merely use the subspace command.

```python
sage: W2 = W.subspace([[3,4,5]]); W2
Vector space of degree 3 and dimension 1 over Finite Field of size 7
Basis matrix:
[1 6 4]
```

Nonetheless the two subspaces are equal (as mathematical objects):

```python
sage: W1 == W2
True
```

**subspaces**(dim)

Iterate over all subspaces of dimension dim.

**INPUT:**

- `dim` - int, dimension of subspaces to be generated

**EXAMPLES:**
```python
sage: V = VectorSpace(GF(3), 5)
sage: len(list(V.subspaces(0)))
1
sage: len(list(V.subspaces(1)))
121
sage: len(list(V.subspaces(2)))
1210
sage: len(list(V.subspaces(3)))
1210
sage: len(list(V.subspaces(4)))
121
sage: len(list(V.subspaces(5)))
1
```

```python
sage: V = VectorSpace(GF(3), 5)
sage: V = V.subspace([V([1,1,0,0,0]),V([0,0,1,1,0])])
sage: list(V.subspaces(1))
[Vector space of degree 5 and dimension 1 over Finite Field of size 3
Basis matrix:
[1 1 0 0 0],
Vector space of degree 5 and dimension 1 over Finite Field of size 3
Basis matrix:
[1 1 1 1 0],
Vector space of degree 5 and dimension 1 over Finite Field of size 3
Basis matrix:
[1 1 2 2 0],
Vector space of degree 5 and dimension 1 over Finite Field of size 3
Basis matrix:
[0 0 1 1 0]]
```

```
vector_space(base_field=None)

Return the vector space associated to self. Since self is a vector space this function simply returns self, unless the base field is different.

EXAMPLES:
```
```
sage: V = span([[1,2,3]],QQ); V
Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[1 2 3]
sage: V.vector_space()
Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[1 2 3]
```

```
zero_submodule()

Return the zero submodule of self.

EXAMPLES:
```
```
sage: (QQ^4).zero submodule()
Vector space of degree 4 and dimension 0 over Rational Field
Basis matrix:
[]
```
zero_subspace()  
Return the zero subspace of self.

EXAMPLES:

```sage
sage: (QQ^4).zero_subspace()
Vector space of degree 4 and dimension 0 over Rational Field
Basis matrix:
[]
```

class sage.modules.free_module.FreeModule_generic_pid(base_ring, rank, degree, sparse=False, coordinate_ring=None)

Bases: sage.modules.free_module.FreeModule_generic

Base class for all free modules over a PID.

denominator()  
The denominator of the basis matrix of self (i.e. the LCM of the coordinate entries with respect to the basis of the ambient space).

EXAMPLES:

```sage
sage: V = QQ^3
sage: L = V.span([[1,1/2,1/3], [-1/5,2/3,3]],ZZ)
sage: L
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[ 1/5 19/6 37/3]
[ 0 23/6 46/3]
sage: L.denominator()
30
```

index_in(other)  
Return the lattice index [other:self] of self in other, as an element of the base field. When self is contained in other, the lattice index is the usual index. If the index is infinite, then this function returns infinity.

EXAMPLES:

```sage
sage: L1 = span([[1,2]], ZZ)
sage: L2 = span([[3,6]], ZZ)
sage: L2.index_in(L1)
3
```

Note that the free modules being compared need not be integral.

```sage
sage: L1 = span(['1/2','1/3'], [4,5]], ZZ)
sage: L2 = span([[1,2], [3,4]], ZZ)
sage: L2.index_in(L1)
12/7
sage: L1.index_in(L2)
7/12
sage: L1.discriminant() / L2.discriminant()
49/144
```

The index of a lattice of infinite index is infinite.
index_in_saturation()  
Return the index of this module in its saturation, i.e., its intersection with $R^n$.

EXAMPLES:

```python
sage: W = span([[2,4,6]], ZZ)
sage: W.index_in_saturation()
2
sage: W = span([[1/2,1/3]], ZZ)
sage: W.index_in_saturation()
1/6
```

intersection(other)  
Return the intersection of self and other.

EXAMPLES:

We intersect two submodules one of which is clearly contained in the other:

```python
sage: A = ZZ^2
sage: M1 = A.span([[1,1]])
sage: M2 = A.span([[3,3]])
sage: M1.intersection(M2)
Free module of degree 2 and rank 1 over Integer Ring
Echelon basis matrix:
[3 3]
sage: M1.intersection(M2) is M2
True
```

We intersect two submodules of $\mathbb{Z}^3$ of rank 2, whose intersection has rank 1:

```python
sage: A = ZZ^3
sage: M1 = A.span([[1,1,1], [1,2,3]])
sage: M2 = A.span([[2,2,2], [1,0,0]])
sage: M1.intersection(M2)
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[2 2 2]
```

We compute an intersection of two $\mathbb{Z}$-modules that are not submodules of $\mathbb{Z}^2$:

```python
sage: A = ZZ^2
sage: M1 = A.span([[1,2]]).scale(1/6)
sage: M2 = A.span([[1,2]]).scale(1/15)
sage: M1.intersection(M2)
Free module of degree 2 and rank 1 over Integer Ring
Echelon basis matrix:
[1/3 2/3]
```

We intersect a $\mathbb{Z}$-module with a $\mathbb{Q}$-vector space:
```
sage: A = ZZ^3
sage: L = ZZ^3
sage: V = QQ^3
sage: W = L.span([[1/2,0,1/2]])
```

```
sage: K = V.span([[1,0,1], [0,0,1]])
```

```
sage: W.intersection(K)
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[1/2 0 1/2]
```

```
sage: K.intersection(W)
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[1/2 0 1/2]
```

We intersect two modules over the ring of integers of a number field:

```
sage: L.<w> = NumberField(x^2 - x + 2)
sage: OL = L.ring_of_integers()
sage: V = L**3
sage: W1 = V.span([[0,w/5,0], [1,0,-1/17]], OL)
sage: W2 = V.span([[0,(1-w)/5,0]], OL)
sage: W1.intersection(W2)
Free module of degree 3 and rank 1 over Maximal Order in
Number Field in w with defining polynomial x^2 - x + 2
Echelon basis matrix:
[0 2/5 0]
```

\textbf{quotient}(\textit{sub}, \textit{check}=True, **\textit{kwds})

Return the quotient of \textit{self} by the given submodule \textit{sub}.

\textbf{INPUT:}

\begin{itemize}
  \item \textit{sub} - a submodule of \textit{self}, or something that can be turned into one via \textit{self.submodule} (\textit{sub}).
  \item \textit{check} - (default: True) whether or not to check that \textit{sub} is a submodule.
  \item further named arguments, that are passed to the constructor of the quotient space.
\end{itemize}

\textbf{EXAMPLES:}

```
sage: A = ZZ^3; V = A.span([[1,2,3], [4,5,6]])
sage: Q = V.quotient( [V.0 + V.1] ); Q
Finitely generated module \textit{V/W} over Integer Ring with invariants (0)
```

\textbf{saturation()} 

Return the saturated submodule of \( R^n \) that spans the same vector space as \textit{self}.

\textbf{EXAMPLES:}

We create a 1-dimensional lattice that is obviously not saturated and saturate it.

```
sage: L = span([[9,9,6]], ZZ); L
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[9 9 6]
sage: L.saturation()
Free module of degree 3 and rank 1 over Integer Ring
```

(continues on next page)
Echelon basis matrix:
[3 3 2]

We create a lattice spanned by two vectors, and saturate. Computation of discriminants shows that the index of lattice in its saturation is 3, which is a prime of congruence between the two generating vectors.

```sage
L = span([[1, 2, 3], [4, 5, 6]], ZZ)
sage: L.saturation()
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[ 1  0 -1]
[ 0  1  2]
sage: L.discriminant()
54
sage: L.saturation().discriminant()
6
```

Notice that the saturation of a non-integral lattice $L$ is defined, but the result is integral hence does not contain $L$:

```sage
L = span([[1/2, 1, 3]], ZZ)
sage: L.saturation()
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[1 2 6]
```

**scale**(other)
Return the product of this module by the number other, which is the module spanned by other times each basis vector.

**EXAMPLES:**

```sage
M = FreeModule(ZZ, 3)
sage: M.scale(2)
Free module of degree 3 and rank 3 over Integer Ring
Echelon basis matrix:
[2 0 0]
[0 2 0]
[0 0 2]
sage: a = QQ('1/3')
sage: M.scale(a)
Free module of degree 3 and rank 3 over Integer Ring
Echelon basis matrix:
[1/3  0  0]
[0  1/3  0]
[0  0  1/3]
```

**span**(gens, base_ring=None, check=True, already_echelonized=False)
Return the R-span of the given list of gens, where R = base_ring. The default R is the base ring of self. Note that this span need not be a submodule of self, nor even of the ambient space. It must, however, be contained in the ambient vector space, i.e., the ambient space tensored with the fraction field of R.

**EXAMPLES:**
sage: V = FreeModule(ZZ,3)
sage: W = V.submodule([V.gen(0)])
sage: W.span([V.gen(1)])
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[0 1 0]
sage: W.submodule([V.gen(1)])
Traceback (most recent call last):
...
ArithmeticError: Argument gens (= [(0, 1, 0)]) does not generate a submodule of self.
sage: V.span([[1,0,0],[1/5,4,0],[6,3/4,0]])
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[1/5 0 0]
[ 0 1/4 0]

It also works with other things than integers:

sage: R.<x>=QQ[]
sage: L=R^1
sage: a=L.span([[1/x,]])
sage: a
Free module of degree 1 and rank 1 over Univariate Polynomial Ring in x over Rational Field
Echelon basis matrix:
[1/x]
sage: b=L.span([[1/x,]])
sage: a(b.gens()[0])
(1/x)
sage: L2 = R^2
sage: L2.span([[x^2+x)/(x^2-3*x+2),1/5],[(x^2+2*x)/(x^2-4*x+3),x]])
Free module of degree 2 and rank 2 over Univariate Polynomial Ring in x over Rational Field
Echelon basis matrix:
[x/(x^3 - 6*x^2 + 11*x - 6) 2/15*x^2 - 17/75*x - 1/75]
[0 x^3 - 11/5*x^2 - 3*x + 4/5]

Note that the base_ring can make a huge difference. We repeat the previous example over the fraction field of R and get a simpler vector space.

sage: L2.span([[x^2+x)/(x^2-3*x+2),1/5],[(x^2+2*x)/(x^2-4*x+3),x]],base_ring=R.fraction_field())
Vector space of degree 2 and dimension 2 over Fraction Field of Univariate Polynomial Ring in x over Rational Field
Basis matrix:
[1 0]
[0 1]

span_of_basis(basis, base_ring=None, check=True, already_echelonized=False)
Return the free R-module with the given basis, where R is the base ring of self or user specified base_ring.

Note that this R-module need not be a submodule of self, nor even of the ambient space. It must, however, be contained in the ambient vector space, i.e., the ambient space tensored with the fraction field of R.
EXAMPLES:

```python
sage: M = FreeModule(ZZ, 3)
sage: W = M.span_of_basis([M([1,2,3])])
```

Next we create two free $\mathbb{Z}$-modules, neither of which is a submodule of $W$.

```python
sage: W.span_of_basis([M([2,4,0])])
Free module of degree 3 and rank 1 over Integer Ring
User basis matrix:
[2 4 0]
```

The following module isn’t in the ambient module $\mathbb{Z}^3$ but is contained in the ambient vector space $\mathbb{Q}^3$:

```python
sage: V = M.ambient_vector_space()
sage: W.span_of_basis([ V([1/5,2/5,0]), V([1/7,1/7,0]) ])
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[1/5 2/5 0]
[1/7 1/7 0]
```

Of course the input basis vectors must be linearly independent:

```python
sage: W.span_of_basis([ [1,2,0], [2,4,0] ])  
Traceback (most recent call last):
...
ValueError: The given basis vectors must be linearly independent.
```

The `submodule` function creates an $R$-submodule of the ambient vector space with given generators, where $R$ is the base ring of self.

**INPUT:**

- `gens` - a list of free module elements or a free module
- `check` - (default: True) whether or not to verify that the gens are in self.

**OUTPUT:**

- `FreeModule` - the submodule spanned by the vectors in the list gens. The basis for the subspace is always put in reduced row echelon form.

**EXAMPLES:**

We create a submodule of $\mathbb{Z}^3$:

```python
sage: N = FreeModule(ZZ, 3)
sage: B = M.basis()
sage: W
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[1 1 0]
[0 2 -1]
```

We create a submodule of a submodule.
We try to create a submodule that isn’t really a submodule, which results in an `ArithmeticError` exception:

```python
sage: W.submodule([B[0] - B[1]])
Traceback (most recent call last):
  ...ArithmeticError: Argument gens (= [(1, -1, 0)]) does not generate a submodule of self.
```

Next we create a submodule of a free module over the principal ideal domain \( \mathbb{Q}[x] \), which uses the general Hermite normal form functionality:

```python
sage: R = PolynomialRing(QQ, 'x'); x = R.gen()
sage: M = FreeModule(R, 3)
sage: B = M.basis()
sage: W = M.submodule([x*B[0], 2*B[1]-x*B[2]]); W
Free module of degree 3 and rank 2 over Univariate Polynomial Ring in x over Rational Field
Echelon basis matrix:
[ x 0 0]
[ 0 2 -x]
sage: W.ambient_module()
Ambient free module of rank 3 over the principal ideal domain Univariate Polynomial Ring in x over Rational Field
```

**submodule_with_basis**(basis, check=True, already_echelonized=False)

Create the \( R \)-submodule of the ambient vector space with given basis, where \( R \) is the base ring of self.

**INPUT:**

- basis – a list of linearly independent vectors
- **check** – whether or not to verify that each gen is in the ambient vector space

**OUTPUT:**

- FreeModule – the \( R \)-submodule with given basis

**EXAMPLES:**

First we create a submodule of \( \mathbb{Z}^3 \):

```python
sage: M = FreeModule(ZZ, 3)
sage: B = M.basis()
sage: N = M.submodule_with_basis([B[0]+B[1], 2*B[1]-B[2]])
sage: N
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[ 1 1 0]
[ 0 2 -1]
```

A list of vectors in the ambient vector space may fail to generate a submodule.
sage: V = M.ambient_vector_space()
sage: X = M.submodule_with_basis([ V(B[0] + B[1])/2, V(B[1] - B[2])/2])
Traceback (most recent call last):
...  
ArithmeticError: The given basis does not generate a submodule of self.

However, we can still determine the \( \mathbb{R} \)-span of vectors in the ambient space, or over-ride the submodule check by setting check to False.

sage: X
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
\[
\begin{bmatrix}
  1/2 & 0 & 1/2 \\
  0 & 1/2 & -1/2 \\
\end{bmatrix}
\]
sage: X == Y
True

Next we try to create a submodule of a free module over the principal ideal domain \( \mathbb{Q}[x] \), using our general Hermite normal form implementation:

sage: R = PolynomialRing(QQ, 'x'); x = R.gen()
sage: M = FreeModule(R, 3)
sage: B = M.basis()
sage: W = M.submodule_with_basis([x*B[0], 2*B[0] - x*B[2]]);
    W
Free module of degree 3 and rank 2 over Univariate Polynomial Ring in x over Rational Field
User basis matrix:
\[
\begin{bmatrix}
  x & 0 & 0 \\
  2 & 0 & -x \\
\end{bmatrix}
\]

vector_space_span(gens, check=True)

Create the vector subspace of the ambient vector space with given generators.

INPUT:

- gens - a list of vector in self
- check - whether or not to verify that each gen is in the ambient vector space

OUTPUT: a vector subspace

EXAMPLES:

We create a 2-dimensional subspace of \( \mathbb{Q}^3 \).

sage: V = VectorSpace(QQ, 3)
sage: B = V.basis()
sage: W
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
\[
\begin{bmatrix}
  1 & 0 & 1/2 \\
  0 & 1 & -1/2 \\
\end{bmatrix}
\]

We create a subspace of a vector space over \( \mathbb{Q}(i) \).
We use the `vector_space_span` command to create a vector subspace of the ambient vector space of a submodule of $\mathbb{Z}^3$.

```python
sage: M = FreeModule(ZZ,3)
sage: W = M.submodule([M([1,2,3])])
sage: W.vector_space_span([M([2,3,4])])
Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[ 1 3/2 2]
```

### `vector_space_span_of_basis(basis, check=True)`

Create the vector subspace of the ambient vector space with given basis.

**INPUT:**

- `basis` – a list of linearly independent vectors
- `check` – whether or not to verify that each gen is in the ambient vector space

**OUTPUT:** a vector subspace with user-specified basis

**EXAMPLES:**

```python
sage: V = VectorSpace(QQ, 3)
sage: B = V.basis()
sage: W
Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[ 1 1 0]
[ 0 2 -1]
```

### `zero_submodule()`

Return the zero submodule of this module.

**EXAMPLES:**

```python
sage: V = FreeModule(ZZ,2)
sage: V.zero submodule()
Free module of degree 2 and rank 0 over Integer Ring
Echelon basis matrix:
[ ]
```

**class** `sage.modules.free_module.FreeModule_submodule_field(ambient, gens, check=True, already_echelonized=False)`

Bases: `sage.modules.free_module.FreeModule_submodule_with_basis_field`

An embedded vector subspace with echelonized basis.
EXAMPLES:

Since this is an embedded vector subspace with echelonized basis, the echelon_coordinates() and user coordinates() agree:

```sage
V = QQ^3
W = V.span([[1,2,3],[4,5,6]])
W
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1 0 -1]
[ 0 1  2]
```

```sage
v = V([1,5,9])
W.echelon_coordinates(v)
[1, 5]
vector(QQ, W.echelon_coordinates(v)) * W.basis_matrix()
(1, 5, 9)
```

```sage
v = V([1,5,9])
W.coordinates(v)
[1, 5]
vector(QQ, W.coordinates(v)) * W.basis_matrix()
(1, 5, 9)
```

`coordinate_vector(v, check=True)`

Write \( v \) in terms of the user basis for self.

INPUT:

- \( v \) – vector
- \( \text{check} \) – bool (default: True); if True, also verify that \( v \) is really in self.

OUTPUT: list

Returns a list \( c \) such that if \( B \) is the basis for self, then

\[
\sum c_i B_i = v.
\]

If \( v \) is not in self, raise an `ArithmeticError` exception.

EXAMPLES:

```sage
V = QQ^3
W = V.span([[1,2,3],[4,5,6]]); W
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1 0 -1]
[ 0 1  2]
v = V([1,5,9])
W.coordinate_vector(v)
(1, 5)
W.coordinates(v)
[1, 5]
vector(QQ, W.coordinates(v)) * W.basis_matrix()
(1, 5, 9)
```
sage: V = VectorSpace(QQ,5, sparse=True)
sage: W = V.subspace([[0,1,2,0,0], [0,-1,0,0,-1/2]])
sage: W.coordinate_vector([0,0,2,0,-1/2])
(0, 2)

`echelon_coordinates(v, check=True)`

Write \( v \) in terms of the echelonized basis of self.

**INPUT:**
- \( v \) - vector
- \( \text{check} \) - bool (default: True); if True, also verify that \( v \) is really in self.

**OUTPUT:** list

Returns a list \( c \) such that if \( B \) is the basis for self, then

\[
\sum c_i B_i = v.
\]

If \( v \) is not in self, raise an \texttt{ArithmeticError} exception.

**EXAMPLES:**

```python
sage: V = QQ^3
sage: W = V.span([[1,2,3],[4,5,6]])
sage: W
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1 0 -1]
[ 0 1 2]
sage: v = V([1,5,9])
sage: W.echelon_coordinates(v)
[1, 5]
sage: vector(QQ, W.echelon_coordinates(v)) * W.basis_matrix()
(1, 5, 9)
```

`has_user_basis()`

Return \texttt{True} if the basis of this free module is specified by the user, as opposed to being the default echelon form.

**EXAMPLES:**

```python
sage: V = QQ^3
sage: W = V.subspace([[2,1/2, 1]])
sage: W.has_user_basis()
False
sage: W = V.subspace_with_basis([[2,1/2,1]])
sage: W.has_user_basis()
True
```

**class** `sage.modules.free_module.FreeModule_submodule_pid(ambient, gens, check=True, already_echelonized=False)`

**Bases:** `sage.modules.free_module.FreeModule_submodule_with_basis_pid`

An \( R \)-submodule of \( K^n \) where \( K \) is the fraction field of a principal ideal domain \( R \).

**EXAMPLES:**
sage: M = ZZ^3
sage: W = M.span_of_basis([[1,2,3],[4,5,19]]); W
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[ 1  2  3]
[ 4  5 19]

Generic tests, including saving and loading submodules and elements:

sage: TestSuite(W).run()
sage: v = W.0 + W.1
sage: TestSuite(v).run()

coordinate_vector(v, check=True)
Write $v$ in terms of the user basis for self.

**INPUT:**

- $v$ – vector
- **check** – bool (default: True); if True, also verify that $v$ is really in self.

**OUTPUT:** list

Returns a list $c$ such that if $B$ is the basis for self, then

$$\sum c_i B_i = v.$$ 

If $v$ is not in self, raise an ArithmeticError exception.

**EXAMPLES:**

sage: V = ZZ^3
sage: W = V.span_of_basis([[1,2,3],[4,5,6]])
sage: W.coordinate_vector([1,5,9])
(5, -1)

has_user_basis()
Return True if the basis of this free module is specified by the user, as opposed to being the default echelon form.

**EXAMPLES:**

sage: A = ZZ^3; A
Ambient free module of rank 3 over the principal ideal domain Integer Ring
sage: A.has_user_basis()
False
sage: W = A.span_of_basis([[2,'1/2',1]])
sage: W.has_user_basis()
True
sage: W = A.span([[2,'1/2',1]])
sage: W.has_user_basis()
False

Chapter 3. Free modules
class sage.modules.free_module.FreeModule_submodule_with_basis_field(ambient, basis, 
check=True, 
echelonize=False, 
echelonized_basis=None, 
already_echelonized=False)

Bases: sage.modules.free_module.FreeModule_generic_field, sage.modules.free_module. 
FreeModule_submodule_with_basis_pid

An embedded vector subspace with a distinguished user basis.

EXAMPLES:

```
sage: M = QQ^3; W = M.submodule_with_basis([[1,2,3], [4,5,19]]); W
Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[ 1  2  3]
[ 4  5 19]
```

Since this is an embedded vector subspace with a distinguished user basis possibly different than the echelonized 
基础，the echelon_coordinates() and user_coordinates() do not agree:

```
sage: V = QQ^3

sage: W = V.submodule_with_basis([[1,2,3], [4,5,6]])

sage: W
Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[1 2 3]
[4 5 6]
```

```
sage: v = V([1,5,9])
sage: W.echelon_coordinates(v)
[1, 5]

sage: vector(QQ, W.echelon_coordinates(v)) * W.echelonized_basis_matrix()
(1, 5, 9)
```

```
sage: v = V([1,5,9])
sage: W.coordinates(v)
[5, -1]

sage: vector(QQ, W.coordinates(v)) * W.basis_matrix()
(1, 5, 9)
```

Generic tests, including saving and loading submodules and elements:

```
sage: TestSuite(W).run()
```

```
sage: K.<x> = FractionField(PolynomialRing(QQ,'x'))
sage: M = K^3; W = M.span_of_basis([[1,1,x]])
sage: TestSuite(W).run()
```

is_ambient()
  Return False since this is not an ambient module.

  EXAMPLES:
```python
sage: V = QQ^3
sage: V.is_ambient()
True
sage: W = V.span_of_basis([[1,2,3],[4,5,6]])
sage: W.is_ambient()
False
```

class `sage.modules.free_module.FreeModule_submodule_with_basis_pid` *(ambient, basis, check=True, echelonize=False, echelonized_basis=None, already_echelonized=False)*

Bases: `sage.modules.free_module.FreeModule_generic_pid`

Construct a submodule of a free module over PID with a distinguished basis.

**INPUT:**
- `ambient` – ambient free module over a principal ideal domain $R$, i.e. $R^n$;
- `basis` – list of elements of $K^n$, where $K$ is the fraction field of $R$. These elements must be linearly independent and will be used as the default basis of the constructed submodule;
- `check` – (default: True) if False, correctness of the input will not be checked and type conversion may be omitted, use with care;
- `echelonize` – (default: False) if True, basis will be echelonized and the result will be used as the default basis of the constructed submodule;
- `echelonized_basis` – (default: None) if not None, must be the echelonized basis spanning the same submodule as `basis`;
- `already_echelonized` – (default: False) if True, basis must be already given in the echelonized form.

**OUTPUT:**
- $R$-submodule of $K^n$ with the user-specified basis.

**EXAMPLES:**
```python
sage: M = ZZ^3
sage: W = M.span_of_basis([[1,2,3],[4,5,6]]); W
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[ 1 2 3]
[ 4 5 6]
```

Now we create a submodule of the ambient vector space, rather than $M$ itself:

```python
sage: W = M.span_of_basis([[1,2,3/2],[4,5,6]]); W
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[  1  2 3/2]
[  4  5  6]
```

`ambient_module()`

Return the ambient module related to the $R$-module self, which was used when creating this module, and is of the form $R^n$. Note that `self` need not be contained in the ambient module, though `self` will be contained in the ambient vector space.

**EXAMPLES:**

```python
sage: A = ZZ^3
sage: M = A.span_of_basis([[1,2,'3/7'],[4,5,6]])
sage: M
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[ 1  2 3/7]
[ 4  5  6]
sage: M.ambient_module()
Ambient free module of rank 3 over the principal ideal domain Integer Ring
sage: M.is_submodule(M.ambient_module())
False

ambient_vector_space()
Return the ambient vector space in which this free module is embedded.

EXAMPLES:
```}
```python
sage: M = ZZ^3;  M.ambient_vector_space()
Vector space of dimension 3 over Rational Field
```
```python
sage: N = M.span_of_basis([[1,2,'1/5']])
sage: N
Free module of degree 3 and rank 1 over Integer Ring
User basis matrix:
[ 1  2 1/5]
sage: N.ambient_vector_space()
Vector space of dimension 3 over Rational Field
sage: N.ambient_vector_space() is N.ambient_vector_space()
True
```

If an inner product on the module is specified, then this is preserved on the ambient vector space.
```python
sage: M = FreeModule(ZZ,4,inner_product_matrix=1)
sage: V = M.ambient_vector_space()
sage: V
Ambient quadratic space of dimension 4 over Rational Field
Inner product matrix:
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
sage: N = M.submodule([[1,-1,0,0],[0,1,-1,0],[0,0,1,-1]])
sage: N.gram_matrix()
[2 1 1]
[1 2 1]
[1 1 2]
sage: V == N.ambient_vector_space()
True
```

basis()
Return the user basis for this free module.

EXAMPLES:
```
```
sage: V = ZZ^3
sage: V.basis()
[(1, 0, 0),
 (0, 1, 0),
 (0, 0, 1)]
sage: M = V.span_of_basis([[1/8, 2, 1]])
sage: M.basis()
[(1/8, 2, 1)]
```

**change_ring(R)**

Return the free module over $R$ obtained by coercing each element of the basis of `self` into a vector over the fraction field of $R$, then taking the resulting $R$-module.

**INPUT:**

- $R$ - a principal ideal domain

**EXAMPLES:**

```
sage: V = QQ^3
sage: W = V.subspace([[2, 1/2, 1]])
sage: W.change_ring(GF(7))
Vector space of degree 3 and dimension 1 over Finite Field of size 7
Basis matrix:
[1 2 4]
```

```
sage: M = (ZZ^2) * (1/2)
sage: N = M.change_ring(QQ)
sage: N
Vector space of degree 2 and dimension 2 over Rational Field
Basis matrix:
[1 0]
[0 1]
sage: N = M.change_ring(QQ['x'])
sage: N
Free module of degree 2 and rank 2 over Univariate Polynomial Ring in x over Rational Field
Echelon basis matrix:
[1/2 0]
[0 1/2]
sage: N.coordinate_ring()
Univariate Polynomial Ring in x over Rational Field
```

The ring must be a principal ideal domain:

```
sage: M.change_ring(ZZ['x'])
Traceback (most recent call last):
...
TypeError: the new ring Univariate Polynomial Ring in x over Integer Ring
should be a principal ideal domain
```
construction()
Returns the functorial construction of self, namely, the subspace of the ambient module spanned by the given basis.

EXAMPLES:
```
sage: M = ZZ^3
sage: W = M.span_of_basis([[1,2,3],[4,5,6]]); W
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[1 2 3]
[4 5 6]
sage: c, V = W.construction()
sage: c(V) == W
True
```

coordinate_vector(v, check=True)
Write \( v \) in terms of the user basis for self.

INPUT:
- \( v \) – vector
- \( \text{check} \) – bool (default: True); if True, also verify that \( v \) is really in self.

OUTPUT: list
Returns a vector \( c \) such that if \( B \) is the basis for self, then
\[
\sum c_i B_i = v.
\]
If \( v \) is not in self, raise an ArithmeticError exception.

EXAMPLES:
```
sage: V = ZZ^3
sage: M = V.span_of_basis([[1/8,2,1]])
sage: M.coordinate_vector([1,16,8])
(8)
```

echelon_coordinate_vector(v, check=True)
Write \( v \) in terms of the echelonized basis for self.

INPUT:
- \( v \) - vector
- \( \text{check} \) - bool (default: True); if True, also verify that \( v \) is really in self.

Returns a list \( c \) such that if \( B \) is the echelonized basis for self, then
\[
\sum c_i B_i = v.
\]
If \( v \) is not in self, raise an ArithmeticError exception.

EXAMPLES:
```
sage: V = ZZ^3
sage: M = V.span_of_basis([[1/2,3,1], [0,'1/6',0]])
sage: B = M.echelonized_basis(); B
```
Modules, Release 9.6

\[
\begin{bmatrix}
(1/2, 0, 1), \\
(0, 1/6, 0)
\end{bmatrix}
\]
sage: M.echelon_coordinate_vector(['1/2', 3, 1])
(1, 18)

**echelon_coordinates(v, check=True)**
Write \( v \) in terms of the echelonized basis for self.

**INPUT:**
- \( v \) - vector
- check - bool (default: True); if True, also verify that \( v \) is really in self.

**OUTPUT:** list
Returns a list \( c \) such that if \( B \) is the basis for self, then
\[
\sum c_i B_i = v.
\]
If \( v \) is not in self, raise an \texttt{ArithmeticError} exception.

**EXAMPLES:**

\[
\begin{align*}
sage: & A = \texttt{ZZ^3} \\
sage: & M = A.\texttt{span_of_basis}([[1,2,'3/7'],[4,5,6]]) \\
sage: & M.\texttt{coordinates([8,10,12])} \\
& [0, 2] \\
sage: & M.\texttt{echelon_coordinates([8,10,12])} \\
& [8, -2] \\
sage: & B = M.\texttt{echelonized_basis()}; B \\
& [ \\
& (1, 2, 3/7), \\
& (0, 3, -30/7) \\
& ] \\
& (8, 10, 12)
\end{align*}
\]

We do an example with a sparse vector space:

\[
\begin{align*}
sage: & V = \texttt{VectorSpace(QQ,5, sparse=True)} \\
sage: & W = V.\texttt{subspace_with_basis([[0,1,2,0,0], [0,-1,0,0,-1/2]])} \\
sage: & W.\texttt{echelonized_basis()} \\
& [ \\
& (0, 1, 0, 0, 1/2), \\
& (0, 0, 1, 0, -1/4) \\
& ] \\
sage: & W.\texttt{echelon_coordinates([0,0,2,0,-1/2])} \\
& [0, 2]
\end{align*}
\]

**echelon_to_user_matrix()**
Return matrix that transforms the echelon basis to the user basis of self. This is a matrix \( A \) such that if \( v \) is a vector written with respect to the echelon basis for \( \texttt{self} \) then \( vA \) is that vector written with respect to the user basis of self.
EXAMPLES:

```
sage: V = QQ^3
sage: W = V.span_of_basis([[1,2,3],[4,5,6]])
sage: W.echelonized_basis()
[ (1, 0, -1),
  (0, 1, 2) ]
sage: A = W.echelon_to_user_matrix(); A
[-5/3 2/3]
[ 4/3 -1/3]
```

The vector \((1, 1, 1)\) has coordinates \(v = (1, 1)\) with respect to the echelonized basis for \(\text{self}\). Multiplying \(vA\) we find the coordinates of this vector with respect to the user basis.

```
sage: v = vector(QQ, [1,1]); v
(1, 1)
sage: v * A
(-1/3, 1/3)
sage: u0, u1 = W.basis()
sage: (-u0 + u1)/3
(1, 1, 1)
```

echelonized_basis()
Return the basis for \(\text{self}\) in echelon form.

EXAMPLES:

```
sage: V = ZZ^3
sage: M = V.span_of_basis([[1/2,3,1], [0,1/6,0]])
sage: M.basis()
[ (1/2, 3, 1),
  (0, 1/6, 0) ]
sage: B = M.echelonized_basis(); B
[ (1/2, 0, 1),
  (0, 1/6, 0) ]
sage: V.span(B) == M
True
```

echelonized_basis_matrix()
Return basis matrix for \(\text{self}\) in row echelon form.

EXAMPLES:

```
sage: V = FreeModule(ZZ, 3).span_of_basis([[1,2,3],[4,5,6]])
sage: V.basis_matrix()
[1 2 3]
[4 5 6]
sage: V.echelonized_basis_matrix()
```

(continues on next page)
has_user_basis()
Return True if the basis of this free module is specified by the user, as opposed to being the default echelon form.

EXAMPLES:

```
sage: V = ZZ^3; V.has_user_basis()
False
sage: M = V.span_of_basis([[1,3,1]]); M.has_user_basis()
True
sage: M = V.span([[1,3,1]]); M.has_user_basis()
False
```

linear_combination_of_basis(v)
Return the linear combination of the basis for self obtained from the coordinates of v.

INPUT:
- v - list

EXAMPLES:

```
sage: V = span([[1,2,3], [4,5,6]], ZZ); V
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[1 2 3]
[0 3 6]
sage: V.linear_combination_of_basis([1,1])
(1, 5, 9)
```

This should raise an error if the resulting element is not in self:

```
sage: W = (QQ**2).span([[2, 0], [0, 8]], ZZ)
sage: W.linear_combination_of_basis([1, -1/2])
Traceback (most recent call last):
  ...
TypeError: element [2, -4] is not in free module
```

user_to_echelon_matrix()
Return matrix that transforms a vector written with respect to the user basis of self to one written with respect to the echelon basis. The matrix acts from the right, as is usual in Sage.

EXAMPLES:

```
sage: A = ZZ^3
sage: M = A.span_of_basis([[1,2,3],[4,5,6]])
sage: M.echelonized_basis()
[ 1 0 0]
[0 1 0]
[0 0 1]
sage: M.user_to_echelon_matrix()
```

The vector $v = (5, 7, 9)$ in $M$ is $(1, 1)$ with respect to the user basis. Multiplying the above matrix on the right by this vector yields $(5, -1)$, which has components the coordinates of $v$ with respect to the echelon basis.

```
sage: v0, v1 = M.basis(); v = v0 + v1
sage: e0, e1 = M.echelonized_basis()
sage: v
(5, 7, 9)
sage: 5*e0 + (-1)*e1
(5, 7, 9)
```

`vector_space(base_field=None)`

Return the vector space associated to this free module via tensor product with the fraction field of the base ring.

**EXAMPLES:**

```
sage: A = ZZ^3; A
Ambient free module of rank 3 over the principal ideal domain Integer Ring
sage: A.vector_space()
Vector space of dimension 3 over Rational Field
sage: M = A.span_of_basis([[1/3, 2, 3/7], [4, 5, 6]]); M
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[1/3 2 3/7]
[4 5 6]
sage: M.vector_space()
Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[1/3 2 3/7]
[4 5 6]
```

class sage.modules.free_module.RealDoubleVectorSpace_class(n)

Bases: sage.modules.free_module.FreeModule_ambient_field

`coordinates(v)`

**EXAMPLES:**

The base can be complicated, as long as it is a field.

```
sage: V = VectorSpace(FractionField(PolynomialRing(ZZ, 'x')), 3)
sage: V
Vector space of dimension 3 over Fraction Field of Univariate Polynomial Ring in x over Integer Ring
sage: V.basis()
[1, (1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 0, 0)]
```
(0, 0, 1)
]

The base must be a field or a TypeError is raised.

```
sage: VectorSpace(ZZ,5)
Traceback (most recent call last):
...  
TypeError: Argument K (= Integer Ring) must be a field.
```

`sage.modules.free_module.basis_seq(V, vecs)`

This converts a list vecs of vectors in V to an Sequence of immutable vectors.

Should it? I.e. in most other parts of the system the return type of basis or generators is a tuple.

**EXAMPLES:**

```
sage: V = VectorSpace(QQ,2)
sage: B = V.gens()
sage: B
((1, 0), (0, 1))
sage: v = B[0]
sage: v[0] = 0  # immutable
Traceback (most recent call last):
...  
ValueError: vector is immutable; please change a copy instead (use copy())
sage: sage.modules.free_module.basis_seq(V, V.gens())
[ 
 (1, 0),
 (0, 1)
]
```

`sage.modules.free_module.element_class(R, is_sparse)`

The class of the vectors (elements of a free module) with base ring R and boolean is_sparse.

**EXAMPLES:**

```
sage: FF = FiniteField(2)
sage: P = PolynomialRing(FF, 'x')
sage: sage.modules.free_module.element_class(QQ, is_sparse=True)
<class 'sage.modules.free_module_element.FreeModuleElement_generic_sparse'>
sage: sage.modules.free_module.element_class(QQ, is_sparse=False)
<class 'sage.modules.vector_rational_dense.Vector_rational_dense'>
sage: sage.modules.free_module.element_class(ZZ, is_sparse=True)
<class 'sage.modules.free_module_element.FreeModuleElement_generic_sparse'>
sage: sage.modules.free_module.element_class(ZZ, is_sparse=False)
<class 'sage.modules.vector_integer_dense.Vector_integer_dense'>
sage: sage.modules.free_module.element_class(FF, is_sparse=True)
<class 'sage.modules.free_module_element.FreeModuleElement_generic_sparse'>
sage: sage.modules.free_module.element_class(FF, is_sparse=False)
<class 'sage.modules.vector_mod2_dense.Vector_mod2_dense'>
sage: sage.modules.free_module.element_class(GF(7), is_sparse=False)
<class 'sage.modules.vector_modn_dense.Vector_modn_dense'>
sage: sage.modules.free_module.element_class(P, is_sparse=True)
```
<class 'sage.modules.free_module_element.FreeModuleElement_generic_sparse'>
sage: sage.modules.free_module.element_class(P, is_sparse=False)
<class 'sage.modules.free_module_element.FreeModuleElement_generic_dense'>

sage.modules.free_module.is_FreeModule(M)
Return True if M inherits from FreeModule_generic.

EXAMPLES:

sage: from sage.modules.free_module import is_FreeModule
sage: V = ZZ^3
sage: is_FreeModule(V)
True
sage: W = V.span([ V.random_element() for i in range(2) ])
sage: is_FreeModule(W)
True

sage.modules.free_module.span(gens, base_ring=None, check=True, already_echelonized=False)
Return the span of the vectors in gens using scalars from base_ring.

INPUT:

• gens - a list of either vectors or lists of ring elements used to generate the span
• base_ring - default: None - a principal ideal domain for the ring of scalars
• check - default: True - passed to the span() method of the ambient module
• already_echelonized - default: False - set to True if the vectors form the rows of a matrix in echelon
  form, in order to skip the computation of an echelonized basis for the span.

OUTPUT:

A module (or vector space) that is all the linear combinations of the free module elements (or vectors) with
scalars from the ring (or field) given by base_ring. See the examples below describing behavior when the base
ring is not specified and/or the module elements are given as lists that do not carry explicit base ring information.

EXAMPLES:

The vectors in the list of generators can be given as lists, provided a base ring is specified and the elements of
the list are in the ring (or the fraction field of the ring). If the base ring is a field, the span is a vector space.

sage: V = span([[1,2,5], [2,2,2]], QQ); V
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1 0 -3]
[ 0 1 4]
sage: span([V.gen(0)], QuadraticField(-7,'a'))
Vector space of degree 3 and dimension 1 over Number Field in a with defining
polynomial x^2 + 7 with a = 2.645751311064591?*I
Basis matrix:
[ 1 0 -3]
sage: span([[1,2,3], [2,2,2], [1,2,5]], GF(2))
Vector space of degree 3 and dimension 1 over Finite Field of size 2
Basis matrix:
[1 0 1]
If the base ring is not a field, then a module is created. The entries of the vectors can lie outside the ring, if they are in the fraction field of the ring.

```
sage: span([[1,2,5], [2,2,2]], ZZ)
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[ 1 0 -3
[ 0 2 8

sage: span([[1,1,1], [1,1/2,1]], ZZ)
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[ 1 0 1
[ 0 1/2 0

sage: R.<x> = QQ[]
sage: M= span( [[x, x^2+1], [1/x, x^3]], R); M
Univariate Polynomial Ring in x over Rational Field
Free module of degree 2 and rank 2 over
Univariate Polynomial Ring in x over Rational Field
Echelon basis matrix:
[ 1/x x^3
[ 0 x^5 - x^2 - 1
sage: M.basis()[0][0].parent()
Fraction Field of Univariate Polynomial Ring in x over Rational Field
```

A base ring can be inferred if the generators are given as a list of vectors.

```
sage: span([vector(QQ, [1,2,3]), vector(QQ, [4,5,6]))
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1 0 -1
[ 0 1 2

sage: span([vector(QQ, [1,2,3]), vector(ZZ, [4,5,6]))
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1 0 -1
[ 0 1 2

sage: span([vector(ZZ, [1,2,3]), vector(ZZ, [4,5,6]))
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[1 2 3
[0 3 6
```
CHAPTER
FOUR

DISCRETE SUBGROUPS OF $\mathbb{Z}^N$

AUTHORS:
• Martin Albrecht (2014-03): initial version
• Jan Pöschko (2012-08): some code in this module was taken from Jan Pöschko’s 2012 GSoC project

class sage.modules.free_module_integer.FreeModule_submodule_with_basis_integer
(ambient, basis, check=True, echelonize=False, echelonized_basis=None, already_echelonized=False, lll_reduce=True)

Bases: sage.modules.free_module.FreeModule_submodule_with_basis_pid

This class represents submodules of $\mathbb{Z}^n$ with a distinguished basis.

However, most functionality in excess of standard submodules over PID is for these submodules considered as discrete subgroups of $\mathbb{Z}^n$, i.e. as lattices. That is, this class provides functions for computing LLL and BKZ reduced bases for this free module with respect to the standard Euclidean norm.

EXAMPLES:

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: L = IntegerLattice(sage.crypto.gen_lattice(type='modular', m=10, seed=1337, dual=True)); L
Free module of degree 10 and rank 10 over Integer Ring
User basis matrix:
[-1  1  2 -2  0  1  0 -1  2  1]
[ 1  0  0 -1 -2  1 -2  3 -1  0]
[ 1  2  0  2 -1  1 -2  2  2  0]
[ 1  0 -1  0  2  3  0  0 -1 -2]
[ 1 -3  0  0  2  1 -2 -1  0  0]
[-3  0 -1  0 -1  2 -2  0  0  2]
[ 0  0  0  1  0  2 -3 -3 -2 -1]
[ 0 -1 -4 -1 -1  1  2 -1  0  1]
[ 1  1 -2  1  1  2  1  1 -2  3]
[ 2 -1  1  2 -3  2  2  1  0  1]
sage: L.shortest_vector()
(-1, 1, 2, -2, 0, 1, 0, -1, 2, 1)
```
**BKZ(***args, **kwds**)

Return a Block Korkine-Zolotareff reduced basis for self.

**INPUT:**

- *args – passed through to `sage.matrix.matrix_integer_dense.Matrix_integer_dense.BKZ()`
- **kwds – passed through to `sage.matrix.matrix_integer_dense.Matrix_integer_dense.BKZ()`

**OUTPUT:**

An integer matrix which is a BKZ-reduced basis for this lattice.

**EXAMPLES:**

```python
sage: from sage.modules.free_module_integer import IntegerLattice
sage: A = sage.crypto.gen_lattice(type='random', n=1, m=60, q=2^60, seed=42)
sage: L = IntegerLattice(A, lll_reduce=False)
sage: min(v.norm().n() for v in L.reduced_basis)
4.17330740711759e15
```

```python
sage: L.LLL()
60 x 60 dense matrix over Integer Ring (use the '.str()' method to see the→ entries)
```

```python
sage: min(v.norm().n() for v in L.reduced_basis)
5.19615242270663
```

```python
sage: L.BKZ(block_size=10)
60 x 60 dense matrix over Integer Ring (use the '.str()' method to see the→ entries)
```

```python
sage: min(v.norm().n() for v in L.reduced_basis)
4.12310562561766
```

**Note:** If block_size == L.rank() where L is this lattice, then this function performs Hermite-Korkine-Zolotareff (HKZ) reduction.

**HKZ(***args, **kwds**)

Hermite-Korkine-Zolotarev (HKZ) reduce the basis.

A basis $B$ of a lattice $L$, with orthogonalized basis $B^*$ such that $B = M \cdot B^*$ is HKZ reduced, if and only if, the following properties are satisfied:

1. The basis $B$ is size-reduced, i.e., all off-diagonal coefficients of $M$ satisfy $|\mu_{i,j}| \leq 1/2$
2. The vector $b_1$ realizes the first minimum $\lambda_1(L)$.
3. The projection of the vectors $b_2, \ldots, b_r$ orthogonally to $b_1$ form an HKZ reduced basis.

**Note:** This is realized by calling `sage.modules.free_module_integer.FreeModule_submodule_with_basis_integer.BKZ()` with block_size == self.rank().

**INPUT:**
- *args – passed through to BKZ()
- *kwargs – passed through to BKZ()

OUTPUT:
An integer matrix which is a HKZ-reduced basis for this lattice.

EXAMPLES:

```python
sage: from sage.modules.free_module_integer import IntegerLattice
code: L = sage.crypto.gen_lattice(type='random', n=1, m=40, q=2^60, seed=1337,
˓→lattice=True)
sage: L.HKZ()
40 x 40 dense matrix over Integer Ring (use the '.str()' method to see the␣entries)
sage: L.reduced_basis[0]
(0, 0, -1, -1, 0, 0, -1, 1, 0, 0, 1, 1, -1, 0, 0, 1, 1, 1, -1, 0, 0, 1, -1, 0,␣0, -1, 0, 0, 1, 0, 0, -1, 0, 0, 0, 1, 1, 0, 0, -2)
```

LLL(*args, **kwargs)

Return an LLL reduced basis for self.

A lattice basis \( (b_1, b_2, \ldots, b_d) \) is \((\delta, \eta)\)-LLL-reduced if the two following conditions hold:

- For any \( i > j \), we have \( |\mu_{i,j}| \leq \eta \).
- For any \( i < d \), we have \( \delta |b_i^*|^2 \leq |b_{i+1}^* + \mu_{i+1,i} b_i^*|^2 \),

where \( \mu_{i,j} = \langle b_i, b_j^* \rangle / \langle b_j^*, b_j^* \rangle \) and \( b_i^* \) is the \( i \)-th vector of the Gram-Schmidt orthogonalisation of \( (b_1, b_2, \ldots, b_d) \).

The default reduction parameters are \( \delta = 3/4 \) and \( \eta = 0.501 \).

The parameters \( \delta \) and \( \eta \) must satisfy: \( 0.25 < \delta \leq 1.0 \) and \( 0.5 \leq \eta < \sqrt{\delta} \). Polynomial time complexity is only guaranteed for \( \delta < 1 \).

INPUT:

- *args – passed through to sage.matrix.matrix_integer_dense.Matrix_integer_dense.LLL()
- **kwargs – passed through to sage.matrix.matrix_integer_dense.Matrix_integer_dense.LLL()

OUTPUT:
An integer matrix which is an LLL-reduced basis for this lattice.

EXAMPLES:

```python
sage: from sage.modules.free_module_integer import IntegerLattice
code: A = random_matrix(ZZ, 10, 10, x=-2000, y=2000)
sage: while A.rank() < 10:
˓→A = random_matrix(ZZ, 10, 10)
sage: L = IntegerLattice(A, lll_reduce=False); L
Free module of degree 10 and rank 10 over Integer Ring
User basis matrix:
...
sage: L.reduced_basis == A
```
True

```
sage: old = L.reduced_basis[0].norm().n()
sage: _ = L.LLL()
sage: new = L.reduced_basis[0].norm().n()
sage: new <= old
```
True

**closest_vector(t)**
Compute the closest vector in the embedded lattice to a given vector.

**INPUT:**

• t – the target vector to compute the closest vector to

**OUTPUT:**

The vector in the lattice closest to t.

**EXAMPLES:**

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: L = IntegerLattice([[1, 0], [0, 1]])
sage: L.closest_vector((-6, 5/3))
```

(-6, 2)

**ALGORITHM:**
Uses the algorithm from [MV2010].

**discriminant()**
Return $|\det(G)|$, i.e. the absolute value of the determinant of the Gram matrix $B \cdot B^T$ for any basis $B$.

**OUTPUT:**
An integer.

**EXAMPLES:**

```
sage: L = sage.crypto.gen_lattice(m=10, seed=1337, lattice=True)
sage: L.discriminant()
```

214358881

**is_unimodular()**
Return True if this lattice is unimodular.

**OUTPUT:**
A boolean.

**EXAMPLES:**

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: L = IntegerLattice([[1, 0], [0, 1]])
sage: L.is_unimodular()
```

True

```
sage: IntegerLattice([[2, 0], [0, 3]]).is_unimodular()
```

False
**reduced_basis**

This attribute caches the currently best known reduced basis for `self`, where “best” is defined by the Euclidean norm of the first row vector.

**EXAMPLES:**

```python
sage: from sage.modules.free_module_integer import IntegerLattice
sage: M = random_matrix(ZZ, 10, 10)
sage: while M.rank() < 10:
    ....:     M = random_matrix(ZZ, 10, 10)
sage: L = IntegerLattice(M, lll_reduce=False)
sage: L.reduced_basis == M
True

sage: LLL = L.LLL()
sage: LLL == L.reduced_basis or bool(LLL[0].norm() >= M[0].norm())
True
```

**shortest_vector(update_reduced_basis=True, algorithm="fplll", *args, **kwds)**

Return a shortest vector.

**INPUT:**

- `update_reduced_basis` – (default: True) set this flag if the found vector should be used to improve the basis
- `algorithm` – (default: "fplll") either "fplll" or "pari"
- `*args` – passed through to underlying implementation
- `**kwds` – passed through to underlying implementation

**OUTPUT:**

A shortest non-zero vector for this lattice.

**EXAMPLES:**

```python
sage: from sage.modules.free_module_integer import IntegerLattice
sage: A = sage.crypto.gen_lattice(type='random', n=1, m=30, q=2^40, seed=42)
sage: L = IntegerLattice(A, lll_reduce=False)
sage: min(v.norm().n() for v in L.reduced_basis)
6.03890756700000e10

sage: L.shortest_vector().norm().n()
3.74165738677394

sage: L = IntegerLattice(A, lll_reduce=False)
sage: min(v.norm().n() for v in L.reduced_basis)
6.03890756700000e10

sage: L.shortest_vector(algorithm="pari").norm().n()
3.74165738677394

sage: L = IntegerLattice(A, lll_reduce=True)
sage: L.shortest_vector(algorithm="pari").norm().n()
3.74165738677394
```
**update_reduced_basis**(*w*)

Inject the vector *w* and run LLL to update the basis.

**INPUT:**
- *w* – a vector

**OUTPUT:**
Nothing is returned but the internal state is modified.

**EXAMPLES:**
```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: A = sage.crypto.gen_lattice(type='random', n=1, m=30, q=2^40, seed=42)
sage: L = IntegerLattice(A)
sage: B = L.reduced_basis
sage: v = L.shortest_vector(update_reduced_basis=False)
sage: L.update_reduced_basis(v)
sage: bool(L.reduced_basis[0].norm() < B[0].norm())
True
```

**volume**()

Return \(\text{vol}(L)\) which is \(\sqrt{\det(B \cdot B^T)}\) for any basis \(B\).

**OUTPUT:**
An integer.

**EXAMPLES:**
```
sage: L = sage.crypto.gen_lattice(m=10, seed=1337, lattice=True)
sage: L.volume()
14641
```

**voronoi_cell**(*radius=None*)

Compute the Voronoi cell of a lattice, returning a Polyhedron.

**INPUT:**
- *radius* – (default: automatic determination) radius of ball containing considered vertices

**OUTPUT:**
The Voronoi cell as a Polyhedron instance.
The result is cached so that subsequent calls to this function return instantly.

**EXAMPLES:**
```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: L = IntegerLattice([[1, 0], [0, 1]])
sage: V = L.voronoi_cell()
sage: V.Vrepresentation()
(A vertex at (1/2, -1/2), A vertex at (1/2, 1/2), A vertex at (-1/2, 1/2), A vertex at (-1/2, -1/2))
```

The volume of the Voronoi cell is the square root of the discriminant of the lattice:
```python
sage: L = IntegerLattice(Matrix(ZZ, 4, 4, [[0,0,1,-1],[1,-1,2,1],[-6,0,3,3],[-6,-24,-6,-5]])); L
Free module of degree 4 and rank 4 over Integer Ring
User basis matrix:
[ 0 0 1 -1]
[ 1 -1 2 1]
[ -6 0 3 3]
[ -6 -24 -6 -5]
sage: V = L.voronoi_cell() # long time
sage: V.volume() # long time
678
sage: sqrt(L.discriminant())
678
```

Lattices not having full dimension are handled as well:

```python
sage: L = IntegerLattice([[2, 0, 0], [0, 2, 0]])
sage: V = L.voronoi_cell()
sage: V.Hrepresentation()
(An inequality (-1, 0, 0) x + 1 >= 0, An inequality (0, -1, 0) x + 1 >= 0, An inequality (1, 0, 0) x + 1 >= 0, An inequality (0, 1, 0) x + 1 >= 0)
```

**ALGORITHM:**

Uses parts of the algorithm from [VB1996].

**voronoi_relevant_vectors()**

Compute the embedded vectors inducing the Voronoi cell.

**OUTPUT:**

The list of Voronoi relevant vectors.

**EXAMPLES:**

```python
sage: from sage.modules.free_module_integer import IntegerLattice
sage: L = IntegerLattice([[3, 0], [4, 0]])
sage: L.voronoi_relevant_vectors()
[(-1, 0), (1, 0)]
```

**sage.modules.free_module_integer.IntegerLattice**(basis, lll_reduce=True)

Construct a new integer lattice from basis.

**INPUT:**

- basis – can be one of the following:
  - a list of vectors
  - a matrix over the integers
  - an element of an absolute order

- lll_reduce – (default: True) run LLL reduction on the basis on construction.

**EXAMPLES:**

We construct a lattice from a list of rows:
Sage includes a generator for hard lattices from cryptography:

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: A = sage.crypto.gen_lattice(type='modular', m=10, seed=1337, dual=True)
sage: IntegerLattice(A)
Free module of degree 10 and rank 10 over Integer Ring
User basis matrix:
[-1 1 2 -2 0 1 0 -1 2 1]
[ 1 0 0 -1 -2 1 -2 3 -1 0]
[ 1 2 0 2 -1 1 -2 2 2 0]
[ 1 0 -1 0 2 3 0 0 -1 -2]
[ 1 -3 0 0 2 1 -2 -1 0 0]
[-3 0 -1 0 -1 2 -2 0 0 2]
[ 0 0 0 1 0 2 -3 -3 -2 -1]
[ 0 -1 -4 -1 -1 1 2 -1 0 1]
[ 1 1 -2 1 1 2 1 1 -2 3]
[ 2 -1 1 2 -3 2 2 1 0 1]
```

You can also construct the lattice directly:

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: from sage.crypto.gen_lattice(type='modular', m=10, seed=1337, dual=True, lattice=True)
Free module of degree 10 and rank 10 over Integer Ring
User basis matrix:
[-1 1 2 -2 0 1 0 -1 2 1]
[ 1 0 0 -1 -2 1 -2 3 -1 0]
[ 1 2 0 2 -1 1 -2 2 2 0]
[ 1 0 -1 0 2 3 0 0 -1 -2]
[ 1 -3 0 0 2 1 -2 -1 0 0]
[-3 0 -1 0 -1 2 -2 0 0 2]
[ 0 0 0 1 0 2 -3 -3 -2 -1]
[ 0 -1 -4 -1 -1 1 2 -1 0 1]
[ 1 1 -2 1 1 2 1 1 -2 3]
[ 2 -1 1 2 -3 2 2 1 0 1]
```

We construct an ideal lattice from an element of an absolute order:

```
sage: K.<a> = CyclotomicField(17)
sage: O = K.ring_of_integers()
sage: f = O(-a^15 + a^13 + 4*a^12 - 12*a^11 - 256*a^10 + a^9 - a^7 - 4*a^6 + a^5 +
-210*a^4 + 2*a^3 - 2*a^2 + 2*a - 2)
sage: from sage.modules.free_module_integer import IntegerLattice
sage: IntegerLattice(f)
Free module of degree 16 and rank 16 over Integer Ring
```

(continues on next page)
We construct $Z^n$:

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: IntegerLattice(ZZ^10)
Free module of degree 10 and rank 10 over Integer Ring
User basis matrix:

[1 0 0 0 0 0 0 0 0 0]
[0 1 0 0 0 0 0 0 0 0]
[0 0 1 0 0 0 0 0 0 0]
[0 0 0 1 0 0 0 0 0 0]
[0 0 0 0 1 0 0 0 0 0]
[0 0 0 0 0 1 0 0 0 0]
[0 0 0 0 0 0 1 0 0 0]
[0 0 0 0 0 0 0 1 0 0]
[0 0 0 0 0 0 0 0 1 0]
[0 0 0 0 0 0 0 0 0 1]
```

Sage also interfaces with fpylll’s lattice generator:

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: from fpylll import IntegerMatrix
sage: A = IntegerMatrix.random(8, "simdioph", bits=20, bits2=10)
sage: A = A.to_matrix(matrix(ZZ, 8, 8))
sage: IntegerLattice(A, lll_reduce=False)
Free module of degree 8 and rank 8 over Integer Ring
User basis matrix:

[ 1024  82956  161099  11567  521155  769480  639201  689979]
[    0  1048576    0    0    0    0    0    0]
[    0    0  1048576    0    0    0    0    0]
[    0    0    0  1048576    0    0    0    0]
[    0    0    0    0  1048576    0    0    0]
[    0    0    0    0    0  1048576    0    0]
[    0    0    0    0    0    0  1048576    0]
[    0    0    0    0    0    0    0  1048576]
```

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CHAPTER FIVE

ELEMENTS OF FREE MODULES

AUTHORS:

• William Stein

• Josh Kantor

• Thomas Feulner (2012-11): Added `FreeModuleElement.hamming_weight()` and `FreeModuleElement_generic_sparse.hamming_weight()`

• Jeroen Demeyer (2015-02-24): Implement fast Cython methods `get_unsafe` and `set_unsafe` similar to other places in Sage (trac ticket #17562)

EXAMPLES: We create a vector space over \( \mathbb{Q} \) and a subspace of this space.

```python
sage: V = QQ^5
sage: W = V.span([V.1, V.2])
```

Arithmetic operations always return something in the ambient space, since there is a canonical map from \( W \) to \( V \) but not from \( V \) to \( W \).

```python
sage: parent(W.0 + V.1)
Vector space of dimension 5 over Rational Field
sage: parent(V.1 + W.0)
Vector space of dimension 5 over Rational Field
sage: W.0 + V.1
(0, 2, 0, 0, 0)
sage: W.0 - V.0
(-1, 1, 0, 0, 0)
```

Next we define modules over \( \mathbb{Z} \) and a finite field.

```python
sage: K = ZZ^5
sage: M = GF(7)^5
```

Arithmetic between the \( \mathbb{Q} \) and \( \mathbb{Z} \) modules is defined, and the result is always over \( \mathbb{Q} \), since there is a canonical coercion map to \( \mathbb{Q} \).

```python
sage: K.0 + V.1
(1, 1, 0, 0, 0)
sage: parent(K.0 + V.1)
Vector space of dimension 5 over Rational Field
```

Since there is no canonical coercion map to the finite field from \( \mathbb{Q} \) the following arithmetic is not defined:
However, there is a map from \( \mathbb{Z} \) to the finite field, so the following is defined, and the result is in the finite field.

```
sage: w = K.0 + M.0; w
(2, 0, 0, 0, 0)
sage: parent(w)
Vector space of dimension 5 over Finite Field of size 7
sage: parent(M.0 + K.0)
Vector space of dimension 5 over Finite Field of size 7
```

Matrix vector multiply:

```
sage: MS = MatrixSpace(QQ,3)
sage: A = MS([[0,1,0],[1,0,0],[0,0,1]])
sage: V = QQ^3
sage: v = V([1,2,3])
sage: v * A
(2, 1, 3)
```

```
class sage.modules.free_module_element.FreeModuleElement
    Bases: sage.structure.element.Vector

    An element of a generic free module.

    \text{Mod}(p)
    EXAMPLES:

    sage: V = vector(ZZ, [5, 9, 13, 15])
sage: V.Mod(7)
(5, 2, 6, 1)
sage: parent(V.Mod(7))
Vector space of dimension 4 over Ring of integers modulo 7

    \text{additive order()}
    Return the additive order of self.

    EXAMPLES:

    sage: v = vector(Integers(4), [1,2])
sage: v.additive_order()
4

    sage: v = vector([1,2,3])
sage: v = v.additive_order()
+Infinity

    sage: v = vector(Integers(30), [6, 15]); v
(6, 15)
sage: v.additive_order()
```

(continues on next page)
apply_map($phi, R=None, sparse=None$)

Apply the given map $phi$ (an arbitrary Python function or callable object) to this free module element. If $R$ is not given, automatically determine the base ring of the resulting element.

**INPUT:**

- `sparse` – True or False will control whether the result is sparse. By default, the result is sparse iff self is sparse.
- `$phi$` - arbitrary Python function or callable object
- `$R$` - (optional) ring

**OUTPUT:** a free module element over $R$

**EXAMPLES:**

```python
sage: m = vector([1,x,sin(x+1)])
sage: m.apply_map(lambda x: x^2)
(1, x^2, sin(x + 1)^2)
sage: m.apply_map(sin)
(sin(1), sin(x), sin(sin(x + 1)))
```

```python
sage: m = vector(ZZ, 9, range(9))
sage: k.<a> = GF(9)
sage: m.apply_map(k)
(0, 1, 2, 0, 1, 2, 0, 1, 2)
```

In this example, we explicitly specify the codomain.

```python
sage: s = GF(3)
sage: f = lambda x: s(x)
sage: n = m.apply_map(f, k); n
(0, 1, 2, 0, 1, 2, 0, 1, 2)
sage: n.parent()
Vector space of dimension 9 over Finite Field in a of size 3^2
```

If your map sends 0 to a non-zero value, then your resulting vector is not mathematically sparse:

```python
sage: v = vector([0] * 6 + [1], sparse=True); v
(0, 0, 0, 0, 0, 0, 1)
sage: v2 = v.apply_map(lambda x: x+1); v2
(1, 1, 1, 1, 1, 1, 2)
```

but it’s still represented with a sparse data type:

```python
sage: parent(v2)
Ambient sparse free module of rank 7 over the principal ideal domain Integer Ring
```

This data type is inefficient for dense vectors, so you may want to specify `sparse=False`:
sage: v2 = v.apply_map(lambda x: x+1, sparse=False); v2
(1, 1, 1, 1, 1, 1, 2)
sage: parent(v2)
Ambient free module of rank 7 over the principal ideal domain Integer Ring

Or if you have a map that will result in mostly zeroes, you may want to specify sparse=True:

sage: v = vector(srange(10))
sage: v2 = v.apply_map(lambda x: 0 if x else 1, sparse=True); v2
(1, 0, 0, 0, 0, 0, 0, 0, 0, 0)
sage: parent(v2)
Ambient sparse free module of rank 10 over the principal ideal domain Integer Ring

change_ring(R)
Change the base ring of this vector.

EXAMPLES:

sage: v = vector(QQ['x,y'], [1..5]); v.change_ring(GF(3))
(1, 2, 0, 1, 2)

column()
Return a matrix with a single column and the same entries as the vector self.

OUTPUT:

A matrix over the same ring as the vector (or free module element), with a single column. The entries of the column are identical to those of the vector, and in the same order.

EXAMPLES:

sage: v = vector(ZZ, [1,2,3])
sage: w = v.column(); w
[1]
[2]
[3]
sage: w.parent()
Full MatrixSpace of 3 by 1 dense matrices over Integer Ring

sage: x = vector(FiniteField(13), [2,4,8,16])
sage: x.column()
[2]
[4]
[8]
[3]

There is more than one way to get one-column matrix from a vector. The column method is about equally efficient to making a row and then taking a transpose. Notice that supplying a vector to the matrix constructor demonstrates Sage's preference for rows.

sage: x = vector(RDF, [sin(i*pi/20) for i in range(10)])
sage: x.column() == matrix(x).transpose()
True
sage: x.column() == x.row().transpose()
True
Sparse or dense implementations are preserved.

```python
sage: d = vector(RR, [1.0, 2.0, 3.0])
sage: s = vector(CDF, {2:5.0+6.0*I})
sage: dm = d.column()
sage: sm = s.column()
sage: all([d.is_dense(), dm.is_dense(), s.is_sparse(), sm.is_sparse()])
True
```

### conjugate()

Returns a vector where every entry has been replaced by its complex conjugate.

**OUTPUT:**

A vector of the same length, over the same ring, but with each entry replaced by the complex conjugate, as implemented by the `conjugate()` method for elements of the base ring, which is presently always complex conjugation.

**EXAMPLES:**

```python
sage: v = vector(CDF, [2.3 - 5.4*I, -1.7 + 3.6*I])
sage: w = v.conjugate(); w
(2.3 + 5.4*I, -1.7 - 3.6*I)
sage: w.parent()
Vector space of dimension 2 over Complex Double Field
```

Even if conjugation seems nonsensical over a certain ring, this method for vectors cooperates silently.

```python
sage: u = vector(ZZ, range(6))
sage: u.conjugate()
(0, 1, 2, 3, 4, 5)
```

Sage implements a few specialized subfields of the complex numbers, such as the cyclotomic fields. This example uses such a field containing a primitive 7-th root of unity named \( a \).

```python
sage: F.<a> = CyclotomicField(7)
sage: v = vector(F, [a^i for i in range(7)])
sage: v
(1, a, a^2, a^3, a^4, a^5, -a^5 - a^4 - a^3 - a^2 - a - 1)
sage: v.conjugate()
(1, -a^5 - a^4 - a^3 - a^2 - a - 1, a^5, a^4, a^3, a^2, a)
```

Sparse vectors are returned as such.

```python
sage: v = vector(CC, {1: 5 - 6*I, 3: -7*I}); v
(0.000000000000000, 5.00000000000000 - 6.00000000000000*I, 0.000000000000000, -7.00000000000000*I)
sage: v.is_sparse()
True
sage: vc = v.conjugate(); vc
(0.000000000000000, 5.00000000000000 + 6.00000000000000*I, 0.000000000000000, 7.00000000000000*I)
sage: vc.conjugate()
(0.000000000000000, 5.00000000000000 - 6.00000000000000*I, 0.000000000000000, -7.00000000000000*I)
```
**coordinate_ring()**

Return the ring from which the coefficients of this vector come.

This is different from `base_ring()`, which returns the ring of scalars.

**EXAMPLES:**

```python
sage: M = (ZZ^2) * (1/2)
sage: v = M([0,1/2])
sage: v.base_ring()
Integer Ring
sage: v.coordinate_ring()
Rational Field
```

**cross_product(right)**

Return the cross product of self and right, which is only defined for vectors of length 3 or 7.

**INPUT:**

- *right* - A vector of the same size as `self`, either degree three or degree seven.

**OUTPUT:**

The cross product (vector product) of `self` and `right`, a vector of the same size of `self` and `right`.

This product is performed under the assumption that the basis vectors are orthonormal. See the method `cross_product()` of vector fields for more general cases.

**EXAMPLES:**

```python
sage: v = vector([1,2,3]); w = vector([0,5,-9])
sage: v.cross_product(v)
(0, 0, 0)
sage: u = v.cross_product(w); u
(-33, 9, 5)
sage: u.dot_product(v)
0
sage: u.dot_product(w)
0
```

The cross product is defined for degree seven vectors as well: see Wikipedia article Cross_product. The 3-D cross product is achieved using the quaternions, whereas the 7-D cross product is achieved using the octonions.

```python
sage: u = vector(QQ, [1, -1/3, 57, -9, 56/4, -4,1])
sage: v = vector(QQ, [37, 55, -99/57, 9, -12, 11/3, 4/98])
sage: u.cross_product(v)
(1394815/2793, -2808401/2793, 39492/49, -48737/399, -9151880/2793, 62513/2793, -
˓→326603/171)
```

The degree seven cross product is anticommutative.

```python
sage: u.cross_product(v) + v.cross_product(u)
(0, 0, 0, 0, 0, 0, 0)
```

The degree seven cross product is distributive across addition.
The degree seven cross product respects scalar multiplication.

```sage
sage: v = vector([2, 17, -11/5, 21, -6, 2/17, 16])
sage: u = vector([-8, 9, -21, -6, -5/3, 12, 99])
sage: (5*v).cross_product(u) - 5*(v.cross_product(u))
(0, 0, 0, 0, 0, 0, 0)
sage: v.cross_product(5*u) - 5*(v.cross_product(u))
(0, 0, 0, 0, 0, 0, 0)
sage: (5*v).cross_product(u) - (v.cross_product(5*u))
(0, 0, 0, 0, 0, 0, 0)
```

The degree seven cross product respects the scalar triple product.

```sage
sage: v = vector([2, 6, -7/4, -9/12, -7, 12, 9])
sage: u = vector([22, -7, -9/11, 12, 15, 15/7, 11])
sage: w = vector([-11, 17, 19, -12/5, 44, 21/56, -8])
sage: v.dot_product(u.cross_product(w)) - w.dot_product(v.cross_product(u))
0
```

AUTHOR:
Billy Wonderly (2010-05-11), Added 7-D Cross Product

cross_product_matrix()
Return the matrix which describes a cross product between self and some other vector.

This operation is sometimes written using the hat operator: see Wikipedia article Hat_operator#Cross_product. It is only defined for vectors of length 3 or 7. For a vector \( \mathbf{v} \) the cross product matrix \( \mathbf{\hat{v}} \) is a matrix which satisfies \( \mathbf{\hat{v}} \cdot \mathbf{w} = \mathbf{v} \times \mathbf{w} \) and also \( \mathbf{w} \cdot \mathbf{\hat{v}} = \mathbf{w} \times \mathbf{v} \) for all vectors \( \mathbf{w} \). The basis vectors are assumed to be orthonormal.

OUTPUT:
The cross product matrix of this vector.

EXAMPLES:

```sage
sage: v = vector([1, 2, 3])
sage: vh = v.cross_product_matrix()
sage: vh
[ 0 -3  2]
[ 3  0 -1]
[-2  1  0]
sage: w = random_vector(3, x=1, y=100)
sage: vh*w == v.cross_product(w)
True
sage: w*vh == w.cross_product(v)
True
sage: vh.is_alternating()
True
```
curl\(\text{(variables=None)}\)

Return the curl of this two-dimensional or three-dimensional vector function.

EXAMPLES:

```python
sage: R.<x,y,z> = QQ[]
sage: vector([-y, x, 0]).curl()
(0, 0, 2)
sage: vector([y, -x, x*y*z]).curl()
(x*z, -y*z, -2)
sage: vector([y^2, 0, 0]).curl()
(0, 0, -2*y)
sage: (R^3).random_element().curl().div()
0
```

For rings where the variable order is not well defined, it must be defined explicitly:

```python
sage: v = vector(SR, [-y, x, 0])
sage: v.curl()
Traceback (most recent call last):
  ... ValueError: Unable to determine ordered variable names for Symbolic Ring
sage: v.curl([x, y, z])
(0, 0, 2)
```

Note that callable vectors have well defined variable orderings:

```python
sage: v(x, y, z) = (-y, x, 0)
sage: v.curl()
(x, y, z) |--> (0, 0, 2)
```

In two-dimensions, this returns a scalar value:

```python
sage: R.<x,y> = QQ[]
sage: vector([-y, x]).curl()
2
```

See also:

curl() of vector fields on Euclidean spaces (and more generally pseudo-Riemannian manifolds), in particular for computing the curl in curvilinear coordinates.

degree()

Return the degree of this vector, which is simply the number of entries.

EXAMPLES:

```python
sage: sage.modules.free_module_element.FreeModuleElement(QQ^389).degree()
389
sage: vector([1/2, 3, 8]).degree()
3
```

denominator()

Return the least common multiple of the denominators of the entries of self.

EXAMPLES:
sage: v = vector([1/2,2/5,3/14])
sage: v.denominator()
70
sage: 2*5*7
70

sage: M = (ZZ^2)*(1/2)
sage: M.basis()[0].denominator()
2

dense_vector()
Return dense version of self. If self is dense, just return self; otherwise, create and return correspond dense vector.

EXAMPLES:

sage: vector([-1,0,3,0,0,0]).dense_vector().is_dense()
True
sage: vector([-1,0,3,0,0,0],sparse=True).dense_vector().is_dense()
True
sage: vector([-1,0,3,0,0,0],sparse=True).dense_vector()
(-1, 0, 3, 0, 0, 0)

derivative(*args)
Derivative with respect to variables supplied in args.

Multiple variables and iteration counts may be supplied; see documentation for the global derivative() function for more details.

diff() is an alias of this function.

EXAMPLES:

sage: v = vector([1,x,x^2])
sage: v.derivative(x)
(0, 1, 2*x)
sage: type(v.derivative(x)) == type(v)
True
sage: v = vector([1,x,x^2], sparse=True)
sage: v.derivative(x)
(0, 1, 2*x)
sage: type(v.derivative(x)) == type(v)
True
sage: v.derivative(x,x)
(0, 0, 2)

dict(copy=True)
Return dictionary of nonzero entries of self.

More precisely, this returns a dictionary whose keys are indices of basis elements in the support of self and whose values are the corresponding coefficients.

INPUT:

• copy – (default: True) if self is internally represented by a dictionary d, then make a copy of d; if False, then this can cause undesired behavior by mutating d

OUTPUT:
• Python dictionary

EXAMPLES:

```
sage: v = vector([0,0,0,1/2,0,3/14])
sage: v.dict()
{4: 1/2, 6: 3/14}
sage: sorted(v.support())
[4, 6]
```

In some cases, when `copy=False`, we get back a dangerous reference:

```
sage: v = vector({0:5, 2:3/7}, sparse=True)
sage: v.dict(copy=False)
{0: 5, 2: 3/7}
sage: v.dict(copy=False)[0] = 18
sage: v
(18, 0, 3/7)
```

diff(*args)
Derivative with respect to variables supplied in args.

Multiple variables and iteration counts may be supplied; see documentation for the global derivative() function for more details.

diff() is an alias of this function.

EXAMPLES:

```
sage: v = vector([1,x,x^2])
sage: v.derivative(x)
(0, 1, 2*x)
sage: type(v.derivative(x)) == type(v)
True
sage: v = vector([1,x,x^2], sparse=True)
sage: v.derivative(x)
(0, 1, 2*x)
sage: type(v.derivative(x)) == type(v)
True
sage: v.derivative(x,x)
(0, 0, 2)
```

div(*variables=None)
Return the divergence of this vector function.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: vector([x, y, z]).div()
3
sage: vector([x*y, y*z, z*x]).div()
x + y + z
sage: R.<x,y,z,w> = QQ[]
sage: vector([x*y, y*z, z*x]).div([x, y, z])
x + y + z
```

(continues on next page)
See also:

divergence() of vector fields on Euclidean spaces (and more generally pseudo-Riemannian manifolds),
in particular for computing the divergence in curvilinear coordinates.

dot_product(right)
Return the dot product of self and right, which is the sum of the product of the corresponding entries.

INPUT:

- right – a vector of the same degree as self. It does not need to belong to the same parent as self,
  so long as the necessary products and sums are defined.

OUTPUT:

If self and right are the vectors \( \vec{x} \) and \( \vec{y} \), of degree \( n \), then this method returns

\[
\sum_{i=1}^{n} x_i y_i
\]

Note: The inner_product() is a more general version of this method, and the
hermitian_inner_product() method may be more appropriate if your vectors have complex
entries.

EXAMPLES:

```
sage: V = FreeModule(ZZ, 3)
sage: v = V([1,2,3])
sage: w = V([4,5,6])
sage: v.dot_product(w)
32
```

```
sage: R.<x> = QQ[]
sage: v = vector([x,x^2,3*x]); w = vector([2*x,x,3+x])
sage: v*w
x^3 + 5*x^2 + 9*x
sage: (x^2*x) + (x^2*x) + (3*x^2(3+x))
x^3 + 5*x^2 + 9*x
sage: w*v
x^3 + 5*x^2 + 9*x
```

The vectors may be from different vector spaces, provided the necessary operations make sense. Notice
that coercion will generate a result of the same type, even if the order of the arguments is reversed.
The dot product of a vector with itself is the 2-norm, squared.

```
sage: v = vector(QQ, [3, 4, 7])
sage: v.dot_product(v) - v.norm()^2
0
```

**element()**

Simply returns self. This is useful, since for many objects, self.element() returns a vector corresponding to self.

**EXAMPLES:**

```
sage: v = vector([1/2, 2/5, 0]); v
(1/2, 2/5, 0)
sage: v.element()
(1/2, 2/5, 0)
```

**get(i)**

Like __getitem__ but without bounds checking: $i$ must satisfy $0 \leq i < \text{self}$.degree.

**EXAMPLES:**

```
sage: vector(SR, [1/2,2/5,0]).get(0)
1/2
```

**hamming_weight()**

Return the number of positions $i$ such that self[i] != 0.

**EXAMPLES:**

```
sage: vector([-1,0,3,0,0,0,0.01]).hamming_weight()
3
```

**hermitian_inner_product(right)**

Returns the dot product, but with the entries of the first vector conjugated beforehand.

**INPUT:**

- right - a vector of the same degree as self

**OUTPUT:**

If self and right are the vectors $\vec{x}$ and $\vec{y}$ of degree $n$ then this routine computes

$$\sum_{i=1}^{n} x_i y_i$$
where the bar indicates complex conjugation.

**Note:** If your vectors do not contain complex entries, then `dot_product()` will return the same result without the overhead of conjugating elements of `self`.

If you are not computing a weighted inner product, and your vectors do not have complex entries, then the `dot_product()` will return the same result.

**EXAMPLES:**

```python
sage: v = vector(CDF, [2+3*I, 5-4*I])
sage: w = vector(CDF, [6-4*I, 2+3*I])
sage: v.hermitian_inner_product(w)  # -2.0 - 3.0*I
```

Sage implements a few specialized fields over the complex numbers, such as cyclotomic fields and quadratic number fields. So long as the base rings have a conjugate method, then the Hermitian inner product will be available.

```python
sage: Q.<a> = QuadraticField(-7)
sage: a^2  # -7
sage: v = vector(Q, [3+a, 5-2*a])
sage: w = vector(Q, [6, 4+3*a])
sage: v.hermitian_inner_product(w)  # 17*a - 4
```

The Hermitian inner product should be additive in each argument (we only need to test one), linear in each argument (with conjugation on the first scalar), and anti-commutative.

```python
sage: alpha = CDF(5.0 + 3.0*I)
sage: u = vector(CDF, [2+4*I, -3+5*I, 2-7*I])
sage: v = vector(CDF, [-1+3*I, 5+4*I, 9-2*I])
sage: w = vector(CDF, [8+3*I, -4+7*I, 3-6*I])
sage: (u+v).hermitian_inner_product(w) == u.hermitian_inner_product(w) + v.hermitian_inner_product(w)
True
sage: (alpha*u).hermitian_inner_product(w) == alpha.conjugate()*u.hermitian_inner_product(w)
True
sage: u.hermitian_inner_product(alpha*w) == alpha*u.hermitian_inner_product(w)
True
sage: u.hermitian_inner_product(v) == v.hermitian_inner_product(u).conjugate()
True
```

For vectors with complex entries, the Hermitian inner product has a more natural relationship with the 2-norm (which is the default for the `norm()` method). The norm squared equals the Hermitian inner product of the vector with itself.

```python
sage: v = vector(CDF, [-0.66+0.47*I, -0.60+0.91*I, -0.62-0.87*I, 0.53+0.32*I])
sage: abs(v.norm()^2 - v.hermitian_inner_product(v)) < 1.0e-10
True
```
inner_product(right)
Returns the inner product of self and right, possibly using an inner product matrix from the parent of self.

INPUT:
  • right - a vector of the same degree as self

OUTPUT:
If the parent vector space does not have an inner product matrix defined, then this is the usual dot product (dot_product()). If self and right are considered as single column matrices, $\vec{x}$ and $\vec{y}$, and $A$ is the inner product matrix, then this method computes

$$(\vec{x})^t A \vec{y}$$

where $t$ indicates the transpose.

Note: If your vectors have complex entries, the hermitian_inner_product() may be more appropriate for your purposes.

EXAMPLES:

```sage
t = vector(QQ, [1,2,3])
w = vector(QQ, [-1,2,-3])
v.inner_product(w)
-6
v.inner_product(w) == v.dot_product(w)
True
```

The vector space or free module that is the parent to self can have an inner product matrix defined, which will be used by this method. This matrix will be passed through to subspaces.

```sage
ipm = matrix(ZZ,[[2,0,-1], [0,2,0], [-1,0,6]])
M = FreeModule(ZZ, 3, inner_product_matrix = ipm)
v = M([1,0,0])
v.inner_product(v)
2
K = M.span_of_basis([[[2/2,-1/2,-1/2], [0,1/2,-1/2],[2,0,0]]])
(K.0).inner_product(K.0)
2
w = M([1,3,-1])
v = M([2,-4,5])
w.row() * ipm * v.column() == w.inner_product(v)
True
```

Note that the inner product matrix comes from the parent of self. So if a vector is not an element of the correct parent, the result could be a source of confusion.

```sage
V = VectorSpace(QQ, 2, inner_product_matrix=[[1,2],[2,1]])
v = V([12, -10])
w = vector(QQ, [10,12])
v.inner_product(w)
88
w.inner_product(v)
```

(continues on next page)
Note: The use of an inner product matrix makes no restrictions on the nature of the matrix. In particular, in this context it need not be Hermitian and positive-definite (as it is in the example above).

integral(*args, **kwds)
Returns a symbolic integral of the vector, component-wise.

integrate() is an alias of the function.

EXAMPLES:

```python
sage: t=var('t')
sage: r=vector([t,t^2,sin(t)])
sage: r.integral(t)
(1/2*t^2, 1/3*t^3, -cos(t))
sage: integrate(r,t)
(1/2*t^2, 1/3*t^3, -cos(t))
sage: r.integrate(t,0,1)
(1/2, 1/3, -cos(1) + 1)
```

is_dense()
Return True if this is a dense vector, which is just a statement about the data structure, not the number of nonzero entries.

EXAMPLES:

```python
sage: vector([1/2,2/5,0]).is_dense()
True
sage: vector([1/2,2/5,0],sparse=True).is_dense()
False
```

is_sparse()
Return True if this is a sparse vector, which is just a statement about the data structure, not the number of nonzero entries.
EXAMPLES:

```
sage: vector([1/2, 2/5, 0]).is_sparse()
False
sage: vector([1/2, 2/5, 0], sparse=True).is_sparse()
True
```

**is_vector()**

Return True, since this is a vector.

EXAMPLES:

```
sage: vector([1/2, 2/5, 0]).is_vector()
True
```

**items()**

Return an iterator over self.

EXAMPLES:

```
sage: v = vector([1, 2/3, pi])
sage: v.items()
<generator object at ...>
sage: list(v.items())
[(0, 1), (1, 2/3), (2, pi)]
```

**iteritems()**

Return an iterator over self.

EXAMPLES:

```
sage: v = vector([1, 2/3, pi])
sage: v.items()
<generator object at ...>
sage: list(v.items())
[(0, 1), (1, 2/3), (2, pi)]
```

**lift()**

Lift self to the cover ring.

**OUTPUT:**

Return a lift of self to the covering ring of the base ring $R$, which is by definition the ring returned by calling `cover_ring()` on $R$, or just $R$ itself if the `cover_ring()` method is not defined.

**EXAMPLES:**

```
sage: V = vector(Integers(7), [5, 9, 13, 15]) ; V
(5, 2, 6, 1)
sage: V.lift()
(5, 2, 6, 1)
sage: parent(V.lift())
Ambient free module of rank 4 over the principal ideal domain Integer Ring
```

If the base ring does not have a cover method, return a copy of the vector:
sage: W = vector(QQ, [1, 2, 3])
sage: W1 = W.lift()
sage: W is W1
False
sage: parent(W1)
Vector space of dimension 3 over Rational Field

lift_centered()
Lift to a congruent, centered vector.

INPUT:
• self A vector with coefficients in \textit{Integers}(n).

OUTPUT:
• The unique integer vector \(v\) such that for each \(i\), \(\text{Mod}(v[i], n) = \text{Mod}(\text{self}[i], n)\) and \(-n/2 < v[i] \leq n/2\).

EXAMPLES:

```python
sage: V = vector(Integers(7), [5, 9, 13, 15]) ; V
(5, 2, 6, 1)
sage: V.lift_centered()
(-2, 2, -1, 1)
sage: parent(V.lift_centered())
Ambient free module of rank 4 over the principal ideal domain Integer Ring
```

list(copy=True)
Return list of elements of self.

INPUT:
• copy – bool, whether returned list is a copy that is safe to change, is ignored.

EXAMPLES:

```python
sage: P.<x,y,z> = QQ[]
sage: v = vector([x,y,z], sparse=True)
sage: type(v)
<class 'sage.modules.free_module_element.FreeModuleElement_generic_sparse'>
sage: a = v.list(); a
[x, y, z]
sage: a[0] = x*y; v
(x, y, z)
```

The optional argument copy is ignored:

```python
sage: a = v.list(copy=False); a
[x, y, z]
sage: a[0] = x*y; v
(x, y, z)
```

list_from_positions(positions)
Return list of elements chosen from this vector using the given positions of this vector.

INPUT:
• positions – iterable of ints
EXAMPLES:

```python
sage: v = vector([1,2/3,pi])
sage: v.list_from_positions([0,0,2,1])
[1, 1, 1, pi, 2/3]
```

**monic()**

Return this vector divided through by the first nonzero entry of this vector.

**EXAMPLES:**

```python
sage: v = vector(QQ, [0, 4/3, 5, 1, 2])
sage: v.monic()
(0, 1, 15/4, 3/4, 3/2)
sage: v = vector(QQ, [])
sage: v.monic()
()
```

**monomial_coefficients**(copy=True)

Return dictionary of nonzero entries of self.

More precisely, this returns a dictionary whose keys are indices of basis elements in the support of self and whose values are the corresponding coefficients.

**INPUT:**

- copy – (default: True) if self is internally represented by a dictionary d, then make a copy of d; if False, then this can cause undesired behavior by mutating d

**OUTPUT:**

- Python dictionary

**EXAMPLES:**

```python
sage: v = vector([0,0,0,0,1/2,0,3/14])
sage: v.dict()
{4: 1/2, 6: 3/14}
sage: sorted(v.support())
[4, 6]
```

In some cases, when copy=False, we get back a dangerous reference:

```python
sage: v = vector({0:5, 2:3/7}, sparse=True)
sage: v.dict(copy=False)
{0: 5, 2: 3/7}
sage: v.dict(copy=False)[0] = 18
sage: v
(18, 0, 3/7)
```

**nintegral**(args, **kwds)

Returns a numeric integral of the vector, component-wise, and the result of the nintegral command on each component of the input.

**nintegrate()** is an alias of the function.

**EXAMPLES:**
```
sage: t=var('t')
sage: r=vector([t,t^2,sin(t)])
sage: vec,answers=r.nintegral(t,0,1)
sage: vec
(0.5, 0.333333333333334, 0.4596976941318602)
sage: type(vec)
<class 'sage.modules.vector_real_double_dense.Vector_real_double_dense'>
sage: answers
[(0.5, 5.55111512312578...e-15, 21, 0), (0.3333333333333..., 3.70074341541719...→e-15, 21, 0), (0.45969769413186...5.10366964392284...e-15, 21, 0)]
sage: r=vector([t,0,1], sparse=True)
sage: r.nintegral(t,0,1)
((0.5, 0.0, 1.0), {0: (0.5, 5.55111512312578...e-15, 21, 0), 2: (1.0, 1.→11022302462515...e-14, 21, 0)})
```

`nintegrate(*args, **kwds)`

Returns a numeric integral of the vector, component-wise, and the result of the nintegral command on each component of the input.

`nintegrate()` is an alias of the function.

**EXAMPLES:**

```
sage: t=var('t')
sage: r=vector([t,t^2,sin(t)])
sage: vec,answers=r.nintegral(t,0,1)
sage: vec
(0.5, 0.333333333333334, 0.4596976941318602)
sage: type(vec)
<class 'sage.modules.vector_real_double_dense.Vector_real_double_dense'>
sage: answers
[(0.5, 5.55111512312578...e-15, 21, 0), (0.3333333333333..., 3.70074341541719...→e-15, 21, 0), (0.45969769413186...5.10366964392284...e-15, 21, 0)]
sage: r=vector([t,0,1], sparse=True)
sage: r.nintegral(t,0,1)
((0.5, 0.0, 1.0), {0: (0.5, 5.55111512312578...e-15, 21, 0), 2: (1.0, 1.→11022302462515...e-14, 21, 0)})
```

`nonzero_positions()`

Return the sorted list of integers $i$ such that `self[i]` != 0.

**EXAMPLES:**

```
sage: vector([-1,0,3,0,0,0,0.01]).nonzero_positions()
[0, 2, 6]
```

`norm(p='__two__')`

Return the $p$-norm of `self`.

**INPUT:**

- `p` - default: 2 - $p$ can be a real number greater than 1, infinity (oo or Infinity), or a symbolic expression.
  - $p = 1$: the taxicab (Manhattan) norm
- \( p = 2 \): the usual Euclidean norm (the default)
- \( p = \infty \): the maximum entry (in absolute value)

**Note:** See also `sage.misc.functional.norm()`

**EXAMPLES:**

```
sage: v = vector([1,2,-3])
sage: v.norm(5)
276**(1/5)
```

The default is the usual Euclidean norm.

```
sage: v.norm()
sqrt(14)
sage: v.norm(2)
sqrt(14)
```

The infinity norm is the maximum size (in absolute value) of the entries.

```
sage: v.norm(Infinity)
3
sage: v.norm(oo)
3
```

Real or symbolic values may be used for \( p \).

```
sage: v=vector(RDF,[1,2,3])
sage: v.norm(5)
3.077384885394063
sage: v.norm(pi/2)  # abs tol 1e-15
4.216595864704748
```

Notice that the result may be a symbolic expression, owing to the necessity of taking a square root (in the default case). These results can be converted to numerical values if needed.

```
sage: v = vector(ZZ, [3,4])
sage: nrm = v.norm(); nrm
5
sage: nrm.parent()
Rational Field
sage: nrm = v.norm(); nrm
sqrt(34)
sage: nrm.parent()
Symbolic Ring
sage: numeric = N(nrm); numeric
5.83095189484...
```

(continues on next page)
sage: numeric.parent()
Real Field with 53 bits of precision

normalized(p='__two__')
Return the input vector divided by the p-norm.

INPUT:
  • “p” - default: 2 - p value for the norm

EXAMPLES:

sage: v = vector(QQ, [4, 1, 3, 2])
sage: v.normalized()
(2/15*sqrt(30), 1/30*sqrt(30), 1/10*sqrt(30), 1/15*sqrt(30))
sage: sum(v.normalized(1))
1

Note that normalizing the vector may change the base ring:

sage: v.base_ring() == v.normalized().base_ring()
False
sage: u = vector(RDF, [-3, 4, 6, 9])
sage: u.base_ring() == u.normalized().base_ring()
True

numerical_approx(prec=None, digits=None, algorithm=None)
Return a numerical approximation of self with prec bits (or decimal digits) of precision, by approxi-
mating all entries.

INPUT:
  • prec – precision in bits
  • digits – precision in decimal digits (only used if prec is not given)
  • algorithm – which algorithm to use to compute the approximation of the entries (the accepted algo-
rithms depend on the object)

If neither prec nor digits is given, the default precision is 53 bits (roughly 16 digits).

EXAMPLES:

sage: v = vector(RealField(212), [1,2,3])
sage: v.n()
(1.0000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000, 2.0000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000, 3.0000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000)
sage: _.parent()
Vector space of dimension 3 over Real Field with 53 bits of precision
sage: numerical_approx(v)
(1.0000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000, 2.0000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000, 3.0000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000)
sage: _.parent()
Vector space of dimension 3 over Real Field with 53 bits of precision
sage: v.n(prec=75)
(1.0000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000, 2.0000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000, 3.0000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000)
sage: _.parent()
Vector space of dimension 3 over Real Field with 53 bits of precision
sage: numerical_approx(v, digits=3)
Both functional and object-oriented usage is possible.

```python
sage: u = vector(QQ, [1/2, 1/3, 1/4])
sage: u.n()
(0.500000000000000, 0.333333333333333, 0.250000000000000)
sage: u.numerical_approx()
(0.500000000000000, 0.333333333333333, 0.250000000000000)
sage: n(u)
(0.500000000000000000000000000000, 0.333333333333333333333333333333, 0.250000000000000000000000000000)
sage: N(u)
(0.500000000000000000000000000000, 0.333333333333333333333333333333, 0.250000000000000000000000000000)
sage: numerical_approx(u)
(0.500000000000000, 0.333333333333333, 0.250000000000000)
```

Precision (bits) and digits (decimal) may be specified. When both are given, prec wins.

```python
sage: u = vector(QQ, [1/2, 1/3, 1/4])
sage: n(u, prec=15)
(0.5000, 0.3333, 0.2500)
sage: n(u, digits=5)
(0.50000, 0.33333, 0.25000)
sage: n(u, prec=30, digits=100)
(0.500000000000000000000000000000, 0.333333333333333333333333333333, 0.250000000000000000000000000000)
```

These are some legacy doctests that were part of various specialized versions of the numerical approximation routine that were removed as part of trac ticket #12195.

```python
sage: v = vector(ZZ, [1, 2, 3])
sage: v.n()
(1.00000000000000, 2.00000000000000, 3.00000000000000)
sage: _.parent()
Vector space of dimension 3 over Real Field with 53 bits of precision
sage: v.n(prec=75)
(1.0000000000000000000000000000000000000000000000000000000000000, 2.0000000000000000000000000000000000000000000000000000000000000, 3.0000000000000000000000000000000000000000000000000000000000000)
sage: _.parent()
Vector space of dimension 3 over Real Field with 75 bits of precision
sage: v = vector(RDF, [1, 2, 3])
sage: v.n()
(1.00000000000000, 2.00000000000000, 3.00000000000000)
sage: _.parent()
Vector space of dimension 3 over Real Field with 53 bits of precision
sage: v = vector(CDF, [1, 2, 3])
sage: v.n()
(1.00000000000000, 2.00000000000000, 3.00000000000000)
sage: _.parent()
Vector space of dimension 3 over Complex Field with 53 bits of precision
sage: v = vector(Integers(8), [1, 2, 3])
```

sage: v.n()
(1.00000000000000, 2.00000000000000, 3.00000000000000)
sage: _.parent()
Vector space of dimension 3 over Real Field with 53 bits of precision
sage: v.n(prec=75)
(1.0000000000000000000000, 2.0000000000000000000000, 3.0000000000000000000000)
sage: _.parent()
Vector space of dimension 3 over Real Field with 75 bits of precision

sage: v = vector(QQ, [1,2,3])
sage: v.n()
(1.00000000000000, 2.00000000000000, 3.00000000000000)
sage: _.parent()
Vector space of dimension 3 over Real Field with 53 bits of precision
sage: v.n(prec=75)
(1.0000000000000000000000, 2.0000000000000000000000, 3.0000000000000000000000)
sage: _.parent()
Vector space of dimension 3 over Real Field with 75 bits of precision

sage: v = vector(GF(2), [1,2,3])
sage: v.n()
(1.00000000000000, 0.000000000000000, 1.00000000000000)
sage: _.parent()
Vector space of dimension 3 over Real Field with 53 bits of precision
sage: v.n(prec=75)
(1.0000000000000000000000, 0.0000000000000000000000, 1.0000000000000000000000)
sage: _.parent()
Vector space of dimension 3 over Real Field with 75 bits of precision

\texttt{numpy}(\texttt{dtype=}'object\texttt{}')

Convert self to a numpy array.

INPUT:

\begin{itemize}
\item \texttt{dtype} – the \texttt{numpy} \texttt{dtype} of the returned array
\end{itemize}

EXAMPLES:

sage: v = vector([1,2,3])
sage: v.numpy()
array([1, 2, 3], dtype=object)
sage: v.numpy() * v.numpy()
array([1, 4, 9], dtype=object)
sage: vector(QQ, [1, 2, 5/6]).numpy()
array([1, 2, 5/6], dtype=object)

By default the \texttt{object} \texttt{dtype} is used. Alternatively, the desired \texttt{dtype} can be passed in as a parameter:

sage: v = vector(QQ, [1, 2, 5/6])
sage: v.numpy()
array([1, 2, 5/6], dtype=object)
sage: v.numpy(dtype=float)
array([1., 2., 0.83333333])
sage: v.numpy(dtype=int)
array([1, 2, 0])
sage: import numpy
sage: v.numpy(dtype=numpy.uint8)
array([1, 2, 0], dtype=uint8)

Passing a dtype of None will let numpy choose a native type, which can be more efficient but may have unintended consequences:

sage: v.numpy(dtype=None)
array([1., 2., 0.83333333])

sage: w = vector(ZZ, [0, 1, 2^63 -1]); w
(0, 1, 9223372036854775807)
sage: wn = w.numpy(dtype=None); wn
array([ 0, 1, 9223372036854775807,...])
sage: wn.dtype
dtype('int64')
sage: w.dot_product(w)
85070591730234615847396907784232501250
sage: wn.dot(wn)  # overflow
2

Numpy can give rather obscure errors; we wrap these to give a bit of context:

sage: vector([1, 1/2, QQ['x'].0]).numpy(dtype=float)
Traceback (most recent call last):
...
ValueError: Could not convert vector over Univariate Polynomial Ring in x over Rational Field to numpy array of type <... 'float'>: setting an array element with a sequence.

outer_product(right)
Returns a matrix, the outer product of two vectors self and right.

INPUT:

- right - a vector (or free module element) of any size, whose elements are compatible (with regard to multiplication) with the elements of self.

OUTPUT:

The outer product of two vectors $x$ and $y$ (respectively self and right) can be described several ways. If we interpret $x$ as a $m \times 1$ matrix and interpret $y$ as a $1 \times n$ matrix, then the outer product is the $m \times n$ matrix from the usual matrix product $xy$. Notice how this is the “opposite” in some ways from an inner product (which would require $m = n$).

If we just consider vectors, use each entry of $x$ to create a scalar multiples of the vector $y$ and use these vectors as the rows of a matrix. Or use each entry of $y$ to create a scalar multiples of $x$ and use these vectors as the columns of a matrix.

EXAMPLES:

sage: u = vector(QQ, [1/2, 1/3, 1/4, 1/5])
sage: v = vector(ZZ, [60, 100, 600])
The more general `sage.matrix.matrix2.tensor_product()` is an operation on a pair of matrices. If we construct a pair of vectors as a column vector and a row vector, then an outer product and a tensor product are identical. Thus `tensor_product` is a synonym for this method.

```python
sage: u = vector(QQ, [1/2, 1/3, 1/4, 1/5])
sage: v = vector(ZZ, [60, 180, 600])
sage: u.tensor_product(v) == (u.column()).tensor_product(v.row())
True
```

The result is always a dense matrix, no matter if the two vectors are, or are not, dense.

```python
sage: d = vector(ZZ, [4, 5], sparse=False)
sage: s = vector(ZZ, [1, 2, 3], sparse=True)
sage: dd = d.outer_product(d)
sage: ds = d.outer_product(s)
sage: sd = s.outer_product(d)
sage: ss = s.outer_product(s)
sage: all([dd.is_dense(), ds.is_dense(), sd.is_dense(), dd.is_dense()])
True
```

Vectors with no entries do the right thing.

```python
sage: v = vector(ZZ, [])
sage: z = v.outer_product(v)
sage: z.parent()
Full MatrixSpace of 0 by 0 dense matrices over Integer Ring
```

There is a fair amount of latitude in the value of the right vector, and the matrix that results can have entries from a new ring large enough to contain the result. If you know better, you can sometimes bring the result down to a less general ring.

```python
sage: R.<t> = ZZ[]
sage: v = vector(R, [12, 24*t])
sage: w = vector(QQ, [1/2, 1/3, 1/4])
sage: op = v.outer_product(w)
sage: op
[ 6  4  3]
[12*t  8*t  6*t]
sage: op.base_ring()
Univariate Polynomial Ring in t over Rational Field
sage: m = op.change_ring(R); m
```
But some inputs are not compatible, even if vectors.

```
sage: w = vector(GF(5), [1,2])
sage: v = vector(GF(7), [1,2,3,4])
sage: z = w.outer_product(v)
```

```
Traceback (most recent call last):
  ...TypeError: unsupported operand parent(s) for *: 'Full MatrixSpace of 2 by 1 → dense matrices over Finite Field of size 5' and 'Full MatrixSpace of 1 by 4 → dense matrices over Finite Field of size 7'
```

And some inputs don’t make any sense at all.

```
sage: w = vector(QQ, [5,10])
sage: z = w.outer_product(6)
```

```
Traceback (most recent call last):
  ...TypeError: right operand in an outer product must be a vector, not an element of Integer Ring
```

**pairwise_product** *(right)*

Return the pairwise product of self and right, which is a vector of the products of the corresponding entries.

**INPUT:**

* right - vector of the same degree as self. It need not be in the same vector space as self, as long as the coefficients can be multiplied.

**EXAMPLES:**

```
sage: V = FreeModule(ZZ, 3)
sage: v = V([1,2,3])
sage: w = V([4,5,6])
sage: v.pairwise_product(w)
(4, 10, 18)
sage: sum(v.pairwise_product(w)) == v.dot_product(w)
True
```

```
sage: W = VectorSpace(GF(3), 3)
sage: w = W([0,1,2])
sage: w.pairwise_product(v)
(0, 2, 0)
sage: w.pairwise_product(v).parent()
Vector space of dimension 3 over Finite Field of size 3
```

Implicit coercion is well defined (regardless of order), so we get 2 even if we do the dot product in the other order.
sage: v.pairwise_product(w).parent()
Vector space of dimension 3 over Finite Field of size 3

plot(plot_type=None, start=None, **kwds)

**INPUT:**

- **plot_type** - (default: ‘arrow’ if \(v\) has 3 or fewer components, otherwise ‘step’) type of plot. Options are:
  - ‘arrow’ to draw an arrow
  - ‘point’ to draw a point at the coordinates specified by the vector
  - ‘step’ to draw a step function representing the coordinates of the vector.

  Both ‘arrow’ and ‘point’ raise exceptions if the vector has more than 3 dimensions.

- **start** - (default: origin in correct dimension) may be a tuple, list, or vector.

**EXAMPLES:**

The following both plot the given vector:

```sage
sage: v = vector(RDF, (1,2))
sage: A = plot(v)
sage: B = v.plot()
sage: A+B # should just show one vector
Graphics object consisting of 2 graphics primitives
```

Examples of the plot types:

```sage
sage: A = plot(v, plot_type='arrow')
sage: B = plot(v, plot_type='point', color='green', size=20)
sage: C = plot(v, plot_type='step') # calls v.plot_step()
sage: A+B+C
Graphics object consisting of 3 graphics primitives
```

You can use the optional arguments for `plot_step()`:

```sage
sage: eps = 0.1
sage: plot(v, plot_type='step', eps=eps, xmax=5, hue=0)
Graphics object consisting of 1 graphics primitive
```

Three-dimensional examples:

```sage
sage: v = vector(RDF, (1,2,1))
sage: plot(v) # defaults to an arrow plot
Graphics3d Object
```

```sage
sage: plot(v, plot_type='arrow')
Graphics3d Object
```

```sage
sage: from sage.plot.plot3d.shapes2 import frame3d
sage: plot(v, plot_type='point')+frame3d((0,0,0), v.list())
Graphics3d Object
```
sage: plot(v, plot_type='step')  # calls v.plot_step()
Graphics object consisting of 1 graphics primitive

sage: plot(v, plot_type='step', eps=eps, xmax=5, hue=0)
Graphics object consisting of 1 graphics primitive

With greater than three coordinates, it defaults to a step plot:

sage: v = vector(RDF, (1,2,3,4))
sage: plot(v)
Graphics object consisting of 1 graphics primitive

One dimensional vectors are plotted along the horizontal axis of the coordinate plane:

sage: plot(vector([1]))
Graphics object consisting of 1 graphics primitive

An optional start argument may also be specified by a tuple, list, or vector:

sage: u = vector([1,2]); v = vector([2,5])
sage: plot(u, start=v)
Graphics object consisting of 1 graphics primitive

plot_step(xmin=0, xmax=1, eps=None, res=None, connect=True, **kwds)
INPUT:
- xmin - (default: 0) start x position to start plotting
- xmax - (default: 1) stop x position to stop plotting
- eps - (default: determined by xmax) we view this vector as defining a function at the points xmin, xmin + eps, xmin + 2*eps,....
- res - (default: all points) total number of points to include in the graph
- connect - (default: True) if True draws a line; otherwise draw a list of points.

EXAMPLES:

sage: eps=0.1
sage: v = vector(RDF, [sin(n*eps) for n in range(100)])
sage: v.plot_step(eps=eps, xmax=5, hue=0)
Graphics object consisting of 1 graphics primitive

row()
Return a matrix with a single row and the same entries as the vector self.

OUTPUT:
A matrix over the same ring as the vector (or free module element), with a single row. The entries of the row are identical to those of the vector, and in the same order.

EXAMPLES:

sage: v = vector(ZZ, [1,2,3])
sage: w = v.row(); w
[1 2 3]
sage: w.parent()
Full MatrixSpace of 1 by 3 dense matrices over Integer Ring

```sage
x = vector(FiniteField(13), [2,4,8,16])
x = x.row()
[2  4  8  3]
```

There is more than one way to get one-row matrix from a vector, but the `row` method is more efficient than making a column and then taking a transpose. Notice that supplying a vector to the matrix constructor demonstrates Sage’s preference for rows.

```sage
x = vector(RDF, [sin(i*pi/20) for i in range(10)])
x = x.row() == matrix(x)
True
x = x.row() == x.column().transpose()
True
```

Sparse or dense implementations are preserved.

```sage
d = vector(RR, [1.0, 2.0, 3.0])
s = vector(CDF, {2:5.0+6.0*I})
dm = d.row()
sm = s.row()
all([d.is_dense(), dm.is_dense(), s.is_sparse(), sm.is_sparse()])
True
```

### set(i, value)
Like `__setitem__` but without type or bounds checking: `i` must satisfy `0 <= i < self.degree` and `value` must be an element of the coordinate ring.

**EXAMPLES:**

```sage:v = vector(SR, [1/2,2/5,0]); v
(1/2, 2/5, 0)
v = v.set(2, pi); v
(1/2, 2/5, pi)
```

### sparse_vector()
Return sparse version of self. If self is sparse, just return self; otherwise, create and return correspond sparse vector.

**EXAMPLES:**

```sage: vector([-1,0,3,0,0,0]).sparse_vector().is_sparse()
True
sage: vector([-1,0,3,0,0,0]).sparse_vector().is_sparse()
True
sage: vector([-1,0,3,0,0,0]).sparse_vector()
(-1, 0, 3, 0, 0, 0)
```

### subs(in_dict=None, **kwds)
**EXAMPLES:**

```sage: var('a,b,d,e')
(a, b, d, e)
```
sage: v = vector([a, b, d, e])
sage: v.substitute(a=1)
(1, b, d, e)
sage: v.subs(a=b, b=d)
(b, d, d, e)

support()
Return the integers i such that self[i] != 0. This is the same as the nonzero_positions function.

EXAMPLES:

sage: vector([-1,0,3,0,0,0,0.01]).support()
[0, 2, 6]

tensor_product(right)
Returns a matrix, the outer product of two vectors self and right.

INPUT:
• right - a vector (or free module element) of any size, whose elements are compatible (with regard to multiplication) with the elements of self.

OUTPUT:
The outer product of two vectors x and y (respectively self and right) can be described several ways. If we interpret x as a m x 1 matrix and interpret y as a 1 x n matrix, then the outer product is the m x n matrix from the usual matrix product xy. Notice how this is the “opposite” in some ways from an inner product (which would require m = n).

If we just consider vectors, use each entry of x to create a scalar multiples of the vector y and use these vectors as the rows of a matrix. Or use each entry of y to create a scalar multiples of x and use these vectors as the columns of a matrix.

EXAMPLES:

sage: u = vector(QQ, [1/2, 1/3, 1/4, 1/5])
sage: v = vector(ZZ, [60, 180, 600])
sage: u.outer_product(v)
[ 30  90  300]
[ 20  60  200]
[ 15  45  150]
[ 12  36  120]
sage: M = v.outer_product(u); M
[30 20 15 12]
[90 60 45 36]
[300 200 150 120]
sage: M.parent()
Full MatrixSpace of 3 by 4 dense matrices over Rational Field

The more general sage.matrix.matrix2.tensor_product() is an operation on a pair of matrices. If we construct a pair of vectors as a column vector and a row vector, then an outer product and a tensor product are identical. Thus tensor_product is a synonym for this method.
The result is always a dense matrix, no matter if the two vectors are, or are not, dense.

```
sage: d = vector(ZZ,[4,5], sparse=False)
sage: s = vector(ZZ,[1,2,3], sparse=True)
sage: dd = d.outer_product(d)
sage: ds = d.outer_product(s)
sage: sd = s.outer_product(d)
sage: ss = s.outer_product(s)
sage: all([dd.is_dense(), ds.is_dense(), sd.is_dense(), dd.is_dense()])
True
```

Vectors with no entries do the right thing.

```
sage: v = vector(ZZ, [])
sage: z = v.outer_product(v)
sage: z.parent()
Full MatrixSpace of 0 by 0 dense matrices over Integer Ring
```

There is a fair amount of latitude in the value of the right vector, and the matrix that results can have entries from a new ring large enough to contain the result. If you know better, you can sometimes bring the result down to a less general ring.

```
sage: R.<t> = ZZ[]
sage: v = vector(R, [12, 24*t])
sage: w = vector(QQ, [1/2, 1/3, 1/4])
sage: op = v.outer_product(w)
sage: op
[ 6 4 3]
[12*t 8*t 6*t]
sage: op.base_ring()
Univariate Polynomial Ring in t over Rational Field
sage: m = op.change_ring(R); m
[ 6 4 3]
[12*t 8*t 6*t]
sage: m.base_ring()
Univariate Polynomial Ring in t over Integer Ring
```

But some inputs are not compatible, even if vectors.

```
sage: w = vector(GF(5), [1,2])
sage: v = vector(GF(7), [1,2,3,4])
sage: z = w.outer_product(v)
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for *: 'Full MatrixSpace of 2 by 1 dense matrices over Finite Field of size 5' and 'Full MatrixSpace of 1 by 4 dense matrices over Finite Field of size 7'
```

And some inputs don’t make any sense at all.
sage: w=vector(QQ, [5,10])
sage: z=w.outer_product(6)
Traceback (most recent call last):
...  
TypeError: right operand in an outer product must be a vector, not an element of Integer Ring

class sage.modules.free_module_element.FreeModuleElement_generic_dense
Bases: sage.modules.free_module_element.FreeModuleElement

A generic dense element of a free module.

function(*args)
Return a vector over a callable symbolic expression ring.

EXAMPLES:

sage: x,y=var('x,y')
sage: v=vector([x,y,x*sin(y)])
sage: w=v.function([x,y]); w
(x, y) |--> (x, y, x*sin(y))
sage: w.coordinate_ring()
Callable function ring with arguments (x, y)
sage: w(1,2)
(1, 2, sin(2))
sage: w(2,1)
(2, 1, 2*sin(1))
sage: w(y=1,x=2)
(2, 1, 2*sin(1))

sage: x,y=var('x,y')
sage: v=vector([x,y,x*sin(y)])
sage: w=v.function([x]); w
x |--> (x, y, x*sin(y))
sage: w.coordinate_ring()
Callable function ring with argument x
sage: w(4)
(4, y, 4*sin(y))

list(copy=True)
Return list of elements of self.

INPUT:

* copy – bool, return list of underlying entries

EXAMPLES:

sage: P.<x,y,z> = QQ[]
sage: v = vector([x,y,z])
sage: type(v)
<class 'sage.modules.free_module_element.FreeModuleElement_generic_dense'>
sage: a = v.list(); a
[x, y, z]
sage: a[0] = x*y; v
(x, y, z)
```python
sage: a = v.list(copy=False); a
[x, y, z]
sage: a[0] = x*y; v
(x^2y, y, z)
```

class `sage.modules.free_module_element.FreeModuleElement_generic_sparse`

Bases: `sage.modules.free_module_element.FreeModuleElement`

A generic sparse free module element is a dictionary with keys ints i and entries in the base ring.

**denominator()**

Return the least common multiple of the denominators of the entries of self.

EXAMPLES:

```python
sage: v = vector([1/2,2/5,3/14], sparse=True)
sage: v.denominator()
70
```

**dict(copy=True)**

Return dictionary of nonzero entries of self.

More precisely, this returns a dictionary whose keys are indices of basis elements in the support of self and whose values are the corresponding coefficients.

INPUT:

- `copy` - (default: True) if self is internally represented by a dictionary d, then make a copy of d; if False, then this can cause undesired behavior by mutating d

OUTPUT:

- Python dictionary

EXAMPLES:

```python
sage: v = vector({1: 1, 3: -2})
sage: w = vector({1: 4, 3: 2})
sage: v+w
(0, 5, 0, 0)
sage: (v+w).hamming_weight()
1

```

**hamming_weight()**

Returns the number of positions i such that self[i] != 0.

EXAMPLES:

```python
sage: v = vector({1: 1, 3: -2})
sage: w = vector({1: 4, 3: 2})
sage: v+w
(0, 5, 0, 0)
sage: (v+w).hamming_weight()
1
```

**items()**

Return an iterator over the entries of self.

EXAMPLES:
sage: v = vector([1, 2/3, pi], sparse=True)
sage: next(v.items())
(0, 1)
sage: list(v.items())
[(0, 1), (1, 2/3), (2, pi)]

iteritems()
Return an iterator over the entries of self.

EXAMPLES:

sage: v = vector([1, 2/3, pi], sparse=True)
sage: next(v.items())
(0, 1)
sage: list(v.items())
[(0, 1), (1, 2/3), (2, pi)]

list(copy=True)
Return list of elements of self.

INPUT:

• copy – ignored for sparse vectors

EXAMPLES:

sage: R.<x> = QQ[]
sage: M = FreeModule(R, 3, sparse=True) * (1/x)
sage: v = M([-x^2, 3/x, 0])
sage: type(v)
<class 'sage.modules.free_module_element.FreeModuleElement_generic_sparse'>
sage: a = v.list()
sage: a
[-x^2, 3/x, 0]
sage: [parent(c) for c in a]
[Fraction Field of Univariate Polynomial Ring in x over Rational Field, Fraction Field of Univariate Polynomial Ring in x over Rational Field, Fraction Field of Univariate Polynomial Ring in x over Rational Field]

monomial_coefficients(copy=True)
Return dictionary of nonzero entries of self.

More precisely, this returns a dictionary whose keys are indices of basis elements in the support of self and whose values are the corresponding coefficients.

INPUT:

• copy – (default: True) if self is internally represented by a dictionary d, then make a copy of d; if False, then this can cause undesired behavior by mutating d

OUTPUT:

• Python dictionary

EXAMPLES:

sage: v = vector([0, 0, 0, 0, 1/2, 0, 3/14], sparse=True)
sage: v.dict()
(continues on next page)
\{4: 1/2, 6: 3/14\}

```python
sage: sorted(v.support())
[4, 6]
```

**nonzero_positions()**

Returns the list of numbers i such that self[i] != 0.

**EXAMPLES:**

```python
sage: v = vector({1: 1, 3: -2})
sage: w = vector({1: 4, 3: 2})
sage: v+w
(0, 5, 0, 0)
sage: (v+w).nonzero_positions()
[1]
```

**numerical_approx**(prec=None, digits=None, algorithm=None)

Return a numerical approximation of self with prec bits (or decimal digits) of precision, by approximating all entries.

**INPUT:**

- **prec** – precision in bits
- **digits** – precision in decimal digits (only used if prec is not given)
- **algorithm** – which algorithm to use to compute the approximation of the entries (the accepted algorithms depend on the object)

If neither prec nor digits is given, the default precision is 53 bits (roughly 16 digits).

**EXAMPLES:**

```python
sage: v = vector(RealField(200), [1,2,3], sparse=True)
sage: v.n()
(1.0000000000000000000000, 2.0000000000000000000000, 3.0000000000000000000000)
sage: _.parent()
Sparse vector space of dimension 3 over Real Field with 53 bits of precision
sage: v.n(prec=75)
(1.0000000000000000000000, 2.0000000000000000000000, 3.0000000000000000000000)
sage: _.parent()
Sparse vector space of dimension 3 over Real Field with 75 bits of precision
```

**sage.modules.free_module_element.free_module_element**(arg0, arg1=None, arg2=None, sparse=None, immutable=False)

Return a vector or free module element with specified entries.

**CALL FORMATS:**

This constructor can be called in several different ways. In each case, sparse=True or sparse=False as well as immutable=True or immutable=False can be supplied as an option. free_module_element() is an alias for vector().

1. vector(object)
2. vector(ring, object)
3. vector(object, ring)
4. vector(ring, degree, object)
5. vector(ring, degree)

INPUT:

- **object** – a list, dictionary, or other iterable containing the entries of the vector, including any object that is palatable to the `Sequence` constructor
- **ring** – a base ring (or field) for the vector space or free module, which contains all of the elements
- **degree** – an integer specifying the number of entries in the vector or free module element
- **sparse** – boolean, whether the result should be a sparse vector
- **immutable** – boolean (default: `False`); whether the result should be an immutable vector

In call format 4, an error is raised if the **degree** does not match the length of **object** so this call can provide some safeguards. Note however that using this format when **object** is a dictionary is unlikely to work properly.

OUTPUT:

An element of the ambient vector space or free module with the given base ring and implied or specified dimension or rank, containing the specified entries and with correct degree.

In call format 5, no entries are specified, so the element is populated with all zeros.

If the **sparse** option is not supplied, the output will generally have a dense representation. The exception is if **object** is a dictionary, then the representation will be sparse.

EXAMPLES:

```
sage: v = vector([1,2,3]); v
(1, 2, 3)
sage: v.parent()
Ambient free module of rank 3 over the principal ideal domain Integer Ring
sage: v = vector([1,2,3/5]); v
(1, 2, 3/5)
sage: v.parent()
Vector space of dimension 3 over Rational Field
```

All entries must *canonically* coerce to some common ring:

```
sage: v = vector([17, GF(11)(5), 19/3]); v
Traceback (most recent call last):
  ...TypeError: unable to find a common ring for all elements
sage: v = vector([17, GF(11)(5), 19]); v
(6, 5, 8)
sage: v.parent()
Vector space of dimension 3 over Finite Field of size 11
sage: v = vector([17, GF(11)(5), 19], QQ); v
(17, 5, 19)
sage: v.parent()
Vector space of dimension 3 over Rational Field
```

(continues on next page)
sage: v.parent()
Vector space of dimension 3 over Rational Field
sage: v = vector(vector([1,2,3])); v
(1, 2, 3)
sage: v.parent()
Ambient free module of rank 3 over the principal ideal domain Integer Ring

You can also use \texttt{free_module_element}, which is the same as \texttt{vector}.

sage: free_module_element([1/3, -4/5])
(1/3, -4/5)

We make a vector mod 3 out of a vector over $\mathbb{Z}$.

sage: vector(vector([1,2,3]), GF(3))
(1, 2, 0)

The degree of a vector may be specified:

sage: vector(QQ, 4, [1,1/2,1/3,1/4])
(1, 1/2, 1/3, 1/4)

But it is an error if the degree and size of the list of entries are mismatched:

sage: vector(QQ, 5, [1,1/2,1/3,1/4])
Traceback (most recent call last):
  ... ValueError: incompatible degrees in vector constructor

Providing no entries populates the vector with zeros, but of course, you must specify the degree since it is not implied. Here we use a finite field as the base ring.

sage: w = vector(FiniteField(7), 4); w
(0, 0, 0, 0)
sage: w.parent()
Vector space of dimension 4 over Finite Field of size 7

The fastest method to construct a zero vector is to call the \texttt{zero_vector()} method directly on a free module or vector space, since \texttt{vector(...)} must do a small amount of type checking. Almost as fast as the \texttt{zero_vector()} method is the \texttt{zero_vector()} constructor, which defaults to the integers.

sage: vector(ZZ, 5)   # works fine
(0, 0, 0, 0, 0)
sage: (ZZ^5).zero_vector()  # very tiny bit faster
(0, 0, 0, 0, 0)
sage: zero_vector(ZZ, 5)  # similar speed to vector(...)
(0, 0, 0, 0, 0)
sage: z = zero_vector(5); z
(0, 0, 0, 0, 0)
sage: z.parent()
Ambient free module of rank 5 over the principal ideal domain Integer Ring
Here we illustrate the creation of sparse vectors by using a dictionary:

```
sage: vector({1:1.1, 3:3.14})
(0.000000000000000, 1.10000000000000, 0.000000000000000, 3.14000000000000)
```

With no degree given, a dictionary of entries implicitly declares a degree by the largest index (key) present. So you can provide a terminal element (perhaps a zero?) to set the degree. But it is probably safer to just include a degree in your construction.

```
sage: v = vector(QQ, {0:1/2, 4:-6, 7:0}); v
(1/2, 0, 0, -6, 0, 0, 0)
sage: v.degree()
8
sage: v.is_sparse()
True
sage: w = vector(QQ, 8, {0:1/2, 4:-6})
sage: w == v
True
```

It is an error to specify a negative degree.

```
sage: vector(RR, -4, [1.0, 2.0, 3.0, 4.0])
Traceback (most recent call last):
...  
ValueError: cannot specify the degree of a vector as a negative integer (-4)
```

It is an error to create a zero vector but not provide a ring as the first argument.

```
sage: vector('junk', 20)
Traceback (most recent call last):
...  
TypeError: first argument must be base ring of zero vector, not junk
```

And it is an error to specify an index in a dictionary that is greater than or equal to a requested degree.

```
sage: vector(ZZ, 10, {3:4, 7:-2, 10:637})
Traceback (most recent call last):
...  
ValueError: dictionary of entries has a key (index) exceeding the requested degree
```

A 1-dimensional numpy array of type float or complex may be passed to vector. Unless an explicit ring is given, the result will be a vector in the appropriate dimensional vector space over the real double field or the complex double field. The data in the array must be contiguous, so column-wise slices of numpy matrices will raise an exception.

```
sage: import numpy
sage: x = numpy.random.randn(10)
sage: y = vector(x)
sage: parent(y)
Vector space of dimension 10 over Real Double Field
sage: parent(vector(RDF, x))
Vector space of dimension 10 over Real Double Field
sage: parent(vector(CDF, x))
Vector space of dimension 10 over Complex Double Field
```

(continues on next page)
sage: parent(vector(RR, x))
Vector space of dimension 10 over Real Field with 53 bits of precision
sage: v = numpy.random.randn(10) * complex(0,1)
sage: w = vector(v)
sage: parent(w)
Vector space of dimension 10 over Complex Double Field

Multi-dimensional arrays are not supported:

sage: import numpy as np
sage: a = np.array([[1, 2, 3], [4, 5, 6]], np.float64)
sage: vector(a)
Traceback (most recent call last):
...
TypeError: cannot convert 2-dimensional array to a vector

If any of the arguments to vector have Python type int, long, real, or complex, they will first be coerced to the appropriate Sage objects. This fixes trac ticket #3847.

sage: v = vector([int(0)]); v
(0)
sage: v[0].parent()
Integer Ring
sage: v = vector(range(10)); v
(0, 1, 2, 3, 4, 5, 6, 7, 8, 9)
sage: v[3].parent()
Integer Ring
sage: v = vector([float(23.4), int(2), complex(2+7*I), 1]); v
(23.4, 2.0, 2.0 + 7.0*I, 1.0)
sage: v[1].parent()
Complex Double Field

If the argument is a vector, it doesn’t change the base ring. This fixes trac ticket #6643:

sage: K.<sqrt3> = QuadraticField(3)
sage: u = vector(K, (1/2, sqrt3/2) )
sage: vector(u).base_ring()
Number Field in sqrt3 with defining polynomial x^2 - 3 with sqrt3 = 1.
˓→7320508075688787?
sage: v = vector(K, (0, 1) )
sage: vector(v).base_ring()
Number Field in sqrt3 with defining polynomial x^2 - 3 with sqrt3 = 1.
˓→73205080756887878?

Constructing a vector from a numpy array behaves as expected:

sage: import numpy
sage: a=numpy.array([1,2,3])
sage: v=vector(a); v
(1, 2, 3)
sage: parent(v)
Ambient free module of rank 3 over the principal ideal domain Integer Ring

Complex numbers can be converted naturally to a sequence of length 2. And then to a vector.
sage: c = CDF(2 + 3*I)
sage: v = vector(c); v
(2.0, 3.0)

A generator, or other iterable, may also be supplied as input. Anything that can be converted to a Sequence is a possible input.

sage: type(i^2 for i in range(3))
<... 'generator'>
sage: v = vector(i^2 for i in range(3)); v
(0, 1, 4)

An empty list, without a ring given, will default to the integers.

sage: x = vector([]); x
()
sage: x.parent()
Ambient free module of rank 0 over the principal ideal domain Integer Ring

The immutable switch allows to create an immutable vector.

sage: v = vector(QQ, {0:1/2, 4:-6, 7:0}, immutable=True); v
(1/2, 0, 0, 0, -6, 0, 0, 0)
sage: v.is Immutable()
True

The immutable switch works regardless of the type of valid input to the constructor.

sage: v = vector(ZZ, 4, immutable=True)
sage: v.isImmutable()
True
sage: w = vector(ZZ, [1,2,3])
sage: v = vector(w, ZZ, immutable=True)
sage: v.isImmutable()
True
sage: v = vector(QQ, w, immutable=True)
sage: v.isImmutable()
True
sage: import numpy as np
sage: w = np.array([1, 2, pi], float)
sage: v = vector(w, immutable=True)
sage: v.isImmutable()
True
sage: w = np.array([i, 2, 3], complex)
sage: v = vector(w, immutable=True)
sage: v.isImmutable()
True

sage.modules.free_module_element.is_FreeModuleElement(x)

EXAMPLES:

sage: sage.modules.free_module_element.is_FreeModuleElement(0)
False

(continues on next page)
sage: sage.modules.free_module_element.is_FreeModuleElement(vector([1,2,3]))
True

sage.modules.free_module_element.make_FreeModuleElement_generic_dense(parent, entries, degree)
EXAMPLES:

sage: sage.modules.free_module_element.make_FreeModuleElement_generic_dense(QQ^3, [1,2,-3/7], 3)
(1, 2, -3/7)

sage.modules.free_module_element.make_FreeModuleElement_generic_dense_v1(parent, entries, degree, is_mutable)
EXAMPLES:

sage: v = sage.modules.free_module_element.make_FreeModuleElement_generic_dense_v1(QQ^3, [1,2,-3/7], 3, True); v
(1, 2, -3/7)
sage: v[0] = 10; v
(10, 2, -3/7)
sage: v = sage.modules.free_module_element.make_FreeModuleElement_generic_dense_v1(QQ^3, [1,2,-3/7], 3, False); v
(1, 2, -3/7)
sage: v[0] = 10
Traceback (most recent call last):
... ValueError: vector is immutable; please change a copy instead (use copy())

sage.modules.free_module_element.make_FreeModuleElement_generic_sparse(parent, entries, degree)
EXAMPLES:

sage: v = sage.modules.free_module_element.make_FreeModuleElement_generic_sparse(QQ^3, {2:5/2}, 3); v
(0, 0, 5/2)

sage.modules.free_module_element.make_FreeModuleElement_generic_sparse_v1(parent, entries, degree, is_mutable)
EXAMPLES:

sage: v = sage.modules.free_module_element.make_FreeModuleElement_generic_sparse_v1(QQ^3, {2:5/2}, 3, False); v
(0, 0, 5/2)
sage: v.is_mutable()
False

sage.modules.free_module_element.prepare(v, R, degree=None)
Converts an object describing elements of a vector into a list of entries in a common ring.

INPUT:

- v - a dictionary with non-negative integers as keys, or a list or other object that can be converted by the
  Sequence constructor
- R - a ring containing all the entries, possibly given as None
- degree - a requested size for the list when the input is a dictionary, otherwise ignored
OUTPUT:

A pair.

The first item is a list of the values specified in the object \( v \). If the object is a dictionary, entries are placed in the list according to the indices that were their keys in the dictionary, and the remainder of the entries are zero. The value of \( \text{degree} \) is assumed to be larger than any index provided in the dictionary and will be used as the number of entries in the returned list.

The second item returned is a ring that contains all of the entries in the list. If \( R \) is given, the entries are coerced in. Otherwise a common ring is found. For more details, see the Sequence object. When \( v \) has no elements and \( R \) is None, the ring returned is the integers.

EXAMPLES:

```python
sage: from sage.modules.free_module_element import prepare
sage: prepare([1,2/3,5],None)
([1, 2/3, 5], Rational Field)

sage: prepare([1,2/3,5],RR)
([1.00000000000000, 0.66666666666667, 5.00000000000000], Real Field with 53 bits of precision)

sage: prepare({1:4, 3:-2}, ZZ, 6)
([0, 4, 0, -2, 0, 0], Integer Ring)

sage: prepare({3:1, 5:3}, QQ, 6)
([0, 0, 1, 0, 3], Rational Field)

sage: prepare([1,2/3,'10',5],RR)
([1.00000000000000, 0.66666666666667, 10.0000000000000, 5.00000000000000], Real Field with 53 bits of precision)

sage: prepare([],QQ, 0)
([], Rational Field)

sage: prepare([1,2/3,'10',5],None)
Traceback (most recent call last):
  ...TypeError: unable to find a common ring for all elements
```

Some objects can be converted to sequences even if they are not always thought of as sequences.

```python
sage: c = CDF(2+3*I)
sage: prepare(c, None)
([2.0, 3.0], Real Double Field)
```

This checks a bug listed at trac ticket #10595. Without good evidence for a ring, the default is the integers.

```python
sage: prepare([], None)
([], Integer Ring)
```

```
sage.modules.free_module_element.random_vector(ring, degree=None, *args, **kwds)

Returns a vector (or module element) with random entries.

INPUT:

* ring – default: ZZ - the base ring for the entries
```
• degree – a non-negative integer for the number of entries in the vector
• sparse – default: False - whether to use a sparse implementation
• args, kwds - additional arguments and keywords are passed to the random_element() method of the ring

OUTPUT:
A vector, or free module element, with degree elements from ring, chosen randomly from the ring according to the ring’s random_element() method.

Note: See below for examples of how random elements are generated by some common base rings.

EXAMPLES:
First, module elements over the integers. The default distribution is tightly clustered around -1, 0, 1. Uniform distributions can be specified by giving bounds, though the upper bound is never met. See sage.rings.integer_ring.IntegerRing_class.random_element() for several other variants.

```
sage: random_vector(10).parent()
Ambient free module of rank 10 over the principal ideal domain Integer Ring
sage: random_vector(20).parent()
Ambient free module of rank 20 over the principal ideal domain Integer Ring

sage: v = random_vector(ZZ, 20, x=4)
sage: all(i in range(4) for i in v)
True

sage: v = random_vector(ZZ, 20, x=-20, y=100)
sage: all(i in range(-20, 100) for i in v)
True
```

If the ring is not specified, the default is the integers, and parameters for the random distribution may be passed without using keywords. This is a random vector with 20 entries uniformly distributed between -20 and 100.

```
sage: random_vector(20, -20, 100).parent()
Ambient free module of rank 20 over the principal ideal domain Integer Ring
```

Now over the rationals. Note that bounds on the numerator and denominator may be specified. See sage.rings.rational_field.RationalField.random_element() for documentation.

```
sage: random_vector(QQ, 10).parent()
Vector space of dimension 10 over Rational Field

sage: v = random_vector(QQ, 10, num_bound=15, den_bound=5)
sage: v.parent()
Vector space of dimension 10 over Rational Field
sage: all(q.numerator() <= 15 and q.denominator() <= 5 for q in v)
True
```

Inexact rings may be used as well. The reals have uniform distributions, with the range (−1,1) as the default. More at: sage.rings.real_mpfr.RealField_class.random_element()

```
sage: v = random_vector(RR, 5)
sage: v.parent()
```

(continues on next page)
Vector space of dimension 5 over Real Field with 53 bits of precision
```
sage: all(-1 <= r <= 1 for r in v)
True
```

```
sage: v = random_vector(RR, 5, min=8, max=14)
sage: v.parent()
Vector space of dimension 5 over Real Field with 53 bits of precision
sage: all(8 <= r <= 14 for r in v)
True
```

Any ring with a `random_element()` method may be used.

```
sage: F = FiniteField(23)
sage: hasattr(F, 'random_element')
True
sage: v = random_vector(F, 10)
sage: v.parent()
Vector space of dimension 10 over Finite Field of size 23
```

The default implementation is a dense representation, equivalent to setting `sparse=False`.

```
sage: v = random_vector(10)
sage: v.is_sparse()
False
```

```
sage: w = random_vector(ZZ, 20, sparse=True)
sage: w.is_sparse()
True
```

The elements are chosen using the ring’s `random_element` method:

```
sage: from sage.misc.randstate import current_randstate
sage: seed = current_randstate().seed()
sage: set_random_seed(seed)

sage: v1 = random_vector(ZZ, 20, distribution="1/n")
sage: v2 = random_vector(ZZ, 15, x=-1000, y=1000)
sage: v3 = random_vector(QQ, 10)
sage: v4 = random_vector(FiniteField(17), 10)
sage: v5 = random_vector(RR, 10)
sage: set_random_seed(seed)

sage: w1 = vector(ZZ.random_element(distribution="1/n") for _ in range(20))
sage: w2 = vector(ZZ.random_element(x=-1000, y=1000) for _ in range(15))
sage: w3 = vector(QQ.random_element() for _ in range(10))
sage: w4 = vector(FiniteField(17).random_element() for _ in range(10))
sage: w5 = vector(RR.random_element() for _ in range(10))

sage: [v1, v2, v3, v4, v5] == [w1, w2, w3, w4, w5]
True
```

Inputs get checked before constructing the vector.

```
sage: random_vector('junk')
Traceback (most recent call last):
```

(continues on next page)
TypeError: degree of a random vector must be an integer, not None

```
sage: random_vector('stuff', 5)
Traceback (most recent call last):
  ...  
TypeError: elements of a vector, or module element, must come from a ring, not stuff
```

```
sage: random_vector(ZZ, -9)
Traceback (most recent call last):
  ...  
ValueError: degree of a random vector must be non-negative, not -9
```

```
sage.modules.free_module_element.vector(arg0, arg1=None, arg2=None, sparse=None, immutable=False)
Return a vector or free module element with specified entries.

CALL FORMATS:

This constructor can be called in several different ways. In each case, sparse=True or sparse=False as well as immutable=True or immutable=False can be supplied as an option. free_module_element() is an alias for vector().

1. vector(object)
2. vector(ring, object)
3. vector(object, ring)
4. vector(ring, degree, object)
5. vector(ring, degree)

INPUT:

- object – a list, dictionary, or other iterable containing the entries of the vector, including any object that is palatable to the Sequence constructor
- ring – a base ring (or field) for the vector space or free module, which contains all of the elements
- degree – an integer specifying the number of entries in the vector or free module element
- sparse – boolean, whether the result should be a sparse vector
- immutable – boolean (default: False); whether the result should be an immutable vector

In call format 4, an error is raised if the degree does not match the length of object so this call can provide some safeguards. Note however that using this format when object is a dictionary is unlikely to work properly.

OUTPUT:

An element of the ambient vector space or free module with the given base ring and implied or specified dimension or rank, containing the specified entries and with correct degree.

In call format 5, no entries are specified, so the element is populated with all zeros.

If the sparse option is not supplied, the output will generally have a dense representation. The exception is if object is a dictionary, then the representation will be sparse.

EXAMPLES:

```
sage: v = vector([1,2,3]); v  
(1, 2, 3)
sage: v.parent()
```
Ambient free module of rank 3 over the principal ideal domain Integer Ring

```sage
v = vector([1,2,3/5]); v
(1, 2, 3/5)
sage: v.parent()
Vector space of dimension 3 over Rational Field
```

All entries must canonically coerce to some common ring:

```sage
v = vector([17, GF(11)(5), 19/3]); v
Traceback (most recent call last):
  ...TypeError: unable to find a common ring for all elements
```

```sage
v = vector([17, GF(11)(5), 19]); v
(6, 5, 8)
sage: v.parent()
Vector space of dimension 3 over Finite Field of size 11
```

```sage
v = vector([17, GF(11)(5), 19], QQ); v
(17, 5, 19)
sage: v.parent()
Vector space of dimension 3 over Rational Field
```

```sage
v = vector((1,2,3), QQ); v
(1, 2, 3)
sage: v.parent()
Vector space of dimension 3 over Rational Field
```

```sage
v = vector(QQ, (1,2,3)); v
(1, 2, 3)
sage: v.parent()
Vector space of dimension 3 over Rational Field
```

```sage
v = vector(vector([1,2,3])); v
(1, 2, 3)
sage: v.parent()
Ambient free module of rank 3 over the principal ideal domain Integer Ring
```

You can also use `free_module_element`, which is the same as `vector`.

```sage
free_module_element([1/3, -4/5])
(1/3, -4/5)
```

We make a vector mod 3 out of a vector over \( \mathbb{Z} \).

```sage
vector(vector([1,2,3]), GF(3))
(1, 2, 0)
```

The degree of a vector may be specified:

```sage
vector(QQ, 4, [1,1/2,1/3,1/4])
(1, 1/2, 1/3, 1/4)
```

But it is an error if the degree and size of the list of entries are mismatched:

```sage
vector(QQ, 5, [1,1/2,1/3,1/4])
Traceback (most recent call last):
```

(continues on next page)
ValueError: incompatible degrees in vector constructor

Providing no entries populates the vector with zeros, but of course, you must specify the degree since it is not implied. Here we use a finite field as the base ring.

```
sage: w = vector(FiniteField(7), 4); w
(0, 0, 0, 0)
sage: w.parent()
Vector space of dimension 4 over Finite Field of size 7
```

The fastest method to construct a zero vector is to call the `zero_vector()` method directly on a free module or vector space, since `vector(...)` must do a small amount of type checking. Almost as fast as the `zero_vector()` method is the `zero_vector()` constructor, which defaults to the integers.

```
sage: vector(ZZ, 5)           # works fine
(0, 0, 0, 0, 0)
sage: (ZZ^5).zero_vector()   # very tiny bit faster
(0, 0, 0, 0, 0)
sage: zero_vector(ZZ, 5)     # similar speed to vector(...)
(0, 0, 0, 0, 0)
sage: z = zero_vector(5); z
(0, 0, 0, 0, 0)
sage: z.parent()
Ambient free module of rank 5 over the principal ideal domain Integer Ring
```

Here we illustrate the creation of sparse vectors by using a dictionary:

```
sage: vector({1:1.1, 3:3.14})
(0.000000000000000, 1.10000000000000, 0.000000000000000, 3.14000000000000)
```

With no degree given, a dictionary of entries implicitly declares a degree by the largest index (key) present. So you can provide a terminal element (perhaps a zero?) to set the degree. But it is probably safer to just include a degree in your construction.

```
sage: v = vector(QQ, {0:1/2, 4:-6, 7:0}); v
(1/2, 0, 0, -6, 0, 0, 0)
sage: v.degree()
8
sage: v.is_sparse()
True
sage: w = vector(QQ, 8, {0:1/2, 4:-6})
sage: w == v
True
```

It is an error to specify a negative degree.

```
sage: vector(RR, -4, [1.0, 2.0, 3.0, 4.0])
Traceback (most recent call last):
  ...
ValueError: cannot specify the degree of a vector as a negative integer (-4)
```

It is an error to create a zero vector but not provide a ring as the first argument.
And it is an error to specify an index in a dictionary that is greater than or equal to a requested degree.

A 1-dimensional numpy array of type float or complex may be passed to vector. Unless an explicit ring is given, the result will be a vector in the appropriate dimensional vector space over the real double field or the complex double field. The data in the array must be contiguous, so column-wise slices of numpy matrices will raise an exception.

Multi-dimensional arrays are not supported:

If any of the arguments to vector have Python type int, long, real, or complex, they will first be coerced to the appropriate Sage objects. This fixes trac ticket #3847.
sage: v = vector([float(23.4), int(2), complex(2+7*I), 1]); v
(23.4, 2.0, 2.0 + 7.0*I, 1.0)
sage: v[1].parent()
Complex Double Field

If the argument is a vector, it doesn’t change the base ring. This fixes trac ticket #6643:
sage: K.<sqrt3> = QuadraticField(3)
sage: u = vector(K, (1/2, sqrt3/2) )
sage: vector(u).base_ring()
Number Field in sqrt3 with defining polynomial x^2 - 3 with sqrt3 = 1.
˓→732050807568878?
sage: v = vector(K, (0, 1) )
sage: vector(v).base_ring()
Number Field in sqrt3 with defining polynomial x^2 - 3 with sqrt3 = 1.
˓→732050807568878?

Constructing a vector from a numpy array behaves as expected:
sage: import numpy
sage: a=numpy.array([1,2,3])
sage: v=vector(a); v
(1, 2, 3)
sage: parent(v)
Ambient free module of rank 3 over the principal ideal domain Integer Ring

Complex numbers can be converted naturally to a sequence of length 2. And then to a vector.
sage: c = CDF(2 + 3*I)
sage: v = vector(c); v
(2.0, 3.0)

A generator, or other iterable, may also be supplied as input. Anything that can be converted to a Sequence is a possible input.
sage: type(i^2 for i in range(3))
<... 'generator'>
sage: v = vector(i^2 for i in range(3)); v
(0, 1, 4)

An empty list, without a ring given, will default to the integers.
sage: x = vector([]); x
()
sage: x.parent()
Ambient free module of rank 0 over the principal ideal domain Integer Ring

The immutable switch allows to create an immutable vector.
sage: v = vector(QQ, {0:1/2, 4:-6, 7:0}, immutable=True); v
(1/2, 0, 0, 0, -6, 0, 0, 0)
sage: v.is Immutable()
True
The \texttt{immutable} switch works regardless of the type of valid input to the constructor.

\begin{verbatim}
sage: v = vector(ZZ, 4, immutable=True)
sage: v.is Immutable()
True
sage: w = vector(ZZ, [1,2,3])
sage: v = vector(w, ZZ, immutable=True)
sage: v.isImmutable()
True
sage: v = vector(QQ, w, immutable=True)
sage: v.isImmutable()
True
sage: import numpy as np
sage: w = np.array([1, 2, pi], float)
sage: v = vector(w, immutable=True)
sage: v.isImmutable()
True
sage: w = np.array([i, 2, 3], complex)
sage: v = vector(w, immutable=True)
sage: v.isImmutable()
True
\end{verbatim}

\texttt{sage.modules.free_module_element.zero_vector(\texttt{arg0, arg1=None})}

Returns a vector or free module element with a specified number of zeros.

\textbf{CALL FORMATS:}

1. \texttt{zero_vector(degree)}
2. \texttt{zero_vector(ring, degree)}

\textbf{INPUT:}

- \texttt{degree} - the number of zero entries in the vector or free module element
- \texttt{ring} - default \texttt{ZZ} - the base ring of the vector space or module containing the constructed zero vector

\textbf{OUTPUT:}

A vector or free module element with \texttt{degree} entries, all equal to zero and belonging to the ring if specified. If no ring is given, a free module element over \texttt{ZZ} is returned.

\textbf{EXAMPLES:}

A zero vector over the field of rationals.

\begin{verbatim}
sage: v = zero_vector(QQ, 5); v
(0, 0, 0, 0, 0)
sage: v.parent()
Vector space of dimension 5 over Rational Field
\end{verbatim}

A free module zero element.

\begin{verbatim}
sage: w = zero_vector(Integers(6), 3); w
(0, 0, 0)
sage: w.parent()
Ambient free module of rank 3 over Ring of integers modulo 6
\end{verbatim}

If no ring is given, the integers are used.
```
sage: u = zero_vector(9); u
(0, 0, 0, 0, 0, 0, 0, 0, 0)
sage: u.parent()
Ambient free module of rank 9 over the principal ideal domain Integer Ring
```

Non-integer degrees produce an error.

```
sage: zero_vector(5.6)
Traceback (most recent call last):
  ...
TypeError: Attempt to coerce non-integral RealNumber to Integer
```

Negative degrees also give an error.

```
sage: zero_vector(-3)
Traceback (most recent call last):
  ...
ValueError: rank (=3) must be nonnegative
```

Garbage instead of a ring will be recognized as such.

```
sage: zero_vector(x^2, 5)
Traceback (most recent call last):
  ...
TypeError: first argument must be a ring
```
The class \texttt{FiniteRankFreeModule} implements free modules of finite rank over a commutative ring.

A \textit{free module of finite rank} over a commutative ring \( R \) is a module \( M \) over \( R \) that admits a \textit{finite basis}, i.e. a finite family of linearly independent generators. Since \( R \) is commutative, it has the invariant basis number property, so that the rank of the free module \( M \) is defined uniquely, as the cardinality of any basis of \( M \).

No distinguished basis of \( M \) is assumed. On the contrary, many bases can be introduced on the free module along with change-of-basis rules (as module automorphisms). Each module element has then various representations over the various bases.

\textbf{Note:} The class \texttt{FiniteRankFreeModule} does not inherit from class \texttt{FreeModule_generic} nor from class \texttt{CombinatorialFreeModule}, since both classes deal with modules with a \textit{distinguished basis} (see details \textit{below}). Accordingly, the class \texttt{FiniteRankFreeModule} inherits directly from the generic class \texttt{Parent} with the category set to \texttt{Modules} (and not to \texttt{ModulesWithBasis}).

\textbf{Todo:}

\begin{itemize}
  \item implement submodules
  \item create a FreeModules category (cf. the \textit{TODO} statement in the documentation of \texttt{Modules}: \textit{Implement a \"FreeModules(R)\" category, when so prompted by a concrete use case})
\end{itemize}

\textbf{AUTHORS:}

\begin{itemize}
  \item Eric Gourgoulhon, Michal Bejger (2014-2015): initial version
  \item Travis Scrimshaw (2016): category set to Modules\text{(ring)\text{.FiniteDimensional()} (trac ticket \#20770)
  \item Michael Jung (2019): improve treatment of the zero element
  \item Eric Gourgoulhon (2021): unicode symbols for tensor and exterior products
\end{itemize}

\textbf{REFERENCES:}

\begin{itemize}
  \item Chap. 10 of R. Godement: \textit{Algebra} [God1968]
  \item Chap. 3 of S. Lang: \textit{Algebra} [Lan2002]
\end{itemize}

\textbf{EXAMPLES:}

Let us define a free module of rank 2 over \( \mathbb{Z} \):

\begin{verbatim}
sage: M = FiniteRankFreeModule(ZZ, 2, name='M') ; M
Rank-2 free module M over the Integer Ring
\end{verbatim}
We introduce a first basis on \( M \):

```python
sage: e = M.basis('e') ; e
Basis \((e_0,e_1)\) on the Rank-2 free module \( M \) over the Integer Ring
```

The elements of the basis are of course module elements:

```python
sage: e[0]
Element \( e_0 \) of the Rank-2 free module \( M \) over the Integer Ring
sage: e[1]
Element \( e_1 \) of the Rank-2 free module \( M \) over the Integer Ring
sage: e[0].parent()
Rank-2 free module \( M \) over the Integer Ring
```

We define a module element by its components w.r.t. basis \( e \):

```python
sage: u = M([2,-3], basis=e, name='u')
sage: u.display(e)
u = 2 e_0 - 3 e_1
```

Module elements can be also be created by arithmetic expressions:

```python
sage: v = -2*u + 4*e[0] ; v
Element of the Rank-2 free module \( M \) over the Integer Ring
sage: v.display(e)
6 e_1
```

We define a second basis on \( M \) from a family of linearly independent elements:

```python
sage: f = M.basis('f', from_family=(e[0]-e[1], -2*e[0]+3*e[1])) ; f
Basis \((f_0,f_1)\) on the Rank-2 free module \( M \) over the Integer Ring
sage: f[0].display(e)
f_0 = e_0 - e_1
sage: f[1].display(e)
f_1 = -2 e_0 + 3 e_1
```

We may of course express the elements of basis \( e \) in terms of basis \( f \):

```python
sage: e[0].display(f)
e_0 = 3 f_0 + f_1
sage: e[1].display(f)
e_1 = 2 f_0 + f_1
```

as well as any module element:

```python
sage: u.display(f)
u = -f_1
sage: v.display(f)
12 f_0 + 6 f_1
```
The two bases are related by a module automorphism:

```
sage: a = M.change_of_basis(e,f) ; a
Automorphism of the Rank-2 free module M over the Integer Ring
sage: a.parent()
General linear group of the Rank-2 free module M over the Integer Ring
sage: a.matrix(e)
[ 1 -2]
[-1  3]
```

Let us check that basis \( f \) is indeed the image of basis \( e \) by \( a \):

```
sage: f[0] == a(e[0])
True
sage: f[1] == a(e[1])
True
```

The reverse change of basis is of course the inverse automorphism:

```
sage: M.change_of_basis(f,e) == a^(-1)
True
```

We introduce a new module element via its components w.r.t. basis \( f \):

```
sage: v = M([2,4], basis=f, name='v')
sage: v.display(f)
v = 2 f_0 + 4 f_1
```

The sum of the two module elements \( u \) and \( v \) can be performed even if they have been defined on different bases, thanks to the known relation between the two bases:

```
sage: s = u + v ; s
Element u+v of the Rank-2 free module M over the Integer Ring
sage: s.display(e)
u+v = -4 e_0 + 7 e_1
sage: s.display(f)
u+v = 2 f_0 + 3 f_1
```

Tensor products of elements are implemented:

```
sage: t = u*v ; t
Type-(2,0) tensor u⊗v on the Rank-2 free module M over the Integer Ring
sage: t.parent()
Free module of type-(2,0) tensors on the
  Rank-2 free module M over the Integer Ring
sage: t.display(e)
u⊗v = -12 e_0⊗e_0 + 20 e_0⊗e_1 + 18 e_1⊗e_0 - 30 e_1⊗e_1
sage: t.display(f)
u⊗v = -2 f_1⊗f_0 - 4 f_1⊗f_1
```

We can access to tensor components w.r.t. to a given basis via the square bracket operator:
The parent of the automorphism $a$ is the group $GL(M)$, but $a$ can also be considered as a tensor of type $(1,1)$ on $M$:

$$
\text{sage: } a.parent()
\text{General linear group of the Rank-2 free module } M \text{ over the Integer Ring}
\text{sage: } a.tensor_type()
(1, 1)
\text{sage: } a.display(e)
e_0 \otimes e^0 - 2 e_0 \otimes e^1 - e_1 \otimes e^0 + 3 e_1 \otimes e^1
\text{sage: } a.display(f)
f_0 \otimes f^0 - 2 f_0 \otimes f^1 - f_1 \otimes f^0 + 3 f_1 \otimes f^1
$$

As such, we can form its tensor product with $t$, yielding a tensor of type $(3,1)$:

$$
\text{sage: } t^a
\text{Type-(3,1) tensor on the Rank-2 free module } M \text{ over the Integer Ring}
\text{sage: } (t^a).display(e)
-12 e_0 \otimes e_0 \otimes e_0 \otimes e^0 + 24 e_0 \otimes e_0 \otimes e_0 \otimes e^1 + 12 e_0 \otimes e_0 \otimes e_1 \otimes e^0 \\
- 36 e_0 \otimes e_0 \otimes e_1 \otimes e^1 + 20 e_0 \otimes e_1 \otimes e_0 \otimes e^0 - 40 e_0 \otimes e_1 \otimes e_0 \otimes e^1 \\
- 20 e_0 \otimes e_1 \otimes e_1 \otimes e^0 + 60 e_0 \otimes e_1 \otimes e_1 \otimes e^1 + 18 e_1 \otimes e_0 \otimes e_0 \otimes e^0 \\
- 36 e_1 \otimes e_0 \otimes e_1 \otimes e^1 - 18 e_1 \otimes e_0 \otimes e_1 \otimes e^0 + 54 e_1 \otimes e_0 \otimes e_0 \otimes e^1 \\
- 30 e_1 \otimes e_1 \otimes e_0 \otimes e^0 + 60 e_1 \otimes e_1 \otimes e_0 \otimes e^1 + 30 e_1 \otimes e_1 \otimes e_1 \otimes e^0 \\
- 90 e_1 \otimes e_1 \otimes e_1 \otimes e^1
$$

The parent of $t \otimes a$ is itself a free module of finite rank over $\mathbb{Z}$:

$$
\text{sage: } T = (t^a).parent() ; T
\text{Free module of type-(3,1) tensors on the Rank-2 free module } M \text{ over the Integer Ring}
\text{sage: } T.base_ring() 
\text{Integer Ring}
\text{sage: } T.rank()
16
$$
Differences between \texttt{FiniteRankFreeModule} and \texttt{FreeModule} (or VectorSpace)

To illustrate the differences, let us create two free modules of rank 3 over $\mathbb{Z}$, one with \texttt{FiniteRankFreeModule} and the other one with \texttt{FreeModule}:

\begin{verbatim}
sage: M = FiniteRankFreeModule(ZZ, 3, name='M') ; M
Rank-3 free module M over the Integer Ring
sage: N = FreeModule(ZZ, 3) ; N
Ambient free module of rank 3 over the principal ideal domain Integer Ring
\end{verbatim}

The main difference is that \texttt{FreeModule} returns a free module with a distinguished basis, while \texttt{FiniteRankFreeModule} does not:

\begin{verbatim}
sage: N.basis()
[(1, 0, 0), (0, 1, 0), (0, 0, 1)]
sage: M.bases()
[]
sage: M.print_bases()
No basis has been defined on the Rank-3 free module M over the Integer Ring
\end{verbatim}

This is also revealed by the category of each module:

\begin{verbatim}
sage: M.category()
Category of finite dimensional modules over Integer Ring
sage: N.category()
Category of finite dimensional modules with basis over
  (euclidean domains and infinite enumerated sets and metric spaces)
\end{verbatim}

In other words, the module created by \texttt{FreeModule} is actually $\mathbb{Z}^3$, while, in the absence of any distinguished basis, no canonical isomorphism relates the module created by \texttt{FiniteRankFreeModule} to $\mathbb{Z}^3$:

\begin{verbatim}
sage: N is ZZ^3
True
sage: M is ZZ^3
False
sage: M == ZZ^3
False
\end{verbatim}

Because it is $\mathbb{Z}^3$, $N$ is unique, while there may be various modules of the same rank over the same ring created by \texttt{FiniteRankFreeModule}; they are then distinguished by their names (actually by the complete sequence of arguments of \texttt{FiniteRankFreeModule}):

\begin{verbatim}
sage: N1 = FreeModule(ZZ, 3) ; N1
Ambient free module of rank 3 over the principal ideal domain Integer Ring
sage: N1 is N # FreeModule(ZZ, 3) is unique
True
sage: M1 = FiniteRankFreeModule(ZZ, 3, name='M_1') ; M1
Rank-3 free module M_1 over the Integer Ring
sage: M1 is M # M1 and M are different rank-3 modules over ZZ
False
\end{verbatim}
As illustrated above, various bases can be introduced on the module created by \texttt{FiniteRankFreeModule}:  

\begin{verbatim}
\texttt{sage: e = M.basis('e') \ ; e}
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring  
\texttt{sage: f = M.basis('f', from_family=(-e[0], e[1]-e[2], -2*e[1]+3*e[2])) \ ; f}
Basis (f_0,f_1,f_2) on the Rank-3 free module M over the Integer Ring  
\texttt{sage: M.bases()}
[\texttt{Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring,}
\texttt{Basis (f_0,f_1,f_2) on the Rank-3 free module M over the Integer Ring}]
\end{verbatim}

Each element of a basis is accessible via its index:

\begin{verbatim}
\texttt{sage: e[0]}
Element e_0 of the Rank-3 free module M over the Integer Ring  
\texttt{sage: e[0].parent()}
Rank-3 free module M over the Integer Ring  
\texttt{sage: f[1]}
Element f_1 of the Rank-3 free module M over the Integer Ring  
\texttt{sage: f[1].parent()}
Rank-3 free module M over the Integer Ring
\end{verbatim}

while on module \texttt{N}, the element of the (unique) basis is accessible directly from the module symbol:

\begin{verbatim}
\texttt{sage: N.0}
(1, 0, 0)  
\texttt{sage: N.1}
(0, 1, 0)  
\texttt{sage: N.0.parent()}
Ambient free module of rank 3 over the principal ideal domain Integer Ring
\end{verbatim}

The arithmetic of elements is similar; the difference lies in the display: a basis has to be specified for elements of \texttt{M}, while elements of \texttt{N} are displayed directly as elements of \(\mathbb{Z}^3\):

\begin{verbatim}
\texttt{sage: u = 2*e[0] - 3*e[2] \ ; u}
Element of the Rank-3 free module M over the Integer Ring  
\texttt{sage: u.display(e)}
2 e_0 - 3 e_2  
\texttt{sage: u.display(f)}
-2 f_0 - 6 f_1 - 3 f_2  
\texttt{sage: u[e,:]}  
[2, 0, -3]  
\texttt{sage: u[f,:]}  
[-2, -6, -3]  
\texttt{sage: v = 2*N.0 - 3*N.2 ; v}
(2, 0, -3)
\end{verbatim}

For the case of \texttt{M}, in order to avoid to specify the basis if the user is always working with the same basis (e.g. only one basis has been defined), the concept of \texttt{default basis} has been introduced:
This is different from the distinguished basis of $N$: it simply means that the mention of the basis can be omitted in function arguments:

```sage
sage: u.display()  # equivalent to u.display(e)
2 e_0 - 3 e_2
sage: u[:]          # equivalent to u[e,:]
[2, 0, -3]
```

At any time, the default basis can be changed:

```sage
sage: M.set_default_basis(f)
sage: u.display()
-2 f_0 - 6 f_1 - 3 f_2
```

Another difference between \texttt{FiniteRankFreeModule} and \texttt{FreeModule} is that for the former the range of indices can be specified (by default, it starts from 0):

```sage
sage: M = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1) ; M
Rank-3 free module $M$ over the Integer Ring
sage: e = M.basis('e') ; e  # compare with (e_0,e_1,e_2) above
Basis (e_1,e_2,e_3) on the Rank-3 free module $M$ over the Integer Ring
sage: e[1], e[2], e[3]
(Element e_1 of the Rank-3 free module $M$ over the Integer Ring,
Element e_2 of the Rank-3 free module $M$ over the Integer Ring,
Element e_3 of the Rank-3 free module $M$ over the Integer Ring)
```

All the above holds for VectorSpace instead of FreeModule: the object created by VectorSpace is actually a Cartesian power of the base field:

```sage
sage: V = VectorSpace(QQ,3) ; V
Vector space of dimension 3 over Rational Field
sage: V.category()
Category of finite dimensional vector spaces with basis
    over (number fields and quotient fields and metric spaces)
sage: V is QQ^3
True
sage: V.basis()
[[(1, 0, 0), (0, 1, 0), (0, 0, 1)]
```

To create a vector space without any distinguished basis, one has to use \texttt{FiniteRankFreeModule}:

```sage
sage: V = FiniteRankFreeModule(QQ, 3, name='V') ; V
3-dimensional vector space $V$ over the Rational Field
```
The class `FiniteRankFreeModule` has been created for the needs of the SageManifolds project, where free modules do not have any distinguished basis. Too kinds of free modules occur in the context of differentiable manifolds (see here for more details):

- the tangent vector space at any point of the manifold (cf. `TangentSpace`);
- the set of vector fields on a parallelizable open subset $U$ of the manifold, which is a free module over the algebra of scalar fields on $U$ (cf. `VectorFieldFreeModule`).

For instance, without any specific coordinate choice, no basis can be distinguished in a tangent space.

On the other side, the modules created by `FreeModule` have much more algebraic functionalities than those created by `FiniteRankFreeModule`. In particular, submodules have not been implemented yet in `FiniteRankFreeModule`. Moreover, modules resulting from `FreeModule` are tailored to the specific kind of their base ring:

- free module over a commutative ring that is not an integral domain ($\mathbb{Z}/6\mathbb{Z}$):

```
sage: R = IntegerModRing(6) ; R
Ring of integers modulo 6
sage: FreeModule(R, 3)
Ambient free module of rank 3 over Ring of integers modulo 6
sage: type(FreeModule(R, 3))
<class 'sage.modules.free_module.FreeModule_ambient_with_category'>
```

- free module over an integral domain that is not principal ($\mathbb{Z}[X]$):

```
sage: R.<X> = ZZ[] ; R
Univariate Polynomial Ring in X over Integer Ring
sage: FreeModule(R, 3)
Ambient free module of rank 3 over the integral domain Univariate Polynomial Ring in X over Integer Ring
sage: type(FreeModule(R, 3))
<class 'sage.modules.free_module.FreeModule_ambient_domain_with_category'>
```

- free module over a principal ideal domain ($\mathbb{Z}$):

```
sage: R = ZZ ; R
Integer Ring
sage: FreeModule(R, 3)
Ambient free module of rank 3 over the principal ideal domain Integer Ring
sage: type(FreeModule(R, 3))
<class 'sage.modules.free_module.FreeModule_ambient_pid_with_category'>
```

On the contrary, all objects constructed with `FiniteRankFreeModule` belong to the same class:

```
sage: R = IntegerModRing(6)
sage: type(FiniteRankFreeModule(R, 3))
(continues on next page)```
Differences between `FiniteRankFreeModule` and `CombinatorialFreeModule`

An alternative to construct free modules in Sage is `CombinatorialFreeModule`. However, as `FreeModule`, it leads to a module with a distinguished basis:

```
sage: N = CombinatorialFreeModule(ZZ, [1, 2, 3]) ; N
Free module generated by {1, 2, 3} over Integer Ring
sage: N.category()
Category of finite dimensional modules with basis over Integer Ring
```

The distinguished basis is returned by the method `basis()`:

```
sage: b = N.basis() ; b
Finite family {1: B[1], 2: B[2], 3: B[3]}
sage: b[1]
B[1]
sage: b[1].parent()
Free module generated by {1, 2, 3} over Integer Ring
```

For the free module $M$ created above with `FiniteRankFreeModule`, the method `basis` has at least one argument: the symbol string that specifies which basis is required:

```
sage: e = M.basis('e') ; e
Basis (e_1,e_2,e_3) on the Rank-3 free module M over the Integer Ring
sage: e[1]
Element e_1 of the Rank-3 free module M over the Integer Ring
sage: e[1].parent()
Rank-3 free module M over the Integer Ring
```

The arithmetic of elements is similar:

```
Element of the Rank-3 free module M over the Integer Ring
```

One notices that elements of $N$ are displayed directly in terms of their expansions on the distinguished basis. For elements of $M$, one has to use the method `display()` in order to specify the basis:

```
sage: u.display(e)
2 e_1 - 5 e_3
```

The components on the basis are returned by the square bracket operator for $M$ and by the method `coefficient` for $N$: 
\[ \text{sage: } [u[e,i] \text{ for } i \text{ in } \{1,2,3\}] \]
\[ [2, 0, -5] \]
\[ \text{sage: } u[e,:] \# \text{a shortcut for the above} \]
\[ [2, 0, -5] \]
\[ \text{sage: } [v.coefficient(i) \text{ for } i \text{ in } \{1,2,3\}] \]
\[ [2, 0, -5] \]

**class** `sage.tensor.modules.finite_rank_free_module.FiniteRankFreeModule`(**ring**, **rank**, **name**=*None*, **latex_name**=*None*, **start_index**=0, **output_formatter**=*None*, **category**=*None*)

Bases: `sage.structure.unique_representation.UniqueRepresentation`, `sage.structure.parent.Parent`

Free module of finite rank over a commutative ring.

A free module of finite rank over a commutative ring \( R \) is a module \( M \) over \( R \) that admits a finite basis, i.e. a finite family of linearly independent generators. Since \( R \) is commutative, it has the invariant basis number property, so that the rank of the free module \( M \) is defined uniquely, as the cardinality of any basis of \( M \).

No distinguished basis of \( M \) is assumed. On the contrary, many bases can be introduced on the free module along with change-of-basis rules (as module automorphisms). Each module element has then various representations over the various bases.

**Note:** The class `FiniteRankFreeModule` does not inherit from class `FreeModule_generic` nor from class `CombinatorialFreeModule`, since both classes deal with modules with a distinguished basis (see details above). Moreover, following the recommendation exposed in trac ticket #16427 the class `FiniteRankFreeModule` inherits directly from `Parent` (with the category set to `Modules`) and not from the Cython class `Module`.

The class `FiniteRankFreeModule` is a Sage parent class, the corresponding element class being `FiniteRankFreeModuleElement`.

**INPUT:**

- **ring** – commutative ring \( R \) over which the free module is constructed
- **rank** – positive integer; rank of the free module
- **name** – (default: `None`) string; name given to the free module
- **latex_name** – (default: `None`) string; \( \LaTeX \) symbol to denote the freemodule; if none is provided, it is set to `name`
- **start_index** – (default: `0`) integer; lower bound of the range of indices in bases defined on the free module
- **output_formatter** – (default: `None`) function or unbound method called to format the output of the tensor components; `output_formatter` must take 1 or 2 arguments: the first argument must be an element of the ring \( R \) and the second one, if any, some format specification

**EXAMPLES:**

Free module of rank 3 over \( \mathbb{Z} \):

\[ \text{sage: } \text{FiniteRankFreeModule\_clear\_cache\_}() \# \text{for doctests only} \]
\[ \text{sage: } M = \text{FiniteRankFreeModule}(\mathbb{Z}Z, \ 3) ; M \]

Rank-3 free module over the Integer Ring

(continues on next page)
sage: M = FiniteRankFreeModule(ZZ, 3, name='M'); M  # declaration with a name
Rank-3 free module M over the Integer Ring
sage: M.category()
Category of finite dimensional modules over Integer Ring
sage: M.base_ring()
Integer Ring
sage: M.rank()
3

If the base ring is a field, the free module is in the category of vector spaces:

sage: V = FiniteRankFreeModule(QQ, 3, name='V'); V
3-dimensional vector space V over the Rational Field
sage: V.category()
Category of finite dimensional vector spaces over Rational Field

The LaTeX output is adjusted via the parameter latex_name:

sage: latex(M)  # the default is the symbol provided in the string "name"
M
sage: M = FiniteRankFreeModule(ZZ, 3, name='M', latex_name=r'\mathcal{M}')
sage: latex(M)
\mathcal{M}

The free module M has no distinguished basis:

sage: M in ModulesWithBasis(ZZ)
False
sage: M in Modules(ZZ)
True

In particular, no basis is initialized at the module construction:

sage: M.print_bases()
No basis has been defined on the Rank-3 free module M over the Integer Ring
sage: M.bases()
[]

Bases have to be introduced by means of the method basis(), the first defined basis being considered as the default basis, meaning it can be skipped in function arguments required a basis (this can be changed by means of the method set_default_basis()):

sage: e = M.basis('e'); e
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
sage: M.default_basis()
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring

A second basis can be created from a family of linearly independent elements expressed in terms of basis e:

sage: f = M.basis('f', from_family=(-e[0], e[1]+e[2], 2*e[1]+3*e[2])); f
Basis (f_0,f_1,f_2) on the Rank-3 free module M over the Integer Ring
sage: M.print_bases()
(continues on next page)
Bases defined on the Rank-3 free module $M$ over the Integer Ring:
- $(e_0, e_1, e_2)$ (default basis)
- $(f_0, f_1, f_2)$
\[ \text{sage: M.bases()} \]
\[
[\text{Basis } (e_0, e_1, e_2) \text{ on the Rank-3 free module } M \text{ over the Integer Ring,}
\text{Basis } (f_0, f_1, f_2) \text{ on the Rank-3 free module } M \text{ over the Integer Ring}] \]

$M$ is a \textit{parent} object, whose elements are instances of \texttt{FiniteRankFreeModuleElement} (actually a dynamically generated subclass of it):

\[ \text{sage: v = M.an_element() ; v} \] Element of the Rank-3 free module $M$ over the Integer Ring
\[ \text{sage: from sage.tensor.modules.free_module_element import FiniteRankFreeModuleElement} \] \[ \text{-->} \]
\[ \text{sage: isinstance(v, FiniteRankFreeModuleElement)} \] True
\[ \text{sage: v in M} \] True
\[ \text{sage: M.is_parent_of(v)} \] True
\[ \text{sage: v.display()} \] # expansion w.r.t. the default basis $(e)$
$e_0 + e_1 + e_2$
\[ \text{sage: v.display(f)} \] 
$-f_0 + f_1$

The test suite of the category of modules is passed:
\[ \text{sage: TestSuite(M).run()} \]

Constructing an element of $M$ from (the integer) 0 yields the zero element of $M$:
\[ \text{sage: M(0)} \] Element zero of the Rank-3 free module $M$ over the Integer Ring
\[ \text{sage: M(0) is M.zero()} \] True

Non-zero elements are constructed by providing their components in a given basis:

\[ \text{sage: v = M([-1,0,3]) ; v} \] # components in the default basis $(e)$ Element of the Rank-3 free module $M$ over the Integer Ring
\[ \text{sage: v.display()} \] # expansion w.r.t. the default basis $(e)$
$-e_0 + 3 e_2$
\[ \text{sage: v.display(f)} \] 
$-f_0 + 3 f_2$
\[ \text{sage: v = M([-1,0,3], basis=f)} ; v \] # components in a specific basis Element of the Rank-3 free module $M$ over the Integer Ring
\[ \text{sage: v.display()} \] 
$-f_0 + 3 f_2$
\[ \text{sage: v = M([-1,0,3], basis=f, name='v')} ; v \] Element $v$ of the Rank-3 free module $M$ over the Integer Ring
\[ \text{sage: v.display()} \]  
(continues on next page)
\( v = -f_0 + 3 f_2 \)

```
sage: v.display()
v = e_0 + 6 e_1 + 9 e_2
```

An alternative is to construct the element from an empty list of components and to set the nonzero components afterwards:

```
sage: v = M([], name='v')
sage: v[e,0] = -1
sage: v[e,2] = 3
sage: v.display(e)
v = -e_0 + 3 e_2
```

Indices on the free module, such as indices labelling the element of a basis, are provided by the generator method \texttt{irange()}. By default, they range from 0 to the module’s rank minus one:

```
sage: list(M.irange())
[0, 1, 2]
```

This can be changed via the parameter \texttt{start_index} in the module construction:

```
sage: M1 = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1)
sage: list(M1.irange())
[1, 2, 3]
```

The parameter \texttt{output_formatter} in the constructor of the free module is used to set the output format of tensor components:

```
sage: N = FiniteRankFreeModule(QQ, 3, output_formatter=Rational.numerical_approx)
sage: e = N.basis('e')
sage: v = N([1/3, 0, -2], basis=e)
sage: v.display(e)
\# default format (53 bits of precision)
0.333333333333333 e_0 - 2.00000000000000 e_2
sage: v.display(e, format_spec=10)
\# 10 bits of precision
0.33 e_0 - 2.0 e_2
```

\textbf{Element}

alias of \texttt{sage.tensor.modules.free_module_element.FiniteRankFreeModuleElement}

\textbf{alternating_contravariant_tensor}(\texttt{(degree, name=None, latex_name=None)})

Construct an alternating contravariant tensor on the free module.

\textbf{INPUT}:

- \texttt{degree} – degree of the alternating contravariant tensor (i.e. its tensor rank)
- \texttt{name} – (default: None) string; name given to the alternating contravariant tensor
- \texttt{latex_name} – (default: None) string; LaTeX symbol to denote the alternating contravariant tensor; if none is provided, the LaTeX symbol is set to \texttt{name}

\textbf{OUTPUT}:

- instance of \texttt{AlternatingContrTensor}
EXAMPLES:

Alternating contravariant tensor on a rank-3 module:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: a = M.alternating_contravariant_tensor(2, 'a'); a
Alternating contravariant tensor a of degree 2 on the
Rank-3 free module M over the Integer Ring
```

The nonzero components in a given basis have to be set in a second step, thereby fully specifying the
alternating form:

```
sage: e = M.basis('e'); e
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
sage: a.set_comp(e)[0,1] = 2
sage: a.set_comp(e)[1,2] = -3
sage: a.display(e)
a = 2 e_0∧e_1 - 3 e_1∧e_2
```

An alternating contravariant tensor of degree 1 is simply an element of the module:

```
sage: a = M.alternating_contravariant_tensor(1, 'a'); a
Element a of the Rank-3 free module M over the Integer Ring
```

See `AlternatingContrTensor` for more documentation.

`alternating_form(degree, name=None, latex_name=None)`

Construct an alternating form on the free module.

**INPUT:**

- `degree` – the degree of the alternating form (i.e. its tensor rank)
- `name` – (default: None) string; name given to the alternating form
- `latex_name` – (default: None) string; LaTeX symbol to denote the alternating form; if none is pro-
  vided, the LaTeX symbol is set to `name`

**OUTPUT:**

- instance of `FreeModuleAltForm`

**EXAMPLES:**

Alternating forms on a rank-3 module:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: a = M.alternating_form(2, 'a'); a
Alternating form a of degree 2 on the
Rank-3 free module M over the Integer Ring
```

The nonzero components in a given basis have to be set in a second step, thereby fully specifying the
alternating form:

```
sage: e = M.basis('e'); e
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
sage: a.set_comp(e)[0,1] = 2
sage: a.set_comp(e)[1,2] = -3
sage: a.display(e)
a = 2 e^0∧e^1 - 3 e^1∧e^2
```
An alternating form of degree 1 is a linear form:

```
sage: a = M.alternating_form(1, 'a') ; a
Linear form a on the Rank-3 free module M over the Integer Ring
```

To construct such a form, it is preferable to call the method `linear_form()` instead:

```
sage: a = M.linear_form('a') ; a
Linear form a on the Rank-3 free module M over the Integer Ring
```

See `FreeModuleAltForm` for more documentation.

**automorphism**(matrix=None, basis=None, name=None, latex_name=None)

Construct a module automorphism of self.

Denoting self by $M$, an automorphism of self is an element of the general linear group $\text{GL}(M)$.

**INPUT:**

- `matrix` – (default: None) matrix of size $\text{rank}(M) \times \text{rank}(M)$ representing the automorphism with respect to basis; this entry can actually be any material from which a matrix of elements of self base ring can be constructed; the columns of matrix must be the components w.r.t. basis of the images of the basis. If `matrix` is None, the automorphism has to be initialized afterwards by method `set_comp()` or via the operator $\mathbf{.}$.
- `basis` – (default: None) basis of self defining the matrix representation; if None the default basis of self is assumed.
- `name` – (default: None) string; name given to the automorphism
- `latex_name` – (default: None) string; LaTeX symbol to denote the automorphism; if none is provided, the LaTeX symbol is set to name

**OUTPUT:**

- instance of `FreeModuleAutomorphism`

**EXAMPLES:**

Automorphism of a rank-2 free $\mathbb{Z}$-module:

```
sage: M = FiniteRankFreeModule(ZZ, 2, name='M')
sage: e = M.basis('e')
sage: a = M.automorphism(matrix=[[1,2],[1,3]], basis=e, name='a') ; a
Automorphism a of the Rank-2 free module M over the Integer Ring
sage: a.parent()
General linear group of the Rank-2 free module M over the Integer Ring
sage: a.matrix(e)
[1 2]
[1 3]
```

An automorphism is a tensor of type (1,1):

```
sage: a.tensor_type()
(1, 1)
sage: a.display(e)
a = e_0 \otimes e^0 + 2 e_0 \otimes e^1 + e_1 \otimes e^0 + 3 e_1 \otimes e^1
```

The automorphism components can be specified in a second step, as components of a type-(1,1) tensor:
\begin{verbatim}
sage: a1 = M.automorphism(name='a')
sage: a1[e,:] = [[1,2],[1,3]]
sage: a1.matrix(e)
[1 2]
[1 3]
sage: a1 == a
True

Component by component specification:

sage: a2 = M.automorphism(name='a')
sage: a2[0,0] = 1  # component set in the module's default basis (e)
sage: a2[0,1] = 2
sage: a2[1,0] = 1
sage: a2[1,1] = 3
sage: a2.matrix(e)
[1 2]
[1 3]
sage: a2 == a
True

See \texttt{FreeModuleAutomorphism} for more documentation.
\end{verbatim}

\textbf{bases()}

Return the list of bases that have been defined on the free module \texttt{self}.

Use the method \texttt{print_bases()} to get a formatted output with more information.

\textbf{OUTPUT:}

\begin{itemize}
  \item list of instances of class \texttt{FreeModuleBasis}
\end{itemize}

\textbf{EXAMPLES:}

Bases on a rank-3 free module:

\begin{verbatim}
sage: M = FiniteRankFreeModule(ZZ, 3, name='M_3', start_index=1)
sage: M.bases()
[]
sage: e = M.basis('e')
sage: M.bases()
[Basis (e_1,e_2,e_3) on the Rank-3 free module M_3 over the Integer Ring]
sage: f = M.basis('f')
sage: M.bases()
[Basis (e_1,e_2,e_3) on the Rank-3 free module M_3 over the Integer Ring, Basis (f_1,f_2,f_3) on the Rank-3 free module M_3 over the Integer Ring]
\end{verbatim}

\textbf{basis(symbol, latex_symbol=None, from_family=None, indices=None, latex_indices=None, symbol_dual=None, latex_symbol_dual=None)}

Define or return a basis of the free module \texttt{self}.

Let \( M \) denotes the free module \texttt{self} and \( n \) its rank.

The basis can be defined from a set of \( n \) linearly independent elements of \( M \) by means of the argument \texttt{from_family}. If \texttt{from_family} is not specified, the basis is created from scratch and, at this stage, is unrelated to bases that could have been defined previously on \( M \). It can be related afterwards by means of the method \texttt{set_change_of_basis()}.\end{itemize}
If the basis specified by the given symbol already exists, it is simply returned, whatever the value of the arguments latex_symbol or from_family.

Note that another way to construct a basis of self is to use the method new_basis() on an existing basis, with the automorphism relating the two bases as an argument.

INPUT:

- symbol – either a string, to be used as a common base for the symbols of the elements of the basis, or a list/tuple of strings, representing the individual symbols of the elements of the basis
- latex_symbol – (default: None) either a string, to be used as a common base for the LaTeX symbols of the elements of the basis, or a list/tuple of strings, representing the individual LaTeX symbols of the elements of the basis; if None, symbol is used in place of latex_symbol
- from_family – (default: None) tuple or list of n linearly independent elements of the free module self (n being the rank of self)
- indices – (default: None; used only if symbol is a single string) list/tuple of strings representing the indices labelling the elements of the basis; if None, the indices will be generated as integers within the range declared on self
- latex_indices – (default: None) list/tuple of strings representing the indices for the LaTeX symbols of the elements of the basis; if None, indices is used instead
- symbol_dual – (default: None) same as symbol but for the dual basis; if None, symbol must be a string and is used for the common base of the symbols of the elements of the dual basis
- latex_symbol_dual – (default: None) same as latex_symbol but for the dual basis

OUTPUT:

- instance of FreeModuleBasis representing a basis on self

EXAMPLES:

Bases on a rank-3 free module:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e') ; e
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
sage: e[0]
Element e_0 of the Rank-3 free module M over the Integer Ring
sage: latex(e)
\left(e_{0},e_{1},e_{2}\right)
```

The LaTeX symbol can be set explicitly:

```
sage: eps = M.basis('eps', latex_symbol=r'\epsilon') ; eps
Basis (eps_0,eps_1,eps_2) on the Rank-3 free module M over the Integer Ring
sage: latex(eps)
\left(\epsilon_{0},\epsilon_{1},\epsilon_{2}\right)
```

The indices can be customized:

```
sage: f = M.basis('f', indices=('x', 'y', 'z')) ; f
Basis (f_x,f_y,f_z) on the Rank-3 free module M over the Integer Ring
sage: latex(f[1])
f_{y}
```
By providing a list or a tuple for the argument symbol, one can have a different symbol for each element of the basis; it is then mandatory to specify some symbols for the dual basis:

```
sage: g = M.basis(('a', 'b', 'c'), symbol_dual=('A', 'B', 'C')); g
g = Basis (a,b,c) on the Rank-3 free module M over the Integer Ring
sage: g.dual_basis()
dg = Dual basis (A,B,C) on the Rank-3 free module M over the Integer Ring
```

If the provided symbol and indices are that of an already defined basis, the latter is returned (no new basis is created):

```
sage: M.basis('e') is e
True
sage: M.basis('eps') is eps
True
sage: M.basis('e', indices=['x', 'y', 'z']) is e
False
sage: M.basis('e', indices=['x', 'y', 'z']) is 
.....: M.basis('e', indices=['x', 'y', 'z'])
True
```

The individual elements of the basis are labelled according the parameter start_index provided at the free module construction:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1)
sage: e = M.basis('e')
Basis (e_1,e_2,e_3) on the Rank-3 free module M over the Integer Ring
sage: e[1]
Element e_1 of the Rank-3 free module M over the Integer Ring
```

Construction of a basis from a spanning family of linearly independent module elements:

```
sage: f1 = -e[2]
sage: f2 = 4*e[1] + 3*e[3]
sage: f3 = 7*e[1] + 5*e[3]
sage: f = M.basis('f', from_family=(f1,f2,f3))
sage: f[1].display()
f_1 = -e_2
sage: f[2].display()
f_2 = 4 e_1 + 3 e_3
sage: f[3].display()
f_3 = 7 e_1 + 5 e_3
```

The change-of-basis automorphisms have been registered:

```
sage: M.change_of_basis(e,f).matrix(e)
[ 0 4 7]
[-1 0 0]
[ 0 3 5]
sage: M.change_of_basis(f,e).matrix(e)
[ 0 -1 0]
[-5 0 7]
[ 3 0 -4]
sage: M.change_of_basis(f,e) == M.change_of_basis(e,f).inverse()
True
```
Check of the change-of-basis $e \to f$:

```python
sage: a = M.change_of_basis(e,f) ; a
Automorphism of the Rank-3 free module M over the Integer Ring
sage: all( f[i] == a(e[i]) for i in M.irange() )
True
```

Providing a family of module elements that are not linearly independent raise an error:

```python
sage: g = M.basis('g', from_family=(f1, f2, f1+f2))
Traceback (most recent call last):
 ... ValueError: the provided module elements are not linearly independent
```

For more documentation on bases see `FreeModuleBasis`.

`change_of_basis(basis1, basis2)`

Return a module automorphism linking two bases defined on the free module `self`.

If the automorphism has not been recorded yet (in the internal dictionary `self._basis_changes`), it is computed by transitivity, i.e. by performing products of recorded changes of basis.

**INPUT:**

- `basis1` – a basis of `self`, denoted $(e_i)$ below
- `basis2` – a basis of `self`, denoted $(f_i)$ below

**OUTPUT:**

- instance of `FreeModuleAutomorphism` describing the automorphism $P$ that relates the basis $(e_i)$ to the basis $(f_i)$ according to $f_i = P(e_i)$

**EXAMPLES:**

Changes of basis on a rank-2 free module:

```python
sage: FiniteRankFreeModule._clear_cache_() # for doctests only
sage: M = FiniteRankFreeModule(ZZ, 2, name='M', start_index=1)
sage: e = M.basis('e')
```

```python
sage: f = M.basis('f', from_family=(e[1]+2*e[2], e[1]+3*e[2]))
```

```python
sage: P = M.change_of_basis(e,f) ; P
Automorphism of the Rank-2 free module M over the Integer Ring
sage: P.matrix(e)
[1 1]
[2 3]
```

Note that the columns of this matrix contain the components of the elements of basis $f$ w.r.t. to basis $e$:

```python
sage: f[1].display(e)
f_1 = e_1 + 2 e_2
sage: f[2].display(e)
f_2 = e_1 + 3 e_2
```

The change of basis is cached:

```python
sage: P is M.change_of_basis(e,f)
True
```
Check of the change-of-basis automorphism:

```python
sage: f[1] == P(e[1])
True
sage: f[2] == P(e[2])
True
```

Check of the reverse change of basis:

```python
sage: M.change_of_basis(f,e) == P^(-1)
True
```

We have of course:

```python
sage: M.change_of_basis(e,e)
Identity map of the Rank-2 free module M over the Integer Ring
sage: M.change_of_basis(e,e) is M.identity_map()
True
```

Let us introduce a third basis on $M$:

```python
sage: h = M.basis('h', from_family=(3*e[1]+4*e[2], 5*e[1]+7*e[2]))
```

The change of basis $e \to h$ has been recorded directly from the definition of $h$:

```python
sage: Q = M.change_of_basis(e,h) ; Q.matrix(e)
[[3, 5]
 [4, 7]
```

The change of basis $f \to h$ is computed by transitivity, i.e. from the changes of basis $f \to e$ and $e \to h$:

```python
sage: R = M.change_of_basis(f,h) ; R
Automorphism of the Rank-2 free module M over the Integer Ring
sage: R.matrix(e)
[[-1, 2]
 [-2, 3]
```

Let us check that $R$ is indeed the change of basis $f \to h$:

```python
sage: h[1] == R(f[1])
True
True
```

A related check is:

```python
sage: R == Q*P^(-1)
True
```

**default_basis()**

Return the default basis of the free module `self`. 

---

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The default basis is simply a basis whose name can be skipped in methods requiring a basis as an argument. By default, it is the first basis introduced on the module. It can be changed by the method set_default_basis().

OUTPUT:

- instance of FreeModuleBasis

EXAMPLES:

At the module construction, no default basis is assumed:

```
sage: M = FiniteRankFreeModule(ZZ, 2, name='M', start_index=1)
sage: M.default_basis()
```

No default basis has been defined on the Rank-2 free module $M$ over the Integer Ring

The first defined basis becomes the default one:

```
sage: e = M.basis('e')
Basis (e_1,e_2) on the Rank-2 free module M over the Integer Ring
sage: M.default_basis()
Basis (e_1,e_2) on the Rank-2 free module M over the Integer Ring
sage: f = M.basis('f')
Basis (f_1,f_2) on the Rank-2 free module M over the Integer Ring
sage: M.default_basis()
Basis (e_1,e_2) on the Rank-2 free module M over the Integer Ring
```

dual()

Return the dual module of self.

EXAMPLES:

Dual of a free module over $\mathbb{Z}$:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: M.dual()
```

Dual of the Rank-3 free module $M$ over the Integer Ring

```
sage: latex(M.dual())
M^*
```

The dual is a free module of the same rank as $M$:

```
sage: isinstance(M.dual(), FiniteRankFreeModule)
True
sage: M.dual().rank()
3
```

It is formed by alternating forms of degree 1, i.e. linear forms:

```
sage: M.dual() is M.dual_exterior_power(1)
True
sage: M.dual().an_element()
Linear form on the Rank-3 free module M over the Integer Ring
sage: a = M.linear_form()
sage: a in M.dual()
True
```
The elements of a dual basis belong of course to the dual module:

```python
sage: e = M.basis('e')
sage: e.dual_basis()[0] in M.dual()
True
```

dual_exterior_power($p$)
Return the $p$-th exterior power of the dual of self.

If $M$ stands for the free module self, the $p$-th exterior power of the dual of $M$ is the set $\Lambda^p(M^*)$ of all alternating forms of degree $p$ on $M$, i.e. of all multilinear maps 
\[ M \times \cdots \times M \rightarrow R \]
$p$ times

that vanish whenever any of two of their arguments are equal. $\Lambda^p(M^*)$ is a free module of rank $\binom{n}{p}$ over the same ring as $M$, where $n$ is the rank of $M$.

INPUT:
- $p$ – non-negative integer

OUTPUT:
- for $p = 0$, the base ring $R$
- for $p \geq 1$, instance of ExtPowerDualFreeModule representing the free module $\Lambda^p(M^*)$

EXAMPLES:

Exterior powers of the dual of a free $\mathbb{Z}$-module of rank 3:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: M.dual_exterior_power(0)  # return the base ring
Integer Ring
sage: M.dual_exterior_power(1)  # return the dual module
Dual of the Rank-3 free module M over the Integer Ring
sage: M.dual_exterior_power(1) is M.dual()
True
sage: M.dual_exterior_power(2)
2nd exterior power of the dual of the Rank-3 free module M over the Integer Ring
sage: M.dual_exterior_power(2).an_element()
Alternating form of degree 2 on the Rank-3 free module M over the Integer Ring
sage: M.dual_exterior_power(2).an_element().display()
e^0\wedge e^1
sage: M.dual_exterior_power(3)
3rd exterior power of the dual of the Rank-3 free module M over the Integer Ring
sage: M.dual_exterior_power(3).an_element()
Alternating form of degree 3 on the Rank-3 free module M over the Integer Ring
sage: M.dual_exterior_power(3).an_element().display()
e^0\wedge e^1\wedge e^2
```

See ExtPowerDualFreeModule for more documentation.

dermorphism($\text{matrix_rep}$, $\text{basis}$=None, $\text{name}$=None, $\text{latex_name}$=None)
Construct an endomorphism of the free module self.

The returned object is a module morphism $\phi : M \rightarrow M$, where $M$ is self.

INPUT:
• **matrix_rep** – matrix of size \( \text{rank}(M) \times \text{rank}(M) \) representing the endomorphism with respect to basis; this entry can actually be any material from which a matrix of elements of self base ring can be constructed; the columns of matrix_rep must be the components w.r.t. basis of the images of the elements of basis.

• **basis** – (default: None) basis of self defining the matrix representation; if None the default basis of self is assumed.

• **name** – (default: None) string; name given to the endomorphism

• **latex_name** – (default: None) string; LaTeX symbol to denote the endomorphism; if none is provided, name will be used.

**OUTPUT:**

• the endomorphism \( \phi : M \to M \) corresponding to the given specifications, as an instance of `FiniteRankFreeModuleMorphism`

**EXAMPLES:**

Construction of an endomorphism with minimal data (module’s default basis and no name):

```python
sage: M = FiniteRankFreeModule(ZZ, 2, name='M')
sage: e = M.basis('e')
sage: phi = M.endomorphism([[1,-2], [-3,4]]) ; phi
Generic endomorphism of Rank-2 free module M over the Integer Ring
sage: phi.matrix() # matrix w.r.t the default basis
[ 1 -2]
[-3 4]
```

Construction with full list of arguments (matrix given a basis different from the default one):

```python
sage: a = M.automorphism() ; a[0,1], a[1,0] = 1, -1
sage: ep = e.new_basis(a, 'ep', latex_symbol="e'")
sage: phi = M.endomorphism([[1,-2], [-3,4]], basis=ep, name='phi', latex_name=r'\phi')
sage: phi
Generic endomorphism of Rank-2 free module M over the Integer Ring
sage: phi.matrix(ep) # the input matrix
[ 1 -2]
[-3 4]
sage: phi.matrix() # matrix w.r.t the default basis
[4 3]
[2 1]
```

See `FiniteRankFreeModuleMorphism` for more documentation.

**exterior_power**(\( p \))

Return the \( p \)-th exterior power of self.

If \( M \) stands for the free module self, the \( p \)-th exterior power of \( M \) is the set \( \Lambda^p(M) \) of all alternating contravariant tensors of rank \( p \), i.e. of all multilinear maps

\[
\underbrace{M^* \times \cdots \times M^*}_{p \text{ times}} \to R
\]

that vanish whenever any of two of their arguments are equal. \( \Lambda^p(M) \) is a free module of rank \( \binom{n}{p} \) over the same ring as \( M \), where \( n \) is the rank of \( M \).

**INPUT:**
• p – non-negative integer

OUTPUT:

• for \( p = 0 \), the base ring \( R \)
• for \( p = 1 \), the free module \( M \), since \( \Lambda^1(M) = M \)
• for \( p \geq 2 \), instance of ExtPowerFreeModule representing the free module \( \Lambda^p(M) \)

EXAMPLES:

Exterior powers of the dual of a free \( \mathbb{Z} \)-module of rank 3:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: M.exterior_power(0)  # return the base ring
Integer Ring
sage: M.exterior_power(1)  # return the module itself
Rank-3 free module M over the Integer Ring
sage: M.exterior_power(1) is M
True
sage: M.exterior_power(2)  # 2nd exterior power of the Rank-3 free module M over the Integer Ring
sage: M.exterior_power(2).an_element()  # Alternating contravariant tensor of degree 2 on the Rank-3 free module M over the Integer Ring
sage: M.exterior_power(2).an_element().display()
e_0 \wedge e_1
sage: M.exterior_power(3)  # 3rd exterior power of the Rank-3 free module M over the Integer Ring
sage: M.exterior_power(3).an_element()  # Alternating contravariant tensor of degree 3 on the Rank-3 free module M over the Integer Ring
sage: M.exterior_power(3).an_element().display()
e_0 \wedge e_1 \wedge e_2
```

See ExtPowerFreeModule for more documentation.

general_linear_group()

Return the general linear group of self.

If self is the free module \( M \), the general linear group is the group \( \text{GL}(M) \) of automorphisms of \( M \).

OUTPUT:

• instance of class FreeModuleLinearGroup representing \( \text{GL}(M) \)

EXAMPLES:

The general linear group of a rank-3 free module:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: GL = M.general_linear_group() ; GL
General linear group of the Rank-3 free module M over the Integer Ring
sage: GL.category()
Category of groups
sage: type(GL)
<class 'sage.tensor.modules.free_module_linear_group.FreeModuleLinearGroup_with_category'>
```
There is a unique instance of the general linear group:

```
sage: M.general_linear_group() is GL
True
```

The group identity element:

```
sage: GL.one()
Identity map of the Rank-3 free module M over the Integer Ring
sage: GL.one().matrix(e)
[1 0 0]
[0 1 0]
[0 0 1]
```

An element:

```
sage: GL.an_element()
Automorphism of the Rank-3 free module M over the Integer Ring
sage: GL.an_element().matrix(e)
[ 1 0 0]
[ 0 -1 0]
[ 0 0 1]
```

See `FreeModuleLinearGroup` for more documentation.

\[ \text{hom}(\text{codomain}, \text{matrix_rep}, \text{bases}=\text{None}, \text{name}=\text{None}, \text{latex_name}=\text{None}) \]

Define a module homomorphism

\[ \phi : M \rightarrow N, \]

where \( M \) is \( \text{self} \) and \( N \) is a free module of finite rank over the same ring \( R \) as \( \text{self} \).

**Note:** This method is a redefinition of `sage.structure.parent.Parent.Parent.hom()` because the latter assumes that \( \text{self} \) has some privileged generators, while an instance of `FiniteRankFreeModule` has no privileged basis.

**INPUT:**

- `codomain` – the target module \( N \)
- `matrix_rep` – matrix of size rank(\( N \))*rank(\( M \)) representing the homomorphism with respect to the pair of bases defined by \( \text{bases} \); this entry can actually be any material from which a matrix of elements of \( R \) can be constructed; the columns of \( \text{matrix_rep} \) must be the components w.r.t. \( \text{basis}_N \) of the images of the elements of \( \text{basis}_M \).
- `bases` – (default: \( \text{None} \)) pair (\( \text{basis}_M \), \( \text{basis}_N \)) defining the matrix representation, \( \text{basis}_M \) being a basis of \( \text{self} \) and \( \text{basis}_N \) a basis of module \( N \); if \( \text{None} \) the pair formed by the default bases of each module is assumed.
- `name` – (default: \( \text{None} \)) string; name given to the homomorphism
- `latex_name` – (default: \( \text{None} \)) string; LaTeX symbol to denote the homomorphism; if \( \text{None} \), name will be used.
• the homomorphism $\phi : M \rightarrow N$ corresponding to the given specifications, as an instance of `FiniteRankFreeModuleMorphism`

**EXAMPLES:**

Homomorphism between two free modules over $\mathbb{Z}$:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: N = FiniteRankFreeModule(ZZ, 2, name='N')
sage: e = M.basis('e')
sage: f = N.basis('f')
sage: phi = M.hom(N, [[-1,2,0], [5,1,2]]) ; phi
Generic morphism:
  From: Rank-3 free module M over the Integer Ring
  To:  Rank-2 free module N over the Integer Ring
```

Homomorphism defined by a matrix w.r.t. bases that are not the default ones:

```python
sage: ep = M.basis('ep', latex_symbol=r"e")
sage: fp = N.basis('fp', latex_symbol=r"f")
sage: phi = M.hom(N, [[3,2,1], [1,2,3]], bases=(ep, fp)) ; phi
```

Call with all arguments specified:

```python
sage: phi = M.hom(N, [[3,2,1], [1,2,3]], bases=(ep, fp),
          ...: name='phi', latex_name=r'\phi')
```

The parent:

```python
sage: phi.parent() is Hom(M,N)
True
```

See class `FiniteRankFreeModuleMorphism` for more documentation.

`identity_map(name='Id', latex_name=None)`

Return the identity map of the free module `self`.

**INPUT:**

- `name` – (string; default: ‘Id’) name given to the identity identity map
- `latex_name` – (string; default: None) LaTeX symbol to denote the identity map; if none is provided, the LaTeX symbol is set to ‘\text{Id}’ if name is ‘Id’ and to `name` otherwise

**OUTPUT:**

- the identity map of `self` as an instance of `FreeModuleAutomorphism`

**EXAMPLES:**

Identity map of a rank-3 $\mathbb{Z}$-module:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: Id = M.identity_map() ; Id
```
The identity map is actually the identity element of GL(M):

```python
sage: Id is M.general_linear_group().one()
True
```

It is also a tensor of type-(1, 1) on M:

```python
sage: Id.tensor_type()
(1, 1)
sage: Id.comp(e)
Kronecker delta of size 3x3
sage: Id[:]
[1 0 0]
[0 1 0]
[0 0 1]
```

Example with a \LaTeX{} symbol different from the default one and set at the creation of the object:

```python
sage: N = FiniteRankFreeModule(ZZ, 3, name='N')
sage: f = N.basis('f')
sage: Id = N.identity_map(name='Id_N', latex_name=r'\mathrm{Id}_N')
sage: Id
Identity map of the Rank-3 free module N over the Integer Ring
sage: latex(Id)
\mathrm{Id}_N
```

### irange(start=None)

Single index generator, labelling the elements of a basis of self.

**INPUT:**

- **start** – (default: None) integer; initial value of the index; if none is provided, self._sindex is assumed

**OUTPUT:**

- an iterable index, starting from start and ending at self._sindex + self.rank() - 1
EXAMPLES:

Index range on a rank-3 module:

```
sage: M = FiniteRankFreeModule(ZZ, 3)
sage: list(M.irange())
[0, 1, 2]
sage: list(M.irange(start=1))
[1, 2]
```

The default starting value corresponds to the parameter `start_index` provided at the module construction (the default value being 0):

```
sage: M1 = FiniteRankFreeModule(ZZ, 3, start_index=1)
sage: list(M1.irange())
[1, 2, 3]
sage: M2 = FiniteRankFreeModule(ZZ, 3, start_index=-4)
sage: list(M2.irange())
[-4, -3, -2]
```

**isomorphism_with_fixed_basis** *(basis, codomain=None)*

Construct the canonical isomorphism from the free module `self` to a free module in which `basis` of `self` is mapped to the distinguished basis of `codomain`.

INPUT:

- `basis` – the basis of `self` which should be mapped to the distinguished basis on `codomain`
- `codomain` – (default: `None`) the codomain of the isomorphism represented by a free module within the category `ModulesWithBasis` with the same rank and base ring as `self`; if `None` a free module represented by `CombinatorialFreeModule` is constructed

OUTPUT:

- a module morphism represented by `ModuleMorphismFromFunction`

EXAMPLES:

```
sage: V = FiniteRankFreeModule(QQ, 3, start_index=1); V
3-dimensional vector space over the Rational Field
sage: basis = e = V.basis("e"); basis
Basis (e_1,e_2,e_3) on the 3-dimensional vector space over the Rational Field
sage: phi_e = V.isomorphism_with_fixed_basis(basis); phi_e
Generic morphism:
  From: 3-dimensional vector space over the Rational Field
  To: Free module generated by {1, 2, 3} over Rational Field
sage: phi_e.category()
Category of finite dimensional vector spaces with basis over Rational Field
sage: phi_e(e[1] + 2 * e[2])
```

```
sage: abc = V.basis(['a', 'b', 'c'], symbol_dual=['d', 'e', 'f']); abc
Basis (a,b,c) on the 3-dimensional vector space over the Rational Field
sage: phi_abc = V.isomorphism_with_fixed_basis(abc); phi_abc
Generic morphism:
```
From: 3-dimensional vector space over the Rational Field
To: Free module generated by \{1, 2, 3\} over Rational Field
`sage: phi_abc(abc[1] + 2 * abc[2])
```

Providing a codomain:
```
sage: W = CombinatorialFreeModule(QQ, ['a', 'b', 'c'])
sage: phi_eW = V.isomorphism_with_fixed_basis(basis, codomain=W); phi_eW
Generic morphism:
From: 3-dimensional vector space over the Rational Field
To: Free module generated by \{'a', 'b', 'c'\} over Rational Field
`sage: phi_eW(e[1] + 2 * e[2])
B['a'] + 2*B['b']
```

**linear_form**(name=None, latex_name=None)

Construct a linear form on the free module `self`.

A linear form on a free module \(M\) over a ring \(R\) is a map \(M \rightarrow R\) that is linear. It can be viewed as a tensor of type \((0, 1)\) on \(M\).

**INPUT:**

- `name` – (default: None) string; name given to the linear form
- `latex_name` – (default: None) string; LaTeX symbol to denote the linear form; if none is provided, the LaTeX symbol is set to `name`

**OUTPUT:**

- instance of `FreeModuleAltForm`

**EXAMPLES:**

Linear form on a rank-3 free module:
```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: a = M.linear_form('A') ; a
Linear form A on the Rank-3 free module M over the Integer Ring
`sage: a[:]= [2,-1,3] # components w.r.t. the module's default basis (e)
sage: a.display()
A = 2 e^0 - e^1 + 3 e^2
```

A linear form maps module elements to ring elements:
```
sage: v = M([1,1,1])
sage: a(v)
4
```

Test of linearity:
```
sage: u = M([-5,-2,7])
sage: a(3*u - 4*v) == 3*a(u) - 4*a(v)
True
```

See `FreeModuleAltForm` for more documentation.
print_bases()
Display the bases that have been defined on the free module self.
Use the method bases() to get the raw list of bases.

EXAMPLES:
Bases on a rank-4 free module:

```python
sage: M = FiniteRankFreeModule(ZZ, 4, name='M', start_index=1)
sage: M.print_bases()
No basis has been defined on the
  Rank-4 free module M over the Integer Ring
sage: e = M.basis('e')
sage: M.print_bases()
Bases defined on the Rank-4 free module M over the Integer Ring:
  - (e_1,e_2,e_3,e_4) (default basis)
sage: f = M.basis('f')
sage: M.print_bases()
Bases defined on the Rank-4 free module M over the Integer Ring:
  - (e_1,e_2,e_3,e_4)
  - (f_1,f_2,f_3,f_4)
sage: M.set_default_basis(f)
sage: M.print_bases()
Bases defined on the Rank-4 free module M over the Integer Ring:
  - (e_1,e_2,e_3,e_4)
  - (f_1,f_2,f_3,f_4) (default basis)
```

rank()
Return the rank of the free module self.
Since the ring over which self is built is assumed to be commutative (and hence has the invariant basis number property), the rank is defined uniquely, as the cardinality of any basis of self.

EXAMPLES:
Rank of free modules over \( \mathbb{Z} \):

```python
sage: M = FiniteRankFreeModule(ZZ, 3)
sage: M.rank()
3
sage: M.tensor_module(0,1).rank()
3
sage: M.tensor_module(0,2).rank()
9
sage: M.tensor_module(1,0).rank()
3
sage: M.tensor_module(1,1).rank()
9
sage: M.tensor_module(1,2).rank()
27
sage: M.tensor_module(2,2).rank()
81
```

set_change_of_basis(basis1, basis2, change_of_basis, compute_inverse=True)
Relates two bases by an automorphism of self.
This updates the internal dictionary self._basis_changes.
INPUT:

• basis1 – basis 1, denoted \( (e_i) \) below
• basis2 – basis 2, denoted \( (f_i) \) below
• change_of_basis – instance of class \texttt{FreeModuleAutomorphism} describing the automorphism \( P \) that relates the basis \( (e_i) \) to the basis \( (f_i) \) according to \( f_i = P(e_i) \)
• compute_inverse (default: True) – if set to True, the inverse automorphism is computed and the change from basis \( (f_i) \) to \( (e_i) \) is set to it in the internal dictionary \texttt{self._basis_changes}

EXAMPLES:

Defining a change of basis on a rank-2 free module:

```python
sage: M = FiniteRankFreeModule(QQ, 2, name='M')
sage: e = M.basis('e')
sage: f = M.basis('f')
sage: a = M.automorphism()
sage: a[:,:] = [[1, 2], [-1, 3]]
sage: M.set_change_of_basis(e, f, a)
```

The change of basis and its inverse have been recorded:

```python
sage: M.change_of_basis(e,f).matrix(e)
[ 1 2]
[-1 3]
sage: M.change_of_basis(f,e).matrix(e)
[ 3/5 -2/5]
[ 1/5 1/5]
```

and are effective:

```python
sage: f[0].display(e)
f_0 = e_0 - e_1
sage: e[0].display(f)
e_0 = 3/5 f_0 + 1/5 f_1
```

\texttt{set_default_basis}(basis)

Sets the default basis of self.

The default basis is simply a basis whose name can be skipped in methods requiring a basis as an argument. By default, it is the first basis introduced on the module.

INPUT:

• basis – instance of \texttt{FreeModuleBasis} representing a basis on self

EXAMPLES:

Changing the default basis on a rank-3 free module:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1)
sage: e = M.basis('e') ; e
Basis \( (e_1,e_2,e_3) \) on the Rank-3 free module \( M \) over the Integer Ring
sage: f = M.basis('f') ; f
Basis \( (f_1,f_2,f_3) \) on the Rank-3 free module \( M \) over the Integer Ring
sage: M.default_basis()
```

(continues on next page)
Basis \((e_1,e_2,e_3)\) on the Rank-3 free module \(M\) over the Integer Ring

```python
sage: M.set_default_basis(f)
sage: M.default_basis()
```
Basis \((f_1,f_2,f_3)\) on the Rank-3 free module \(M\) over the Integer Ring

```python
sage: sym_bilinear_form(name=None, latex_name=None)
```
Construct a symmetric bilinear form on the free module `self`.  

**INPUT:**

- `name` – (default: `None`) string; name given to the symmetric bilinear form
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the symmetric bilinear form; if none is provided, the LaTeX symbol is set to `name`

**OUTPUT:**

- instance of `FreeModuleTensor` of tensor type \((0, 2)\) and symmetric

**EXAMPLES:**

Symmetric bilinear form on a rank-3 free module:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: a = M.sym_bilinear_form('A') ; a
Symmetric bilinear form A on the
Rank-3 free module M over the Integer Ring
```
A symmetric bilinear form is a type-\((0, 2)\) tensor that is symmetric:

```python
sage: a.parent()
Free module of type-\((0, 2)\) tensors on the
Rank-3 free module M over the Integer Ring
sage: a.tensor_type()
(0, 2)
sage: a.tensor_rank()
2
sage: a.symmetries()
symmetry: (0, 1); no antisymmetry
```
Components with respect to a given basis:

```python
sage: e = M.basis('e')
sage: a[0,0], a[0,1], a[0,2] = 1, 2, 3
sage: a[1,1], a[1,2] = 4, 5
sage: a[2,2] = 6
```
Only independent components have been set; the other ones are deduced by symmetry:

```python
sage: a[1,0], a[2,0], a[2,1]
(2, 3, 5)
sage: a[:]
[1 2 3]
[2 4 5]
[3 5 6]
```
A symmetric bilinear form acts on pairs of module elements:
The sum of two symmetric bilinear forms is another symmetric bilinear form:

```
sage: b = M.sym_bilinear_form('B')
sage: b[0,0], b[0,1], b[1,2] = -2, 1, -3
sage: s = a + b ; s
Symmetric bilinear form A+B on the
  Rank-3 free module M over the Integer Ring
sage: a[:], b[:], s[:]
([1 2 3] [-2 1 0] [-1 3 3]
 [2 4 5] [ 1 0 -3] [ 3 4 2]
[3 5 6], [ 0 -3 0], [ 3 2 6]
```

Adding a symmetric bilinear from with a non-symmetric one results in a generic type-(0, 2) tensor:

```
sage: c = M.tensor((0,2), name='C')
sage: c[0,1] = 4
sage: s = a + c ; s
Type-(0,2) tensor A+C on the Rank-3 free module M over the Integer Ring
sage: s.symmetries()
no symmetry; no antisymmetry
sage: s[:]
[1 6 3]
[2 4 5]
[3 5 6]
```

See `FreeModuleTensor` for more documentation.

```python
tensor(tensor_type, name=None, latex_name=None, sym=None, antisym=None)
```

Construct a tensor on the free module `self`.

**INPUT:**

- `tensor_type` – pair `(k, l)` with `k` being the contravariant rank and `l` the covariant rank
- `name` – (default: `None`) string; name given to the tensor
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the tensor; if none is provided, the LaTeX symbol is set to `name`
- `sym` – (default: `None`) a symmetry or a list of symmetries among the tensor arguments: each symmetry is described by a tuple containing the positions of the involved arguments, with the convention `position = 0` for the first argument. For instance:
  - `sym = (0,1)` for a symmetry between the 1st and 2nd arguments
  - `sym = [(0,2), (1,3,4)]` for a symmetry between the 1st and 3rd arguments and a symmetry between the 2nd, 4th and 5th arguments.
- `antisym` – (default: `None`) antisymmetry or list of antisymmetries among the arguments, with the same convention as for `sym`
OUTPUT:

- instance of `FreeModuleTensor` representing the tensor defined on `self` with the provided characteristics

EXAMPLES:

Tensors on a rank-3 free module:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: t = M.tensor((1,0), name='t') ; t
Element t of the Rank-3 free module M over the Integer Ring
sage: t = M.tensor((0,1), name='t') ; t
Linear form t on the Rank-3 free module M over the Integer Ring
sage: t = M.tensor((1,1), name='t') ; t
Type-(1,1) tensor t on the Rank-3 free module M over the Integer Ring
sage: t = M.tensor((0,2), name='t', sym=(0,1)) ; t
Symmetric bilinear form t on the Rank-3 free module M over the Integer Ring
sage: t = M.tensor((0,2), name='t', antisym=(0,1)) ; t
Alternating form t of degree 2 on the Rank-3 free module M over the Integer Ring
sage: t = M.tensor((1,2), name='t') ; t
Type-(1,2) tensor t on the Rank-3 free module M over the Integer Ring
```

See `FreeModuleTensor` for more examples and documentation.

`tensor_from_comp(tensor_type, comp, name=None, latex_name=None)`

Construct a tensor on `self` from a set of components.

The tensor symmetries are deduced from those of the components.

INPUT:

- `tensor_type` – pair `(k, l)` with `k` being the contravariant rank and `l` the covariant rank
- `comp` – instance of `Components` representing the tensor components in a given basis
- `name` – (default: None) string; name given to the tensor
- `latex_name` – (default: None) string; LaTeX symbol to denote the tensor; if none is provided, the LaTeX symbol is set to `name`

OUTPUT:

- instance of `FreeModuleTensor` representing the tensor defined on `self` with the provided characteristics.

EXAMPLES:

Construction of a tensor of rank 1:

```python
sage: from sage.tensor.modules.comp import Components, CompWithSym,
    CompFullySym, CompFullyAntiSym
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e') ; e
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
sage: c = Components(ZZ, e, 1)
sage: c[:]
[0, 0, 0]
```
Construction of a tensor of rank 2:

```python
sage: c = CompFullySym(ZZ, e, 2)
sage: c[0,0], c[1,2] = 4, 5
sage: t = M.tensor_from_comp((0,2), c) ; t
Symmetric bilinear form on the Rank-3 free module M over the Integer Ring
sage: t.symmetries()
symmetry: (0, 1); no antisymmetry
sage: t.display(e)
4 e^0 \otimes e^0 + 5 e^1 \otimes e^2 + 5 e^2 \otimes e^1
```

```
sage: c = CompFullyAntiSym(ZZ, e, 2)
sage: c[0,1], c[1,2] = 4, 5
sage: t = M.tensor_from_comp((0,2), c) ; t
Alternating form of degree 2 on the Rank-3 free module M over the Integer Ring
sage: t.display(e)
4 e^0 \wedge e^1 + 5 e^1 \wedge e^2
```

**tensor_module**(*k*, *l*)

Return the free module of all tensors of type (*k*, *l*) defined on self.

**INPUT:**

- *k* – non-negative integer; the contravariant rank, the tensor type being (*k*, *l*)
- *l* – non-negative integer; the covariant rank, the tensor type being (*k*, *l*)

**OUTPUT:**

- instance of **TensorFreeModule** representing the free module $T^{(k,l)}(M)$ of type-(*k*, *l*) tensors on the free module self

**EXAMPLES:**

Tensor modules over a free module over $\mathbb{Z}$:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: T = M.tensor_module(1,2) ; T
Free module of type-(1,2) tensors on the Rank-3 free module M over the Integer Ring
sage: T.an_element()
Type-(1,2) tensor on the Rank-3 free module M over the Integer Ring
```

Tensor modules are unique:
The base module is itself the module of all type-(1,0) tensors:

```
sage: M.tensor_module(1,0) is M
True
```

See `TensorFreeModule` for more documentation.

### zero()

Return the zero element of `self`.

**EXAMPLES:**

Zero elements of free modules over \(\mathbb{Z}\):

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: M.zero()
Element zero of the Rank-3 free module M over the Integer Ring
sage: M.zero().parent() is M
True
sage: M.zero() is M(0)
True
sage: T = M.tensor_module(1,1)
sage: T.zero()
Type-(1,1) tensor zero on the Rank-3 free module M over the Integer Ring
sage: T.zero().parent() is T
True
sage: T.zero() is T(0)
True
```

Components of the zero element with respect to some basis:

```
sage: e = M.basis('e')
sage: M.zero()[e,:
[0, 0, 0]
sage: all(M.zero()[:,i] == M.base_ring().zero() for i in M.irange())
True
sage: T.zero()[:]
[0 0 0]
[0 0 0]
[0 0 0]
sage: M.tensor_module(1,2).zero()[:]
[[(0, 0, 0), (0, 0, 0), (0, 0, 0)],
 [(0, 0, 0), (0, 0, 0), (0, 0, 0)],
 [(0, 0, 0), (0, 0, 0), (0, 0, 0)]]
```
CHAPTER
SEVEN

PICKLING FOR THE OLD CDF VECTOR CLASS

AUTHORS:
  • Jason Grout
CHAPTER EIGHT

PICKLING FOR THE OLD RDF VECTOR CLASS

AUTHORS:

• Jason Grout
VECTORS OVER CALLABLE SYMBOLIC RINGS

AUTHOR: Jason Grout (2010)

EXAMPLES:

```
sage: f(r, theta, z) = (r*cos(theta), r*sin(theta), z)
sage: f.parent()
Vector space of dimension 3 over Callable function ring with arguments (r, theta, z)
sage: f
(r, theta, z) |--> (r*cos(theta), r*sin(theta), z)
sage: f[0]
(r, theta, z) |--> r*cos(theta)
sage: f+f
(r, theta, z) |--> (2*r*cos(theta), 2*r*sin(theta), 2*z)
sage: 3*f
(r, theta, z) |--> (3*r*cos(theta), 3*r*sin(theta), 3*z)
sage: f*f # dot product
(r, theta, z) |--> r^2*cos(theta)^2 + r^2*sin(theta)^2 + z^2
sage: f.diff()(0,1,2) # the matrix derivative
[cos(1) 0 0]
[sin(1) 0 0]
[ 0 0 1]
```

class `sage.modules.vector_callable_symbolic_dense.Vector_callable_symbolic_dense`
Bases: `sage.modules.free_module_element.FreeModuleElement_generic_dense`
SPACE OF MORPHISMS OF VECTOR SPACES (LINEAR TRANSFORMATIONS)

AUTHOR:

• Rob Beezer: (2011-06-29)

A VectorSpaceHomSpace object represents the set of all possible homomorphisms from one vector space to another. These mappings are usually known as linear transformations.

For more information on the use of linear transformations, consult the documentation for vector space morphisms at sage.modules.vector_space_morphism. Also, this is an extremely thin veneer on free module homspaces (sage.modules.free_module_homspace) and free module morphisms (sage.modules.free_module_morphism) - objects which might also be useful, and places where much of the documentation resides.

EXAMPLES:

Creation and basic examination is simple.

```
sage: V = QQ^3
sage: W = QQ^2
sage: H = Hom(V, W)
```
```
Set of Morphisms (Linear Transformations) from
Vector space of dimension 3 over Rational Field to
Vector space of dimension 2 over Rational Field
```
```
sage: H.domain()
Vector space of dimension 3 over Rational Field
```
```
sage: H.codomain()
Vector space of dimension 2 over Rational Field
```

Homspaces have a few useful properties. A basis is provided by a list of matrix representations, where these matrix representatives are relative to the bases of the domain and codomain.

```
sage: K = Hom(GF(3)^2, GF(3)^2)
sage: B = K.basis()
sage: for f in B:
    ....:     print(f)
    ....:     print("\n")
```
```
Vector space morphism represented by the matrix:
[1 0]
[0 0]
Domain: Vector space of dimension 2 over Finite Field of size 3
Codomain: Vector space of dimension 2 over Finite Field of size 3
```

(continues on next page)
Vector space morphism represented by the matrix:
\[
\begin{bmatrix}
0 & 1 \\
0 & 0 \\
\end{bmatrix}
\]
Domain: Vector space of dimension 2 over Finite Field of size 3
Codomain: Vector space of dimension 2 over Finite Field of size 3

Vector space morphism represented by the matrix:
\[
\begin{bmatrix}
0 & 0 \\
1 & 0 \\
\end{bmatrix}
\]
Domain: Vector space of dimension 2 over Finite Field of size 3
Codomain: Vector space of dimension 2 over Finite Field of size 3

Vector space morphism represented by the matrix:
\[
\begin{bmatrix}
0 & 0 \\
0 & 1 \\
\end{bmatrix}
\]
Domain: Vector space of dimension 2 over Finite Field of size 3
Codomain: Vector space of dimension 2 over Finite Field of size 3

The zero and identity mappings are properties of the space. The identity mapping will only be available if the domain and codomain allow for endomorphisms (equal vector spaces with equal bases).

```python
sage: H = Hom(QQ^3, QQ^3)
sage: g = H.zero()
sage: g([1, 1/2, -3])
(0, 0, 0)
sage: f = H.identity()
sage: f([1, 1/2, -3])
(1, 1/2, -3)
```

The homspace may be used with various representations of a morphism in the space to create the morphism. We demonstrate three ways to create the same linear transformation between two two-dimensional subspaces of \( \mathbb{Q}^3 \). The \( V.n \) notation is a shortcut to the generators of each vector space, better known as the basis elements. Note that the matrix representations are relative to the bases, which are purposely fixed when the subspaces are created (“user bases”).

```python
sage: U = QQ^3
sage: V = U.subspace_with_basis([U.0+U.1, U.1-U.2])
sage: W = U.subspace_with_basis([U.0, U.1+U.2])
sage: H = Hom(V, W)
```

First, with a matrix. Note that the matrix representation acts by matrix multiplication with the vector on the left. The input to the linear transformation, \((3, 1, 2)\), is converted to the coordinate vector \((3, -2)\), then matrix multiplication yields the vector \((-3, -2, -2)\), which represents the vector \((-3, -2, -2)\) in the codomain.

```python
sage: m = matrix(QQ, [[1, 2], [3, 4]])
sage: f1 = H(m)
sage: f1
Vector space morphism represented by the matrix:
\[
\begin{bmatrix}
1 & 2 \\
3 & 4 \\
\end{bmatrix}
\]
Domain: Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
\[
\begin{bmatrix}
1 & 1 & 0 \\
\end{bmatrix}
\]
```

(continues on next page)
Second, with a list of images of the domain’s basis elements.

```python
sage: img = [1*(U.0) + 2*(U.1+U.2), 3*U.0 + 4*(U.1+U.2)]
sage: f2 = H(img)
sage: f2
Vector space morphism represented by the matrix:
[1 2]
[3 4]
Domain: Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[ 1 1 0]
[ 0 1 -1]
Codomain: Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[1 0 0]
[0 1 1]
sage: f2([3,1,2])
(-3, -2, -2)
```

Third, with a linear function taking the domain to the codomain.

```python
sage: g = lambda x: vector(QQ, [-2*x[0]+3*x[1], -2*x[0]+4*x[1], -2*x[0]+4*x[1]])
sage: f3 = H(g)
sage: f3
Vector space morphism represented by the matrix:
[1 2]
[3 4]
Domain: Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[ 1 1 0]
[ 0 1 -1]
Codomain: Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[1 0 0]
[0 1 1]
sage: f3([3,1,2])
(-3, -2, -2)
```

The three linear transformations look the same, and are the same.

```python
sage: f1 == f2
True
sage: f2 == f3
True
```
class sage.modules.vector_space_homspace.VectorSpaceHomspace(X, Y, category=None, check=True, base=None):
    Bases: sage.modules.free_module_homspace.FreeModuleHomspace

sage.modules.vector_space_homspace.is_VectorSpaceHomspace(x)
Return True if x is a vector space homspace.

INPUT:
  x - anything

EXAMPLES:
To be a vector space morphism, the domain and codomain must both be vector spaces, in other words, modules over fields. If either set is just a module, then the Hom() constructor will build a space of free module morphisms.

    sage: H = Hom(QQ^3, QQ^2)
    sage: type(H)
    <class 'sage.modules.vector_space_homspace.VectorSpaceHomspace_with_category'>
    sage: sage.modules.vector_space_homspace.is_VectorSpaceHomspace(H)
    True
    sage: K = Hom(QQ^3, ZZ^2)
    sage: type(K)
    <class 'sage.modules.free_module_homspace.FreeModuleHomspace_with_category'>
    sage: sage.modules.vector_space_homspace.is_VectorSpaceHomspace(K)
    False
    sage: L = Hom(ZZ^3, QQ^2)
    sage: type(L)
    <class 'sage.modules.free_module_homspace.FreeModuleHomspace_with_category'>
    sage: sage.modules.vector_space_homspace.is_VectorSpaceHomspace(L)
    False
    sage: sage.modules.vector_space_homspace.is_VectorSpaceHomspace('junk')
    False
VECTOR SPACE MORPHISMS (AKA LINEAR TRANSFORMATIONS)

AUTHOR:

- Rob Beezer: (2011-06-29)

A vector space morphism is a homomorphism between vector spaces, better known as a linear transformation. These are a specialization of Sage’s free module homomorphisms. (A free module is like a vector space, but with scalars from a ring that may not be a field.) So references to free modules in the documentation or error messages should be understood as simply reflecting a more general situation.

11.1 Creation

The constructor `linear_transformation()` is designed to accept a variety of inputs that can define a linear transformation. See the documentation of the function for all the possibilities. Here we give two.

First a matrix representation. By default input matrices are understood to act on vectors placed to left of the matrix. Optionally, an input matrix can be described as acting on vectors placed to the right.

```sage
sage: A = matrix(QQ, [[-1, 2, 3], [4, 2, 0]])
sage: phi = linear_transformation(A)
sage: phi
Vector space morphism represented by the matrix:
[-1  2  3]
[ 4  2  0]
Domain: Vector space of dimension 2 over Rational Field
Codomain: Vector space of dimension 3 over Rational Field
sage: phi([2, -3])
(-14, -2, 6)
```

A symbolic function can be used to specify the “rule” for a linear transformation, along with explicit descriptions of the domain and codomain.

```sage
sage: F = Integers(13)
sage: D = F^3
sage: C = F^2
sage: x, y, z = var('x y z')
sage: f(x, y, z) = [2*x + 3*y + 5*z, x + z]
sage: rho = linear_transformation(D, C, f)
sage: f(1, 2, 3)
(23, 4)
sage: rho([1, 2, 3])
(10, 4)
```
A “vector space homspace” is the set of all linear transformations between two vector spaces. Various input can be coerced into a homspace to create a linear transformation. See `sage.modules.vector_space_homspace` for more.

```python
sage: D = QQ^4
sage: C = QQ^2
sage: hom_space = Hom(D, C)
sage: images = [[1, 3], [2, -1], [4, 0], [3, 7]]
sage: zeta = hom_space(images)
sage: zeta
Vector space morphism represented by the matrix:
[ 1  3]
[ 2 -1]
[ 4  0]
[ 3  7]
Domain: Vector space of dimension 4 over Rational Field
Codomain: Vector space of dimension 2 over Rational Field
```

A homomorphism may also be created via a method on the domain.

```python
sage: F = QQ[sqrt(3)]
sage: a = F.gen(0)
sage: D = F^2
sage: C = F^2
sage: A = matrix(F, [[a, 1], [2*a, 2]])
sage: psi = D.hom(A, C)
sage: psi
Vector space morphism represented by the matrix:
[ sqrt3  1]
[2*sqrt3  2]
Domain: Vector space of dimension 2 over Number Field in sqrt3 with defining polynomial x^2 - 3 with sqrt3 = 1.732050807568878?
Codomain: Vector space of dimension 2 over Number Field in sqrt3 with defining polynomial x^2 - 3 with sqrt3 = 1.732050807568878?
sage: psi([1, 4])
(9*sqrt3, 9)
```

### 11.2 Properties

Many natural properties of a linear transformation can be computed. Some of these are more general methods of objects in the classes `sage.modules.free_module_morphism.FreeModuleMorphism` and `sage.modules.matrix_morphism.MatrixMorphism`.

Values are computed in a natural way, an inverse image of an element can be computed with the `lift()` method, when the inverse image actually exists.

```python
sage: A = matrix(QQ, [[1,2], [2,4], [3,6]])
sage: phi = linear_transformation(A)
sage: phi([1,2,0])
(5, 10)
sage: phi.lift([10, 20])
(10, 0, 0)
sage: phi.lift([100, 100])
```

(continues on next page)
Images and pre-images can be computed as vector spaces.

```
sage: A = matrix(QQ, [[1,2], [2,4], [3,6]])
sage: phi = linear_transformation(A)
sage: phi.image()
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[1 2]
sage: phi.inverse_image( (QQ^2).span([[1,2]]) )
Vector space of degree 3 and dimension 3 over Rational Field
Basis matrix:
[1 0 0]
[0 1 0]
[0 0 1]
sage: phi.inverse_image( (QQ^2).span([[1,1]]) )
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[1 0 -1/3]
[0 1 -2/3]
```

Injectivity and surjectivity can be checked.

```
sage: A = matrix(QQ, [[1,2], [2,4], [3,6]])
sage: phi = linear_transformation(A)
sage: phi.is_injective()
False
sage: phi.is_surjective()
False
```

### 11.3 Restrictions and Representations

It is possible to restrict the domain and codomain of a linear transformation to make a new linear transformation. We will use those commands to replace the domain and codomain by equal vector spaces, but with alternate bases. The point here is that the matrix representation used to represent linear transformations are relative to the bases of both the domain and codomain.

```
sage: A = graphs.PetersenGraph().adjacency_matrix()
sage: V = QQ^10
sage: phi = linear_transformation(V, V, A)
sage: phi
Vector space morphism represented by the matrix:
[0 1 0 0 1 1 0 0 0 0]
[1 0 1 0 0 0 1 0 0 0]
[0 1 0 0 0 0 1 0 0 0]
```

(continues on next page)
\[
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
\end{bmatrix}
\]

Domain: Vector space of dimension 10 over Rational Field
Codomain: Vector space of dimension 10 over Rational Field

```python
sage: B1 = [V.gen(i) + V.gen(i+1) for i in range(9)] + [V.gen(9)]
sage: B2 = [V.gen(0)] + [-V.gen(i-1) + V.gen(i) for i in range(1,10)]
sage: D = V.subspace_with_basis(B1)
sage: C = V.subspace_with_basis(B2)
sage: rho = phi.restrict_codomain(C)
sage: zeta = rho.restrict_domain(D)
sage: zeta
Vector space morphism represented by the matrix:
\[
\begin{bmatrix}
6 & 5 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\
6 & 5 & 4 & 3 & 2 & 2 & 2 & 1 & 0 & 0 \\
6 & 6 & 5 & 4 & 3 & 2 & 2 & 2 & 1 & 0 \\
6 & 5 & 5 & 4 & 3 & 2 & 2 & 2 & 1 & 0 \\
6 & 4 & 4 & 3 & 3 & 3 & 3 & 2 & 1 & 0 \\
6 & 5 & 4 & 4 & 4 & 4 & 4 & 3 & 1 & 0 \\
6 & 6 & 5 & 4 & 4 & 3 & 3 & 3 & 2 & 1 \\
6 & 6 & 6 & 5 & 4 & 4 & 2 & 1 & 1 & 1 \\
6 & 6 & 6 & 6 & 5 & 4 & 3 & 1 & 0 & 0 \\
3 & 3 & 3 & 3 & 3 & 2 & 2 & 1 & 0 & 0 \\
\end{bmatrix}
\]
Domain: Vector space of degree 10 and dimension 10 over Rational Field
User basis matrix:
\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
Codomain: Vector space of degree 10 and dimension 10 over Rational Field
User basis matrix:
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
\end{bmatrix}
\]
```
An endomorphism is a linear transformation with an equal domain and codomain, and here each needs to have the same basis. We are using a matrix that has well-behaved eigenvalues, as part of showing that these do not change as the representation changes.

```sage
sage: A = graphs.PetersenGraph().adjacency_matrix()
sage: V = QQ^10
sage: phi = linear_transformation(V, V, A)
sage: phi.eigenvalues()
[3, -2, -2, -2, -2, 1, 1, 1, 1, 1]

sage: B1 = [V.gen(i) + V.gen(i+1) for i in range(9)] + [V.gen(9)]
sage: C = V.subspace_with_basis(B1)
sage: zeta = phi.restrict(C)
sage: zeta
Vector space morphism represented by the matrix:
[ 1 0 1 -1 2 -1 2 -2 2 -2]
[ 1 0 1 0 0 0 1 0 0 0]
[ 0 1 0 1 0 0 0 1 0 0]
[ 1 -1 2 -1 2 -2 2 -2 3 -2]
[ 2 -2 2 -1 1 -1 1 0 1 0]
[ 1 0 0 0 0 0 0 1 1 0]
[ 0 1 0 0 0 1 -1 1 0 2]
[ 0 0 1 0 0 2 -1 1 -1 2]
[ 0 0 0 1 0 1 1 0 0 0]
[ 0 0 0 0 1 -1 2 -1 1 -1]
Domain: Vector space of degree 10 and dimension 10 over Rational Field
User basis matrix:
[1 1 0 0 0 0 0 0 0 0]
[0 1 1 0 0 0 0 0 0 0]
[0 0 1 1 0 0 0 0 0 0]
[0 0 0 1 1 0 0 0 0 0]
[0 0 0 0 1 1 0 0 0 0]
[0 0 0 0 0 1 1 0 0 0]
[0 0 0 0 0 0 1 1 0 0]
[0 0 0 0 0 0 0 1 1]
[0 0 0 0 0 0 0 0 1]

Codomain: Vector space of degree 10 and dimension 10 over Rational Field
User basis matrix:
[1 1 0 0 0 0 0 0 0 0]
[0 1 1 0 0 0 0 0 0 0]
[0 0 1 1 0 0 0 0 0 0]
[0 0 0 1 1 0 0 0 0 0]
[0 0 0 0 1 1 0 0 0 0]
[0 0 0 0 0 1 1 0 0 0]
[0 0 0 0 0 0 1 1 0 0]
[0 0 0 0 0 0 0 1 1]
[0 0 0 0 0 0 0 0 1]

sage: zeta.eigenvalues()
[3, -2, -2, -2, -2, 1, 1, 1, 1, 1]
```
11.4 Equality

Equality of linear transformations is a bit nuanced. The equality operator == tests if two linear transformations have equal matrix representations, while we determine if two linear transformations are the same function with the .is_equal_function() method. Notice in this example that the function never changes, just the representations.

```
sage: f = lambda x: vector(QQ, [x[1], x[0]+x[1], x[0]])
sage: H = Hom(QQ^2, QQ^3)
sage: phi = H(f)
```

```
sage: rho = linear_transformation(QQ^2, QQ^3, matrix(QQ,2, 3, [[0,1,1], [1,1,0]]))
sage: phi == rho
True
```

```
sage: U = (QQ^2).subspace_with_basis([[1, 2], [-3, 1]])
sage: V = (QQ^3).subspace_with_basis([[0, 1, 0], [2, 3, 1], [-1, 1, 6]])
sage: K = Hom(U, V)
sage: zeta = K(f)
```

```
sage: zeta == phi
False
```

```
sage: zeta.is_equal_function(phi)
True
```

```
sage: zeta.is_equal_function(rho)
True
```

```
class sage.modules.vector_space_morphism.VectorSpaceMorphism(homspace, A, side='left')
Bases: sage.modules.free_module_morphism.FreeModuleMorphism

Create a linear transformation, a morphism between vector spaces.

INPUT:

- homspace - a homspace (of vector spaces) to serve as a parent for the linear transformation and a home for the domain and codomain of the morphism
- A - a matrix representing the linear transformation, which will act on vectors placed to the left of the matrix

EXAMPLES:

Nominally, we require a homspace to hold the domain and codomain and a matrix representation of the morphism (linear transformation).

```
sage: from sage.modules.vector_space_homspace import VectorSpaceHomspace
sage: from sage.modules.vector_space_morphism import VectorSpaceMorphism
sage: H = VectorSpaceHomspace(QQ^3, QQ^2)
sage: A = matrix(QQ, 3, 2, range(6))
sage: zeta = VectorSpaceMorphism(H, A)
sage: zeta
Vector space morphism represented by the matrix:
[0 1]
[2 3]
[4 5]
Domain: Vector space of dimension 3 over Rational Field
Codomain: Vector space of dimension 2 over Rational Field
```
```
See the constructor, \texttt{sage.modules.vector_space_morphism.linear_transformation()} for another way to create linear transformations.

The .\texttt{hom()} method of a vector space will create a vector space morphism.

\begin{verbatim}
sage: V = QQ^3; W = V.subspace_with_basis([[1,2,3], [-1,2,5/3], [0,1,-1]])
sage: phi = V.hom(matrix(QQ, 3, range(9)), codomain=W)  # indirect doctest
sage: type(phi)
<class 'sage.modules.vector_space_morphism.VectorSpaceMorphism'>
\end{verbatim}

A matrix may be coerced into a vector space homspace to create a vector space morphism.

\begin{verbatim}
sage: from sage.modules.vector_space_homspace import VectorSpaceHomspace
sage: H = VectorSpaceHomspace(QQ^3, QQ^2)
sage: A = matrix(QQ, 3, 2, range(6))
sage: rho = H(A)  # indirect doctest
sage: type(rho)
<class 'sage.modules.vector_space_morphism.VectorSpaceMorphism'>
\end{verbatim}

\texttt{is_invertible()}  
Determines if the vector space morphism has an inverse.  

\begin{verbatim}
OUTPUT:

True if the vector space morphism is invertible, otherwise False.

EXAMPLES:

If the dimension of the domain does not match the dimension of the codomain, then the morphism cannot be invertible.

\begin{verbatim}
sage: V = QQ^3
sage: U = V.subspace_with_basis([V.0 + V.1, 2*V.1 + 3*V.2])
sage: phi = V.hom([U.0, U.0 + U.1, U.0 - U.1], U)
sage: phi.is_invertible()
False
\end{verbatim}

An invertible linear transformation.

\begin{verbatim}
sage: A = matrix(QQ, 3, [[-3, 5, -5], [4, -7, 7], [6, -8, 10]])
sage: A.determinant()
2
sage: H = Hom(QQ^3, QQ^3)
sage: rho = H(A)
sage: rho.is_invertible()
True
\end{verbatim}

A non-invertible linear transformation, an endomorphism of a vector space over a finite field.

\begin{verbatim}
sage: F.<a> = GF(11^2)
sage: A = matrix(F, [[6*a + 3, 8*a + 2, 10*a + 3], ....: [2*a + 7, 4*a + 3, 2*a + 3], ....: [9*a + 2, 10*a + 10, 3*a + 3]])
sage: A.nullity()
1
sage: E = End(F^3)
sage: zeta = E(A)
\end{verbatim}
sage: zeta.is_invertible()
False

sage.modules.vector_space_morphism.is_VectorSpaceMorphism(x)
Returns True if x is a vector space morphism (a linear transformation).

INPUT:
- x - anything

OUTPUT:
True only if x is an instance of a vector space morphism, which are also known as linear transformations.

EXAMPLES:

sage: V = QQ^2; f = V.hom([V.1,-2*V.0])
sage: sage.modules.vector_space_morphism.is_VectorSpaceMorphism(f)
True
sage: sage.modules.vector_space_morphism.is_VectorSpaceMorphism('junk')
False

sage.modules.vector_space_morphism.linear_transformation(arg0, arg1=None, arg2=None, side='left')
Create a linear transformation from a variety of possible inputs.

FORMATS:
In the following, D and C are vector spaces over the same field that are the domain and codomain (respectively) of the linear transformation.

side is a keyword that is either 'left' or 'right'. When a matrix is used to specify a linear transformation, as in the first two call formats below, you may specify if the function is given by matrix multiplication with the vector on the left, or the vector on the right. The default is 'left'. The matrix representation may be obtained as either version, no matter how it is created.

• linear_transformation(A, side='left')
  Where A is a matrix. The domain and codomain are inferred from the dimension of the matrix and the base ring of the matrix. The base ring must be a field, or have its fraction field implemented in Sage.

• linear_transformation(D, C, A, side='left')
  A is a matrix that behaves as above. However, now the domain and codomain are given explicitly. The matrix is checked for compatibility with the domain and codomain. Additionally, the domain and codomain may be supplied with alternate ("user") bases and the matrix is interpreted as being a representation relative to those bases.

• linear_transformation(D, C, f)
  f is any function that can be applied to the basis elements of the domain and that produces elements of the codomain. The linear transformation returned is the unique linear transformation that extends this mapping on the basis elements. f may come from a function defined by a Python def statement, or may be defined as a lambda function.
  Alternatively, f may be specified by a callable symbolic function, see the examples below for a demonstration.

• linear_transformation(D, C, images)
images is a list, or tuple, of codomain elements, equal in number to the size of the basis of the domain. Each basis element of the domain is mapped to the corresponding element of the images list, and the linear transformation returned is the unique linear transformation that extends this mapping.

OUTPUT:

A linear transformation described by the input. This is a “vector space morphism”, an object of the class `sage.modules.vector_space_morphism`.

EXAMPLES:

We can define a linear transformation with just a matrix, understood to act on a vector placed on one side or the other. The field for the vector spaces used as domain and codomain is obtained from the base ring of the matrix, possibly promoting to a fraction field.

```
sage: A = matrix(ZZ, [[1, -1, 4], [2, 0, 5]])
sage: phi = linear_transformation(A)
sage: phi
Vector space morphism represented by the matrix:
[ 1 -1 4]
[ 2 0 5]
Domain: Vector space of dimension 2 over Rational Field
Codomain: Vector space of dimension 3 over Rational Field
sage: phi([1/2, 5])
(21/2, -1/2, 27)
sage: B = matrix(Integers(7), [[1, 2, 1], [3, 5, 6]])
sage: rho = linear_transformation(B, side='right')
sage: rho
Vector space morphism represented by the matrix:
[1 3]
[2 5]
[1 6]
Domain: Vector space of dimension 3 over Ring of integers modulo 7
Codomain: Vector space of dimension 2 over Ring of integers modulo 7
sage: rho([2, 4, 6])
(2, 6)
```

We can define a linear transformation with a matrix, while explicitly giving the domain and codomain. Matrix entries will be coerced into the common field of scalars for the vector spaces.

```
sage: D = QQ^3
sage: C = QQ^2
sage: A = matrix([[1, 7], [2, -1], [0, 5]])
sage: A.parent()
Full MatrixSpace of 3 by 2 dense matrices over Integer Ring
sage: zeta = linear_transformation(D, C, A)
sage: zeta.matrix().parent()
Full MatrixSpace of 3 by 2 dense matrices over Rational Field
sage: zeta
Vector space morphism represented by the matrix:
[ 1 7]
[ 2 -1]
[ 0 5]
Domain: Vector space of dimension 3 over Rational Field
Codomain: Vector space of dimension 2 over Rational Field
```

11.4. Equality
Matrix representations are relative to the bases for the domain and codomain.

```
sage: u = vector(QQ, [1, -1])
sage: v = vector(QQ, [2, 3])
sage: D = (QQ^2).subspace_with_basis([u, v])
sage: x = vector(QQ, [2, 1])
sage: y = vector(QQ, [-1, 4])
sage: C = (QQ^2).subspace_with_basis([x, y])
sage: A = matrix(QQ, [[2, 5], [3, 7]])
sage: psi = linear_transformation(D, C, A)
sage: psi
Vector space morphism represented by the matrix:
[2 5]
[3 7]
Domain: Vector space of degree 2 and dimension 2 over Rational Field
User basis matrix:
[ 1 -1]
[ 2 3]
Codomain: Vector space of degree 2 and dimension 2 over Rational Field
User basis matrix:
[ 2 1]
[-1 4]
sage: psi(u) == 2*x + 5*y
True
sage: psi(v) == 3*x + 7*y
True
```

Functions that act on the domain may be used to compute images of the domain’s basis elements, and this mapping can be extended to a unique linear transformation. The function may be a Python function (via `def` or `lambda`) or a Sage symbolic function.

```
sage: def g(x):
    ....:     return vector(QQ, [2*x[0]+x[2], 5*x[1]])
sage: phi = linear_transformation(QQ^3, QQ^2, g)
sage: phi
Vector space morphism represented by the matrix:
[2 0]
[0 5]
[1 0]
Domain: Vector space of dimension 3 over Rational Field
Codomain: Vector space of dimension 2 over Rational Field
sage: f = lambda x: vector(QQ, [2*x[0]+x[2], 5*x[1]])
sage: rho = linear_transformation(QQ^3, QQ^2, f)
sage: rho
Vector space morphism represented by the matrix:
[2 0]
[0 5]
[1 0]
Domain: Vector space of dimension 3 over Rational Field
Codomain: Vector space of dimension 2 over Rational Field
sage: x, y, z = var('x y z')
sage: h(x, y, z) = [2*x + z, 5*y]
sage: ```
We create a linear transformation relative to non-standard bases, and capture its representation relative to standard bases. With this, we can build functions that create the same linear transformation relative to the nonstandard bases.

Finally, we can give an explicit list of images for the basis elements of the domain.
sage: D = F^3
sage: C = F^3
sage: rho = linear_transformation(D, C, [u, v, w])
sage: rho.matrix()
[ 1    a a^2]
[ a    a^2 2 ]
[ a + 1 a^2 + a a^2 + 2 ]
sage: C = (F^3).subspace_with_basis([u, v])
sage: D = (F^3).subspace_with_basis([u, v])
sage: psi = linear_transformation(C, D, [u+v, u-v])
sage: psi.matrix()
[ 1    1]
[ 1   -1]
EXAMPLES: We create \( \text{End}(\mathbb{Z}^2) \) and compute a basis.

```python
sage: M = FreeModule(IntegerRing(), 2)
sage: E = End(M)
sage: B = E.basis()
sage: len(B)
4
sage: B[0]
Free module morphism defined by the matrix
[1 0]
[0 0]
Domain: Ambient free module of rank 2 over the principal ideal domain ...
Codomain: Ambient free module of rank 2 over the principal ideal domain ...
```

We create \( \text{Hom}(\mathbb{Z}^3, \mathbb{Z}^2) \) and compute a basis.

```python
sage: V3 = FreeModule(IntegerRing(), 3)
sage: V2 = FreeModule(IntegerRing(), 2)
sage: H = Hom(V3, V2)
sage: H
Set of Morphisms from Ambient free module of rank 3 over
the principal ideal domain Integer Ring
to Ambient free module of rank 2
over the principal ideal domain Integer Ring
in Category of finite dimensional modules with basis over
(euclidean domains and infinite enumerated sets and metric spaces)
sage: B = H.basis()
sage: len(B)
6
sage: B[0]
Free module morphism defined by the matrix
[1 0]
[0 0]
[0 0]...
```

```python
class sage.modules.free_module_homspace.FreeModuleHomspace(X, Y, category=None, check=True, base=None)
Bases: sage.categories.homset.HomsetWithBase

basis(side='left')
    Return a basis for this space of free module homomorphisms.
```
INPUT:

• side – side of the vectors acted on by the matrix (default: left)

OUTPUT:

• tuple

EXAMPLES:

```
sage: H = Hom(ZZ^2, ZZ^1)
sage: H.basis()
(Free module morphism defined by the matrix
[1]
[0]
Domain: Ambient free module of rank 2 over the principal ideal domain ...
Codomain: Ambient free module of rank 1 over the principal ideal domain ...,
Free module morphism defined by the matrix
[0]
[1]
Domain: Ambient free module of rank 2 over the principal ideal domain ...
Codomain: Ambient free module of rank 1 over the principal ideal domain ...)  
sage: H.basis("right")
(Free module morphism defined as left-multiplication by the matrix
[1 0]
[0 1]
Domain: Ambient free module of rank 2 over the principal ideal domain ...
Codomain: Ambient free module of rank 1 over the principal ideal domain ...,
Free module morphism defined as left-multiplication by the matrix
[0 1]
[1 0]
Domain: Ambient free module of rank 2 over the principal ideal domain ...
Codomain: Ambient free module of rank 1 over the principal ideal domain ...)
```

\texttt{identity} (\texttt{side='left'})

Return identity morphism in an endomorphism ring.

INPUT:

• side – side of the vectors acted on by the matrix (default: left)

EXAMPLES:

```
sage: V=FreeModule(ZZ,5)
sage: H=V.Hom(V)
sage: H.identity()
Free module morphism defined by the matrix
[1 0 0 0 0]
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 0 1]
Domain: Ambient free module of rank 5 over the principal ideal domain ...
Codomain: Ambient free module of rank 5 over the principal ideal domain ...)
```

\texttt{zero} (\texttt{side='left'})

INPUT:

• side – side of the vectors acted on by the matrix (default: left)

EXAMPLES:
sage: E = ZZ^2
sage: F = ZZ^3
sage: H = Hom(E, F)
sage: f = H.zero()
sage: f
Free module morphism defined by the matrix
[0 0 0]
[0 0 0]
Domain: Ambient free module of rank 2 over the principal ideal domain Integer
--> Ring
Codomain: Ambient free module of rank 3 over the principal ideal domain Integer
--> Ring
sage: f(E.an_element())
(0, 0, 0)
sage: f(E.an_element()) == F.zero()
True
sage: H.zero("right")
Free module morphism defined as left-multiplication by the matrix
[0 0]
[0 0]
[0 0]
Domain: Ambient free module of rank 2 over the principal ideal domain Integer
--> Ring
Codomain: Ambient free module of rank 3 over the principal ideal domain Integer
--> Ring

sage.modules.free_module_homspace.is_FreeModuleHomspace(x)
Return True if x is a free module homspace.

EXAMPLES:
Notice that every vector space is a free module, but when we construct a set of morphisms between two vector spaces, it is a VectorSpaceHomspace, which qualifies as a FreeModuleHomspace, since the former is a special case of the latter.

sage: H = Hom(ZZ^3, ZZ^2) sage: type(H) <class 'sage.modules.free_module_homspace.FreeModuleHomspace_with_category'>
sage: sage.modules.free_module_homspace.is_FreeModuleHomspace(H) True
sage: K = Hom(QQ^3, ZZ^2) sage: type(K) <class 'sage.modules.free_module_homspace.FreeModuleHomspace_with_category'>
sage: sage.modules.free_module_homspace.is_FreeModuleHomspace(K) True
sage: L = Hom(ZZ^3, QQ^2) sage: type(L) <class 'sage.modules.free_module_homspace.FreeModuleHomspace_with_category'>
sage: sage.modules.free_module_homspace.is_FreeModuleHomspace(L) True
sage: P = Hom(QQ^3, QQ^2) sage: type(P) <class 'sage.modules.vector_space_homspace.VectorSpaceHomspace_with_category'>
sage: sage.modules.free_module_homspace.is_FreeModuleHomspace(P) True
sage: sage.modules.free_module_homspace.is_FreeModuleHomspace('junk') False
MORPHISMS OF FREE MODULES

AUTHORS:
• William Stein: initial version
• Miguel Marco (2010-06-19): added eigenvalues, eigenvectors and minpoly functions

class sage.modules.free_module_morphism.BaseIsomorphism1D
Bases: sage.categories.morphism.Morphism
An isomorphism between a ring and a free rank-1 module over the ring.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: V, from_V, to_V = R.free_module(R)
sage: from_V
Isomorphism morphism:
  From: Ambient free module of rank 1 over the integral domain Multivariate...
  To: Multivariate Polynomial Ring in x, y over Rational Field
```

```
sage: from_V.is_injective()  
True
```

```
sage: from_V.is_surjective()
True
```

class sage.modules.free_module_morphism.BaseIsomorphism1D_from_FM(parent, basis=None)
Bases: sage.modules.free_module_morphism.BaseIsomorphism1D
An isomorphism to a ring from its 1-dimensional free module

INPUT:
• parent – the homset
• basis – (default 1) an invertible element of the ring

EXAMPLES:

```python
sage: R.<x> = QQ[[[]]
sage: V, from_V, to_V = R.free_module(R)
sage: v = to_V(1+x); v
(1 + x)
sage: from_V(v)
1 + x
sage: W, from_W, to_W = R.free_module(R, basis=(1-x))
sage: W is V
True
sage: w = to_W(1+x); w
(1 - x^2)
sage: from_W(w)
1 + x + O(x^20)
```

The basis vector has to be a unit so that the map is an isomorphism:

```python
sage: W, from_W, to_W = R.free_module(R, basis=x)
Traceback (most recent call last):
  ... ValueError: Basis element must be a unit
```

class sage.modules.free_module_morphism.BaseIsomorphism1D_to_FM(parent, basis=None)

Bases: sage.modules.free_module_morphism.BaseIsomorphism1D

An isomorphism from a ring to its 1-dimensional free module

INPUT:

• parent – the homset

• basis – (default 1) an invertible element of the ring

EXAMPLES:

```python
sage: R = Zmod(8)
sage: V, from_V, to_V = R.free_module(R)
sage: v = to_V(2); v
(2)
sage: from_V(v)
2
sage: W, from_W, to_W = R.free_module(R, basis=3)
sage: W is V
True
sage: w = to_W(2); w
(6)
sage: from_W(w)
2
```

The basis vector has to be a unit so that the map is an isomorphism:

```python
sage: W, from_W, to_W = R.free_module(R, basis=4)
Traceback (most recent call last):
(continues on next page)```
... 

ValueError: Basis element must be a unit

class sage.modules.free_module_morphism.FreeModuleMorphism(parent, A, side='left')

Bases: sage.modules.matrix_morphism.MatrixMorphism

INPUT:

• parent - a homspace in a (sub) category of free modules

• A - matrix

• side – side of the vectors acted on by the matrix (default: "left")

EXAMPLES:

```python
sage: V = ZZ^3; W = span([[1,2,3],[-1,2,8]], ZZ)
sage: phi = V.hom(matrix(ZZ,3,[1..9]))
sage: type(phi)
<class 'sage.modules.free_module_morphism.FreeModuleMorphism'>
```

change_ring(R)

Change the ring over which this morphism is defined. This changes the ring of the domain, codomain, and underlying matrix.

EXAMPLES:

```python
sage: V0 = span([[0,0,1],[0,2,0]],ZZ); V1 = span([[1/2,0],[0,2]],ZZ); W = ...
sage: h = V0.hom([-3*V1.0-3*V1.1, -3*V1.0-3*V1.1])
sage: h.base_ring()
Integer Ring
sage: h
Free module morphism defined by the matrix
[-3 -3]
[-3 -3]...
```

```python
sage: h.change_ring(QQ).base_ring()
Rational Field
sage: f = h.change_ring(QQ); f
Vector space morphism represented by the matrix:
[-3 -3]
[-3 -3]
Domain: Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[0 1 0]
[0 0 1]
Codomain: Vector space of degree 2 and dimension 2 over Rational Field
Basis matrix:
[1 0]
[0 1]
```

```python
sage: f = h.change_ring(GF(7)); f
Vector space morphism represented by the matrix:
[4 4]
[4 4]
Domain: Vector space of degree 3 and dimension 2 over Finite Field of size 7
```
Basis matrix:
[0 1 0]
[0 0 1]
Codomain: Vector space of degree 2 and dimension 2 over Finite Field of size 7
Basis matrix:
[1 0]
[0 1]

**eigenspaces**(extend=True)

Compute a list of subspaces formed by eigenvectors of self.

**INPUT:**

- extend – (default: True) determines if field extensions should be considered

**OUTPUT:**

- a list of pairs (eigenvalue, eigenspace)

**EXAMPLES:**

```sage
sage: V = QQ^3
sage: h = V.hom([[1,0,0],[0,0,1],[0,-1,0]], V)
sage: h.eigenspaces()
[(1, Vector space of degree 3 and dimension 1 over Rational Field
  Basis matrix:
  [1 0 0]),
 (-1*I, Vector space of degree 3 and dimension 1 over Algebraic Field
  Basis matrix:
  [ 0 1 1*I]),
 (1*I, Vector space of degree 3 and dimension 1 over Algebraic Field
  Basis matrix:
  [ 0 1 -1*I])]
sage: h.eigenspaces(extend=False)
[(1, Vector space of degree 3 and dimension 1 over Rational Field
  Basis matrix:
  [1 0 0])]
sage: h = V.hom([[2,1,0], [0,2,0], [0,0,-1]], V)
sage: h.eigenspaces()
[(-1, Vector space of degree 3 and dimension 1 over Rational Field
  Basis matrix:
  [0 0 1]),
 (2, Vector space of degree 3 and dimension 1 over Rational Field
  Basis matrix:
  [0 1 0])]
sage: h = V.hom([[2,1,0], [0,2,0], [0,0,2]], V)
sage: h.eigenspaces()
[(2, Vector space of degree 3 and dimension 2 over Rational Field
  Basis matrix:
  [1 0 0]])
```

(continues on next page)
Basis matrix:
\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
sage: V = \mathbb{Q}^2; m = \text{matrix}(2, [1, 1, 0, 1])
\]
\[
sage: V.\text{hom}(m, \text{side}="right").\text{eigenspaces}()
\]

\[
(1,
\text{Vector space of degree 2 and dimension 1 over Rational Field}
\text{Basis matrix:}
\begin{bmatrix}
1 & 0
\end{bmatrix}
\]

\[
sage: V.\text{hom}(m).\text{eigenspaces}()
\]

\[
(1,
\text{Vector space of degree 2 and dimension 1 over Rational Field}
\text{Basis matrix:}
\begin{bmatrix}
0 & 1
\end{bmatrix}
\]

\[
eigenvalues(\text{extend}=True)
\]

Returns a list with the eigenvalues of the endomorphism of vector spaces.

**INPUT:**

- `extend` – boolean (default: True) decides if base field extensions should be considered or not.

**EXAMPLES:**

We compute the eigenvalues of an endomorphism of $\mathbb{Q}^3$:

\[
sage: V = \mathbb{Q}^3
\]
\[
sage: H = V.\text{endomorphism_ring}()([[1,-1,0],[-1,1,1],[0,3,1]])
\]
\[
sage: H.\text{eigenvalues}()
\]

\[
[3, 1, -1]
\]

Note the effect of the `extend` option:

\[
sage: V = \mathbb{Q}^2
\]
\[
sage: H = V.\text{endomorphism_ring}()([[0,-1],[1,0]])
\]
\[
sage: H.\text{eigenvalues}()
\]

\[
[-1*I, 1*I]
\]
\[
sage: H.\text{eigenvalues}(\text{extend}=\text{False})
\]

\[
[]
\]

\[
eigenvectors(\text{extend}=True)
\]

Computes the subspace of eigenvectors of a given eigenvalue.

**INPUT:**

- `extend` – boolean (default: True) decides if base field extensions should be considered or not.

**OUTPUT:**

A sequence of tuples. Each tuple contains an eigenvalue, a sequence with a basis of the corresponding subspace of eigenvectors, and the algebraic multiplicity of the eigenvalue.

**EXAMPLES:**

\[
sage: V = (\mathbb{Q}^4).\text{subspace}([[0,2,1,4],[1,2,5,0],[1,1,1,1]])
\]
\[
sage: H = (V.\text{Hom}(V))(\text{matrix}(\mathbb{Q}, [[0,1,0],[-1,0,0],[0,0,3]]))
\]

(continues on next page)
inverse_image(V)

Given a submodule V of the codomain of self, return the inverse image of V under self, i.e., the biggest submodule of the domain of self that maps into V.

EXAMPLES:

We test computing inverse images over a field:

```
sage: V = QQ^3; W = span([[1,2,3],[-1,2,5/3]], QQ)
sage: phi = V.hom(matrix(QQ,3,[[1,0,0],[0,1,0],[0,0,1]]))
sage: phi.rank()
2
```
We test computing inverse images between two spaces embedded in different ambient spaces:

```python
sage: V0 = span([[0,0,1],[0,2,0]],ZZ); V1 = span([[1/2,0],[0,2]],ZZ); W = V0 + V1
sage: h = V0.hom([-3*V1.0-3*V1.1, -3*V1.0-3*V1.1])
```

```
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[0 2 1]
[0 0 2]
sage: h(h.inverse_image(W)).is_submodule(W)
True
sage: h(h.inverse_image(W)).index_in(W)
+Infinity
sage: h(h.inverse_image(W))
Free module of degree 2 and rank 1 over Integer Ring
Echelon basis matrix:
[3 12]
```

We test computing inverse images over the integers:

```python
sage: V = QQ^3; W = V.span_of_basis([[2,2,3],[-1,2,5/3]], ZZ)
sage: phi = W.hom([W.0, W.0-W.1])
sage: Z = W.span([2*W.1]); Z
```

```
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[2 -4 -10/3]
sage: Y = phi.inverse_image(Z); Y
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[6 0 8/3]
sage: phi(Y) == Z
True
```

We test that trac ticket #24590 is resolved:

```python
sage: A = FreeQuadraticModule(ZZ,1,matrix([[2]]))
sage: f = A.Hom(A).an_element()
sage: f.inverse_image(A)
```

```
Free module of degree 1 and rank 1 over Integer Ring
```
We test that it respects the side:

```
sage: V = ZZ^2
sage: m = matrix(2, \[1, 1, 0, 1\])
sage: h = V.hom(m, side="right")
sage: h
Free module morphism defined as left-multiplication by the matrix
[1 1]
[0 1]...
sage: SV = V.span(\[V.0\])
sage: h.inverse_image(SV)
Free module of degree 2 and rank 1 over Integer Ring
Echelon basis matrix:
[1 0]
sage: V.hom(m).inverse_image(SV)
Free module of degree 2 and rank 1 over Integer Ring
Echelon basis matrix:
[ 1 -1]
```

**`lift(x)`**

Given an element of the image, return an element of the codomain that maps onto it.

Note that `lift` and `preimage_representative` are equivalent names for this method, with the latter suggesting that the return value is a coset representative of the domain modulo the kernel of the morphism.

**EXAMPLES:**

```
sage: X = QQ**2
sage: V = X.span([[2, 0], [0, 8]], ZZ)
sage: W = (QQ**1).span([[1/12]], ZZ)
sage: f = V.hom([W([1/3]), W([1/2])], W)
sage: l=f.lift([1/3]); l  # random
(8, -16)
sage: f(l)
(1/3)
sage: f(f.lift([1/2]))
(1/2)
sage: f(f.lift([1/6]))
(1/6)
sage: f.lift([1/12])
Traceback (most recent call last):
...  
ValueError: element is not in the image
sage: f.lift([1/24])
Traceback (most recent call last):
...  
TypeError: element [1/24] is not in free module
```

This works for vector spaces, too:
This example illustrates the use of the `preimage_representative` as an equivalent name for this method.

```python
sage: V = ZZ^3
sage: W = ZZ^2
sage: w = vector(ZZ, [1, 2])
sage: f = V.hom([w, w, w], W)
sage: f.preimage_representative(vector(ZZ, [10, 20]))
(0, 0, 10)
```

### minimal_polynomial(var='x')

Computes the minimal polynomial. `minpoly()` and `minimal_polynomial()` are the same method.

**INPUT:**
- `var` - string (default: `x`) a variable name

**OUTPUT:**
polynomial in `var` - the minimal polynomial of the endomorphism.

**EXAMPLES:**

Compute the minimal polynomial, and check it.

```python
sage: V=GF(7)^3
sage: H=V.Hom(V)([[0,1,2],[-1,0,3],[2,4,1]])
sage: H
Vector space morphism represented by the matrix:
[0 1 2]
[6 0 3]
[2 4 1]
Domain: Vector space of dimension 3 over Finite Field of size 7
Codomain: Vector space of dimension 3 over Finite Field of size 7
sage: H.minpoly()
```

(continues on next page)
\[ x^3 + 6x^2 + 6x + 1 \]

```
sage: H.minimal_polynomial()
x^3 + 6x^2 + 6x + 1
```

```
sage: H^3 + (H^2)*6 + H*6 + 1
```

Vector space morphism represented by the matrix:

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Domain: Vector space of dimension 3 over Finite Field of size 7
Codomain: Vector space of dimension 3 over Finite Field of size 7

**minpoly**(var='x')

Computes the minimal polynomial.

\textit{minpoly()} and \textit{minimal_polynomial()} are the same method.

**INPUT:**

- \textit{var} - string (default: `'x') a variable name

**OUTPUT:**

polynomial in \textit{var} - the minimal polynomial of the endomorphism.

**EXAMPLES:**

Compute the minimal polynomial, and check it.

```
sage: V=GF(7)^3
sage: H=V.Hom(V)([[0,1,2],[-1,0,3],[2,4,1]])
sage: H
Vector space morphism represented by the matrix:
\[
\begin{bmatrix}
0 & 1 & 2 \\
6 & 0 & 3 \\
2 & 4 & 1
\end{bmatrix}
\]
Domain: Vector space of dimension 3 over Finite Field of size 7
Codomain: Vector space of dimension 3 over Finite Field of size 7
```

```
sage: H.minpoly()
x^3 + 6x^2 + 6x + 1
```

```
sage: H.minimal_polynomial()
x^3 + 6x^2 + 6x + 1
```

```
sage: H^3 + (H^2)*6 + H*6 + 1
```

Vector space morphism represented by the matrix:

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Domain: Vector space of dimension 3 over Finite Field of size 7
Codomain: Vector space of dimension 3 over Finite Field of size 7

**preimage_representative**(x)

Given an element of the image, return an element of the codomain that maps onto it.
Note that `lift` and `preimage_representative` are equivalent names for this method, with the latter suggesting that the return value is a coset representative of the domain modulo the kernel of the morphism.

**EXAMPLES:**

```python
sage: X = QQ^2
sage: V = X.span([[2, 0], [0, 8]], ZZ)
sage: W = (QQ^1).span([[1/12]], ZZ)
sage: f = V.hom([W([1/3]), W([1/2])], W)
sage: l=f.lift([1/3]); l # random
     (8, -16)
sage: f(l)
     (1/3)
sage: f(f.lift([1/2]))
     (1/2)
sage: f(f.lift([1/6]))
     (1/6)
sage: f.lift([1/12])
Traceback (most recent call last):
... ValueError: element is not in the image
```

This works for vector spaces, too:

```python
sage: V = VectorSpace(GF(3), 2)
sage: W = VectorSpace(GF(3), 3)
sage: f = V.hom([W.1, W.1 - W.0])
sage: f.lift(W.1)
     (1, 0)
sage: f.lift(W.2)
Traceback (most recent call last):
... ValueError: element is not in the image
```

This example illustrates the use of the `preimage_representative` as an equivalent name for this method.

```python
sage: V = ZZ^3
sage: W = ZZ^2
sage: w = vector(ZZ, [1,2])
sage: f = V.hom([w, w, w], W)
sage: f.preimage_representative(vector(ZZ, [10, 20]))
     (0, 0, 10)
```

```python
sage: V = QQ^2; m = matrix(2, [1, 1, 0, 1])
sage: V.hom(m, side="right").lift(V.0+V.1)
     (0, 1)
sage: V.hom(m).lift(V.0+V.1)
```

(continues on next page)
pushforward(x)
Compute the image of a sub-module of the domain.

EXAMPLES:

```sage
V = QQ^3; W = span([[1,2,3],[-1,2,5/3]], QQ)
sage: phi = V.hom(matrix(QQ,3,[1..9]))
sage: phi.rank()
2
sage: phi(V)  #indirect doctest
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1 0 -1]
[ 0 1 2]
```

We compute the image of a submodule of a ZZ-module embedded in a rational vector space:

```sage
V = QQ^3; W = V.span_of_basis([[2,2,3],[0,-1,5/3]], ZZ)
sage: phi = W.hom([W.0, W.0-W.1]); phi
Free module morphism defined by the matrix
[ 1 0]
[ 1 -1]...
sage: phi(span([2*W.1],ZZ))
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[ 6 0 8/3]
sage: phi(2*W.1)
(6, 0, 8/3)
```

`sage.modules.free_module_morphism.is_FreeModuleMorphism(x)
EXAMPLES:

```sage
V = ZZ^2; f = V.hom([V.1,-2*V.0])
sage: sage.modules.free_module_morphism.is_FreeModuleMorphism(f)
True
sage: sage.modules.free_module_morphism.is_FreeModuleMorphism(0)
False
```
A matrix morphism is a morphism that is defined by multiplication by a matrix. Elements of domain must either have a method `vector()` that returns a vector that the defining matrix can hit from the left, or be coercible into vector space of appropriate dimension.

EXAMPLES:

```python
sage: from sage.modules.matrix_morphism import MatrixMorphism, is_MatrixMorphism
sage: V = QQ^3
sage: T = End(V)
```

```python
code:
sage: M = MatrixSpace(QQ,3)
sage: I = M.identity_matrix()
sage: m = MatrixMorphism(T, I); m
Morphism defined by the matrix
[1 0 0]
[0 1 0]
[0 0 1]
sage: is_MatrixMorphism(m)
True
sage: m.charpoly('x')
x^3 - 3*x^2 + 3*x - 1
sage: m.base_ring()
Rational Field
sage: m.det()
1
sage: m.fcp('x')
(x - 1)^3
sage: m.matrix()
[1 0 0]
[0 1 0]
[0 0 1]
sage: m.rank()
3
sage: m.trace()
3
```

**AUTHOR:**

- William Stein: initial versions
- David Joyner (2005-12-17): added examples
- William Stein (2005-01-07): added `__reduce__`
- Craig Citro (2008-03-18): refactored `MatrixMorphism` class
class sage.modules.matrix_morphism.MatrixMorphism(parent, A, copy_matrix=True, side='left')

Bases: sage.modules.matrix_morphism.MatrixMorphism_abstract

A morphism defined by a matrix.

INPUT:

- **parent** – a homspace
- **A** – matrix or a *MatrixMorphism_abstract* instance
- **copy_matrix** – (default: True) make an immutable copy of the matrix A if it is mutable; if False, then this makes A immutable

is_injective()

Tell whether *self* is injective.

EXAMPLES:

```python
sage: V1 = QQ^2
sage: V2 = QQ^3
sage: phi = V1.hom(Matrix([[1,2,3],[4,5,6]]), V2)
sage: phi.is_injective()
True
sage: psi = V2.hom(Matrix([[1,2],[3,4],[5,6]]), V1)
sage: psi.is_injective()
False
```
This tests if trac ticket #11552 is fixed.

AUTHORS:

• Simon King (2010-05)
• Rob Beezer (2011-06-28)

matrix(side=None)
Return a matrix that defines this morphism.

INPUT:

• side – (default: 'None') the side of the matrix where a vector is placed to effect the morphism

OUTPUT:

A matrix which represents the morphism, relative to bases for the domain and codomain. If the modules
are provided with user bases, then the representation is relative to these bases.

Internally, Sage represents a matrix morphism with the matrix multiplying a row vector placed to the left of
the matrix. If the option side='right' is used, then a matrix is returned that acts on a vector to the right
of the matrix. These two matrices are just transposes of each other and the difference is just a preference
for the style of representation.

EXAMPLES:

class sage.modules.matrix_morphism.MatrixMorphism_abstract(parent, side='left')
Bases: sage.categories.morphism.Morphism

INPUT:
• parent - a homspace
• A - matrix

EXAMPLES:

```python
sage: from sage.modules.matrix_morphism import MatrixMorphism
sage: T = End(ZZ^3)
sage: M = MatrixSpace(ZZ,3)
sage: I = M.identity_matrix()
sage: A = MatrixMorphism(T, I)
sage: loads(A.dumps()) == A
True
```

**base_ring()**

Return the base ring of self, that is, the ring over which self is given by a matrix.

EXAMPLES:

```python
sage: sage.modules.matrix_morphism.MatrixMorphism((ZZ**2).endomorphism_ring(),
˓→Matrix(ZZ,2,[3..6])).base_ring()
Integer Ring
```

**characteristic_polynomial(\texttt{var}='x')**

Return the characteristic polynomial of this endomorphism.

characteristic_polynomial and char_poly are the same method.

**INPUT:**

• \texttt{var} – variable

**EXAMPLES:**

```python
sage: V = ZZ^2; phi = V.hom([V.0+V.1, 2*V.1])
sage: phi.characteristic_polynomial()
x^2 - 3*x + 2
sage: phi.charpoly()
x^2 - 3*x + 2
sage: phi.matrix().charpoly()
x^2 - 3*x + 2
sage: phi.charpoly('T')
T^2 - 3*T + 2
```

**charpoly(\texttt{var}='x')**

Return the characteristic polynomial of this endomorphism.

characteristic_polynomial and char_poly are the same method.

**INPUT:**

• \texttt{var} – variable

**EXAMPLES:**

```python
sage: V = ZZ^2; phi = V.hom([V.0+V.1, 2*V.1])
sage: phi.characteristic_polynomial()
x^2 - 3*x + 2
sage: phi.charpoly()
x^2 - 3*x + 2
```
decomposition(*args, **kwds)

Return decomposition of this endomorphism, i.e., sequence of subspaces obtained by finding invariant subspaces of self.

See the documentation for self.matrix().decomposition for more details. All inputs to this function are passed onto the matrix one.

EXAMPLES:

```python
sage: V = ZZ^2; phi = V.hom([V.0+V.1, 2*V.1])
sage: phi.decomposition()
[
    Free module of degree 2 and rank 1 over Integer Ring
    Echelon basis matrix:
    [0 1],
    Free module of degree 2 and rank 1 over Integer Ring
    Echelon basis matrix:
    [ 1 -1]
]
sage: phi2 = V.hom(phi.matrix(), side="right")
sage: phi2.decomposition()
[
    Free module of degree 2 and rank 1 over Integer Ring
    Echelon basis matrix:
    [1 1],
    Free module of degree 2 and rank 1 over Integer Ring
    Echelon basis matrix:
    [1 0]
]
```

det()

Return the determinant of this endomorphism.

EXAMPLES:

```python
sage: V = ZZ^2; phi = V.hom([V.0+V.1, 2*V.1])
sage: phi.det()
2
```

fcp(var='x')

Return the factorization of the characteristic polynomial.

EXAMPLES:

```python
sage: V = ZZ^2; phi = V.hom([V.0+V.1, 2*V.1])
sage: phi.fcp()
(x - 2) * (x - 1)
sage: phi.fcp('T')
(T - 2) * (T - 1)
```
image()
Compute the image of this morphism.

EXAMPLES:

```
sage: V = VectorSpace(QQ,3)
sage: phi = V.Hom(V)(matrix(QQ, 3, range(9)))
sage: phi.image()
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1 0 -1]
[ 0 1 2]
sage: hom(GF(7)^3, GF(7)^2, zero_matrix(GF(7), 3, 2)).image()
Vector space of degree 2 and dimension 0 over Finite Field of size 7
Basis matrix: []
sage: m = matrix(3, [1, 0, 0, 1, 0, 0, 0, 0, 1]); m
[1 0 0]
[1 0 0]
[0 0 1]
sage: f1 = V.hom(m)
sage: f2 = V.hom(m, side="right")
sage: f1.image()
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[1 0 0]
[0 0 1]
sage: f2.image()
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[1 1 0]
[0 0 1]
```

Compute the image of the identity map on a ZZ-submodule:

```
sage: V = (ZZ^2).span([[1,2],[3,4]])
sage: phi = V.Hom(V)(identity_matrix(ZZ,2))
sage: phi(V.0) == V.0
True
sage: phi(V.1) == V.1
True
sage: phi.image()
Free module of degree 2 and rank 2 over Integer Ring
Echelon basis matrix:
[1 0]
[0 2]
sage: phi.image() == V
True
```

inverse()
Return the inverse of this matrix morphism, if the inverse exists.

Raises a ZeroDivisionError if the inverse does not exist.

EXAMPLES:
An invertible morphism created as a restriction of a non-invertible morphism, and which has an unequal domain and codomain.

```
sage: V = QQ^4
sage: W = QQ^3
sage: m = matrix(QQ, [[2, 0, 3], [-6, 1, 4], [1, 2, -4], [1, 0, 1]])
sage: phi = V.hom(m, W)
sage: rho = phi.restrict_domain(V.span([V.0, V.3]))
sage: zeta = rho.restrict_codomain(W.span([W.0, W.2]))
sage: x = vector(QQ, [2, 0, 0, -7])
sage: y = zeta(x); y
(-3, 0, -1)
sage: inv = zeta.inverse(); inv
Vector space morphism represented by the matrix:
[-1 3]
[ 1 -2]
Domain: Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[1 0 0]
[0 0 1]
Codomain: Vector space of degree 4 and dimension 2 over Rational Field
Basis matrix:
[1 0 0 0]
[0 0 0 1]
sage: inv(y) == x
True
```

An example of an invertible morphism between modules, (rather than between vector spaces).

```
sage: M = ZZ^4
sage: p = matrix(ZZ, [[ 0, -1, 1, -2],
                ....: [ 1, -3, 2, -3],
                ....: [ 0, 4, -3, 4],
                ....: [-2, 8, -4, 3]])
```

```
sage: phi = M.hom(p, M)
sage: x = vector(ZZ, [1, -3, 5, -2])
sage: y = phi(x); y
(1, 12, -12, 21)
sage: rho = phi.inverse(); rho
Free module morphism defined by the matrix
[ -5  3 -1  1]
[ -9  4 -3  2]
[-20  8 -7  4]
[-16  2 -2  1]
Domain: Ambient free module of rank 4 over the principal ideal domain ...
Codomain: Ambient free module of rank 4 over the principal ideal domain ...
sage: rho(y) == x
True
```

A non-invertible morphism, despite having an appropriate domain and codomain.

```
sage: V = QQ^2
sage: m = matrix(QQ, [[1, 2], [20, 40]])
sage: phi = V.hom(m, V)
```

(continues on next page)
The matrix representation of this morphism is invertible over the rationals, but not over the integers, thus the morphism is not invertible as a map between modules. It is easy to notice from the definition that every vector of the image will have a second entry that is an even integer.

```
sage: V = ZZ^2
sage: q = matrix(ZZ, [[1, 2], [3, 4]])
sage: phi = V.hom(q, V)
sage: phi.matrix().change_ring(QQ).inverse()
[ -2  1]
[ 3/2 -1/2]
sage: phi.is_bijective()
False
sage: phi.image()
Free module of degree 2 and rank 2 over Integer Ring
Echelon basis matrix:
[1 0]
[0 2]
sage: phi.lift(vector(ZZ, [1, 1]))
Traceback (most recent call last):
...  
ValueError: element is not in the image
sage: phi.inverse()
Traceback (most recent call last):
...  
ZeroDivisionError: matrix morphism not invertible
```

The unary invert operator (~, tilde, "wiggle") is synonymous with the inverse() method (and a lot easier to type).

```
sage: V = QQ^2
sage: r = matrix(QQ, [[4, 3], [-2, 5]])
sage: phi = V.hom(r, V)
sage: rho = phi.inverse()
sage: zeta = ~phi
sage: rho.is_equal_function(zeta)
True
```

is_bijective()  
Tell whether self is bijective.

EXAMPLES:

Two morphisms that are obviously not bijective, simply on considerations of the dimensions. However, each fullfills half of the requirements to be a bijection.

```
sage: V1 = QQ^2
sage: V2 = QQ^3
```

We construct a simple bijection between two one-dimensional vector spaces.

```
sage: V1 = QQ^3
sage: V2 = QQ^2
sage: phi = V1.hom(matrix(QQ, [[1, 2, 3], [4, 5, 6]]), V2)
sage: x = vector(QQ, [1, -1, 4])
sage: y = phi(x) ; y
(18, 22)
sage: rho = phi.restrict_domain(V1.span([x]))
sage: zeta = rho.restrict_codomain(V2.span([y]))
sage: zeta.is_bijective()
True
```

AUTHOR:

• Rob Beezer (2011-06-28)

**is_equal_function**(other)

Determines if two morphisms are equal functions.

INPUT:

• other - a morphism to compare with self

OUTPUT:

Returns True precisely when the two morphisms have equal domains and codomains (as sets) and produce identical output when given the same input. Otherwise returns False.

This is useful when self and other may have different representations.

Sage’s default comparison of matrix morphisms requires the domains to have the same bases and the codomains to have the same bases, and then compares the matrix representations. This notion of equality is more permissive (it will return True “more often”), but is more correct mathematically.

EXAMPLES:

Three morphisms defined by combinations of different bases for the domain and codomain and different functions. Two are equal, the third is different from both of the others.

```
sage: B = matrix(QQ, [[-3, 5, -4, 2], [-1, 2, -1, 4], [4, -6, 5, -1], [-5, 7, -6, 1]])
sage: U = (QQ^4).subspace_with_basis(B.rows())
(continues on next page)```
sage: C = matrix(QQ, [[-1, -6, -4], [3, -5, 6], [1, 2, 3]])
sage: V = (QQ^3).subspace_with_basis(C.rows())
sage: H = Hom(U, V)

sage: D = matrix(QQ, [[-7, -2, -5, 2], [-5, 1, -4, -8], [1, -1, 1, 4], [-4, -1, -3, 1]])
sage: X = (QQ^4).subspace_with_basis(D.rows())
sage: E = matrix(QQ, [[4, -1, 4], [5, -4, -5], [-1, 0, -2]])
sage: Y = (QQ^3).subspace_with_basis(E.rows())
sage: K = Hom(X, Y)

sage: f = lambda x: vector(QQ, [x[0]+x[1], 2*x[1]-4*x[2], 5*x[3]])
sage: g = lambda x: vector(QQ, [x[0]-x[2], 2*x[1]-4*x[2], 5*x[3]])

sage: rho = H(f)
sage: phi = K(f)
sage: zeta = H(g)

sage: rho.is_equal_function(phi)
True
sage: phi.is_equal_function(rho)
True
sage: zeta.is_equal_function(rho)
False
sage: phi.is_equal_function(zeta)
False

AUTHOR:

- Rob Beezer (2011-07-15)

is_identity()

Determines if this morphism is an identity function or not.

EXAMPLES:

A homomorphism that cannot possibly be the identity due to an unequal domain and codomain.

sage: V = QQ^3
sage: W = QQ^2
sage: m = matrix(QQ, [[1, 2], [3, 4], [5, 6]])
sage: phi = V.hom(m, W)
sage: phi.is_identity()
False

A bijection, but not the identity.
A restriction that is the identity.

```
sage: V = QQ^3
sage: p = matrix(QQ, [[1, 0, 0], [5, 8, 3], [0, 0, 1]])
sage: phi = V.hom(p, V)
sage: rho = phi.restrict(V.span([V.0, V.2]))
sage: rho.is_identity()
True
```

An identity linear transformation that is defined with a domain and codomain with wildly different bases, so that the matrix representation is not simply the identity matrix.

```
sage: A = matrix(QQ, [[1, 1, 0], [2, 3, -4], [2, 4, -7]])
sage: B = matrix(QQ, [[2, 7, -2], [-1, -3, 1], [-1, -6, 2]])
sage: U = (QQ^3).subspace_with_basis(A.rows())
sage: V = (QQ^3).subspace_with_basis(B.rows())
sage: H = Hom(U, V)
sage: id = lambda x: x
sage: phi = H(id)
sage: phi([203, -179, 34])
(203, -179, 34)
sage: phi.matrix()
[ 1 0 1]
[ -9 -18 -2]
[-17 -31 -5]
sage: phi.is_identity()
True
```

AUTHOR:

- Rob Beezer (2011-06-28)

**is_zero()**

Determines if this morphism is a zero function or not.

**EXAMPLES:**

A zero morphism created from a function.

```
sage: V = ZZ^5
sage: W = ZZ^3
sage: z = lambda x: zero_vector(ZZ, 3)
sage: phi = V.hom(z, W)
sage: phi.is_zero()
True
```

An image list that just barely makes a non-zero morphism.
AUTHOR:

- Rob Beezer (2011-07-15)

**kernel()**

Compute the kernel of this morphism.

EXAMPLES:

```python
sage: V = VectorSpace(QQ,3)
sage: id = V.Hom(V)(identity_matrix(QQ,3))
sage: null = V.Hom(V)(0*identity_matrix(QQ,3))
sage: id.kernel()
Vector space of degree 3 and dimension 0 over Rational Field
Basis matrix:
[]
sage: phi = V.Hom(V)(matrix(QQ,3,range(9)))
sage: phi.kernel()
Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[ 1 -2 1]
sage: hom(CC^2, CC^2, matrix(CC, [1,0], [0,1])).kernel()
Vector space of degree 2 and dimension 0 over Complex Field with 53 bits of →
˓precision
Basis matrix:
[]
sage: m = matrix(3, [1, 0, 0, 1, 0, 0, 0, 0, 1]); m
[1 0 0]
[1 0 0]
[0 0 1]
sage: f1 = V.hom(m)
sage: f2 = V.hom(m, side="right")
sage: f1.kernel()
Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[ 1 -1 0]
sage: f2.kernel()
Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[0 1 0]
```

**matrix()**

EXAMPLES:

```python
sage: V = ZZ^2; phi = V.hom(V.basis())
sage: phi.matrix()
```
nullity()  
Returns the nullity of the matrix representing this morphism, which is the dimension of its kernel.

EXAMPLES:

```sage
sage: V = ZZ^2; phi = V.hom(V.basis())
sage: phi.nullity()
0
sage: V = ZZ^2; phi = V.hom([V.0, V.0])
sage: phi.nullity()
1
```

rank()  
Returns the rank of the matrix representing this morphism.

EXAMPLES:

```sage
sage: V = ZZ^2; phi = V.hom(V.basis())
sage: phi.rank()
2
sage: V = ZZ^2; phi = V.hom([V.0, V.0])
sage: phi.rank()
1
```

restrict(sub)  
Restrict this matrix morphism to a subspace sub of the domain.

The codomain and domain of the resulting matrix are both sub.

EXAMPLES:

```sage
sage: V = ZZ^2; phi = V.hom([3*V.0, 2*V.1])
sage: phi.restrict(V.span([V.0]))
Free module morphism defined by the matrix
[3]
Domain: Free module of degree 2 and rank 1 over Integer Ring
Echelon ... 
```
Codomain: Free module of degree 2 and rank 1 over Integer Ring
Echelon ...

sage: V = (QQ^2).span_of_basis([[1,2],[3,4]])
sage: phi = V.hom([V.0+V.1, 2*V.1])
sage: phi(V.1) == 2*V.1
True
sage: W = span([V.1])
sage: phi(W)
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[ 1 4/3]
sage: psi = phi.restrict(W); psi
Vector space morphism represented by the matrix:
[2]
Domain: Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[ 1 4/3]
Codomain: Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[ 1 4/3]
sage: psi.domain() == W
True
sage: psi(W.0) == 2*W.0
True

sage: V = ZZ^3
sage: h1 = V.hom([V.0, V.1+V.2, -V.1+V.2])
sage: h2 = h1.side_switch()
sage: SV = V.span([2*V.1,2*V.2])
sage: h1.restrict(SV)
Free module morphism defined by the matrix
[ 1 1]
[-1 1]
Domain: Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[0 2 0]
[0 0 2]
Codomain: Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[0 2 0]
[0 0 2]
sage: h2.restrict(SV)
Free module morphism defined as left-multiplication by the matrix
[ 1 -1]
[ 1 1]
Domain: Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[0 2 0]
[0 0 2]
Codomain: Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
restrict_codomain(sub)
Restrict this matrix morphism to a subspace sub of the codomain.

The resulting morphism has the same domain as before, but a new codomain.

EXAMPLES:

```
sage: V = ZZ^2; phi = V.hom([4*(V.0+V.1),0])
sage: W = V.span([2*(V.0+V.1)])
sage: phi
Free module morphism defined by the matrix
[4 4]
[0 0]
Domain: Ambient free module of rank 2 over the principal ideal domain ...
Codomain: Ambient free module of rank 2 over the principal ideal domain ...
sage: psi = phi.restrict_codomain(W); psi
Free module morphism defined by the matrix
[2]
[0]
Domain: Ambient free module of rank 2 over the principal ideal domain ...
Codomain: Free module of degree 2 and rank 1 over Integer Ring
Echelon ...
sage: phi2 = phi.side_switch(); phi2.restrict_codomain(W)
Free module morphism defined as left-multiplication by the matrix
[2 0]
[0 0]
Domain: Ambient free module of rank 2 over the principal ideal domain Integer...
Codomain: Free module of degree 2 and rank 1 over Integer Ring
Echelon ...
```

An example in which the codomain equals the full ambient space, but with a different basis:

```
sage: V = QQ^2
sage: W = V.span_of_basis([\[1,2\], [3,4]])
sage: phi = V.hom(matrix(QQ,2,[1,0,2,0]),W)
sage: phi.matrix()
[1 0]
[2 0]
sage: phi(V.0)  
(1, 2)
sage: phi(V.1)  
(2, 4)
sage: X = V.span([[1,2]]); X
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[1 2]
sage: phi(V.0) in X
True
sage: phi(V.1) in X
True
```
sage: psi = phi.restrict_codomain(X); psi
Vector space morphism represented by the matrix:
[1]
[2]
Domain: Vector space of dimension 2 over Rational Field
Codomain: Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[1 2]
sage: psi(V.0)
(1, 2)
sage: psi(V.1)
(2, 4)
sage: psi(V.0).parent() is X
True

**restrict_domain**(sub)
Restrict this matrix morphism to a subspace sub of the domain. The subspace sub should have a basis() method and elements of the basis should be coercible into domain.

The resulting morphism has the same codomain as before, but a new domain.

EXAMPLES:

```
sage: V = ZZ^2; phi = V.hom([3*V.0, 2*V.1])
sage: phi.restrict_domain(V.span([V.0]))
Free module morphism defined by the matrix
[3 0]
Domain: Free module of degree 2 and rank 1 over Integer Ring
Echelon ... 
Codomain: Ambient free module of rank 2 over the principal ideal domain ...
sage: phi.restrict_domain(V.span([V.1]))
Free module morphism defined by the matrix
[0 2]...
sage: m = matrix(2, range(1,5))
sage: f1 = V.hom(m); f2 = V.hom(m, side="right")
sage: SV = V.span([V.0])
sage: f1.restrict_domain(SV)
Free module morphism defined by the matrix
[1 2]...
sage: f2.restrict_domain(SV)
Free module morphism defined as left-multiplication by the matrix
[1]
[3]...
```

**side()**
Return the side of vectors acted on, relative to the matrix.

EXAMPLES:

```
sage: m = matrix(2, [1, 1, 0, 1])
sage: V = ZZ^2
sage: h1 = V.hom(m); h2 = V.hom(m, side="right")
sage: h1.side()
'left'
```
\begin{verbatim}
sage: h1([1, 0])
(1, 1)
sage: h2.side()
'right'
sage: h2([1, 0])
(1, 0)
\end{verbatim}

side_switch()

Return the same morphism, acting on vectors on the opposite side

EXAMPLES:

\begin{verbatim}
sage: m = matrix(2, [1,1,0,1]); m
[1 1]
[0 1]
sage: V = ZZ^2
sage: h = V.hom(m); h.side()
'left'
sage: h2 = h.side_switch(); h2
Free module morphism defined as left-multiplication by the matrix
[1 0]
[1 1]
Domain: Ambient free module of rank 2 over the principal ideal domain Integer...
Codomain: Ambient free module of rank 2 over the principal ideal domain Integer...
sage: h2.side()
'right'
sage: h2.side_switch().matrix()
[1 1]
[0 1]
\end{verbatim}

trace()

Return the trace of this endomorphism.

EXAMPLES:

\begin{verbatim}
sage: V = ZZ^2; phi = V.hom([V.0+V.1, 2*V.1])
sage: phi.trace()
3
\end{verbatim}

sage.modules.matrix_morphism.is_MatrixMorphism(x)

Return True if x is a Matrix morphism of free modules.

EXAMPLES:

\begin{verbatim}
sage: V = ZZ^2; phi = V.hom([3*V.0, 2*V.1])
sage: sage.modules.matrix_morphism.is_MatrixMorphism(phi)
True
sage: sage.modules.matrix_morphism.is_MatrixMorphism(3)
False
\end{verbatim}
Finitely generated modules over a PID

You can use Sage to compute with finitely generated modules (FGM’s) over a principal ideal domain \( R \) presented as a quotient \( V/W \), where \( V \) and \( W \) are free.

**Note:** Currently this is only enabled over \( R=\mathbb{Z} \), since it has not been tested and debugged over more general PIDs. All algorithms make sense whenever there is a Hermite form implementation. In theory the obstruction to extending the implementation is only that one has to decide how elements print. If you’re annoyed that by this, fix things and post a patch!

We represent \( M = V/ W \) as a pair \((V, W)\) with \( W \) contained in \( V \), and we internally represent elements of \( M \) non-canonically as elements \( x \) of \( V \). We also fix independent generators \( g[i] \) for \( M \) in \( V \), and when we print out elements of \( V \) we print their coordinates with respect to the \( g[i] \); over \( \mathbb{Z} \) this is canonical, since each coefficient is reduce modulo the additive order of \( g[i] \). To obtain the vector in \( V \) corresponding to \( x \) in \( M \), use \( x\text{.lift()} \).

Morphisms between finitely generated \( R \) modules are well supported. You create a homomorphism by simply giving the images of generators of \( M_0 \) in \( M_1 \). Given a morphism \( \phi:M_0\to M_1 \), you can compute the image of \( \phi \), the kernel of \( \phi \), and using \( y=\phi\text{.lift}(x) \) you can lift an elements \( x \) in \( M_1 \) to an element \( y \) in \( M_0 \), if such a \( y \) exists.

**TECHNICAL NOTE:** For efficiency, we introduce a notion of optimized representation for quotient modules. The optimized representation of \( M=V/W \) is the quotient \( V'/W' \) where \( V' \) has as basis lifts of the generators \( g[i] \) for \( M \). We internally store a morphism from \( M_0=V_0/W_0 \) to \( M_1=V_1/W_1 \) by giving a morphism from the optimized representation \( V' \) of \( M_0 \) to \( V_1 \) that sends \( W_0 \) into \( W_1 \).

The following TUTORIAL illustrates several of the above points.

First we create free modules \( V_0 \) and \( W_0 \) and the quotient module \( M_0 \). Notice that everything works fine even though \( V_0 \) and \( W_0 \) are not contained inside \( \mathbb{Z}^n \), which is extremely convenient.

```python
sage: V0 = span([[1/2,0,0],[3/2,2,1],[0,0,1]],ZZ); W0 = V0.span([V0.0+2*V0.1, 9*V0.0+2*V0.1, 4*V0.2])
sage: M0 = V0/W0; M0
Finitely generated module V/W over Integer Ring with invariants (4, 16)
```

The invariants are computed using the Smith normal form algorithm, and determine the structure of this finitely generated module.

You can get the \( V \) and \( W \) used in constructing the quotient module using \( V() \) and \( W() \) methods:

```python
sage: M0.V()
Free module of degree 3 and rank 3 over Integer Ring
Echelon basis matrix:
[1/2 0 0]
[ 0 2 0]
```

(continues on next page)
We note that the optimized representation of \( M_0 \), mentioned above in the technical note has a \( V \) that need not be equal to \( V_0 \), in general.

```
sage: M0.optimized()[0].V()
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[ 0  8  1]
[ 0 -2  0]
```

Create elements of \( M_0 \) either by coercing in elements of \( V_0 \), getting generators, or coercing in a list or tuple or coercing in 0. Finally, one can express an element as a linear combination of the smith form generators

```
sage: M0(V0.0)
(0, 2)
sage: M0(V0.0 + W0.0)  # no difference modulo W0
(0, 2)
sage: M0.linear_combination_of_smith_form_gens([3,20])
(3, 4)
sage: 3*M0.0 + 20*M0.1
(3, 4)
```

We make an element of \( M_0 \) by taking a difference of two generators, and lift it. We also illustrate making an element from a list, which coerces to \( V_0 \), then take the equivalence class modulo \( W_0 \).

```
sage: x = M0.0 - M0.1; x
(1, 15)
sage: x.lift()
(0, 10, 1)
sage: M0(vector([1/2,0,0]))
(0, 2)
sage: x.additive_order()
16
```

Similarly, we construct \( V_1 \) and \( W_1 \), and the quotient \( M_1 \), in a completely different 2-dimensional ambient space.

```
sage: V1 = span([[1/2,0],[3/2,2]],ZZ); W1 = V1.span([2*V1.0, 3*V1.1])
sage: M1 = V1/W1; M1
Finitely generated module V/W over Integer Ring with invariants (6)
```

We create the homomorphism from \( M_0 \) to \( M_1 \) that sends both generators of \( M_0 \) to 3 times the generator of \( M_1 \). This is well defined since 3 times the generator has order 2.

```
sage: f = M0.hom([3*M1.0, 3*M1.0]); f
Morphism from module over Integer Ring with invariants (4, 16) to module with invariants (6,) that sends the generators to [(3), (3)]
```
We evaluate the homomorphism on our element $x$ of the domain, and on the first generator of the domain. We also evaluate at an element of $V_0$, which is coerced into $M_0$.

```sage
sage: f(x)
(0)
sage: f(M0.0)
(3)
sage: f(V0.1)
(3)
```

Here we illustrate lifting an element of the image of $f$, i.e., finding an element of $M_0$ that maps to a given element of $M_1$:

```sage
sage: y = f.lift(3*M1.0)
sage: y
# random
(0, 13)
sage: f(y)
(3)
```

We compute the kernel of $f$, i.e., the submodule of elements of $M_0$ that map to 0. Note that the kernel is not explicitly represented as a submodule, but as another quotient $V/W$ where $V$ is contained in $V_0$. You can explicitly coerce elements of the kernel into $M_0$ though.

```sage
sage: K = f.kernel(); K
Finitely generated module $V/W$ over Integer Ring with invariants $(2, 16)$

sage: M0(K.0)
(2, 8)
sage: M0(K.1)
(1, 5)
sage: f(M0(K.0))
(0)
sage: f(M0(K.1))
(0)
```

We compute the image of $f$.

```sage
sage: f.image()
Finitely generated module $V/W$ over Integer Ring with invariants $(2)$
```

Notice how the elements of the image are written as $(0)$ and $(1)$, despite the image being naturally a submodule of $M_1$, which has elements $(0), (1), (2), (3), (4), (5)$. However, below we coerce the element $(1)$ of the image into the codomain, and get $(3)$:

```sage
sage: list(f.image())
[(0), (1)]
sage: list(M1)
[(0), (1), (2), (3), (4), (5)]
sage: x = f.image().0; x
(1)
sage: M1(x)
(3)
```

**AUTHOR:**

- William Stein, 2009
sage.modules.fg_pid.fgp_module.FGP_Module(V, W, check=True)

INPUT:

• V – a free R-module
• W – a free R-submodule of V
• check – bool (default: True); if True, more checks on correctness are performed; in particular, we check the data types of V and W, and that W is a submodule of V with the same base ring.

OUTPUT:

• the quotient V / W as a finitely generated R-module

EXAMPLES:

```python
class sage.modules.fg_pid.fgp_module.FGP_Module_class(V, W, check=True)
Bases: sage.modules.module.Module
A finitely generated module over a PID presented as a quotient V / W.

INPUT:

• V – an R-module
• W – an R-submodule of V
• check – bool (default: True)

EXAMPLES:

```
EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: Q = V/W
sage: Q.V()
Free module of degree 3 and rank 3 over Integer Ring
Echelon basis matrix:
[1/2  0  0]
[ 0  1  0]
[ 0  0  1]
```

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: Q = V/W
sage: Q.W()
Free module of degree 3 and rank 3 over Integer Ring
Echelon basis matrix:
[1/2  8  0]
[ 0 12  0]
[ 0  0  4]
```

```
sage: V = span([[1/2,0,0],[3/2,2,1],[0,0,1]],ZZ); W = V.span([V.0+2*V.1, 9*V.0+2*V.1, 4*V.2])
sage: Q = V/W; Q.annihilator()
Principal ideal (16) of Integer Ring
sage: Q.annihilator().gen()
16
sage: Q = V/V.span([V.0]); Q
Finitely generated module V/W over Integer Ring with invariants (0, 0)
sage: Q.annihilator()
Principal ideal (0) of Integer Ring
```

We check that trac ticket #22720 is resolved:

```
sage: H=AdditiveAbelianGroup([])
sage: H.annihilator()
Principal ideal (1) of Integer Ring
```

```
sage: H=AdditiveAbelianGroup([])
sage: H.annihilator()
Principal ideal (1) of Integer Ring
```

```
sage: H=AdditiveAbelianGroup([])
sage: H.annihilator()
Principal ideal (1) of Integer Ring
```

```
sage: H=AdditiveAbelianGroup([])
sage: H.annihilator()
Principal ideal (1) of Integer Ring
```

```
sage: H=AdditiveAbelianGroup([])
sage: H.annihilator()
Principal ideal (1) of Integer Ring
```

base_ring()  
Return the base ring of self.

EXAMPLES:
Return the cardinality of this module as a set.

EXAMPLES:

```
sage: V = ZZ^2; W = V.span([[1,2],[3,4]]); A = V/W; A
Finitely generated module V/W over Integer Ring with invariants (2)
sage: A.cardinality()
2
sage: V = QQ^2; W = V.span([[1,2]]); A = V/W; A
Vector space quotient V/W of dimension 1 over Rational Field where
V: Vector space of dimension 2 over Rational Field
W: Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[1 2]
sage: A.cardinality()
+Infinity
```

cardinality()

The construction functor and ambient module for self.

EXAMPLES:

```
sage: W = ZZ^2
sage: A1 = W.submodule([[1,0]])
sage: B1 = W.submodule([[2,0]])
sage: T1 = A1 / B1
sage: T1.construction()
(QuotientModuleFunctor,
 Free module of degree 2 and rank 1 over Integer Ring
 Echelon basis matrix:
 [1 0])
```

code_vector(x, reduce=False)

Return coordinates of x with respect to the optimized representation of self.

INPUT:

- x – element of self
- reduce – (default: False); if True, reduce coefficients modulo invariants; this is ignored if the base
  ring is not ZZ.

OUTPUT:

The coordinates as a vector. That is, the same type as self.V(), but in general with fewer entries.

EXAMPLES:
```python
sage: V = span([[1/4,0,0],[3/4,4,2],[0,0,2]],ZZ); W = V.span([4*V.0+12*V.1])
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 0, 0)
sage: Q.coordinate_vector(-Q.0)
(-1, 0, 0)
sage: Q.coordinate_vector(-Q.0, reduce=True)
(3, 0, 0)
```

If x is not in self, it is coerced in:

```python
sage: Q.coordinate_vector(V.0)
(1, -3, 0)
sage: Q.coordinate_vector(Q(V.0))
(1, -3, 0)
```

**cover()**

If this module was constructed as V/W, returns the cover module V.

This is the same as self.V().

**EXAMPLES:**

```python
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: Q = V/W
sage: Q.V()
Free module of degree 3 and rank 3 over Integer Ring
Echelon basis matrix:
[1/2 0 0]
[0 1 0]
[0 0 1]
```

**gen(i)**

Return the i-th generator of self.

**INPUT:**

- i – integer

**EXAMPLES:**

```python
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: Q = V/W
sage: Q.gen(0)
(1, 0)
sage: Q.gen(1)
(0, 1)
sage: Q.gen(2)
Traceback (most recent call last):
...
ValueError: Generator 2 not defined
sage: Q.gen(-1)
Traceback (most recent call last):
```

(continues on next page)
ValueError: Generator -1 not defined

gens()
Returns tuple of elements $g_0, \ldots, g_n$ of self such that the module generated by the $g_i$ is isomorphic to the direct sum of $R/e_i^*R$, where $e_i$ are the invariants of self and $R$ is the base ring.

Note that these are not generally uniquely determined, and depending on how Smith normal form is implemented for the base ring, they may not even be deterministic.

This can safely be overridden in all derived classes.

EXAMPLES:

```python
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.\rightarrow 0+12*V.1, 4*V.2])
sage: Q = V/W
go: Q gens()
((1, 0), (0, 1))
sage: Q.0
(1, 0)
```

gens_to_smith()
Return the transformation matrix from the user to Smith form generators.

To go in the other direction use `smith_to_gens()`.

OUTPUT:

• a matrix over the base ring

EXAMPLES:

```python
sage: L2 = IntegralLattice(3 * matrix([[[-2,0,0],[0,1,0],[0,0,4]])))
sage: D = L2.discriminant_group().normal_form()
sage: D
Finite quadratic module over Integer Ring with invariants (3, 6, 12)
Gram matrix of the quadratic form with values in Q/Z:
[1/2 0 0 0 0]
[ 0 1/4 0 0 0]
[ 0 0 1/3 0 0]
[ 0 0 3 0]
[ 0 0 4 2/3]
sage: D.gens_to_smith()
[0 3 0]
[0 0 3]
[0 4 0]
[1 2 0]
[0 0 4]
sage: T = D.gens_to_smith()*D.smith_to_gens()
sage: T
[ 3 0 3 0 0]
[ 0 33 0 0 3]
[ 4 0 4 0 0]
[ 2 0 3 1 0]
[ 0 44 0 0 4]
```
The matrix $T$ now satisfies a certain congruence:

```
sage: for i in range(T.nrows()):
    ....:     T[:,i] = T[:,i] % D.gens()[i].order()
sage: T
[1 0 0 0 0]
[0 1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```

**gens_vector**($x$, reduce=False)

Return coordinates of $x$ with respect to the generators.

**INPUT:**

- $x$ – element of self
- reduce – (default: False); if True, reduce coefficients modulo invariants; this is ignored if the base ring is not $\mathbb{Z}$

**EXAMPLES:**

We create a derived class and overwrite $\texttt{gens()}$:

```
sage: from sage.modules.fg_pid.fgp_module import FGP_Module_class
sage: W = ZZ^3
sage: V = W.span(matrix.diagonal([1/6,1/3,1/12]))
sage: class FGP_with_gens(FGP_Module_class):
    ....: def __init__(self, V, W, gens):
    ....:     FGP_Module_class.__init__(self, V, W, gens):
    ....:         self._gens = tuple([self(g) for g in gens])
    .....:
    ....:    def gens(self):
    ....:        return self._gens
sage: gens = [(1/2, 0, 0), (0, 0, 1/4), (1/3, 0, 0), (0, 1/3, 0), (0, 0, 2/3)]
sage: gens = [V(g) for g in gens]
sage: D = FGP_with_gens(V, W, gens)
sage: D.gens()
((0, 3, 0), (0, 0, 3), (0, 4, 0), (1, 2, 0), (0, 0, 8))
```

We create some element of $D$:

```
sage: x = D.linear_combination_of_smith_form_gens((1,2,3))
sage: x
(1, 2, 3)
```

In our generators:

```
sage: v = D.gens_vector(x)
sage: v
(2, 9, 3, 1, 33)
```

The output can be further reduced:

```
sage: D.gens_vector(x, reduce=True)
(0, 1, 0, 1, 0)
```

Let us check:
\textbf{sage: } x == \text{sum}(v[i]^\ast D$.\text{gen}(i) \text{ for } i \text{ in } \text{range}(\text{len}(D$.\text{gens}())))
\text{True}

\textbf{has\_canonical\_map\_to(A)}
Return True if self has a canonical map to A, relative to the given presentation of A.

This means that A is a finitely generated quotient module, self.V() is a submodule of A.V() and self.W() is a submodule of A.W(), i.e., that there is a natural map induced by inclusion of the V's. Note that we do not require that this natural map be injective; for this use \textit{is\_submodule()}.

\textbf{EXAMPLES:}

\textbf{sage: } V = \text{span}([[1/2,1,1],[3/2,2,1],[0,0,1]],\mathbb{Z}); W = V$.\text{span}([2^\ast 0.0+4^\ast 0.1, 9^\ast 0.0+12^\ast 0.1, 4^\ast 0.2])
\textbf{sage: } Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 12)
\textbf{sage: } A = Q$.\text{submodule}((Q.0, Q.0 + 3^\ast Q.1)); A
Finitely generated module V/W over Integer Ring with invariants (4, 4)
\textbf{sage: } A$.\text{has\_canonical\_map\_to}(Q)
\text{True}
\textbf{sage: } Q$.\text{has\_canonical\_map\_to}(A)
\text{False}

\textbf{hom(im\_gens, codomain=None, check=True)}
Homomorphism defined by giving the images of self.gens() in some fixed fg R-module.

\textbf{Note:} We do not assume that the generators given by self.gens() are the same as the Smith form generators, since this may not be true for a general derived class.

\textbf{INPUT:}
\begin{itemize}
  \item \texttt{im\_gens} – a list of the images of self.gens() in some R-module
\end{itemize}

\textbf{EXAMPLES:}

\textbf{sage: } V = \text{span}([[1/2,1,1],[3/2,2,1],[0,0,1]],\mathbb{Z}); W = V$.\text{span}([2^\ast 0.0+4^\ast 0.1, 9^\ast 0.0+12^\ast 0.1, 4^\ast 0.2])
\textbf{sage: } Q = V/W; Q

This example illustrates creating a morphism to a free module. The free module is turned into an FGP module (i.e., quotient V/W with W=0), and the morphism is constructed:

\textbf{sage: } V = \text{span}([[1/2,0,0],[3/2,2,1],[0,0,1]],\mathbb{Z}); W = V$.\text{span}([2^\ast 0.0+4^\ast 0.1])
\textbf{sage: } Q = V/W; Q

(continues on next page)
Finitely generated module $V/W$ over Integer Ring with invariants $(2, 0, 0)$

```
sage: phi = Q.hom([0, V.0, V.1]); phi
Morphism from module over Integer Ring with invariants $(2, 0, 0)$ to module with invariants $(0, 0, 0)$ that sends the generators to $[(0, 0, 0), (1, 0, 0), (0, 1, 0)]$
```

```
sage: phi.domain()
Finitely generated module $V/W$ over Integer Ring with invariants $(2, 0, 0)$
```

```
sage: phi.codomain()
Finitely generated module $V/W$ over Integer Ring with invariants $(0, 0, 0)$
```

```
sage: phi(Q.0)
(0, 0, 0)
sage: phi(Q.1)
(1, 0, 0)
sage: phi(Q.2) == V.1
True
```

Constructing two zero maps from the zero module:

```
sage: A = (ZZ^2)/(ZZ^2); A
Finitely generated module $V/W$ over Integer Ring with invariants ()
```

```
sage: A.hom([])
Morphism from module over Integer Ring with invariants () to module with invariants () that sends the generators to []
```

```
sage: A.hom([]).codomain() is A
True
```

```
sage: B = (ZZ^3)/(ZZ^3)
sage: A.hom([], codomain=B)
Morphism from module over Integer Ring with invariants () to module with invariants () that sends the generators to []
```

```
sage: phi = A.hom([], codomain=B); phi
Morphism from module over Integer Ring with invariants () to module with invariants () that sends the generators to []
```

```
sage: phi(A(0))
()
sage: phi(A(0)) == B(0)
True
```

A degenerate case:

```
sage: A = (ZZ^2)/(ZZ^2)
sage: phi = A.hom([]); phi
Morphism from module over Integer Ring with invariants () to module with invariants () that sends the generators to []
```

```
sage: phi(A(0))
()
```

The code checks that the morphism is valid. In the example below we try to send a generator of order 2 to an element of order 14:

```
sage: V = span([[1/14,3/14],[0,1/2]],ZZ); W = ZZ^2
sage: Q = V/W; Q
Finitely generated module $V/W$ over Integer Ring with invariants $(2, 14)$
```

```
sage: Q.linear_combination_of_smith_form_gens([1,11]).additive_order()
(continues on next page)```
sage: f = Q.hom([Q.linear_combination_of_smith_form_gens([1,11]), Q.linear_combination_of_smith_form_gens([1,3])]); f
Traceback (most recent call last):
  ... ValueError: phi must send optimized submodule of M.W() into N.W()

invariants(include_ones=False)

Return the diagonal entries of the smith form of the relative matrix that defines self (see _relative_matrix()) padded with zeros, excluding 1’s by default. Thus if v is the list of integers returned, then self is abstractly isomorphic to the product of cyclic groups $\mathbb{Z}/n\mathbb{Z}$ where $n$ is in $v$.

INPUT:
- include_ones – bool (default: False); if True, also include 1’s in the output list.

EXAMPLES:

sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.1+12*V.1, 4*V.2])
sage: Q = V/W
sage: Q.invariants()
(4, 12)

An example with 1 and 0 rows:

sage: V = ZZ^3; W = V.span([[1,2,0],[0,1,0],[0,2,0]]); Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (0, 0, 0)
sage: Q.is_finite()
False

is_finite()

Return True if self is finite and False otherwise.

EXAMPLES:

sage: V = span([[1/2,0,0],[3/2,2,1],[0,0,1]],ZZ); W = V.span([V.0+2*V.1, 9*V.1+2*V.1, 4*V.2])
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 16)
sage: Q.is_finite()
True
sage: Q = V/V.zero_submodule(); Q
Finitely generated module V/W over Integer Ring with invariants (0, 0, 0)
sage: Q.is_finite()
False

is_submodule(A)

Return True if self is a submodule of A.

More precisely, this returns True if self.V() is a submodule of A.V(), with self.W() equal to A.W().

Compare has_canonical_map_to().

EXAMPLES:
This example illustrates that this command works in some subtle cases:

```
sage: A = ZZ^1
sage: Q3 = A / A.span([3])
```

```
sage: Q6 = A / A.span([6])
```

```
sage: Q6.is_submodule(Q3)
False
```

```
sage: Q6.has_canonical_map_to(Q3)
True
```

```
sage: Q = A.span([2]) / A.span([6])
```

```
sage: Q.is_submodule(Q6)
True
```

### linear_combination_of_smith_form_gens(x)

Compute a linear combination of the optimised generators of this module as returned by `smith_form_gens()`.

**EXAMPLES:**

```
sage: X = ZZ**2 / span([[3,0],[0,2]], ZZ)
```

```
sage: X.linear_combination_of_smith_form_gens([1])
(1)
```

### list()

Return a list of the elements of self.

**EXAMPLES:**

```
sage: V = ZZ^2; W = V.span([[1,2],[3,4]])
```

```
sage: list(V/W)
[(0), (1)]
```

### ngens()

Return the number of generators of self.

(Note for developers: This is just the length of `gens()`, rather than of the minimal set of generators as returned by `smith_form_gens()`; these are the same in the `FGP_Module` class, but not necessarily in derived classes.)

**EXAMPLES:**

```
sage: A = (ZZ**2) / span([[4,0],[0,3]], ZZ)
```

```
sage: A.ngens()
1
```

This works (but please do not do it in production code!)
sage: A.gens = lambda: [1,2,"Barcelona!"]
sage: A.ngens()
3

optimized()
Return a module isomorphic to this one, but with \( V \) replaced by a submodule of \( V \) such that the generators of self all lift trivially to generators of \( V \). Replace \( W \) by the intersection of \( V \) and \( W \). This has the advantage that \( V \) has small dimension and any homomorphism from self trivially extends to a homomorphism from \( V \).

OUTPUT:

• \( Q \) – an optimized quotient \( V_0/W_0 \) with \( V_0 \) a submodule of \( V \) such that \( \phi: V_0/W_0 \rightarrow V/W \) is an isomorphism
• \( Z \) – matrix such that if \( x \) is in \( \text{self.V}() \) and \( c \) gives the coordinates of \( x \) in terms of the basis for \( \text{self.V}() \), then \( c^*Z \) is in \( V_0 \) and \( c^*Z \) maps to \( x \) via \( \phi \) above.

EXAMPLES:

sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: Q = V/W
sage: O, X = Q.optimized(); O
Finitely generated module \( V/W \) over Integer Ring with invariants \((4, 12)\)
User basis matrix:
[ 0 3 1]
[ 0 -1 0]
sage: O.V()
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[ 0 3 1]
[ 0 -1 0]
sage: O.W()
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[ 0 12 0]
[ 0 0 4]
sage: X
[0 4 0]
[0 1 0]
[0 0 1]
sage: OV = O.V()
sage: Q(OV([0,-8,0])) == V.0
True
sage: Q(OV([0,1,0])) == V.1
True
sage: Q(OV([0,0,1])) == V.2
True

quotient_map()
Given this quotient space \( Q = V/W \), return the natural quotient map from \( V \) to \( Q \).

EXAMPLES:

sage: A = (ZZ**2) / span([[4,0],[0,3]], ZZ)
sage: A.quotient_map()
Coercion map:
random_element(*args, **kwds)
Create a random element of self=V/W, by creating a random element of V and reducing it modulo W.

All arguments are passed onto the random_element method of V.

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: Q = V/W
sage: Q.random_element().parent() is Q
True
sage: Q.cardinality()
48
sage: S = set()
sage: while len(S) < 48:
    ....:     S.add(Q.random_element())
```

relations()
If self was constructed as V / W, return the relations module W.

This is the same as self.W().

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ)
sage: W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: Q = V/W
sage: Q.relations()
Free module of degree 3 and rank 3 over Integer Ring
Echelon basis matrix:
[1/2 8 0]
[ 0 12 0]
[ 0 0 4]
```

smith_form_gen(i)
Return the i-th generator of self.

This is a separate method so we can freely override gen() in derived classes.

INPUT:

- i – integer

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ)
sage: W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 12)
sage: Q.smith_form_gen(0)
(1, 0)
```
sage: Q.smith_form_gen(1)
(0, 1)

\textbf{smith\_form\_gens}()

Return a set of generators for self which are in Smith normal form.

EXAMPLES:

sage: V = span([[1/2, 1, 1], [3/2, 2, 1], [0, 0, 1]], ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: Q = V/W
sage: Q.smith_form_gens()
((1, 0), (0, 1))

sage: [x.lift() for x in Q.smith_form_gens()]
[(0, 3, 1), (0, -1, 0)]

\textbf{smith\_to\_gens}()

Return the transformation matrix from Smith form to user generators.

To go in the other direction use \texttt{gens\_to\_smith}().

OUTPUT:

- a matrix over the base ring

EXAMPLES:

sage: L2 = IntegralLattice(3 * matrix([[1/2, 0, 0], [0, 1, 0], [0, 0, -1/2]]))
sage: D = L2.discriminant_group().normal_form()
sage: D
Finite quadratic module over Integer Ring with invariants (3, 6, 12)
Gram matrix of the quadratic form with values in Q/Z:

\begin{bmatrix}
1/2 & 0 & 0 & 0 & 0 \\
0 & 1/4 & 0 & 0 & 0 \\
0 & 0 & 1/3 & 0 & 0 \\
0 & 0 & 0 & 1/3 & 0 \\
0 & 0 & 0 & 0 & 2/3
\end{bmatrix}

sage: D.smith_to_gens()

\begin{bmatrix}
0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 11 & 0 & 0 & 1
\end{bmatrix}

sage: T = D.smith_to_gens()*D.gens_to_smith()
sage: T

\begin{bmatrix}
1 & 6 & 0 \\
0 & 7 & 0 \\
0 & 37 & 0
\end{bmatrix}

This matrix satisfies the congruence:

sage: for i in range(T.ncols()):
....: T[:, i] = T[:, i] % D.smith_form_gens()[i].order()

sage: T

\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
We create some element of our FGP_module:

```
sage: x = D.linear_combination_of_smith_form_gens((1,2,3))
sage: x
(1, 2, 3)
```

and want to know some (it is not unique) linear combination of the user defined generators that is x:

```
sage: x.vector() * D.smith_to_gens()
(2, 33, 3, 1, 3)
```

**submodule(x)**

Return the submodule defined by x.

**INPUT:**

- `x` – list, tuple, or FGP module

**EXAMPLES:**

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 12)
sage: Q.gens()
((1, 0), (0, 1))
```

We create submodules generated by a list or tuple of elements:

```
sage: Q.submodule([Q.0])
Finitely generated module V/W over Integer Ring with invariants (4)
sage: Q.submodule([Q.1])
Finitely generated module V/W over Integer Ring with invariants (12)
sage: Q.submodule((Q.0, Q.0 + 3*Q.1))
Finitely generated module V/W over Integer Ring with invariants (4, 4)
```

A submodule defined by a submodule:

```
sage: A = Q.submodule((Q.0, Q.0 + 3*Q.1)); A
Finitely generated module V/W over Integer Ring with invariants (4, 4)
sage: Q.submodule(A)
Finitely generated module V/W over Integer Ring with invariants (4, 4)
```

Inclusion is checked:

```
sage: A.submodule(Q)
Traceback (most recent call last):
... ValueError: x.V() must be contained in self's V.
```

**sage.modules.fg_pid.fgp_module.is_FGP_Module(x)**

Return true of x is an FGP module, i.e., a finitely generated module over a PID represented as a quotient of finitely generated free modules over a PID.

**EXAMPLES:**
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2]); Q = V/W
sage: sage.modules.fg_pid.fgp_module.is_FGP_Module(V)
False
sage: sage.modules.fg_pid.fgp_module.is_FGP_Module(Q)
True

sage.modules.fg_pid.fgp_module.random_fgp_module(n, R=Integer Ring, finite=False)
Return a random FGP module inside a rank n free module over R.

INPUT:
- n – nonnegative integer
- R – base ring (default: ZZ)
- finite – bool (default: True); if True, make the random module finite.

EXAMPLES:

sage: import sage.modules.fg_pid.fgp_module as fgp
sage: fgp.random_fgp_module(4)
Finitely generated module V/W over Integer Ring with invariants (...)

In most cases the cardinality is small or infinite:

sage: for g in (1, 2, 3, +Infinity):
    ....:     while fgp.random_fgp_module(4).cardinality() != 1:
    ....:         pass

One can force a finite module:

sage: fgp.random_fgp_module(4, finite=True).is_finite()
True

Larger finite modules appear:

sage: while fgp.random_fgp_module(4, finite=True).cardinality() < 100:
    ....:     pass

sage.modules.fg_pid.fgp_module.random_fgp_morphism_0(*args, **kwds)
Construct a random fgp module using random_fgp_module, then construct a random morphism that sends each generator to a random multiple of itself.

Inputs are the same as to random_fgp_module().

EXAMPLES:

sage: import sage.modules.fg_pid.fgp_module as fgp
sage: mor = fgp.random_fgp_morphism_0(4)
sage: mor.domain() == mor.codomain()
True
sage: fgp.is_FGP_Module(mor.domain())
True

Each generator is sent to a random multiple of itself:
```
sage: gens = mor.domain().gens()
sage: im_gens = mor.im_gens()
sage: all(im_gens[i] == sum(im_gens[i])*gens[i] for i in range(len(gens)))
True
```

```
sage.modules.fg_pid.fgp_module.test_morphism_0(*args,**kwds)

EXAMPLES:
```
sage: import sage.modules.fg_pid.fgp_module as fgp
sage: s = 0  # we set a seed so results clearly and easily reproducible across runs.
sage: set_random_seed(s); v = [fgp.test_morphism_0(1) for _ in range(30)]
sage: set_random_seed(s); v = [fgp.test_morphism_0(2) for _ in range(30)]
sage: set_random_seed(s); v = [fgp.test_morphism_0(3) for _ in range(10)]
sage: set_random_seed(s); v = [fgp.test_morphism_0(i) for i in range(1,20)]
sage: set_random_seed(s); v = [fgp.test_morphism_0(4) for _ in range(50)]  # long...
```
CHAPTER

SIXTEEN

ELEMENTS OF FINITELY GENERATED MODULES OVER A PID

AUTHOR:

• William Stein, 2009

class sage.modules.fg_pid.fgp_element.FGP_Element(parent, x, check=True)
Bases: sage.structure.element.ModuleElement

An element of a finitely generated module over a PID.

INPUT:

• parent – parent module M
• x – element of M.V()

EXAMPLES:

```python
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: Q = V/W
sage: x = Q(V.0-V.1); x #indirect doctest
(0, 9)
sage: isinstance(x, sage.modules.fg_pid.fgp_element.FGP_Element)
True
sage: type(x)
<class 'sage.modules.fg_pid.fgp_module.FGP_Module_class_with_category.element_class'>
sage: x is Q(x)
True
sage: x.parent() is Q
True
```

additive_order()

Return the additive order of this element.

EXAMPLES:

```python
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 12)
sage: Q.0.additive_order()
4
sage: Q.1.additive_order()
12
```

(continues on next page)
sage: (Q.0+Q.1).additive_order()
12
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1])
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (12, 0)
sage: Q.0.additive_order()
12
sage: type(Q.0.additive_order())
<class 'sage.rings.integer.Integer'>
sage: Q.1.additive_order()
+Infinity

lift()
Lift self to an element of V, where the parent of self is the quotient module V/W.

EXAMPLES:

sage: V = span([[1/2,0,0],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 12)
sage: Q.0
(1, 0)
sage: Q.1
(0, 1)
sage: Q.0.lift()
(0, 6, 1)
sage: Q.1.lift()
(0, -2, 0)
sage: x = Q(V.0); x
(0, 8)
sage: x.lift()
(1/2, 0, 0)
sage: x == 8*Q.1
True
sage: x.lift().parent() == V
True

A silly version of the integers modulo 100:

sage: A = (ZZ^1)/span([[100]], ZZ); A
Finitely generated module V/W over Integer Ring with invariants (100)
sage: x = A([5]); x
(5)
sage: v = x.lift(); v
(5)
sage: v.parent()
Ambient free module of rank 1 over the principal ideal domain Integer Ring

vector()
EXAMPLES:
sage: V = span([[1/2,0,0],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])

sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 12)

sage: x = Q.0 + 3*Q.1; x
(1, 3)

sage: x.vector()
(1, 3)

sage: tuple(x)
(1, 3)

sage: list(x)
[1, 3]

sage: x.vector().parent()
Ambient free module of rank 2 over the principal ideal domain Integer Ring
CHAPTER

SEVENTEEN

MORPHISMS BETWEEN FINITELY GENERATED MODULES OVER A PID

AUTHOR:

• William Stein, 2009

```
sage.modules.fg_pid.fgp_morphism.FGP_Homset(X, Y)
EXAMPLES:

    sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.1+12*V.1, 4*V.2]); Q = V/W
    sage: Q.Hom(Q) # indirect doctest
    Set of Morphisms from Finitely generated module V/W over Integer Ring with invariants (4, 12) to Finitely generated module V/W over Integer Ring with invariants (4, 12) in Category of modules over Integer Ring
    sage: True # Q.Hom(Q) is Q.Hom(Q)
    True
    sage: type(Q.Hom(Q))
    <class 'sage.modules.fg_pid.fgp_morphism.FGP_Homset_class_with_category'>
```

class sage.modules.fg_pid.fgp_morphism.FGP_Homset_class(X, Y, category=None)
Bases: sage.categories.homset.Homset

Homsets of FGP_Module

Element

alias of FGP_Morphism

class sage.modules.fg_pid.fgp_morphism.FGP_Morphism(parent, phi, check=True)
Bases: sage.categories.morphism.Morphism

A morphism between finitely generated modules over a PID.

EXAMPLES:

An endomorphism:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.1+12*V.1, 4*V.2])
    sage: Q = V/W; Q
    Finitely generated module V/W over Integer Ring with invariants (4, 12)
    sage: phi = Q.hom([Q.0+3*Q.1, -Q.1]); phi
    Morphism from module over Integer Ring with invariants (4, 12) to module with invariants (4, 12) that sends the generators to [(1, 3), (0, 11)]
    sage: phi(Q.0) == Q.0 + 3*Q.1
    (continues on next page)
```
A morphism between different modules $V_1/W_1 \rightarrow V_2/W_2$ in different ambient spaces:

```sage
sage: V1 = ZZ^2; W1 = V1.span([[1,2],[3,4]]); A1 = V1/W1
sage: V2 = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W2 = V2.span([2*V2.0+4*V2.1, 9*V2.0+12*V2.1, 4*V2.2]); A2=V2/W2
sage: A1
Finitely generated module $V_1/W_1$ over Integer Ring with invariants (2)
sage: A2
Finitely generated module $V_2/W_2$ over Integer Ring with invariants (4, 12)
sage: phi = A1.hom([2*A2.0])
sage: phi(A1.0)
(2, 0)
sage: 2*A2.0
(2, 0)
sage: phi(2*A1.0)
(0, 0)
```

**im_gens()**

Return tuple of the images of the generators of the domain under this morphism.

**EXAMPLES:**

```sage
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2]); Q = V/W
sage: phi = Q.hom([Q.0,Q.0 + 2*Q.1])
sage: phi.im_gens()
((1, 0), (1, 2))
sage: phi.im_gens() is phi.im_gens()
True
```

**image()**

Compute the image of this homomorphism.

**EXAMPLES:**

```sage
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: Q = V/W; Q
Finitely generated module $V/W$ over Integer Ring with invariants (4, 12)
sage: Q.hom([Q.0+3*Q.1, -Q.1]).image()
Finitely generated module $V/W$ over Integer Ring with invariants (4, 12)
sage: Q.hom([3*Q.1, Q.1]).image()
Finitely generated module $V/W$ over Integer Ring with invariants (12)
```

**inverse_image(A)**

Given a submodule $A$ of the codomain of this morphism, return the inverse image of $A$ under this morphism.

**EXAMPLES:**
Compute the kernel of this homomorphism.

EXAMPLES:

```python
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.
˓→0+12*V.1, 4*V.2]); Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 12)
sage: phi = Q.hom([0, Q.1])
sage: phi.kernel()
Finitely generated module V/W over Integer Ring with invariants (4)
sage: Q.1 in phi.kernel()
True
sage: phi = Q.hom([Q.0-3*Q.1, Q.0+Q.1])
sage: A = phi.kernel(); A
Finitely generated module V/W over Integer Ring with invariants (4)
sage: phi(A)
Finitely generated module V/W over Integer Ring with invariants ()
```

lift(x)

Given an element x in the codomain of self, if possible find an element y in the domain such that self(y) == x. Raise a ValueError if no such y exists.

INPUT:

* x – element of the codomain of self.

EXAMPLES:
\[
V = \text{span}([[1/2, 1, 1], [3/2, 2, 1], [0, 0, 1]], \mathbb{Z}); \ W = V.\text{span}([2^*V.0 + 4^*V.1, 9^*V.
\rightarrow 0 + 12^*V.1, 4^*V.2])
\]
\[
sage: Q = V/W; \ phi = Q.\text{hom}([2^*Q.0, Q.1])
\]
\[
sage: \phi.\text{lift}(Q.1)
(0, 1)
\]
\[
sage: \phi.\text{lift}(Q.0)
\text{Traceback (most recent call last):}
\]
\[
ValueError: no lift of element to domain
\]
\[
sage: \phi.\text{lift}(2^*Q.0)
(1, 0)
\]
\[
sage: \phi.\text{lift}(2^*Q.0 + Q.1)
(1, 1)
\]
\[
sage: V = \text{span}([[5, -1/2]], \mathbb{Z}); \ W = \text{span}([[2^*0, -2]], \mathbb{Z}); \ Q = V/W; \phi = Q.\text{hom}([2^*Q.0])
\]
\[
sage: x = \phi.\text{image}.0; \ phi(\phi.\text{lift}(x)) == x
\text{True}
\]
CHAPTER
EIGHTEEN

FINITE \(\mathbb{Z}\)-MODULES WITH WITH BILINEAR AND QUADRATIC FORMS.

AUTHORS:
• Simon Brandhorst (2017-09): First created

sage.modules.torsion_quadratic_module.TorsionQuadraticForm(q)
Create a torsion quadratic form module from a rational matrix.

The resulting quadratic form takes values in \(\mathbb{Q}/\mathbb{Z}\) or \(\mathbb{Q}/2\mathbb{Z}\) (depending on \(q\)). If it takes values modulo 2, then it is non-degenerate. In any case the bilinear form is non-degenerate.

INPUT:
• \(q\) – a symmetric rational matrix

EXAMPLES:

```
sage: q1 = Matrix(QQ,2,[1,1/2,1/2,1])
sage: TorsionQuadraticForm(q1)
Finite quadratic module over Integer Ring with invariants (2, 2)
Gram matrix of the quadratic form with values in \(\mathbb{Q}/2\mathbb{Z}\):
[ 1 1/2]
[1/2 1]
```

In the following example the quadratic form is degenerate. But the bilinear form is still non-degenerate:

```
sage: q2 = diagonal_matrix(QQ,[1/4,1/3])
sage: TorsionQuadraticForm(q2)
Finite quadratic module over Integer Ring with invariants (12,)
Gram matrix of the quadratic form with values in \(\mathbb{Q}/\mathbb{Z}\):
[7/12]
```

class sage.modules.torsion_quadratic_module.TorsionQuadraticModule(V, W, gens, modulus, modulus_qf)
Bases:
  sage.modules.fg_pid.fgp_module.FGP_Module_class
  sage.structure.unique_representation.CachedRepresentation

Finite quotients with a bilinear and a quadratic form.

Let \(V\) be a symmetric FreeQuadraticModule and \(W \subseteq V\) a submodule of the same rank as \(V\). The quotient \(V/W\) is a torsion quadratic module. It inherits a bilinear form \(b\) and a quadratic form \(q\).

\(b: V \times V \to \mathbb{Q}/m\mathbb{Z}\), where \(m\mathbb{Z} = (V, W)\) and \(b(x, y) = (x, y) + m\mathbb{Z}\)

\(q: V \to \mathbb{Q}/n\mathbb{Z}\), where \(n\mathbb{Z} = 2(V, W) + \mathbb{Z}\{w, w\mid w \in W\}\)
INPUT:

- \( V \) – a `FreeModule` with a symmetric inner product matrix
- \( W \) – a submodule of \( V \) of the same rank as \( V \)
- `check` – bool (default: True)
- `modulus` – a rational number dividing \( m \) (default: \( m \)); the inner product \( b \) is defined in \( \mathbb{Q}/\text{modulus} \mathbb{Z} \)
- `modulus_qf` – a rational number dividing \( n \) (default: \( n \)); the quadratic form \( q \) is defined in \( \mathbb{Q}/\text{modulus}_q \mathbb{Z} \)

EXAMPLES:

```python
sage: from sage.modules.torsion_quadratic_module import TorsionQuadraticModule
g sage: V = FreeModule(ZZ, 3)
g sage: T = TorsionQuadraticModule(V, 5*V)
g sage: T
Finite quadratic module over Integer Ring with invariants (5, 5, 5)
Gram matrix of the quadratic form with values in \( \mathbb{Q}/5\mathbb{Z} \):
[1 0 0]
[0 1 0]
[0 0 1]

Element

alias of `TorsionQuadraticModuleElement`

all_submodules()

Return a list of all submodules of `self`.

Warning: This method creates all submodules in memory. The number of submodules grows rapidly with the number of generators. For example consider a vector space of dimension \( n \) over a finite field of prime order \( p \). The number of subspaces is (very) roughly \( p^{(n^2-n)/2} \).

EXAMPLES:

```python
sage: D = IntegralLattice("D4").discriminant_group()
sage: D.all_submodules()
[Finite quadratic module over Integer Ring with invariants (),
 Gram matrix of the quadratic form with values in \( \mathbb{Q}/2\mathbb{Z} \):
 [],
 Finite quadratic module over Integer Ring with invariants (2,),
 Gram matrix of the quadratic form with values in \( \mathbb{Q}/2\mathbb{Z} \):
 [1],
 Finite quadratic module over Integer Ring with invariants (2,),
 Gram matrix of the quadratic form with values in \( \mathbb{Q}/2\mathbb{Z} \):
 [1],
 Finite quadratic module over Integer Ring with invariants (2,),
 Gram matrix of the quadratic form with values in \( \mathbb{Q}/2\mathbb{Z} \):
 [1],
 Finite quadratic module over Integer Ring with invariants (2, 2)
 Gram matrix of the quadratic form with values in \( \mathbb{Q}/2\mathbb{Z} \):
 [ 1 1/2]
 [1/2 1]]
```
brown_invariant()
Return the Brown invariant of this torsion quadratic form.

Let \((D, q)\) be a torsion quadratic module with values in \(\mathbb{Q}/2\mathbb{Z}\). The Brown invariant \(Br(D, q) \in \mathbb{Z}/8\mathbb{Z}\) is defined by the equation

\[
\exp\left(\frac{2\pi i}{8} Br(q)\right) = \frac{1}{\sqrt{D}} \sum_{x \in D} \exp(i\pi q(x)).
\]

The Brown invariant is additive with respect to direct sums of torsion quadratic modules.

OUTPUT:
* an element of \(\mathbb{Z}/8\mathbb{Z}\)

EXAMPLES:
```
sage: L = IntegralLattice("D4")
sage: D = L.discriminant_group()
sage: D.brown_invariant()
4
```

We require the quadratic form to be defined modulo \(2\mathbb{Z}\):
```
sage: from sage.modules.torsion_quadratic_module import TorsionQuadraticModule
sage: V = FreeQuadraticModule(ZZ, 3, matrix.identity(3))
sage: T = TorsionQuadraticModule((1/10)*V, V)
sage: T.brown_invariant()
Traceback (most recent call last):
...
ValueError: the torsion quadratic form must have values in \(\mathbb{Q}/2\mathbb{Z}\)
```

gens()
Return generators of self.

There is no assumption on the generators except that they generate the module.

EXAMPLES:
```
sage: from sage.modules.torsion_quadratic_module import TorsionQuadraticModule
sage: V = FreeModule(ZZ, 3)
sage: T = TorsionQuadraticModule(V, 5*V)
sage: T.gens()
((1, 0, 0), (0, 1, 0), (0, 0, 1))
```

genus(signature_pair)
Return the genus defined by self and the signature_pair.

If no such genus exists, raise a ValueError.

REFERENCES:
[Nik1977] Corollary 1.9.4 and 1.16.3.

EXAMPLES:
```
sage: L = IntegralLattice("D4").direct_sum(IntegralLattice("A2"))
sage: D = L.discriminant_group()
sage: genus = D.genus(L.signature_pair())
```

(continues on next page)
Let $H$ be an even unimodular lattice of signature $(9,1)$. Then $L = D_4 + A_2$ is primitively embedded in $H$. We compute the discriminant form of the orthogonal complement of $L$ in $H$:

```
sage: DK = D.twist(-1)
sage: DK
Finite quadratic module over Integer Ring with invariants (2, 6)
Gram matrix of the quadratic form with values in Q/2Z:

[ 1 1/2]
[1/2 1/3]
```

We know that $K$ has signature $(5,1)$ and thus we can compute the genus of $K$ as:

```
sage: DK.genus((3,1))
Genus of None
Signature: (3, 1)
Genus symbol at 2: 1^2:2^-2
Genus symbol at 3: 1^-3 3^1
```

We can also compute the genus of an odd lattice from its discriminant form:

```
sage: L = IntegralLattice(matrix.diagonal(range(1,5)))
sage: D = L.discriminant_group()
sage: D.genus((4,0))
Genus of None
Signature: (4, 0)
Genus symbol at 2: [1^-2 2^1 4^1]_6
Genus symbol at 3: 1^-3 3^1
```

\textbf{gram_matrix_bilinear()}

Return the Gram matrix with respect to the generators.

\textbf{OUTPUT:}

A rational matrix $G$ with $G[i,j]$ given by the inner product of the $i$-th and $j$-th generator. Its entries are only well defined $\mod (V, W)$.

\textbf{EXAMPLES:}

```
sage: from sage.modules.torsion_quadratic_module import TorsionQuadraticModule
sage: V = FreeQuadraticModule(ZZ, 3, matrix.identity(3)*5)
sage: T = TorsionQuadraticModule((1/5)*V, V)
sage: T.gram_matrix_bilinear()
[[1/5 0 0]
[0 0 0]
[0 0 0]]
```
gram_matrix_quadratic()
   The Gram matrix of the quadratic form with respect to the generators.

OUTPUT:
   * a rational matrix $G_q$ with $G_q[i,j] = \text{gens}[i] \ast \text{gens}[j]$ and $G[i,i] = \text{gens}[i].q()$

EXAMPLES:
>>> from sage.modules.torsion_quadratic_module import TorsionQuadraticModule
>>> D4_gram = Matrix(ZZ, [[2,0,0,-1],[0,2,0,-1],[0,0,2,-1],[-1,-1,-1,2]])
>>> D4 = FreeQuadraticModule(ZZ, 4, D4_gram)
>>> D4dual = D4.span(D4_gram.inverse())
>>> discrForm = TorsionQuadraticModule(D4dual, D4)
>>> discrForm.gram_matrix_quadratic()
[ 1 1/2]
[1/2 1]
>>> discrForm.gram_matrix_bilinear()
[ 0 1/2]
[1/2 0]

is_genus(signature_pair, even=True)
   Return True if there is a lattice with this signature and discriminant form.

Todo: implement the same for odd lattices

INPUT:
   * signature_pair – a tuple of non negative integers (s_plus, s_minus)
   * even – bool (default: True)

EXAMPLES:
>>> L = IntegralLattice("D4").direct_sum(IntegralLattice(3 * Matrix(ZZ,2,[2,1,-1,2])))
>>> D = L.discriminant_group()
>>> D.is_genus((6,0))
True

Let us see if there is a lattice in the genus defined by the same discriminant form but with a different signature:

>>> D.is_genus((4,2))
False
>>> D.is_genus((16,2))
True

normal_form(partial=False)
   Return the normal form of this torsion quadratic module.
Two torsion quadratic modules are isomorphic if and only if they have the same value modules and the same normal form.

A torsion quadratic module \((T, q)\) with values in \(\mathbb{Q}/n\mathbb{Z}\) is in normal form if the rescaled quadratic module \((T, q/n)\) with values in \(\mathbb{Q}/\mathbb{Z}\) is in normal form.

For the definition of normal form see [MirMor2009] IV Definition 4.6. Below are some of its properties. Let \(p\) be odd and \(u\) be the smallest non-square modulo \(p\). The normal form is a diagonal matrix with diagonal entries either \(p^n\) or \(up^n\).

If \(p = 2\) is even, then the normal form consists of 1 x 1 blocks of the form

\[
(0), \quad 2^n(1), \quad 2^n(3), \quad 2^n(5), \quad 2^n(7)
\]

or of 2 x 2 blocks of the form

\[
\begin{pmatrix}
2 & 1 \\
1 & 2
\end{pmatrix}, \quad \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

The blocks are ordered by their valuation.

**INPUT:**

- partial - bool (default: False) return only a partial normal form it is not unique but still useful to extract invariants

**OUTPUT:**

- a torsion quadratic module

**EXAMPLES:**

```
sage: L1=IntegerLattice(matrix([[-2,0,0],[0,1,0],[0,0,4]]))
sage: L1.discriminant_group().normal_form()
Finite quadratic module over Integer Ring with invariants (2, 4)
Gram matrix of the quadratic form with values in Q/Z:
[1/2  0]
[ 0  1/4]
sage: L2=IntegerLattice(matrix([[-2,0,0],[0,1,0],[0,0,-4]]))
sage: L2.discriminant_group().normal_form()
Finite quadratic module over Integer Ring with invariants (2, 4)
Gram matrix of the quadratic form with values in Q/Z:
[1/2  0]
[ 0  1/4]
```

We check that trac ticket \#24864 is fixed:

```
sage: L1=IntegerLattice(matrix([[-4,0,0],[0,4,0],[0,0,-2]]))
sage: AL1=L1.discriminant_group()
sage: AL1.normal_form()
Finite quadratic module over Integer Ring with invariants (2, 4, 4)
Gram matrix of the quadratic form with values in Q/Z:
[1/2  0  0]
[ 0  1/4  0]
[ 0   0 5/4]
sage: AL2.normal_form()
```

(continues on next page)
Finite quadratic module over Integer Ring with invariants $(2, 4, 4)$
Gram matrix of the quadratic form with values in $\mathbb{Q}/2\mathbb{Z}$:
\[
\begin{bmatrix}
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{4} & 0 \\
0 & 0 & \frac{5}{4}
\end{bmatrix}
\]

Some exotic cases:

```python
sage: from sage.modules.torsion_quadratic_module import TorsionQuadraticModule
sage: D4_gram = Matrix(ZZ,4,4,[2,0,0,-1,0,2,0,-1,0,0,2,-1,-1,-1,-1,2])
sage: D4 = FreeQuadraticModule(ZZ,4,D4_gram)
sage: D4dual = D4.span(D4_gram.inverse())
sage: T = TorsionQuadraticModule((1/6)*D4dual,D4)
sage: T
Finite quadratic module over Integer Ring with invariants $(6, 6, 12, 12)$
Gram matrix of the quadratic form with values in $\mathbb{Q}/(1/3)\mathbb{Z}$:
\[
\begin{bmatrix}
\frac{1}{18} & \frac{1}{12} & \frac{5}{36} & \frac{1}{36} \\
\frac{1}{12} & \frac{1}{6} & \frac{1}{36} & \frac{1}{9} \\
\frac{5}{36} & \frac{1}{36} & \frac{1}{36} & \frac{11}{72} \\
\frac{1}{36} & \frac{1}{9} & \frac{11}{72} & \frac{1}{36}
\end{bmatrix}
\]
sage: T.normal_form()
Finite quadratic module over Integer Ring with invariants $(6, 6, 12, 12)$
Gram matrix of the quadratic form with values in $\mathbb{Q}/(1/3)\mathbb{Z}$:
\[
\begin{bmatrix}
\frac{1}{6} & 1/12 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/12 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/12 & 1/24 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/24 & 1/12 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/9 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/9 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/9 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/9
\end{bmatrix}
\]

`orthogonal_group(gens=None, check=False)`
Orthogonal group of the associated torsion quadratic form.

**Warning:** This is can be smaller than the orthogonal group of the bilinear form.

**INPUT:**
- `gens` – a list of generators, for instance square matrices, something that acts on `self`, or an automorphism of the underlying abelian group
- `check` – perform additional checks on the generators

**EXAMPLES:**
You can provide generators to obtain a subgroup of the full orthogonal group:

```python
sage: D = TorsionQuadraticForm(matrix.identity(2)/2)
sage: f = matrix(2,[0,1,1,0])
sage: D.orthogonal_group(gens=[f]).order()
sage: 2
```
If no generators are given a slow brute force approach is used to calculate the full orthogonal group:

```
sage: D = TorsionQuadraticForm(matrix.identity(3)/2)
sage: OD = D.orthogonal_group()
sage: OD.order()
6
sage: fd = D.hom([D.1,D.0,D.2])
sage: OD(fd)
[[0 1 0]
 [1 0 0]
 [0 0 1]]
```

We compute the kernel of the action of the orthogonal group of \( L \) on the discriminant group.

```
sage: L=IntegralLattice('A4')
sage: O=L.orthogonal_group()
sage: D=L.discriminant_group()
sage: Obar = D.orthogonal_group(O.gens())
sage: O.order() 240
sage: Obar.order() 2
sage: phi = O.hom(Obar.gens())
sage: phi.kernel().order() 120
```

**orthogonal submodule_to**(*S*)

Return the submodule orthogonal to \( S \).

**INPUT:**

- \( S \) – a submodule, list, or tuple of generators

**EXAMPLES:**

```
sage: from sage.modules.torsion_quadratic_module import TorsionQuadraticModule
sage: V = FreeModule(ZZ, 10)
sage: T = TorsionQuadraticModule(V, 3*V)
sage: S = T.submodule(T.gens()[[:5]])
sage: O = T.orthogonal_submodule_to(S)
sage: O
Finite quadratic module over Integer Ring with invariants (3, 3, 3, 3, 3)
Gram matrix of the quadratic form with values in Q/3Z:
[1 0 0 0 0]
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 0 1]
sage: O.V() + S.V() == T.V()
True
```

**primary_part**(*m*)

Return the \( m \)-primary part of this torsion quadratic module as a submodule.

**INPUT:**

- \( m \) – an integer

**OUTPUT:**

- a submodule

**EXAMPLES:**

```
sage: from sage.modules.torsion_quadratic_module import TorsionQuadraticModule
sage: T = TorsionQuadraticModule((1/6)*ZZ^3,ZZ^3)
sage: T
```

(continues on next page)
Finite quadratic module over Integer Ring with invariants (6, 6, 6)
Gram matrix of the quadratic form with values in $\mathbb{Q}/(1/3)\mathbb{Z}$:
\[
\begin{bmatrix}
 1/36 & 0 & 0 \\
 0 & 1/36 & 0 \\
 0 & 0 & 1/36 \\
\end{bmatrix}
\]
sage: T.primary_part(2)
Finite quadratic module over Integer Ring with invariants (2, 2, 2)
Gram matrix of the quadratic form with values in $\mathbb{Q}/(1/3)\mathbb{Z}$:
\[
\begin{bmatrix}
 1/4 & 0 & 0 \\
 0 & 1/4 & 0 \\
 0 & 0 & 1/4 \\
\end{bmatrix}
\]

**submodule_with_gens(gens)**

Return a submodule with generators given by gens.

**INPUT:**

- gens – a list of generators that convert into self

**OUTPUT:**

- a submodule with the specified generators

**EXAMPLES:**

```python
sage: from sage.modules.torsion_quadratic_module import TorsionQuadraticModule
sage: V = FreeQuadraticModule(ZZ,3,matrix.identity(3)*10)
sage: T = TorsionQuadraticModule((1/10)*V, V)
sage: g = T.gens()
sage: new_gens = [2*g[0], 5*g[0]]

```

Finite quadratic module over Integer Ring with invariants (10,)
Gram matrix of the quadratic form with values in $\mathbb{Q}/2\mathbb{Z}$:
\[
\begin{bmatrix}
 1/2 & 0 \\
 0 & 1/2 \\
\end{bmatrix}
\]
The generators do not need to be independent:

```python
sage: new_gens = [g[0], 2*g[1], g[0], g[1]]
```

Finite quadratic module over Integer Ring with invariants (10, 10)
Gram matrix of the quadratic form with values in $\mathbb{Q}/2\mathbb{Z}$:
\[
\begin{bmatrix}
 1/10 & 0 & 1/10 & 0 \\
 0 & 1/5 & 0 & 1/5 \\
 1/10 & 0 & 1/10 & 0 \\
 0 & 1/5 & 0 & 1/10 \\
\end{bmatrix}
\]

**twist(s)**

Return the torsion quadratic module with quadratic form scaled by $s$.

If the old form was defined modulo $n$, then the new form is defined modulo $ns$.

**INPUT:**

- $s$ - a rational number

**EXAMPLES:**

```python
```
```python
sage: q = TorsionQuadraticForm(matrix.diagonal([3/9, 1/9]))
sage: q.twist(-1)
Finite quadratic module over Integer Ring with invariants (3, 9)
Gram matrix of the quadratic form with values in Q/Z:
[2/3 0]
[0 8/9]
```

This form is defined modulo 3:

```python
sage: q.twist(3)
Finite quadratic module over Integer Ring with invariants (3, 9)
Gram matrix of the quadratic form with values in Q/3Z:
[1 0]
[0 1/3]
```

The next form is defined modulo 4:

```python
sage: q.twist(4)
Finite quadratic module over Integer Ring with invariants (3, 9)
Gram matrix of the quadratic form with values in Q/4Z:
[4/3 0]
[0 4/9]
```

`value_module()`

Return \(\mathbb{Q}/m\mathbb{Z}\) with \(m = (V, W)\).

This is where the inner product takes values.

**EXAMPLES:**

```python
sage: A2 = Matrix(ZZ, 2, 2, [2,-1,-1,2])
sage: L = IntegralLattice(2*A2)
sage: D = L.discriminant_group()
sage: D
Finite quadratic module over Integer Ring with invariants (2, 6)
Gram matrix of the quadratic form with values in Q/2Z:
[1 1/2]
[1/2 1/3]
sage: D.value_module()
\mathbb{Q}/\mathbb{Z}
```

`value_module_qf()`

Return \(\mathbb{Q}/n\mathbb{Z}\) with \(n\mathbb{Z} = (V, W) + \mathbb{Z}\{(w, w) \mid w \in W\}\).

This is where the torsion quadratic form takes values.

**EXAMPLES:**

```python
sage: A2 = Matrix(ZZ, 2, 2, [2,-1,-1,2])
sage: L = IntegralLattice(2*A2)
sage: D = L.discriminant_group()
sage: D
Finite quadratic module over Integer Ring with invariants (2, 6)
Gram matrix of the quadratic form with values in Q/2Z:
[1 1/2]
```

(continues on next page)
class sage.modules.torsion_quadratic_module.TorsionQuadraticModuleElement(parent, x, check=True)

Bases: sage.modules.fg_pid.fgp_element.FGP_Element

An element of a torsion quadratic module.

INPUT:

• parent -- parent
• x -- element of parent.V()
• check -- bool (default: True)

b(other)

Compute the inner product of two elements.

OUTPUT:

• an element of \( \mathbb{Q}/m\mathbb{Z} \) with \( m\mathbb{Z} = (V, W) \)

EXAMPLES:

```
sage: from sage.modules.torsion_quadratic_module import TorsionQuadraticModule
sage: V = (1/2)*ZZ^2; W = ZZ^2
sage: T = TorsionQuadraticModule(V, W)
sage: g = T.gens()
sage: x = g[0]
sage: y = g[0] + g[1]
sage: x
(1, 0)
sage: x*y
1/4
```

The inner product has further aliases:

```
sage: x.inner_product(y)
1/4
sage: x.b(y)
1/4
```

inner_product(other)

Compute the inner product of two elements.

OUTPUT:

• an element of \( \mathbb{Q}/m\mathbb{Z} \) with \( m\mathbb{Z} = (V, W) \)

EXAMPLES:

```
sage: from sage.modules.torsion_quadratic_module import TorsionQuadraticModule
sage: V = (1/2)*ZZ^2; W = ZZ^2
sage: T = TorsionQuadraticModule(V, W)
sage: g = T.gens()
```
sage: x = g[0]
sage: y = g[0] + g[1]
sage: x
(1, 0)
sage: x*y
1/4

The inner product has further aliases:

sage: x.inner_product(y)
1/4
sage: x.b(y)
1/4

q()
Compute the quadratic_product of self.

OUTPUT:
• an element of $\mathbb{Q}/n\mathbb{Z}$ where $n\mathbb{Z} = 2(V,W) + \mathbb{Z}\{(w,w) | w \in W\}$

EXAMPLES:

sage: from sage.modules.torsion_quadratic_module import TorsionQuadraticModule
sage: W = FreeQuadraticModule(ZZ, 2, 2*matrix.identity(2))
sage: V = (1/2) * W
sage: T = TorsionQuadraticModule(V,W)
sage: x = T.gen(0)
sage: x
(1, 0)
sage: x.quadratic_product()
1/2
sage: x.quadratic_product().parent()
$\mathbb{Q}/\mathbb{Z}$

quadratic_product()
Compute the quadratic_product of self.

OUTPUT:
• an element of $\mathbb{Q}/n\mathbb{Z}$ where $n\mathbb{Z} = 2(V,W) + \mathbb{Z}\{(w,w) | w \in W\}$

EXAMPLES:

sage: from sage.modules.torsion_quadratic_module import TorsionQuadraticModule
sage: W = FreeQuadraticModule(ZZ, 2, 2*matrix.identity(2))
sage: V = (1/2) * W
sage: T = TorsionQuadraticModule(V,W)
sage: x = T.gen(0)
sage: x
(1, 0)
\begin{verbatim}
sage: x.quadratic_product()
1/2
sage: x.quadratic_product().parent()
Q/2Z
sage: x*x
1/2
sage: (x*x).parent()
Q/Z
\end{verbatim}
DIAMOND CUTTING IMPLEMENTATION

AUTHORS:

- Jan Poeschko (2012-07-02): initial version

`sage.modules.diamond_cutting.calculate_voronoi_cell(basis, radius=None, verbose=False)`
Calculate the Voronoi cell of the lattice defined by basis

**INPUT:**

- `basis` – embedded basis matrix of the lattice
- `radius` – radius of basis vectors to consider
- `verbose` – whether to print debug information

**OUTPUT:**

A Polyhedron instance.

**EXAMPLES:**

```python
sage: from sage.modules.diamond_cutting import calculate_voronoi_cell
sage: V = calculate_voronoi_cell(matrix([[1, 0], [0, 1]]))
```

`sage.modules.diamond_cutting.diamond_cut(V, GM, C, verbose=False)`
Perform diamond cutting on polyhedron V with basis matrix GM and radius C.

**INPUT:**

- `V` – polyhedron to cut from
- `GM` – half of the basis matrix of the lattice
- `C` – radius to use in cutting algorithm
- `verbose` – (default: False) whether to print debug information

**OUTPUT:**

A Polyhedron instance.

**EXAMPLES:**

```python
sage: from sage.modules.diamond_cutting import diamond_cut
sage: V = Polyhedron([[0], [2]])
sage: GM = matrix([2])
sage: V = diamond_cut(V, GM, 4)
```
sage: V.vertices()
(A vertex at (2), A vertex at (0))

```
sage.modules.diamond_cutting.jacobi(M)

Compute the upper-triangular part of the Cholesky/Jacobi decomposition of the symmetric matrix M.

Let $M$ be a symmetric $n \times n$-matrix over a field $F$. Let $m_{i,j}$ denote the $(i,j)$-th entry of $M$ for any $1 \leq i \leq n$ and $1 \leq j \leq n$. Then, the upper-triangular part computed by this method is the upper-triangular $n \times n$-matrix $Q$ whose $(i,j)$-th entry $q_{i,j}$ satisfies

$$q_{i,j} = \begin{cases} 
\frac{1}{q_{i,i}} \left( m_{i,j} - \sum_{r < i} q_{r,r} q_{r,i} q_{r,j} \right) & i < j, \\
q_{i,i} - \sum_{r < i} q_{r,r} q_{r,i}^2 & i = j, \\
0 & i > j,
\end{cases}$$

for all $1 \leq i \leq n$ and $1 \leq j \leq n$. (These equalities determine the entries of $Q$ uniquely by recursion.) This matrix $Q$ is defined for all $M$ in a certain Zariski-dense open subset of the set of all $n \times n$-matrices.

**Note:** This should be a method of matrices.

**EXAMPLES:**

```
sage: from sage.modules.diamond_cutting import jacobi
sage: jacobi(identity_matrix(3) * 4)
[4 0 0]
[0 4 0]
[0 0 4]
sage: def testall(M):
....:     Q = jacobi(M)
....:     for j in range(3):
....:         for i in range(j):
....:             if Q[i,j] * Q[i,i] != M[i,j] - sum(Q[r,i] * Q[r,j] * Q[r,r] for r in range(i)):
....:                 return False
....:     for i in range(3):
....:         if M[i,i] != Q[i,i] - sum(Q[r,i] ** 2 * Q[r,r] for r in range(i)):
....:             return False
....:     return True
sage: M = Matrix(QQ, [[8,1,5], [1,6,0], [5,0,3]])
sage: Q = jacobi(M); Q
[ 8 1/8 5/8]
[ 0 47/8 -5/47]
[ 0 0 -9/47]
sage: testall(M)
True
```

(continues on next page)
sage: testall(M)
True

sage.modules.diamond_cutting.plane_inequality(v)
Return the inequality for points on the same side as the origin with respect to the plane through v normal to v.

EXAMPLES:

sage: from sage.modules.diamond_cutting import plane_inequality
sage: ieq = plane_inequality([1, -1]); ieq
[2, -1, 1]
sage: ieq[0] + vector(ieq[1:]) * vector([1, -1])
0
CONCRETE CLASSES RELATED TO MODULES WITH A DISTINGUISHED BASIS.

This module provides concrete classes for various constructions related to modules with a distinguished basis:

- **morphism** – Concrete classes for morphisms of modules with basis

**See also:**
The category `ModulesWithBasis`
class sage.modules.with_basis.cell_module.CellModule(A, mu, **kwds)
Bases: sage.combinat.free_module.CombinatorialFreeModule

A cell module.

Let \( R \) be a commutative ring. Let \( A \) be a cellular \( R \)-algebra with cell datum \((\Lambda, i, M, C)\). A cell module \( W(\lambda) \) is the \( R \)-module given by \( R\{C_s \mid s \in M(\lambda)\} \) with an action of \( a \in A \) given by
\[
aC_s^\lambda = \sum_{u \in M(\lambda)} r_a(u, s) C_u + \sum_{\mu < \lambda, x, y \in M(\mu)} R C_x^\mu C_y^\mu.
\]

INPUT:
- \( A \) – a cellular algebra
- \( mu \) – an element of the cellular poset of \( A \)

See also:
CellularBasis

AUTHORS:
- Travis Scrimshaw (2015-11-5): Initial version

REFERENCES:
- [GrLe1996]
- [KX1998]
- [Mat1999]
- Wikipedia article Cellular_algebra

class Element
Bases: sage.modules.with_basis.indexed_element.IndexedFreeModuleElement

bilinear_form(x, y)

Return the bilinear form on \( x \) and \( y \).

The cell module \( W(\lambda) \) has a canonical bilinear form \( \Phi_\lambda : W(\lambda) \times W(\lambda) \to W(\lambda) \) given by
\[
C_s^\lambda C_t^\lambda = \Phi_\lambda(C_s, C_t) + \sum_{\mu < \lambda, x, y \in M(\mu)} R C_x^\mu C_y^\mu.
\]

EXAMPLES:
sage: S = SymmetricGroupAlgebra(QQ, 3)
sage: W = S.cell_module([2,1])
sage: elt = W.an_element(); elt
2*W[[1, 2], [3]] + 2*W[[1, 3], [2]]
sage: W.bilinear_form(elt, elt)
8

bilinear_form_matrix(ordering=None)

Return the matrix corresponding to the bilinear form of self.

INPUT:

• ordering – (optional) an ordering of the indices

EXAMPLES:

sage: S = SymmetricGroupAlgebra(QQ, 3)
sage: W = S.cell_module([2,1])
sage: W.bilinear_form_matrix()
[1 0]
[0 1]

cellular_algebra()

Return the cellular algebra of self.

EXAMPLES:

sage: S = SymmetricGroupAlgebra(QQ, 3)
sage: W = S.cell_module([2,1])
sage: W.cellular_algebra() == S.cellular_basis()
True
sage: S.has_coerce_map_from(W.cellular_algebra())
True

nonzero_bilinear_form()

Return True if the bilinear form of self is non-zero.

EXAMPLES:

sage: S = SymmetricGroupAlgebra(QQ, 3)
sage: W = S.cell_module([2,1])
sage: W.nonzero_bilinear_form()
True

radical()

Return the radical of self.

Let \( W(\lambda) \) denote a cell module. The radical of \( W(\lambda) \) is defined as

\[
\text{rad}(\lambda) := \{ x \in W(\lambda) \mid \Phi(\lambda)(x, y) \},
\]

and note that it is a submodule of \( W(\lambda) \).

EXAMPLES:

sage: S = SymmetricGroupAlgebra(QQ, 3)
sage: W = S.cell_module([2,1])
sage: R = W.radical(); R
Radical of Cell module indexed by [2, 1] of Cellular basis of
  Symmetric group algebra of order 3 over Rational Field
sage: R.basis()
Finite family {}

radical_basis()
Return a basis of the radical of self.

EXAMPLES:

sage: S = SymmetricGroupAlgebra(QQ, 3)
sage: W = S.cell_module([2,1])
sage: W.radical_basis()

simple_module()
Return the corresponding simple module of self.

Let $W(\lambda)$ denote a cell module. The simple module $L(\lambda)$ is defined as $W(\lambda)/\text{rad}(\lambda)$, where $\text{rad}(\lambda)$ is
the radical of the bilinear form $\Phi_\lambda$.

See also:
  radical()

EXAMPLES:

class sage.modules.with_basis.cell_module.SimpleModule(submodule)
Bases: sage.modules.with_basis.subquotient.QuotientModuleWithBasis

A simple module of a cellular algebra.

Let $W(\lambda)$ denote a cell module. The simple module $L(\lambda)$ is defined as $W(\lambda)/\text{rad}(\lambda)$, where $\text{rad}(\lambda)$ is the
radical of the bilinear form $\Phi_\lambda$.

class Element
    Bases: sage.modules.with_basis.indexed_element.IndexedFreeModuleElement
AN ELEMENT IN AN INDEXED FREE MODULE.

AUTHORS:

- Travis Scrimshaw (03-2017): Moved code from \texttt{sage.combinat.free_module}.

**class** \texttt{sage.modules.with\_basis.indexed\_element.IndexedFreeModuleElement}

Bases: \texttt{sage.structure.element.ModuleElement}

Create a combinatorial module element. This should never be called directly, but only through the parent combinatorial free module's \texttt{\_call\_}() method.

**\texttt{monomial\_coefficients}(\texttt{copy=\texttt{True}})**

Return the internal dictionary which has the combinatorial objects indexing the basis as keys and their corresponding coefficients as values.

INPUT:

- \texttt{copy} – (default: \texttt{True}) if \texttt{self} is internally represented by a dictionary \(d\), then make a copy of \(d\); if \texttt{False}, then this can cause undesired behavior by mutating \(d\)

**EXAMPLES:**

\begin{verbatim}
sage: F = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: B = F.basis()
sage: f = B['a'] + 3*B['c']
sage: d = f.monomial_coefficients()
sage: d['a']
1
sage: d['c']
3
\end{verbatim}

To run through the monomials of an element, it is better to use the idiom:

\begin{verbatim}
sage: for (t,c) in f:
....:     print("{} {}\".format(t,c))
a 1
c 3
\end{verbatim}

\begin{verbatim}
sage: s = SymmetricFunctions(QQ).schur()
sage: a = s([2,1])+2*s([3,2])
sage: d = a.monomial_coefficients()
sage: type(d)
<... 'dict'>
sage: d[ Partition([2,1]) ]
1
\end{verbatim}

(continues on next page)
\texttt{sage: \ d[ \ \text{Partition([3,2])} \ ]}
\texttt{2}

\texttt{to\_vector(new\_base\_ring=\text{None}, order=\text{None}, sparse=\text{False})}

Return \texttt{self} as a vector.

**INPUT:**

- \texttt{new\_base\_ring} – a ring (default: \text{None})
- \texttt{order} – (optional) an ordering of the support of \texttt{self}
- \texttt{sparse} – (default: \text{False}) whether to return a sparse vector or a dense vector

**OUTPUT:** a \texttt{FreeModule()} vector

**Warning:** This will crash/run forever if \texttt{self} is infinite dimensional!

**See also:**

- \texttt{vector()}
- \texttt{CombinatorialFreeModule.get\_order()}
- \texttt{CombinatorialFreeModule.from\_vector()}
- \texttt{CombinatorialFreeModule._dense\_free\_module()}

**EXAMPLES:**

\begin{verbatim}
sage: F = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: B = F.basis()
sage: f = B['a'] - 3*B['c']
sage: f._vector_()
(1, 0, -3)
\end{verbatim}

One can use equivalently:

\begin{verbatim}
sage: f.to\_vector()
(1, 0, -3)
sage: vector(f)
(1, 0, -3)
\end{verbatim}

More examples:

\begin{verbatim}
sage: QS3 = SymmetricGroupAlgebra(QQ, 3)
sage: a = 2*QS3([1,2,3]) + 4*QS3([3,2,1])
sage: a._vector_()
(2, 0, 0, 0, 0, 4)
sage: a.to\_vector()
(2, 0, 0, 0, 0, 4)
sage: vector(a)
(2, 0, 0, 0, 0, 4)
sage: a == QS3.from\_vector(a.to\_vector())
True
\end{verbatim}
If `new_base_ring` is specified, then a vector over `new_base_ring` is returned:

```
sage: a._vector_(RDF)
(2.0, 0.0, 0.0, 0.0, 0.0, 4.0)
```

Note: trac ticket #13406: the current implementation has been optimized, at the price of breaking the encapsulation for FreeModule elements creation, with the following use case as metric, on a 2008’ Macbook Pro:

```
sage: F = CombinatorialFreeModule(QQ, range(10))
sage: f = F.an_element()
sage: %timeit f._vector_()  # not tested
625 loops, best of 3: 17.5 micros per loop
```

Other use cases may call for different or further optimizations.
class sage.modules.with_basis.invariant.FiniteDimensionalInvariantModule(M, S,
    action=<built-in function mul>,
    side='left', *args, **kwargs)

Bases: sage.modules.with_basis.subquotient.SubmoduleWithBasis

The invariant submodule under a semigroup action.

When a semigroup $S$ acts on a module $M$, the invariant module is the set of elements $m \in M$ such that $s \cdot m = m$ for all $s \in S$:

$$M^S := \{m \in M : s \cdot m = m, \forall s \in S\}.$$

INPUT:

- $M$ – a module in the category of FiniteDimensionalModulesWithBasis
- $S$ – a semigroup in the category of FinitelyGeneratedSemigroups
- action – (default: operator.mul) the action of $S$ on $M$
- side – (default: 'left') the side on which $S$ acts

EXAMPLES:

First, we create the invariant defined by the cyclic group action on the free module with basis $\{1, 2, 3\}$:

```sage
sage: G = CyclicPermutationGroup(3)
sage: M = CombinatorialFreeModule(QQ, [1,2,3], prefix='M')
sage: action = lambda g, m: M.monomial(g(m))  # cyclically permute coordinates
sage: R = Representation(G, M, action)
sage: I = R.invariant_module()
sage: [I.lift(b) for b in I.basis()]
```

In order to give the module an action of $G$, we create a `Representation`:

```sage
sage: from sage.modules.with_basis.representation import Representation
sage: R = Representation(G, M, action)
```

Then we can lift the basis from the invariant to the original module:

```sage
sage: [I.lift(b) for b in I.basis()]
```

The we could also have the action be a right-action, instead of the default left-action:
sage: def rt_action(g, m):
    return M.monomial(g(m))  # cyclically permute coordinates
sage: R = Representation(G, M, rt_action, side='right')  # same as last but on right
sage: g = G.an_element(); g
(1,2,3)
sage: r = R.an_element(); r
sage: R.side()
'right'

So now we can see that multiplication with \(g\) on the right sends \(M[1]\) to \(M[2]\) and so on:

\[
\begin{align*}
    I &= R.invariant_module()
\end{align*}
\]

Now we will take the regular representation of the symmetric group on three elements to be the module, and compute its invariant submodule:

\[
\begin{align*}
    G &= SymmetricGroup(3) \\
    R &= G.regular_representation(QQ) \\
    I &= R.invariant_module()
\end{align*}
\]

We can also check the scalar multiplication by elements of the base ring (for this example, the rational field):

\[
\begin{align*}
    I &= R.invariant_module() \\
    \text{basis} &= [(1, 1, 1, 1, 1, 1)]
\end{align*}
\]

A more subtle example is the invariant submodule of a skew-commutative module, for example the exterior algebra \(\mathcal{E}[x_0, x_1, x_2]\) generated by three elements:

\[
\begin{align*}
    G &= CyclicPermutationGroup(3) \\
    M &= algebras.Exterior(QQ, 'x', 3) \\
    def cyclic_ext_action(g, m):
        # cyclically permute generators
        return M.prod([M.monomial((g(j+1)-1,)) for j in m])
\end{align*}
\]

If you care about being able to exploit the algebra structure of the exterior algebra (i.e. if you want to multiply elements together), you should make sure the representation knows it is also an algebra with the semigroup action being by algebra endomorphisms:

\[
\begin{align*}
    cat &= Algebras(QQ).WithBasis().FiniteDimensional() \\
    R &= Representation(G, M, cyclic_ext_action, category=cat) \\
    I &= R.invariant_module()
\end{align*}
\]

We can express the basis in the ambient algebra \(\mathcal{E}[x_0, x_1, x_2]\):

\[
\begin{align*}
    I &= R.invariant_module() \\
    \text{basis} &= [1, x0 + x1 + x2, x0^2 - x0*x2 + x1*x2, x0*x1*x2]
\end{align*}
\]

or we can express the basis intrinsically to the invariant \(\mathcal{I}\):

\[
\begin{align*}
    \text{basis} &= [1, x0 + x1 + x2, x0^2 - x0*x2 + x1*x2, x0*x1*x2]
\end{align*}
\]
sage: B = I.basis()

This lifts to the exterior algebra:

sage: I.lift(m)
3 + 2*x0 + 7*x0*x1*x2 + 2*x1 + 2*x2

We can also check using the invariant element m that arithmetic works:

sage: m^2
sage: m+m

To see the actual elements expressed in the exterior algebra, we lift them again:

sage: I.lift(m+m)
6 + 4*x0 + 14*x0*x1*x2 + 4*x1 + 4*x2
sage: 7*m
sage: I.lift(7*m)
21 + 14*x0 + 49*x0*x1*x2 + 14*x1 + 14*x2

The classic example of an invariant module is the module of symmetric functions, which is the invariant module of polynomials whose variables are acted upon by permutation. We can create a module isomorphic to the homogeneous component of a polynomial ring in n variable of a fixed degree d by looking at weak compositions of d of length n, which we consider as the exponent vector. For example, \( x^2yz \in \mathbb{Q}[x,y,z] \) would have the exponent vector (2,1,1). The vector (2,1,1) is a weak composition of 4, with length 3, and so we can think of it as being in the degree-4 homogeneous component of a polynomial ring in three variables:

sage: C = IntegerVectors(4, length=3, min_part=0)  # representing degree-4 monomials
sage: M = CombinatorialFreeModule(QQ, C)  # isomorphic to deg-4 homog. polynomials
sage: G = SymmetricGroup(3)
sage: def perm_action(g,x):
    return M.monomial(C(g(list(x))))
sage: perm_action(G((1,2,3)), C([4,3,2]))
B[[3, 2, 4]]
sage: R = Representation(G, M, perm_action)
sage: I = R.invariant_module()

These are the monomial symmetric functions, which are a well-known basis for the symmetric functions. For comparison:

sage: Sym = SymmetricFunctions(QQ)
sage: m = Sym.monomial()
sage: [m(mu).expand(3) for mu in Partitions(4)]
[x0^4 + x1^4 + x2^4,
 x0^3*x1 + x0*x1^3 + x0^3*x2 + x1^3*x2 + x0*x2^3 + x1*x2^3,
(continues on next page)
x0^2*x1^2 + x0^2*x2^2 + x1^2*x2^2,
x0^2*x1*x2 + x0*x1^2*x2 + x0*x1*x2^2,
0]

Note: The current implementation works when $S$ is a finitely-generated semigroup, and when $M$ is a finite-dimensional free module with a distinguished basis.

Todo: Extend this to have multiple actions, including actions on both sides.

Todo: Extend when $M$ does not have a basis and $S$ is a permutation group using:

- arXiv 0812.3082

class Element

Bases: `sage.modules.with_basis.indexed_element.IndexedFreeModuleElement`

`semigroup()`

Return the semigroup $S$ whose action `self` is invariant under.

EXAMPLES:

```
sage: G = SymmetricGroup(3)
sage: M = CombinatorialFreeModule(QQ, [1,2,3], prefix='M')
sage: def action(g,x):
    return M.monomial(g(x))
sage: I = M.invariant_module(G, action_on_basis=action)
sage: I.semigroup()
Symmetric group of order 3! as a permutation group
```

`semigroup_representation()`

Return the ambient space of `self`.

EXAMPLES:

```
sage: X = CombinatorialFreeModule(QQ, range(3)); x = X.basis()
sage: Y = X.submodule((x[0]-x[1], x[1]-x[2]))
sage: Y.ambient() is X
True
```

class `sage.modules.with_basis.invariant.FiniteDimensionalTwistedInvariantModule`

`FiniteDimensionalTwistedInvariantModule(M, G, chi, action=<built-in function mul>, side='left', **kwargs)`

Bases: `sage.modules.with_basis.subquotient.SubmoduleWithBasis`

Construct the $\chi$-twisted invariant submodule of $M$.

When a group $G$ acts on a module $M$, the $\chi$-twisted invariant submodule of $M$ is the isotypic component of the representation $M$ corresponding to the irreducible character $\chi$.
For more information, see [Sta1979].

INPUT:

- \( M \) – a module in the category of \texttt{FiniteDimensionalModulesWithBasis} and whose base ring contains all the values passed to \( \text{chi} \) and \( 1/|G| \)
- \( G \) – a finitely generated group
- \( \text{chi} \) – list/tuple of the character values of the irreducible representation onto which you want to project.
  The order of values of \( \text{chi} \) must agree with the order of \( G.conjugacy\_classes() \)
- \( \text{action} \) – (default: \texttt{operator.mul}) the action of \( G \) on \( M \)
- \( \text{side} \) – (default: \'left\') the side on which \( G \) acts

**Warning:** The current implementation does not check if \( \text{chi} \) is irreducible. Passing character values of non-irreducible representations may lead to mathematically incorrect results.

EXAMPLES:

Suppose that the symmetric group \( S_3 \) acts on a four dimensional vector space by permuting the first three coordinates only:

```sage
M = CombinatorialFreeModule(QQ, [1,2,3,4], prefix='M')
G = SymmetricGroup(3)
sage: action = lambda g,x: M.term(g(x))
sage: chi = ClassFunction(G, (1,1,1))
sage: T = M.twisted_invariant_module(G, chi, action_on_basis=action)
sage: type(T)
<class 'sage.modules.with_basis.invariant.FiniteDimensionalInvariantModule_with_category'>
```

The trivial representation corresponds to the usual invariant module, so trying to create the twisted invariant module when there is no twist returns a \texttt{FiniteDimensionalInvariantModule}:

```sage
sage: T.basis()
Finite family {0: B[0], 1: B[1]}
sage: [T.lift(b) for b in T.basis()]
[M[1], M[2], M[3]]
```

In this case, there are two copies of the trivial representation, one coming from the first three coordinates and the other coming from the fact that \( S_3 \) does not touch the fourth coordinate:

```sage
sage: T.basis()
Finite family {0: B[0], 1: B[1]}
sage: [T.lift(b) for b in T.basis()]
```

The character values of the standard representation are 2, 0, −1:

```sage
sage: chi = ClassFunction(G, [2,0,-1])
sage: T = M.twisted_invariant_module(G, chi, action_on_basis=action)
sage: type(T)
<class 'sage.modules.with_basis.invariant.FiniteDimensionalTwistedInvariantModule_with_category'>
```

```sage
sage: T.basis()
Finite family {0: B[0], 1: B[1]}
sage: [T.lift(b) for b in T.basis()]
```
The permutation representation is the direct sum of the standard representation with the trivial representation, and the action on the basis element $B[4]$ is itself a copy of the trivial representation, so the sign representation does not appear in the decomposition:

```
sage: T = M.twisted_invariant_module(G, [1,-1,1], action_on_basis=action)
sage: T.basis()
Finite family {}
```

We can also get two copies of the standard representation by looking at two copies of the permutation representation, found by reduction modulo three on the indices of a six-dimensional module:

```
sage: M = CombinatorialFreeModule(QQ, [0,1,2,3,4,5], prefix='M')
sage: action = lambda g,x: M.term(g(x%3 + 1)-1 + (x>=3)*3)
sage: T = M.twisted_invariant_module(G, [2,0,-1], action_on_basis=action)
sage: T.basis()
Finite family {0: B[0], 1: B[1], 2: B[2], 3: B[3]}
sage: [T.lift(b) for b in T.basis()]
```

There are still no copies of the sign representation:

```
sage: T = M.twisted_invariant_module(G, [1,-1,1], action_on_basis=action)
sage: T.basis()
Finite family {}
```

The trivial representation also contains no copies of the sign representation:

```
sage: R = G.trivial_representation(QQ)
sage: T = R.twisted_invariant_module([1,-1,1])
sage: T.basis()
Finite family {}
```

The regular representation contains two copies of the standard representation and one copy each of the trivial and the sign:

```
sage: R = G.regular_representation(QQ)
sage: std = R.twisted_invariant_module([2,0,-1])
sage: std.basis()
Finite family {0: B[0], 1: B[1], 2: B[2], 3: B[3]}
sage: [std.lift(b) for b in std.basis()]
[() - (1,2,3), -(1,2,3) + (1,3,2), (2,3) - (1,2), -(1,2) + (1,3)]
```

```
sage: triv = R.twisted_invariant_module([1,1,1])
sage: triv.basis()
Finite family {0: B[0]}
sage: [triv.lift(b) for b in triv.basis()]
[() + (2,3) + (1,2) + (1,2,3) + (1,3,2) + (1,3)]
```
sage: sgn = R.twisted_invariant_module([1,-1,1])
sage: sgn.basis()
Finite family {0: B[0]}
sage: [sgn.lift(b) for b in sgn.basis()]
[(0) - (2,3) - (1,2) + (1,2,3) + (1,3,2) - (1,3)]

For the next example, we construct a twisted invariant by the character for the 2 dimensional representation of $S_3$ on the natural action on the exterior algebra. While $S_3$ acts by automorphisms, the twisted invariants do not form an algebra in this case:

sage: G = SymmetricGroup(3); G.rename('S3')
sage: E = algebras.Exterior(QQ, 'x', 3); E.rename('E')
sage: def action(g,m):
    return E.prod([E.monomial((g(j+1)-1,)) for j in m])
sage: from sage.modules.with_basis.representation import Representation
sage: EA = Representation(G, E, action, category=Algebras(QQ).WithBasis()).
    FiniteDimensional()
Sage: T = EA.twisted_invariant_module([2,0,-1])
sage: t = T.an_element(); t

We can still get meaningful information about the product by taking the product in the ambient space:

sage: T.lift(t) * T.lift(t)
-36*x0*x1*x2

We can see this does not lie in this twisted invariant algebra:

sage: T.retract(T.lift(t) * T.lift(t))
Traceback (most recent call last):
  ... ValueError: -36*x0*x1*x2 is not in the image

It happens to be in the trivial isotypic component (equivalently in the usual invariant algebra) but Sage does not know this.

sage: G.rename(); E.rename()  # reset the names

Todo:

- Replace G by S in FinitelyGeneratedSemigroups
- Allow for chi to be a Representation
- Add check for irreducibility of chi

class Element
    Bases: sage.modules.with_basis.indexed_element.IndexedFreeModuleElement

project(x)
    Project x in the ambient module onto self.

EXAMPLES:
The standard representation is the orthogonal complement of the trivial representation inside of the permutation representation, so the basis for the trivial representation projects to 0:

```python
sage: M = CombinatorialFreeModule(QQ, [1,2,3]); M.rename('M')
sage: B = M.basis()
sage: G = SymmetricGroup(3); G.rename('S3')
sage: def action(g,x): return M.term(g(x))
sage: T = M.twisted_invariant_module(G, [2,0,-1], action_on_basis=action)
sage: parent(m) M
sage: t = T.project(m); t
0
sage: parent(t)
Twist of (S3)-invariant submodule of M by character [2, 0, -1]
sage: G.rename(); M.rename() # reset names
```

**project_ambient(x)**

Project x in the ambient representation onto the submodule of the ambient representation to which self is isomorphic as a module.

**Note:** The image of self.project_ambient is not in self but rather is in self.ambient().

**EXAMPLES:**

```python
sage: M = CombinatorialFreeModule(QQ, [1,2,3]); M.rename('M')
sage: B = M.basis()
sage: G = SymmetricGroup(3); G.rename('S3')
sage: def action(g,x): return M.term(g(x))
sage: T = M.twisted_invariant_module(G, [2,0,-1], action_on_basis=action)
To compare with self.project, we can inspect the parents. The image of self.project is in self, while the image of self.project_ambient is in self._ambient:

```python
0
sage: parent(t)
Twist of (S3)-invariant submodule of M by character [2, 0, -1]
0
sage: parent(s)
Representation of S3 indexed by {1, 2, 3} over Rational Field
```

Note that because of the construction of T, self._ambient is an instance of Representation, but you still may pass elements of M, which is an instance of CombinatorialFreeModule, because the underlying Representation is built off of M and we can canonically construct elements of the Representation from elements of M.

```python
sage: G.rename(); M.rename() # reset names
```

**projection_matrix()**

Return the matrix defining the projection map from the ambient representation onto self.
EXAMPLES:

```python
sage: M = CombinatorialFreeModule(QQ, [1,2,3])
sage: def action(g,x): return M.term(g(x))
sage: G = SymmetricGroup(3)
```

If the matrix $A$ has columns form a basis for the subspace onto which we are trying to project, then we can find the projection matrix via the formula $P = A(A^T A)^{-1} A^T$. Recall that the standard representation twisted invariant has basis $(B[1] - B[3], B[2] - B[3])$, hence:

```python
sage: A = Matrix([[1,0],[0,1],[-1,-1]])
sage: P = A*(A.transpose()*A).inverse()*A.transpose()
sage: T = M.twisted_invariant_module(G, [2,0,-1], action_on_basis=action)
sage: P == T.projection_matrix()
True
```

Moreover, since there is no component of the sign representation in this representation, the projection matrix is just the zero matrix:

```python
sage: T = M.twisted_invariant_module(G, [1,-1,1], action_on_basis=action)
sage: T.projection_matrix()
[0 0 0]
[0 0 0]
[0 0 0]
```
This module contains a hierarchy of classes for morphisms of modules with a basis (category `Modules.WithBasis`):

- `ModuleMorphism`
- `ModuleMorphismByLinearity`
- `ModuleMorphismFromMatrix`
- `ModuleMorphismFromFunction`
- `TriangularModuleMorphism`
- `TriangularModuleMorphismByLinearity`
- `TriangularModuleMorphismFromFunction`

These are internal classes; it is recommended not to use them directly, and instead to construct morphisms through the `ModulesWithBasis.ParentMethods.module_morphism()` method of the domain, or through the homset. See the former for an overview of the possible arguments.

**EXAMPLES:**

We construct a morphism through the method `ModulesWithBasis.ParentMethods.module_morphism()`, by specifying the image of each element of the distinguished basis:

```python
sage: X = CombinatorialFreeModule(QQ, [1,2,3]); x = X.basis()
sage: Y = CombinatorialFreeModule(QQ, [1,2,3,4]); y = Y.basis()
sage: on_basis = lambda i: Y.monomial(i) + 2*Y.monomial(i+1)
sage: phi1 = X.module_morphism(on_basis, codomain=Y)
sage: phi1(x[1])
sage: phi1
Generic morphism:
  From: Free module generated by {1, 2, 3} over Rational Field
  To:   Free module generated by {1, 2, 3, 4} over Rational Field
sage: phi1.parent()
Set of Morphisms from Free module generated by {1, 2, 3} over Rational Field to Free...
  in Category of finite dimensional vector spaces with basis over Rational Field
sage: phi1.__class__
<class 'sage.modules.with_basis.morphism.ModuleMorphismByLinearity_with_category'>
```

Constructing the same morphism from the homset:
Constructing the same morphism directly using the class; no backward compatibility is guaranteed in this case:

```python
sage: from sage.modules.with_basis.morphism import ModuleMorphismByLinearity
sage: phi3 = ModuleMorphismByLinearity(X, on_basis, codomain=Y)
sage: phi3 == phi1
True
```

**Warning:** The hierarchy of classes implemented in this module is one of the first non-trivial hierarchies of classes for morphisms. It is hitting a couple scaling issues:

- There are many independent properties from which module morphisms can get code (being defined by linearity, from a matrix, or a function; being triangular, being diagonal, ...). How to mitigate the class hierarchy growth?

  This will become even more stringent as more properties are added (e.g. being defined from generators for an algebra morphism, ...)

- Categories, whose primary purpose is to provide infrastructure for handling such large hierarchy of classes, can't help at this point: there is no category whose morphisms are triangular morphisms, and it's not clear such a category would be sensible.

- How to properly handle `__init__` method calls and multiple inheritance?

- Who should be in charge of setting the default category: the classes themselves, or `ModulesWithBasis.ParentMethods.module_morphism()`?

Because of this, the hierarchy of classes, and the specific APIs, is likely to be refactored as better infrastructure and best practices emerge.

**AUTHORS:**

- Jason Bandlow and Florent Hivert (2010): Triangular Morphisms
- Christian Stump (2010): trac ticket #9648 module_morphism’s to a wider class of codomains

Before trac ticket #8678, this hierarchy of classes used to be in sage.categories.modules_with_basis; see trac ticket #8678 for the complete log.

```python
class sage.modules.with_basis.morphism.DiagonalModuleMorphism(domain, diagonal, codomain=None, category=None):
    Bases: sage.modules.with_basis.morphism.ModuleMorphismByLinearity
    A class for diagonal module morphisms.
    See ModulesWithBasis.ParentMethods.module_morphism().

    INPUT:
    - domain, codomain – two modules with basis $F$ and $G$, respectively
    - diagonal – a function $d$

    Assumptions:
```
• domain and codomain have the same base ring $R$,
• their respective bases $F$ and $G$ have the same index set $I$,
• $d$ is a function $I \to R$.

Return the diagonal module morphism from domain to codomain sending $F(i) \mapsto d(i)G(i)$ for all $i \in I$.

By default, codomain is currently assumed to be domain. (Todo: make a consistent choice with *ModuleMorphism.)

Todo:
• Implement an optimized __call__() function.
• Generalize to a mapcoeffs.
• Generalize to a mapterms.

EXAMPLES:

```
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); X.rename("X")
sage: phi = X.module_morphism(diagonal=factorial, codomain=X)
sage: x = X.basis()
sage: phi(x[1]), phi(x[2]), phi(x[3])
(B[1], 2*B[2], 6*B[3])
```

class sage.modules.with_basis.morphism.ModuleMorphism(domain, codomain=None, category=None, affine=False)

Bases: sage.categories.morphism.Morphism

The top abstract base class for module with basis morphisms.

INPUT:

• domain – a parent in ModulesWithBasis(...)
• codomain – a parent in Modules(...);
• category – a category or None (default: None)
• affine – whether we define an affine module morphism (default: False).

Construct a module morphism from domain to codomain in the category category. By default, the category is the first of Modules(R).WithBasis().FiniteDimensional(), Modules(R).WithBasis(), Modules(R), CommutativeAdditiveMonoids() that contains both the domain and the codomain. If initializing an affine morphism, then $Sets()$ is used instead.

See also:

• ModulesWithBasis.ParentMethods.module_morphism() for usage information and examples;
• sage.modules.with_basis.morphism for a technical overview of the classes for module morphisms;
• ModuleMorphismFromFunction and TriangularModuleMorphism.

The role of this class is minimal: it provides an __init__() method which:

• handles the choice of the default category
• handles the proper inheritance from categories by updating the class of self upon construction.
class sage.modules.with_basis.morphism.ModuleMorphismByLinearity(domain, on_basis=None, codomain=None, category=None, zero=None, position=0)

Bases: sage.modules.with_basis.morphism.ModuleMorphism

A class for module morphisms obtained by extending a function by linearity.

INPUT:

• domain, codomain, category – as for ModuleMorphism
• on_basis – a function which accepts indices of the basis of domain as position-th argument
• codomain – a parent in Modules(...) (default: on_basis.codomain())
• position – a non-negative integer (default: 0)
• zero – the zero of the codomain (defaults: codomain.zero())

See also:
• ModulesWithBasis.ParentMethods.module_morphism() for usage information and examples;
• sage.modules.with_basis.morphism for a technical overview of the classes for module morphisms;
• ModuleMorphismFromFunction and TriangularModuleMorphism.

Note: on_basis may alternatively be provided in derived classes by passing None as argument, and implementing or setting the attribute _on_basis

on_basis()

Return the action of this morphism on basis elements, as per ModulesWithBasis.Homsets.ElementMethods.on_basis().

OUTPUT:

• a function from the indices of the basis of the domain to the codomain

EXAMPLES:

```
sage: X = CombinatorialFreeModule(ZZ, [-2, -1, 1, 2])
sage: Y = CombinatorialFreeModule(ZZ, [1, 2])
sage: phi_on_basis = Y.monomial * abs
sage: phi = sage.modules.with_basis.morphism.ModuleMorphismByLinearity(X, on_˓
˓→basis = phi_on_basis, codomain=Y)
sage: x = X.basis()
sage: phi.on_basis()(-2)
B[2]
sage: phi.on_basis() == phi_on_basis
True
```
INPUT:

- domain, codomain, category – as for `ModuleMorphism`
- function – any function or callable from domain to codomain

See also:

- `ModulesWithBasis.ParentMethods.module_morphism()` for usage information and examples;
- `sage.modules.with_basis.morphism` for a technical overview of the classes for module morphisms;
- `ModuleMorphismFromFunction` and `TriangularModuleMorphism`.

```python
class sage.modules.with_basis.morphism.ModuleMorphismFromMatrix(domain, matrix, codomain=None, category=None, side='left', **kwargs):

    Bases: sage.modules.with_basis.morphism.ModuleMorphismByLinearity

    A class for module morphisms built from a matrix in the distinguished bases of the domain and codomain.

    See also:

    - `ModulesWithBasis.ParentMethods.module_morphism()`
    - `ModulesWithBasis.FiniteDimensional.MorphismMethods.matrix()`

    INPUT:

    - domain, codomain – two finite dimensional modules over the same base ring \( R \) with basis \( F \) and \( G \), respectively
    - matrix – a matrix with base ring \( R \) and dimensions matching that of \( F \) and \( G \), respectively
    - side – “left” or “right” (default: “left”)
        If side is “left”, this morphism is considered as acting on the left; i.e. each column of the matrix
        represents the image of an element of the basis of the domain.
    - category – a category or None (default: None)

    EXAMPLES:

    ```python
    sage: X = CombinatorialFreeModule(ZZ, [1,2]); X.rename("X"); x = X.basis()
    sage: Y = CombinatorialFreeModule(ZZ, [3,4]); Y.rename("Y"); y = Y.basis()
    sage: m = matrix([[1,2],[3,5]])
    sage: phi = X.module_morphism(matrix=m, codomain=Y)
    sage: phi.parent()
    Set of Morphisms from X to Y in Category of finite dimensional modules with basis \( \rightarrow \) over Integer Ring
    sage: phi.__class__
    <class 'sage.modules.with_basis.morphism.ModuleMorphismFromMatrix_with_category'>
    sage: phi(x[1])
    sage: phi(x[2])
    sage: m = matrix([[1,2],[3,5]])
    sage: phi = X.module_morphism(matrix=m, codomain=Y, side="right",
    ....: category=Modules(ZZ).WithBasis())
    ```
```
```
Todo: Possibly implement rank, addition, multiplication, matrix, etc, from the stored matrix.

class sage.modules.with_basis.morphism.PointwiseInverseFunction(f)
Bases: sage.structure.sage_object.SageObject
A class for pointwise inverse functions.
The pointwise inverse function of a function \( f \) is the function sending every \( x \) to \( 1/f(x) \).

EXEMPLARY:
sage: from sage.modules.with_basis.morphism import PointwiseInverseFunction
sage: f = PointwiseInverseFunction(factorial)
sage: f(0), f(1), f(2), f(3)
(1, 1, 1/2, 1/6)

pointwise_inverse()

class sage.modules.with_basis.morphism.TriangularModuleMorphism(triangular='upper',
unitriangular=False, key=None,
inverse=None,
inverse_on_support=<built-in function identity>,
invertible=None)
Bases: sage.modules.with_basis.morphism.ModuleMorphism
An abstract class for triangular module morphisms

Let \( X \) and \( Y \) be modules over the same base ring, with distinguished bases \( F \) indexed by \( I \) and \( G \) indexed by \( J \), respectively.

A module morphism \( \phi \) from \( X \) to \( Y \) is triangular if its representing matrix in the distinguished bases of \( X \) and \( Y \) is upper triangular (echelon form).

More precisely, \( \phi \) is upper triangular w.r.t. a total order \( < \) on \( J \) if, for any \( j \in J \), there exists at most one index \( i \in I \) such that the leading support of \( \phi(F_i) \) is \( j \) (see leading_support()). We denote by \( r(j) \) this index, setting \( r(j) \) to None if it does not exist.

Lower triangular morphisms are defined similarly, taking the trailing support instead (see trailing_support()).

A triangular morphism is unitriangular if all its pivots (i.e. coefficient of \( j \) in each \( \phi(F[r(j)]) \)) are 1.

INPUT:

* domain – a module with basis \( X \)
* codomain – a module with basis \( Y \) (default: \( X \))
* category – a category, as for ModuleMorphism
• **triangular** – "upper" or "lower" (default: "upper")

• **unitriangular** – boolean (default: False) As a shorthand, one may use `unitriangular="lower"` for `triangular="lower"`, `unitriangular=True`.

• **key** – a comparison key on \( J \) (default: the usual comparison of elements of \( J \))

• **inverse_on_support** – a function \( J \rightarrow I \cup \{ \text{None} \} \) implementing \( r \) (default: the identity function). If set to “compute”, the values of \( r(j) \) are precomputed by running through the index set \( I \) of the basis of the domain. This of course requires the domain to be finite dimensional.

• **invertible** – a boolean or None (default: None); can be set to specify that \( \phi \) is known to be (or not to be) invertible. If the domain and codomain share the same indexing set, this is by default automatically set to True if `inverse_on_support` is the identity, or in the finite dimensional case.

**See also:**

• `ModulesWithBasis.ParentMethods.module_morphism()` for usage information and examples;

• `sage.modules.with_basis.morphism` for a technical overview of the classes for module morphisms;

• `ModuleMorphismFromFunction` and `TriangularModuleMorphism`.

**OUTPUT:**

A morphism from \( X \) to \( Y \).

**Warning:** This class is meant to be used as a complement for a concrete morphism class. In particular, the `__init__()` method focuses on setting up the data structure describing the triangularity of the morphism. It purposely does not call `ModuleMorphism.__init__()` which should be called (directly or indirectly) beforehand.

**EXAMPLES:**

We construct and invert an upper unitriangular module morphism between two free \( \mathbb{Q} \)-modules:

```
sage: I = range(1,200)
sage: X = CombinatorialFreeModule(QQ, I); X.rename("X"); x = X.basis()
sage: Y = CombinatorialFreeModule(QQ, I); Y.rename("Y"); y = Y.basis()
sage: ut = Y.sum_of_monomials * divisors  # This * is map composition.
sage: phi = X.module_morphism(ut, unitriangular="upper", codomain=Y)
sage: phi(x[2])
sage: phi(x[6])
sage: phi(x[30])
sage: phi.preimage(y[2])
sage: phi.preimage(y[6])
sage: phi.preimage(y[30])
sage: (phi^-1)(y[30])
```

A lower triangular (but not unitriangular) morphism:
Using the key keyword, we can use triangularity even if the map becomes triangular only after a permutation of the basis:

```
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); X.rename("X"); x = X.basis()
sage: perm = [0, 2, 1, 3]
sage: phi = X.module_morphism(ut, triangular="upper", codomain=X,
....:                      key=lambda a: perm[a])
sage: [phi(x[i]) for i in range(1, 4)]
sage: [phi.preimage(x[i]) for i in range(1, 4)]
```

The same works in the lower-triangular case:

```
sage: phi = X.module_morphism(lt, triangular="lower", codomain=X,
....:                     key=lambda a: perm[a])
sage: [phi(x[i]) for i in range(1, 4)]
sage: [phi.preimage(x[i]) for i in range(1, 4)]
```

An injective but not surjective morphism cannot be inverted, but the inverse_on_support keyword allows Sage to find a partial inverse:

```
sage: X = CombinatorialFreeModule(QQ, [1,2,3]); x = X.basis()
sage: Y = CombinatorialFreeModule(QQ, [1,2,3,4,5]); y = Y.basis()
sage: ult = lambda i: sum( y[j] for j in range(i+1,6) )
sage: phi = X.module_morphism(ult, unitriangular="lower", codomain=Y,
....:                     inverse_on_support=lambda i: i-1 if i in [2,3,4] else None)
sage: phi(x[2])
sage: phi.preimage(y[3])
```

The inverse_on_support keyword can also be used if the bases of the domain and the codomain are identical but one of them has to be permuted in order to render the morphism triangular. For example:

```
sage: def ut(i):
....:     return (x[3] if i == 1 else x[1] if i == 2
....:              else x[1] + x[2])
```

(continues on next page)
sage: def perm(i):
    ....:     return (2 if i == 1 else 3 if i == 2 else 1)
sage: phi = X.module_morphism(ut, triangular="upper", codomain=X,
    ....:     inverse_on_support=perm)
sage: [phi(x[i]) for i in range(1, 4)]
sage: [phi.preimage(x[i]) for i in range(1, 4)]

The same works if the permutation induces lower triangularity:

sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); X.rename("X"); x = X.basis()
sage: def lt(i):
    ....:     return (x[3] if i == 1 else x[2] if i == 2
    ....:            else x[1] + x[2])
sage: def perm(i):
    ....:     return 4 - i
sage: phi = X.module_morphism(lt, triangular="lower", codomain=X,
    ....:     inverse_on_support=perm)
sage: [phi(x[i]) for i in range(1, 4)]
sage: [phi.preimage(x[i]) for i in range(1, 4)]

In the finite dimensional case, one can ask Sage to recover \( \text{inverse}_\text{on-support} \) by a precomputation:

sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); x = X.basis()
sage: Y = CombinatorialFreeModule(QQ, [1, 2, 3, 4]); y = Y.basis()
sage: ut = lambda i: sum( y[j] for j in range(1,i+2) )
sage: phi = X.module_morphism(ut, triangular="upper", codomain=Y,
    ....:     inverse_on_support="compute")
sage: tx = "{} {} {}"
sage: for j in Y.basis().keys():
    ....:     i = phi._inverse_on_support(j)
    ....:     print(tx.format(j, i, phi(x[i]) if i is not None else None))
1 None None

The \( \text{inverse}_\text{on-basis} \) and key keywords can be combined:

sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); X.rename("X")
sage: x = X.basis()
sage: def ut(i):
    ....:     return (2*x[2] + 3*x[3] if i == 1
    ....:            else 4*x[2])
sage: def perm(i):
    ....:     return (2 if i == 1 else 3 if i == 2 else 1)
sage: perverse_key = lambda a: (a - 2) % 3
sage: phi = X.module_morphism(ut, triangular="upper", codomain=X,
    ....:     inverse_on_support=perm, key=perverse_key)
cokernel_basis_indices()

Return the indices of the natural monomial basis of the cokernel of self.

INPUT:

- self – a triangular morphism over a field or a unitriangular morphism over a ring, with a finite dimensional codomain.

OUTPUT:

A list $E$ of indices of the basis $(B_e)_e$ of the codomain of self so that $(B_e)_e \in E$ forms a basis of a supplementary of the image set of self.

Thinking of this triangular morphism as a row echelon matrix, this returns the complementary of the characteristic columns. Namely $E$ is the set of indices which do not appear as leading support of some element of the image set of self.

EXAMPLES:

```python
sage: X = CombinatorialFreeModule(ZZ, [1,2,3]); x = X.basis()
sage: Y = CombinatorialFreeModule(ZZ, [1,2,3,4,5]); y = Y.basis()
sage: uut = lambda i: sum( y[j] for j in range(i+1,6) ) # uni-upper
    inverse_on_support=lambda i: i-1 if i in [2,3,4] else None
sage: phi = X.module_morphism(uut, unitriangular="upper", codomain=Y,
    ....:     inverse_on_support=lambda i: i-1 if i in [2,3,4] else None)
sage: phi.cokernel_basis_indices()
[1, 5]
```

```python
sage: phi = X.module_morphism(uut, triangular="upper", codomain=Y,
    ....:     inverse_on_support=lambda i: i-1 if i in [2,3,4] else None)
sage: phi.cokernel_basis_indices()
Traceback (most recent call last):
  ... Not Implemented Error: cokernel_basis_indices for a triangular but not.unitriangular morphism over a ring
```

cokernel_projection(category=None)

Return a projection on the co-kernel of self.

INPUT:

- category – the category of the result

EXAMPLES:
sage: X = CombinatorialFreeModule(QQ, [1,2,3]); x = X.basis()
sage: Y = CombinatorialFreeModule(QQ, [1,2,3,4,5]); y = Y.basis()
sage: lt = lambda i: sum( y[j] for j in range(i+1,6) ) # lower
sage: phi = X.module_morphism(lt, triangular="lower", codomain=Y, ....: inverse_on_support=lambda i: i-1 if i in [2,3,4] else None)
sage: phipro = phi.cokernel_projection()
sage: phipro(y[1] + y[2])
B[1]
sage: all(phipro(phi(x)).is_zero() for x in X.basis())
True
sage: phipro(y[1])
B[1]
sage: phipro(y[4])
-B[5]
sage: phipro(y[5])
B[5]

Return $y$ reduced w.r.t. the image of self.

INPUT:

- self – a triangular morphism over a field, or a unitriangular morphism over a ring
- $y$ – an element of the codomain of self

Suppose that self is a morphism from $X$ to $Y$. Then, for any $y \in Y$, the call self.coreduced($y$) returns a normal form for $y$ in the quotient $Y/I$ where $I$ is the image of self.

EXAMPLES:

```
sage: X = CombinatorialFreeModule(QQ, [1,2,3]); x = X.basis()
sage: Y = CombinatorialFreeModule(QQ, [1,2,3,4,5]); y = Y.basis()
sage: ult = lambda i: sum( y[j] for j in range(i+1,6) )
sage: phi = X.module_morphism(ult, unitriangular="lower", codomain=Y, ....: inverse_on_support=lambda i: i-1 if i in [2,3,4] else None)
sage: [phi(v) for v in X.basis()]
sage: [phi.coreduced(y[1]-2*y[4])]
sage: [phi.coreduced(v) for v in y]
[B[1], 0, 0, -B[5], B[5]]
```

Now with a non unitriangular morphism:

```
sage: lt = lambda i: sum( j*y[j] for j in range(i+1,6) )
sage: phi = X.module_morphism(lt, triangular="lower", codomain=Y, ....: inverse_on_support=lambda i: i-1 if i in [2,3,4] else None)
sage: [phi(v) for v in X.basis()]
sage: [phi.coreduced(y[1]-2*y[4])]
```

(continues on next page)
For general rings, this method is only implemented for unitriangular morphisms:

```python
sage: X = CombinatorialFreeModule(ZZ, [1,2,3]); x = X.basis()

sage: X = CombinatorialFreeModule(ZZ, [1,2,3,4,5]); y = X.basis()

sage: phi = X.module_morphism(ult, unitriangular="lower", codomain=X,
                                 ....: inverse_on_support=lambda i: i-1 if i in [2,3,4] else None)

sage: phi.coreduced(y[1] - 2*y[4])
[0, 0, 0, -B[5], B[5]]

sage: phi.coreduced(v) for v in y
[0, 0, 0, -B[5], B[5]]

sage: phi = X.module_morphism(lt, triangular="lower", codomain=Y,
                                 ....: inverse_on_support=lambda i: i-1 if i in [2,3,4] else None)

sage: phi.coreduced(v) for v in y
Traceback (most recent call last):
  ... NotImplementedError: coreduce for a triangular but not unitriangular morphism...

```

Note: Before trac ticket #8678 this method used to be called co_reduced.

**preimage(f)**

Return the preimage of \( f \) under \( self \).

**EXAMPLES:**

```python
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); x = X.basis()

sage: Y = CombinatorialFreeModule(QQ, [1, 2, 3]); y = Y.basis()

sage: ult = lambda i: sum( y[j] for j in range(i,4) ) # uni-lower
sage: phi = X.module_morphism(ult, triangular="lower", codomain=Y)

sage: phi.preimage(y[1] + y[2])
```

The morphism need not be surjective. In the following example, the codomain is of larger dimension than the domain:

```python
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); x = X.basis()

sage: Y = CombinatorialFreeModule(QQ, [1, 2, 3, 4, 5]); y = Y.basis()

sage: lt = lambda i: sum( y[j] for j in range(i,5) )

sage: phi = X.module_morphism(lt, triangular="lower", codomain=Y)

sage: phi.preimage(y[1] + y[2])
```

Here are examples using `inverse_on_support` to handle a morphism that shifts the leading indices by 1:

```python
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); x = X.basis()

sage: Y = CombinatorialFreeModule(QQ, [1, 2, 3, 4, 5]); y = Y.basis()

sage: lt = lambda i: sum( y[j] for j in range(i+1,6) ) # lower

sage: phi.preimage(y[1] + y[2])
```
sage: phi = X.module_morphism(lt, triangular="lower", codomain=Y, .....: inverse_on_support=lambda i: i-1 if i in [2,3,4] else None)
sage: phi(x[1])
sage: phi(x[3])
sage: phi.preimage(y[2] + y[3])
True
sage: phi.preimage(phi(el)) == el
True

sage: phi.preimage(y[1])
Traceback (most recent call last):
...
ValueError: B[1] is not in the image
sage: phi.preimage(y[4])
Traceback (most recent call last):
...
ValueError: B[4] is not in the image

Over a base ring like \(\mathbb{Z}\), the morphism need not be surjective even when the dimensions match:

sage: X = CombinatorialFreeModule(ZZ, [1, 2, 3]); x = X.basis()
sage: Y = CombinatorialFreeModule(ZZ, [1, 2, 3]); y = Y.basis()
sage: lt = lambda i: sum( 2* y[j] for j in range(i+1,6) ) # uni-lower
sage: phi = X.module_morphism(lt, triangular="lower", codomain=Y)
sage: phi.preimage(2*y[1] + 2*y[2])

The error message in case of failure could be more specific though:

sage: phi.preimage(y[1] + y[2])
Traceback (most recent call last):
...
TypeError: no conversion of this rational to integer

section()

Return the section (partial inverse) of self.

Return a partial triangular morphism which is a section of self. The section morphism raise a ValueError if asked to apply on an element which is not in the image of self.

EXAMPLES:

sage: X = CombinatorialFreeModule(QQ, [1,2,3]); x = X.basis()
sage: X.rename("X")
sage: Y = CombinatorialFreeModule(QQ, [1,2,3,4,5]); y = Y.basis()
sage: ult = lambda i: sum( y[j] for j in range(i+1,6) ) # uni-lower
sage: phi = X.module_morphism(ult, triangular="lower", codomain=Y, .....: inverse_on_support=lambda i: i-1 if i in [2,3,4] else None)
sage: -phi
class sage.modules.with_basis.morphism.TriangularModuleMorphismByLinearity(domain, on_basis, codomain=None, category=None, **keywords)

Bases: sage.modules.with_basis.morphism.ModuleMorphismByLinearity, sage.modules.with_basis.morphism.TriangularModuleMorphism

A concrete class for triangular module morphisms obtained by extending a function by linearity.

See also:

• ModulesWithBasis.ParentMethods.module_morphism() for usage information and examples;
• sage.modules.with_basis.morphism for a technical overview of the classes for module morphisms;
• ModuleMorphismByLinearity and TriangularModuleMorphism.

class sage.modules.with_basis.morphism.TriangularModuleMorphismFromFunction(domain, function, codomain=None, category=None, **keywords)

Bases: sage.modules.with_basis.morphism.ModuleMorphismFromFunction, sage.modules.with_basis.morphism.TriangularModuleMorphism

A concrete class for triangular module morphisms implemented by a function.

See also:

• ModulesWithBasis.ParentMethods.module_morphism() for usage information and examples;
• sage.modules.with_basis.morphism for a technical overview of the classes for module morphisms;
• ModuleMorphismFromFunction and TriangularModuleMorphism.

sage.modules.with_basis.morphism.pointwise_inverse_function(f)
Return the function \( x \mapsto 1/f(x) \).

INPUT:

• \( f \) – a function

EXAMPLES:

sage: from sage.modules.with_basis.morphism import pointwise_inverse_function
sage: def f(x): return x
sage: g = pointwise_inverse_function(f)
\texttt{sage: g(1), g(2), g(3)}
\begin{verbatim}
(1, 1/2, 1/3)
\end{verbatim}

\textit{pointwise\_inverse\_function()} is an involution:

\texttt{sage: f is pointwise\_inverse\_function(g)}
\begin{verbatim}
True
\end{verbatim}

\textbf{Todo:} This has nothing to do here!!! Should there be a library for pointwise operations on functions somewhere in Sage?
class sage.modules.with_basis.subquotient.QuotientModuleWithBasis(submodule, category)

Bases: sage.combinat.free_module.CombinatorialFreeModule

A class for quotients of a module with basis by a submodule.

INPUT:

• submodule – a submodule of self

• category – a category (default: ModulesWithBasis(submodule.base_ring()))

submodule should be a free submodule admitting a basis in unitriangular echelon form. Typically submodule is a SubmoduleWithBasis as returned by Modules.WithBasis.ParentMethods.submodule().

The lift method should have a method .cokernel_basis_indices that computes the indexing set of a subset $B$ of the basis of self that spans some supplementary of submodule in self (typically the non characteristic columns of the aforementioned echelon form). submodule should further implement a submodule.reduce(x) method that returns the unique element in the span of $B$ which is equivalent to $x$ modulo submodule.

This is meant to be constructed via Modules.WithBasis.FiniteDimensional.ParentMethods.quotient_module() This differs from sage.rings.quotient_ring.QuotientRing in the following ways:

• submodule needs not be an ideal. If it is, the transportation of the ring structure is taken care of by the Subquotients categories.

• Thanks to .cokernel_basis_indices, we know the indices of a basis of the quotient, and elements are represented directly in the free module spanned by those indices rather than by wrapping elements of the ambient space.

There is room for sharing more code between those two implementations and generalizing them. See trac ticket #18204.

See also:

• Modules.WithBasis.ParentMethods.submodule()

• Modules.WithBasis.FiniteDimensional.ParentMethods.quotient_module()

• SubmoduleWithBasis

• sage.rings.quotient_ring.QuotientRing

ambient()

Return the ambient space of self.

EXAMPLES:
sage: X = CombinatorialFreeModule(QQ, range(3), prefix="x"); x = X.basis()
sage: Y = X.quotient_module((x[0]-x[1], x[1]-x[2]))
sage: Y.ambient() is X
True

lift(x)
Lift x to the ambient space of self.

INPUT:
• x – an element of self

EXAMPLES:

```
sage: X = CombinatorialFreeModule(QQ, range(3), prefix="x"); x = X.basis()
sage: Y = X.quotient_module((x[0]-x[1], x[1]-x[2])); y = Y.basis()
sage: Y.lift(y[2])
x[2]
```

retract(x)
Retract an element of the ambient space by projecting it back to self.

INPUT:
• x – an element of the ambient space of self

EXAMPLES:

```
sage: X = CombinatorialFreeModule(QQ, range(3), prefix="x"); x = X.basis()
sage: Y = X.quotient_module((x[0]-x[1], x[1]-x[2])); y = Y.basis()
sage: Y.print_options(prefix='y')
sage: Y.retract(x[0])
y[2]
sage: Y.retract(x[1])
y[2]
sage: Y.retract(x[2])
y[2]
```

class sage.modules.with_basis.subquotient.SubmoduleWithBasis(basis, support_order, ambient, unitriangular, category, *args, **opts)

Bases: sage.combinat.free_module.CombinatorialFreeModule

A base class for submodules of a ModuleWithBasis spanned by a (possibly infinite) basis in echelon form.

INPUT:
• basis – a family of elements in echelon form in some module with basis V, or data that can be converted into such a family
• support_order – an ordering of the support of basis expressed in ambient given as a list
• unitriangular – if the lift morphism is unitriangular
• ambient – the ambient space V
• category – a category

Further arguments are passed down to CombinatorialFreeModule.

This is meant to be constructed via Modules.WithBasis.ParentMethods.submodule().
See also:

- `Modules.WithBasis.ParentMethods.submodule()`
- `QuotientModuleWithBasis`

`ambient()`
Return the ambient space of `self`.

```python
sage: X = CombinatorialFreeModule(QQ, range(3)); x = X.basis()
sage: Y = X.submodule((x[0]-x[1], x[1]-x[2]))
sage: Y.ambient() is X
True
```

`is_submodule(other)`
Return whether `self` is a submodule of `other`.

INPUT:

- `other` – another submodule of the same ambient module, or the ambient module itself

```python
sage: X = CombinatorialFreeModule(QQ, range(4)); x = X.basis()
sage: F = X.submodule([x[0]-x[1], x[1]-x[2], x[2]-x[3]])
sage: G = X.submodule([x[0]-x[2]])
sage: H = X.submodule([x[0]-x[1], x[2]])
sage: F.is_submodule(X)
True
sage: G.is_submodule(F)
True
sage: H.is_submodule(F)
False
```

`lift()`
The lift (embedding) map from `self` to the ambient space.

```python
sage: X = CombinatorialFreeModule(QQ, range(3), prefix="x"); x = X.basis()
sage: Y = X.submodule((x[0]-x[1], x[1]-x[2]), already_echelonized=True); y = Y.basis()
sage: Y.lift
Generic morphism:
  From: Free module generated by {0, 1} over Rational Field
  To:   Free module generated by {0, 1, 2} over Rational Field
sage: [Y.lift(u) for u in y]
[x[0] - x[1], x[1] - x[2]]
sage: (y[0] + y[1]).lift()
x[0] - x[2]
```

`reduce()`
The reduce map.
This map reduces elements of the ambient space modulo this submodule.
EXAMPLES:

```python
sage: X = CombinatorialFreeModule(QQ, range(3), prefix="x"); x = X.basis()
sage: Y = X.submodule((x[0]-x[1], x[1]-x[2]), already_echelonized=True)
sage: Y.reduce
Generic endomorphism of Free module generated by {0, 1, 2} over Rational Field
sage: Y.reduce(x[1])
x[2]
sage: Y.reduce(2*x[0] + x[1])
3*x[2]
```

`retract()`
The retract map from the ambient space.

EXAMPLES:

```python
sage: X = CombinatorialFreeModule(QQ, range(3), prefix="x"); x = X.basis()
sage: Y = X.submodule((x[0]-x[1], x[1]-x[2]), already_echelonized=True)
sage: Y.print_options(prefix='y')
sage: Y.retract
Generic morphism:  
  From: Free module generated by {0, 1, 2} over Rational Field  
  To:   Free module generated by {0, 1} over Rational Field
sage: Y.retract(x[0] - x[2])
y[0] + y[1]
```
REPRESENTATIONS OF A SEMIGROUP

AUTHORS:

• Travis Scrimshaw (2015-11-21): Initial version
• Siddharth Singh (2020-03-21): Signed Representation

```python
class sage.modules.with_basis.representation.RegularRepresentation(semigroup, base_ring, side='left')
```

The regular representation of a semigroup.

The left regular representation of a semigroup $S$ over a commutative ring $R$ is the semigroup ring $R[S]$ equipped with the left $S$-action $xb = bxy$, where $(b_z)_{z \in S}$ is the natural basis of $R[S]$ and $x, y \in S$.

INPUT:

- `semigroup` – a semigroup
- `base_ring` – the base ring for the representation
- `side` – (default: "left") whether this is a "left" or "right" representation

REFERENCES:

- Wikipedia article Regular_representation

```python
class sage.modules.with_basis.representation.Representation(semigroup, module, on_basis, side='left', **kwargs)
```

Representation of a semigroup.

INPUT:

- `semigroup` – a semigroup
- `module` – a module with a basis
- `on_basis` – function which takes as input $g, m$, where $g$ is an element of the semigroup and $m$ is an element of the indexing set for the basis, and returns the result of $g$ acting on $m$
- `side` – (default: "left") whether this is a "left" or "right" representation

EXAMPLES:

We construct the sign representation of a symmetric group:
sage: G = SymmetricGroup(4)
sage: M = CombinatorialFreeModule(QQ, ['v'])
sage: from sage.modules.with_basis.representation import Representation
sage: on_basis = lambda g,m: M.term(m, g.sign())
sage: R = Representation(G, M, on_basis)
sage: x = R.an_element(); x
2*B['v']
sage: c,s = G.gens()
sage: c,s
((1,2,3,4), (1,2))
sage: c * x
-2*B['v']
sage: s * x
-2*B['v']
sage: c * s * x
2*B['v']
sage: (c * s) * x
2*B['v']

This extends naturally to the corresponding group algebra:

sage: A = G.algebra(QQ)
sage: s,c = A.algebra_generators()
sage: c,s
((1,2,3,4), (1,2))
sage: c * x
-2*B['v']
sage: s * x
-2*B['v']
sage: c * s * x
2*B['v']
sage: (c * s) * x
2*B['v']
sage: (c + s) * x
-4*B['v']

REFERENCES:

- Wikipedia article Group_representation

class Element

    Bases: sage.modules.with_basis.indexed_element.IndexedFreeModuleElement

product_by_coercion(left, right)

    Return the product of left and right by passing to self._module and then building a new element of self.

    EXAMPLES:

    sage: G = groups.permutation.KleinFour()
sage: E = algebras.Exterior(QQ, 'e', 4)
sage: on_basis = lambda g,m: E.monomial(m) # the trivial representation
sage: from sage.modules.with_basis.representation import Representation
sage: R = Representation(G, E, on_basis)
sage: r = R.an_element(); r

(continues on next page)
1 + 2*e0 + 3*e1 + e1*e2
sage: g = G.an_element();
sage: g*r == r
True
sage: r*r
Traceback (most recent call last):
...TypeError: unsupported operand parent(s) for *:
'Representation of The Klein 4 group of order 4, as a permutation group indexed by Subsets of {0, 1, 2, 3} over Rational Field'
and
'Representation of The Klein 4 group of order 4, as a permutation group indexed by Subsets of {0, 1, 2, 3} over Rational Field'
sage: from sage.categories.algebras import Algebras
sage: category = Algebras(QQ).FiniteDimensional().WithBasis()
sage: T = Representation(G, E, on_basis, category=category)
sage: t = T.an_element(); t
1 + 2*e0 + 3*e1 + e1*e2
sage: g*t == t
True
sage: t*t
1 + 4*e0 + 4*e0*e1*e2 + 6*e1 + 2*e1*e2

side()
Return whether self is a left or a right representation.

OUTPUT:

• the string "left" or "right"

EXAMPLES:

sage: G = groups.permutation.Dihedral(4)
sage: R = G.regular_representation()
sage: R.side()
'left'
sage: S = G.regular_representation(side="right")
sage: S.side()
'right'

class sage.modules.with_basis.representation.Representation_abstract(semigroup, base_ring, *args, **opts)

Bases: sage.combinat.free_module.CombinatorialFreeModule

Abstract base class for representations of semigroups.

INPUT:

• semigroup – a semigroup
• base_ring – a commutative ring

invariant_module(S=None, **kwars)
Return the submodule of self invariant under the action of S.

For a semigroup S acting on a module M, the invariant submodule is given by

\[ M^S = \{ m \in M : s \cdot m = m \forall s \in S \}. \]
INPUT:

- \( S \) – a finitely-generated semigroup (default: the semigroup this is a representation of)
- action – a function (default: \texttt{operator.mul})
- side – 'left' or 'right' (default: \texttt{side()}); which side of self the elements of S acts

\textbf{Note:} Two sided actions are considered as left actions for the invariant module.

OUTPUT:

- \texttt{FiniteDimensionalInvariantModule}

\textbf{EXAMPLES:}

```sage
sage: S3 = SymmetricGroup(3)
sage: M = S3.regular_representation()
sage: I = M.invariant_module()
sage: [I.lift(b) for b in I.basis()]
[(0) + (2,3) + (1,2) + (1,2,3) + (1,3,2) + (1,3)]
```

We build the \( D_4 \)-invariant representation inside of the regular representation of \( S_4 \):

```sage
sage: D4 = groups.permutation.Dihedral(4)
sage: S4 = SymmetricGroup(4)
sage: R = S4.regular_representation()
sage: I = R.invariant_module(D4)
sage: [I.lift(b) for b in I.basis()]
[(0) + (2,4) + (1,2)(3,4) + (1,3) + (1,3)(2,4) + (1,4,3,2) + (1,4)(2, \rightarrow 3),
 (3,4) + (2,3,4) + (1,2) + (1,2,4) + (1,3,2) + (1,3,2,4) + (1,4,3) + (1,4,2,3),
 (2,3) + (2,4,3) + (1,2,3) + (1,2,4,3) + (1,3,4,2) + (1,3,4) + (1,4,2) + (1,4)]
```

\texttt{semigroup()}

Return the semigroup whose representation \texttt{self} is.

\textbf{EXAMPLES:}

```sage
sage: G = SymmetricGroup(4)
sage: M = CombinatorialFreeModule(QQ, ['v'])
sage: from sage.modules.with_basis.representation import Representation
sage: on_basis = lambda g,m: M.term(m, g.sign())
sage: R = Representation(G, M, on_basis)
sage: R.semigroup()
Symmetric group of order 4! as a permutation group
```

\texttt{semigroup_algebra()}

Return the semigroup algebra whose representation \texttt{self} is.

\textbf{EXAMPLES:}

```sage
sage: G = SymmetricGroup(4)
sage: M = CombinatorialFreeModule(QQ, ['v'])
sage: from sage.modules.with_basis.representation import Representation
sage: on_basis = lambda g,m: M.term(m, g.sign())
(continues on next page)
sage: R = Representation(G, M, on_basis)
sage: R.semigroup_algebra()
Symmetric group algebra of order 4 over Rational Field

side()
Return whether self is a left, right, or two-sided representation.

OUTPUT:
• the string "left", "right", or "twosided"

EXAMPLES:

sage: G = groups.permutation.Dihedral(4)
sage: R = G.regular_representation()
sage: R.side()
'left'

twisted_invariant_module(chi, G=None, **kwargs)
Create the isotypic component of the action of G on self with irreducible character given by chi.

See also:
• FiniteDimensionalTwistedInvariantModule

INPUT:
• chi – a list/tuple of character values or an instance of ClassFunction_gap
• G – a finitely-generated semigroup (default: the semigroup this is a representation of)

This also accepts the group to be the first argument to be the group.

OUTPUT:
• FiniteDimensionalTwistedInvariantModule

EXAMPLES:

sage: G = SymmetricGroup(3)
sage: R = G.regular_representation(QQ)
sage: T = R.twisted_invariant_module([2,0,-1])
sage: T.basis()
Finite family {0: B[0], 1: B[1], 2: B[2], 3: B[3]}
sage: [T.lift(b) for b in T.basis()]
[(0) - (1,2,3), -(1,2,3) + (1,3,2), (2,3) - (1,2), -(1,2) + (1,3)]

We check the different inputs work

sage: R.twisted_invariant_module([2,0,-1], G) is T True
definition sage:  R.twisted_invariant_module(G, [2,0,-1]) is T True
class sage.modules.with_basis.representation.SignRepresentationCoxeterGroup(group, base_ring, sign_function=None)

Bases: sage.modules.with_basis.representation.SignRepresentation_abstract

The sign representation for a Coxeter group.

EXAMPLES:
```python
sage: G = WeylGroup(["A", 1, 1])
sage: V = G.sign_representation()
sage: TestSuite(V).run()
```

class `sage.modules.with_basis.representation.SignRepresentationMatrixGroup(group, base_ring, sign_function=None)`

Bases: `sage.modules.with_basis.representation.SignRepresentation_abstract`

The sign representation for a matrix group.

EXAMPLES:

```python
sage: G = groups.permutation.PGL(2, 3)
sage: V = G.sign_representation()
sage: TestSuite(V).run()
```

class `sage.modules.with_basis.representation.SignRepresentationPermgroup(group, base_ring, sign_function=None)`

Bases: `sage.modules.with_basis.representation.SignRepresentation_abstract`

The sign representation for a permutation group.

EXAMPLES:

```python
sage: G = groups.permutation.PGL(2, 3)
sage: V = G.sign_representation()
sage: TestSuite(V).run()
```

class `sage.modules.with_basis.representation.SignRepresentation_abstract(group, base_ring, sign_function=None)`

Bases: `sage.modules.with_basis.representation.Representation_abstract`

Generic implementation of a sign representation.

The sign representation of a semigroup $S$ over a commutative ring $R$ is the 1-dimensional $R$-module on which every element of $S$ acts by 1 if order of element is even (including 0) or -1 if order of element if odd.

This is simultaneously a left and right representation.

INPUT:

- `permgroup` – a permgroup
- `base_ring` – the base ring for the representation
- `sign_function` – a function which returns 1 or -1 depending on the elements sign

REFERENCES:

- Wikipedia article Representation_theory_of_the_symmetric_group

class `Element`

Bases: `sage.modules.with_basis.indexed_element.IndexedFreeModuleElement`

```python
side()
```

Return that `self` is a two-sided representation.

OUTPUT:

- the string "twosided"

EXAMPLES:
```
sage: G = groups.permutation.Dihedral(4)
sage: R = G.sign_representation()
sage: R.side()
'twosided'
```

class sage.modules.with_basis.representation.TrivialRepresentation(semigroup, base_ring)
Bases: sage.modules.with_basis.representation.Representation_abstract

The trivial representation of a semigroup.

The trivial representation of a semigroup $S$ over a commutative ring $R$ is the 1-dimensional $R$-module on which every element of $S$ acts by the identity.

This is simultaneously a left and right representation.

INPUT:

- `semigroup` – a semigroup
- `base_ring` – the base ring for the representation

REFERENCES:

- Wikipedia article Trivial_representation

class Element
Bases: sage.modules.with_basis.indexed_element.IndexedFreeModuleElement

side()
Return that `self` is a two-sided representation.

OUTPUT:

- the string "twosided"

EXAMPLES:
```
sage: G = groups.permutation.Dihedral(4)
sage: R = G.trivial_representation()
sage: R.side()
'twosided'
```
ITERATORS OVER FINITE SUBMODULES OF A \( \mathbb{Z} \)-MODULE

We iterate over the elements of a finite \( \mathbb{Z} \)-module. The action of \( \mathbb{Z} \) must be the natural one.

This class is intended to provide optimizations for the sage.free_module.FreeModule_generic:__iter__() method.

AUTHORS:

• Thomas Feulner (2012-08-31): initial version
• Punarbasu Purkayastha (2012-11-09): replaced the loop with recursion
• Thomas Feulner (2012-11-09): added functionality to enumerate cosets, FiniteFieldsubspace_projPoint_iterator

EXAMPLES:

```python
sage: from sage.modules.finite_submodule_iter import FiniteZZsubmodule_iterator
sage: F.<x,y,z> = FreeAlgebra(GF(3),3)
sage: iter = FiniteZZsubmodule_iterator([x,y], [3,3])
sage: list(iter)
[0, x, 2*x, y, x + y, 2*x + y, 2*y, x + 2*y, 2*x + 2*y]
```

There is a specialization for subspaces over finite fields:

```python
sage: from sage.modules.finite_submodule_iter import FiniteFieldsubspace_iterator
sage: A = random_matrix(GF(4, 'a'), 5, 100)
sage: iter = FiniteFieldsubspace_iterator(A)
sage: len(list(iter))
1024
```

The module also allows the iteration over cosets:

```python
sage: from sage.modules.finite_submodule_iter import FiniteFieldsubspace_iterator
sage: A = random_matrix(GF(4, 'a'), 5, 100)
sage: v = random_vector(GF(4, 'a'), 100)
sage: iter = FiniteFieldsubspace_iterator(A, v)
sage: len(list(iter))
1024
```

class sage.modules.finite_submodule_iter.FiniteFieldsubspace_iterator

    Bases: sage.modules.finite_submodule_iter.FiniteZZsubmodule_iterator

    This class implements an iterator over the subspace of a vector space over a finite field. The subspace is generated by basis.

    INPUT:
• **basis** – a list of vectors or a matrix with elements over a finite field. If a matrix is provided then it is not checked whether the matrix is full ranked. Similarly, if a list of vectors is provided, then the linear independence of the vectors is not checked.

• **coset_rep** (optional) – a vector in the same ambient space, if one aims to compute a coset of the vector space given by **basis**.

• **immutable** (optional; default: `False`) – set it to `True` to return immutable vectors.

EXAMPLES:

```python
sage: from sage.modules.finite_submodule_iter import FiniteFieldsubspace_iterator
sage: A = random_matrix(GF(2), 10, 100)
sage: iter = FiniteFieldsubspace_iterator(A)
sage: len(list(iter))
1024
sage: X = random_matrix(GF(4, 'a'), 7, 100).row_space()
sage: s = list(X) # long time (5s on sage.math, 2013)
sage: t = list(FiniteFieldsubspace_iterator(X.basis())) # takes 0.31s
sage: sorted(t) == sorted(s) # long time
True
```

class sage.modules.finite_submodule_iter.FiniteFieldsubspace_projPoint_iterator

Bases: object

This class implements an iterator over the projective points of a vector space over a finite field. The vector space is generated by **basis** and need not to be equal to the full ambient space.

A projective point (= one dimensional subspace) \( P \) will be represented by a generator \( p \). To ensure that all \( p \) will be normalized you can set the optional argument **normalize** to `True`.

INPUT:

• **basis** – a list of vectors or a matrix with elements over a finite field. If a matrix is provided then it is not checked whether the matrix is full ranked. Similarly, if a list of vectors is provided, then the linear independence of the vectors is not checked.

• **normalize** (optional; default: `False`) – boolean which indicates if the returned vectors should be normalized, i.e. the first nonzero coordinate is equal to 1.

• **immutable** (optional; default: `False`) – set it to `True` to return immutable vectors.

EXAMPLES:

```python
sage: from sage.modules.finite_submodule_iter import FiniteFieldsubspace_iterator
    # FiniteFieldsubspace_projPoint_iterator
sage: A = random_matrix(GF(4, 'a'), 5, 100)
sage: a = len(list(FiniteFieldsubspace_iterator(A)))
sage: b = len(list(FiniteFieldsubspace_projPoint_iterator(A)))
sage: b == (a-1)/3
True
```

Prove that the option **normalize** == `True` will only return normalized vectors.

```python
sage: all(x.monic() == x for x in FiniteFieldsubspace_projPoint_iterator(A, True)) True
```

class sage.modules.finite_submodule_iter.FiniteZZsubmodule_iterator

Bases: object

Let \( G \) be an abelian group and suppose that \((g_0, \ldots, g_n)\) is a list of elements of \( G \), whose additive orders are equal to \( m_i \) and \( \sum_{i=0}^{n} x_i g_i = 0 \) for \( x_i \in \mathbb{Z}_{m_i} \) for \( i \in \{0, \ldots, n\} \) implies \( x_i = 0 \) for all \( i \).
This class implements an iterator over the $\mathbb{Z}$-submodule $M = \{ \sum_{i=0}^{n} x_i g_i \}$. If the independence condition from above is not fulfilled, we can still use this iterator to run over the elements. In this case the elements will occur multiple times.

Getting from one element of the submodule to another is performed by one single addition in $G$.

INPUT:

- `basis` - the elements $(g_0, \ldots, g_n)$
- `order` (optional) - the additive_orders $m_i$ of $g_i$.
- `coset_rep` (optional) – an element of $g$, if one aims to compute a coset of the $\mathbb{Z}$-submodule $M$.
- `immutable` (optional; default: False) – set it to True to return immutable elements. Setting this to True makes sense if the elements are vectors. See `FiniteFieldSubspace_iterator` for examples.

EXAMPLES:

```python
sage: from sage.modules.finite_submodule_iter import FiniteZZsubmodule_iterator
sage: F.<x,y,z> = FreeAlgebra(GF(3),3)
(sage: iter = FiniteZZsubmodule_iterator([x,y], [3,3])
(sage: list(iter)
[0, x, 2*x, y, x + y, 2*x + y, 2*y, x + 2*y, 2*x + 2*y]
(sage: iter = FiniteZZsubmodule_iterator([x,y], [3,3], z)
(sage: list(iter)
[z, x + z, 2*x + z, y + z, x + y + z, 2*x + y + z, x + 2*y + z, 2*x + 2*y + z]
```
FREE QUADRATIC MODULES

Sage supports computation with free quadratic modules over an arbitrary commutative ring. Nontrivial functionality is available over \( \mathbb{Z} \) and fields. All free modules over an integral domain are equipped with an embedding in an ambient vector space and an inner product, which you can specify and change.

Create the free module of rank \( n \) over an arbitrary commutative ring \( R \) using the command \texttt{FreeModule}(R,n) with a given \texttt{inner_product_matrix}.

The following example illustrates the creation of both a vector spaces and a free module over the integers and a submodule of it. Use the functions \texttt{FreeModule}, \texttt{span} and member functions of free modules to create free modules.

“Do not use the \texttt{FreeModule}_{xxx} constructors directly.”

EXAMPLES:

```python
sage: M = Matrix(QQ,[[2,1,0],[1,2,1],[0,1,2]])
sage: V = VectorSpace(QQ,3,inner_product_matrix=M)
sage: type(V)
<class 'sage.modules.free_quadratic_module.FreeQuadraticModule_ambient_field_with_
˓→category'>
sage: V.inner_product_matrix()
\[
\begin{bmatrix}
2 & 1 & 0 \\
1 & 2 & 1 \\
0 & 1 & 2
\end{bmatrix}
\]
sage: W = V.subspace([[1,2,7], [1,1,0]])
sage: type(W)
<class 'sage.modules.free_quadratic_module.FreeQuadraticModule_submodule_field_with_
˓→category'>
sage: W
Quadratic space of degree 3 and dimension 2 over Rational Field
Basis matrix:
\[
\begin{bmatrix}
1 & 0 & -7 \\
0 & 1 & 7
\end{bmatrix}
\]
Inner product matrix:
\[
\begin{bmatrix}
2 & 1 & 0 \\
1 & 2 & 1 \\
0 & 1 & 2
\end{bmatrix}
\]
sage: W.gram_matrix()
\[
\begin{bmatrix}
100 & -104 \\
-104 & 114
\end{bmatrix}
\]
```

AUTHORS:

- David Kohel (2008-06): First created (based on free_module.py)
sage.modules.free_quadratic_module.FreeQuadraticModule(base_ring, rank, inner_product_matrix, sparse=False, inner_product_ring=None)

Create the free quadratic module over the given commutative ring of the given rank.

**INPUT:**
- `base_ring` – a commutative ring
- `rank` – a nonnegative integer
- `inner_product_matrix` – the inner product matrix
- `sparse` – bool; (default False)
- `inner_product_ring` – the inner product codomain ring; (default None)

**OUTPUT:**
A free quadratic module (with given inner product matrix).

**Note:** In Sage it is the case that there is only one dense and one sparse free ambient quadratic module of rank $n$ over $R$ and given inner product matrix.

**EXAMPLES:**
```
sage: M2 = FreeQuadraticModule(ZZ, 2, inner_product_matrix=[1, 2, 3, 4])
sage: M2 is FreeQuadraticModule(ZZ, 2, inner_product_matrix=[1, 2, 3, 4])
True
sage: M2.inner_product_matrix()
[1 2]
[3 4]
sage: M3 = FreeModule(ZZ, 2, inner_product_matrix=[[1, 2], [3, 4]])
sage: M3 is M2
True
```

class sage.modules.free_quadratic_module.FreeQuadraticModule_ambient(base_ring, rank, inner_product_matrix, sparse=False)

Bases: sage.modules.free_module.FreeModule_ambient, sage.modules.free_quadratic_module.FreeQuadraticModule_generic

Ambient free module over a commutative ring.

class sage.modules.free_quadratic_module.FreeQuadraticModule_ambient_domain(base_ring, rank, inner_product_matrix, sparse=False)

Bases: sage.modules.free_module.FreeModule_ambient_domain, sage.modules.free_quadratic_module.FreeQuadraticModule_ambient

Ambient free quadratic module over an integral domain.

**ambient_vector_space()**

Returns the ambient vector space, which is this free module tensored with its fraction field.

**EXAMPLES:**
```
sage: M = ZZ^3; M.ambient_vector_space()
Vector space of dimension 3 over Rational Field
```

Chapter 28. Free quadratic modules
class sage.modules.free_quadratic_module.FreeQuadraticModule_ambient_field(base_field, dimension, inner_product_matrix, sparse=False)

Bases:
  sage.modules.free_module.FreeModule_ambient_field,
  sage.modules.free_quadratic_module.FreeQuadraticModule_generic_field,
  sage.modules.free_quadratic_module.FreeQuadraticModule_ambient_pid

Create the ambient vector space of given dimension over the given field.

INPUT:
  • base_field – a field
  • dimension – a non-negative integer
  • sparse – bool (default: False)

EXAMPLES:

```
sage: VectorSpace(QQ,3,inner_product_matrix=[[2,1,0],[1,2,0],[0,1,2]])
Ambient quadratic space of dimension 3 over Rational Field
Inner product matrix:
[2 1 0]
[1 2 0]
[0 1 2]
```

class sage.modules.free_quadratic_module.FreeQuadraticModule_ambient_pid(base_ring, rank, inner_product_matrix, sparse=False)

Bases:
  sage.modules.free_module.FreeModule_ambient_pid,
  sage.modules.free_quadratic_module.FreeQuadraticModule_generic_pid,
  sage.modules.free_quadratic_module.FreeQuadraticModule_ambient_domain

Ambient free quadratic module over a principal ideal domain.

class sage.modules.free_quadratic_module.FreeQuadraticModule_generic(base_ring, rank, degree, inner_product_matrix, sparse=False)

Bases: sage.modules.free_module.FreeModule_generic

Base class for all free quadratic modules.

Modules are ordered by inclusion in the same ambient space.

ambient_module()

Return the ambient module associated to this module.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: N = FreeModule(R,2)
sage: N.ambient_module()
Ambient free module of rank 2 over the integral domain Multivariate Polynomial Ring in x, y over Rational Field
```

```
sage: V = FreeModule(QQ, 4).span([[1,2,3,4], [1,0,0,0]]); V
Vector space of degree 4 and dimension 2 over Rational Field
Basis matrix:
(continues on next page)
determinant()

Return the determinant of this free module.

EXAMPLES:

```
sage: M = FreeModule(ZZ, 3, inner_product_matrix=1)
sage: M.determinant()
1
sage: N = M.span([[1,2,3]])
sage: N.determinant()
14
sage: P = M.span([[1,2,3], [1,1,1]])
sage: P.determinant()
6
```

discriminant()

Return the discriminant of this free module, defined to be \((-1)^r\) of the determinant, where \(r = n/2\) (n even) or \((n-1)/2\) (n odd) for a module of rank n.

EXAMPLES:

```
sage: M = FreeModule(ZZ, 3)
sage: M.discriminant()
1
sage: N = M.span([[1,2,3]])
sage: N.discriminant()
14
sage: P = M.span([[1,2,3], [1,1,1]])
sage: P.discriminant()
6
```

gram_matrix()

Return the gram matrix associated to this free module, defined to be \(G = B^T A B\), where A is the inner product matrix (induced from the ambient space), and B the basis matrix.

EXAMPLES:

```
sage: V = VectorSpace(QQ,4)
sage: u = V([1/2,1/2,1/2,1/2])
sage: v = V([0,1,1,0])
sage: w = V([0,0,1,1])
sage: M = span([u,v,w], ZZ)
sage: M.inner_product_matrix() == V.inner_product_matrix()
True
sage: L = M.submodule_with_basis([u,v,w])
sage: L.inner_product_matrix() == M.inner_product_matrix()
True
sage: L.gram_matrix()
[1 1 1]
```

(continues on next page)
inner_product_matrix()

Return the inner product matrix associated to this module. By definition this is the inner product matrix of the ambient space, hence may be of degree greater than the rank of the module.

N.B. The inner product does not have to be symmetric (see examples).

TODO: Differentiate the image ring of the inner product from the base ring of the module and/or ambient space. E.g. On an integral module over ZZ the inner product pairing could naturally take values in ZZ, QQ, RR, or CC.

EXAMPLES:

```
sage: N = FreeModule(ZZ, 3)
sage: N.inner_product_matrix()
[1 0 0]
[0 1 0]
[0 0 1]
```

The inner product does not have to be symmetric or definite:

```
sage: N = FreeModule(ZZ, 2, inner_product_matrix=[[1,-1],[2,5]])
sage: N.inner_product_matrix()
[ 1 -1]
[ 2  5]
sage: u, v = N.basis()
sage: u.inner_product(v)
-1
sage: v.inner_product(u)
 2
```

The inner product matrix is defined with respect to the ambient space.

```
sage: V = QQ^3 sage: u = V([1/2,1,1]) sage: v = V([1,1,1/2]) sage: M = span([u,v], ZZ)
sage: M.inner_product_matrix() == V.inner_product_matrix() True sage: M.gram_matrix() 
[ 1/2 -3/4]
[-3/4 13/4]
```

class sage.modules.free_quadratic_module.FreeQuadraticModule_generic_field(base_field, dimension, degree, inner_product_matrix, sparse=False)

Base class for all free modules over fields.

span(gens, check=True, already_echelonized=False)

Return the K-span of the given list of gens, where K is the base field of self. Note that this span is a subspace of the ambient vector space, but need not be a subspace of self.

INPUT:

- gens – list of vectors
- check – bool (default: True): whether or not to coerce entries of gens into base field
• already_echelonized – bool (default: False): set this if you know the gens are already in echelon form

EXAMPLES:

```python
sage: V = VectorSpace(GF(7), 3)
sage: W = V.subspace([[2,3,4]]); W
Vector space of degree 3 and dimension 1 over Finite Field of size 7
Basis matrix:
[1 5 2]
sage: W.span([[1,1,1]])
Vector space of degree 3 and dimension 1 over Finite Field of size 7
Basis matrix:
[1 1 1]
```

**span_of_basis**(basis, check=True, already_echelonized=False)

Return the free $K$-module with the given basis, where $K$ is the base field of self. Note that this span is a subspace of the ambient vector space, but need not be a subspace of self.

INPUT:

• basis – list of vectors

• check – bool (default: True): whether or not to coerce entries of gens into base field

• already_echelonized – bool (default: False): set this if you know the gens are already in echelon form

EXAMPLES:

```python
sage: V = VectorSpace(GF(7), 3)
sage: W = V.subspace([[2,3,4]]); W
Vector space of degree 3 and dimension 1 over Finite Field of size 7
Basis matrix:
[1 5 2]
sage: W.span_of_basis([[2,2,2], [3,3,0]])
Vector space of degree 3 and dimension 2 over Finite Field of size 7
User basis matrix:
[2 2 2]
[3 3 0]
```

The basis vectors must be linearly independent or a `ValueError` exception is raised:

```python
sage: W.span_of_basis([[2,2,2], [3,3,3]])
Traceback (most recent call last):
  ...
ValueError: The given basis vectors must be linearly independent.
```

class `sage.modules.free_quadratic_module.FreeQuadraticModule_generic_pid`(base_ring, rank, degree, inner_product_matrix, sparse=False)

Class of all free modules over a PID.

**span**(gens, check=True, already_echelonized=False)

Return the R-span of the given list of gens, where R is the base ring of self. Note that this span need not be a submodule of self, nor even of the ambient space. It must, however, be contained in the ambient vector space, i.e., the ambient space tensored with the fraction field of R.
EXAMPLES:

```python
sage: V = FreeModule(ZZ, 3)
sage: W = V.submodule([V.gen(0)])
sage: W.span([V.gen(1)])
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[0 1 0]
sage: W.submodule([V.gen(1)])
Traceback (most recent call last):
...
ArithmeticError: Argument gens (= [(0, 1, 0)]) does not generate a submodule of...
˓→self.
```

**span_of_basis** *(basis, check=True, already_echelonized=False)*  
Return the free $R$-module with the given basis, where $R$ is the base ring of self. Note that this $R$-module need not be a submodule of self, nor even of the ambient space. It must, however, be contained in the ambient vector space, i.e., the ambient space tensored with the fraction field of $R$.

EXAMPLES:

```python
sage: M = FreeModule(ZZ, 3)
sage: W = M.span_of_basis([M([1,2,3])])
Next we create two free $\mathbb{Z}$-modules, neither of which is a submodule of $W$:

```python
sage: W.span_of_basis([M([2,4,0])])
Free module of degree 3 and rank 1 over Integer Ring
User basis matrix:
[2 4 0]
The following module is not even in the ambient space:

```python
sage: Q = QQ
sage: W.span_of_basis([ Q([1/5])*M([1,2,0]), Q([1/7])*M([1,1,0]) ])  
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[1/5 2/5 0]
[1/7 1/7 0]
Of course the input basis vectors must be linearly independent:

```python
sage: W.span_of_basis([ [1,2,0], [2,4,0] ])  
Traceback (most recent call last):
...
ValueError: The given basis vectors must be linearly independent.
```

**zero_submodule()**  
Return the zero submodule of this module.

EXAMPLES:

```python
sage: V = FreeModule(ZZ, 2)
sage: V.zero_submodule()
Free module of degree 2 and rank 0 over Integer Ring
```
Echelon basis matrix:
[]

class sage.modules.free_quadratic_module.FreeQuadraticModule_submodule_field(ambient, gens, inner_product_matrix, check=True, already_echelonized=False)

Bases: sage.modules.free_module.FreeModule_submodule_field, free_quadratic_module.FreeQuadraticModule_submodule_with_basis_field

An embedded vector subspace with echelonized basis.

EXAMPLES:
Since this is an embedded vector subspace with echelonized basis, the echelon_coordinates() and user_coordinates() agree:

```
sage: V = QQ^3
sage: W = V.span([[1,2,3],[4,5,6]])
sage: W
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1 0 -1]
[ 0 1 2]
sage: v = V([1,5,9])
sage: W.echelon_coordinates(v)
[1, 5]
sage: vector(QQ, W.echelon_coordinates(v)) * W.basis_matrix()
(1, 5, 9)
sage: v = V([1,5,9])
sage: W.coordinates(v)
[1, 5]
sage: vector(QQ, W.coordinates(v)) * W.basis_matrix()
(1, 5, 9)
```

class sage.modules.free_quadratic_module.FreeQuadraticModule_submodule_pid(ambient, gens, inner_product_matrix, check=True, already_echelonized=False)

Bases: sage.modules.free_module.FreeModule_submodule_pid, free_quadratic_module.FreeQuadraticModule_submodule_with_basis_pid

An $\mathcal{R}$-submodule of $\mathcal{K}^n$ where $\mathcal{K}$ is the fraction field of a principal ideal domain $\mathcal{R}$.

EXAMPLES:

```
sage: M = ZZ^3
sage: W = M.span_of_basis([[1,2,3],[4,5,19]]); W
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[ 1 2 3]
[ 4 5 19]`
We can save and load submodules and elements:

```python
sage: loads(W.dumps()) == W
True
sage: v = W.0 + W.1
sage: loads(v.dumps()) == v
True
```

class `sage.modules.free_quadratic_module.FreeQuadraticModule_submodule_with_basis_field`

```python
(ambient, basis, inner_product_matrix, check=True, echelonized=False, echelonized_basis=None, already_echelonized=None)
```


An embedded vector subspace with a distinguished user basis.

**EXAMPLES:**

```python
sage: M = QQ^3; W = M.submodule_with_basis([[1,2,3], [4,5,19]]); W
Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[ 1 2 3]
[ 4 5 19]
```

Since this is an embedded vector subspace with a distinguished user basis possibly different than the echelonized basis, the `echelon_coordinates()` and `user_coordinates()` do not agree:

```python
sage: V = QQ^3
sage: W = V.submodule_with_basis([[1,2,3], [4,5,6]])
```

```python
sage: W
Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[1 2 3]
[4 5 6]
```

```python
sage: v = V([1,5,9])
sage: W.echelon_coordinates(v)
[1, 5]
sage: vector(QQ, W.echelon_coordinates(v)) * W.echelonized_basis_matrix()
```

(continues on next page)
We can load and save submodules:

```python
sage: loads(W.dumps()) == W
True
sage: K.<x> = FractionField(PolynomialRing(QQ, 'x'))
```

The class `sage.modules.free_quadratic_module.FreeQuadraticModule_submodule_with_basis_pid` is defined as follows:

```python
class sage.modules.free_quadratic_module.FreeQuadraticModule_submodule_with_basis_pid(ambient, basis, inner_product_matrix, check=True, echelonize=False, echelonized_basis=None, already_echelonized=False)
```

Bases: `sage.modules.free_module.FreeModule_submodule_with_basis_pid`, `sage.modules.free_quadratic_module.FreeQuadraticModule_generic_pid`

An $R$-submodule of $K^n$ with distinguished basis, where $K$ is the fraction field of a principal ideal domain $R$. Modules are ordered by inclusion.

**EXAMPLES:**

First we compare two equal vector spaces:

```python
sage: A = FreeQuadraticModule(QQ,3,2*matrix.identity(3))
sage: V = A.span([[1,2,3], [5,6,7], [8,9,10]])
sage: W = A.span([[5,6,7], [8,9,10]])
sage: V == W
True
```

Next we compare a one dimensional space to the two dimensional space defined above:

```python
sage: M = A.span([[5,6,7]])
sage: V == M
False
sage: M < V
(continues on next page)
We compare a $\mathbb{Z}$-module to the one-dimensional space above:

```
sage: V = A.span([[5,6,7]])
sage: V = V.change_ring(ZZ).scale(1/11)
sage: V < M
True
sage: M < V
False
```

`change_ring(R)`
Return the free module over $R$ obtained by coercing each element of self into a vector over the fraction field of $R$, then taking the resulting $R$-module. Raises a TypeError if coercion is not possible.

**INPUT:**
- $R$ – a principal ideal domain

**EXAMPLES:**
Changing rings preserves the inner product and the user basis:

```
sage: V = QQ^3
sage: W = V.subspace([[2, '1/2', 1]]); W
Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[ 1 1/4 1/2]
sage: W.change_ring(GF(7))
Vector space of degree 3 and dimension 1 over Finite Field of size 7
Basis matrix:
[1 2 4]
sage: N = FreeModule(ZZ, 2, inner_product_matrix=[[1,-1],[2,5]])
sage: N.inner_product_matrix()
[ 1 -1]
[ 2  5]
sage: Np = N.change_ring(RDF)
sage: Np.inner_product_matrix()
[ 1.0 -1.0]
[ 2.0  5.0]
```

`sage.modules.free_quadratic_module.InnerProductSpace(K, dimension, inner_product_matrix, sparse=False)`

**EXAMPLES:**
The base can be complicated, as long as it is a field:

```
sage: F.<x> = FractionField(PolynomialRing(ZZ,'x'))
sage: D = diagonal_matrix([x,x-1,x+1])
sage: V = QuadraticSpace(F,3,D)
sage: V
Ambient quadratic space of dimension 3 over Fraction Field of Univariate Polynomial Ring in x over Integer Ring
```

(continues on next page)
Inner product matrix:
\[
\begin{bmatrix}
  x & 0 & 0 \\
  0 & x - 1 & 0 \\
  0 & 0 & x + 1
\end{bmatrix}
\]

```
sage: V.basis()
[
(1, 0, 0),
(0, 1, 0),
(0, 0, 1)
]
```

The base must be a field or a TypeError is raised:

```
sage: QuadraticSpace(ZZ,5,identity_matrix(ZZ,2))
Traceback (most recent call last):
...
TypeError: Argument K (= Integer Ring) must be a field.
```

```
sage.modules.free_quadratic_module.QuadraticSpace(K, dimension, inner_product_matrix, sparse=False)
```

EXAMPLES:

The base can be complicated, as long as it is a field:

```
sage: F.<x> = FractionField(PolynomialRing(ZZ,'x'))
sage: D = diagonal_matrix([x,x-1,x+1])
sage: V = QuadraticSpace(F,3,D)
sage: V
Ambient quadratic space of dimension 3 over Fraction Field of Univariate Polynomial Ring in x over Integer Ring
Inner product matrix:
\[
\begin{bmatrix}
  x & 0 & 0 \\
  0 & x - 1 & 0 \\
  0 & 0 & x + 1
\end{bmatrix}
\]

```

```
sage: V.basis()
[
(1, 0, 0),
(0, 1, 0),
(0, 0, 1)
]
```

The base must be a field or a TypeError is raised:

```
sage: QuadraticSpace(ZZ,5,identity_matrix(ZZ,2))
Traceback (most recent call last):
...
TypeError: Argument K (= Integer Ring) must be a field.
```

```
sage.modules.free_quadratic_module.is_FreeQuadraticModule(M)
```

Return True if $M$ is a free quadratic module.

EXAMPLES:
```python
sage: from sage.modules.free_quadratic_module import is_FreeQuadraticModule
sage: U = FreeModule(QQ,3)
sage: is_FreeQuadraticModule(U)
False
sage: V = FreeModule(QQ,3,inner_product_matrix=diagonal_matrix([1,1,1]))
sage: is_FreeQuadraticModule(V)
True
sage: W = FreeModule(QQ,3,inner_product_matrix=diagonal_matrix([2,3,3]))
sage: is_FreeQuadraticModule(W)
True
```
An integral lattice is a finitely generated free abelian group \( L \cong \mathbb{Z}^r \) equipped with a non-degenerate, symmetric bilinear form \( L \times L: \rightarrow \mathbb{Z} \).

Here, lattices have an ambient quadratic space \( Q^n \) and a distinguished basis.

**EXAMPLES:**

```sage
sage: M = Matrix(ZZ, [[0,1], [1,0]])
sage: IntegralLattice(M)
Lattice of degree 2 and rank 2 over Integer Ring
Standard basis
Inner product matrix:
[0 1]
[1 0]
```

A lattice can be defined by an inner product matrix of the ambient space and a basis:

```sage
sage: G = matrix.identity(3)
sage: basis = [[1,-1,0], [0,1,-1]]
sage: L = IntegralLattice(G, basis)
sage: L
Lattice of degree 3 and rank 2 over Integer Ring
Basis matrix:
[ 1 -1 0]
[ 0 1 -1]
Standard scalar product
```

**AUTHORS:**

- Simon Brandhorst (2017-09): First created
- Paolo Menegatti (2018-03): Added `IntegralLatticeDirectSum`, `IntegralLatticeGluing`
class sage.modules.free_quadratic_module_integer_symmetric.FreeQuadraticModule_integer_symmetric(ambient, basis, inner_product_matrix, check=True, already_echelonized=False)

Bases: sage.modules.free_quadratic_module.FreeQuadraticModule_submodule_with_basis_pid

This class represents non-degenerate, integral, symmetric free quadratic \( \mathbb{Z} \)-modules.

INPUT:

- ambient – an ambient free quadratic module
- basis – a list of elements of ambient or a matrix
- inner_product_matrix – a symmetric matrix over the rationals

EXAMPLES:

```python
sage: IntegralLattice("U",basis=[vector([1,1])])
Lattice of degree 2 and rank 1 over Integer Ring
Basis matrix:
[1 1]
Inner product matrix:
[0 1]
[1 0]
```

LLL()

Return this lattice with an LLL reduced basis.

EXAMPLES:

```python
sage: L = IntegralLattice('A2')
sage: L.lll() == L
True
sage: G = matrix(ZZ,3,[0,1,0, 1,0,0, 0,0,7])
sage: V = matrix(ZZ,3,[-14,-15,-15, -4,1,16, -5,-5,-4])
sage: L = IntegralLattice(V * G * V.T)
sage: L.lll().gram_matrix()
[0 0 1]
[0 7 0]
[1 0 0]
```

automorphisms(gens=None, is_finite=None)

Return the orthogonal group of this lattice as a matrix group.

The elements are isometries of the ambient vector space which preserve this lattice. They are represented by matrices with respect to the standard basis.

INPUT:

- gens – a list of matrices (default: None)
- is_finite – bool (default: None) If set to True, then the group is placed in the category of finite groups. Sage does not check this.

OUTPUT:
The matrix group generated by gens. If gens is not specified, then generators of the full orthogonal group of this lattice are computed. They are continued as the identity on the orthogonal complement of the lattice in its ambient space. Currently, we can only compute the orthogonal group for positive definite lattices.

EXAMPLES:

```
sage: A4 = IntegralLattice("A4")
sage: Aut = A4.orthogonal_group()
sage: Aut
Group of isometries with 5 generators (  
[-1 0 0 0] [0 0 0 1] [-1 -1 -1 0] [ 1 0 0 0] [ 1 0 0 0]  
[ 0 0 1 0] [0 0 0 -1] [-1 -1 -1 -1] [0 1 0 0] [ 0 1 0 0]  
[0 0 -1 0] [0 1 0 0] [0 0 1 1] [0 0 0 1] [0 0 1 1]  
[ 0 0 0 -1], [1 0 0 0], [0 1 0 0], [0 0 1 0], [0 0 0 -1]  
)

The group acts from the right on the lattice and its discriminant group:

```
sage: x = A4.an_element()
sage: g = Aut.an_element()
sage: g
[ 1 1 1 0]  
[ 0 0 -1 0]  
[ 0 0 1 1]  
[ 0 -1 -1 -1]
sage: x*g  
(1, 1, 1, 0)
sage: (x*g).parent()==A4  
True
sage: (g*x).parent()  
Vector space of dimension 4 over Rational Field
sage: y = A4.discriminant_group().an_element()
sage: y*g  
(1)
```

If the group is finite we can compute the usual things:

```
sage: Aut.order()  
240
sage: conj = Aut.conjugacy_classes_representatives()
sage: len(conj)  
14
sage: Aut.structure_description()  
'C2 x S5'
```

The lattice can live in a larger ambient space:

```
sage: A2 = IntegralLattice(matrix.identity(3),Matrix(ZZ,2,3,[1,-1,0,0,1,-1]))
sage: A2.orthogonal_group()
Group of isometries with 3 generators (  
[-1/3 2/3 2/3] [2/3 2/3 -1/3] [1 0 0]  
[2/3 -1/3 2/3] [2/3 -1/3 2/3] [0 0 1]  
[2/3 2/3 -1/3], [-1/3 2/3 2/3], [0 1 0]  
)
It can be negative definite as well:

```
sage: A2m = IntegralLattice(-Matrix(ZZ,2,[[2,1],[1,2]]))
sage: G = A2m.orthogonal_group()
sage: G.order()
12
```

If the lattice is indefinite, sage does not know how to compute generators. Can you teach it?

```
sage: U = IntegralLattice(Matrix(ZZ,2,[[0,1],[1,0]]))
sage: U.orthogonal_group()
Traceback (most recent call last):
...  
NotImplementedError: currently, we can only compute generators for orthogonal groups over definite lattices.
```

But we can define subgroups:

```
sage: S = IntegralLattice(Matrix(ZZ,2,[[2,3],[3,2]]))
sage: f = Matrix(ZZ,2,[[0,1],[-1,3]])
sage: S.orthogonal_group([f])
Group of isometries with 1 generator ( 
 [ 0 1]
[-1 3]
)
```

def direct_sum(M)

Return the direct sum of this lattice with M.

INPUT:

• M -- a module over \( \mathbb{Z} \)

EXAMPLES:

```
sage: A = IntegralLattice(1)
sage: A.direct_sum(A)
Lattice of degree 2 and rank 2 over Integer Ring
Standard basis
Standard scalar product
```

def discriminant_group(s=0)

Return the discriminant group \( L^\vee/L \) of this lattice.

INPUT:

• s -- an integer (default: 0)

OUTPUT:

The \( s \) primary part of the discriminant group. If \( s = 0 \), returns the whole discriminant group.

EXAMPLES:

```
sage: L = IntegralLattice(Matrix(ZZ,2,2,[[2,1,1,-2]]))
sage: L.discriminant_group()
Finite quadratic module over Integer Ring with invariants (2, 10)
Gram matrix of the quadratic form with values in \( \mathbb{Q}/2\mathbb{Z} \):
```

(continues on next page)
sage: L.discriminant_group(2)
Finite quadratic module over Integer Ring with invariants (2, 2)
Gram matrix of the quadratic form with values in $\mathbb{Q}/2\mathbb{Z}$:

\[
\begin{bmatrix}
1 & 1/2 \\
1/2 & 1
\end{bmatrix}
\]

sage: L.discriminant_group(5)
Finite quadratic module over Integer Ring with invariants (5,)
Gram matrix of the quadratic form with values in $\mathbb{Q}/2\mathbb{Z}$:

\[
\begin{bmatrix}
4/5 \\
0 & 1
\end{bmatrix}
\]

dual_lattice()

Return the dual lattice as a FreeQuadraticModule

Let $L$ be a lattice. Its dual lattice is

$$L^\vee = \{x \in L \otimes \mathbb{Q} : (x, l) \in \mathbb{Z} \forall l \in L\}.$$

EXAMPLES:

sage: L = IntegralLattice("A2")
sage: Ldual = L.dual_lattice()
sage: Ldual
Free module of degree 2 and rank 2 over Integer Ring
Echelon basis matrix:

\[
\begin{bmatrix}
1/3 & 2/3 \\
0 & 1
\end{bmatrix}
\]

Since our lattices are always integral, a lattice is contained in its dual:

sage: L.is_submodule(Ldual)
True

genus()

Return the genus of this lattice.

EXAMPLES:

sage: L = IntegralLattice("U")
sage: L.genus()
Genus of

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

Signature: (1, 1)
Genus symbol at 2: $1^2$

is_even()

Return whether the diagonal entries of the Gram matrix are even.

EXAMPLES:

sage: G = Matrix(ZZ, 2, 2, [-1, 1, 1, 2])
sage: L = IntegralLattice(G)
is_primitive(M)
Return whether \( M \) is a primitive submodule of this lattice.

A \( \mathbb{Z} \)-submodule \( M \) of a \( \mathbb{Z} \)-module \( L \) is called primitive if the quotient \( L/M \) is torsion free.

INPUT:

* \( M \) – a submodule of this lattice

EXAMPLES:

```python
sage: U = IntegralLattice("U")
sage: L1 = U.span([vector([1,1])])
sage: L2 = U.span([vector([1,-1])])
sage: U.is_primitive(L1)
True
sage: U.is_primitive(L2)
True
sage: U.is_primitive(L1+L2)
False
```

We can also compute the index:

```python
sage: (L1+L2).index_in(U)
2
```

lll()
Return this lattice with an LLL reduced basis.

EXAMPLES:

```python
sage: L = IntegralLattice('A2')
sage: L.lll() == L
True
sage: G = matrix(ZZ,3,[-14,15,-15, -4,1,16, -5,-5,-4])
sage: V = matrix(ZZ,3,[-14,-15,-15, -4,1,16, -5,-5,-4])
sage: L = IntegralLattice(V * G * V.T)
sage: L.lll().gram_matrix()
[0 0 1]
[0 7 0]
[1 0 0]
```

max()
Return the maximum of this lattice.

\[
\text{max}\{x^2 | x \in L \setminus \{0\}\}
\]

EXAMPLES:
sage: L = IntegralLattice('A2')
sage: L.maximum()
+Infinity
sage: L.twist(-1).maximum()
-2

**maximal_overlattice** *(p=None)*
Return a maximal even integral overlattice of this lattice.

**INPUT:**

- *p* – (default: None) if given return an overlattice *M* of this lattice *L* that is maximal at *p* and the completions *M*ₚ = *L*ₚ are equal for all primes *q* ≠ *p*.

If *p* is 2 or None, then the lattice must be even.

**EXAMPLES:**

```
sage: L = IntegralLattice("A4").twist(25*89)
sage: L.maximal_overlattice().determinant()
5
sage: L.maximal_overlattice(89).determinant().factor()
5^9
sage: L.maximal_overlattice(5).determinant().factor()
5 * 89^4
```

**maximum**
Return the maximum of this lattice.

\[
\max \{ x^2 | x \in L \setminus \{0\} \}
\]

**EXAMPLES:**

```
sage: L = IntegralLattice('A2')
sage: L.maximum()
+Infinity
sage: L.twist(-1).maximum()
-2
```

**min**
Return the minimum of this lattice.

\[
\min \{ x^2 | x \in L \setminus \{0\} \}
\]

**EXAMPLES:**

```
sage: L = IntegralLattice('A2')
sage: L.minimum()
2
sage: L.twist(-1).minimum()
-Infinity
```

**minimum**
Return the minimum of this lattice.

\[
\min \{ x^2 | x \in L \setminus \{0\} \}
\]

**EXAMPLES:**
orthogonal_complement($M$)
Return the orthogonal complement of $M$ in this lattice.

INPUT:

- $M$ – a module in the same ambient space or a list of elements of the ambient space

EXAMPLES:

```python
sage: H5 = Matrix(ZZ,2,[2,1,1,-2])
sage: L = IntegralLattice(H5)
sage: S = L.span([vector([1,1])])
sage: L.orthogonal_complement(S)
Lattice of degree 2 and rank 1 over Integer Ring
Basis matrix:
[1 3]
Inner product matrix:
[2 1]
[1 -2]
```

```python
sage: L = IntegralLattice(2)
sage: L.orthogonal_complement([vector(ZZ,[1,0])])
Lattice of degree 2 and rank 1 over Integer Ring
Basis matrix:
[0 1]
Standard scalar product
```

orthogonal_group($gens$=None, $is_finite$=None)
Return the orthogonal group of this lattice as a matrix group.

The elements are isometries of the ambient vector space which preserve this lattice. They are represented by matrices with respect to the standard basis.

INPUT:

- $gens$ – a list of matrices (default:None)
  
- $is_finite$ – bool (default: None) If set to True, then the group is placed in the category of finite groups. Sage does not check this.

OUTPUT:

The matrix group generated by $gens$. If $gens$ is not specified, then generators of the full orthogonal group of this lattice are computed. They are continued as the identity on the orthogonal complement of the lattice in its ambient space. Currently, we can only compute the orthogonal group for positive definite lattices.

EXAMPLES:

```python
sage: A4 = IntegralLattice("A4")
sage: Aut = A4.orthogonal_group()
sage: Aut
Group of isometries with 5 generators ( (continues on next page)```
The group acts from the right on the lattice and its discriminant group:

\[
\begin{align*}
[&-1 & 0 & 0 & 0] & [0 & 0 & 0 & 1] & [-1 & -1 & -1 & 0] & [1 & 0 & 0 & 0] & [1 & 0 & 0 & 0] \\
[&0 & -1 & 0 & 0] & [0 & 1 & 0] & [0 & 0 & 0 & -1] & [-1 & -1 & -1 & 1] & [0 & 1 & 0 & 0] \\
[&0 & 0 & -1 & 0] & [0 & 1 & 0] & [0 & 0 & 1 & 1] & [0 & 0 & 0 & 1] & [0 & 0 & 1 & 1] \\
[&0 & 0 & 0 & -1], & [1 & 0 & 0 & 0], & [0 & 1 & 0 & 0], & [0 & 0 & 1 & 0], & [0 & 0 & 0 & -1]
\end{align*}
\]

\[
\begin{align*}
\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{pmatrix}
\end{align*}
\]

If the group is finite we can compute the usual things:

\[
\begin{align*}
\text{sage: } & \text{A4.an_element()} \\
\text{sage: } & \text{Aut.an_element()} \\
\text{sage: } & \text{g} \\
& \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{pmatrix} \\
\text{sage: } & \text{x*g} \\
& (1, 1, 1, 0) \\
\text{sage: } & \text{(x*g).parent() == A4} \\
& \text{True} \\
\text{sage: } & \text{(g*x).parent()} \\
& \text{Vector space of dimension 4 over Rational Field} \\
\text{sage: } & \text{y = A4.discriminant_group().an_element()} \\
\text{sage: } & \text{y*g} \\
& (1)
\end{align*}
\]

The lattice can live in a larger ambient space:

\[
\begin{align*}
\text{sage: } & \text{A2 = IntegralLattice(matrix.identity(3), Matrix(ZZ,2,3,[1,-1,0,0,1,-1]))} \\
\text{sage: } & \text{A2.orthogonal_group()} \\
\end{align*}
\]

If the lattice is indefinite, sage does not know how to compute generators. Can you teach it?:
sage: U = IntegralLattice(Matrix(ZZ,2,\([0,1,1,0]\)))
sage: U.orthogonal_group()
Traceback (most recent call last):
...
NotImplementedError: currently, we can only compute generators for orthogonal→groups over definite lattices.

But we can define subgroups:

sage: S = IntegralLattice(Matrix(ZZ,2,\([2, 3, 3, 2]\)))
sage: f = Matrix(ZZ,2,\([0,1,-1,3]\))
sage: S.orthogonal_group([f])
Group of isometries with 1 generator (\[
\begin{bmatrix}
0 & 1 \\
-1 & 3
\end{bmatrix}
\])

overlattice(gens)
Return the lattice spanned by this lattice and gens.

INPUT:

• gens – a list of elements or a rational matrix

EXAMPLES:

sage: L = IntegralLattice(Matrix(ZZ,2,2,\([2,0,0,2]\)))
sage: M = L.overlattice([vector([1,1])/2])
sage: M.gram_matrix()
\[
\begin{bmatrix}
1 & 1 \\
1 & 2
\end{bmatrix}
\]

quadratic_form()
Return the quadratic form given by \(q(x) = (x, x)\).

EXAMPLES:

sage: L = IntegralLattice("A2")
sage: q = L.quadratic_form()
sage: q
Quadratic form in 2 variables over Integer Ring with coefficients:
\[
\begin{bmatrix}
2 & -2 \\
* & 2
\end{bmatrix}
\]

short_vectors(n, **kwargs)
Return the short vectors of length < \(n\).

INPUT:

• n – an integer

• further key word arguments are passed on to \texttt{sage.quadratic_forms.short_vector_list_up_to_length()}.

OUTPUT:

• a list \(L\) where \(L[k]\) is the list of vectors of lengths \(k\)

EXAMPLES:
sage: A2 = IntegralLattice('A2')
sage: A2.short_vectors(3)
[[0, 0], [], [(1, 1), (-1, -1), (0, 1), (0, -1), (1, 0), (-1, 0)]]
sage: A2.short_vectors(3,up_to_sign_flag=True)
[[0, 0], [], [(1, 1), (0, 1), (1, 0)]]

signature()
Return the signature of this lattice, which is defined as the difference between the number of positive eigenvalues and the number of negative eigenvalues in the Gram matrix.

EXAMPLES:

sage: U = IntegralLattice("U")
sage: U.signature()
0

signature_pair()
Return the signature tuple \((n_+, n_-)\) of this lattice.
Here \(n_+\) (resp. \(n_-\)) is the number of positive (resp. negative) eigenvalues of the Gram matrix.

EXAMPLES:

sage: A2 = IntegralLattice("A2")
sage: A2.signature_pair()
(2, 0)

sublattice(basis)
Return the sublattice spanned by \(basis\).

INPUT:

* basis – A list of elements of this lattice.

EXAMPLES:

sage: U = IntegralLattice("U")
sage: S = U.sublattice([vector([1,1])])
sage: S
Lattice of degree 2 and rank 1 over Integer Ring
Basis matrix:
[1 1]
Inner product matrix:
[0 1]
[1 0]
sage: U.sublattice([vector([1,-1])/2])
Traceback (most recent call last):
... ValueError: lattices must be integral; use FreeQuadraticModule instead
sage: S.sublattice([vector([1,-1])])
Traceback (most recent call last):
... ValueError: the basis (= [(1, -1)]) does not span a submodule

tensor_product(other, discard_basis=False)
Return the tensor product of \(self\) and \(other\).

INPUT:
• **other** – an integral lattice

• **discard_basis** – a boolean (default: False). If True, then the lattice returned is equipped with the standard basis.

**EXAMPLES:**

```python
sage: L = IntegralLattice("D3", [[1, -1, 0], [0, 1, -1]])
sage: L1 = L.tensor_product(L)
sage: L2 = L.tensor_product(L, True)
sage: L1
Lattice of degree 9 and rank 4 over Integer Ring
Basis matrix:
[ 1 -1 0 -1 1 0 0 0 0]
[ 0 1 -1 0 -1 1 0 0 0]
[ 0 0 1 -1 0 -1 1 0]
[ 0 0 0 1 -1 0 -1 1]
Inner product matrix:
[ 4 -2 -2 -2 1 1 -2 1 1]
[-2 4 0 1 -2 0 1 -2 0]
[-2 0 4 1 -2 1 0 -2]
[-2 1 1 4 -2 -2 0 0 0]
[ 1 -2 0 -2 4 0 0 0 0]
[ 1 0 -2 -2 0 4 0 0 0]
[-2 1 1 0 0 0 4 -2 -2]
[ 1 -2 0 0 0 -2 4 0]
[ 1 0 -2 0 0 0 -2 0 4]
sage: L1.gram_matrix()
[ 36 -12 -12 4]
[-12 24 4 -8]
[-12 4 24 -8]
[ 4 -8 -8 16]
sage: L2
Lattice of degree 4 and rank 4 over Integer Ring
Standard basis
Inner product matrix:
[ 36 -12 -12 4]
[-12 24 4 -8]
[-12 4 24 -8]
[ 4 -8 -8 16]
```

**twist**(s, **discard_basis**=False)

Return the lattice with inner product matrix scaled by s.

**INPUT:**

• s – a nonzero integer

• discard_basis – a boolean (default: False). If True, then the lattice returned is equipped with the standard basis.

**EXAMPLES:**

```python
sage: L = IntegralLattice("A4")
sage: L.twist(3)
Lattice of degree 4 and rank 4 over Integer Ring
Standard basis
```
Inner product matrix:
\[
\begin{bmatrix}
6 & -3 & 0 & 0 \\
-3 & 6 & -3 & 0 \\
0 & -3 & 6 & -3 \\
0 & 0 & -3 & 6 \\
\end{bmatrix}
\]

```
sage: L = IntegralLattice([2,1,0],[0,1,1])
sage: L
Lattice of degree 3 and rank 2 over Integer Ring
Basis matrix:
[2 1 0]
[0 1 1]
Standard scalar product
sage: L.twist(1)
Lattice of degree 3 and rank 2 over Integer Ring
Basis matrix:
[2 1 0]
[0 1 1]
Standard scalar product
sage: L.twist(1, True)
Lattice of degree 2 and rank 2 over Integer Ring
Standard basis
Inner product matrix:
[5 1]
[1 2]
```

```
sage.modules.free_quadratic_module_integer_symmetric.IntegralLattice(data, basis=None)

Return the integral lattice spanned by basis in the ambient space.

A lattice is a finitely generated free abelian group $L \cong \mathbb{Z}^r$ equipped with a non-degenerate, symmetric bilinear form $L \times L: \rightarrow \mathbb{Z}$. Here, lattices have an ambient quadratic space $\mathbb{Q}^n$ and a distinguished basis.

INPUT:

The input is a descriptor of the lattice and a (optional) basis. - data – can be one of the following:

- a symmetric matrix over the rationals – the inner product matrix
- an integer – the dimension for an Euclidean lattice
- a symmetric Cartan type or anything recognized by CartanMatrix (see also Cartan types) – for a root lattice
- the string "U" or "H" – for hyperbolic lattices
- basis – (optional) a matrix whose rows form a basis of the lattice, or a list of module elements forming a basis

OUTPUT:

A lattice in the ambient space defined by the inner_product_matrix. Unless specified, the basis of the lattice is the standard basis.

EXAMPLES:

```
sage: H5 = Matrix(ZZ, 2, [2,1,1,-2])
sage: IntegralLattice(H5)
Lattice of degree 2 and rank 2 over Integer Ring
```

(continues on next page)
A basis can be specified too:

\begin{verbatim}
sage: IntegralLattice(H5, Matrix([[1,1]]))
Lattice of degree 2 and rank 1 over Integer Ring
Basis matrix:
[1 1]
Inner product matrix:
[2 1]
[1 -2]
\end{verbatim}

We can define an Euclidean lattice just by its dimension:

\begin{verbatim}
sage: IntegralLattice(3)
Lattice of degree 3 and rank 3 over Integer Ring
Standard basis
Standard scalar product
\end{verbatim}

Here is an example of the $A_2$ root lattice in Euclidean space:

\begin{verbatim}
sage: basis = Matrix([[1,-1,0], [0,1,-1]])
sage: A2 = IntegralLattice(3, basis)
sage: A2
Lattice of degree 3 and rank 2 over Integer Ring
Basis matrix:
[ 1 -1  0]
[ 0  1 -1]
Standard scalar product
sage: A2.gram_matrix()
[ 2 -1]
[-1  2]
\end{verbatim}

We use "U" or "H" for defining a hyperbolic lattice:

\begin{verbatim}
sage: L1 = IntegralLattice("U")
sage: L1
Lattice of degree 2 and rank 2 over Integer Ring
Standard basis
Inner product matrix:
[0 1]
[1 0]
sage: L1 == IntegralLattice("H")
True
\end{verbatim}

We can construct root lattices by specifying their type (see Cartan types and CartanMatrix):

\begin{verbatim}
sage: IntegralLattice(["E", 7])
Lattice of degree 7 and rank 7 over Integer Ring
Standard basis
\end{verbatim}
Inner product matrix:
\[
\begin{bmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2
\end{bmatrix}
\]

\textbf{sage}: \texttt{IntegralLattice(["A", 2])}
Lattice of degree 2 and rank 2 over Integer Ring
Standard basis
Inner product matrix:
\[
\begin{bmatrix}
2 & -1 \\
-1 & 2
\end{bmatrix}
\]
\textbf{sage}: \texttt{IntegralLattice("D3")}
Lattice of degree 3 and rank 3 over Integer Ring
Standard basis
Inner product matrix:
\[
\begin{bmatrix}
2 & -1 & -1 \\
-1 & 2 & 0 \\
-1 & 0 & 2
\end{bmatrix}
\]
\textbf{sage}: \texttt{IntegralLattice(["D", 4])}
Lattice of degree 4 and rank 4 over Integer Ring
Standard basis
Inner product matrix:
\[
\begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & -1 \\
0 & -1 & 2 & 0 \\
0 & -1 & 0 & 2
\end{bmatrix}
\]

We can specify a basis as well:

\textbf{sage}: \texttt{G = Matrix(ZZ, 2, [0,1,1,0])}
\textbf{sage}: \texttt{B = [vector([1,1])]}
\textbf{sage}: \texttt{IntegralLattice(G, basis=B)}
Lattice of degree 2 and rank 1 over Integer Ring
Basis matrix:
\[
\begin{bmatrix}
1 & 1 \\
0 & 1 \\
1 & 0
\end{bmatrix}
\]
Inner product matrix:
\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]
\textbf{sage}: \texttt{IntegralLattice(["A", 3], [[1,1,1]])}
Lattice of degree 3 and rank 1 over Integer Ring
Basis matrix:
\[
\begin{bmatrix}
1 & 1 & 1 \\
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{bmatrix}
\]
\textbf{sage}: \texttt{IntegralLattice(4, [[1,1,1,1]])}
Lattice of degree 4 and rank 1 over Integer Ring
Basis matrix:
Standard scalar product

```python
sage: IntegralLattice("A2", [[1,1]])
```

Lattice of degree 2 and rank 1 over Integer Ring
Basis matrix:

```
[1 1]
```
Inner product matrix:

```
[ 2 -1]
[-1 2]
```

sage.modules.free_quadratic_module_integer_symmetric.IntegralLatticeDirectSum(Lattices, return_embeddings=False)

Return the direct sum of the lattices contained in the list Lattices.

INPUT:

- Lattices – a list of lattices \([L_1, \ldots, L_n]\)
- return_embeddings – (default: False) a boolean

OUTPUT:

The direct sum of the \(L_i\) if return_embeddings is False or the tuple \([L, \phi]\) where \(L\) is the direct sum of \(L_i\) and \(\phi\) is the list of embeddings from \(L_i\) to \(L\).

EXAMPLES:

```python
sage: from sage.modules.free_quadratic_module_integer_symmetric import IntegralLatticeDirectSum
sage: L1 = IntegralLattice("D4")
```

```
[1 0 0 0 0 0 0 0 0 0 0]
[0 1 0 0 0 0 0 0 0 0 0]
[0 0 1 0 0 0 0 0 0 0 0]
[0 0 0 1 0 0 0 0 0 0 0]
[0 0 0 0 1 1 2 0 0 0 0]
[0 0 0 0 0 0 0 1 1 2]
[0 0 0 0 0 0 1 2 3 1]
```

Inner product matrix:

```
[ 2 -1 0 0 0 0 0 0 0 0 0]
[-1 2 -1 0 0 0 0 0 0 0 0]
[ 0 -1 0 0 0 0 0 0 0 0 0]
[ 0 -1 0 0 0 0 0 0 0 0 0]
[ 0 0 0 0 -2 -1 0 0 0 0 0]
[ 0 0 0 0 1 -2 -1 0 0 0 0]
[ 0 0 0 0 -1 2 -1 0 0 0 0]
[ 0 0 0 0 0 -1 2 -1 0 0 0]
[ 0 0 0 0 0 0 2 -1 0 0 0]
[ 0 0 0 0 0 0 0 2 -1 0 0]
[ 0 0 0 0 0 0 0 0 -1 2 -1]
[ 0 0 0 0 0 0 0 0 0 -1 2]
```

(continues on next page)
sage: [L, phi] = IntegralLatticeDirectSum([L1, L2, L3], True)
sage: LL3 = L.sublattice(phi[2].image().basis_matrix())
sage: L3.discriminant() == LL3.discriminant()
True
sage: x = L3([1, 2, 3, 1])
sage: phi[2](x).inner_product(phi[2](x)) == x.inner_product(x)
True

sage.modules.free_quadratic_module_integer_symmetric.IntegralLatticeGluing(Lattices, glue, return_embeddings=False)

Return an overlattice of the direct sum as defined by glue.

INPUT:

- Lattices – a list of lattices $[L_1, ..., L_n]$
- glue – a list where the elements are lists in the form $[g_1, ..., g_n]$; here $g_i$ is an element of the discriminant group of $L_i$
- return_embeddings – (default: False) a boolean

OUTPUT:

The glued lattice given by $L_i$ and $glue$ if return_embeddings is False or the tuple $[L, phi]$ where $L$ is the glued lattice and $phi$ the list of embeddings from $L_i$ to $L$

EXAMPLES:

A single lattice can be glued. This is the same as taking an overlattice:

sage: from sage.modules.free_quadratic_module_integer_symmetric import...
...
→IntegralLatticeGluing
sage: L1 = IntegralLattice(matrix([[4]]))
sage: g1 = L1.discriminant_group().gens()[0]
sage: glue = [[2 * g1]]
sage: L = IntegralLatticeGluing([L1], glue)
sage: L
Lattice of degree 1 and rank 1 over Integer Ring
Basis matrix:
[1/2]
Inner product matrix:
[4]
sage: L.gram_matrix()
[1]
sage: IntegralLatticeGluing([L1], glue, return_embeddings=True)
[Lattice of degree 1 and rank 1 over Integer Ring
Basis matrix:
[1/2]
Inner product matrix:
[4], [Free module morphism defined by the matrix
[2]
Domain: Lattice of degree 1 and rank 1 over Integer Ring
Standard basis
Inner product matrix:
[4]
Codomain: Lattice of degree 1 and rank 1 over Integer Ring

(continues on next page)
Basis matrix:
[[1/2]]
Inner product matrix:
[[4]]]

\[\text{sage: } L1 = \text{IntegralLattice}([[2]])\]
\[\text{sage: } L2 = \text{IntegralLattice}([[2]])\]
\[\text{sage: } AL1 = L1.\text{discriminant\_group}()\]
\[\text{sage: } AL2 = L2.\text{discriminant\_group}()\]
\[\text{sage: } AL1\]
Finite quadratic module over Integer Ring with invariants (2,)
Gram matrix of the quadratic form with values in \(\mathbb{Q}/2\mathbb{Z}\):
[[1/2]]
\[\text{sage: } g1 = L1.\text{discriminant\_group}().\text{gens}()[0]\]
\[\text{sage: } g2 = L2.\text{discriminant\_group}().\text{gens}()[0]\]
\[\text{sage: } \text{glue} = [[g1, g2]]\]
\[\text{sage: } \text{IntegralLatticeGluing([L1, L2], glue)}\]
Lattice of degree 2 and rank 2 over Integer Ring
Basis matrix:
[[1/2 1/2]
 [ 0  1]]
Inner product matrix:
[[2 0]
 [0 2]]

\[\text{sage: } L1 = \text{IntegralLattice}(\text{"A4"})\]
\[\text{sage: } L2 = \text{IntegralLattice}(\text{"A4"})\]
\[\text{sage: } g1 = L1.\text{discriminant\_group}().\text{gens}()[0]\]
\[\text{sage: } g2 = L2.\text{discriminant\_group}().\text{gens}()[0]\]
\[\text{sage: } \text{glue} = [[g1, 2 * g2]]\]
\[\text{sage: } \text{[V, phi]} = \text{IntegralLatticeGluing([L1, L2], glue, True)}\]
\[\text{sage: } V\]
Lattice of degree 8 and rank 8 over Integer Ring
Basis matrix:
[[1/5 2/5 3/5 4/5 2/5 4/5 1/5 3/5]
 [ 0  1  0  0  0  0  0  0]
 [ 0  0  1  0  0  0  0  0]
 [ 0  0  0  1  0  0  0  0]
 [ 0  0  0  0  1  0  0  0]
 [ 0  0  0  0  0  1  0  0]
 [ 0  0  0  0  0  0  1  0]
 [ 0  0  0  0  0  0  0  1]]
Inner product matrix:
[[ 2 -1  0  0  0  0  0  0]
 [-1  2 -1  0  0  0  0  0]
 [ 0 -1  2 -1  0  0  0  0]
 [ 0  0 -1  2  0  0  0  0]
 [ 0  0  0  0  2 -1  0  0]
 [ 0  0  0 -1  0  2 -1  0]
 [ 0  0  0  0 -1  2 -1  0]
 [ 0  0  0  0  0  0  2 -1]]
\[\text{sage: } V.\text{sublattice}(\text{phi}[0].\text{image}().\text{basis\_matrix}())\]
Lattice of degree 8 and rank 4 over Integer Ring
Basis matrix:
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
Inner product matrix:
\[
\begin{bmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{bmatrix}
\]

Different gluings can be composed:

```sage
sage: D4 = IntegralLattice("D4")
sage: D4.discriminant_group()
Finite quadratic module over Integer Ring with invariants (2, 2)
Gram matrix of the quadratic form with values in Q/2Z:
\[
\begin{bmatrix}
1 & 1/2 \\
1/2 & 1
\end{bmatrix}
\]
sage: L2 = IntegralLattice(2 * matrix.identity(2))
sage: L2.discriminant_group()
Finite quadratic module over Integer Ring with invariants (2, 2)
Gram matrix of the quadratic form with values in Q/2Z:
\[
\begin{bmatrix}
1/2 & 0 \\
0 & 1/2
\end{bmatrix}
\]
sage: g1 = D4.discriminant_group().gens()[0]
sage: g2 = L2.discriminant_group().gens()[0] + L2.discriminant_group().gens()[1]
sage: D6, phi = IntegralLatticeGluing([D4, L2], [[g1, g2]], True)
sage: AD6 = D6.discriminant_group()
sage: AD6.normal_form()
Finite quadratic module over Integer Ring with invariants (2, 2)
Gram matrix of the quadratic form with values in Q/2Z:
\[
\begin{bmatrix}
3/2 & 0 \\
0 & 3/2
\end{bmatrix}
\]
sage: f1, g1 = AD6.normal_form().gens()
sage: f2, g2 = L2.discriminant_group().gens()
sage: E8, psi = IntegralLatticeGluing([D6, L2], [[f1, f2], [g1, g2]], True)
sage: D4embed = E8.sublattice(psi[0](phi[0].image()).basis_matrix())
sage: x = D4([1, 0, 0, 0])
sage: psi[0](phi[0](x)).inner_product(psi[0](phi[0](x)))==x.inner_product(x)
True
sage: D4embed
```
Lattice of degree 8 and rank 4 over Integer Ring
Basis matrix:
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
The inner product matrix:

\[
\begin{bmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
\end{bmatrix}
\]

The input may be a list of three or more lattices:

```
sage: A7 = IntegralLattice("A7")
sage: D5 = IntegralLattice("D5")
sage: gA7 = A7.discriminant_group().gens()[0]
sage: gD5 = D5.discriminant_group().gens()[0]
sage: [L, phi] = IntegralLatticeGluing([A7, A7, D5, D5],
....:   [[gA7, gA7, gD5, 2 * gD5],
....:    [gA7, 7 * gA7, 2 * gD5, gD5]], True)
sage: L.determinant() 1
sage: B = phi[0].matrix()
sage: B*L.gram_matrix()*B.transpose()==A7.gram_matrix() True
```

The gluing takes place in the direct sum of the respective ambient spaces:

```
sage: L1 = IntegralLattice("D4", [[1, 1, 0, 0], [0, 1, 1, 0]])
sage: L2 = IntegralLattice("E6", [[0, 2, 0, 0, 0, 0], [0, 0, 0, 0, 1, 1]])
sage: [f1, f2] = L1.discriminant_group().gens()
sage: [g1, g2] = L2.discriminant_group().gens()
sage: [L, phi] = IntegralLatticeGluing([L1, L2], [[f1, g1], [f2, 2 * g2]], True)
sage: phi[0]
Free module morphism defined by the matrix
[ 2  2 -2 -1]
[ 0  2 -1  0]
Domain: Lattice of degree 4 and rank 2 over Integer Ring
Basis matrix:
[1 1 0 0]
[0 1 1 0]
Inner product matrix:
[ 2 -1 0 0]
[-1 2 -1 -1]
[ 0 -1 2 0]
[ 0 -1 0 2]
Codomain: Lattice of degree 10 and rank 4 over Integer Ring
Basis matrix:
[1/2 0 -1/2 0 0 1/2 0 0 1/2 1/2]
[0 1/2 1/2 0 0 0 1/2 0 0 0]
[0 0 0 0 0 1 0 0 0 0]
[0 0 0 0 0 0 0 0 1 1]
```

(continues on next page)
Inner product matrix:
\[
\begin{bmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix}
\]

sage: B = phi[0].matrix()
sage: B * L.gram_matrix() * B.transpose()==L1.gram_matrix()
True

`sage.modules.free_quadratic_module_integer_symmetric.local_modification(M, G, p, check=True)`
Return a local modification of $M$ that matches $G$ at $p$.

**INPUT:**
- $M$ – a $\mathbb{Z}_p$-maximal lattice
- $G$ – the gram matrix of a lattice isomorphic to $M$ over $\mathbb{Q}_p$
- $p$ – a prime number

**OUTPUT:**
an integral lattice $M'$ in the ambient space of $M$ such that $M$ and $M'$ are locally equal at all completions except at $p$ where $M'$ is locally equivalent to the lattice with gram matrix $G$

**EXAMPLES:**

```python
sage: from sage.modules.free_quadratic_module_integer_symmetric import local_modification
sage: L = IntegralLattice("A3").twist(15)
sage: M = L.maximal_overlattice()
sage: for p in prime_divisors(L.determinant()):
    ....:     M = local_modification(M, L.gram_matrix(), p)
sage: M.genus() == L.genus()
True
sage: L = IntegralLattice("D4").twist(3*4)
sage: M = L.maximal_overlattice()
sage: local_modification(M, L.gram_matrix(), 2)
Lattice of degree 4 and rank 4 over Integer Ring
Basis matrix:
\[
\begin{bmatrix}
1/3 & 0 & 2/3 & 2/3 \\
0 & 1/3 & 0 & 2/3 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Inner product matrix:
\[
\begin{bmatrix}
24 & -12 & 0 & 0 \\
-12 & 24 & -12 & -12 \\
0 & -12 & 24 & 0 \\
0 & -12 & 0 & 24
\end{bmatrix}
\]
```
AUTHORS:

- William Stein (2007-11-18)

`sage.modules.misc.gram_schmidt(B)`

Return the Gram-Schmidt orthogonalization of the entries in the list B of vectors, along with the matrix mu of Gram-Schmidt coefficients.

Note that the output vectors need not have unit length. We do this to avoid having to extract square roots.

Note: Use of this function is discouraged. It fails on linearly dependent input and its output format is not as natural as it could be. Instead, see `sage.matrix.matrix2.Matrix2.gram_schmidt()` which is safer and more general-purpose.

EXAMPLES:

```
sage: B = [vector([1,2,1/5]), vector([1,2,3]), vector([-1,0,0])]
sage: from sage.modules.misc import gram_schmidt
sage: G, mu = gram_schmidt(B)
sage: G
[(1, 2, 1/5), (-1/9, -2/9, 25/9), (-4/5, 2/5, 0)]
sage: G[0] * G[1]
0
sage: G[0] * G[2]
0
0
sage: mu
[ 0 0 0]
[10/9 0 0]
[-25/126 1/70 0]
sage: a = matrix([])
sage: a.gram_schmidt()
([], [])
sage: a = matrix([[],[],[],[]])
sage: a.gram_schmidt()
([], []
```

Linearly dependent input leads to a zero dot product in a denominator. This shows that trac ticket #10791 is fixed.
sage: from sage.modules.misc import gram_schmidt
sage: V = [vector(ZZ,[1,1]), vector(ZZ,[2,2]), vector(ZZ,[1,2])]
sage: gram_schmidt(V)
Traceback (most recent call last):
...  
ValueError: linearly dependent input for module version of Gram-Schmidt
QUOTIENTS OF FINITE RANK FREE MODULES OVER A FIELD.

class sage.modules.quotient_module.FreeModule_ambient_field_quotient(domain, sub, quotient_matrix, lift_matrix, inner_product_matrix=None)

Bases: sage.modules.free_module.FreeModule_ambient_field

A quotient \( V/W \) of two vector spaces as a vector space.

To obtain \( V \) or \( W \) use self.V() and self.W().

EXAMPLES:

```python
sage: k.<i> = QuadraticField(-1)
sage: A = k^3; V = A.span([[1,0,i], [2,i,0]])
sage: W = A.span([[3,i,i]])
sage: U = V/W; U
Vector space quotient V/W of dimension 1 over Number Field in i with defining polynomial x^2 + 1 with i = 1*I where
V: Vector space of degree 3 and dimension 2 over Number Field in i with defining polynomial x^2 + 1 with i = 1*I
Basis matrix:
[ 1 0 i]
[ 0 1 -2]
W: Vector space of degree 3 and dimension 1 over Number Field in i with defining polynomial x^2 + 1 with i = 1*I
Basis matrix:
[ 1 1/3*i 1/3*i]
sage: U.V()
Vector space of degree 3 and dimension 2 over Number Field in i with defining polynomial x^2 + 1 with i = 1*I
Basis matrix:
[ 1 0 i]
[ 0 1 -2]
sage: U.W()
Vector space of degree 3 and dimension 1 over Number Field in i with defining polynomial x^2 + 1 with i = 1*I
Basis matrix:
[ 1 1/3*i 1/3*i]
sage: U.quotient_map()
Vector space morphism represented by the matrix:
[ 1]
[3*i]
```
We create three quotient spaces and compare them:

```python
sage: A = QQ^2
sage: V = A.span_of_basis([[1,0], [1,1]])
sage: W0 = V.span([V.1, V.0])
sage: W1 = V.span([V.1])
sage: W2 = V.span([V.1])
sage: Q0 = V/W0
sage: Q1 = V/W1
sage: Q2 = V/W2

sage: Q0 == Q1
False
sage: Q1 == Q2
True
```

**V()**

Given this quotient space $Q = V/W$, return $V$.

EXAMPLES:

```python
sage: M = QQ^10 / [list(range(10)), list(range(2,12))]
sage: M.V()
Vector space of dimension 10 over Rational Field
```

**W()**

Given this quotient space $Q = V/W$, return $W$.

EXAMPLES:

```python
sage: M = QQ^10 / [list(range(10)), list(range(2,12))]
sage: M.W()
Vector space of degree 10 and dimension 2 over Rational Field
```

(continues on next page)
Given this quotient space $Q = V/W$, return $V$.

This is the same as $V()$.

EXAMPLES:

```sage
M = QQ^10 / [list(range(10)), list(range(2,12))]
sage: M.cover()
Vector space of dimension 10 over Rational Field
```

`lift(x)`

Lift element of this quotient $V/W$ to $V$ by applying the fixed lift homomorphism.

The lift is a fixed homomorphism.

EXAMPLES:

```sage
M = QQ^3 / [[1,2,3]]
sage: M.lift(M.0)
(1, 0, 0)
sage: M.lift(M.1)
(0, 1, 0)
sage: M.lift(M.0 - 2*M.1)
(1, -2, 0)
```

`lift_map()`

Given this quotient space $Q = V/W$, return a fixed choice of linear homomorphism (a section) from $Q$ to $V$.

EXAMPLES:

```sage
M = QQ^3 / [[1,2,3]]
sage: M.lift_map()
Vector space morphism represented by the matrix:
[1 0 0]
[0 1 0]
Domain: Vector space quotient V/W of dimension 2 over Rational Field where V: Vector space of dimension 3 over Rational Field
W: Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[1 2 3]
Codomain: Vector space of dimension 3 over Rational Field
```

`quotient_map()`

Given this quotient space $Q = V/W$, return the natural quotient map from $V$ to $Q$.

EXAMPLES:

```sage
M = QQ^3 / [[1,2,3]]
sage: M.quotient_map()
Vector space morphism represented by the matrix:
[ 1 0]
```

(continues on next page)
Domain: Vector space of dimension 3 over Rational Field
Codomain: Vector space quotient V/W of dimension 2 over Rational Field where
V: Vector space of dimension 3 over Rational Field
W: Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
\[
\begin{bmatrix}
1 & 2 & 3 \\
-1/3 & -2/3 & \\
\end{bmatrix}
\]

```
sage: M.quotient_map()( (QQ^3)([1,2,3]) )
(0, 0)
```

\textbf{relations()}

Given this quotient space \( Q = V/W \), return \( W \).

This is the same as \( W() \).

\textbf{EXAMPLES:}

```
sage: M = QQ^10 / [list(range(10)), list(range(2,12))]
sage: M.relations()
```

Vector space of degree 10 and dimension 2 over Rational Field
Basis matrix:
\[
\begin{bmatrix}
1 & 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{bmatrix}
\]
DENSE COMPLEX DOUBLE VECTORS USING A NUMPY BACKEND.

EXAMPLES:

```
sage: v = vector(CDF,[(1,-1), (2,pi), (3,5)])
sage: v
(1.0 - 1.0*I, 2.0 + 3.141592653589793*I, 3.0 + 5.0*I)
sage: type(v)
<class 'sage.modules.vector_complex_double_dense.Vector_complex_double_dense'>
sage: parent(v)
Vector space of dimension 3 over Complex Double Field
sage: v[0] = 5
sage: v
(5.0, 2.0 + 3.141592653589793*I, 3.0 + 5.0*I)
sage: loads(dumps(v)) == v
True
```

AUTHORS:

– Jason Grout, Oct 2008: switch to NumPy backend, factored out Vector_double_dense class

class sage.modules.vector_complex_double_dense.Vector_complex_double_dense

Bases: sage.modules.vector_double_dense.Vector_double_dense

Vectors over the Complex Double Field. These are supposed to be fast vector operations using C doubles. Most operations are implemented using numpy which will call the underlying BLAS, if needed, on the system.

EXAMPLES:

```
sage: v = vector(CDF,[(1,-1), (2,pi), (3,5)])
sage: v
(1.0 - 1.0*I, 2.0 + 3.141592653589793*I, 3.0 + 5.0*I)
sage: v*v
# rel tol 1e-15
-21.86960440108936 + 40.56637061435917*I
```

sage.modules.vector_complex_double_dense.unpickle_v0(parent, entries, degree)

Create a complex double vector containing the entries.

EXAMPLES:

```
sage: v = vector(CDF, [1,2,3])
sage: w = sage.modules.vector_complex_double_dense.unpickle_v0(v.parent(), list(v), v.degree())
sage: v == w
True
```
sage.modules.vector_complex_double_dense.unpickle_v1(parent, entries, degree, is_mutable=None)
Create a complex double vector with the given parent, entries, degree, and mutability.

EXAMPLES:

```python
sage: v = vector(CDF, [1,2,3])
sage: w = sage.modules.vector_complex_double_dense.unpickle_v1(v.parent(), list(v),
               →v.degree(), v.is_immutable())
sage: v == w
True
```
DENSE VECTORS USING A NUMPY BACKEND.

This serves as a base class for dense vectors over Real Double Field and Complex Double Field.

EXAMPLES:

```python
sage: v = vector(CDF, [(1,-1), (2,pi), (3,5)])
sage: v
(1.0 - 1.0*I, 2.0 + 3.141592653589793*I, 3.0 + 5.0*I)
sage: type(v)
<class 'sage.modules.vector_complex_double_dense.Vector_complex_double_dense'>
sage: parent(v)
Vector space of dimension 3 over Complex Double Field
sage: v[0] = 5
sage: v
(5.0, 2.0 + 3.141592653589793*I, 3.0 + 5.0*I)
sage: loads(dumps(v)) == v
True
sage: v = vector(RDF, [1,2,3,4]); v
(1.0, 2.0, 3.0, 4.0)
sage: loads(dumps(v)) == v
True
```

AUTHORS:

- Jason Grout, Oct 2008: switch to numpy backend, factored out `Vector_double_dense` class
- Josh Kantor
- William Stein

**class** `sage.modules.vector_double_dense.Vector_double_dense`

Bases: `sage.modules.vector_numpy_dense.Vector_numpy_dense`

Base class for vectors over the Real Double Field and the Complex Double Field. These are supposed to be fast vector operations using C doubles. Most operations are implemented using numpy which will call the underlying BLAS, if needed, on the system.

This class cannot be instantiated on its own. The numpy vector creation depends on several variables that are set in the subclasses.

EXAMPLES:

```python
sage: v = vector(RDF, [1,2,3,4]); v
(1.0, 2.0, 3.0, 4.0)
sage: v*v
30.0
```
complex_vector()
Return the associated complex vector, i.e., this vector but with coefficients viewed as complex numbers.

EXAMPLES:

```python
sage: v = vector(RDF,4,range(4)); v
(0.0, 1.0, 2.0, 3.0)
sage: v.complex_vector()
(0.0, 1.0, 2.0, 3.0)
sage: v = vector(RDF,0)
sage: v.complex_vector()
()
```

fft(
*direction='forward', algorithm='radix2', inplace=False*)
This performs a fast Fourier transform on the vector.

INPUT:

- direction – ‘forward’ (default) or ‘backward’

The algorithm and inplace arguments are ignored.

This function is fastest if the vector’s length is a power of 2.

EXAMPLES:

```python
sage: v = vector(CDF,[1+2*I,2,3*I,4])
sage: v.fft()
(7.0 + 5.0*I, 1.0 + 1.0*I, -5.0 + 5.0*I, 1.0 - 3.0*I)
sage: v.fft(direction='backward')
(1.75 + 1.25*I, 0.25 - 0.75*I, -1.25 + 1.25*I, 0.25 + 0.25*I)
sage: v.fft().fft(direction='backward')
(1.0 + 2.0*I, 2.0, 3.0*I, 4.0)
sage: v.fft(inplace=True)
sage: v
(7.0 + 5.0*I, 1.0 + 1.0*I, -5.0 + 5.0*I, 1.0 - 3.0*I)
sage: v = vector(RDF,4,range(4)); v
(0.0, 1.0, 2.0, 3.0)
sage: v.fft()
(6.0, -2.0 + 2.0*I, -2.0, -2.0 - 2.0*I)
sage: v.fft(inplace=True)
```

inv_fft(
*algorithm=’radix2’, inplace=False*)
This performs the inverse fast Fourier transform on the vector.

The Fourier transform can be done in place using the keyword inplace=True
This will be fastest if the vector’s length is a power of 2.

EXAMPLES:

```
sage: v = vector(CDF,[1,2,3,4])
sage: w = v.fft()
sage: max(v - w.inv_fft()) < 1e-12
True
```

**mean()**

Calculate the arithmetic mean of the vector.

EXAMPLES:

```
sage: v = vector(RDF, range(9))
sage: w = vector(CDF, [k+(9-k)*I for k in range(9)])
sage: v.mean()
4.0
sage: w.mean()
4.0 + 5.0*I
```

**norm**(p=2)

Returns the norm (or related computations) of the vector.

INPUT:

- p - default: 2 - controls which norm is computed, allowable values are any real number and positive and negative infinity. See output discussion for specifics.

OUTPUT:

Returned value is a double precision floating point value in RDF (or an integer when p=0). The default value of p = 2 is the “usual” Euclidean norm. For other values:

- p = Infinity or p = oo: the maximum of the absolute values of the entries, where the absolute value of the complex number $a + bi$ is $\sqrt{a^2 + b^2}$.
- p = -Infinity or p = -oo: the minimum of the absolute values of the entries.
- p = 0: the number of nonzero entries in the vector.
- p is any other real number: for a vector $\vec{x}$ this method computes

$$\left( \sum_{i} x_i^p \right)^{1/p}$$

For $p < 0$ this function is not a norm, but the above computation may be useful for other purposes.

ALGORITHM:

Computation is performed by the norm() function of the SciPy/NumPy library.

EXAMPLES:

First over the reals.

```
sage: v = vector(RDF, range(9))
sage: v.norm()  
14.28285685...
sage: v.norm(p=2)  
14.28285685...
```
sage: v.norm(p=6)
8.744039097...
sage: v.norm(p=Infinity)
8.0
sage: v.norm(p=-oo)
0.0
sage: v.norm(p=0)
8.0
sage: v.norm(p=0.3)
4099.153615...

And over the complex numbers.

sage: w = vector(CDF, [3-4*I, 0, 5+12*I])
sage: w.norm()
13.9283882...
sage: w.norm(p=2)
13.9283882...
sage: w.norm(p=0)
2.0
sage: w.norm(p=4.2)
13.0555695...
sage: w.norm(p=oo)
13.0

Negative values of $p$ are allowed and will provide the same computation as for positive values. A zero entry in the vector will raise a warning and return zero.

sage: v = vector(CDF, range(1,10))
sage: v.norm(p=-3.2)
0.953760808...
sage: w = vector(CDF, [-1,0,1])
sage: w.norm(p=-1.6)
Traceback (most recent call last):
  ...: RuntimeWarning: divide by zero encountered in power
0.0

Return values are in RDF, or an integer when $p = 0$.

sage: v = vector(RDF, [1,2,4,8])
sage: v.norm(p=0) in RDF
True
sage: v.norm(p=0) in ZZ
True

Improper values of $p$ are caught.

sage: w = vector(CDF, [-1,0,1])
sage: w.norm(p='junk')
Traceback (most recent call last):
  ...: ValueError: vector norm 'p' must be +/- infinity or a real number, not junk

prod()
Return the product of the entries of self.
EXAMPLES:

```python
sage: v = vector(RDF, range(9))
sage: w = vector(CDF, [k+(9-k)*I for k in range(9)])
sage: v.prod()
0.0
sage: w.prod()
57204225.0*I
```

`standard_deviation(population=True)`

Calculate the standard deviation of entries of the vector.

**INPUT:** population – If False, calculate the sample standard deviation.

**EXAMPLES:**

```python
sage: v = vector(RDF, range(9))
sage: w = vector(CDF, [k+(9-k)*I for k in range(9)])
sage: v.standard_deviation()
2.7386127875258306
sage: v.standard_deviation(population=False)
2.581988897471611
sage: w.standard_deviation()
3.872983346207417
sage: w.standard_deviation(population=False)
3.6514837167011076
```

`stats_kurtosis()`

Compute the kurtosis of a dataset.

Kurtosis is the fourth central moment divided by the square of the variance. Since we use Fisher's definition, 3.0 is subtracted from the result to give 0.0 for a normal distribution. (Paragraph from the scipy.stats docstring.)

**EXAMPLES:**

```python
sage: v = vector(RDF, range(9))
sage: w = vector(CDF, [k+(9-k)*I for k in range(9)])
sage: v.stats_kurtosis()  # rel tol 5e-15
-1.2300000000000000
sage: w.stats_kurtosis()  # rel tol 5e-15
-1.2300000000000000
```

`sum()`

Return the sum of the entries of self.

**EXAMPLES:**

```python
sage: v = vector(RDF, range(9))
sage: w = vector(CDF, [k+(9-k)*I for k in range(9)])
sage: v.sum()
36.0
sage: w.sum()
36.0 + 45.0*I
```

`variance(population=True)`

Calculate the variance of entries of the vector.
INPUT:

- **population** – If False, calculate the sample variance.

EXAMPLES:

```python
sage: v = vector(RDF, range(9))
sage: w = vector(CDF, [k+(9-k)*I for k in range(9)])
sage: v.variance()
7.5
sage: v.variance(population=False)
6.666666666666667
sage: w.variance()
15.0
sage: w.variance(population=False)
13.333333333333334
```

**zero_at(eps)**

Returns a copy with small entries replaced by zeros.

This is useful for modifying output from algorithms which have large relative errors when producing zero elements, e.g. to create reliable doctests.

INPUT:

- **eps** - cutoff value

OUTPUT:

A modified copy of the vector. Elements smaller than or equal to eps are replaced with zeroes. For complex vectors, the real and imaginary parts are considered individually.

EXAMPLES:

```python
sage: v = vector(RDF, [1.0, 2.0, 10^-10, 3.0])
sage: v.zero_at(1e-8)
(1.0, 2.0, 0.0, 3.0)
sage: v.zero_at(1e-12)
(1.0, 2.0, 1e-10, 3.0)
```

For complex numbers the real and imaginary parts are considered separately.

```python
sage: w = vector(CDF, [10^-6 + 5*I, 5 + 10^-6*I, 5 + 5*I, 10^-6 + 10^-6*I])
sage: w.zero_at(1.0e-4)
(5.0*I, 5.0, 5.0 + 5.0*I, 0.0)
sage: w.zero_at(1.0e-8)
(1e-06 + 5.0*I, 5.0 + 1e-06*I, 5.0 + 5.0*I, 1e-06 + 1e-06*I)
```
VECTORS WITH INTEGER ENTRIES

AUTHOR:
• William Stein (2007)

EXAMPLES:

```sage
sage: v = vector(ZZ,[1,2,3,4,5])
sage: v
(1, 2, 3, 4, 5)
sage: 3*v
(3, 6, 9, 12, 15)
sage: v*7
(7, 14, 21, 28, 35)
sage: -v
(-1, -2, -3, -4, -5)
sage: v - v
(0, 0, 0, 0, 0)
sage: v + v
(2, 4, 6, 8, 10)
sage: v * v  # dot product.
```

We make a large zero vector:

```sage
sage: k = ZZ^100000; k
Ambient free module of rank 100000 over the principal ideal domain Integer Ring
sage: v = k(0)
sage: v[:10]
(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
```

```class
sage.modules.vector_integer_dense.Vector_integer_dense

Bases: sage.modules.free_module_element.FreeModuleElement

list(copy=True)
The list of entries of the vector.

INPUT:

• copy, ignored optional argument.

EXAMPLES:

```sage
sage: v = vector([1,2,3,4])
sage: a = v.list(copy=False); a
```
```
\begin{verbatim}
[1, 2, 3, 4]
sage: a[0] = 0
sage: v
(1, 2, 3, 4)
\end{verbatim}

\begin{verbatim}
sage.modules.vector_integer_dense.unpickle_v0(parent, entries, degree)
sage.modules.vector_integer_dense.unpickle_v1(parent, entries, degree, is_mutable)
\end{verbatim}
VECTORS WITH ELEMENTS IN GF(2)

AUTHOR:

- Martin Albrecht (2009-12): initial implementation
- Thomas Feulner (2012-11): added \texttt{Vector\_mod2\_dense.hamming\_weight()}

EXAMPLES:

```sage
sage: VS = GF(2)^3
sage: e = VS.random_element()
```

```sage
e.parent() is VS
```

```sage
True
```

```sage
S = set(vector(v, immutable=True) for v in VS)
```

```sage
S1 = set()
```

```sage
while S != S1:
    ...:
    S1.add(vector(VS.random_element(), immutable=True))
```

```sage
class sage.modules.vector_mod2_dense.Vector_mod2_dense
    Bases: sage.modules.free_module_element.FreeModuleElement
```

EXAMPLES:

```sage
sage: VS = VectorSpace(GF(2),3)
sage: VS((0,0,1/3))
(0, 0, 1)
```

```sage
type(_)
```

```sage
<class 'sage.modules.vector_mod2_dense.Vector\_mod2\_dense'>
```

```sage
sage: VS((0,0,int(3)))
(0, 0, 1)
```

```sage
sage: VS((0,0,3))
(0, 0, 1)
```

```sage
sage: VS((0,0,GF(2)(1)))
(0, 0, 1)
```

```sage
hamming_weight()
```

Return the number of positions \(i\) such that \(\text{self}[i] \neq 0\).

EXAMPLES:

```sage
sage: vector(GF(2), [1,1,0]).hamming_weight()
```

```sage
2
```

```sage
list(copy=True)
```

Return a list of entries in \texttt{self}.
INPUT:

- copy - always True

EXAMPLES:

```python
sage: VS = VectorSpace(GF(2), 10)
sage: entries = [GF(2).random_element() for _ in range(10)]
sage: e = VS(entries)
sage: e.list() == entries
True
```

```
sage.modules.vector_mod2_dense.unpickle_v0(parent, entries, degree, is_immutable)
EXAMPLES:

```python
sage: from sage.modules.vector_mod2_dense import unpickle_v0
sage: VS = VectorSpace(GF(2), 10)
sage: unpickle_v0(VS, [0,1,2,3,4,5,6,7,8,9], 10, 0)
(0, 1, 0, 1, 0, 1, 0, 1, 0, 1)
```

```
CHAPTER

THIRTY SIX

VECTORS WITH INTEGER MOD N ENTRIES, WITH N SMALL

EXAMPLES:

```python
sage: v = vector(Integers(8),[1,2,3,4,5])
sage: type(v)
<class 'sage.modules.vector_modn_dense.Vector_modn_dense'>
sage: v
(1, 2, 3, 4, 5)
sage: 3*v
(3, 6, 1, 4, 7)
sage: v^7
(7, 6, 5, 4, 3)
sage: -v
(7, 6, 5, 4, 3)
sage: v - v
(0, 0, 0, 0, 0)
sage: v + v
(2, 4, 6, 0, 2)
sage: v * v
7

sage: v = vector(Integers(8),[1,2,3,4,5])
sage: u = vector(Integers(8),[1,2,3,4,4])
sage: v - u
(0, 0, 0, 0, 1)
sage: u - v
(0, 0, 0, 0, 4)
```

We make a large zero vector:

```python
sage: k = Integers(8)^100000; k
Ambient free module of rank 100000 over Ring of integers modulo 8
sage: v = k(0)
sage: v[:10]
(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
```
We multiply a vector by a matrix:

```
sage: a = (GF(97)^5)(range(5))
sage: m = matrix(GF(97),5,range(25))
sage: a*m
(53, 63, 73, 83, 93)
```

AUTHOR:

- William Stein (2007)

```python
class sage.modules.vector_modn_dense.Vector_modn_dense
    Bases: sage.modules.free_module_element.FreeModuleElement
sage.modules.vector_modn_dense.pickle_v0(parent, entries, degree, p)
sage.modules.vector_modn_dense.pickle_v1(parent, entries, degree, p, is_mutable)
```
VECTORS WITH RATIONAL ENTRIES

AUTHOR:

- William Stein (2007)
- Soroosh Yazdani (2007)

EXAMPLES:

```
sage: v = vector(QQ,[1,2,3,4,5])
sage: v
(1, 2, 3, 4, 5)
sage: 3*v
(3, 6, 9, 12, 15)
sage: v/2
(1/2, 1, 3/2, 2, 5/2)
sage: -v
(-1, -2, -3, -4, -5)
sage: v - v
(0, 0, 0, 0, 0)
sage: v + v
(2, 4, 6, 8, 10)
sage: v * v
55
```

We make a large zero vector:

```
sage: k = QQ^100000; k
Vector space of dimension 100000 over Rational Field
sage: v = k(0)
sage: v[:10]
(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
```

class sage.modules.vector_rational_dense.Vector_rational_dense

Bases: sage.modules.free_module_element.FreeModuleElement

```
list(copy=True)
The list of entries of the vector.

INPUT:

- copy, ignored optional argument.

EXAMPLES:
```
sage: v = vector(QQ,[1,2,3,4])
sage: a = v.list(copy=False); a
[1, 2, 3, 4]
sage: a[0] = 0
sage: v
(1, 2, 3, 4)

sage.modules.vector_rational_dense.unpickle_v0(parent, entries, degree)
sage.modules.vector_rational_dense.unpickle_v1(parent, entries, degree, is_mutable)
DENSE REAL DOUBLE VECTORS USING A NUMPY BACKEND.

EXAMPLES:

```
sage: v = vector(RDF,[1, pi, sqrt(2)])
sage: v
(1.0, 3.141592653589793, 1.414213562373095)
sage: type(v)
<class 'sage.modules.vector_real_double_dense.Vector_real_double_dense'>
sage: parent(v)
Vector space of dimension 3 over Real Double Field
sage: v[0] = 5
sage: v
(5.0, 3.141592653589793, 1.414213562373095)
sage: loads(dumps(v)) == v
True
```

AUTHORS:

– Jason Grout, Oct 2008: switch to numpy backend, factored out Vector_double_dense class

### class sage.modules.vector_real_double_dense.Vector_real_double_dense

Bases: `sage.modules.vector_double_dense.Vector_double_dense`

Vectors over the Real Double Field. These are supposed to be fast vector operations using C doubles. Most operations are implemented using numpy which will call the underlying BLAS, if needed, on the system.

EXAMPLES:

```
sage: v = vector(RDF, [1,2,3,4]); v
(1.0, 2.0, 3.0, 4.0)
sage: v*v
30.0
```

```python
stats_skew()
```

Computes the skewness of a data set.

For normally distributed data, the skewness should be about 0. A skewness value > 0 means that there is more weight in the left tail of the distribution. (Paragraph from the scipy.stats docstring.)

EXAMPLES:

```
sage: v = vector(RDF, range(9))
sage: v.stats_skew()
0.0
```
sage.modules.vector_real_double_dense.unpickle_v0(parent, entries, degree)
Create a real double vector containing the entries.

EXAMPLES:
```
sage: v = vector(RDF, [1,2,3])
sage: w = sage.modules.vector_real_double_dense.unpickle_v0(v.parent(), list(v), v.
˓→degree())
sage: v == w
True
```

sage.modules.vector_real_double_dense.unpickle_v1(parent, entries, degree, is_mutable=None)
Create a real double vector with the given parent, entries, degree, and mutability.

EXAMPLES:
```
sage: v = vector(RDF, [1,2,3])
sage: w = sage.modules.vector_real_double_dense.unpickle_v1(v.parent(), list(v), v.
˓→degree(), v.is_immutable())
sage: v == w
True
```
VECTORS OVER THE SYMBOLIC RING

Implements vectors over the symbolic ring.

AUTHORS:

EXAMPLES:

```sage
sage: x, y = var('x, y')
sage: u = vector([sin(x)^2 + cos(x)^2, log(2*y) + log(3*y)]); u
(cos(x)^2 + sin(x)^2, log(3*y) + log(2*y))
sage: type(u)
<class 'sage.modules.free_module.FreeModule_ambient_field_with_category.element_class'>
sage: u.simplify_full()
(1, log(3*y) + log(2*y))
```

```python
class sage.modules.vector_symbolic_dense.Vector_symbolic_dense
    Bases: sage.modules.free_module_element.FreeModuleElement_generic_dense

canonicalize_radical(*args, **kwds)

```
Generic function used to implement common symbolic operations elementwise as methods of a vector.

EXAMPLES:

```sage
sage: var('x,y')
(x, y)
sage: v = vector([sin(x)^2 + cos(x)^2, log(x*y), sin(x/(x^2 + x)),-factorial(x+1)/factorial(x)]);

sage: v.simplify_trig()
(1, log(x*y), sin(1/(x + 1)), factorial(x + 1)/factorial(x))
sage: v.canonicalize_radical()
(cos(x)^2 + sin(x)^2, log(x) + log(y), sin(1/(x + 1)), factorial(x + 1)/-
→factorial(x))
sage: v.simplify_rational()
(cos(x)^2 + sin(x)^2, log(x*y), sin(1/(x + 1)), factorial(x + 1)/-
→factorial(x))
sage: v.simplify_factorial()
(cos(x)^2 + sin(x)^2, log(x*y), sin(x/(x^2 + x)), x + 1)
sage: v.simplify_full()
(1, log(x*y), sin(1/(x + 1)), x + 1)
```

(continues on next page)
sage: v = vector([sin(2*x), sin(3*x)])
sage: v.simplify_trig()
(2*cos(x)*sin(x), (4*cos(x)^2 - 1)*sin(x))
sage: v.simplify_trig(False)
(sin(2*x), sin(3*x))
sage: v.simplify_trig(expand=False)
(sin(2*x), sin(3*x))

See Expression.canonicalize_radical() for optional arguments.

`simplify(*args, **kwds)`

Generic function used to implement common symbolic operations elementwise as methods of a vector.

EXAMPLES:

sage: var('x,y')
(x, y)
sage: v = vector([sin(x)^2 + cos(x)^2, log(x*y), sin(x/(x^2 + x)), factorial(x+1)/factorial(x)])
sage: v.simplify_trig()
(1, log(x*y), sin(1/(x + 1)), factorial(x + 1)/factorial(x))
sage: v.canonicalize_radical()
(cos(x)^2 + sin(x)^2, log(x) + log(y), sin(1/(x + 1)), factorial(x + 1)/factorial(x))
sage: v.simplify_rational()
(cos(x)^2 + sin(x)^2, log(x*y), sin(1/(x + 1)), factorial(x + 1)/factorial(x))
sage: v.simplify_factorial()
(cos(x)^2 + sin(x)^2, log(x*y), sin(x/(x^2 + x)), x + 1)
sage: v.simplify_full()
(1, log(x*y), sin(1/(x + 1)), x + 1)

sage: v = vector([sin(2*x), sin(3*x)])
sage: v.simplify_trig()
(2*cos(x)*sin(x), (4*cos(x)^2 - 1)*sin(x))
sage: v.simplify_trig(False)
(sin(2*x), sin(3*x))
sage: v.simplify_trig(expand=False)
(sin(2*x), sin(3*x))

See Expression.simplify() for optional arguments.

`simplify_factorial(*args, **kwds)`

Generic function used to implement common symbolic operations elementwise as methods of a vector.

EXAMPLES:
sage: var('x,y')
(x, y)
sage: v = vector([sin(x)^2 + cos(x)^2, log(x*y), sin(x/(x^2 + x)), factorial(x+1)/factorial(x)])
sage: v.simplify_trig()
(1, log(x*y), sin(1/(x + 1)), factorial(x + 1)/factorial(x))
sage: v.canonicalize_radical()
(cos(x)^2 + sin(x)^2, log(x) + log(y), sin(1/(x + 1)), factorial(x + 1)/factorial(x))
sage: v.simplify_rational()
(cos(x)^2 + sin(x)^2, log(x*y), sin(1/(x + 1)), factorial(x + 1)/factorial(x))
sage: v.simplify_factorial()
(cos(x)^2 + sin(x)^2, log(x*y), sin(x/(x^2 + x)), x + 1)
sage: v.simplify_full()
(1, log(x*y), sin(1/(x + 1)), x + 1)
sage: v = vector([sin(2*x), sin(3*x)])
sage: v.simplify_trig()
(2*cos(x)*sin(x), (4*cos(x)^2 - 1)*sin(x))
sage: v.simplify_trig(expand=False)
(sin(2*x), sin(3*x))
See Expression.simplify_factorial() for optional arguments.

simplify_full(*args, **kwds)

Generic function used to implement common symbolic operations elementwise as methods of a vector.

EXAMPLES:

sage: var('x,y')
(x, y)
sage: v = vector([sin(x)^2 + cos(x)^2, log(x*y), sin(x/(x^2 + x)), factorial(x+1)/factorial(x)])
sage: v.simplify_trig()
(1, log(x*y), sin(1/(x + 1)), factorial(x + 1)/factorial(x))
sage: v.canonicalize_radical()
(cos(x)^2 + sin(x)^2, log(x) + log(y), sin(1/(x + 1)), factorial(x + 1)/factorial(x))
sage: v.simplify_rational()
(cos(x)^2 + sin(x)^2, log(x*y), sin(1/(x + 1)), factorial(x + 1)/factorial(x))
sage: v.simplify_factorial()
(cos(x)^2 + sin(x)^2, log(x*y), sin(x/(x^2 + x)), x + 1)
sage: v.simplify_full()
(1, log(x*y), sin(1/(x + 1)), x + 1)
sage: v = vector([sin(2*x), sin(3*x)])
sage: v.simplify_trig()
\[(2\cos(x)\sin(x), (4\cos(x)^2 - 1)\sin(x))\]

```
sage: v.simplify_trig(False)
(sin(2*x), sin(3*x))
sage: v.simplify_trig(expand=False)
(sin(2*x), sin(3*x))
```

See Expression.simplify_full() for optional arguments.

```
simplify_log(*args, **kwds)
```

Generic function used to implement common symbolic operations elementwise as methods of a vector.

**EXAMPLES:**

```
sage: var('x,y')
(x, y)
sage: v = vector([sin(x)^2 + cos(x)^2, log(x*y), sin(x/(x^2 + x)), factorial(x+1)/factorial(x)])
sage: v.simplify_trig()
(1, log(x*y), sin(1/(x + 1)), factorial(x + 1)/factorial(x))
sage: v.canonicalize_radical()
(cos(x)^2 + sin(x)^2, log(x) + log(y), sin(1/(x + 1)), factorial(x + 1)/factorial(x))
sage: v.simplify_rational()
(cos(x)^2 + sin(x)^2, log(x*y), sin(1/(x + 1)), factorial(x + 1)/factorial(x))
sage: v.simplify_factorial()
(cos(x)^2 + sin(x)^2, log(x*y), sin(x/(x^2 + x)), x + 1)
sage: v.simplify_full()
(1, log(x*y), sin(1/(x + 1)), x + 1)
```

See Expression.simplify_log() for optional arguments.

```
simplify_rational(*args, **kwds)
```

Generic function used to implement common symbolic operations elementwise as methods of a vector.

**EXAMPLES:**

```
sage: var('x,y')
(x, y)
sage: v = vector([sin(2*x), sin(3*x)])
sage: v.simplify_trig()
(2*cos(x)*sin(x), (4*cos(x)^2 - 1)*sin(x))
sage: v.simplify_trig(False)
(sin(2*x), sin(3*x))
sage: v.simplify_trig(expand=False)
(sin(2*x), sin(3*x))
```

(continues on next page)
sage: v.simplify_trig()
(1, log(x*y), sin(1/(x + 1)), factorial(x + 1)/factorial(x))
sage: v.canonicalize_radical()
(cos(x)^2 + sin(x)^2, log(x) + log(y), sin(1/(x + 1)), factorial(x + 1)/
factorial(x))
sage: v.simplify_rational()
(cos(x)^2 + sin(x)^2, log(x*y), sin(1/(x + 1)), factorial(x + 1)/
factorial(x))
sage: v.simplify_factorial()
(cos(x)^2 + sin(x)^2, log(x*y), sin(x/(x^2 + x)), x + 1)
sage: v.simplify_full()
(1, log(x*y), sin(1/(x + 1)), x + 1)
sage: v = vector([sin(2*x), sin(3*x)])
sage: v.simplify_trig()  # simplify_trig(*args, **kwds)
(2*cos(x)*sin(x), (4*cos(x)^2 - 1)*sin(x))
sage: v.simplify_trig(expand=False)
(simplify_trig)
(sin(2*x), sin(3*x))

See Expression.simplify_rational() for optional arguments.

simplify_trig(*args, **kwds)

Generic function used to implement common symbolic operations elementwise as methods of a vector.

EXAMPLES:

sage: var('x,y')
(x, y)
sage: v = vector([sin(x)^2 + cos(x)^2, log(x^y), sin(1/(x^2 + x)), factorial(x+1)/factorial(x)])
sage: v.simplify_trig()
(1, log(x*y), sin(1/(x + 1)), factorial(x + 1)/factorial(x))
sage: v.canonicalize_radical()
(cos(x)^2 + sin(x)^2, log(x) + log(y), sin(1/(x + 1)), factorial(x + 1)/
factorial(x))
sage: v.simplify_rational()
(cos(x)^2 + sin(x)^2, log(x*y), sin(1/(x + 1)), factorial(x + 1)/
factorial(x))
sage: v.simplify_factorial()
(cos(x)^2 + sin(x)^2, log(x*y), sin(x/(x^2 + x)), x + 1)
sage: v.simplify_full()
(1, log(x*y), sin(1/(x + 1)), x + 1)
sage: v = vector([sin(2*x), sin(3*x)])
sage: v.simplify_trig()  # simplify_trig(*args, **kwds)
(2*cos(x)*sin(x), (4*cos(x)^2 - 1)*sin(x))
sage: v.simplify_trig(expand=False)
(simplify_trig)
(sin(2*x), sin(3*x))
sage: v.simplify_trig(expand=False)
(sin(2*x), sin(3*x))

See Expression.simplify_trig() for optional arguments.

**trig_expand**(*args, **kwds)

Generic function used to implement common symbolic operations elementwise as methods of a vector.

EXAMPLES:

```python
sage: var('x,y')
(x, y)
sage: v = vector([sin(x)^2 + cos(x)^2, log(x*y), sin(x/(x^2 + x)),
                factorial(x+1)/factorial(x)])
sage: v.simplify_trig()
(1, log(x*y), sin(1/(x + 1)), factorial(x + 1)/factorial(x))
sage: v.canonicalize_radical()
(cos(x)*2 + sin(x)^2, log(x) + log(y), sin(1/(x + 1)), factorial(x + 1)/
  ...factorial(x))
sage: v.simplify_rational()
(cos(x)*2 + sin(x)^2, log(x*y), sin(1/(x + 1)), factorial(x + 1)/
  ...factorial(x))
sage: v.simplify_factorial()
(cos(x)*2 + sin(x)^2, log(x^2 + x), x + 1)
sage: v.simplify_full()
(1, log(x*y), sin(1/(x + 1)), x + 1)
```

See Expression.expand_trig() for optional arguments.

**trig_reduce**(*args, **kwds)

Generic function used to implement common symbolic operations elementwise as methods of a vector.

EXAMPLES:

```python
sage: var('x,y')
(x, y)
sage: v = vector([sin(2*x), sin(3*x)])
sage: v.simplify_trig()
(2*cos(x)*sin(x), (4*cos(x)^2 - 1)*sin(x))
sage: v.simplify_trig(False)
(sin(2*x), sin(3*x))
sage: v.simplify_trig(expand=False)
(sin(2*x), sin(3*x))
```
(cos(x)^2 + sin(x)^2, log(x) + log(y), sin(1/(x + 1)), factorial(x + 1)/
  factorial(x))

sage: v.simplify_rational()
(cos(x)^2 + sin(x)^2, log(x*y), sin(1/(x + 1)), factorial(x + 1)/
  factorial(x))

sage: v.simplify_factorial()
(cos(x)^2 + sin(x)^2, log(x*y), sin(x/(x^2 + x)), x + 1)

sage: v.simplify_full()
(1, log(x*y), sin(1/(x + 1)), x + 1)

sage: v = vector([sin(2*x), sin(3*x)])

sage: v.simplify_trig()
(2*cos(x)*sin(x), (4*cos(x)^2 - 1)*sin(x))

sage: v.simplify_trig(False)
(sin(2*x), sin(3*x))

sage: v.simplify_trig(expand=False)
(sin(2*x), sin(3*x))

See Expression.reduce_trig() for optional arguments.

sage.modules.vector_symbolic_dense.apply_map(phi)
Returns a function that applies phi to its argument.

EXAMPLES:

sage: from sage.modules.vector_symbolic_dense import apply_map
sage: v = vector([1,2,3])
sage: f = apply_map(lambda x: x+1)
sage: f(v)
(2, 3, 4)
This module implements filtered vector spaces, that is, a descending sequence of vector spaces

\[ \cdots \supset F_d \supset F_{d+1} \supset F_{d+2} \supset \cdots \]

with degrees \( d \in \mathbb{Z} \). It is not required that \( F_d \) is the entire ambient space for \( d \ll 0 \) (see \texttt{is_exhaustive()}\) nor that \( F_d = 0 \) for \( d \gg 0 \) (see \texttt{is_separating()}\). To construct a filtered vector space, use the \texttt{FilteredVectorSpace()} command. It supports easy creation of simple filtrations, for example the trivial one:

```sage
sage: FilteredVectorSpace(2, base_ring=RDF)
RDF^2
```

The next-simplest filtration has a single non-trivial inclusion between \( V_d \) and \( V_{d+1} \):

```sage
d = 1
sage: V = FilteredVectorSpace(2, d); V
QQ^2 >= 0
sage: [V.get_degree(i).dimension() for i in range(0,4)]
[2, 2, 0, 0]
```

To construct general filtrations, you need to tell Sage about generating vectors for the nested subspaces. For example, a dictionary whose keys are the degrees and values are a list of generators:

```sage
r1 = (1, 0, 5)
r2 = (0, 1, 2)
r3 = (1, 2, 1)
sage: V = FilteredVectorSpace({0:[r1, r2, r3], 1:[r1, r2], 3:[r1]}); V
QQ^3 >= QQ^2 >= QQ^1 >= QQ^1 >= 0
```

For degrees \( d \) that are not specified, the associated vector subspace is the same as the next-lower degree, that is, \( V_d \simeq V_{d-1} \). In the above example, this means that

- \( V_d \simeq \mathbb{Q}^3 \) for \( d < 0 \)
- \( V_0 = \text{span}(r_1, r_2) \simeq \mathbb{Q}^2 \)
- \( V_1 = V_2 = \text{span}(r_3) \simeq \mathbb{Q} \)
- \( V_d = 0 \) for \( d \geq 3 \)

That is:

```sage
sage: V.get_degree(0) == V
True
sage: V.get_degree(1) == V.span([r1, r2])
```

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If you have many generators you can just pass the generators once and then refer to them by index:

\[
\text{sage: FilteredVectorSpace([r1, r2, r3], \{0:[0,1,2], 1:[1,2], 3:[1]\})}
\]
\[
\text{QQ^3} \supseteq \text{QQ^2} \supseteq \text{QQ^1} \supseteq \text{QQ^1} \supseteq 0
\]

Note that generators for the degree-\(d\) subspace of the filtration are automatically generators for all lower degrees. For example, here we do not have to specify the ray \(r_2\) separately in degree 1:

\[
\text{sage: FilteredVectorSpace([r1, r2, r3], \{0:[0\ ], 1:[1]\})}
\]
\[
\text{QQ^2} \supseteq \text{QQ^1} \supseteq 0 \text{ in } \text{QQ^3}
\]

The degree can be infinite (plus infinity), this allows construction of filtered vector spaces that are not eventually zero in high degree:

\[
\text{sage: FilteredVectorSpace([r1, r2, r3], \{0:[0,1], oo:[1]\})}
\]
\[
\text{QQ^2} \supseteq \text{QQ^1} \text{ in } \text{QQ^3}
\]

Any field can be used as the vector space base. For example a finite field:

\[
\text{sage: F.<a> = GF(5^3)}
\]
\[
\text{sage: r1 = (a, 0, F(5)); r1}
\]
\[
(a, 0, 0)
\]
\[
\text{sage: FilteredVectorSpace([r1, r2, r3], \{0:[0,1], oo:[1]\}, base_ring=F)}
\]
\[
\text{GF(125)^2} \supseteq \text{GF(125)^1} \text{ in } \text{GF(125)^3}
\]

Or the algebraic field:

\[
\text{sage: r1 = (1, 0, 1+QQbar(I)); r1}
\]
\[
(1, 0, I + 1)
\]
\[
\text{sage: FilteredVectorSpace([r1, r2, r3], \{0:[0,1], oo:[1]\}, base_ring=QQbar)}
\]

Construct a filtered vector space.

INPUT:

This function accepts various input that determines the vector space and filtration.

- Just the dimension\texttt{FilteredVectorSpace(dimension)}: Return the trivial filtration (where all vector spaces are isomorphic).
- Dimension and maximal degree, see \texttt{constructor_from_dim_degree()} for arguments. Construct a filtration with only one non-trivial step \(V \supset 0\) at the given cutoff degree.
• A dictionary containing the degrees as keys and a list of vector space generators as values, see
  \texttt{FilteredVectorSpace\_from\_generators()}

• Generators and a dictionary containing the degrees as keys and the indices of vector space generators as
  values, see \texttt{FilteredVectorSpace\_from\_generators\_indices()}

In addition, the following keyword arguments are supported:

• \texttt{base\_ring} – a field (optional, default \texttt{Q}). The base field of the vector space. Must be a field.

**EXAMPLES:**

Just the dimension for the trivial filtration:

\begin{verbatim}
sage: FilteredVectorSpace(2)
sage: QQ^2
\end{verbatim}

Dimension and degree:

\begin{verbatim}
sage: FilteredVectorSpace(2, 1)
sage: QQ^2 >= 0
\end{verbatim}

Dictionary of generators:

\begin{verbatim}
sage: FilteredVectorSpace({1:[(1,0), (0,1)], 3:[(1,0)]})
sage: QQ^2 >= QQ^1 >= QQ^1 >= 0
\end{verbatim}

Generators and a dictionary referring to them by index:

\begin{verbatim}
sage: FilteredVectorSpace([(1,0), (0,1)], {1:[0,1], 3:[0]})
sage: QQ^2 >= QQ^1 >= QQ^1 >= 0
\end{verbatim}

**class** \texttt{sage.modules.filtered\_vector\_space.FilteredVectorSpace\_class}(base\_ring, dim, generators, filtration, check=True)

Bases: \texttt{sage.modules.free\_module.FreeModule\_ambient\_field}

A descending filtration of a vector space

**INPUT:**

• \texttt{base\_ring} – a field. The base field of the ambient vector space.

• \texttt{dim} – integer. The dimension of the ambient vector space.

• \texttt{generators} – tuple of generators for the ambient vector space. These will be used to span the subspaces
  of the filtration.

• \texttt{filtration} – a dictionary of filtration steps in ray index notation. See \texttt{construct\_from\_generators\_indices()} for details.

• \texttt{check} – boolean (optional; default: True). Whether to perform consistency checks.

**ambient\_vector\_space()**

Return the ambient (unfiltered) vector space.

**OUTPUT:**

A vector space.

**EXAMPLES:**
**change_ring**(base\_ring)
Return the same filtration over a different base ring.

**INPUT:**

**OUTPUT:**
This method returns a new filtered vector space whose subspaces are defined by the same generators but over a different base ring.

**EXAMPLES:**

```
sage: V = FilteredVectorSpace(1, 0); V
QQ^1 >= 0
sage: V.change_ring(RDF)
RDF^1 >= 0
```

**direct\_sum**(other)
Return the direct sum.

**INPUT:**
- other – a filtered vector space.

**OUTPUT:**
The direct sum as a filtered vector space.

**EXAMPLES:**

```
sage: V = FilteredVectorSpace(2, 0)
sage: W = FilteredVectorSpace({0:[(1,-1),(2,1)], 1:[(1,1)]})
sage: V.direct_sum(W)
QQ^4 >= QQ^1 >= 0
sage: V + W  # syntactic sugar
QQ^4 >= QQ^1 >= 0
sage: V + V == FilteredVectorSpace(4, 0)
True
sage: W = FilteredVectorSpace([(1,-1),(2,1)], {1:[0,1], 2:[1]})
sage: V + W
QQ^4 >= QQ^2 >= QQ^1 >= 0
```

A suitable base ring is chosen if they do not match:

```
sage: v = [(1,0), (0,1)]
sage: F1 = FilteredVectorSpace(v, {0:[0], 1:[1]}, base\_ring=QQ)
sage: F2 = FilteredVectorSpace(v, {0:[0], 1:[1]}, base\_ring=RDF)
sage: F1 + F2
RDF^4 >= RDF^2 >= 0
```

**dual()**
Return the dual filtered vector space.
OUTPUT:

The graded dual, that is, the dual of a degree-$d$ subspace is a set of linear constraints in degree $-d + 1$. That is, the dual generators live in degree $-d$.

EXAMPLES:

```
sage: gens = identity_matrix(3).rows()
sage: F = FilteredVectorSpace(gens, {0:[0,1,2], 2:[0]}); F
QQ^3 >= QQ^1 >= QQ^1 >= 0
sage: F.support()
(0, 2)
sage: F.dual()
QQ^3 >= QQ^2 >= QQ^2 >= 0
sage: F.dual().support()
(-2, 0)
```

`exterior_power(n)`

Return the $n$-th graded exterior power.

INPUT:


OUTPUT:

The graded exterior product, that is, the wedge product of a generator of degree $d_1$ with a generator in degree $d_2$ has degree $d_1 + d_2$.

EXAMPLES:

```
sage: F = FilteredVectorSpace(1, 1) + FilteredVectorSpace(1, 2); F
QQ^2 >= QQ^1 >= 0
sage: F.exterior_power(1)
QQ^2 >= QQ^1 >= 0
sage: F.exterior_power(2)
QQ^1 >= 0
sage: F.exterior_power(3)
0
sage: F.wedge(2)
QQ^1 >= 0
```

`get_degree(d)`

Return the degree-$d$ entry of the filtration.

INPUT:

- `d` – Integer. The desired degree of the filtration.

OUTPUT:

The degree-$d$ vector space in the filtration as subspace of the ambient space.

EXAMPLES:

```
sage: rays = [(1,0), (1,1), (1,2), (-1,-1)]
sage: F = FilteredVectorSpace(rays, {3:[1], 1:[1,2]})
sage: F.get_degree(2)
Vector space of degree 2 and dimension 1 over Rational Field
```
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(continued from previous page)

Basis matrix:
\[
\begin{bmatrix}
1 & 1
\end{bmatrix}
\]
sage: F.get_degree(oo)
Vector space of degree 2 and dimension 0 over Rational Field
Basis matrix:
\[
[
]
\]
sage: F.get_degree(-oo)
Vector space of degree 2 and dimension 2 over Rational Field
Basis matrix:
\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

graded\(d)\)
Return the associated graded vectorspace.

INPUT:

- \(d\) – integer. The degree.

OUTPUT:
The quotient \(G_d = F_d/F_{d+1}\).

EXAMPLES:

sage: rays = [(1,0), (1,1), (1,2)]
sage: F = FilteredVectorSpace(rays, {3:[1], 1:[1,2]})
sage: F.graded(1)
Vector space quotient V/W of dimension 1 over Rational Field where
V: Vector space of degree 2 and dimension 2 over Rational Field
Basis matrix:
\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]
W: Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
\[
\begin{bmatrix}
1 & 1
\end{bmatrix}
\]

is_constant()
Return whether the filtration is constant.

OUTPUT:

Boolean. Whether the filtered vector spaces are identical in all degrees.

EXAMPLES:

sage: V = FilteredVectorSpace(2); V
QQ^2
sage: V.is_constant()
True

sage: V = FilteredVectorSpace(1, 0); V
QQ^1 \geq 0
sage: V.is_constant()
False

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```
sage: V = FilteredVectorSpace({0:[(1,)]}); V
QQ^1 >= 0
sage: V.is_constant()
False
```

**is_exhaustive()**

Return whether the filtration is exhaustive.

A filtration \(\{F_d\}\) in an ambient vector space \(V\) is exhaustive if \(\bigcup F_d = V\). See also `is_separating()`.

**OUTPUT:**

Boolean.

**EXAMPLES:**

```
sage: F = FilteredVectorSpace({0:[(1,1)]}); F
QQ^1 >= 0 in QQ^2
sage: F.is_exhaustive()
False
sage: G = FilteredVectorSpace(2, 0); G
QQ^2 >= 0
sage: G.is_exhaustive()
True
```

**is_separating()**

Return whether the filtration is separating.

A filtration \(\{F_d\}\) in an ambient vector space \(V\) is exhaustive if \(\bigcap F_d = 0\). See also `is_exhaustive()`.

**OUTPUT:**

Boolean.

**EXAMPLES:**

```
sage: F = FilteredVectorSpace({0:[(1,1)]}); F
QQ^1 >= 0 in QQ^2
sage: F.is_separating()
True
sage: G = FilteredVectorSpace({0:[(1,1,0)], oo:[(0,0,1)]}); G
QQ^2 >= QQ^1 in QQ^3
sage: G.is_separating()
False
```

**max_degree()**

Return the highest degree of the filtration.

**OUTPUT:**

Integer or minus infinity. The smallest degree of the filtration such that the filtration is constant to the right.

**EXAMPLES:**

```
sage: FilteredVectorSpace(1, 3).max_degree()
4
sage: FilteredVectorSpace({0:[[1]]}).max_degree()
1
```

(continues on next page)
sage: FilteredVectorSpace(3).max_degree()
-Infinity

\section*{min_degree()}

Return the lowest degree of the filtration.

\textbf{OUTPUT:}

Integer or plus infinity. The largest degree \(d\) of the (descending) filtration such that the filtered vector space \(F_d\) is still equal to \(F_{-\infty}\).

\textbf{EXAMPLES:}

sage: FilteredVectorSpace(1, 3).min_degree()
3
sage: FilteredVectorSpace(2).min_degree()
+Infinity

\section*{presentation()}

Return a presentation in terms of generators of various degrees.

\textbf{OUTPUT:}

A pair consisting of generators and a filtration suitable as input to \texttt{construct_from_generators_indices()}.

\textbf{EXAMPLES:}

sage: rays = [(1,0), (1,1), (1,2), (-1,-1)]
sage: F = FilteredVectorSpace(rays, {0:[1, 2], 2:[3]}); F
QQ^2 \geq QQ^1 \geq QQ^1 \geq 0
sage: F.presentation()
(((0, 1), (1, 0), (1, 1)), {0: (1, 0), 2: (2,), +Infinity: ()})

\section*{random_deformation(\texttt{epsilon}=None)}

Return a random deformation.

\textbf{INPUT:}

- \texttt{epsilon} – a number in the base ring.

\textbf{OUTPUT:}

A new filtered vector space where the generators of the subspaces are moved by \texttt{epsilon} times a random vector.

\textbf{EXAMPLES:}

sage: gens = identity_matrix(3).rows()
sage: F = FilteredVectorSpace(gens, {0:[0,1,2], 2:[0]}); F
QQ^3 \geq QQ^1 \geq QQ^1 \geq 0
sage: F.get_degree(2)
Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[1 0 0]
sage: G = F.random_deformation(1/50); G
QQ^3 \geq QQ^1 \geq QQ^1 \geq 0
sage: D = G.get_degree(2)
shift(deg)
Return a filtered vector space with degrees shifted by a constant.

EXAMPLES:

\begin{verbatim}
sage: gens = identity_matrix(3).rows()
sage: F = FilteredVectorSpace(gens, {0:[0,1,2], 2:[0]}); F
QQ^3 >= QQ^1 >= QQ^1 >= 0
sage: F.support()
(0, 2)
sage: F.shift(-5).support()
(-5, -3)
\end{verbatim}

support()
Return the degrees in which there are non-trivial generators.

OUTPUT:
A tuple of integers (and plus infinity) in ascending order. The last entry is plus infinity if and only if the filtration is not separating (see is_separating()).

EXAMPLES:

\begin{verbatim}
sage: G = FilteredVectorSpace({0:[(1,1,0)], 3:[0,1,0]}); G
QQ^2 >= QQ^1 >= QQ^1 >= QQ^1 >= QQ^1 in QQ^3
sage: G.support()
(0, 3)
sage: G = FilteredVectorSpace({0:[(1,1,0)], 3:[0,1,0], oo:[(0,0,1)]}); G
QQ^3 >= QQ^2 >= QQ^2 >= QQ^1
sage: G.support()
(0, 3, +Infinity)
\end{verbatim}

symmetric_power(n)
Return the \(n\)-th graded symmetric power.

INPUT:

- \(n\) – integer. Symmetric product of how many copies of self.

OUTPUT:

The graded symmetric product, that is, the symmetrization of a generator of degree \(d_1\) with a generator in degree \(d_2\) has degree \(d_1 + d_2\).

EXAMPLES:
sage: F = FilteredVectorSpace(1, 1) + FilteredVectorSpace(1, 2); F
QQ^2 >= QQ^1 >= 0
sage: F.symmetric_power(2)
QQ^3 >= QQ^2 >= QQ^1 >= 0
tensor_product(other)
Return the graded tensor product.

INPUT:
• other – a filtered vector space.

OUTPUT:
The graded tensor product, that is, the tensor product of a generator of degree \(d_1\) with a generator in degree \(d_2\) has degree \(d_1 + d_2\).

EXAMPLES:

sage: F1 = FilteredVectorSpace(1, 1)
sage: F2 = FilteredVectorSpace(1, 2)
sage: F1.tensor_product(F2)
QQ^1 >= 0
sage: F1 * F2
QQ^1 >= 0
sage: F1.min_degree()
1
sage: F2.min_degree()
2
sage: (F1*F2).min_degree()
3

A suitable base ring is chosen if they do not match:

sage: v = [(1,0), (0,1)]
sage: F1 = FilteredVectorSpace(v, {0:[0], 1:[1]}, base_ring=QQ)
sage: F2 = FilteredVectorSpace(v, {0:[0], 1:[1]}, base_ring=RDF)
sage: F1 * F2
RDF^4 >= RDF^3 >= RDF^1 >= 0

wedge(n)
Return the \(n\)-th graded exterior power.

INPUT:
• n – integer. Exterior product of how many copies of self.

OUTPUT:
The graded exterior product, that is, the wedge product of a generator of degree \(d_1\) with a generator in degree \(d_2\) has degree \(d_1 + d_2\).

EXAMPLES:

sage: F = FilteredVectorSpace(1, 1) + FilteredVectorSpace(1, 2); F
QQ^2 >= QQ^1 >= 0
sage: F.exterior_power(1)
QQ^2 >= QQ^1 >= 0

sage: F.exterior_power(2)
QQ^1 >= 0
sage: F.exterior_power(3)
0
sage: F.wedge(2)
QQ^1 >= 0

sage.modules.filtered_vector_space.construct_from_dim_degree(dim, max_degree, base_ring, check)
Construct a filtered vector space.

INPUT:

- `dim` – integer. The dimension.
- `max_degree` – integer or infinity. The maximal degree where the vector subspace of the filtration is still the entire space.

EXAMPLES:

sage: V = FilteredVectorSpace(2, 5); V
Vector space of degree 2 and dimension 2 over Rational Field
Basis matrix:
[1 0]
[0 1]
sage: V.get_degree(5)
Vector space of degree 2 and dimension 2 over Rational Field
Basis matrix:
[1 0]
[0 1]
sage: V.get_degree(6)
Vector space of degree 2 and dimension 0 over Rational Field
Basis matrix:
[]
sage: FilteredVectorSpace(2, oo)
QQ^2
sage: FilteredVectorSpace(2, -oo)
0 in QQ^2

sage.modules.filtered_vector_space.construct_from_generators(filtration, base_ring, check)
Construct a filtered vector space.

INPUT:

- `filtration` – a dictionary of filtration steps. Each filtration step is a pair consisting of an integer degree and a list/tuple/iterable of vector space generators. The integer `degree` stipulates that all filtration steps of degree higher or equal than `degree` (up to the next filtration step) are said subspace.

EXAMPLES:

sage: from sage.modules.filtered_vector_space import construct_from_generators
sage: r = [1, 2]
sage: construct_from_generators({1:[r]}, QQ, True)
QQ^1 >= 0 in QQ^2

sage.modules.filtered_vector_space.construct_from_generators_indices(generators, filtration, base_ring, check)
Construct a filtered vector space.
INPUT:

- **generators** – a list/tuple/iterable of vectors, or something convertible to them. The generators spanning various subspaces.
- **filtration** – a list or iterable of filtration steps. Each filtration step is a pair (degree, ray_indices). The ray_indices are a list or iterable of ray indices, which span a subspace of the vector space. The integer degree stipulates that all filtration steps of degree higher or equal than degree (up to the next filtration step) are said subspace.

EXAMPLES:

```python
sage: from sage.modules.filtered_vector_space import construct_from_generators_indices
sage: gens = [(1,0), (0,1), (-1,-1)]

sage: V = construct_from_generators_indices(gens, {1:[0,1], 3:[1]}, QQ, True); V
QQ^2 >= QQ^1 >= QQ^1 >= 0
```

`sage.modules.filtered_vector_space.is_FilteredVectorSpace(X)`

Test whether X is a filtered vector space.

INPUT:

- X – anything.

OUTPUT:

Boolean.

EXAMPLES:

```python
sage: from sage.modules.filtered_vector_space import is_FilteredVectorSpace
sage: V = FilteredVectorSpace(2, 1)

sage: is_FilteredVectorSpace(V)
True
sage: is_FilteredVectorSpace('ceci n’est pas une pipe')
False
```

`sage.modules.filtered_vector_space.normalize_degree(deg)`

Normalized the degree

- deg – something that defines the degree (either integer or infinity).

OUTPUT:

Plus/minus infinity or a Sage integer.

EXAMPLES:

```python
sage: from sage.modules.filtered_vector_space import normalize_degree
sage: type(normalize_degree(int(1)))
<class 'sage.rings.integer.Integer'>

sage: normalize_degree(oo)
+Infinity
```
MULTIPLE $\mathbb{Z}$-GRADED FILTRATIONS OF A SINGLE VECTOR SPACE

See filtered_vector_space for simply graded vector spaces. This module implements the analog but for a collection of filtrations of the same vector space.

The basic syntax to use it is a dictionary whose keys are some arbitrary indexing set and values are FilteredVectorSpace()

```python
sage: F1 = FilteredVectorSpace(2, 1)
sage: F2 = FilteredVectorSpace({0:[(1,0), 2:[(2,3)]})
sage: V = MultiFilteredVectorSpace({'first':F1, 'second':F2})
sage: V
Filtrations
    first: $\mathbb{Q}^2 \geq \mathbb{Q}^2 \geq 0 \geq 0$
    second: $\mathbb{Q}^2 \geq \mathbb{Q}^1 \geq \mathbb{Q}^1 \geq 0$

sage: V.index_set() # random output
{'second', 'first'}
sage: sorted(V.index_set())
['first', 'second']

sage: V.get_filtration('first')
$\mathbb{Q}^2 \geq 0$

sage: V.get_degree('second', 1)
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[ 1 3/2]
```

sage.modules.multi_filtered_vector_space.MultiFilteredVectorSpace(arg, base_ring=None, check=True)

Construct a multi-filtered vector space.

INPUT:

- arg – either a non-empty dictionary of filtrations or an integer. The latter is interpreted as the vector space dimension, and the indexing set of the filtrations is empty.

- base_ring – a field (optional, default 'None'). The base field of the vector space. Must be a field. If not specified, the base field is derived from the filtrations.

- check – boolean (optional; default: True). Whether to perform consistency checks.

EXAMPLES:

```python
sage: MultiFilteredVectorSpace(3, QQ)
Unfiltered $\mathbb{Q}^3$
```

(continues on next page)
sage: F1 = FilteredVectorSpace(2, 1)
sage: F2 = FilteredVectorSpace(2, 3)
sage: V = MultiFilteredVectorSpace({1:F1, 2:F2}); V

Filtrations
    1: QQ^2 >= 0 >= 0 >= 0
    2: QQ^2 >= QQ^2 >= QQ^2 >= 0

class sage.modules.multi_filtered_vector_space.MultiFilteredVectorSpace_class(base_ring, dim, filtrations, check=True)

Bases: sage.modules.free_module.FreeModule_ambient_field

Python constructor.

Warning: Use MultiFilteredVectorSpace() to construct multi-filtered vector spaces.

INPUT:

• base_ring – a ring. the base ring.
• dim – integer. The dimension of the ambient vector space.
• filtrations – a dictionary whose values are filtrations.
• check – boolean (optional). Whether to perform additional consistency checks.

EXAMPLES:

sage: F1 = FilteredVectorSpace(2, 1)
sage: F2 = FilteredVectorSpace(2, 3)
sage: V = MultiFilteredVectorSpace({1:F1, 2:F2}); V

Filtrations
    1: QQ^2 >= 0 >= 0 >= 0
    2: QQ^2 >= QQ^2 >= QQ^2 >= 0

ambient_vector_space()

Return the ambient (unfiltered) vector space.

OUTPUT:

A vector space.

EXAMPLES:

sage: V = FilteredVectorSpace(2, 0)
sage: W = FilteredVectorSpace(2, 2)
sage: F = MultiFilteredVectorSpace({'a':V, 'b':W})
sage: F.ambient_vector_space()

Vector space of dimension 2 over Rational Field

change_ring(base_ring)

Return the same multi-filtration over a different base ring.

INPUT:
• `base_ring` – a ring. The new base ring.

OUT\textsc{P}UT:

This method returns a new multi-filtered vector space whose subspaces are defined by the same generators but over a different base ring.

\textsc{E}X\textsc{A}M\textsc{P}L\textsc{E}\textsc{S}\textsc{S}:

\begin{verbatim}
sage: V = FilteredVectorSpace(2, 0)
sage: W = FilteredVectorSpace(2, 2)
sage: F = MultiFilteredVectorSpace({'a':V, 'b':W}); F
Filtrations
  a: QQ^2 >= 0 >= 0 >= 0
  b: QQ^2 >= QQ^2 >= QQ^2 >= 0
sage: F.change_ring(RDF)
Filtrations
  a: RDF^2 >= 0 >= 0 >= 0
  b: RDF^2 >= RDF^2 >= RDF^2 >= 0
sage: MultiFilteredVectorSpace(3, base_ring=QQ).change_ring(RR)
Unfiltered RR^3
\end{verbatim}

\texttt{d\textasciitilde r\textasciitilde i\textasciitilde c\textasciitilde t\textasciitilde \textunderscore \textasciitilde s\textasciitilde u\textasciitilde m}(\texttt{o\textasciitilde t\texttt{h\textasciitilde \texttt{e\texttt{r\textasciitilde}}}\texttt{r\textasciitilde \texttt{e\texttt{ller}}})

Return the direct sum.

IN\textsc{P}UT:

• `other` – a multi-filtered vector space with the same \texttt{index\textunderscore set()}.\n
OUT\textsc{P}UT:

The direct sum as a multi-filtered vector space. See \texttt{d\textasciitilde r\textasciitilde c\textasciitilde t\textasciitilde \textunderscore \textasciitilde s\textasciitilde u\textasciitilde m()}.\n
\textsc{E}X\textsc{A}M\textsc{P}L\textsc{E}\textsc{S}\textsc{S}:

\begin{verbatim}
sage: F1 = FilteredVectorSpace(2, 1)
sage: F2 = FilteredVectorSpace(1, 3) + FilteredVectorSpace(1,0)
sage: V = MultiFilteredVectorSpace({'a':F1, 'b':F2})
sage: G1 = FilteredVectorSpace(1, 1)
sage: G2 = FilteredVectorSpace(1, 3)
sage: W = MultiFilteredVectorSpace({'a':G1, 'b':G2})
sage: V.direct_sum(W)
  Filtrations
    a: QQ^3 >= QQ^3 >= 0 >= 0 >= 0
    b: QQ^3 >= QQ^2 >= QQ^2 >= QQ^2 >= 0
sage: V + W  # syntactic sugar
  Filtrations
    a: QQ^3 >= QQ^3 >= 0 >= 0 >= 0
    b: QQ^3 >= QQ^2 >= QQ^2 >= QQ^2 >= 0
\end{verbatim}

d\texttt{u\textasciitilde a\textasciitilde l()}\n
Return the dual.

OUT\textsc{P}UT:

The dual as a multi-filtered vector space. See \texttt{d\textasciitilde u\textasciitilde a\textasciitilde l()}.\n
\textsc{E}X\textsc{A}M\textsc{P}L\textsc{E}\textsc{S}\textsc{S}:
sage: F1 = FilteredVectorSpace(2, 1)
sage: F2 = FilteredVectorSpace(1, 3) + FilteredVectorSpace(1,0)
sage: V = MultiFilteredVectorSpace({'a':F1, 'b':F2})
sage: V.dual()
Filtrations
  a: QQ^2 >= QQ^2 >= QQ^2 >= 0 >= 0
  b: QQ^2 >= QQ^1 >= QQ^1 >= QQ^1 >= 0

**exterior_power(n)**

Return the \( n \)-th graded exterior power.

**INPUT:**


**OUTPUT:**

The exterior power as a multi-filtered vector space. See `exterior_power()`.

**EXAMPLES:**

sage: F1 = FilteredVectorSpace(2, 1)
sage: F2 = FilteredVectorSpace(1, 3) + FilteredVectorSpace(1,0)
sage: V = MultiFilteredVectorSpace({'a':F1, 'b':F2})
sage: V.exterior_power(2)  # long time
Filtrations
  a: QQ^1 >= 0 >= 0
  b: QQ^1 >= QQ^1 >= 0

**get_degree(key, deg)**

Return one filtered vector space.

**INPUT:**

- `key` – an element of the `index_set()`. Specifies which filtration.
- `deg` – Integer. The desired degree of the filtration.

**OUTPUT:**

The vector space of degree `deg` in the filtration indexed by `key` as subspace of the ambient space.

**EXAMPLES:**

sage: F1 = FilteredVectorSpace(2, 1)
sage: F2 = FilteredVectorSpace(2, 3)
sage: V = MultiFilteredVectorSpace({1:F1, 2:F2})
sage: V.get_degree(2, 0)
Vector space of degree 2 and dimension 2 over Rational Field
Basis matrix:
[1 0]
[0 1]

**get_filtration(key)**

Return the filtration indexed by `key`.

**OUTPUT:**

A filtered vector space.

**EXAMPLES:**
sage: F1 = FilteredVectorSpace(2, 1)
sage: F2 = FilteredVectorSpace(2, 3)
sage: V = MultiFilteredVectorSpace({1:F1, 2:F2})
sage: V.get_filtration(2)
QQ^2 >= 0
graded(key, deg)
Return the associated graded vector space.

INPUT:

• key – an element of the index_set(). Specifies which filtration.
• d – Integer. The desired degree of the filtration.

OUTPUT:

The quotient \( G_d = F_d/F_{d+1} \) of the filtration \( F \) corresponding to key.

EXAMPLES:
sage: F1 = FilteredVectorSpace(2, 1)
sage: F2 = FilteredVectorSpace(1, 3) + FilteredVectorSpace(1,0)
sage: V = MultiFilteredVectorSpace({1:F1, 2:F2})
sage: V.graded(2, 3)
Vector space quotient V/W of dimension 1 over Rational Field where
V: Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[1 0]
W: Vector space of degree 2 and dimension 0 over Rational Field
Basis matrix:
[]

index_set()
Return the allowed indices for the different filtrations.

OUTPUT:

Set.

EXAMPLES:
sage: F1 = FilteredVectorSpace(2, 1)
sage: F2 = FilteredVectorSpace(2, 3)
sage: V = MultiFilteredVectorSpace({1:F1, 2:F2})
sage: V.index_set()
{1, 2}

is_constant()
Return whether the multi-filtration is constant.

OUTPUT:

Boolean. Whether the each filtration is constant, see is_constant().

EXAMPLES:
sage: V = FilteredVectorSpace(2, 0)
sage: W = FilteredVectorSpace(2, 2)
sage: F = MultiFilteredVectorSpace({'a':V, 'b':W}); F
Filtrations
  a: QQ^2 >= 0 >= 0 >= 0
  b: QQ^2 >= QQ^2 >= QQ^2 >= 0
sage: F.is_constant()
False

is_exhaustive()
Return whether the multi-filtration is exhaustive.

A filtration \( \{F_d\} \) in an ambient vector space \( V \) is exhaustive if \( \bigcup F_d = V \). See also \is_separating().

OUTPUT:
Boolean. Whether each filtration is constant, see \is_exhaustive().

EXAMPLES:

```sage
sage: F1 = FilteredVectorSpace(2, 1)
sage: F2 = FilteredVectorSpace(2, 3)
sage: V = MultiFilteredVectorSpace({1:F1, 2:F2})
sage: V.is_exhaustive()
True
```

is_separating()
Return whether the multi-filtration is separating.

A filtration \( \{F_d\} \) in an ambient vector space \( V \) is exhaustive if \( \bigcap F_d = 0 \). See also \is_exhaustive().

OUTPUT:
Boolean. Whether each filtration is separating, see \is_separating().

EXAMPLES:

```sage
sage: F1 = FilteredVectorSpace(2, 1)
sage: F2 = FilteredVectorSpace(2, 3)
sage: V = MultiFilteredVectorSpace({1:F1, 2:F2})
sage: V.is_separating()
True
```

max_degree()
Return the highest degree of the filtration.

OUTPUT:
Integer or minus infinity. The smallest degree of the filtrations such that the filtrations are constant to the right.

EXAMPLES:

```sage
sage: F1 = FilteredVectorSpace(2, 1)
sage: F2 = FilteredVectorSpace(2, 3)
sage: V = MultiFilteredVectorSpace({1:F1, 2:F2})
sage: V.max_degree()
4
```

min_degree()
Return the lowest degree of the filtration.
OUTPUT:

Integer or plus infinity. The largest degree $d$ of the (descending) filtrations such that, for each individual filtration, the filtered vector space $F_d$ still equal to $F_{-\infty}$.

EXAMPLES:

```python
sage: F1 = FilteredVectorSpace(2, 1)
sage: F2 = FilteredVectorSpace(2, 3)
sage: V = MultiFilteredVectorSpace({1:F1, 2:F2})
sage: V.min_degree()
1
```

`random_deformation(epsilon=None)`

Return a random deformation

INPUT:

- `epsilon` – a number in the base ring.

OUTPUT:

A new multi-filtered vector space where the generating vectors of subspaces are moved by `epsilon` times a random vector.

EXAMPLES:

```python
sage: F1 = FilteredVectorSpace(2, 1)
sage: F2 = FilteredVectorSpace(1, 3) + FilteredVectorSpace(1,0)
sage: V = MultiFilteredVectorSpace({'a':F1, 'b':F2})
sage: V.get_degree('b',1)
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[1 0]
sage: D = V.random_deformation(1/100).get_degree('b',1)
sage: D.degree()
2
sage: D.dimension()
1
sage: D.matrix()[0, 0]
1

sage: while V.random_deformation(1/100).get_degree('b',1).matrix() == matrix([1, ˓→ 0]):
    ....:     pass
```

`shift(deg)`

Return a filtered vector space with degrees shifted by a constant.

OUTPUT:

The shift of `self`. See `shift()`.

EXAMPLES:

```python
sage: F1 = FilteredVectorSpace(2, 1)
sage: F2 = FilteredVectorSpace(1, 3) + FilteredVectorSpace(1,0)
sage: V = MultiFilteredVectorSpace({'a':F1, 'b':F2})
sage: V.support()
```
support()

Return the degrees in which there are non-trivial generators.

OUTPUT:

A tuple of integers (and plus infinity) in ascending order. The last entry is plus infinity if and only if the filtration is not separating (see `is_separating()`).

EXAMPLES:

```python
sage: F1 = FilteredVectorSpace(2, 1)
sage: F2 = FilteredVectorSpace(2, 3)
sage: V = MultiFilteredVectorSpace({'1':F1, '2':F2})
sage: V.support()
(1, 3)
```

symmetric_power(n)

Return the \(n\)-th graded symmetric power.

INPUT:

- \(n\) – integer. Symmetric product of how many copies of \(self\).

OUTPUT:

The symmetric power as a multi-filtered vector space. See `symmetric_power()`.

EXAMPLES:

```python
sage: F1 = FilteredVectorSpace(2, 1)
sage: F2 = FilteredVectorSpace(1, 3) + FilteredVectorSpace(1,0)
sage: V = MultiFilteredVectorSpace({'a':F1, 'b':F2})
sage: V.symmetric_power(2)
```

tensor_product(other)

Return the graded tensor product.

INPUT:

- \(other\) – a multi-filtered vector space with the same `index_set()`.

OUTPUT:

The tensor product of \(self\) and \(other\) as a multi-filtered vector space. See `tensor_product()`.

EXAMPLES:

```python
sage: F1 = FilteredVectorSpace(2, 1)
sage: F2 = FilteredVectorSpace(1, 3) + FilteredVectorSpace(1,0)
sage: V = MultiFilteredVectorSpace({'a':F1, 'b':F2})
sage: G1 = FilteredVectorSpace(1, 1)
sage: G2 = FilteredVectorSpace(1, 3)
```
```python
sage: W = MultiFilteredVectorSpace({'a':G1, 'b':G2})
sage: V.tensor_product(W)
Filtrations
  a: QQ^2 >= 0 >= 0 >= 0 >= 0 >= 0
  b: QQ^2 >= QQ^2 >= QQ^1 >= QQ^1 >= QQ^1 >= 0
sage: V * W  # syntactic sugar
Filtrations
  a: QQ^2 >= 0 >= 0 >= 0 >= 0 >= 0
  b: QQ^2 >= QQ^2 >= QQ^1 >= QQ^1 >= QQ^1 >= 0
```

### wedge(n)

Return the $n$-th graded exterior power.

**INPUT:**


**OUTPUT:**

The exterior power as a multi-filtered vector space. See `exterior_power()`.

**EXAMPLES:**

```python
sage: F1 = FilteredVectorSpace(2, 1)
sage: F2 = FilteredVectorSpace(1, 3) + FilteredVectorSpace(1,0)
sage: V = MultiFilteredVectorSpace({'a':F1, 'b':F2})
sage: V.exterior_power(2)  # long time
Filtrations
  a: QQ^1 >= 0 >= 0
  b: QQ^1 >= QQ^1 >= 0
```

CHAPTER

FORTYTW0

HELPER CLASSES TO IMPLEMENT TENSOR OPERATIONS

**Warning:** This module is not meant to be used directly. It just provides functionality for other classes to implement tensor operations.

The `VectorCollection` constructs the basis of tensor products (and symmetric/exterior powers) in terms of a chosen collection of vectors that generate the vector space(s).

**EXAMPLES:**

```sage
from sage.modules.tensor_operations import VectorCollection, TensorOperation
sage: V = VectorCollection([(1,0), (-1, 0), (1,2)], QQ, 2)
sage: W = VectorCollection([(1,1), (1,-1), (-1, 1)], QQ, 2)
sage: VW = TensorOperation([V, W], operation='product')
```

Here is the tensor product of two vectors:

```sage
sage: V.vectors()[0]
(1, 0)
sage: W.vectors()[1]
(1, -1)
```

In a convenient choice of basis, the tensor product is \((a, b) \otimes (c, d) = (ac, ad, bc, bd)\). In this example, it is one of the vectors of the vector collection `VW`

```sage
sage: VW.index_map(0, 1)
1
sage: VW.vectors()[VW.index_map(0, 1)]
(1, -1, 0, 0)
```

```sage
sage: rows = []
sage: for i, j in cartesian_product((range(3), range(3))):
    ....:    v = V.vectors()[i]
    ....:    w = W.vectors()[j]
    ....:    i_tensor_j = VW.index_map(i, j)
    ....:    vw = VW.vectors()[i_tensor_j]
    ....:    rows.append([i, v, j, w, i_tensor_j, vw])
sage: table(rows)
```

(continues on next page)
class sage.modules.tensor_operations.TensorOperation(vector_collections, operation='product')

Bases: sage.modules.tensor_operations.VectorCollection

Auxiliary class to compute the tensor product of two VectorCollection objects.

**Warning:** This class is only used as a base class for filtered vector spaces. You should not use it yourself.

**INPUT:**
- `vector_collections` – a nonempty list/tuple/iterable of VectorCollection objects.
- `operation` – string. The tensor operation. Currently allowed values are product, symmetric, and antisymmetric.

**Todo:** More general tensor operations (specified by Young tableaux) should be implemented.

**EXAMPLES:**

```python
sage: from sage.modules.tensor_operations import VectorCollection, TensorOperation
sage: R = VectorCollection([(1,0), (1,2), (-1,-2)], QQ, 2)
sage: S = VectorCollection([(1,), (-1,)], QQ, 1)
sage: R_tensor_S = TensorOperation([R, S])
```

```python
sage: R_tensor_S.index_map(0, 0)
0
sage: matrix(ZZ, 3, 2, lambda i,j: R_tensor_S.index_map(i, j))
[[0 1]
 [2 3]
 [3 2]]
```

```python
sage: R_tensor_S.vectors()
((1, 0), (-1, 0), (1, 2), (-1, -2))
```

**codomain()**

The codomain of the index map.

**OUTPUT:**

A list of integers. The image of `index_map()`.

**EXAMPLES:**

```python
sage: from sage.modules.tensor_operations import VectorCollection, TensorOperation
sage: R = VectorCollection([(1,0), (0,1), (-2,-3)], QQ, 2)
```

```python
sage: detR = TensorOperation([R]*2, 'antisymmetric')
```

```python
sage: sorted(detR.preimage())
[(0, 1), (0, 2), (1, 2)]
```
index_map(*i)
Return the result of the tensor operation.

INPUT:
• *i – list of integers. The indices (in the corresponding factor of the tensor operation) of the domain vector.

OUTPUT:
The index (in vectors()) of the image of the tensor product/operation acting on the domain vectors indexed by i.
None is returned if the tensor operation maps the generators to zero (usually because of antisymmetry).

EXAMPLES:

```python
sage: from sage.modules.tensor_operations import VectorCollection, TensorOperation
sage: R = VectorCollection([(1,0), (1,2), (-1,-2)], QQ, 2)

sage: Sym3_R = TensorOperation([R]*3, 'symmetric')

sage: Sym3_R.index_map(0, 1, 2)
4

sage: Sym3_R.vectors()[4]
(-1, -4, -4, 0)
```

The symmetric product of the first vector (1,0), the second vector (1,2), and the third vector (-1,-2) equals the vector with index number 4 (that is, the fifth) in the symmetric product vector collection:

```python
sage: Sym3_R.index_map(2, 0, 1)
4
sage: Sym3_R.index_map(2, 1, 0)
4
```

As another example, here is the rank-2 determinant:

```python
sage: from sage.modules.tensor_operations import VectorCollection, TensorOperation
sage: R = VectorCollection([(1,0), (0,1), (-2,-3)], QQ, 2)

sage: detR = TensorOperation([R]*2, 'antisymmetric')

sage: detR.index_map(1, 0)
0
sage: detR.index_map(0, 1)
```

preimage()
A choice of pre-image multi-indices.
OUTPUT:
A list of multi-indices (tuples of integers) whose image is the entire image under the \texttt{index\_map()}. 

EXAMPLES:

```python
sage: from sage.modules.tensor_operations import ...:
   VectorCollection, TensorOperation
sage: R = VectorCollection([(1,0), (0,1), (-2,-3)], QQ, 2)
sage: detR = TensorOperation([R]^2, 'antisymmetric')
```

```python
dsage: sorted(detR.preimage())
[(0, 1), (0, 2), (1, 2)]
dsage: sorted(detR.codomain())
[0, 1, 2]
```

class \texttt{sage.modules.tensor\_operations.VectorCollection}(\texttt{vector\_collection, base\_ring, dim})

\texttt{Bases}: \texttt{sage.modules.free\_module.FreeModule\_ambient\_field}

An ordered collection of generators of a vector space.

This is like a list of vectors, but with extra argument checking.

\textbf{Warning:} This class is only used as a base class for filtered vector spaces. You should not use it yourself.

INPUT:

- \texttt{dim} – integer. The dimension of the ambient vector space.
- \texttt{base\_ring} – a field. The base field of the ambient vector space.
- \texttt{rays} – any list/iterable of things than can be converted into vectors of the ambient vector space. These will be used to span the subspaces of the filtration. Must span the ambient vector space.

EXAMPLES:

```python
sage: from sage.modules.tensor_operations import VectorCollection
sage: R = VectorCollection([(1,0), (0,1), (1,2)], QQ, 2); R
Vector space of dimension 2 over Rational Field
```

\texttt{n\_vectors()}
Return the number of vectors

OUTPUT:

Integer.

EXAMPLES:

```python
sage: from sage.modules.tensor_operations import VectorCollection
sage: V = VectorCollection([(1,0), (0,1), (1,2)], QQ, 2)
sage: V.n\_vectors()
3
```

\texttt{vectors()}
Return the collection of vectors

OUTPUT:

A tuple of vectors. The vectors that were specified in the constructor, in the same order.
EXAMPLES:

```python
sage: from sage.modules.tensor_operations import VectorCollection
sage: V = VectorCollection([(1,0), (0,1), (1,2)], QQ, 2)
sage: V.vectors()
((1, 0), (0, 1), (1, 2))
```

```
sage.modules.tensor_operations.antisymmetrized_coordinate_sums(dim, n)

Return formal anti-symmetrized sum of multi-indices

INPUT:

• dim – integer. The dimension (range of each index).
• n – integer. The total number of indices.

OUTPUT:

An anti-symmetrized formal sum of multi-indices (tuples of integers)

EXAMPLES:

```python
sage: from sage.modules.tensor_operations import antisymmetrized_coordinate_sums
sage: antisymmetrized_coordinate_sums(3, 2)
((0, 1) - (1, 0), (0, 2) - (2, 0), (1, 2) - (2, 1))
```
```

```
sage.modules.tensor_operations.symmetrized_coordinate_sums(dim, n)

Return formal symmetrized sum of multi-indices

INPUT:

• dim – integer. The dimension (range of each index).
• n – integer. The total number of indices.

OUTPUT:

A symmetrized formal sum of multi-indices (tuples of integers)

EXAMPLES:

```python
sage: from sage.modules.tensor_operations import symmetrized_coordinate_sums
sage: symmetrized_coordinate_sums(2, 2)
((0, 0), (0, 1) + (1, 0), (1, 1))
```
43.1 Finitely generated free graded left modules over connected graded algebras

Let $A$ be a connected graded algebra. Some methods here require in addition that $A$ be an algebra over a field or a PID and that Sage has a description of a basis for $A$.

For example, let $p$ be a prime number. The mod $p$ Steenrod algebra $A_p$ is a connected algebra over the finite field of $p$ elements. Many of the modules presented here will be defined over $A_p$, or one of its sub-Hopf algebras. E.g.:

```
sage: A = SteenrodAlgebra(p=2)
```

However, the current implementation can use any connected graded algebra that has a graded basis where each graded part is finite dimensional. Another good family is the exterior algebras:

```
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
```

A free module is defined by the graded algebra and an ordered tuple of degrees for the generators:

```
sage: M = A.free_graded_module(generator_degrees=(0,1))
sage: M
```
Free graded left module on 2 generators over mod 2 Steenrod algebra, milnor basis

```
sage: F.<a,b,c> = E.free_graded_module((0,3,6))
sage: F
```
Free graded left module on 3 generators over The exterior algebra of rank 3 over Rational Field

The resulting free modules will have generators in the degrees as specified:

```
sage: M.generator_degrees()
(0, 1)
sage: F.generator_degrees()
(0, 3, 6)
```

The default names for the generators are $g[\text{degree}]$ if they are in distinct degrees, $g[\text{degree, i}]$ otherwise. They can be given other names, as was done when creating the module $F$:

```
sage: M.generators()
(g[0], g[1])
```
(continues on next page)
The connectivity of a module over a connected graded algebra is the minimum degree of all its module generators. Thus, if the module is non-trivial, the connectivity is an integer:

```
sage: M.connectivity()
0
```

### 43.1.1 Module elements

For an $A$-module with generators $\{g_i\}_{i=1}^N$, any homogeneous element of degree $n$ has the form

$$x = \sum_{i=1}^N a_i \cdot g_i,$$

where $a_i \in A_{n - \deg(g_i)}$ for all $i$. The ordered set $\{a_i\}$ is referred to as the coefficients of $x$.

You can produce module elements from a given set of coefficients:

```
sage: coeffs = [Sq(5), Sq(1,1)]
sage: x = M(coeffs); x
Sq(5)*g[0] + Sq(1,1)*g[1]
```

You can also use the module action:

```
sage: Sq(2) * x
(Sq(4,1)+Sq(7))*g[0] + Sq(3,1)*g[1]
```

Each non-zero element has a well-defined degree:

```
sage: x.degree()
5
```

However the zero element does not:

```
sage: zero = M.zero(); zero
0
sage: zero.degree()
Traceback (most recent call last):
  ...
ValueError: the zero element does not have a well-defined degree
```

Any two elements can be added as long as they are in the same degree:

```
sage: y = M.an_element(5); y
Sq(2,1)*g[0] + Sq(4)*g[1]
sage: x + y
(Sq(2,1)+Sq(5))*g[0] + (Sq(1,1)+Sq(4))*g[1]
```

or when at least one of them is zero:
Finally, additive inverses exist:

```
sage: x - x
0
```

For every integer $n$, the set of module elements of degree $n$ form a free module over the ground ring $k$. A basis for this free module can be computed:

```
sage: M.basis_elements(5)
(Sq(2,1)*g[0], Sq(5)*g[0], Sq(1,1)*g[1], Sq(4)*g[1])
```

together with a corresponding free module presentation:

```
sage: M.vector_presentation(5)
Vector space of dimension 4 over Finite Field of size 2
```

Given any element, its coordinates with respect to this basis can be computed:

```
sage: v = x.vector_presentation(); v
(0, 1, 1, 0)
```

Going the other way, any element can be constructed by specifying its coordinates:

```
sage: x_ = M.element_from_coordinates((0,1,1,0), 5)
sage: x_
Sq(5)*g[0] + Sq(1,1)*g[1]
sage: x_ == x
True
```

### 43.1.2 Module homomorphisms

Homomorphisms of free graded $A$-modules $M \rightarrow N$ are linear maps of their underlying free $k$-module which commute with the $A$-module structure.

To create a homomorphism, first create the object modeling the set of all such homomorphisms using the free function `Hom`:

```
sage: M = A.free_graded_module((0,1))
sage: N.<c2> = A.free_graded_module((2,))
sage: homspace = Hom(M, N); homspace
Set of Morphisms from Free graded left module on 2 generators
   over mod 2 Steenrod algebra, milnor basis
to Free graded left module on 1 generator
   over mod 2 Steenrod algebra, milnor basis
in Category of finite dimensional graded modules with basis
   over mod 2 Steenrod algebra, milnor basis
```

Just as module elements, homomorphisms are created using the `homspace` object. The only argument is a list of module elements in the codomain, corresponding to the module generators of the domain:
The resulting homomorphism is the one sending the $i$-th generator of the domain to the $i$-th codomain value given:

```
sage: f
Module morphism:
    From: Free graded left module on 2 generators over mod 2 Steenrod algebra, milnor basis
    To:   Free graded left module on 1 generator over mod 2 Steenrod algebra, milnor basis
    Defn: g[0] |--> Sq(2)*c2
          g[1] |--> (Sq(0,1)+Sq(3))*c2
```

Convenience methods exist for creating the trivial morphism:

```
sage: homspace.zero()
Module morphism:
    From: Free graded left module on 2 generators over mod 2 Steenrod algebra, milnor basis
    To:   Free graded left module on 1 generator over mod 2 Steenrod algebra, milnor basis
    Defn: g[0] |--> 0
          g[1] |--> 0
```

as well as the identity endomorphism:

```
sage: Hom(M, M).identity()
Module endomorphism of Free graded left module on 2 generators over mod 2 Steenrod algebra, milnor basis
    Defn: g[0] |--> g[0]
          g[1] |--> g[1]
```

Homomorphisms can be evaluated on elements of the domain module:

```
sage: v1 = f(Sq(7)*M.generator(0)); v1
Sq(3,2)*c2
sage: v2 = f(Sq(17)*M.generator(1)); v2
(Sq(11,3)+Sq(13,0,1)+Sq(17,1))*c2
```

and they respect the module action:

```
sage: v1 == Sq(7)*f(M.generator(0))
True
sage: v2 == Sq(17)*f(M.generator(1))
True
```

Any non-trivial homomorphism has a well-defined degree:

```
sage: f.degree()
4
```

but just as module elements, the trivial homomorphism does not:

```
sage: zero_map = homspace.zero()
sage: zero_map.degree()
(continues on next page)
```
Any two homomorphisms can be added as long as they are of the same degree:

```python
sage: f2 = homspace([Sq(2)*c2, Sq(3)*c2])
sage: f + f2
Module morphism:
  From: Free graded left module on 2 generators over mod 2 Steenrod algebra, milnor basis
  To:   Free graded left module on 1 generator over mod 2 Steenrod algebra, milnor basis
  Defn: g[0] |--> 0
         g[1] |--> Sq(0,1)*c2
```
or when at least one of them is zero:

```python
sage: f + zero_map == f
True
```
Finally, additive inverses exist:

```python
sage: f - f == 0
True
```
The restriction of a homomorphism to the free module of \( n \)-dimensional module elements is a linear transformation:

```python
sage: f_4 = f.vector_presentation(4); f_4
Vector space morphism represented by the matrix:
[0 1 0]
[1 1 1]
[0 1 0]
[0 0 0]
Domain: Vector space of dimension 4 over Finite Field of size 2
Codomain: Vector space of dimension 3 over Finite Field of size 2
```
This is compatible with the vector presentations of its domain and codomain modules:

```python
sage: f.domain() is M
True
sage: f.codomain() is N
True
sage: f_4.domain() is M.vector_presentation(4)
True
sage: f_4.codomain() is N.vector_presentation(4 + f.degree())
True
```

**AUTHORS:**

- Sverre Lunoe–Nielsen (2020-07-01): Refactored the code and added new documentation and tests.
class sage.modules.fp_graded.free_module.FreeGradedModule(algebra, generator_degrees, category, names=None, **kwds)

Bases: sage.combinat.free_module.CombinatorialFreeModule

Create a finitely generated free graded module over a connected graded algebra, with generators in specified degrees.

INPUT:

• `algebra` – the graded connected algebra over which the module is defined; this algebra must be equipped with a graded basis
• `generator_degrees` – tuple of integers defining the number of generators of the module, and their degrees
• `names` – optional, the names of the generators. If `names` is a comma-separated string like 'a, b, c', then those will be the names. Otherwise, for example if `names` is abc, then the names will be abc(d, i).

By default, if all generators are in distinct degrees, then the `names` of the generators will have the form g_{d} where d is the degree of the generator. If the degrees are not distinct, then the generators will be called g_{d,i} where d is the degree and i is its index in the list of generators in that degree.

EXAMPLES:

```python
sage: from sage.modules.fp_graded.free_module import FreeGradedModule
sage: E.<x,y,z> = ExteriorAlgebra(QQ)

sage: M = FreeGradedModule(E, (-1,3))

sage: M
Free graded left module on 2 generators over
The exterior algebra of rank 3 over Rational Field

sage: M.generator_degrees()
(-1, 3)

sage: a, b = M.generators()

sage: (x*y*b).degree()
5

sage: M.generators()
(g[-1], g[3])

sage: FreeGradedModule(E, (0, 0, 2)).generators()
(g[0, 0], g[0, 1], g[2, 0])

sage: FreeGradedModule(E, (0, 0, 2), names='x, y, z').generators()
(x, y, z)

sage: FreeGradedModule(E, (0, 0, 2), names='xyz').generators()
(xyz[0, 0], xyz[0, 1], xyz[2, 0])
```

names of generators:

```python
sage: M.generators()
(g[-1], g[3])

sage: FreeGradedModule(E, (0, 0, 2)).generators()
(g[0, 0], g[0, 1], g[2, 0])

sage: FreeGradedModule(E, (0, 0, 2), names='x, y, z').generators()
(x, y, z)

sage: FreeGradedModule(E, (0, 0, 2), names='xyz').generators()
(xyz[0, 0], xyz[0, 1], xyz[2, 0])
```

names can also be defined implicitly using Sage's M.<...> syntax:

```python
sage: A = SteenrodAlgebra(2)

sage: M.<x,y,z> = FreeGradedModule(A, (-2,2,4))

sage: M
Free graded left module on 3 generators over
mod 2 Steenrod algebra, milnor basis

sage: M.gens()
(x, y, z)
```

Element

alias of sage.modules.fp_graded.free_element.FreeGradedModuleElement
**an_element** *(n=None)*

Return an element of self.

This function chooses deterministically an element of the module in the given degree.

**INPUT:**

* n – (optional) the degree of the element to construct

**OUTPUT:**

An element (of the given degree if specified).

**EXAMPLES:**

```python
sage: from sage.modules.fp_graded.free_module import FreeGradedModule
gsage: A = SteenrodAlgebra(2)
gsage: M = FreeGradedModule(A, (0,2,4))
gsage: M.an_element(172)
Sq(0,0,2,0,1,0,1)*g[0] + Sq(0,4,0,0,1,0,1)*g[2] + Sq(7,1,0,0,1,0,1)*g[4]
```

Zero is the only element in the trivial module:

```python
sage: FreeGradedModule(A, ()).an_element()
0
```

**basis_elements** *(n=None)*

Return a basis for the free module of degree n module elements.

**Note:** Suppose self is a module over the graded algebra A with base ring R. This returns a basis as a free module over R, not a basis as a free module over A.

**INPUT:**

* n – an integer

**OUTPUT:**

A sequence of homogeneous module elements of degree n, which is a basis for the free module of all degree n module elements.

**See also:**

'\texttt{vector\_presentation()}', '\texttt{element\_from\_coordinates()}'

**EXAMPLES:**

```python
sage: A = SteenrodAlgebra(2)
sage: M.<m0, m2, m4> = A.free_graded_module((0,2,4))
sage: M.basis_elements(8)
(Sq(1,0,1)*m0,
 Sq(2,2)*m0,
 Sq(5,1)*m0,
 Sq(8)*m0,
 Sq(0,2)*m2,
 Sq(3,1)*m2,
 Sq(6)*m2,
 Sq(1,1)*m4,
 Sq(4)*m4)
```
**change_ring(algebra)**
Change the base ring of self.

**INPUT:**
- `algebra` – a connected graded algebra

**OUTPUT:**
The free graded module over `algebra` defined with the same number of generators of the same degrees as `self`.

**EXAMPLES:**
```
sage: from sage.modules.fp_graded.free_module import FreeGradedModule
c sage: A = SteenrodAlgebra(2)
c sage: A2 = SteenrodAlgebra(2, profile=(3,2,1))
c sage: M = FreeGradedModule(A, [0,1])
c sage: N = M.change_ring(A2); N
Free graded left module on 2 generators over sub-Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [3, 2, 1]
c```
Changing back yields the original module:
```
c sage: N.change_ring(A) is M
True
c```

**connectivity()**
The connectivity of `self`.

**OUTPUT:**
An integer equal to the minimal degree of all the generators, if this module is non-trivial. Otherwise, $+\infty$.

**EXAMPLES:**
```
sage: from sage.modules.fp_graded.free_module import FreeGradedModule
c sage: A = SteenrodAlgebra(2)
c sage: M = FreeGradedModule(A, (-2,2,4))
c sage: M.connectivity()
-2
c```

**element_from_coordinates(coordinates, n)**
The module element of degree `n` having the given coordinates with respect to the basis of module elements given by `basis_elements()`.

**INPUT:**
- `coordinates` – a sequence of elements of the ground ring
- `n` – an integer

**OUTPUT:**
A module element of degree `n`.

**See also:**
`vector_presentation()`, and `basis_elements()`.

**EXAMPLES:**
 sage: A = SteenrodAlgebra(2)
sage: M = A.free_graded_module((0,1))
sage: x = M.element_from_coordinates((0,1,0,1), 5); x
Sq(5)*g[0] + Sq(4)*g[1]
sage: basis = M.basis_elements(5)
sage: x == y
True
 sage: M.element_from_coordinates((0,0,0,0), 5)
0

\textbf{gen}(\textit{index})

Return the module generator with the given index.

EXAMPLES:

 sage: \textbf{from sage.modules.fp_graded.free_module import FreeGradedModule}
 sage: A = SteenrodAlgebra(2)
sage: M = FreeGradedModule(A, (0,2,4))
sage: M.generator(0)
g[0]
sage: M.generator(1)
g[2]
sage: M.generator(2)
g[4]

generator\textunderscore degrees()\

The degrees of the module generators.

OUTPUT:

A tuple containing the degrees of the generators for this module, in the order that the generators were given when \textit{self} was constructed.

EXAMPLES:

 sage: \textbf{from sage.modules.fp_graded.free_module import FreeGradedModule}
 sage: A = SteenrodAlgebra(2)
sage: M = FreeGradedModule(A, (-2,2,4))

generators()
Return all the module generators.

EXAMPLES:

```
sage: from sage.modules.fp_graded.free_module import FreeGradedModule
sage: A = SteenrodAlgebra(2)
sage: M = FreeGradedModule(A, (-2,1))
sage: M.generators()
(g[-2], g[1])
```

has_relations()
Return False as this has no relations.
This is for compatibility with `FPModule`.

EXAMPLES:

```
sage: from sage.modules.fp_graded.free_module import FreeGradedModule
sage: A = SteenrodAlgebra(2)
sage: F = FreeGradedModule(A, (-2,2,4))
sage: F.has_relations()
False
```

is_trivial()
Return True if this module is trivial and False otherwise.

EXAMPLES:

```
sage: from sage.modules.fp_graded.free_module import FreeGradedModule
sage: A = SteenrodAlgebra(2)
sage: FreeGradedModule(A, (-2,2,4)).is_trivial()
False
sage: FreeGradedModule(A, ()).is_trivial()
True
```

minimal_presentation(top_dim=None, verbose=False)
Return a minimal presentation of self.

OUTPUT:
The identity morphism as self is free.

EXAMPLES:

```
sage: from sage.modules.fp_graded.module import FPModule
sage: A2 = SteenrodAlgebra(2)
sage: M = A2.free_graded_module([0,1])
sage: M.minimal_presentation().is_identity()
True
```

relations()
Return the relations of self, which is ().
This is for compatibility with `FPModule`.

EXAMPLES:

```python
sage: from sage.modules.fp_graded.free_module import FreeGradedModule
sage: A = SteenrodAlgebra(2)
sage: F = FreeGradedModule(A, (-2,2,4))
sage: F.relations()
()  
```

```python
resolution(k, top_dim=None, verbose=False)
```

Return a free resolution of `self` of length `k`.

Since `self` is free, the initial map in the resolution will be the identity, and the rest of the maps will be zero.

**INPUT:**

- `k` – an non-negative integer
- `top_dim` – stop the computation at this degree. Ignored, for compatibility with `sage.modules.fp_graded.module.FPModule.resolution()`.
- `verbose` – (default: `False`) a boolean to control if log messages should be emitted

**OUTPUT:**

A list of homomorphisms `[1_M, 0, 0, ..., 0]` consisting of the identity map on this module followed by zero maps. Other than this module, the other modules in the resolution will be zero.

**EXAMPLES:**

```python
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: M = E.free_graded_module((1,2))
sage: M.resolution(0)
[Module endomorphism of Free graded left module on 2 generators over The exterior algebra of rank 3 over Rational Field
  Defn: g[1] |--> g[1]
  g[2] |--> g[2]]
sage: M.resolution(1)
[Module endomorphism of Free graded left module on 2 generators over The exterior algebra of rank 3 over Rational Field
  Defn: g[1] |--> g[1]
  g[2] |--> g[2],
  Module morphism:
  From: Free graded left module on 0 generators over The exterior algebra of rank 3 over Rational Field
  To: Free graded left module on 2 generators over The exterior algebra of rank 3 over Rational Field]
sage: M.resolution(4)
[Module endomorphism of Free graded left module on 2 generators over The exterior algebra of rank 3 over Rational Field
  Defn: g[1] |--> g[1]
  g[2] |--> g[2],
  Module morphism:
  From: Free graded left module on 0 generators over The exterior algebra of rank 3 over Rational Field
  To: Free graded left module on 2 generators over The exterior algebra of rank 3 over Rational Field]
```

(continues on next page)
Module endomorphism of Free graded left module on 0 generators over The exterior algebra of rank 3 over Rational Field,
Module endomorphism of Free graded left module on 0 generators over The exterior algebra of rank 3 over Rational Field,
Module endomorphism of Free graded left module on 0 generators over The exterior algebra of rank 3 over Rational Field]

suspension(t)
Suspend self by the given degree t.

INPUT:
• t – an integer

OUTPUT:
A module which is isomorphic to this module by a shift of degrees by the integer t.

EXAMPLES:

sage: from sage.modules.fp_graded.free_module import FreeGradedModule
gsage: A = SteenrodAlgebra(2)
sage: M = FreeGradedModule(A, (0,2,4))
sage: M.suspension(4).generator_degrees()
(4, 6, 8)
sage: M.suspension(-4).generator_degrees()
(-4, -2, 0)

vector_presentation(n)
Return a free module over the ground ring of the module algebra isomorphic to the degree n elements of self.

Let || be the ground ring of the algebra over this module is defined, and let \( M_n \) be the free module of module elements of degree n.

The return value of this function is the free module \( ||^r \) where \( r = \text{dim}(M_n) \).

The isomorphism between \( k^r \) and \( M_n \) is given by the bijection taking the standard basis element \( e_i \) to the \( i \)-th element of the array returned by basis_elements().

INPUT:
• n – an integer degree

OUTPUT:
A free module over the ground ring of the algebra over which self is defined, isomorphic to the free module of module elements of degree n.

See also:
basis_elements(), element_from_coordinates()

EXAMPLES:

sage: A1 = SteenrodAlgebra(2, profile=[2,1])
sage: M.<x> = A1.free_graded_module((0,))
sage: M.vector_presentation(3)
Vector space of dimension 2 over Finite Field of size 2
sage: M.basis_elements(3)
43.2 Elements of finitely generated free graded left modules

For an overview, see the free graded modules documentation.

AUTHORS:

• Sverre Lunoe–Nielsen and Koen van Woerden (2019-11-29): Updated the original software to Sage version 8.9.
• Sverre Lunoe–Nielsen (2020-07-01): Refactored the code and added new documentation and tests.

class sage.modules.fp_graded.free_element.FreeGradedModuleElement
Bases: sage.modules.with_basis.indexed_element.IndexedFreeModuleElement

Create a module element of a finitely generated free graded left module over a connected graded algebra.

EXAMPLES:

sage: from sage.modules.fp_graded.free_module import FreeGradedModule
sage: M = FreeGradedModule(SteenrodAlgebra(2), (0, 1))

sage: M([0, 0])
0

sage: M([1, 0])
g[0]

sage: M([0, 1])
g[1]

sage: M([Sq(1), 1])
Sq(1)*g[0] + g[1]

degree()
The degree of self.

OUTPUT:

The integer degree of this element, or raise an error if this is the zero element.

EXAMPLES:

sage: from sage.modules.fp_graded.free_module import *
sage: A = SteenrodAlgebra(2)
sage: M = FreeGradedModule(A, (0, 1))
sage: x = M.an_element(7); x
Sq(0,0,1)*g[0] + Sq(3,1)*g[1]
sage: x.degree()
7
The zero element has no degree:

```
sage: (x-x).degree()
Traceback (most recent call last):
...  
ValueError: the zero element does not have a well-defined degree
```

Neither do non-homogeneous elements:

```
sage: y = M.an_element(4)
sage: (x+y).degree()
Traceback (most recent call last):
...  
ValueError: this is a nonhomogeneous element, no well-defined degree
```

dense_coefficient_list(order=None)
Return a list of all coefficients of self.

INPUT:

* order – (optional) an ordering of the basis indexing set

Note that this includes all of the coefficients, not just the nonzero ones. By default they appear in the same order as the module generators.

EXAMPLES:

```
sage: from sage.modules.fp_graded.free_module import FreeGradedModule
sage: A = SteenrodAlgebra()
sage: M.<Y,Z> = FreeGradedModule(A, (0, 1))
sage: x = M.an_element(7); x
Sq(0,0,1)*Y + Sq(3,1)*Z
sage: x.dense_coefficient_list()
[Sq(0,0,1), Sq(3,1)]
```

lift_to_free()
Return self.

It is provided for compatibility with the method of the same name for sage.modules.fp_graded.FPModule.

EXAMPLES:

```
sage: from sage.modules.fp_graded.free_module import FreeGradedModule
sage: A = SteenrodAlgebra(2)
sage: M = FreeGradedModule(A, (0,1))
sage: x = M.an_element()  
sage: x.lift_to_free() == x
True
sage: x.lift_to_free() is x
True
```

vector_representation()
A coordinate vector representing self when it is a non-zero homogeneous element.

These are coordinates with respect to the basis chosen by basis_elements(). When the element is zero, it has no well defined degree, and this function returns None.

OUTPUT:
A vector of elements in the ground ring of the algebra for this module when this element is non-zero. Otherwise, the value None.

See also:

- *sage.modules.fp_graded.free_module.FreeGradedModule.vector_presentation()*
- *sage.modules.fp_graded.free_module.FreeGradedModule.basis_elements()*
- *sage.modules.fp_graded.free_module.FreeGradedModule.element_from_coordinates()*

EXAMPLES:

```python
sage: A2 = SteenrodAlgebra(2, profile=(3,2,1))
sage: M = A2.free_graded_module((0,1))
sage: x = M.an_element(7)
sage: v = x.vector_presentation(); v
(1, 0, 0, 0, 0, 1, 0)
sage: type(v)
<class 'sage.modules.vector_mod2_dense.Vector_mod2_dense'>
sage: M.gen(0).vector_presentation()
(1)
sage: M.gen(1).vector_presentation()
(0, 1)
sage: V = M.vector_presentation(7)
sage: v in V
True
```

We can use the basis for the module elements in the degree of \(x\), together with the coefficients \(v\) to recreate the element \(x\):

```python
sage: basis = M.basis_elements(7)
sage: x_ = sum( [c*b for (c,b) in zip(v, basis)] ); x_
Sq(0,0,1)*g[0] + Sq(3,1)*g[1]
sage: x__ = M.linear_combination(zip(basis, v)); x__
Sq(0,0,1)*g[0] + Sq(3,1)*g[1]
sage: x == x_ == x__
True
```

This is not defined for elements that are not homogeneous:

```python
sage: sum(M.basis()).vector_presentation()
Traceback (most recent call last):
...
ValueError: this is a nonhomogeneous element, no well-defined degree
```
43.3 Homomorphisms of finitely generated free graded left modules

AUTHORS:

• Sverre Lunøe–Nielsen and Koen van Woerden (2019-11-29): Updated the original software to Sage version 8.9.
• Sverre Lunøe–Nielsen (2020-07-01): Refactored the code and added new documentation and tests.

class sage.modules.fp_graded.free_morphism.FreeGradedModuleMorphism(parent, values)

Bases: sage.modules.fp_graded.morphism.FPModuleMorphism

Create a homomorphism from a finitely generated free graded module to a graded module.

INPUT:

• parent – a homspace in the category of finitely generated free modules
• values – a list of elements in the codomain; each element corresponds (by their ordering) to a module generator in the domain

EXAMPLES:

sage: from sage.modules.fp_graded.free_module import FreeGradedModule
sage: A = SteenrodAlgebra(2)
sage: F1 = FreeGradedModule(A, (4,5), names='b')
sage: F2 = FreeGradedModule(A, (3,4), names='c')
sage: F3 = FreeGradedModule(A, (2,3), names='d')
sage: H1 = Hom(F1, F2)
sage: H2 = Hom(F2, F3)
sage: f = H1((F2((Sq(4), 0)), F2((0, Sq(4)))))
sage: g = H2((F3((Sq(2), 0)), F3((Sq(3), Sq(2)))))
sage: g*f

Module morphism:
From: Free graded left module on 2 generators over mod 2 Steenrod algebra, milnor_˓→basis
To: Free graded left module on 2 generators over mod 2 Steenrod algebra, milnor_˓→basis
Defn: b[4] |--> (Sq(0,2)+Sq(3,1)+Sq(6))*d[2]
 b[5] |--> (Sq(1,2)+Sq(7))*d[2] + (Sq(0,2)+Sq(3,1)+Sq(6))*d[3]

degree()

The degree of self.

OUTPUT:

The degree of this homomorphism. Raise an error if this is the trivial homomorphism.

EXAMPLES:

sage: from sage.modules.fp_graded.free_module import FreeGradedModule
sage: A = SteenrodAlgebra(2)
sage: homspace = Hom(FreeGradedModule(A, (0,1)), FreeGradedModule(A, (0,)))
sage: N = homspace.codomain()
sage: values = [Sq(5)^N.generator(0), Sq(3,1)^N.generator(0)]
sage: f = homspace(values)

(continues on next page)
The zero homomorphism has no degree:

```
sage: homspace.zero().degree()
Traceback (most recent call last):
  ... 
ValueError: the zero morphism does not have a well-defined degree
```

**fp_module()**

Create a finitely presented module from `self`.

**OUTPUT:**

The finitely presented module with presentation equal to `self`.

**EXAMPLES:**

```
sage: A = SteenrodAlgebra(2)
sage: F1 = A.free_graded_module([2])
sage: F2 = A.free_graded_module([0])
sage: v = F2([Sq(2)])
sage: pres = Hom(F1, F2)([v])
sage: M = pres.fp_module(); M
Finitely presented left module on 1 generator and 1 relation over mod 2 Steenrod algebra, milnor basis
sage: M.generator_degrees()
(0,)
sage: M.relations()
(Sq(2))^g[0],)
sage: from sage.modules.fp_graded.module import FPModule
sage: A = SteenrodAlgebra(2)
sage: F1 = A.free_graded_module((2,))
sage: F2 = FPModule(A, (0,), [[Sq(4)]])
sage: v = F2([Sq(2)])
sage: pres = Hom(F1, F2)([v])
sage: pres.fp_module()
Traceback (most recent call last):
  ... 
ValueError: this is not a morphism between free modules
```

### 43.4 Homsets of finitely generated free graded left modules

For an overview, see the `free graded modules documentation`.

**EXAMPLES:**

```
sage: from sage.modules.fp_graded.module import FPModule
sage: A = SteenrodAlgebra(2)
sage: F1 = A.free_graded_module((2,))
sage: F2 = FPModule(A, (0,), [[Sq(4)]])
sage: v = F2([Sq(2)])
sage: pres = Hom(F1, F2)([v])
sage: pres.fp_module()
Traceback (most recent call last):
  ... 
ValueError: this is not a morphism between free modules
```

(continues on next page)
sage: F2 = FreeGradedModule(A, (2,3), names='h')
sage: homset = Hom(F1, F2)
sage: homset
Set of Morphisms from Free graded left module on 2 generators ...

sage: homset([F2((Sq(1), 1)), F2((0, Sq(2)))])
Module morphism:
    From: Free graded left module on 2 generators over mod 2 Steenrod algebra, milnor basis
    To:   Free graded left module on 2 generators over mod 2 Steenrod algebra, milnor basis
    g[3] |--> Sq(2)*h[3]

sage: TestSuite(homset).run()
class sage.modules.fp_graded.module.FPModule(j, names)


A finitely presented module over a connected graded algebra.

INPUT:

One of the following:

• arg0 – a morphism such that the module is the cokernel, or a free graded module, in which case the output is the same module, viewed as finitely presented

Otherwise:

• arg0 – the graded connected algebra over which the module is defined; this algebra must be equipped with a graded basis
• generator_degrees – tuple of integer degrees
• relations – tuple of relations; a relation is a tuple of coefficients \((c_1, \ldots, c_n)\), ordered so that they correspond to the module generators, that is, such a tuple corresponds to the relation

\[ c_1 g_1 + \ldots + c_n g_n = 0 \]

if the generators are \((g_1, \ldots, g_n)\)

EXAMPLES:

```python
sage: from sage.modules.fp_graded.module import FPModule

sage: E.<x,y> = ExteriorAlgebra(QQ)
sage: M = FPModule(E, [0, 1], [[x, 1]])
sage: a, b = M.generators()
sage: x*a + b == 0
True
sage: (x*a + b).normalize()
0

sage: A3 = SteenrodAlgebra(2, profile=(4,3,2,1))
sage: M = FPModule(A3, [0, 1], [[Sq(2), Sq(1)]])

sage: M.generators()
(g[0], g[1])
sage: M.relations()
(Sq(2)*g[0] + Sq(1)*g[1],)
sage: M.is_trivial()
False

sage: Z = FPModule(A3, [])
sage: Z.generators()
()
sage: Z.relations()
()
sage: Z.is_trivial()
True
```

From sage.modules.fp_graded.free_module import FreeGradedModule

```python
sage: F = FreeGradedModule(E, [0, 1])
```

(continues on next page)
sage: one = Hom(F, F).identity()
sage: Z = FPModule(one)
sage: Z.is_trivial()
True

sage: FPModule(E.free_graded_module([0, 1]))
Free graded left module on 2 generators over The exterior algebra of rank 2 over Rational Field

Element
alias of sage.modules.fp_graded.element.FPElement

an_element(n=None)
An element of this module.

This function chooses deterministically an element, i.e. the output depends only on the module and its input n.

INPUT:
• n – (optional) the degree of the element to construct

OUTPUT:
A module element of the given degree.

EXAMPLES:

sage: from sage.modules.fp_graded.module import FPModule
sage: A2 = SteenrodAlgebra(2, profile=(3,2,1))
sage: M = FPModule(A2, [0,2,4], [[0, Sq(5), Sq(3)], [Sq(7), 0, Sq(2)*Sq(1)]])
sage: [M.an_element(i) for i in range(10)]
g[0],
Sq(1)*g[0],
Sq(2)*g[0] + g[2],
Sq(0,1)*g[0] + Sq(1)*g[2],
Sq(1,1)*g[0] + Sq(2)*g[2] + g[4],
Sq(2,1)*g[0] + Sq(0,1)*g[2] + Sq(1)*g[4],
Sq(0,2)*g[0] + Sq(1,1)*g[2] + Sq(2)*g[4],
Sq(0,0,1)*g[0] + Sq(2,1)*g[2] + Sq(0,1)*g[4],
Sq(1,0,1)*g[0] + Sq(6)*g[2] + Sq(1,1)*g[4],
Sq(2,0,1)*g[0] + Sq(4,1)*g[2] + Sq(2,1)*g[4]]

basis_elements(n, verbose=False)
Return a basis for the free module of degree n module elements.

Note: Suppose self is a module over the graded algebra A with base ring R. This returns a basis as a free module over R.

INPUT:
• n – an integer
• verbose – (default: False) a boolean to control if log messages should be emitted

OUTPUT:
A list of homogeneous module elements of degree $n$ which is a basis for the free module of all degree $n$ module elements.

See also:

vector_presentation(), element_from_coordinates()

EXAMPLES:

```sage
sage: from sage.modules.fp_graded.module import FPModule
sage: A2 = SteenrodAlgebra(2, profile=(3,2,1))
sage: M.<m0,m2> = FPModule(A2, [0,2], [[Sq(4), Sq(2)], [0, Sq(6)]])

sage: M.basis_elements(4)
(Sq(1,1)*m0, Sq(4)*m0)

sage: M.basis_elements(5)
(Sq(2,1)*m0, Sq(5)*m0, Sq(0,1)*m2)

sage: M.basis_elements(25)
()

sage: M.basis_elements(0)
(m0,)

sage: M.basis_elements(2)
(Sq(2)*m0, m2)
```

change_ring(algebra)

Change the base ring of self.

INPUT:

• algebra – a connected graded algebra

OUTPUT:

The finitely presented module over algebra defined with the exact same number of generators of the same degrees and relations as self.

EXAMPLES:

```sage
sage: from sage.modules.fp_graded.module import FPModule
sage: A = SteenrodAlgebra(2)
sage: A2 = SteenrodAlgebra(2, profile=(3,2,1))

sage: M = FPModule(A, [0,1], [[Sq(2), Sq(1)]])

sage: N = M.basis_elements(2)
(sq(2)*m0, m2)

sage: M.basis_elements(2)
(sq(2)*m0, m2)
```

Changing back yields the original module:

```sage
sage: M.basis_elements(2)
(sq(2)*m0, m2)
```

connectivity()

The connectivity of self.
Since a finitely presented module over a connected algebra is in particular bounded below, the connectivity is an integer when the module is non-trivial, and $+\infty$ when the module is trivial.

**EXAMPLES:**

```python
sage: from sage.modules.fp_graded.module import FPModulesage: A = SteenrodAlgebra(2)
sage: M = FPModule(A, [0,2,4], [[0, Sq(5), Sq(3)], [Sq(7), 0, Sq(2)*Sq(1)]])
sage: M.connectivity()
0
sage: G = FPModule(A, [0,2], [[1,0]])
sage: G.connectivity()
2
```

**defining_homomorphism()**

Return the homomorphism defining self.

self is a finitely presented module defined as the cokernel of a map $j : F_0 \otimes F_1$ of free modules, and this returns that map.

**EXAMPLES:**

```python
sage: from sage.modules.fp_graded.module import FPModulesage: E.<x,y> = ExteriorAlgebra(QQ)
sage: M = FPModule(E, [0, 1], [[x, 1]])
sage: M.defining_homomorphism()
Module morphism:
  From: Free graded left module on 1 generator over The exterior algebra of
  --> rank 2 over Rational Field
  To:   Free graded left module on 2 generators over The exterior algebra of
  --> rank 2 over Rational Field
  Defn: g[1] |--> x*g[0] + g[1]
```

**element_from_coordinates***(coordinates, n)**

Return the module element in degree $n$ having the given coordinates with respect to the basis returned by `basis_elements()`.

This function is inverse to `sage.modules.fp_graded.element.FPElement.vector_projection()`.

**INPUT:**

- coordinates – a vector of coordinates
- $n$ – the degree of the element to construct

**OUTPUT:**

A module element of degree $n$ having the given coordinates with respect to the basis returned by `basis_elements()`.

**EXAMPLES:**

```python
sage: from sage.modules.fp_graded.module import FPModulesage: A = SteenrodAlgebra(2)
sage: M = FPModule(A, [0], [[Sq(4)], [Sq(7)], [Sq(4)*Sq(9)]])
```

(continues on next page)
sage: M.vector_presentation(12).dimension()
3
sage: x = M.element_from_coordinates((0, 1, 0), 12); x
Sq(0,4)*g[0]

Applying the inverse function brings us back to the coordinate form:

sage: x.vector_presentation()
(0, 1, 0)

See also:

`sage.modules.fp_graded.module.FPModule.vector_presentation()`

gen\( (\text{index}) \)

Return the module generator with the given index.

EXAMPLES:

```python
sage: from sage.modules.fp_graded.module import FPModule
sage: A4 = SteenrodAlgebra(2, profile=(4,3,2,1))
sage: M = FPModule(A4, [0,2,3])
sage: M.generator(0)
g[0]
sage: N = FPModule(A4, [0, 1], [[Sq(2), Sq(1)]], names='h')
sage: N.generator(1)
h[1]
```

gen\(\text{degrees} ()\)

Return the degrees of the generators for self.

EXAMPLES:

```python
sage: from sage.modules.fp_graded.module import FPModule
sage: A4 = SteenrodAlgebra(2, profile=(4,3,2,1))
sage: M = FPModule(A4, [0,2,3])
sage: M.generator(0)
g[0]
sage: N = FPModule(A4, [0, 1], [[Sq(2), Sq(1)]], names='h')
sage: N.generator(1)
h[1]
```
sage: N.generator_degrees()
(0, 1)

generators()

Return the generators of self.

EXAMPLES:

sage: from sage.modules.fp_graded.module import FPModule
sage: A4 = SteenrodAlgebra(2, profile=(4,3,2,1))

sage: M = FPModule(A4, [0,0,2,3])

sage: N = FPModule(A4, [0, 1], [[Sq(2), Sq(1)]], names='h')

sage: Z = FPModule(A4, [])

gens()

Return the generators of self.

EXAMPLES:

sage: from sage.modules.fp_graded.module import FPModule
sage: A4 = SteenrodAlgebra(2, profile=(4,3,2,1))

sage: M = FPModule(A4, [0,0,2,3])

sage: N = FPModule(A4, [0, 1], [[Sq(2), Sq(1)]], names='h')

sage: Z = FPModule(A4, [])

has_relations()

Return True if no relations are defined, and False otherwise.

Note: This module is free if this function returns False, but a free module can have (redundant) relations.

EXAMPLES:

sage: from sage.modules.fp_graded.module import FPModule
sage: A2 = SteenrodAlgebra(2, profile=(3,2,1))
sage: F = FPModule(A2, [1,2])
sage: F.has_relations()
False

sage: M = FPModule(A2, [1,2], [[Sq(2), Sq(1)]]))
sage: M.has_relations()
True

A free module constructed with a redundant generator and relation:

sage: N = FPModule(A2, [0, 0], [[0, 1]])
sage: N.has_relations()
True

sage: # Computing a minimal presentation reveals an
....: # isomorphic module with no relations.
sage: N_min = N.minimal_presentation().domain()
sage: N_min.has_relations()
False

is_trivial()
Return True if self is isomorphic to the trivial module and False otherwise.

EXAMPLES:

sage: from sage.modules.fp_graded.module import FPModule
sage: A = SteenrodAlgebra(2)
sage: A2 = SteenrodAlgebra(2, profile=(3,2,1))
sage: M = FPModule(A2, [])
sage: M.is_trivial()
True

sage: N = FPModule(A, [1,2])
sage: N.is_trivial()
False

sage: P = FPModule(A, [1,2], [[1,0], [0,1]])
sage: P.is_trivial()
True

minimal_presentation(top_dim=None, verbose=False)
Return a minimal presentation of self.

OUTPUT:
An isomorphism $M \rightarrow S$, where $M$ has minimal presentation and $S$ is self.

EXAMPLES:

sage: from sage.modules.fp_graded.module import FPModule
sage: A2 = SteenrodAlgebra(2, profile=(3,2,1))
sage: M = FPModule(A2, [0,1], [[Sq(2), Sq(1), [0,Sq(2),[Sq(3),0]]))
sage: i = M.minimal_presentation()
sage: M_min = i.domain()
i is an isomorphism between $M_{\text{min}}$ and $M$:

```
sage: i.codomain() is M
True
sage: i.is_injective()
True
sage: i.is_surjective()
True
```

There are more relations in $M$ than in $M_{\text{min}}$:

```
sage: M.relations()
(Sq(2)*g[0] + Sq(1)*g[1], Sq(2)*g[1], Sq(3)*g[0])
sage: M_min.relations()
(Sq(2)*g[0] + Sq(1)*g[1], Sq(2)*g[1])
```

`monomial()`

Return the basis element indexed by $i$.

**INPUT:**

- $i$ – an element of the index set

**EXAMPLES:**

```
sage: from sage.modules.fp_graded.module import FPModule
sage: M = FPModule(SteenrodAlgebra(2), [0,1], [[Sq(4), Sq(3)]])
sage: M.monomial(0)
g[0]
sage: M.monomial(1)
g[1]
```

`relation(index)`

Return the module relation of the given index.

**EXAMPLES:**

```
sage: from sage.modules.fp_graded.module import FPModule
sage: A4 = SteenrodAlgebra(2, profile=(4,3,2,1))
sage: N = FPModule(A4, [0, 1], [[Sq(2), Sq(1)]])
sage: N.relation(0)
Sq(2)*g[0] + Sq(1)*g[1]
```

`relations()`

Return the relations of $self$.

**EXAMPLES:**

```
sage: from sage.modules.fp_graded.module import FPModule
sage: A4 = SteenrodAlgebra(2, profile=(4,3,2,1))
sage: M = FPModule(A4, [0,2,3])
sage: M.relations()
()  # empty relation
sage: N = FPModule(A4, [0, 1], [[Sq(2), Sq(1)]])
sage: N.relations()
```

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\[
(Sq(2)^0g[0] + Sq(1)^0g[1],)
\]

\texttt{sage: Z = FPModule(A4, [])}
\texttt{sage: Z.relations()}
\[
()
\]

\texttt{resolution}(k, top\_dim=None, verbose=False)

Return a free resolution of this module of length \( k \).

\textbf{INPUT:}

- \( k \) – an non-negative integer
- \( \text{top\_dim} \) – stop the computation at this degree (optional, default None, but required if the algebra is not finite-dimensional)
- \( \text{verbose} \) – (default: False) a boolean to control if log messages should be emitted

\textbf{OUTPUT:}

A list of homomorphisms \([\epsilon, f_1, \ldots, f_k]\) such that

\[
f_i : F_i \to F_{i-1} \text{ for } 1 < i \leq k, \quad \epsilon : F_0 \to M,
\]

where each \( F_i \) is a finitely generated free module, and the sequence

\[
F_k \xrightarrow{f_k} F_{k-1} \xrightarrow{f_{k-1}} \ldots \xrightarrow{f_0} F_0 \xrightarrow{\epsilon} M \to 0
\]
is exact. Note that the 0th element in this list is the map \( \epsilon \), and the rest of the maps are between free modules.

\textbf{EXAMPLES:}

\texttt{sage: from sage.modules.fp_graded.module import FPModule}
\texttt{sage: E.<x,y> = ExteriorAlgebra(QQ)}
\texttt{sage: M = FPModule(E, [0], [[x], [y]])}
\texttt{sage: res = M.resolution(3); res}

[Module morphism:
  \text{From: Free graded left module on 1 generator over The exterior algebra of the rank 2 over Rational Field}
  \text{To: Finitely presented left module on 1 generator and 2 relations over The exterior algebra of rank 2 over Rational Field}
  \text{Defn: g[0] |--> g[0]},

Module morphism:
  \text{From: Free graded left module on 2 generators over The exterior algebra of the rank 2 over Rational Field}
  \text{To: Free graded left module on 1 generator over The exterior algebra of the rank 2 over Rational Field}
  \text{Defn: g[1, 0] |--> x*g[0]
  g[1, 1] |--> y*g[0]},

Module morphism:
  \text{From: Free graded left module on 3 generators over The exterior algebra of the rank 2 over Rational Field}
  \text{To: Free graded left module on 2 generators over The exterior algebra of the rank 2 over Rational Field}
  \text{Defn: g[2, 0] |--> x*g[1, 0]}

(continues on next page)
\[
g[2, 1] \mapsto y^*g[1, 0] + x^*g[1, 1]
g[2, 2] \mapsto y^*g[1, 1],
\]
Module morphism:
From: Free graded left module on 4 generators over The exterior algebra of
\(\rightarrow\) rank 2 over Rational Field
To: Free graded left module on 3 generators over The exterior algebra of
\(\rightarrow\) rank 2 over Rational Field
Defn: 
\[
\begin{align*}
g[3, 0] & \mapsto x^*g[2, 0] \\
g[3, 1] & \mapsto y^*g[2, 0] + x^*g[2, 1] \\
g[3, 2] & \mapsto y^*g[2, 1] + x^*g[2, 2] \\
g[3, 3] & \mapsto y^*g[2, 2]
\end{align*}
\]
sage: all((res[i] * res[i+1]).is_zero() for i in range(len(res)-1))
True
sage: e = SymmetricFunctions(QQ).e()
sage: M = FPModule(e, [0], [[e[2]+e[1,1], [e[1,1]]]])
sage: res = M.resolution(3, top_dim=10)
sage: all((res[i] * res[i+1]).is_zero() for i in range(2))
True
sage: res[-1].domain().is_trivial()
True
sage: M = FPModule(e, [0,2], [[e[2]+e[1,1], [e[1,1]]]])
sage: res = M.resolution(3, top_dim=10)
sage: all((res[i] * res[i+1]).is_zero() for i in range(2))
True
sage: res[-1].domain().is_trivial()
True
sage: A2 = SteenrodAlgebra(2, profile=(3,2,1))
sage: M = FPModule(A2, [0,1], [[Sq(2), Sq(1)]])
sage: M.resolution(0)
[Module morphism:
From: Free graded left module on 2 generators over sub-Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [3, 2, 1]
To: Finitely presented left module on 2 generators and 1 relation over sub-
Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [3, 2, 1]
Defn: 
\[
\begin{align*}
g[0] & \mapsto g[0] \\
g[1] & \mapsto g[1]
\end{align*}
\]
]sage: res = M.resolution(4, verbose=True)
Computing f_1 (1/4)
Computing f_2 (2/4)
Computing using the profile:
(2, 1)
Resolving the kernel in the range of dimensions [2, 8]: 2 3 4 5 6 7 8.
Computing f_3 (3/4)
Computing using the profile:
(2, 1)
Resolving the kernel in the range of dimensions [8, 14]: 8 9 10 11 12 13 14.
Computing f_4 (4/4)
Computing using the profile:
(2, 1)
Resolving the kernel in the range of dimensions [9, 16]: 9 10 11 12 13 14 15 16.
sage: len(res)
5
sage: res
[Module morphism:
 From: Free graded left module on 2 generators over sub-Hopf algebra of mod 2 → Steenrod algebra, milnor basis, profile function [3, 2, 1]
 To: Finitely presented left module on 2 generators and 1 relation over sub-
 → Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [3, 2, 1]
 Defn: g[0] |--> g[0]
g[1] |--> g[1],
Module morphism:
 From: Free graded left module on 1 generator over sub-Hopf algebra of mod 2 → Steenrod algebra, milnor basis, profile function [3, 2, 1]
 To: Finitely presented left module on 2 generators over sub-
 → Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [3, 2, 1]
 Defn: g[2] |--> Sq(2)*g[0] + Sq(1)*g[1],
Module morphism:
 From: Free graded left module on 1 generator over sub-Hopf algebra of mod 2 → Steenrod algebra, milnor basis, profile function [3, 2, 1]
 To: Free graded left module on 1 generator over sub-Hopf algebra of mod 2 → Steenrod algebra, milnor basis, profile function [3, 2, 1]
 Defn: g[8] |--> Sq(3,1)*g[2],
Module morphism:
 From: Free graded left module on 2 generators over sub-Hopf algebra of mod 2 → Steenrod algebra, milnor basis, profile function [3, 2, 1]
 To: Free graded left module on 1 generator over sub-Hopf algebra of mod 2 → Steenrod algebra, milnor basis, profile function [3, 2, 1]
 Defn: g[9] |--> Sq(1)*g[8]
g[10] |--> Sq(2)*g[8],
Module morphism:
 From: Free graded left module on 2 generators over sub-Hopf algebra of mod 2 → Steenrod algebra, milnor basis, profile function [3, 2, 1]
 To: Free graded left module on 2 generators over sub-Hopf algebra of mod 2 → Steenrod algebra, milnor basis, profile function [3, 2, 1]
 Defn: g[10] |--> Sq(1)*g[9]
g[12] |--> Sq(0,1)*g[9] + Sq(2)*g[10]]
sage: for i in range(len(res)-1):
    ....:   assert (res[i] * res[i+1]).is_zero(), 'the result is not a complex'

We construct $F_3$ as a $\mathbb{Z}$-module (with trivial grading concentrated in degree 0) and compute its resolution:

```
sage: E = ExteriorAlgebra(ZZ, 0)
sage: M = FPModule(E, [0], [[3]])
sage: res = M.resolution(3)
sage: res
[Module morphism:
 From: Free graded left module on 1 generator over The exterior algebra of
 → rank 0 over Integer Ring
 To: Finitely presented left module on 1 generator and 1 relation over The,
 → exterior algebra of rank 0 over Integer Ring
```

(continues on next page)
Defn: $g[0] \rightarrow g[0]$,  
Module endomorphism of Free graded left module on 1 generator over The\_  
\_exterior algebra of rank 0 over Integer Ring  
Defn: $g[0] \rightarrow 3\cdot g[0]$,  
Module morphism:  
  From: Free graded left module on 0 generators over The exterior algebra of\_  
\_rank 0 over Integer Ring  
  To: Free graded left module on 1 generator over The exterior algebra of\_  
\_rank 0 over Integer Ring,  
Module endomorphism of Free graded left module on 0 generators over The\_  
\_exterior algebra of rank 0 over Integer Ring]

**submodule_inclusion**(spanning_elements)

Return the inclusion morphism of the submodule of self spanned by the given elements.

**INPUT:**

- spanning_elements – an iterable of elements

**OUTPUT:**

The inclusion of the submodule into this module.

Because a submodule of a finitely presented module need not be finitely presented, this method will only work if the underlying algebra is finite-dimensional. Indeed, the current implementation only works if the algebra has a `top_class` method, which gets used in `sage.modules.fp_graded.morphism._resolve_kernel()`.

**EXAMPLES:**

```
sage: from sage.modules.fp_graded.module import FPModule
sage: A2 = SteenrodAlgebra(2, profile=(3,2,1))

sage: M = FPModule(A2, [0,1], [[Sq(2),Sq(1)]])
sage: i = M.submodule_inclusion([M.generator(0)])
sage: i.codomain() is M
True
sage: i.is_injective()
True
sage: i.domain().generator_degrees()
(0,)
sage: i.domain().relations()
(Sq(3)^*g[0],)
```

**suspension**(t)

Return the suspension of self by degree t.

**INPUT:**

- t – an integer degree by which the module is suspended

**OUTPUT:**

A module which is identical to this module by a shift of degrees by the integer t.

**EXAMPLES:**
```python
sage: from sage.modules.fp_graded.module import FPModule
sage: A = SteenrodAlgebra(2)
sage: A2 = SteenrodAlgebra(2, profile=(3,2,1))

sage: Y = FPModule(A2, [0], [[Sq(1)]])
sage: X = Y.suspension(4)

sage: X.generator_degrees()
(4,)
sage: X.relations()
(Sq(1)*g[4],)

sage: M = FPModule(A, [2,3], [[Sq(2), Sq(1)], [0, Sq(2)]]

sage: Q = M.suspension(1)

sage: Q.generator_degrees()
(3, 4)
sage: Q.relations()
(Sq(2)*g[3] + Sq(1)*g[4], Sq(2)*g[4])

sage: Q = M.suspension(-3)

sage: Q.generator_degrees()
(-1, 0)
sage: Q = M.suspension(0)

sage: Q.generator_degrees()
(2, 3)
```

**vector_presentation**($n$, verbose=False)

Return a free module isomorphic to the free module of module elements of degree $n$.

**INPUT:**

- $n$ – the degree of the presentation

**OUTPUT:**

A vector space.

**See also:**

`basis_elements()`, `element_from_coordinates()`

**EXAMPLES:**

```python
sage: from sage.modules.fp_graded.module import FPModule
sage: A = SteenrodAlgebra(2)

sage: M = FPModule(A, [0,2,4], [[Sq(4),Sq(2),0]])

sage: V = M.vector_presentation(4)

sage: V.dimension()
3

sage: len(M.basis_elements(4))
3
```

**zero()**

Return the zero element.

**EXAMPLES:**
43.6 Elements of finitely presented graded modules

AUTHORS:

- Sverre Lunøe-Nielsen (2020-07-01): Refactored the code and added new documentation and tests.

```python
sage: from sage.modules.fp_graded.module import FPModule
sage: M = FPModule(SteenrodAlgebra(2), [0,1], [[Sq(4), Sq(3)]])
sage: M.zero()
0
```

```python
sage: from sage.modules.fp_graded.module import FPModule
sage: M = FPModule(SteenrodAlgebra(2), [0,1], [[Sq(4), Sq(3)]])
sage: x = M.an_element(7)
sage: x
Sq(0,0,1)*g[0] + Sq(3,1)*g[1]
sage: x.degree()
7
```

The zero element has no degree:

```python
sage: (x-x).degree()
Traceback (most recent call last):
  ... ValueError: the zero element does not have a well-defined degree
```

```python
dense_coefficient_list(order=None)
Return a list of all coefficients of self.
```

INPUT:

- order – (optional) an ordering of the basis indexing set

Note that this includes all of the coefficients, not just the nonzero ones. By default they appear in the same order as the module generators.

EXAMPLES:
sage: from sage.modules.fp_graded.module import FPModule
sage: A = SteenrodAlgebra()
sage: M = FPModule(SteenrodAlgebra(2), [0,1], [[Sq(4), Sq(3)]])
sage: x = M([Sq(1), 1])
sage: x.dense_coefficient_list()
[Sq(1), 1]
sage: y = Sq(2) * M.generator(1)
sage: y.dense_coefficient_list()
[0, Sq(2)]

lift_to_free()
Return the lift of self to the free module F, where self is in a quotient of F.

EXAMPLES:

sage: from sage.modules.fp_graded.module import FPModule
sage: M = FPModule(SteenrodAlgebra(2), [0,1], [[Sq(4), Sq(3)]])
sage: x = M([Sq(1), 1])
sage: x
Sq(1)*g[0] + g[1]
sage: x.parent()  # doctest: +NORMALIZE_WHITESPACE
Finitely presented left module on 2 generators and 1 relation over mod 2 → Steenrod algebra, milnor basis
sage: x.lift_to_free()
Sq(1)*g[0] + g[1]
sage: x.lift_to_free().parent()  # doctest: +NORMALIZE_WHITESPACE
Free graded left module on 2 generators over mod 2 Steenrod algebra, milnor basis

normalize()
A normalized form of self.

OUTPUT:
An instance representing the same module element as self in normalized form.

EXAMPLES:

sage: from sage.modules.fp_graded.module import FPModule
sage: M.<a0,b2,c4> = FPModule(SteenrodAlgebra(2), [0,2,4], [[Sq(4),Sq(2),0]])
sage: m = M((Sq(6), 0, Sq(2))); m
Sq(6)*a0 + Sq(2)*c4
sage: m.normalize()
Sq(6)*a0 + Sq(2)*c4
sage: m == m.normalize()
True
sage: n = M((Sq(4), Sq(2), 0)); n
Sq(4)*a0 + Sq(2)*b2
sage: n.normalize()
0
sage: n == n.normalize()
True
vector_presentation()

A coordinate vector representing self when it is non-zero.

These are coordinates with respect to the basis chosen by basis_elements(). When the element is zero, it has no well defined degree, and this function returns None.

OUTPUT:

A vector of elements in the ground ring of the algebra for this module when this element is non-zero. Otherwise, the value None.

See also:

• sage.modules.fp_graded.module.FPModule.vector_presentation()
• sage.modules.fp_graded.module.FPModule.basis_elements()
• sage.modules.fp_graded.module.FPModule.element_from_coordinates()

EXAMPLES:

```python
sage: from sage.modules.fp_graded.module import FPModule
sage: A2 = SteenrodAlgebra(2, profile=(3,2,1))
sage: M.<m0,m1> = FPModule(A2, (0,1))

sage: x = M.an_element(7)
sage: v = x.vector_presentation(); v
(1, 0, 0, 0, 0, 1, 0)

sage: type(v)
<class 'sage.modules.vector_mod2_dense.Vector_mod2_dense'>

sage: V = M.vector_presentation(7)
sage: v in V
True

sage: M.element_from_coordinates(v, 7) == x
True
```

We can use the basis for the module elements in the degree of \( x \), together with the coefficients \( v \) to recreate the element \( x \):

```python
sage: basis = M.basis_elements(7)
sage: x_ = sum([c*b for (c,b) in zip(v, basis)]); x_
Sq(0,0,1)*m0 + Sq(3,1)*m1

sage: x__ = M.linear_combination(zip(basis, v)); x__
Sq(0,0,1)*m0 + Sq(3,1)*m1

sage: x == x_ == x__
True
```
43.7 Homomorphisms of finitely presented graded modules

AUTHORS:

• Sverre Lunøe–Nielsen and Koen van Woerden (2019-11-29): Updated the original software to Sage version 8.9.
• Sverre Lunøe–Nielsen (2020-07-01): Refactored the code and added new documentation and tests.

class sage.modules.fp_graded.morphism.FPModuleMorphism(parent, values, check=True)
Bases: sage.categories.morphism.Morphism

Create a homomorphism between finitely presented graded modules.

INPUT:

• parent – a homspace of finitely presented graded modules
• values – a list of elements in the codomain; each element corresponds to a module generator in the domain
• check – boolean (default: True); if True, check that the morphism is well-defined

change_ring(algebra)
Change the base ring of self.

INPUT:

• algebra – a graded algebra

EXAMPLES:

    sage: from sage.modules.fp_graded.module import FPModule
    sage: A2 = SteenrodAlgebra(2, profile=(3,2,1))
    sage: A3 = SteenrodAlgebra(2, profile=(4,3,2,1))
    sage: M = FPModule(A2, [0], relations=[[Sq(1)]])
    sage: N = FPModule(A2, [0], relations=[[Sq(4)],[Sq(1)]]))

    sage: f = Hom(M,N)([A2.Sq(3)*N.generator(0)]); f
    Module morphism:
    From: Finitely presented left module on 1 generator and 1 relation over sub-
   Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [3, 2,␣
   → 1]
    To: Finitely presented left module on 1 generator and 2 relations over sub-
   Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [3, 2,␣
   → 1]
    Defn: g[0] |--> Sq(3)*g[0]

    sage: f.base_ring()
    sub-Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [3,␣
   → 2, 1]

    sage: g = f.change_ring(A3)
    sage: g.base_ring()
    sub-Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [4,␣
   → 3, 2, 1]

cokernel_projection()
Return the map to the cokernel of self.
The natural projection from the codomain of this homomorphism to its cokernel.

```python
sage: from sage.modules.fp_graded.module import FPModule
sage: A1 = SteenrodAlgebra(2, profile=(2,1))
sage: M = FPModule(A1, [0], [[Sq(2)]])
sage: F = FPModule(A1, [0])

sage: r = Hom(F, M)([A1.Sq(1)*M.generator(0)])
sage: co = r.cokernel_projection(); co
Module morphism:
  From: Finitely presented left module on 1 generator and 1 relation over sub-
  Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [2, 1]
  To: Finitely presented left module on 1 generator and 2 relations over sub-
  Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [2, 1]
  Defn: g[0] |--> g[0]

sage: co.domain().is_trivial()
False
```

**degree()**

The degree of self.

**OUTPUT:**

The integer degree of self.

**EXAMPLES:**

```python
sage: from sage.modules.fp_graded.module import *
sage: A = SteenrodAlgebra(2)
sage: M = FPModule(A, [0,1], [[Sq(2), Sq(1)]])
sage: N = FPModule(A, [2], [[Sq(4)]])
sage: homspace = Hom(M, N)

sage: values = [Sq(5)*N.generator(0), Sq(3,1)*N.generator(0)]
sage: f = homspace(values)
sage: f.degree()
7

The trivial homomorphism has no degree:

```python
sage: homspace.zero().degree()
Traceback (most recent call last):
  ...
ValueError: the zero morphism does not have a well-defined degree
```

**fp_module()**

Create a finitely presented module from self.

**OUTPUT:**

The finitely presented module having presentation equal to self as long as the domain and codomain are free.

**EXAMPLES:**
We construct examples with free modules that are presented with a redundant relation:

```python
sage: from sage.modules.fp_graded.module import FPModule
sage: A = SteenrodAlgebra(2)
sage: F1 = FPModule(A, (2,), [[0]])
sage: F2 = FPModule(A, (0,), [[0]])
sage: v = F2([Sq(2)])
sage: pres = Hom(F1, F2)([v])
sage: M = pres.fp_module(); M
Finitely presented left module on 1 generator and 1 relation over mod 2 Steenrod algebra, milnor basis
sage: M.generator_degrees()
(0,)
sage: M.relations()
(Sq(2)*g[0],)
sage: F2 = A.free_graded_module((0,))
sage: v = F2([Sq(2)])
sage: pres = Hom(F1, F2)([v])
sage: M = pres.fp_module(); M
Finitely presented left module on 1 generator and 1 relation over mod 2 Steenrod algebra, milnor basis
sage: M.generator_degrees()
(0,)
sage: M.relations()
(Sq(2)*g[0],)
sage: F3 = FPModule(A, (0,), [[Sq(4)])
sage: v = F3([Sq(2)])
sage: pres = Hom(F1, F3)([v])
sage: pres.fp_module()
Traceback (most recent call last):
  ... ValueError: this is not a morphism between free modules
```

**homology** *(f, top_dim=None, verbose=False)*

Compute the sub-quotient module of \( H(\text{self}, f) = \ker(\text{self})/\im(f) \) in a range of degrees.

For a pair of composable morphisms \( f : M \to N \) and \( g : N \to Q \) of finitely presented modules, the homology module is a finitely presented quotient of the kernel sub module \( \ker(g) \subset N \).

**INPUT:**

- \( f \) – a homomorphism with codomain equal to the domain of \( \text{self} \) and image contained in the kernel of this homomorphism
- \( \text{top_dim} \) – integer (optional); used by this function to stop the computation at the given degree
- \( \text{verbose} \) – boolean (default: False); enable progress messages

**OUTPUT:**

A quotient homomorphism \( \ker(\text{self}) \to H \), where \( H \) is isomorphic to \( H(\text{self}, f) \) in degrees less than or equal to \( \text{top_dim} \).

**Note:** If the algebra for this module is finite, then no \( \text{top_dim} \) needs to be specified in order to ensure that
this function terminates.

EXAMPLES:

```python
sage: from sage.modules.fp_graded.module import FPModule
sage: A = SteenrodAlgebra(2, profile=(3,2,1))
sage: M = FPModule(A, [0], [[Sq(3)]])
sage: N = FPModule(A, [0], [[Sq(2,2)]])
sage: F = FPModule(A, [0])
sage: f = Hom(M,N)([A.Sq(2)*N.generator(0)])
sage: g = Hom(F, M)([A.Sq(4)*A.Sq(1,2)*M.generator(0)])
sage: ho = f.homology(g)
sage: ho.codomain()
Finitely presented left module on 1 generator and 5 relations over
sub-Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [3,➞2, 1]
sage: ho.codomain().is_trivial()
False
```

`image(top_dim=None, verbose=False)`
Compute the image of `self`.

INPUT:

- `top_dim` – integer (optional); used by this function to stop the computation at the given degree
- `verbose` – boolean (default: `False`); enable progress messages

OUTPUT:

A homomorphism into `im(self)` that is an isomorphism in degrees less than or equal to `top_dim`

**Note:** If the algebra for this module is finite, then no `top_dim` needs to be specified in order to ensure that this function terminates.

EXAMPLES:

```python
sage: from sage.modules.fp_graded.module import FPModule
sage: A3 = SteenrodAlgebra(2, profile=(4,3,2,1))
sage: F = FPModule(A3, [1,3]);
sage: L = FPModule(A3, [2,3], [[Sq(2),Sq(1)], [0,Sq(2)]]);
sage: H = Hom(F, L);
sage: H([L((A3.Sq(1), 1)), L((0, A3.Sq(2)))]).image() # long time
Module morphism:
    From: Finitely presented left module on 1 generator and 1 relation over sub-
    Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [4, 3,➞2, 1]
    To:   Finitely presented left module on 2 generators and 2 relations over sub-
    Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [4, 3,➞2, 1]
    Defn: g[3] |--> Sq(1)*g[2] + g[3]
sage: M = FPModule(A3, [0,7], [[Sq(1), 0], [Sq(2), 0], [Sq(4), 0], [Sq(8), ◆
    →Sq(1)], [0, Sq(7)], [0, Sq(0,1)]]+Sq(4,2)])
```

(continues on next page)
sage: F2 = FPMModule(A3, [0], [[Sq(1)], [Sq(2)], [Sq(4)], [Sq(8)], [Sq(15)]]))
sage: H = Hom(M, F2)
sage: f = H([F2([1]), F2([0])])
sage: K = f.image(\text{verbose=True, top_dim=17})

1. Computing the generators of the image presentation:
Resolving the image in the range of dimensions $[0, 17]$:
\begin{align*}
0 & \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 & \quad 7 & \quad 8 & \quad 9 & \quad 10 & \quad 11 & \quad 12 & \quad 13 & \quad 14 & \quad 15 & \quad 16 & \quad 17.
\end{align*}

2. Computing the relations of the image presentation:
Computing using the profile:
\begin{align*}
(4, 3, 2, 1) \\
\end{align*}
Resolving the kernel in the range of dimensions $[0, 17]$:
\begin{align*}
0 & \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 & \quad 7 & \quad 8 & \quad 9 & \quad 10 & \quad 11 & \quad 12 & \quad 13 & \quad 14 & \quad 15 & \quad 16 & \quad 17.
\end{align*}

sage: K.is_injective()  \# long time
True
sage: K.domain().relations()
(0,)
(\text{Sq(1)} \cdot g[0], \text{Sq(2)} \cdot g[0], \text{Sq(4)} \cdot g[0], \text{Sq(8)} \cdot g[0])
sage: K.domain().is_trivial()
False

\textbf{is\_identity()}
Decide if self is the identity endomorphism.

\textbf{EXAMPLES:}

sage: from sage.modules.fp_graded.module import *
sage: A = SteenrodAlgebra(2)
sage: M = FPMModule(A, [0,1], [[Sq(2), Sq(1)]]))
sage: N = FPMModule(A, [2], [[Sq(4)]]))
sage: values = [Sq(5)*N.generator(0), Sq(3,1)*N.generator(0)]
sage: f = Hom(M, N)(values)
sage: f.is_identity()
False

sage: one = Hom(M, M)(M.generators()); one
Module endomorphism of Finitely presented left module on 2 generators and 1 → relation over mod 2 Steenrod algebra, milnor basis
  Defn: g[0] |--> g[0]
  g[1] |--> g[1]

sage: one.is_identity()
True

sage: M = A.free_graded_module((0,1))
sage: N = A.free_graded_module((2,))
sage: v = N.generator(0)
sage: values = [Sq(5)\cdot v, Sq(3,1)\cdot v]
sage: f = Hom(M, N)(values)
sage: f.is_identity()
False
sage: one = Hom(M, M)(M.generators()); one
Module endomorphism of Free graded left module on 2 generators over mod 2
˓→Steenrod algebra, milnor basis
    Defn: g[0] |--> g[0]
    g[1] |--> g[1]
sage: one.is_identity()
True

is_injective(top_dim=None, verbose=False)
Return True if and only if self has a trivial kernel.

INPUT:

- top_dim – integer (optional); used by this function to stop the computation at the given degree

- verbose – boolean (default: False); enable progress messages

EXAMPLES:

sage: from sage.modules.fp_graded.module import FPModule
sage: A = SteenrodAlgebra(2)
sage: K = FPModule(A, [2], [[Sq(2)]])
sage: HZ = FPModule(A, [0], [[Sq(1)]])
sage: f = Hom(K, HZ)([Sq(2)*HZ([1])])
sage: f.is_injective(top_dim=23)
True

is_surjective()
Return True if and only if self has a trivial cokernel.

EXAMPLES:

sage: from sage.modules.fp_graded.module import FPModule
sage: A = SteenrodAlgebra(2)
sage: F = FPModule(A, [0])
sage: f = Hom(F,F)([Sq(1)*F.generator(0)])
sage: f.is_surjective()
False

is_zero()
Decide if self is the zero homomorphism.

OUTPUT:

The boolean value True if self is trivial and False otherwise.

EXAMPLES:

sage: from sage.modules.fp_graded.module import FPModule
sage: A = SteenrodAlgebra(2)
sage: M = FPModule(A, [0,1], [[Sq(2), Sq(1)]])
sage: N = FPModule(A, [2], [[Sq(4)]])
sage: values = [Sq(5)*N.generator(0), Sq(3,1)*N.generator(0)]
sage: f = Hom(M, N)(values)
sage: f.is_zero()
False
sage: (f-f).is_zero()
True

sage: M = A.free_graded_module((0,1))
sage: N = A.free_graded_module((2,))
sage: v = N.generator(0)
sage: values = [Sq(5)*v, Sq(3,1)*v]
sage: f = Hom(M, N)(values)
sage: f.is_zero()
False
sage: (f-f).is_zero()
True

\textbf{kernel\_inclusion}(top\_dim=None, verbose=False)

Return the kernel of \textit{self}.

\textbf{INPUT}:

\begin{itemize}
  \item top\_dim -- integer (optional); used by this function to stop the computation at the given degree
  \item verbose -- boolean (default: False); enable progress messages
\end{itemize}

\textbf{OUTPUT}:

A homomorphism into $\ker(self)$ which is an isomorphism in degrees less than or equal to \texttt{top\_dim}.

\textbf{Note}: If the algebra for this module is finite, then no \texttt{top\_dim} needs to be specified in order to ensure that this function terminates.

\textbf{EXAMPLES}:

\begin{verbatim}
sage: from sage.modules.fp_graded.module import FPModule
sage: A3 = SteenrodAlgebra(2, profile=(4,3,2,1))
sage: F = FPModule(A3, [1,3]);
sage: L = FPModule(A3, [2,3], [[Sq(2),Sq(1)], [0,Sq(2)]]);
sage: H = Hom(F, L);
sage: H([L((A3.Sq(1), 1)), L((0, A3.Sq(2)))]).kernel_inclusion() # long time
Module morphism:
  From: Finitely presented left module on 2 generators and 1 relation over sub-
       \longrightarrow\text{Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [4, 3,}
       \longrightarrow\text{2, 1]}
  To: Free graded left module on 2 generators over sub-Hopf algebra of mod 2-
       \longrightarrow\text{Steenrod algebra, milnor basis, profile function [4, 3, 2, 1]}
  Defn: g[3] |--> g[3]
        g[4] |--> Sq(0,1)*g[1]

sage: M = FPModule(A3, [0,7], [[Sq(1), 0], [Sq(2), 0], [Sq(4), 0], [Sq(8),
       \longrightarrow\text{0}, Sq(7)], [0, Sq(7)], [0, Sq(0,1,1)+Sq(4,2)]])
\end{verbatim}
\texttt{sage: F2 = FPMODULE(A3, [0], [[Sq(1)], [Sq(2)], [Sq(4)], [Sq(8)], [Sq(15)]])}
\texttt{sage: H = Hom(M, F2)}
\texttt{sage: f = H([F2([1]), F2([0])])}
\texttt{sage: K = f.kernel_inclusion(\texttt{verbose=True}, \texttt{top_dim=17})}
1. Computing the generators of the kernel presentation:
   Resolving the kernel in the range of dimensions [0, 17]: 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17.
2. Computing the relations of the kernel presentation:
   Computing using the profile:
   (4, 3, 2, 1)
   Resolving the kernel in the range of dimensions [7, 17]: 7 8 9 10 11 12 13 14 15 16 17.
\texttt{sage: K.domain().generators()}
\texttt{(g[7],)}
\texttt{sage: K.domain().relations()}
\texttt{((Sq(0,1)+Sq(3))*g[7],}
\texttt{(Sq(0,0,1)+Sq(1,2)+Sq(4,1))*g[7],}
\texttt{Sq(9)*g[7],}
\texttt{(Sq(0,1,1)+Sq(4,2))*g[7])}
\texttt{sage: K}
Module morphism:
   From: Finitely presented left module on 1 generator and 4 relations over sub-
   \rightarrow Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [4, 3,
   \rightarrow 2, 1]
   To: Finitely presented left module on 2 generators and 6 relations over sub-
   \rightarrow Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [4, 3,
   \rightarrow 2, 1]
   Defn: g[7] |--> g[7]

**lift(f, verbose=False)**
Return a lift of this homomorphism over the given homomorphism \( f \).

**INPUT:**

- \( f \) – a homomorphism with codomain equal to the codomain of \( \text{self} \)
- \( \text{verbose} \) – boolean (default: False); enable progress messages

**OUTPUT:**
A homomorphism \( g \) with the property that \( \text{self} \) equals \( f \circ g \). If no lift exist, \( \text{None} \) is returned.

**ALGORITHM:**
Let \( s \) be this homomorphism with \( L \) the domain of \( s \). Choose \( x_1, \ldots, x_N \) such that \( f(x_i) = s(g_i) \), where
the \( g_i \)'s are the module generators of \( L \).

The linear function sending \( g_i \) to \( x_i \) for every \( i \) is well defined if and only if the vector \( x = (x_1, \ldots, x_N) \) lies in the nullspace of the coefficient matrix \( R = (r_{ij}) \) given by the relations of \( L \).

Let \( k \in \ker(f) \) solve the matrix equation:

\[
    R \cdot k = R \cdot x.
\]
Define a module homomorphism by sending the generators of \( L \) to \( x_1 - k_1, \ldots, x_N - k_N \). This is well defined, and is also a lift of this homomorphism over \( f \).

Note that it does not matter how we choose the initial elements \( x_i \): If \( x' \) is another choice then \( x' - x \in \ker(f) \) and \( R \cdot k = R \cdot x \) if and only if \( R \cdot (k + x' - x) = R \cdot x' \).

**EXAMPLES:**

```
sage: from sage.modules.fp_graded.module import FPModule
sage: A = SteenrodAlgebra(2)
```

Lifting a map from a free module is always possible:

```
sage: M = FPModule(A, [0], [[Sq(3)]])
sage: N = FPModule(A, [0], [[Sq(2,2)]])
sage: F = FPModule(A, [0])
sage: f = Hom(M,N)\([\text{[[Sq}(3)]]])
sage: k = Hom(F,N)\([\text{[[Sq}(2,2)]])
```

```
sage: f_ = k.lift(f)
sage: f_ == k
```

```
True
```

```
sage: A_plus_HZ = FPModule(A, [0,0], [[0, Sq(1)]])
sage: HZ = FPModule(A, [0], [[Sq(1)]])
sage: q = Hom(A_plus_HZ, HZ)([HZ([1]), HZ([1])])
```

```
# We can construct a splitting of 'q' manually:
sage: split = Hom(HZ,A_plus_HZ)([A_plus_HZ.generator(1)])
```

```
(q*split).is_identity()
```

```
True
```

Thus, lifting the identity homomorphism over \( q \) should be possible:

```
sage: one = Hom(HZ,HZ).identity()
sage: j = one.lift(q)
```

```
Module morphism:
  From: Finitely presented left module on 1 generator and 1 relation over mod 2 Steenrod algebra, milnor basis
  To:   Finitely presented left module on 2 generators and 1 relation over mod 2 Steenrod algebra, milnor basis
  Defn: g[0] |--> g[0, 1]
```

```
sage: q^j
```

```
Module endomorphism of Finitely presented left module on 1 generator and 1 relation over mod 2 Steenrod algebra, milnor basis
  Defn: g[0] |--> g[0]
```

Lifting over the inclusion of the image sub module:
When a lift cannot be found, the None value is returned. By setting the verbose argument to True, an explanation of why the lifting failed will be displayed:

```
sage: F2 = FPModule(A, [0,0])
sage: non_surjection = Hom(F2, F2)([F2([1, 0]), F2([0, 0])])
sage: lift = Hom(F2, F2).identity().lift(non_surjection, verbose=True)
The generators of the domain of this homomorphism do not map into
the image of the homomorphism we are lifting over.
sage: lift is None
True
```

See also:

* `split()`
* `solve(x)`

`solve(x)`
Return an element in the inverse image of `x`.

INPUT:

* `x` – an element of the codomain of this morphism

OUTPUT:

An element of the domain which maps to `x` under this morphism or `None` if `x` was not in the image of this morphism.

EXAMPLES:

```
sage: from sage.modules.fp_graded.module import FPModule
sage: A = SteenrodAlgebra(2)
sage: M = FPModule(A, [0], [[Sq(0,1)]])
sage: N = FPModule(A, [0], [[Sq(2,2)]])
sage: f = Hom(M,N)([Sq(2)*N.generator(0)])
sage: y = Sq(1,1)*N.generator(0); y
Sq(1,1)*g[0]
sage: x = f.solve(y); x
Sq(2)*g[0]
sage: y == f(x)
True
```

Trying to lift an element which is not in the image results in a None value:
```python
sage: z = f.solve(Sq(1)^N.generator(0))
sage: z is None
True
```

**split**(verbose=False)

Return a split of `self`.

**INPUT:**

- `verbose` – boolean (default: False); enable progress messages

**OUTPUT:**

A homomorphism with the property that the composite homomorphism $S \circ f = id$, where $S$ is `self`, is the identity homomorphism if no such split exist, `None` is returned.

**EXAMPLES:**

```python
sage: from sage.modules.fp_graded.module import FPModule
sage: A = SteenrodAlgebra(2)
sage: M = FPModule(A, [0,0], [[0, Sq(1)]]),
sage: N = FPModule(A, [0], [[Sq(1)]]),
sage: p = Hom(M, N)([N.generator(0), N.generator(0)])
sage: s = p.split(); s
Module morphism:
    From: Finitely presented left module on 1 generator and 1 relation over mod 2 Steenrod algebra, milnor basis
    To: Finitely presented left module on 2 generators and 1 relation over mod 2 Steenrod algebra, milnor basis
    Defn: g[0] |--> g[0, 1]
sage: # Verify that `s` is a splitting:
sage: p*s
Module endomorphism of Finitely presented left module on 1 generator and 1 relation over mod 2 Steenrod algebra, milnor basis
    Defn: g[0] |--> g[0]
```

**See also:**

`lift()`

**suspension**(t)

The suspension of this morphism by the given degree `t`.

**INPUT:**

- `t` – an integer by which the morphism is suspended

**OUTPUT:**

The morphism which is the suspension of `self` by the degree `t`.

**EXAMPLES:**

```python
sage: from sage.modules.fp_graded.module import FPModule
sage: A = SteenrodAlgebra(2)
sage: F1 = FPModule(A, [4,5])
sage: F2 = FPModule(A, [5])
sage: f = Hom(F1, F2)( ( F2([Sq(4)]), F2([Sq(5)]) ) ); f
(continues on next page)
```
Module morphism:

From: Free graded left module on 2 generators over mod 2 Steenrod algebra, milnor basis
To: Free graded left module on 1 generator over mod 2 Steenrod algebra, milnor basis
Defn: g[4] |--> Sq(4)*g[5]
      g[5] |--> Sq(5)*g[5]

```
sage: e1 = F1([1, 0])
sage: e2 = F1([0, 1])
sage: f(e1)
Sq(4)*g[5]
sage: f(e2)
Sq(5)*g[5]
```

```
sage: sf = f.suspension(4); sf
Module morphism:
From: Free graded left module on 2 generators over mod 2 Steenrod algebra, milnor basis
To: Free graded left module on 1 generator over mod 2 Steenrod algebra, milnor basis
Defn: g[8] |--> Sq(4)*g[9]
      g[9] |--> Sq(5)*g[9]
```

```
sage: sf.domain() is f.domain().suspension(4)
True
sage: sf.codomain() is f.codomain().suspension(4)
True
```

values()

The values under self of the module generators of the domain module.

OUTPUT:

A sequence of module elements of the codomain.

EXAMPLES:

```
sage: from sage.modules.fp_graded.module import FPModule
sage: A = SteenrodAlgebra(2)
sage: N = FPModule(A, [0,1], [[Sq(2), Sq(1)]])
sage: N = FPModule(A, [2], [[Sq(4)]])
sage: homspace = Hom(M, N)

sage: values = [Sq(5)*N.generator(0), Sq(3,1)*N.generator(0)]
sage: f = homspace(values)

sage: f.values()
(Sq(5)*g[2], Sq(3,1)*g[2])
sage: homspace.zero().values()
(0, 0)
```
vector\_presentation\( (n)\)  
Return the restriction of \texttt{self} to the domain module elements of degree \(n\).

The restriction of a non-zero module homomorphism to the free module of module elements of degree \(n\) is a linear function into the free module of elements of degree \(n + d\) belonging to the codomain. Here \(d\) is the degree of this homomorphism.

When this homomorphism is zero, it has no well defined degree so the function cannot be presented since we do not know the degree of its codomain. In this case, an error is raised.

INPUT:

• \(n\) – an integer degree

OUTPUT:

A linear function of finite dimensional free modules over the ground ring of the algebra for this module.

The domain is isomorphic to the free module of domain elements of degree \(n\) of \texttt{self} via the choice of basis given by \texttt{basis\_elements()}.

If the morphism is zero, an error is raised.

See also:

• \texttt{sage.modules.fp\_graded.module.FPModule.vector\_presentation()}  
• \texttt{sage.modules.fp\_graded.module.FPModule.basis\_elements()}  
• \texttt{sage.modules.fp\_graded.free\_module.FreeGradedModule.vector\_presentation()}  
• \texttt{sage.modules.fp\_graded.free\_module.FreeGradedModule.basis\_elements()}

EXAMPLES:

\begin{verbatim}sage: from sage.modules.fp_graded.module import FPModule sage: A = SteenrodAlgebra(2) sage: M = FPModule(A, [0,1], [[Sq(2), Sq(1)]]) sage: N = FPModule(A, [2], [[Sq(4)]]) sage: values = [Sq(5)*N.generator(0), Sq(3,1)*N.generator(0)] sage: f = Hom(M, N)(values) sage: f.vector_presentation(0) Vector space morphism represented by the matrix: [0] Domain: Vector space quotient V/W of dimension 1 over Finite Field of size 2 \(\rightarrow\) where  
V: Vector space of dimension 1 over Finite Field of size 2  
W: Vector space of degree 1 and dimension 0 over Finite Field of size 2  
Basis matrix: []  
Codomain: Vector space quotient V/W of dimension 1 over Finite Field of size 2 \(\rightarrow\) where  
\end{verbatim}  

V: Vector space of dimension 2 over Finite Field of size 2
W: Vector space of degree 2 and dimension 1 over Finite Field of size 2
Basis matrix:
\[
\begin{bmatrix}
0 & 1
\end{bmatrix}
\]

```
sage: f.vector_presentation(1)
Vector space morphism represented by the matrix:
\[
\begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}
\]
Domain: Vector space quotient V/W of dimension 2 over Finite Field of size 2
where
V: Vector space of dimension 2 over Finite Field of size 2
W: Vector space of degree 2 and dimension 0 over Finite Field of size 2
Basis matrix:
\[
[
]
\]
Codomain: Vector space quotient V/W of dimension 2 over Finite Field of size 2
where
V: Vector space of dimension 3 over Finite Field of size 2
W: Vector space of degree 3 and dimension 1 over Finite Field of size 2
Basis matrix:
\[
[0 1 1]
\]
```

```
sage: f.vector_presentation(2)
Vector space morphism represented by the matrix:
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]
Domain: Vector space of dimension 2 over Finite Field of size 2
Codomain: Vector space of dimension 2 over Finite Field of size 2
```

```
sage: M = A.free_graded_module((0,1))
sage: N = A.free_graded_module((2,))
sage: v = N.generator(0)
sage: values = [Sq(5)*v, Sq(3,1)*v]
sage: f = Hom(M, N)(values)
sage: f.vector_presentation(0)
Vector space morphism represented by the matrix:
\[
\begin{bmatrix}
0 & 1
\end{bmatrix}
\]
```

```
sage: f.vector_presentation(1)
Vector space morphism represented by the matrix:
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]
```

```
(continues on next page)
```
Codomain: Vector space of dimension 3 over Finite Field of size 2
\[ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]
Domain: Vector space of dimension 2 over Finite Field of size 2
Codomain: Vector space of dimension 4 over Finite Field of size 2

43.8 Homsets of finitely presented graded modules

AUTHORS:

- Sverre Lunøe–Nielsen (2020-07-01): Refactored the code and added new documentation and tests.

class `sage.modules.fp_graded.homspace.FPModuleHomspace(X, Y, category=None, base=None, check=True)`

Bases: `sage.categories.homset.Homset`

Element

alias of `sage.modules.fp_graded.morphism.FPModuleMorphism`

`an_element(n=0)`

Create a homomorphism belonging to `self`.

INPUT:

- `n` – (default: 0) an integer degree

OUTPUT:

A module homomorphism of degree `n`.

EXAMPLES:

```python
sage: from sage.modules.fp_graded.module import FPModule
sage: A = SteenrodAlgebra(2)
sage: HZ = FPModule(A, [0], relations=[[Sq(1)]])

sage: Hom(HZ, HZ).an_element(3)
Module endomorphism of Finitely presented left module on 1 generator and 1 → relation over mod 2 Steenrod algebra, milnor basis
  Defn: g[0] |--> Sq(0,1)*g[0]
```

`basis_elements(n)`

Return a basis for the free module of degree `n` morphisms.

INPUT:

- `n` – an integer degree

OUTPUT:

A basis for the set of all module homomorphisms of degree `n`.

```python
sage: from sage.modules.fp_graded.module import FPModule
sage: A = SteenrodAlgebra(2)
sage: HZ = FPModule(A, [0], relations=[[Sq(1)]])

sage: Hom(HZ, HZ).basis_elements(3)
```

43.8. Homsets of finitely presented graded modules 467
EXAMPLES:

```python
sage: from sage.modules.fp_graded.module import FPModule
sage: A = SteenrodAlgebra(2)
sage: Hko = FPModule(A, [0], relations=[[Sq(2)], [Sq(1)]])

sage: Hom(Hko, Hko).basis_elements(21)
[Module endomorphism of Finitely presented left module on 1 generator and 2
 → relations over mod 2 Steenrod algebra, milnor basis
  Defn: g[0] |--> (Sq(0,0,3)+Sq(0,2,0,1))*g[0],
Module endomorphism of Finitely presented left module on 1 generator and 2
 → relations over mod 2 Steenrod algebra, milnor basis
  Defn: g[0] |--> Sq(8,2,1)*g[0]]
```

`identity()`

Return the identity homomorphism.

EXAMPLES:

```python
sage: from sage.modules.fp_graded.module import FPModule
sage: A2 = SteenrodAlgebra(2, profile=(3,2,1))
sage: L = FPModule(A2, [2,3], [[Sq(2),Sq(1)], [0,Sq(2)]])

sage: one = Hom(L, L).identity(); one
Module endomorphism of Finitely presented left module on 2 generators and 2
 → relations over sub-Hopf algebra of mod 2 Steenrod algebra, milnor basis,
 → profile function [3, 2, 1]
  Defn: g[2] |--> g[2]
  g[3] |--> g[3]

sage: e = L.an_element(5)
sage: e == one(e)
True
```

It is an error to call this function when the homset is not a set of endomorphisms:

```python
sage: F = FPModule(A2, [1,3])
sage: Hom(F,L).identity()
Traceback (most recent call last):
  ...
TypeError: this homspace does not consist of endomorphisms
```

An example with free graded modules:

```python
sage: A2 = SteenrodAlgebra(2, profile=(3,2,1))
sage: L = A2.free_graded_module((2,3))
sage: H = Hom(L, L)
sage: H.identity()
Module endomorphism of Free graded left module on 2 generators
 → over sub-Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile
 → function [3, 2, 1]
  Defn: g[2] |--> g[2]
  g[3] |--> g[3]
```
zero()

Create the trivial homomorphism in self.

EXAMPLES:

```python
sage: from sage.modules.fp_graded.module import FPModule
sage: A2 = SteenrodAlgebra(2, profile=(3,2,1))
sage: F = FPModule(A2, [1,3])
sage: L = FPModule(A2, [2,3], [[Sq(2),Sq(1)], [0,Sq(2)]])

sage: z = Hom(F, L).zero()
sage: z(F.an_element(5))
0
sage: z(F.an_element(23))
0
```

Example with free modules:

```python
sage: A2 = SteenrodAlgebra(2, profile=(3,2,1))
sage: F = A2.free_graded_module((1,3))
sage: L = A2.free_graded_module((2,3))
sage: H = Hom(F, L)
sage: H.zero()

Module morphism:
  From: Free graded left module on 2 generators over sub-Hopf algebra of mod 2 → Steenrod algebra, milnor basis, profile function [3, 2, 1]
  To:  Free graded left module on 2 generators over sub-Hopf algebra of mod 2 → Steenrod algebra, milnor basis, profile function [3, 2, 1]
  Defn: g[1] |--> 0
g[3] |--> 0
```

43.9 Finitely presented graded modules over the Steenrod algebra

This package allows the user to define finitely presented modules over the mod $p$ Steenrod algebra, elements of them, and morphisms between them. Methods are provided for doing basic homological algebra, e.g. computing kernels and images of homomorphisms, and finding free resolutions of modules.

Theoretical background

The category of finitely presented graded modules over an arbitrary non-Noetherian graded ring $R$ is not abelian in general, since kernels of homomorphisms are not necessarily finitely presented.

The mod $p$ Steenrod algebra is non-Noetherian, but it is the union of a countable set of finite sub-Hopf algebras ([Mar1983] Ch. 15, Sect. 1, Prop 7). It is therefore an example of a $P$-algebra ([Mar1983] Ch. 13).

Any finitely presented module over the Steenrod algebra is defined over one of these finite sub-Hopf algebras. Similarly, any homomorphism between finitely presented modules over the Steenrod algebra is defined over a finite sub-Hopf algebra of the Steenrod algebra. Further, tensoring up from the category of modules over a sub-Hopf algebra is an exact functor, since the Steenrod algebra is free over any sub-Hopf algebra.

It follows that kernels, cokernels, images, and, more generally, any finite limits or colimits can be computed in the category of modules over the Steenrod algebra by working in the category of modules over an appropriate finite sub-Hopf algebra.
It is also the case that presentations of modules and the images of the generators of the domain of a homomorphism are the same over the sub-Hopf algebra and over the whole Steenrod algebra, so that the tensoring up is entirely implicit and requires no computation.

The definitions and computations carried out by this package are thus done as follows. First, find a small finite sub-Hopf algebra over which the computation can be done. Then, carry out the calculation there, where it is a finite problem and can be reduced to linear algebra over a finite prime field.

For examples, see the Steenrod algebra modules thematic tutorial.

AUTHORS:

• Sverre Lunoe–Nielsen and Koen van Woerden (2019-11-29): Updated the original software to Sage version 8.9.
• Sverre Lunoe–Nielsen (2020-07-01): Refactored the code and added new documentation and tests.

```python
class sage.modules.fp_graded.steenrod.module.SteenrodFPModule(j, names):
    Bases: sage.modules.fp_graded.module.FPModule, sage.modules.fp_graded.steenrod.module.SteenrodModuleMixin
    Create a finitely presented module over the Steenrod algebra.
    See also:
    The thematic tutorial on Steenrod algebra modules.
    INPUT:
    One of the following:
    • arg0 – a morphism such that the module is the cokernel, or a free graded module, in which case the output is the same module, viewed as finitely presented
    Otherwise:
    • arg0 – the graded connected algebra over which the module is defined; this algebra must be equipped with a graded basis
    • generator_degrees – tuple of integer degrees
    • relations – tuple of relations; a relation is a tuple of coefficients \((c_1, \ldots, c_n)\), ordered so that they correspond to the module generators, that is, such a tuple corresponds to the relation
      \[ c_1 g_1 + \cdots + c_n g_n = 0 \]
      if the generators are \((g_1, \ldots, g_n)\)
    resolution(k, top_dim=None, verbose=False)
    A free resolution of self of the given length.
    INPUT:
    • k – non-negative integer
    • top_dim – (optional) stop the computation at this degree
    • verbose – (default: False) whether log messages are printed
    OUTPUT:
    A list of homomorphisms \([\epsilon, f_1, \ldots, f_k]\) such that
    \[ f_i : F_i \rightarrow F_{i-1} \text{ for } 1 \leq i \leq k, \]
    \[ \epsilon : F_0 \rightarrow M, \]
```
where each $F_i$ is a finitely generated free module, and the sequence

$$F_k \xrightarrow{f_k} F_{k-1} \xrightarrow{f_{k-1}} \ldots \rightarrow F_0 \xrightarrow{\epsilon} M \rightarrow 0$$

is exact. Note that the 0th element in this list is the map $\epsilon$, and the rest of the maps are between free modules.

**EXAMPLES:**

```python
sage: from sage.modules.fp_graded.steenrod.module import SteenrodFPModule
sage: A = SteenrodAlgebra(2)
sage: Hko = SteenrodFPModule(A, [0], [[Sq(1)], [Sq(2)]]

sage: res = Hko.resolution(5, verbose=True)
Computing $f_1$ (1/5)
Computing $f_2$ (2/5)
Computing using the profile:
(2, 1)
Resolving the kernel in the range of dimensions [1, 8]: 1 2 3 4 5 6 7 8.
Computing $f_3$ (3/5)
Computing using the profile:
(2, 1)
Resolving the kernel in the range of dimensions [2, 10]: 2 3 4 5 6 7 8 9 10.
Computing $f_4$ (4/5)
Computing using the profile:
(2, 1)
Resolving the kernel in the range of dimensions [3, 13]: 3 4 5 6 7 8 9 10 11 12
\rightarrow 13.
Computing $f_5$ (5/5)
Computing using the profile:
(2, 1)
Resolving the kernel in the range of dimensions [4, 18]: 4 5 6 7 8 9 10 11 12
\rightarrow 13 14 15 16 17 18.
sage: [x.domain() for x in res]
[Free graded left module on 1 generator over mod 2 Steenrod algebra, milnor
basis,
Free graded left module on 2 generators over mod 2 Steenrod algebra, milnor
basis,
Free graded left module on 2 generators over mod 2 Steenrod algebra, milnor
basis,
Free graded left module on 2 generators over mod 2 Steenrod algebra, milnor
basis,
Free graded left module on 3 generators over mod 2 Steenrod algebra, milnor
basis,
Free graded left module on 4 generators over mod 2 Steenrod algebra, milnor
basis]
```

When there are no relations, the resolution is trivial:

```python
sage: M = SteenrodFPModule(A, [0])
sage: M.resolution(4)
[Module endomorphism of Free graded left module on 1 generator over mod 2 Steenrod algebra, milnor
basis
  Defn: g[0] \rightarrow g[0],
Module morphism:
  From: Free graded left module on 0 generators over mod 2 Steenrod algebra,
  To: Free graded left module on 0 generators over mod 2 Steenrod algebra,
(continues on next page)
```

43.9. Finitely presented graded modules over the Steenrod algebra
To: Free graded left module on 1 generator over mod 2 Steenrod algebra, milnor basis,
Module endomorphism of Free graded left module on 0 generators over mod 2 Steenrod algebra, milnor basis,
Module endomorphism of Free graded left module on 0 generators over mod 2 Steenrod algebra, milnor basis,
Module endomorphism of Free graded left module on 0 generators over mod 2 Steenrod algebra, milnor basis

class sage.modules.fp_graded.steenrod.module.SteenrodFreeModule(algebra, generator_degrees, category, names=None, **kwds)

class sage.modules.fp_graded.steenrod.module.SteenrodModuleMixin
Bases: object

Mixin class for common methods of the Steenrod algebra modules.

export_module_definition(powers_of_two_only=True)
Return the module to the input format used by R. Bruner's Ext software as a string.

INPUT:

• powers_of_two_only – boolean (default: True); if the output should contain the action of all Steenrod squaring operations (restricted by the profile), or only the action of the operations of degree equal to a power of two

EXAMPLES:

sage: from sage.modules.fp_graded.steenrod.module import SteenrodFPModule
sage: A1 = algebra=SteenrodAlgebra(p=2, profile=[2,1])
sage: M = SteenrodFPModule(A1, [0])
sage: print(M.export_module_definition())
8 0 1 2 3 3 4 5 6
0 1 1 1
2 1 1 4
3 1 1 5
6 1 1 7
0 2 1 2
1 2 2 3 4
2 2 1 5
3 2 1 6
4 2 1 6
5 2 1 7
sage: N = SteenrodFPModule(A1, [0], [[Sq(1)]])
sage: print(N.export_module_definition())
4 0 2 3 5
1 1 1 2
0 2 1 1
2 2 1 3
sage: print(N.export_module_definition(powers_of_two_only=False))
4 0 2 3 5
1 1 1 2
0 2 1 1
profile()
Return a finite profile over which self can be defined.

Any finitely presented module over the Steenrod algebra can be defined over a finite-dimensional sub-Hopf algebra, and this method identifies such a sub-Hopf algebra and returns its profile function.

**Note:** The profile produced by this function is reasonably small but is not guaranteed to be minimal.

**EXAMPLES:**

```python
sage: from sage.modules.fp_graded.steenrod.module import SteenrodFPModule
sage: A = SteenrodAlgebra(2)
sage: M = SteenrodFPModule(A, [0,1], [[Sq(2),Sq(1)],[0,Sq(2)],[Sq(3),0]])
sage: M.profile()
(2, 1)
```

### 43.10 Homomorphisms of finitely presented modules over the Steenrod algebra

This class implements construction and basic manipulation of homomorphisms between *finitely presented graded modules* over the mod $p$ Steenrod algebra.

**AUTHORS:**

- Sverre Lunøe–Nielsen (2020-07-01): Refactored the code and added new documentation and tests.

**class** `sage.modules.fp_graded.steenrod.morphism.SteenrodFPModuleMorphism(parent, values, check=True)`

**Bases:** `sage.modules.fp_graded.morphism.FPModuleMorphism`

**cokernel_projection**(verbose=False)
Compute the map to the cokernel of self.
INPUT:

• verbose – (default: False) whether log messages are printed

OUTPUT:

The natural projection from the codomain of this homomorphism to its cokernel.

EXAMPLES:

```python
sage: from sage.modules.fp_graded.steenrod.module import SteenrodFPModule
sage: A1 = SteenrodAlgebra(2, profile=(2,1))
sage: M = SteenrodFPModule(A1, [0], [[Sq(2)]])
sage: F = SteenrodFPModule(A1, [0])

sage: r = Hom(F, M)([A1.Sq(1)*M.generator(0)])
sage: co = r.cokernel_projection(); co
Module morphism:
   From: Finitely presented left module on 1 generator and 1 relation over sub-
  → Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [2, 1]
   To: Finitely presented left module on 1 generator and 2 relations over sub-
  → Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [2, 1]
   Defn: g[0] |--> g[0]

sage: co.domain().is_trivial()
False
```

image(
top_dim=None, verbose=False
)

Return the image of self.

INPUT:

• top_dim – integer (optional); used by this function to stop the computation at the given degree

• verbose – (default: False) whether log messages are printed

OUTPUT:

An injective homomorphism into the codomain of self which is onto the image of self.

EXAMPLES:

```python
sage: from sage.modules.fp_graded.steenrod.module import SteenrodFPModule
sage: A = SteenrodAlgebra(2)
sage: M = SteenrodFPModule(A, [0,1], [[Sq(2),Sq(1)], [0,Sq(2)]])
sage: S = SteenrodFPModule(A, [0], [[Sq(2)]])
sage: f = Hom(S, M)([M([0,1])])

sage: f.is_injective()
True

sage: i = f.image(); i
Module morphism:
   From: Finitely presented left module on 1 generator and 1 relation over mod 2 Steenrod algebra, milnor basis
   To: Finitely presented left module on 2 generators and 2 relations over mod 2 Steenrod algebra, milnor basis
   Defn: g[1] |--> g[1]

sage: i.codomain() is M
True
```
Lift the map $f$ over the inclusion $i$:

```
sage: f_ = f.lift(i)
sage: f_.is_injective()
True
sage: f_.is_surjective()
True
```

```
sage: g = Hom(S, M)([M([Sq(3),Sq(2)])])
sage: j = g.image(); j
Module morphism:
  From: Free graded left module on 0 generators over mod 2 Steenrod algebra, ⬠milnor basis
  To:  Finitely presented left module on 2 generators and 2 relations over mod_, ⬠2 Steenrod algebra, milnor basis
```

So $g$ had to be trivial:

```
sage: g.is_zero()
True
```

**is_injective**(*top_dim=None, verbose=False*)

Return True if self is injective.

**INPUT:**

- `top_dim` – (optional) stop the computation at this degree; if not specified, this is determined using `profile()`
- `verbose` – (default: False) whether log messages are printed

**EXAMPLES:**

```
sage: from sage.modules.fp_graded.steenrod.module import SteenrodFPModule
sage: A = SteenrodAlgebra(2)
sage: M = SteenrodFPModule(A, [0,1], [[Sq(2),Sq(1)], [0,Sq(2)]])
sage: S = SteenrodFPModule(A, [0], [[Sq(2)]])
sage: f = Hom(S, M)([M([0,1])])
sage: f.is_injective()
True
sage: g = Hom(S, M).zero()
sage: g.is_injective()
False
sage: z = Hom(SteenrodFPModule(A, []), M).zero()
sage: z.is_injective()
True
sage: z.is_zero()
True
```

**kernel_inclusion**(*top_dim=None, verbose=False*)

Return the kernel of self as a morphism.

**INPUT:**

- `top_dim` – (optional) stop the computation at this degree; if not specified, this is determined using `profile()`
- `verbose` – (default: False) whether log messages are printed
OUTPUT: An injective homomorphism into the domain \texttt{self} which is onto the kernel of this homomorphism.

EXAMPLES:

```python
sage: from sage.modules.fp_graded.steenrod.module import SteenrodFPModule
sage: A = SteenrodAlgebra(2)
sage: M = SteenrodFPModule(A, [0,1], [[Sq(2),Sq(1)], [0,Sq(2)]]

sage: S = SteenrodFPModule(A, [0], [[Sq(2)]])
sage: f = Hom(S, M)([M([0,1])])
sage: f.is_injective()
True
sage: k = f.kernel_inclusion()
sage: k == 0
True
```

Since \( k \) is both trivial and injective, its domain should be the zero module:

```python
sage: k.domain().is_trivial()
True
```

```python
sage: g = Hom(S, M)([M([Sq(3),Sq(2)])])
sage: h = g.kernel_inclusion()
sage: h.is_identity()
True
sage: ker = h.domain();
sage: ker
```

```python
True
```

So \( g \) had to be trivial:

```python
sage: g.is_zero()
True
```

\texttt{profile()}:

Return a finite profile over which \texttt{self} can be defined.

This is in some ways the key method for these morphisms. As discussed in the “Theoretical background” section of \texttt{sage.modules.fp_graded.steenrod.module}, any homomorphism of finitely presented modules over the Steenrod algebra can be defined over a finite-dimensional sub-Hopf algebra, and this method identifies such a sub-Hopf algebra and returns its profile function.

EXAMPLES:

```python
sage: from sage.modules.fp_graded.steenrod.module import SteenrodFPModule
sage: A = SteenrodAlgebra(2)
sage: M = SteenrodFPModule(A, [0,1], [[Sq(2),Sq(1)], [0,Sq(2)]]

sage: one = Hom(M,M).identity()
sage: one.profile()
(2, 1)
sage: zero = Hom(M,M).zero()
sage: zero.profile()
(2, 1)
sage: A_fin = SteenrodAlgebra(2, profile=(2,1))
sage: M_fin = M.change_ring(A_fin)
```
Change the ring of the module $M$:

```
sage: M_fin.change_ring(A) == M
True
```

We can change rings to the finite sub-Hopf algebra defined by the profile we just computed:

```
sage: one_fin = one.change_ring(A_fin)
sage: one_fin.domain()
Finitely presented left module on 2 generators and 2 relations over
  sub-Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [2,\rightarrow 1]
```

If we change back to the full Steenrod algebra, we are back where we started:

```
sage: one_fin.change_ring(A) == one
True
```
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