Noncommutative Polynomials

Release 10.0

The Sage Development Team

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CHAPTER
ONE

UNIVARIATE ORE POLYNOMIALS

1.1 Univariate Ore polynomial rings

This module provides the OrePolynomialRing, which constructs a general dense univariate Ore polynomial ring over a commutative base with equipped with an endomorphism and/or a derivation.

**AUTHOR:**
- Xavier Caruso (2020-04)

```python
class sage.rings.polynomial.ore_polynomial_ring.OrePolynomialRing(base_ring, morphism, derivation, name, sparse, category=None)
```

Bases: UniqueRepresentation, Algebra

Construct and return the globally unique Ore polynomial ring with the given properties and variable names.

Given a ring \( R \) and a ring automorphism \( \sigma \) of \( R \) and a \( \sigma \)-derivation \( \partial \), the ring of Ore polynomials \( R[X; \sigma, \partial] \) is the usual abelian group polynomial \( R[X] \) equipped with the modification multiplication deduced from the rule \( Xa = \sigma(a)X + \partial(a) \). We refer to [Ore1933] for more material on Ore polynomials.

**INPUT:**
- `base_ring` – a commutative ring
- `twisting_map` – either an endomorphism of the base ring, or a (twisted) derivation of it
- `names` – a string or a list of strings
- `sparse` – a boolean (default: False); currently not supported

**EXAMPLES:**

The case of a twisting endomorphism

We create the Ore ring \( \mathbb{F}_{5^3}[x, \text{Frob}] \) where Frob is the Frobenius endomorphism:

```python
sage: k.<a> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S = OrePolynomialRing(k, Frob, 'x')
sage: S
Ore Polynomial Ring in x over Finite Field in a of size 5^3 twisted by a |
```

In particular, observe that it is not needed to create and pass in the twisting derivation (which is 0 in our example). As a shortcut, we can use the square brackets notation as follow:
Noncommutative Polynomials, Release 10.0

```
sage: T.<x> = k['x', Frob]
sage: T
Ore Polynomial Ring in x over Finite Field in a of size 5^3 twisted by a ↦ a^5
sage: T is S
True
```

We emphasize that it is necessary to repeat the name of the variable in the right hand side. Indeed, the following fails (it is interpreted by Sage as a classical polynomial ring with variable name Frob):

```
sage: T.<x> = k[Frob]
Traceback (most recent call last):
...
ValueError: variable name 'Frobenius endomorphism a ↦ a^5 on Finite Field in a˓→of size 5^3' is not alphanumeric
```

Note moreover that, similarly to the classical case, using the brackets notation also sets the variable:

```
sage: x.parent() is S
True
```

We are now ready to carry on computations in the Ore ring:

```
sage: x*a
(2*a^2 + 4*a + 4)*x
sage: Frob(a)*x
(2*a^2 + 4*a + 4)*x
```

**The case of a twisting derivation**

We can similarly create the Ore ring of differential operators over $\mathbb{Q}[t]$, namely $\mathbb{Q}[t][d, \frac{d}{dt}]$:

```
sage: R.<t> = QQ[]
sage: der = R.derivation(); der
d/dt
sage: A = OrePolynomialRing(R, der, 'd')
sage: A
Ore Polynomial Ring in d over Univariate Polynomial Ring in t over Rational Field˓→twisted by d/dt
```

Again, the brackets notation is available:

```
sage: B.<d> = R['d', der]
sage: A is B
True
```

and computations can be carried out:

```
sage: d^t
t*d + 1
```
The combined case

Ore polynomial rings involving at the same time a twisting morphism $\sigma$ and a twisting $\sigma$-derivation can be created as well as follows:

```sage
sage: F.<u> = Qq(3^2)
sage: sigma = F.frobenius_endomorphism(); sigma
Frobenius endomorphism on 3-adic Unramified Extension Field in u
defined by x^2 + 2*x + 2 lifting u -> u^3 on the residue field
sage: der = F.derivation(3, twist=sigma); der
(3 + O(3^21))*(\text{Frob}) - id)
sage: M.<X> = F[X, der]
sage: M
Ore Polynomial Ring in X over 3-adic Unramified Extension Field in u
defined by x^2 + 2*x + 2 twisted by Frob and (3 + O(3^21))*(\text{Frob} - id)
```

We emphasize that we only need to pass in the twisted derivation as it already contains in it the datum of the twisting endomorphism. Actually, passing in both twisting maps results in an error:

```sage
sage: F.<X, sigma, der]
Traceback (most recent call last):
...:
ValueError: variable name 'Frobenius endomorphism ...' is not alphanumeric
```

Examples of variable name context

Consider the following:

```sage
sage: R.<t> = ZZ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = SkewPolynomialRing(R, sigma); S
Ore Polynomial Ring in x over Univariate Polynomial Ring in t over Integer Ring
twisted by t |--> t + 1
```

The names of the variables defined above cannot be arbitrarily modified because each Ore polynomial ring is unique in Sage and other objects in Sage could have pointers to that Ore polynomial ring.

However, the variable can be changed within the scope of a `with` block using the `localvars` context:

```sage
sage: with localvars(S, ['y']):
.....: print(S)
Ore Polynomial Ring in y over Univariate Polynomial Ring in t over Integer Ring
twisted by t |--> t + 1
```
**Uniqueness and immutability**

In Sage, there is exactly one Ore polynomial ring for each quadruple (base ring, twisting morphism, twisting derivation, name of the variable):

```
sage: k.<a> = GF(7^3)
sage: Frob = k.frobenius_endomorphism()
sage: S = k['x'], Frob
sage: T = k['x'], Frob
sage: S is T
True
```

Rings with different variables names are different:

```
sage: S is k['y'], Frob
False
```

Similarly, varying the twisting morphisms yields to different Ore rings (expect when the morphism coincide):

```
sage: S is k['x'], Frob^2
False
sage: S is k['x'], Frob^3
False
sage: S is k['x'], Frob^4
True
```

**Todo:**

- Sparse Ore Polynomial Ring
- Multivariate Ore Polynomial Ring

---

**Element** = None

**change_var**(var)

Return the Ore polynomial ring in variable var with the same base ring, twisting morphism and twisting derivation as self.

**INPUT:**

- var – a string representing the name of the new variable

**EXAMPLES:**

```
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: R.<x> = OrePolynomialRing(k,Frob); R
Ore Polynomial Ring in x over Finite Field in t of size 5^3 twisted by t |--> t^5
sage: Ry = R.change_var('y'); Ry
Ore Polynomial Ring in y over Finite Field in t of size 5^3 twisted by t |--> t^5
sage: Ry is R.change_var('y')
True
```
**characteristic()**

Return the characteristic of the base ring of self.

**EXAMPLES:**

```sage
sage: R.<t> = QQ[]
sage: sigma = R.hom([t+1])
sage: R['x',sigma].characteristic()
0
```

```sage
sage: k.<u> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: k['y',Frob].characteristic()
5
```

**fraction_field()**

Return the fraction field of this skew ring.

**EXAMPLES:**

```sage
sage: k.<a> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'], Frob
sage: K = S.fraction_field(); K
Ore Function Field in x over Finite Field in a of size 5^3 twisted by a |--> a^5
sage: f = 1/(x + a); f
(x + a)^(-1)
sage: f.parent() is K
True
```

Below is another example with differential operators:

```sage
sage: R.<t> = QQ[]
sage: der = R.derivation()
sage: A.<d> = R['d', der]
sage: A.fraction_field()
Ore Function Field in d over Fraction Field of Univariate Polynomial Ring in t over Rational Field twisted by d/dt
sage: f = t/d; f
(d - 1/t)^(-1) * t
sage: f^d
t
```

**See also:**

`sage.rings.polynomial.ore_function_field`

**gen(n=0)**

Return the indeterminate generator of this Ore polynomial ring.

**INPUT:**

- `n` – index of generator to return (default: 0); exists for compatibility with other polynomial rings

**EXAMPLES:**
sage: R.<t> = QQ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x',sigma]; S
Ore Polynomial Ring in x over Univariate Polynomial Ring in t over Rational
˓→Field twisted by t |--> t + 1
sage: y = S.gen(); y
x
sage: y == x
True
sage: y is x
True
sage: S.gen(0)
x
This is also known as the parameter:

sage: S.parameter() is S.gen()
True

gens_dict()

Return a {name: variable} dictionary of the generators of this Ore polynomial ring.

EXAMPLES:

sage: R.<t> = ZZ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = SkewPolynomialRing(R,sigma)
sage: S.gens_dict()
{'x': x}

is_commutative()

Return True if this Ore polynomial ring is commutative, i.e. if the twisting morphism is the identity and the twisting derivation vanishes.

EXAMPLES:

sage: k.<a> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x', Frob]
sage: S.is_commutative()
False
sage: T.<y> = k['y', Frob^3]
sage: T.is_commutative()
True
sage: R.<t> = GF(5)[]
sage: der = R.derivation()
sage: A.<d> = R['d', der]
sage: A.is_commutative()
False
sage: B.<b> = R['b', 5*der]

(continues on next page)
is_exact()

Return True if elements of this Ore polynomial ring are exact. This happens if and only if elements of the base ring are exact.

EXAMPLES:

```sage
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x', Frob]
sage: S.is_exact()
True
sage: S.base_ring().is_exact()
True
sage: R.<u> = k[[u]]
sage: sigma = R.hom([u+u^2])
sage: T.<y> = R['y', sigma]
sage: T.is_exact()
False
sage: T.base_ring().is_exact()
False
```

is_field(proof=False)

Return always False since Ore polynomial rings are never fields.

EXAMPLES:

```sage
sage: k.<a> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x', Frob]
sage: S.is_field()
False
```

is_finite()

Return False since Ore polynomial rings are not finite (unless the base ring is 0).

EXAMPLES:

```sage
sage: k.<t> = GF(5^3)
sage: k.is_finite()
True
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x', Frob]
sage: S.is_finite()
False
```

is_sparse()

Return True if the elements of this Ore polynomial ring are sparsely represented.
**Warning:** Since sparse Ore polynomials are not yet implemented, this function always returns False.

### EXAMPLES:
```
sage: R.<t> = RR[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x',sigma]
sage: S.is_sparse()
False
```

### **ngens()**
Return the number of generators of this Ore polynomial ring, which is 1.

### EXAMPLES:
```
sage: R.<t> = RR[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x',sigma]
sage: S.ngens()
1
```

### **parameter(n=0)**
Return the indeterminate generator of this Ore polynomial ring.

**INPUT:**
- *n* – index of generator to return (default: 0); exists for compatibility with other polynomial rings

### EXAMPLES:
```
sage: R.<t> = QQ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x',sigma]; S
Ore Polynomial Ring in x over Univariate Polynomial Ring in t over Rational
˓→Field twisted by t |--> t + 1
sage: y = S.gen(); y
x
sage: y == x
True
sage: y is x
True
sage: S.gen(0)
x
```

This is also known as the parameter:
```
sage: S.parameter() is S.gen()
True
```

### **random_element**(degree=(-1, 2), monic=False, *args, **kwds)
Return a random Ore polynomial in this ring.

**INPUT:**
- *degree* – (default: (-1, 2)) integer with degree or a tuple of integers with minimum and maximum degrees

---

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• monic – (default: False) if True, return a monic Ore polynomial

• *args, **kwds – passed on to the random_element method for the base ring

OUTPUT:
Ore polynomial such that the coefficients of $x^i$, for $i$ up to degree, are random elements from the base ring, randomized subject to the arguments *args and **kwds.

EXAMPLES:

```python
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x', Frob]
sage: S.random_element()  # random
(2*t^2 + 3)*x^2 + (4*t^2 + t + 4)*x + 2*t^2 + 2
sage: S.random_element(monic=True)  # random
x^2 + (2*t^2 + t + 1)*x + 3*t^2 + 3*t + 2
```

Use degree to obtain polynomials of higher degree:

```python
sage: p = S.random_element(degree=5)  # random
(t^2 + 3*t)*x^5 + (4*t + 4)*x^3 + (4*t^2 + 4*t)*x^2 + (2*t^2 + 1)*x + 3
sage: p.degree() == 5
True
```

If a tuple of two integers is given for the degree argument, a random integer will be chosen between the first and second element of the tuple as the degree, both inclusive:

```python
sage: S.random_element(degree=(2,7))  # random
(3*t^2 + 1)*x^4 + (4*t + 2)*x^3 + (4*t + 1)*x^2 + (t^2 + 3*t + 3)*x + 3*t^2 + 2*t + 2
```

**random_irreducible**(degree=2, monic=True, *args, **kwds)

Return a random irreducible Ore polynomial.

**Warning:** Elements of this Ore polynomial ring need to have a method is_irreducible(). Currently, this method is implemented only when the base ring is a finite field.

INPUT:

• degree - Integer with degree (default: 2) or a tuple of integers with minimum and maximum degrees

• monic - if True, returns a monic Ore polynomial (default: True)

• *args, **kwds - passed in to the random_element method for the base ring

EXAMPLES:

```python
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x', Frob]
sage: A = S.random_irreducible()
sage: A.is_irreducible()
True
sage: B = S.random_irreducible(degree=3, monic=False)
```
twisting_derivation()  
Return the twisting derivation defining this Ore polynomial ring or None if this Ore polynomial ring is not twisted by a derivation.

EXAMPLES:

```sage
sage: R.<t> = QQ[]
sage: der = R.derivation(); der
d/dt
sage: A.<d> = R['d', der]
sage: A.twisting_derivation()
d/dt
```

See also:

twisting_morphism()

twisting_morphism(n=1)  
Return the twisting endomorphism defining this Ore polynomial ring iterated n times or None if this Ore polynomial ring is not twisted by an endomorphism.

INPUT:

* n - an integer (default: 1)

EXAMPLES:

```sage
sage: R.<t> = QQ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x', sigma]
sage: S.twisting_morphism() == sigma
True
sage: S.twisting_morphism(10)
```

If n in negative, Sage tries to compute the inverse of the twisting morphism:

```sage
sage: k.<a> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: T.<y> = k['y',Frob]
sage: T.twisting_morphism(-1)
```

Sometimes it fails, even if the twisting morphism is actually invertible:
sage: K = R.fraction_field()
sage: phi = K.hom([(t+1)/(t-1)])
sage: T.<y> = K['y'], phi]
sage: T.twisting_morphism(-1)
Traceback (most recent call last):
...  
NotImplementedError: inverse not implemented for morphisms of Fraction Field of...
Univariate Polynomial Ring in t over Rational Field

When the Ore polynomial ring is only twisted by a derivation, this method returns nothing:

sage: der = R.derivation()
sage: A.<d> = R['x', der]
sage: A
Ore Polynomial Ring in x over Univariate Polynomial Ring in t over Rational...
 Field twisted by d/dt
sage: A.twisting_morphism()

Here is an example where the twisting morphism is automatically inferred from the derivation:

sage: k.<a> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: der = k.derivation(1, twist=Frob)
sage: der
[a |--> a^5] - id
sage: S.<x> = k['x', der]
sage: S.twisting_morphism()
Frobenius endomorphism a |--> a^5 on Finite Field in a of size 5^3

See also:

twisting_derivation()

1.2 Univariate Ore polynomials

This module provides the OrePolynomial, which constructs a single univariate Ore polynomial over a commutative base equipped with an endomorphism and/or a derivation. It provides generic implementation of standard arithmetical operations on Ore polynomials as addition, multiplication, gcd, lcm, etc.

The generic implementation of dense Ore polynomials is OrePolynomial_generic_dense. The classes ConstantOrePolynomialSection and OrePolynomialBaseringInjection handle conversion from a Ore polynomial ring to its base ring and vice versa.

AUTHORS:

• Xavier Caruso (2020-05)

class sage.rings.polynomial.ore_polynomial_element.ConstantOrePolynomialSection

  Bases: Map

  Representation of the canonical homomorphism from the constants of a Ore polynomial ring to the base ring.

  This class is necessary for automatic coercion from zero-degree Ore polynomial ring into the base ring.

  EXAMPLES:
sage: from sage.rings.polynomial.ore_polynomial_element import \nConstantOrePolynomialSection
sage: R.<t> = QQ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x'],sigma
sage: m = ConstantOrePolynomialSection(S, R); m
Generic map:
    From: Ore Polynomial Ring in x over Univariate Polynomial Ring in t over Rational Field twisted by t |--> t + 1
    To:   Univariate Polynomial Ring in t over Rational Field

class sage.rings.polynomial.ore_polynomial_element.OrePolynomial

Bases: AlgebraElement

Abstract base class for Ore polynomials.
This class must be inherited from and have key methods overridden.

Definition

Let \( R \) be a commutative ring equipped with an automorphism \( \sigma \) and a \( \sigma \)-derivation \( \partial \).
A Ore polynomial is given by the equation:

\[ F(X) = a_n X^n + \cdots + a_0, \]

where the coefficients \( a_i \in R \) and \( X \) is a formal variable.

Addition between two Ore polynomials is defined by the usual addition operation and the modified multiplication is defined by the rule \( X a = \sigma(a)X + \partial(a) \) for all \( a \) in \( R \). Ore polynomials are thus non-commutative and the degree of a product is equal to the sum of the degrees of the factors.

Let \( a \) and \( b \) be two Ore polynomials in the same ring \( S \). The left (resp. right) euclidean division of \( a \) by \( b \) is a couple \((q, r)\) of elements in \( S \) such that

- \( a = qb + r \) (resp. \( a = bq + r \))
- the degree of \( r \) is less than the degree of \( b \)

\( q \) (resp. \( r \)) is called the quotient (resp. the remainder) of this euclidean division.

Properties

Keeping the previous notation, if the leading coefficient of \( b \) is a unit (e.g. if \( b \) is monic) then the quotient and the remainder in the right euclidean division exist and are unique.

The same result holds for the left euclidean division if in addition the twisting morphism defining the Ore polynomial ring is invertible.

EXAMPLES:
We illustrate some functionalities implemented in this class.
We create the Ore polynomial ring (here the derivation is zero):
sage: R.<t> = ZZ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x',sigma]; S
Ore Polynomial Ring in x over Univariate Polynomial Ring in t over Integer Ring twisted by t |--> t + 1

and some elements in it:

sage: a = t + x + 1; a
x + t + 1
sage: b = S([t^2,t+1,1]); b
x^2 + (t + 1)*x + t^2
sage: c = S.random_element(degree=3,monic=True)
sage: c.parent() is S
True

Ring operations are supported:

sage: a + b
x^2 + (t + 2)*x + t^2 + t + 1
sage: a - b
-x^2 - t*x - t^2 + t + 1
sage: a * b
x^3 + (2*t + 3)*x^2 + (2*t^2 + 4*t + 2)*x + t^3 + t^2
sage: b * a
x^3 + (2*t + 4)*x^2 + (2*t^2 + 3*t + 2)*x + t^3 + t^2
sage: a * b == b * a
False
sage: b^2
x^4 + (2*t + 4)*x^3 + (3*t^2 + 7*t + 6)*x^2 + (2*t^3 + 4*t^2 + 3*t + 1)*x + t^4
sage: b^2 == b*b
True

Sage also implements arithmetic over Ore polynomial rings. You will find below a short panorama:

sage: q,r = c.right_quo_rem(b)
sage: c == q*b + r
True

The operators // and % give respectively the quotient and the remainder of the right euclidean division:

sage: q == c // b
True
sage: r == c % b
True

Here we can see the effect of the operator evaluation compared to the usual polynomial evaluation:

sage: a = x^2
sage: a(t)
doctest:...: FutureWarning: This class/method/function is marked as experimental.
It, its functionality or its interface might change without a formal deprecation. See https://github.com/sagemath/sage/issues/13215 for details.

Here is another example over a finite field:

```
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x', Frob]
sage: a = x^4 + (4*t + 1)*x^3 + (t^2 + 3*t + 3)*x^2 + (3*t^2 + 2^t + 2)*x + (3*t^2 + t + 1)
sage: b = (2*t^2 + 3)*x^2 + (3*t^2 + 1)*x + 4*t + 2
sage: q, r = a.left_quo_rem(b)
sage: q
(4*t^2 + t + 1)*x^2 + (2*t^2 + 2*t + 2)*x + 2*t^2 + 4*t + 3
sage: r
(t + 2)*x + 3*t^2 + 2*t + 4
sage: a == b*q + r
True
```

Once we have euclidean divisions, we have for free gcd and lcm (at least if the base ring is a field):

```
sage: a = (x + t) * (x + t^2)^2
sage: b = (x + t) * (t*x + t + 1) * (x + t^2)
sage: a.right_gcd(b)
x + t^2
sage: a.left_gcd(b)
x + t
```

The left lcm has the following meaning: given Ore polynomials $a$ and $b$, their left lcm is the least degree polynomial $c = u_a = v_b$ for some Ore polynomials $u, v$. Such a $c$ always exist if the base ring is a field:

```
sage: c = a.left_lcm(b); c
x^5 + (4*t^2 + t + 3)*x^4 + (3*t^2 + 4*t)*x^3 + 2*t^2*x^2 + (2*t^2 + t)*x + 4*t^2 + 4
sage: c.is_right_divisible_by(a)
True
sage: c.is_right_divisible_by(b)
True
```

The right lcm is defined similarly as the least degree polynomial $c = a_u = b_v$ for some $u, v$:

```
sage: d = a.right_lcm(b); d
x^5 + (t^2 + 1)*x^4 + (3*t^2 + 3*t + 3)*x^3 + (3*t^2 + t + 2)*x^2 + (4*t^2 + 3*t)*x + 4*t + 4
sage: d.is_left_divisible_by(a)
True
sage: d.is_left_divisible_by(b)
True
```

See also:

- `sage.rings.polynomial.ore_polynomial_ring`
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1.2. Univariate Ore polynomials

\textbf{base_ring()}

Return the base ring of self.

EXAMPLES:

\begin{verbatim}
sage: R.<t> = ZZ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x'],sigma
sage: a = S.random_element()
sage: a.base_ring()
Univariate Polynomial Ring in t over Integer Ring
sage: a.base_ring() is R
True
\end{verbatim}

\textbf{change_variable_name(var)}

Change the name of the variable of self.

This will create the Ore polynomial ring with the new name but same base ring, twisting morphism and twisting derivation. The returned Ore polynomial will be an element of that Ore polynomial ring.

INPUT:

• var – the name of the new variable

EXAMPLES:

\begin{verbatim}
sage: R.<t> = ZZ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x'], sigma
sage: a = x^3 + (2*t + 1)*x + t^2 + 3*t + 5
sage: b = a.change_variable_name('y'); b
y^3 + (2*t + 1)*y + t^2 + 3*t + 5
\end{verbatim}

Note that a new parent is created at the same time:

\begin{verbatim}
sage: b.parent()
Ore Polynomial Ring in y over Univariate Polynomial Ring in t over Integer Ring
twisted by t |--> t + 1
\end{verbatim}

\textbf{coefficients(sparse=True)}

Return the coefficients of the monomials appearing in self.

If sparse=True (the default), return only the non-zero coefficients. Otherwise, return the same value as self.list().

\textbf{Note:} This should be overridden in subclasses.

EXAMPLES:

\begin{verbatim}
sage: R.<t> = QQ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x'], sigma
sage: a = 1 + x^4 + (t+1)*x^2 + t^2
sage: a.coefficients()
[t^2 + 1, t + 1, 1]
\end{verbatim}

(continues on next page)
**constant_coefficient()**

Return the constant coefficient (i.e. the coefficient of term of degree 0) of `self`.

**EXAMPLES:**

```python
sage: R.<t> = ZZ[
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x',sigma]
sage: a = x + t^2 + 2
sage: a.constant_coefficient()
t^2 + 2
```

**degree()**

Return the degree of `self`.

By convention, the zero Ore polynomial has degree $-1$.

**EXAMPLES:**

```python
sage: R.<t> = ZZ[
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x',sigma]
sage: a = x^2 + t*x^3 + t^2*x + 1
sage: a.degree()
3
sage: S.zero().degree()
-1
sage: S(5).degree()
0
```

**exponents()**

Return the exponents of the monomials appearing in `self`.

**EXAMPLES:**

```python
sage: R.<t> = QQ[
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x',sigma]
sage: a = 1 + x^4 + (t+1)*x^2 + t^2
sage: a.exponents()
[0, 2, 4]
```

**hamming_weight()**

Return the number of non-zero coefficients of `self`.

This is also known as the weight, hamming weight or sparsity.

**EXAMPLES:**

```python
sage: R.<t> = QQ[
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x',sigma]
```

(continues on next page)
sage: a = 1 + x^4 + (t+1)*x^2 + t^2
sage: a.number_of_terms()
3

This is also an alias for hamming_weight:

sage: a.hamming_weight()
3

is_constant()

Return whether self is a constant polynomial.

EXAMPLES:

sage: R.<t> = QQ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x'],sigma
sage: R(2).is_constant()
True
sage: (x + 1).is_constant()
False

is_left_divisible_by(other)

Check if self is divisible by other on the left.

INPUT:

• other – a Ore polynomial in the same ring as self

OUTPUT:

Return True or False.

EXAMPLES:

sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'],Frob
sage: a = x^2 + t*x + t^2 + 3
sage: b = x^3 + (t + 1)*x^2 + 1
sage: c = a*b
sage: c.is_left_divisible_by(a)
True
sage: c.is_left_divisible_by(b)
False

Divisibility by 0 does not make sense:

sage: c.is_left_divisible_by(S(0))
Traceback (most recent call last):
...
ZeroDivisionError: division by zero is not valid

is_monic()

Return True if this Ore polynomial is monic.
The zero polynomial is by definition not monic.

**EXAMPLES:**

```
sage: R.<t> = ZZ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x',sigma]
sage: a = x + t
sage: a.is_monic()
True
sage: a = 0*x
sage: a.is_monic()
False
sage: a = t*x^3 + x^4 + (t+1)*x^2
sage: a.is_monic()
True
sage: a = (t^2 + 2*t)*x^2 + x^3 + t^10*x^5
sage: a.is_monic()
False
```

**is_monomial()**

Return True if self is a monomial, i.e., a power of the generator.

**EXAMPLES:**

```
sage: R.<t> = ZZ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x',sigma]
sage: x.is_monomial()
True
sage: (x+1).is_monomial()
False
sage: (x^2).is_monomial()
True
sage: S(1).is_monomial()
True
```

The coefficient must be 1:

```
sage: (2*x^5).is_monomial()
False
sage: S(t).is_monomial()
False
```

To allow a non-1 leading coefficient, use is_term():

```
sage: (2*x^5).is_term()
True
sage: S(t).is_term()
True
```

**is_nilpotent()**

Check if self is nilpotent.

**Note:** The paper “Nilpotents and units in skew polynomial rings over commutative rings” by M. Rimmer
and K.R. Pearson describes a method to check whether a given skew polynomial is nilpotent. That method however, requires one to know the order of the automorphism which is not available in Sage. This method is thus not yet implemented.

EXAMPLES:

```sage
def check_skepolynomial_neiltype(R, sigma):
    S.<x> = R['x'], sigma
    x.is_nilpotent()
```

`is_one()`
Test whether this polynomial is 1.

EXAMPLES:

```sage
def check_is_one(R):
    sigma = R.hom([R(1)])
    S.<x> = R['x'], sigma
    R(1).is_one()
    (x + 3).is_one()
```

`is_right_divisible_by(other)`
Check if self is divisible by other on the right.

INPUT:
- other – a Ore polynomial in the same ring as self

OUTPUT:
Return True or False.

EXAMPLES:

```sage
def check_divisibility_by(R, a, b):
    k.<t> = GF(5^3)
    Frob = k.frobenius_endomorphism()
    S.<x> = k['x'], Frob
    a = x^2 + t*x + t^2 + 3
    b = x^3 + (t + 1)*x^2 + 1
    c = a*b
    c.is_right_divisible_by(a)
    c.is_right_divisible_by(b)
```

Divisibility by 0 does not make sense:

```sage
c.is_right_divisible_by(S(0))
```

1.2. Univariate Ore polynomials
This function does not work if the leading coefficient of the divisor is not a unit:

```python
sage: R.<t> = QQ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x'],sigma
sage: a = x^2 + 2*x + t
sage: b = (t+1)*x + t^2
sage: c = a*b
sage: c.is_right_divisible_by(b)
Traceback (most recent call last):
  ... Not ImplementedError: the leading coefficient of the divisor is not invertible
```
EXAMPLES:

```python
sage: R.<t> = ZZ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x',sigma]
sage: a = x + 1
sage: a.is_zero()
False
sage: b = S.zero()
sage: b.is_zero()
True
```

`leading_coefficient()`

Return the coefficient of the highest-degree monomial of `self`.

EXAMPLES:

```python
sage: R.<t> = ZZ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x',sigma]
sage: a = (t+1)*x^5 + t^2*x^3 + x
sage: a.leading_coefficient()
t + 1
```

By convention, the leading coefficient to the zero polynomial is zero:

```python
sage: S(0).leading_coefficient()
0
```

`left_divides(other)`

Check if `self` divides `other` on the left.

INPUT:

- `other` – a Ore polynomial in the same ring as `self`

OUTPUT:

Return True or False.

EXAMPLES:

```python
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x',Frob]
sage: a = x^2 + t*x + t^2 + 3
sage: b = x^3 + (t + 1)*x^2 + 1
sage: c = a*b
sage: a.left_divides(c)
True
sage: b.left_divides(c)
False
```

Divisibility by 0 does not make sense:

```python
sage: S(0).left_divides(c)
Traceback (most recent call last):
```

(continues on next page)
... ZeroDivisionError: division by zero is not valid

**left_gcd**(other, monic=True)

Return the left gcd of self and other.

**INPUT:**

- other – a Ore polynomial in the same ring as self
- monic – boolean (default: True); return whether the left gcd should be normalized to be monic

**OUTPUT:**

The left gcd of self and other, that is a Ore polynomial $g$ with the following property: any Ore polynomial is divisible on the left by $g$ iff it is divisible on the left by both self and other. If monic is True, $g$ is in addition monic. (With this extra condition, it is uniquely determined.)

**Note:** Works only if following two conditions are fulfilled (otherwise left gcd do not exist in general): 1) the base ring is a field and 2) the twisting morphism is bijective.

**EXAMPLES:**

```sage
k.<t> = GF(5^3)
k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x',Frob]
sage: a = (x + t) * (x^2 + t*x + 1)
sage: b = 2 * (x + t) * (x^3 + (t+1)*x^2 + t^2)
sage: a.left_gcd(b)
sage: a.left_gcd(b,monic=False)
sage: a.left_gcd(b,monic=False)
```

$2t^2x + 4t + 2$

The base ring needs to be a field:

```sage
R.<t> = QQ[]
R.<t> = QQ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x',sigma]
sage: a = (x + t) * (x^2 + t*x + 1)
sage: b = 2 * (x + t) * (x^3 + (t+1)*x^2 + t^2)
sage: a.left_gcd(b)
Traceback (most recent call last):
...
```

TypeError: the base ring must be a field

And the twisting morphism needs to be bijective:

```sage
FR = R.fraction_field()
FR = R.fraction_field()
sage: sigma = FR.hom([FR(t)^2])
sage: S.<x> = FR['x',sigma]
sage: a = (x + t) * (x^2 + t*x + 1)
```

 sage: b = 2 * (x + t) * (x^3 + (t+1)*x^2 + t^2)
sage: a.left_gcd(b)
Traceback (most recent call last):
  ... 
NotImplementedError: inversion of the twisting morphism Ring endomorphism of
→Fraction Field of Univariate Polynomial Ring in t over Rational Field
  Defn: t |--> t^2

left_lcm(other, monic=True)

Return the left lcm of self and other.

INPUT:

• other – a Ore polynomial in the same ring as self
• monic – boolean (default: True); return whether the left lcm should be normalized to be monic

OUTPUT:

The left lcm of self and other, that is a Ore polynomial g with the following property: any Ore polynomial divides g on the right iff it divides both self and other on the right. If monic is True, g is in addition monic. (With this extra condition, it is uniquely determined.)

Note: Works only if the base ring is a field (otherwise left lcm do not exist in general).

EXAMPLES:

 sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'],Frob
sage: a = (x + t^2) * (x + t)
sage: b = 2 * (x^2 + t + 1) * (x * t)
sage: c = a.left_lcm(b); c
x^5 + (2*t^2 + t + 4)*x^4 + (3*t^2 + 4)*x^3 + (3*t^2 + 3*t + 2)*x^2 + (t^2 + t →+ 2)*x
sage: c.is_right_divisible_by(a)
True
sage: c.is_right_divisible_by(b)
True
sage: a.degree() + b.degree() == c.degree() + a.right_gcd(b).degree()
True

Specifying monic=False, we can get a nonmonic gcd:

 sage: a.left_lcm(b,monic=False)
(t^2 + t)*x^5 + (4*t^2 + 4*t + 1)*x^4 + (t + 1)*x^3 + (t^2 + 2)*x^2 + (3*t + →4)*x

The base ring needs to be a field:

 sage: R.<t> = QQ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x'],sigma
sage: a = (x + t^2) * (x + t)

(continues on next page)
sage: b = 2 * (x^2 + t + 1) * (x * t)
sage: a.left_lcm(b)
Traceback (most recent call last):
...
TypeError: the base ring must be a field

left_mod(other)

Return the remainder of left division of self by other.

EXAMPLES:

sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'],Frob
sage: a = 1 + t*x^2
sage: b = x + 1
sage: a.left_mod(b)
2*t^2 + 4*t

left_monic()

Return the unique monic Ore polynomial \( m \) which divides this polynomial on the left and has the same degree.

Given a Ore polynomial \( P \) of degree \( n \), its left monic is given by \( P \cdot \sigma^{-n}(1/k) \), where \( k \) is the leading coefficient of \( P \) and \( \sigma \) is the twisting morphism.

EXAMPLES:

sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'],Frob
sage: a = (3*t^2 + 3*t + 2)*x^3 + (2*t^2 + 3)*x^2 + (4*t^2 + t + 4)*x + 2*t^2 +
˓→2
sage: b = a.left_monic(); b
x^3 + (4*t^2 + 3*t)*x^2 + (4*t + 2)*x + 2*t^2 + 4*t + 3

Check list:

sage: b.degree() == a.degree()
True
sage: b.is_left_divisible_by(a)
True
sage: twist = S.twisting_morphism(-a.degree())
sage: a == b * twist(a.leading_coefficient())
True

Note that \( b \) does not divide \( a \) on the right:

sage: a.is_right_divisible_by(b)
False

This function does not work if the leading coefficient is not a unit:
sage: R.<t> = QQ[]
sage: der = R.derivation()
sage: S.<x> = R['x', der]
sage: a = t*x
sage: a.left_monic()
Traceback (most recent call last):
  ... NotImplementedError: the leading coefficient is not a unit

left_quo_rem(other)
Return the quotient and remainder of the left euclidean division of self by other.

INPUT:
• other – a Ore polynomial in the same ring as self

OUTPUT:
• the quotient and the remainder of the left euclidean division of this Ore polynomial by other

Note: This will fail if the leading coefficient of other is not a unit or if Sage can’t invert the twisting morphism.

EXAMPLES:

sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'], Frob
sage: a = (3*t^2 + 3*t + 2)*x^3 + (2*t^2 + 3)*x^2 + (4*t^2 + t + 4)*x + 2*t^2 +
  → 2
sage: b = (3*t^2 + 4*t + 2)*x^2 + (2*t^2 + 4*t + 3)*x + 2*t^2 + t + 1
sage: q,r = a.left_quo_rem(b)
sage: a == b*q + r
True

In the following example, Sage does not know the inverse of the twisting morphism:

sage: R.<t> = QQ[]
sage: K = R.fraction_field()
sage: sigma = K.hom([(t+1)/(t-1)])
sage: S.<x> = K['x'], sigma
sage: a = (-2*t^2 - t + 1)*x^3 + (-t^2 + t)*x^2 + (-12*t - 2)*x - t^2 - 95*t + 1
sage: b = x^2 + (5*t - 6)*x - 4*t^2 + 4*t - 1
sage: a.left_quo_rem(b)
Traceback (most recent call last):
  ... NotImplementedError: inversion of the twisting morphism Ring endomorphism of
  → Fraction Field of Univariate Polynomial Ring in t over Rational Field
  Defn: t |--> (t + 1)/(t - 1)

left_xgcd(other, monic=True)
Return the left gcd of self and other along with the coefficients for the linear combination.

If a is self and b is other, then there are Ore polynomials u and v such that g = au + bv, where g is the left gcd of a and b. This method returns (g, u, v).
INPUT:
- **other** – a Ore polynomial in the same ring as **self**
- **monic** – boolean (default: True); return whether the left gcd should be normalized to be monic

OUTPUT:
- The left gcd of **self** and **other**, that is a Ore polynomial $g$ with the following property: any Ore polynomial is divisible on the left by $g$ iff it is divisible on the left by both **self** and **other**. If monic is **True**, $g$ is in addition monic. (With this extra condition, it is uniquely determined.)
- Two Ore polynomials $u$ and $v$ such that:

$$g = a \ast u + b \ast v,$$

where $s$ is **self** and $b$ is **other**.

**Note:** Works only if following two conditions are fulfilled (otherwise left gcd do not exist in general): 1) the base ring is a field and 2) the twisting morphism is bijective.

**EXAMPLES:**

```sage
k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'],Frob
sage: a = (x + t) * (x^2 + t*x + 1)
sage: b = 2 * (x + t) * (x^3 + (t+1)*x^2 + t^2)
sage: g,u,v = a.left_xgcd(b); g
x + t
sage: a*u + b*v == g
True
```

Specifying monic=False, we can get a nonmonic gcd:

```sage
g,u,v = a.left_xgcd(b, monic=False); g
2*t*x + 4*t + 2
sage: a*u + b*v == g
True
```

The base ring must be a field:

```sage
R.<t> = QQ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x'],sigma
sage: a = (x + t) * (x^2 + t*x + 1)
sage: b = 2 * (x + t) * (x^3 + (t+1)*x^2 + t^2)
sage: a.left_xgcd(b)
Traceback (most recent call last):
...
TypeError: the base ring must be a field
```

And the twisting morphism must be bijective:

```sage
FR = R.fraction_field()
sage: f = FR.hom([FR(t)^2])
```
Noncommutative Polynomials, Release 10.0

sage: S.<x> = FR['x',f]
sage: a = (x + t) * (x^2 + t*x + 1)
sage: b = 2 * (x + t) * (x^3 + (t+1)*x^2 + t^2)
sage: a.left_xgcd(b)
Traceback (most recent call last):
  ...  
NotImplementedError: inversion of the twisting morphism
    Ring endomorphism of
    → Fraction Field of Univariate Polynomial Ring in t over Rational Field
    Defn: t |--> t^2

left_xlcm(other, monic=True)

Return the left lcm $L$ of self and other together with two Ore polynomials $U$ and $V$ such that

$$ U \cdot \text{self} = V \cdot \text{other} = L. $$

EXAMPLES:

sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x',Frob]
sage: P = (x + t^2) * (x + t)
sage: Q = 2 * (x^2 + t + 1) * (x * t)
sage: L, U, V = P.left_xlcm(Q)
sage: L
x^5 + (2*t^2 + t + 4)*x^4 + (3*t^2 + 4)*x^3 + (3*t^2 + 3*t + 2)*x^2 + (t^2 + t + 2)*x

sage: U*P == L
True
sage: V*Q == L
True

number_of_terms()

Return the number of non-zero coefficients of self.

This is also known as the weight, hamming weight or sparsity.

EXAMPLES:

sage: R.<t> = QQ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x',sigma]
sage: a = 1 + x^4 + (t+1)*x^2 + t^2
sage: a.number_of_terms()
3

This is also an alias for hamming_weight:

sage: a.hamming_weight()
3

padded_list(n=None)

Return list of coefficients of self up to (but not including) degree $n$.

Includes 0' if the list has length exactly $n$.  

1.2. Univariate Ore polynomials
INPUT:

- $n$ – (default: None); if given, an integer that is at least 0

EXAMPLES:

```
sage: R.<t> = QQ[

sage: sigma = R.hom([t+1])

sage: S.<x> = R['x',sigma]

sage: a = 1 + t*x^3 + t^2*x^5

sage: a.padded_list()
[1, 0, 0, t, 0, t^2]

sage: a.padded_list(10)
[1, 0, 0, t, 0, t^2, 0, 0, 0, 0]

sage: len(a.padded_list(10))
10

sage: a.padded_list(3)
[1, 0, 0]

sage: a.padded_list(0)
[]

sage: a.padded_list(-1)
Traceback (most recent call last):
...
ValueError: n must be at least 0
```
False
\texttt{sage: b.right_divides(c)}
True

Divisibility by 0 does not make sense:

\texttt{sage: S(0).right_divides(c)}

Traceback (most recent call last):
...
ZeroDivisionError: division by zero is not valid

This function does not work if the leading coefficient of the divisor is not a unit:

\texttt{sage: }\texttt{R.<t> = QQ[]}\texttt{\[}
\texttt{sage: sigma = R.hom([t+1])}
\texttt{sage: S.<x> = R['x',sigma]}
\texttt{sage: a = x^2 + 2*x + t}
\texttt{sage: b = (t+1)*x + t^2}
\texttt{sage: c = a*b}
\texttt{sage: b.right_divides(c)}

Traceback (most recent call last):
...
NotImplementedError: the leading coefficient of the divisor is not invertible

\texttt{right\_gcd} (other, monic=True)

Return the right gcd of self and other.

INPUT:

* other – a Ore polynomial in the same ring as self
* monic – boolean (default: True); return whether the right gcd should be normalized to be monic

OUTPUT:

The right gcd of self and other, that is a Ore polynomial g with the following property: any Ore polynomial is divisible on the right by g iff it is divisible on the right by both self and other. If monic is True, g is in addition monic. (With this extra condition, it is uniquely determined.)

Note: Works only if the base ring is a field (otherwise right gcd do not exist in general).

EXAMPLES:

\texttt{sage: k.<t> = GF(5^3)}
\texttt{sage: Frob = k.frobenius_endomorphism()}
\texttt{sage: S.<x> = k['x',Frob]}
\texttt{sage: a = (x^2 + t^2*x + 1) * (x + t)}
\texttt{sage: b = 2 * (x^3 + (t+1)*x^2 + t^2) * (x + t)}
\texttt{sage: a.right_gcd(b)}
\texttt{x + t}

Specifying monic=False, we can get a nonmonic gcd:
```python
sage: a.right_gcd(b,monic=False)
(4*t^2 + 4*t + 1)*x + 4*t^2 + 4*t + 3
```

The base ring need to be a field:

```python
sage: R.<t> = QQ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x'],sigma
sage: a = (x^2 + t*x + 1) * (x + t)
sage: b = 2 * (x^3 + (t+1)*x^2 + t^2) * (x + t)
sage: a.right_gcd(b)
Traceback (most recent call last):
...
TypeError: the base ring must be a field
```

**right_lcm**(other, monic=True)

Return the right lcm of self and other.

**INPUT:**

- other – a Ore polynomial in the same ring as self
- monic – boolean (default: True); return whether the right lcm should be normalized to be monic

**OUTPUT:**

The right lcm of self and other, that is a Ore polynomial \( g \) with the following property: any Ore polynomial divides \( g \) on the left iff it divides both self and other on the left. If monic is True, \( g \) is in addition monic. (With this extra condition, it is uniquely determined.)

**Note:** Works only if two following conditions are fulfilled (otherwise right lcm do not exist in general): 1) the base ring is a field and 2) the twisting morphism on this field is bijective.

**EXAMPLES:**

```python
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'],Frob
sage: a = (x + t) * (x + t^2)
sage: b = 2 * (x + t) * (x^2 + t + 1)
sage: c = a.right_lcm(b); c
x^4 + (2*t^2 + t + 2)*x^3 + (3*t^2 + 4*t + 1)*x^2 + (3*t^2 + 4*t + 1)*x + t^2 +
˓→4
sage: c.is_left_divisible_by(a)
True
sage: c.is_left_divisible_by(b)
True
sage: a.degree() + b.degree() == c.degree() + a.left_gcd(b).degree()
True
```

Specifying monic=False, we can get a nonmonic gcd:

```python
sage: a.right_lcm(b,monic=False)
2*t*x^4 + (3*t + 1)*x^3 + (4*t^2 + 4*t + 3)*x^2 + (3*t^2 + 4*t + 2)*x + 3*t^2 + 2*t + 3
```
The base ring needs to be a field:

```
sage: R.<t> = QQ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x',sigma]
sage: a = (x + t) * (x + t^2)
sage: b = 2 * (x + t) * (x^2 + t + 1)
sage: a.right_lcm(b)
Traceback (most recent call last):
  ...
TypeError: the base ring must be a field
```

And the twisting morphism needs to be bijective:

```
sage: FR = R.fraction_field()
sage: f = FR.hom([FR(t)^2])
sage: S.<x> = FR['x',f]
sage: a = (x + t) * (x + t^2)
sage: b = 2 * (x + t) * (x^2 + t + 1)
sage: a.right_lcm(b)
Traceback (most recent call last):
  ...
NotImplementedError: inversion of the twisting morphism
```

```
right_mod(other)

Return the remainder of right division of self by other.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x',sigma]
sage: a = 1 + t*x^2
sage: b = x + 1
sage: a % b
t + 1
```

```
right_monic()

Return the unique monic Ore polynomial which divides this polynomial on the right and has the same
degree.

Given an Ore polynomial \( P \) of degree \( n \), its left monic is given by \((1/k) \cdot P\), where \( k \) is the leading coefficient of \( p \).

EXAMPLES:

```
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x',Frob]
sage: a = (3*t^2 + 3*t + 2)*x^3 + (2*t^2 + 3)*x^2 + (4*t^2 + t + 4)*x + 2*t^2 + ...
```

(continues on next page)
Check list:

```
sage: b.degree() == a.degree()
True
sage: b.is_right_divisible_by(a)
True
sage: a == a.leading_coefficient() * b
True
```

Note that $b$ does not divide $a$ on the right:

```
sage: a.is_left_divisible_by(b)
False
```

This function does not work if the leading coefficient is not a unit:

```
sage: R.<t> = QQ[]
sage: der = R.derivation()
sage: S.<x> = R[x], der
sage: a = t*x
sage: a.right_monic()
Traceback (most recent call last):
  ...,
NotImplementedError: the leading coefficient is not a unit
```

**right_quo_rem**(*other*)

Return the quotient and remainder of the right euclidean division of self by other.

**INPUT:**

- *other* – a Ore polynomial in the same ring as self

**OUTPUT:**

- the quotient and the remainder of the left euclidean division of this Ore polynomial by other

**Note:** This will fail if the leading coefficient of the divisor is not a unit.

**EXAMPLES:**

```
sage: R.<t> = ZZ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R[x], sigma
sage: a = S.random_element(degree=4)
sage: b = S.random_element(monic=True)
sage: q,r = a.right_quo_rem(b)
sage: a == q*b + r
True
```

The leading coefficient of the divisor need to be invertible:
```python
sage: a = S.random_element()
sage: c = S.random_element()
sage: while not c or c.leading_coefficient().is_unit():
    c = S.random_element()

sage: a = S.random_element(degree=4)
sage: a.right_quo_rem(c)
Traceback (most recent call last):
...  
NotImplementedError: the leading coefficient of the divisor is not invertible
```

**right_xgcd**(other, monic=True)

Return the right gcd of `self` and `other` along with the coefficients for the linear combination.

If `a` is `self` and `b` is `other`, then there are Ore polynomials `u` and `v` such that `g = ua + vb`, where `g` is the right gcd of `a` and `b`. This method returns `(g, u, v)`.

**INPUT:**

- `other` – a Ore polynomial in the same ring as `self`
- `monic` – boolean (default: `True`); return whether the right gcd should be normalized to be monic

**OUTPUT:**

- The right gcd of `self` and `other`, that is a Ore polynomial `g` with the following property: any Ore polynomial is divisible on the right by `g` iff it is divisible on the right by both `self` and `other`. If `monic` is `True`, `g` is in addition monic. (With this extra condition, it is uniquely determined.)
- Two Ore polynomials `u` and `v` such that:

\[ g = u \cdot a + v \cdot b \]

where `a` is `self` and `b` is `other`.

**Note:** Works only if the base ring is a field (otherwise right gcd do not exist in general).

**EXAMPLES:**

```python
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'],Frob
sage: a = (x^2 + t*x + 1) * (x + t)
sage: b = 2 * (x^3 + (t+1)*x^2 + t^2) * (x + t)
sage: g,u,v = a.right_xgcd(b); g
x + t
sage: u*a + v*b == g
True
```

Specifying `monic=False`, we can get a nonmonic gcd:
sage: g,u,v = a.right_xgcd(b,monic=False); g
(4*t^2 + 4*t + 1)*x + 4*t^2 + 4*t + 3
sage: u*a + v*b == g
True

The base ring must be a field:

sage: R.<t> = QQ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x',sigma]
sage: a = (x^2 + t*x + 1) * (x + t)
sage: b = 2 * (x^3 + (t+1)*x^2 + t^2) * (x + t)
sage: a.right_xgcd(b)
Traceback (most recent call last):
...TypeError: the base ring must be a field

right_xlcm(other, monic=True)

Return the right lcm \( L \) of \( self \) and \( other \) together with two Ore polynomials \( U \) and \( V \) such that

\[
self \cdot U = other \cdot V = L.
\]

INPUT:

- \( other \) – a Ore polynomial in the same ring as \( self \)
- \( monic \) – a boolean (default: True); whether the right lcm should be normalized to be monic

EXAMPLES:

sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x',Frob]
sage: P = (x + t) * (x + t^2)
sage: Q = 2 * (x + t) * (x^2 + t + 1)
sage: L, U, V = P.right_xlcm(Q)
sage: L
x^4 + (2*t^2 + t + 2)*x^3 + (3*t^2 + 4*t + 1)*x^2 + (3*t^2 + 4*t + 1)*x + t^2 + 4
sage: P*U == L
True
sage: Q*V == L
True

shift(n)

Return \( self \) multiplied on the right by the power \( x^n \).

If \( n \) is negative, terms below \( x^n \) will be discarded.

EXAMPLES:

sage: R.<t> = QQ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x',sigma]
sage: a = x^5 + t^4*x^4 + t^2*x^2 + t^10

(continues on next page)
One can also use the infix shift operator:

\[
\begin{align*}
sage: & a >> 2 \\
& x^3 + t^4*x^2 + t^2 \\
sage: & a << 2 \\
& x^7 + t^4*x^6 + t^2*x^4 + t^10*x^2
\end{align*}
\]

**square()**

Return the square of self.

**EXAMPLES:**

\[
\begin{align*}
sage: & R.<t> = QQ[] \\
sage: & sigma = R.hom([t+1]) \\
sage: & S.<x> = R['x', sigma] \\
sage: & a = x + t; a \\
& x + t \\
sage: & a.square() \\
& x^2 + (2*t + 1)*x + t^2 \\
sage: & a.square() == a^a \\
& True
\end{align*}
\]

**variable_name()**

Return the string name of the variable used in self.

**EXAMPLES:**

\[
\begin{align*}
sage: & R.<t> = QQ[] \\
sage: & sigma = R.hom([t+1]) \\
sage: & S.<x> = R['x', sigma] \\
sage: & a = x + t \\
sage: & a.variable_name() \\
& 'x'
\end{align*}
\]

**class** sage.rings.polynomial.ore_polynomial_element.OrePolynomialBaseringInjection

Bases: Morphism

Representation of the canonical homomorphism from a ring $R$ into a Ore polynomial ring over $R$.

This class is necessary for automatic coercion from the base ring to the Ore polynomial ring.
See also:

PolynomialBaseringInjection

EXAMPLES:

```sage
R.<t> = QQ[]
sigma = R.hom([t+1])
S.<x> = R['x',sigma]
s.coerce_map_from(S.base_ring()) # indirect doctest
```

Ore Polynomial base injection morphism:

From: Univariate Polynomial Ring in t over Rational Field
To: Ore Polynomial Ring in x over Univariate Polynomial Ring in t over Rational Field
Twisted by t |---> t + 1

```sage
an_element()

Return an element of the codomain of the ring homomorphism.

EXAMPLES:

```sage
from sage.rings.polynomial.ore_polynomial_element import OrePolynomialBaseringInjection
k.<t> = GF(5^3)
Frob = k.frobenius_endomorphism()
S.<x> = k['x'], Frob
m = OrePolynomialBaseringInjection(k, k['x'], Frob)
m.an_element()
```

section()

Return the canonical homomorphism from the constants of a Ore polynomial ring to the base ring according to self.

```sage
coefficients(sparse=True)

Return the coefficients of the monomials appearing in self.

If sparse=True (the default), return only the non-zero coefficients. Otherwise, return the same value as self.list().

EXAMPLES:

```sage
R.<t> = QQ[]
sigma = R.hom([t+1])
S.<x> = R['x',sigma]
a = 1 + x^4 + (t+1)*x^2 + t^2
a.coefficients()
[a^2 + 1, t + 1, 1]
a.coefficients(sparse=False)
[a^2 + 1, 0, t + 1, 0, 1]```

36 Chapter 1. Univariate Ore Polynomials
degree()

Return the degree of self.

By convention, the zero Ore polynomial has degree $-1$.

**EXAMPLES:**

```
sage: R.<t> = ZZ[]  
sage: sigma = R.hom([t+1])  
sage: S.<x> = R['x',sigma]  
sage: a = x^2 + t*x^3 + t^2*x + 1  
sage: a.degree()  
3
```

By convention, the degree of 0 is $-1$:

```
sage: S(0).degree()  
-1
```

dict()

Return a dictionary representation of self.

**EXAMPLES:**

```
sage: R.<t> = ZZ[]  
sage: sigma = R.hom([t+1])  
sage: S.<x> = R['x',sigma]  
sage: a = x^2012 + t*x^1006 + t^3 + 2*t  
sage: a.dict()  
{0: t^3 + 2*t, 1006: t, 2012: 1}
```

hilbert_shift(s, var=None)

Return this Ore polynomial with variable shifted by $s$, i.e. if this Ore polynomial is $P(x)$, return $P(x+s)$.

**INPUT:**

- $s$ – an element in the base ring
- $var$ – a string; the variable name

**EXAMPLES:**

```
sage: R.<t> = GF(7)[]  
sage: der = R.derivation()  
sage: A.<d> = R['d', der]  
sage: L = d^3 + t*d^2  
sage: L.hilbert_shift(t)  
d^3 + 4*t*d^2 + (5*t^2 + 3)*d + 2*t^3 + 4*t  
sage: (d+t)^3 + t*(d+t)^2  
d^3 + 4*t*d^2 + (5*t^2 + 3)*d + 2*t^3 + 4*t
```

One can specify another variable name:

```
sage: L.hilbert_shift(t, var='x')  
x^3 + 4*t*x^2 + (5*t^2 + 3)*x + 2*t^3 + 4*t
```

When the twisting morphism is not trivial, the output lies in a different Ore polynomial ring.
Noncommutative Polynomials, Release 10.0

```
sage: k.<a> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x', Frob]
sage: P = x^2 + a*x + a^2
sage: Q = P.hilbert_shift(a); Q
x^2 + (2*a^2 + a + 4)*x + a^2 + 3*a + 4
sage: Q.parent()
Ore Polynomial Ring in x over Finite Field in a of size 5^3 twisted by a |--> a^5 and a*(a |--> a^5) - id
sage: Q.parent() is S
False
```

This behavior ensures that the Hilbert shift by a fixed element defines an homomorphism of rings:

```
sage: U = S.random_element(degree=5)
sage: V = S.random_element(degree=5)
sage: s = k.random_element()
sage: (U+V).hilbert_shift(s) == U.hilbert_shift(s) + V.hilbert_shift(s)
True
sage: (U*V).hilbert_shift(s) == U.hilbert_shift(s) * V.hilbert_shift(s)
True
```

We check that shifting by an element and then by its opposite gives back the initial Ore polynomial:

```
sage: P = S.random_element(degree=10)
sage: s = k.random_element()
sage: P.hilbert_shift(s).hilbert_shift(-s) == P
True
```

```
list(copy=True)

Return a list of the coefficients of self.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x', sigma]
sage: a = 1 + x^4 + (t+1)*x^2 + t^2
sage: l = a.list(); l
[t^2 + 1, 0, t + 1, 0, 1]
```

Note that \( l \) is a list, it is mutable, and each call to the list method returns a new list:

```
sage: type(l)
<... 'list'>
sage: l[0] = 5
sage: a.list()
[t^2 + 1, 0, t + 1, 0, 1]
```

```
truncate(n)

Return the polynomial resulting from discarding all monomials of degree at least \( n \).

EXAMPLES:
```
valuation()

Return the minimal degree of a non-zero monomial of self.

By convention, the zero Ore polynomial has valuation $+\infty$.

EXAMPLES:

```
sage: R.<t> = ZZ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x'],sigma
sage: a = x^2 + t*x^3 + t^2*x
sage: a.valuation()
1
```

By convention, the valuation of 0 is $+\infty$:

```
sage: S(0).valuation()
+Infinity
```

1.3 Univariate skew polynomial rings

This module provides the `SkewPolynomialRing`. In the class hierarchy in Sage, the locution `Skew Polynomial` is used for a Ore polynomial without twisting derivation.

This module also provides:

- the class `SkewPolynomialRing_finite_order`, which is a specialized class for skew polynomial rings over fields equipped with an automorphism of finite order. It inherits from `SkewPolynomialRing` but contains more methods and provides better algorithms.

- the class `SkewPolynomialRing_finite_field`, which is a specialized class for skew polynomial rings over finite fields.

See also:

`OrePolynomialRing`

AUTHOR:

- Xavier Caruso (2012-06-29): initial version
- Arpit Merchant (2016-08-04): improved docstrings, fixed doctests and refactored classes and methods
- Johan Rosenkilde (2016-08-03): changes for bug fixes, docstring and doctest errors

class sage.rings.polynomial.skew_polynomial_ring.SkewPolynomialRing

SectionSkewPolynomialCenterInjection

Bases: Section

Section of the canonical injection of the center of a skew polynomial ring into this ring.
class sage.rings.polynomial.skew_polynomial_ring.SkewPolynomialCenterInjection(domain, codomain, embed, order)

Bases: RingHomomorphism

Canonical injection of the center of a skew polynomial ring into this ring.

section()

Return a section of this morphism.

EXAMPLES:

sage: k.<a> = GF(5^3)
sage: S.<x> = SkewPolynomialRing(k, k.frobenius_endomorphism())
sage: Z = S.center()
sage: iota = S.convert_map_from(Z)
sage: sigma = iota.section()
sage: sigma(x^3)

class sage.rings.polynomial.skew_polynomial_ring.SkewPolynomialRing(base_ring, twisting_morphism, derivation, name, sparse, category=None)

Bases: OrePolynomialRing

Initialize self.

INPUT:

• base_ring – a commutative ring
• twisting_morphism – an automorphism of the base ring
• name – string or list of strings representing the name of the variables of ring
• sparse – boolean (default: False)
• category – a category

EXAMPLES:

sage: R.<t> = ZZ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = SkewPolynomialRing(R, sigma)
sage: S.category()
Category of algebras over Univariate Polynomial Ring in t over Integer Ring
sage: S([[1]]) + S([-1])
0
sage: TestSuite(S).run()

lagrange_polynomial(points)

Return the minimal-degree polynomial which interpolates the given points.

More precisely, given n pairs \((x_1, y_1), \ldots, (x_n, y_n)\) ∈ \(R^2\), where \(R\) is self.base_ring(), compute a skew polynomial \(p(x)\) such that \(p(x_i) = y_i\) for each \(i\), under the condition that the \(x_i\) are linearly independent over the fixed field of self.twisting_morphism().
If the $x_i$ are linearly independent over the fixed field of `self.twisting_morphism()` then such a polynomial is guaranteed to exist. Otherwise, it might exist depending on the $y_i$, but the algorithm used in this implementation does not support that, and so an error is always raised.

**INPUT:**

- **points** – a list of pairs $(x_1, y_1), \ldots, (x_n, y_n)$ of elements of the base ring of `self`; the $x_i$ should be linearly independent over the fixed field of `self.twisting_morphism()`

**OUTPUT:**

The Lagrange polynomial.

**EXAMPLES:**

```python
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'], Frob
sage: points = [(t, 3*t^2 + 4*t + 4), (t^2, 4*t)]
sage: d = S.lagrange_polynomial(points); d
x + t
sage: R.<t> = ZZ[

sage: sigma = R.hom([t+1])
sage: T.<x> = R['x'], sigma
sage: points = [ (1, t^2 + 3*t + 4), (t, 2*t^2 + 3*t + 1), (t^2, t^2 + 3*t + 4)]

sage: p = T.lagrange_polynomial(points); p
((-t^4 - 2*t - 3)/-2)*x^2 + (-t^4 - t^3 - t^2 - 3*t - 2)*x + (-t^4 - 2*t^3 - 4*t^2 - 10*t - 9)/-2

sage: p.multi_point_evaluation([1, t, t^2]) == [ t^2 + 3*t + 4, 2*t^2 + 3*t + 1, t^2 + 3*t + 4 ]
True
```

If the $x_i$ are linearly dependent over the fixed field of `self.twisting_morphism()`, then an error is raised:

```python
sage: T.lagrange_polynomial([ (t, 1), (2*t, 3) ])
Traceback (most recent call last):
...
ValueError: the given evaluation points are linearly dependent over the fixed
\ldots field of the twisting morphism,
so a Lagrange polynomial could not be determined (and might not exist)
```

**minimal_vanishing_polynomial**(eval_pts)

Return the minimal-degree, monic skew polynomial which vanishes at all the given evaluation points.

The degree of the vanishing polynomial is at most the length of `eval_pts`. Equality holds if and only if the elements of `eval_pts` are linearly independent over the fixed field of `self.twisting_morphism()`.

- **eval_pts** – list of evaluation points which are linearly independent over the fixed field of the twisting morphism of the associated skew polynomial ring

**OUTPUT:**

The minimal vanishing polynomial.

**EXAMPLES:**
```python
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k[x], Frob
sage: eval_pts = [1, t, t^2]
sage: b = S.minimal_vanishing_polynomial(eval_pts); b
x^3 + 4
```

The minimal vanishing polynomial evaluates to 0 at each of the evaluation points:

```python
sage: eval = b.multi_point_evaluation(eval_pts); eval
[0, 0, 0]
```

If the evaluation points are linearly dependent over the fixed field of the twisting morphism, then the returned polynomial has lower degree than the number of evaluation points:

```python
sage: S.minimal_vanishing_polynomial([t])
x + 3*t^2 + 3*t
sage: S.minimal_vanishing_polynomial([t, 3*t])
x + 3*t^2 + 3*t
```

**class** `sage.rings.polynomial.skew_polynomial_ring.SkewPolynomialRing_finite_field` (**base_ring**, **morphism**, **derivation**, **names**, **sparse**, **category=**None)

**Bases:** `SkewPolynomialRing_finite_order`

A specialized class for skew polynomial rings over finite fields.

**See also:**

- `sage.rings.polynomial.skew_polynomial_ring.SkewPolynomialRing`
- `sage.rings.polynomial.skew_polynomial_finite_field`

**Todo:** Add methods related to center of skew polynomial ring, irreducibility, karatsuba multiplication and factorization.

**class** `sage.rings.polynomial.skew_polynomial_ring.SkewPolynomialRing_finite_order` (**base_ring**, **morphism**, **derivation**, **name**, **sparse**, **category=**None)

**Bases:** `SkewPolynomialRing`

A specialized class for skew polynomial rings whose twisting morphism has finite order.

**See also:**

- `sage.rings.polynomial.skew_polynomial_ring.SkewPolynomialRing`
center(name=None, names=None, default=False)
    Return the center of this skew polynomial ring.

    Note: If $F$ denotes the subring of $R$ fixed by $\sigma$ and $\sigma$ has order $r$, the center of $K[x, \sigma]$ is $F[x^r]$, that is a univariate polynomial ring over $F$.

INPUT:
    • name – a string or None (default: None); the name for the central variable (namely $x^r$)
    • default – a boolean (default: False); if True, set the default variable name for the center to name

EXAMPLES:

```python
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x',Frob]; S
Ore Polynomial Ring in x over Finite Field in t of size 5^3 twisted by t |--> t^5
sage: Z = S.center(); Z
Univariate Polynomial Ring in z over Finite Field of size 5
sage: Z.gen()
z
```

We can pass in another variable name:

```python
sage: S.center(name='y')
Univariate Polynomial Ring in y over Finite Field of size 5
```

or use the bracket notation:

```python
sage: Zy.<y> = S.center(); Zy
Univariate Polynomial Ring in y over Finite Field of size 5
sage: y.parent() is Zy
True
```

A coercion map from the center to the skew polynomial ring is set:

```python
sage: S.has_coerce_map_from(Zy)
True
sage: P = y + x; P
x^3 + x
sage: P.parent()
Ore Polynomial Ring in x over Finite Field in t of size 5^3 twisted by t |--> t^5
sage: P.parent() is S
True
```

together with a conversion map in the reverse direction:
Two different skew polynomial rings can share the same center:

<table>
<thead>
<tr>
<th>Command</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>sage: S1.&lt;x1&gt; = k['x1'], Frob</code></td>
<td>Define skew polynomial ring</td>
</tr>
<tr>
<td><code>sage: S2.&lt;x2&gt; = k['x2'], Frob</code></td>
<td>Define skew polynomial ring</td>
</tr>
<tr>
<td><code>sage: S1.center() is S2.center()</code></td>
<td>Check if centers are the same</td>
</tr>
<tr>
<td><code>True</code></td>
<td>Result is true</td>
</tr>
</tbody>
</table>

### About the default name of the central variable

A priori, the default is \( z \).

However, a variable name is given the first time this method is called, the given name become the default for the next calls:

<table>
<thead>
<tr>
<th>Command</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>sage: K.&lt;t&gt; = GF(11^3)</code></td>
<td>Define finite field</td>
</tr>
<tr>
<td><code>sage: phi = K.frobenius_endomorphism()</code></td>
<td>Define Frobenius endomorphism</td>
</tr>
<tr>
<td><code>sage: A.&lt;X&gt; = K['X'], phi</code></td>
<td>Define skew polynomial ring</td>
</tr>
<tr>
<td><code>sage: C.&lt;u&gt; = A.center() # first call</code></td>
<td>Define center</td>
</tr>
<tr>
<td><code>sage: C</code></td>
<td>Result is the center</td>
</tr>
<tr>
<td><code>sage: A.center() is C</code></td>
<td>Check if center is the same</td>
</tr>
<tr>
<td><code>True</code></td>
<td>Result is true</td>
</tr>
</tbody>
</table>

We can update the default variable name by passing in the argument `default=True`:

<table>
<thead>
<tr>
<th>Command</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>sage: D.&lt;v&gt; = A.center(default=True)</code></td>
<td>Define center with default name <code>v</code></td>
</tr>
<tr>
<td><code>sage: D</code></td>
<td>Result is the center</td>
</tr>
<tr>
<td><code>sage: A.center()</code></td>
<td>Check if center is the same</td>
</tr>
<tr>
<td><code>Univariate Polynomial Ring in v over Finite Field of size 11</code></td>
<td>Result</td>
</tr>
<tr>
<td><code>sage: A.center() is D</code></td>
<td>Check if center is the same</td>
</tr>
<tr>
<td><code>True</code></td>
<td>Result is true</td>
</tr>
</tbody>
</table>
1.4 Univariate skew polynomials

This module provides the SkewPolynomial. In the class hierarchy in Sage, the locution Skew Polynomial is used for a Ore polynomial without twisting derivation.

Warning: The current semantics of __call__() are experimental, so a warning is thrown when a skew polynomial is evaluated for the first time in a session. See the method documentation for details.

AUTHORS:
• Xavier Caruso (2012-06-29): initial version
• Arpit Merchant (2016-08-04): improved docstrings, fixed doctests and refactored classes and methods
• Johan Rosenkilde (2016-08-03): changes for bug fixes, docstring and doctest errors

class sage.rings.polynomial.skew_polynomial_element.SkewPolynomial_generic_dense

Bases: OrePolynomial_generic_dense

Generic implementation of dense skew polynomial supporting any valid base ring and twisting morphism.

conjugate(n)

Return self conjugated by $x^n$, where x is the variable of self.

The conjugate is obtained from self by applying the $n$-th iterate of the twisting morphism to each of its coefficients.

INPUT:

• $n$ – an integer, the power of conjugation

EXAMPLES:

sage: R.<t> = QQ[]
sage: K = R.fraction_field()
sage: sigma = K.hom([1 + 1/t])
sage: S.<x> = K['x',sigma]
sage: a = t*x^3 + (t^2 + 1)*x^2 + 2*t
sage: b = a.conjugate(2); b
((2*t + 1)/(t + 1))*x^3 + ((5*t^2 + 6*t + 2)/(t^2 + 2*t + 1))*x^2 + (4*t + 2)/→(t + 1)
sage: x^2*a == b*x^2
True

In principle, negative values for $n$ are allowed, but Sage needs to be able to invert the twisting morphism:

sage: b = a.conjugate(-1)
Traceback (most recent call last):
  ...\NotImplementedError: inverse not implemented for morphisms of Fraction Field of...

Here is a working example:

sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
Noncommutative Polynomials, Release 10.0

sage: T.<y> = k['y'],Frob
sage: u = T.random_element()
Sage: v = u.conjugate(-1)
Sage: u*y == y*v
True

\textit{left\_power\_mod}(\textit{exp}, \textit{modulus})

Return the remainder of \texttt{self}**\textit{exp} in the left euclidean division by \textit{modulus}.

INPUT:

- \textit{exp} – an Integer
- \textit{modulus} – a skew polynomial in the same ring as \textit{self}

OUTPUT:

Remainder of \texttt{self}**\textit{exp} in the left euclidean division by \textit{modulus}.

REMARK:

The quotient of the underlying skew polynomial ring by the principal ideal generated by \textit{modulus} is in general not a ring.

As a consequence, Sage first computes exactly \texttt{self}**\textit{exp} and then reduce it modulo \textit{modulus}.

EXAMPLES:

\begin{verbatim}
sage: k, <t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'],Frob
sage: a = x + t
sage: modulus = x^3 + t*x^2 + (t+3)*x - 2
sage: a.left_power_mod(100,modulus)
(4*t^2 + t + 1)*x^2 + (t^2 + 4*t + 1)*x + 3*t^2 + 3*t
\end{verbatim}

\textit{multi\_point\_evaluation}(\textit{eval\_pts})

Evaluate \texttt{self} at list of evaluation points.

INPUT:

- \textit{eval\_pts} – list of points at which \texttt{self} is to be evaluated

OUTPUT:

List of values of \texttt{self} at the \textit{eval\_pts}.

Todo: This method currently trivially calls the evaluation function repeatedly. If fast skew polynomial multiplication is available, an asymptotically faster method is possible using standard divide and conquer techniques and \texttt{minimal\_vanishing\_polynomial()}.

EXAMPLES:

\begin{verbatim}
sage: k, <t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'],Frob
sage: a = x + t
\end{verbatim}
sage: eval_pts = [1, t, t^2]
sage: c = a.multi_point_evaluation(eval_pts); c
[t + 1, 3*t^2 + 4*t + 4, 4*t]
sage: c == [ a(e) for e in eval_pts ]
True

**operator_eval**(eval_pt)

Evaluate self at eval_pt by the operator evaluation method.

INPUT:

• eval_pt – element of the base ring of self

OUTPUT:

The value of the polynomial at the point specified by the argument.

EXAMPLES:

```python
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: T.<x> = k['x'],Frob
sage: a = 3*t^2*x^2 + (t + 1)*x + 2
sage: a(t) #indirect test
2*t^2 + 2*t + 3
sage: a.operator_eval(t)
2*t^2 + 2*t + 3
```

Evaluation points outside the base ring is usually not possible due to the twisting morphism:

```python
sage: R.<t> = QQ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x'],sigma
sage: a = t*x + 1
sage: a.operator_eval(1/t)
Traceback (most recent call last):
... TypeError: 1/t fails to convert into the map's domain Univariate Polynomial Ring in t over Rational Field, but a 'pushforward' method is not properly implemented
```

**right_power_mod**(exp, modulus)

Return the remainder of self**exp in the right euclidean division by modulus.

INPUT:

• exp – an integer
• modulus – a skew polynomial in the same ring as self

OUTPUT:

Remainder of self**exp in the right euclidean division by modulus.

REMARK:

The quotient of the underlying skew polynomial ring by the principal ideal generated by modulus is in general not a ring.
As a consequence, Sage first computes exactly `self**exp` and then reduce it modulo `modulus`.

EXAMPLES:

```python
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x',Frob]
sage: a = x + t
sage: b = a^5 # indirect doctest
sage: b
x^5 + (2*t^2 + 4)*x^4 + (t^2 + 2)*x^3 + 2*x^2 + (4*t^2 + 2)*x + 2*t^2 + 4*t + 4
sage: b == a * a * a * a * a
True
sage: modulus = x^3 + t*x^2 + (t+3)*x - 2
sage: br = a.right_power_mod(5, modulus); br
(t + 1)*x^2 + (2*t^2 + t + 1)*x + 2*t^2 + 4*t + 2
sage: br == b % modulus
True
sage: a.right_power_mod(100, modulus)
(2*t^2 + 3)*x^2 + (t^2 + 4*t + 2)*x + t^2 + 2*t + 1
```

Negative exponents are supported:

```python
sage: a^(-5) (x^5 + (2*t^2 + 4)*x^4 + (t^2 + 2)*x^3 + 2*x^2 + (4*t^2 + 2)*x + 2*t^2 + 4*t + 4)^(-1)
sage: b * a^(-5)
1
```

However, they cannot be combined with modulus:

```python
sage: a.right_power_mod(-10, modulus)
Traceback (most recent call last):
...
ValueError: modulus cannot be combined with negative exponent
```

1.5 Univariate dense skew polynomials over a field with a finite order automorphism

AUTHOR:

- Xavier Caruso (2012-06-29): initial version
- Arpit Merchant (2016-08-04): improved docstrings, fixed doctests and refactored classes and methods

```python
class sage.rings.polynomial.skew_polynomial_finite_order.SkewPolynomial_finite_order_dense
```

This method constructs a generic dense skew polynomial over a field equipped with an automorphism of finite order.

INPUT:

- `parent` - parent of `self`
- `x` - list of coefficients from which `self` can be constructed
• check – flag variable to normalize the polynomial
• construct – boolean (default: False)

\texttt{bound()}

Return a bound of this skew polynomial (i.e. a multiple of this skew polynomial lying in the center).

\textbf{Note:} Since \(b\) is central, it divides a skew polynomial on the left iff it divides it on the right

\textbf{ALGORITHM:}

1. Sage first checks whether \texttt{self} is itself in the center. It if is, it returns \texttt{self}
2. If an optimal bound was previously computed and cached, Sage returns it
3. Otherwise, Sage returns the reduced norm of \texttt{self}

As a consequence, the output of this function may depend on previous computations (an example is given below).

\textbf{EXAMPLES:}

\begin{verbatim}
\texttt{sage:} \texttt{k.<t> = GF(5^3)}
\texttt{sage:} \texttt{Frob = k.frobenius_endomorphism()}
\texttt{sage:} \texttt{S.<x> = k['x'], Frob}
\texttt{sage:} \texttt{Z = S.center(); Z}
Univariate Polynomial Ring in z over Finite Field of size 5
\texttt{sage:} \texttt{a = x^2 + (4*t + 2)*x + 4*t^2 + 3}
\texttt{sage:} \texttt{b = a.bound(); b}
z^2 + z + 4
\end{verbatim}

We observe that the bound is explicitly given as an element of the center (which is a univariate polynomial ring in the variable \(z\)). We can use conversion to send it in the skew polynomial ring:

\begin{verbatim}
\texttt{sage:} \texttt{S(b)}
x^6 + x^3 + 4
\end{verbatim}

We check that \(b\) is divisible by \(a\):

\begin{verbatim}
\texttt{sage:} \texttt{S(b).is_right_divisible_by(a)}
True
\texttt{sage:} \texttt{S(b).is_left_divisible_by(a)}
True
\end{verbatim}

Actually, \(b\) is the reduced norm of \(a\):

\begin{verbatim}
\texttt{sage:} \texttt{b == a.reduced_norm()}
True
\end{verbatim}

Now, we compute the optimal bound of \(a\) and see that it affects the behaviour of \texttt{bound()}:

\begin{verbatim}
\texttt{sage:} \texttt{a.optimal_bound()}
z + 3
\texttt{sage:} \texttt{a.bound()}
z + 3
\end{verbatim}
is_central()  
Return True if this skew polynomial lies in the center.  

EXAMPLES:

```
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x', Frob]

sage: x.is_central()
False
sage: (t^2*x^3).is_central()
False
sage: (x^6 + x^3).is_central()
True
```

optimal_bound()  
Return the optimal bound of this skew polynomial (i.e. the monic multiple of this skew polynomial of minimal degree lying in the center).  

**Note:** The result is cached.  

EXAMPLES:

```
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x', Frob]

sage: Z = S.center(); Z
Univariate Polynomial Ring in z over Finite Field of size 5

sage: a = x^2 + (4*t + 2)*x + 4*t^2 + 3
sage: b = a.optimal_bound(); b
z + 3
```

We observe that the bound is explicitly given as an element of the center (which is a univariate polynomial ring in the variable \(z\)). We can use conversion to send it in the skew polynomial ring:

```
sage: S(b)
x^3 + 3
```

We check that \(b\) is divisible by \(a\):

```
sage: S(b).is_right_divisible_by(a)
True
sage: S(b).is_left_divisible_by(a)
True
```

reduced_charpoly(var=None)  
Return the reduced characteristic polynomial of this skew polynomial.  

**INPUT:**  
- var – a string, a pair of strings or None (default: None); the variable names used for the characteristic polynomial and the center
Examples:

```python
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<u> = k['u'], Frob
sage: a = u^3 + (2*t^2 + 3)*u^2 + (4*t^2 + t + 4)*u + 2*t^2 + 2
sage: chi = a.reduced_charpoly()
sage: chi
x^3 + (2*z + 1)*x^2 + (3*z^2 + 4*z)*x + 4*z^3 + z^2 + 1
```

The reduced characteristic polynomial has coefficients in the center of $S$, which is itself a univariate polynomial ring in the variable $z = u^3$ over $F_5$. Hence it appears as a bivariate polynomial:

```python
sage: chi.parent()
Univariate Polynomial Ring in x over Univariate Polynomial Ring in z over Finite Field of size 5
```

The constant coefficient of the reduced characteristic polynomial is the reduced norm, up to a sign:

```python
sage: chi[0] == -a.reduced_norm()
True
```

Its coefficient of degree $\deg(a) - 1$ is the opposite of the reduced trace:

```python
True
```

By default, the name of the variable of the reduced characteristic polynomial is $x$ and the name of central variable is usually $z$ (see `center()` for more details about this). The user can specify different names if desired:

```python
sage: a.reduced_charpoly(var='T')  # variable name for the characteristic polynomial
T^3 + (2*z + 1)*T^2 + (3*z^2 + 4*z)*T + 4*z^3 + z^2 + 1
sage: a.reduced_charpoly(var=('T', 'c'))
T^3 + (2*c + 1)*T^2 + (3*c^2 + 4*c)*T + 4*c^3 + c^2 + 1
```

See also:

- `reduced_trace()`, `reduced_norm()`
- `reduced_norm(var=None)`

Return the reduced norm of this skew polynomial.

**INPUT:**

- `var` -- a string or `False` or `None` (default: `None`); the variable name; if `False`, return the list of coefficients

**Note:** The result is cached.
EXAMPLES:

```
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x', Frob]
sage: a = x^3 + (2*t^2 + 3)*x^2 + (4*t^2 + t + 4)*x + 2*t^2 + 2
sage: N = a.reduced_norm(); N
z^3 + 4*z^2 + 4
```

The reduced norm lies in the center of $S$, which is a univariate polynomial ring in the variable $z = x^3$ over $F_5$:

```
sage: N.parent()
Univariate Polynomial Ring in z over Finite Field of size 5
sage: N.parent() is S.center()
True
```

We can use explicit conversion to view $N$ as a skew polynomial:

```
sage: S(N)
x^9 + 4*x^6 + 4
```

By default, the name of the central variable is usually $z$ (see `center()` for more details about this). However, the user can specify a different variable name if desired:

```
sage: a.reduced_norm(var='u')
u^3 + 4*u^2 + 4
```

When passing in `var=False`, a tuple of coefficients (instead of an actual polynomial) is returned:

```
sage: a.reduced_norm(var=False)
(4, 0, 4, 1)
```

**ALGORITHM:**

If $r$ (= the order of the twist map) is small compared to $d$ (= the degree of this skew polynomial), the reduced norm is computed as the determinant of the multiplication by $P$ (= this skew polynomial) acting on $K[X, \sigma]$ (= the underlying skew ring) viewed as a free module of rank $r$ over $K[X^r]$.

Otherwise, the reduced norm is computed as the characteristic polynomial of the left multiplication by $X$ on the quotient $K[X, \sigma]/K[X, \sigma]P$ (which is a $K$-vector space of dimension $d$).

See also:

`reduced_trace()`, `reduced_charpoly()`

**reduced_trace(var=None)**

Return the reduced trace of this skew polynomial.

**INPUT:**

- `var` – a string or `False` or `None` (default: `None`); the variable name; if `False`, return the list of coefficients

**EXAMPLES:**

```
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
```
\begin{verbatim}
  sage: S.<x> = k['x', Frob]
  sage: a = x^3 + (2*t^2 + 3)*x^2 + (4*t^2 + t + 4)*x + 2*t^2 + 2
  sage: tr = a.reduced_trace(); tr
  3*z + 4

  The reduced trace lies in the center of \( S \), which is a univariate polynomial ring in the variable \( z = x^3 \) over \( \mathbb{F}_5 \):

  sage: tr.parent()
  Univariate Polynomial Ring in z over Finite Field of size 5
  sage: tr.parent() is S.center()
  True

  We can use explicit conversion to view \( \text{tr} \) as a skew polynomial:

  sage: S(tr)
  3*x^3 + 4

  By default, the name of the central variable is usually \( z \) (see \text{center()} \) for more details about this). However, the user can specify a different variable name if desired:

  sage: a.reduced_trace(var='u')
  3*u + 4

  When passing in \text{var=False}, a tuple of coefficients (instead of an actual polynomial) is returned:

  sage: a.reduced_trace(var=False)
  (4, 3)

  See also:
  \text{reduced_norm()}, \text{reduced_charpoly()}
\end{verbatim}

\section*{1.6 Univariate dense skew polynomials over finite fields}

This module provides the class: \textit{sage.rings.polynomial.skew_polynomial_finite_field.SkewPolynomial_finite_field_dense}, which constructs a single univariate skew polynomial over a finite field equipped with the Frobenius endomorphism. Among other things, it implements the fast factorization algorithm designed in [CL2017].

**AUTHOR:**

- Xavier Caruso (2012-06-29): initial version

  - Arpit Merchant (2016-08-04): improved docstrings, fixed doctests and refactored classes and methods

**class**

\texttt{sage.rings.polynomial.skew_polynomial_finite_field.SkewPolynomial\_finite\_field\_dense}

\texttt{Bases: SkewPolynomial\_finite\_order\_dense}

\texttt{count\_factorizations()}

Return the number of factorizations (as a product of a unit and a product of irreducible monic factors) of this skew polynomial.

\texttt{EXAMPLES:}
We illustrate that an irreducible polynomial in the center have in general a lot of distinct factorizations in the skew polynomial ring:

```sage
sage: Z.<x3> = S.center()
sage: N = x3^5 + 4*x3^4 + 4*x3^2 + 4*x3 + 3
sage: N.is_irreducible()
True
sage: S(N).count_factorizations()
30537115626
```

### count_irreducible_divisors()

Return the number of irreducible monic divisors of this skew polynomial.

**Note:** One can prove that there are always as many left irreducible monic divisors as right irreducible monic divisors.

**EXAMPLES:**

```sage
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x',Frob]
```

We illustrate that a skew polynomial may have a number of irreducible divisors greater than its degree:

```sage
sage: a = x^4 + (4*t + 3)*x^3 + t^2*x^2 + (4*t^2 + 3*t)*x + 3*t
sage: a.count_irreducible_divisors()
12
```

We illustrate that an irreducible polynomial in the center have in general a lot of irreducible divisors in the skew polynomial ring:

```sage
sage: Z.<x3> = S.center()
sage: N = x3^5 + 4*x3^4 + 4*x3^2 + 4*x3 + 3; N
x3^5 + 4*x3^4 + 4*x3^2 + 4*x3 + 3
sage: N.is_irreducible()
True
sage: S(N).count_irreducible_divisors()
9768751
```

### factor

Return a factorization of this skew polynomial.

**INPUT:**

- `uniform` — a boolean (default: `False`); whether the output irreducible divisor should be uniformly distributed among all possibilities
Noncommutative Polynomials, Release 10.0

EXAMPLES:

sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'], Frob
sage: a = x^3 + (t^2 + 4*t + 2)*x^2 + (3*t + 3)*x + t^2 + 1
sage: F = a.factor(); F  # random
(x + t^2 + 4) * (x + t + 3) * (x + t)
sage: F.value() == a
True

The result of the factorization is cached. Hence, if we try again to factor \( a \), we will get the same answer:

sage: a.factor()  # random
(x + t^2 + 4) * (x + t + 3) * (x + t)

However, the algorithm is probabilistic. Hence if we first reinitialize \( a \), we may get a different answer:

sage: a = x^3 + (t^2 + 1)*x^2 + (2*t + 3)*x + t^2 + t + 2
sage: F = a.factor(); F  # random
(x + 2*t^2 + t + 2) * (x + 2*t^2 + t + 4) * (x + t)
sage: F.value() == a
True

There is a priori no guarantee on the distribution of the factorizations we get. Passing in the keyword \( \text{uniform=True} \) ensures the output is uniformly distributed among all factorizations:

sage: a.factor(uniform=True)  # random
(x + 3*t^2 + 4*t) * (x + 2*t^2) * (x + 4*t^2 + 4*t + 2)

By convention, the zero skew polynomial has no factorization:

sage: S(0).factor()
Traceback (most recent call last):
...
ValueError: factorization of 0 not defined

\textbf{factorizations()}

Return an iterator over all factorizations (as a product of a unit and a product of irreducible monic factors) of this skew polynomial.

EXAMPLES:

sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'], Frob
sage: a = x^3 + (t^2 + 1)*x^2 + (2*t + 3)*x + t^2 + t + 2
sage: iter = a.factorizations(); iter
<generator object at 0x...>
sage: next(iter)  # random
(x + 3*t^2 + 4*t) * (x + 2*t^2) * (x + 4*t^2 + 4*t + 2)

(continues on next page)
We can use this function to build the list of factorizations of $a$:

\[
\text{sage: factorizations = [ F for F in a.factorizations() ]}
\]

We do some checks:

\[
\begin{align*}
\text{sage: len(factorizations)} &= a.count_factorizations() \\
&= \text{a.count_factorizations()} \\
\text{sage: len(factorizations)} &= \text{Set(factorizations).cardinality()} \quad \# \text{check no duplicates} \\
&= \text{Set(factorizations).cardinality()} \\
\text{sage: for F in factorizations:} \\
&\quad \begin{align*}
&\text{....: assert F.value() == a, "factorization has a different value"} \\
&\text{....: for d, _ in F:} \\
&\quad \begin{align*}
&\text{....: assert d.is_irreducible(), "a factor is not irreducible"}
\end{align*}
\end{align*}
\]

Note that the algorithm used in this method is probabilistic. As a consequence, if we call it two times with the same input, we can get different orderings:

\[
\begin{align*}
\text{sage: factorizations2 = [ F for F in a.factorizations() ]} \\
\text{sage: factorizations == factorizations2} \\
&= \text{False} \\
\text{sage: sorted(factorizations) == sorted(factorizations2)} \\
&= \text{True}
\end{align*}
\]

**is_irreducible()**

Return True if this skew polynomial is irreducible.

**EXAMPLES:**

\[
\begin{align*}
\text{sage: k.<t> = GF(5^3)} \\
\text{sage: Frob = k.frobenius_endomorphism()} \\
\text{sage: S.<x> = k['x'], Frob} \\
\text{sage: a = x^2 + t*x + 1} \\
\text{sage: a.is_irreducible()} \\
&= \text{False} \\
\text{sage: a.factor()} \\
&= (x + 4*t^2 + 4*t + 1) * (x + 3*t^2 + t + 2) \\
\text{sage: a = x^2 + t*x + t + 1} \\
\text{sage: a.is_irreducible()} \\
&= \text{True} \\
\text{sage: a.factor()} \\
&= x^2 + t*x + t + 1
\end{align*}
\]

Skew polynomials of degree 1 are of course irreducible:

\[
\begin{align*}
\text{sage: a = x + t} \\
\text{sage: a.is_irreducible()} \\
&= \text{True}
\end{align*}
\]
A random irreducible skew polynomial is irreducible:

```sage
sage: a = S.random_irreducible(degree=4,monic=True); a  # random
x^4 + (t + 1)*x^3 + (3*t^2 + 2*t + 3)*x^2 + 3*t^2*x + 3*t
sage: a.is_irreducible()
True
```

By convention, constant skew polynomials are not irreducible:

```sage
sage: S(1).is_irreducible()
False
sage: S(0).is_irreducible()
False
```

`left_irreducible_divisor(uniform=False)`

Return a left irreducible divisor of this skew polynomial.

INPUT:

- `uniform` – a boolean (default: False); whether the output irreducible divisor should be uniformly distributed among all possibilities

EXAMPLES:

```sage
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k[t][x],Frob]
sage: a = x^6 + 3*t*x^5 + (3*t + 1)*x^3 + (4*t^2 + 3*t + 4)*x^2 + (t^2 + 2)*x + 4*t^2 + 3_t + 3
sage: dl = a.left_irreducible_divisor(); dl  # random
x^3 + (t^2 + t + 2)*x^2 + (t + 2)*x + 3*t^2 + t + 4
sage: a.is_left_divisible_by(dl)
True
```

The algorithm is probabilistic. Hence, if we ask again for a left irreducible divisor of \(a\), we may get a different answer:

```sage
sage: a.left_irreducible_divisor()  # random
x^3 + (4*t^2 + 3)*x^2 + (2*t^2 + 3*t + 4)*x + 4*t^2 + 2*t
```

We can also generate uniformly distributed irreducible monic divisors as follows:

```sage
sage: a.left_irreducible_divisor(uniform=True)  # random
x^3 + (4*t^2 + 3)*x^2 + (2*t^2 + 3*t + 4)*x + 4*t^2 + 2*t
sage: a.left_irreducible_divisor(uniform=True)  # random
x^3 + (2*t^2 + t + 4)*x^2 + (2*t^2 + 4*t + 4)*x + 2*t + 3
sage: a.left_irreducible_divisor(uniform=True)  # random
x^3 + (3*t^2 + t)*x^2 + (3*t^2 + t)*x + 2*t + 1
```

By convention, the zero skew polynomial has no irreducible divisor:

```sage
sage: S(0).left_irreducible_divisor()
Traceback (most recent call last):
  ...
ValueError: 0 has no irreducible divisor
```

1.6. Univariate dense skew polynomials over finite fields
**left_irreducible_divisors()**

Return an iterator over all irreducible monic left divisors of this skew polynomial.

**EXAMPLES:**

```python
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'], Frob
sage: a = x^4 + 2*t*x^3 + 3*t^2*x^2 + (t^2 + t + 1)*x + 4*t + 3
sage: iter = a.left_irreducible_divisors(); iter
<generator object at 0x...>
sage: next(iter)  # random
x + 3*t + 3
sage: next(iter)  # random
x + 4*t + 2
```

We can use this function to build the list of all monic irreducible divisors of \( a \):

```python
sage: leftdiv = [ d for d in a.left_irreducible_divisors() ]
```

Note that the algorithm is probabilistic. As a consequence, if we build again the list of left monic irreducible divisors of \( a \), we may get a different ordering:

```python
sage: leftdiv2 = [ d for d in a.left_irreducible_divisors() ]
sage: Set(leftdiv) == Set(leftdiv2)
True
```

**right_irreducible_divisor**(uniform=False)

Return a right irreducible divisor of this skew polynomial.

**INPUT:**

- `uniform` - a boolean (default: False); whether the output irreducible divisor should be uniformly distributed among all possibilities

**EXAMPLES:**

```python
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'], Frob
sage: a = x^6 + 3*t*x^5 + (3*t + 1)*x^3 + (4*t^2 + 3*t + 4)*x^2 + (t^2 + 2)*x + 4*t^2 + 3
sage: dr = a.right_irreducible_divisor(); dr  # random
x^3 + (2*t^2 + t + 4)*x^2 + (4*t + 1)*x + 4*t^2 + t + 1
sage: a.is_right_divisible_by(dr)
True
```

Right divisors are cached. Hence, if we ask again for a right divisor, we will get the same answer:

```python
sage: a.right_irreducible_divisor()  # random
x^3 + (2*t^2 + t + 4)*x^2 + (4*t + 1)*x + 4*t^2 + t + 1
```

However the algorithm is probabilistic. Hence, if we first reinitialize \( a \), we may get a different answer:
We can also generate uniformly distributed irreducible monic divisors as follows:

```
sage: a.right_irreducible_divisor(uniform=True)  # random
x^3 + (4*t + 2)*x^2 + (2*t^2 + 2*t + 2)*x + 2*t^2 + 2
sage: a.right_irreducible_divisor(uniform=True)  # random
x^3 + x^2 + (4*t^2 + 2*t + 4)*x + t^2 + 3
```

By convention, the zero skew polynomial has no irreducible divisor:

```
sage: S(0).right_irreducible_divisor()
Traceback (most recent call last):
  ... ValueError: 0 has no irreducible divisor
```

Return an iterator over all irreducible monic right divisors of this skew polynomial.

```
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'], Frob
sage: a = x^4 + 2*t*x^3 + 3*t^2*x^2 + (t^2 + t + 1)*x + 4*t + 3
sage: iter = a.right_irreducible_divisors(); iter
<generator object at 0x...>
sage: next(iter)  # random
x + 2*t^2 + 4*t + 4
sage: next(iter)  # random
x + 3*t^2 + 4*t + 1
```

We can use this function to build the list of all monic irreducible divisors of \(a\):

```
sage: rightdiv = [ d for d in a.right_irreducible_divisors() ]
```

Note that the algorithm is probabilistic. As a consequence, if we build again the list of right monic irreducible divisors of \(a\), we may get a different ordering:

```
sage: rightdiv2 = [ d for d in a.right_irreducible_divisors() ]
sage: rightdiv == rightdiv2
False
sage: Set(rightdiv) == Set(rightdiv2)
True
```

Return the \(N\)-type of this skew polynomial (see definition below).

```
return N
```

\(N\) – an irreducible polynomial in the center of the underlying skew polynomial ring
DEFINITION:

The $N$-type of a skew polynomial $a$ is the Partition $(t_0, t_1, t_2, \ldots)$ defined by

$$t_0 + \cdots + t_i = \frac{\deg \gcd(a, N^i)}{\deg N},$$

where $\deg N$ is the degree of $N$ considered as an element in the center.

This notion has an important mathematical interest because it corresponds to the Jordan type of the $N$-typical part of the associated Galois representation.

EXAMPLES:

```sage
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'],Frob
sage: Z = S.center(); x3 = Z.gen()
sage: a = x^4 + x^3 + (4*t^2 + 4)*x^2 + (t^2 + 2)*x + 4*t^2
sage: N = x3^2 + x3 + 1
sage: a.type(N)
[1]
sage: N = x3 + 1
sage: a.type(N)
[2]
sage: a = x^3 + (3*t^2 + 1)*x^2 + (3*t^2 + t + 1)*x + t + 1
sage: N = x3 + 1
sage: a.type(N)
[2, 1]
```

If $N$ does not divide the reduced map of $a$, the type is empty:

```sage
sage: N = x3 + 2
sage: a.type(N)
[]
```

If $a = N$, the type is just $[r]$ where $r$ is the order of the twisting morphism $\text{Frob}$:

```sage
sage: N = x3^2 + x3 + 1
sage: S(N).type(N)
[3]
```

$N$ must be irreducible:

```sage
sage: N = (x3 + 1) * (x3 + 2)
sage: a.type(N)
Traceback (most recent call last):
... ValueError: N is not irreducible
```
1.7 Fraction fields of Ore polynomial rings

Sage provides support for building the fraction field of any Ore polynomial ring and performing basic operations in it. The fraction field is constructed by the method `sage.rings.polynomial.ore_polynomial_ring.OrePolynomialRing.fraction_field()` as demonstrated below:

```python
sage: R.<t> = QQ[]
sage: der = R.derivation()
sage: A.<d> = R['d', der]
sage: K = A.fraction_field()
sage: K
Ore Function Field in d over Fraction Field of Univariate Polynomial Ring in t over Rational Field twisted by d/dt
```

The simplest way to build elements in \( K \) is to use the division operator over Ore polynomial rings:

```python
sage: f = 1/d
sage: f
d^(-1)
sage: f.parent() is K
True
```

**REPRESENTATION OF ELEMENTS:**

Elements in \( K \) are internally represented by fractions of the form \( s^{-1}t \) with the denominator on the left. Notice that, because of noncommutativity, this is not the same that fractions with denominator on the right. For example, a fraction created by the division operator is usually displayed with a different numerator and/or a different denominator:

```python
sage: g = t / d
sage: g
(d - 1/t)^(-1) * t
sage: g.left_numerator()
t
sage: g.right_denominator()
d
```

The left numerator and right denominator are accessible as follows:

```python
sage: g.left_denominator()
d - 1/t
sage: g.right_numerator()
t
```

Similarly the methods `OrePolynomial.left_denominator()` and `OrePolynomial.right_numerator()` give access to the Ore polynomials \( s \) and \( t \) in the representation \( s^{-1}t \):

```python
sage: g.left_denominator()
d - 1/t
sage: g.right_numerator()
t
```

We favored the writing \( s^{-1}t \) because it always exists. On the contrary, the writing \( st^{-1} \) is only guaranteed when the twisting morphism defining the Ore polynomial ring is bijective. As a consequence, when the latter assumption is not fulfilled (or actually if Sage cannot invert the twisting morphism), computing the left numerator and the right denominator fails:

```python
sage: sigma = R.hom([t^2])
sage: S.<x> = R['x', sigma]
```

(continues on next page)
```plaintext
sage: F = S.fraction_field()
sage: f = F.random_element()
sage: while not f:
    ....: f = F.random_element()
sage: f.left_numerator()
Traceback (most recent call last):
   ...:
NotImplementedError: inversion of the twisting morphism
Ring endomorphism of Fraction
    → Field of Univariate Polynomial Ring in t over Rational Field
    Defn: t |--> t^2

On a related note, fractions are systematically simplified when the twisting morphism is bijective but they are not
otherwise. As an example, compare the two following computations:
```
```plaintext
sage: P = d^2 + t*d + 1
sage: Q = d + t^2
sage: D = d^3 + t^2 + 1
sage: f = P^(-1) * Q
```
```plaintext
sage: f
(d^2 + t*d + 1)^(-1) * (d + t^2)
```
```plaintext
sage: g = (D*P)^(-1) * (D*Q)
```
```plaintext
sage: g
(d^2 + t*d + 1)^(-1) * (d + t^2)
```
```plaintext
sage: P = x^2 + t*x + 1
sage: Q = x + t^2
sage: D = x^3 + t^2 + 1
sage: f = P^(-1) * Q
```
```plaintext
sage: f
(x^2 + t*x + 1)^(-1) * (x + t^2)
```
```plaintext
sage: g = (D*P)^(-1) * (D*Q)
```
```plaintext
sage: g
(x^5 + t^8*x^4 + x^3 + (t^2 + 1)*x^2 + (t^3 + t)*x + t^2 + 1)^(-1) * (x^4 + t^16*x^3 +
   →(t^2 + 1)*x + t^4 + t^2)
```
```plaintext
sage: f == g
True
```

OPERATIONS:

Basic arithmetical operations are available:
```
```
\[ d^{-1} \ast (d + t) \]

Of course, multiplication remains noncommutative:

\[
\text{sage: } g \ast f \\
(d^2 + t^2 d + 1)^{-1} \\
\text{sage: } g^{-1} \ast f \\
(d - 1/t)^{-1} \ast (d + (t^2 - 1)/t)
\]

AUTHOR:
• Xavier Caruso (2020-05)

class sage.rings.polynomial.ore_function_field.OreFunctionCenterInjection(domain, codomain, ringembed)

Bases: RingHomomorphism

Canonical injection of the center of a Ore function field into this field.

section()

Return a section of this morphism.

EXAMPLES:

\[
\text{sage: } k.<a> = GF(5^3) \\
\text{sage: } S.<x> = SkewPolynomialRing(k, k.frobenius_endomorphism()) \\
\text{sage: } K = S.fraction_field() \\
\text{sage: } Z = K.center() \\
\text{sage: } iota = K.coerce_map_from(Z) \\
\text{sage: } sigma = iota.section() \\
\text{sage: } sigma(x^3 / (x^6 + 1)) \\
z / (z^2 + 1)
\]

class sage.rings.polynomial.ore_function_field.OreFunctionField(ring, category=None)

Bases: Algebra, UniqueRepresentation

A class for fraction fields of Ore polynomial rings.

Element = None

change_var(var)

Return the Ore function field in variable var with the same base ring, twisting morphism and twisting derivation as self.

INPUT:

• var – a string representing the name of the new variable.

EXAMPLES:

\[
\text{sage: } k.<t> = GF(5^3) \\
\text{sage: } Frob = k.frobenius_endomorphism() \\
\text{sage: } R.<x> = OrePolynomialRing(k,Frob) \\
\text{sage: } K = R.fraction_field() \\
\text{sage: } K \\
\text{Ore Function Field in x over Finite Field in t of size 5^3 twisted by t \mapsto t^5}
\]
Noncommutative Polynomials, Release 10.0

(continued from previous page)

```sage
K = K.change_var('y'); Ky
Ore Function Field in y over Finite Field in t of size 5^3 twisted by t |---> t^5
sage: Ky is K.change_var('y')
True
```

class `characteristic`

Return the characteristic of this Ore function field.

EXAMPLES:

```sage
R.<t> = QQ[
S = R['x'],sigma
S.fraction_field().characteristic()
0
k.<u> = GF(5^3)
Frob = k.frobenius_endomorphism()
S = k['y'],Frob
S.fraction_field().characteristic()
5
```

```sage
fractions_field()

Return the fraction field of this Ore function field, i.e. this Ore function field itself.

EXAMPLES:

```sage
R.<t> = QQ[
A.<d> = R['d', der]
A.fraction_field()
K = K.fraction_field()
Ore Function Field in d over Fraction Field of Univariate Polynomial Ring in t |-->
over Rational Field twisted by d/dt
K.fraction_field() is K
True
```

class `gen`(n=0)

Return the indeterminate generator of this Ore function field.

INPUT:

• n – index of generator to return (default: 0). Exists for compatibility with other polynomial rings.

EXAMPLES:

```sage
k.<a> = GF(5^4)
Frob = k.frobenius_endomorphism()
S.<x> = k['x', Frob]
S.fraction_field()
```

(continues on next page)
sage: K.gen()
x

gens_dict()

Return a {name: variable} dictionary of the generators of this Ore function field.

EXAMPLES:

sage: R.<t> = ZZ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = OrePolynomialRing(R, sigma)
sage: K = S.fraction_field()
sage: K.gens_dict()
{'x': x}

is_commutative()

Return True if this Ore function field is commutative, i.e. if the twisting morphism is the identity and the twisting derivation vanishes.

EXAMPLES:

sage: k.<a> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'], Frob
sage: K = S.fraction_field()
sage: K.is_commutative()
False

sage: T.<y> = k['y', Frob^3]
sage: L = T.fraction_field()
sage: L.is_commutative()
True

is_exact()

Return True if elements of this Ore function field are exact. This happens if and only if elements of the base ring are exact.

EXAMPLES:

sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x', Frob]
sage: K = S.fraction_field()
sage: K.is_exact()
True

sage: k.<u> = Qq(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x', Frob]
sage: K = S.fraction_field()
sage: K.is_exact()
False
is_field(proof=False)
Return always True since Ore function field are (skew) fields.

EXAMPLES:

```sage
sage: k.<a> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x', Frob]
sage: K = S.fraction_field()

sage: S.is_field()
False
sage: K.is_field()
True
```

is_finite()
Return False since Ore function field are not finite.

EXAMPLES:

```sage
sage: k.<t> = GF(5^3)
sage: k.is_finite()
True
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x', Frob]
sage: K = S.fraction_field()
sage: K.is_finite()
False
```

is_sparse()
Return True if the elements of this Ore function field are sparsely represented.

**Warning:** Since sparse Ore polynomials are not yet implemented, this function always returns False.

EXAMPLES:

```sage
sage: R.<t> = RR[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x', sigma]
sage: K = S.fraction_field()
sage: K.is_sparse()
False
```

gens()
Return the number of generators of this Ore function field, which is 1.

EXAMPLES:

```sage
sage: R.<t> = RR[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x', sigma]
sage: K = S.fraction_field()
sage: K.ngens()
1
```
**parameter** (*n=0*)

Return the indeterminate generator of this Ore function field.

**INPUT:**

- *n* – index of generator to return (default: 0). Exists for compatibility with other polynomial rings.

**EXAMPLES:**

```python
sage: k.<a> = GF(5^4)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x', Frob]
sage: K = S.fraction_field()
sage: K.gen()
x
```

**random_element** (*degree=2, monic=False, *args, **kwargs*)

Return a random Ore function in this field.

**INPUT:**

- *degree* – (default: 2) an integer or a list of two integers; the degrees of the denominator and numerator
- *monic* – (default: False) if True, return a monic Ore function with monic numerator and denominator
- *args, kwargs* – passed in to the random_element method for the base ring

**EXAMPLES:**

```python
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x', Frob]
sage: K = S.fraction_field()
sage: K.random_element()
# random (x^2 + (2*t^2 + t + 1)*x + 2*t^2 + 2*t + 3)^(-1) * ((2*t^2 + 3)*x^2 + (4*t^2 + t + 4)*x + 2*t^2 + 2)
sage: K.random_element(monic=True)
# random (x^2 + (4*t^2 + 3*t + 4)*x + 4*t^2 + t)^(-1) * (x^2 + (2*t^2 + t + 3)*x + 3*t^2 + t + 2)
sage: K.random_element(degree=3)
# random (x^3 + (2*t^2 + 3)*x^2 + (2*t^2 + 4)*x + t + 3)^(-1) * ((t + 4)*x^3 + (4*t^2 + t + 2)*x^2 + (2*t^2 + 3*t + 3)*x + 3*t^2 + 3*t + 1)
sage: K.random_element(degree=[2,5])
# random (x^2 + (4*t^2 + 2*t + 2)*x + 4*t^2 + t + 2)^(-1) * ((3*t^2 + t + 1)*x^5 + (2*t^2 + 2*t + 2*t)*x + (t^2 + t + 4)*x^2 + (t^2 + t + 4)*x)
```

**twisting_derivation()**

Return the twisting derivation defining this Ore function field or None if this Ore function field is not twisted by a derivation.

**EXAMPLES:**

```python
sage: R.<t> = QQ[]
sage: der = R.derivation(); der
d/dt
sage: A.<d> = R['d', der]
sage: F = A.fraction_field()
```
sage: F.twisting_derivation()
d/dt

sage: k.<a> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'], Frob
sage: K = S.fraction_field()
sage: K.twisting_derivation()

See also:
sage.rings.polynomial.ore_polynomial_element.OrePolynomial.twisting_derivation(),
twisting_morphism()

twisting_morphism(n=1)
Return the twisting endomorphism defining this Ore function field iterated n times or None if this Ore function field is not twisted by an endomorphism.

INPUT:

* n - an integer (default: 1)

EXAMPLES:

sage: R.<t> = QQ[]
sage: sigma = R.hom([t+1])
sage: S.<x> = R['x'], sigma
sage: K = S.fraction_field()
sage: K.twisting_morphism()
Ring endomorphism of Fraction Field of Univariate Polynomial Ring in t over Rational Field
  Defn: t |--> t + 1

When the Ore polynomial ring is only twisted by a derivation, this method returns nothing:

sage: der = R.derivation()
sage: A.<d> = R['x'], der
sage: F = A.fraction_field()
sage: F.twisting_morphism()

See also:
sage.rings.polynomial.ore_polynomial_element.OrePolynomial.twisting_morphism(),
twisting_derivation()

class sage.rings.polynomial.ore_function_field.OreFunctionField_with_large_center(ring, category=None)

Bases: OreFunctionField
A specialized class for Ore polynomial fields whose center has finite index.

center(name=None, names=None, default=False)
Return the center of this Ore function field.
Note: One can prove that the center is a field of rational functions over a subfield of the base ring of this Ore function field.

INPUT:

- `name` – a string or `None` (default: `None`); the name for the central variable
- `default` – a boolean (default: `False`); if `True`, set the default variable name for the center to `name`

EXAMPLES:

```
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x',Frob]
sage: K = S.fraction_field()
sage: Z = K.center(); Z
Fraction Field of Univariate Polynomial Ring in z over Finite Field of size 5
```

We can pass in another variable name:

```
sage: K.center(name='y')
Fraction Field of Univariate Polynomial Ring in y over Finite Field of size 5
```

or use the bracket notation:

```
sage: Zy.<y> = K.center(); Zy
Fraction Field of Univariate Polynomial Ring in y over Finite Field of size 5
```

A coercion map from the center to the Ore function field is set:

```
sage: K.has_coerce_map_from(Zy)
True
```

and pushout works:

```
sage: x.parent()
Ore Polynomial Ring in x over Finite Field in t of size 5^3 twisted by t |--> t^5
sage: y.parent()
Fraction Field of Univariate Polynomial Ring in y over Finite Field of size 5
sage: P = x + y; P
x^3 + x
sage: P.parent()
Ore Function Field in x over Finite Field in t of size 5^3 twisted by t |--> t^5
```

A conversion map in the reverse direction is also set:

```
sage: Zy(x^(-6) + 2)
(2*y^2 + 1)/y^2
sage: Zy(1/x^2)
Traceback (most recent call last):
  ... ValueError: x^(−2) is not in the center
```
ABOUT THE DEFAULT NAME OF THE CENTRAL VARIABLE:
A priori, the default is \(z\).

However, a variable name is given the first time this method is called, the given name become the default for the next calls:

```
sage: k.<t> = GF(11^3)
sage: phi = k.frobenius_endomorphism()
sage: S.<X> = k['X', phi]
sage: K = S.fraction_field()
sage: C.<u> = K.center()  # first call
sage: C
Fraction Field of Univariate Polynomial Ring in u over Finite Field of size 11
sage: K.center()
Fraction Field of Univariate Polynomial Ring in u over Finite Field of size 11
```

We can update the default variable name by passing in the argument `default=True`:

```
sage: D.<v> = K.center(default=True)
sage: D
Fraction Field of Univariate Polynomial Ring in v over Finite Field of size 11
sage: K.center()
Fraction Field of Univariate Polynomial Ring in v over Finite Field of size 11
```

1.8 Fraction field elements of Ore polynomial rings

AUTHOR:

- Xavier Caruso (2020-05)

```python
sage: from sage.rings.polynomial.ore_polynomial_element import ConstantOrePolynomialSection
sage: k.<a> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x', Frob]
sage: K = S.fraction_field()
sage: iota = K.coerce_map_from(k)
sage: sigma = iota.section()
sage: sigma
```

(continues on next page)
class sage.rings.polynomial.ore_function_element.OreFunction(parent, numerator, denominator=None, simplify=True)

Bases: AlgebraElement

An element in a Ore function field.

hilbert_shift(s, var=None)

Return this Ore function with variable shifted by s, i.e. if this Ore function is f(x), return f(x + s).

INPUT:

• s – an element in the base ring

• var – a string; the variable name

EXAMPLES:

sage: R.<t> = GF(7)[]
sage: der = R.derivation()
sage: A.<d> = R['d', der]
sage: K = A.fraction_field()
sage: f = 1 / (d-t)
sage: f.hilbert_shift(t)
d^(-1)

One can specify another variable name:

sage: f.hilbert_shift(t, var='x')
x^(-1)

When the twisting morphism is not trivial, the output lies in a different Ore polynomial ring:

sage: k.<a> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x', Frob]
sage: K = S.fraction_field()
sage: f = (x-a)^(-2)
sage: g = f.hilbert_shift(a); g
x^(-2)
sage: g.parent()
Ore Function Field in x over Finite Field in a of size 5^3 twisted by a |--> a^5 and a^5([a |--> a^5] - id)
sage: g.parent() is S
False

This behavior ensures that the Hilbert shift by a fixed element defines an homomorphism of fields:
Noncommutative Polynomials, Release 10.0

```python
sage: U = K.random_element(degree=5)
sage: V = K.random_element(degree=5)
sage: s = k.random_element()
sage: (U+V).hilbert_shift(s) == U.hilbert_shift(s) + V.hilbert_shift(s)
True
sage: (U*V).hilbert_shift(s) == U.hilbert_shift(s) * V.hilbert_shift(s)
True
```

**is_zero()**

Return True if this element is equal to zero.

**EXAMPLES:**

```python
sage: R.<t> = GF(3)[]
sage: der = R.derivation()
sage: A.<d> = R['x'], der
sage: f = t/d
sage: f.is_zero()
False
sage: (f-f).is_zero()
True
```

**left_denominator()**

Return $s$ if this element reads $s^{-1}t$.

**WARNING:**

When the twisting morphism is bijective, there is a unique irreducible fraction of the form $s^{-1}t$ representing this element. Here irreducible means that $s$ and $t$ have no nontrivial common left divisor. Under this additional assumption, this method always returns this distinguished denominator $s$.

On the contrary, when the twisting morphism is not bijective, this method returns the denominator of some fraction representing the input element. However, the software guarantees that the method `right_numerator()` outputs the numerator of the same fraction.

**EXAMPLES:**

```python
sage: k.<a> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'], Frob
sage: s = x + a
sage: t = x^2 + a*x + a^2
sage: f = s^(-1) * t
sage: f.left_denominator()
x + a
```

In the example below, a simplification occurs:

```python
sage: u = S.random_element(degree=2)
sage: g = (u*s)^(-1) * (u*t)
sage: g.left_denominator()
x + a
```

When the twisting morphism is not invertible, simplifications do not occur in general:
sage: R.<z> = GF(11)[]
sage: sigma = R.hom([z^2])
sage: S.<x> = R['x', sigma]
sage: s = (x + z)^2
sage: t = (x + z) * (x^2 + z^2)
sage: f = s^(-1) * t
sage: f.left_denominator()
x^2 + (z^2 + z)*x + z^2

However, the following always holds true:

sage: f == f.left_denominator()^(-1) * f.right_numerator()
True

See also:

right_numerator(), left_numerator(), right_denominator()

left_numerator()

Return $t$ if this element reads $t s^{-1}$.

WARNING:

When the twisting morphism is bijective, there is a unique irreducible fraction of the form $t s^{-1}$ representing this element. Here irreducible means that $s$ and $t$ have no nontrivial common right divisor. Under this additional assumption, this method always returns this distinguished numerator $t$.

On the contrary, when the twisting morphism is not bijective, the existence of the writing $t s^{-1}$ is not guaranteed in general. In this case, this method raises an error.

EXAMPLES:

sage: k.<a> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x', Frob]
sage: s = x + a
sage: t = x^2 + a*x + a^2
sage: f = t/s
sage: f.left_numerator()
x^2 + a*x + a^2

In the example below, a simplification occurs:

sage: u = S.random_element(degree=2)
sage: g = (t*u) / (s*u)
sage: g.left_numerator()
x^2 + a*x + a^2

right_denominator()

Return $s$ if this element reads $t s^{-1}$.

WARNING:

When the twisting morphism is bijective, there is a unique irreducible fraction of the form $t s^{-1}$ representing this element. Here irreducible means that $s$ and $t$ have no nontrivial common right divisor. Under this additional assumption, this method always returns this distinguished denominator $s$. 
On the contrary, when the twisting morphism is not bijective, the existence of the writing \( ts^{-1} \) is not guaranteed in general. In this case, this method raises an error.

**EXAMPLES:**

```
sage: k.<a> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'], Frob
sage: s = x + a
sage: t = x^2 + a*x + a^2
sage: f = t/s
sage: f.right_denominator()
x + a
```

In the example below, a simplification occurs:

```
sage: u = S.random_element(degree=2)
sage: g = (t*u) / (s*u)
sage: g.right_denominator()
x + a
```

See also:
- `left_numerator()`, `left_denominator()`, `right_numerator()`

**right_numerator()**

Return \( t \) if this element reads \( s^{-1}t \).

**WARNING:**

When the twisting morphism is bijective, there is a unique irreducible fraction of the form \( s^{-1}t \) representing this element. Here irreducible means that \( s \) and \( t \) have no nontrivial common left divisor. Under this additional assumption, this method always returns this distinguished numerator \( t \).

On the contrary, when the twisting morphism is not bijective, this method returns the numerator of some fraction representing the input element. However, the software guarantees that the method `left_denominator()` outputs the numerator of the same fraction.

**EXAMPLES:**

```
sage: k.<a> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'], Frob
sage: s = x + a
sage: t = x^2 + a*x + a^2
sage: f = s^(-1) * t
sage: f.right_numerator()
x^2 + a*x + a^2
```

In the example below, a simplification occurs:

```
sage: u = S.random_element(degree=2)
sage: g = (u*s)^(-1) * (u*t)
sage: g.right_numerator()
x^2 + a*x + a^2
```
See also:

left_denominator(), left_numerator(), right_denominator()

class sage.rings.polynomial.ore_function_element.OreFunctionBaseringInjection(domain, codomain)

Bases: Morphism

Representation of the canonical homomorphism from a field $k$ into a Ore function field over $k$.

This class is needed by the coercion system.

an_element()

Return an element of the codomain of the ring homomorphism.

EXAMPLES:

```
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'], Frob
sage: K = S.fraction_field()
sage: m = K.coerce_map_from(k)
sage: m.an_element()
x
```

section()

Return the canonical homomorphism from the constants of a Ore function fielded to its base field.

class sage.rings.polynomial.ore_function_element.OreFunction_with_large_center

Bases: OreFunction

A special class for elements of Ore function fields whose center has finite index.

reduced_norm(var=None)

Return the reduced norm of this Ore function.

INPUT:

- var – a string or None (default: None); the name of the central variable

EXAMPLES:

```
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'], Frob
sage: K = S.fraction_field()
sage: m = K.coerce_map_from(k)
sage: a = (x + t) / (x^2 + t^2)
sage: N = a.reduced_norm(); N
(z + 2)/(z^2 + 4)
```

The reduced norm lies in the center of $S$, which is the fraction field of a univariate polynomial ring in the variable $z = x^3$ over $GF(5)$. 

1.8. Fraction field elements of Ore polynomial rings
Noncommutative Polynomials, Release 10.0

```python
sage: N.parent()
Fraction Field of Univariate Polynomial Ring in z over Finite Field of size 5
sage: N.parent() is K.center()
True
```

We can use explicit conversion to view $N$ as a skew polynomial:

```python
sage: K(N)
\((x^6 + 4)^{-1} * (x^3 + 2)\)
```

By default, the name of the central variable is usually $z$ (see `sage.rings.polynomial.skew_polynomial_ring.SkewPolynomialRing_finite_order.center()` for more details about this). However, the user can specify a different variable name if desired:

```python
sage: a.reduced_norm(var='u')
\((u + 2)/(u^2 + 4)\)
```

**reduced_trace**(var=None)

Return the reduced trace of this element.

**INPUT:**

- var -- a string or None (default: None); the name of the central variable

**EXAMPLES:**

```python
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: S.<x> = k['x'], Frob]
sage: K = S.fraction_field()
sage: a = 1 / (x^2 + t)
sage: tr = a.reduced_trace(); tr
3/(z^2 + 2)
```

The reduced trace lies in the center of $S$, which is the fraction field of a univariate polynomial ring in the variable $z = x^3$ over $GF(5)$:

```python
sage: tr.parent()
Fraction Field of Univariate Polynomial Ring in z over Finite Field of size 5
sage: tr.parent() is K.center()
True
```

We can use explicit conversion to view $tr$ as a Ore function:

```python
sage: K(tr)
\((x^6 + 2)^{-1} * 3\)
```

By default, the name of the central variable is usually $z$ (see `sage.rings.polynomial.skew_polynomial_ring.OreFunctionField_with_large_center.center()` for more details about this). However, the user can specify a different variable name if desired:

```python
sage: a.reduced_trace(var='u')
3/(u^2 + 2)
```
CHAPTER TWO

NONCOMMUTATIVE MULTIVARIATE POLYNOMIALS

2.1 Noncommutative polynomials via libSINGULAR/Plural

This module provides specialized and optimized implementations for noncommutative multivariate polynomials over many coefficient rings, via the shared library interface to SINGULAR. In particular, the following coefficient rings are supported by this implementation:

• the rational numbers \(\mathbb{Q}\) and
• finite fields \(\mathbb{F}_p\) for \(p\) prime

AUTHORS:

The PLURAL wrapper is due to

• Burcin Erocal (2008-11 and 2010-07): initial implementation and concept
• Michael Brickenstein (2008-11 and 2010-07): initial implementation and concept
• Oleksandr Motsak (2010-07): complete overall noncommutative functionality and first release
• Alexander Dreyer (2010-07): noncommutative ring functionality and documentation
• Simon King (2011-09): left and two-sided ideals; normal forms; pickling; documentation

The underlying libSINGULAR interface was implemented by

• Martin Albrecht (2007-01): initial implementation
• Joel Mohler (2008-01): misc improvements, polishing
• Martin Albrecht (2008-08): added \(\mathbb{Q}(a)\) and \(\mathbb{Z}\) support
• Simon King (2009-04): improved coercion
• Martin Albrecht (2009-05): added \(\mathbb{Z}/n\mathbb{Z}\) support, refactoring
• Martin Albrecht (2009-06): refactored the code to allow better re-use

Todo: extend functionality towards those of libSINGULAR's commutative part

EXAMPLES:

We show how to construct various noncommutative polynomial rings:

\[\text{sage: } A.<x,y,z> = \text{FreeAlgebra}(\mathbb{Q}, 3)\]
\[\text{sage: } P.<x,y,z> = A.g_algebra(relations={y*x:-x*y}, order = 'lex')\]
sage: P
Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field, nc-
relations: {y*x: -x*y}

sage: y*x + 1/2
-x*y + 1/2

sage: A.<x,y,z> = FreeAlgebra(GF(17), 3)
sage: P.<x,y,z> = A.g_algebra(relations={y*x:-x*y}, order = 'lex')
sage: P
Noncommutative Multivariate Polynomial Ring in x, y, z over Finite Field of size 17, nc-
relations: {y*x: -x*y}

sage: y*x + 7
-x*y + 7

Raw use of this class; this is not the intended use!

sage: from sage.matrix.constructor import Matrix
sage: c = Matrix(3)
sage: c[0,1] = -2
sage: c[0,2] = 1
sage: c[1,2] = 1

sage: d = Matrix(3)
sage: d[0,1] = 17
sage: P = QQ['x','y','z']
sage: c = c.change_ring(P)
sage: d = d.change_ring(P)

sage: from sage.rings.polynomial.plural import NCPolynomialRing_plural
sage: R.<x,y,z> = NCPolynomialRing_plural(QQ, c = c, d = d, order=TermOrder('lex',3),
category=Algebras(QQ))
sage: R
Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field, nc-
relations: {y*x: -2*x*y + 17}

sage: R.term_order()
Lexicographic term order

sage: a,b,c = R.gens()
sage: f = 57 + a^2*b + 43 * c + 1; f
57*x^2*y + 43*z + 1

Return the exterior algebra on some generators

This is also known as a Grassmann algebra. This is a finite dimensional algebra, where all generators anti-
commute.

See Wikipedia article Exterior algebra

INPUT:

* base_ring – the ground ring
* names – a list of variable names

EXAMPLES:

```python
sage: from sage.rings.polynomial.plural import ExteriorAlgebra
sage: E = ExteriorAlgebra(QQ, ['x', 'y', 'z'])
#random
Quotient of Noncommutative Multivariate Polynomial Ring in x, y, z over Rational
˓→Field, nc-relations: {z*x: -x*z, z*y: -y*z, y*x: -x*y} by the ideal (z^2, y^2, x^˓→2)
sage: sorted(E.cover().domain().relations().items(), key=str)
[(y*x, -x*y), (z*x, -x*z), (z*y, -y*z)]
sage: sorted(E.cover().kernel().gens(), key=str)
[x^2, y^2, z^2]
sage: E.inject_variables()
Defining xbar, ybar, zbar
sage: x,y,z = (xbar,ybar,zbar)
sage: y*x
-x*y
sage: all(v^2==0 for v in E.gens())
True
sage: E.one()
1
```

class `sage.rings.polynomial.plural.ExteriorAlgebra_plural`

Bases: `NCPolynomialRing_plural`

```python
class sage.rings.polynomial.plural.G_AlgFactory

Bases: UniqueFactory

A factory for the creation of g-algebras as unique parents.

create_key_and_extra_args(base_ring, c, d, names=None, order=None, category=None, check=None)

Create a unique key for g-algebras.

INPUT:

• base_ring - a ring
• c, d - two matrices
• names - a tuple or list of names
• order - (optional) term order
• category - (optional) category
• check - optional bool

create_object(version, key, **extra_args)

Create a g-algebra to a given unique key.

INPUT:

• key - a 6-tuple, formed by a base ring, a tuple of names, two matrices over a polynomial ring over the base ring with the given variable names, a term order, and a category
• extra_args - a dictionary, whose only relevant key is ‘check’.
```

class `sage.rings.polynomial.plural.NCPolynomialRing_plural`

Bases: `Ring`

A non-commutative polynomial ring.
EXAMPLES:

```
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H = A.g_algebra({y*x:x*y-z, z*x:x*z+2*x, z*y:y*z-2*y})
sage: H._is_category_initialized()  
True
sage: H.category()  
Category of algebras over Rational Field
sage: TestSuite(H).run()
```

Note that two variables commute if they are not part of the given relations:

```
sage: H.<x,y,z> = A.g_algebra({z*x:x*z+2*x, z*y:y*z-2*y})
sage: x*y == y*x  
True
```

### free_algebra()

The free algebra of which this is the quotient.

**EXAMPLES:**

```
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: P = A.g_algebra(relations={y*x:-x*y}, order = 'lex')
sage: B = P.free_algebra()
sage: A == B  
True
```

### gen(n=0)

Returns the n-th generator of this noncommutative polynomial ring.

**INPUT:**

- `n` – an integer \( \geq 0 \)

**EXAMPLES:**

```
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: P = A.g_algebra(relations={y*x:-x*y}, order = 'lex')
sage: P.gen(),P.gen(1)  
(x, y)
```

Note that the generators are not cached:

```
sage: P.gen(1) is P.gen(1)  
False
```

### ideal(*gens, **kwds)

Create an ideal in this polynomial ring.

**INPUT:**

- `*gens` - list or tuple of generators (or several input arguments)
- `coerce` - bool (default: True); this must be a keyword argument. Only set it to `False` if you are certain that each generator is already in the ring.
- `side` - string (either “left”, which is the default, or “twosided”) Must be a keyword argument. Defines whether the ideal is a left ideal or a two-sided ideal. Right ideals are not implemented.
EXAMPLES:

```python
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: P.<x,y,z> = A.g_algebra(relations={y*x:-x*y}, order = 'lex')

sage: P.ideal([x + 2*y + 2*z-1, 2*x*y + 2*y*z-y, x^2 + 2*y^2 + 2*z^2-x])
Left Ideal (x + 2*y + 2*z - 1, 2*x*y + 2*y*z - y, x^2 - x + 2*y^2 + 2*z^2) of
Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field, nc-relations: {y*x: -x*y}

sage: P.ideal([x + 2*y + 2*z-1, 2*x*y + 2*y*z-y, x^2 + 2*y^2 + 2*z^2-x], side='twosided')
Twosided Ideal (x + 2*y + 2*z - 1, 2*x*y + 2*y*z - y, x^2 - x + 2*y^2 + 2*z^2) of
Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field, nc-relations: {y*x: -x*y}
```

\texttt{is\_commutative()}

Return False.

\textbf{Todo:} Provide a mathematically correct answer.

EXAMPLES:

```python
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: P = A.g_algebra(relations={y*x:-x*y}, order = 'lex')
sage: P.is_commutative()
False
```

\texttt{is\_field(*args, **kwargs)}

Return False.

EXAMPLES:

```python
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: P = A.g_algebra(relations={y*x:-x*y}, order = 'lex')
sage: P.is_field()
False
```

\texttt{monomial\_all\_divisors(t)}

Return a list of all monomials that divide \( t \).

Coefficients are ignored.

\textbf{INPUT:}

\begin{itemize}
  \item \( t \) - a monomial
\end{itemize}

\textbf{OUTPUT:}

a list of monomials

EXAMPLES:

```python
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: P = A.g_algebra(relations={y*x:-x*y}, order = 'lex')
sage: P.inject_variables()
Defining x, y, z
```

(continues on next page)
sage: P.monomial_all_divisors(x^2*z^3)
[x, x^2, z, x*z, x^2*z, z^2, x*z^2, x^2*z^2, z^3, x*z^3, x^2*z^3]

ALGORITHM: addwithcarry idea by Toon Segers

\textbf{monomial\_divides}(a, b)

Return False if a does not divide b and True otherwise.

Coefficients are ignored.

\textbf{INPUT}:

\begin{itemize}
\item a – monomial
\item b – monomial
\end{itemize}

\textbf{EXAMPLES}:

\begin{verbatim}
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: P = A.g_algebra(relations={y*x:-x*y}, order='lex')
sage: P.inject_variables()
Defining x, y, z

sage: P.monomial_divides(x*y*z, x^3*y^2*z^4)
True
sage: P.monomial_divides(x^3*y^2*z^4, x*y*z)
False
\end{verbatim}

\textbf{monomial\_lcm}(f, g)

LCM for monomials. Coefficients are ignored.

\textbf{INPUT}:

\begin{itemize}
\item f - monomial
\item g - monomial
\end{itemize}

\textbf{EXAMPLES}:

\begin{verbatim}
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: P = A.g_algebra(relations={y*x:-x*y}, order='lex')
sage: P.inject_variables()
Defining x, y, z

sage: P.monomial_lcm(3/2*x*y, x)
x*y
\end{verbatim}

\textbf{monomial\_pairwise\_prime}(g, h)

Return True if h and g are pairwise prime.

Both h and g are treated as monomials.

Coefficients are ignored.

\textbf{INPUT}:

\begin{itemize}
\item h - monomial
\item g - monomial
\end{itemize}
EXAMPLES:

```sage
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: P = A.g_algebra(relations={y*x:-x*y}, order='lex')
sage: P.inject_variables()
Defining x, y, z

sage: P.monomial_pairwise_prime(x^2*z^3, y^4)
True

sage: P.monomial_pairwise_prime(1/2*x^3*y^2, 3/4*y^3)
False
```

```
monomial_quotient(f, g, coeff=False)
Return f/g, where both f and g are treated as monomials.
Coefficients are ignored by default.

INPUT:
• f - monomial
• g - monomial
• coeff - divide coefficients as well (default: False)

EXAMPLES:

```sage
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: P = A.g_algebra(relations={y*x:-x*y}, order='lex')
sage: P.inject_variables()
Defining x, y, z

sage: P.monomial_quotient(3/2*x*y,x,coeff=True)
3/2*y
```

Note that Z behaves differently if coeff=True:

```
sage: P.monomial_quotient(2*x,3*x)
1
sage: P.monomial_quotient(2*x,3*x,coeff=True)
2/3
```

**Warning:** Assumes that the head term of f is a multiple of the head term of g and return the multiplicant m. If this rule is violated, funny things may happen.

```
monomial_reduce(f, G)
Try to find a g in G where g.lm() divides f. If found (flt, g) is returned, (0, 0) otherwise, where flt is f/g.lm().

It is assumed that G is iterable and contains only elements in this polynomial ring.
Coefficients are ignored.

INPUT:
• f - monomial
```
• G - list/set of mpolynomials

EXAMPLES:

```
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: P = A.g_algebra(relations={y*x:-x*y}, order='lex')
sage: P.inject_variables()
Defining x, y, z
sage: f = x*y^2
sage: G = [ 3/2*x^3 + y^2 + 1/2, 1/4*x*y + 2/7, 1/2 ]
sage: P.monomial_reduce(f,G)
(y, 1/4*x*y + 2/7)
```

ngens()
Returns the number of variables in this noncommutative polynomial ring.

EXAMPLES:

```
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: P.<x,y,z> = A.g_algebra(relations={y*x:-x*y}, order='lex')
sage: P.ngens()
3
```

relations(add_commutative=False)
Return the relations of this g-algebra.

INPUT:
add_commutative (optional bool, default False)

OUTPUT:
The defining relations. There are some implicit relations: Two generators commute if they are not part of any given relation. The implicit relations are not provided, unless add_commutative==True.

EXAMPLES:

```
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H.<x,y,z> = A.g_algebra({z*x:x*z+2*x, z*y:y*z-2*y})
sage: x*y == y*x
True
sage: H.relations()
{z*x: x*z + 2*x, z*y: y*z - 2*y}
sage: H.relations(add_commutative=True)
{y*x: x*y, z*x: x*z + 2*x, z*y: y*z - 2*y}
```

term_order()
Return the term ordering of the noncommutative ring.

EXAMPLES:

```
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: P = A.g_algebra(relations={y*x:-x*y}, order='lex')
sage: P.term_order()
Lexicographic term order
```

(continues on next page)
sage: P = A.g_algebra(relations={y*x:-x*y})
sage: P.term_order()
Degree reverse lexicographic term order

class sage.rings.polynomial.plural.NCPolynomial_plural
    Bases: RingElement

A noncommutative multivariate polynomial implemented using libSINGULAR.

coefficient(degrees)
    Return the coefficient of the variables with the degrees specified in the python dictionary degrees.

Mathematically, this is the coefficient in the base ring adjoined by the variables of this ring not listed in degrees. However, the result has the same parent as this polynomial.

This function contrasts with the function monomial_coefficient() which returns the coefficient in the base ring of a monomial.

INPUT:

• degrees - Can be any of:
  – a dictionary of degree restrictions
  – a list of degree restrictions (with None in the unrestricted variables)
  – a monomial (very fast, but not as flexible)

OUTPUT:

element of the parent of this element.

Note: For coefficients of specific monomials, look at monomial_coefficient().

EXAMPLES:

sage: A.<x,z,y> = FreeAlgebra(QQ, 3)
sage: R = A.g_algebra(relations={y*x:-x*y + z}, order='lex')
sage: R.inject_variables()
Defining x, z, y
sage: f=x*y+y+5
sage: f.coefficient({x:0,y:1})
1
sage: f.coefficient({x:0})
y + 5
sage: f=1+y+y^2*(1+x+x^2)
sage: f.coefficient({x:0})
z + y^2 + y + 1
sage: f.coefficient(x)
y^2 - y + 1
sage: f.coefficient([0,None]) # not tested
y^2 + y + 1

Be aware that this may not be what you think! The physical appearance of the variable x is deceiving – particularly if the exponent would be a variable.
sage: f.coefficient(x^0)  # outputs the full polynomial
x^2*y^2 + x^2*y + x^2 + x*y^2 - x*y + x + z + y^2 + y + 1

sage: A.<x,z,y> = FreeAlgebra(GF(389), 3)
sage: R = A.g_algebra(relations={y*x:-x*y + z}, order='lex')
sage: R.inject_variables()
Defining x, z, y
sage: f=x*y+5
sage: c=f.coefficient({x:0,y:0}); c
5
sage: parent(c)
Noncommutative Multivariate Polynomial Ring in x, z, y over Finite Field of size 389, nc-relations: {y*x: -x*y + z}

AUTHOR:
• Joel B. Mohler (2007-10-31)

custom_coefficient()

Return the constant coefficient of this multivariate polynomial.

EXAMPLES:

sage: A.<x,z,y> = FreeAlgebra(GF(389), 3)
sage: P = A.g_algebra(relations={y*x:-x*y + z}, order='lex')
sage: P.inject_variables()
Defining x, z, y
sage: f = y^2 - x^9 - x
sage: f.degree(x)
9

degree(x=None)

Return the maximal degree of this polynomial in x, where x must be one of the generators for the parent of this polynomial.

INPUT:
• x – multivariate polynomial (a generator of the parent of self) If x is not specified (or is None), return the total degree, which is the maximum degree of any monomial.

OUTPUT:
integer

EXAMPLES:

sage: A.<x,z,y> = FreeAlgebra(QQ, 3)
sage: R = A.g_algebra(relations={y*x:-x*y + z}, order='lex')
sage: R.inject_variables()
Defining x, z, y
sage: f = y^2 - x^9 - x
sage: f.degree(x)
9

(continues on next page)
degrees()
Returns a tuple with the maximal degree of each variable in this polynomial. The list of degrees is ordered by the order of the generators.
EXAMPLES:

```sage
sage: A.<y0,y1,y2> = FreeAlgebra(QQ, 3)
sage: R = A.g_algebra(relations={y1*y0:-y0*y1 + y2}, order='lex')
sage: R.inject_variables()
Defining y0, y1, y2
sage: q = 3*y0*y1*y1*y2; q
3*y0*y1^2*y2
sage: q.degrees()
(1, 2, 1)
sage: (q + y0^5).degrees()
(5, 2, 1)
```

dict()
Return a dictionary representing self. This dictionary is in the same format as the generic MPolynomial: The dictionary consists of ETuple:coefficient pairs.
EXAMPLES:

```sage
sage: A.<x,z,y> = FreeAlgebra(GF(389), 3)
sage: R = A.g_algebra(relations={y*x:-x*y + z}, order='lex')
sage: R.inject_variables()
Defining x, z, y
sage: f = x^3 + y + 2*z^2
sage: f.dict()
{(0, 0, 0): 2, (1, 2, 3): 2, (2, 0, 0): 7}
```

exponents(as_ETuples=True)
Return the exponents of the monomials appearing in this polynomial.
INPUT:
• as_ETuples - (default: True) if True returns the result as an list of ETuples otherwise returns a list of tuples
EXAMPLES:

```sage
sage: A.<x,z,y> = FreeAlgebra(GF(389), 3)
sage: R = A.g_algebra(relations={y*x:-x*y + z}, order='lex')
sage: R.inject_variables()
Defining x, z, y
sage: f = x^3 + y + 2*z^2
sage: f.exponents(as_ETuples=True)
[((0, 0, 0), 3), ((1, 2, 3), 2), ((2, 0, 0), 7)]
```
is_constant()  
Return True if this polynomial is constant.

EXAMPLES:

```python
sage: A.<x,z,y> = FreeAlgebra(GF(389), 3)
sage: P = A.g_algebra(relations={y*x:-x*y + z}, order='lex')
sage: P.inject_variables()
Defining x, z, y
sage: x.is_constant()
False
sage: P(1).is_constant()
True
```

is_homogeneous()  
Return True if this polynomial is homogeneous.

EXAMPLES:

```python
sage: A.<x,z,y> = FreeAlgebra(GF(389), 3)
sage: P = A.g_algebra(relations={y*x:-x*y + z}, order='lex')
sage: P.inject_variables()
Defining x, z, y
sage: (x+y+z).is_homogeneous()
True
sage: (x.parent()(0)).is_homogeneous()
True
sage: (x+y^2+z^3).is_homogeneous()
False
sage: (x^2 + y^2).is_homogeneous()
True
sage: (x^2 + y^2*x).is_homogeneous()
False
sage: (x^2*y + y^2*x).is_homogeneous()
True
```

is_monomial()  
Return True if this polynomial is a monomial.

A monomial is defined to be a product of generators with coefficient 1.

EXAMPLES:

```python
sage: A.<x,z,y> = FreeAlgebra(GF(389), 3)
sage: P = A.g_algebra(relations={y*x:-x*y + z}, order='lex')
sage: P.inject_variables()
Defining x, z, y
sage: x.is_monomial()
True
```
is_zero()  
Return True if this polynomial is zero.

EXAMPLES:

```sage
A.<x,z,y> = FreeAlgebra(QQ, 3)
R = A.g_algebra(relations={y*x:-x*y + z}, order='lex')
R.inject_variables()
Defining x, z, y
sage: x.is_zero()
False
sage: (x-x).is_zero()
True
```

lc()  
Leading coefficient of this polynomial with respect to the term order of self.parent().

EXAMPLES:

```sage
A.<x,y,z> = FreeAlgebra(GF(7), 3)
R = A.g_algebra(relations={y*x:-x*y + z}, order='lex')
R.inject_variables()
Defining x, y, z
sage: f = 3*x^1*y^2 + 2*y^3*z^4
sage: f.lc()
3
sage: f = 5*x^3*y^2*z^4 + 4*x^3*y^2*z^1
sage: f.lc()
5
```

lm()  
Returns the lead monomial of self with respect to the term order of self.parent().

In Sage a monomial is a product of variables in some power without a coefficient.

EXAMPLES:

```sage
A.<x,y,z> = FreeAlgebra(GF(7), 3)
R = A.g_algebra(relations={y*x:-x*y + z}, order='lex')
R.inject_variables()
Defining x, y, z
sage: f = x^1*y^2 + y^3*z^4
sage: f.lm()
x*y^2
```
lt()  
Leading term of this polynomial.

In Sage a term is a product of variables in some power and a coefficient.

EXAMPLES:

```
sage: A.<x,y,z> = FreeAlgebra(GF(7), 3)
sage: R = A.g_algebra(relations={y*x:-x*y + z}, order='lex')
sage: R.inject_variables()
Defining x, y, z
sage: f = 3*x^1*y^2 + 2*y^3*z^4
sage: f.lt()
3*x*y^2
sage: f = 5*x^3*y^2*z^4 + 4*x^3*y^2*z^1
sage: f.lt()
-2*x^3*y^2*z^4
```

monomial_coefficient (mon)

Return the coefficient in the base ring of the monomial mon in self, where mon must have the same parent as self.

This function contrasts with the function coefficient() which returns the coefficient of a monomial viewing this polynomial in a polynomial ring over a base ring having fewer variables.

INPUT:
**mon** - a monomial

OUTPUT:

coefficient in base ring

See also:

For coefficients in a base ring of fewer variables, look at `coefficient()`

EXAMPLES:

```python
sage: A.<x,z,y> = FreeAlgebra(GF(389), 3)
sage: P = A.g_algebra(relations={y*x:-x*y + z}, order='lex')
sage: P.inject_variables()
Defining x, z, y

The parent of the return is a member of the base ring.
sage: f = 2 * x * y
sage: c = f.monomial_coefficient(x*y); c
2
sage: c.parent()
Finite Field of size 389

sage: f = y^42 + y^2*x - x^9 - 7^x + 5^x*y
sage: f.monomial_coefficient(y^2)
1
sage: f.monomial_coefficient(x*y)
5
sage: f.monomial_coefficient(x^9)
388
sage: f.monomial_coefficient(x^10)
0
```

`monomials()`

Return the list of monomials in `self`

The returned list is decreasingly ordered by the term ordering of `self.parent()`.

EXAMPLES:

```python
sage: A.<x,z,y> = FreeAlgebra(GF(389), 3)
sage: P = A.g_algebra(relations={y*x:-x*y + z}, order='lex')
sage: P.inject_variables()
Defining x, z, y

sage: f = x + (3^2)*y*z^2 + (2+3)
sage: f.monomials()
[x, z^2*y, 1]
sage: f = P(3^2)
sage: f.monomials()
[1]
```

`reduce(I)`

EXAMPLES:
The result of reduction is not the normal form, if one reduces by a list of polynomials:

```
sage: (x*z).reduce(I.gens())
x*z
```

However, if the argument is an ideal, then a normal form (reduction with respect to a two-sided Groebner basis) is returned:

```
sage: (x*z).reduce(I)
-x
```

The Groebner basis shows that the result is correct:

```
sage: I.std() #random
Left Ideal (z^2 - 1, y*z - y, x*z + x, y^2, 2*x*y - z - 1, x^2) of
Noncommutative Multivariate Polynomial Ring in x, y, z over Rational
Field, nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - y}
sage: sorted(I.std().gens(),key=str)
[2*x*y - z - 1, x*z + x, x^2, y*z - y, y^2, z^2 - 1]
```

**total_degree()**

Return the total degree of self, which is the maximum degree of all monomials in self.

**EXAMPLES:**

```
sage: A.<x,z,y> = FreeAlgebra(QQ, 3)
sage: R = A.g_algebra(relations={y*x:-x*y + z}, order='lex')
sage: R.inject_variables()
Defining x, z, y
sage: f=2*x*y^3*z^2
sage: f.total_degree()
6
sage: f=4*x^2*y^2*z^3
sage: f.total_degree()
7
sage: f=99*x^6*y^3*z^9
sage: f.total_degree()
18
sage: f=x*y^3*z^6+3*x^2
sage: f.total_degree()
10
sage: f=z^3+8*x^4+4*y^5*z
sage: f.total_degree()
10
sage: f=z^9+10*x^4+y^8*x^2
sage: f.total_degree()
10
```

The Groebner basis shows that the result is correct:

```
sage: I.std() #random
Left Ideal (z^2 - 1, y*z - y, x*z + x, y^2, 2*x*y - z - 1, x^2) of
Noncommutative Multivariate Polynomial Ring in x, y, z over Rational
Field, nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - y}
sage: sorted(I.std().gens(),key=str)
[2*x*y - z - 1, x*z + x, x^2, y*z - y, y^2, z^2 - 1]
```

sage.rings.polynomial.plural.SCA(base_ring, names, alt_vars, order='degrevlex')

Return a free graded-commutative algebra
This is also known as a free super-commutative algebra.

**INPUT:**

- `base_ring` – the ground field
- `names` – a list of variable names
- `alt_vars` – a list of indices of to be anti-commutative variables (odd variables)
- `order` – ordering to be used for the constructed algebra

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.plural import SCA
sage: E = SCA(QQ, ['x', 'y', 'z'], [0, 1], order = 'degrevlex')
sage: E
Quotient of Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field, nc-relations: {y*x: -x*y} by the ideal (y^2, x^2)
sage: E.inject_variables()
Defining xbar, ybar, zbar
sage: x,y,z = (xbar,ybar,zbar)
sage: y*x
-x*y
sage: z*x
x*z
sage: x^2
0
sage: y^2
0
sage: z^2
z^2
sage: E.one()
1
```

`sage.rings.polynomial.plural.new_CRing(rw, base_ring)`

Construct MPolynomialRing_libsingular from ringWrap, assuming the ground field to be `base_ring`

**EXAMPLES:**

```python
sage: H.<x,y,z> = PolynomialRing(QQ, 3)
sage: from sage.libs.singular.function import singular_function
sage: ringlist = singular_function('ringlist')
sage: ring = singular_function("ring")
sage: L = ringlist(H, ring=H); L
[0, ['x', 'y', 'z'], [['dp', (1, 1, 1)], ['C', (0,)]], [0]]
sage: len(L)
4
sage: W = ring(L, ring=H); W
<RingWrap>
sage: from sage.rings.polynomial.plural import new_CRing
sage: R = new_CRing(W, H.base_ring())
```

(continues on next page)
sage: R # indirect doctest
Multivariate Polynomial Ring in x, y, z over Rational Field

Check that github issue #13145 has been resolved:

sage: h = hash(R.gen() + 1) # sets currRing
sage: from sage.libs.singular.ring import ring_refcount_dict, currRing_wrapper
sage: curcnt = ring_refcount_dict[currRing_wrapper()]
sage: newR = new_CRing(W, H.base_ring())
sage: ring_refcount_dict[currRing_wrapper()] - curcnt
2

Check that github issue #29311 is fixed:

sage: R.<x,y,z> = QQ[]
sage: from sage.libs.singular.function_factory import ff
sage: W = ff.ring(ff.ringlist(R), ring=R)

sage: C = sage.rings.polynomial.plural.new_CRing(W, R.base_ring())

sage: C.one()
1

sage.rings.polynomial.plural.new_NRing(rw, base_ring)

Construct NCPolynomialRing_plural from ringWrap, assuming the ground field to be base_ring

EXAMPLES:

sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H = A.g_algebra({y*x:x*y-1})
sage: H.inject_variables()
Defining x, y, z
sage: z^x
x^z
sage: z^y
y^z
sage: y^x
x^y - 1
sage: I = H.ideal([y^2, x^2, z^2-1])
sage: I._groebner_basis_libsingular()
[1]

sage: from sage.libs.singular.function import singular_function
sage: ringlist = singular_function('ringlist')
sage: ring = singular_function("ring")

sage: L = ringlist(H, ring=H); L
[
[0 1 1]
[0 0 1]
0, ['x', 'y', 'z'], [['dp', (1, 1, 1)], ['C', (0,)], [0], [0 0 0],
[ 0 -1 0] 0 0 0]
Constructs a Sage ring out of low level RingWrap, which wraps a pointer to a Singular ring.
The constructed ring is either commutative or noncommutative depending on the Singular ring.

EXAMPLES:

```python
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H = A.g_algebra({y*x:x*y-1})
sage: H.inject_variables()
Defining x, y, z
sage: z*x
x*z
sage: z*y
y*z
sage: y*x
x*y
sage: I = H.ideal([y^2, x^2, z^2-1])
sage: I._groebner_basis_libsingular()
[1]
sage: from sage.libs.singular.function import singular_function
sage: ringlist = singular_function('ringlist')
sage: ring = singular_function("ring")
sage: L = ringlist(H, ring=H); L
[ 0 0 0]
[ 0 -1 0]
[ 0 0 0]
[0 1 1]
[0 0 1]
0, ['x', 'y', 'z'], [['dp', (1, 1, 1)], ['C', (0,)], [0], [0 0 0],
[0 0 0],
]
sage: len(L)
6
```
sage: W = ring(L, ring=H); W
<noncommutative RingWrap>

sage: from sage.rings.polynomial.plural import new_Ring
sage: R = new_Ring(W, H.base_ring()); R
Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field, nc-
relations: {y*x: x*y - 1}

sage.rings.polynomial.plural.unpickle_NCPolynomial_plural(R, d)

Auxiliary function to unpickle a non-commutative polynomial.
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