Polynomials

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The Sage Development Team

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CONTENTS

CHAPTER

ONE

POLYNOMIAL RINGS

1.1 Constructors for polynomial rings

This module provides the function `PolynomialRing()`, which constructs rings of univariate and multivariate polynomials, and implements caching to prevent the same ring being created in memory multiple times (which is wasteful and breaks the general assumption in Sage that parents are unique).

There is also a function `BooleanPolynomialRing_constructor()`, used for constructing Boolean polynomial rings, which are not technically polynomial rings but rather quotients of them (see module `sage.rings.polynomial.pbori` for more details).

`sage.rings.polynomial.polynomial_ring_constructor.`**`BooleanPolynomialRing_constructor`**(*n=None*, *names=None*, *order='lex'*)

Construct a boolean polynomial ring with the following parameters:

INPUT:

- `n` – number of variables (an integer > 1)
- `names` – names of ring variables, may be a string or list/tuple of strings
- `order` – term order (default: `'lex'`)

EXAMPLES:

```
sage: # needs sage.rings.polynomial.pbori
sage: R.<x, y, z> = BooleanPolynomialRing(); R  # indirect doctest
Boolean PolynomialRing in x, y, z
sage: p = x*y + x*z + y*z
sage: x*p
x*y*z + x*y + x*z
sage: R.term_order()
Lexicographic term order

sage: R = BooleanPolynomialRing(5, 'x', order='deglex(3),deglex(2)')            #␣
↪needs sage.rings.polynomial.pbori
sage: R.term_order()                                                            #␣
↪needs sage.rings.polynomial.pbori
Block term order with blocks:
(Degree lexicographic term order of length 3,
 Degree lexicographic term order of length 2)
```

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```
sage: R = BooleanPolynomialRing(3, 'x', order='degneglex')                     #␣
↪needs sage.rings.polynomial.pbori
sage: R.term_order()                                                           #␣
↪needs sage.rings.polynomial.pbori
Degree negative lexicographic term order

sage: BooleanPolynomialRing(names=('x','y'))                                   #␣
↪needs sage.rings.polynomial.pbori
Boolean PolynomialRing in x, y

sage: BooleanPolynomialRing(names='x,y')                                       #␣
↪needs sage.rings.polynomial.pbori
Boolean PolynomialRing in x, y
```

```
>>> from sage.all import *
>>> # needs sage.rings.polynomial.pbori
>>> R = BooleanPolynomialRing(names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3); R   # indirect doctest
Boolean PolynomialRing in x, y, z
>>> p = x*y + x*z + y*z
>>> x*p
x*y*z + x*y + x*z
>>> R.term_order()
Lexicographic term order

>>> R = BooleanPolynomialRing(Integer(5), 'x', order='deglex(3),deglex(2)')      ␣
↪     # needs sage.rings.polynomial.pbori
>>> R.term_order()                                                            #␣
↪needs sage.rings.polynomial.pbori
Block term order with blocks:
(Degree lexicographic term order of length 3,
 Degree lexicographic term order of length 2)

>>> R = BooleanPolynomialRing(Integer(3), 'x', order='degneglex')                ␣
↪     # needs sage.rings.polynomial.pbori
>>> R.term_order()                                                            #␣
↪needs sage.rings.polynomial.pbori
Degree negative lexicographic term order

>>> BooleanPolynomialRing(names=('x','y'))                                    #␣
↪needs sage.rings.polynomial.pbori
Boolean PolynomialRing in x, y

>>> BooleanPolynomialRing(names='x,y')                                        #␣
↪needs sage.rings.polynomial.pbori
Boolean PolynomialRing in x, y
```

`sage.rings.polynomial.polynomial_ring_constructor.`**`PolynomialRing`**(*base_ring*, **args*, ***kwds*)

Return the globally unique univariate or multivariate polynomial ring with given properties and variable name or names.

There are many ways to specify the variables for the polynomial ring:

1. `PolynomialRing(base_ring, name, ...)`
2. `PolynomialRing(base_ring, names, ...)`
3. `PolynomialRing(base_ring, n, names, ...)`
4. `PolynomialRing(base_ring, n, ..., var_array=var_array, ...)`

The `...` at the end of these commands stands for additional keywords, like `sparse` or `order`.

INPUT:

- `base_ring` – a ring
- `n` – integer
- `name` – string
- `names` – list or tuple of names (strings), or a comma separated string
- `var_array` – list or tuple of names, or a comma separated string
- `sparse` – boolean; whether or not elements are sparse. The default is a dense representation (`sparse=False`) for univariate rings and a sparse representation (`sparse=True`) for multivariate rings.
- `order` – string or `TermOrder` object, e.g.,
  - `'degrevlex'` – default; degree reverse lexicographic
  - `'lex'` – lexicographic
  - `'deglex'` – degree lexicographic
  - `TermOrder('deglex',3) + TermOrder('deglex',3)` – block ordering
- `implementation` – string or None; selects an implementation in cases where Sage includes multiple choices (currently $\mathbf{Z}[x]$ can be implemented with `'NTL'` or `'FLINT'`; default is `'FLINT'`). For many base rings, the `'singular'` implementation is available. One can always specify `implementation="generic"` for a generic Sage implementation which does not use any specialized library.

Note

If the given implementation does not exist for rings with the given number of generators and the given sparsity, then an error results.

OUTPUT:

`PolynomialRing(base_ring, name, sparse=False)` returns a univariate polynomial ring; also, PolynomialRing(base_ring, names, sparse=False) yields a univariate polynomial ring, if names is a list or tuple providing exactly one name. All other input formats return a multivariate polynomial ring.

UNIQUENESS and IMMUTABILITY: In Sage there is exactly one single-variate polynomial ring over each base ring in each choice of variable, sparseness, and implementation. There is also exactly one multivariate polynomial ring over each base ring for each choice of names of variables and term order. The names of the generators can only be temporarily changed after the ring has been created. Do this using the `localvars()` context.

EXAMPLES:

1. PolynomialRing(base_ring, name, …)

```
sage: PolynomialRing(QQ, 'w')
Univariate Polynomial Ring in w over Rational Field
```

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```
sage: PolynomialRing(QQ, name='w')
Univariate Polynomial Ring in w over Rational Field
```

```
>>> from sage.all import *
>>> PolynomialRing(QQ, 'w')
Univariate Polynomial Ring in w over Rational Field
>>> PolynomialRing(QQ, name='w')
Univariate Polynomial Ring in w over Rational Field
```

Use the diamond brackets notation to make the variable ready for use after you define the ring:

```
sage: R.<w> = PolynomialRing(QQ)
sage: (1 + w)^3
w^3 + 3*w^2 + 3*w + 1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('w',)); (w,) = R._first_ngens(1)
>>> (Integer(1) + w)**Integer(3)
w^3 + 3*w^2 + 3*w + 1
```

You must specify a name:

```
sage: PolynomialRing(QQ)
Traceback (most recent call last):
...
TypeError: you must specify the names of the variables

sage: R.<abc> = PolynomialRing(QQ, sparse=True); R
Sparse Univariate Polynomial Ring in abc over Rational Field

sage: R.<w> = PolynomialRing(PolynomialRing(GF(7),'k')); R
Univariate Polynomial Ring in w over
 Univariate Polynomial Ring in k over Finite Field of size 7
```

```
>>> from sage.all import *
>>> PolynomialRing(QQ)
Traceback (most recent call last):
...
TypeError: you must specify the names of the variables

>>> R = PolynomialRing(QQ, sparse=True, names=('abc',)); (abc,) = R._first_
↪ngens(1); R
Sparse Univariate Polynomial Ring in abc over Rational Field

>>> R = PolynomialRing(PolynomialRing(GF(Integer(7)),'k'), names=('w',)); (w,) =␣
↪R._first_ngens(1); R
Univariate Polynomial Ring in w over
 Univariate Polynomial Ring in k over Finite Field of size 7
```

The square bracket notation:

```
sage: R.<y> = QQ['y']; R
Univariate Polynomial Ring in y over Rational Field
sage: y^2 + y
y^2 + y
```

```
>>> from sage.all import *
>>> R = QQ['y']; (y,) = R._first_ngens(1); R
Univariate Polynomial Ring in y over Rational Field
>>> y**Integer(2) + y
y^2 + y
```

In fact, since the diamond brackets on the left determine the variable name, you can omit the variable from the square brackets:

```
sage: R.<zz> = QQ[]; R
Univariate Polynomial Ring in zz over Rational Field
sage: (zz + 1)^2
zz^2 + 2*zz + 1
```

```
>>> from sage.all import *
>>> R = QQ['zz']; (zz,) = R._first_ngens(1); R
Univariate Polynomial Ring in zz over Rational Field
>>> (zz + Integer(1))**Integer(2)
zz^2 + 2*zz + 1
```

This is exactly the same ring as what PolynomialRing returns:

```
sage: R is PolynomialRing(QQ, 'zz')
True
```

```
>>> from sage.all import *
>>> R is PolynomialRing(QQ, 'zz')
True
```

However, rings with different variables are different:

```
sage: QQ['x'] == QQ['y']
False
```

```
>>> from sage.all import *
>>> QQ['x'] == QQ['y']
False
```

Sage has two implementations of univariate polynomials over the integers, one based on NTL and one based on FLINT. The default is FLINT. Note that FLINT uses a “more dense” representation for its polynomials than NTL, so in particular, creating a polynomial like 2^1000000 * x^1000000 in FLINT may be unwise.

```
sage: ZxNTL = PolynomialRing(ZZ, 'x', implementation='NTL'); ZxNTL              #␣
↪needs sage.libs.ntl
Univariate Polynomial Ring in x over Integer Ring (using NTL)
sage: ZxFLINT = PolynomialRing(ZZ, 'x', implementation='FLINT'); ZxFLINT        #␣
↪needs sage.libs.flint
```

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```
Univariate Polynomial Ring in x over Integer Ring
sage: ZxFLINT is ZZ['x']                                              #␣
↪needs sage.libs.flint
True
sage: ZxFLINT is PolynomialRing(ZZ, 'x')                              #␣
↪needs sage.libs.flint
True
sage: xNTL = ZxNTL.gen()                                              #␣
↪needs sage.libs.ntl
sage: xFLINT = ZxFLINT.gen()                                          #␣
↪needs sage.libs.flint
sage: xNTL.parent()                                                   #␣
↪needs sage.libs.ntl
Univariate Polynomial Ring in x over Integer Ring (using NTL)
sage: xFLINT.parent()                                                 #␣
↪needs sage.libs.flint
Univariate Polynomial Ring in x over Integer Ring
```

```
>>> from sage.all import *
>>> ZxNTL = PolynomialRing(ZZ, 'x', implementation='NTL'); ZxNTL     #␣
↪needs sage.libs.ntl
Univariate Polynomial Ring in x over Integer Ring (using NTL)
>>> ZxFLINT = PolynomialRing(ZZ, 'x', implementation='FLINT'); ZxFLINT      #␣
↪needs sage.libs.flint
Univariate Polynomial Ring in x over Integer Ring
>>> ZxFLINT is ZZ['x']                                               #␣
↪needs sage.libs.flint
True
>>> ZxFLINT is PolynomialRing(ZZ, 'x')                               #␣
↪needs sage.libs.flint
True
>>> xNTL = ZxNTL.gen()                                               #␣
↪needs sage.libs.ntl
>>> xFLINT = ZxFLINT.gen()                                           #␣
↪needs sage.libs.flint
>>> xNTL.parent()                                                    #␣
↪needs sage.libs.ntl
Univariate Polynomial Ring in x over Integer Ring (using NTL)
>>> xFLINT.parent()                                                  #␣
↪needs sage.libs.flint
Univariate Polynomial Ring in x over Integer Ring
```

There is a coercion from the non-default to the default implementation, so the values can be mixed in a single expression:

```
sage: (xNTL + xFLINT^2)                                               #␣
↪needs sage.libs.flint sage.libs.ntl
x^2 + x
```

```
>>> from sage.all import *
>>> (xNTL + xFLINT**Integer(2))                                        ␣
↪       # needs sage.libs.flint sage.libs.ntl
```

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```
x^2 + x
```

The result of such an expression will use the default, i.e., the FLINT implementation:

```
sage: (xNTL + xFLINT^2).parent()                                          #␣
↪needs sage.libs.flint sage.libs.ntl
Univariate Polynomial Ring in x over Integer Ring
```

```
>>> from sage.all import *
>>> (xNTL + xFLINT**Integer(2)).parent()                                   ␣
↪      # needs sage.libs.flint sage.libs.ntl
Univariate Polynomial Ring in x over Integer Ring
```

The generic implementation uses neither NTL nor FLINT:

```
sage: Zx = PolynomialRing(ZZ, 'x', implementation='generic'); Zx
Univariate Polynomial Ring in x over Integer Ring
sage: Zx.element_class
<... 'sage.rings.polynomial.polynomial_element.Polynomial_generic_dense'>
```

```
>>> from sage.all import *
>>> Zx = PolynomialRing(ZZ, 'x', implementation='generic'); Zx
Univariate Polynomial Ring in x over Integer Ring
>>> Zx.element_class
<... 'sage.rings.polynomial.polynomial_element.Polynomial_generic_dense'>
```

2. PolynomialRing(base_ring, names, ...)

```
sage: R = PolynomialRing(QQ, 'a,b,c'); R
Multivariate Polynomial Ring in a, b, c over Rational Field

sage: S = PolynomialRing(QQ, ['a','b','c']); S
Multivariate Polynomial Ring in a, b, c over Rational Field

sage: T = PolynomialRing(QQ, ('a','b','c')); T
Multivariate Polynomial Ring in a, b, c over Rational Field
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, 'a,b,c'); R
Multivariate Polynomial Ring in a, b, c over Rational Field

>>> S = PolynomialRing(QQ, ['a','b','c']); S
Multivariate Polynomial Ring in a, b, c over Rational Field

>>> T = PolynomialRing(QQ, ('a','b','c')); T
Multivariate Polynomial Ring in a, b, c over Rational Field
```

All three rings are identical:

```
sage: R is S
True
```

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```
sage: S is T
True
```

```
>>> from sage.all import *
>>> R is S
True
>>> S is T
True
```

There is a unique polynomial ring with each term order:

```
sage: R = PolynomialRing(QQ, 'x,y,z', order='degrevlex'); R
Multivariate Polynomial Ring in x, y, z over Rational Field
sage: S = PolynomialRing(QQ, 'x,y,z', order='invlex'); S
Multivariate Polynomial Ring in x, y, z over Rational Field
sage: S is PolynomialRing(QQ, 'x,y,z', order='invlex')
True
sage: R == S
False
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, 'x,y,z', order='degrevlex'); R
Multivariate Polynomial Ring in x, y, z over Rational Field
>>> S = PolynomialRing(QQ, 'x,y,z', order='invlex'); S
Multivariate Polynomial Ring in x, y, z over Rational Field
>>> S is PolynomialRing(QQ, 'x,y,z', order='invlex')
True
>>> R == S
False
```

Note that a univariate polynomial ring is returned, if the list of names is of length one. If it is of length zero, a multivariate polynomial ring with no variables is returned.

```
sage: PolynomialRing(QQ,["x"])
Univariate Polynomial Ring in x over Rational Field
sage: PolynomialRing(QQ,[])
Multivariate Polynomial Ring in no variables over Rational Field
```

```
>>> from sage.all import *
>>> PolynomialRing(QQ,["x"])
Univariate Polynomial Ring in x over Rational Field
>>> PolynomialRing(QQ,[])
Multivariate Polynomial Ring in no variables over Rational Field
```

The Singular implementation always returns a multivariate ring, even for 1 variable:

```
sage: PolynomialRing(QQ, "x", implementation='singular')                    #␣
↪needs sage.libs.singular
Multivariate Polynomial Ring in x over Rational Field
sage: P.<x> = PolynomialRing(QQ, implementation='singular'); P              #␣
↪needs sage.libs.singular
Multivariate Polynomial Ring in x over Rational Field
```

```
>>> from sage.all import *
>>> PolynomialRing(QQ, "x", implementation='singular')                          #␣
↪needs sage.libs.singular
Multivariate Polynomial Ring in x over Rational Field
>>> P = PolynomialRing(QQ, implementation='singular', names=('x',)); (x,) = P._
↪first_ngens(1); P                    # needs sage.libs.singular
Multivariate Polynomial Ring in x over Rational Field
```

3. PolynomialRing(base_ring, n, names, ...) (where the arguments `n` and `names` may be reversed)

If you specify a single name as a string and a number of variables, then variables labeled with numbers are created.

```
sage: PolynomialRing(QQ, 'x', 10)
Multivariate Polynomial Ring in x0, x1, x2, x3, x4, x5, x6, x7, x8, x9 over␣
↪Rational Field

sage: PolynomialRing(QQ, 2, 'alpha0')
Multivariate Polynomial Ring in alpha00, alpha01 over Rational Field

sage: PolynomialRing(GF(7), 'y', 5)
Multivariate Polynomial Ring in y0, y1, y2, y3, y4 over Finite Field of size 7

sage: PolynomialRing(QQ, 'y', 3, sparse=True)
Multivariate Polynomial Ring in y0, y1, y2 over Rational Field
```

```
>>> from sage.all import *
>>> PolynomialRing(QQ, 'x', Integer(10))
Multivariate Polynomial Ring in x0, x1, x2, x3, x4, x5, x6, x7, x8, x9 over␣
↪Rational Field

>>> PolynomialRing(QQ, Integer(2), 'alpha0')
Multivariate Polynomial Ring in alpha00, alpha01 over Rational Field

>>> PolynomialRing(GF(Integer(7)), 'y', Integer(5))
Multivariate Polynomial Ring in y0, y1, y2, y3, y4 over Finite Field of size 7

>>> PolynomialRing(QQ, 'y', Integer(3), sparse=True)
Multivariate Polynomial Ring in y0, y1, y2 over Rational Field
```

Note that a multivariate polynomial ring is returned when an explicit number is given.

```
sage: PolynomialRing(QQ,"x",1)
Multivariate Polynomial Ring in x over Rational Field
sage: PolynomialRing(QQ,"x",0)
Multivariate Polynomial Ring in no variables over Rational Field
```

```
>>> from sage.all import *
>>> PolynomialRing(QQ,"x",Integer(1))
Multivariate Polynomial Ring in x over Rational Field
>>> PolynomialRing(QQ,"x",Integer(0))
Multivariate Polynomial Ring in no variables over Rational Field
```

It is easy in Python to create fairly arbitrary variable names. For example, here is a ring with generators labeled by the primes less than 100:

```
sage: R = PolynomialRing(ZZ, ['x%s'%p for p in primes(100)]); R              #␣
↪needs sage.libs.pari
Multivariate Polynomial Ring in x2, x3, x5, x7, x11, x13, x17, x19, x23, x29,
 x31, x37, x41, x43, x47, x53, x59, x61, x67, x71, x73, x79, x83, x89, x97
 over Integer Ring
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, ['x%s'%p for p in primes(Integer(100))]); R      ␣
↪     # needs sage.libs.pari
Multivariate Polynomial Ring in x2, x3, x5, x7, x11, x13, x17, x19, x23, x29,
 x31, x37, x41, x43, x47, x53, x59, x61, x67, x71, x73, x79, x83, x89, x97
 over Integer Ring
```

By calling the `inject_variables()` method, all those variable names are available for interactive use:

```
sage: R.inject_variables()                                                   #␣
↪needs sage.libs.pari
Defining x2, x3, x5, x7, x11, x13, x17, x19, x23, x29, x31, x37, x41, x43,
x47, x53, x59, x61, x67, x71, x73, x79, x83, x89, x97
sage: (x2 + x41 + x71)^2                                                     #␣
↪needs sage.libs.pari
x2^2 + 2*x2*x41 + x41^2 + 2*x2*x71 + 2*x41*x71 + x71^2
```

```
>>> from sage.all import *
>>> R.inject_variables()                                                   #␣
↪needs sage.libs.pari
Defining x2, x3, x5, x7, x11, x13, x17, x19, x23, x29, x31, x37, x41, x43,
x47, x53, x59, x61, x67, x71, x73, x79, x83, x89, x97
>>> (x2 + x41 + x71)**Integer(2)                                             ␣
↪      # needs sage.libs.pari
x2^2 + 2*x2*x41 + x41^2 + 2*x2*x71 + 2*x41*x71 + x71^2
```

4. PolynomialRing(base_ring, n, …, var_array=var_array, …)

This creates an array of variables where each variables begins with an entry in `var_array` and is indexed from 0 to $n-1$.

```
sage: PolynomialRing(ZZ, 3, var_array=['x','y'])
Multivariate Polynomial Ring in x0, y0, x1, y1, x2, y2 over Integer Ring
sage: PolynomialRing(ZZ, 3, var_array='a,b')
Multivariate Polynomial Ring in a0, b0, a1, b1, a2, b2 over Integer Ring
```

```
>>> from sage.all import *
>>> PolynomialRing(ZZ, Integer(3), var_array=['x','y'])
Multivariate Polynomial Ring in x0, y0, x1, y1, x2, y2 over Integer Ring
>>> PolynomialRing(ZZ, Integer(3), var_array='a,b')
Multivariate Polynomial Ring in a0, b0, a1, b1, a2, b2 over Integer Ring
```

It is possible to create higher-dimensional arrays:

```
sage: PolynomialRing(ZZ, 2, 3, var_array=('p', 'q'))
Multivariate Polynomial Ring
 in p00, q00, p01, q01, p02, q02, p10, q10, p11, q11, p12, q12
```

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```
 over Integer Ring
sage: PolynomialRing(ZZ, 2, 3, 4, var_array='m')
Multivariate Polynomial Ring in m000, m001, m002, m003, m010, m011,
 m012, m013, m020, m021, m022, m023, m100, m101, m102, m103, m110,
 m111, m112, m113, m120, m121, m122, m123 over Integer Ring
```

```
>>> from sage.all import *
>>> PolynomialRing(ZZ, Integer(2), Integer(3), var_array=('p', 'q'))
Multivariate Polynomial Ring
 in p00, q00, p01, q01, p02, q02, p10, q10, p11, q11, p12, q12
 over Integer Ring
>>> PolynomialRing(ZZ, Integer(2), Integer(3), Integer(4), var_array='m')
Multivariate Polynomial Ring in m000, m001, m002, m003, m010, m011,
 m012, m013, m020, m021, m022, m023, m100, m101, m102, m103, m110,
 m111, m112, m113, m120, m121, m122, m123 over Integer Ring
```

The array is always at least 2-dimensional. So, if `var_array` is a single string and only a single number n is given, this creates an $n \times n$ array of variables:

```
sage: PolynomialRing(ZZ, 2, var_array='m')
Multivariate Polynomial Ring in m00, m01, m10, m11 over Integer Ring
```

```
>>> from sage.all import *
>>> PolynomialRing(ZZ, Integer(2), var_array='m')
Multivariate Polynomial Ring in m00, m01, m10, m11 over Integer Ring
```

Square brackets notation

You can alternatively create a polynomial ring over a ring R with square brackets:

```
sage: # needs sage.rings.real_mpfr
sage: RR["x"]
Univariate Polynomial Ring in x over Real Field with 53 bits of precision
sage: RR["x,y"]
Multivariate Polynomial Ring in x, y over Real Field with 53 bits of precision
sage: P.<x,y> = RR[]; P
Multivariate Polynomial Ring in x, y over Real Field with 53 bits of precision
```

```
>>> from sage.all import *
>>> # needs sage.rings.real_mpfr
>>> RR["x"]
Univariate Polynomial Ring in x over Real Field with 53 bits of precision
>>> RR["x,y"]
Multivariate Polynomial Ring in x, y over Real Field with 53 bits of precision
>>> P = RR['x, y']; (x, y,) = P._first_ngens(2); P
Multivariate Polynomial Ring in x, y over Real Field with 53 bits of precision
```

This notation does not allow to set any of the optional arguments.

Changing variable names

Consider

```
sage: R.<x,y> = PolynomialRing(QQ, 2); R
Multivariate Polynomial Ring in x, y over Rational Field
sage: f = x^2 - 2*y^2
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(2), names=('x', 'y',)); (x, y,) = R._first_
↪ngens(2); R
Multivariate Polynomial Ring in x, y over Rational Field
>>> f = x**Integer(2) - Integer(2)*y**Integer(2)
```

You can't just globally change the names of those variables. This is because objects all over Sage could have pointers to that polynomial ring.

```
sage: R._assign_names(['z','w'])
Traceback (most recent call last):
...
ValueError: variable names cannot be changed after object creation.
```

```
>>> from sage.all import *
>>> R._assign_names(['z','w'])
Traceback (most recent call last):
...
ValueError: variable names cannot be changed after object creation.
```

However, you can very easily change the names within a `with` block:

```
sage: with localvars(R, ['z','w']):
....:     print(f)
z^2 - 2*w^2
```

```
>>> from sage.all import *
>>> with localvars(R, ['z','w']):
...     print(f)
z^2 - 2*w^2
```

After the `with` block the names revert to what they were before:

```
sage: print(f)
x^2 - 2*y^2
```

```
>>> from sage.all import *
>>> print(f)
x^2 - 2*y^2
```

`sage.rings.polynomial.polynomial_ring_constructor.`**`polynomial_default_category`**`(`*n_variables*`)`

Choose an appropriate category for a polynomial ring.

It is assumed that the corresponding base ring is nonzero.

INPUT:

- `base_ring_category` – the category of ring over which the polynomial ring shall be defined
- `n_variables` – number of variables

EXAMPLES:

```
sage: from sage.rings.polynomial.polynomial_ring_constructor import polynomial_
↪default_category
sage: polynomial_default_category(Rings(), 1)
Category of infinite algebras with basis over rings
sage: polynomial_default_category(Rings().Commutative(), 1)
Category of infinite commutative algebras with basis
    over commutative rings
sage: polynomial_default_category(Fields(), 1)
Join of Category of euclidean domains
    and Category of algebras with basis over fields
    and Category of commutative algebras over fields
    and Category of infinite sets
sage: polynomial_default_category(Fields(), 2)
Join of Category of unique factorization domains
    and Category of algebras with basis over fields
    and Category of commutative algebras over fields
    and Category of infinite sets

sage: QQ['t'].category() is EuclideanDomains() & CommutativeAlgebras(QQ.
↪category()).WithBasis().Infinite()
True
sage: QQ['s','t'].category() is UniqueFactorizationDomains() &␣
↪CommutativeAlgebras(QQ.category()).WithBasis().Infinite()
True
sage: QQ['s']['t'].category() is UniqueFactorizationDomains() &␣
↪CommutativeAlgebras(QQ['s'].category()).WithBasis().Infinite()
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polynomial_ring_constructor import polynomial_
↪default_category
>>> polynomial_default_category(Rings(), Integer(1))
Category of infinite algebras with basis over rings
>>> polynomial_default_category(Rings().Commutative(), Integer(1))
Category of infinite commutative algebras with basis
    over commutative rings
>>> polynomial_default_category(Fields(), Integer(1))
Join of Category of euclidean domains
    and Category of algebras with basis over fields
    and Category of commutative algebras over fields
    and Category of infinite sets
>>> polynomial_default_category(Fields(), Integer(2))
Join of Category of unique factorization domains
    and Category of algebras with basis over fields
    and Category of commutative algebras over fields
    and Category of infinite sets

>>> QQ['t'].category() is EuclideanDomains() & CommutativeAlgebras(QQ.category()).
↪WithBasis().Infinite()
True
>>> QQ['s','t'].category() is UniqueFactorizationDomains() &␣
```

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```
↪CommutativeAlgebras(QQ.category()).WithBasis().Infinite()
True
>>> QQ['s']['t'].category() is UniqueFactorizationDomains() &␣
↪CommutativeAlgebras(QQ['s'].category()).WithBasis().Infinite()
True
```

sage.rings.polynomial.polynomial_ring_constructor.**unpickle_PolynomialRing**(*base_ring*, *arg1=None*, *arg2=None*, *sparse=False*)

Custom unpickling function for polynomial rings.

This has the same positional arguments as the old `PolynomialRing` constructor before Issue #23338.

CHAPTER

TWO

UNIVARIATE POLYNOMIALS

2.1 Univariate Polynomials and Polynomial Rings

Sage's architecture for polynomials 'under the hood' is complex, interfacing to a variety of C/C++ libraries for polynomials over specific rings. In practice, the user rarely has to worry about which backend is being used.

The hierarchy of class inheritance is somewhat confusing, since most of the polynomial element classes are implemented as Cython extension types rather than pure Python classes and thus can only inherit from a single base class, whereas others have multiple bases.

2.1.1 Univariate Polynomial Rings

Sage implements sparse and dense polynomials over commutative and non-commutative rings. In the non-commutative case, the polynomial variable commutes with the elements of the base ring.

AUTHOR:

- William Stein
- Kiran Kedlaya (2006-02-13): added macaulay2 option
- Martin Albrecht (2006-08-25): removed it again as it isn't needed anymore
- Simon King (2011-05): Dense and sparse polynomial rings must not be equal.
- Simon King (2011-10): Choice of categories for polynomial rings.

EXAMPLES:

```
sage: z = QQ['z'].0
sage: (z^3 + z - 1)^3
z^9 + 3*z^7 - 3*z^6 + 3*z^5 - 6*z^4 + 4*z^3 - 3*z^2 + 3*z - 1
```

```
>>> from sage.all import *
>>> z = QQ['z'].gen(0)
>>> (z**Integer(3) + z - Integer(1))**Integer(3)
z^9 + 3*z^7 - 3*z^6 + 3*z^5 - 6*z^4 + 4*z^3 - 3*z^2 + 3*z - 1
```

Saving and loading of polynomial rings works:

```
sage: loads(dumps(QQ['x'])) == QQ['x']
True
sage: k = PolynomialRing(QQ['x'],'y'); loads(dumps(k))==k
True
sage: k = PolynomialRing(ZZ,'y'); loads(dumps(k)) == k
```

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```
True
sage: k = PolynomialRing(ZZ,'y', sparse=True); loads(dumps(k))
Sparse Univariate Polynomial Ring in y over Integer Ring
```

```
>>> from sage.all import *
>>> loads(dumps(QQ['x'])) == QQ['x']
True
>>> k = PolynomialRing(QQ['x'],'y'); loads(dumps(k))==k
True
>>> k = PolynomialRing(ZZ,'y'); loads(dumps(k)) == k
True
>>> k = PolynomialRing(ZZ,'y', sparse=True); loads(dumps(k))
Sparse Univariate Polynomial Ring in y over Integer Ring
```

Rings with different variable names are not equal; in fact, by Issue #9944, polynomial rings are equal if and only if they are identical (which should be the case for all parent structures in Sage):

```
sage: QQ['y'] != QQ['x']
True
sage: QQ['y'] != QQ['z']
True
```

```
>>> from sage.all import *
>>> QQ['y'] != QQ['x']
True
>>> QQ['y'] != QQ['z']
True
```

We create a polynomial ring over a quaternion algebra:

```
sage: # needs sage.combinat sage.modules
sage: A.<i,j,k> = QuaternionAlgebra(QQ, -1,-1)
sage: R.<w> = PolynomialRing(A, sparse=True)
sage: f = w^3 + (i+j)*w + 1
sage: f
w^3 + (i + j)*w + 1
sage: f^2
w^6 + (2*i + 2*j)*w^4 + 2*w^3 - 2*w^2 + (2*i + 2*j)*w + 1
sage: f = w + i ; g = w + j
sage: f * g
w^2 + (i + j)*w + k
sage: g * f
w^2 + (i + j)*w - k
```

```
>>> from sage.all import *
>>> # needs sage.combinat sage.modules
>>> A = QuaternionAlgebra(QQ, -Integer(1),-Integer(1), names=('i', 'j', 'k',)); (i, j,
↪ k,) = A._first_ngens(3)
>>> R = PolynomialRing(A, sparse=True, names=('w',)); (w,) = R._first_ngens(1)
>>> f = w**Integer(3) + (i+j)*w + Integer(1)
>>> f
w^3 + (i + j)*w + 1
```

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```
>>> f**Integer(2)
w^6 + (2*i + 2*j)*w^4 + 2*w^3 - 2*w^2 + (2*i + 2*j)*w + 1
>>> f = w + i ; g = w + j
>>> f * g
w^2 + (i + j)*w + k
>>> g * f
w^2 + (i + j)*w - k
```

Issue #9944 introduced some changes related with coercion. Previously, a dense and a sparse polynomial ring with the same variable name over the same base ring evaluated equal, but of course they were not identical. Coercion maps are cached - but if a coercion to a dense ring is requested and a coercion to a sparse ring is returned instead (since the cache keys are equal!), all hell breaks loose.

Therefore, the coercion between rings of sparse and dense polynomials works as follows:

```
sage: R.<x> = PolynomialRing(QQ, sparse=True)
sage: S.<x> = QQ[]
sage: S == R
False
sage: S.has_coerce_map_from(R)
True
sage: R.has_coerce_map_from(S)
False
sage: (R.0 + S.0).parent()
Univariate Polynomial Ring in x over Rational Field
sage: (S.0 + R.0).parent()
Univariate Polynomial Ring in x over Rational Field
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, sparse=True, names=('x',)); (x,) = R._first_ngens(1)
>>> S = QQ['x']; (x,) = S._first_ngens(1)
>>> S == R
False
>>> S.has_coerce_map_from(R)
True
>>> R.has_coerce_map_from(S)
False
>>> (R.gen(0) + S.gen(0)).parent()
Univariate Polynomial Ring in x over Rational Field
>>> (S.gen(0) + R.gen(0)).parent()
Univariate Polynomial Ring in x over Rational Field
```

It may be that one has rings of dense or sparse polynomials over different base rings. In that situation, coercion works by means of the `pushout()` formalism:

```
sage: R.<x> = PolynomialRing(GF(5), sparse=True)
sage: S.<x> = PolynomialRing(ZZ)
sage: R.has_coerce_map_from(S)
False
sage: S.has_coerce_map_from(R)
False
sage: S.0 + R.0
2*x
```

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```
sage: (S.0 + R.0).parent()
Univariate Polynomial Ring in x over Finite Field of size 5
sage: (S.0 + R.0).parent().is_sparse()
False
```

```
>>> from sage.all import *
>>> R = PolynomialRing(GF(Integer(5)), sparse=True, names=('x',)); (x,) = R._first_
↪ngens(1)
>>> S = PolynomialRing(ZZ, names=('x',)); (x,) = S._first_ngens(1)
>>> R.has_coerce_map_from(S)
False
>>> S.has_coerce_map_from(R)
False
>>> S.gen(0) + R.gen(0)
2*x
>>> (S.gen(0) + R.gen(0)).parent()
Univariate Polynomial Ring in x over Finite Field of size 5
>>> (S.gen(0) + R.gen(0)).parent().is_sparse()
False
```

Similarly, there is a coercion from the (non-default) NTL implementation for univariate polynomials over the integers to the default FLINT implementation, but not vice versa:

```
sage: R.<x> = PolynomialRing(ZZ, implementation='NTL')                         #␣
↪needs sage.libs.ntl
sage: S.<x> = PolynomialRing(ZZ, implementation='FLINT')
sage: (S.0 + R.0).parent() is S                                                #␣
↪needs sage.libs.flint sage.libs.ntl
True
sage: (R.0 + S.0).parent() is S                                                #␣
↪needs sage.libs.flint sage.libs.ntl
True
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, implementation='NTL', names=('x',)); (x,) = R._first_
↪ngens(1)# needs sage.libs.ntl
>>> S = PolynomialRing(ZZ, implementation='FLINT', names=('x',)); (x,) = S._first_
↪ngens(1)
>>> (S.gen(0) + R.gen(0)).parent() is S                                        ␣
↪       # needs sage.libs.flint sage.libs.ntl
True
>>> (R.gen(0) + S.gen(0)).parent() is S                                        ␣
↪       # needs sage.libs.flint sage.libs.ntl
True
```

class sage.rings.polynomial.polynomial_ring.**PolynomialRing_cdvf**(*base_ring*, *name=None*, *sparse=False*, *implementation=None*, *element_class=None*, *category=None*)

Bases: *PolynomialRing_cdvr*, *PolynomialRing_field*

A class for polynomial ring over complete discrete valuation fields

class sage.rings.polynomial.polynomial_ring.**PolynomialRing_cdvr**(*base_ring*, *name=None*, *sparse=False*, *implementation=None*, *element_class=None*, *category=None*)

Bases: *PolynomialRing_integral_domain*

A class for polynomial ring over complete discrete valuation rings

class sage.rings.polynomial.polynomial_ring.**PolynomialRing_commutative**(*base_ring*, *name=None*, *sparse=False*, *implementation=None*, *element_class=None*, *category=None*)

Bases: *PolynomialRing_generic*

Univariate polynomial ring over a commutative ring.

quotient_by_principal_ideal(*f*, *names=None*, ***kwds*)

Return the quotient of this polynomial ring by the principal ideal (generated by) f.

INPUT:

- f – either a polynomial in self, or a principal ideal of self
- further named arguments that are passed to the quotient constructor

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: I = (x^2 - 1) * R
sage: R.quotient_by_principal_ideal(I)                                       #␣
↪needs sage.libs.pari
Univariate Quotient Polynomial Ring in xbar
 over Rational Field with modulus x^2 - 1
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> I = (x**Integer(2) - Integer(1)) * R
>>> R.quotient_by_principal_ideal(I)                                         #␣
↪needs sage.libs.pari
Univariate Quotient Polynomial Ring in xbar
 over Rational Field with modulus x^2 - 1
```

The same example, using the polynomial instead of the ideal, and customizing the variable name:

```
sage: R.<x> = QQ[]
sage: R.quotient_by_principal_ideal(x^2 - 1, names=('foo',))                 #␣
↪needs sage.libs.pari
Univariate Quotient Polynomial Ring in foo
 over Rational Field with modulus x^2 - 1
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> R.quotient_by_principal_ideal(x**Integer(2) - Integer(1), names=('foo',))␣
```

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```
↪                # needs sage.libs.pari
Univariate Quotient Polynomial Ring in foo
 over Rational Field with modulus x^2 - 1
```

weyl_algebra()

Return the Weyl algebra generated from `self`.

EXAMPLES:

```
sage: R = QQ['x']
sage: W = R.weyl_algebra(); W                                                   #␣
↪needs sage.modules
Differential Weyl algebra of polynomials in x over Rational Field
sage: W.polynomial_ring() == R                                                  #␣
↪needs sage.modules
True
```

```
>>> from sage.all import *
>>> R = QQ['x']
>>> W = R.weyl_algebra(); W                                                   #␣
↪needs sage.modules
Differential Weyl algebra of polynomials in x over Rational Field
>>> W.polynomial_ring() == R                                                  #␣
↪needs sage.modules
True
```

class sage.rings.polynomial.polynomial_ring.**PolynomialRing_dense_finite_field**(*base_ring*, *name='x'*, *element_class=None*, *implementation=None*)

Bases: *PolynomialRing_field*

Univariate polynomial ring over a finite field.

EXAMPLES:

```
sage: R = PolynomialRing(GF(27, 'a'), 'x')                                      #␣
↪needs sage.rings.finite_rings
sage: type(R)                                                                   #␣
↪needs sage.rings.finite_rings
<class 'sage.rings.polynomial.polynomial_ring.PolynomialRing_dense_finite_field_
↪with_category'>
```

```
>>> from sage.all import *
>>> R = PolynomialRing(GF(Integer(27), 'a'), 'x')                              ␣
↪     # needs sage.rings.finite_rings
>>> type(R)                                                                   #␣
↪needs sage.rings.finite_rings
<class 'sage.rings.polynomial.polynomial_ring.PolynomialRing_dense_finite_field_
↪with_category'>
```

irreducible_element(*n*, *algorithm=None*)

Construct a monic irreducible polynomial of degree n.

INPUT:

- `n` – integer; degree of the polynomial to construct
- `algorithm` – string (algorithm to use) or `None`:
  - `'random'` or `None`: try random polynomials until an irreducible one is found
  - `'first_lexicographic'`: try polynomials in lexicographic order until an irreducible one is found

OUTPUT: a monic irreducible polynomial of degree n in `self`

EXAMPLES:

```
sage: # needs sage.modules sage.rings.finite_rings
sage: f = GF(5^3, 'a')['x'].irreducible_element(2)
sage: f.degree()
2
sage: f.is_irreducible()
True
sage: R = GF(19)['x']
sage: R.irreducible_element(21, algorithm='first_lexicographic')
x^21 + x + 5
sage: R = GF(5**2, 'a')['x']
sage: R.irreducible_element(17, algorithm='first_lexicographic')
x^17 + a*x + 4*a + 3
```

```
>>> from sage.all import *
>>> # needs sage.modules sage.rings.finite_rings
>>> f = GF(Integer(5)**Integer(3), 'a')['x'].irreducible_element(Integer(2))
>>> f.degree()
2
>>> f.is_irreducible()
True
>>> R = GF(Integer(19))['x']
>>> R.irreducible_element(Integer(21), algorithm='first_lexicographic')
x^21 + x + 5
>>> R = GF(Integer(5)**Integer(2), 'a')['x']
>>> R.irreducible_element(Integer(17), algorithm='first_lexicographic')
x^17 + a*x + 4*a + 3
```

AUTHORS:

- Peter Bruin (June 2013)
- Jean-Pierre Flori (May 2014)

class sage.rings.polynomial.polynomial_ring.**PolynomialRing_dense_mod_n**(*base_ring*, *name=None*, *element_class=None*, *implementation=None*, *category=None*)

Bases: *PolynomialRing_commutative*

modulus()

EXAMPLES:

```
sage: R.<x> = Zmod(15)[]
sage: R.modulus()
15
```

```
>>> from sage.all import *
>>> R = Zmod(Integer(15))['x']; (x,) = R._first_ngens(1)
>>> R.modulus()
15
```

residue_field(*ideal*, *names=None*)

Return the residue finite field at the given ideal.

EXAMPLES:

```
sage: # needs sage.libs.ntl
sage: R.<t> = GF(2)[]
sage: k.<a> = R.residue_field(t^3 + t + 1); k
Residue field in a
 of Principal ideal (t^3 + t + 1) of Univariate Polynomial Ring in t
 over Finite Field of size 2 (using GF2X)
sage: k.list()
[0, a, a^2, a + 1, a^2 + a, a^2 + a + 1, a^2 + 1, 1]
sage: R.residue_field(t)
Residue field of Principal ideal (t) of Univariate Polynomial Ring in t
 over Finite Field of size 2 (using GF2X)
sage: P = R.irreducible_element(8) * R
sage: P
Principal ideal (t^8 + t^4 + t^3 + t^2 + 1) of Univariate Polynomial Ring in t
 over Finite Field of size 2 (using GF2X)
sage: k.<a> = R.residue_field(P); k
Residue field in a
 of Principal ideal (t^8 + t^4 + t^3 + t^2 + 1) of Univariate Polynomial Ring␣
↪in t
 over Finite Field of size 2 (using GF2X)
sage: k.cardinality()
256
```

```
>>> from sage.all import *
>>> # needs sage.libs.ntl
>>> R = GF(Integer(2))['t']; (t,) = R._first_ngens(1)
>>> k = R.residue_field(t**Integer(3) + t + Integer(1), names=('a',)); (a,) =␣
↪k._first_ngens(1); k
Residue field in a
 of Principal ideal (t^3 + t + 1) of Univariate Polynomial Ring in t
 over Finite Field of size 2 (using GF2X)
>>> k.list()
[0, a, a^2, a + 1, a^2 + a, a^2 + a + 1, a^2 + 1, 1]
>>> R.residue_field(t)
Residue field of Principal ideal (t) of Univariate Polynomial Ring in t
 over Finite Field of size 2 (using GF2X)
```

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```
>>> P = R.irreducible_element(Integer(8)) * R
>>> P
Principal ideal (t^8 + t^4 + t^3 + t^2 + 1) of Univariate Polynomial Ring in t
 over Finite Field of size 2 (using GF2X)
>>> k = R.residue_field(P, names=('a',)); (a,) = k._first_ngens(1); k
Residue field in a
 of Principal ideal (t^8 + t^4 + t^3 + t^2 + 1) of Univariate Polynomial Ring␣
↪in t
 over Finite Field of size 2 (using GF2X)
>>> k.cardinality()
256
```

Non-maximal ideals are not accepted:

```
sage: # needs sage.libs.ntl
sage: R.residue_field(t^2 + 1)
Traceback (most recent call last):
...
ArithmeticError: ideal is not maximal
sage: R.residue_field(0)
Traceback (most recent call last):
...
ArithmeticError: ideal is not maximal
sage: R.residue_field(1)
Traceback (most recent call last):
...
ArithmeticError: ideal is not maximal
```

```
>>> from sage.all import *
>>> # needs sage.libs.ntl
>>> R.residue_field(t**Integer(2) + Integer(1))
Traceback (most recent call last):
...
ArithmeticError: ideal is not maximal
>>> R.residue_field(Integer(0))
Traceback (most recent call last):
...
ArithmeticError: ideal is not maximal
>>> R.residue_field(Integer(1))
Traceback (most recent call last):
...
ArithmeticError: ideal is not maximal
```

class sage.rings.polynomial.polynomial_ring.**PolynomialRing_dense_mod_p**(*base_ring*, *name='x'*, *implementation=None*, *element_class=None*, *category=None*)

Bases: *PolynomialRing_dense_finite_field*, *PolynomialRing_dense_mod_n*, *PolynomialRing_singular_repr*

fraction_field()

Return the fraction field of `self`.

EXAMPLES:

```
sage: R.<t> = GF(5)[]
sage: R.fraction_field()
Fraction Field of Univariate Polynomial Ring in t
 over Finite Field of size 5
```

```
>>> from sage.all import *
>>> R = GF(Integer(5))['t']; (t,) = R._first_ngens(1)
>>> R.fraction_field()
Fraction Field of Univariate Polynomial Ring in t
 over Finite Field of size 5
```

irreducible_element(*n*, *algorithm=None*)

Construct a monic irreducible polynomial of degree n.

INPUT:

- `n` – integer; the degree of the polynomial to construct
- `algorithm` – string (algorithm to use) or `None`; currently available options are:
  - `'adleman-lenstra'`: a variant of the Adleman–Lenstra algorithm as implemented in PARI.
  - `'conway'`: look up the Conway polynomial of degree n over the field of p elements in the database; raise a `RuntimeError` if it is not found.
  - `'ffprimroot'`: use the pari:ffprimroot function from PARI.
  - `'first_lexicographic'`: return the lexicographically smallest irreducible polynomial of degree n.
  - `'minimal_weight'`: return an irreducible polynomial of degree n with minimal number of non-zero coefficients. Only implemented for $p = 2$.
  - `'primitive'`: return a polynomial f such that a root of f generates the multiplicative group of the finite field extension defined by f. This uses the Conway polynomial if possible, otherwise it uses `'ffprimroot'`.
  - `'random'`: try random polynomials until an irreducible one is found.

  If `algorithm` is `None`, use $x - 1$ in degree 1. In degree > 1, the Conway polynomial is used if it is found in the database. Otherwise, the algorithm `minimal_weight` is used if $p = 2$, and the algorithm `'adleman-lenstra'` if $p > 2$.

OUTPUT: a monic irreducible polynomial of degree n in `self`

EXAMPLES:

```
sage: # needs sage.rings.finite_rings
sage: GF(5)['x'].irreducible_element(2)
x^2 + 4*x + 2
sage: GF(5)['x'].irreducible_element(2, algorithm='adleman-lenstra')
x^2 + x + 1
sage: GF(5)['x'].irreducible_element(2, algorithm='primitive')
x^2 + 4*x + 2
sage: GF(5)['x'].irreducible_element(32, algorithm='first_lexicographic')
x^32 + 2
sage: GF(5)['x'].irreducible_element(32, algorithm='conway')
```

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```
Traceback (most recent call last):
...
RuntimeError: requested Conway polynomial not in database.
sage: GF(5)['x'].irreducible_element(32, algorithm='primitive')
x^32 + ...
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> GF(Integer(5))['x'].irreducible_element(Integer(2))
x^2 + 4*x + 2
>>> GF(Integer(5))['x'].irreducible_element(Integer(2), algorithm='adleman-
↪lenstra')
x^2 + x + 1
>>> GF(Integer(5))['x'].irreducible_element(Integer(2), algorithm='primitive')
x^2 + 4*x + 2
>>> GF(Integer(5))['x'].irreducible_element(Integer(32), algorithm='first_
↪lexicographic')
x^32 + 2
>>> GF(Integer(5))['x'].irreducible_element(Integer(32), algorithm='conway')
Traceback (most recent call last):
...
RuntimeError: requested Conway polynomial not in database.
>>> GF(Integer(5))['x'].irreducible_element(Integer(32), algorithm='primitive
↪')
x^32 + ...
```

In characteristic 2:

```
sage: GF(2)['x'].irreducible_element(33)                                    #␣
↪needs sage.rings.finite_rings
x^33 + x^13 + x^12 + x^11 + x^10 + x^8 + x^6 + x^3 + 1
sage: GF(2)['x'].irreducible_element(33, algorithm='minimal_weight')        #␣
↪needs sage.rings.finite_rings
x^33 + x^10 + 1
```

```
>>> from sage.all import *
>>> GF(Integer(2))['x'].irreducible_element(Integer(33))                     ␣
↪              # needs sage.rings.finite_rings
x^33 + x^13 + x^12 + x^11 + x^10 + x^8 + x^6 + x^3 + 1
>>> GF(Integer(2))['x'].irreducible_element(Integer(33), algorithm='minimal_
↪weight')         # needs sage.rings.finite_rings
x^33 + x^10 + 1
```

In degree 1:

```
sage: GF(97)['x'].irreducible_element(1)                                    #␣
↪needs sage.rings.finite_rings
x + 96
sage: GF(97)['x'].irreducible_element(1, algorithm='conway')                #␣
↪needs sage.rings.finite_rings
x + 92
sage: GF(97)['x'].irreducible_element(1, algorithm='adleman-lenstra')       #␣
```

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```
↪needs sage.rings.finite_rings
x
```

```
>>> from sage.all import *
>>> GF(Integer(97))['x'].irreducible_element(Integer(1))                     ␣
↪               # needs sage.rings.finite_rings
x + 96
>>> GF(Integer(97))['x'].irreducible_element(Integer(1), algorithm='conway') ␣
↪               # needs sage.rings.finite_rings
x + 92
>>> GF(Integer(97))['x'].irreducible_element(Integer(1), algorithm='adleman-
↪lenstra')        # needs sage.rings.finite_rings
x
```

AUTHORS:

- Peter Bruin (June 2013)
- Jeroen Demeyer (September 2014): add "ffprimroot" algorithm, see Issue #8373.

class sage.rings.polynomial.polynomial_ring.**PolynomialRing_dense_padic_field_capped_relative**(*base_ring, name=None, implementation=None, element_class=None, category=None*)

Bases: *PolynomialRing_dense_padic_field_generic*

class sage.rings.polynomial.polynomial_ring.**PolynomialRing_dense_padic_field_generic**(*base_ring, name=None, implementation=None, element_class=None, category=None*)

Bases: *PolynomialRing_cdvf*

A class for dense polynomial ring over p-adic fields

class sage.rings.polynomial.polynomial_ring.**PolynomialRing_dense_padic_ring_capped_absolute**(*base_ring*, *name=None*, *implementation=None*, *element_class=None*, *category=None*)

Bases: `PolynomialRing_dense_padic_ring_generic`

class sage.rings.polynomial.polynomial_ring.**PolynomialRing_dense_padic_ring_capped_relative**(*base_ring*, *name=None*, *implementation=None*, *element_class=None*, *category=None*)

Bases: `PolynomialRing_dense_padic_ring_generic`

class sage.rings.polynomial.polynomial_ring.**PolynomialRing_dense_padic_ring_fixed_mod**(*base_ring*, *name=None*, *implementation=None*, *element_class=None*, *category=None*)

Bases: `PolynomialRing_dense_padic_ring_generic`

class sage.rings.polynomial.polynomial_ring.**PolynomialRing_dense_padic_ring_generic**(*base_ring*, *name=None*, *implementation=None*, *element_class=None*, *category=None*)

Bases: *PolynomialRing_cdvr*

A class for dense polynomial ring over p-adic rings

class sage.rings.polynomial.polynomial_ring.**PolynomialRing_field**(*base_ring*, *name='x'*, *sparse=False*, *implementation=None*, *element_class=None*, *category=None*)

Bases: *PolynomialRing_integral_domain*

divided_difference(*points*, *full_table=False*)

Return the Newton divided-difference coefficients of the Lagrange interpolation polynomial through `points`.

INPUT:

- `points` – list of pairs $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ of elements of the base ring of `self`, where $x_i - x_j$ is invertible for $i \neq j$. This method converts the x_i and y_i into the base ring of `self`.
- `full_table` – boolean (default: `False`); if `True`, return the full divided-difference table. If `False`, only return entries along the main diagonal; these are the Newton divided-difference coefficients $F_{i,i}$.

OUTPUT:

The Newton divided-difference coefficients of the n-th Lagrange interpolation polynomial $P_n(x)$ that passes through the points in `points` (see *lagrange_polynomial()*). These are the coefficients $F_{0,0}, F_{1,1}, \dots, F_{n,n}$ in the base ring of `self` such that

$$P_n(x) = \sum_{i=0}^{n} F_{i,i} \prod_{j=0}^{i-1} (x - x_j)$$

EXAMPLES:

Only return the divided-difference coefficients $F_{i,i}$. This example is taken from Example 1, page 121 of [BF2005]:

```
sage: # needs sage.rings.real_mpfr
sage: points = [(1.0, 0.7651977), (1.3, 0.6200860), (1.6, 0.4554022),
....:           (1.9, 0.2818186), (2.2, 0.1103623)]
sage: R = PolynomialRing(RR, "x")
sage: R.divided_difference(points)
[0.765197700000000,
 -0.483705666666666,
 -0.108733888888889,
 0.0658783950617283,
 0.00182510288066044]
```

```
>>> from sage.all import *
>>> # needs sage.rings.real_mpfr
>>> points = [(RealNumber('1.0'), RealNumber('0.7651977')), (RealNumber('1.3
↪'), RealNumber('0.6200860')), (RealNumber('1.6'), RealNumber('0.4554022')),
...           (RealNumber('1.9'), RealNumber('0.2818186')), (RealNumber('2.2
↪'), RealNumber('0.1103623'))]
>>> R = PolynomialRing(RR, "x")
>>> R.divided_difference(points)
[0.765197700000000,
 -0.483705666666666,
 -0.108733888888889,
 0.0658783950617283,
 0.00182510288066044]
```

Now return the full divided-difference table:

```
sage: # needs sage.rings.real_mpfr
sage: points = [(1.0, 0.7651977), (1.3, 0.6200860), (1.6, 0.4554022),
....:           (1.9, 0.2818186), (2.2, 0.1103623)]
sage: R = PolynomialRing(RR, "x")
sage: R.divided_difference(points, full_table=True)
[[0.765197700000000],
 [0.620086000000000, -0.483705666666666],
 [0.455402200000000, -0.548946000000000, -0.108733888888889],
 [0.281818600000000, -0.578612000000000,
                     -0.0494433333333339, 0.0658783950617283],
 [0.110362300000000, -0.571520999999999, 0.0118183333333349,
                     0.0680685185185209, 0.00182510288066044]]
```

```
>>> from sage.all import *
>>> # needs sage.rings.real_mpfr
>>> points = [(RealNumber('1.0'), RealNumber('0.7651977')), (RealNumber('1.3
↪'), RealNumber('0.6200860')), (RealNumber('1.6'), RealNumber('0.4554022')),
...           (RealNumber('1.9'), RealNumber('0.2818186')), (RealNumber('2.2
↪'), RealNumber('0.1103623'))]
>>> R = PolynomialRing(RR, "x")
>>> R.divided_difference(points, full_table=True)
[[0.765197700000000],
 [0.620086000000000, -0.483705666666666],
 [0.455402200000000, -0.548946000000000, -0.108733888888889],
 [0.281818600000000, -0.578612000000000,
                     -0.0494433333333339, 0.0658783950617283],
 [0.110362300000000, -0.571520999999999, 0.0118183333333349,
                     0.0680685185185209, 0.00182510288066044]]
```

The following example is taken from Example 4.12, page 225 of [MF1999]:

```
sage: points = [(1, -3), (2, 0), (3, 15), (4, 48), (5, 105), (6, 192)]
sage: R = PolynomialRing(QQ, "x")
sage: R.divided_difference(points)
[-3, 3, 6, 1, 0, 0]
sage: R.divided_difference(points, full_table=True)
[[-3],
```

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```
 [0, 3],
 [15, 15, 6],
 [48, 33, 9, 1],
 [105, 57, 12, 1, 0],
 [192, 87, 15, 1, 0, 0]]
```

```
>>> from sage.all import *
>>> points = [(Integer(1), -Integer(3)), (Integer(2), Integer(0)),
↪(Integer(3), Integer(15)), (Integer(4), Integer(48)), (Integer(5),
↪Integer(105)), (Integer(6), Integer(192))]
>>> R = PolynomialRing(QQ, "x")
>>> R.divided_difference(points)
[-3, 3, 6, 1, 0, 0]
>>> R.divided_difference(points, full_table=True)
[[-3],
 [0, 3],
 [15, 15, 6],
 [48, 33, 9, 1],
 [105, 57, 12, 1, 0],
 [192, 87, 15, 1, 0, 0]]
```

fraction_field()

Return the fraction field of `self`.

EXAMPLES:

```
sage: QQbar['x'].fraction_field()
Fraction Field of Univariate Polynomial Ring in x over Algebraic
Field
```

```
>>> from sage.all import *
>>> QQbar['x'].fraction_field()
Fraction Field of Univariate Polynomial Ring in x over Algebraic
Field
```

lagrange_polynomial(*points*, *algorithm='divided_difference'*, *previous_row=None*)

Return the Lagrange interpolation polynomial through the given points.

INPUT:

- `points` – list of pairs $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ of elements of the base ring of `self`, where $x_i - x_j$ is invertible for $i \neq j$. This method converts the x_i and y_i into the base ring of `self`.
- `algorithm` – (default: `'divided_difference'`) one of the following:
  - `'divided_difference'`: use the method of divided differences.
  - `'neville'`: adapt Neville's method as described on page 144 of [BF2005] to recursively generate the Lagrange interpolation polynomial. Neville's method generates a table of approximating polynomials, where the last row of that table contains the n-th Lagrange interpolation polynomial. The adaptation implemented by this method is to only generate the last row of this table, instead of the full table itself. Generating the full table can be memory inefficient.
- `previous_row` – (default: `None`) this option is only relevant if used with `algorithm='neville'`. If provided, this should be the last row of the table resulting from a previous use of Neville's method.

If such a row is passed, then `points` should consist of both previous and new interpolating points. Neville's method will then use that last row and the interpolating points to generate a new row containing an interpolation polynomial for the new points.

OUTPUT:

The Lagrange interpolation polynomial through the points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$. This is the unique polynomial P_n of degree at most n in `self` satisfying $P_n(x_i) = y_i$ for $0 \leq i \leq n$.

EXAMPLES:

By default, we use the method of divided differences:

```
sage: R = PolynomialRing(QQ, 'x')
sage: f = R.lagrange_polynomial([(0,1), (2,2), (3,-2), (-4,9)]); f
-23/84*x^3 - 11/84*x^2 + 13/7*x + 1
sage: f(0)
1
sage: f(2)
2
sage: f(3)
-2
sage: f(-4)
9

sage: # needs sage.rings.finite_rings
sage: R = PolynomialRing(GF(2**3, 'a'), 'x')
sage: a = R.base_ring().gen()
sage: f = R.lagrange_polynomial([(a^2+a, a), (a, 1), (a^2, a^2+a+1)]); f
a^2*x^2 + a^2*x + a^2
sage: f(a^2 + a)
a
sage: f(a)
1
sage: f(a^2)
a^2 + a + 1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, 'x')
>>> f = R.lagrange_polynomial([(Integer(0),Integer(1)), (Integer(2),
↪Integer(2)), (Integer(3),-Integer(2)), (-Integer(4),Integer(9))]); f
-23/84*x^3 - 11/84*x^2 + 13/7*x + 1
>>> f(Integer(0))
1
>>> f(Integer(2))
2
>>> f(Integer(3))
-2
>>> f(-Integer(4))
9

>>> # needs sage.rings.finite_rings
>>> R = PolynomialRing(GF(Integer(2)**Integer(3), 'a'), 'x')
>>> a = R.base_ring().gen()
>>> f = R.lagrange_polynomial([(a**Integer(2)+a, a), (a, Integer(1)),␣
```

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```
↪(a**Integer(2), a**Integer(2)+a+Integer(1))]); f
a^2*x^2 + a^2*x + a^2
>>> f(a**Integer(2) + a)
a
>>> f(a)
1
>>> f(a**Integer(2))
a^2 + a + 1
```

Now use a memory efficient version of Neville's method:

```
sage: R = PolynomialRing(QQ, 'x')
sage: R.lagrange_polynomial([(0,1), (2,2), (3,-2), (-4,9)],
....:                       algorithm='neville')
[9,
-11/7*x + 19/7,
-17/42*x^2 - 83/42*x + 53/7,
-23/84*x^3 - 11/84*x^2 + 13/7*x + 1]

sage: # needs sage.rings.finite_rings
sage: R = PolynomialRing(GF(2**3, 'a'), 'x')
sage: a = R.base_ring().gen()
sage: R.lagrange_polynomial([(a^2+a, a), (a, 1), (a^2, a^2+a+1)],
....:                       algorithm='neville')
[a^2 + a + 1, x + a + 1, a^2*x^2 + a^2*x + a^2]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, 'x')
>>> R.lagrange_polynomial([(Integer(0),Integer(1)), (Integer(2),Integer(2)),␣
↪(Integer(3),-Integer(2)), (-Integer(4),Integer(9))],
...                       algorithm='neville')
[9,
-11/7*x + 19/7,
-17/42*x^2 - 83/42*x + 53/7,
-23/84*x^3 - 11/84*x^2 + 13/7*x + 1]

>>> # needs sage.rings.finite_rings
>>> R = PolynomialRing(GF(Integer(2)**Integer(3), 'a'), 'x')
>>> a = R.base_ring().gen()
>>> R.lagrange_polynomial([(a**Integer(2)+a, a), (a, Integer(1)),␣
↪(a**Integer(2), a**Integer(2)+a+Integer(1))],
...                       algorithm='neville')
[a^2 + a + 1, x + a + 1, a^2*x^2 + a^2*x + a^2]
```

Repeated use of Neville's method to get better Lagrange interpolation polynomials:

```
sage: R = PolynomialRing(QQ, 'x')
sage: p = R.lagrange_polynomial([(0,1), (2,2)], algorithm='neville')
sage: R.lagrange_polynomial([(0,1), (2,2), (3,-2), (-4,9)],
....:                       algorithm='neville', previous_row=p)[-1]
-23/84*x^3 - 11/84*x^2 + 13/7*x + 1
```

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```
sage: # needs sage.rings.finite_rings
sage: R = PolynomialRing(GF(2**3, 'a'), 'x')
sage: a = R.base_ring().gen()
sage: p = R.lagrange_polynomial([(a^2+a, a), (a, 1)], algorithm='neville')
sage: R.lagrange_polynomial([(a^2+a, a), (a, 1), (a^2, a^2+a+1)],
....:                       algorithm='neville', previous_row=p)[-1]
a^2*x^2 + a^2*x + a^2
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, 'x')
>>> p = R.lagrange_polynomial([(Integer(0),Integer(1)), (Integer(2),
↪Integer(2))], algorithm='neville')
>>> R.lagrange_polynomial([(Integer(0),Integer(1)), (Integer(2),Integer(2)),␣
↪(Integer(3),-Integer(2)), (-Integer(4),Integer(9))],
...                       algorithm='neville', previous_row=p)[-Integer(1)]
-23/84*x^3 - 11/84*x^2 + 13/7*x + 1

>>> # needs sage.rings.finite_rings
>>> R = PolynomialRing(GF(Integer(2)**Integer(3), 'a'), 'x')
>>> a = R.base_ring().gen()
>>> p = R.lagrange_polynomial([(a**Integer(2)+a, a), (a, Integer(1))],␣
↪algorithm='neville')
>>> R.lagrange_polynomial([(a**Integer(2)+a, a), (a, Integer(1)),␣
↪(a**Integer(2), a**Integer(2)+a+Integer(1))],
...                       algorithm='neville', previous_row=p)[-Integer(1)]
a^2*x^2 + a^2*x + a^2
```

sage.rings.polynomial.polynomial_ring.**PolynomialRing_general**

alias of *PolynomialRing_generic*

class sage.rings.polynomial.polynomial_ring.**PolynomialRing_generic**(*base_ring*, *name=None*, *sparse=False*, *implementation=None*, *element_class=None*, *category=None*)

Bases: Ring

Univariate polynomial ring over a ring.

base_extend(R)

Return the base extension of this polynomial ring to R.

EXAMPLES:

```
sage: # needs sage.rings.real_mpfr
sage: R.<x> = RR[]; R
Univariate Polynomial Ring in x over Real Field with 53 bits of precision
sage: R.base_extend(CC)
Univariate Polynomial Ring in x over Complex Field with 53 bits of precision
sage: R.base_extend(QQ)
Traceback (most recent call last):
...
TypeError: no such base extension
```

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```
sage: R.change_ring(QQ)
Univariate Polynomial Ring in x over Rational Field
```

```
>>> from sage.all import *
>>> # needs sage.rings.real_mpfr
>>> R = RR['x']; (x,) = R._first_ngens(1); R
Univariate Polynomial Ring in x over Real Field with 53 bits of precision
>>> R.base_extend(CC)
Univariate Polynomial Ring in x over Complex Field with 53 bits of precision
>>> R.base_extend(QQ)
Traceback (most recent call last):
...
TypeError: no such base extension
>>> R.change_ring(QQ)
Univariate Polynomial Ring in x over Rational Field
```

change_ring(*R*)

Return the polynomial ring in the same variable as `self` over R.

EXAMPLES:

```
sage: # needs sage.rings.finite_rings sage.rings.real_interval_field
sage: R.<ZZZ> = RealIntervalField()[]; R
Univariate Polynomial Ring in ZZZ over
 Real Interval Field with 53 bits of precision
sage: R.change_ring(GF(19^2, 'b'))
Univariate Polynomial Ring in ZZZ over Finite Field in b of size 19^2
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings sage.rings.real_interval_field
>>> R = RealIntervalField()['ZZZ']; (ZZZ,) = R._first_ngens(1); R
Univariate Polynomial Ring in ZZZ over
 Real Interval Field with 53 bits of precision
>>> R.change_ring(GF(Integer(19)**Integer(2), 'b'))
Univariate Polynomial Ring in ZZZ over Finite Field in b of size 19^2
```

change_var(*var*)

Return the polynomial ring in variable `var` over the same base ring.

EXAMPLES:

```
sage: R.<x> = ZZ[]; R
Univariate Polynomial Ring in x over Integer Ring
sage: R.change_var('y')
Univariate Polynomial Ring in y over Integer Ring
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1); R
Univariate Polynomial Ring in x over Integer Ring
>>> R.change_var('y')
Univariate Polynomial Ring in y over Integer Ring
```

characteristic()

Return the characteristic of this polynomial ring, which is the same as that of its base ring.

EXAMPLES:

```
sage: # needs sage.rings.real_interval_field
sage: R.<ZZZ> = RealIntervalField()[]; R
Univariate Polynomial Ring in ZZZ over Real Interval Field with 53 bits of
↪precision
sage: R.characteristic()
0
sage: S = R.change_ring(GF(19^2, 'b')); S                                    #
↪needs sage.rings.finite_rings
Univariate Polynomial Ring in ZZZ over Finite Field in b of size 19^2
sage: S.characteristic()                                                      #
↪needs sage.rings.finite_rings
19
```

```
>>> from sage.all import *
>>> # needs sage.rings.real_interval_field
>>> R = RealIntervalField()['ZZZ']; (ZZZ,) = R._first_ngens(1); R
Univariate Polynomial Ring in ZZZ over Real Interval Field with 53 bits of
↪precision
>>> R.characteristic()
0
>>> S = R.change_ring(GF(Integer(19)**Integer(2), 'b')); S
↪                # needs sage.rings.finite_rings
Univariate Polynomial Ring in ZZZ over Finite Field in b of size 19^2
>>> S.characteristic()                                                      #
↪needs sage.rings.finite_rings
19
```

completion(*p=None*, *prec=20*, *extras=None*)

Return the completion of `self` with respect to the irreducible polynomial `p`.

Currently only implemented for `p=self.gen()` (the default), i.e. you can only complete $R[x]$ with respect to x, the result being a ring of power series in x. The `prec` variable controls the precision used in the power series ring. If `prec` is ∞, then this returns a `LazyPowerSeriesRing`.

EXAMPLES:

```
sage: P.<x> = PolynomialRing(QQ)
sage: P
Univariate Polynomial Ring in x over Rational Field
sage: PP = P.completion(x)
sage: PP
Power Series Ring in x over Rational Field
sage: f = 1 - x
sage: PP(f)
1 - x
sage: 1 / f
-1/(x - 1)
sage: g = 1 / PP(f); g
1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^10 + x^11
```

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```
 + x^12 + x^13 + x^14 + x^15 + x^16 + x^17 + x^18 + x^19 + O(x^20)
sage: 1 / g
1 - x + O(x^20)

sage: # needs sage.combinat
sage: PP = P.completion(x, prec=oo); PP
Lazy Taylor Series Ring in x over Rational Field
sage: g = 1 / PP(f); g
1 + x + x^2 + O(x^3)
sage: 1 / g == f
True
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, names=('x',)); (x,) = P._first_ngens(1)
>>> P
Univariate Polynomial Ring in x over Rational Field
>>> PP = P.completion(x)
>>> PP
Power Series Ring in x over Rational Field
>>> f = Integer(1) - x
>>> PP(f)
1 - x
>>> Integer(1) / f
-1/(x - 1)
>>> g = Integer(1) / PP(f); g
1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^10 + x^11
 + x^12 + x^13 + x^14 + x^15 + x^16 + x^17 + x^18 + x^19 + O(x^20)
>>> Integer(1) / g
1 - x + O(x^20)

>>> # needs sage.combinat
>>> PP = P.completion(x, prec=oo); PP
Lazy Taylor Series Ring in x over Rational Field
>>> g = Integer(1) / PP(f); g
1 + x + x^2 + O(x^3)
>>> Integer(1) / g == f
True
```

construction()

Return the construction functor.

cyclotomic_polynomial(n)

Return the n-th cyclotomic polynomial as a polynomial in this polynomial ring. For details of the implementation, see the documentation for *sage.rings.polynomial.cyclotomic.cyclotomic_coeffs()*.

EXAMPLES:

```
sage: R = ZZ['x']
sage: R.cyclotomic_polynomial(8)
x^4 + 1
sage: R.cyclotomic_polynomial(12)
x^4 - x^2 + 1
```

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```
sage: S = PolynomialRing(FiniteField(7), 'x')
sage: S.cyclotomic_polynomial(12)
x^4 + 6*x^2 + 1
sage: S.cyclotomic_polynomial(1)
x + 6
```

```
>>> from sage.all import *
>>> R = ZZ['x']
>>> R.cyclotomic_polynomial(Integer(8))
x^4 + 1
>>> R.cyclotomic_polynomial(Integer(12))
x^4 - x^2 + 1

>>> S = PolynomialRing(FiniteField(Integer(7)), 'x')
>>> S.cyclotomic_polynomial(Integer(12))
x^4 + 6*x^2 + 1
>>> S.cyclotomic_polynomial(Integer(1))
x + 6
```

extend_variables(*added_names*, *order='degrevlex'*)

Return a multivariate polynomial ring with the same base ring but with `added_names` as additional variables.

EXAMPLES:

```
sage: R.<x> = ZZ[]; R
Univariate Polynomial Ring in x over Integer Ring
sage: R.extend_variables('y, z')
Multivariate Polynomial Ring in x, y, z over Integer Ring
sage: R.extend_variables(('y', 'z'))
Multivariate Polynomial Ring in x, y, z over Integer Ring
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1); R
Univariate Polynomial Ring in x over Integer Ring
>>> R.extend_variables('y, z')
Multivariate Polynomial Ring in x, y, z over Integer Ring
>>> R.extend_variables(('y', 'z'))
Multivariate Polynomial Ring in x, y, z over Integer Ring
```

flattening_morphism()

Return the flattening morphism of this polynomial ring.

EXAMPLES:

```
sage: QQ['a','b']['x'].flattening_morphism()
Flattening morphism:
  From: Univariate Polynomial Ring in x over
        Multivariate Polynomial Ring in a, b over Rational Field
  To:   Multivariate Polynomial Ring in a, b, x over Rational Field

sage: QQ['x'].flattening_morphism()
Identity endomorphism of Univariate Polynomial Ring in x over Rational Field
```

```
>>> from sage.all import *
>>> QQ['a','b']['x'].flattening_morphism()
Flattening morphism:
  From: Univariate Polynomial Ring in x over
        Multivariate Polynomial Ring in a, b over Rational Field
  To:   Multivariate Polynomial Ring in a, b, x over Rational Field

>>> QQ['x'].flattening_morphism()
Identity endomorphism of Univariate Polynomial Ring in x over Rational Field
```

gen (*n=0*)

Return the indeterminate generator of this polynomial ring.

EXAMPLES:

```
sage: R.<abc> = Integers(8)[]; R
Univariate Polynomial Ring in abc over Ring of integers modulo 8
sage: t = R.gen(); t
abc
sage: t.is_gen()
True
```

```
>>> from sage.all import *
>>> R = Integers(Integer(8))['abc']; (abc,) = R._first_ngens(1); R
Univariate Polynomial Ring in abc over Ring of integers modulo 8
>>> t = R.gen(); t
abc
>>> t.is_gen()
True
```

An identical generator is always returned.

```
sage: t is R.gen()
True
```

```
>>> from sage.all import *
>>> t is R.gen()
True
```

gens_dict ()

Return a dictionary whose entries are `{name:variable,...}`, where `name` stands for the variable names of this object (as strings) and `variable` stands for the corresponding generators (as elements of this object).

EXAMPLES:

```
sage: R.<y,x,a42> = RR[]
sage: R.gens_dict()
{'a42': a42, 'x': x, 'y': y}
```

```
>>> from sage.all import *
>>> R = RR['y, x, a42']; (y, x, a42,) = R._first_ngens(3)
>>> R.gens_dict()
{'a42': a42, 'x': x, 'y': y}
```

is_exact()

EXAMPLES:

```
sage: class Foo:
....:     def __init__(self, x):
....:         self._x = x
....:     @cached_method
....:     def f(self):
....:         return self._x^2
sage: a = Foo(2)
sage: print(a.f.cache)
None
sage: a.f()
4
sage: a.f.cache
4
```

```
>>> from sage.all import *
>>> class Foo:
...     def __init__(self, x):
...         self._x = x
...     @cached_method
...     def f(self):
...         return self._x**Integer(2)
>>> a = Foo(Integer(2))
>>> print(a.f.cache)
None
>>> a.f()
4
>>> a.f.cache
4
```

is_field(*proof=True*)

Return `False`, since polynomial rings are never fields.

EXAMPLES:

```
sage: # needs sage.libs.ntl
sage: R.<z> = Integers(2)[]; R
Univariate Polynomial Ring in z over Ring of integers modulo 2 (using GF2X)
sage: R.is_field()
False
```

```
>>> from sage.all import *
>>> # needs sage.libs.ntl
>>> R = Integers(Integer(2))['z']; (z,) = R._first_ngens(1); R
Univariate Polynomial Ring in z over Ring of integers modulo 2 (using GF2X)
>>> R.is_field()
False
```

is_integral_domain(*proof=True*)

EXAMPLES:

```
sage: ZZ['x'].is_integral_domain()
True
sage: Integers(8)['x'].is_integral_domain()
False
```

```
>>> from sage.all import *
>>> ZZ['x'].is_integral_domain()
True
>>> Integers(Integer(8))['x'].is_integral_domain()
False
```

is_noetherian()

is_sparse()

Return `True` if elements of this polynomial ring have a sparse representation.

EXAMPLES:

```
sage: R.<z> = Integers(8)[]; R
Univariate Polynomial Ring in z over Ring of integers modulo 8
sage: R.is_sparse()
False
sage: R.<W> = PolynomialRing(QQ, sparse=True); R
Sparse Univariate Polynomial Ring in W over Rational Field
sage: R.is_sparse()
True
```

```
>>> from sage.all import *
>>> R = Integers(Integer(8))['z']; (z,) = R._first_ngens(1); R
Univariate Polynomial Ring in z over Ring of integers modulo 8
>>> R.is_sparse()
False
>>> R = PolynomialRing(QQ, sparse=True, names=('W',)); (W,) = R._first_
↪ngens(1); R
Sparse Univariate Polynomial Ring in W over Rational Field
>>> R.is_sparse()
True
```

is_unique_factorization_domain(*proof=True*)

EXAMPLES:

```
sage: ZZ['x'].is_unique_factorization_domain()
True
sage: Integers(8)['x'].is_unique_factorization_domain()
False
```

```
>>> from sage.all import *
>>> ZZ['x'].is_unique_factorization_domain()
True
>>> Integers(Integer(8))['x'].is_unique_factorization_domain()
False
```

karatsuba_threshold()

Return the Karatsuba threshold used for this ring by the method `_mul_karatsuba()` to fall back to the schoolbook algorithm.

EXAMPLES:

```
sage: K = QQ['x']
sage: K.karatsuba_threshold()
8
sage: K = QQ['x']['y']
sage: K.karatsuba_threshold()
0
```

```
>>> from sage.all import *
>>> K = QQ['x']
>>> K.karatsuba_threshold()
8
>>> K = QQ['x']['y']
>>> K.karatsuba_threshold()
0
```

krull_dimension()

Return the Krull dimension of this polynomial ring, which is one more than the Krull dimension of the base ring.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: R.krull_dimension()
1

sage: # needs sage.rings.finite_rings
sage: R.<z> = GF(9, 'a')[]; R
Univariate Polynomial Ring in z over Finite Field in a of size 3^2
sage: R.krull_dimension()
1
sage: S.<t> = R[]
sage: S.krull_dimension()
2
sage: for n in range(10):
....:     S = PolynomialRing(S, 'w')
sage: S.krull_dimension()
12
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> R.krull_dimension()
1

>>> # needs sage.rings.finite_rings
>>> R = GF(Integer(9), 'a')['z']; (z,) = R._first_ngens(1); R
Univariate Polynomial Ring in z over Finite Field in a of size 3^2
>>> R.krull_dimension()
1
```

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```
>>> S = R['t']; (t,) = S._first_ngens(1)
>>> S.krull_dimension()
2
>>> for n in range(Integer(10)):
...     S = PolynomialRing(S, 'w')
>>> S.krull_dimension()
12
```

monics(*of_degree=None*, *max_degree=None*)

Return an iterator over the monic polynomials of specified degree.

INPUT: Pass exactly one of:

- `max_degree` – an int; the iterator will generate all monic polynomials which have degree less than or equal to `max_degree`
- `of_degree` – an int; the iterator will generate all monic polynomials which have degree `of_degree`

OUTPUT: an iterator

EXAMPLES:

```
sage: # needs sage.rings.finite_rings
sage: P = PolynomialRing(GF(4, 'a'), 'y')
sage: for p in P.monics(of_degree=2): print(p)
y^2
y^2 + a
y^2 + a + 1
y^2 + 1
y^2 + a*y
y^2 + a*y + a
y^2 + a*y + a + 1
y^2 + a*y + 1
y^2 + (a + 1)*y
y^2 + (a + 1)*y + a
y^2 + (a + 1)*y + a + 1
y^2 + (a + 1)*y + 1
y^2 + y
y^2 + y + a
y^2 + y + a + 1
y^2 + y + 1
sage: for p in P.monics(max_degree=1): print(p)
1
y
y + a
y + a + 1
y + 1
sage: for p in P.monics(max_degree=1, of_degree=3): print(p)
Traceback (most recent call last):
...
ValueError: you should pass exactly one of of_degree and max_degree
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
```

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```
>>> P = PolynomialRing(GF(Integer(4), 'a'), 'y')
>>> for p in P.monics(of_degree=Integer(2)): print(p)
y^2
y^2 + a
y^2 + a + 1
y^2 + 1
y^2 + a*y
y^2 + a*y + a
y^2 + a*y + a + 1
y^2 + a*y + 1
y^2 + (a + 1)*y
y^2 + (a + 1)*y + a
y^2 + (a + 1)*y + a + 1
y^2 + (a + 1)*y + 1
y^2 + y
y^2 + y + a
y^2 + y + a + 1
y^2 + y + 1
>>> for p in P.monics(max_degree=Integer(1)): print(p)
1
y
y + a
y + a + 1
y + 1
>>> for p in P.monics(max_degree=Integer(1), of_degree=Integer(3)): print(p)
Traceback (most recent call last):
...
ValueError: you should pass exactly one of of_degree and max_degree
```

AUTHORS:

- Joel B. Mohler

monomial(*exponent*)

Return the monomial with the `exponent`.

INPUT:

- `exponent` – nonnegative integer

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ)
sage: R.monomial(5)
x^5
sage: e=(10,)
sage: R.monomial(*e)
x^10
sage: m = R.monomial(100)
sage: R.monomial(m.degree()) == m
True
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
```

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```
>>> R.monomial(Integer(5))
x^5
>>> e=(Integer(10),)
>>> R.monomial(*e)
x^10
>>> m = R.monomial(Integer(100))
>>> R.monomial(m.degree()) == m
True
```

ngens()

Return the number of generators of this polynomial ring, which is 1 since it is a univariate polynomial ring.

EXAMPLES:

```
sage: R.<z> = Integers(8)[]; R
Univariate Polynomial Ring in z over Ring of integers modulo 8
sage: R.ngens()
1
```

```
>>> from sage.all import *
>>> R = Integers(Integer(8))['z']; (z,) = R._first_ngens(1); R
Univariate Polynomial Ring in z over Ring of integers modulo 8
>>> R.ngens()
1
```

parameter()

Return the generator of this polynomial ring.

This is the same as `self.gen()`.

polynomials(*of_degree=None*, *max_degree=None*)

Return an iterator over the polynomials of specified degree.

INPUT: Pass exactly one of:

- `max_degree` – an int; the iterator will generate all polynomials which have degree less than or equal to `max_degree`
- `of_degree` – an int; the iterator will generate all polynomials which have degree `of_degree`

OUTPUT: an iterator

EXAMPLES:

```
sage: P = PolynomialRing(GF(3), 'y')
sage: for p in P.polynomials(of_degree=2): print(p)
y^2
y^2 + 1
y^2 + 2
y^2 + y
y^2 + y + 1
y^2 + y + 2
y^2 + 2*y
y^2 + 2*y + 1
y^2 + 2*y + 2
```

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```
2*y^2
2*y^2 + 1
2*y^2 + 2
2*y^2 + y
2*y^2 + y + 1
2*y^2 + y + 2
2*y^2 + 2*y
2*y^2 + 2*y + 1
2*y^2 + 2*y + 2
sage: for p in P.polynomials(max_degree=1): print(p)
0
1
2
y
y + 1
y + 2
2*y
2*y + 1
2*y + 2
sage: for p in P.polynomials(max_degree=1, of_degree=3): print(p)
Traceback (most recent call last):
...
ValueError: you should pass exactly one of of_degree and max_degree
```

```
>>> from sage.all import *
>>> P = PolynomialRing(GF(Integer(3)), 'y')
>>> for p in P.polynomials(of_degree=Integer(2)): print(p)
y^2
y^2 + 1
y^2 + 2
y^2 + y
y^2 + y + 1
y^2 + y + 2
y^2 + 2*y
y^2 + 2*y + 1
y^2 + 2*y + 2
2*y^2
2*y^2 + 1
2*y^2 + 2
2*y^2 + y
2*y^2 + y + 1
2*y^2 + y + 2
2*y^2 + 2*y
2*y^2 + 2*y + 1
2*y^2 + 2*y + 2
>>> for p in P.polynomials(max_degree=Integer(1)): print(p)
0
1
2
y
y + 1
y + 2
```

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```
2*y
2*y + 1
2*y + 2
>>> for p in P.polynomials(max_degree=Integer(1), of_degree=Integer(3)):␣
↪print(p)
Traceback (most recent call last):
...
ValueError: you should pass exactly one of of_degree and max_degree
```

AUTHORS:

- Joel B. Mohler

random_element (*degree=(-1, 2)*, *monic=False*, **args*, ***kwds*)

Return a random polynomial of given degree (bounds).

INPUT:

- `degree` – (default: `(-1, 2)`) integer for fixing the degree or a tuple of minimum and maximum degrees
- `monic` – boolean (default: `False`); indicate whether the sampled polynomial should be monic
- `*args, **kwds` – additional keyword parameters passed on to the `random_element` method for the base ring

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: f = R.random_element(10, x=5, y=10)
sage: f.degree()
10
sage: f.parent() is R
True
sage: all(a in range(5, 10) for a in f.coefficients())
True
sage: R.random_element(6).degree()
6
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> f = R.random_element(Integer(10), x=Integer(5), y=Integer(10))
>>> f.degree()
10
>>> f.parent() is R
True
>>> all(a in range(Integer(5), Integer(10)) for a in f.coefficients())
True
>>> R.random_element(Integer(6)).degree()
6
```

If a tuple of two integers is given for the `degree` argument, a polynomial is chosen among all polynomials with degree between them. If the base ring can be sampled uniformly, then this method also samples uniformly:

```
sage: R.random_element(degree=(0, 4)).degree() in range(0, 5)
True
```

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```
sage: found = [False]*5
sage: while not all(found):
....:     found[R.random_element(degree=(0, 4)).degree()] = True
```

```
>>> from sage.all import *
>>> R.random_element(degree=(Integer(0), Integer(4))).degree() in␣
↪range(Integer(0), Integer(5))
True
>>> found = [False]*Integer(5)
>>> while not all(found):
...     found[R.random_element(degree=(Integer(0), Integer(4))).degree()] =␣
↪True
```

Note that the zero polynomial has degree -1, so if you want to consider it set the minimum degree to -1:

```
sage: while R.random_element(degree=(-1,2), x=-1, y=1) != R.zero():
....:     pass
```

```
>>> from sage.all import *
>>> while R.random_element(degree=(-Integer(1),Integer(2)), x=-Integer(1),␣
↪y=Integer(1)) != R.zero():
...     pass
```

Monic polynomials are chosen among all monic polynomials with degree between the given `degree` argument:

```
sage: all(R.random_element(degree=(-1, 1), monic=True).is_monic() for _ in␣
↪range(10^3))
True
sage: all(R.random_element(degree=(0, 1), monic=True).is_monic() for _ in␣
↪range(10^3))
True
```

```
>>> from sage.all import *
>>> all(R.random_element(degree=(-Integer(1), Integer(1)), monic=True).is_
↪monic() for _ in range(Integer(10)**Integer(3)))
True
>>> all(R.random_element(degree=(Integer(0), Integer(1)), monic=True).is_
↪monic() for _ in range(Integer(10)**Integer(3)))
True
```

set_karatsuba_threshold(*Karatsuba_threshold*)

Changes the default threshold for this ring in the method `_mul_karatsuba()` to fall back to the schoolbook algorithm.

> **Warning**
>
> This method may have a negative performance impact in polynomial arithmetic. So use it at your own risk.

EXAMPLES:

```
sage: K = QQ['x']
sage: K.karatsuba_threshold()
8
sage: K.set_karatsuba_threshold(0)
sage: K.karatsuba_threshold()
0
```

```
>>> from sage.all import *
>>> K = QQ['x']
>>> K.karatsuba_threshold()
8
>>> K.set_karatsuba_threshold(Integer(0))
>>> K.karatsuba_threshold()
0
```

some_elements()

Return a list of polynomials.

This is typically used for running generic tests.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: R.some_elements()
[x, 0, 1, 1/2, x^2 + 2*x + 1, x^3, x^2 - 1, x^2 + 1, 2*x^2 + 2]
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> R.some_elements()
[x, 0, 1, 1/2, x^2 + 2*x + 1, x^3, x^2 - 1, x^2 + 1, 2*x^2 + 2]
```

variable_names_recursive(*depth=+Infinity*)

Return the list of variable names of this ring and its base rings, as if it were a single multi-variate polynomial.

INPUT:

- `depth` – integer or `Infinity`

OUTPUT: a tuple of strings

EXAMPLES:

```
sage: R = QQ['x']['y']['z']
sage: R.variable_names_recursive()
('x', 'y', 'z')
sage: R.variable_names_recursive(2)
('y', 'z')
```

```
>>> from sage.all import *
>>> R = QQ['x']['y']['z']
>>> R.variable_names_recursive()
('x', 'y', 'z')
>>> R.variable_names_recursive(Integer(2))
('y', 'z')
```

class sage.rings.polynomial.polynomial_ring.**PolynomialRing_integral_domain**(*base_ring*, *name='x'*, *sparse=False*, *implementation=None*, *element_class=None*, *category=None*)

Bases: *PolynomialRing_commutative*, *PolynomialRing_singular_repr*, CommutativeRing

construction()

Return the construction functor.

EXAMPLES:

```
sage: from sage.rings.polynomial.polynomial_ring import PolynomialRing_
↪integral_domain as PRing
sage: R = PRing(ZZ, 'x'); R
Univariate Polynomial Ring in x over Integer Ring
sage: functor, arg = R.construction(); functor, arg
(Poly[x], Integer Ring)
sage: functor.implementation is None
True

sage: # needs sage.libs.ntl
sage: R = PRing(ZZ, 'x', implementation='NTL'); R
Univariate Polynomial Ring in x over Integer Ring (using NTL)
sage: functor, arg = R.construction(); functor, arg
(Poly[x], Integer Ring)
sage: functor.implementation
'NTL'
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polynomial_ring import PolynomialRing_integral_
↪domain as PRing
>>> R = PRing(ZZ, 'x'); R
Univariate Polynomial Ring in x over Integer Ring
>>> functor, arg = R.construction(); functor, arg
(Poly[x], Integer Ring)
>>> functor.implementation is None
True

>>> # needs sage.libs.ntl
>>> R = PRing(ZZ, 'x', implementation='NTL'); R
Univariate Polynomial Ring in x over Integer Ring (using NTL)
>>> functor, arg = R.construction(); functor, arg
(Poly[x], Integer Ring)
>>> functor.implementation
'NTL'
```

weil_polynomials(*d*, *q*, *sign=1*, *lead=1*)

Return all integer polynomials whose complex roots all have a specified absolute value.

Such polynomials f satisfy a functional equation

$$T^d f(q/T) = sq^{d/2} f(T)$$

where d is the degree of f, s is a sign and $q^{1/2}$ is the absolute value of the roots of f.

INPUT:

- `d` – integer; the degree of the polynomials
- `q` – integer; the square of the complex absolute value of the roots
- `sign` – integer (default: 1); the sign s of the functional equation
- `lead` – integer; list of integers or list of pairs of integers (default: 1), constraints on the leading few coefficients of the generated polynomials. If pairs (a, b) of integers are given, they are treated as a constraint of the form $\equiv a \pmod b$; the moduli must be in decreasing order by divisibility, and the modulus of the leading coefficient must be 0.

See also

More documentation and additional options are available using the iterator `sage.rings.polynomial.weil.weil_polynomials.WeilPolynomials` directly. In addition, polynomials have a method `is_weil_polynomial()` to test whether or not the given polynomial is a Weil polynomial.

EXAMPLES:

```
sage: # needs sage.libs.flint
sage: R.<T> = ZZ[]
sage: L = R.weil_polynomials(4, 2)
sage: len(L)
35
sage: L[9]
T^4 + T^3 + 2*T^2 + 2*T + 4
sage: all(p.is_weil_polynomial() for p in L)
True
```

```
>>> from sage.all import *
>>> # needs sage.libs.flint
>>> R = ZZ['T']; (T,) = R._first_ngens(1)
>>> L = R.weil_polynomials(Integer(4), Integer(2))
>>> len(L)
35
>>> L[Integer(9)]
T^4 + T^3 + 2*T^2 + 2*T + 4
>>> all(p.is_weil_polynomial() for p in L)
True
```

Setting multiple leading coefficients:

```
sage: R.<T> = QQ[]
sage: l = R.weil_polynomials(4, 2, lead=((1,0), (2,4), (1,2))); l              #␣
↪needs sage.libs.flint
[T^4 + 2*T^3 + 5*T^2 + 4*T + 4,
 T^4 + 2*T^3 + 3*T^2 + 4*T + 4,
```

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```
 T^4 - 2*T^3 + 5*T^2 - 4*T + 4,
 T^4 - 2*T^3 + 3*T^2 - 4*T + 4]
```

```
>>> from sage.all import *
>>> R = QQ['T']; (T,) = R._first_ngens(1)
>>> l = R.weil_polynomials(Integer(4), Integer(2), lead=((Integer(1),
↪Integer(0)), (Integer(2),Integer(4)), (Integer(1),Integer(2)))); l                ␣
↪ # needs sage.libs.flint
[T^4 + 2*T^3 + 5*T^2 + 4*T + 4,
 T^4 + 2*T^3 + 3*T^2 + 4*T + 4,
 T^4 - 2*T^3 + 5*T^2 - 4*T + 4,
 T^4 - 2*T^3 + 3*T^2 - 4*T + 4]
```

We do not require Weil polynomials to be monic. This example generates Weil polynomials associated to K3 surfaces over $\mathbf{F}_2$ of Picard number at least 12:

```
sage: R.<T> = QQ[]
sage: l = R.weil_polynomials(10, 1, lead=2)                                        #␣
↪needs sage.libs.flint
sage: len(l)                                                                       #␣
↪needs sage.libs.flint
4865
sage: l[len(l)//2]                                                                 #␣
↪needs sage.libs.flint
2*T^10 + T^8 + T^6 + T^4 + T^2 + 2
```

```
>>> from sage.all import *
>>> R = QQ['T']; (T,) = R._first_ngens(1)
>>> l = R.weil_polynomials(Integer(10), Integer(1), lead=Integer(2))               ␣
↪                         # needs sage.libs.flint
>>> len(l)                                                                       #␣
↪needs sage.libs.flint
4865
>>> l[len(l)//Integer(2)]                                                          ␣
↪      # needs sage.libs.flint
2*T^10 + T^8 + T^6 + T^4 + T^2 + 2
```

`sage.rings.polynomial.polynomial_ring.`**`is_PolynomialRing`**`(x)`

Return `True` if `x` is a *univariate* polynomial ring (and not a sparse multivariate polynomial ring in one variable).

EXAMPLES:

```
sage: from sage.rings.polynomial.polynomial_ring import is_PolynomialRing
sage: from sage.rings.polynomial.multi_polynomial_ring import is_MPolynomialRing
sage: is_PolynomialRing(2)
doctest:warning...
DeprecationWarning: The function is_PolynomialRing is deprecated;
use 'isinstance(..., PolynomialRing_generic)' instead.
See https://github.com/sagemath/sage/issues/38266 for details.
False
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polynomial_ring import is_PolynomialRing
>>> from sage.rings.polynomial.multi_polynomial_ring import is_MPolynomialRing
>>> is_PolynomialRing(Integer(2))
doctest:warning...
DeprecationWarning: The function is_PolynomialRing is deprecated;
use 'isinstance(..., PolynomialRing_generic)' instead.
See https://github.com/sagemath/sage/issues/38266 for details.
False
```

This polynomial ring is not univariate.

```
sage: is_PolynomialRing(ZZ['x,y,z'])
False
sage: is_MPolynomialRing(ZZ['x,y,z'])
doctest:warning...
DeprecationWarning: The function is_MPolynomialRing is deprecated;
use 'isinstance(..., MPolynomialRing_base)' instead.
See https://github.com/sagemath/sage/issues/38266 for details.
True
```

```
>>> from sage.all import *
>>> is_PolynomialRing(ZZ['x,y,z'])
False
>>> is_MPolynomialRing(ZZ['x,y,z'])
doctest:warning...
DeprecationWarning: The function is_MPolynomialRing is deprecated;
use 'isinstance(..., MPolynomialRing_base)' instead.
See https://github.com/sagemath/sage/issues/38266 for details.
True
```

```
sage: is_PolynomialRing(ZZ['w'])
True
```

```
>>> from sage.all import *
>>> is_PolynomialRing(ZZ['w'])
True
```

Univariate means not only in one variable, but is a specific data type. There is a multivariate (sparse) polynomial ring data type, which supports a single variable as a special case.

```
sage: # needs sage.libs.singular
sage: R.<w> = PolynomialRing(ZZ, implementation='singular'); R
Multivariate Polynomial Ring in w over Integer Ring
sage: is_PolynomialRing(R)
False
sage: type(R)
<class 'sage.rings.polynomial.multi_polynomial_libsingular.MPolynomialRing_
↪libsingular'>
```

```
>>> from sage.all import *
>>> # needs sage.libs.singular
```

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```
>>> R = PolynomialRing(ZZ, implementation='singular', names=('w',)); (w,) = R._
↪first_ngens(1); R
Multivariate Polynomial Ring in w over Integer Ring
>>> is_PolynomialRing(R)
False
>>> type(R)
<class 'sage.rings.polynomial.multi_polynomial_libsingular.MPolynomialRing_
↪libsingular'>
```

`sage.rings.polynomial.polynomial_ring.`**`polygen`**`(`*ring_or_element*, *name='x'*`)`

Return a polynomial indeterminate.

INPUT:

- `polygen(base_ring, name='x')`
- `polygen(ring_element, name='x')`

If the first input is a ring, return a polynomial generator over that ring. If it is a ring element, return a polynomial generator over the parent of the element.

EXAMPLES:

```
sage: z = polygen(QQ, 'z')
sage: z^3 + z +1
z^3 + z + 1
sage: parent(z)
Univariate Polynomial Ring in z over Rational Field
```

```
>>> from sage.all import *
>>> z = polygen(QQ, 'z')
>>> z**Integer(3) + z +Integer(1)
z^3 + z + 1
>>> parent(z)
Univariate Polynomial Ring in z over Rational Field
```

Note

If you give a list or comma-separated string to *`polygen()`*, you'll get a tuple of indeterminates, exactly as if you called *`polygens()`*.

`sage.rings.polynomial.polynomial_ring.`**`polygens`**`(`*base_ring*, *names='x'*, **args*`)`

Return indeterminates over the given base ring with the given names.

EXAMPLES:

```
sage: x,y,z = polygens(QQ,'x,y,z')
sage: (x+y+z)^2
x^2 + 2*x*y + y^2 + 2*x*z + 2*y*z + z^2
sage: parent(x)
Multivariate Polynomial Ring in x, y, z over Rational Field
sage: t = polygens(QQ, ['x','yz','abc'])
sage: t
(x, yz, abc)
```

```
>>> from sage.all import *
>>> x,y,z = polygens(QQ,'x,y,z')
>>> (x+y+z)**Integer(2)
x^2 + 2*x*y + y^2 + 2*x*z + 2*y*z + z^2
>>> parent(x)
Multivariate Polynomial Ring in x, y, z over Rational Field
>>> t = polygens(QQ, ['x','yz','abc'])
>>> t
(x, yz, abc)
```

The number of generators can be passed as a third argument:

```
sage: polygens(QQ, 'x', 4)
(x0, x1, x2, x3)
```

```
>>> from sage.all import *
>>> polygens(QQ, 'x', Integer(4))
(x0, x1, x2, x3)
```

2.1.2 Ring homomorphisms from a polynomial ring to another ring

This module currently implements the canonical ring homomorphism from $A[x]$ to $B[x]$ induced by a ring homomorphism from A to B.

Todo

Implement homomorphisms from $A[x]$ to an arbitrary ring R, given by a ring homomorphism from A to R and the image of x in R.

AUTHORS:

- Peter Bruin (March 2014): initial version

class sage.rings.polynomial.polynomial_ring_homomorphism.**PolynomialRingHomomorphism_from_base**

Bases: RingHomomorphism_from_base

The canonical ring homomorphism from $R[x]$ to $S[x]$ induced by a ring homomorphism from R to S.

EXAMPLES:

```
sage: QQ['x'].coerce_map_from(ZZ['x'])
Ring morphism:
  From: Univariate Polynomial Ring in x over Integer Ring
  To:   Univariate Polynomial Ring in x over Rational Field
  Defn: Induced from base ring by
        Natural morphism:
          From: Integer Ring
          To:   Rational Field
```

```
>>> from sage.all import *
>>> QQ['x'].coerce_map_from(ZZ['x'])
Ring morphism:
```

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```
From: Univariate Polynomial Ring in x over Integer Ring
To:   Univariate Polynomial Ring in x over Rational Field
Defn: Induced from base ring by
      Natural morphism:
        From: Integer Ring
        To:   Rational Field
```

is_injective()

Return whether this morphism is injective.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: S.<x> = QQ[]
sage: R.hom(S).is_injective()
True
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> S = QQ['x']; (x,) = S._first_ngens(1)
>>> R.hom(S).is_injective()
True
```

is_surjective()

Return whether this morphism is surjective.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: S.<x> = Zmod(2)[]
sage: R.hom(S).is_surjective()
True
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> S = Zmod(Integer(2))['x']; (x,) = S._first_ngens(1)
>>> R.hom(S).is_surjective()
True
```

2.1.3 Univariate polynomial base class

AUTHORS:

- William Stein: first version
- Martin Albrecht: added singular coercion
- Robert Bradshaw: moved Polynomial_generic_dense to Cython
- Miguel Marco: implemented resultant in the case where PARI fails
- Simon King: used a faster way of conversion from the base ring
- Kwankyu Lee (2013-06-02): enhanced `quo_rem()`

- Julian Rueth (2012-05-25,2014-05-09): fixed is_squarefree() for imperfect fields, fixed division without remainder over QQbar; added `_cache_key` for polynomials with unhashable coefficients
- Simon King (2013-10): implemented copying of `PolynomialBaseringInjection`
- Bruno Grenet (2014-07-13): enhanced `quo_rem()`
- Kiran Kedlaya (2016-03): added root counting
- Edgar Costa (2017-07): added rational reconstruction
- Kiran Kedlaya (2017-09): added reciprocal transform, trace polynomial
- David Zureick-Brown (2017-09): added is_weil_polynomial
- Sebastian Oehms (2018-10): made `roots()` and `factor()` work over more cases of proper integral domains (see Issue #26421)

class sage.rings.polynomial.polynomial_element.**ConstantPolynomialSection**

Bases: `Map`

This class is used for conversion from a polynomial ring to its base ring.

Since Issue #9944, it calls the `constant_coefficient()` method, which can be optimized for a particular polynomial type.

EXAMPLES:

```
sage: P0.<y_1> = GF(3)[]
sage: P1.<y_2,y_1,y_0> = GF(3)[]
sage: P0(-y_1)
2*y_1

sage: phi = GF(3).convert_map_from(P0); phi
Generic map:
  From: Univariate Polynomial Ring in y_1 over Finite Field of size 3
  To:   Finite Field of size 3
sage: type(phi)
<class 'sage.rings.polynomial.polynomial_element.ConstantPolynomialSection'>
sage: phi(P0.one())
1
sage: phi(y_1)
Traceback (most recent call last):
...
TypeError: y_1 is not a constant polynomial
```

```
>>> from sage.all import *
>>> P0 = GF(Integer(3))['y_1']; (y_1,) = P0._first_ngens(1)
>>> P1 = GF(Integer(3))['y_2, y_1, y_0']; (y_2, y_1, y_0,) = P1._first_ngens(3)
>>> P0(-y_1)
2*y_1

>>> phi = GF(Integer(3)).convert_map_from(P0); phi
Generic map:
  From: Univariate Polynomial Ring in y_1 over Finite Field of size 3
  To:   Finite Field of size 3
>>> type(phi)
<class 'sage.rings.polynomial.polynomial_element.ConstantPolynomialSection'>
```

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```
>>> phi(P0.one())
1
>>> phi(y_1)
Traceback (most recent call last):
...
TypeError: y_1 is not a constant polynomial
```

class sage.rings.polynomial.polynomial_element.**Polynomial**

Bases: CommutativePolynomial

A polynomial.

EXAMPLES:

```
sage: R.<y> = QQ['y']
sage: S.<x> = R['x']
sage: S
Univariate Polynomial Ring in x over Univariate Polynomial Ring in y
over Rational Field
sage: f = x*y; f
y*x
sage: type(f)
<class 'sage.rings.polynomial.polynomial_element.Polynomial_generic_dense'>
sage: p = (y+1)^10; p(1)
1024
```

```
>>> from sage.all import *
>>> R = QQ['y']; (y,) = R._first_ngens(1)
>>> S = R['x']; (x,) = S._first_ngens(1)
>>> S
Univariate Polynomial Ring in x over Univariate Polynomial Ring in y
over Rational Field
>>> f = x*y; f
y*x
>>> type(f)
<class 'sage.rings.polynomial.polynomial_element.Polynomial_generic_dense'>
>>> p = (y+Integer(1))**Integer(10); p(Integer(1))
1024
```

add(*right*)

Add two polynomials.

EXAMPLES:

```
sage: R = ZZ['x']
sage: p = R([1,2,3,4])
sage: q = R([4,-3,2,-1])
sage: p + q     # indirect doctest
3*x^3 + 5*x^2 - x + 5
```

```
>>> from sage.all import *
>>> R = ZZ['x']
>>> p = R([Integer(1),Integer(2),Integer(3),Integer(4)])
```

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```
>>> q = R([Integer(4),-Integer(3),Integer(2),-Integer(1)])
>>> p + q     # indirect doctest
3*x^3 + 5*x^2 - x + 5
```

sub(*other*)

Default implementation of subtraction using addition and negation.

lmul(*left*)

Multiply `self` on the left by a scalar.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: f = (x^3 + x + 5)
sage: f._lmul_(7)
7*x^3 + 7*x + 35
sage: 7*f
7*x^3 + 7*x + 35
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> f = (x**Integer(3) + x + Integer(5))
>>> f._lmul_(Integer(7))
7*x^3 + 7*x + 35
>>> Integer(7)*f
7*x^3 + 7*x + 35
```

rmul(*right*)

Multiply `self` on the right by a scalar.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: f = (x^3 + x + 5)
sage: f._rmul_(7)
7*x^3 + 7*x + 35
sage: f*7
7*x^3 + 7*x + 35
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> f = (x**Integer(3) + x + Integer(5))
>>> f._rmul_(Integer(7))
7*x^3 + 7*x + 35
>>> f*Integer(7)
7*x^3 + 7*x + 35
```

mul(*right*)

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: (x - 4) * (x^2 - 8*x + 16)
x^3 - 12*x^2 + 48*x - 64
```

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```
sage: C.<t> = PowerSeriesRing(ZZ)
sage: D.<s> = PolynomialRing(C)
sage: z = (1 + O(t)) + t*s^2
sage: z*z
t^2*s^4 + (2*t + O(t^2))*s^2 + 1 + O(t)

## More examples from trac 2943, added by Kiran S. Kedlaya 2 Dec 09
sage: C.<t> = PowerSeriesRing(Integers())
sage: D.<s> = PolynomialRing(C)
sage: z = 1 + (t + O(t^2))*s + (t^2 + O(t^3))*s^2
sage: z*z
(t^4 + O(t^5))*s^4 + (2*t^3 + O(t^4))*s^3 + (3*t^2 + O(t^3))*s^2 + (2*t + O(t^
↪2))*s + 1
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> (x - Integer(4)) * (x**Integer(2) - Integer(8)*x + Integer(16))
x^3 - 12*x^2 + 48*x - 64
>>> C = PowerSeriesRing(ZZ, names=('t',)); (t,) = C._first_ngens(1)
>>> D = PolynomialRing(C, names=('s',)); (s,) = D._first_ngens(1)
>>> z = (Integer(1) + O(t)) + t*s**Integer(2)
>>> z*z
t^2*s^4 + (2*t + O(t^2))*s^2 + 1 + O(t)

## More examples from trac 2943, added by Kiran S. Kedlaya 2 Dec 09
>>> C = PowerSeriesRing(Integers(), names=('t',)); (t,) = C._first_ngens(1)
>>> D = PolynomialRing(C, names=('s',)); (s,) = D._first_ngens(1)
>>> z = Integer(1) + (t + O(t**Integer(2)))*s + (t**Integer(2) +␣
↪O(t**Integer(3)))*s**Integer(2)
>>> z*z
(t^4 + O(t^5))*s^4 + (2*t^3 + O(t^4))*s^3 + (3*t^2 + O(t^3))*s^2 + (2*t + O(t^
↪2))*s + 1
```

_mul_trunc_(*right*, *n*)

Return the truncated multiplication of two polynomials up to `n`.

This is the default implementation that does the multiplication and then truncate! There are custom implementations in several subclasses:

- *on dense polynomial over integers (via FLINT)*
- *on dense polynomial over Z/nZ (via FLINT)*
- *on dense rational polynomial (via FLINT)*
- *on dense polynomial on Z/nZ (via NTL)*

EXAMPLES:

```
sage: R = QQ['x']['y']
sage: y = R.gen()
sage: x = R.base_ring().gen()
sage: p1 = 1 - x*y + 2*y**3
sage: p2 = -1/3 + y**5
```

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```
sage: p1._mul_trunc_(p2, 5)
-2/3*y^3 + 1/3*x*y - 1/3
```

```
>>> from sage.all import *
>>> R = QQ['x']['y']
>>> y = R.gen()
>>> x = R.base_ring().gen()
>>> p1 = Integer(1) - x*y + Integer(2)*y**Integer(3)
>>> p2 = -Integer(1)/Integer(3) + y**Integer(5)
>>> p1._mul_trunc_(p2, Integer(5))
-2/3*y^3 + 1/3*x*y - 1/3
```

Todo

implement a generic truncated Karatsuba and use it here.

adams_operator(**args*, ***kwds*)

Deprecated: Use `adams_operator_on_roots()` instead. See Issue #36396 for details.

adams_operator_on_roots(*n*, *monic=False*)

Return the polynomial whose roots are the n-th powers of the roots of `self`.

INPUT:

- `n` – integer
- `monic` – boolean (default: `False`) if set to `True`, force the output to be monic

EXAMPLES:

```
sage: # needs sage.libs.pari sage.libs.singular
sage: f = cyclotomic_polynomial(30)
sage: f.adams_operator_on_roots(7) == f
True
sage: f.adams_operator_on_roots(6) == cyclotomic_polynomial(5)**2
True
sage: f.adams_operator_on_roots(10) == cyclotomic_polynomial(3)**4
True
sage: f.adams_operator_on_roots(15) == cyclotomic_polynomial(2)**8
True
sage: f.adams_operator_on_roots(30) == cyclotomic_polynomial(1)**8
True

sage: x = polygen(QQ)
sage: f = x^2 - 2*x + 2
sage: f.adams_operator_on_roots(10)                                     #␣
↪needs sage.libs.singular
x^2 + 1024
```

```
>>> from sage.all import *
>>> # needs sage.libs.pari sage.libs.singular
>>> f = cyclotomic_polynomial(Integer(30))
```

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```
>>> f.adams_operator_on_roots(Integer(7)) == f
True
>>> f.adams_operator_on_roots(Integer(6)) == cyclotomic_
↪polynomial(Integer(5))**Integer(2)
True
>>> f.adams_operator_on_roots(Integer(10)) == cyclotomic_
↪polynomial(Integer(3))**Integer(4)
True
>>> f.adams_operator_on_roots(Integer(15)) == cyclotomic_
↪polynomial(Integer(2))**Integer(8)
True
>>> f.adams_operator_on_roots(Integer(30)) == cyclotomic_
↪polynomial(Integer(1))**Integer(8)
True

>>> x = polygen(QQ)
>>> f = x**Integer(2) - Integer(2)*x + Integer(2)
>>> f.adams_operator_on_roots(Integer(10))                                      
↪     # needs sage.libs.singular
x^2 + 1024
```

When `self` is monic, the output will have leading coefficient ± 1 depending on the degree, but we can force it to be monic:

```
sage: R.<a,b,c> = ZZ[]
sage: x = polygen(R)
sage: f = (x - a) * (x - b) * (x - c)
sage: f.adams_operator_on_roots(3).factor()                                    #
↪needs sage.libs.singular
(-1) * (x - a^3) * (x - b^3) * (x - c^3)
sage: f.adams_operator_on_roots(3, monic=True).factor()                        #
↪needs sage.libs.singular
(x - a^3) * (x - b^3) * (x - c^3)
```

```
>>> from sage.all import *
>>> R = ZZ['a, b, c']; (a, b, c,) = R._first_ngens(3)
>>> x = polygen(R)
>>> f = (x - a) * (x - b) * (x - c)
>>> f.adams_operator_on_roots(Integer(3)).factor()                             
↪     # needs sage.libs.singular
(-1) * (x - a^3) * (x - b^3) * (x - c^3)
>>> f.adams_operator_on_roots(Integer(3), monic=True).factor()                 
↪     # needs sage.libs.singular
(x - a^3) * (x - b^3) * (x - c^3)
```

add_bigoh(*prec*)

Return the power series of precision at most `prec` got by adding $O(q^{\text{prec}})$ to self, where q is its variable.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: f = 1 + 4*x + x^3
```

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```
sage: f.add_bigoh(7)
1 + 4*x + x^3 + O(x^7)
sage: f.add_bigoh(2)
1 + 4*x + O(x^2)
sage: f.add_bigoh(2).parent()
Power Series Ring in x over Integer Ring
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> f = Integer(1) + Integer(4)*x + x**Integer(3)
>>> f.add_bigoh(Integer(7))
1 + 4*x + x^3 + O(x^7)
>>> f.add_bigoh(Integer(2))
1 + 4*x + O(x^2)
>>> f.add_bigoh(Integer(2)).parent()
Power Series Ring in x over Integer Ring
```

all_roots_in_interval(*a=None*, *b=None*)

Return `True` if the roots of this polynomial are all real and contained in the given interval.

EXAMPLES:

```
sage: # needs sage.libs.pari
sage: R.<x> = PolynomialRing(ZZ)
sage: pol = (x - 1)^2 * (x - 2)^2 * (x - 3)
sage: pol.all_roots_in_interval(1, 3)
True
sage: pol.all_roots_in_interval(1.01, 3)
False
sage: pol = chebyshev_T(5, x)
sage: pol.all_roots_in_interval(-1, 1)
True
sage: pol = chebyshev_T(5, x/2)
sage: pol.all_roots_in_interval(-1, 1)
False
sage: pol.all_roots_in_interval()
True
```

```
>>> from sage.all import *
>>> # needs sage.libs.pari
>>> R = PolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> pol = (x - Integer(1))**Integer(2) * (x - Integer(2))**Integer(2) * (x -
→Integer(3))
>>> pol.all_roots_in_interval(Integer(1), Integer(3))
True
>>> pol.all_roots_in_interval(RealNumber('1.01'), Integer(3))
False
>>> pol = chebyshev_T(Integer(5), x)
>>> pol.all_roots_in_interval(-Integer(1), Integer(1))
True
>>> pol = chebyshev_T(Integer(5), x/Integer(2))
>>> pol.all_roots_in_interval(-Integer(1), Integer(1))
```

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```
False
>>> pol.all_roots_in_interval()
True
```

any_irreducible_factor(*degree=None*, *assume_squarefree=False*, *assume_equal_deg=False*, *ext_degree=None*)

Return an irreducible factor of this polynomial.

INPUT:

- `degree` – `None` or positive integer (default: `None`). Used for polynomials over finite fields. If `None`, returns the first factor found (usually the smallest). Otherwise, attempts to return an irreducible factor of `self` of chosen degree `degree`.
- `assume_squarefree` – boolean (default: `False`); Used for polynomials over finite fields. If `True`, this polynomial is assumed to be squarefree.
- `assume_equal_deg` – boolean (default: `False`); Used for polynomials over finite fields. If `True`, this polynomial is assumed to be the product of irreducible polynomials of degree equal to `degree`.
- `ext_degree` – positive integer or `None` (default); used for polynomials over finite fields. If not `None` only returns irreducible factors of `self` whose degree divides `ext_degree`.

EXAMPLES:

```
sage: # needs sage.rings.finite_rings
sage: F = GF(163)
sage: R.<x> = F[]
sage: f = (x + 40)^3 * (x^5 + 76*x^4 + 93*x^3 + 112*x^2 + 22*x + 27)^2 * (x^6␣
↪+ 50*x^5 + 143*x^4 + 162*x^2 + 109*x + 140)
sage: f.any_irreducible_factor()
x + 40
sage: f.any_irreducible_factor(degree=5)
x^5 + 76*x^4 + 93*x^3 + 112*x^2 + 22*x + 27
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> F = GF(Integer(163))
>>> R = F['x']; (x,) = R._first_ngens(1)
>>> f = (x + Integer(40))**Integer(3) * (x**Integer(5) +␣
↪Integer(76)*x**Integer(4) + Integer(93)*x**Integer(3) +␣
↪Integer(112)*x**Integer(2) + Integer(22)*x + Integer(27))**Integer(2) *␣
↪(x**Integer(6) + Integer(50)*x**Integer(5) + Integer(143)*x**Integer(4) +␣
↪Integer(162)*x**Integer(2) + Integer(109)*x + Integer(140))
>>> f.any_irreducible_factor()
x + 40
>>> f.any_irreducible_factor(degree=Integer(5))
x^5 + 76*x^4 + 93*x^3 + 112*x^2 + 22*x + 27
```

When the polynomial is known to be squarefree we can optimise the call by setting `assume_squarefree` to be `True`:

```
sage: # needs sage.rings.finite_rings
sage: F = GF(163)
sage: R.<x> = F[]
```

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```
sage: g = (x - 1) * (x^3 + 7*x + 161)
sage: g.any_irreducible_factor(assume_squarefree=True)
x + 162
sage: g.any_irreducible_factor(degree=3, assume_squarefree=True)
x^3 + 7*x + 161
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> F = GF(Integer(163))
>>> R = F['x']; (x,) = R._first_ngens(1)
>>> g = (x - Integer(1)) * (x**Integer(3) + Integer(7)*x + Integer(161))
>>> g.any_irreducible_factor(assume_squarefree=True)
x + 162
>>> g.any_irreducible_factor(degree=Integer(3), assume_squarefree=True)
x^3 + 7*x + 161
```

If we ask for an irreducible factor which does not exist, the function will throw a `ValueError`:

```
sage: # needs sage.rings.finite_rings
sage: F = GF(163)
sage: R.<x> = F[]
sage: g = (x - 1) * (x^3 + 7*x + 161)
sage: g.any_irreducible_factor(degree=2, assume_squarefree=True)
Traceback (most recent call last):
...
ValueError: no irreducible factor of degree 2 could be computed from x^4 +␣
↪162*x^3 + 7*x^2 + 154*x + 2
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> F = GF(Integer(163))
>>> R = F['x']; (x,) = R._first_ngens(1)
>>> g = (x - Integer(1)) * (x**Integer(3) + Integer(7)*x + Integer(161))
>>> g.any_irreducible_factor(degree=Integer(2), assume_squarefree=True)
Traceback (most recent call last):
...
ValueError: no irreducible factor of degree 2 could be computed from x^4 +␣
↪162*x^3 + 7*x^2 + 154*x + 2
```

If we assume that the polynomial is product of irreducible polynomials of the same degree, we must also supply the degree:

```
sage: # needs sage.rings.finite_rings
sage: F = GF(163)
sage: R.<x> = F[]
sage: h = (x + 57) * (x + 98) * (x + 117) * (x + 145)
sage: h.any_irreducible_factor(degree=1, assume_equal_deg=True)    # random
x + 98
sage: h.any_irreducible_factor(assume_equal_deg=True)
Traceback (most recent call last):
...
ValueError: degree must be known if distinct degree factorisation is assumed
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> F = GF(Integer(163))
>>> R = F['x']; (x,) = R._first_ngens(1)
>>> h = (x + Integer(57)) * (x + Integer(98)) * (x + Integer(117)) * (x +␣
↪Integer(145))
>>> h.any_irreducible_factor(degree=Integer(1), assume_equal_deg=True)    #␣
↪random
x + 98
>>> h.any_irreducible_factor(assume_equal_deg=True)
Traceback (most recent call last):
...
ValueError: degree must be known if distinct degree factorisation is assumed
```

Also works for extension fields and even characteristic:

```
sage: F.<z4> = GF(2^4)
sage: R.<x> = F[]
sage: f = (x + z4^3 + z4^2)^4 * (x^2 + z4*x + z4) * (x^2 + (z4^3 + z4^2 +␣
↪z4)*x + z4^2 + z4 + 1)
sage: f.any_irreducible_factor()
x + z4^3 + z4^2
sage: f.any_irreducible_factor(degree=2)   # random
x^2 + (z4^3 + z4^2 + z4)*x + z4^2 + z4 + 1
```

```
>>> from sage.all import *
>>> F = GF(Integer(2)**Integer(4), names=('z4',)); (z4,) = F._first_ngens(1)
>>> R = F['x']; (x,) = R._first_ngens(1)
>>> f = (x + z4**Integer(3) + z4**Integer(2))**Integer(4) * (x**Integer(2) +␣
↪z4*x + z4) * (x**Integer(2) + (z4**Integer(3) + z4**Integer(2) + z4)*x +␣
↪z4**Integer(2) + z4 + Integer(1))
>>> f.any_irreducible_factor()
x + z4^3 + z4^2
>>> f.any_irreducible_factor(degree=Integer(2))   # random
x^2 + (z4^3 + z4^2 + z4)*x + z4^2 + z4 + 1
```

We can also use this function for polynomials which are not defined over finite fields, but this simply falls back to a slow method of factorisation:

```
sage: R.<x> = ZZ[]
sage: f = 3*x^4 + 2*x^3
sage: f.any_irreducible_factor()
3*x + 2
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> f = Integer(3)*x**Integer(4) + Integer(2)*x**Integer(3)
>>> f.any_irreducible_factor()
3*x + 2
```

any_root(*ring=None*, *degree=None*, *assume_squarefree=False*, *assume_equal_deg=False*)

Return a root of this polynomial in the given ring.

INPUT:

- `ring` – the ring in which a root is sought; by default this is the coefficient ring
- `degree` – `None` or nonzero integer; used for polynomials over finite fields. Return a root of degree `abs(degree)` over the ground field. If negative, also assumes that all factors of this polynomial are of degree `abs(degree)`. If `None`, returns a root of minimal degree contained within the given ring.
- `assume_squarefree` – boolean; used for polynomials over finite fields. If `True`, this polynomial is assumed to be squarefree.
- `assume_equal_deg` – boolean; used for polynomials over finite fields. If `True`, all factors of this polynomial are assumed to have degree `degree`. Note that `degree` must be set.

Warning

Negative degree input will be deprecated. Instead use `assume_equal_deg`.

Note

For finite fields, `any_root()` is non-deterministic when finding linear roots of a polynomial over the base ring. However, if `degree` is greater than one, or `ring` is an extension of the base ring, then the root computed is found by attempting to return a root after factorisation. Roots found in this way are deterministic. This may change in the future. For all other rings or fields, roots are found by first fully-factoring `self` and the output is deterministic.

EXAMPLES:

```
sage: # needs sage.rings.finite_rings
sage: R.<x> = GF(11)[]
sage: f = 7*x^7 + 8*x^6 + 4*x^5 + x^4 + 6*x^3 + 10*x^2 + 8*x + 5
sage: f.any_root()
2
sage: f.factor()
(7) * (x + 9) * (x^6 + 10*x^4 + 6*x^3 + 5*x^2 + 2*x + 2)
sage: f = x^6 + 10*x^4 + 6*x^3 + 5*x^2 + 2*x + 2
sage: root = f.any_root(GF(11^6, 'a'))
sage: roots = sorted(f.roots(GF(11^6, 'a'), multiplicities=False))
sage: roots
[10*a^5 + 2*a^4 + 8*a^3 + 9*a^2 + a,
a^5 + a^4 + 7*a^3 + 2*a^2 + 10*a,
9*a^5 + 5*a^4 + 10*a^3 + 8*a^2 + 3*a + 1,
2*a^5 + 8*a^4 + 3*a^3 + 6*a + 2,
a^5 + 3*a^4 + 8*a^3 + 2*a^2 + 3*a + 4,
10*a^5 + 3*a^4 + 8*a^3 + a^2 + 10*a + 4]
sage: root in roots
True

sage: # needs sage.rings.finite_rings
sage: g = (x-1) * (x^2 + 3*x + 9) * (x^5 + 5*x^4 + 8*x^3 + 5*x^2 + 3*x + 5)
sage: g.any_root(ring=GF(11^10, 'b'), degree=1)
1
sage: root = g.any_root(ring=GF(11^10, 'b'), degree=2)
sage: roots = (x^2 + 3*x + 9).roots(ring=GF(11^10, 'b'), multiplicities=False)
```

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```
sage: root in roots
True
sage: root = g.any_root(ring=GF(11^10, 'b'), degree=5)
sage: roots = (x^5 + 5*x^4 + 8*x^3 + 5*x^2 + 3*x + 5).roots(ring=GF(11^10, 'b
↪'), multiplicities=False)
sage: root in roots
True
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> R = GF(Integer(11))['x']; (x,) = R._first_ngens(1)
>>> f = Integer(7)*x**Integer(7) + Integer(8)*x**Integer(6) +␣
↪Integer(4)*x**Integer(5) + x**Integer(4) + Integer(6)*x**Integer(3) +␣
↪Integer(10)*x**Integer(2) + Integer(8)*x + Integer(5)
>>> f.any_root()
2
>>> f.factor()
(7) * (x + 9) * (x^6 + 10*x^4 + 6*x^3 + 5*x^2 + 2*x + 2)
>>> f = x**Integer(6) + Integer(10)*x**Integer(4) + Integer(6)*x**Integer(3)␣
↪+ Integer(5)*x**Integer(2) + Integer(2)*x + Integer(2)
>>> root = f.any_root(GF(Integer(11)**Integer(6), 'a'))
>>> roots = sorted(f.roots(GF(Integer(11)**Integer(6), 'a'),␣
↪multiplicities=False))
>>> roots
[10*a^5 + 2*a^4 + 8*a^3 + 9*a^2 + a,
a^5 + a^4 + 7*a^3 + 2*a^2 + 10*a,
9*a^5 + 5*a^4 + 10*a^3 + 8*a^2 + 3*a + 1,
2*a^5 + 8*a^4 + 3*a^3 + 6*a + 2,
a^5 + 3*a^4 + 8*a^3 + 2*a^2 + 3*a + 4,
10*a^5 + 3*a^4 + 8*a^3 + a^2 + 10*a + 4]
>>> root in roots
True

>>> # needs sage.rings.finite_rings
>>> g = (x-Integer(1)) * (x**Integer(2) + Integer(3)*x + Integer(9)) *␣
↪(x**Integer(5) + Integer(5)*x**Integer(4) + Integer(8)*x**Integer(3) +␣
↪Integer(5)*x**Integer(2) + Integer(3)*x + Integer(5))
>>> g.any_root(ring=GF(Integer(11)**Integer(10), 'b'), degree=Integer(1))
1
>>> root = g.any_root(ring=GF(Integer(11)**Integer(10), 'b'),␣
↪degree=Integer(2))
>>> roots = (x**Integer(2) + Integer(3)*x + Integer(9)).
↪roots(ring=GF(Integer(11)**Integer(10), 'b'), multiplicities=False)
>>> root in roots
True
>>> root = g.any_root(ring=GF(Integer(11)**Integer(10), 'b'),␣
↪degree=Integer(5))
>>> roots = (x**Integer(5) + Integer(5)*x**Integer(4) +␣
↪Integer(8)*x**Integer(3) + Integer(5)*x**Integer(2) + Integer(3)*x +␣
↪Integer(5)).roots(ring=GF(Integer(11)**Integer(10), 'b'),␣
↪multiplicities=False)
>>> root in roots
```

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```
True
```

args()

Return the generator of this polynomial ring, which is the (only) argument used when calling `self`.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: x.args()
(x,)
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> x.args()
(x,)
```

A constant polynomial has no variables, but still takes a single argument.

```
sage: R(2).args()
(x,)
```

```
>>> from sage.all import *
>>> R(Integer(2)).args()
(x,)
```

base_extend(*R*)

Return a copy of this polynomial but with coefficients in `R`, if there is a natural map from the coefficient ring of `self` to `R`.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: f = x^3 - 17*x + 3
sage: f.base_extend(GF(7))
Traceback (most recent call last):
...
TypeError: no such base extension
sage: f.change_ring(GF(7))
x^3 + 4*x + 3
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(3) - Integer(17)*x + Integer(3)
>>> f.base_extend(GF(Integer(7)))
Traceback (most recent call last):
...
TypeError: no such base extension
>>> f.change_ring(GF(Integer(7)))
x^3 + 4*x + 3
```

base_ring()

Return the base ring of the parent of `self`.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: x.base_ring()
Integer Ring
sage: (2*x + 3).base_ring()
Integer Ring
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> x.base_ring()
Integer Ring
>>> (Integer(2)*x + Integer(3)).base_ring()
Integer Ring
```

change_ring(*R*)

Return a copy of this polynomial but with coefficients in R, if at all possible.

INPUT:

- R – a ring or morphism

EXAMPLES:

```
sage: K.<z> = CyclotomicField(3)                                            #␣
↪needs sage.rings.number_field
sage: f = K.defining_polynomial()                                           #␣
↪needs sage.rings.number_field
sage: f.change_ring(GF(7))                                                  #␣
↪needs sage.rings.finite_rings sage.rings.number_field
x^2 + x + 1
```

```
>>> from sage.all import *
>>> K = CyclotomicField(Integer(3), names=('z',)); (z,) = K._first_ngens(1)#␣
↪needs sage.rings.number_field
>>> f = K.defining_polynomial()                                           #␣
↪needs sage.rings.number_field
>>> f.change_ring(GF(Integer(7)))                                           ␣
↪     # needs sage.rings.finite_rings sage.rings.number_field
x^2 + x + 1
```

```
sage: # needs sage.rings.number_field
sage: K.<z> = CyclotomicField(3)
sage: R.<x> = K[]
sage: f = x^2 + z
sage: f.change_ring(K.embeddings(CC)[1])                                    #␣
↪needs sage.rings.real_mpfr
x^2 - 0.500000000000000 - 0.866025403784438*I
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> K = CyclotomicField(Integer(3), names=('z',)); (z,) = K._first_ngens(1)
>>> R = K['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(2) + z
>>> f.change_ring(K.embeddings(CC)[Integer(1)])                             ␣
```

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```
↪     # needs sage.rings.real_mpfr
x^2 - 0.500000000000000 - 0.866025403784438*I
```

```
sage: R.<x> = QQ[]
sage: f = x^2 + 1
sage: f.change_ring(QQ.embeddings(CC)[0])                                    #␣
↪needs sage.rings.real_mpfr
x^2 + 1.00000000000000
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(2) + Integer(1)
>>> f.change_ring(QQ.embeddings(CC)[Integer(0)])                               ␣
↪     # needs sage.rings.real_mpfr
x^2 + 1.00000000000000
```

change_variable_name(*var*)

Return a new polynomial over the same base ring but in a different variable.

EXAMPLES:

```
sage: x = polygen(QQ,'x')
sage: f = -2/7*x^3 + (2/3)*x - 19/993; f
-2/7*x^3 + 2/3*x - 19/993
sage: f.change_variable_name('theta')
-2/7*theta^3 + 2/3*theta - 19/993
```

```
>>> from sage.all import *
>>> x = polygen(QQ,'x')
>>> f = -Integer(2)/Integer(7)*x**Integer(3) + (Integer(2)/Integer(3))*x -␣
↪Integer(19)/Integer(993); f
-2/7*x^3 + 2/3*x - 19/993
>>> f.change_variable_name('theta')
-2/7*theta^3 + 2/3*theta - 19/993
```

coefficients(*sparse=True*)

Return the coefficients of the monomials appearing in `self`.

If `sparse=True` (the default), it returns only the nonzero coefficients. Otherwise, it returns the same value as `self.list()`. (In this case, it may be slightly faster to invoke `self.list()` directly.) In either case, the coefficients are ordered by increasing degree.

EXAMPLES:

```
sage: _.<x> = PolynomialRing(ZZ)
sage: f = 3*x^4 + 2*x^2 + 1
sage: f.coefficients()
[1, 2, 3]
sage: f.coefficients(sparse=False)
[1, 0, 2, 0, 3]
```

```
>>> from sage.all import *
>>> _ = PolynomialRing(ZZ, names=('x',)); (x,) = _._first_ngens(1)
>>> f = Integer(3)*x**Integer(4) + Integer(2)*x**Integer(2) + Integer(1)
>>> f.coefficients()
[1, 2, 3]
>>> f.coefficients(sparse=False)
[1, 0, 2, 0, 3]
```

complex_roots()

Return the complex roots of this polynomial, without multiplicities.

Calls `self.roots(ring=CC)`, unless this is a polynomial with floating-point coefficients, in which case it is uses the appropriate precision from the input coefficients.

EXAMPLES:

```
sage: # needs sage.libs.pari sage.rings.real_mpfr
sage: x = polygen(ZZ)
sage: (x^3 - 1).complex_roots()   # note: low order bits slightly different␣
↪on ppc.
[1.00000000000000,
 -0.500000000000000 - 0.86602540378443...*I,
 -0.500000000000000 + 0.86602540378443...*I]
```

```
>>> from sage.all import *
>>> # needs sage.libs.pari sage.rings.real_mpfr
>>> x = polygen(ZZ)
>>> (x**Integer(3) - Integer(1)).complex_roots()   # note: low order bits␣
↪slightly different on ppc.
[1.00000000000000,
 -0.500000000000000 - 0.86602540378443...*I,
 -0.500000000000000 + 0.86602540378443...*I]
```

compose_power(*k*, *algorithm=None*, *monic=False*)

Return the k-th iterate of the composed product of this polynomial with itself.

INPUT:

- `k` – nonnegative integer
- `algorithm` – `None` (default), `'resultant'`, or `'BFSS'`; see *composed_op()*
- `monic` – boolean (default: `False`); see *composed_op()*

OUTPUT:

The polynomial of degree d^k where d is the degree, whose roots are all k-fold products of roots of this polynomial. That is, $f * f * \cdots * f$ where this is f and $f * f =$ `f.composed_op(f, operator.mul)`.

EXAMPLES:

```
sage: R.<a,b,c> = ZZ[]
sage: x = polygen(R)
sage: f = (x - a) * (x - b) * (x - c)
sage: f.compose_power(2).factor()                                          #␣
↪needs sage.libs.singular sage.modules
(x - a^2) * (x - b^2) * (x - c^2) * (x - a*b)^2 * (x - a*c)^2 * (x - b*c)^2
```

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```
sage: # needs sage.libs.singular sage.modules
sage: x = polygen(QQ)
sage: f = x^2 - 2*x + 2
sage: f2 = f.compose_power(2); f2
x^4 - 4*x^3 + 8*x^2 - 16*x + 16
sage: f2 == f.composed_op(f, operator.mul)
True
sage: f3 = f.compose_power(3); f3
x^8 - 8*x^7 + 32*x^6 - 64*x^5 + 128*x^4 - 512*x^3 + 2048*x^2 - 4096*x + 4096
sage: f3 == f2.composed_op(f, operator.mul)
True
sage: f4 = f.compose_power(4)
sage: f4 == f3.composed_op(f, operator.mul)
True
```

```
>>> from sage.all import *
>>> R = ZZ['a, b, c']; (a, b, c,) = R._first_ngens(3)
>>> x = polygen(R)
>>> f = (x - a) * (x - b) * (x - c)
>>> f.compose_power(Integer(2)).factor()
↪     # needs sage.libs.singular sage.modules
(x - a^2) * (x - b^2) * (x - c^2) * (x - a*b)^2 * (x - a*c)^2 * (x - b*c)^2

>>> # needs sage.libs.singular sage.modules
>>> x = polygen(QQ)
>>> f = x**Integer(2) - Integer(2)*x + Integer(2)
>>> f2 = f.compose_power(Integer(2)); f2
x^4 - 4*x^3 + 8*x^2 - 16*x + 16
>>> f2 == f.composed_op(f, operator.mul)
True
>>> f3 = f.compose_power(Integer(3)); f3
x^8 - 8*x^7 + 32*x^6 - 64*x^5 + 128*x^4 - 512*x^3 + 2048*x^2 - 4096*x + 4096
>>> f3 == f2.composed_op(f, operator.mul)
True
>>> f4 = f.compose_power(Integer(4))
>>> f4 == f3.composed_op(f, operator.mul)
True
```

compose_trunc(*other*, *n*)

Return the composition of `self` and `other`, truncated to $O(x^n)$.

This method currently works for some specific coefficient rings only.

EXAMPLES:

```
sage: Pol.<x> = CBF[]                                                      #
↪needs sage.libs.flint
sage: (1 + x + x^2/2 + x^3/6 + x^4/24 + x^5/120).compose_trunc(1 + x, 2)    #
↪needs sage.libs.flint
([2.708333333333333 +/- ...e-16])*x + [2.716666666666667 +/- ...e-15]

sage: Pol.<x> = QQ['y'][]
```

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```
sage: (1 + x + x^2/2 + x^3/6 + x^4/24 + x^5/120).compose_trunc(1 + x, 2)
Traceback (most recent call last):
...
NotImplementedError: truncated composition is not implemented
for this subclass of polynomials
```

```
>>> from sage.all import *
>>> Pol = CBF['x']; (x,) = Pol._first_ngens(1)# needs sage.libs.flint
>>> (Integer(1) + x + x**Integer(2)/Integer(2) + x**Integer(3)/Integer(6) +␣
↪x**Integer(4)/Integer(24) + x**Integer(5)/Integer(120)).compose_
↪trunc(Integer(1) + x, Integer(2))     # needs sage.libs.flint
([2.708333333333333 +/- ...e-16])*x + [2.716666666666667 +/- ...e-15]

>>> Pol = QQ['y']['x']; (x,) = Pol._first_ngens(1)
>>> (Integer(1) + x + x**Integer(2)/Integer(2) + x**Integer(3)/Integer(6) +␣
↪x**Integer(4)/Integer(24) + x**Integer(5)/Integer(120)).compose_
↪trunc(Integer(1) + x, Integer(2))
Traceback (most recent call last):
...
NotImplementedError: truncated composition is not implemented
for this subclass of polynomials
```

composed_op(*p1*, *p2*, *op*, *algorithm=None*, *monic=False*)

Return the composed sum, difference, product or quotient of this polynomial with another one.

In the case of two monic polynomials p_1 and p_2 over an integral domain, the composed sum, difference, etc. are given by

$$\prod_{p_1(a)=p_2(b)=0} (x - (a * b)), \qquad * \in \{+, -, \times, /\}$$

where the roots a and b are to be considered in the algebraic closure of the fraction field of the coefficients and counted with multiplicities. If the polynomials are not monic this quantity is multiplied by $\alpha_1^{\deg(p_2)} \alpha_2^{\deg(p_1)}$ where α_1 and α_2 are the leading coefficients of p_1 and p_2 respectively.

INPUT:

- `p2` – univariate polynomial belonging to the same polynomial ring as this polynomial
- `op` – `operator.OP` where `OP=add` or `sub` or `mul` or `truediv`
- `algorithm` – can be `'resultant'` or `'BFSS'`; by default the former is used when the polynomials have few nonzero coefficients and small degrees or if the base ring is not $\mathbf{Z}$ or $\mathbf{Q}$. Otherwise the latter is used.
- `monic` – whether to return a monic polynomial. If `True` the coefficients of the result belong to the fraction field of the coefficients.

ALGORITHM:

The computation is straightforward using resultants. Indeed for the composed sum it would be $Res_y(p_1(x - y), p_2(y))$. However, the method from [BFSS2006] using series expansions is asymptotically much faster.

Note that the algorithm `BFSS` with polynomials with coefficients in $\mathbf{Z}$ needs to perform operations over $\mathbf{Q}$.

Todo

- The [BFSS2006] algorithm has been implemented here only in the case of polynomials over rationals. For other rings of zero characteristic (or if the characteristic is larger than the product of the degrees), one needs to implement a generic method `_exp_series`. In the general case of nonzero characteristic there is an alternative algorithm in the same paper.
- The Newton series computation can be done much more efficiently! See [BFSS2006].

EXAMPLES:

```
sage: x = polygen(ZZ)
sage: p1 = x^2 - 1
sage: p2 = x^4 - 1
sage: p1.composed_op(p2, operator.add)                                    #␣
↪needs sage.libs.singular
x^8 - 4*x^6 + 4*x^4 - 16*x^2
sage: p1.composed_op(p2, operator.mul)                                    #␣
↪needs sage.libs.singular
x^8 - 2*x^4 + 1
sage: p1.composed_op(p2, operator.truediv)                                #␣
↪needs sage.libs.singular
x^8 - 2*x^4 + 1
```

```
>>> from sage.all import *
>>> x = polygen(ZZ)
>>> p1 = x**Integer(2) - Integer(1)
>>> p2 = x**Integer(4) - Integer(1)
>>> p1.composed_op(p2, operator.add)                                    #␣
↪needs sage.libs.singular
x^8 - 4*x^6 + 4*x^4 - 16*x^2
>>> p1.composed_op(p2, operator.mul)                                    #␣
↪needs sage.libs.singular
x^8 - 2*x^4 + 1
>>> p1.composed_op(p2, operator.truediv)                                #␣
↪needs sage.libs.singular
x^8 - 2*x^4 + 1
```

This function works over any field. However for base rings other than **Z** and **Q** only the resultant algorithm is available:

```
sage: # needs sage.rings.number_field
sage: x = polygen(QQbar)
sage: p1 = x**2 - AA(2).sqrt()
sage: p2 = x**3 - AA(3).sqrt()
sage: r1 = p1.roots(multiplicities=False)
sage: r2 = p2.roots(multiplicities=False)
sage: p = p1.composed_op(p2, operator.add); p
x^6 - 4.242640687119285?*x^4 - 3.464101615137755?*x^3 + 6*x^2
 - 14.69693845669907?*x + 0.1715728752538099?
sage: all(p(x+y).is_zero() for x in r1 for y in r2)
True
```

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```
sage: x = polygen(GF(2))
sage: p1 = x**2 + x - 1
sage: p2 = x**3 + x - 1
sage: p_add = p1.composed_op(p2, operator.add); p_add                        #␣
↪needs sage.libs.singular
x^6 + x^5 + x^3 + x^2 + 1
sage: p_mul = p1.composed_op(p2, operator.mul); p_mul                        #␣
↪needs sage.libs.singular
x^6 + x^4 + x^2 + x + 1
sage: p_div = p1.composed_op(p2, operator.truediv); p_div                    #␣
↪needs sage.libs.singular
x^6 + x^5 + x^4 + x^2 + 1

sage: # needs sage.rings.finite_rings
sage: K = GF(2**6, 'a')
sage: r1 = p1.roots(K, multiplicities=False)
sage: r2 = p2.roots(K, multiplicities=False)
sage: all(p_add(x1+x2).is_zero() for x1 in r1 for x2 in r2)                  #␣
↪needs sage.libs.singular
True
sage: all(p_mul(x1*x2).is_zero() for x1 in r1 for x2 in r2)                  #␣
↪needs sage.libs.singular
True
sage: all(p_div(x1/x2).is_zero() for x1 in r1 for x2 in r2)                  #␣
↪needs sage.libs.singular
True
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> x = polygen(QQbar)
>>> p1 = x**Integer(2) - AA(Integer(2)).sqrt()
>>> p2 = x**Integer(3) - AA(Integer(3)).sqrt()
>>> r1 = p1.roots(multiplicities=False)
>>> r2 = p2.roots(multiplicities=False)
>>> p = p1.composed_op(p2, operator.add); p
x^6 - 4.242640687119285?*x^4 - 3.464101615137755?*x^3 + 6*x^2
 - 14.69693845669907?*x + 0.1715728752538099?
>>> all(p(x+y).is_zero() for x in r1 for y in r2)
True

>>> x = polygen(GF(Integer(2)))
>>> p1 = x**Integer(2) + x - Integer(1)
>>> p2 = x**Integer(3) + x - Integer(1)
>>> p_add = p1.composed_op(p2, operator.add); p_add                          #␣
↪needs sage.libs.singular
x^6 + x^5 + x^3 + x^2 + 1
>>> p_mul = p1.composed_op(p2, operator.mul); p_mul                          #␣
↪needs sage.libs.singular
x^6 + x^4 + x^2 + x + 1
>>> p_div = p1.composed_op(p2, operator.truediv); p_div                      #␣
↪needs sage.libs.singular
```

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```
x^6 + x^5 + x^4 + x^2 + 1

>>> # needs sage.rings.finite_rings
>>> K = GF(Integer(2)**Integer(6), 'a')
>>> r1 = p1.roots(K, multiplicities=False)
>>> r2 = p2.roots(K, multiplicities=False)
>>> all(p_add(x1+x2).is_zero() for x1 in r1 for x2 in r2)                    #␣
↪needs sage.libs.singular
True
>>> all(p_mul(x1*x2).is_zero() for x1 in r1 for x2 in r2)                    #␣
↪needs sage.libs.singular
True
>>> all(p_div(x1/x2).is_zero() for x1 in r1 for x2 in r2)                    #␣
↪needs sage.libs.singular
True
```

constant_coefficient()

Return the constant coefficient of this polynomial.

OUTPUT: element of base ring

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: f = -2*x^3 + 2*x - 1/3
sage: f.constant_coefficient()
-1/3
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = -Integer(2)*x**Integer(3) + Integer(2)*x - Integer(1)/Integer(3)
>>> f.constant_coefficient()
-1/3
```

content_ideal()

Return the content ideal of this polynomial, defined as the ideal generated by its coefficients.

EXAMPLES:

```
sage: R.<x> = IntegerModRing(4)[]
sage: f = x^4 + 3*x^2 + 2
sage: f.content_ideal()
Ideal (2, 3, 1) of Ring of integers modulo 4
```

```
>>> from sage.all import *
>>> R = IntegerModRing(Integer(4))['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(4) + Integer(3)*x**Integer(2) + Integer(2)
>>> f.content_ideal()
Ideal (2, 3, 1) of Ring of integers modulo 4
```

When the base ring is a gcd ring, the content as a ring element is the generator of the content ideal:

```
sage: R.<x> = ZZ[]
sage: f = 2*x^3 - 4*x^2 + 6*x - 10
sage: f.content_ideal().gen()
2
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> f = Integer(2)*x**Integer(3) - Integer(4)*x**Integer(2) + Integer(6)*x -␣
↪Integer(10)
>>> f.content_ideal().gen()
2
```

cyclotomic_part()

Return the product of the irreducible factors of this polynomial which are cyclotomic polynomials.

The algorithm assumes that the polynomial has rational coefficients.

See also

is_cyclotomic() is_cyclotomic_product() has_cyclotomic_factor()

EXAMPLES:

```
sage: P.<x> = PolynomialRing(Integers())
sage: pol = 2*(x^4 + 1)
sage: pol.cyclotomic_part()
x^4 + 1
sage: pol = x^4 + 2
sage: pol.cyclotomic_part()
1
sage: pol = (x^4 + 1)^2 * (x^4 + 2)
sage: pol.cyclotomic_part()
x^8 + 2*x^4 + 1

sage: P.<x> = PolynomialRing(QQ)
sage: pol = (x^4 + 1)^2 * (x^4 + 2)
sage: pol.cyclotomic_part()
x^8 + 2*x^4 + 1

sage: pol = (x - 1) * x * (x + 2)
sage: pol.cyclotomic_part()
x - 1
```

```
>>> from sage.all import *
>>> P = PolynomialRing(Integers(), names=('x',)); (x,) = P._first_ngens(1)
>>> pol = Integer(2)*(x**Integer(4) + Integer(1))
>>> pol.cyclotomic_part()
x^4 + 1
>>> pol = x**Integer(4) + Integer(2)
>>> pol.cyclotomic_part()
1
>>> pol = (x**Integer(4) + Integer(1))**Integer(2) * (x**Integer(4) +␣
```

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```
↪Integer(2))
>>> pol.cyclotomic_part()
x^8 + 2*x^4 + 1

>>> P = PolynomialRing(QQ, names=('x',)); (x,) = P._first_ngens(1)
>>> pol = (x**Integer(4) + Integer(1))**Integer(2) * (x**Integer(4) +␣
↪Integer(2))
>>> pol.cyclotomic_part()
x^8 + 2*x^4 + 1

>>> pol = (x - Integer(1)) * x * (x + Integer(2))
>>> pol.cyclotomic_part()
x - 1
```

degree(*gen=None*)

Return the degree of this polynomial. The zero polynomial has degree -1.

EXAMPLES:

```
sage: x = ZZ['x'].0
sage: f = x^93 + 2*x + 1
sage: f.degree()
93
sage: x = PolynomialRing(QQ, 'x', sparse=True).0
sage: f = x^100000
sage: f.degree()
100000
```

```
>>> from sage.all import *
>>> x = ZZ['x'].gen(0)
>>> f = x**Integer(93) + Integer(2)*x + Integer(1)
>>> f.degree()
93
>>> x = PolynomialRing(QQ, 'x', sparse=True).gen(0)
>>> f = x**Integer(100000)
>>> f.degree()
100000
```

```
sage: x = QQ['x'].0
sage: f = 2006*x^2006 - x^2 + 3
sage: f.degree()
2006
sage: f = 0*x
sage: f.degree()
-1
sage: f = x + 33
sage: f.degree()
1
```

```
>>> from sage.all import *
>>> x = QQ['x'].gen(0)
>>> f = Integer(2006)*x**Integer(2006) - x**Integer(2) + Integer(3)
```

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```
>>> f.degree()
2006
>>> f = Integer(0)*x
>>> f.degree()
-1
>>> f = x + Integer(33)
>>> f.degree()
1
```

AUTHORS:

- Naqi Jaffery (2006-01-24): examples

denominator()

Return a denominator of `self`.

First, the lcm of the denominators of the entries of `self` is computed and returned. If this computation fails, the unit of the parent of `self` is returned.

Note that some subclasses may implement their own *denominator()* method. For example, see *sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint*

Warning

This is not the denominator of the rational function defined by `self`, which would always be 1 since `self` is a polynomial.

EXAMPLES:

First we compute the denominator of a polynomial with integer coefficients, which is of course 1.

```
sage: R.<x> = ZZ[]
sage: f = x^3 + 17*x + 1
sage: f.denominator()
1
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(3) + Integer(17)*x + Integer(1)
>>> f.denominator()
1
```

Next we compute the denominator of a polynomial with rational coefficients.

```
sage: R.<x> = PolynomialRing(QQ)
sage: f = (1/17)*x^19 - (2/3)*x + 1/3; f
1/17*x^19 - 2/3*x + 1/3
sage: f.denominator()
51
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = (Integer(1)/Integer(17))*x**Integer(19) - (Integer(2)/Integer(3))*x +␣
```

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```
↪Integer(1)/Integer(3); f
1/17*x^19 - 2/3*x + 1/3
>>> f.denominator()
51
```

Finally, we try to compute the denominator of a polynomial with coefficients in the real numbers, which is a ring whose elements do not have a `denominator()` method.

```
sage: # needs sage.rings.real_mpfr
sage: R.<x> = RR[]
sage: f = x + RR('0.3'); f
x + 0.300000000000000
sage: f.denominator()
1.00000000000000
```

```
>>> from sage.all import *
>>> # needs sage.rings.real_mpfr
>>> R = RR['x']; (x,) = R._first_ngens(1)
>>> f = x + RR('0.3'); f
x + 0.300000000000000
>>> f.denominator()
1.00000000000000
```

Check that the denominator is an element over the base whenever the base has no `denominator()` method. This closes Issue #9063.

```
sage: R.<a> = GF(5)[]
sage: x = R(0)
sage: x.denominator()
1
sage: type(x.denominator())
<class 'sage.rings.finite_rings.integer_mod.IntegerMod_int'>
sage: isinstance(x.numerator() / x.denominator(), Polynomial)
True
sage: isinstance(x.numerator() / R(1), Polynomial)
False
```

```
>>> from sage.all import *
>>> R = GF(Integer(5))['a']; (a,) = R._first_ngens(1)
>>> x = R(Integer(0))
>>> x.denominator()
1
>>> type(x.denominator())
<class 'sage.rings.finite_rings.integer_mod.IntegerMod_int'>
>>> isinstance(x.numerator() / x.denominator(), Polynomial)
True
>>> isinstance(x.numerator() / R(Integer(1)), Polynomial)
False
```

derivative(**args*)

The formal derivative of this polynomial, with respect to variables supplied in `args`.

Multiple variables and iteration counts may be supplied; see documentation for the global `derivative()`

function for more details.

See also

`_derivative()`

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: g = -x^4 + x^2/2 - x
sage: g.derivative()
-4*x^3 + x - 1
sage: g.derivative(x)
-4*x^3 + x - 1
sage: g.derivative(x, x)
-12*x^2 + 1
sage: g.derivative(x, 2)
-12*x^2 + 1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> g = -x**Integer(4) + x**Integer(2)/Integer(2) - x
>>> g.derivative()
-4*x^3 + x - 1
>>> g.derivative(x)
-4*x^3 + x - 1
>>> g.derivative(x, x)
-12*x^2 + 1
>>> g.derivative(x, Integer(2))
-12*x^2 + 1
```

```
sage: R.<t> = PolynomialRing(ZZ)
sage: S.<x> = PolynomialRing(R)
sage: f = t^3*x^2 + t^4*x^3
sage: f.derivative()
3*t^4*x^2 + 2*t^3*x
sage: f.derivative(x)
3*t^4*x^2 + 2*t^3*x
sage: f.derivative(t)
4*t^3*x^3 + 3*t^2*x^2
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('t',)); (t,) = R._first_ngens(1)
>>> S = PolynomialRing(R, names=('x',)); (x,) = S._first_ngens(1)
>>> f = t**Integer(3)*x**Integer(2) + t**Integer(4)*x**Integer(3)
>>> f.derivative()
3*t^4*x^2 + 2*t^3*x
>>> f.derivative(x)
3*t^4*x^2 + 2*t^3*x
>>> f.derivative(t)
4*t^3*x^3 + 3*t^2*x^2
```

dict(*copy=None*)

Return a sparse dictionary representation of this univariate polynomial.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: f = x^3 + -1/7*x + 13
sage: f.monomial_coefficients()
{0: 13, 1: -1/7, 3: 1}
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(3) + -Integer(1)/Integer(7)*x + Integer(13)
>>> f.monomial_coefficients()
{0: 13, 1: -1/7, 3: 1}
```

`dict` is an alias:

```
sage: f.dict()
{0: 13, 1: -1/7, 3: 1}
```

```
>>> from sage.all import *
>>> f.dict()
{0: 13, 1: -1/7, 3: 1}
```

diff(**args*)

The formal derivative of this polynomial, with respect to variables supplied in `args`.

Multiple variables and iteration counts may be supplied; see documentation for the global *derivative()* function for more details.

See also

`_derivative()`

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: g = -x^4 + x^2/2 - x
sage: g.derivative()
-4*x^3 + x - 1
sage: g.derivative(x)
-4*x^3 + x - 1
sage: g.derivative(x, x)
-12*x^2 + 1
sage: g.derivative(x, 2)
-12*x^2 + 1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> g = -x**Integer(4) + x**Integer(2)/Integer(2) - x
>>> g.derivative()
-4*x^3 + x - 1
```

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```
>>> g.derivative(x)
-4*x^3 + x - 1
>>> g.derivative(x, x)
-12*x^2 + 1
>>> g.derivative(x, Integer(2))
-12*x^2 + 1
```

```
sage: R.<t> = PolynomialRing(ZZ)
sage: S.<x> = PolynomialRing(R)
sage: f = t^3*x^2 + t^4*x^3
sage: f.derivative()
3*t^4*x^2 + 2*t^3*x
sage: f.derivative(x)
3*t^4*x^2 + 2*t^3*x
sage: f.derivative(t)
4*t^3*x^3 + 3*t^2*x^2
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('t',)); (t,) = R._first_ngens(1)
>>> S = PolynomialRing(R, names=('x',)); (x,) = S._first_ngens(1)
>>> f = t**Integer(3)*x**Integer(2) + t**Integer(4)*x**Integer(3)
>>> f.derivative()
3*t^4*x^2 + 2*t^3*x
>>> f.derivative(x)
3*t^4*x^2 + 2*t^3*x
>>> f.derivative(t)
4*t^3*x^3 + 3*t^2*x^2
```

differentiate(**args*)

The formal derivative of this polynomial, with respect to variables supplied in `args`.

Multiple variables and iteration counts may be supplied; see documentation for the global `derivative()` function for more details.

See also

`_derivative()`

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: g = -x^4 + x^2/2 - x
sage: g.derivative()
-4*x^3 + x - 1
sage: g.derivative(x)
-4*x^3 + x - 1
sage: g.derivative(x, x)
-12*x^2 + 1
sage: g.derivative(x, 2)
-12*x^2 + 1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> g = -x**Integer(4) + x**Integer(2)/Integer(2) - x
>>> g.derivative()
-4*x^3 + x - 1
>>> g.derivative(x)
-4*x^3 + x - 1
>>> g.derivative(x, x)
-12*x^2 + 1
>>> g.derivative(x, Integer(2))
-12*x^2 + 1
```

```
sage: R.<t> = PolynomialRing(ZZ)
sage: S.<x> = PolynomialRing(R)
sage: f = t^3*x^2 + t^4*x^3
sage: f.derivative()
3*t^4*x^2 + 2*t^3*x
sage: f.derivative(x)
3*t^4*x^2 + 2*t^3*x
sage: f.derivative(t)
4*t^3*x^3 + 3*t^2*x^2
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('t',)); (t,) = R._first_ngens(1)
>>> S = PolynomialRing(R, names=('x',)); (x,) = S._first_ngens(1)
>>> f = t**Integer(3)*x**Integer(2) + t**Integer(4)*x**Integer(3)
>>> f.derivative()
3*t^4*x^2 + 2*t^3*x
>>> f.derivative(x)
3*t^4*x^2 + 2*t^3*x
>>> f.derivative(t)
4*t^3*x^3 + 3*t^2*x^2
```

discriminant()

Return the discriminant of `self`.

The discriminant is

$$R_n := a_n^{2n-2} \prod_{1<i<j<n} (r_i - r_j)^2,$$

where n is the degree of `self`, a_n is the leading coefficient of `self`, and the roots of `self` are $r_1, \ldots, r_n$.

OUTPUT: an element of the base ring of the polynomial ring

ALGORITHM:

Uses the identity $R_n(f) := (-1)^{n(n-1)/2} R(f, f') a_n^{n-k-2}$, where n is the degree of `self`, a_n is the leading coefficient of `self`, f' is the derivative of f, and k is the degree of f'. Calls `resultant()`.

EXAMPLES:

In the case of elliptic curves in special form, the discriminant is easy to calculate:

```
sage: R.<x> = QQ[]
sage: f = x^3 + x + 1
```

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```
sage: d = f.discriminant(); d                                                # needs sage.libs.pari
-31
sage: d.parent() is QQ                                                       # needs sage.libs.pari
True
sage: EllipticCurve([1, 1]).discriminant()/16                                # needs sage.libs.pari sage.schemes
-31
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(3) + x + Integer(1)
>>> d = f.discriminant(); d                                                  # needs sage.libs.pari
-31
>>> d.parent() is QQ                                                         # needs sage.libs.pari
True
>>> EllipticCurve([Integer(1), Integer(1)]).discriminant()/Integer(16)                         # needs sage.libs.pari sage.schemes
-31
```

```
sage: R.<x> = QQ[]
sage: f = 2*x^3 + x + 1
sage: d = f.discriminant(); d                                                # needs sage.libs.pari
-116
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = Integer(2)*x**Integer(3) + x + Integer(1)
>>> d = f.discriminant(); d                                                  # needs sage.libs.pari
-116
```

We can compute discriminants over univariate and multivariate polynomial rings:

```
sage: R.<a> = QQ[]
sage: S.<x> = R[]
sage: f = a*x + x + a + 1
sage: d = f.discriminant(); d                                                # needs sage.libs.pari
1
sage: d.parent() is R                                                        # needs sage.libs.pari
True
```

```
>>> from sage.all import *
>>> R = QQ['a']; (a,) = R._first_ngens(1)
>>> S = R['x']; (x,) = S._first_ngens(1)
```

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```
>>> f = a*x + x + a + Integer(1)
>>> d = f.discriminant(); d                                                 #␣
↪needs sage.libs.pari
1
>>> d.parent() is R                                                         #␣
↪needs sage.libs.pari
True
```

```
sage: R.<a, b> = QQ[]
sage: S.<x> = R[]
sage: f = x^2 + a + b
sage: d = f.discriminant(); d                                               #␣
↪needs sage.libs.pari
-4*a - 4*b
sage: d.parent() is R                                                       #␣
↪needs sage.libs.pari
True
```

```
>>> from sage.all import *
>>> R = QQ['a, b']; (a, b,) = R._first_ngens(2)
>>> S = R['x']; (x,) = S._first_ngens(1)
>>> f = x**Integer(2) + a + b
>>> d = f.discriminant(); d                                                 #␣
↪needs sage.libs.pari
-4*a - 4*b
>>> d.parent() is R                                                         #␣
↪needs sage.libs.pari
True
```

dispersion(*other=None*)

Compute the dispersion of a pair of polynomials.

The dispersion of f and g is the largest nonnegative integer n such that $f(x+n)$ and $g(x)$ have a nonconstant common factor.

When `other` is `None`, compute the auto-dispersion of `self`, i.e., its dispersion with itself.

See also

dispersion_set()

EXAMPLES:

```
sage: Pol.<x> = QQ[]
sage: x.dispersion(x + 1)                                                   #␣
↪needs sage.libs.pari
1
sage: (x + 1).dispersion(x)                                                 #␣
↪needs sage.libs.pari
-Infinity

sage: # needs sage.libs.pari sage.rings.number_field sage.symbolic
```

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```
sage: Pol.<x> = QQbar[]
sage: pol = Pol([sqrt(5), 1, 3/2])
sage: pol.dispersion()
0
sage: (pol*pol(x+3)).dispersion()
3
```

```
>>> from sage.all import *
>>> Pol = QQ['x']; (x,) = Pol._first_ngens(1)
>>> x.dispersion(x + Integer(1))                                          ␣
↪     # needs sage.libs.pari
1
>>> (x + Integer(1)).dispersion(x)                                        ␣
↪     # needs sage.libs.pari
-Infinity

>>> # needs sage.libs.pari sage.rings.number_field sage.symbolic
>>> Pol = QQbar['x']; (x,) = Pol._first_ngens(1)
>>> pol = Pol([sqrt(Integer(5)), Integer(1), Integer(3)/Integer(2)])
>>> pol.dispersion()
0
>>> (pol*pol(x+Integer(3))).dispersion()
3
```

dispersion_set(*other=None*)

Compute the dispersion set of two polynomials.

The dispersion set of f and g is the set of nonnegative integers n such that $f(x + n)$ and $g(x)$ have a nonconstant common factor.

When `other` is `None`, compute the auto-dispersion set of `self`, i.e., its dispersion set with itself.

ALGORITHM:

See Section 4 of Man & Wright [MW1994].

See also

dispersion()

EXAMPLES:

```
sage: Pol.<x> = QQ[]
sage: x.dispersion_set(x + 1)                                            #␣
↪needs sage.libs.pari
[1]
sage: (x + 1).dispersion_set(x)                                          #␣
↪needs sage.libs.pari
[]

sage: pol = x^3 + x - 7
sage: (pol*pol(x+3)^2).dispersion_set()                                  #␣
```

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```
↪needs sage.libs.pari
[0, 3]
```

```
>>> from sage.all import *
>>> Pol = QQ['x']; (x,) = Pol._first_ngens(1)
>>> x.dispersion_set(x + Integer(1))                                         ␣
↪     # needs sage.libs.pari
[1]
>>> (x + Integer(1)).dispersion_set(x)                                       ␣
↪     # needs sage.libs.pari
[]

>>> pol = x**Integer(3) + x - Integer(7)
>>> (pol*pol(x+Integer(3))**Integer(2)).dispersion_set()                     ␣
↪               # needs sage.libs.pari
[0, 3]
```

divides(*p*)

Return `True` if this polynomial divides p.

This method is only implemented for polynomials over an integral domain.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: (2*x + 1).divides(4*x**2 - 1)
True
sage: (2*x + 1).divides(4*x**2 + 1)
False
sage: (2*x + 1).divides(R(0))
True
sage: R(0).divides(2*x + 1)
False
sage: R(0).divides(R(0))
True
sage: S.<y> = R[]
sage: p = x * y**2 + (2*x + 1) * y + x + 1
sage: q = (x + 1) * y + (3*x + 2)
sage: q.divides(p)
False
sage: q.divides(p * q)
True
sage: R.<x> = Zmod(6)[]
sage: p = 4*x + 3
sage: q = 5*x**2 + x + 2
sage: q.divides(p)
False
sage: p.divides(q)
False
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
```

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```
>>> (Integer(2)*x + Integer(1)).divides(Integer(4)*x**Integer(2) - Integer(1))
True
>>> (Integer(2)*x + Integer(1)).divides(Integer(4)*x**Integer(2) + Integer(1))
False
>>> (Integer(2)*x + Integer(1)).divides(R(Integer(0)))
True
>>> R(Integer(0)).divides(Integer(2)*x + Integer(1))
False
>>> R(Integer(0)).divides(R(Integer(0)))
True
>>> S = R['y']; (y,) = S._first_ngens(1)
>>> p = x * y**Integer(2) + (Integer(2)*x + Integer(1)) * y + x + Integer(1)
>>> q = (x + Integer(1)) * y + (Integer(3)*x + Integer(2))
>>> q.divides(p)
False
>>> q.divides(p * q)
True
>>> R = Zmod(Integer(6))['x']; (x,) = R._first_ngens(1)
>>> p = Integer(4)*x + Integer(3)
>>> q = Integer(5)*x**Integer(2) + x + Integer(2)
>>> q.divides(p)
False
>>> p.divides(q)
False
```

euclidean_degree()

Return the degree of this element as an element of a Euclidean domain.

If this polynomial is defined over a field, this is simply its *degree()*.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: x.euclidean_degree()
1
sage: R.<x> = ZZ[]
sage: x.euclidean_degree()
Traceback (most recent call last):
...
NotImplementedError
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> x.euclidean_degree()
1
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> x.euclidean_degree()
Traceback (most recent call last):
...
NotImplementedError
```

exponents()

Return the exponents of the monomials appearing in `self`.

EXAMPLES:

```
sage: _.<x> = PolynomialRing(ZZ)
sage: f = x^4 + 2*x^2 + 1
sage: f.exponents()
[0, 2, 4]
```

```
>>> from sage.all import *
>>> _ = PolynomialRing(ZZ, names=('x',)); (x,) = _._first_ngens(1)
>>> f = x**Integer(4) + Integer(2)*x**Integer(2) + Integer(1)
>>> f.exponents()
[0, 2, 4]
```

factor(***kwargs*)

Return the factorization of `self` over its base ring.

INPUT:

- `kwargs` – any keyword arguments are passed to the method `_factor_univariate_polynomial()` of the base ring if it defines such a method.

OUTPUT:

A factorization of `self` over its parent into a unit and irreducible factors. If the parent is a polynomial ring over a field, these factors are monic.

EXAMPLES:

Factorization is implemented over various rings. Over **Q**:

```
sage: x = QQ['x'].0
sage: f = (x^3 - 1)^2
sage: f.factor()                                                            #␣
↪needs sage.libs.pari
(x - 1)^2 * (x^2 + x + 1)^2
```

```
>>> from sage.all import *
>>> x = QQ['x'].gen(0)
>>> f = (x**Integer(3) - Integer(1))**Integer(2)
>>> f.factor()                                                            #␣
↪needs sage.libs.pari
(x - 1)^2 * (x^2 + x + 1)^2
```

Since **Q** is a field, the irreducible factors are monic:

```
sage: f = 10*x^5 - 1
sage: f.factor()                                                            #␣
↪needs sage.libs.pari
(10) * (x^5 - 1/10)
sage: f = 10*x^5 - 10
sage: f.factor()                                                            #␣
↪needs sage.libs.pari
(10) * (x - 1) * (x^4 + x^3 + x^2 + x + 1)
```

```
>>> from sage.all import *
>>> f = Integer(10)*x**Integer(5) - Integer(1)
```

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```
>>> f.factor()                                                    #␣
↪needs sage.libs.pari
(10) * (x^5 - 1/10)
>>> f = Integer(10)*x**Integer(5) - Integer(10)
>>> f.factor()                                                    #␣
↪needs sage.libs.pari
(10) * (x - 1) * (x^4 + x^3 + x^2 + x + 1)
```

Over **Z** the irreducible factors need not be monic:

```
sage: x = ZZ['x'].0
sage: f = 10*x^5 - 1
sage: f.factor()                                                  #␣
↪needs sage.libs.pari
10*x^5 - 1
```

```
>>> from sage.all import *
>>> x = ZZ['x'].gen(0)
>>> f = Integer(10)*x**Integer(5) - Integer(1)
>>> f.factor()                                                    #␣
↪needs sage.libs.pari
10*x^5 - 1
```

We factor a non-monic polynomial over a finite field of 25 elements:

```
sage: # needs sage.rings.finite_rings
sage: k.<a> = GF(25)
sage: R.<x> = k[]
sage: f = 2*x^10 + 2*x + 2*a
sage: F = f.factor(); F
(2) * (x + a + 2) * (x^2 + 3*x + 4*a + 4) * (x^2 + (a + 1)*x + a + 2)
* (x^5 + (3*a + 4)*x^4 + (3*a + 3)*x^3 + 2*a*x^2 + (3*a + 1)*x + 3*a + 1)
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> k = GF(Integer(25), names=('a',)); (a,) = k._first_ngens(1)
>>> R = k['x']; (x,) = R._first_ngens(1)
>>> f = Integer(2)*x**Integer(10) + Integer(2)*x + Integer(2)*a
>>> F = f.factor(); F
(2) * (x + a + 2) * (x^2 + 3*x + 4*a + 4) * (x^2 + (a + 1)*x + a + 2)
* (x^5 + (3*a + 4)*x^4 + (3*a + 3)*x^3 + 2*a*x^2 + (3*a + 1)*x + 3*a + 1)
```

Notice that the unit factor is included when we multiply F back out:

```
sage: expand(F)                                                   #␣
↪needs sage.rings.finite_rings sage.symbolic
2*x^10 + 2*x + 2*a
```

```
>>> from sage.all import *
>>> expand(F)                                                     #␣
↪needs sage.rings.finite_rings sage.symbolic
2*x^10 + 2*x + 2*a
```

A new ring. In the example below, we set the special method `_factor_univariate_polynomial()` in the base ring which is called to factor univariate polynomials. This facility can be used to easily extend polynomial factorization to work over new rings you introduce:

```
sage: # needs sage.libs.ntl
sage: R.<x> = PolynomialRing(IntegerModRing(4), implementation="NTL")
sage: (x^2).factor()
Traceback (most recent call last):
...
NotImplementedError: factorization of polynomials over rings with
composite characteristic is not implemented
sage: def my_factor(f):
....:     return f.change_ring(ZZ).factor()
sage: R.base_ring()._factor_univariate_polynomial = my_factor
sage: (x^2).factor()                                                    #␣
↪needs sage.libs.pari
x^2
sage: del R.base_ring()._factor_univariate_polynomial  # clean up
```

```
>>> from sage.all import *
>>> # needs sage.libs.ntl
>>> R = PolynomialRing(IntegerModRing(Integer(4)), implementation="NTL",␣
↪names=('x',)); (x,) = R._first_ngens(1)
>>> (x**Integer(2)).factor()
Traceback (most recent call last):
...
NotImplementedError: factorization of polynomials over rings with
composite characteristic is not implemented
>>> def my_factor(f):
...     return f.change_ring(ZZ).factor()
>>> R.base_ring()._factor_univariate_polynomial = my_factor
>>> (x**Integer(2)).factor()                                              ␣
↪      # needs sage.libs.pari
x^2
>>> del R.base_ring()._factor_univariate_polynomial  # clean up
```

Arbitrary precision real and complex factorization:

```
sage: # needs sage.libs.pari sage.rings.real_mpfr
sage: R.<x> = RealField(100)[]
sage: F = factor(x^2 - 3); F
(x - 1.7320508075688772935274463415) * (x + 1.7320508075688772935274463415)
sage: expand(F)
x^2 - 3.0000000000000000000000000000
sage: factor(x^2 + 1)
x^2 + 1.0000000000000000000000000000

sage: # needs sage.libs.pari sage.rings.real_mpfr
sage: R.<x> = ComplexField(100)[]
sage: F = factor(x^2 + 3); F
(x - 1.7320508075688772935274463415*I) * (x + 1.
↪7320508075688772935274463415*I)
sage: expand(F)
```

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```
x^2 + 3.0000000000000000000000000000
sage: factor(x^2 + 1)
(x - I) * (x + I)
sage: f = R(I) * (x^2 + 1) ; f
I*x^2 + I
sage: F = factor(f); F
(1.0000000000000000000000000000*I) * (x - I) * (x + I)
sage: expand(F)
I*x^2 + I
```

```
>>> from sage.all import *
>>> # needs sage.libs.pari sage.rings.real_mpfr
>>> R = RealField(Integer(100))['x']; (x,) = R._first_ngens(1)
>>> F = factor(x**Integer(2) - Integer(3)); F
(x - 1.7320508075688772935274463415) * (x + 1.7320508075688772935274463415)
>>> expand(F)
x^2 - 3.0000000000000000000000000000
>>> factor(x**Integer(2) + Integer(1))
x^2 + 1.0000000000000000000000000000

>>> # needs sage.libs.pari sage.rings.real_mpfr
>>> R = ComplexField(Integer(100))['x']; (x,) = R._first_ngens(1)
>>> F = factor(x**Integer(2) + Integer(3)); F
(x - 1.7320508075688772935274463415*I) * (x + 1.
↪7320508075688772935274463415*I)
>>> expand(F)
x^2 + 3.0000000000000000000000000000
>>> factor(x**Integer(2) + Integer(1))
(x - I) * (x + I)
>>> f = R(I) * (x**Integer(2) + Integer(1)) ; f
I*x^2 + I
>>> F = factor(f); F
(1.0000000000000000000000000000*I) * (x - I) * (x + I)
>>> expand(F)
I*x^2 + I
```

Over a number field:

```
sage: # needs sage.rings.number_field
sage: K.<z> = CyclotomicField(15)
sage: x = polygen(K)
sage: ((x^3 + z*x + 1)^3 * (x - z)).factor()
(x - z) * (x^3 + z*x + 1)^3
sage: cyclotomic_polynomial(12).change_ring(K).factor()
(x^2 - z^5 - 1) * (x^2 + z^5)
sage: ((x^3 + z*x + 1)^3 * (x/(z+2) - 1/3)).factor()
(-1/331*z^7 + 3/331*z^6 - 6/331*z^5 + 11/331*z^4
    - 21/331*z^3 + 41/331*z^2 - 82/331*z + 165/331)
* (x - 1/3*z - 2/3) * (x^3 + z*x + 1)^3
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
```

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```
>>> K = CyclotomicField(Integer(15), names=('z',)); (z,) = K._first_ngens(1)
>>> x = polygen(K)
>>> ((x**Integer(3) + z*x + Integer(1))**Integer(3) * (x - z)).factor()
(x - z) * (x^3 + z*x + 1)^3
>>> cyclotomic_polynomial(Integer(12)).change_ring(K).factor()
(x^2 - z^5 - 1) * (x^2 + z^5)
>>> ((x**Integer(3) + z*x + Integer(1))**Integer(3) * (x/(z+Integer(2)) -␣
↪Integer(1)/Integer(3))).factor()
(-1/331*z^7 + 3/331*z^6 - 6/331*z^5 + 11/331*z^4
    - 21/331*z^3 + 41/331*z^2 - 82/331*z + 165/331)
* (x - 1/3*z - 2/3) * (x^3 + z*x + 1)^3
```

Over a relative number field:

```
sage: # needs sage.rings.number_field
sage: x = polygen(QQ)
sage: K.<z> = CyclotomicField(3)
sage: L.<a> = K.extension(x^3 - 2)
sage: t = polygen(L, 't')
sage: f = (t^3 + t + a) * (t^5 + t + z); f
t^8 + t^6 + a*t^5 + t^4 + z*t^3 + t^2 + (a + z)*t + z*a
sage: f.factor()
(t^3 + t + a) * (t^5 + t + z)
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> x = polygen(QQ)
>>> K = CyclotomicField(Integer(3), names=('z',)); (z,) = K._first_ngens(1)
>>> L = K.extension(x**Integer(3) - Integer(2), names=('a',)); (a,) = L._
↪first_ngens(1)
>>> t = polygen(L, 't')
>>> f = (t**Integer(3) + t + a) * (t**Integer(5) + t + z); f
t^8 + t^6 + a*t^5 + t^4 + z*t^3 + t^2 + (a + z)*t + z*a
>>> f.factor()
(t^3 + t + a) * (t^5 + t + z)
```

Over the real double field:

```
sage: # needs numpy
sage: R.<x> = RDF[]
sage: (-2*x^2 - 1).factor()
(-2.0) * (x^2 + 0.5000000000000001)
sage: (-2*x^2 - 1).factor().expand()
-2.0*x^2 - 1.0000000000000002
sage: f = (x - 1)^3
sage: f.factor()  # abs tol 2e-5
(x - 1.0000065719436413) * (x^2 - 1.9999934280563585*x + 0.9999934280995487)
```

```
>>> from sage.all import *
>>> # needs numpy
>>> R = RDF['x']; (x,) = R._first_ngens(1)
>>> (-Integer(2)*x**Integer(2) - Integer(1)).factor()
```

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```
(-2.0) * (x^2 + 0.5000000000000001)
>>> (-Integer(2)*x**Integer(2) - Integer(1)).factor().expand()
-2.0*x^2 - 1.0000000000000002
>>> f = (x - Integer(1))**Integer(3)
>>> f.factor()  # abs tol 2e-5
(x - 1.0000065719436413) * (x^2 - 1.9999934280563585*x + 0.9999934280995487)
```

The above output is incorrect because it relies on the `roots()` method, which does not detect that all the roots are real:

```
sage: f.roots()  # abs tol 2e-5                                          #␣
↪needs numpy
[(1.0000065719436413, 1)]
```

```
>>> from sage.all import *
>>> f.roots()  # abs tol 2e-5                                        #␣
↪needs numpy
[(1.0000065719436413, 1)]
```

Over the complex double field the factors are approximate and therefore occur with multiplicity 1:

```
sage: # needs numpy sage.rings.complex_double
sage: R.<x> = CDF[]
sage: f = (x^2 + 2*R(I))^3
sage: F = f.factor()
sage: F  # abs tol 3e-5
(x - 1.0000138879287663 + 1.0000013435286879*I)
* (x - 0.9999942196864997 + 0.9999873009803959*I)
* (x - 0.9999918923847313 + 1.0000113554909125*I)
* (x + 0.9999908759550227 - 1.0000069659624138*I)
* (x + 0.9999985293216753 - 0.9999886153831807*I)
* (x + 1.0000105947233 - 1.0000044186544053*I)
sage: [f(t[0][0]).abs() for t in F]  # abs tol 1e-13
[1.979365054e-14, 1.97936298566e-14, 1.97936990747e-14,
 3.6812407475e-14, 3.65211563729e-14, 3.65220890052e-14]
```

```
>>> from sage.all import *
>>> # needs numpy sage.rings.complex_double
>>> R = CDF['x']; (x,) = R._first_ngens(1)
>>> f = (x**Integer(2) + Integer(2)*R(I))**Integer(3)
>>> F = f.factor()
>>> F  # abs tol 3e-5
(x - 1.0000138879287663 + 1.0000013435286879*I)
* (x - 0.9999942196864997 + 0.9999873009803959*I)
* (x - 0.9999918923847313 + 1.0000113554909125*I)
* (x + 0.9999908759550227 - 1.0000069659624138*I)
* (x + 0.9999985293216753 - 0.9999886153831807*I)
* (x + 1.0000105947233 - 1.0000044186544053*I)
>>> [f(t[Integer(0)][Integer(0)]).abs() for t in F]  # abs tol 1e-13
[1.979365054e-14, 1.97936298566e-14, 1.97936990747e-14,
 3.6812407475e-14, 3.65211563729e-14, 3.65220890052e-14]
```

Factoring polynomials over $\mathbf{Z}/n\mathbf{Z}$ for composite n is not implemented:

```
sage: R.<x> = PolynomialRing(Integers(35))
sage: f = (x^2 + 2*x + 2) * (x^2 + 3*x + 9)
sage: f.factor()
Traceback (most recent call last):
...
NotImplementedError: factorization of polynomials over
rings with composite characteristic is not implemented
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(35)), names=('x',)); (x,) = R._first_
↪ngens(1)
>>> f = (x**Integer(2) + Integer(2)*x + Integer(2)) * (x**Integer(2) +␣
↪Integer(3)*x + Integer(9))
>>> f.factor()
Traceback (most recent call last):
...
NotImplementedError: factorization of polynomials over
rings with composite characteristic is not implemented
```

Factoring polynomials over the algebraic numbers (see Issue #8544):

```
sage: R.<x> = QQbar[]                                                        #␣
↪needs sage.rings.number_field
sage: (x^8 - 1).factor()                                                     #␣
↪needs sage.rings.number_field
(x - 1) * (x - 0.7071067811865475? - 0.7071067811865475?*I)
* (x - 0.7071067811865475? + 0.7071067811865475?*I) * (x - I) * (x + I)
* (x + 0.7071067811865475? - 0.7071067811865475?*I)
* (x + 0.7071067811865475? + 0.7071067811865475?*I) * (x + 1)
```

```
>>> from sage.all import *
>>> R = QQbar['x']; (x,) = R._first_ngens(1)# needs sage.rings.number_field
>>> (x**Integer(8) - Integer(1)).factor()                                     ␣
↪                # needs sage.rings.number_field
(x - 1) * (x - 0.7071067811865475? - 0.7071067811865475?*I)
* (x - 0.7071067811865475? + 0.7071067811865475?*I) * (x - I) * (x + I)
* (x + 0.7071067811865475? - 0.7071067811865475?*I)
* (x + 0.7071067811865475? + 0.7071067811865475?*I) * (x + 1)
```

Factoring polynomials over the algebraic reals (see Issue #8544):

```
sage: R.<x> = AA[]                                                           #␣
↪needs sage.rings.number_field
sage: (x^8 + 1).factor()                                                     #␣
↪needs sage.rings.number_field
(x^2 - 1.847759065022574?*x + 1.000000000000000?)
* (x^2 - 0.7653668647301795?*x + 1.000000000000000?)
* (x^2 + 0.7653668647301795?*x + 1.000000000000000?)
* (x^2 + 1.847759065022574?*x + 1.000000000000000?)
```

```
>>> from sage.all import *
>>> R = AA['x']; (x,) = R._first_ngens(1)# needs sage.rings.number_field
```

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```
>>> (x**Integer(8) + Integer(1)).factor()
↪                # needs sage.rings.number_field
(x^2 - 1.847759065022574?*x + 1.000000000000000?)
* (x^2 - 0.7653668647301795?*x + 1.000000000000000?)
* (x^2 + 0.7653668647301795?*x + 1.000000000000000?)
* (x^2 + 1.847759065022574?*x + 1.000000000000000?)
```

gcd(*other*)

Return a greatest common divisor of this polynomial and `other`.

INPUT:

- `other` – a polynomial in the same ring as this polynomial

OUTPUT:

A greatest common divisor as a polynomial in the same ring as this polynomial. If the base ring is a field, the return value is a monic polynomial.

> **Note**
>
> The actual algorithm for computing greatest common divisors depends on the base ring underlying the polynomial ring. If the base ring defines a method `_gcd_univariate_polynomial()`, then this method will be called (see examples below).

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: (2*x^2).gcd(2*x)
x
sage: R.zero().gcd(0)
0
sage: (2*x).gcd(0)
x
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> (Integer(2)*x**Integer(2)).gcd(Integer(2)*x)
x
>>> R.zero().gcd(Integer(0))
0
>>> (Integer(2)*x).gcd(Integer(0))
x
```

One can easily add gcd functionality to new rings by providing a method `_gcd_univariate_polynomial`:

```
sage: # needs sage.rings.number_field sage.symbolic
sage: O = ZZ[-sqrt(5)]
sage: R.<x> = O[]
sage: a = O.1
sage: p = x + a
sage: q = x^2 - 5
sage: p.gcd(q)
```

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```
Traceback (most recent call last):
...
NotImplementedError: Order of conductor 2 generated by a in Number
Field in a with defining polynomial x^2 - 5 with a = -2.236067977499790?
does not provide a gcd implementation for univariate polynomials
sage: S.<x> = O.number_field()[]
sage: O._gcd_univariate_polynomial = lambda f, g: R(S(f).gcd(S(g)))
sage: p.gcd(q)
x + a
sage: del O._gcd_univariate_polynomial
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field sage.symbolic
>>> O = ZZ[-sqrt(Integer(5))]
>>> R = O['x']; (x,) = R._first_ngens(1)
>>> a = O.gen(1)
>>> p = x + a
>>> q = x**Integer(2) - Integer(5)
>>> p.gcd(q)
Traceback (most recent call last):
...
NotImplementedError: Order of conductor 2 generated by a in Number
Field in a with defining polynomial x^2 - 5 with a = -2.236067977499790?
does not provide a gcd implementation for univariate polynomials
>>> S = O.number_field()['x']; (x,) = S._first_ngens(1)
>>> O._gcd_univariate_polynomial = lambda f, g: R(S(f).gcd(S(g)))
>>> p.gcd(q)
x + a
>>> del O._gcd_univariate_polynomial
```

Use multivariate implementation for polynomials over polynomials rings:

```
sage: R.<x> = ZZ[]
sage: S.<y> = R[]
sage: T.<z> = S[]
sage: r = 2*x*y + z
sage: p = r * (3*x*y*z - 1)
sage: q = r * (x + y + z - 2)
sage: p.gcd(q)                                                          #␣
↪needs sage.libs.singular
z + 2*x*y

sage: R.<x> = QQ[]
sage: S.<y> = R[]
sage: r = 2*x*y + 1
sage: p = r * (x - 1/2 * y)
sage: q = r * (x*y^2 - x + 1/3)
sage: p.gcd(q)                                                          #␣
↪needs sage.libs.singular
2*x*y + 1
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> S = R['y']; (y,) = S._first_ngens(1)
>>> T = S['z']; (z,) = T._first_ngens(1)
>>> r = Integer(2)*x*y + z
>>> p = r * (Integer(3)*x*y*z - Integer(1))
>>> q = r * (x + y + z - Integer(2))
>>> p.gcd(q)                                                                    #␣
↪needs sage.libs.singular
z + 2*x*y

>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> S = R['y']; (y,) = S._first_ngens(1)
>>> r = Integer(2)*x*y + Integer(1)
>>> p = r * (x - Integer(1)/Integer(2) * y)
>>> q = r * (x*y**Integer(2) - x + Integer(1)/Integer(3))
>>> p.gcd(q)                                                                    #␣
↪needs sage.libs.singular
2*x*y + 1
```

global_height(*prec=None*)

Return the (projective) global height of the polynomial.

This returns the absolute logarithmic height of the coefficients thought of as a projective point.

INPUT:

- `prec` – desired floating point precision (default: default `RealField` precision)

OUTPUT: a real number

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: f = 3*x^3 + 2*x^2 + x
sage: exp(f.global_height())                                                    #␣
↪needs sage.symbolic
3.00000000000000
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = Integer(3)*x**Integer(3) + Integer(2)*x**Integer(2) + x
>>> exp(f.global_height())                                                      #␣
↪needs sage.symbolic
3.00000000000000
```

Scaling should not change the result:

```
sage: R.<x> = PolynomialRing(QQ)
sage: f = 1/25*x^2 + 25/3*x + 1
sage: f.global_height()                                                         #␣
↪needs sage.symbolic
6.43775164973640
sage: g = 100 * f
sage: g.global_height()                                                         #␣
```

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```
↪needs sage.symbolic
6.43775164973640
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = Integer(1)/Integer(25)*x**Integer(2) + Integer(25)/Integer(3)*x +␣
↪Integer(1)
>>> f.global_height()                                                     #␣
↪needs sage.symbolic
6.43775164973640
>>> g = Integer(100) * f
>>> g.global_height()                                                     #␣
↪needs sage.symbolic
6.43775164973640
```

```
sage: R.<x> = PolynomialRing(QQbar)                                         #␣
↪needs sage.rings.number_field
sage: f = QQbar(i)*x^2 + 3*x                                                #␣
↪needs sage.rings.number_field
sage: f.global_height()                                                     #␣
↪needs sage.rings.number_field
1.09861228866811
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQbar, names=('x',)); (x,) = R._first_ngens(1)# needs␣
↪sage.rings.number_field
>>> f = QQbar(i)*x**Integer(2) + Integer(3)*x                              ␣
↪               # needs sage.rings.number_field
>>> f.global_height()                                                     #␣
↪needs sage.rings.number_field
1.09861228866811
```

```
sage: # needs sage.rings.number_field
sage: R.<x> = PolynomialRing(QQ)
sage: K.<k> = NumberField(x^2 + 5)
sage: T.<t> = PolynomialRing(K)
sage: f = 1/1331 * t^2 + 5 * t + 7
sage: f.global_height()
9.13959596745043
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> K = NumberField(x**Integer(2) + Integer(5), names=('k',)); (k,) = K._
↪first_ngens(1)
>>> T = PolynomialRing(K, names=('t',)); (t,) = T._first_ngens(1)
>>> f = Integer(1)/Integer(1331) * t**Integer(2) + Integer(5) * t + Integer(7)
>>> f.global_height()
9.13959596745043
```

```
sage: R.<x> = QQ[]
sage: f = 1/123*x^2 + 12
sage: f.global_height(prec=2)                                                    #␣
↪needs sage.symbolic
8.0
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = Integer(1)/Integer(123)*x**Integer(2) + Integer(12)
>>> f.global_height(prec=Integer(2))                                              ␣
↪     # needs sage.symbolic
8.0
```

```
sage: R.<x> = QQ[]
sage: f = 0*x
sage: f.global_height()                                                          #␣
↪needs sage.rings.real_mpfr
0.000000000000000
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = Integer(0)*x
>>> f.global_height()                                                          #␣
↪needs sage.rings.real_mpfr
0.000000000000000
```

gradient()

Return a list of the partial derivative of `self` with respect to the variable of this univariate polynomial.

There is only one partial derivative.

EXAMPLES:

```
sage: P.<x> = QQ[]
sage: f = x^2 + (2/3)*x + 1
sage: f.gradient()
[2*x + 2/3]
sage: f = P(1)
sage: f.gradient()
[0]
```

```
>>> from sage.all import *
>>> P = QQ['x']; (x,) = P._first_ngens(1)
>>> f = x**Integer(2) + (Integer(2)/Integer(3))*x + Integer(1)
>>> f.gradient()
[2*x + 2/3]
>>> f = P(Integer(1))
>>> f.gradient()
[0]
```

hamming_weight()

Return the number of nonzero coefficients of `self`.

Also called weight, Hamming weight or sparsity.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: f = x^3 - x
sage: f.number_of_terms()
2
sage: R(0).number_of_terms()
0
sage: f = (x + 1)^100
sage: f.number_of_terms()
101
sage: S = GF(5)['y']
sage: S(f).number_of_terms()
5
sage: cyclotomic_polynomial(105).number_of_terms()
33
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(3) - x
>>> f.number_of_terms()
2
>>> R(Integer(0)).number_of_terms()
0
>>> f = (x + Integer(1))**Integer(100)
>>> f.number_of_terms()
101
>>> S = GF(Integer(5))['y']
>>> S(f).number_of_terms()
5
>>> cyclotomic_polynomial(Integer(105)).number_of_terms()
33
```

The method `hamming_weight()` is an alias:

```
sage: f.hamming_weight()
101
```

```
>>> from sage.all import *
>>> f.hamming_weight()
101
```

has_cyclotomic_factor()

Return `True` if the given polynomial has a nontrivial cyclotomic factor.

The algorithm assumes that the polynomial has rational coefficients.

If the polynomial is known to be irreducible, it may be slightly more efficient to call `is_cyclotomic()` instead.

See also

`is_cyclotomic()` `is_cyclotomic_product()` `cyclotomic_part()`

EXAMPLES:

```
sage: pol.<x> = PolynomialRing(Rationals())
sage: u = x^5 - 1; u.has_cyclotomic_factor()
True
sage: u = x^5 - 2; u.has_cyclotomic_factor()
False
sage: u = pol(cyclotomic_polynomial(7)) * pol.random_element()  # random
sage: u.has_cyclotomic_factor()                                  # random
True
```

```
>>> from sage.all import *
>>> pol = PolynomialRing(Rationals(), names=('x',)); (x,) = pol._first_
↪ngens(1)
>>> u = x**Integer(5) - Integer(1); u.has_cyclotomic_factor()
True
>>> u = x**Integer(5) - Integer(2); u.has_cyclotomic_factor()
False
>>> u = pol(cyclotomic_polynomial(Integer(7))) * pol.random_element()  #␣
↪random
>>> u.has_cyclotomic_factor()                                  # random
True
```

homogenize(*var='h'*)

Return the homogenization of this polynomial.

The polynomial itself is returned if it is homogeneous already. Otherwise, its monomials are multiplied with the smallest powers of `var` such that they all have the same total degree.

INPUT:

- `var` – a variable in the polynomial ring (as a string, an element of the ring, or `0`) or a name for a new variable (default: `'h'`)

OUTPUT:

If `var` specifies the variable in the polynomial ring, then a homogeneous element in that ring is returned. Otherwise, a homogeneous element is returned in a polynomial ring with an extra last variable `var`.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: f = x^2 + 1
sage: f.homogenize()
x^2 + h^2
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(2) + Integer(1)
>>> f.homogenize()
x^2 + h^2
```

The parameter `var` can be used to specify the name of the variable:

```
sage: g = f.homogenize('z'); g
x^2 + z^2
```

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```
sage: g.parent()
Multivariate Polynomial Ring in x, z over Rational Field
```

```
>>> from sage.all import *
>>> g = f.homogenize('z'); g
x^2 + z^2
>>> g.parent()
Multivariate Polynomial Ring in x, z over Rational Field
```

However, if the polynomial is homogeneous already, then that parameter is ignored and no extra variable is added to the polynomial ring:

```
sage: f = x^2
sage: g = f.homogenize('z'); g
x^2
sage: g.parent()
Univariate Polynomial Ring in x over Rational Field
```

```
>>> from sage.all import *
>>> f = x**Integer(2)
>>> g = f.homogenize('z'); g
x^2
>>> g.parent()
Univariate Polynomial Ring in x over Rational Field
```

For compatibility with the multivariate case, if `var` specifies the variable of the polynomial ring, then the monomials are multiplied with the smallest powers of `var` such that the result is homogeneous; in other words, we end up with a monomial whose leading coefficient is the sum of the coefficients of the polynomial:

```
sage: f = x^2 + x + 1
sage: f.homogenize('x')
3*x^2
```

```
>>> from sage.all import *
>>> f = x**Integer(2) + x + Integer(1)
>>> f.homogenize('x')
3*x^2
```

In positive characteristic, the degree can drop in this case:

```
sage: R.<x> = GF(2)[]
sage: f = x + 1
sage: f.homogenize(x)
0
```

```
>>> from sage.all import *
>>> R = GF(Integer(2))['x']; (x,) = R._first_ngens(1)
>>> f = x + Integer(1)
>>> f.homogenize(x)
0
```

For compatibility with the multivariate case, the parameter `var` can also be 0 to specify the variable in the polynomial ring:

```
sage: R.<x> = QQ[]
sage: f = x^2 + x + 1
sage: f.homogenize(0)
3*x^2
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(2) + x + Integer(1)
>>> f.homogenize(Integer(0))
3*x^2
```

integral(*var=None*)

Return the integral of this polynomial.

By default, the integration variable is the variable of the polynomial.

Otherwise, the integration variable is the optional parameter `var`

> **Note**
>
> The integral is always chosen so that the constant term is 0.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: R(0).integral()
0
sage: f = R(2).integral(); f
2*x
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> R(Integer(0)).integral()
0
>>> f = R(Integer(2)).integral(); f
2*x
```

Note that the integral lives over the fraction field of the scalar coefficients:

```
sage: f.parent()
Univariate Polynomial Ring in x over Rational Field
sage: R(0).integral().parent()
Univariate Polynomial Ring in x over Rational Field

sage: f = x^3 + x - 2
sage: g = f.integral(); g
1/4*x^4 + 1/2*x^2 - 2*x
sage: g.parent()
Univariate Polynomial Ring in x over Rational Field
```

```
>>> from sage.all import *
>>> f.parent()
```

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```
Univariate Polynomial Ring in x over Rational Field
>>> R(Integer(0)).integral().parent()
Univariate Polynomial Ring in x over Rational Field

>>> f = x**Integer(3) + x - Integer(2)
>>> g = f.integral(); g
1/4*x^4 + 1/2*x^2 - 2*x
>>> g.parent()
Univariate Polynomial Ring in x over Rational Field
```

This shows that the issue at Issue #7711 is resolved:

```
sage: # needs sage.rings.finite_rings
sage: P.<x,z> = PolynomialRing(GF(2147483647))
sage: Q.<y> = PolynomialRing(P)
sage: p = x + y + z
sage: p.integral()
-1073741823*y^2 + (x + z)*y

sage: # needs sage.rings.finite_rings
sage: P.<x,z> = PolynomialRing(GF(next_prime(2147483647)))
sage: Q.<y> = PolynomialRing(P)
sage: p = x + y + z
sage: p.integral()
1073741830*y^2 + (x + z)*y
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> P = PolynomialRing(GF(Integer(2147483647)), names=('x', 'z',)); (x, z,) =␣
↪P._first_ngens(2)
>>> Q = PolynomialRing(P, names=('y',)); (y,) = Q._first_ngens(1)
>>> p = x + y + z
>>> p.integral()
-1073741823*y^2 + (x + z)*y

>>> # needs sage.rings.finite_rings
>>> P = PolynomialRing(GF(next_prime(Integer(2147483647))), names=('x', 'z',
↪)); (x, z,) = P._first_ngens(2)
>>> Q = PolynomialRing(P, names=('y',)); (y,) = Q._first_ngens(1)
>>> p = x + y + z
>>> p.integral()
1073741830*y^2 + (x + z)*y
```

A truly convoluted example:

```
sage: A.<a1, a2> = PolynomialRing(ZZ)
sage: B.<b> = PolynomialRing(A)
sage: C.<c> = PowerSeriesRing(B)
sage: R.<x> = PolynomialRing(C)
sage: f = a2*x^2 + c*x - a1*b
sage: f.parent()
Univariate Polynomial Ring in x over Power Series Ring in c
```

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```
over Univariate Polynomial Ring in b over Multivariate Polynomial
Ring in a1, a2 over Integer Ring
sage: f.integral()
1/3*a2*x^3 + 1/2*c*x^2 - a1*b*x
sage: f.integral().parent()
Univariate Polynomial Ring in x over Power Series Ring in c
over Univariate Polynomial Ring in b over Multivariate Polynomial
Ring in a1, a2 over Rational Field
sage: g = 3*a2*x^2 + 2*c*x - a1*b
sage: g.integral()
a2*x^3 + c*x^2 - a1*b*x
sage: g.integral().parent()
Univariate Polynomial Ring in x over Power Series Ring in c
over Univariate Polynomial Ring in b over Multivariate Polynomial
Ring in a1, a2 over Rational Field
```

```
>>> from sage.all import *
>>> A = PolynomialRing(ZZ, names=('a1', 'a2',)); (a1, a2,) = A._first_ngens(2)
>>> B = PolynomialRing(A, names=('b',)); (b,) = B._first_ngens(1)
>>> C = PowerSeriesRing(B, names=('c',)); (c,) = C._first_ngens(1)
>>> R = PolynomialRing(C, names=('x',)); (x,) = R._first_ngens(1)
>>> f = a2*x**Integer(2) + c*x - a1*b
>>> f.parent()
Univariate Polynomial Ring in x over Power Series Ring in c
over Univariate Polynomial Ring in b over Multivariate Polynomial
Ring in a1, a2 over Integer Ring
>>> f.integral()
1/3*a2*x^3 + 1/2*c*x^2 - a1*b*x
>>> f.integral().parent()
Univariate Polynomial Ring in x over Power Series Ring in c
over Univariate Polynomial Ring in b over Multivariate Polynomial
Ring in a1, a2 over Rational Field
>>> g = Integer(3)*a2*x**Integer(2) + Integer(2)*c*x - a1*b
>>> g.integral()
a2*x^3 + c*x^2 - a1*b*x
>>> g.integral().parent()
Univariate Polynomial Ring in x over Power Series Ring in c
over Univariate Polynomial Ring in b over Multivariate Polynomial
Ring in a1, a2 over Rational Field
```

Integration with respect to a variable in the base ring:

```
sage: R.<x> = QQ[]
sage: t = PolynomialRing(R,'t').gen()
sage: f = x*t + 5*t^2
sage: f.integral(x)
5*x*t^2 + 1/2*x^2*t
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> t = PolynomialRing(R,'t').gen()
>>> f = x*t + Integer(5)*t**Integer(2)
```

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```
>>> f.integral(x)
5*x*t^2 + 1/2*x^2*t
```

inverse_mod(*a*, *m*)

Invert the polynomial a with respect to m, or raise a ValueError if no such inverse exists.

The parameter m may be either a single polynomial or an ideal (for consistency with *inverse_mod()* in other rings).

See also

If you are only interested in the inverse modulo a monomial x^k then you might use the specialized method *inverse_series_trunc()* which is much faster.

EXAMPLES:

```
sage: S.<t> = QQ[]
sage: f = inverse_mod(t^2 + 1, t^3 + 1); f
-1/2*t^2 - 1/2*t + 1/2
sage: f * (t^2 + 1) % (t^3 + 1)
1
sage: f = t.inverse_mod((t + 1)^7); f
-t^6 - 7*t^5 - 21*t^4 - 35*t^3 - 35*t^2 - 21*t - 7
sage: (f * t) + (t + 1)^7
1
sage: t.inverse_mod(S.ideal((t + 1)^7)) == f
True
```

```
>>> from sage.all import *
>>> S = QQ['t']; (t,) = S._first_ngens(1)
>>> f = inverse_mod(t**Integer(2) + Integer(1), t**Integer(3) + Integer(1)); f
-1/2*t^2 - 1/2*t + 1/2
>>> f * (t**Integer(2) + Integer(1)) % (t**Integer(3) + Integer(1))
1
>>> f = t.inverse_mod((t + Integer(1))**Integer(7)); f
-t^6 - 7*t^5 - 21*t^4 - 35*t^3 - 35*t^2 - 21*t - 7
>>> (f * t) + (t + Integer(1))**Integer(7)
1
>>> t.inverse_mod(S.ideal((t + Integer(1))**Integer(7))) == f
True
```

This also works over inexact rings, but note that due to rounding error the product may not always exactly equal the constant polynomial 1 and have extra terms with coefficients close to zero.

```
sage: # needs scipy sage.modules
sage: R.<x> = RDF[]
sage: epsilon = RDF(1).ulp()*50   # Allow an error of up to 50 ulp
sage: f = inverse_mod(x^2 + 1, x^5 + x + 1); f  # abs tol 1e-14
0.4*x^4 - 0.2*x^3 - 0.4*x^2 + 0.2*x + 0.8
sage: poly = f * (x^2 + 1) % (x^5 + x + 1)
sage: # Remove noisy zero terms:
```

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```
sage: parent(poly)([0.0 if abs(c) <= epsilon else c
....:              for c in poly.coefficients(sparse=False)])
1.0
sage: f = inverse_mod(x^3 - x + 1, x - 2); f
0.14285714285714285
sage: f * (x^3 - x + 1) % (x - 2)
1.0
sage: g = 5*x^3 + x - 7; m = x^4 - 12*x + 13; f = inverse_mod(g, m); f
-0.0319636125...*x^3 - 0.0383269759...*x^2 - 0.0463050900...*x + 0.346479687..
↪.
sage: poly = f*g % m
sage: # Remove noisy zero terms:
sage: parent(poly)([0.0 if abs(c) <= epsilon else c  # abs tol 1e-14
....:              for c in poly.coefficients(sparse=False)])
1.0000000000000004
```

```
>>> from sage.all import *
>>> # needs scipy sage.modules
>>> R = RDF['x']; (x,) = R._first_ngens(1)
>>> epsilon = RDF(Integer(1)).ulp()*Integer(50)   # Allow an error of up to␣
↪50 ulp
>>> f = inverse_mod(x**Integer(2) + Integer(1), x**Integer(5) + x +␣
↪Integer(1)); f  # abs tol 1e-14
0.4*x^4 - 0.2*x^3 - 0.4*x^2 + 0.2*x + 0.8
>>> poly = f * (x**Integer(2) + Integer(1)) % (x**Integer(5) + x + Integer(1))
>>> # Remove noisy zero terms:
>>> parent(poly)([RealNumber('0.0') if abs(c) <= epsilon else c
...              for c in poly.coefficients(sparse=False)])
1.0
>>> f = inverse_mod(x**Integer(3) - x + Integer(1), x - Integer(2)); f
0.14285714285714285
>>> f * (x**Integer(3) - x + Integer(1)) % (x - Integer(2))
1.0
>>> g = Integer(5)*x**Integer(3) + x - Integer(7); m = x**Integer(4) -␣
↪Integer(12)*x + Integer(13); f = inverse_mod(g, m); f
-0.0319636125...*x^3 - 0.0383269759...*x^2 - 0.0463050900...*x + 0.346479687..
↪.
>>> poly = f*g % m
>>> # Remove noisy zero terms:
>>> parent(poly)([RealNumber('0.0') if abs(c) <= epsilon else c  # abs tol 1e-
↪14
...              for c in poly.coefficients(sparse=False)])
1.0000000000000004
```

ALGORITHM: Solve the system $as + mt = 1$, returning s as the inverse of a mod m.

Uses the Euclidean algorithm for exact rings, and solves a linear system for the coefficients of s and t for inexact rings (as the Euclidean algorithm may not converge in that case).

AUTHORS:

- Robert Bradshaw (2007-05-31)

inverse_of_unit()

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: f = x - 90283
sage: f.inverse_of_unit()
Traceback (most recent call last):
...
ArithmeticError: x - 90283 is not a unit
in Univariate Polynomial Ring in x over Rational Field
sage: f = R(-90283); g = f.inverse_of_unit(); g
-1/90283
sage: parent(g)
Univariate Polynomial Ring in x over Rational Field
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = x - Integer(90283)
>>> f.inverse_of_unit()
Traceback (most recent call last):
...
ArithmeticError: x - 90283 is not a unit
in Univariate Polynomial Ring in x over Rational Field
>>> f = R(-Integer(90283)); g = f.inverse_of_unit(); g
-1/90283
>>> parent(g)
Univariate Polynomial Ring in x over Rational Field
```

inverse_series_trunc(*prec*)

Return a polynomial approximation of precision `prec` of the inverse series of this polynomial.

See also

The method *inverse_mod()* allows more generally to invert this polynomial with respect to any ideal.

EXAMPLES:

```
sage: x = polygen(ZZ)
sage: s = (1 + x).inverse_series_trunc(5)
sage: s
x^4 - x^3 + x^2 - x + 1
sage: s * (1 + x)
x^5 + 1
```

```
>>> from sage.all import *
>>> x = polygen(ZZ)
>>> s = (Integer(1) + x).inverse_series_trunc(Integer(5))
>>> s
x^4 - x^3 + x^2 - x + 1
>>> s * (Integer(1) + x)
x^5 + 1
```

Note that the constant coefficient needs to be a unit:

```
sage: ZZx.<x> = ZZ[]
sage: ZZxy.<y> = ZZx[]
sage: (1+x + y**2).inverse_series_trunc(4)
Traceback (most recent call last):
...
ValueError: constant term x + 1 is not a unit
sage: (1+x + y**2).change_ring(ZZx.fraction_field()).inverse_series_trunc(4)
(-1/(x^2 + 2*x + 1))*y^2 + 1/(x + 1)
```

```
>>> from sage.all import *
>>> ZZx = ZZ['x']; (x,) = ZZx._first_ngens(1)
>>> ZZxy = ZZx['y']; (y,) = ZZxy._first_ngens(1)
>>> (Integer(1)+x + y**Integer(2)).inverse_series_trunc(Integer(4))
Traceback (most recent call last):
...
ValueError: constant term x + 1 is not a unit
>>> (Integer(1)+x + y**Integer(2)).change_ring(ZZx.fraction_field()).inverse_
↪series_trunc(Integer(4))
(-1/(x^2 + 2*x + 1))*y^2 + 1/(x + 1)
```

The method works over any polynomial ring:

```
sage: R = Zmod(4)
sage: Rx.<x> = R[]
sage: Rxy.<y> = Rx[]

sage: p = 1 + (1+2*x)*y + x**2*y**4
sage: q = p.inverse_series_trunc(10)
sage: (p*q).truncate(11)
(2*x^4 + 3*x^2 + 3)*y^10 + 1
```

```
>>> from sage.all import *
>>> R = Zmod(Integer(4))
>>> Rx = R['x']; (x,) = Rx._first_ngens(1)
>>> Rxy = Rx['y']; (y,) = Rxy._first_ngens(1)

>>> p = Integer(1) + (Integer(1)+Integer(2)*x)*y + x**Integer(2)*y**Integer(4)
>>> q = p.inverse_series_trunc(Integer(10))
>>> (p*q).truncate(Integer(11))
(2*x^4 + 3*x^2 + 3)*y^10 + 1
```

Even noncommutative ones:

```
sage: # needs sage.modules
sage: M = MatrixSpace(ZZ, 2)
sage: x = polygen(M)
sage: p = M([1,2,3,4])*x^3 + M([-1,0,0,1])*x^2 + M([1,3,-1,0])*x + M.one()
sage: q = p.inverse_series_trunc(5)
sage: (p*q).truncate(5) == M.one()
True
sage: q = p.inverse_series_trunc(13)
sage: (p*q).truncate(13) == M.one()
True
```

```
>>> from sage.all import *
>>> # needs sage.modules
>>> M = MatrixSpace(ZZ, Integer(2))
>>> x = polygen(M)
>>> p = M([Integer(1),Integer(2),Integer(3),Integer(4)])*x**Integer(3) + M([-
↪Integer(1),Integer(0),Integer(0),Integer(1)])*x**Integer(2) + M([Integer(1),
↪Integer(3),-Integer(1),Integer(0)])*x + M.one()
>>> q = p.inverse_series_trunc(Integer(5))
>>> (p*q).truncate(Integer(5)) == M.one()
True
>>> q = p.inverse_series_trunc(Integer(13))
>>> (p*q).truncate(Integer(13)) == M.one()
True
```

AUTHORS:

- David Harvey (2006-09-09): Newton's method implementation for power series
- Vincent Delecroix (2014-2015): move the implementation directly in polynomial

is_constant()

Return `True` if this is a constant polynomial.

OUTPUT: boolean; `True` if and only if this polynomial is constant

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: x.is_constant()
False
sage: R(2).is_constant()
True
sage: R(0).is_constant()
True
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> x.is_constant()
False
>>> R(Integer(2)).is_constant()
True
>>> R(Integer(0)).is_constant()
True
```

is_cyclotomic(*certificate=False*, *algorithm='pari'*)

Test if this polynomial is a cyclotomic polynomial.

A *cyclotomic polynomial* is a monic, irreducible polynomial such that all roots are roots of unity.

By default the answer is a boolean. But if `certificate` is `True`, the result is a nonnegative integer: it is `0` if `self` is not cyclotomic, and a positive integer `n` if `self` is the n-th cyclotomic polynomial.

See also

is_cyclotomic_product() cyclotomic_part() has_cyclotomic_factor()

INPUT:

- `certificate` – boolean (default: `False`); only works with `algorithm` set to `'pari'`
- `algorithm` – either `'pari'` (default) or `'sage'`

ALGORITHM:

The native algorithm implemented in Sage uses the first algorithm of [BD1989]. The algorithm in PARI (using pari:poliscyclo) is more subtle since it does compute the inverse of the Euler ϕ function to determine the n such that the polynomial is the n-th cyclotomic polynomial.

EXAMPLES:

Quick tests:

```
sage: # needs sage.libs.pari
sage: P.<x> = ZZ['x']
sage: (x - 1).is_cyclotomic()
True
sage: (x + 1).is_cyclotomic()
True
sage: (x^2 - 1).is_cyclotomic()
False
sage: (x^2 + x + 1).is_cyclotomic(certificate=True)
3
sage: (x^2 + 2*x + 1).is_cyclotomic(certificate=True)
0
```

```
>>> from sage.all import *
>>> # needs sage.libs.pari
>>> P = ZZ['x']; (x,) = P._first_ngens(1)
>>> (x - Integer(1)).is_cyclotomic()
True
>>> (x + Integer(1)).is_cyclotomic()
True
>>> (x**Integer(2) - Integer(1)).is_cyclotomic()
False
>>> (x**Integer(2) + x + Integer(1)).is_cyclotomic(certificate=True)
3
>>> (x**Integer(2) + Integer(2)*x + Integer(1)).is_
→cyclotomic(certificate=True)
0
```

Test first 100 cyclotomic polynomials:

```
sage: all(cyclotomic_polynomial(i).is_cyclotomic() for i in range(1, 101))   #␣
→needs sage.libs.pari
True
```

```
>>> from sage.all import *
>>> all(cyclotomic_polynomial(i).is_cyclotomic() for i in range(Integer(1),␣
→Integer(101)))   # needs sage.libs.pari
True
```

Some more tests:

```
sage: # needs sage.libs.pari
sage: f = x^16 + x^14 - x^10 + x^8 - x^6 + x^2 + 1
sage: f.is_cyclotomic(algorithm='pari')
False
sage: f.is_cyclotomic(algorithm='sage')
False
sage: g = x^16 + x^14 - x^10 - x^8 - x^6 + x^2 + 1
sage: g.is_cyclotomic(algorithm='pari')
True
sage: g.is_cyclotomic(algorithm='sage')
True

sage: y = polygen(QQ)
sage: (y/2 - 1/2).is_cyclotomic()
False
sage: (2*(y/2 - 1/2)).is_cyclotomic()                                    #␣
↪needs sage.libs.pari
True
```

```
>>> from sage.all import *
>>> # needs sage.libs.pari
>>> f = x**Integer(16) + x**Integer(14) - x**Integer(10) + x**Integer(8) -␣
↪x**Integer(6) + x**Integer(2) + Integer(1)
>>> f.is_cyclotomic(algorithm='pari')
False
>>> f.is_cyclotomic(algorithm='sage')
False
>>> g = x**Integer(16) + x**Integer(14) - x**Integer(10) - x**Integer(8) -␣
↪x**Integer(6) + x**Integer(2) + Integer(1)
>>> g.is_cyclotomic(algorithm='pari')
True
>>> g.is_cyclotomic(algorithm='sage')
True

>>> y = polygen(QQ)
>>> (y/Integer(2) - Integer(1)/Integer(2)).is_cyclotomic()
False
>>> (Integer(2)*(y/Integer(2) - Integer(1)/Integer(2))).is_cyclotomic()     ␣
↪                                  # needs sage.libs.pari
True
```

Invalid arguments:

```
sage: (x - 3).is_cyclotomic(algorithm='sage', certificate=True)          #␣
↪needs sage.libs.pari
Traceback (most recent call last):
...
ValueError: no implementation of the certificate within Sage
```

```
>>> from sage.all import *
>>> (x - Integer(3)).is_cyclotomic(algorithm='sage', certificate=True)     ␣
↪     # needs sage.libs.pari
```

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```
Traceback (most recent call last):
...
ValueError: no implementation of the certificate within Sage
```

Test using other rings:

```
sage: z = polygen(GF(5))
sage: (z - 1).is_cyclotomic()
Traceback (most recent call last):
...
NotImplementedError: not implemented in nonzero characteristic
```

```
>>> from sage.all import *
>>> z = polygen(GF(Integer(5)))
>>> (z - Integer(1)).is_cyclotomic()
Traceback (most recent call last):
...
NotImplementedError: not implemented in nonzero characteristic
```

is_cyclotomic_product()

Test whether this polynomial is a product of cyclotomic polynomials.

This method simply calls the function pari:poliscycloprod from the Pari library.

See also

is_cyclotomic() cyclotomic_part() has_cyclotomic_factor()

EXAMPLES:

```
sage: x = polygen(ZZ)
sage: (x^5 - 1).is_cyclotomic_product()                                   #␣
↪needs sage.libs.pari
True
sage: (x^5 + x^4 - x^2 + 1).is_cyclotomic_product()                       #␣
↪needs sage.libs.pari
False

sage: p = prod(cyclotomic_polynomial(i) for i in [2, 5, 7, 12])
sage: p.is_cyclotomic_product()                                           #␣
↪needs sage.libs.pari
True

sage: (x^5 - 1/3).is_cyclotomic_product()
False

sage: x = polygen(Zmod(5))
sage: (x - 1).is_cyclotomic_product()
Traceback (most recent call last):
...
NotImplementedError: not implemented in nonzero characteristic
```

```
>>> from sage.all import *
>>> x = polygen(ZZ)
>>> (x**Integer(5) - Integer(1)).is_cyclotomic_product()                    ␣
↪               # needs sage.libs.pari
True
>>> (x**Integer(5) + x**Integer(4) - x**Integer(2) + Integer(1)).is_
↪cyclotomic_product()                             # needs sage.libs.pari
False

>>> p = prod(cyclotomic_polynomial(i) for i in [Integer(2), Integer(5),␣
↪Integer(7), Integer(12)])
>>> p.is_cyclotomic_product()                                           #␣
↪needs sage.libs.pari
True

>>> (x**Integer(5) - Integer(1)/Integer(3)).is_cyclotomic_product()
False

>>> x = polygen(Zmod(Integer(5)))
>>> (x - Integer(1)).is_cyclotomic_product()
Traceback (most recent call last):
...
NotImplementedError: not implemented in nonzero characteristic
```

is_gen()

Return `True` if this polynomial is the distinguished generator of the parent polynomial ring.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: R(1).is_gen()
False
sage: R(x).is_gen()
True
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> R(Integer(1)).is_gen()
False
>>> R(x).is_gen()
True
```

Important - this function doesn't return `True` if `self` equals the generator; it returns `True` if `self` *is* the generator.

```
sage: f = R([0,1]); f
x
sage: f.is_gen()
False
sage: f is x
False
sage: f == x
True
```

```
>>> from sage.all import *
>>> f = R([Integer(0),Integer(1)]); f
x
>>> f.is_gen()
False
>>> f is x
False
>>> f == x
True
```

is_homogeneous()

Return `True` if this polynomial is homogeneous.

EXAMPLES:

```
sage: P.<x> = PolynomialRing(QQ)
sage: x.is_homogeneous()
True
sage: P(0).is_homogeneous()
True
sage: (x + 1).is_homogeneous()
False
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, names=('x',)); (x,) = P._first_ngens(1)
>>> x.is_homogeneous()
True
>>> P(Integer(0)).is_homogeneous()
True
>>> (x + Integer(1)).is_homogeneous()
False
```

is_irreducible()

Return whether this polynomial is irreducible.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: (x^3 + 1).is_irreducible()                                            #␣
↪needs sage.libs.pari
False
sage: (x^2 - 1).is_irreducible()                                            #␣
↪needs sage.libs.pari
False
sage: (x^3 + 2).is_irreducible()                                            #␣
↪needs sage.libs.pari
True
sage: R(0).is_irreducible()
False
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> (x**Integer(3) + Integer(1)).is_irreducible()                           ␣
```

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```
↪                 # needs sage.libs.pari
False
>>> (x**Integer(2) - Integer(1)).is_irreducible()                              ␣
↪                 # needs sage.libs.pari
False
>>> (x**Integer(3) + Integer(2)).is_irreducible()                              ␣
↪                 # needs sage.libs.pari
True
>>> R(Integer(0)).is_irreducible()
False
```

The base ring does matter: for example, $2x$ is irreducible as a polynomial in $\mathbf{Q}[x]$, but not in $\mathbf{Z}[x]$:

```
sage: R.<x> = ZZ[]
sage: R(2*x).is_irreducible()                                                  #␣
↪needs sage.libs.pari
False
sage: R.<x> = QQ[]
sage: R(2*x).is_irreducible()                                                  #␣
↪needs sage.libs.pari
True
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> R(Integer(2)*x).is_irreducible()                                           ␣
↪      # needs sage.libs.pari
False
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> R(Integer(2)*x).is_irreducible()                                           ␣
↪      # needs sage.libs.pari
True
```

is_lorentzian(*explain=False*)

Return `True` if this is a Lorentzian polynomial.

A univariate real polynomial is Lorentzian if and only if it is a monomial with positive coefficient, or zero. The definition is more involved for multivariate real polynomials.

INPUT:

- `explain` – boolean (default: `False`); if `True` return a tuple whose first element is the boolean result of the test, and the second element is a string describing the reason the test failed, or `None` if the test succeeded

EXAMPLES:

```
sage: P.<x> = QQ[]
sage: p1 = x^2
sage: p1.is_lorentzian()
True
sage: p2 = 1 + x^2
sage: p2.is_lorentzian()
False
sage: p3 = P.zero()
```

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```
sage: p3.is_lorentzian()
True
sage: p4 = -2*x^3
sage: p4.is_lorentzian()
False
```

```
>>> from sage.all import *
>>> P = QQ['x']; (x,) = P._first_ngens(1)
>>> p1 = x**Integer(2)
>>> p1.is_lorentzian()
True
>>> p2 = Integer(1) + x**Integer(2)
>>> p2.is_lorentzian()
False
>>> p3 = P.zero()
>>> p3.is_lorentzian()
True
>>> p4 = -Integer(2)*x**Integer(3)
>>> p4.is_lorentzian()
False
```

It is an error to check if a polynomial is Lorentzian if its base ring is not a subring of the real numbers, as the notion is not defined in this case:

```
sage: # needs sage.rings.real_mpfr
sage: Q.<y> = CC[]
sage: q = y^2
sage: q.is_lorentzian()
Traceback (most recent call last):
...
NotImplementedError: is_lorentzian only implemented for real polynomials
```

```
>>> from sage.all import *
>>> # needs sage.rings.real_mpfr
>>> Q = CC['y']; (y,) = Q._first_ngens(1)
>>> q = y**Integer(2)
>>> q.is_lorentzian()
Traceback (most recent call last):
...
NotImplementedError: is_lorentzian only implemented for real polynomials
```

The method can give a reason for a polynomial failing to be Lorentzian:

```
sage: p = x^2 + 2*x
sage: p.is_lorentzian(explain=True)
(False, 'inhomogeneous')
```

```
>>> from sage.all import *
>>> p = x**Integer(2) + Integer(2)*x
>>> p.is_lorentzian(explain=True)
(False, 'inhomogeneous')
```

REFERENCES:

For full definitions and related discussion, see [BrHu2019] and [HMMS2019].

is_monic()

Return `True` if this polynomial is monic. The zero polynomial is by definition not monic.

EXAMPLES:

```
sage: x = QQ['x'].0
sage: f = x + 33
sage: f.is_monic()
True
sage: f = 0*x
sage: f.is_monic()
False
sage: f = 3*x^3 + x^4 + x^2
sage: f.is_monic()
True
sage: f = 2*x^2 + x^3 + 56*x^5
sage: f.is_monic()
False
```

```
>>> from sage.all import *
>>> x = QQ['x'].gen(0)
>>> f = x + Integer(33)
>>> f.is_monic()
True
>>> f = Integer(0)*x
>>> f.is_monic()
False
>>> f = Integer(3)*x**Integer(3) + x**Integer(4) + x**Integer(2)
>>> f.is_monic()
True
>>> f = Integer(2)*x**Integer(2) + x**Integer(3) + Integer(56)*x**Integer(5)
>>> f.is_monic()
False
```

AUTHORS:

- Naqi Jaffery (2006-01-24): examples

is_monomial()

Return `True` if `self` is a monomial, i.e., a power of the generator.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: x.is_monomial()
True
sage: (x + 1).is_monomial()
False
sage: (x^2).is_monomial()
True
sage: R(1).is_monomial()
True
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> x.is_monomial()
True
>>> (x + Integer(1)).is_monomial()
False
>>> (x**Integer(2)).is_monomial()
True
>>> R(Integer(1)).is_monomial()
True
```

The coefficient must be 1:

```
sage: (2*x^5).is_monomial()
False
```

```
>>> from sage.all import *
>>> (Integer(2)*x**Integer(5)).is_monomial()
False
```

To allow a non-1 leading coefficient, use `is_term()`:

```
sage: (2*x^5).is_term()
True
```

```
>>> from sage.all import *
>>> (Integer(2)*x**Integer(5)).is_term()
True
```

Warning

The definition of `is_monomial()` in Sage up to 4.7.1 was the same as `is_term()`, i.e., it allowed a coefficient not equal to 1.

is_nilpotent()

Return `True` if this polynomial is nilpotent.

EXAMPLES:

```
sage: R = Integers(12)
sage: S.<x> = R[]
sage: f = 5 + 6*x
sage: f.is_nilpotent()
False
sage: f = 6 + 6*x^2
sage: f.is_nilpotent()
True
sage: f^2
0
```

```
>>> from sage.all import *
>>> R = Integers(Integer(12))
>>> S = R['x']; (x,) = S._first_ngens(1)
>>> f = Integer(5) + Integer(6)*x
>>> f.is_nilpotent()
False
>>> f = Integer(6) + Integer(6)*x**Integer(2)
>>> f.is_nilpotent()
True
>>> f**Integer(2)
0
```

EXERCISE (Atiyah-McDonald, Ch 1): Let $A[x]$ be a polynomial ring in one variable. Then $f = \sum a_i x^i \in A[x]$ is nilpotent if and only if every a_i is nilpotent.

is_one()

Test whether this polynomial is 1.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: (x - 3).is_one()
False
sage: R(1).is_one()
True

sage: R2.<y> = R[]
sage: R2(x).is_one()
False
sage: R2(1).is_one()
True
sage: R2(-1).is_one()
False
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> (x - Integer(3)).is_one()
False
>>> R(Integer(1)).is_one()
True

>>> R2 = R['y']; (y,) = R2._first_ngens(1)
>>> R2(x).is_one()
False
>>> R2(Integer(1)).is_one()
True
>>> R2(-Integer(1)).is_one()
False
```

is_primitive(*n=None*, *n_prime_divs=None*)

Return `True` if the polynomial is primitive.

The semantics of "primitive" depend on the polynomial coefficients.

- (field theory) A polynomial of degree m over a finite field $\mathbf{F}_q$ is primitive if it is irreducible and its root in $\mathbf{F}_{q^m}$ generates the multiplicative group $\mathbf{F}_{q^m}^*$.
- (ring theory) A polynomial over a ring is primitive if its coefficients generate the unit ideal.

Calling `is_primitive()` on a polynomial over an infinite field will raise an error.

The additional inputs to this function are to speed up computation for field semantics (see note).

INPUT:

- `n` – (default: `None`) if provided, should equal $q - 1$ where `self.parent()` is the field with q elements; otherwise it will be computed
- `n_prime_divs` – (default: `None`) if provided, should be a list of the prime divisors of n; otherwise it will be computed

Note

Computation of the prime divisors of n can dominate the running time of this method, so performing this computation externally (e.g., `pdivs = n.prime_divisors()`) is a good idea for repeated calls to `is_primitive()` for polynomials of the same degree.

Results may be incorrect if the wrong n and/or factorization are provided.

EXAMPLES:

Field semantics examples.

```
sage: # needs sage.rings.finite_rings
sage: R.<x> = GF(2)['x']
sage: f = x^4 + x^3 + x^2 + x + 1
sage: f.is_irreducible(), f.is_primitive()
(True, False)
sage: f = x^3 + x + 1
sage: f.is_irreducible(), f.is_primitive()
(True, True)
sage: R.<x> = GF(3)[]
sage: f = x^3 - x + 1
sage: f.is_irreducible(), f.is_primitive()
(True, True)
sage: f = x^2 + 1
sage: f.is_irreducible(), f.is_primitive()
(True, False)
sage: R.<x> = GF(5)[]
sage: f = x^2 + x + 1
sage: f.is_primitive()
False
sage: f = x^2 - x + 2
sage: f.is_primitive()
True
sage: x = polygen(QQ); f = x^2 + 1
sage: f.is_primitive()
Traceback (most recent call last):
...
NotImplementedError: is_primitive() not defined for polynomials over infinite␣
↪fields.
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> R = GF(Integer(2))['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(4) + x**Integer(3) + x**Integer(2) + x + Integer(1)
>>> f.is_irreducible(), f.is_primitive()
(True, False)
>>> f = x**Integer(3) + x + Integer(1)
>>> f.is_irreducible(), f.is_primitive()
(True, True)
>>> R = GF(Integer(3))['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(3) - x + Integer(1)
>>> f.is_irreducible(), f.is_primitive()
(True, True)
>>> f = x**Integer(2) + Integer(1)
>>> f.is_irreducible(), f.is_primitive()
(True, False)
>>> R = GF(Integer(5))['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(2) + x + Integer(1)
>>> f.is_primitive()
False
>>> f = x**Integer(2) - x + Integer(2)
>>> f.is_primitive()
True
>>> x = polygen(QQ); f = x**Integer(2) + Integer(1)
>>> f.is_primitive()
Traceback (most recent call last):
...
NotImplementedError: is_primitive() not defined for polynomials over infinite
↪fields.
```

Ring semantics examples.

```
sage: x = polygen(ZZ)
sage: f = 5*x^2 + 2
sage: f.is_primitive()
True
sage: f = 5*x^2 + 5
sage: f.is_primitive()
False

sage: # needs sage.rings.number_field
sage: K = NumberField(x^2 + 5, 'a')
sage: R = K.ring_of_integers()
sage: a = R.gen(1)
sage: a^2
-5
sage: f = a*x + 2
sage: f.is_primitive()
True
sage: f = (1+a)*x + 2
sage: f.is_primitive()
False
```

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```
sage: x = polygen(Integers(10))
sage: f = 5*x^2 + 2
sage: #f.is_primitive()  #BUG:: elsewhere in Sage, should return True
sage: f = 4*x^2 + 2
sage: #f.is_primitive()  #BUG:: elsewhere in Sage, should return False
```

```
>>> from sage.all import *
>>> x = polygen(ZZ)
>>> f = Integer(5)*x**Integer(2) + Integer(2)
>>> f.is_primitive()
True
>>> f = Integer(5)*x**Integer(2) + Integer(5)
>>> f.is_primitive()
False

>>> # needs sage.rings.number_field
>>> K = NumberField(x**Integer(2) + Integer(5), 'a')
>>> R = K.ring_of_integers()
>>> a = R.gen(Integer(1))
>>> a**Integer(2)
-5
>>> f = a*x + Integer(2)
>>> f.is_primitive()
True
>>> f = (Integer(1)+a)*x + Integer(2)
>>> f.is_primitive()
False

>>> x = polygen(Integers(Integer(10)))
>>> f = Integer(5)*x**Integer(2) + Integer(2)
>>> #f.is_primitive()  #BUG:: elsewhere in Sage, should return True
>>> f = Integer(4)*x**Integer(2) + Integer(2)
>>> #f.is_primitive()  #BUG:: elsewhere in Sage, should return False
```

is_real_rooted()

Return `True` if the roots of this polynomial are all real.

EXAMPLES:

```
sage: # needs sage.libs.pari
sage: R.<x> = PolynomialRing(ZZ)
sage: pol = chebyshev_T(5, x)
sage: pol.is_real_rooted()
True
sage: pol = x^2 + 1
sage: pol.is_real_rooted()
False
```

```
>>> from sage.all import *
>>> # needs sage.libs.pari
>>> R = PolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> pol = chebyshev_T(Integer(5), x)
```

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```
>>> pol.is_real_rooted()
True
>>> pol = x**Integer(2) + Integer(1)
>>> pol.is_real_rooted()
False
```

is_square(*root=False*)

Return whether or not polynomial is square.

If the optional argument `root` is set to `True`, then also returns the square root (or `None`, if the polynomial is not square).

INPUT:

- `root` – whether or not to also return a square root (default: `False`)

OUTPUT:

- boolean; whether or not a square
- (optional) an actual square root if found, and `None` otherwise

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: (x^2 + 2*x + 1).is_square()
True
sage: (x^4 + 2*x^3 - x^2 - 2*x + 1).is_square(root=True)
(True, x^2 + x - 1)

sage: f = 12 * (x + 1)^2 * (x + 3)^2
sage: f.is_square()
False
sage: f.is_square(root=True)
(False, None)

sage: h = f/3; h
4*x^4 + 32*x^3 + 88*x^2 + 96*x + 36
sage: h.is_square(root=True)
(True, 2*x^2 + 8*x + 6)

sage: S.<y> = PolynomialRing(RR)
sage: g = 12 * (y + 1)^2 * (y + 3)^2

sage: g.is_square()
True
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> (x**Integer(2) + Integer(2)*x + Integer(1)).is_square()
True
>>> (x**Integer(4) + Integer(2)*x**Integer(3) - x**Integer(2) - Integer(2)*x␣
↪+ Integer(1)).is_square(root=True)
(True, x^2 + x - 1)
```

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```
>>> f = Integer(12) * (x + Integer(1))**Integer(2) * (x +
↪Integer(3))**Integer(2)
>>> f.is_square()
False
>>> f.is_square(root=True)
(False, None)

>>> h = f/Integer(3); h
4*x^4 + 32*x^3 + 88*x^2 + 96*x + 36
>>> h.is_square(root=True)
(True, 2*x^2 + 8*x + 6)

>>> S = PolynomialRing(RR, names=('y',)); (y,) = S._first_ngens(1)
>>> g = Integer(12) * (y + Integer(1))**Integer(2) * (y +
↪Integer(3))**Integer(2)

>>> g.is_square()
True
```

is_squarefree()

Return `False` if this polynomial is not square-free, i.e., if there is a non-unit g in the polynomial ring such that g^2 divides `self`.

> **Warning**
>
> This method is not consistent with *squarefree_decomposition()* since the latter does not factor the content of a polynomial. See the examples below.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: f = (x-1) * (x-2) * (x^2-5) * (x^17-3); f
x^21 - 3*x^20 - 3*x^19 + 15*x^18 - 10*x^17 - 3*x^4 + 9*x^3 + 9*x^2 - 45*x + 30
sage: f.is_squarefree()
True
sage: (f * (x^2-5)).is_squarefree()
False
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = (x-Integer(1)) * (x-Integer(2)) * (x**Integer(2)-Integer(5)) *
↪(x**Integer(17)-Integer(3)); f
x^21 - 3*x^20 - 3*x^19 + 15*x^18 - 10*x^17 - 3*x^4 + 9*x^3 + 9*x^2 - 45*x + 30
>>> f.is_squarefree()
True
>>> (f * (x**Integer(2)-Integer(5))).is_squarefree()
False
```

A generic implementation is available, which relies on gcd computations:

```
sage: # needs sage.libs.pari
sage: R.<x> = ZZ[]
sage: (2*x).is_squarefree()
True
sage: (4*x).is_squarefree()
False
sage: (2*x^2).is_squarefree()
False
sage: R(0).is_squarefree()
False

sage: S.<y> = QQ[]
sage: R.<x> = S[]
sage: (2*x*y).is_squarefree()
True
sage: (2*x*y^2).is_squarefree()
False
```

```
>>> from sage.all import *
>>> # needs sage.libs.pari
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> (Integer(2)*x).is_squarefree()
True
>>> (Integer(4)*x).is_squarefree()
False
>>> (Integer(2)*x**Integer(2)).is_squarefree()
False
>>> R(Integer(0)).is_squarefree()
False

>>> S = QQ['y']; (y,) = S._first_ngens(1)
>>> R = S['x']; (x,) = R._first_ngens(1)
>>> (Integer(2)*x*y).is_squarefree()
True
>>> (Integer(2)*x*y**Integer(2)).is_squarefree()
False
```

In positive characteristic, we compute the square-free decomposition or a full factorization, depending on which is available:

```
sage: K.<t> = FunctionField(GF(3))
sage: R.<x> = K[]
sage: (x^3 - x).is_squarefree()
True
sage: (x^3 - 1).is_squarefree()                                         #␣
↪needs sage.libs.pari
False
sage: (x^3 + t).is_squarefree()                                         #␣
↪needs sage.libs.pari
True
sage: (x^3 + t^3).is_squarefree()                                       #␣
↪needs sage.libs.pari
False
```

```
>>> from sage.all import *
>>> K = FunctionField(GF(Integer(3)), names=('t',)); (t,) = K._first_ngens(1)
>>> R = K['x']; (x,) = R._first_ngens(1)
>>> (x**Integer(3) - x).is_squarefree()
True
>>> (x**Integer(3) - Integer(1)).is_squarefree()
↪                # needs sage.libs.pari
False
>>> (x**Integer(3) + t).is_squarefree()
↪        # needs sage.libs.pari
True
>>> (x**Integer(3) + t**Integer(3)).is_squarefree()
↪                 # needs sage.libs.pari
False
```

In the following example, t^2 is a unit in the base field:

```
sage: R(t^2).is_squarefree()
True
```

```
>>> from sage.all import *
>>> R(t**Integer(2)).is_squarefree()
True
```

This method is not consistent with `squarefree_decomposition()`:

```
sage: R.<x> = ZZ[]
sage: f = 4 * x
sage: f.is_squarefree()                                                    #
↪needs sage.libs.pari
False
sage: f.squarefree_decomposition()                                         #
↪needs sage.libs.pari
(4) * x
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> f = Integer(4) * x
>>> f.is_squarefree()                                                    #
↪needs sage.libs.pari
False
>>> f.squarefree_decomposition()                                         #
↪needs sage.libs.pari
(4) * x
```

If you want this method equally not to consider the content, you can remove it as in the following example:

```
sage: c = f.content()
sage: (f/c).is_squarefree()                                                #
↪needs sage.libs.pari
True
```

```
>>> from sage.all import *
>>> c = f.content()
>>> (f/c).is_squarefree()                                                    #␣
↪needs sage.libs.pari
True
```

If the base ring is not an integral domain, the question is not mathematically well-defined:

```
sage: R.<x> = IntegerModRing(9)[]
sage: pol = (x + 3) * (x + 6); pol
x^2
sage: pol.is_squarefree()
Traceback (most recent call last):
...
TypeError: is_squarefree() is not defined for
polynomials over Ring of integers modulo 9
```

```
>>> from sage.all import *
>>> R = IntegerModRing(Integer(9))['x']; (x,) = R._first_ngens(1)
>>> pol = (x + Integer(3)) * (x + Integer(6)); pol
x^2
>>> pol.is_squarefree()
Traceback (most recent call last):
...
TypeError: is_squarefree() is not defined for
polynomials over Ring of integers modulo 9
```

is_term()

Return `True` if this polynomial is a nonzero element of the base ring times a power of the variable.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: x.is_term()
True
sage: R(0).is_term()
False
sage: R(1).is_term()
True
sage: (3*x^5).is_term()
True
sage: (1 + 3*x^5).is_term()
False
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> x.is_term()
True
>>> R(Integer(0)).is_term()
False
>>> R(Integer(1)).is_term()
True
>>> (Integer(3)*x**Integer(5)).is_term()
```

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```
True
>>> (Integer(1) + Integer(3)*x**Integer(5)).is_term()
False
```

To require that the coefficient is 1, use *is_monomial()* instead:

```
sage: (3*x^5).is_monomial()
False
```

```
>>> from sage.all import *
>>> (Integer(3)*x**Integer(5)).is_monomial()
False
```

is_unit()

Return `True` if this polynomial is a unit.

EXAMPLES:

```
sage: a = Integers(90384098234^3)
sage: b = a(2*191*236607587)
sage: b.is_nilpotent()
True

sage: # needs sage.libs.pari
sage: R.<x> = a[]
sage: f = 3 + b*x + b^2*x^2
sage: f.is_unit()
True
sage: f = 3 + b*x + b^2*x^2 + 17*x^3
sage: f.is_unit()
False
```

```
>>> from sage.all import *
>>> a = Integers(Integer(90384098234)**Integer(3))
>>> b = a(Integer(2)*Integer(191)*Integer(236607587))
>>> b.is_nilpotent()
True

>>> # needs sage.libs.pari
>>> R = a['x']; (x,) = R._first_ngens(1)
>>> f = Integer(3) + b*x + b**Integer(2)*x**Integer(2)
>>> f.is_unit()
True
>>> f = Integer(3) + b*x + b**Integer(2)*x**Integer(2) +␣
↪Integer(17)*x**Integer(3)
>>> f.is_unit()
False
```

EXERCISE (Atiyah-McDonald, Ch 1): Let $A[x]$ be a polynomial ring in one variable. Then $f = \sum a_i x^i \in A[x]$ is a unit if and only if a_0 is a unit and $a_1, \ldots, a_n$ are nilpotent.

is_weil_polynomial(*return_q=False*)

Return `True` if this is a Weil polynomial.

This polynomial must have rational or integer coefficients.

INPUT:

- `self` – polynomial with rational or integer coefficients
- `return_q` – (default: `False`) if `True`, return a second value q which is the prime power with respect to which this is q-Weil, or 0 if there is no such value

EXAMPLES:

```
sage: polRing.<x> = PolynomialRing(Rationals())
sage: P0 = x^4 + 5*x^3 + 15*x^2 + 25*x + 25
sage: P1 = x^4 + 25*x^3 + 15*x^2 + 5*x + 25
sage: P2 = x^4 + 5*x^3 + 25*x^2 + 25*x + 25
sage: P0.is_weil_polynomial(return_q=True)                              #␣
↪needs sage.libs.pari
(True, 5)
sage: P0.is_weil_polynomial(return_q=False)                             #␣
↪needs sage.libs.pari
True
sage: P1.is_weil_polynomial(return_q=True)
(False, 0)
sage: P1.is_weil_polynomial(return_q=False)
False
sage: P2.is_weil_polynomial()                                           #␣
↪needs sage.libs.pari
False
```

```
>>> from sage.all import *
>>> polRing = PolynomialRing(Rationals(), names=('x',)); (x,) = polRing._
↪first_ngens(1)
>>> P0 = x**Integer(4) + Integer(5)*x**Integer(3) + Integer(15)*x**Integer(2)␣
↪+ Integer(25)*x + Integer(25)
>>> P1 = x**Integer(4) + Integer(25)*x**Integer(3) +␣
↪Integer(15)*x**Integer(2) + Integer(5)*x + Integer(25)
>>> P2 = x**Integer(4) + Integer(5)*x**Integer(3) + Integer(25)*x**Integer(2)␣
↪+ Integer(25)*x + Integer(25)
>>> P0.is_weil_polynomial(return_q=True)                              #␣
↪needs sage.libs.pari
(True, 5)
>>> P0.is_weil_polynomial(return_q=False)                             #␣
↪needs sage.libs.pari
True
>>> P1.is_weil_polynomial(return_q=True)
(False, 0)
>>> P1.is_weil_polynomial(return_q=False)
False
>>> P2.is_weil_polynomial()                                           #␣
↪needs sage.libs.pari
False
```

See also

Polynomial rings have a method `weil_polynomials()` to compute sets of Weil polynomials. This computation uses the iterator `sage.rings.polynomial.weil.weil_polynomials.WeilPolynomials`.

AUTHORS:

David Zureick-Brown (2017-10-01)

is_zero()

Test whether this polynomial is zero.

EXAMPLES:

```
sage: R = GF(2)['x']['y']
sage: R([0,1]).is_zero()
False
sage: R([0]).is_zero()
True
sage: R([-1]).is_zero()
False
```

```
>>> from sage.all import *
>>> R = GF(Integer(2))['x']['y']
>>> R([Integer(0),Integer(1)]).is_zero()
False
>>> R([Integer(0)]).is_zero()
True
>>> R([-Integer(1)]).is_zero()
False
```

lc()

Return the leading coefficient of this polynomial.

OUTPUT: element of the base ring

This method is the same as *leading_coefficient()*.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: f = (-2/5)*x^3 + 2*x - 1/3
sage: f.lc()
-2/5
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = (-Integer(2)/Integer(5))*x**Integer(3) + Integer(2)*x - Integer(1)/
→Integer(3)
>>> f.lc()
-2/5
```

lcm(*other*)

Let f and g be two polynomials. Then this function returns the monic least common multiple of f and g.

leading_coefficient()

Return the leading coefficient of this polynomial.

OUTPUT: element of the base ring

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: f = (-2/5)*x^3 + 2*x - 1/3
sage: f.leading_coefficient()
-2/5
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = (-Integer(2)/Integer(5))*x**Integer(3) + Integer(2)*x - Integer(1)/
↪Integer(3)
>>> f.leading_coefficient()
-2/5
```

list(*copy=True*)

Return a new copy of the list of the underlying elements of `self`.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: f = (-2/5)*x^3 + 2*x - 1/3
sage: v = f.list(); v
[-1/3, 2, 0, -2/5]
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = (-Integer(2)/Integer(5))*x**Integer(3) + Integer(2)*x - Integer(1)/
↪Integer(3)
>>> v = f.list(); v
[-1/3, 2, 0, -2/5]
```

Note that `v` is a list, it is mutable, and each call to the `list()` method returns a new list:

```
sage: type(v)
<... 'list'>
sage: v[0] = 5
sage: f.list()
[-1/3, 2, 0, -2/5]
```

```
>>> from sage.all import *
>>> type(v)
<... 'list'>
>>> v[Integer(0)] = Integer(5)
>>> f.list()
[-1/3, 2, 0, -2/5]
```

Here is an example with a generic polynomial ring:

```
sage: R.<x> = QQ[]
sage: S.<y> = R[]
sage: f = y^3 + x*y - 3*x; f
y^3 + x*y - 3*x
```

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```
sage: type(f)
<class 'sage.rings.polynomial.polynomial_element.Polynomial_generic_dense'>
sage: v = f.list(); v
[-3*x, x, 0, 1]
sage: v[0] = 10
sage: f.list()
[-3*x, x, 0, 1]
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> S = R['y']; (y,) = S._first_ngens(1)
>>> f = y**Integer(3) + x*y - Integer(3)*x; f
y^3 + x*y - 3*x
>>> type(f)
<class 'sage.rings.polynomial.polynomial_element.Polynomial_generic_dense'>
>>> v = f.list(); v
[-3*x, x, 0, 1]
>>> v[Integer(0)] = Integer(10)
>>> f.list()
[-3*x, x, 0, 1]
```

lm()

Return the leading monomial of this polynomial.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: f = (-2/5)*x^3 + 2*x - 1/3
sage: f.lm()
x^3
sage: R(5).lm()
1
sage: R(0).lm()
0
sage: R(0).lm().parent() is R
True
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = (-Integer(2)/Integer(5))*x**Integer(3) + Integer(2)*x - Integer(1)/
↪Integer(3)
>>> f.lm()
x^3
>>> R(Integer(5)).lm()
1
>>> R(Integer(0)).lm()
0
>>> R(Integer(0)).lm().parent() is R
True
```

local_height(*v*, *prec=None*)

Return the maximum of the local height of the coefficients of this polynomial.

INPUT:

- `v` – a prime or prime ideal of the base ring
- `prec` – desired floating point precision (default: default `RealField` precision)

OUTPUT: a real number

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: f = 1/1331*x^2 + 1/4000*x
sage: f.local_height(1331)                                              #␣
↪needs sage.rings.real_mpfr
7.19368581839511
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = Integer(1)/Integer(1331)*x**Integer(2) + Integer(1)/Integer(4000)*x
>>> f.local_height(Integer(1331))                                       ␣
↪     # needs sage.rings.real_mpfr
7.19368581839511
```

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: K.<k> = NumberField(x^2 - 5)
sage: T.<t> = K[]
sage: I = K.ideal(3)
sage: f = 1/3*t^2 + 3
sage: f.local_height(I)
1.09861228866811
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> K = NumberField(x**Integer(2) - Integer(5), names=('k',)); (k,) = K._
↪first_ngens(1)
>>> T = K['t']; (t,) = T._first_ngens(1)
>>> I = K.ideal(Integer(3))
>>> f = Integer(1)/Integer(3)*t**Integer(2) + Integer(3)
>>> f.local_height(I)
1.09861228866811
```

```
sage: R.<x> = QQ[]
sage: f = 1/2*x^2 + 2
sage: f.local_height(2, prec=2)                                         #␣
↪needs sage.rings.real_mpfr
0.75
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = Integer(1)/Integer(2)*x**Integer(2) + Integer(2)
>>> f.local_height(Integer(2), prec=Integer(2))                         ␣
```

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```
↪               # needs sage.rings.real_mpfr
0.75
```

local_height_arch(*i*, *prec=None*)

Return the maximum of the local height at the `i`-th infinite place of the coefficients of this polynomial.

INPUT:

- `i` – integer
- `prec` – desired floating point precision (default: default `RealField` precision)

OUTPUT: a real number

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: f = 210*x^2
sage: f.local_height_arch(0)                                               #␣
↪needs sage.rings.real_mpfr
5.34710753071747
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = Integer(210)*x**Integer(2)
>>> f.local_height_arch(Integer(0))                                        ␣
↪     # needs sage.rings.real_mpfr
5.34710753071747
```

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: K.<k> = NumberField(x^2 - 5)
sage: T.<t> = K[]
sage: f = 1/2*t^2 + 3
sage: f.local_height_arch(1, prec=52)
1.09861228866811
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> K = NumberField(x**Integer(2) - Integer(5), names=('k',)); (k,) = K._
↪first_ngens(1)
>>> T = K['t']; (t,) = T._first_ngens(1)
>>> f = Integer(1)/Integer(2)*t**Integer(2) + Integer(3)
>>> f.local_height_arch(Integer(1), prec=Integer(52))
1.09861228866811
```

```
sage: R.<x> = QQ[]
sage: f = 1/2*x^2 + 3
sage: f.local_height_arch(0, prec=2)                                       #␣
↪needs sage.rings.real_mpfr
1.0
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = Integer(1)/Integer(2)*x**Integer(2) + Integer(3)
>>> f.local_height_arch(Integer(0), prec=Integer(2))                     ␣
↪               # needs sage.rings.real_mpfr
1.0
```

lt()

Return the leading term of this polynomial.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: f = (-2/5)*x^3 + 2*x - 1/3
sage: f.lt()
-2/5*x^3
sage: R(5).lt()
5
sage: R(0).lt()
0
sage: R(0).lt().parent() is R
True
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = (-Integer(2)/Integer(5))*x**Integer(3) + Integer(2)*x - Integer(1)/
↪Integer(3)
>>> f.lt()
-2/5*x^3
>>> R(Integer(5)).lt()
5
>>> R(Integer(0)).lt()
0
>>> R(Integer(0)).lt().parent() is R
True
```

map_coefficients(*f*, *new_base_ring=None*)

Return the polynomial obtained by applying `f` to the nonzero coefficients of `self`.

If `f` is a `sage.categories.map.Map`, then the resulting polynomial will be defined over the codomain of `f`. Otherwise, the resulting polynomial will be over the same ring as `self`. Set `new_base_ring` to override this behaviour.

INPUT:

- `f` – a callable that will be applied to the coefficients of `self`
- `new_base_ring` – (optional) if given, the resulting polynomial will be defined over this ring

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: f = x^2 + 2
sage: f.map_coefficients(lambda a: a + 42)
43*x^2 + 44
```

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```
sage: R.<x> = PolynomialRing(ZZ, sparse=True)
sage: f = x^(2^32) + 2
sage: f.map_coefficients(lambda a: a + 42)
43*x^4294967296 + 44

sage: # needs sage.symbolic
sage: R.<x> = SR[]
sage: f = (1+I)*x^2 + 3*x - I
sage: f.map_coefficients(lambda z: z.conjugate())
(-I + 1)*x^2 + 3*x + I
sage: R.<x> = PolynomialRing(SR, sparse=True)
sage: f = (1+I)*x^(2^32) - I
sage: f.map_coefficients(lambda z: z.conjugate())
(-I + 1)*x^4294967296 + I
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(2) + Integer(2)
>>> f.map_coefficients(lambda a: a + Integer(42))
43*x^2 + 44
>>> R = PolynomialRing(ZZ, sparse=True, names=('x',)); (x,) = R._first_
↪ngens(1)
>>> f = x**(Integer(2)**Integer(32)) + Integer(2)
>>> f.map_coefficients(lambda a: a + Integer(42))
43*x^4294967296 + 44

>>> # needs sage.symbolic
>>> R = SR['x']; (x,) = R._first_ngens(1)
>>> f = (Integer(1)+I)*x**Integer(2) + Integer(3)*x - I
>>> f.map_coefficients(lambda z: z.conjugate())
(-I + 1)*x^2 + 3*x + I
>>> R = PolynomialRing(SR, sparse=True, names=('x',)); (x,) = R._first_
↪ngens(1)
>>> f = (Integer(1)+I)*x**(Integer(2)**Integer(32)) - I
>>> f.map_coefficients(lambda z: z.conjugate())
(-I + 1)*x^4294967296 + I
```

Examples with different base ring:

```
sage: R.<x> = ZZ[]
sage: k = GF(2)
sage: residue = lambda x: k(x)
sage: f = 4*x^2 + x + 3
sage: g = f.map_coefficients(residue); g
x + 1
sage: g.parent()
Univariate Polynomial Ring in x over Integer Ring
sage: g = f.map_coefficients(residue, new_base_ring=k); g
x + 1
sage: g.parent()                                                        #␣
↪needs sage.libs.ntl
Univariate Polynomial Ring in x over Finite Field of size 2 (using GF2X)
```

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```
sage: residue = k.coerce_map_from(ZZ)
sage: g = f.map_coefficients(residue); g
x + 1
sage: g.parent()                                                        #␣
↪needs sage.libs.ntl
Univariate Polynomial Ring in x over Finite Field of size 2 (using GF2X)
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> k = GF(Integer(2))
>>> residue = lambda x: k(x)
>>> f = Integer(4)*x**Integer(2) + x + Integer(3)
>>> g = f.map_coefficients(residue); g
x + 1
>>> g.parent()
Univariate Polynomial Ring in x over Integer Ring
>>> g = f.map_coefficients(residue, new_base_ring=k); g
x + 1
>>> g.parent()                                                      #␣
↪needs sage.libs.ntl
Univariate Polynomial Ring in x over Finite Field of size 2 (using GF2X)
>>> residue = k.coerce_map_from(ZZ)
>>> g = f.map_coefficients(residue); g
x + 1
>>> g.parent()                                                      #␣
↪needs sage.libs.ntl
Univariate Polynomial Ring in x over Finite Field of size 2 (using GF2X)
```

mod(*other*)

Remainder of division of `self` by `other`.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: x % (x+1)
-1
sage: (x^3 + x - 1) % (x^2 - 1)
2*x - 1
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> x % (x+Integer(1))
-1
>>> (x**Integer(3) + x - Integer(1)) % (x**Integer(2) - Integer(1))
2*x - 1
```

monic()

Return this polynomial divided by its leading coefficient. Does not change this polynomial.

EXAMPLES:

```
sage: x = QQ['x'].0
sage: f = 2*x^2 + x^3 + 56*x^5
```

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```
sage: f.monic()
x^5 + 1/56*x^3 + 1/28*x^2
sage: f = (1/4)*x^2 + 3*x + 1
sage: f.monic()
x^2 + 12*x + 4
```

```
>>> from sage.all import *
>>> x = QQ['x'].gen(0)
>>> f = Integer(2)*x**Integer(2) + x**Integer(3) + Integer(56)*x**Integer(5)
>>> f.monic()
x^5 + 1/56*x^3 + 1/28*x^2
>>> f = (Integer(1)/Integer(4))*x**Integer(2) + Integer(3)*x + Integer(1)
>>> f.monic()
x^2 + 12*x + 4
```

The following happens because $f = 0$ cannot be made into a monic polynomial

```
sage: f = 0*x
sage: f.monic()
Traceback (most recent call last):
...
ZeroDivisionError: rational division by zero
```

```
>>> from sage.all import *
>>> f = Integer(0)*x
>>> f.monic()
Traceback (most recent call last):
...
ZeroDivisionError: rational division by zero
```

Notice that the monic version of a polynomial over the integers is defined over the rationals.

```
sage: x = ZZ['x'].0
sage: f = 3*x^19 + x^2 - 37
sage: g = f.monic(); g
x^19 + 1/3*x^2 - 37/3
sage: g.parent()
Univariate Polynomial Ring in x over Rational Field
```

```
>>> from sage.all import *
>>> x = ZZ['x'].gen(0)
>>> f = Integer(3)*x**Integer(19) + x**Integer(2) - Integer(37)
>>> g = f.monic(); g
x^19 + 1/3*x^2 - 37/3
>>> g.parent()
Univariate Polynomial Ring in x over Rational Field
```

AUTHORS:

- Naqi Jaffery (2006-01-24): examples

monomial_coefficient(*m*)

Return the coefficient in the base ring of the monomial `m` in `self`, where `m` must have the same parent as self.

INPUT:

- m – a monomial

OUTPUT: coefficient in base ring

EXAMPLES:

```
sage: P.<x> = QQ[]
sage: f = 2 * x
sage: c = f.monomial_coefficient(x); c
2
sage: c.parent()
Rational Field

sage: f = x^9 - 1/2*x^2 + 7*x + 5/11
sage: f.monomial_coefficient(x^9)
1
sage: f.monomial_coefficient(x^2)
-1/2
sage: f.monomial_coefficient(x)
7
sage: f.monomial_coefficient(x^0)
5/11
sage: f.monomial_coefficient(x^3)
0
```

```
>>> from sage.all import *
>>> P = QQ['x']; (x,) = P._first_ngens(1)
>>> f = Integer(2) * x
>>> c = f.monomial_coefficient(x); c
2
>>> c.parent()
Rational Field

>>> f = x**Integer(9) - Integer(1)/Integer(2)*x**Integer(2) + Integer(7)*x +␣
↪Integer(5)/Integer(11)
>>> f.monomial_coefficient(x**Integer(9))
1
>>> f.monomial_coefficient(x**Integer(2))
-1/2
>>> f.monomial_coefficient(x)
7
>>> f.monomial_coefficient(x**Integer(0))
5/11
>>> f.monomial_coefficient(x**Integer(3))
0
```

monomial_coefficients(*copy=None*)

Return a sparse dictionary representation of this univariate polynomial.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: f = x^3 + -1/7*x + 13
sage: f.monomial_coefficients()
{0: 13, 1: -1/7, 3: 1}
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(3) + -Integer(1)/Integer(7)*x + Integer(13)
>>> f.monomial_coefficients()
{0: 13, 1: -1/7, 3: 1}
```

`dict` is an alias:

```
sage: f.dict()
{0: 13, 1: -1/7, 3: 1}
```

```
>>> from sage.all import *
>>> f.dict()
{0: 13, 1: -1/7, 3: 1}
```

monomials()

Return the list of the monomials in `self` in a decreasing order of their degrees.

EXAMPLES:

```
sage: P.<x> = QQ[]
sage: f = x^2 + (2/3)*x + 1
sage: f.monomials()
[x^2, x, 1]
sage: f = P(3/2)
sage: f.monomials()
[1]
sage: f = P(0)
sage: f.monomials()
[]
sage: f = x
sage: f.monomials()
[x]
sage: f = - 1/2*x^2 + x^9 + 7*x + 5/11
sage: f.monomials()
[x^9, x^2, x, 1]

sage: # needs sage.rings.number_field
sage: x = polygen(ZZ, 'x')
sage: K.<rho> = NumberField(x**2 + 1)
sage: R.<y> = QQ[]
sage: p = rho * y
sage: p.monomials()
[y]
```

```
>>> from sage.all import *
>>> P = QQ['x']; (x,) = P._first_ngens(1)
```

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```
>>> f = x**Integer(2) + (Integer(2)/Integer(3))*x + Integer(1)
>>> f.monomials()
[x^2, x, 1]
>>> f = P(Integer(3)/Integer(2))
>>> f.monomials()
[1]
>>> f = P(Integer(0))
>>> f.monomials()
[]
>>> f = x
>>> f.monomials()
[x]
>>> f = - Integer(1)/Integer(2)*x**Integer(2) + x**Integer(9) + Integer(7)*x␣
↪+ Integer(5)/Integer(11)
>>> f.monomials()
[x^9, x^2, x, 1]

>>> # needs sage.rings.number_field
>>> x = polygen(ZZ, 'x')
>>> K = NumberField(x**Integer(2) + Integer(1), names=('rho',)); (rho,) = K._
↪first_ngens(1)
>>> R = QQ['y']; (y,) = R._first_ngens(1)
>>> p = rho * y
>>> p.monomials()
[y]
```

multiplication_trunc(*other*, *n*)

Truncated multiplication.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: (x^10 + 5*x^5 + x^2 - 3).multiplication_trunc(x^7 - 3*x^3 + 1, 11)
x^10 + x^9 - 15*x^8 - 3*x^7 + 2*x^5 + 9*x^3 + x^2 - 3
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> (x**Integer(10) + Integer(5)*x**Integer(5) + x**Integer(2) - Integer(3)).
↪multiplication_trunc(x**Integer(7) - Integer(3)*x**Integer(3) + Integer(1),␣
↪Integer(11))
x^10 + x^9 - 15*x^8 - 3*x^7 + 2*x^5 + 9*x^3 + x^2 - 3
```

Check that coercion is working:

```
sage: R2 = QQ['x']
sage: x2 = R2.gen()
sage: p1 = (x^3 + 1).multiplication_trunc(x2^3 - 2, 5); p1
-x^3 - 2
sage: p2 = (x2^3 + 1).multiplication_trunc(x^3 - 2, 5); p2
-x^3 - 2
sage: parent(p1) == parent(p2) == R2
True
```

```
>>> from sage.all import *
>>> R2 = QQ['x']
>>> x2 = R2.gen()
>>> p1 = (x**Integer(3) + Integer(1)).multiplication_trunc(x2**Integer(3) -␣
↪Integer(2), Integer(5)); p1
-x^3 - 2
>>> p2 = (x2**Integer(3) + Integer(1)).multiplication_trunc(x**Integer(3) -␣
↪Integer(2), Integer(5)); p2
-x^3 - 2
>>> parent(p1) == parent(p2) == R2
True
```

newton_raphson (*n*, *x0*)

Return a list of n iterative approximations to a root of this polynomial, computed using the Newton-Raphson method.

The Newton-Raphson method is an iterative root-finding algorithm. For $f(x)$ a polynomial, as is the case here, this is essentially the same as Horner's method.

INPUT:

- `n` – integer; the number of iterations
- `x0` – an initial guess x_0

OUTPUT: list of numbers hopefully approximating a root of $f(x) = 0$

If one of the iterates is a critical point of f, a `ZeroDivisionError` exception is raised.

EXAMPLES:

```
sage: x = PolynomialRing(RealField(), 'x').gen()                            #␣
↪needs sage.rings.real_mpfr
sage: f = x^2 - 2                                                            #␣
↪needs sage.rings.real_mpfr
sage: f.newton_raphson(4, 1)                                                 #␣
↪needs sage.rings.real_mpfr
[1.50000000000000, 1.41666666666667, 1.41421568627451, 1.41421356237469]
```

```
>>> from sage.all import *
>>> x = PolynomialRing(RealField(), 'x').gen()                            #␣
↪needs sage.rings.real_mpfr
>>> f = x**Integer(2) - Integer(2)                                           ␣
↪               # needs sage.rings.real_mpfr
>>> f.newton_raphson(Integer(4), Integer(1))                                 ␣
↪              # needs sage.rings.real_mpfr
[1.50000000000000, 1.41666666666667, 1.41421568627451, 1.41421356237469]
```

AUTHORS:

- David Joyner and William Stein (2005-11-28)

newton_slopes (*p*, *lengths=False*)

Return the p-adic slopes of the Newton polygon of `self`, when this makes sense.

OUTPUT:

If `lengths` is `False`, a list of rational numbers. If `lengths` is `True`, a list of couples (s, l) where s is the slope and l the length of the corresponding segment in the Newton polygon.

EXAMPLES:

```
sage: x = QQ['x'].0
sage: f = x^3 + 2
sage: f.newton_slopes(2)                                                      #␣
↪needs sage.libs.pari
[1/3, 1/3, 1/3]
sage: R.<x> = PolynomialRing(ZZ, sparse=True)
sage: p = x^5 + 6*x^2 + 4
sage: p.newton_slopes(2)                                                      #␣
↪needs sage.libs.pari
[1/2, 1/2, 1/3, 1/3, 1/3]
sage: p.newton_slopes(2, lengths=True)
[(1/2, 2), (1/3, 3)]
sage: (x^2^100 + 27).newton_slopes(3, lengths=True)
[(3/1267650600228229401496703205376, 1267650600228229401496703205376)]
```

```
>>> from sage.all import *
>>> x = QQ['x'].gen(0)
>>> f = x**Integer(3) + Integer(2)
>>> f.newton_slopes(Integer(2))                                               ␣
↪     # needs sage.libs.pari
[1/3, 1/3, 1/3]
>>> R = PolynomialRing(ZZ, sparse=True, names=('x',)); (x,) = R._first_
↪ngens(1)
>>> p = x**Integer(5) + Integer(6)*x**Integer(2) + Integer(4)
>>> p.newton_slopes(Integer(2))                                               ␣
↪     # needs sage.libs.pari
[1/2, 1/2, 1/3, 1/3, 1/3]
>>> p.newton_slopes(Integer(2), lengths=True)
[(1/2, 2), (1/3, 3)]
>>> (x**Integer(2)**Integer(100) + Integer(27)).newton_slopes(Integer(3),␣
↪lengths=True)
[(3/1267650600228229401496703205376, 1267650600228229401496703205376)]
```

ALGORITHM: Uses PARI if `lengths` is `False`.

norm(*p*)

Return the p-norm of this polynomial.

DEFINITION: For integer p, the p-norm of a polynomial is the pth root of the sum of the pth powers of the absolute values of the coefficients of the polynomial.

INPUT:

- `p` – positive integer or +infinity; the degree of the norm

EXAMPLES:

```
sage: # needs sage.rings.real_mpfr
sage: R.<x> = RR[]
sage: f = x^6 + x^2 + -x^4 - 2*x^3
sage: f.norm(2)
```

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```
2.64575131106459
sage: (sqrt(1^2 + 1^2 + (-1)^2 + (-2)^2)).n()                                  #␣
↪needs sage.symbolic
2.64575131106459
```

```
>>> from sage.all import *
>>> # needs sage.rings.real_mpfr
>>> R = RR['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(6) + x**Integer(2) + -x**Integer(4) -␣
↪Integer(2)*x**Integer(3)
>>> f.norm(Integer(2))
2.64575131106459
>>> (sqrt(Integer(1)**Integer(2) + Integer(1)**Integer(2) + (-
↪Integer(1))**Integer(2) + (-Integer(2))**Integer(2))).n()                     ␣
↪              # needs sage.symbolic
2.64575131106459
```

```
sage: f.norm(1)                                                                #␣
↪needs sage.rings.real_mpfr
5.00000000000000
sage: f.norm(infinity)                                                         #␣
↪needs sage.rings.real_mpfr
2.00000000000000
```

```
>>> from sage.all import *
>>> f.norm(Integer(1))                                                         ␣
↪       # needs sage.rings.real_mpfr
5.00000000000000
>>> f.norm(infinity)                                                         #␣
↪needs sage.rings.real_mpfr
2.00000000000000
```

```
sage: f.norm(-1)                                                               #␣
↪needs sage.rings.real_mpfr
Traceback (most recent call last):
...
ValueError: The degree of the norm must be positive
```

```
>>> from sage.all import *
>>> f.norm(-Integer(1))                                                        ␣
↪       # needs sage.rings.real_mpfr
Traceback (most recent call last):
...
ValueError: The degree of the norm must be positive
```

AUTHORS:

- Didier Deshommes
- William Stein: fix bugs, add definition, etc.

nth_root(n)

Return a n-th root of this polynomial.

This is computed using Newton method in the ring of power series. This method works only when the base ring is an integral domain. Moreover, for polynomial whose coefficient of lower degree is different from 1, the elements of the base ring should have a method `nth_root()` implemented.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: a = 27 * (x+3)**6 * (x+5)**3
sage: a.nth_root(3)
3*x^3 + 33*x^2 + 117*x + 135

sage: b = 25 * (x^2 + x + 1)
sage: b.nth_root(2)
Traceback (most recent call last):
...
ValueError: not a 2nd power
sage: R(0).nth_root(3)
0
sage: R.<x> = QQ[]
sage: a = 1/4 * (x/7 + 3/2)^2 * (x/2 + 5/3)^4
sage: a.nth_root(2)
1/56*x^3 + 103/336*x^2 + 365/252*x + 25/12

sage: # needs sage.rings.number_field
sage: K.<sqrt2> = QuadraticField(2)
sage: R.<x> = K[]
sage: a = (x + sqrt2)^3 * ((1+sqrt2)*x - 1/sqrt2)^6
sage: b = a.nth_root(3); b
(2*sqrt2 + 3)*x^3 + (2*sqrt2 + 2)*x^2 + (-2*sqrt2 - 3/2)*x + 1/2*sqrt2
sage: b^3 == a
True

sage: # needs sage.rings.number_field
sage: R.<x> = QQbar[]
sage: p = x**3 + QQbar(2).sqrt() * x - QQbar(3).sqrt()
sage: r = (p**5).nth_root(5)
sage: r * p[0] == p * r[0]
True
sage: p = (x+1)^20 + x^20
sage: p.nth_root(20)
Traceback (most recent call last):
...
ValueError: not a 20th power

sage: # needs sage.rings.finite_rings
sage: z = GF(4).gen()
sage: R.<x> = GF(4)[]
sage: p = z*x**4 + 2*x - 1
sage: r = (p**15).nth_root(15)
sage: r * p[0] == p * r[0]
True
sage: ((x+1)**2).nth_root(2)
x + 1
sage: ((x+1)**4).nth_root(4)
```

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```
x + 1
sage: ((x+1)**12).nth_root(12)
x + 1
sage: (x^4 + x^3 + 1).nth_root(2)
Traceback (most recent call last):
...
ValueError: not a 2nd power
sage: p = (x+1)^17 + x^17
sage: r = p.nth_root(17)
Traceback (most recent call last):
...
ValueError: not a 17th power

sage: R1.<x> = QQ[]
sage: R2.<y> = R1[]
sage: R3.<z> = R2[]
sage: (((y**2+x)*z^2 + x*y*z + 2*x)**3).nth_root(3)
(y^2 + x)*z^2 + x*y*z + 2*x
sage: ((x+y+z)**5).nth_root(5)
z + y + x
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> a = Integer(27) * (x+Integer(3))**Integer(6) * (x+Integer(5))**Integer(3)
>>> a.nth_root(Integer(3))
3*x^3 + 33*x^2 + 117*x + 135

>>> b = Integer(25) * (x**Integer(2) + x + Integer(1))
>>> b.nth_root(Integer(2))
Traceback (most recent call last):
...
ValueError: not a 2nd power
>>> R(Integer(0)).nth_root(Integer(3))
0
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> a = Integer(1)/Integer(4) * (x/Integer(7) + Integer(3)/
↪Integer(2))**Integer(2) * (x/Integer(2) + Integer(5)/Integer(3))**Integer(4)
>>> a.nth_root(Integer(2))
1/56*x^3 + 103/336*x^2 + 365/252*x + 25/12

>>> # needs sage.rings.number_field
>>> K = QuadraticField(Integer(2), names=('sqrt2',)); (sqrt2,) = K._first_
↪ngens(1)
>>> R = K['x']; (x,) = R._first_ngens(1)
>>> a = (x + sqrt2)**Integer(3) * ((Integer(1)+sqrt2)*x - Integer(1)/
↪sqrt2)**Integer(6)
>>> b = a.nth_root(Integer(3)); b
(2*sqrt2 + 3)*x^3 + (2*sqrt2 + 2)*x^2 + (-2*sqrt2 - 3/2)*x + 1/2*sqrt2
>>> b**Integer(3) == a
True

>>> # needs sage.rings.number_field
```

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```
>>> R = QQbar['x']; (x,) = R._first_ngens(1)
>>> p = x**Integer(3) + QQbar(Integer(2)).sqrt() * x - QQbar(Integer(3)).
↪sqrt()
>>> r = (p**Integer(5)).nth_root(Integer(5))
>>> r * p[Integer(0)] == p * r[Integer(0)]
True
>>> p = (x+Integer(1))**Integer(20) + x**Integer(20)
>>> p.nth_root(Integer(20))
Traceback (most recent call last):
...
ValueError: not a 20th power

>>> # needs sage.rings.finite_rings
>>> z = GF(Integer(4)).gen()
>>> R = GF(Integer(4))['x']; (x,) = R._first_ngens(1)
>>> p = z*x**Integer(4) + Integer(2)*x - Integer(1)
>>> r = (p**Integer(15)).nth_root(Integer(15))
>>> r * p[Integer(0)] == p * r[Integer(0)]
True
>>> ((x+Integer(1))**Integer(2)).nth_root(Integer(2))
x + 1
>>> ((x+Integer(1))**Integer(4)).nth_root(Integer(4))
x + 1
>>> ((x+Integer(1))**Integer(12)).nth_root(Integer(12))
x + 1
>>> (x**Integer(4) + x**Integer(3) + Integer(1)).nth_root(Integer(2))
Traceback (most recent call last):
...
ValueError: not a 2nd power
>>> p = (x+Integer(1))**Integer(17) + x**Integer(17)
>>> r = p.nth_root(Integer(17))
Traceback (most recent call last):
...
ValueError: not a 17th power

>>> R1 = QQ['x']; (x,) = R1._first_ngens(1)
>>> R2 = R1['y']; (y,) = R2._first_ngens(1)
>>> R3 = R2['z']; (z,) = R3._first_ngens(1)
>>> (((y**Integer(2)+x)*z**Integer(2) + x*y*z + Integer(2)*x)**Integer(3)).
↪nth_root(Integer(3))
(y^2 + x)*z^2 + x*y*z + 2*x
>>> ((x+y+z)**Integer(5)).nth_root(Integer(5))
z + y + x
```

Here we consider a base ring without `nth_root` method. The third example with a non-trivial coefficient of lowest degree raises an error:

```
sage: # needs sage.libs.pari
sage: R.<x> = QQ[]
sage: R2 = R.quotient(x**2 + 1)
sage: x = R2.gen()
sage: R3.<y> = R2[]
```

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```
sage: (y**2 - 2*y + 1).nth_root(2)
-y + 1
sage: (y**3).nth_root(3)
y
sage: (y**2 + x).nth_root(2)
Traceback (most recent call last):
...
AttributeError: ... has no attribute 'nth_root'...
```

```
>>> from sage.all import *
>>> # needs sage.libs.pari
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> R2 = R.quotient(x**Integer(2) + Integer(1))
>>> x = R2.gen()
>>> R3 = R2['y']; (y,) = R3._first_ngens(1)
>>> (y**Integer(2) - Integer(2)*y + Integer(1)).nth_root(Integer(2))
-y + 1
>>> (y**Integer(3)).nth_root(Integer(3))
y
>>> (y**Integer(2) + x).nth_root(Integer(2))
Traceback (most recent call last):
...
AttributeError: ... has no attribute 'nth_root'...
```

number_of_real_roots()

Return the number of real roots of this polynomial, counted without multiplicity.

EXAMPLES:

```
sage: # needs sage.libs.pari
sage: R.<x> = PolynomialRing(ZZ)
sage: pol = (x - 1)^2 * (x - 2)^2 * (x - 3)
sage: pol.number_of_real_roots()
3
sage: pol = (x - 1) * (x - 2) * (x - 3)

sage: # needs sage.libs.pari sage.rings.real_mpfr
sage: pol2 = pol.change_ring(CC)
sage: pol2.number_of_real_roots()
3
sage: R.<x> = PolynomialRing(CC)
sage: pol = (x - 1) * (x - CC(I))
sage: pol.number_of_real_roots()
1
```

```
>>> from sage.all import *
>>> # needs sage.libs.pari
>>> R = PolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> pol = (x - Integer(1))**Integer(2) * (x - Integer(2))**Integer(2) * (x -
↪Integer(3))
>>> pol.number_of_real_roots()
3
```

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```
>>> pol = (x - Integer(1)) * (x - Integer(2)) * (x - Integer(3))

>>> # needs sage.libs.pari sage.rings.real_mpfr
>>> pol2 = pol.change_ring(CC)
>>> pol2.number_of_real_roots()
3
>>> R = PolynomialRing(CC, names=('x',)); (x,) = R._first_ngens(1)
>>> pol = (x - Integer(1)) * (x - CC(I))
>>> pol.number_of_real_roots()
1
```

number_of_roots_in_interval(*a=None*, *b=None*)

Return the number of roots of this polynomial in the interval $[a, b]$, counted without multiplicity. The endpoints a, b default to `-Infinity`, `Infinity` (which are also valid input values).

Calls the PARI routine pari:polsturm.

Note that as of version 2.8, PARI includes the left endpoint of the interval (and no longer uses Sturm's algorithm on exact inputs). pari:polsturm requires a polynomial with real coefficients; in case PARI returns an error, we try again after taking the GCD of `self` with its complex conjugate.

EXAMPLES:

```
sage: # needs sage.libs.pari
sage: R.<x> = PolynomialRing(ZZ)
sage: pol = (x - 1)^2 * (x - 2)^2 * (x - 3)
sage: pol.number_of_roots_in_interval(1, 2)
2
sage: pol.number_of_roots_in_interval(1.01, 2)
1
sage: pol.number_of_roots_in_interval(None, 2)
2
sage: pol.number_of_roots_in_interval(1, Infinity)
3
sage: pol.number_of_roots_in_interval()
3
sage: pol = (x - 1) * (x - 2) * (x - 3)

sage: # needs sage.libs.pari sage.rings.real_mpfr
sage: pol2 = pol.change_ring(CC)
sage: pol2.number_of_roots_in_interval()
3
sage: R.<x> = PolynomialRing(CC)
sage: pol = (x - 1) * (x - CC(I))
sage: pol.number_of_roots_in_interval(0, 2)
1
```

```
>>> from sage.all import *
>>> # needs sage.libs.pari
>>> R = PolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> pol = (x - Integer(1))**Integer(2) * (x - Integer(2))**Integer(2) * (x -␣
↪Integer(3))
>>> pol.number_of_roots_in_interval(Integer(1), Integer(2))
```

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```
2
>>> pol.number_of_roots_in_interval(RealNumber('1.01'), Integer(2))
1
>>> pol.number_of_roots_in_interval(None, Integer(2))
2
>>> pol.number_of_roots_in_interval(Integer(1), Infinity)
3
>>> pol.number_of_roots_in_interval()
3
>>> pol = (x - Integer(1)) * (x - Integer(2)) * (x - Integer(3))

>>> # needs sage.libs.pari sage.rings.real_mpfr
>>> pol2 = pol.change_ring(CC)
>>> pol2.number_of_roots_in_interval()
3
>>> R = PolynomialRing(CC, names=('x',)); (x,) = R._first_ngens(1)
>>> pol = (x - Integer(1)) * (x - CC(I))
>>> pol.number_of_roots_in_interval(Integer(0), Integer(2))
1
```

number_of_terms()

Return the number of nonzero coefficients of `self`.

Also called weight, Hamming weight or sparsity.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: f = x^3 - x
sage: f.number_of_terms()
2
sage: R(0).number_of_terms()
0
sage: f = (x + 1)^100
sage: f.number_of_terms()
101
sage: S = GF(5)['y']
sage: S(f).number_of_terms()
5
sage: cyclotomic_polynomial(105).number_of_terms()
33
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(3) - x
>>> f.number_of_terms()
2
>>> R(Integer(0)).number_of_terms()
0
>>> f = (x + Integer(1))**Integer(100)
>>> f.number_of_terms()
101
>>> S = GF(Integer(5))['y']
```

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```
>>> S(f).number_of_terms()
5
>>> cyclotomic_polynomial(Integer(105)).number_of_terms()
33
```

The method *hamming_weight()* is an alias:

```
sage: f.hamming_weight()
101
```

```
>>> from sage.all import *
>>> f.hamming_weight()
101
```

numerator()

Return a numerator of `self`, computed as `self * self.denominator()`.

Note that some subclasses may implement its own numerator function. For example, see *sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint*

> **Warning**
>
> This is not the numerator of the rational function defined by `self`, which would always be `self` since `self` is a polynomial.

EXAMPLES:

First we compute the numerator of a polynomial with integer coefficients, which is of course `self`.

```
sage: R.<x> = ZZ[]
sage: f = x^3 + 17*x + 1
sage: f.numerator()
x^3 + 17*x + 1
sage: f == f.numerator()
True
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(3) + Integer(17)*x + Integer(1)
>>> f.numerator()
x^3 + 17*x + 1
>>> f == f.numerator()
True
```

Next we compute the numerator of a polynomial with rational coefficients.

```
sage: R.<x> = PolynomialRing(QQ)
sage: f = (1/17)*x^19 - (2/3)*x + 1/3; f
1/17*x^19 - 2/3*x + 1/3
sage: f.numerator()
3*x^19 - 34*x + 17
```

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```
sage: f == f.numerator()
False
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = (Integer(1)/Integer(17))*x**Integer(19) - (Integer(2)/Integer(3))*x +␣
↪Integer(1)/Integer(3); f
1/17*x^19 - 2/3*x + 1/3
>>> f.numerator()
3*x^19 - 34*x + 17
>>> f == f.numerator()
False
```

We try to compute the denominator of a polynomial with coefficients in the real numbers, which is a ring whose elements do not have a denominator method.

```
sage: # needs sage.rings.real_mpfr
sage: R.<x> = RR[]
sage: f = x + RR('0.3'); f
x + 0.300000000000000
sage: f.numerator()
x + 0.300000000000000
```

```
>>> from sage.all import *
>>> # needs sage.rings.real_mpfr
>>> R = RR['x']; (x,) = R._first_ngens(1)
>>> f = x + RR('0.3'); f
x + 0.300000000000000
>>> f.numerator()
x + 0.300000000000000
```

We check that the computation of the numerator and denominator are valid.

```
sage: # needs sage.rings.number_field sage.symbolic
sage: K = NumberField(symbolic_expression('x^3+2'), 'a')['s,t']['x']
sage: f = K.random_element()
sage: f.numerator() / f.denominator() == f
True

sage: R = RR['x']
sage: f = R.random_element()
sage: f.numerator() / f.denominator() == f
True
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field sage.symbolic
>>> K = NumberField(symbolic_expression('x^3+2'), 'a')['s,t']['x']
>>> f = K.random_element()
>>> f.numerator() / f.denominator() == f
True

>>> R = RR['x']
```

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```
>>> f = R.random_element()
>>> f.numerator() / f.denominator() == f
True
```

ord(*p=None*)

This is the same as the valuation of `self` at p. See the documentation for *valuation()*.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: (x^2 + x).ord(x + 1)
1
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> (x**Integer(2) + x).ord(x + Integer(1))
1
```

padded_list(*n=None*)

Return list of coefficients of `self` up to (but not including) q^n.

Includes 0s in the list on the right so that the list has length n.

INPUT:

- `n` – (default: `None`) if given, an integer that is at least 0

EXAMPLES:

```
sage: x = polygen(QQ)
sage: f = 1 + x^3 + 23*x^5
sage: f.padded_list()
[1, 0, 0, 1, 0, 23]
sage: f.padded_list(10)
[1, 0, 0, 1, 0, 23, 0, 0, 0, 0]
sage: len(f.padded_list(10))
10
sage: f.padded_list(3)
[1, 0, 0]
sage: f.padded_list(0)
[]
sage: f.padded_list(-1)
Traceback (most recent call last):
...
ValueError: n must be at least 0
```

```
>>> from sage.all import *
>>> x = polygen(QQ)
>>> f = Integer(1) + x**Integer(3) + Integer(23)*x**Integer(5)
>>> f.padded_list()
[1, 0, 0, 1, 0, 23]
>>> f.padded_list(Integer(10))
[1, 0, 0, 1, 0, 23, 0, 0, 0, 0]
>>> len(f.padded_list(Integer(10)))
```

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```
10
>>> f.padded_list(Integer(3))
[1, 0, 0]
>>> f.padded_list(Integer(0))
[]
>>> f.padded_list(-Integer(1))
Traceback (most recent call last):
...
ValueError: n must be at least 0
```

plot(*xmin=None*, *xmax=None*, **args*, ***kwds*)

Return a plot of this polynomial.

INPUT:

- `xmin` – float
- `xmax` – float
- `*args, **kwds` – passed to either plot or point

OUTPUT: a graphic object

EXAMPLES:

```
sage: x = polygen(GF(389))
sage: plot(x^2 + 1, rgbcolor=(0,0,1))                                       #␣
↪needs sage.plot
Graphics object consisting of 1 graphics primitive
sage: x = polygen(QQ)
sage: plot(x^2 + 1, rgbcolor=(1,0,0))                                       #␣
↪needs sage.plot
Graphics object consisting of 1 graphics primitive
```

```
>>> from sage.all import *
>>> x = polygen(GF(Integer(389)))
>>> plot(x**Integer(2) + Integer(1), rgbcolor=(Integer(0),Integer(0),
↪Integer(1)))                                         # needs sage.plot
Graphics object consisting of 1 graphics primitive
>>> x = polygen(QQ)
>>> plot(x**Integer(2) + Integer(1), rgbcolor=(Integer(1),Integer(0),
↪Integer(0)))                                         # needs sage.plot
Graphics object consisting of 1 graphics primitive
```

polynomial(*var*)

Let `var` be one of the variables of the parent of `self`. This returns `self` viewed as a univariate polynomial in `var` over the polynomial ring generated by all the other variables of the parent.

For univariate polynomials, if `var` is the generator of the parent ring, we return this polynomial, otherwise raise an error.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: (x + 1).polynomial(x)
x + 1
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> (x + Integer(1)).polynomial(x)
x + 1
```

power_trunc(*n*, *prec*)

Truncated n-th power of this polynomial up to precision prec.

INPUT:

- n – nonnegative integer; power to be taken
- prec – integer; the precision

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: (3*x^2 - 2*x + 1).power_trunc(5, 8)
-1800*x^7 + 1590*x^6 - 1052*x^5 + 530*x^4 - 200*x^3 + 55*x^2 - 10*x + 1
sage: ((3*x^2 - 2*x + 1)^5).truncate(8)
-1800*x^7 + 1590*x^6 - 1052*x^5 + 530*x^4 - 200*x^3 + 55*x^2 - 10*x + 1

sage: S.<y> = R[]
sage: (x + y).power_trunc(5,5)
5*x*y^4 + 10*x^2*y^3 + 10*x^3*y^2 + 5*x^4*y + x^5
sage: ((x + y)^5).truncate(5)
5*x*y^4 + 10*x^2*y^3 + 10*x^3*y^2 + 5*x^4*y + x^5

sage: R.<x> = GF(3)[]
sage: p = x^2 - x + 1
sage: q = p.power_trunc(80, 20); q
x^19 + x^18 + ... + 2*x^4 + 2*x^3 + x + 1
sage: (p^80).truncate(20) == q
True

sage: R.<x> = GF(7)[]
sage: p = (x^2 + x + 1).power_trunc(2^100, 100); p
2*x^99 + x^98 + x^95 + 2*x^94 + ... + 3*x^2 + 2*x + 1

sage: for i in range(100):
....:     q1 = (x^2 + x + 1).power_trunc(2^100 + i, 100)
....:     q2 = p * (x^2 + x + 1).power_trunc(i, 100)
....:     q2 = q2.truncate(100)
....:     assert q1 == q2, "i = {}".format(i)
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> (Integer(3)*x**Integer(2) - Integer(2)*x + Integer(1)).power_
↪trunc(Integer(5), Integer(8))
-1800*x^7 + 1590*x^6 - 1052*x^5 + 530*x^4 - 200*x^3 + 55*x^2 - 10*x + 1
>>> ((Integer(3)*x**Integer(2) - Integer(2)*x + Integer(1))**Integer(5)).
↪truncate(Integer(8))
-1800*x^7 + 1590*x^6 - 1052*x^5 + 530*x^4 - 200*x^3 + 55*x^2 - 10*x + 1
```

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```
>>> S = R['y']; (y,) = S._first_ngens(1)
>>> (x + y).power_trunc(Integer(5),Integer(5))
5*x*y^4 + 10*x^2*y^3 + 10*x^3*y^2 + 5*x^4*y + x^5
>>> ((x + y)**Integer(5)).truncate(Integer(5))
5*x*y^4 + 10*x^2*y^3 + 10*x^3*y^2 + 5*x^4*y + x^5

>>> R = GF(Integer(3))['x']; (x,) = R._first_ngens(1)
>>> p = x**Integer(2) - x + Integer(1)
>>> q = p.power_trunc(Integer(80), Integer(20)); q
x^19 + x^18 + ... + 2*x^4 + 2*x^3 + x + 1
>>> (p**Integer(80)).truncate(Integer(20)) == q
True

>>> R = GF(Integer(7))['x']; (x,) = R._first_ngens(1)
>>> p = (x**Integer(2) + x + Integer(1)).power_trunc(Integer(2)**Integer(100),
→ Integer(100)); p
2*x^99 + x^98 + x^95 + 2*x^94 + ... + 3*x^2 + 2*x + 1

>>> for i in range(Integer(100)):
...     q1 = (x**Integer(2) + x + Integer(1)).power_
→trunc(Integer(2)**Integer(100) + i, Integer(100))
...     q2 = p * (x**Integer(2) + x + Integer(1)).power_trunc(i, Integer(100))
...     q2 = q2.truncate(Integer(100))
...     assert q1 == q2, "i = {}".format(i)
```

prec()

Return the precision of this polynomial. This is always infinity, since polynomials are of infinite precision by definition (there is no big-oh).

EXAMPLES:

```
sage: x = polygen(ZZ)
sage: (x^5 + x + 1).prec()
+Infinity
sage: x.prec()
+Infinity
```

```
>>> from sage.all import *
>>> x = polygen(ZZ)
>>> (x**Integer(5) + x + Integer(1)).prec()
+Infinity
>>> x.prec()
+Infinity
```

pseudo_quo_rem(*other*)

Compute the pseudo-division of two polynomials.

INPUT:

- `other` – a nonzero polynomial

OUTPUT:

Q and R such that $l^{m-n+1}\mathrm{self} = Q \cdot \mathrm{other} + R$ where m is the degree of this polynomial, n is the degree of `other`, l is the leading coefficient of `other`. The result is such that $\deg(R) < \deg(\mathrm{other})$.

ALGORITHM:

Algorithm 3.1.2 in [Coh1993].

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ, sparse=True)
sage: p = x^4 + 6*x^3 + x^2 - x + 2
sage: q = 2*x^2 - 3*x - 1
sage: quo, rem = p.pseudo_quo_rem(q); quo, rem
(4*x^2 + 30*x + 51, 175*x + 67)
sage: 2^(4-2+1)*p == quo*q + rem
True

sage: S.<T> = R[]
sage: p = (-3*x^2 - x)*T^3 - 3*x*T^2 + (x^2 - x)*T + 2*x^2 + 3*x - 2
sage: q = (-x^2 - 4*x - 5)*T^2 + (6*x^2 + x + 1)*T + 2*x^2 - x
sage: quo, rem = p.pseudo_quo_rem(q); quo, rem
((3*x^4 + 13*x^3 + 19*x^2 + 5*x)*T + 18*x^4 + 12*x^3 + 16*x^2 + 16*x,
 (-113*x^6 - 106*x^5 - 133*x^4 - 101*x^3 - 42*x^2 - 41*x)*T
      - 34*x^6 + 13*x^5 + 54*x^4 + 126*x^3 + 134*x^2 - 5*x - 50)
sage: (-x^2 - 4*x - 5)^(3-2+1) * p == quo*q + rem
True
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, sparse=True, names=('x',)); (x,) = R._first_
↪ngens(1)
>>> p = x**Integer(4) + Integer(6)*x**Integer(3) + x**Integer(2) - x +␣
↪Integer(2)
>>> q = Integer(2)*x**Integer(2) - Integer(3)*x - Integer(1)
>>> quo, rem = p.pseudo_quo_rem(q); quo, rem
(4*x^2 + 30*x + 51, 175*x + 67)
>>> Integer(2)**(Integer(4)-Integer(2)+Integer(1))*p == quo*q + rem
True

>>> S = R['T']; (T,) = S._first_ngens(1)
>>> p = (-Integer(3)*x**Integer(2) - x)*T**Integer(3) -␣
↪Integer(3)*x*T**Integer(2) + (x**Integer(2) - x)*T +␣
↪Integer(2)*x**Integer(2) + Integer(3)*x - Integer(2)
>>> q = (-x**Integer(2) - Integer(4)*x - Integer(5))*T**Integer(2) +␣
↪(Integer(6)*x**Integer(2) + x + Integer(1))*T + Integer(2)*x**Integer(2) - x
>>> quo, rem = p.pseudo_quo_rem(q); quo, rem
((3*x^4 + 13*x^3 + 19*x^2 + 5*x)*T + 18*x^4 + 12*x^3 + 16*x^2 + 16*x,
 (-113*x^6 - 106*x^5 - 133*x^4 - 101*x^3 - 42*x^2 - 41*x)*T
      - 34*x^6 + 13*x^5 + 54*x^4 + 126*x^3 + 134*x^2 - 5*x - 50)
>>> (-x**Integer(2) - Integer(4)*x - Integer(5))**(Integer(3)-
↪Integer(2)+Integer(1)) * p == quo*q + rem
True
```

radical()

Return the radical of `self`.

Over a field, this is the product of the distinct irreducible factors of `self`. (This is also sometimes called the "square-free part" of `self`, but that term is ambiguous; it is sometimes used to mean the quotient of `self` by its maximal square factor.)

EXAMPLES:

```
sage: P.<x> = ZZ[]
sage: t = (x^2-x+1)^3 * (3*x-1)^2
sage: t.radical()
3*x^3 - 4*x^2 + 4*x - 1
sage: radical(12 * x^5)
6*x
```

```
>>> from sage.all import *
>>> P = ZZ['x']; (x,) = P._first_ngens(1)
>>> t = (x**Integer(2)-x+Integer(1))**Integer(3) * (Integer(3)*x-
↪Integer(1))**Integer(2)
>>> t.radical()
3*x^3 - 4*x^2 + 4*x - 1
>>> radical(Integer(12) * x**Integer(5))
6*x
```

If `self` has a factor of multiplicity divisible by the characteristic (see Issue #8736):

```
sage: P.<x> = GF(2)[]
sage: (x^3 + x^2).radical()                                             #␣
↪needs sage.rings.finite_rings
x^2 + x
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> (x**Integer(3) + x**Integer(2)).radical()                           ␣
↪                # needs sage.rings.finite_rings
x^2 + x
```

rational_reconstruct(**args, **kwds*)

Deprecated: Use `rational_reconstruction()` instead. See Issue #12696 for details.

rational_reconstruction(*m, n_deg=None, d_deg=None*)

Return a tuple of two polynomials `(n, d)` where `self * d` is congruent to `n` modulo `m` and `n.degree() <= n_deg` and `d.degree() <= d_deg`.

INPUT:

- `m` – a univariate polynomial
- `n_deg` – (optional) an integer; the default is $\lfloor(\deg(m) - 1)/2\rfloor$
- `d_deg` – (optional) an integer; the default is $\lfloor(\deg(m) - 1)/2\rfloor$

ALGORITHM:

The algorithm is based on the extended Euclidean algorithm for the polynomial greatest common divisor.

EXAMPLES:

Over $\mathbf{Q}[z]$:

```
sage: z  = PolynomialRing(QQ, 'z').gen()
sage: p = -z**16 - z**15 - z**14 + z**13 + z**12 + z**11 - z**5 - z**4 - z**3␣
↪+ z**2 + z + 1
```

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```
sage: m = z**21
sage: n, d = p.rational_reconstruction(m); n, d
(z^4 + 2*z^3 + 3*z^2 + 2*z + 1,
 z^10 + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)
sage: ((p*d - n) % m).is_zero()
True
```

```
>>> from sage.all import *
>>> z  = PolynomialRing(QQ, 'z').gen()
>>> p = -z**Integer(16) - z**Integer(15) - z**Integer(14) + z**Integer(13) +␣
↪z**Integer(12) + z**Integer(11) - z**Integer(5) - z**Integer(4) -␣
↪z**Integer(3) + z**Integer(2) + z + Integer(1)
>>> m = z**Integer(21)
>>> n, d = p.rational_reconstruction(m); n, d
(z^4 + 2*z^3 + 3*z^2 + 2*z + 1,
 z^10 + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)
>>> ((p*d - n) % m).is_zero()
True
```

Over $\mathbf{Z}[z]$:

```
sage: z  = PolynomialRing(ZZ, 'z').gen()
sage: p = -z**16 - z**15 - z**14 + z**13 + z**12 + z**11 - z**5 - z**4 - z**3␣
↪+ z**2 + z + 1
sage: m = z**21
sage: n, d = p.rational_reconstruction(m); n, d
(z^4 + 2*z^3 + 3*z^2 + 2*z + 1,
 z^10 + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)
sage: ((p*d - n) % m).is_zero()
True
```

```
>>> from sage.all import *
>>> z  = PolynomialRing(ZZ, 'z').gen()
>>> p = -z**Integer(16) - z**Integer(15) - z**Integer(14) + z**Integer(13) +␣
↪z**Integer(12) + z**Integer(11) - z**Integer(5) - z**Integer(4) -␣
↪z**Integer(3) + z**Integer(2) + z + Integer(1)
>>> m = z**Integer(21)
>>> n, d = p.rational_reconstruction(m); n, d
(z^4 + 2*z^3 + 3*z^2 + 2*z + 1,
 z^10 + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)
>>> ((p*d - n) % m).is_zero()
True
```

Over an integral domain, `d` might not be monic:

```
sage: P = PolynomialRing(ZZ, 'x')
sage: x = P.gen()
sage: p = 7*x^5 - 10*x^4 + 16*x^3 - 32*x^2 + 128*x + 256
sage: m = x^5
sage: n, d = p.rational_reconstruction(m, 3, 2); n, d
(-32*x^3 + 384*x^2 + 2304*x + 2048, 5*x + 8)
sage: ((p*d - n) % m).is_zero()
```

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```
True
sage: n, d = p.rational_reconstruction(m, 4, 0); n, d
(-10*x^4 + 16*x^3 - 32*x^2 + 128*x + 256, 1)
sage: ((p*d - n) % m).is_zero()
True
```

```
>>> from sage.all import *
>>> P = PolynomialRing(ZZ, 'x')
>>> x = P.gen()
>>> p = Integer(7)*x**Integer(5) - Integer(10)*x**Integer(4) +␣
↪Integer(16)*x**Integer(3) - Integer(32)*x**Integer(2) + Integer(128)*x +␣
↪Integer(256)
>>> m = x**Integer(5)
>>> n, d = p.rational_reconstruction(m, Integer(3), Integer(2)); n, d
(-32*x^3 + 384*x^2 + 2304*x + 2048, 5*x + 8)
>>> ((p*d - n) % m).is_zero()
True
>>> n, d = p.rational_reconstruction(m, Integer(4), Integer(0)); n, d
(-10*x^4 + 16*x^3 - 32*x^2 + 128*x + 256, 1)
>>> ((p*d - n) % m).is_zero()
True
```

Over $\mathbf{Q}(t)[z]$:

```
sage: P = PolynomialRing(QQ, 't')
sage: t = P.gen()
sage: Pz = PolynomialRing(P.fraction_field(), 'z')
sage: z = Pz.gen()
sage: # p = (1 + t^2*z + z^4) / (1 - t*z)
sage: p = (1 + t^2*z + z^4)*(1 - t*z).inverse_mod(z^9)
sage: m = z^9
sage: n, d = p.rational_reconstruction(m); n, d
(-1/t*z^4 - t*z - 1/t, z - 1/t)
sage: ((p*d - n) % m).is_zero()
True
sage: w = PowerSeriesRing(P.fraction_field(), 'w').gen()
sage: n = -10*t^2*z^4 + (-t^2 + t - 1)*z^3 + (-t - 8)*z^2 + z + 2*t^2 - t
sage: d = z^4 + (2*t + 4)*z^3 + (-t + 5)*z^2 + (t^2 + 2)*z + t^2 + 2*t + 1
sage: prec = 9
sage: x = n.subs(z=w)/d.subs(z=w) + O(w^prec)

sage: # needs sage.libs.flint (otherwise timeout)
sage: nc, dc = Pz(x.list()).rational_reconstruction(z^prec)
sage: (nc, dc) == (n, d)
True
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, 't')
>>> t = P.gen()
>>> Pz = PolynomialRing(P.fraction_field(), 'z')
>>> z = Pz.gen()
>>> # p = (1 + t^2*z + z^4) / (1 - t*z)
```

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```
>>> p = (Integer(1) + t**Integer(2)*z + z**Integer(4))*(Integer(1) - t*z).
↪inverse_mod(z**Integer(9))
>>> m = z**Integer(9)
>>> n, d = p.rational_reconstruction(m); n, d
(-1/t*z^4 - t*z - 1/t, z - 1/t)
>>> ((p*d - n) % m).is_zero()
True
>>> w = PowerSeriesRing(P.fraction_field(), 'w').gen()
>>> n = -Integer(10)*t**Integer(2)*z**Integer(4) + (-t**Integer(2) + t -
↪Integer(1))*z**Integer(3) + (-t - Integer(8))*z**Integer(2) + z +
↪Integer(2)*t**Integer(2) - t
>>> d = z**Integer(4) + (Integer(2)*t + Integer(4))*z**Integer(3) + (-t +
↪Integer(5))*z**Integer(2) + (t**Integer(2) + Integer(2))*z + t**Integer(2)
↪+ Integer(2)*t + Integer(1)
>>> prec = Integer(9)
>>> x = n.subs(z=w)/d.subs(z=w) + O(w**prec)

>>> # needs sage.libs.flint (otherwise timeout)
>>> nc, dc = Pz(x.list()).rational_reconstruction(z**prec)
>>> (nc, dc) == (n, d)
True
```

Over $\mathbf{Q}[t][z]$:

```
sage: P = PolynomialRing(QQ, 't')
sage: t = P.gen()
sage: z = PolynomialRing(P, 'z').gen()
sage: # p = (1 + t^2*z + z^4) / (1 - t*z) mod z^9
sage: p = (1 + t^2*z + z^4) * sum((t*z)**i for i in range(9))
sage: m = z^9
sage: n, d = p.rational_reconstruction(m,); n, d
(-z^4 - t^2*z - 1, t*z - 1)
sage: ((p*d - n) % m).is_zero()
True
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, 't')
>>> t = P.gen()
>>> z = PolynomialRing(P, 'z').gen()
>>> # p = (1 + t^2*z + z^4) / (1 - t*z) mod z^9
>>> p = (Integer(1) + t**Integer(2)*z + z**Integer(4)) * sum((t*z)**i for i
↪in range(Integer(9)))
>>> m = z**Integer(9)
>>> n, d = p.rational_reconstruction(m,); n, d
(-z^4 - t^2*z - 1, t*z - 1)
>>> ((p*d - n) % m).is_zero()
True
```

Over $\mathbf{Q}_5$:

```
sage: # needs sage.rings.padics
sage: x = PolynomialRing(Qp(5), 'x').gen()
```

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```
sage: p = 4*x^5 + 3*x^4 + 2*x^3 + 2*x^2 + 4*x + 2
sage: m = x^6
sage: n, d = p.rational_reconstruction(m, 3, 2)
sage: ((p*d - n) % m).is_zero()
True
```

```
>>> from sage.all import *
>>> # needs sage.rings.padics
>>> x = PolynomialRing(Qp(Integer(5)), 'x').gen()
>>> p = Integer(4)*x**Integer(5) + Integer(3)*x**Integer(4) +␣
↪Integer(2)*x**Integer(3) + Integer(2)*x**Integer(2) + Integer(4)*x +␣
↪Integer(2)
>>> m = x**Integer(6)
>>> n, d = p.rational_reconstruction(m, Integer(3), Integer(2))
>>> ((p*d - n) % m).is_zero()
True
```

Can also be used to obtain known Padé approximations:

```
sage: z = PowerSeriesRing(QQ, 'z').gen()
sage: P = PolynomialRing(QQ, 'x')
sage: x = P.gen()
sage: p = P(z.exp().list())
sage: m = x^5
sage: n, d = p.rational_reconstruction(m, 4, 0); n, d
(1/24*x^4 + 1/6*x^3 + 1/2*x^2 + x + 1, 1)
sage: ((p*d - n) % m).is_zero()
True
sage: m = x^3
sage: n, d = p.rational_reconstruction(m, 1, 1); n, d
(-x - 2, x - 2)
sage: ((p*d - n) % m).is_zero()
True
sage: p = P(log(1-z).list())
sage: m = x^9
sage: n, d = p.rational_reconstruction(m, 4, 4); n, d
(25/6*x^4 - 130/3*x^3 + 105*x^2 - 70*x, x^4 - 20*x^3 + 90*x^2 - 140*x + 70)
sage: ((p*d - n) % m).is_zero()
True
sage: p = P(sqrt(1+z).list())
sage: m = x^6
sage: n, d = p.rational_reconstruction(m, 3, 2); n, d
(1/6*x^3 + 3*x^2 + 8*x + 16/3, x^2 + 16/3*x + 16/3)
sage: ((p*d - n) % m).is_zero()
True
sage: p = P((2*z).exp().list())
sage: m = x^7
sage: n, d = p.rational_reconstruction(m, 3, 3); n, d
(-x^3 - 6*x^2 - 15*x - 15, x^3 - 6*x^2 + 15*x - 15)
sage: ((p*d - n) % m).is_zero()
True
```

```
>>> from sage.all import *
>>> z = PowerSeriesRing(QQ, 'z').gen()
>>> P = PolynomialRing(QQ, 'x')
>>> x = P.gen()
>>> p = P(z.exp().list())
>>> m = x**Integer(5)
>>> n, d = p.rational_reconstruction(m, Integer(4), Integer(0)); n, d
(1/24*x^4 + 1/6*x^3 + 1/2*x^2 + x + 1, 1)
>>> ((p*d - n) % m).is_zero()
True
>>> m = x**Integer(3)
>>> n, d = p.rational_reconstruction(m, Integer(1), Integer(1)); n, d
(-x - 2, x - 2)
>>> ((p*d - n) % m).is_zero()
True
>>> p = P(log(Integer(1)-z).list())
>>> m = x**Integer(9)
>>> n, d = p.rational_reconstruction(m, Integer(4), Integer(4)); n, d
(25/6*x^4 - 130/3*x^3 + 105*x^2 - 70*x, x^4 - 20*x^3 + 90*x^2 - 140*x + 70)
>>> ((p*d - n) % m).is_zero()
True
>>> p = P(sqrt(Integer(1)+z).list())
>>> m = x**Integer(6)
>>> n, d = p.rational_reconstruction(m, Integer(3), Integer(2)); n, d
(1/6*x^3 + 3*x^2 + 8*x + 16/3, x^2 + 16/3*x + 16/3)
>>> ((p*d - n) % m).is_zero()
True
>>> p = P((Integer(2)*z).exp().list())
>>> m = x**Integer(7)
>>> n, d = p.rational_reconstruction(m, Integer(3), Integer(3)); n, d
(-x^3 - 6*x^2 - 15*x - 15, x^3 - 6*x^2 + 15*x - 15)
>>> ((p*d - n) % m).is_zero()
True
```

Over $\mathbf{R}[z]$:

```
sage: # needs sage.rings.real_mpfr
sage: z = PowerSeriesRing(RR, 'z').gen()
sage: P = PolynomialRing(RR, 'x')
sage: x = P.gen()
sage: p = P((2*z).exp().list())
sage: m = x^7
sage: n, d = p.rational_reconstruction(m, 3, 3); n, d  # absolute tolerance␣
↪1e-10
(-x^3 - 6.0*x^2 - 15.0*x - 15.0, x^3 - 6.0*x^2 + 15.0*x - 15.0)
```

```
>>> from sage.all import *
>>> # needs sage.rings.real_mpfr
>>> z = PowerSeriesRing(RR, 'z').gen()
>>> P = PolynomialRing(RR, 'x')
>>> x = P.gen()
>>> p = P((Integer(2)*z).exp().list())
>>> m = x**Integer(7)
```

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```
>>> n, d = p.rational_reconstruction(m, Integer(3), Integer(3)); n, d  #␣
↪absolute tolerance 1e-10
(-x^3 - 6.0*x^2 - 15.0*x - 15.0, x^3 - 6.0*x^2 + 15.0*x - 15.0)
```

See also:

- sage.matrix.berlekamp_massey,
- *sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint.rational_reconstruction()*

real_roots()

Return the real roots of this polynomial, without multiplicities.

Calls `self.roots(ring=RR)`, unless this is a polynomial with floating-point real coefficients, in which case it calls `self.roots()`.

EXAMPLES:

```
sage: x = polygen(ZZ)
sage: (x^2 - x - 1).real_roots()                                          #␣
↪needs sage.libs.pari sage.rings.real_mpfr
[-0.618033988749895, 1.61803398874989]
```

```
>>> from sage.all import *
>>> x = polygen(ZZ)
>>> (x**Integer(2) - x - Integer(1)).real_roots()                        ␣
↪                # needs sage.libs.pari sage.rings.real_mpfr
[-0.618033988749895, 1.61803398874989]
```

reciprocal_transform(*R=1*, *q=1*)

Transform a general polynomial into a self-reciprocal polynomial.

The input Q and output P satisfy the relation

$$P(x) = Q(x + q/x)x^{\deg(Q)}R(x).$$

In this relation, Q has all roots in the real interval $[-2\sqrt{q}, 2\sqrt{q}]$ if and only if P has all roots on the circle $|x| = \sqrt{q}$ and R divides $x^2 - q$.

See also:

The inverse operation is *trace_polynomial()*.

INPUT:

- `R` – polynomial
- `q` – scalar (default: 1)

EXAMPLES:

```
sage: pol.<x> = PolynomialRing(Rationals())
sage: u = x^2 + x - 1
sage: u.reciprocal_transform()
x^4 + x^3 + x^2 + x + 1
sage: u.reciprocal_transform(R=x-1)
x^5 - 1
sage: u.reciprocal_transform(q=3)
x^4 + x^3 + 5*x^2 + 3*x + 9
```

```
>>> from sage.all import *
>>> pol = PolynomialRing(Rationals(), names=('x',)); (x,) = pol._first_
↪ngens(1)
>>> u = x**Integer(2) + x - Integer(1)
>>> u.reciprocal_transform()
x^4 + x^3 + x^2 + x + 1
>>> u.reciprocal_transform(R=x-Integer(1))
x^5 - 1
>>> u.reciprocal_transform(q=Integer(3))
x^4 + x^3 + 5*x^2 + 3*x + 9
```

resultant(*other*)

Return the resultant of `self` and `other`.

INPUT:

- `other` – a polynomial

OUTPUT: an element of the base ring of the polynomial ring

ALGORITHM:

Uses PARI's function pari:polresultant. For base rings that are not supported by PARI, the resultant is computed as the determinant of the Sylvester matrix.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: f = x^3 + x + 1;  g = x^3 - x - 1
sage: r = f.resultant(g); r                                             #␣
↪needs sage.libs.pari
-8
sage: r.parent() is QQ                                                  #␣
↪needs sage.libs.pari
True
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(3) + x + Integer(1);  g = x**Integer(3) - x - Integer(1)
>>> r = f.resultant(g); r                                             #␣
↪needs sage.libs.pari
-8
>>> r.parent() is QQ                                                  #␣
↪needs sage.libs.pari
True
```

We can compute resultants over univariate and multivariate polynomial rings:

```
sage: R.<a> = QQ[]
sage: S.<x> = R[]
sage: f = x^2 + a; g = x^3 + a
sage: r = f.resultant(g); r                                          #␣
↪needs sage.libs.pari
a^3 + a^2
sage: r.parent() is R                                                #␣
↪needs sage.libs.pari
True
```

```
>>> from sage.all import *
>>> R = QQ['a']; (a,) = R._first_ngens(1)
>>> S = R['x']; (x,) = S._first_ngens(1)
>>> f = x**Integer(2) + a; g = x**Integer(3) + a
>>> r = f.resultant(g); r                                          #␣
↪needs sage.libs.pari
a^3 + a^2
>>> r.parent() is R                                                #␣
↪needs sage.libs.pari
True
```

```
sage: R.<a, b> = QQ[]
sage: S.<x> = R[]
sage: f = x^2 + a; g = x^3 + b
sage: r = f.resultant(g); r                                          #␣
↪needs sage.libs.pari
a^3 + b^2
sage: r.parent() is R                                                #␣
↪needs sage.libs.pari
True
```

```
>>> from sage.all import *
>>> R = QQ['a, b']; (a, b,) = R._first_ngens(2)
>>> S = R['x']; (x,) = S._first_ngens(1)
>>> f = x**Integer(2) + a; g = x**Integer(3) + b
>>> r = f.resultant(g); r                                          #␣
↪needs sage.libs.pari
a^3 + b^2
>>> r.parent() is R                                                #␣
↪needs sage.libs.pari
True
```

reverse(*degree=None*)

Return polynomial but with the coefficients reversed.

If an optional `degree` argument is given, the coefficient list will be truncated or zero padded as necessary before reversing it. Assuming that the constant coefficient of `self` is nonzero, the reverse polynomial will have the specified degree.

EXAMPLES:

```
sage: R.<x> = ZZ[]; S.<y> = R[]
sage: f = y^3 + x*y - 3*x; f
```

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```
y^3 + x*y - 3*x
sage: f.reverse()
-3*x*y^3 + x*y^2 + 1
sage: f.reverse(degree=2)
-3*x*y^2 + x*y
sage: f.reverse(degree=5)
-3*x*y^5 + x*y^4 + y^2
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1); S = R['y']; (y,) = S._first_
↪ngens(1)
>>> f = y**Integer(3) + x*y - Integer(3)*x; f
y^3 + x*y - 3*x
>>> f.reverse()
-3*x*y^3 + x*y^2 + 1
>>> f.reverse(degree=Integer(2))
-3*x*y^2 + x*y
>>> f.reverse(degree=Integer(5))
-3*x*y^5 + x*y^4 + y^2
```

revert_series(*n*)

Return a polynomial `f` such that `f(self(x)) = self(f(x)) = x mod x^n`.

Currently, this is only implemented over some coefficient rings.

EXAMPLES:

```
sage: Pol.<x> = QQ[]
sage: (x + x^3/6 + x^5/120).revert_series(6)
3/40*x^5 - 1/6*x^3 + x
sage: Pol.<x> = CBF[]                                                     #␣
↪needs sage.libs.flint
sage: (x + x^3/6 + x^5/120).revert_series(6)                              #␣
↪needs sage.libs.flint
([0.075000000000000 +/- ...e-17])*x^5 + ([-0.166666666666667 +/- ...e-16])*x^
↪3 + x

sage: # needs sage.symbolic
sage: Pol.<x> = SR[]
sage: x.revert_series(6)
Traceback (most recent call last):
...
NotImplementedError: only implemented for certain base rings
```

```
>>> from sage.all import *
>>> Pol = QQ['x']; (x,) = Pol._first_ngens(1)
>>> (x + x**Integer(3)/Integer(6) + x**Integer(5)/Integer(120)).revert_
↪series(Integer(6))
3/40*x^5 - 1/6*x^3 + x
>>> Pol = CBF['x']; (x,) = Pol._first_ngens(1)# needs sage.libs.flint
>>> (x + x**Integer(3)/Integer(6) + x**Integer(5)/Integer(120)).revert_
↪series(Integer(6))                                 # needs sage.libs.flint
```

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```
([0.075000000000000 +/- ...e-17])*x^5 + ([-0.166666666666667 +/- ...e-16])*x^
↪3 + x

>>> # needs sage.symbolic
>>> Pol = SR['x']; (x,) = Pol._first_ngens(1)
>>> x.revert_series(Integer(6))
Traceback (most recent call last):
...
NotImplementedError: only implemented for certain base rings
```

root_field(*names*, *check_irreducible=True*)

Return the field generated by the roots of the irreducible polynomial `self`. The output is either a number field, relative number field, a quotient of a polynomial ring over a field, or the fraction field of the base ring.

EXAMPLES:

```
sage: R.<x> = QQ['x']
sage: f = x^3 + x + 17
sage: f.root_field('a')                                                   #␣
↪needs sage.rings.number_field
Number Field in a with defining polynomial x^3 + x + 17
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(3) + x + Integer(17)
>>> f.root_field('a')                                                   #␣
↪needs sage.rings.number_field
Number Field in a with defining polynomial x^3 + x + 17
```

```
sage: R.<x> = QQ['x']
sage: f = x - 3
sage: f.root_field('b')                                                   #␣
↪needs sage.rings.number_field
Rational Field
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = x - Integer(3)
>>> f.root_field('b')                                                   #␣
↪needs sage.rings.number_field
Rational Field
```

```
sage: R.<x> = ZZ['x']
sage: f = x^3 + x + 17
sage: f.root_field('b')                                                   #␣
↪needs sage.rings.number_field
Number Field in b with defining polynomial x^3 + x + 17
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(3) + x + Integer(17)
```

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```
>>> f.root_field('b')                                                    #␣
↪needs sage.rings.number_field
Number Field in b with defining polynomial x^3 + x + 17
```

```
sage: # needs sage.rings.number_field
sage: y = QQ['x'].0
sage: L.<a> = NumberField(y^3 - 2)
sage: R.<x> = L['x']
sage: f = x^3 + x + 17
sage: f.root_field('c')
Number Field in c with defining polynomial x^3 + x + 17 over its base field
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> y = QQ['x'].gen(0)
>>> L = NumberField(y**Integer(3) - Integer(2), names=('a',)); (a,) = L._
↪first_ngens(1)
>>> R = L['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(3) + x + Integer(17)
>>> f.root_field('c')
Number Field in c with defining polynomial x^3 + x + 17 over its base field
```

```
sage: # needs sage.rings.finite_rings
sage: R.<x> = PolynomialRing(GF(9, 'a'))
sage: f = x^3 + x^2 + 8
sage: K.<alpha> = f.root_field(); K
Univariate Quotient Polynomial Ring in alpha
 over Finite Field in a of size 3^2 with modulus x^3 + x^2 + 2
sage: alpha^2 + 1
alpha^2 + 1
sage: alpha^3 + alpha^2
1
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> R = PolynomialRing(GF(Integer(9), 'a'), names=('x',)); (x,) = R._first_
↪ngens(1)
>>> f = x**Integer(3) + x**Integer(2) + Integer(8)
>>> K = f.root_field(names=('alpha',)); (alpha,) = K._first_ngens(1); K
Univariate Quotient Polynomial Ring in alpha
 over Finite Field in a of size 3^2 with modulus x^3 + x^2 + 2
>>> alpha**Integer(2) + Integer(1)
alpha^2 + 1
>>> alpha**Integer(3) + alpha**Integer(2)
1
```

```
sage: R.<x> = QQ[]
sage: f = x^2
sage: K.<alpha> = f.root_field()                                         #␣
↪needs sage.libs.pari
Traceback (most recent call last):
```

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```
...
ValueError: polynomial must be irreducible
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(2)
>>> K = f.root_field(names=('alpha',)); (alpha,) = K._first_ngens(1)# needs␣
↪sage.libs.pari
Traceback (most recent call last):
...
ValueError: polynomial must be irreducible
```

roots(*ring=None*, *multiplicities=True*, *algorithm=None*, ***kwds*)

Return the roots of this polynomial (by default, in the base ring of this polynomial).

INPUT:

- `ring` – the ring to find roots in
- `multiplicities` – boolean (default: `True`); if `True` return list of pairs (r, n), where r is the root and n is the multiplicity. If `False`, just return the unique roots, with no information about multiplicities.
- `algorithm` – the root-finding algorithm to use. We attempt to select a reasonable algorithm by default, but this lets the caller override our choice.

By default, this finds all the roots that lie in the base ring of the polynomial. However, the ring parameter can be used to specify a ring to look for roots in.

If the polynomial and the output ring are both exact (integers, rationals, finite fields, etc.), then the output should always be correct (or raise an exception, if that case is not yet handled).

If the output ring is approximate (floating-point real or complex numbers), then the answer will be estimated numerically, using floating-point arithmetic of at least the precision of the output ring. If the polynomial is ill-conditioned, meaning that a small change in the coefficients of the polynomial will lead to a relatively large change in the location of the roots, this may give poor results. Distinct roots may be returned as multiple roots, multiple roots may be returned as distinct roots, real roots may be lost entirely (because the numerical estimate thinks they are complex roots). Note that polynomials with multiple roots are always ill-conditioned; there's a footnote at the end of the docstring about this.

If the output ring is a `RealIntervalField` or `ComplexIntervalField` of a given precision, then the answer will always be correct (or an exception will be raised, if a case is not implemented). Each root will be contained in one of the returned intervals, and the intervals will be disjoint. (The returned intervals may be of higher precision than the specified output ring.)

At the end of this docstring (after the examples) is a description of all the cases implemented in this function, and the algorithms used. That section also describes the possibilities for the `algorithm` keyword, for the cases where multiple algorithms exist.

EXAMPLES:

```
sage: # needs sage.libs.pari
sage: x = QQ['x'].0
sage: f = x^3 - 1
sage: f.roots()
[(1, 1)]
sage: f.roots(ring=CC)   # ... - low order bits slightly different on ppc   #␣
↪needs sage.rings.real_mpfr
```

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```
[(1.00000000000000, 1),
 (-0.500000000000000 - 0.86602540378443...*I, 1),
 (-0.500000000000000 + 0.86602540378443...*I, 1)]
sage: f = (x^3 - 1)^2
sage: f.roots()
[(1, 2)]
sage: f = -19*x + 884736
sage: f.roots()
[(884736/19, 1)]
sage: (f^20).roots()
[(884736/19, 20)]
```

```
>>> from sage.all import *
>>> # needs sage.libs.pari
>>> x = QQ['x'].gen(0)
>>> f = x**Integer(3) - Integer(1)
>>> f.roots()
[(1, 1)]
>>> f.roots(ring=CC)   # ... - low order bits slightly different on ppc   #␣
↪needs sage.rings.real_mpfr
[(1.00000000000000, 1),
 (-0.500000000000000 - 0.86602540378443...*I, 1),
 (-0.500000000000000 + 0.86602540378443...*I, 1)]
>>> f = (x**Integer(3) - Integer(1))**Integer(2)
>>> f.roots()
[(1, 2)]
>>> f = -Integer(19)*x + Integer(884736)
>>> f.roots()
[(884736/19, 1)]
>>> (f**Integer(20)).roots()
[(884736/19, 20)]
```

```
sage: # needs sage.rings.number_field
sage: K.<z> = CyclotomicField(3)
sage: f = K.defining_polynomial()
sage: f.roots(ring=GF(7))
[(4, 1), (2, 1)]
sage: g = f.change_ring(GF(7))
sage: g.roots()
[(4, 1), (2, 1)]
sage: g.roots(multiplicities=False)
[4, 2]
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> K = CyclotomicField(Integer(3), names=('z',)); (z,) = K._first_ngens(1)
>>> f = K.defining_polynomial()
>>> f.roots(ring=GF(Integer(7)))
[(4, 1), (2, 1)]
>>> g = f.change_ring(GF(Integer(7)))
>>> g.roots()
```

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```
[(4, 1), (2, 1)]
>>> g.roots(multiplicities=False)
[4, 2]
```

A new ring. In the example below, we add the special method `_roots_univariate_polynomial()` to the base ring, and observe that this method is called instead to find roots of polynomials over this ring. This facility can be used to easily extend root finding to work over new rings you introduce:

```
sage: R.<x> = QQ[]
sage: (x^2 + 1).roots()                                                    #␣
↪needs sage.libs.pari
[]
sage: def my_roots(f, *args, **kwds):
....:     return f.change_ring(CDF).roots()
sage: QQ._roots_univariate_polynomial = my_roots
sage: (x^2 + 1).roots()   # abs tol 1e-14                                  #␣
↪needs numpy
[(2.7755575615628914e-17 - 1.0*I, 1), (0.9999999999999997*I, 1)]
sage: del QQ._roots_univariate_polynomial
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> (x**Integer(2) + Integer(1)).roots()                                    ␣
↪              # needs sage.libs.pari
[]
>>> def my_roots(f, *args, **kwds):
...     return f.change_ring(CDF).roots()
>>> QQ._roots_univariate_polynomial = my_roots
>>> (x**Integer(2) + Integer(1)).roots()   # abs tol 1e-14                  ␣
↪              # needs numpy
[(2.7755575615628914e-17 - 1.0*I, 1), (0.9999999999999997*I, 1)]
>>> del QQ._roots_univariate_polynomial
```

An example over RR, which illustrates that only the roots in RR are returned:

```
sage: # needs numpy sage.rings.real_mpfr
sage: x = RR['x'].0
sage: f = x^3 - 2
sage: f.roots()
[(1.25992104989487, 1)]
sage: f.factor()
(x - 1.25992104989487) * (x^2 + 1.25992104989487*x + 1.58740105196820)
sage: x = RealField(100)['x'].0
sage: f = x^3 - 2
sage: f.roots()
[(1.2599210498948731647672106073, 1)]
```

```
>>> from sage.all import *
>>> # needs numpy sage.rings.real_mpfr
>>> x = RR['x'].gen(0)
>>> f = x**Integer(3) - Integer(2)
>>> f.roots()
```

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```
[(1.25992104989487, 1)]
>>> f.factor()
(x - 1.25992104989487) * (x^2 + 1.25992104989487*x + 1.58740105196820)
>>> x = RealField(Integer(100))['x'].gen(0)
>>> f = x**Integer(3) - Integer(2)
>>> f.roots()
[(1.2599210498948731647672106073, 1)]
```

```
sage: # needs sage.rings.real_mpfr
sage: x = CC['x'].0
sage: f = x^3 - 2
sage: f.roots()                                                              #␣
↪needs numpy
[(1.25992104989487, 1),
 (-0.62996052494743... - 1.09112363597172*I, 1),
 (-0.62996052494743... + 1.09112363597172*I, 1)]
sage: f.roots(algorithm='pari')                                              #␣
↪needs sage.libs.pari
[(1.25992104989487, 1),
 (-0.629960524947437 - 1.09112363597172*I, 1),
 (-0.629960524947437 + 1.09112363597172*I, 1)]
```

```
>>> from sage.all import *
>>> # needs sage.rings.real_mpfr
>>> x = CC['x'].gen(0)
>>> f = x**Integer(3) - Integer(2)
>>> f.roots()                                                              #␣
↪needs numpy
[(1.25992104989487, 1),
 (-0.62996052494743... - 1.09112363597172*I, 1),
 (-0.62996052494743... + 1.09112363597172*I, 1)]
>>> f.roots(algorithm='pari')                                              #␣
↪needs sage.libs.pari
[(1.25992104989487, 1),
 (-0.629960524947437 - 1.09112363597172*I, 1),
 (-0.629960524947437 + 1.09112363597172*I, 1)]
```

Another example showing that only roots in the base ring are returned:

```
sage: x = polygen(ZZ)
sage: f = (2*x - 3) * (x - 1) * (x + 1)
sage: f.roots()                                                              #␣
↪needs sage.libs.pari
[(1, 1), (-1, 1)]
sage: f.roots(ring=QQ)                                                       #␣
↪needs sage.libs.pari
[(3/2, 1), (1, 1), (-1, 1)]
```

```
>>> from sage.all import *
>>> x = polygen(ZZ)
>>> f = (Integer(2)*x - Integer(3)) * (x - Integer(1)) * (x + Integer(1))
```

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```
>>> f.roots()                                                          #␣
↪needs sage.libs.pari
[(1, 1), (-1, 1)]
>>> f.roots(ring=QQ)                                                   #␣
↪needs sage.libs.pari
[(3/2, 1), (1, 1), (-1, 1)]
```

An example where we compute the roots lying in a subring of the base ring:

```
sage: Pols.<n> = QQ[]
sage: pol = (n - 1/2)^2 * (n - 1)^2 * (n - 2)
sage: pol.roots(ZZ)                                                    #␣
↪needs sage.libs.pari
[(2, 1), (1, 2)]
```

```
>>> from sage.all import *
>>> Pols = QQ['n']; (n,) = Pols._first_ngens(1)
>>> pol = (n - Integer(1)/Integer(2))**Integer(2) * (n -␣
↪Integer(1))**Integer(2) * (n - Integer(2))
>>> pol.roots(ZZ)                                                      #␣
↪needs sage.libs.pari
[(2, 1), (1, 2)]
```

An example involving large numbers:

```
sage: # needs numpy sage.rings.real_mpfr
sage: x = RR['x'].0
sage: f = x^2 - 1e100
sage: f.roots()
[(-1.00000000000000e50, 1), (1.00000000000000e50, 1)]
sage: f = x^10 - 2 * (5*x - 1)^2
sage: f.roots(multiplicities=False)
[-1.6772670339941..., 0.19995479628..., 0.20004530611..., 1.5763035161844...]
```

```
>>> from sage.all import *
>>> # needs numpy sage.rings.real_mpfr
>>> x = RR['x'].gen(0)
>>> f = x**Integer(2) - RealNumber('1e100')
>>> f.roots()
[(-1.00000000000000e50, 1), (1.00000000000000e50, 1)]
>>> f = x**Integer(10) - Integer(2) * (Integer(5)*x - Integer(1))**Integer(2)
>>> f.roots(multiplicities=False)
[-1.6772670339941..., 0.19995479628..., 0.20004530611..., 1.5763035161844...]
```

```
sage: # needs numpy sage.rings.real_mpfr
sage: x = CC['x'].0
sage: i = CC.0
sage: f = (x - 1) * (x - i)
sage: f.roots(multiplicities=False)
[1.00000000000000, 1.00000000000000*I]
sage: g = (x - 1.33 + 1.33*i) * (x - 2.66 - 2.66*i)
```

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```
sage: g.roots(multiplicities=False)
[1.33000000000000 - 1.33000000000000*I, 2.66000000000000 + 2.66000000000000*I]
```

```
>>> from sage.all import *
>>> # needs numpy sage.rings.real_mpfr
>>> x = CC['x'].gen(0)
>>> i = CC.gen(0)
>>> f = (x - Integer(1)) * (x - i)
>>> f.roots(multiplicities=False)
[1.00000000000000, 1.00000000000000*I]
>>> g = (x - RealNumber('1.33') + RealNumber('1.33')*i) * (x - RealNumber('2.
↪66') - RealNumber('2.66')*i)
>>> g.roots(multiplicities=False)
[1.33000000000000 - 1.33000000000000*I, 2.66000000000000 + 2.66000000000000*I]
```

Describing roots using radical expressions:

```
sage: x = QQ['x'].0
sage: f = x^2 + 2
sage: f.roots(SR)                                                          #␣
↪needs sage.symbolic
[(-I*sqrt(2), 1), (I*sqrt(2), 1)]
sage: f.roots(SR, multiplicities=False)                                    #␣
↪needs sage.symbolic
[-I*sqrt(2), I*sqrt(2)]
```

```
>>> from sage.all import *
>>> x = QQ['x'].gen(0)
>>> f = x**Integer(2) + Integer(2)
>>> f.roots(SR)                                                          #␣
↪needs sage.symbolic
[(-I*sqrt(2), 1), (I*sqrt(2), 1)]
>>> f.roots(SR, multiplicities=False)                                    #␣
↪needs sage.symbolic
[-I*sqrt(2), I*sqrt(2)]
```

The roots of some polynomials cannot be described using radical expressions:

```
sage: (x^5 - x + 1).roots(SR)                                              #␣
↪needs sage.symbolic
[]
```

```
>>> from sage.all import *
>>> (x**Integer(5) - x + Integer(1)).roots(SR)                              ␣
↪                # needs sage.symbolic
[]
```

For some other polynomials, no roots can be found at the moment due to the way roots are computed. Issue #17516 addresses these defects. Until that gets implemented, one such example is the following:

```
sage: f = x^6 - 300*x^5 + 30361*x^4 - 1061610*x^3 + 1141893*x^2 - 915320*x +␣
↪101724
```

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```
sage: f.roots()                                                                    #␣
↪needs sage.libs.pari
[]
```

```
>>> from sage.all import *
>>> f = x**Integer(6) - Integer(300)*x**Integer(5) +␣
↪Integer(30361)*x**Integer(4) - Integer(1061610)*x**Integer(3) +␣
↪Integer(1141893)*x**Integer(2) - Integer(915320)*x + Integer(101724)
>>> f.roots()                                                                      #␣
↪needs sage.libs.pari
[]
```

A purely symbolic roots example:

```
sage: # needs sage.symbolic
sage: X = var('X')
sage: f = expand((X - 1) * (X - I)^3 * (X^2 - sqrt(2))); f
X^6 - (3*I + 1)*X^5 - sqrt(2)*X^4 + (3*I - 3)*X^4 + (3*I + 1)*sqrt(2)*X^3
+ (I + 3)*X^3 - (3*I - 3)*sqrt(2)*X^2 - I*X^2 - (I + 3)*sqrt(2)*X + I*sqrt(2)
sage: f.roots()
[(I, 3), (-2^(1/4), 1), (2^(1/4), 1), (1, 1)]
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> X = var('X')
>>> f = expand((X - Integer(1)) * (X - I)**Integer(3) * (X**Integer(2) -␣
↪sqrt(Integer(2)))); f
X^6 - (3*I + 1)*X^5 - sqrt(2)*X^4 + (3*I - 3)*X^4 + (3*I + 1)*sqrt(2)*X^3
+ (I + 3)*X^3 - (3*I - 3)*sqrt(2)*X^2 - I*X^2 - (I + 3)*sqrt(2)*X + I*sqrt(2)
>>> f.roots()
[(I, 3), (-2^(1/4), 1), (2^(1/4), 1), (1, 1)]
```

The same operation, performed over a polynomial ring with symbolic coefficients:

```
sage: # needs sage.symbolic
sage: X = SR['X'].0
sage: f = (X - 1) * (X - I)^3 * (X^2 - sqrt(2)); f
X^6 + (-3*I - 1)*X^5 + (-sqrt(2) + 3*I - 3)*X^4 + ((3*I + 1)*sqrt(2) + I +␣
↪3)*X^3
+ (-(3*I - 3)*sqrt(2) - I)*X^2 + (-(I + 3)*sqrt(2))*X + I*sqrt(2)
sage: f.roots()
[(I, 3), (-2^(1/4), 1), (2^(1/4), 1), (1, 1)]
sage: f.roots(multiplicities=False)
[I, -2^(1/4), 2^(1/4), 1]
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> X = SR['X'].gen(0)
>>> f = (X - Integer(1)) * (X - I)**Integer(3) * (X**Integer(2) -␣
↪sqrt(Integer(2))); f
X^6 + (-3*I - 1)*X^5 + (-sqrt(2) + 3*I - 3)*X^4 + ((3*I + 1)*sqrt(2) + I +␣
↪3)*X^3
```

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```
+ (-(3*I - 3)*sqrt(2) - I)*X^2 + (-(I + 3)*sqrt(2))*X + I*sqrt(2)
>>> f.roots()
[(I, 3), (-2^(1/4), 1), (2^(1/4), 1), (1, 1)]
>>> f.roots(multiplicities=False)
[I, -2^(1/4), 2^(1/4), 1]
```

A couple of examples where the base ring does not have a factorization algorithm (yet). Note that this is currently done via a rather naive enumeration, so could be very slow:

```
sage: R = Integers(6)
sage: S.<x> = R['x']
sage: p = x^2 - 1
sage: p.roots()
Traceback (most recent call last):
...
NotImplementedError: root finding with multiplicities for this polynomial
not implemented (try the multiplicities=False option)
sage: p.roots(multiplicities=False)                                          #␣
↪needs sage.libs.pari
[5, 1]
sage: R = Integers(9)
sage: A = PolynomialRing(R, 'y')
sage: y = A.gen()
sage: f = 10*y^2 - y^3 - 9
sage: f.roots(multiplicities=False)                                          #␣
↪needs sage.libs.pari
[1, 0, 3, 6]
```

```
>>> from sage.all import *
>>> R = Integers(Integer(6))
>>> S = R['x']; (x,) = S._first_ngens(1)
>>> p = x**Integer(2) - Integer(1)
>>> p.roots()
Traceback (most recent call last):
...
NotImplementedError: root finding with multiplicities for this polynomial
not implemented (try the multiplicities=False option)
>>> p.roots(multiplicities=False)                                          #␣
↪needs sage.libs.pari
[5, 1]
>>> R = Integers(Integer(9))
>>> A = PolynomialRing(R, 'y')
>>> y = A.gen()
>>> f = Integer(10)*y**Integer(2) - y**Integer(3) - Integer(9)
>>> f.roots(multiplicities=False)                                          #␣
↪needs sage.libs.pari
[1, 0, 3, 6]
```

An example over the complex double field (where root finding is fast, thanks to NumPy):

```
sage: # needs numpy sage.rings.complex_double
sage: R.<x> = CDF[]
```

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```
sage: f = R.cyclotomic_polynomial(5); f
x^4 + x^3 + x^2 + x + 1.0
sage: f.roots(multiplicities=False)  # abs tol 1e-9
[-0.8090169943749469 - 0.5877852522924724*I, -0.8090169943749473 + 0.
↪5877852522924724*I,
 0.30901699437494773 - 0.951056516295154*I, 0.30901699437494756 + 0.
↪9510565162951525*I]
sage: [z^5 for z in f.roots(multiplicities=False)]  # abs tol 2e-14
[0.9999999999999957 - 1.2864981197413038e-15*I, 0.9999999999999976 + 3.
↪062854959141552e-15*I,
 1.0000000000000024 + 1.1331077795295987e-15*I, 0.9999999999999953 - 2.
↪0212861992297117e-15*I]
sage: f = CDF['x']([1,2,3,4]); f
4.0*x^3 + 3.0*x^2 + 2.0*x + 1.0
sage: r = f.roots(multiplicities=False)
sage: [f(a).abs() for a in r]  # abs tol 1e-14
[2.574630599127759e-15, 1.457101633618084e-15, 1.1443916996305594e-15]
```

```
>>> from sage.all import *
>>> # needs numpy sage.rings.complex_double
>>> R = CDF['x']; (x,) = R._first_ngens(1)
>>> f = R.cyclotomic_polynomial(Integer(5)); f
x^4 + x^3 + x^2 + x + 1.0
>>> f.roots(multiplicities=False)  # abs tol 1e-9
[-0.8090169943749469 - 0.5877852522924724*I, -0.8090169943749473 + 0.
↪5877852522924724*I,
 0.30901699437494773 - 0.951056516295154*I, 0.30901699437494756 + 0.
↪9510565162951525*I]
>>> [z**Integer(5) for z in f.roots(multiplicities=False)]  # abs tol 2e-14
[0.9999999999999957 - 1.2864981197413038e-15*I, 0.9999999999999976 + 3.
↪062854959141552e-15*I,
 1.0000000000000024 + 1.1331077795295987e-15*I, 0.9999999999999953 - 2.
↪0212861992297117e-15*I]
>>> f = CDF['x']([Integer(1),Integer(2),Integer(3),Integer(4)]); f
4.0*x^3 + 3.0*x^2 + 2.0*x + 1.0
>>> r = f.roots(multiplicities=False)
>>> [f(a).abs() for a in r]  # abs tol 1e-14
[2.574630599127759e-15, 1.457101633618084e-15, 1.1443916996305594e-15]
```

Another example over RDF:

```
sage: x = RDF['x'].0
sage: ((x^3 - 1)).roots()  # abs tol 4e-16                                  #␣
↪needs numpy
[(1.0000000000000002, 1)]
sage: ((x^3 - 1)).roots(multiplicities=False)  # abs tol 4e-16              #␣
↪needs numpy
[1.0000000000000002]
```

```
>>> from sage.all import *
>>> x = RDF['x'].gen(0)
>>> ((x**Integer(3) - Integer(1))).roots()  # abs tol 4e-16                 ␣
```

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```
↪                # needs numpy
[(1.0000000000000002, 1)]
>>> ((x**Integer(3) - Integer(1))).roots(multiplicities=False)   # abs tol 4e-
↪16              # needs numpy
[1.0000000000000002]
```

More examples involving the complex double field:

```
sage: # needs numpy sage.rings.complex_double sage.rings.real_mpfr
sage: x = CDF['x'].0
sage: i = CDF.0
sage: f = x^3 + 2*i; f
x^3 + 2.0*I
sage: f.roots()
[(-1.09112363597172... - 0.62996052494743...*I, 1),
 (...1.25992104989487...*I, 1),
 (1.09112363597172... - 0.62996052494743...*I, 1)]
sage: f.roots(multiplicities=False)
[-1.09112363597172... - 0.62996052494743...*I, ...1.25992104989487...*I,
  1.09112363597172... - 0.62996052494743...*I]
sage: [abs(f(z)) for z in f.roots(multiplicities=False)]   # abs tol 1e-14
[8.95090418262362e-16, 8.728374398092689e-16, 1.0235750533041806e-15]
sage: f = i*x^3 + 2; f
I*x^3 + 2.0
sage: f.roots()
[(-1.09112363597172... + 0.62996052494743...*I, 1),
 (...1.25992104989487...*I, 1),
 (1.09112363597172... + 0.62996052494743...*I, 1)]
sage: abs(f(f.roots()[0][0]))   # abs tol 1e-13
1.1102230246251565e-16
```

```
>>> from sage.all import *
>>> # needs numpy sage.rings.complex_double sage.rings.real_mpfr
>>> x = CDF['x'].gen(0)
>>> i = CDF.gen(0)
>>> f = x**Integer(3) + Integer(2)*i; f
x^3 + 2.0*I
>>> f.roots()
[(-1.09112363597172... - 0.62996052494743...*I, 1),
 (...1.25992104989487...*I, 1),
 (1.09112363597172... - 0.62996052494743...*I, 1)]
>>> f.roots(multiplicities=False)
[-1.09112363597172... - 0.62996052494743...*I, ...1.25992104989487...*I,
  1.09112363597172... - 0.62996052494743...*I]
>>> [abs(f(z)) for z in f.roots(multiplicities=False)]   # abs tol 1e-14
[8.95090418262362e-16, 8.728374398092689e-16, 1.0235750533041806e-15]
>>> f = i*x**Integer(3) + Integer(2); f
I*x^3 + 2.0
>>> f.roots()
[(-1.09112363597172... + 0.62996052494743...*I, 1),
 (...1.25992104989487...*I, 1),
 (1.09112363597172... + 0.62996052494743...*I, 1)]
```

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```
>>> abs(f(f.roots()[Integer(0)][Integer(0)]))  # abs tol 1e-13
1.1102230246251565e-16
```

Examples using real root isolation:

```
sage: x = polygen(ZZ)
sage: f = x^2 - x - 1
sage: f.roots()                                                             #␣
↪needs sage.libs.pari
[]
sage: f.roots(ring=AA)                                                      #␣
↪needs sage.rings.number_field
[(-0.618033988749895?, 1), (1.618033988749895?, 1)]

sage: # needs sage.rings.real_interval_field
sage: f.roots(ring=RIF)
[(-0.6180339887498948482045868343657?, 1), (1.6180339887498948482045868343657?
↪, 1)]
sage: f.roots(ring=RIF, multiplicities=False)
[-0.6180339887498948482045868343657?, 1.6180339887498948482045868343657?]
sage: f.roots(ring=RealIntervalField(150))
[(-0.6180339887498948482045868343656381177203091798057628621354486227?, 1),
 (1.618033988749894848204586834365638117720309179805762862135448623?, 1)]
sage: f = f^2 * (x - 1)
sage: f.roots(ring=RIF)
[(-0.6180339887498948482045868343657?, 2),
 (1.0000000000000000000000000000000?, 1),
 (1.6180339887498948482045868343657?, 2)]
sage: f.roots(ring=RIF, multiplicities=False)
[-0.6180339887498948482045868343657?,
 1.0000000000000000000000000000000?,
 1.6180339887498948482045868343657?]
```

```
>>> from sage.all import *
>>> x = polygen(ZZ)
>>> f = x**Integer(2) - x - Integer(1)
>>> f.roots()                                                             #␣
↪needs sage.libs.pari
[]
>>> f.roots(ring=AA)                                                      #␣
↪needs sage.rings.number_field
[(-0.618033988749895?, 1), (1.618033988749895?, 1)]

>>> # needs sage.rings.real_interval_field
>>> f.roots(ring=RIF)
[(-0.6180339887498948482045868343657?, 1), (1.6180339887498948482045868343657?
↪, 1)]
>>> f.roots(ring=RIF, multiplicities=False)
[-0.6180339887498948482045868343657?, 1.6180339887498948482045868343657?]
>>> f.roots(ring=RealIntervalField(Integer(150)))
[(-0.6180339887498948482045868343656381177203091798057628621354486227?, 1),
 (1.618033988749894848204586834365638117720309179805762862135448623?, 1)]
```

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```
>>> f = f**Integer(2) * (x - Integer(1))
>>> f.roots(ring=RIF)
[(-0.6180339887498948482045868343657?, 2),
 (1.0000000000000000000000000000000?, 1),
 (1.6180339887498948482045868343657?, 2)]
>>> f.roots(ring=RIF, multiplicities=False)
[-0.6180339887498948482045868343657?,
 1.0000000000000000000000000000000?,
 1.6180339887498948482045868343657?]
```

Examples using complex root isolation:

```
sage: x = polygen(ZZ)
sage: p = x^5 - x - 1
sage: p.roots()                                                              #␣
↪needs sage.libs.pari
[]
sage: p.roots(ring=CIF)                                                      #␣
↪needs sage.rings.complex_interval_field
[(1.167303978261419?, 1),
 (-0.764884433600585? - 0.352471546031727?*I, 1),
 (-0.764884433600585? + 0.352471546031727?*I, 1),
 (0.181232444469876? - 1.083954101317711?*I, 1),
 (0.181232444469876? + 1.083954101317711?*I, 1)]
sage: p.roots(ring=ComplexIntervalField(200))                                #␣
↪needs sage.rings.complex_interval_field
[(1.167303978261418684256045899854842180720560371525489039140082?, 1),
 (-0.76488443360058472602982318770854173032899665194736756700778? - 0.
↪35247154603172624931794709140258105439420648082424733283770?*I, 1),
 (-0.76488443360058472602982318770854173032899665194736756700778? + 0.
↪35247154603172624931794709140258105439420648082424733283770?*I, 1),
 (0.18123244446987538390180023778112063996871646618462304743774? - 1.
↪08395410131771066843034449298076657427364024315511565430114?*I, 1),
 (0.18123244446987538390180023778112063996871646618462304743774? + 1.
↪08395410131771066843034449298076657427364024315511565430114?*I, 1)]
sage: rts = p.roots(ring=QQbar); rts                                         #␣
↪needs sage.rings.number_field
[(1.167303978261419?, 1),
 (-0.7648844336005847? - 0.3524715460317263?*I, 1),
 (-0.7648844336005847? + 0.3524715460317263?*I, 1),
 (0.1812324444698754? - 1.083954101317711?*I, 1),
 (0.1812324444698754? + 1.083954101317711?*I, 1)]
sage: p.roots(ring=AA)                                                       #␣
↪needs sage.rings.number_field
[(1.167303978261419?, 1)]
sage: p = (x - rts[4][0])^2 * (3*x^2 + x + 1)                                #␣
↪needs sage.rings.number_field
sage: p.roots(ring=QQbar)                                                    #␣
↪needs sage.rings.number_field
[(-0.1666666666666667? - 0.552770798392567?*I, 1),
 (-0.1666666666666667? + 0.552770798392567?*I, 1),
 (0.1812324444698754? + 1.083954101317711?*I, 2)]
```

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```
sage: p.roots(ring=CIF)                                                    #␣
↪needs sage.rings.complex_interval_field
[(-0.1666666666666667? - 0.552770798392567?*I, 1),
 (-0.1666666666666667? + 0.552770798392567?*I, 1),
 (0.1812324444698754? + 1.083954101317711?*I, 2)]
```

```
>>> from sage.all import *
>>> x = polygen(ZZ)
>>> p = x**Integer(5) - x - Integer(1)
>>> p.roots()                                                              #␣
↪needs sage.libs.pari
[]
>>> p.roots(ring=CIF)                                                      #␣
↪needs sage.rings.complex_interval_field
[(1.167303978261419?, 1),
 (-0.764884433600585? - 0.352471546031727?*I, 1),
 (-0.764884433600585? + 0.352471546031727?*I, 1),
 (0.181232444469876? - 1.083954101317711?*I, 1),
 (0.181232444469876? + 1.083954101317711?*I, 1)]
>>> p.roots(ring=ComplexIntervalField(Integer(200)))                         ␣
↪     # needs sage.rings.complex_interval_field
[(1.167303978261418684256045899854842180720560371525489039140082?, 1),
 (-0.76488443360058472602982318770854173032899665194736756700778? - 0.
↪35247154603172624931794709140258105439420648082424733283770?*I, 1),
 (-0.76488443360058472602982318770854173032899665194736756700778? + 0.
↪35247154603172624931794709140258105439420648082424733283770?*I, 1),
 (0.18123244446987538390180023778112063996871646618462304743774? - 1.
↪08395410131771066843034449298076657427364024315511565430114?*I, 1),
 (0.18123244446987538390180023778112063996871646618462304743774? + 1.
↪08395410131771066843034449298076657427364024315511565430114?*I, 1)]
>>> rts = p.roots(ring=QQbar); rts                                         #␣
↪needs sage.rings.number_field
[(1.167303978261419?, 1),
 (-0.7648844336005847? - 0.3524715460317263?*I, 1),
 (-0.7648844336005847? + 0.3524715460317263?*I, 1),
 (0.1812324444698754? - 1.083954101317711?*I, 1),
 (0.1812324444698754? + 1.083954101317711?*I, 1)]
>>> p.roots(ring=AA)                                                       #␣
↪needs sage.rings.number_field
[(1.167303978261419?, 1)]
>>> p = (x - rts[Integer(4)][Integer(0)])**Integer(2) *␣
↪(Integer(3)*x**Integer(2) + x + Integer(1))                                 #␣
↪needs sage.rings.number_field
>>> p.roots(ring=QQbar)                                                    #␣
↪needs sage.rings.number_field
[(-0.1666666666666667? - 0.552770798392567?*I, 1),
 (-0.1666666666666667? + 0.552770798392567?*I, 1),
 (0.1812324444698754? + 1.083954101317711?*I, 2)]
>>> p.roots(ring=CIF)                                                      #␣
↪needs sage.rings.complex_interval_field
[(-0.1666666666666667? - 0.552770798392567?*I, 1),
 (-0.1666666666666667? + 0.552770798392567?*I, 1),
```

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```
(0.1812324444698754? + 1.083954101317711?*I, 2)]
```

In some cases, it is possible to isolate the roots of polynomials over complex ball fields:

```
sage: # needs sage.libs.flint
sage: Pol.<x> = CBF[]
sage: set((x^2 + 2).roots(multiplicities=False))
{[+/- ...e-19] + [-1.414213562373095 +/- ...e-17]*I,
 [+/- ...e-19] + [1.414213562373095 +/- ...e-17]*I}
sage: (x^3 - 1/2).roots(RBF, multiplicities=False)
[[0.7937005259840997 +/- ...e-17]]
sage: ((x - 1)^2).roots(multiplicities=False, proof=False)
doctest:...
UserWarning: roots may have been lost...
[[1.000000000... +/- ...] + [+/- ...]*I,
 [1.000000000... +/- ...] + [+/- ...]*I]
sage: ((x - 1)^2).roots(multiplicities=False, proof=False, warn=False)
[[1.000000000... +/- ...] + [+/- ...]*I,
 [1.000000000... +/- ...] + [+/- ...]*I]
```

```
>>> from sage.all import *
>>> # needs sage.libs.flint
>>> Pol = CBF['x']; (x,) = Pol._first_ngens(1)
>>> set((x**Integer(2) + Integer(2)).roots(multiplicities=False))
{[+/- ...e-19] + [-1.414213562373095 +/- ...e-17]*I,
 [+/- ...e-19] + [1.414213562373095 +/- ...e-17]*I}
>>> (x**Integer(3) - Integer(1)/Integer(2)).roots(RBF, multiplicities=False)
[[0.7937005259840997 +/- ...e-17]]
>>> ((x - Integer(1))**Integer(2)).roots(multiplicities=False, proof=False)
doctest:...
UserWarning: roots may have been lost...
[[1.000000000... +/- ...] + [+/- ...]*I,
 [1.000000000... +/- ...] + [+/- ...]*I]
>>> ((x - Integer(1))**Integer(2)).roots(multiplicities=False, proof=False,
↪warn=False)
[[1.000000000... +/- ...] + [+/- ...]*I,
 [1.000000000... +/- ...] + [+/- ...]*I]
```

Note that coefficients in a number field with defining polynomial x^2+1 are considered to be Gaussian rationals (with the generator mapping to $+I$), if you ask for complex roots.

```
sage: # needs sage.rings.number_field
sage: K.<im> = QuadraticField(-1)
sage: y = polygen(K)
sage: p = y^4 - 2 - im
sage: p.roots(ring=CC)
[(-1.2146389322441... - 0.14142505258239...*I, 1),
 (-0.14142505258239... + 1.2146389322441...*I, 1),
 (0.14142505258239... - 1.2146389322441...*I, 1),
 (1.2146389322441... + 0.14142505258239...*I, 1)]
sage: p = p^2 * (y^2 - 2)
sage: p.roots(ring=CIF)
```

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```
[(-1.414213562373095?, 1), (1.414213562373095?, 1),
 (-1.214638932244183? - 0.141425052582394?*I, 2),
 (-0.141425052582394? + 1.214638932244183?*I, 2),
 (0.141425052582394? - 1.214638932244183?*I, 2),
 (1.214638932244183? + 0.141425052582394?*I, 2)]
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> K = QuadraticField(-Integer(1), names=('im',)); (im,) = K._first_ngens(1)
>>> y = polygen(K)
>>> p = y**Integer(4) - Integer(2) - im
>>> p.roots(ring=CC)
[(-1.2146389322441... - 0.14142505258239...*I, 1),
 (-0.14142505258239... + 1.2146389322441...*I, 1),
 (0.14142505258239... - 1.2146389322441...*I, 1),
 (1.2146389322441... + 0.14142505258239...*I, 1)]
>>> p = p**Integer(2) * (y**Integer(2) - Integer(2))
>>> p.roots(ring=CIF)
[(-1.414213562373095?, 1), (1.414213562373095?, 1),
 (-1.214638932244183? - 0.141425052582394?*I, 2),
 (-0.141425052582394? + 1.214638932244183?*I, 2),
 (0.141425052582394? - 1.214638932244183?*I, 2),
 (1.214638932244183? + 0.141425052582394?*I, 2)]
```

Note that one should not use NumPy when wanting high precision output as it does not support any of the high precision types:

```
sage: # needs numpy sage.rings.real_mpfr sage.symbolic
sage: R.<x> = RealField(200)[]
sage: f = x^2 - R(pi)
sage: f.roots()
[(-1.7724538509055160272981674833411451827975494561223871282138, 1),
 (1.7724538509055160272981674833411451827975494561223871282138, 1)]
sage: f.roots(algorithm='numpy')
doctest... UserWarning: NumPy does not support arbitrary precision arithmetic.
The roots found will likely have less precision than you expect.
[(-1.77245385090551..., 1), (1.77245385090551..., 1)]
```

```
>>> from sage.all import *
>>> # needs numpy sage.rings.real_mpfr sage.symbolic
>>> R = RealField(Integer(200))['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(2) - R(pi)
>>> f.roots()
[(-1.7724538509055160272981674833411451827975494561223871282138, 1),
 (1.7724538509055160272981674833411451827975494561223871282138, 1)]
>>> f.roots(algorithm='numpy')
doctest... UserWarning: NumPy does not support arbitrary precision arithmetic.
The roots found will likely have less precision than you expect.
[(-1.77245385090551..., 1), (1.77245385090551..., 1)]
```

We can also find roots over number fields:

```
sage: K.<z> = CyclotomicField(15)                                             #␣
↪needs sage.rings.number_field
sage: R.<x> = PolynomialRing(K)                                               #␣
↪needs sage.rings.number_field
sage: (x^2 + x + 1).roots()                                                   #␣
↪needs sage.rings.number_field
[(z^5, 1), (-z^5 - 1, 1)]
```

```
>>> from sage.all import *
>>> K = CyclotomicField(Integer(15), names=('z',)); (z,) = K._first_ngens(1)#␣
↪needs sage.rings.number_field
>>> R = PolynomialRing(K, names=('x',)); (x,) = R._first_ngens(1)# needs sage.
↪rings.number_field
>>> (x**Integer(2) + x + Integer(1)).roots()                                  ␣
↪                # needs sage.rings.number_field
[(z^5, 1), (-z^5 - 1, 1)]
```

There are many combinations of floating-point input and output types that work. (Note that some of them are quite pointless like using `algorithm='numpy'` with high-precision types.)

```
sage: # needs numpy sage.rings.complex_double sage.rings.real_mpfr
sage: rflds = (RR, RDF, RealField(100))
sage: cflds = (CC, CDF, ComplexField(100))
sage: def cross(a, b):
....:     return list(cartesian_product_iterator([a, b]))
sage: flds = cross(rflds, rflds) + cross(rflds, cflds) + cross(cflds, cflds)
sage: for (fld_in, fld_out) in flds:
....:     x = polygen(fld_in)
....:     f = x^3 - fld_in(2)
....:     x2 = polygen(fld_out)
....:     f2 = x2^3 - fld_out(2)
....:     for algo in (None, 'pari', 'numpy'):
....:         rts = f.roots(ring=fld_out, multiplicities=False)
....:         rts = sorted(rts, key=lambda x: x.imag())
....:         if fld_in == fld_out and algo is None:
....:             print("{} {}".format(fld_in, rts))
....:         for rt in rts:
....:             assert(abs(f2(rt)) <= 1e-10)
....:             assert(rt.parent() == fld_out)
Real Field with 53 bits of precision [1.25992104989487]
Real Double Field [1.25992104989...]
Real Field with 100 bits of precision [1.2599210498948731647672106073]
Complex Field with 53 bits of precision [-0.62996052494743... - 1.
↪09112363597172*I, 1.25992104989487, -0.62996052494743... + 1.
↪09112363597172*I]
Complex Double Field [-0.629960524947... - 1.0911236359717...*I, 1.
↪25992104989487..., -0.629960524947... + 1.0911236359717...*I]
Complex Field with 100 bits of precision [-0.62996052494743658238360530364 -␣
↪1.0911236359717214035600726142*I, 1.2599210498948731647672106073, -0.
↪62996052494743658238360530364 + 1.0911236359717214035600726142*I]
```

```
>>> from sage.all import *
```

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```
>>> # needs numpy sage.rings.complex_double sage.rings.real_mpfr
>>> rflds = (RR, RDF, RealField(Integer(100)))
>>> cflds = (CC, CDF, ComplexField(Integer(100)))
>>> def cross(a, b):
...     return list(cartesian_product_iterator([a, b]))
>>> flds = cross(rflds, rflds) + cross(rflds, cflds) + cross(cflds, cflds)
>>> for (fld_in, fld_out) in flds:
...     x = polygen(fld_in)
...     f = x**Integer(3) - fld_in(Integer(2))
...     x2 = polygen(fld_out)
...     f2 = x2**Integer(3) - fld_out(Integer(2))
...     for algo in (None, 'pari', 'numpy'):
...         rts = f.roots(ring=fld_out, multiplicities=False)
...         rts = sorted(rts, key=lambda x: x.imag())
...         if fld_in == fld_out and algo is None:
...             print("{} {}".format(fld_in, rts))
...         for rt in rts:
...             assert(abs(f2(rt)) <= RealNumber('1e-10'))
...             assert(rt.parent() == fld_out)
Real Field with 53 bits of precision [1.25992104989487]
Real Double Field [1.25992104989...]
Real Field with 100 bits of precision [1.2599210498948731647672106073]
Complex Field with 53 bits of precision [-0.62996052494743... - 1.
↪09112363597172*I, 1.25992104989487, -0.62996052494743... + 1.
↪09112363597172*I]
Complex Double Field [-0.629960524947... - 1.0911236359717...*I, 1.
↪25992104989487..., -0.629960524947... + 1.0911236359717...*I]
Complex Field with 100 bits of precision [-0.62996052494743658238360530364 -␣
↪1.0911236359717214035600726142*I, 1.2599210498948731647672106073, -0.
↪62996052494743658238360530364 + 1.0911236359717214035600726142*I]
```

Note that we can find the roots of a polynomial with algebraic coefficients:

```
sage: # needs sage.rings.number_field
sage: rt2 = sqrt(AA(2))
sage: rt3 = sqrt(AA(3))
sage: x = polygen(AA)
sage: f = (x - rt2) * (x - rt3); f
x^2 - 3.146264369941973?*x + 2.449489742783178?
sage: rts = f.roots(); rts
[(1.414213562373095?, 1), (1.732050807568878?, 1)]
sage: rts[0][0] == rt2
True
sage: f.roots(ring=RealIntervalField(150))
[(1.414213562373095048801688724209698078569671875376948073176679738?, 1),
 (1.732050807568877293527446341505872366942805253810380628055806980?, 1)]
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> rt2 = sqrt(AA(Integer(2)))
>>> rt3 = sqrt(AA(Integer(3)))
>>> x = polygen(AA)
```

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```
>>> f = (x - rt2) * (x - rt3); f
x^2 - 3.146264369941973?*x + 2.449489742783178?
>>> rts = f.roots(); rts
[(1.414213562373095?, 1), (1.732050807568878?, 1)]
>>> rts[Integer(0)][Integer(0)] == rt2
True
>>> f.roots(ring=RealIntervalField(Integer(150)))
[(1.414213562373095048801688724209698078569671875376948073176679738?, 1),
 (1.732050807568877293527446341505872366942805253810380628055806980?, 1)]
```

We can handle polynomials with huge coefficients.

This number doesn't even fit in an IEEE double-precision float, but `RR` and `CC` allow a much larger range of floating-point numbers:

```
sage: bigc = 2^1500
sage: CDF(bigc)                                                         #␣
↪needs sage.rings.complex_double
+infinity
sage: CC(bigc)                                                          #␣
↪needs sage.rings.real_mpfr
3.50746621104340e451
```

```
>>> from sage.all import *
>>> bigc = Integer(2)**Integer(1500)
>>> CDF(bigc)                                                         #␣
↪needs sage.rings.complex_double
+infinity
>>> CC(bigc)                                                          #␣
↪needs sage.rings.real_mpfr
3.50746621104340e451
```

Polynomials using such large coefficients can't be handled by numpy, but pari can deal with them:

```
sage: x = polygen(QQ)
sage: p = x + bigc
sage: p.roots(ring=RR, algorithm='numpy')                               #␣
↪needs numpy sage.rings.real_mpfr
Traceback (most recent call last):
...
LinAlgError: Array must not contain infs or NaNs
sage: p.roots(ring=RR, algorithm='pari')                                #␣
↪needs sage.libs.pari sage.rings.real_mpfr
[(-3.50746621104340e451, 1)]
sage: p.roots(ring=AA)                                                  #␣
↪needs sage.rings.number_field
[(-3.507466211043404?e451, 1)]
sage: p.roots(ring=QQbar)                                               #␣
↪needs sage.rings.number_field
[(-3.507466211043404?e451, 1)]
sage: p = bigc*x + 1
sage: p.roots(ring=RR)                                                  #␣
↪needs numpy
```

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```
[(-2.85106096489671e-452, 1)]
sage: p.roots(ring=AA)                                                     #␣
↪needs sage.rings.number_field
[(-2.851060964896705?e-452, 1)]
sage: p.roots(ring=QQbar)                                                  #␣
↪needs sage.rings.number_field
[(-2.851060964896705?e-452, 1)]
sage: p = x^2 - bigc
sage: p.roots(ring=RR)                                                     #␣
↪needs numpy
[(-5.92238652153286e225, 1), (5.92238652153286e225, 1)]
sage: p.roots(ring=QQbar)                                                  #␣
↪needs sage.rings.number_field
[(-5.922386521532855?e225, 1), (5.922386521532855?e225, 1)]
```

```
>>> from sage.all import *
>>> x = polygen(QQ)
>>> p = x + bigc
>>> p.roots(ring=RR, algorithm='numpy')                                  #␣
↪needs numpy sage.rings.real_mpfr
Traceback (most recent call last):
...
LinAlgError: Array must not contain infs or NaNs
>>> p.roots(ring=RR, algorithm='pari')                                   #␣
↪needs sage.libs.pari sage.rings.real_mpfr
[(-3.50746621104340e451, 1)]
>>> p.roots(ring=AA)                                                     #␣
↪needs sage.rings.number_field
[(-3.507466211043403?e451, 1)]
>>> p.roots(ring=QQbar)                                                  #␣
↪needs sage.rings.number_field
[(-3.507466211043403?e451, 1)]
>>> p = bigc*x + Integer(1)
>>> p.roots(ring=RR)                                                     #␣
↪needs numpy
[(-2.85106096489671e-452, 1)]
>>> p.roots(ring=AA)                                                     #␣
↪needs sage.rings.number_field
[(-2.851060964896705?e-452, 1)]
>>> p.roots(ring=QQbar)                                                  #␣
↪needs sage.rings.number_field
[(-2.851060964896705?e-452, 1)]
>>> p = x**Integer(2) - bigc
>>> p.roots(ring=RR)                                                     #␣
↪needs numpy
[(-5.92238652153286e225, 1), (5.92238652153286e225, 1)]
>>> p.roots(ring=QQbar)                                                  #␣
↪needs sage.rings.number_field
[(-5.922386521532855?e225, 1), (5.922386521532855?e225, 1)]
```

Check that Issue #30522 is fixed:

```
sage: PolynomialRing(SR, names="x")("x^2").roots()                     #␣
↪needs sage.symbolic
[(0, 2)]
```

```
>>> from sage.all import *
>>> PolynomialRing(SR, names="x")("x^2").roots()                    #␣
↪needs sage.symbolic
[(0, 2)]
```

Check that Issue #30523 is fixed:

```
sage: PolynomialRing(SR, names="x")("x^2 + q").roots()                 #␣
↪needs sage.symbolic
[(-sqrt(-q), 1), (sqrt(-q), 1)]
```

```
>>> from sage.all import *
>>> PolynomialRing(SR, names="x")("x^2 + q").roots()                #␣
↪needs sage.symbolic
[(-sqrt(-q), 1), (sqrt(-q), 1)]
```

ALGORITHM:

For brevity, we will use `RR` to mean any `RealField` of any precision; similarly for `RIF`, `CC`, and `CIF`. Since Sage has no specific implementation of Gaussian rationals (or of number fields with embedding, at all), when we refer to Gaussian rationals below we will accept any number field with defining polynomial x^2+1, mapping the field generator to +I.

We call the base ring of the polynomial K, and the ring given by the `ring` argument L. (If `ring` is not specified, then L is the same as K.)

If K and L are floating-point (`RDF`, `CDF`, `RR`, or `CC`), then a floating-point root-finder is used. If L is `RDF` or `CDF`, then we default to using NumPy's `roots()`; otherwise, we use PARI's function pari:polroots. This choice can be overridden with `algorithm='pari'` or `algorithm='numpy'`. If the algorithm is unspecified and NumPy's `roots()` algorithm fails, then we fall back to PARI (NumPy will fail if some coefficient is infinite, for instance).

If L is `SR` (or one of its subrings), then the roots will be radical expressions, computed as the solutions of a symbolic polynomial expression. At the moment this delegates to `sage.symbolic.expression.Expression.solve()` which in turn uses Maxima to find radical solutions. Some solutions may be lost in this approach. Once Issue #17516 gets implemented, all possible radical solutions should become available.

If L is `AA` or `RIF`, and K is `ZZ`, `QQ`, or `AA`, then the root isolation algorithm `sage.rings.polynomial.real_roots.real_roots()` is used. (You can call `real_roots()` directly to get more control than this method gives.)

If L is `QQbar` or `CIF`, and K is `ZZ`, `QQ`, `AA`, `QQbar`, or the Gaussian rationals, then the root isolation algorithm `sage.rings.polynomial.complex_roots.complex_roots()` is used. (You can call `complex_roots()` directly to get more control than this method gives.)

If L is `AA` and K is `QQbar` or the Gaussian rationals, then `complex_roots()` is used (as above) to find roots in `QQbar`, then these roots are filtered to select only the real roots.

If L is floating-point and K is not, then we attempt to change the polynomial ring to L (using `change_ring()`) (or, if L is complex and K is not, to the corresponding real field). Then we use either PARI or NumPy as specified above.

For all other cases where K is different from L, we attempt to use `.change_ring(L)`. When that fails but L is a subring of K, we also attempt to compute the roots over K and filter the ones belonging to L.

The next method, which is used if K is an integral domain, is to attempt to factor the polynomial. If this succeeds, then for every degree-one factor $ax + b$, we add $-b/a$ as a root (as long as this quotient is actually in the desired ring).

If factoring over K is not implemented (or K is not an integral domain), and K is finite, then we find the roots by enumerating all elements of K and checking whether the polynomial evaluates to zero at that value.

Note

We mentioned above that polynomials with multiple roots are always ill-conditioned; if your input is given to n bits of precision, you should not expect more than n/k good bits for a k-fold root. (You can get solutions that make the polynomial evaluate to a number very close to zero; basically the problem is that with a multiple root, there are many such numbers, and it's difficult to choose between them.)

To see why this is true, consider the naive floating-point error analysis model where you just pretend that all floating-point numbers are somewhat imprecise - a little 'fuzzy', if you will. Then the graph of a floating-point polynomial will be a fuzzy line. Consider the graph of $(x - 1)^3$; this will be a fuzzy line with a horizontal tangent at $x = 1$, $y = 0$. If the fuzziness extends up and down by about j, then it will extend left and right by about cube_root(j).

shift(*n*)

Return this polynomial multiplied by the power x^n. If n is negative, terms below x^n will be discarded. Does not change this polynomial (since polynomials are immutable).

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: p = x^2 + 2*x + 4
sage: p.shift(0)
 x^2 + 2*x + 4
sage: p.shift(-1)
 x + 2
sage: p.shift(-5)
 0
sage: p.shift(2)
 x^4 + 2*x^3 + 4*x^2
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> p = x**Integer(2) + Integer(2)*x + Integer(4)
>>> p.shift(Integer(0))
 x^2 + 2*x + 4
>>> p.shift(-Integer(1))
 x + 2
>>> p.shift(-Integer(5))
 0
>>> p.shift(Integer(2))
 x^4 + 2*x^3 + 4*x^2
```

One can also use the infix shift operator:

```
sage: f = x^3 + x
sage: f >> 2
x
```

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```
sage: f << 2
x^5 + x^3
```

```
>>> from sage.all import *
>>> f = x**Integer(3) + x
>>> f >> Integer(2)
x
>>> f << Integer(2)
x^5 + x^3
```

AUTHORS:

- David Harvey (2006-08-06)
- Robert Bradshaw (2007-04-18): Added support for infix operator.

specialization(*D=None*, *phi=None*)

Specialization of this polynomial.

Given a family of polynomials defined over a polynomial ring. A specialization is a particular member of that family. The specialization can be specified either by a dictionary or a `SpecializationMorphism`.

INPUT:

- `D` – dictionary (optional)
- `phi` – `SpecializationMorphism` (optional)

OUTPUT: a new polynomial

EXAMPLES:

```
sage: R.<c> = PolynomialRing(ZZ)
sage: S.<z> = PolynomialRing(R)
sage: F = c*z^2 + c^2
sage: F.specialization({c:2})
2*z^2 + 4
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('c',)); (c,) = R._first_ngens(1)
>>> S = PolynomialRing(R, names=('z',)); (z,) = S._first_ngens(1)
>>> F = c*z**Integer(2) + c**Integer(2)
>>> F.specialization({c:Integer(2)})
2*z^2 + 4
```

```
sage: A.<c> = QQ[]
sage: R.<x> = Frac(A)[]
sage: X = (1 + x/c).specialization({c:20})
sage: X
1/20*x + 1
sage: X.parent()
Univariate Polynomial Ring in x over Rational Field
```

```
>>> from sage.all import *
>>> A = QQ['c']; (c,) = A._first_ngens(1)
```

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```
>>> R = Frac(A)['x']; (x,) = R._first_ngens(1)
>>> X = (Integer(1) + x/c).specialization({c:Integer(20)})
>>> X
1/20*x + 1
>>> X.parent()
Univariate Polynomial Ring in x over Rational Field
```

splitting_field(*names=None*, *map=False*, ***kwds*)

Compute the absolute splitting field of a given polynomial.

INPUT:

- `names` – (default: `None`) a variable name for the splitting field
- `map` – boolean (default: `False`); also return an embedding of `self` into the resulting field
- `kwds` – additional keywords depending on the type. Currently, only number fields are implemented. See `sage.rings.number_field.splitting_field.splitting_field()` for the documentation of these keywords.

OUTPUT:

If `map` is `False`, the splitting field as an absolute field. If `map` is `True`, a tuple `(K, phi)` where `phi` is an embedding of the base field of `self` in `K`.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ)
sage: K.<a> = (x^3 + 2).splitting_field(); K                                    #␣
↪needs sage.rings.number_field
Number Field in a with defining polynomial
 x^6 + 3*x^5 + 6*x^4 + 11*x^3 + 12*x^2 - 3*x + 1
sage: K.<a> = (x^3 - 3*x + 1).splitting_field(); K                              #␣
↪needs sage.rings.number_field
Number Field in a with defining polynomial x^3 - 3*x + 1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> K = (x**Integer(3) + Integer(2)).splitting_field(names=('a',)); (a,) = K._
↪first_ngens(1); K                                   # needs sage.rings.number_
↪field
Number Field in a with defining polynomial
 x^6 + 3*x^5 + 6*x^4 + 11*x^3 + 12*x^2 - 3*x + 1
>>> K = (x**Integer(3) - Integer(3)*x + Integer(1)).splitting_field(names=('a
↪',)); (a,) = K._first_ngens(1); K                              # needs sage.
↪rings.number_field
Number Field in a with defining polynomial x^3 - 3*x + 1
```

Relative situation:

```
sage: # needs sage.rings.number_field
sage: R.<x> = PolynomialRing(QQ)
sage: K.<a> = NumberField(x^3 + 2)
sage: S.<t> = PolynomialRing(K)
sage: L.<b> = (t^2 - a).splitting_field()
```

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```
sage: L
Number Field in b with defining polynomial t^6 + 2
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> K = NumberField(x**Integer(3) + Integer(2), names=('a',)); (a,) = K._
↪first_ngens(1)
>>> S = PolynomialRing(K, names=('t',)); (t,) = S._first_ngens(1)
>>> L = (t**Integer(2) - a).splitting_field(names=('b',)); (b,) = L._first_
↪ngens(1)
>>> L
Number Field in b with defining polynomial t^6 + 2
```

With `map=True`, we also get the embedding of the base field into the splitting field:

```
sage: L.<b>, phi = (t^2 - a).splitting_field(map=True)                     #␣
↪needs sage.rings.number_field
sage: phi                                                                  #␣
↪needs sage.rings.number_field
Ring morphism:
  From: Number Field in a with defining polynomial x^3 + 2
  To:   Number Field in b with defining polynomial t^6 + 2
  Defn: a |--> b^2
```

```
>>> from sage.all import *
>>> L, phi  = (t**Integer(2) - a).splitting_field(map=True, names=('b',)); (b,
↪) = L._first_ngens(1)# needs sage.rings.number_field
>>> phi                                                                  #␣
↪needs sage.rings.number_field
Ring morphism:
  From: Number Field in a with defining polynomial x^3 + 2
  To:   Number Field in b with defining polynomial t^6 + 2
  Defn: a |--> b^2
```

An example over a finite field:

```
sage: P.<x> = PolynomialRing(GF(7))
sage: t = x^2 + 1
sage: t.splitting_field('b')                                               #␣
↪needs sage.rings.finite_rings
Finite Field in b of size 7^2

sage: P.<x> = PolynomialRing(GF(7^3, 'a'))                                 #␣
↪needs sage.rings.finite_rings
sage: t = x^2 + 1
sage: t.splitting_field('b', map=True)                                     #␣
↪needs sage.rings.finite_rings
(Finite Field in b of size 7^6,
 Ring morphism:
   From: Finite Field in a of size 7^3
   To:   Finite Field in b of size 7^6
```

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```
    Defn: a |--> 2*b^4 + 6*b^3 + 2*b^2 + 3*b + 2)
```

```
>>> from sage.all import *
>>> P = PolynomialRing(GF(Integer(7)), names=('x',)); (x,) = P._first_ngens(1)
>>> t = x**Integer(2) + Integer(1)
>>> t.splitting_field('b')                                                  #␣
↪needs sage.rings.finite_rings
Finite Field in b of size 7^2

>>> P = PolynomialRing(GF(Integer(7)**Integer(3), 'a'), names=('x',)); (x,) =␣
↪P._first_ngens(1)# needs sage.rings.finite_rings
>>> t = x**Integer(2) + Integer(1)
>>> t.splitting_field('b', map=True)                                        #␣
↪needs sage.rings.finite_rings
(Finite Field in b of size 7^6,
 Ring morphism:
   From: Finite Field in a of size 7^3
   To:   Finite Field in b of size 7^6
   Defn: a |--> 2*b^4 + 6*b^3 + 2*b^2 + 3*b + 2)
```

If the extension is trivial and the generators have the same name, the map will be the identity:

```
sage: t = 24*x^13 + 2*x^12 + 14
sage: t.splitting_field('a', map=True)                                        #␣
↪needs sage.rings.finite_rings
(Finite Field in a of size 7^3,
 Identity endomorphism of Finite Field in a of size 7^3)

sage: t = x^56 - 14*x^3
sage: t.splitting_field('b', map=True)                                        #␣
↪needs sage.rings.finite_rings
(Finite Field in b of size 7^3,
 Ring morphism:
 From: Finite Field in a of size 7^3
   To:   Finite Field in b of size 7^3
   Defn: a |--> b)
```

```
>>> from sage.all import *
>>> t = Integer(24)*x**Integer(13) + Integer(2)*x**Integer(12) + Integer(14)
>>> t.splitting_field('a', map=True)                                        #␣
↪needs sage.rings.finite_rings
(Finite Field in a of size 7^3,
 Identity endomorphism of Finite Field in a of size 7^3)

>>> t = x**Integer(56) - Integer(14)*x**Integer(3)
>>> t.splitting_field('b', map=True)                                        #␣
↪needs sage.rings.finite_rings
(Finite Field in b of size 7^3,
 Ring morphism:
 From: Finite Field in a of size 7^3
   To:   Finite Field in b of size 7^3
   Defn: a |--> b)
```

See also

`sage.rings.number_field.splitting_field.splitting_field()` for more examples over number fields

square()

Return the square of this polynomial.

Todo

- This is just a placeholder; for now it just uses ordinary multiplication. But generally speaking, squaring is faster than ordinary multiplication, and it's frequently used, so subclasses may choose to provide a specialised squaring routine.
- Perhaps this even belongs at a lower level? RingElement or something?

AUTHORS:

- David Harvey (2006-09-09)

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: f = x^3 + 1
sage: f.square()
x^6 + 2*x^3 + 1
sage: f*f
x^6 + 2*x^3 + 1
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(3) + Integer(1)
>>> f.square()
x^6 + 2*x^3 + 1
>>> f*f
x^6 + 2*x^3 + 1
```

squarefree_decomposition()

Return the square-free decomposition of this polynomial. This is a partial factorization into square-free, coprime polynomials.

EXAMPLES:

```
sage: x = polygen(QQ)
sage: p = 37 * (x - 1)^3 * (x - 2)^3 * (x - 1/3)^7 * (x - 3/7)
sage: p.squarefree_decomposition()
(37*x - 111/7) * (x^2 - 3*x + 2)^3 * (x - 1/3)^7
sage: p = 37 * (x - 2/3)^2
sage: p.squarefree_decomposition()
(37) * (x - 2/3)^2
sage: x = polygen(GF(3))
sage: x.squarefree_decomposition()
```

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```
x
sage: f = QQbar['x'](1)                                                    #␣
↪needs sage.rings.number_field
sage: f.squarefree_decomposition()                                         #␣
↪needs sage.rings.number_field
1
```

```
>>> from sage.all import *
>>> x = polygen(QQ)
>>> p = Integer(37) * (x - Integer(1))**Integer(3) * (x -␣
↪Integer(2))**Integer(3) * (x - Integer(1)/Integer(3))**Integer(7) * (x -␣
↪Integer(3)/Integer(7))
>>> p.squarefree_decomposition()
(37*x - 111/7) * (x^2 - 3*x + 2)^3 * (x - 1/3)^7
>>> p = Integer(37) * (x - Integer(2)/Integer(3))**Integer(2)
>>> p.squarefree_decomposition()
(37) * (x - 2/3)^2
>>> x = polygen(GF(Integer(3)))
>>> x.squarefree_decomposition()
x
>>> f = QQbar['x'](Integer(1))                                             ␣
↪     # needs sage.rings.number_field
>>> f.squarefree_decomposition()                                         #␣
↪needs sage.rings.number_field
1
```

subresultants(*other*)

Return the nonzero subresultant polynomials of `self` and `other`.

INPUT:

- `other` – a polynomial

OUTPUT: list of polynomials in the same ring as `self`

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: f = x^8 + x^6 - 3*x^4 - 3*x^3 + 8*x^2 + 2*x - 5
sage: g = 3*x^6 + 5*x^4 - 4*x^2 - 9*x + 21
sage: f.subresultants(g)
[260708,
 9326*x - 12300,
 169*x^2 + 325*x - 637,
 65*x^2 + 125*x - 245,
 25*x^4 - 5*x^2 + 15,
 15*x^4 - 3*x^2 + 9]
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(8) + x**Integer(6) - Integer(3)*x**Integer(4) -␣
↪Integer(3)*x**Integer(3) + Integer(8)*x**Integer(2) + Integer(2)*x -␣
↪Integer(5)
```

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```
>>> g = Integer(3)*x**Integer(6) + Integer(5)*x**Integer(4) -␣
↪Integer(4)*x**Integer(2) - Integer(9)*x + Integer(21)
>>> f.subresultants(g)
[260708,
 9326*x - 12300,
 169*x^2 + 325*x - 637,
 65*x^2 + 125*x - 245,
 25*x^4 - 5*x^2 + 15,
 15*x^4 - 3*x^2 + 9]
```

ALGORITHM:

We use the schoolbook algorithm with Lazard's optimization described in [Duc1998]

REFERENCES:

Wikipedia article Polynomial_greatest_common_divisor#Subresultants

subs(*in_dict=None*, **args*, ***kwds*)

Substitute the variable in `self`.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: f = x^3 + x - 3
sage: f.subs(x=5)
127
sage: f.subs(5)
127
sage: f.subs({x:2})
7
sage: f.subs({})
x^3 + x - 3
sage: f.subs({'x':2})
Traceback (most recent call last):
...
TypeError: keys do not match self's parent
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(3) + x - Integer(3)
>>> f.subs(x=Integer(5))
127
>>> f.subs(Integer(5))
127
>>> f.subs({x:Integer(2)})
7
>>> f.subs({})
x^3 + x - 3
>>> f.subs({'x':Integer(2)})
Traceback (most recent call last):
...
TypeError: keys do not match self's parent
```

sylvester_matrix(*right*, *variable=None*)

Return the Sylvester matrix of `self` and `right`.

Note that the Sylvester matrix is not defined if one of the polynomials is zero.

INPUT:

- `right` – a polynomial in the same ring as `self`
- `variable` – (optional) included for compatibility with the multivariate case only; the variable of the polynomials

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ)
sage: f = (6*x + 47) * (7*x^2 - 2*x + 38)
sage: g = (6*x + 47) * (3*x^3 + 2*x + 1)
sage: M = f.sylvester_matrix(g); M                                    #␣
↪needs sage.modules
[  42  317  134 1786    0    0    0]
[   0   42  317  134 1786    0    0]
[   0    0   42  317  134 1786    0]
[   0    0    0   42  317  134 1786]
[  18  141   12  100   47    0    0]
[   0   18  141   12  100   47    0]
[   0    0   18  141   12  100   47]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = (Integer(6)*x + Integer(47)) * (Integer(7)*x**Integer(2) -␣
↪Integer(2)*x + Integer(38))
>>> g = (Integer(6)*x + Integer(47)) * (Integer(3)*x**Integer(3) +␣
↪Integer(2)*x + Integer(1))
>>> M = f.sylvester_matrix(g); M                                    #␣
↪needs sage.modules
[  42  317  134 1786    0    0    0]
[   0   42  317  134 1786    0    0]
[   0    0   42  317  134 1786    0]
[   0    0    0   42  317  134 1786]
[  18  141   12  100   47    0    0]
[   0   18  141   12  100   47    0]
[   0    0   18  141   12  100   47]
```

If the polynomials share a non-constant common factor then the determinant of the Sylvester matrix will be zero:

```
sage: M.determinant()                                                 #␣
↪needs sage.modules
0
```

```
>>> from sage.all import *
>>> M.determinant()                                                 #␣
↪needs sage.modules
0
```

If `self` and `right` are polynomials of positive degree, the determinant of the Sylvester matrix is the resultant of the polynomials.:

```
sage: h1 = R._random_nonzero_element()
sage: h2 = R._random_nonzero_element()
sage: M1 = h1.sylvester_matrix(h2)                                          #␣
↪needs sage.modules
sage: M1.determinant() == h1.resultant(h2)                                  #␣
↪needs sage.libs.pari sage.modules
True
```

```
>>> from sage.all import *
>>> h1 = R._random_nonzero_element()
>>> h2 = R._random_nonzero_element()
>>> M1 = h1.sylvester_matrix(h2)                                          #␣
↪needs sage.modules
>>> M1.determinant() == h1.resultant(h2)                                  #␣
↪needs sage.libs.pari sage.modules
True
```

The rank of the Sylvester matrix is related to the degree of the gcd of `self` and `right`:

```
sage: f.gcd(g).degree() == f.degree() + g.degree() - M.rank()               #␣
↪needs sage.modules
True
sage: h1.gcd(h2).degree() == h1.degree() + h2.degree() - M1.rank()          #␣
↪needs sage.modules
True
```

```
>>> from sage.all import *
>>> f.gcd(g).degree() == f.degree() + g.degree() - M.rank()               #␣
↪needs sage.modules
True
>>> h1.gcd(h2).degree() == h1.degree() + h2.degree() - M1.rank()          #␣
↪needs sage.modules
True
```

symmetric_power(*k*, *monic=False*)

Return the polynomial whose roots are products of k-th distinct roots of this.

EXAMPLES:

```
sage: x = polygen(QQ)
sage: f = x^4 - x + 2
sage: [f.symmetric_power(k) for k in range(5)]                              #␣
↪needs sage.libs.singular
[x - 1, x^4 - x + 2, x^6 - 2*x^4 - x^3 - 4*x^2 + 8, x^4 - x^3 + 8, x - 2]

sage: f = x^5 - 2*x + 2
sage: [f.symmetric_power(k) for k in range(6)]                              #␣
↪needs sage.libs.singular
[x - 1,
 x^5 - 2*x + 2,
 x^10 + 2*x^8 - 4*x^6 - 8*x^5 - 8*x^4 - 8*x^3 + 16,
```

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```
 x^10 + 4*x^7 - 8*x^6 + 16*x^5 - 16*x^4 + 32*x^2 + 64,
 x^5 + 2*x^4 - 16,
 x + 2]

sage: R.<a,b,c,d> = ZZ[]
sage: x = polygen(R)
sage: f = (x - a) * (x - b) * (x - c) * (x - d)
sage: [f.symmetric_power(k).factor() for k in range(5)]                     #␣
↪needs sage.libs.singular
[x - 1,
 (-x + d) * (-x + c) * (-x + b) * (-x + a),
 (x - a*b) * (x - a*c) * (x - b*c) * (x - a*d) * (x - b*d) * (x - c*d),
 (x - a*b*c) * (x - a*b*d) * (x - a*c*d) * (x - b*c*d),
 x - a*b*c*d]
```

```
>>> from sage.all import *
>>> x = polygen(QQ)
>>> f = x**Integer(4) - x + Integer(2)
>>> [f.symmetric_power(k) for k in range(Integer(5))]                        ␣
↪     # needs sage.libs.singular
[x - 1, x^4 - x + 2, x^6 - 2*x^4 - x^3 - 4*x^2 + 8, x^4 - x^3 + 8, x - 2]

>>> f = x**Integer(5) - Integer(2)*x + Integer(2)
>>> [f.symmetric_power(k) for k in range(Integer(6))]                        ␣
↪     # needs sage.libs.singular
[x - 1,
 x^5 - 2*x + 2,
 x^10 + 2*x^8 - 4*x^6 - 8*x^5 - 8*x^4 - 8*x^3 + 16,
 x^10 + 4*x^7 - 8*x^6 + 16*x^5 - 16*x^4 + 32*x^2 + 64,
 x^5 + 2*x^4 - 16,
 x + 2]

>>> R = ZZ['a, b, c, d']; (a, b, c, d,) = R._first_ngens(4)
>>> x = polygen(R)
>>> f = (x - a) * (x - b) * (x - c) * (x - d)
>>> [f.symmetric_power(k).factor() for k in range(Integer(5))]               ␣
↪     # needs sage.libs.singular
[x - 1,
 (-x + d) * (-x + c) * (-x + b) * (-x + a),
 (x - a*b) * (x - a*c) * (x - b*c) * (x - a*d) * (x - b*d) * (x - c*d),
 (x - a*b*c) * (x - a*b*d) * (x - a*c*d) * (x - b*c*d),
 x - a*b*c*d]
```

trace_polynomial()

Compute the trace polynomial and cofactor.

The input P and output Q satisfy the relation

$$P(x) = Q(x + q/x)x^{\deg(Q)}R(x).$$

In this relation, Q has all roots in the real interval $[-2\sqrt{q}, 2\sqrt{q}]$ if and only if P has all roots on the circle $|x| = \sqrt{q}$ and R divides $x^2 - q$. We thus require that the base ring of this polynomial have a coercion to the real numbers.

See also

The inverse operation is *reciprocal_transform()*.

OUTPUT:

- `Q` – trace polynomial
- `R` – cofactor
- `q` – scaling factor

EXAMPLES:

```
sage: pol.<x> = PolynomialRing(Rationals())
sage: u = x^5 - 1; u.trace_polynomial()
(x^2 + x - 1, x - 1, 1)
sage: u = x^4 + x^3 + 5*x^2 + 3*x + 9
sage: u.trace_polynomial()
(x^2 + x - 1, 1, 3)
```

```
>>> from sage.all import *
>>> pol = PolynomialRing(Rationals(), names=('x',)); (x,) = pol._first_
↪ngens(1)
>>> u = x**Integer(5) - Integer(1); u.trace_polynomial()
(x^2 + x - 1, x - 1, 1)
>>> u = x**Integer(4) + x**Integer(3) + Integer(5)*x**Integer(2) +␣
↪Integer(3)*x + Integer(9)
>>> u.trace_polynomial()
(x^2 + x - 1, 1, 3)
```

We check that this function works for rings that have a coercion to the reals:

```
sage: # needs sage.rings.number_field
sage: K.<a> = NumberField(x^2 - 2, embedding=1.4)
sage: u = x^4 + a*x^3 + 3*x^2 + 2*a*x + 4
sage: u.trace_polynomial()
(x^2 + a*x - 1, 1, 2)
sage: (u*(x^2-2)).trace_polynomial()
(x^2 + a*x - 1, x^2 - 2, 2)
sage: (u*(x^2-2)^2).trace_polynomial()
(x^4 + a*x^3 - 9*x^2 - 8*a*x + 8, 1, 2)
sage: (u*(x^2-2)^3).trace_polynomial()
(x^4 + a*x^3 - 9*x^2 - 8*a*x + 8, x^2 - 2, 2)
sage: u = x^4 + a*x^3 + 3*x^2 + 4*a*x + 16
sage: u.trace_polynomial()
(x^2 + a*x - 5, 1, 4)
sage: (u*(x-2)).trace_polynomial()
(x^2 + a*x - 5, x - 2, 4)
sage: (u*(x+2)).trace_polynomial()
(x^2 + a*x - 5, x + 2, 4)
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
```

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```
>>> K = NumberField(x**Integer(2) - Integer(2), embedding=RealNumber('1.4'),␣
↪names=('a',)); (a,) = K._first_ngens(1)
>>> u = x**Integer(4) + a*x**Integer(3) + Integer(3)*x**Integer(2) +␣
↪Integer(2)*a*x + Integer(4)
>>> u.trace_polynomial()
(x^2 + a*x - 1, 1, 2)
>>> (u*(x**Integer(2)-Integer(2))).trace_polynomial()
(x^2 + a*x - 1, x^2 - 2, 2)
>>> (u*(x**Integer(2)-Integer(2))**Integer(2)).trace_polynomial()
(x^4 + a*x^3 - 9*x^2 - 8*a*x + 8, 1, 2)
>>> (u*(x**Integer(2)-Integer(2))**Integer(3)).trace_polynomial()
(x^4 + a*x^3 - 9*x^2 - 8*a*x + 8, x^2 - 2, 2)
>>> u = x**Integer(4) + a*x**Integer(3) + Integer(3)*x**Integer(2) +␣
↪Integer(4)*a*x + Integer(16)
>>> u.trace_polynomial()
(x^2 + a*x - 5, 1, 4)
>>> (u*(x-Integer(2))).trace_polynomial()
(x^2 + a*x - 5, x - 2, 4)
>>> (u*(x+Integer(2))).trace_polynomial()
(x^2 + a*x - 5, x + 2, 4)
```

truncate(*n*)

Return the polynomial of degree $< n$ which is equivalent to `self` modulo x^n.

EXAMPLES:

```
sage: R.<x> = ZZ[]; S.<y> = PolynomialRing(R, sparse=True)
sage: f = y^3 + x*y - 3*x; f
y^3 + x*y - 3*x
sage: f.truncate(2)
x*y - 3*x
sage: f.truncate(1)
-3*x
sage: f.truncate(0)
0
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1); S = PolynomialRing(R, sparse=True,␣
↪names=('y',)); (y,) = S._first_ngens(1)
>>> f = y**Integer(3) + x*y - Integer(3)*x; f
y^3 + x*y - 3*x
>>> f.truncate(Integer(2))
x*y - 3*x
>>> f.truncate(Integer(1))
-3*x
>>> f.truncate(Integer(0))
0
```

valuation(*p=None*)

If $f = a_r x^r + a_{r+1} x^{r+1} + \cdots$, with a_r nonzero, then the valuation of f is r. The valuation of the zero polynomial is ∞.

If a prime (or non-prime) p is given, then the valuation is the largest power of p which divides `self`.

The valuation at ∞ is `-self.degree()`.

EXAMPLES:

```
sage: P.<x> = ZZ[]
sage: (x^2 + x).valuation()
1
sage: (x^2 + x).valuation(x + 1)
1
sage: (x^2 + 1).valuation()
0
sage: (x^3 + 1).valuation(infinity)
-3
sage: P(0).valuation()
+Infinity
```

```
>>> from sage.all import *
>>> P = ZZ['x']; (x,) = P._first_ngens(1)
>>> (x**Integer(2) + x).valuation()
1
>>> (x**Integer(2) + x).valuation(x + Integer(1))
1
>>> (x**Integer(2) + Integer(1)).valuation()
0
>>> (x**Integer(3) + Integer(1)).valuation(infinity)
-3
>>> P(Integer(0)).valuation()
+Infinity
```

variable_name()

Return name of variable used in this polynomial as a string.

OUTPUT: string

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: f = t^3 + 3/2*t + 5
sage: f.variable_name()
't'
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = t**Integer(3) + Integer(3)/Integer(2)*t + Integer(5)
>>> f.variable_name()
't'
```

variables()

Return the tuple of variables occurring in this polynomial.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: x.variables()
(x,)
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> x.variables()
(x,)
```

A constant polynomial has no variables.

```
sage: R(2).variables()
()
```

```
>>> from sage.all import *
>>> R(Integer(2)).variables()
()
```

xgcd(*other*)

Return an extended gcd of this polynomial and `other`.

INPUT:

- `other` – a polynomial in the same ring as this polynomial

OUTPUT:

A tuple `(r, s, t)` where `r` is a greatest common divisor of this polynomial and `other`, and `s` and `t` are such that `r = s*self + t*other` holds.

Note: The actual algorithm for computing the extended gcd depends on the base ring underlying the polynomial ring. If the base ring defines a method `_xgcd_univariate_polynomial()`, then this method will be called (see examples below).

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQbar[]
sage: (2*x^2).gcd(2*x)
x
sage: R.zero().gcd(0)
0
sage: (2*x).gcd(0)
x
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQbar['x']; (x,) = R._first_ngens(1)
>>> (Integer(2)*x**Integer(2)).gcd(Integer(2)*x)
x
>>> R.zero().gcd(Integer(0))
0
>>> (Integer(2)*x).gcd(Integer(0))
x
```

One can easily add xgcd functionality to new rings by providing a method `_xgcd_univariate_polynomial()`:

```
sage: R.<x> = QQ[]
sage: S.<y> = R[]
sage: h1 = y*x
sage: h2 = y^2*x^2
sage: h1.xgcd(h2)
Traceback (most recent call last):
...
NotImplementedError: Univariate Polynomial Ring in x over Rational Field
does not provide an xgcd implementation for univariate polynomials
sage: T.<x,y> = QQ[]
sage: def poor_xgcd(f, g):
....:     ret = S(T(f).gcd(g))
....:     if ret == f: return ret, S.one(), S.zero()
....:     if ret == g: return ret, S.zero(), S.one()
....:     raise NotImplementedError
sage: R._xgcd_univariate_polynomial = poor_xgcd
sage: h1.xgcd(h2)
(x*y, 1, 0)
sage: del R._xgcd_univariate_polynomial
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> S = R['y']; (y,) = S._first_ngens(1)
>>> h1 = y*x
>>> h2 = y**Integer(2)*x**Integer(2)
>>> h1.xgcd(h2)
Traceback (most recent call last):
...
NotImplementedError: Univariate Polynomial Ring in x over Rational Field
does not provide an xgcd implementation for univariate polynomials
>>> T = QQ['x, y']; (x, y,) = T._first_ngens(2)
>>> def poor_xgcd(f, g):
...     ret = S(T(f).gcd(g))
...     if ret == f: return ret, S.one(), S.zero()
...     if ret == g: return ret, S.zero(), S.one()
...     raise NotImplementedError
>>> R._xgcd_univariate_polynomial = poor_xgcd
>>> h1.xgcd(h2)
(x*y, 1, 0)
>>> del R._xgcd_univariate_polynomial
```

class `sage.rings.polynomial.polynomial_element.`**`PolynomialBaseringInjection`**

Bases: `Morphism`

This class is used for conversion from a ring to a polynomial over that ring.

It calls the `_new_constant_poly()` method on the generator, which should be optimized for a particular polynomial type.

Technically, it should be a method of the polynomial ring, but few polynomial rings are Cython classes, and so, as a method of a Cython polynomial class, it is faster.

EXAMPLES:

We demonstrate that most polynomial ring classes use polynomial base injection maps for coercion. They are supposed to be the fastest maps for that purpose. See Issue #9944.

```
sage: # needs sage.rings.padics
sage: R.<x> = Qp(3)[]
sage: R.coerce_map_from(R.base_ring())
Polynomial base injection morphism:
  From: 3-adic Field with capped relative precision 20
  To:   Univariate Polynomial Ring in x over
        3-adic Field with capped relative precision 20
sage: R.<x,y> = Qp(3)[]
sage: R.coerce_map_from(R.base_ring())
Polynomial base injection morphism:
  From: 3-adic Field with capped relative precision 20
  To:   Multivariate Polynomial Ring in x, y over
        3-adic Field with capped relative precision 20

sage: R.<x,y> = QQ[]
sage: R.coerce_map_from(R.base_ring())
Polynomial base injection morphism:
  From: Rational Field
  To:   Multivariate Polynomial Ring in x, y over Rational Field
sage: R.<x> = QQ[]
sage: R.coerce_map_from(R.base_ring())
Polynomial base injection morphism:
  From: Rational Field
  To:   Univariate Polynomial Ring in x over Rational Field
```

```
>>> from sage.all import *
>>> # needs sage.rings.padics
>>> R = Qp(Integer(3))['x']; (x,) = R._first_ngens(1)
>>> R.coerce_map_from(R.base_ring())
Polynomial base injection morphism:
  From: 3-adic Field with capped relative precision 20
  To:   Univariate Polynomial Ring in x over
        3-adic Field with capped relative precision 20
>>> R = Qp(Integer(3))['x, y']; (x, y,) = R._first_ngens(2)
>>> R.coerce_map_from(R.base_ring())
Polynomial base injection morphism:
  From: 3-adic Field with capped relative precision 20
  To:   Multivariate Polynomial Ring in x, y over
        3-adic Field with capped relative precision 20

>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> R.coerce_map_from(R.base_ring())
Polynomial base injection morphism:
  From: Rational Field
  To:   Multivariate Polynomial Ring in x, y over Rational Field
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> R.coerce_map_from(R.base_ring())
Polynomial base injection morphism:
  From: Rational Field
  To:   Univariate Polynomial Ring in x over Rational Field
```

By Issue #9944, there are now only very few exceptions:

```
sage: PolynomialRing(QQ,names=[]).coerce_map_from(QQ)
Call morphism:
  From: Rational Field
  To:   Multivariate Polynomial Ring in no variables over Rational Field
```

```
>>> from sage.all import *
>>> PolynomialRing(QQ,names=[]).coerce_map_from(QQ)
Call morphism:
  From: Rational Field
  To:   Multivariate Polynomial Ring in no variables over Rational Field
```

is_injective()

Return whether this morphism is injective.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: S.<y> = R[]
sage: S.coerce_map_from(R).is_injective()
True
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> S = R['y']; (y,) = S._first_ngens(1)
>>> S.coerce_map_from(R).is_injective()
True
```

Check that Issue #23203 has been resolved:

```
sage: R.is_subring(S)   # indirect doctest
True
```

```
>>> from sage.all import *
>>> R.is_subring(S)   # indirect doctest
True
```

is_surjective()

Return whether this morphism is surjective.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: R.coerce_map_from(ZZ).is_surjective()
False
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> R.coerce_map_from(ZZ).is_surjective()
False
```

section()

class sage.rings.polynomial.polynomial_element.**Polynomial_generic_dense**

Bases: *Polynomial*

A generic dense polynomial.

EXAMPLES:

```
sage: f = QQ['x']['y'].random_element()
sage: loads(f.dumps()) == f
True
```

```
>>> from sage.all import *
>>> f = QQ['x']['y'].random_element()
>>> loads(f.dumps()) == f
True
```

constant_coefficient()

Return the constant coefficient of this polynomial.

OUTPUT: element of base ring

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: S.<x> = R[]
sage: f = x*t + x + t
sage: f.constant_coefficient()
t
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> S = R['x']; (x,) = S._first_ngens(1)
>>> f = x*t + x + t
>>> f.constant_coefficient()
t
```

degree(*gen=None*)

EXAMPLES:

```
sage: R.<x> = RDF[]
sage: f = (1+2*x^7)^5
sage: f.degree()
35
```

```
>>> from sage.all import *
>>> R = RDF['x']; (x,) = R._first_ngens(1)
>>> f = (Integer(1)+Integer(2)*x**Integer(7))**Integer(5)
>>> f.degree()
35
```

is_term()

Return `True` if this polynomial is a nonzero element of the base ring times a power of the variable.

EXAMPLES:

```
sage: # needs sage.symbolic
sage: R.<x> = SR[]
sage: R(0).is_term()
False
sage: R(1).is_term()
True
sage: (3*x^5).is_term()
True
sage: (1 + 3*x^5).is_term()
False
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> R = SR['x']; (x,) = R._first_ngens(1)
>>> R(Integer(0)).is_term()
False
>>> R(Integer(1)).is_term()
True
>>> (Integer(3)*x**Integer(5)).is_term()
True
>>> (Integer(1) + Integer(3)*x**Integer(5)).is_term()
False
```

list(*copy=True*)

Return a new copy of the list of the underlying elements of `self`.

EXAMPLES:

```
sage: R.<x> = GF(17)[]
sage: f = (1+2*x)^3 + 3*x; f
8*x^3 + 12*x^2 + 9*x + 1
sage: f.list()
[1, 9, 12, 8]
```

```
>>> from sage.all import *
>>> R = GF(Integer(17))['x']; (x,) = R._first_ngens(1)
>>> f = (Integer(1)+Integer(2)*x)**Integer(3) + Integer(3)*x; f
8*x^3 + 12*x^2 + 9*x + 1
>>> f.list()
[1, 9, 12, 8]
```

quo_rem(*other*)

Return the quotient and remainder of the Euclidean division of `self` and `other`.

Raises a `ZeroDivisionError` if `other` is zero. Raises an `ArithmeticError` if the division is not exact.

EXAMPLES:

```
sage: P.<x> = QQ[]
sage: R.<y> = P[]
sage: f = y^10 + R.random_element(9)
sage: g = y^5 + R.random_element(4)
sage: q, r = f.quo_rem(g)
sage: f == q*g + r
```

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```
True
sage: g = x*y^5
sage: f.quo_rem(g)
Traceback (most recent call last):
...
ArithmeticError: division non exact (consider coercing
to polynomials over the fraction field)
sage: g = 0
sage: f.quo_rem(g)
Traceback (most recent call last):
...
ZeroDivisionError: division by zero polynomial
```

```
>>> from sage.all import *
>>> P = QQ['x']; (x,) = P._first_ngens(1)
>>> R = P['y']; (y,) = R._first_ngens(1)
>>> f = y**Integer(10) + R.random_element(Integer(9))
>>> g = y**Integer(5) + R.random_element(Integer(4))
>>> q, r = f.quo_rem(g)
>>> f == q*g + r
True
>>> g = x*y**Integer(5)
>>> f.quo_rem(g)
Traceback (most recent call last):
...
ArithmeticError: division non exact (consider coercing
to polynomials over the fraction field)
>>> g = Integer(0)
>>> f.quo_rem(g)
Traceback (most recent call last):
...
ZeroDivisionError: division by zero polynomial
```

Polynomials over noncommutative rings are also allowed (after Issue #34733):

```
sage: # needs sage.combinat sage.modules
sage: HH = QuaternionAlgebra(QQ, -1, -1)
sage: P.<x> = HH[]
sage: f = P.random_element(5)
sage: g = P.random_element((0, 5))
sage: q, r = f.quo_rem(g)
sage: f == q*g + r
True
```

```
>>> from sage.all import *
>>> # needs sage.combinat sage.modules
>>> HH = QuaternionAlgebra(QQ, -Integer(1), -Integer(1))
>>> P = HH['x']; (x,) = P._first_ngens(1)
>>> f = P.random_element(Integer(5))
>>> g = P.random_element((Integer(0), Integer(5)))
>>> q, r = f.quo_rem(g)
>>> f == q*g + r
```

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```
True
```

shift(n)

Return this polynomial multiplied by the power x^n.

If n is negative, terms below x^n will be discarded. Does not change this polynomial.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(PolynomialRing(QQ,'y'), 'x')
sage: p = x^2 + 2*x + 4
sage: type(p)
<class 'sage.rings.polynomial.polynomial_element.Polynomial_generic_dense'>
sage: p.shift(0)
 x^2 + 2*x + 4
sage: p.shift(-1)
 x + 2
sage: p.shift(2)
 x^4 + 2*x^3 + 4*x^2
```

```
>>> from sage.all import *
>>> R = PolynomialRing(PolynomialRing(QQ,'y'), 'x', names=('x',)); (x,) = R._
↪first_ngens(1)
>>> p = x**Integer(2) + Integer(2)*x + Integer(4)
>>> type(p)
<class 'sage.rings.polynomial.polynomial_element.Polynomial_generic_dense'>
>>> p.shift(Integer(0))
 x^2 + 2*x + 4
>>> p.shift(-Integer(1))
 x + 2
>>> p.shift(Integer(2))
 x^4 + 2*x^3 + 4*x^2
```

AUTHORS:

- David Harvey (2006-08-06)

truncate(n)

Return the polynomial of degree $< n$ which is equivalent to `self` modulo x^n.

EXAMPLES:

```
sage: S.<q> = QQ['t']['q']
sage: f = (1 + q^10 + q^11 + q^12).truncate(11); f
q^10 + 1
sage: f = (1 + q^10 + q^100).truncate(50); f
q^10 + 1
sage: f.degree()
10
sage: f = (1 + q^10 + q^100).truncate(500); f
q^100 + q^10 + 1
```

```
>>> from sage.all import *
>>> S = QQ['t']['q']; (q,) = S._first_ngens(1)
```

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```
>>> f = (Integer(1) + q**Integer(10) + q**Integer(11) + q**Integer(12)).
↪truncate(Integer(11)); f
q^10 + 1
>>> f = (Integer(1) + q**Integer(10) + q**Integer(100)).truncate(Integer(50));
↪ f
q^10 + 1
>>> f.degree()
10
>>> f = (Integer(1) + q**Integer(10) + q**Integer(100)).
↪truncate(Integer(500)); f
q^100 + q^10 + 1
```

class sage.rings.polynomial.polynomial_element.**Polynomial_generic_dense_inexact**

Bases: *Polynomial_generic_dense*

A dense polynomial over an inexact ring.

AUTHOR:

- Xavier Caruso (2013-03)

degree(*secure=False*)

INPUT:

- `secure` – boolean (default: `False`)

OUTPUT: the degree of `self`

If `secure` is `True` and the degree of this polynomial is not determined (because the leading coefficient is indistinguishable from 0), an error is raised

If `secure` is `False`, the returned value is the largest n so that the coefficient of x^n does not compare equal to 0.

EXAMPLES:

```
sage: # needs sage.rings.padics
sage: K = Qp(3, 10)
sage: R.<T> = K[]
sage: f = T + 2; f
(1 + O(3^10))*T + 2 + O(3^10)
sage: f.degree()
1
sage: (f - T).degree()
0
sage: (f - T).degree(secure=True)
Traceback (most recent call last):
...
PrecisionError: the leading coefficient is indistinguishable from 0

sage: # needs sage.rings.padics
sage: x = O(3^5)
sage: li = [3^i * x for i in range(0,5)]; li
[O(3^5), O(3^6), O(3^7), O(3^8), O(3^9)]
sage: f = R(li); f
O(3^9)*T^4 + O(3^8)*T^3 + O(3^7)*T^2 + O(3^6)*T + O(3^5)
```

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```
sage: f.degree()
-1
sage: f.degree(secure=True)
Traceback (most recent call last):
...
PrecisionError: the leading coefficient is indistinguishable from 0
```

```
>>> from sage.all import *
>>> # needs sage.rings.padics
>>> K = Qp(Integer(3), Integer(10))
>>> R = K['T']; (T,) = R._first_ngens(1)
>>> f = T + Integer(2); f
(1 + O(3^10))*T + 2 + O(3^10)
>>> f.degree()
1
>>> (f - T).degree()
0
>>> (f - T).degree(secure=True)
Traceback (most recent call last):
...
PrecisionError: the leading coefficient is indistinguishable from 0

>>> # needs sage.rings.padics
>>> x = O(Integer(3)**Integer(5))
>>> li = [Integer(3)**i * x for i in range(Integer(0),Integer(5))]; li
[O(3^5), O(3^6), O(3^7), O(3^8), O(3^9)]
>>> f = R(li); f
O(3^9)*T^4 + O(3^8)*T^3 + O(3^7)*T^2 + O(3^6)*T + O(3^5)
>>> f.degree()
-1
>>> f.degree(secure=True)
Traceback (most recent call last):
...
PrecisionError: the leading coefficient is indistinguishable from 0
```

AUTHOR:

- Xavier Caruso (2013-03)

prec_degree()

Return the largest n so that precision information is stored about the coefficient of x^n.

Always greater than or equal to degree.

EXAMPLES:

```
sage: # needs sage.rings.padics
sage: K = Qp(3, 10)
sage: R.<T> = K[]
sage: f = T + 2; f
(1 + O(3^10))*T + 2 + O(3^10)
sage: f.degree()
1
sage: f.prec_degree()
```

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```
1

sage: g = f - T; g                                                     #␣
↪needs sage.rings.padics
O(3^10)*T + 2 + O(3^10)
sage: g.degree()                                                       #␣
↪needs sage.rings.padics
0
sage: g.prec_degree()                                                  #␣
↪needs sage.rings.padics
1
```

```
>>> from sage.all import *
>>> # needs sage.rings.padics
>>> K = Qp(Integer(3), Integer(10))
>>> R = K['T']; (T,) = R._first_ngens(1)
>>> f = T + Integer(2); f
(1 + O(3^10))*T + 2 + O(3^10)
>>> f.degree()
1
>>> f.prec_degree()
1

>>> g = f - T; g                                                     #␣
↪needs sage.rings.padics
O(3^10)*T + 2 + O(3^10)
>>> g.degree()                                                       #␣
↪needs sage.rings.padics
0
>>> g.prec_degree()                                                  #␣
↪needs sage.rings.padics
1
```

AUTHOR:

- Xavier Caruso (2013-03)

`sage.rings.polynomial.polynomial_element.`**`generic_power_trunc`**`(`*p*, *n*, *prec*`)`

Generic truncated power algorithm.

INPUT:

- `p` – a polynomial
- `n` – integer (of type `sage.rings.integer.Integer`)
- `prec` – a precision (should fit into a C `long`)

`sage.rings.polynomial.polynomial_element.`**`is_Polynomial`**`(`*f*`)`

Return `True` if `f` is of type univariate polynomial.

This function is deprecated.

INPUT:

- `f` – an object

EXAMPLES:

```
sage: from sage.rings.polynomial.polynomial_element import is_Polynomial
sage: R.<x> = ZZ[]
sage: is_Polynomial(x^3 + x + 1)
doctest:...: DeprecationWarning: the function is_Polynomial is deprecated;
use isinstance(x, sage.rings.polynomial.polynomial_element.Polynomial) instead
See https://github.com/sagemath/sage/issues/32709 for details.
True
sage: S.<y> = R[]
sage: f = y^3 + x*y - 3*x; f
y^3 + x*y - 3*x
sage: is_Polynomial(f)
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polynomial_element import is_Polynomial
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> is_Polynomial(x**Integer(3) + x + Integer(1))
doctest:...: DeprecationWarning: the function is_Polynomial is deprecated;
use isinstance(x, sage.rings.polynomial.polynomial_element.Polynomial) instead
See https://github.com/sagemath/sage/issues/32709 for details.
True
>>> S = R['y']; (y,) = S._first_ngens(1)
>>> f = y**Integer(3) + x*y - Integer(3)*x; f
y^3 + x*y - 3*x
>>> is_Polynomial(f)
True
```

However this function does not return `True` for genuine multivariate polynomial type objects or symbolic polynomials, since those are not of the same data type as univariate polynomials:

```
sage: R.<x,y> = QQ[]
sage: f = y^3 + x*y - 3*x; f
y^3 + x*y - 3*x
sage: is_Polynomial(f)
False

sage: # needs sage.symbolic
sage: var('x,y')
(x, y)
sage: f = y^3 + x*y - 3*x; f
y^3 + x*y - 3*x
sage: is_Polynomial(f)
False
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = y**Integer(3) + x*y - Integer(3)*x; f
y^3 + x*y - 3*x
>>> is_Polynomial(f)
False
```

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```
>>> # needs sage.symbolic
>>> var('x,y')
(x, y)
>>> f = y**Integer(3) + x*y - Integer(3)*x; f
y^3 + x*y - 3*x
>>> is_Polynomial(f)
False
```

sage.rings.polynomial.polynomial_element.**make_generic_polynomial**(*parent*, *coeffs*)

sage.rings.polynomial.polynomial_element.**polynomial_is_variable**(*x*)

Test whether the given polynomial is a variable of its parent ring.

Implemented for instances of *Polynomial* and MPolynomial.

See also

- *sage.rings.polynomial.polynomial_element.Polynomial.is_gen()*
- *sage.rings.polynomial.multi_polynomial.MPolynomial.is_gen()*

EXAMPLES:

```
sage: from sage.rings.polynomial.polynomial_element import polynomial_is_variable
sage: R.<x> = QQ[]
sage: polynomial_is_variable(x)
True
sage: polynomial_is_variable(R([0,1]))
True
sage: polynomial_is_variable(x^2)
False
sage: polynomial_is_variable(R(42))
False
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polynomial_element import polynomial_is_variable
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> polynomial_is_variable(x)
True
>>> polynomial_is_variable(R([Integer(0),Integer(1)]))
True
>>> polynomial_is_variable(x**Integer(2))
False
>>> polynomial_is_variable(R(Integer(42)))
False
```

```
sage: R.<y,z> = QQ[]
sage: polynomial_is_variable(y)
True
sage: polynomial_is_variable(z)
True
```

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```
sage: polynomial_is_variable(y^2)
False
sage: polynomial_is_variable(y+z)
False
sage: polynomial_is_variable(R(42))
False
```

```
>>> from sage.all import *
>>> R = QQ['y, z']; (y, z,) = R._first_ngens(2)
>>> polynomial_is_variable(y)
True
>>> polynomial_is_variable(z)
True
>>> polynomial_is_variable(y**Integer(2))
False
>>> polynomial_is_variable(y+z)
False
>>> polynomial_is_variable(R(Integer(42)))
False
```

```
sage: polynomial_is_variable(42)
False
```

```
>>> from sage.all import *
>>> polynomial_is_variable(Integer(42))
False
```

sage.rings.polynomial.polynomial_element.**universal_discriminant**()

Return the discriminant of the 'universal' univariate polynomial $a_n x^n + \cdots + a_1 x + a_0$ in $\mathbf{Z}[a_0, \ldots, a_n][x]$.

INPUT:

- n – degree of the polynomial

OUTPUT:

The discriminant as a polynomial in $n + 1$ variables over $\mathbf{Z}$. The result will be cached, so subsequent computations of discriminants of the same degree will be faster.

EXAMPLES:

```
sage: # needs sage.libs.pari
sage: from sage.rings.polynomial.polynomial_element import universal_discriminant
sage: universal_discriminant(1)
1
sage: universal_discriminant(2)
a1^2 - 4*a0*a2
sage: universal_discriminant(3)
a1^2*a2^2 - 4*a0*a2^3 - 4*a1^3*a3 + 18*a0*a1*a2*a3 - 27*a0^2*a3^2
sage: universal_discriminant(4).degrees()
(3, 4, 4, 4, 3)
```

```
>>> from sage.all import *
>>> # needs sage.libs.pari
>>> from sage.rings.polynomial.polynomial_element import universal_discriminant
>>> universal_discriminant(Integer(1))
1
>>> universal_discriminant(Integer(2))
a1^2 - 4*a0*a2
>>> universal_discriminant(Integer(3))
a1^2*a2^2 - 4*a0*a2^3 - 4*a1^3*a3 + 18*a0*a1*a2*a3 - 27*a0^2*a3^2
>>> universal_discriminant(Integer(4)).degrees()
(3, 4, 4, 4, 3)
```

See also

`Polynomial.discriminant()`

2.1.4 Univariate Polynomials over domains and fields

AUTHORS:

- William Stein: first version
- Martin Albrecht: Added singular coercion.
- David Harvey: split off polynomial_integer_dense_ntl.pyx (2007-09)
- Robert Bradshaw: split off polynomial_modn_dense_ntl.pyx (2007-09)

class `sage.rings.polynomial.polynomial_element_generic.`**`Polynomial_generic_cdv`**(*parent*, *is_gen=False*, *construct=False*)

Bases: `Polynomial_generic_domain`

A generic class for polynomials over complete discrete valuation domains and fields.

AUTHOR:

- Xavier Caruso (2013-03)

`factor_of_slope`(*slope=None*)

INPUT:

- `slope` – a rational number (default: the first slope in the Newton polygon of `self`)

OUTPUT:

The factor of `self` corresponding to the slope `slope` (i.e. the unique monic divisor of `self` whose slope is `slope` and degree is the length of `slope` in the Newton polygon).

EXAMPLES:

```
sage: # needs sage.geometry.polyhedron sage.rings.padics
sage: K = Qp(5)
sage: R.<x> = K[]
sage: K = Qp(5)
sage: R.<t> = K[]
```

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```
sage: f = 5 + 3*t + t^4 + 25*t^10
sage: f.newton_slopes()
[1, 0, 0, 0, -1/3, -1/3, -1/3, -1/3, -1/3, -1/3]
sage: g = f.factor_of_slope(0)
sage: g.newton_slopes()
[0, 0, 0]
sage: (f % g).is_zero()
True
sage: h = f.factor_of_slope()
sage: h.newton_slopes()
[1]
sage: (f % h).is_zero()
True
```

```
>>> from sage.all import *
>>> # needs sage.geometry.polyhedron sage.rings.padics
>>> K = Qp(Integer(5))
>>> R = K['x']; (x,) = R._first_ngens(1)
>>> K = Qp(Integer(5))
>>> R = K['t']; (t,) = R._first_ngens(1)
>>> f = Integer(5) + Integer(3)*t + t**Integer(4) + Integer(25)*t**Integer(10)
>>> f.newton_slopes()
[1, 0, 0, 0, -1/3, -1/3, -1/3, -1/3, -1/3, -1/3]
>>> g = f.factor_of_slope(Integer(0))
>>> g.newton_slopes()
[0, 0, 0]
>>> (f % g).is_zero()
True
>>> h = f.factor_of_slope()
>>> h.newton_slopes()
[1]
>>> (f % h).is_zero()
True
```

If `slope` is not a slope of `self`, the corresponding factor is 1:

```
sage: f.factor_of_slope(-1)                                                    #␣
↪needs sage.geometry.polyhedron sage.rings.padics
1 + O(5^20)
```

```
>>> from sage.all import *
>>> f.factor_of_slope(-Integer(1))                                              ␣
↪     # needs sage.geometry.polyhedron sage.rings.padics
1 + O(5^20)
```

AUTHOR:

- Xavier Caruso (2013-03-20)

hensel_lift(a)

Lift a to a root of this polynomial (using Newton iteration).

If a is not close enough to a root (so that Newton iteration does not converge), an error is raised.

EXAMPLES:

```
sage: # needs sage.rings.padics
sage: K = Qp(5, 10)
sage: P.<x> = PolynomialRing(K)
sage: f = x^2 + 1
sage: root = f.hensel_lift(2); root
2 + 5 + 2*5^2 + 5^3 + 3*5^4 + 4*5^5 + 2*5^6 + 3*5^7 + 3*5^9 + O(5^10)
sage: f(root)
O(5^10)

sage: g = (x^2 + 1) * (x - 7)                                                  #␣
↪needs sage.rings.padics
sage: g.hensel_lift(2)  # here, 2 is a multiple root modulo p                  #␣
↪needs sage.rings.padics
Traceback (most recent call last):
...
ValueError: a is not close enough to a root of this polynomial
```

```
>>> from sage.all import *
>>> # needs sage.rings.padics
>>> K = Qp(Integer(5), Integer(10))
>>> P = PolynomialRing(K, names=('x',)); (x,) = P._first_ngens(1)
>>> f = x**Integer(2) + Integer(1)
>>> root = f.hensel_lift(Integer(2)); root
2 + 5 + 2*5^2 + 5^3 + 3*5^4 + 4*5^5 + 2*5^6 + 3*5^7 + 3*5^9 + O(5^10)
>>> f(root)
O(5^10)

>>> g = (x**Integer(2) + Integer(1)) * (x - Integer(7))                         ␣
↪                          # needs sage.rings.padics
>>> g.hensel_lift(Integer(2))  # here, 2 is a multiple root modulo p            ␣
↪     # needs sage.rings.padics
Traceback (most recent call last):
...
ValueError: a is not close enough to a root of this polynomial
```

AUTHOR:

- Xavier Caruso (2013-03-23)

newton_polygon()

Return a list of vertices of the Newton polygon of this polynomial.

> **Note**
>
> If some coefficients have not enough precision an error is raised.

EXAMPLES:

```
sage: # needs sage.geometry.polyhedron sage.rings.padics
sage: K = Qp(5)
sage: R.<t> = K[]
```

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```
sage: f = 5 + 3*t + t^4 + 25*t^10
sage: f.newton_polygon()
Finite Newton polygon with 4 vertices: (0, 1), (1, 0), (4, 0), (10, 2)
sage: g = f + K(0,0)*t^4; g
(5^2 + O(5^22))*t^10 + O(5^0)*t^4 + (3 + O(5^20))*t + 5 + O(5^21)
sage: g.newton_polygon()
Traceback (most recent call last):
...
PrecisionError: The coefficient of t^4 has not enough precision
```

```
>>> from sage.all import *
>>> # needs sage.geometry.polyhedron sage.rings.padics
>>> K = Qp(Integer(5))
>>> R = K['t']; (t,) = R._first_ngens(1)
>>> f = Integer(5) + Integer(3)*t + t**Integer(4) + Integer(25)*t**Integer(10)
>>> f.newton_polygon()
Finite Newton polygon with 4 vertices: (0, 1), (1, 0), (4, 0), (10, 2)
>>> g = f + K(Integer(0),Integer(0))*t**Integer(4); g
(5^2 + O(5^22))*t^10 + O(5^0)*t^4 + (3 + O(5^20))*t + 5 + O(5^21)
>>> g.newton_polygon()
Traceback (most recent call last):
...
PrecisionError: The coefficient of t^4 has not enough precision
```

AUTHOR:

- Xavier Caruso (2013-03-20)

newton_slopes(*repetition=True*)

Return a list of the Newton slopes of this polynomial.

These are the valuations of the roots of this polynomial.

If `repetition` is `True`, each slope is repeated a number of times equal to its multiplicity. Otherwise it appears only one time.

EXAMPLES:

```
sage: # needs sage.geometry.polyhedron sage.rings.padics
sage: K = Qp(5)
sage: R.<t> = K[]
sage: f = 5 + 3*t + t^4 + 25*t^10
sage: f.newton_polygon()
Finite Newton polygon with 4 vertices: (0, 1), (1, 0), (4, 0), (10, 2)
sage: f.newton_slopes()
[1, 0, 0, 0, -1/3, -1/3, -1/3, -1/3, -1/3, -1/3]
sage: f.newton_slopes(repetition=False)
[1, 0, -1/3]
```

```
>>> from sage.all import *
>>> # needs sage.geometry.polyhedron sage.rings.padics
>>> K = Qp(Integer(5))
>>> R = K['t']; (t,) = R._first_ngens(1)
>>> f = Integer(5) + Integer(3)*t + t**Integer(4) + Integer(25)*t**Integer(10)
```

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```
>>> f.newton_polygon()
Finite Newton polygon with 4 vertices: (0, 1), (1, 0), (4, 0), (10, 2)
>>> f.newton_slopes()
[1, 0, 0, 0, -1/3, -1/3, -1/3, -1/3, -1/3, -1/3]
>>> f.newton_slopes(repetition=False)
[1, 0, -1/3]
```

AUTHOR:

- Xavier Caruso (2013-03-20)

slope_factorization()

Return a factorization of `self` into a product of factors corresponding to each slope in the Newton polygon.

EXAMPLES:

```
sage: # needs sage.geometry.polyhedron sage.rings.padics
sage: K = Qp(5)
sage: R.<x> = K[]
sage: K = Qp(5)
sage: R.<t> = K[]
sage: f = 5 + 3*t + t^4 + 25*t^10
sage: f.newton_slopes()
[1, 0, 0, 0, -1/3, -1/3, -1/3, -1/3, -1/3, -1/3]
sage: F = f.slope_factorization()
sage: F.prod() == f
True
sage: for (f,_) in F:
....:     print(f.newton_slopes())
[-1/3, -1/3, -1/3, -1/3, -1/3, -1/3]
[0, 0, 0]
[1]
```

```
>>> from sage.all import *
>>> # needs sage.geometry.polyhedron sage.rings.padics
>>> K = Qp(Integer(5))
>>> R = K['x']; (x,) = R._first_ngens(1)
>>> K = Qp(Integer(5))
>>> R = K['t']; (t,) = R._first_ngens(1)
>>> f = Integer(5) + Integer(3)*t + t**Integer(4) + Integer(25)*t**Integer(10)
>>> f.newton_slopes()
[1, 0, 0, 0, -1/3, -1/3, -1/3, -1/3, -1/3, -1/3]
>>> F = f.slope_factorization()
>>> F.prod() == f
True
>>> for (f,_) in F:
...     print(f.newton_slopes())
[-1/3, -1/3, -1/3, -1/3, -1/3, -1/3]
[0, 0, 0]
[1]
```

AUTHOR:

- Xavier Caruso (2013-03-20)

class sage.rings.polynomial.polynomial_element_generic.**Polynomial_generic_cdvf**(*parent*, *is_gen=False*, *construct=False*)

Bases: *Polynomial_generic_cdv*, *Polynomial_generic_field*

class sage.rings.polynomial.polynomial_element_generic.**Polynomial_generic_cdvr**(*parent*, *is_gen=False*, *construct=False*)

Bases: *Polynomial_generic_cdv*

class sage.rings.polynomial.polynomial_element_generic.**Polynomial_generic_dense_cdv**

Bases: *Polynomial_generic_dense_inexact*, *Polynomial_generic_cdv*

class sage.rings.polynomial.polynomial_element_generic.**Polynomial_generic_dense_cdvf**

Bases: *Polynomial_generic_dense_cdv*, *Polynomial_generic_cdvf*

class sage.rings.polynomial.polynomial_element_generic.**Polynomial_generic_dense_cdvr**

Bases: *Polynomial_generic_dense_cdv*, *Polynomial_generic_cdvr*

class sage.rings.polynomial.polynomial_element_generic.**Polynomial_generic_dense_field**(*parent*, *x=None*, *check=True*, *is_gen=False*, *construct=False*)

Bases: *Polynomial_generic_dense*, *Polynomial_generic_field*

class sage.rings.polynomial.polynomial_element_generic.**Polynomial_generic_domain**(*parent*, *is_gen=False*, *construct=False*)

Bases: *Polynomial*, IntegralDomainElement

is_unit()

Return `True` if this polynomial is a unit.

EXERCISE (Atiyah-McDonald, Ch 1): Let $A[x]$ be a polynomial ring in one variable. Then $f = \sum a_i x^i \in A[x]$ is a unit if and only if a_0 is a unit and $a_1, \ldots, a_n$ are nilpotent.

EXAMPLES:

```
sage: R.<z> = PolynomialRing(ZZ, sparse=True)
sage: (2 + z^3).is_unit()
False
sage: f = -1 + 3*z^3; f
3*z^3 - 1
sage: f.is_unit()
False
sage: R(-3).is_unit()
False
sage: R(-1).is_unit()
True
```

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```
sage: R(0).is_unit()
False
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, sparse=True, names=('z',)); (z,) = R._first_
↪ngens(1)
>>> (Integer(2) + z**Integer(3)).is_unit()
False
>>> f = -Integer(1) + Integer(3)*z**Integer(3); f
3*z^3 - 1
>>> f.is_unit()
False
>>> R(-Integer(3)).is_unit()
False
>>> R(-Integer(1)).is_unit()
True
>>> R(Integer(0)).is_unit()
False
```

class sage.rings.polynomial.polynomial_element_generic.**Polynomial_generic_field**(*parent*, *is_gen=False*, *construct=False*)

Bases: *Polynomial_singular_repr*, *Polynomial_generic_domain*, EuclideanDomainElement

quo_rem(*other*)

Return a tuple (quotient, remainder) where self = quotient * other + remainder.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<y> = PolynomialRing(QQ)
sage: K.<t> = NumberField(y^2 - 2)
sage: P.<x> = PolynomialRing(K)
sage: x.quo_rem(K(1))
(x, 0)
sage: x.xgcd(K(1))
(1, 0, 1)
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = PolynomialRing(QQ, names=('y',)); (y,) = R._first_ngens(1)
>>> K = NumberField(y**Integer(2) - Integer(2), names=('t',)); (t,) = K._
↪first_ngens(1)
>>> P = PolynomialRing(K, names=('x',)); (x,) = P._first_ngens(1)
>>> x.quo_rem(K(Integer(1)))
(x, 0)
>>> x.xgcd(K(Integer(1)))
(1, 0, 1)
```

class sage.rings.polynomial.polynomial_element_generic.**Polynomial_generic_sparse**(*parent*, *x=None*, *check=True*, *is_gen=False*, *construct=False*)

Bases: *Polynomial*

A generic sparse polynomial.

The Polynomial_generic_sparse class defines functionality for sparse polynomials over any base ring. A sparse polynomial is represented using a dictionary which maps each exponent to the corresponding coefficient. The coefficients must never be zero.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(PolynomialRing(QQ, 'y'), sparse=True)
sage: f = x^3 - x + 17
sage: type(f)
<class 'sage.rings.polynomial.polynomial_ring.PolynomialRing_integral_domain_with_
↪category.element_class'>
sage: loads(f.dumps()) == f
True
```

```
>>> from sage.all import *
>>> R = PolynomialRing(PolynomialRing(QQ, 'y'), sparse=True, names=('x',)); (x,)␣
↪= R._first_ngens(1)
>>> f = x**Integer(3) - x + Integer(17)
>>> type(f)
<class 'sage.rings.polynomial.polynomial_ring.PolynomialRing_integral_domain_with_
↪category.element_class'>
>>> loads(f.dumps()) == f
True
```

A more extensive example:

```
sage: # needs sage.libs.pari
sage: A.<T> = PolynomialRing(Integers(5), sparse=True)
sage: f = T^2 + 1; B = A.quo(f)
sage: C.<s> = PolynomialRing(B)
sage: C
Univariate Polynomial Ring in s over Univariate Quotient Polynomial Ring in Tbar
 over Ring of integers modulo 5 with modulus T^2 + 1
sage: s + T
s + Tbar
sage: (s + T)**2
s^2 + 2*Tbar*s + 4
```

```
>>> from sage.all import *
>>> # needs sage.libs.pari
>>> A = PolynomialRing(Integers(Integer(5)), sparse=True, names=('T',)); (T,) = A.
↪_first_ngens(1)
>>> f = T**Integer(2) + Integer(1); B = A.quo(f)
>>> C = PolynomialRing(B, names=('s',)); (s,) = C._first_ngens(1)
```

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```
>>> C
Univariate Polynomial Ring in s over Univariate Quotient Polynomial Ring in Tbar
 over Ring of integers modulo 5 with modulus T^2 + 1
>>> s + T
s + Tbar
>>> (s + T)**Integer(2)
s^2 + 2*Tbar*s + 4
```

coefficients(*sparse=True*)

Return the coefficients of the monomials appearing in `self`.

EXAMPLES:

```
sage: R.<w> = PolynomialRing(Integers(8), sparse=True)
sage: f = 5 + w^1997 - w^10000; f
7*w^10000 + w^1997 + 5
sage: f.coefficients()
[5, 1, 7]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(8)), sparse=True, names=('w',)); (w,)␣
↪= R._first_ngens(1)
>>> f = Integer(5) + w**Integer(1997) - w**Integer(10000); f
7*w^10000 + w^1997 + 5
>>> f.coefficients()
[5, 1, 7]
```

degree(*gen=None*)

Return the degree of this sparse polynomial.

EXAMPLES:

```
sage: R.<z> = PolynomialRing(ZZ, sparse=True)
sage: f = 13*z^50000 + 15*z^2 + 17*z
sage: f.degree()
50000
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, sparse=True, names=('z',)); (z,) = R._first_
↪ngens(1)
>>> f = Integer(13)*z**Integer(50000) + Integer(15)*z**Integer(2) +␣
↪Integer(17)*z
>>> f.degree()
50000
```

dict(*copy=None*)

Return a new copy of the dict of the underlying elements of `self`.

EXAMPLES:

```
sage: R.<w> = PolynomialRing(Integers(8), sparse=True)
sage: f = 5 + w^1997 - w^10000; f
7*w^10000 + w^1997 + 5
```

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```
sage: d = f.monomial_coefficients(); d
{0: 5, 1997: 1, 10000: 7}
sage: d[0] = 10
sage: f.monomial_coefficients()
{0: 5, 1997: 1, 10000: 7}
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(8)), sparse=True, names=('w',)); (w,)␣
↪= R._first_ngens(1)
>>> f = Integer(5) + w**Integer(1997) - w**Integer(10000); f
7*w^10000 + w^1997 + 5
>>> d = f.monomial_coefficients(); d
{0: 5, 1997: 1, 10000: 7}
>>> d[Integer(0)] = Integer(10)
>>> f.monomial_coefficients()
{0: 5, 1997: 1, 10000: 7}
```

`dict` is an alias:

```
sage: f.dict()
{0: 5, 1997: 1, 10000: 7}
```

```
>>> from sage.all import *
>>> f.dict()
{0: 5, 1997: 1, 10000: 7}
```

exponents()

Return the exponents of the monomials appearing in `self`.

EXAMPLES:

```
sage: R.<w> = PolynomialRing(Integers(8), sparse=True)
sage: f = 5 + w^1997 - w^10000; f
7*w^10000 + w^1997 + 5
sage: f.exponents()
[0, 1997, 10000]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(8)), sparse=True, names=('w',)); (w,)␣
↪= R._first_ngens(1)
>>> f = Integer(5) + w**Integer(1997) - w**Integer(10000); f
7*w^10000 + w^1997 + 5
>>> f.exponents()
[0, 1997, 10000]
```

gcd(*other*, *algorithm=None*)

Return the gcd of this polynomial and `other`.

INPUT:

- `other` – a polynomial defined over the same ring as this polynomial

ALGORITHM:

Two algorithms are provided:

- `'generic'` – uses the generic implementation, which depends on the base ring being a UFD or a field
- `'dense'` – the polynomials are converted to the dense representation, their gcd is computed and is converted back to the sparse representation

Default is `'dense'` for polynomials over **Z** and `'generic'` in the other cases.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ, sparse=True)
sage: p = x^6 + 7*x^5 + 8*x^4 + 6*x^3 + 2*x^2 + x + 2
sage: q = 2*x^4 - x^3 - 2*x^2 - 4*x - 1
sage: gcd(p, q)
x^2 + x + 1
sage: gcd(p, q, algorithm='dense')
x^2 + x + 1
sage: gcd(p, q, algorithm='generic')
x^2 + x + 1
sage: gcd(p, q, algorithm='foobar')
Traceback (most recent call last):
...
ValueError: Unknown algorithm 'foobar'
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, sparse=True, names=('x',)); (x,) = R._first_
↪ngens(1)
>>> p = x**Integer(6) + Integer(7)*x**Integer(5) + Integer(8)*x**Integer(4) +␣
↪Integer(6)*x**Integer(3) + Integer(2)*x**Integer(2) + x + Integer(2)
>>> q = Integer(2)*x**Integer(4) - x**Integer(3) - Integer(2)*x**Integer(2) -␣
↪Integer(4)*x - Integer(1)
>>> gcd(p, q)
x^2 + x + 1
>>> gcd(p, q, algorithm='dense')
x^2 + x + 1
>>> gcd(p, q, algorithm='generic')
x^2 + x + 1
>>> gcd(p, q, algorithm='foobar')
Traceback (most recent call last):
...
ValueError: Unknown algorithm 'foobar'
```

integral(*var=None*)

Return the integral of this polynomial.

By default, the integration variable is the variable of the polynomial.

Otherwise, the integration variable is the optional parameter `var`

> **Note**
>
> The integral is always chosen so that the constant term is 0.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ, sparse=True)
sage: (1 + 3*x^10 - 2*x^100).integral()
-2/101*x^101 + 3/11*x^11 + x
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, sparse=True, names=('x',)); (x,) = R._first_
↪ngens(1)
>>> (Integer(1) + Integer(3)*x**Integer(10) - Integer(2)*x**Integer(100)).
↪integral()
-2/101*x^101 + 3/11*x^11 + x
```

list(*copy=True*)

Return a new copy of the list of the underlying elements of `self`.

EXAMPLES:

```
sage: R.<z> = PolynomialRing(Integers(100), sparse=True)
sage: f = 13*z^5 + 15*z^2 + 17*z
sage: f.list()
[0, 17, 15, 0, 0, 13]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(100)), sparse=True, names=('z',)); (z,
↪) = R._first_ngens(1)
>>> f = Integer(13)*z**Integer(5) + Integer(15)*z**Integer(2) + Integer(17)*z
>>> f.list()
[0, 17, 15, 0, 0, 13]
```

monomial_coefficients(*copy=None*)

Return a new copy of the dict of the underlying elements of `self`.

EXAMPLES:

```
sage: R.<w> = PolynomialRing(Integers(8), sparse=True)
sage: f = 5 + w^1997 - w^10000; f
7*w^10000 + w^1997 + 5
sage: d = f.monomial_coefficients(); d
{0: 5, 1997: 1, 10000: 7}
sage: d[0] = 10
sage: f.monomial_coefficients()
{0: 5, 1997: 1, 10000: 7}
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(8)), sparse=True, names=('w',)); (w,)␣
↪= R._first_ngens(1)
>>> f = Integer(5) + w**Integer(1997) - w**Integer(10000); f
7*w^10000 + w^1997 + 5
>>> d = f.monomial_coefficients(); d
{0: 5, 1997: 1, 10000: 7}
>>> d[Integer(0)] = Integer(10)
>>> f.monomial_coefficients()
{0: 5, 1997: 1, 10000: 7}
```

`dict` is an alias:

```
sage: f.dict()
{0: 5, 1997: 1, 10000: 7}
```

```
>>> from sage.all import *
>>> f.dict()
{0: 5, 1997: 1, 10000: 7}
```

number_of_terms()

Return the number of nonzero terms.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ, sparse=True)
sage: p = x^100 - 3*x^10 + 12
sage: p.number_of_terms()
3
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, sparse=True, names=('x',)); (x,) = R._first_
↪ngens(1)
>>> p = x**Integer(100) - Integer(3)*x**Integer(10) + Integer(12)
>>> p.number_of_terms()
3
```

quo_rem(*other*)

Return the quotient and remainder of the Euclidean division of `self` and `other`.

Raises `ZeroDivisionError` if `other` is zero.

Raises `ArithmeticError` if `other` has a nonunit leading coefficient and this causes the Euclidean division to fail.

EXAMPLES:

```
sage: P.<x> = PolynomialRing(ZZ, sparse=True)
sage: R.<y> = PolynomialRing(P, sparse=True)
sage: f = R.random_element(10)
sage: while x.divides(f.leading_coefficient()):
....:     f = R.random_element(10)
sage: g = y^5 + R.random_element(4)
sage: q, r = f.quo_rem(g)
sage: f == q*g + r and r.degree() < g.degree()
True
sage: g = x*y^5
sage: f.quo_rem(g)
Traceback (most recent call last):
...
ArithmeticError: Division non exact
(consider coercing to polynomials over the fraction field)
sage: g = 0
sage: f.quo_rem(g)
Traceback (most recent call last):
...
ZeroDivisionError: Division by zero polynomial
```

```
>>> from sage.all import *
>>> P = PolynomialRing(ZZ, sparse=True, names=('x',)); (x,) = P._first_
↪ngens(1)
>>> R = PolynomialRing(P, sparse=True, names=('y',)); (y,) = R._first_ngens(1)
>>> f = R.random_element(Integer(10))
>>> while x.divides(f.leading_coefficient()):
...     f = R.random_element(Integer(10))
>>> g = y**Integer(5) + R.random_element(Integer(4))
>>> q, r = f.quo_rem(g)
>>> f == q*g + r and r.degree() < g.degree()
True
>>> g = x*y**Integer(5)
>>> f.quo_rem(g)
Traceback (most recent call last):
...
ArithmeticError: Division non exact
(consider coercing to polynomials over the fraction field)
>>> g = Integer(0)
>>> f.quo_rem(g)
Traceback (most recent call last):
...
ZeroDivisionError: Division by zero polynomial
```

If the leading coefficient of `other` is not a unit, Euclidean division may still work:

```
sage: f = -x*y^10 + 2*x*y^7 + y^3 - 2*x^2*y^2 - y
sage: g = x*y^5
sage: f.quo_rem(g)
(-y^5 + 2*y^2, y^3 - 2*x^2*y^2 - y)
```

```
>>> from sage.all import *
>>> f = -x*y**Integer(10) + Integer(2)*x*y**Integer(7) + y**Integer(3) -␣
↪Integer(2)*x**Integer(2)*y**Integer(2) - y
>>> g = x*y**Integer(5)
>>> f.quo_rem(g)
(-y^5 + 2*y^2, y^3 - 2*x^2*y^2 - y)
```

Polynomials over noncommutative rings are also allowed:

```
sage: # needs sage.combinat sage.modules
sage: HH = QuaternionAlgebra(QQ, -1, -1)
sage: P.<x> = PolynomialRing(HH, sparse=True)
sage: f = P.random_element(5)
sage: g = P.random_element((0, 5))
sage: q, r = f.quo_rem(g)
sage: f == q*g + r
True
```

```
>>> from sage.all import *
>>> # needs sage.combinat sage.modules
>>> HH = QuaternionAlgebra(QQ, -Integer(1), -Integer(1))
>>> P = PolynomialRing(HH, sparse=True, names=('x',)); (x,) = P._first_
↪ngens(1)
```

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```
>>> f = P.random_element(Integer(5))
>>> g = P.random_element((Integer(0), Integer(5)))
>>> q, r = f.quo_rem(g)
>>> f == q*g + r
True
```

AUTHORS:

- Bruno Grenet (2014-07-09)

reverse(*degree=None*)

Return this polynomial but with the coefficients reversed.

If an optional degree argument is given, the coefficient list will be truncated or zero padded as necessary and the reverse polynomial will have the specified degree.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ, sparse=True)
sage: p = x^4 + 2*x^2^100
sage: p.reverse()
x^1267650600228229401496703205372 + 2
sage: p.reverse(10)
x^6
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, sparse=True, names=('x',)); (x,) = R._first_
↪ngens(1)
>>> p = x**Integer(4) + Integer(2)*x**Integer(2)**Integer(100)
>>> p.reverse()
x^1267650600228229401496703205372 + 2
>>> p.reverse(Integer(10))
x^6
```

shift(*n*)

Return this polynomial multiplied by the power x^n.

If n is negative, terms below x^n will be discarded. Does not change this polynomial.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ, sparse=True)
sage: p = x^100000 + 2*x + 4
sage: type(p)
<class 'sage.rings.polynomial.polynomial_ring.PolynomialRing_integral_domain_
↪with_category.element_class'>
sage: p.shift(0)
 x^100000 + 2*x + 4
sage: p.shift(-1)
 x^99999 + 2
sage: p.shift(-100002)
 0
sage: p.shift(2)
 x^100002 + 2*x^3 + 4*x^2
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, sparse=True, names=('x',)); (x,) = R._first_
↪ngens(1)
>>> p = x**Integer(100000) + Integer(2)*x + Integer(4)
>>> type(p)
<class 'sage.rings.polynomial.polynomial_ring.PolynomialRing_integral_domain_
↪with_category.element_class'>
>>> p.shift(Integer(0))
 x^100000 + 2*x + 4
>>> p.shift(-Integer(1))
 x^99999 + 2
>>> p.shift(-Integer(100002))
 0
>>> p.shift(Integer(2))
 x^100002 + 2*x^3 + 4*x^2
```

AUTHOR: - David Harvey (2006-08-06)

truncate(*n*)

Return the polynomial of degree $< n$ equal to `self` modulo x^n.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ, sparse=True)
sage: (x^11 + x^10 + 1).truncate(11)
x^10 + 1
sage: (x^2^500 + x^2^100 + 1).truncate(2^101)
x^1267650600228229401496703205376 + 1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, sparse=True, names=('x',)); (x,) = R._first_
↪ngens(1)
>>> (x**Integer(11) + x**Integer(10) + Integer(1)).truncate(Integer(11))
x^10 + 1
>>> (x**Integer(2)**Integer(500) + x**Integer(2)**Integer(100) + Integer(1)).
↪truncate(Integer(2)**Integer(101))
x^1267650600228229401496703205376 + 1
```

valuation(*p=None*)

Return the valuation of `self`.

EXAMPLES:

```
sage: # needs sage.rings.finite_rings
sage: R.<w> = PolynomialRing(GF(9, 'a'), sparse=True)
sage: f = w^1997 - w^10000
sage: f.valuation()
1997
sage: R(19).valuation()
0
sage: R(0).valuation()
+Infinity
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> R = PolynomialRing(GF(Integer(9), 'a'), sparse=True, names=('w',)); (w,)
↪= R._first_ngens(1)
>>> f = w**Integer(1997) - w**Integer(10000)
>>> f.valuation()
1997
>>> R(Integer(19)).valuation()
0
>>> R(Integer(0)).valuation()
+Infinity
```

class sage.rings.polynomial.polynomial_element_generic.**Polynomial_generic_sparse_cdv**(*parent*, *x=None*, *check=True*, *is_gen=False*, *construct=False*)

Bases: *Polynomial_generic_sparse*, *Polynomial_generic_cdv*

class sage.rings.polynomial.polynomial_element_generic.**Polynomial_generic_sparse_cdvf**(*parent*, *x=None*, *check=True*, *is_gen=False*, *construct=False*)

Bases: *Polynomial_generic_sparse_cdv*, *Polynomial_generic_cdvf*

class sage.rings.polynomial.polynomial_element_generic.**Polynomial_generic_sparse_cdvr**(*parent*, *x=None*, *check=True*, *is_gen=False*, *construct=False*)

Bases: *Polynomial_generic_sparse_cdv*, *Polynomial_generic_cdvr*

class sage.rings.polynomial.polynomial_element_generic.**Polynomial_generic_sparse_field**(*parent*, *x=None*, *check=True*, *is_gen=False*, *construct=False*)

Bases: *Polynomial_generic_sparse*, *Polynomial_generic_field*

EXAMPLES:

```
sage: R.<x> = PolynomialRing(Frac(RR['t']), sparse=True)
sage: f = x^3 - x + 17
sage: type(f)
<class 'sage.rings.polynomial.polynomial_ring.PolynomialRing_field_with_category.
```

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```
↪element_class'>
sage: loads(f.dumps()) == f
True
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Frac(RR['t']), sparse=True, names=('x',)); (x,) = R._first_
↪ngens(1)
>>> f = x**Integer(3) - x + Integer(17)
>>> type(f)
<class 'sage.rings.polynomial.polynomial_ring.PolynomialRing_field_with_category.
↪element_class'>
>>> loads(f.dumps()) == f
True
```

2.1.5 Univariate Polynomials over GF(2) via NTL's GF2X

AUTHOR: - Martin Albrecht (2008-10) initial implementation

sage.rings.polynomial.polynomial_gf2x.**GF2X_BuildIrred_list**(*n*)

Return the list of coefficients of the lexicographically smallest irreducible polynomial of degree n over the field of 2 elements.

EXAMPLES:

```
sage: from sage.rings.polynomial.polynomial_gf2x import GF2X_BuildIrred_list
sage: GF2X_BuildIrred_list(2)
[1, 1, 1]
sage: GF2X_BuildIrred_list(3)
[1, 1, 0, 1]
sage: GF2X_BuildIrred_list(4)
[1, 1, 0, 0, 1]
sage: GF(2)['x'](GF2X_BuildIrred_list(33))
x^33 + x^6 + x^3 + x + 1
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polynomial_gf2x import GF2X_BuildIrred_list
>>> GF2X_BuildIrred_list(Integer(2))
[1, 1, 1]
>>> GF2X_BuildIrred_list(Integer(3))
[1, 1, 0, 1]
>>> GF2X_BuildIrred_list(Integer(4))
[1, 1, 0, 0, 1]
>>> GF(Integer(2))['x'](GF2X_BuildIrred_list(Integer(33)))
x^33 + x^6 + x^3 + x + 1
```

sage.rings.polynomial.polynomial_gf2x.**GF2X_BuildRandomIrred_list**(*n*)

Return the list of coefficients of an irreducible polynomial of degree n of minimal weight over the field of 2 elements.

EXAMPLES:

```
sage: from sage.rings.polynomial.polynomial_gf2x import GF2X_BuildRandomIrred_list
sage: GF2X_BuildRandomIrred_list(2)
[1, 1, 1]
sage: GF2X_BuildRandomIrred_list(3) in [[1, 1, 0, 1], [1, 0, 1, 1]]
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polynomial_gf2x import GF2X_BuildRandomIrred_list
>>> GF2X_BuildRandomIrred_list(Integer(2))
[1, 1, 1]
>>> GF2X_BuildRandomIrred_list(Integer(3)) in [[Integer(1), Integer(1),␣
↪Integer(0), Integer(1)], [Integer(1), Integer(0), Integer(1), Integer(1)]]
True
```

sage.rings.polynomial.polynomial_gf2x.**GF2X_BuildSparseIrred_list**(*n*)

Return the list of coefficients of an irreducible polynomial of degree n of minimal weight over the field of 2 elements.

EXAMPLES:

```
sage: from sage.rings.polynomial.polynomial_gf2x import GF2X_BuildIrred_list,␣
↪GF2X_BuildSparseIrred_list
sage: all([GF2X_BuildSparseIrred_list(n) == GF2X_BuildIrred_list(n)
....:      for n in range(1,33)])
True
sage: GF(2)['x'](GF2X_BuildSparseIrred_list(33))
x^33 + x^10 + 1
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polynomial_gf2x import GF2X_BuildIrred_list, GF2X_
↪BuildSparseIrred_list
>>> all([GF2X_BuildSparseIrred_list(n) == GF2X_BuildIrred_list(n)
...      for n in range(Integer(1),Integer(33))])
True
>>> GF(Integer(2))['x'](GF2X_BuildSparseIrred_list(Integer(33)))
x^33 + x^10 + 1
```

class sage.rings.polynomial.polynomial_gf2x.**Polynomial_GF2X**

Bases: *Polynomial_template*

Univariate Polynomials over $\mathbf{F}_2$ via NTL's GF2X.

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: x^3 + x^2 + 1
x^3 + x^2 + 1
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> x**Integer(3) + x**Integer(2) + Integer(1)
x^3 + x^2 + 1
```

compose_mod(*g*, *h*, *algorithm=None*)

Compute $f(g) \pmod{h}$.

Both implementations use Brent-Kung's Algorithm 2.1 (*Fast Algorithms for Manipulation of Formal Power Series*, JACM 1978).

INPUT:

- `g` – a polynomial
- `h` – a polynomial
- `algorithm` – either `'native'` or `'ntl'` (default: `'native'`)

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: r = 279
sage: f = x^r + x +1
sage: g = x^r
sage: g.modular_composition(g, f) == g(g) % f
True

sage: P.<x> = GF(2)[]
sage: f = x^29 + x^24 + x^22 + x^21 + x^20 + x^16 + x^15 + x^14 + x^10 + x^9␣
↪+ x^8 + x^7 + x^6 + x^5 + x^2
sage: g = x^31 + x^30 + x^28 + x^26 + x^24 + x^21 + x^19 + x^18 + x^11 + x^10␣
↪+ x^9 + x^8 + x^5 + x^2 + 1
sage: h = x^30 + x^28 + x^26 + x^25 + x^24 + x^22 + x^21 + x^18 + x^17 + x^15␣
↪+ x^13 + x^12 + x^11 + x^10 + x^9 + x^4
sage: f.modular_composition(g, h) == f(g) % h
True
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> r = Integer(279)
>>> f = x**r + x +Integer(1)
>>> g = x**r
>>> g.modular_composition(g, f) == g(g) % f
True

>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> f = x**Integer(29) + x**Integer(24) + x**Integer(22) + x**Integer(21) +␣
↪x**Integer(20) + x**Integer(16) + x**Integer(15) + x**Integer(14) +␣
↪x**Integer(10) + x**Integer(9) + x**Integer(8) + x**Integer(7) +␣
↪x**Integer(6) + x**Integer(5) + x**Integer(2)
>>> g = x**Integer(31) + x**Integer(30) + x**Integer(28) + x**Integer(26) +␣
↪x**Integer(24) + x**Integer(21) + x**Integer(19) + x**Integer(18) +␣
↪x**Integer(11) + x**Integer(10) + x**Integer(9) + x**Integer(8) +␣
↪x**Integer(5) + x**Integer(2) + Integer(1)
>>> h = x**Integer(30) + x**Integer(28) + x**Integer(26) + x**Integer(25) +␣
↪x**Integer(24) + x**Integer(22) + x**Integer(21) + x**Integer(18) +␣
↪x**Integer(17) + x**Integer(15) + x**Integer(13) + x**Integer(12) +␣
↪x**Integer(11) + x**Integer(10) + x**Integer(9) + x**Integer(4)
>>> f.modular_composition(g, h) == f(g) % h
True
```

AUTHORS:

- Paul Zimmermann (2008-10) initial implementation
- Martin Albrecht (2008-10) performance improvements

is_irreducible()

Return whether this polynomial is irreducible over $\mathbf{F}_2$.

EXAMPLES:

```
sage: R.<x> = GF(2)[]
sage: (x^2 + 1).is_irreducible()
False
sage: (x^3 + x + 1).is_irreducible()
True
```

```
>>> from sage.all import *
>>> R = GF(Integer(2))['x']; (x,) = R._first_ngens(1)
>>> (x**Integer(2) + Integer(1)).is_irreducible()
False
>>> (x**Integer(3) + x + Integer(1)).is_irreducible()
True
```

Test that caching works:

```
sage: R.<x> = GF(2)[]
sage: f = x^2 + 1
sage: f.is_irreducible()
False
sage: f.is_irreducible.cache
False
```

```
>>> from sage.all import *
>>> R = GF(Integer(2))['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(2) + Integer(1)
>>> f.is_irreducible()
False
>>> f.is_irreducible.cache
False
```

modular_composition(*g*, *h*, *algorithm=None*)

Compute $f(g) \pmod h$.

Both implementations use Brent-Kung's Algorithm 2.1 (*Fast Algorithms for Manipulation of Formal Power Series*, JACM 1978).

INPUT:

- `g` – a polynomial
- `h` – a polynomial
- `algorithm` – either `'native'` or `'ntl'` (default: `'native'`)

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: r = 279
sage: f = x^r + x +1
sage: g = x^r
sage: g.modular_composition(g, f) == g(g) % f
True

sage: P.<x> = GF(2)[]
sage: f = x^29 + x^24 + x^22 + x^21 + x^20 + x^16 + x^15 + x^14 + x^10 + x^9
↪+ x^8 + x^7 + x^6 + x^5 + x^2
sage: g = x^31 + x^30 + x^28 + x^26 + x^24 + x^21 + x^19 + x^18 + x^11 + x^10
↪+ x^9 + x^8 + x^5 + x^2 + 1
sage: h = x^30 + x^28 + x^26 + x^25 + x^24 + x^22 + x^21 + x^18 + x^17 + x^15
↪+ x^13 + x^12 + x^11 + x^10 + x^9 + x^4
sage: f.modular_composition(g, h) == f(g) % h
True
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> r = Integer(279)
>>> f = x**r + x +Integer(1)
>>> g = x**r
>>> g.modular_composition(g, f) == g(g) % f
True

>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> f = x**Integer(29) + x**Integer(24) + x**Integer(22) + x**Integer(21) +
↪x**Integer(20) + x**Integer(16) + x**Integer(15) + x**Integer(14) +
↪x**Integer(10) + x**Integer(9) + x**Integer(8) + x**Integer(7) +
↪x**Integer(6) + x**Integer(5) + x**Integer(2)
>>> g = x**Integer(31) + x**Integer(30) + x**Integer(28) + x**Integer(26) +
↪x**Integer(24) + x**Integer(21) + x**Integer(19) + x**Integer(18) +
↪x**Integer(11) + x**Integer(10) + x**Integer(9) + x**Integer(8) +
↪x**Integer(5) + x**Integer(2) + Integer(1)
>>> h = x**Integer(30) + x**Integer(28) + x**Integer(26) + x**Integer(25) +
↪x**Integer(24) + x**Integer(22) + x**Integer(21) + x**Integer(18) +
↪x**Integer(17) + x**Integer(15) + x**Integer(13) + x**Integer(12) +
↪x**Integer(11) + x**Integer(10) + x**Integer(9) + x**Integer(4)
>>> f.modular_composition(g, h) == f(g) % h
True
```

AUTHORS:

- Paul Zimmermann (2008-10) initial implementation
- Martin Albrecht (2008-10) performance improvements

class sage.rings.polynomial.polynomial_gf2x.**Polynomial_template**

Bases: *Polynomial*

Template for interfacing to external C / C++ libraries for implementations of polynomials.

AUTHORS:

- Robert Bradshaw (2008-10): original idea for templating

- Martin Albrecht (2008-10): initial implementation

This file implements a simple templating engine for linking univariate polynomials to their C/C++ library implementations. It requires a 'linkage' file which implements the `celement_` functions (see `sage.libs.ntl.ntl_GF2X_linkage` for an example). Both parts are then plugged together by inclusion of the linkage file when inheriting from this class. See `sage.rings.polynomial.polynomial_gf2x` for an example.

We illustrate the generic glueing using univariate polynomials over $\mathrm{GF}(2)$.

Note

Implementations using this template MUST implement coercion from base ring elements and `get_unsafe()`. See `Polynomial_GF2X` for an example.

degree()

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: x.degree()
1
sage: P(1).degree()
0
sage: P(0).degree()
-1
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> x.degree()
1
>>> P(Integer(1)).degree()
0
>>> P(Integer(0)).degree()
-1
```

gcd(*other*)

Return the greatest common divisor of `self` and `other`.

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: f = x*(x+1)
sage: f.gcd(x+1)
x + 1
sage: f.gcd(x^2)
x
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> f = x*(x+Integer(1))
>>> f.gcd(x+Integer(1))
x + 1
>>> f.gcd(x**Integer(2))
x
```

get_cparent()

is_gen()

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: x.is_gen()
True
sage: (x+1).is_gen()
False
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> x.is_gen()
True
>>> (x+Integer(1)).is_gen()
False
```

is_one()

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: P(1).is_one()
True
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> P(Integer(1)).is_one()
True
```

is_zero()

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: x.is_zero()
False
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> x.is_zero()
False
```

list(*copy=True*)

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: x.list()
[0, 1]
sage: list(x)
[0, 1]
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> x.list()
[0, 1]
>>> list(x)
[0, 1]
```

quo_rem(*right*)

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: f = x^2 + x + 1
sage: f.quo_rem(x + 1)
(x, 1)
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> f = x**Integer(2) + x + Integer(1)
>>> f.quo_rem(x + Integer(1))
(x, 1)
```

shift(*n*)

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: f = x^3 + x^2 + 1
sage: f.shift(1)
x^4 + x^3 + x
sage: f.shift(-1)
x^2 + x
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> f = x**Integer(3) + x**Integer(2) + Integer(1)
>>> f.shift(Integer(1))
x^4 + x^3 + x
>>> f.shift(-Integer(1))
x^2 + x
```

truncate(*n*)

Return this polynomial mod x^n.

EXAMPLES:

```
sage: R.<x> =GF(2)[]
sage: f = sum(x^n for n in range(10)); f
x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1
sage: f.truncate(6)
x^5 + x^4 + x^3 + x^2 + x + 1
```

```
>>> from sage.all import *
>>> R = GF(Integer(2))['x']; (x,) = R._first_ngens(1)
```

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```
>>> f = sum(x**n for n in range(Integer(10))); f
x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1
>>> f.truncate(Integer(6))
x^5 + x^4 + x^3 + x^2 + x + 1
```

If the precision is higher than the degree of the polynomial then the polynomial itself is returned:

```
sage: f.truncate(10) is f
True
```

```
>>> from sage.all import *
>>> f.truncate(Integer(10)) is f
True
```

If the precision is negative, the zero polynomial is returned:

```
sage: f.truncate(-1)
0
```

```
>>> from sage.all import *
>>> f.truncate(-Integer(1))
0
```

xgcd(*other*)

Compute extended gcd of `self` and `other`.

EXAMPLES:

```
sage: P.<x> = GF(7)[]
sage: f = x*(x+1)
sage: f.xgcd(x+1)
(x + 1, 0, 1)
sage: f.xgcd(x^2)
(x, 1, 6)
```

```
>>> from sage.all import *
>>> P = GF(Integer(7))['x']; (x,) = P._first_ngens(1)
>>> f = x*(x+Integer(1))
>>> f.xgcd(x+Integer(1))
(x + 1, 0, 1)
>>> f.xgcd(x**Integer(2))
(x, 1, 6)
```

`sage.rings.polynomial.polynomial_gf2x.`**make_element**(*parent*, *args*)

2.1.6 Univariate polynomials over number fields

AUTHOR:

- Luis Felipe Tabera Alonso (2014-02): initial version.

EXAMPLES:

Define a polynomial over an absolute number field and perform basic operations with them:

```
sage: x = polygen(ZZ, 'x')
sage: N.<a> = NumberField(x^2 - 2)
sage: K.<x> = N[]
sage: f = x - a
sage: g = x^3 - 2*a + 1
sage: f * (x + a)
x^2 - 2
sage: f + g
x^3 + x - 3*a + 1
sage: g // f
x^2 + a*x + 2
sage: g % f
1
sage: factor(x^3 - 2*a*x^2 - 2*x + 4*a)
(x - 2*a) * (x - a) * (x + a)
sage: gcd(f, x - a)
x - a
```

```
>>> from sage.all import *
>>> x = polygen(ZZ, 'x')
>>> N = NumberField(x**Integer(2) - Integer(2), names=('a',)); (a,) = N._first_
↪ngens(1)
>>> K = N['x']; (x,) = K._first_ngens(1)
>>> f = x - a
>>> g = x**Integer(3) - Integer(2)*a + Integer(1)
>>> f * (x + a)
x^2 - 2
>>> f + g
x^3 + x - 3*a + 1
>>> g // f
x^2 + a*x + 2
>>> g % f
1
>>> factor(x**Integer(3) - Integer(2)*a*x**Integer(2) - Integer(2)*x + Integer(4)*a)
(x - 2*a) * (x - a) * (x + a)
>>> gcd(f, x - a)
x - a
```

Polynomials are aware of embeddings of the underlying field:

```
sage: # needs sage.rings.padics
sage: x = polygen(ZZ, 'x')
sage: Q7 = Qp(7)
sage: r1 = Q7(3 + 7 + 2*7^2 + 6*7^3 + 7^4 + 2*7^5 + 7^6 + 2*7^7 + 4*7^8
....:          + 6*7^9 + 6*7^10 + 2*7^11 + 7^12 + 7^13 + 2*7^15 + 7^16 + 7^17
....:          + 4*7^18 + 6*7^19)
sage: N.<b> = NumberField(x^2 - 2, embedding=r1)
sage: K.<t> = N[]
sage: f = t^3 - 2*t + 1
sage: f(r1)
1 + O(7^20)
```

```
>>> from sage.all import *
>>> # needs sage.rings.padics
>>> x = polygen(ZZ, 'x')
>>> Q7 = Qp(Integer(7))
>>> r1 = Q7(Integer(3) + Integer(7) + Integer(2)*Integer(7)**Integer(2) +␣
↪Integer(6)*Integer(7)**Integer(3) + Integer(7)**Integer(4) +␣
↪Integer(2)*Integer(7)**Integer(5) + Integer(7)**Integer(6) +␣
↪Integer(2)*Integer(7)**Integer(7) + Integer(4)*Integer(7)**Integer(8)
...          + Integer(6)*Integer(7)**Integer(9) + Integer(6)*Integer(7)**Integer(10)␣
↪+ Integer(2)*Integer(7)**Integer(11) + Integer(7)**Integer(12) +␣
↪Integer(7)**Integer(13) + Integer(2)*Integer(7)**Integer(15) +␣
↪Integer(7)**Integer(16) + Integer(7)**Integer(17)
...          + Integer(4)*Integer(7)**Integer(18) +␣
↪Integer(6)*Integer(7)**Integer(19))
>>> N = NumberField(x**Integer(2) - Integer(2), embedding=r1, names=('b',)); (b,) = N.
↪_first_ngens(1)
>>> K = N['t']; (t,) = K._first_ngens(1)
>>> f = t**Integer(3) - Integer(2)*t + Integer(1)
>>> f(r1)
1 + O(7^20)
```

We can also construct polynomials over relative number fields:

```
sage: # needs sage.symbolic
sage: N.<i, s2> = QQ[I, sqrt(2)]
sage: K.<x> = N[]
sage: f = x - s2
sage: g = x^3 - 2*i*x^2 + s2*x
sage: f * (x + s2)
x^2 - 2
sage: f + g
x^3 - 2*I*x^2 + (sqrt2 + 1)*x - sqrt2
sage: g // f
x^2 + (-2*I + sqrt2)*x - 2*sqrt2*I + sqrt2 + 2
sage: g % f
-4*I + 2*sqrt2 + 2
sage: factor(i*x^4 - 2*i*x^2 + 9*i)
(I) * (x - I + sqrt2) * (x + I - sqrt2) * (x - I - sqrt2) * (x + I + sqrt2)
sage: gcd(f, x - i)
1
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> N = QQ[I, sqrt(Integer(2))]; (i, s2,) = N._first_ngens(2)
>>> K = N['x']; (x,) = K._first_ngens(1)
>>> f = x - s2
>>> g = x**Integer(3) - Integer(2)*i*x**Integer(2) + s2*x
>>> f * (x + s2)
x^2 - 2
>>> f + g
x^3 - 2*I*x^2 + (sqrt2 + 1)*x - sqrt2
>>> g // f
x^2 + (-2*I + sqrt2)*x - 2*sqrt2*I + sqrt2 + 2
```

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```
>>> g % f
-4*I + 2*sqrt2 + 2
>>> factor(i*x**Integer(4) - Integer(2)*i*x**Integer(2) + Integer(9)*i)
(I) * (x - I + sqrt2) * (x + I - sqrt2) * (x - I - sqrt2) * (x + I + sqrt2)
>>> gcd(f, x - i)
1
```

class sage.rings.polynomial.polynomial_number_field.**Polynomial_absolute_number_field_dense**(*parent*, *x=None*, *check=True*, *is_gen=False*, *construct=False*)

Bases: *Polynomial_generic_dense_field*

Class of dense univariate polynomials over an absolute number field.

gcd(*other*)

Compute the monic gcd of two univariate polynomials using PARI.

INPUT:

- `other` – a polynomial with the same parent as `self`

OUTPUT: the monic gcd of `self` and `other`

EXAMPLES:

```
sage: x = polygen(ZZ, 'x')
sage: N.<a> = NumberField(x^3 - 1/2, 'a')
sage: R.<r> = N['r']
sage: f = (5/4*a^2 - 2*a + 4)*r^2 + (5*a^2 - 81/5*a - 17/2)*r + 4/5*a^2 +␣
↪24*a + 6
sage: g = (5/4*a^2 - 2*a + 4)*r^2 + (-11*a^2 + 79/5*a - 7/2)*r - 4/5*a^2 -␣
↪24*a - 6
sage: gcd(f, g**2)
r - 60808/96625*a^2 - 69936/96625*a - 149212/96625
sage: R = QQ[I]['x']
sage: f = R.random_element(2)
sage: g = f + 1
sage: h = R.random_element(2).monic()
sage: f *= h
sage: g *= h
sage: gcd(f, g) - h
0
sage: f.gcd(g) - h
0
```

```
>>> from sage.all import *
>>> x = polygen(ZZ, 'x')
>>> N = NumberField(x**Integer(3) - Integer(1)/Integer(2), 'a', names=('a',));
↪ (a,) = N._first_ngens(1)
>>> R = N['r']; (r,) = R._first_ngens(1)
```

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```
>>> f = (Integer(5)/Integer(4)*a**Integer(2) - Integer(2)*a +
↪Integer(4))*r**Integer(2) + (Integer(5)*a**Integer(2) - Integer(81)/
↪Integer(5)*a - Integer(17)/Integer(2))*r + Integer(4)/
↪Integer(5)*a**Integer(2) + Integer(24)*a + Integer(6)
>>> g = (Integer(5)/Integer(4)*a**Integer(2) - Integer(2)*a +
↪Integer(4))*r**Integer(2) + (-Integer(11)*a**Integer(2) + Integer(79)/
↪Integer(5)*a - Integer(7)/Integer(2))*r - Integer(4)/
↪Integer(5)*a**Integer(2) - Integer(24)*a - Integer(6)
>>> gcd(f, g**Integer(2))
r - 60808/96625*a^2 - 69936/96625*a - 149212/96625
>>> R = QQ[I]['x']
>>> f = R.random_element(Integer(2))
>>> g = f + Integer(1)
>>> h = R.random_element(Integer(2)).monic()
>>> f *= h
>>> g *= h
>>> gcd(f, g) - h
0
>>> f.gcd(g) - h
0
```

class sage.rings.polynomial.polynomial_number_field.**Polynomial_relative_number_field_dense**(*parent*, *x=None*, *check=True*, *is_gen=False*, *construct=False*)

Bases: *Polynomial_generic_dense_field*

Class of dense univariate polynomials over a relative number field.

gcd(*other*)

Compute the monic gcd of two polynomials.

Currently, the method checks corner cases in which one of the polynomials is zero or a constant. Then, computes an absolute extension and performs the computations there.

INPUT:

- other – a polynomial with the same parent as self

OUTPUT: the monic gcd of self and other

See *Polynomial_absolute_number_field_dense.gcd()* for more details.

EXAMPLES:

```
sage: # needs sage.symbolic
sage: N = QQ[sqrt(2), sqrt(3)]
sage: s2, s3 = N.gens()
sage: x = polygen(N)
sage: f = x^4 - 5*x^2 + 6
sage: g = x^3 + (-2*s2 + s3)*x^2 + (-2*s3*s2 + 2)*x + 2*s3
sage: gcd(f, g)
x^2 + (-sqrt2 + sqrt3)*x - sqrt3*sqrt2
```

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```
sage: f.gcd(g)
x^2 + (-sqrt2 + sqrt3)*x - sqrt3*sqrt2
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> N = QQ[sqrt(Integer(2)), sqrt(Integer(3))]
>>> s2, s3 = N.gens()
>>> x = polygen(N)
>>> f = x**Integer(4) - Integer(5)*x**Integer(2) + Integer(6)
>>> g = x**Integer(3) + (-Integer(2)*s2 + s3)*x**Integer(2) + (-
↪Integer(2)*s3*s2 + Integer(2))*x + Integer(2)*s3
>>> gcd(f, g)
x^2 + (-sqrt2 + sqrt3)*x - sqrt3*sqrt2
>>> f.gcd(g)
x^2 + (-sqrt2 + sqrt3)*x - sqrt3*sqrt2
```

2.1.7 Dense univariate polynomials over $\mathbf{Z}$, implemented using FLINT

AUTHORS:

- David Harvey: rewrote to talk to NTL directly, instead of via ntl.pyx (2007-09); a lot of this was based on Joel Mohler's recent rewrite of the NTL wrapper
- David Harvey: split off from polynomial_element_generic.py (2007-09)
- Burcin Erocal: rewrote to use FLINT (2008-06-16)

class sage.rings.polynomial.polynomial_integer_dense_flint.**Polynomial_integer_dense_flint**

Bases: *Polynomial*

A dense polynomial over the integers, implemented via FLINT.

add(*right*)

Return `self` plus `right`.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ)
sage: f = 2*x + 1
sage: g = -3*x^2 + 6
sage: f + g
-3*x^2 + 2*x + 7
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = Integer(2)*x + Integer(1)
>>> g = -Integer(3)*x**Integer(2) + Integer(6)
>>> f + g
-3*x^2 + 2*x + 7
```

sub(*right*)

Return `self` minus `right`.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ)
sage: f = 2*x + 1
sage: g = -3*x^2 + 6
sage: f - g
3*x^2 + 2*x - 5
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = Integer(2)*x + Integer(1)
>>> g = -Integer(3)*x**Integer(2) + Integer(6)
>>> f - g
3*x^2 + 2*x - 5
```

lmul(*right*)

Return `self` multiplied by `right`, where `right` is a scalar (integer).

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ)
sage: x*3
3*x
sage: (2*x^2 + 4)*3
6*x^2 + 12
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> x*Integer(3)
3*x
>>> (Integer(2)*x**Integer(2) + Integer(4))*Integer(3)
6*x^2 + 12
```

rmul(*right*)

Return `self` multiplied by `right`, where right is a scalar (integer).

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ)
sage: 3*x
3*x
sage: 3*(2*x^2 + 4)
6*x^2 + 12
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> Integer(3)*x
3*x
>>> Integer(3)*(Integer(2)*x**Integer(2) + Integer(4))
6*x^2 + 12
```

mul(*right*)

Return `self` multiplied by `right`.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ)
sage: (x - 2)*(x^2 - 8*x + 16)
x^3 - 10*x^2 + 32*x - 32
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> (x - Integer(2))*(x**Integer(2) - Integer(8)*x + Integer(16))
x^3 - 10*x^2 + 32*x - 32
```

_mul_trunc_(*right*, *n*)

Truncated multiplication.

> **See also**
>
> *_mul_()* for standard multiplication

EXAMPLES:

```
sage: x = polygen(ZZ)
sage: p1 = 1 + x + x^2 + x^4
sage: p2 = -2 + 3*x^2 + 5*x^4
sage: p1._mul_trunc_(p2, 4)
3*x^3 + x^2 - 2*x - 2
sage: (p1*p2).truncate(4)
3*x^3 + x^2 - 2*x - 2
sage: p1._mul_trunc_(p2, 6)
5*x^5 + 6*x^4 + 3*x^3 + x^2 - 2*x - 2
```

```
>>> from sage.all import *
>>> x = polygen(ZZ)
>>> p1 = Integer(1) + x + x**Integer(2) + x**Integer(4)
>>> p2 = -Integer(2) + Integer(3)*x**Integer(2) + Integer(5)*x**Integer(4)
>>> p1._mul_trunc_(p2, Integer(4))
3*x^3 + x^2 - 2*x - 2
>>> (p1*p2).truncate(Integer(4))
3*x^3 + x^2 - 2*x - 2
>>> p1._mul_trunc_(p2, Integer(6))
5*x^5 + 6*x^4 + 3*x^3 + x^2 - 2*x - 2
```

content()

Return the greatest common divisor of the coefficients of this polynomial. The sign is the sign of the leading coefficient. The content of the zero polynomial is zero.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ)
sage: (2*x^2 - 4*x^4 + 14*x^7).content()
2
sage: x.content()
1
sage: R(1).content()
1
```

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```
sage: R(0).content()
0
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> (Integer(2)*x**Integer(2) - Integer(4)*x**Integer(4) +␣
↪Integer(14)*x**Integer(7)).content()
2
>>> x.content()
1
>>> R(Integer(1)).content()
1
>>> R(Integer(0)).content()
0
```

degree(*gen=None*)

Return the degree of this polynomial.

The zero polynomial has degree -1.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ)
sage: x.degree()
1
sage: (x^2).degree()
2
sage: R(1).degree()
0
sage: R(0).degree()
-1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> x.degree()
1
>>> (x**Integer(2)).degree()
2
>>> R(Integer(1)).degree()
0
>>> R(Integer(0)).degree()
-1
```

disc(*proof=True*)

Return the discriminant of `self`, which is by definition

$$(-1)^{m(m-1)/2}\,\mathrm{resultant}(a, a')/\,\mathrm{lc}(a),$$

where $m = \deg(a)$, and $\mathrm{lc}(a)$ is the leading coefficient of a. If `proof` is False (the default is True), then this function may use a randomized strategy that errors with probability no more than 2^{-80}.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: f = 3*x^3 + 2*x + 1
sage: f.discriminant()
-339
sage: f.discriminant(proof=False)
-339
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> f = Integer(3)*x**Integer(3) + Integer(2)*x + Integer(1)
>>> f.discriminant()
-339
>>> f.discriminant(proof=False)
-339
```

discriminant(*proof=True*)

Return the discriminant of `self`, which is by definition

$$(-1)^{m(m-1)/2} \operatorname{resultant}(a, a')/\operatorname{lc}(a),$$

where $m = \deg(a)$, and $\operatorname{lc}(a)$ is the leading coefficient of a. If `proof` is False (the default is True), then this function may use a randomized strategy that errors with probability no more than 2^{-80}.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: f = 3*x^3 + 2*x + 1
sage: f.discriminant()
-339
sage: f.discriminant(proof=False)
-339
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> f = Integer(3)*x**Integer(3) + Integer(2)*x + Integer(1)
>>> f.discriminant()
-339
>>> f.discriminant(proof=False)
-339
```

factor()

This function overrides the generic polynomial factorization to make a somewhat intelligent decision to use PARI or NTL based on some benchmarking.

Note: This function factors the content of the polynomial, which can take very long if it's a really big integer. If you do not need the content factored, divide it out of your polynomial before calling this function.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: f = x^4 - 1
sage: f.factor()
(x - 1) * (x + 1) * (x^2 + 1)
sage: f = 1 - x
```

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```
sage: f.factor()
(-1) * (x - 1)
sage: f.factor().unit()
-1
sage: f = -30*x; f.factor()
(-1) * 2 * 3 * 5 * x
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(4) - Integer(1)
>>> f.factor()
(x - 1) * (x + 1) * (x^2 + 1)
>>> f = Integer(1) - x
>>> f.factor()
(-1) * (x - 1)
>>> f.factor().unit()
-1
>>> f = -Integer(30)*x; f.factor()
(-1) * 2 * 3 * 5 * x
```

factor_mod(*p*)

Return the factorization of `self` modulo the prime p.

INPUT:

- `p` – prime

OUTPUT: factorization of `self` reduced modulo p

EXAMPLES:

```
sage: R.<x> = ZZ['x']
sage: f = -3*x*(x-2)*(x-9) + x
sage: f.factor_mod(3)
x
sage: f = -3 * x * (x - 2) * (x - 9)
sage: f.factor_mod(3)
Traceback (most recent call last):
...
ArithmeticError: factorization of 0 is not defined

sage: f = 2 * x * (x - 2) * (x - 9)
sage: f.factor_mod(7)
(2) * x * (x + 5)^2
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> f = -Integer(3)*x*(x-Integer(2))*(x-Integer(9)) + x
>>> f.factor_mod(Integer(3))
x
>>> f = -Integer(3) * x * (x - Integer(2)) * (x - Integer(9))
>>> f.factor_mod(Integer(3))
Traceback (most recent call last):
```

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```
...
ArithmeticError: factorization of 0 is not defined

>>> f = Integer(2) * x * (x - Integer(2)) * (x - Integer(9))
>>> f.factor_mod(Integer(7))
(2) * x * (x + 5)^2
```

factor_padic(*p*, *prec=10*)

Return p-adic factorization of `self` to given precision.

INPUT:

- `p` – prime
- `prec` – integer; the precision

OUTPUT: factorization of `self` over the completion at p

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ)
sage: f = x^2 + 1
sage: f.factor_padic(5, 4)
((1 + O(5^4))*x + 2 + 5 + 2*5^2 + 5^3 + O(5^4))
* ((1 + O(5^4))*x + 3 + 3*5 + 2*5^2 + 3*5^3 + O(5^4))
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = x**Integer(2) + Integer(1)
>>> f.factor_padic(Integer(5), Integer(4))
((1 + O(5^4))*x + 2 + 5 + 2*5^2 + 5^3 + O(5^4))
* ((1 + O(5^4))*x + 3 + 3*5 + 2*5^2 + 3*5^3 + O(5^4))
```

A more difficult example:

```
sage: f = 100 * (5*x + 1)^2 * (x + 5)^2
sage: f.factor_padic(5, 10)
(4 + O(5^10)) * (5 + O(5^11))^2 * ((1 + O(5^10))*x + 5 + O(5^10))^2
* ((5 + O(5^10))*x + 1 + O(5^10))^2
```

```
>>> from sage.all import *
>>> f = Integer(100) * (Integer(5)*x + Integer(1))**Integer(2) * (x +␣
↪Integer(5))**Integer(2)
>>> f.factor_padic(Integer(5), Integer(10))
(4 + O(5^10)) * (5 + O(5^11))^2 * ((1 + O(5^10))*x + 5 + O(5^10))^2
* ((5 + O(5^10))*x + 1 + O(5^10))^2
```

gcd(*right*)

Return the GCD of `self` and `right`. The leading coefficient need not be 1.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ)
sage: f = (6*x + 47) * (7*x^2 - 2*x + 38)
```

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```
sage: g = (6*x + 47) * (3*x^3 + 2*x + 1)
sage: f.gcd(g)
6*x + 47
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = (Integer(6)*x + Integer(47)) * (Integer(7)*x**Integer(2) -␣
↪Integer(2)*x + Integer(38))
>>> g = (Integer(6)*x + Integer(47)) * (Integer(3)*x**Integer(3) +␣
↪Integer(2)*x + Integer(1))
>>> f.gcd(g)
6*x + 47
```

inverse_series_trunc(*prec*)

Return a polynomial approximation of precision `prec` of the inverse series of this polynomial.

EXAMPLES:

```
sage: x = polygen(ZZ)
sage: p = 1 + x + 2*x^2
sage: q5 = p.inverse_series_trunc(5)
sage: q5
-x^4 + 3*x^3 - x^2 - x + 1
sage: p*q5
-2*x^6 + 5*x^5 + 1

sage: (x-1).inverse_series_trunc(5)
-x^4 - x^3 - x^2 - x - 1

sage: q100 = p.inverse_series_trunc(100)
sage: (q100 * p).truncate(100)
1
```

```
>>> from sage.all import *
>>> x = polygen(ZZ)
>>> p = Integer(1) + x + Integer(2)*x**Integer(2)
>>> q5 = p.inverse_series_trunc(Integer(5))
>>> q5
-x^4 + 3*x^3 - x^2 - x + 1
>>> p*q5
-2*x^6 + 5*x^5 + 1

>>> (x-Integer(1)).inverse_series_trunc(Integer(5))
-x^4 - x^3 - x^2 - x - 1

>>> q100 = p.inverse_series_trunc(Integer(100))
>>> (q100 * p).truncate(Integer(100))
1
```

is_one()

Return `True` if `self` is equal to one.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: R(0).is_one()
False
sage: R(1).is_one()
True
sage: x.is_one()
False
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> R(Integer(0)).is_one()
False
>>> R(Integer(1)).is_one()
True
>>> x.is_one()
False
```

is_zero()

Return `True` if `self` is equal to zero.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: R(0).is_zero()
True
sage: R(1).is_zero()
False
sage: x.is_zero()
False
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> R(Integer(0)).is_zero()
True
>>> R(Integer(1)).is_zero()
False
>>> x.is_zero()
False
```

lcm(*right*)

Return the LCM of `self` and `right`.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ)
sage: f = (6*x + 47) * (7*x^2 - 2*x + 38)
sage: g = (6*x + 47) * (3*x^3 + 2*x + 1)
sage: h = f.lcm(g); h
126*x^6 + 951*x^5 + 486*x^4 + 6034*x^3 + 585*x^2 + 3706*x + 1786
sage: h == (6*x + 47) * (7*x^2 - 2*x + 38) * (3*x^3 + 2*x + 1)
True
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = (Integer(6)*x + Integer(47)) * (Integer(7)*x**Integer(2) -
↪Integer(2)*x + Integer(38))
>>> g = (Integer(6)*x + Integer(47)) * (Integer(3)*x**Integer(3) +
↪Integer(2)*x + Integer(1))
>>> h = f.lcm(g); h
126*x^6 + 951*x^5 + 486*x^4 + 6034*x^3 + 585*x^2 + 3706*x + 1786
>>> h == (Integer(6)*x + Integer(47)) * (Integer(7)*x**Integer(2) -
↪Integer(2)*x + Integer(38)) * (Integer(3)*x**Integer(3) + Integer(2)*x +
↪Integer(1))
True
```

list(*copy=True*)

Return a new copy of the list of the underlying elements of `self`.

EXAMPLES:

```
sage: x = PolynomialRing(ZZ,'x').0
sage: f = x^3 + 3*x - 17
sage: f.list()
[-17, 3, 0, 1]
sage: f = PolynomialRing(ZZ,'x')(0)
sage: f.list()
[]
```

```
>>> from sage.all import *
>>> x = PolynomialRing(ZZ,'x').gen(0)
>>> f = x**Integer(3) + Integer(3)*x - Integer(17)
>>> f.list()
[-17, 3, 0, 1]
>>> f = PolynomialRing(ZZ,'x')(Integer(0))
>>> f.list()
[]
```

pseudo_divrem(*B*)

Write `A = self`. This function computes polynomials Q and R and an integer d such that

$$\mathrm{lead}(B)^d A = BQ + R$$

where R has degree less than that of B.

INPUT:

- `B` – a polynomial over $\mathbf{Z}$

OUTPUT:

- `Q`, `R` – polynomials
- `d` – nonnegative integer

EXAMPLES:

```
sage: R.<x> = ZZ['x']
sage: A = R(range(10))
```

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```
sage: B = 3*R([-1, 0, 1])
sage: Q, R, d = A.pseudo_divrem(B)
sage: Q, R, d
(9*x^7 + 8*x^6 + 16*x^5 + 14*x^4 + 21*x^3 + 18*x^2 + 24*x + 20, 75*x + 60, 1)
sage: B.leading_coefficient()^d * A == B*Q + R
True
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> A = R(range(Integer(10)))
>>> B = Integer(3)*R([-Integer(1), Integer(0), Integer(1)])
>>> Q, R, d = A.pseudo_divrem(B)
>>> Q, R, d
(9*x^7 + 8*x^6 + 16*x^5 + 14*x^4 + 21*x^3 + 18*x^2 + 24*x + 20, 75*x + 60, 1)
>>> B.leading_coefficient()**d * A == B*Q + R
True
```

quo_rem(*right*)

Attempts to divide `self` by `right`, and return a quotient and remainder.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ)
sage: f = R(range(10)); g = R([-1, 0, 1])
sage: q, r = f.quo_rem(g)
sage: q, r
(9*x^7 + 8*x^6 + 16*x^5 + 14*x^4 + 21*x^3 + 18*x^2 + 24*x + 20, 25*x + 20)
sage: q*g + r == f
True

sage: f = x^2
sage: f.quo_rem(0)
Traceback (most recent call last):
...
ZeroDivisionError: division by zero polynomial

sage: f = (x^2 + 3) * (2*x - 1)
sage: f.quo_rem(2*x - 1)
(x^2 + 3, 0)

sage: f = x^2
sage: f.quo_rem(2*x - 1)
(0, x^2)
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = R(range(Integer(10))); g = R([-Integer(1), Integer(0), Integer(1)])
>>> q, r = f.quo_rem(g)
>>> q, r
(9*x^7 + 8*x^6 + 16*x^5 + 14*x^4 + 21*x^3 + 18*x^2 + 24*x + 20, 25*x + 20)
>>> q*g + r == f
True
```

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```
>>> f = x**Integer(2)
>>> f.quo_rem(Integer(0))
Traceback (most recent call last):
...
ZeroDivisionError: division by zero polynomial

>>> f = (x**Integer(2) + Integer(3)) * (Integer(2)*x - Integer(1))
>>> f.quo_rem(Integer(2)*x - Integer(1))
(x^2 + 3, 0)

>>> f = x**Integer(2)
>>> f.quo_rem(Integer(2)*x - Integer(1))
(0, x^2)
```

real_root_intervals()

Return isolating intervals for the real roots of this polynomial.

EXAMPLES: We compute the roots of the characteristic polynomial of some Salem numbers:

```
sage: R.<x> = PolynomialRing(ZZ)
sage: f = 1 - x^2 - x^3 - x^4 + x^6
sage: f.real_root_intervals()                                        #␣
↪needs sage.libs.linbox
[((1/2, 3/4), 1), ((1, 3/2), 1)]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = Integer(1) - x**Integer(2) - x**Integer(3) - x**Integer(4) +␣
↪x**Integer(6)
>>> f.real_root_intervals()                                        #␣
↪needs sage.libs.linbox
[((1/2, 3/4), 1), ((1, 3/2), 1)]
```

resultant(*other*, *proof=True*)

Return the resultant of `self` and `other`, which must lie in the same polynomial ring.

If `proof=False` (the default is `proof=True`), then this function may use a randomized strategy that errors with probability no more than 2^{-80}.

INPUT:

- `other` – a polynomial

OUTPUT: an element of the base ring of the polynomial ring

EXAMPLES:

```
sage: x = PolynomialRing(ZZ,'x').0
sage: f = x^3 + x + 1;  g = x^3 - x - 1
sage: r = f.resultant(g); r
-8
sage: r.parent() is ZZ
True
```

```
>>> from sage.all import *
>>> x = PolynomialRing(ZZ,'x').gen(0)
>>> f = x**Integer(3) + x + Integer(1);  g = x**Integer(3) - x - Integer(1)
>>> r = f.resultant(g); r
-8
>>> r.parent() is ZZ
True
```

reverse(*degree=None*)

Return a polynomial with the coefficients of this polynomial reversed.

If an optional degree argument is given the coefficient list will be truncated or zero padded as necessary before computing the reverse.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: p = R([1,2,3,4]); p
4*x^3 + 3*x^2 + 2*x + 1
sage: p.reverse()
x^3 + 2*x^2 + 3*x + 4
sage: p.reverse(degree=6)
x^6 + 2*x^5 + 3*x^4 + 4*x^3
sage: p.reverse(degree=2)
x^2 + 2*x + 3
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> p = R([Integer(1),Integer(2),Integer(3),Integer(4)]); p
4*x^3 + 3*x^2 + 2*x + 1
>>> p.reverse()
x^3 + 2*x^2 + 3*x + 4
>>> p.reverse(degree=Integer(6))
x^6 + 2*x^5 + 3*x^4 + 4*x^3
>>> p.reverse(degree=Integer(2))
x^2 + 2*x + 3
```

revert_series(*n*)

Return a polynomial f such that $f($ `self` $(x)) =$ `self` $(f(x)) = x$ (mod x^n).

EXAMPLES:

```
sage: R.<t> = ZZ[]
sage: f = t - t^3 + t^5
sage: f.revert_series(6)
2*t^5 + t^3 + t

sage: f.revert_series(-1)
Traceback (most recent call last):
...
ValueError: argument n must be a nonnegative integer, got -1

sage: g = - t^3 + t^5
sage: g.revert_series(6)
```

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```
Traceback (most recent call last):
...
ValueError: self must have constant coefficient 0 and a unit for coefficient␣
↪t^1
```

```
>>> from sage.all import *
>>> R = ZZ['t']; (t,) = R._first_ngens(1)
>>> f = t - t**Integer(3) + t**Integer(5)
>>> f.revert_series(Integer(6))
2*t^5 + t^3 + t

>>> f.revert_series(-Integer(1))
Traceback (most recent call last):
...
ValueError: argument n must be a nonnegative integer, got -1

>>> g = - t**Integer(3) + t**Integer(5)
>>> g.revert_series(Integer(6))
Traceback (most recent call last):
...
ValueError: self must have constant coefficient 0 and a unit for coefficient␣
↪t^1
```

squarefree_decomposition()

Return the square-free decomposition of `self`. This is a partial factorization of `self` into square-free, relatively prime polynomials.

This is a wrapper for the NTL function `SquareFreeDecomp`.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ)
sage: p = (x-1)^2 * (x-2)^2 * (x-3)^3 * (x-4)
sage: p.squarefree_decomposition()
(x - 4) * (x^2 - 3*x + 2)^2 * (x - 3)^3
sage: p = 37 * (x-1)^2 * (x-2)^2 * (x-3)^3 * (x-4)
sage: p.squarefree_decomposition()
(37) * (x - 4) * (x^2 - 3*x + 2)^2 * (x - 3)^3
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> p = (x-Integer(1))**Integer(2) * (x-Integer(2))**Integer(2) * (x-
↪Integer(3))**Integer(3) * (x-Integer(4))
>>> p.squarefree_decomposition()
(x - 4) * (x^2 - 3*x + 2)^2 * (x - 3)^3
>>> p = Integer(37) * (x-Integer(1))**Integer(2) * (x-Integer(2))**Integer(2)␣
↪* (x-Integer(3))**Integer(3) * (x-Integer(4))
>>> p.squarefree_decomposition()
(37) * (x - 4) * (x^2 - 3*x + 2)^2 * (x - 3)^3
```

xgcd(*right*)

Return a triple (g, s, t) such that $g = s \cdot$ `self` $+ t \cdot$ `right` and such that g is the gcd of `self` and `right` up to a divisor of the resultant of `self` and `other`.

As integer polynomials do not form a principal ideal domain, it is not always possible given a and b to find a pair s, t such that $gcd(a, b) = sa + tb$. Take $a = x + 2$ and $b = x + 4$ as an example for which the gcd is 1 but the best you can achieve in the Bezout identity is 2.

If `self` and `right` are coprime as polynomials over the rationals, then `g` is guaranteed to be the resultant of `self` and `right`, as a constant polynomial.

EXAMPLES:

```
sage: P.<x> = PolynomialRing(ZZ)

sage: (x + 2).xgcd(x + 4)
(2, -1, 1)
sage: (x + 2).resultant(x + 4)
2
sage: (x + 2).gcd(x + 4)
1

sage: F = (x^2 + 2)*x^3; G = (x^2 + 2) * (x - 3)
sage: g, u, v = F.xgcd(G)
sage: g, u, v
(27*x^2 + 54, 1, -x^2 - 3*x - 9)
sage: u*F + v*G
27*x^2 + 54

sage: zero = P(0)
sage: x.xgcd(zero)
(x, 1, 0)
sage: zero.xgcd(x)
(x, 0, 1)

sage: F = (x - 3)^3; G = (x - 15)^2
sage: g, u, v = F.xgcd(G)
sage: g, u, v
(2985984, -432*x + 8208, 432*x^2 + 864*x + 14256)
sage: u*F + v*G
2985984
```

```
>>> from sage.all import *
>>> P = PolynomialRing(ZZ, names=('x',)); (x,) = P._first_ngens(1)

>>> (x + Integer(2)).xgcd(x + Integer(4))
(2, -1, 1)
>>> (x + Integer(2)).resultant(x + Integer(4))
2
>>> (x + Integer(2)).gcd(x + Integer(4))
1

>>> F = (x**Integer(2) + Integer(2))*x**Integer(3); G = (x**Integer(2) +␣
→Integer(2)) * (x - Integer(3))
>>> g, u, v = F.xgcd(G)
>>> g, u, v
(27*x^2 + 54, 1, -x^2 - 3*x - 9)
>>> u*F + v*G
```

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```
27*x^2 + 54

>>> zero = P(Integer(0))
>>> x.xgcd(zero)
(x, 1, 0)
>>> zero.xgcd(x)
(x, 0, 1)

>>> F = (x - Integer(3))**Integer(3); G = (x - Integer(15))**Integer(2)
>>> g, u, v = F.xgcd(G)
>>> g, u, v
(2985984, -432*x + 8208, 432*x^2 + 864*x + 14256)
>>> u*F + v*G
2985984
```

2.1.8 Dense univariate polynomials over $\mathbf{Z}$, implemented using NTL.

AUTHORS:

- David Harvey: split off from polynomial_element_generic.py (2007-09)
- David Harvey: rewrote to talk to NTL directly, instead of via ntl.pyx (2007-09); a lot of this was based on Joel Mohler's recent rewrite of the NTL wrapper

Sage includes two implementations of dense univariate polynomials over $\mathbf{Z}$; this file contains the implementation based on NTL, but there is also an implementation based on FLINT in *sage.rings.polynomial.polynomial_integer_dense_flint*.

The FLINT implementation is preferred (FLINT's arithmetic operations are generally faster), so it is the default; to use the NTL implementation, you can do:

```
sage: K.<x> = PolynomialRing(ZZ, implementation='NTL')
sage: K
Univariate Polynomial Ring in x over Integer Ring (using NTL)
```

```
>>> from sage.all import *
>>> K = PolynomialRing(ZZ, implementation='NTL', names=('x',)); (x,) = K._first_
↪ngens(1)
>>> K
Univariate Polynomial Ring in x over Integer Ring (using NTL)
```

class sage.rings.polynomial.polynomial_integer_dense_ntl.**Polynomial_integer_dense_ntl**

Bases: *Polynomial*

A dense polynomial over the integers, implemented via NTL.

content()

Return the greatest common divisor of the coefficients of this polynomial. The sign is the sign of the leading coefficient. The content of the zero polynomial is zero.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ, implementation='NTL')
sage: (2*x^2 - 4*x^4 + 14*x^7).content()
2
```

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```
sage: (2*x^2 - 4*x^4 - 14*x^7).content()
-2
sage: x.content()
1
sage: R(1).content()
1
sage: R(0).content()
0
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, implementation='NTL', names=('x',)); (x,) = R._
↪first_ngens(1)
>>> (Integer(2)*x**Integer(2) - Integer(4)*x**Integer(4) +␣
↪Integer(14)*x**Integer(7)).content()
2
>>> (Integer(2)*x**Integer(2) - Integer(4)*x**Integer(4) -␣
↪Integer(14)*x**Integer(7)).content()
-2
>>> x.content()
1
>>> R(Integer(1)).content()
1
>>> R(Integer(0)).content()
0
```

degree(*gen=None*)

Return the degree of this polynomial. The zero polynomial has degree -1.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ, implementation='NTL')
sage: x.degree()
1
sage: (x^2).degree()
2
sage: R(1).degree()
0
sage: R(0).degree()
-1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, implementation='NTL', names=('x',)); (x,) = R._
↪first_ngens(1)
>>> x.degree()
1
>>> (x**Integer(2)).degree()
2
>>> R(Integer(1)).degree()
0
>>> R(Integer(0)).degree()
-1
```

discriminant(*proof=True*)

Return the discriminant of `self`, which is by definition

$$(-1)^{m(m-1)/2}\text{resultant}(a, a')/lc(a),$$

where $m = deg(a)$, and $lc(a)$ is the leading coefficient of a. If `proof` is False (the default is True), then this function may use a randomized strategy that errors with probability no more than 2^{-80}.

EXAMPLES:

```
sage: f = ntl.ZZX([1,2,0,3])
sage: f.discriminant()
-339
sage: f.discriminant(proof=False)
-339
```

```
>>> from sage.all import *
>>> f = ntl.ZZX([Integer(1),Integer(2),Integer(0),Integer(3)])
>>> f.discriminant()
-339
>>> f.discriminant(proof=False)
-339
```

factor()

This function overrides the generic polynomial factorization to make a somewhat intelligent decision to use PARI or NTL based on some benchmarking.

Note: This function factors the content of the polynomial, which can take very long if it's a really big integer. If you do not need the content factored, divide it out of your polynomial before calling this function.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: f = x^4 - 1
sage: f.factor()
(x - 1) * (x + 1) * (x^2 + 1)
sage: f = 1 - x
sage: f.factor()
(-1) * (x - 1)
sage: f.factor().unit()
-1
sage: f = -30*x; f.factor()
(-1) * 2 * 3 * 5 * x
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(4) - Integer(1)
>>> f.factor()
(x - 1) * (x + 1) * (x^2 + 1)
>>> f = Integer(1) - x
>>> f.factor()
(-1) * (x - 1)
>>> f.factor().unit()
-1
```

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```
>>> f = -Integer(30)*x; f.factor()
(-1) * 2 * 3 * 5 * x
```

factor_mod(*p*)

Return the factorization of `self` modulo the prime p.

INPUT:

- `p` – prime

OUTPUT: factorization of `self` reduced modulo p

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ, 'x', implementation='NTL')
sage: f = -3*x*(x-2)*(x-9) + x
sage: f.factor_mod(3)
x
sage: f = -3*x*(x-2)*(x-9)
sage: f.factor_mod(3)
Traceback (most recent call last):
...
ArithmeticError: factorization of 0 is not defined

sage: f = 2*x*(x-2)*(x-9)
sage: f.factor_mod(7)
(2) * x * (x + 5)^2
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, 'x', implementation='NTL', names=('x',)); (x,) = R.
↪_first_ngens(1)
>>> f = -Integer(3)*x*(x-Integer(2))*(x-Integer(9)) + x
>>> f.factor_mod(Integer(3))
x
>>> f = -Integer(3)*x*(x-Integer(2))*(x-Integer(9))
>>> f.factor_mod(Integer(3))
Traceback (most recent call last):
...
ArithmeticError: factorization of 0 is not defined

>>> f = Integer(2)*x*(x-Integer(2))*(x-Integer(9))
>>> f.factor_mod(Integer(7))
(2) * x * (x + 5)^2
```

factor_padic(*p*, *prec=10*)

Return p-adic factorization of `self` to given precision.

INPUT:

- `p` – prime
- `prec` – integer; the precision

OUTPUT: factorization of `self` over the completion at p

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ, implementation='NTL')
sage: f = x^2 + 1
sage: f.factor_padic(5, 4)
((1 + O(5^4))*x + 2 + 5 + 2*5^2 + 5^3 + O(5^4))
* ((1 + O(5^4))*x + 3 + 3*5 + 2*5^2 + 3*5^3 + O(5^4))
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, implementation='NTL', names=('x',)); (x,) = R._
↪first_ngens(1)
>>> f = x**Integer(2) + Integer(1)
>>> f.factor_padic(Integer(5), Integer(4))
((1 + O(5^4))*x + 2 + 5 + 2*5^2 + 5^3 + O(5^4))
* ((1 + O(5^4))*x + 3 + 3*5 + 2*5^2 + 3*5^3 + O(5^4))
```

A more difficult example:

```
sage: f = 100 * (5*x + 1)^2 * (x + 5)^2
sage: f.factor_padic(5, 10)
(4 + O(5^10)) * (5 + O(5^11))^2 * ((1 + O(5^10))*x + 5 + O(5^10))^2
* ((5 + O(5^10))*x + 1 + O(5^10))^2
```

```
>>> from sage.all import *
>>> f = Integer(100) * (Integer(5)*x + Integer(1))**Integer(2) * (x +␣
↪Integer(5))**Integer(2)
>>> f.factor_padic(Integer(5), Integer(10))
(4 + O(5^10)) * (5 + O(5^11))^2 * ((1 + O(5^10))*x + 5 + O(5^10))^2
* ((5 + O(5^10))*x + 1 + O(5^10))^2
```

gcd(*right*)

Return the GCD of `self` and `right`. The leading coefficient need not be 1.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ, implementation='NTL')
sage: f = (6*x + 47) * (7*x^2 - 2*x + 38)
sage: g = (6*x + 47) * (3*x^3 + 2*x + 1)
sage: f.gcd(g)
6*x + 47
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, implementation='NTL', names=('x',)); (x,) = R._
↪first_ngens(1)
>>> f = (Integer(6)*x + Integer(47)) * (Integer(7)*x**Integer(2) -␣
↪Integer(2)*x + Integer(38))
>>> g = (Integer(6)*x + Integer(47)) * (Integer(3)*x**Integer(3) +␣
↪Integer(2)*x + Integer(1))
>>> f.gcd(g)
6*x + 47
```

lcm(*right*)

Return the LCM of `self` and `right`.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ, implementation='NTL')
sage: f = (6*x + 47) * (7*x^2 - 2*x + 38)
sage: g = (6*x + 47) * (3*x^3 + 2*x + 1)
sage: h = f.lcm(g); h
126*x^6 + 951*x^5 + 486*x^4 + 6034*x^3 + 585*x^2 + 3706*x + 1786
sage: h == (6*x + 47) * (7*x^2 - 2*x + 38) * (3*x^3 + 2*x + 1)
True
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, implementation='NTL', names=('x',)); (x,) = R._
↪first_ngens(1)
>>> f = (Integer(6)*x + Integer(47)) * (Integer(7)*x**Integer(2) -␣
↪Integer(2)*x + Integer(38))
>>> g = (Integer(6)*x + Integer(47)) * (Integer(3)*x**Integer(3) +␣
↪Integer(2)*x + Integer(1))
>>> h = f.lcm(g); h
126*x^6 + 951*x^5 + 486*x^4 + 6034*x^3 + 585*x^2 + 3706*x + 1786
>>> h == (Integer(6)*x + Integer(47)) * (Integer(7)*x**Integer(2) -␣
↪Integer(2)*x + Integer(38)) * (Integer(3)*x**Integer(3) + Integer(2)*x +␣
↪Integer(1))
True
```

list(*copy=True*)

Return a new copy of the list of the underlying elements of `self`.

EXAMPLES:

```
sage: x = PolynomialRing(ZZ, 'x', implementation='NTL').0
sage: f = x^3 + 3*x - 17
sage: f.list()
[-17, 3, 0, 1]
sage: f = PolynomialRing(ZZ, 'x', implementation='NTL')(0)
sage: f.list()
[]
```

```
>>> from sage.all import *
>>> x = PolynomialRing(ZZ, 'x', implementation='NTL').gen(0)
>>> f = x**Integer(3) + Integer(3)*x - Integer(17)
>>> f.list()
[-17, 3, 0, 1]
>>> f = PolynomialRing(ZZ, 'x', implementation='NTL')(Integer(0))
>>> f.list()
[]
```

quo_rem(*right*)

Attempt to divide `self` by `right`, and return a quotient and remainder.

If right is monic, then it returns `(q, r)` where `self = q * right + r` and `deg(r) < deg(right)`.

If right is not monic, then it returns $(q, 0)$ where `q = self/right` if `right` exactly divides `self`, otherwise it raises an exception.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ, implementation='NTL')
sage: f = R(range(10)); g = R([-1, 0, 1])
sage: q, r = f.quo_rem(g)
sage: q, r
(9*x^7 + 8*x^6 + 16*x^5 + 14*x^4 + 21*x^3 + 18*x^2 + 24*x + 20, 25*x + 20)
sage: q*g + r == f
True

sage: 0//(2*x)
0

sage: f = x^2
sage: f.quo_rem(0)
Traceback (most recent call last):
...
ArithmeticError: division by zero polynomial

sage: f = (x^2 + 3) * (2*x - 1)
sage: f.quo_rem(2*x - 1)
(x^2 + 3, 0)

sage: f = x^2
sage: f.quo_rem(2*x - 1)
Traceback (most recent call last):
...
ArithmeticError: division not exact in Z[x] (consider coercing to Q[x] first)
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, implementation='NTL', names=('x',)); (x,) = R._
↪first_ngens(1)
>>> f = R(range(Integer(10))); g = R([-Integer(1), Integer(0), Integer(1)])
>>> q, r = f.quo_rem(g)
>>> q, r
(9*x^7 + 8*x^6 + 16*x^5 + 14*x^4 + 21*x^3 + 18*x^2 + 24*x + 20, 25*x + 20)
>>> q*g + r == f
True

>>> Integer(0)//(Integer(2)*x)
0

>>> f = x**Integer(2)
>>> f.quo_rem(Integer(0))
Traceback (most recent call last):
...
ArithmeticError: division by zero polynomial

>>> f = (x**Integer(2) + Integer(3)) * (Integer(2)*x - Integer(1))
>>> f.quo_rem(Integer(2)*x - Integer(1))
(x^2 + 3, 0)

>>> f = x**Integer(2)
>>> f.quo_rem(Integer(2)*x - Integer(1))
```

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```
Traceback (most recent call last):
...
ArithmeticError: division not exact in Z[x] (consider coercing to Q[x] first)
```

real_root_intervals()

Return isolating intervals for the real roots of this polynomial.

EXAMPLES: We compute the roots of the characteristic polynomial of some Salem numbers:

```
sage: R.<x> = PolynomialRing(ZZ, implementation='NTL')
sage: f = 1 - x^2 - x^3 - x^4 + x^6
sage: f.real_root_intervals()                                           #␣
↪needs sage.libs.linbox
[((1/2, 3/4), 1), ((1, 3/2), 1)]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, implementation='NTL', names=('x',)); (x,) = R._
↪first_ngens(1)
>>> f = Integer(1) - x**Integer(2) - x**Integer(3) - x**Integer(4) +␣
↪x**Integer(6)
>>> f.real_root_intervals()                                             #␣
↪needs sage.libs.linbox
[((1/2, 3/4), 1), ((1, 3/2), 1)]
```

resultant(*other*, *proof=True*)

Return the resultant of `self` and `other`, which must lie in the same polynomial ring.

If `proof=False` (the default is `proof=True`), then this function may use a randomized strategy that errors with probability no more than 2^{-80}.

INPUT:

- `other` – a polynomial

OUTPUT: an element of the base ring of the polynomial ring

EXAMPLES:

```
sage: x = PolynomialRing(ZZ, 'x', implementation='NTL').0
sage: f = x^3 + x + 1;  g = x^3 - x - 1
sage: r = f.resultant(g); r
-8
sage: r.parent() is ZZ
True
```

```
>>> from sage.all import *
>>> x = PolynomialRing(ZZ, 'x', implementation='NTL').gen(0)
>>> f = x**Integer(3) + x + Integer(1);  g = x**Integer(3) - x - Integer(1)
>>> r = f.resultant(g); r
-8
>>> r.parent() is ZZ
True
```

squarefree_decomposition()

Return the square-free decomposition of `self`. This is a partial factorization of `self` into square-free, relatively prime polynomials.

This is a wrapper for the NTL function `SquareFreeDecomp`.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ, implementation='NTL')
sage: p = 37 * (x-1)^2 * (x-2)^2 * (x-3)^3 * (x-4)
sage: p.squarefree_decomposition()
(37) * (x - 4) * (x^2 - 3*x + 2)^2 * (x - 3)^3
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, implementation='NTL', names=('x',)); (x,) = R._
↪first_ngens(1)
>>> p = Integer(37) * (x-Integer(1))**Integer(2) * (x-Integer(2))**Integer(2)␣
↪* (x-Integer(3))**Integer(3) * (x-Integer(4))
>>> p.squarefree_decomposition()
(37) * (x - 4) * (x^2 - 3*x + 2)^2 * (x - 3)^3
```

xgcd(*right*)

This function can't in general return (g, s, t) as above, since they need not exist. Instead, over the integers, we first multiply g by a divisor of the resultant of a/g and b/g, up to sign, and return `g, u, v` such that `g = u*self + v*right`. But note that this g may be a multiple of the gcd.

If `self` and `right` are coprime as polynomials over the rationals, then `g` is guaranteed to be the resultant of `self` and `right`, as a constant polynomial.

EXAMPLES:

```
sage: P.<x> = PolynomialRing(ZZ, implementation='NTL')
sage: F = (x^2 + 2)*x^3; G = (x^2+2)*(x-3)
sage: g, u, v = F.xgcd(G)
sage: g, u, v
(27*x^2 + 54, 1, -x^2 - 3*x - 9)
sage: u*F + v*G
27*x^2 + 54
sage: x.xgcd(P(0))
(x, 1, 0)
sage: f = P(0)
sage: f.xgcd(x)
(x, 0, 1)
sage: F = (x-3)^3; G = (x-15)^2
sage: g, u, v = F.xgcd(G)
sage: g, u, v
(2985984, -432*x + 8208, 432*x^2 + 864*x + 14256)
sage: u*F + v*G
2985984
```

```
>>> from sage.all import *
>>> P = PolynomialRing(ZZ, implementation='NTL', names=('x',)); (x,) = P._
↪first_ngens(1)
>>> F = (x**Integer(2) + Integer(2))*x**Integer(3); G =␣
↪(x**Integer(2)+Integer(2))*(x-Integer(3))
```

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```
>>> g, u, v = F.xgcd(G)
>>> g, u, v
(27*x^2 + 54, 1, -x^2 - 3*x - 9)
>>> u*F + v*G
27*x^2 + 54
>>> x.xgcd(P(Integer(0)))
(x, 1, 0)
>>> f = P(Integer(0))
>>> f.xgcd(x)
(x, 0, 1)
>>> F = (x-Integer(3))**Integer(3); G = (x-Integer(15))**Integer(2)
>>> g, u, v = F.xgcd(G)
>>> g, u, v
(2985984, -432*x + 8208, 432*x^2 + 864*x + 14256)
>>> u*F + v*G
2985984
```

2.1.9 Univariate polynomials over $\mathbf{Q}$ implemented via FLINT

AUTHOR:

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class sage.rings.polynomial.polynomial_rational_flint.**Polynomial_rational_flint**

Bases: *Polynomial*

Univariate polynomials over the rationals, implemented via FLINT.

Internally, we represent rational polynomial as the quotient of an integer polynomial and a positive denominator which is coprime to the content of the numerator.

add(*right*)

Return the sum of two rational polynomials.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: f = 2/3 + t + 2*t^3
sage: g = -1 + t/3 - 10/11*t^4
sage: f + g
-10/11*t^4 + 2*t^3 + 4/3*t - 1/3
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = Integer(2)/Integer(3) + t + Integer(2)*t**Integer(3)
>>> g = -Integer(1) + t/Integer(3) - Integer(10)/Integer(11)*t**Integer(4)
>>> f + g
-10/11*t^4 + 2*t^3 + 4/3*t - 1/3
```

sub(*right*)

Return the difference of two rational polynomials.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: f = -10/11*t^4 + 2*t^3 + 4/3*t - 1/3
sage: g = 2*t^3
sage: f - g                                    # indirect doctest
-10/11*t^4 + 4/3*t - 1/3
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = -Integer(10)/Integer(11)*t**Integer(4) + Integer(2)*t**Integer(3) +␣
↪Integer(4)/Integer(3)*t - Integer(1)/Integer(3)
>>> g = Integer(2)*t**Integer(3)
>>> f - g                                    # indirect doctest
-10/11*t^4 + 4/3*t - 1/3
```

lmul(*right*)

Return `self * right`, where `right` is a rational number.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: f = 3/2*t^3 - t + 1/3
sage: f * 6                        # indirect doctest
9*t^3 - 6*t + 2
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = Integer(3)/Integer(2)*t**Integer(3) - t + Integer(1)/Integer(3)
>>> f * Integer(6)                        # indirect doctest
9*t^3 - 6*t + 2
```

rmul(*left*)

Return `left * self`, where `left` is a rational number.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: f = 3/2*t^3 - t + 1/3
sage: 6 * f                        # indirect doctest
9*t^3 - 6*t + 2
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = Integer(3)/Integer(2)*t**Integer(3) - t + Integer(1)/Integer(3)
>>> Integer(6) * f                        # indirect doctest
9*t^3 - 6*t + 2
```

mul(*right*)

Return the product of `self` and `right`.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: f = -1 + 3*t/2 - t^3
```

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```
sage: g = 2/3 + 7/3*t + 3*t^2
sage: f * g                                  # indirect doctest
-3*t^5 - 7/3*t^4 + 23/6*t^3 + 1/2*t^2 - 4/3*t - 2/3
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = -Integer(1) + Integer(3)*t/Integer(2) - t**Integer(3)
>>> g = Integer(2)/Integer(3) + Integer(7)/Integer(3)*t +␣
↪Integer(3)*t**Integer(2)
>>> f * g                                  # indirect doctest
-3*t^5 - 7/3*t^4 + 23/6*t^3 + 1/2*t^2 - 4/3*t - 2/3
```

_mul_trunc_(*right*, *n*)

Truncated multiplication.

EXAMPLES:

```
sage: x = polygen(QQ)
sage: p1 = 1/2 - 3*x + 2/7*x**3
sage: p2 = x + 2/5*x**5 + x**7
sage: p1._mul_trunc_(p2, 5)
2/7*x^4 - 3*x^2 + 1/2*x
sage: (p1*p2).truncate(5)
2/7*x^4 - 3*x^2 + 1/2*x

sage: p1._mul_trunc_(p2, 1)
0
sage: p1._mul_trunc_(p2, 0)
Traceback (most recent call last):
...
ValueError: n must be > 0
```

```
>>> from sage.all import *
>>> x = polygen(QQ)
>>> p1 = Integer(1)/Integer(2) - Integer(3)*x + Integer(2)/
↪Integer(7)*x**Integer(3)
>>> p2 = x + Integer(2)/Integer(5)*x**Integer(5) + x**Integer(7)
>>> p1._mul_trunc_(p2, Integer(5))
2/7*x^4 - 3*x^2 + 1/2*x
>>> (p1*p2).truncate(Integer(5))
2/7*x^4 - 3*x^2 + 1/2*x

>>> p1._mul_trunc_(p2, Integer(1))
0
>>> p1._mul_trunc_(p2, Integer(0))
Traceback (most recent call last):
...
ValueError: n must be > 0
```

ALGORITHM:

Call the FLINT method `fmpq_poly_mullow`.

degree()

Return the degree of `self`.

By convention, the degree of the zero polynomial is -1.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: f = 1 + t + t^2/2 + t^3/3 + t^4/4
sage: f.degree()
4
sage: g = R(0)
sage: g.degree()
-1
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = Integer(1) + t + t**Integer(2)/Integer(2) + t**Integer(3)/Integer(3)␣
↪+ t**Integer(4)/Integer(4)
>>> f.degree()
4
>>> g = R(Integer(0))
>>> g.degree()
-1
```

denominator()

Return the denominator of `self`.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: f = (3 * t^3 + 1) / -3
sage: f.denominator()
3
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = (Integer(3) * t**Integer(3) + Integer(1)) / -Integer(3)
>>> f.denominator()
3
```

disc()

Return the discriminant of this polynomial.

The discriminant R_n is defined as

$$R_n = a_n^{2n-2} \prod_{1 \le i < j \le n} (r_i - r_j)^2,$$

where n is the degree of this polynomial, a_n is the leading coefficient and the roots over $\overline{\mathbf{Q}}$ are $r_1, \ldots, r_n$.

The discriminant of constant polynomials is defined to be 0.

OUTPUT: discriminant, an element of the base ring of the polynomial ring

Note

Note the identity $R_n(f) := (-1)^{(n(n-1)/2)} R(f, f') a_n^{(n-k-2)}$, where n is the degree of this polynomial, a_n is the leading coefficient, f' is the derivative of f, and k is the degree of f'. Calls `resultant()`.

ALGORITHM:

Use PARI.

EXAMPLES:

In the case of elliptic curves in special form, the discriminant is easy to calculate:

```
sage: R.<t> = QQ[]
sage: f = t^3 + t + 1
sage: d = f.discriminant(); d
-31
sage: d.parent() is QQ
True
sage: EllipticCurve([1, 1]).discriminant() / 16                              #␣
↪needs sage.schemes
-31
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = t**Integer(3) + t + Integer(1)
>>> d = f.discriminant(); d
-31
>>> d.parent() is QQ
True
>>> EllipticCurve([Integer(1), Integer(1)]).discriminant() / Integer(16)      ␣
↪                           # needs sage.schemes
-31
```

```
sage: R.<t> = QQ[]
sage: f = 2*t^3 + t + 1
sage: d = f.discriminant(); d
-116
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = Integer(2)*t**Integer(3) + t + Integer(1)
>>> d = f.discriminant(); d
-116
```

```
sage: R.<t> = QQ[]
sage: f = t^3 + 3*t - 17
sage: f.discriminant()
-7911
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
```

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```
>>> f = t**Integer(3) + Integer(3)*t - Integer(17)
>>> f.discriminant()
-7911
```

discriminant()

Return the discriminant of this polynomial.

The discriminant R_n is defined as

$$R_n = a_n^{2n-2} \prod_{1 \le i < j \le n} (r_i - r_j)^2,$$

where n is the degree of this polynomial, a_n is the leading coefficient and the roots over $\overline{\mathbf{Q}}$ are $r_1, \ldots, r_n$.

The discriminant of constant polynomials is defined to be 0.

OUTPUT: discriminant, an element of the base ring of the polynomial ring

Note

Note the identity $R_n(f) := (-1)^{(n(n-1)/2)} R(f, f') a_n^{(n-k-2)}$, where n is the degree of this polynomial, a_n is the leading coefficient, f' is the derivative of f, and k is the degree of f'. Calls `resultant()`.

ALGORITHM:

Use PARI.

EXAMPLES:

In the case of elliptic curves in special form, the discriminant is easy to calculate:

```
sage: R.<t> = QQ[]
sage: f = t^3 + t + 1
sage: d = f.discriminant(); d
-31
sage: d.parent() is QQ
True
sage: EllipticCurve([1, 1]).discriminant() / 16                              #␣
↪needs sage.schemes
-31
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = t**Integer(3) + t + Integer(1)
>>> d = f.discriminant(); d
-31
>>> d.parent() is QQ
True
>>> EllipticCurve([Integer(1), Integer(1)]).discriminant() / Integer(16)      ␣
↪                       # needs sage.schemes
-31
```

```
sage: R.<t> = QQ[]
sage: f = 2*t^3 + t + 1
sage: d = f.discriminant(); d
-116
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = Integer(2)*t**Integer(3) + t + Integer(1)
>>> d = f.discriminant(); d
-116
```

```
sage: R.<t> = QQ[]
sage: f = t^3 + 3*t - 17
sage: f.discriminant()
-7911
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = t**Integer(3) + Integer(3)*t - Integer(17)
>>> f.discriminant()
-7911
```

factor_mod(p)

Return the factorization of `self` modulo the prime p.

Assumes that the degree of this polynomial is at least one, and raises a `ValueError` otherwise.

INPUT:

- `p` – prime number

OUTPUT: factorization of this polynomial modulo p

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: (x^5 + 17*x^3 + x + 3).factor_mod(3)
x * (x^2 + 1)^2
sage: (x^5 + 2).factor_mod(5)
(x + 2)^5
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> (x**Integer(5) + Integer(17)*x**Integer(3) + x + Integer(3)).factor_
↪mod(Integer(3))
x * (x^2 + 1)^2
>>> (x**Integer(5) + Integer(2)).factor_mod(Integer(5))
(x + 2)^5
```

Variable names that are reserved in PARI, such as `zeta`, are supported (see Issue #20631):

```
sage: R.<zeta> = QQ[]
sage: (zeta^2 + zeta + 1).factor_mod(7)
(zeta + 3) * (zeta + 5)
```

```
>>> from sage.all import *
>>> R = QQ['zeta']; (zeta,) = R._first_ngens(1)
>>> (zeta**Integer(2) + zeta + Integer(1)).factor_mod(Integer(7))
(zeta + 3) * (zeta + 5)
```

factor_padic(*p*, *prec=10*)

Return the p-adic factorization of this polynomial to the given precision.

INPUT:

- `p` – prime number
- `prec` – integer; the precision

OUTPUT: factorization of `self` viewed as a p-adic polynomial

EXAMPLES:

```
sage: # needs sage.rings.padic
sage: R.<x> = QQ[]
sage: f = x^3 - 2
sage: f.factor_padic(2)
(1 + O(2^10))*x^3 + O(2^10)*x^2 + O(2^10)*x
+ 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + O(2^10)
sage: f.factor_padic(3)
(1 + O(3^10))*x^3 + O(3^10)*x^2 + O(3^10)*x
+ 1 + 2*3 + 2*3^2 + 2*3^3 + 2*3^4 + 2*3^5 + 2*3^6 + 2*3^7 + 2*3^8 + 2*3^9 +␣
↪O(3^10)
sage: f.factor_padic(5)
((1 + O(5^10))*x
 + 2 + 4*5 + 2*5^2 + 2*5^3 + 5^4 + 3*5^5 + 4*5^7 + 2*5^8 + 5^9 + O(5^10))
* ((1 + O(5^10))*x^2
   + (3 + 2*5^2 + 2*5^3 + 3*5^4 + 5^5 + 4*5^6 + 2*5^8 + 3*5^9 + O(5^10))*x
   + 4 + 5 + 2*5^2 + 4*5^3 + 4*5^4 + 3*5^5 + 3*5^6 + 4*5^7 + 4*5^9 + O(5^10))
```

```
>>> from sage.all import *
>>> # needs sage.rings.padic
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(3) - Integer(2)
>>> f.factor_padic(Integer(2))
(1 + O(2^10))*x^3 + O(2^10)*x^2 + O(2^10)*x
+ 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + O(2^10)
>>> f.factor_padic(Integer(3))
(1 + O(3^10))*x^3 + O(3^10)*x^2 + O(3^10)*x
+ 1 + 2*3 + 2*3^2 + 2*3^3 + 2*3^4 + 2*3^5 + 2*3^6 + 2*3^7 + 2*3^8 + 2*3^9 +␣
↪O(3^10)
>>> f.factor_padic(Integer(5))
((1 + O(5^10))*x
 + 2 + 4*5 + 2*5^2 + 2*5^3 + 5^4 + 3*5^5 + 4*5^7 + 2*5^8 + 5^9 + O(5^10))
* ((1 + O(5^10))*x^2
   + (3 + 2*5^2 + 2*5^3 + 3*5^4 + 5^5 + 4*5^6 + 2*5^8 + 3*5^9 + O(5^10))*x
   + 4 + 5 + 2*5^2 + 4*5^3 + 4*5^4 + 3*5^5 + 3*5^6 + 4*5^7 + 4*5^9 + O(5^10))
```

The input polynomial is considered to have "infinite" precision, therefore the p-adic factorization of the polynomial is not the same as first coercing to $\mathbf{Q}_p$ and then factoring (see also Issue #15422):

```
sage: # needs sage.rings.padic
sage: f = x^2 - 3^6
sage: f.factor_padic(3, 5)
((1 + O(3^5))*x + 3^3 + O(3^5)) * ((1 + O(3^5))*x + 2*3^3 + 2*3^4 + O(3^5))
sage: f.change_ring(Qp(3,5)).factor()
Traceback (most recent call last):
...
PrecisionError: p-adic factorization not well-defined since
the discriminant is zero up to the requestion p-adic precision
```

```
>>> from sage.all import *
>>> # needs sage.rings.padic
>>> f = x**Integer(2) - Integer(3)**Integer(6)
>>> f.factor_padic(Integer(3), Integer(5))
((1 + O(3^5))*x + 3^3 + O(3^5)) * ((1 + O(3^5))*x + 2*3^3 + 2*3^4 + O(3^5))
>>> f.change_ring(Qp(Integer(3),Integer(5))).factor()
Traceback (most recent call last):
...
PrecisionError: p-adic factorization not well-defined since
the discriminant is zero up to the requestion p-adic precision
```

A more difficult example:

```
sage: R.<x> = QQ[]
sage: f = 100 * (5*x + 1)^2 * (x + 5)^2
sage: f.factor_padic(5, 10)                                                  #␣
↪needs sage.rings.padic
(4*5^4 + O(5^14)) * ((1 + O(5^9))*x + 5^-1 + O(5^9))^2
* ((1 + O(5^10))*x + 5 + O(5^10))^2
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = Integer(100) * (Integer(5)*x + Integer(1))**Integer(2) * (x +␣
↪Integer(5))**Integer(2)
>>> f.factor_padic(Integer(5), Integer(10))                                  ␣
↪              # needs sage.rings.padic
(4*5^4 + O(5^14)) * ((1 + O(5^9))*x + 5^-1 + O(5^9))^2
* ((1 + O(5^10))*x + 5 + O(5^10))^2
```

Try some bogus inputs:

```
sage: # needs sage.rings.padic
sage: f.factor_padic(3, -1)
Traceback (most recent call last):
...
ValueError: prec_cap must be nonnegative
sage: f.factor_padic(6, 10)
Traceback (most recent call last):
...
ValueError: p must be prime
sage: f.factor_padic('hello', 'world')
Traceback (most recent call last):
```

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```
...
TypeError: unable to convert 'hello' to an integer
```

```
>>> from sage.all import *
>>> # needs sage.rings.padic
>>> f.factor_padic(Integer(3), -Integer(1))
Traceback (most recent call last):
...
ValueError: prec_cap must be nonnegative
>>> f.factor_padic(Integer(6), Integer(10))
Traceback (most recent call last):
...
ValueError: p must be prime
>>> f.factor_padic('hello', 'world')
Traceback (most recent call last):
...
TypeError: unable to convert 'hello' to an integer
```

galois_group(*pari_group=False*, *algorithm='pari'*)

Return the Galois group of this polynomial as a permutation group.

INPUT:

- `self` – irreducible polynomial
- `pari_group` – boolean (default: `False`); if `True` instead return the Galois group as a PARI group. This has a useful label in it, and may be slightly faster since it doesn't require looking up a group in GAP. To get a permutation group from a PARI group `P`, type `PermutationGroup(P)`.
- `algorithm` – `'pari'`, `'gap'`, `'kash'`, `'magma'` (default: `'pari'`, for degrees is at most 11; `'gap'`, for degrees from 12 to 15; `'kash'`, for degrees from 16 or more).

OUTPUT: Galois group

ALGORITHM:

The Galois group is computed using PARI in C library mode, or possibly GAP, KASH, or MAGMA.

Note

The PARI documentation contains the following warning: The method used is that of resolvent polynomials and is sensitive to the current precision. The precision is updated internally but, in very rare cases, a wrong result may be returned if the initial precision was not sufficient.

GAP uses the "Transitive Groups Libraries" from the "TransGrp" GAP package which comes installed with the "gap" Sage package.

MAGMA does not return a provably correct result. Please see the MAGMA documentation for how to obtain a provably correct result.

EXAMPLES:

```
sage: # needs sage.groups sage.libs.pari
sage: R.<x> = QQ[]
sage: f = x^4 - 17*x^3 - 2*x + 1
```

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```
sage: G = f.galois_group(); G
Transitive group number 5 of degree 4
sage: G.gens()
((1,2,3,4), (1,2))
sage: G.order()
24
```

```
>>> from sage.all import *
>>> # needs sage.groups sage.libs.pari
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(4) - Integer(17)*x**Integer(3) - Integer(2)*x + Integer(1)
>>> G = f.galois_group(); G
Transitive group number 5 of degree 4
>>> G.gens()
((1,2,3,4), (1,2))
>>> G.order()
24
```

It is potentially useful to instead obtain the corresponding PARI group, which is little more than a 4-tuple. See the PARI manual for the exact details. (Note that the third entry in the tuple is in the new standard ordering.)

```
sage: # needs sage.groups sage.libs.pari
sage: f = x^4 - 17*x^3 - 2*x + 1
sage: G = f.galois_group(pari_group=True); G
PARI group [24, -1, 5, "S4"] of degree 4
sage: PermutationGroup(G)
Transitive group number 5 of degree 4
```

```
>>> from sage.all import *
>>> # needs sage.groups sage.libs.pari
>>> f = x**Integer(4) - Integer(17)*x**Integer(3) - Integer(2)*x + Integer(1)
>>> G = f.galois_group(pari_group=True); G
PARI group [24, -1, 5, "S4"] of degree 4
>>> PermutationGroup(G)
Transitive group number 5 of degree 4
```

You can use KASH or GAP to compute Galois groups as well. The advantage is that KASH (resp. GAP) can compute Galois groups of fields up to degree 23 (resp. 15), whereas PARI only goes to degree 11. (In my not-so-thorough experiments PARI is faster than KASH.)

```
sage: R.<x> = QQ[]
sage: f = x^4 - 17*x^3 - 2*x + 1
sage: f.galois_group(algorithm='kash')    # optional - kash
Transitive group number 5 of degree 4

sage: # needs sage.libs.gap
sage: f = x^4 - 17*x^3 - 2*x + 1
sage: f.galois_group(algorithm='gap')
Transitive group number 5 of degree 4
sage: f = x^13 - 17*x^3 - 2*x + 1
sage: f.galois_group(algorithm='gap')
Transitive group number 9 of degree 13
```

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```
sage: f = x^12 - 2*x^8 - x^7 + 2*x^6 + 4*x^4 - 2*x^3 - x^2 - x + 1
sage: f.galois_group(algorithm='gap')
Transitive group number 183 of degree 12

sage: f.galois_group(algorithm='magma')  # optional - magma
Transitive group number 183 of degree 12
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(4) - Integer(17)*x**Integer(3) - Integer(2)*x + Integer(1)
>>> f.galois_group(algorithm='kash')   # optional - kash
Transitive group number 5 of degree 4

>>> # needs sage.libs.gap
>>> f = x**Integer(4) - Integer(17)*x**Integer(3) - Integer(2)*x + Integer(1)
>>> f.galois_group(algorithm='gap')
Transitive group number 5 of degree 4
>>> f = x**Integer(13) - Integer(17)*x**Integer(3) - Integer(2)*x + Integer(1)
>>> f.galois_group(algorithm='gap')
Transitive group number 9 of degree 13
>>> f = x**Integer(12) - Integer(2)*x**Integer(8) - x**Integer(7) +␣
↪Integer(2)*x**Integer(6) + Integer(4)*x**Integer(4) -␣
↪Integer(2)*x**Integer(3) - x**Integer(2) - x + Integer(1)
>>> f.galois_group(algorithm='gap')
Transitive group number 183 of degree 12

>>> f.galois_group(algorithm='magma')  # optional - magma
Transitive group number 183 of degree 12
```

galois_group_davenport_smith_test(*num_trials=50*, *assume_irreducible=False*)

Use the Davenport-Smith test to attempt to certify that f has Galois group A_n or S_n.

Return 1 if the Galois group is certified as S_n, 2 if A_n, or 0 if no conclusion is reached.

By default, we first check that f is irreducible. For extra efficiency, one can override this by specifying `assume_irreducible=True`; this yields undefined results if f is not irreducible.

A corresponding function in Magma is `IsEasySnAn`.

EXAMPLES:

```
sage: P.<x> = QQ[]
sage: u = x^7 + x + 1
sage: u.galois_group_davenport_smith_test()
1
sage: u = x^7 - x^4 - x^3 + 3*x^2 - 1
sage: u.galois_group_davenport_smith_test()
2
sage: u = x^7 - 2
sage: u.galois_group_davenport_smith_test()
0
```

```
>>> from sage.all import *
>>> P = QQ['x']; (x,) = P._first_ngens(1)
>>> u = x**Integer(7) + x + Integer(1)
>>> u.galois_group_davenport_smith_test()
1
>>> u = x**Integer(7) - x**Integer(4) - x**Integer(3) +␣
↪Integer(3)*x**Integer(2) - Integer(1)
>>> u.galois_group_davenport_smith_test()
2
>>> u = x**Integer(7) - Integer(2)
>>> u.galois_group_davenport_smith_test()
0
```

gcd(*right*)

Return the (monic) greatest common divisor of `self` and `right`.

Corner cases: if `self` and `right` are both zero, returns zero. If only one of them is zero, returns the other polynomial, up to normalisation.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: f = -2 + 3*t/2 + 4*t^2/7 - t^3
sage: g = 1/2 + 4*t + 2*t^4/3
sage: f.gcd(g)
1
sage: f = (-3*t + 1/2) * f
sage: g = (-3*t + 1/2) * (4*t^2/3 - 1) * g
sage: f.gcd(g)
t - 1/6
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = -Integer(2) + Integer(3)*t/Integer(2) + Integer(4)*t**Integer(2)/
↪Integer(7) - t**Integer(3)
>>> g = Integer(1)/Integer(2) + Integer(4)*t + Integer(2)*t**Integer(4)/
↪Integer(3)
>>> f.gcd(g)
1
>>> f = (-Integer(3)*t + Integer(1)/Integer(2)) * f
>>> g = (-Integer(3)*t + Integer(1)/Integer(2)) * (Integer(4)*t**Integer(2)/
↪Integer(3) - Integer(1)) * g
>>> f.gcd(g)
t - 1/6
```

hensel_lift(p, e)

Assuming that this polynomial factors modulo p into distinct monic factors, computes the Hensel lifts of these factors modulo p^e. We assume that `self` has integer coefficients.

Return an empty list if this polynomial has degree less than one.

INPUT:

- `p` – prime number; coerceable to `Integer`
- `e` – exponent; coerceable to `Integer`

OUTPUT: Hensel lifts; list of polynomials over $\mathbf{Z}/p^e\mathbf{Z}$

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: R((x-1)*(x+1)).hensel_lift(7, 2)
[x + 1, x + 48]
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> R((x-Integer(1))*(x+Integer(1))).hensel_lift(Integer(7), Integer(2))
[x + 1, x + 48]
```

If the input polynomial f is not monic, we get a factorization of $f/lc(f)$:

```
sage: R(2*x^2 - 2).hensel_lift(7, 2)
[x + 1, x + 48]
```

```
>>> from sage.all import *
>>> R(Integer(2)*x**Integer(2) - Integer(2)).hensel_lift(Integer(7),␣
↪Integer(2))
[x + 1, x + 48]
```

inverse_series_trunc(*prec*)

Return a polynomial approximation of precision `prec` of the inverse series of this polynomial.

EXAMPLES:

```
sage: x = polygen(QQ)
sage: p = 2 + x - 3/5*x**2
sage: q5 = p.inverse_series_trunc(5)
sage: q5
151/800*x^4 - 17/80*x^3 + 11/40*x^2 - 1/4*x + 1/2
sage: q5 * p
-453/4000*x^6 + 253/800*x^5 + 1

sage: q100 = p.inverse_series_trunc(100)
sage: (q100 * p).truncate(100)
1
```

```
>>> from sage.all import *
>>> x = polygen(QQ)
>>> p = Integer(2) + x - Integer(3)/Integer(5)*x**Integer(2)
>>> q5 = p.inverse_series_trunc(Integer(5))
>>> q5
151/800*x^4 - 17/80*x^3 + 11/40*x^2 - 1/4*x + 1/2
>>> q5 * p
-453/4000*x^6 + 253/800*x^5 + 1

>>> q100 = p.inverse_series_trunc(Integer(100))
>>> (q100 * p).truncate(Integer(100))
1
```

is_irreducible()

Return whether this polynomial is irreducible.

This method computes the primitive part as an element of $\mathbf{Z}[t]$ and calls the method `is_irreducible` for elements of that polynomial ring.

By definition, over any integral domain, an element r is irreducible if and only if it is nonzero, not a unit and whenever $r = ab$ then a or b is a unit.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: (t^2 + 2).is_irreducible()
True
sage: (t^2 - 1).is_irreducible()
False
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> (t**Integer(2) + Integer(2)).is_irreducible()
True
>>> (t**Integer(2) - Integer(1)).is_irreducible()
False
```

is_one()

Return whether or not this polynomial is one.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: R([0,1]).is_one()
False
sage: R([1]).is_one()
True
sage: R([0]).is_one()
False
sage: R([-1]).is_one()
False
sage: R([1,1]).is_one()
False
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> R([Integer(0),Integer(1)]).is_one()
False
>>> R([Integer(1)]).is_one()
True
>>> R([Integer(0)]).is_one()
False
>>> R([-Integer(1)]).is_one()
False
>>> R([Integer(1),Integer(1)]).is_one()
False
```

is_zero()

Return whether or not `self` is the zero polynomial.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: f = 1 - t + 1/2*t^2 - 1/3*t^3
sage: f.is_zero()
False
sage: R(0).is_zero()
True
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = Integer(1) - t + Integer(1)/Integer(2)*t**Integer(2) - Integer(1)/
↪Integer(3)*t**Integer(3)
>>> f.is_zero()
False
>>> R(Integer(0)).is_zero()
True
```

lcm(*right*)

Return the monic (or zero) least common multiple of `self` and `right`.

Corner cases: if either of `self` and `right` are zero, returns zero. This behaviour is ensures that the relation $\operatorname{lcm}(a, b) \cdot \gcd(a, b) = a \cdot b$ holds up to multiplication by rationals.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: f = -2 + 3*t/2 + 4*t^2/7 - t^3
sage: g = 1/2 + 4*t + 2*t^4/3
sage: f.lcm(g)
t^7 - 4/7*t^6 - 3/2*t^5 + 8*t^4 - 75/28*t^3 - 66/7*t^2 + 87/8*t + 3/2
sage: f.lcm(g) * f.gcd(g) // (f * g)
-3/2
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = -Integer(2) + Integer(3)*t/Integer(2) + Integer(4)*t**Integer(2)/
↪Integer(7) - t**Integer(3)
>>> g = Integer(1)/Integer(2) + Integer(4)*t + Integer(2)*t**Integer(4)/
↪Integer(3)
>>> f.lcm(g)
t^7 - 4/7*t^6 - 3/2*t^5 + 8*t^4 - 75/28*t^3 - 66/7*t^2 + 87/8*t + 3/2
>>> f.lcm(g) * f.gcd(g) // (f * g)
-3/2
```

list(*copy=True*)

Return a list with the coefficients of `self`.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: f = 1 + t + t^2/2 + t^3/3 + t^4/4
sage: f.list()
[1, 1, 1/2, 1/3, 1/4]
```

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```
sage: g = R(0)
sage: g.list()
[]
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = Integer(1) + t + t**Integer(2)/Integer(2) + t**Integer(3)/Integer(3)␣
↪+ t**Integer(4)/Integer(4)
>>> f.list()
[1, 1, 1/2, 1/3, 1/4]
>>> g = R(Integer(0))
>>> g.list()
[]
```

numerator()

Return the numerator of `self`.

Representing `self` as the quotient of an integer polynomial and a positive integer denominator (coprime to the content of the polynomial), returns the integer polynomial.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: f = (3 * t^3 + 1) / -3
sage: f.numerator()
-3*t^3 - 1
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = (Integer(3) * t**Integer(3) + Integer(1)) / -Integer(3)
>>> f.numerator()
-3*t^3 - 1
```

quo_rem(*right*)

Return the quotient and remainder of the Euclidean division of `self` and `right`.

Raises a `ZeroDivisionError` if `right` is zero.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: f = R.random_element(2000)
sage: g = R.random_element(1000)
sage: q, r = f.quo_rem(g)
sage: f == q*g + r
True
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = R.random_element(Integer(2000))
>>> g = R.random_element(Integer(1000))
>>> q, r = f.quo_rem(g)
>>> f == q*g + r
True
```

real_root_intervals()

Return isolating intervals for the real roots of `self`.

EXAMPLES:

We compute the roots of the characteristic polynomial of some Salem numbers:

```
sage: R.<t> = QQ[]
sage: f = 1 - t^2 - t^3 - t^4 + t^6
sage: f.real_root_intervals()
[((1/2, 3/4), 1), ((1, 3/2), 1)]
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = Integer(1) - t**Integer(2) - t**Integer(3) - t**Integer(4) +␣
↪t**Integer(6)
>>> f.real_root_intervals()
[((1/2, 3/4), 1), ((1, 3/2), 1)]
```

resultant(*right*)

Return the resultant of `self` and `right`.

Enumerating the roots over **Q** as $r_1, \dots, r_m$ and $s_1, \dots, s_n$ and letting x and y denote the leading coefficients of f and g, the resultant of the two polynomials is defined by

$$x^{\deg g} y^{\deg f} \prod_{i,j} (r_i - s_j).$$

Corner cases: if one of the polynomials is zero, the resultant is zero. Note that otherwise if one of the polynomials is constant, the last term in the above is the empty product.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: f = (t - 2/3) * (t + 4/5) * (t - 1)
sage: g = (t - 1/3) * (t + 1/2) * (t + 1)
sage: f.resultant(g)
119/1350
sage: h = (t - 1/3) * (t + 1/2) * (t - 1)
sage: f.resultant(h)
0
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = (t - Integer(2)/Integer(3)) * (t + Integer(4)/Integer(5)) * (t -␣
↪Integer(1))
>>> g = (t - Integer(1)/Integer(3)) * (t + Integer(1)/Integer(2)) * (t +␣
↪Integer(1))
>>> f.resultant(g)
119/1350
>>> h = (t - Integer(1)/Integer(3)) * (t + Integer(1)/Integer(2)) * (t -␣
↪Integer(1))
>>> f.resultant(h)
0
```

reverse(*degree=None*)

Reverse the coefficients of this polynomial (thought of as a polynomial of degree `degree`).

INPUT:

- `degree` – `None` or integral value that fits in an `unsigned long` (default: degree of `self`); if specified, truncate or zero pad the list of coefficients to this degree before reversing it

EXAMPLES:

We first consider the simplest case, where we reverse all coefficients of a polynomial and obtain a polynomial of the same degree:

```
sage: R.<t> = QQ[]
sage: f = 1 + t + t^2 / 2 + t^3 / 3 + t^4 / 4
sage: f.reverse()
t^4 + t^3 + 1/2*t^2 + 1/3*t + 1/4
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = Integer(1) + t + t**Integer(2) / Integer(2) + t**Integer(3) /␣
↪Integer(3) + t**Integer(4) / Integer(4)
>>> f.reverse()
t^4 + t^3 + 1/2*t^2 + 1/3*t + 1/4
```

Next, an example where the returned polynomial has lower degree because the original polynomial has low coefficients equal to zero:

```
sage: R.<t> = QQ[]
sage: f = 3/4*t^2 + 6*t^7
sage: f.reverse()
3/4*t^5 + 6
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = Integer(3)/Integer(4)*t**Integer(2) + Integer(6)*t**Integer(7)
>>> f.reverse()
3/4*t^5 + 6
```

The next example illustrates the passing of a value for `degree` less than the length of `self`, notationally resulting in truncation prior to reversing:

```
sage: R.<t> = QQ[]
sage: f = 1 + t + t^2 / 2 + t^3 / 3 + t^4 / 4
sage: f.reverse(2)
t^2 + t + 1/2
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = Integer(1) + t + t**Integer(2) / Integer(2) + t**Integer(3) /␣
↪Integer(3) + t**Integer(4) / Integer(4)
>>> f.reverse(Integer(2))
t^2 + t + 1/2
```

Now we illustrate the passing of a value for `degree` greater than the length of `self`, notationally resulting in zero padding at the top end prior to reversing:

```
sage: R.<t> = QQ[]
sage: f = 1 + t + t^2 / 2 + t^3 / 3
sage: f.reverse(4)
t^4 + t^3 + 1/2*t^2 + 1/3*t
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = Integer(1) + t + t**Integer(2) / Integer(2) + t**Integer(3) /␣
↪Integer(3)
>>> f.reverse(Integer(4))
t^4 + t^3 + 1/2*t^2 + 1/3*t
```

revert_series(*n*)

Return a polynomial f such that `f(self(x)) = self(f(x)) = x mod x^n`.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: f = t - t^3/6 + t^5/120
sage: f.revert_series(6)
3/40*t^5 + 1/6*t^3 + t

sage: f.revert_series(-1)
Traceback (most recent call last):
ValueError: argument n must be a nonnegative integer, got -1

sage: g = - t^3/3 + t^5/5
sage: g.revert_series(6)
Traceback (most recent call last):
...
ValueError: self must have constant coefficient 0 and a unit for coefficient␣
↪t^1
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = t - t**Integer(3)/Integer(6) + t**Integer(5)/Integer(120)
>>> f.revert_series(Integer(6))
3/40*t^5 + 1/6*t^3 + t

>>> f.revert_series(-Integer(1))
Traceback (most recent call last):
ValueError: argument n must be a nonnegative integer, got -1

>>> g = - t**Integer(3)/Integer(3) + t**Integer(5)/Integer(5)
>>> g.revert_series(Integer(6))
Traceback (most recent call last):
...
ValueError: self must have constant coefficient 0 and a unit for coefficient␣
↪t^1
```

truncate(*n*)

Return `self` truncated modulo t^n.

INPUT:

- n – the power of t modulo which self is truncated

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: f = 1 - t + 1/2*t^2 - 1/3*t^3
sage: f.truncate(0)
0
sage: f.truncate(2)
-t + 1
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = Integer(1) - t + Integer(1)/Integer(2)*t**Integer(2) - Integer(1)/
↪Integer(3)*t**Integer(3)
>>> f.truncate(Integer(0))
0
>>> f.truncate(Integer(2))
-t + 1
```

xgcd(*right*)

Return polynomials d, s, and t such that d == s * self + t * right, where d is the (monic) greatest common divisor of self and right. The choice of s and t is not specified any further.

Corner cases: if self and right are zero, returns zero polynomials. Otherwise, if only self is zero, returns (d, s, t) = (right, 0, 1) up to normalisation, and similarly if only right is zero.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: f = 2/3 + 3/4 * t - t^2
sage: g = -3 + 1/7 * t
sage: f.xgcd(g)
(1, -12/5095, -84/5095*t - 1701/5095)
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> f = Integer(2)/Integer(3) + Integer(3)/Integer(4) * t - t**Integer(2)
>>> g = -Integer(3) + Integer(1)/Integer(7) * t
>>> f.xgcd(g)
(1, -12/5095, -84/5095*t - 1701/5095)
```

2.1.10 Dense univariate polynomials over $\mathbf{Z}/n\mathbf{Z}$, implemented using FLINT

This module gives a fast implementation of $(\mathbf{Z}/n\mathbf{Z})[x]$ whenever n is at most sys.maxsize. We use it by default in preference to NTL when the modulus is small, falling back to NTL if the modulus is too large, as in the example below.

EXAMPLES:

```
sage: R.<a> = PolynomialRing(Integers(100))
sage: type(a)
<class 'sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint'>
sage: R.<a> = PolynomialRing(Integers(5*2^64))
sage: type(a)
<class 'sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_dense_modn_ntl_ZZ'>
```

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```
sage: R.<a> = PolynomialRing(Integers(5*2^64), implementation="FLINT")
Traceback (most recent call last):
...
ValueError: FLINT does not support modulus 92233720368547758080
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(100)), names=('a',)); (a,) = R._first_ngens(1)
>>> type(a)
<class 'sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint'>
>>> R = PolynomialRing(Integers(Integer(5)*Integer(2)**Integer(64)), names=('a',));␣
↪(a,) = R._first_ngens(1)
>>> type(a)
<class 'sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_dense_modn_ntl_ZZ'>
>>> R = PolynomialRing(Integers(Integer(5)*Integer(2)**Integer(64)), implementation=
↪"FLINT", names=('a',)); (a,) = R._first_ngens(1)
Traceback (most recent call last):
...
ValueError: FLINT does not support modulus 92233720368547758080
```

AUTHORS:

- Burcin Erocal (2008-11) initial implementation
- Martin Albrecht (2009-01) another initial implementation

class sage.rings.polynomial.polynomial_zmod_flint.**Polynomial_template**

Bases: *Polynomial*

Template for interfacing to external C / C++ libraries for implementations of polynomials.

AUTHORS:

- Robert Bradshaw (2008-10): original idea for templating
- Martin Albrecht (2008-10): initial implementation

This file implements a simple templating engine for linking univariate polynomials to their C/C++ library implementations. It requires a 'linkage' file which implements the `celement_` functions (see `sage.libs.ntl.ntl_GF2X_linkage` for an example). Both parts are then plugged together by inclusion of the linkage file when inheriting from this class. See *sage.rings.polynomial.polynomial_gf2x* for an example.

We illustrate the generic glueing using univariate polynomials over $GF(2)$.

Note

Implementations using this template MUST implement coercion from base ring elements and `get_unsafe()`. See *Polynomial_GF2X* for an example.

degree()

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: x.degree()
1
sage: P(1).degree()
```

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```
0
sage: P(0).degree()
-1
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> x.degree()
1
>>> P(Integer(1)).degree()
0
>>> P(Integer(0)).degree()
-1
```

gcd(*other*)

Return the greatest common divisor of `self` and `other`.

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: f = x*(x+1)
sage: f.gcd(x+1)
x + 1
sage: f.gcd(x^2)
x
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> f = x*(x+Integer(1))
>>> f.gcd(x+Integer(1))
x + 1
>>> f.gcd(x**Integer(2))
x
```

get_cparent()

is_gen()

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: x.is_gen()
True
sage: (x+1).is_gen()
False
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> x.is_gen()
True
>>> (x+Integer(1)).is_gen()
False
```

is_one()

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: P(1).is_one()
True
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> P(Integer(1)).is_one()
True
```

is_zero()

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: x.is_zero()
False
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> x.is_zero()
False
```

list(*copy=True*)

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: x.list()
[0, 1]
sage: list(x)
[0, 1]
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> x.list()
[0, 1]
>>> list(x)
[0, 1]
```

quo_rem(*right*)

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: f = x^2 + x + 1
sage: f.quo_rem(x + 1)
(x, 1)
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> f = x**Integer(2) + x + Integer(1)
>>> f.quo_rem(x + Integer(1))
(x, 1)
```

shift(*n*)

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: f = x^3 + x^2 + 1
sage: f.shift(1)
x^4 + x^3 + x
sage: f.shift(-1)
x^2 + x
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> f = x**Integer(3) + x**Integer(2) + Integer(1)
>>> f.shift(Integer(1))
x^4 + x^3 + x
>>> f.shift(-Integer(1))
x^2 + x
```

truncate(*n*)

Return this polynomial mod x^n.

EXAMPLES:

```
sage: R.<x> =GF(2)[]
sage: f = sum(x^n for n in range(10)); f
x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1
sage: f.truncate(6)
x^5 + x^4 + x^3 + x^2 + x + 1
```

```
>>> from sage.all import *
>>> R = GF(Integer(2))['x']; (x,) = R._first_ngens(1)
>>> f = sum(x**n for n in range(Integer(10))); f
x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1
>>> f.truncate(Integer(6))
x^5 + x^4 + x^3 + x^2 + x + 1
```

If the precision is higher than the degree of the polynomial then the polynomial itself is returned:

```
sage: f.truncate(10) is f
True
```

```
>>> from sage.all import *
>>> f.truncate(Integer(10)) is f
True
```

If the precision is negative, the zero polynomial is returned:

```
sage: f.truncate(-1)
0
```

```
>>> from sage.all import *
>>> f.truncate(-Integer(1))
0
```

xgcd(*other*)

Compute extended gcd of `self` and `other`.

EXAMPLES:

```
sage: P.<x> = GF(7)[]
sage: f = x*(x+1)
sage: f.xgcd(x+1)
(x + 1, 0, 1)
sage: f.xgcd(x^2)
(x, 1, 6)
```

```
>>> from sage.all import *
>>> P = GF(Integer(7))['x']; (x,) = P._first_ngens(1)
>>> f = x*(x+Integer(1))
>>> f.xgcd(x+Integer(1))
(x + 1, 0, 1)
>>> f.xgcd(x**Integer(2))
(x, 1, 6)
```

class sage.rings.polynomial.polynomial_zmod_flint.**Polynomial_zmod_flint**

Bases: *Polynomial_template*

Polynomial on $\mathbf{Z}/n\mathbf{Z}$ implemented via FLINT.

add(*right*)

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: x + 1
x + 1
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> x + Integer(1)
x + 1
```

sub(*right*)

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: x - 1
x + 1
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> x - Integer(1)
x + 1
```

lmul(*left*)

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: t = x^2 + x + 1
sage: 0*t
0
sage: 1*t
x^2 + x + 1

sage: R.<y> = GF(5)[]
sage: u = y^2 + y + 1
sage: 3*u
3*y^2 + 3*y + 3
sage: 5*u
0
sage: (2^81)*u
2*y^2 + 2*y + 2
sage: (-2^81)*u
3*y^2 + 3*y + 3
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> t = x**Integer(2) + x + Integer(1)
>>> Integer(0)*t
0
>>> Integer(1)*t
x^2 + x + 1

>>> R = GF(Integer(5))['y']; (y,) = R._first_ngens(1)
>>> u = y**Integer(2) + y + Integer(1)
>>> Integer(3)*u
3*y^2 + 3*y + 3
>>> Integer(5)*u
0
>>> (Integer(2)**Integer(81))*u
2*y^2 + 2*y + 2
>>> (-Integer(2)**Integer(81))*u
3*y^2 + 3*y + 3
```

```
sage: P.<x> = GF(2)[]
sage: t = x^2 + x + 1
sage: t*0
0
sage: t*1
x^2 + x + 1

sage: R.<y> = GF(5)[]
sage: u = y^2 + y + 1
sage: u*3
3*y^2 + 3*y + 3
sage: u*5
0
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> t = x**Integer(2) + x + Integer(1)
>>> t*Integer(0)
0
>>> t*Integer(1)
x^2 + x + 1

>>> R = GF(Integer(5))['y']; (y,) = R._first_ngens(1)
>>> u = y**Integer(2) + y + Integer(1)
>>> u*Integer(3)
3*y^2 + 3*y + 3
>>> u*Integer(5)
0
```

rmul(*right*)

Multiply `self` on the right by a scalar.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: f = (x^3 + x + 5)
sage: f._rmul_(7)
7*x^3 + 7*x + 35
sage: f*7
7*x^3 + 7*x + 35
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> f = (x**Integer(3) + x + Integer(5))
>>> f._rmul_(Integer(7))
7*x^3 + 7*x + 35
>>> f*Integer(7)
7*x^3 + 7*x + 35
```

mul(*right*)

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: x*(x+1)
x^2 + x
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> x*(x+Integer(1))
x^2 + x
```

_mul_trunc_(*right*, *n*)

Return the product of this polynomial and other truncated to the given length n.

This function is usually more efficient than simply doing the multiplication and then truncating. The function is tuned for length n about half the length of a full product.

EXAMPLES:

```
sage: P.<a> = GF(7)[]
sage: a = P(range(10)); b = P(range(5, 15))
sage: a._mul_trunc_(b, 5)
4*a^4 + 6*a^3 + 2*a^2 + 5*a
```

```
>>> from sage.all import *
>>> P = GF(Integer(7))['a']; (a,) = P._first_ngens(1)
>>> a = P(range(Integer(10))); b = P(range(Integer(5), Integer(15)))
>>> a._mul_trunc_(b, Integer(5))
4*a^4 + 6*a^3 + 2*a^2 + 5*a
```

compose_mod(*other*, *modulus*)

Compute $f(g) \bmod h$.

To be precise about the order fo compostion, given `self`, `other` and `modulus` as $f(x)$, $g(x)$ and $h(x)$ compute $f(g(x)) \bmod h(x)$.

INPUT:

- `other` – a polynomial $g(x)$
- `modulus` – a polynomial $h(x)$

EXAMPLES:

```
sage: R.<x> = GF(163)[]
sage: f = R.random_element()
sage: g = R.random_element()
sage: g.compose_mod(g, f) == g(g) % f
True

sage: f = R([i for i in range(100)])
sage: g = R([i**2 for i in range(100)])
sage: h = 1 + x + x**5
sage: f.compose_mod(g, h)
82*x^4 + 56*x^3 + 45*x^2 + 60*x + 127
sage: f.compose_mod(g, h) == f(g) % h
True
```

```
>>> from sage.all import *
>>> R = GF(Integer(163))['x']; (x,) = R._first_ngens(1)
>>> f = R.random_element()
>>> g = R.random_element()
>>> g.compose_mod(g, f) == g(g) % f
True

>>> f = R([i for i in range(Integer(100))])
>>> g = R([i**Integer(2) for i in range(Integer(100))])
>>> h = Integer(1) + x + x**Integer(5)
>>> f.compose_mod(g, h)
82*x^4 + 56*x^3 + 45*x^2 + 60*x + 127
>>> f.compose_mod(g, h) == f(g) % h
True
```

AUTHORS:

- Giacomo Pope (2024-08) initial implementation

factor()

Return the factorization of the polynomial.

EXAMPLES:

```
sage: R.<x> = GF(5)[]
sage: (x^2 + 1).factor()
(x + 2) * (x + 3)
```

```
>>> from sage.all import *
>>> R = GF(Integer(5))['x']; (x,) = R._first_ngens(1)
>>> (x**Integer(2) + Integer(1)).factor()
(x + 2) * (x + 3)
```

It also works for prime-power moduli:

```
sage: R.<x> = Zmod(23^5)[]
sage: (x^3 + 1).factor()
(x + 1) * (x^2 + 6436342*x + 1)
```

```
>>> from sage.all import *
>>> R = Zmod(Integer(23)**Integer(5))['x']; (x,) = R._first_ngens(1)
>>> (x**Integer(3) + Integer(1)).factor()
(x + 1) * (x^2 + 6436342*x + 1)
```

is_irreducible()

Return whether this polynomial is irreducible.

EXAMPLES:

```
sage: R.<x> = GF(5)[]
sage: (x^2 + 1).is_irreducible()
False
sage: (x^3 + x + 1).is_irreducible()
True
```

```
>>> from sage.all import *
>>> R = GF(Integer(5))['x']; (x,) = R._first_ngens(1)
>>> (x**Integer(2) + Integer(1)).is_irreducible()
False
>>> (x**Integer(3) + x + Integer(1)).is_irreducible()
True
```

Not implemented when the base ring is not a field:

```
sage: S.<s> = Zmod(10)[]
sage: (s^2).is_irreducible()
Traceback (most recent call last):
...
NotImplementedError: checking irreducibility of polynomials
over rings with composite characteristic is not implemented
```

```
>>> from sage.all import *
>>> S = Zmod(Integer(10))['s']; (s,) = S._first_ngens(1)
>>> (s**Integer(2)).is_irreducible()
Traceback (most recent call last):
...
NotImplementedError: checking irreducibility of polynomials
over rings with composite characteristic is not implemented
```

minpoly_mod(*other*)

Thin wrapper for *sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_dense_mod_n.minpoly_mod()*.

EXAMPLES:

```
sage: R.<x> = GF(127)[]
sage: type(x)
<class 'sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint'>
sage: (x^5 - 3).minpoly_mod(x^3 + 5*x - 1)
x^3 + 34*x^2 + 125*x + 95
```

```
>>> from sage.all import *
>>> R = GF(Integer(127))['x']; (x,) = R._first_ngens(1)
>>> type(x)
<class 'sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint'>
>>> (x**Integer(5) - Integer(3)).minpoly_mod(x**Integer(3) + Integer(5)*x -
↪Integer(1))
x^3 + 34*x^2 + 125*x + 95
```

modular_composition(*other*, *modulus*)

Compute $f(g)$ mod h.

To be precise about the order fo compostion, given `self`, `other` and `modulus` as $f(x)$, $g(x)$ and $h(x)$ compute $f(g(x))$ mod $h(x)$.

INPUT:

- `other` – a polynomial $g(x)$
- `modulus` – a polynomial $h(x)$

EXAMPLES:

```
sage: R.<x> = GF(163)[]
sage: f = R.random_element()
sage: g = R.random_element()
sage: g.compose_mod(g, f) == g(g) % f
True

sage: f = R([i for i in range(100)])
sage: g = R([i**2 for i in range(100)])
sage: h = 1 + x + x**5
sage: f.compose_mod(g, h)
82*x^4 + 56*x^3 + 45*x^2 + 60*x + 127
sage: f.compose_mod(g, h) == f(g) % h
True
```

```
>>> from sage.all import *
>>> R = GF(Integer(163))['x']; (x,) = R._first_ngens(1)
>>> f = R.random_element()
>>> g = R.random_element()
>>> g.compose_mod(g, f) == g(g) % f
True

>>> f = R([i for i in range(Integer(100))])
>>> g = R([i**Integer(2) for i in range(Integer(100))])
>>> h = Integer(1) + x + x**Integer(5)
>>> f.compose_mod(g, h)
82*x^4 + 56*x^3 + 45*x^2 + 60*x + 127
>>> f.compose_mod(g, h) == f(g) % h
True
```

AUTHORS:

- Giacomo Pope (2024-08) initial implementation

monic()

Return this polynomial divided by its leading coefficient.

Raises `ValueError` if the leading coefficient is not invertible in the base ring.

EXAMPLES:

```
sage: R.<x> = GF(5)[]
sage: (2*x^2 + 1).monic()
x^2 + 3
```

```
>>> from sage.all import *
>>> R = GF(Integer(5))['x']; (x,) = R._first_ngens(1)
>>> (Integer(2)*x**Integer(2) + Integer(1)).monic()
x^2 + 3
```

rational_reconstruct(**args*, ***kwds*)

Deprecated: Use `rational_reconstruction()` instead. See Issue #12696 for details.

rational_reconstruction(*m*, *n_deg=0*, *d_deg=0*)

Construct a rational function n/d such that $p * d$ is equivalent to n modulo m where p is this polynomial.

EXAMPLES:

```
sage: P.<x> = GF(5)[]
sage: p = 4*x^5 + 3*x^4 + 2*x^3 + 2*x^2 + 4*x + 2
sage: n, d = p.rational_reconstruction(x^9, 4, 4); n, d
(3*x^4 + 2*x^3 + x^2 + 2*x, x^4 + 3*x^3 + x^2 + x)
sage: (p*d % x^9) == n
True
```

```
>>> from sage.all import *
>>> P = GF(Integer(5))['x']; (x,) = P._first_ngens(1)
>>> p = Integer(4)*x**Integer(5) + Integer(3)*x**Integer(4) +
↪Integer(2)*x**Integer(3) + Integer(2)*x**Integer(2) + Integer(4)*x +
↪Integer(2)
```

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```
>>> n, d = p.rational_reconstruction(x**Integer(9), Integer(4), Integer(4));␣
↪n, d
(3*x^4 + 2*x^3 + x^2 + 2*x, x^4 + 3*x^3 + x^2 + x)
>>> (p*d % x**Integer(9)) == n
True
```

Check that Issue #37169 is fixed - it does not throw an error:

```
sage: R.<x> = Zmod(4)[]
sage: _.<z> = R.quotient_ring(x^2 - 1)
sage: c = 2 * z + 1
sage: c * Zmod(2).zero()
Traceback (most recent call last):
...
RuntimeError: Aborted
```

```
>>> from sage.all import *
>>> R = Zmod(Integer(4))['x']; (x,) = R._first_ngens(1)
>>> _ = R.quotient_ring(x**Integer(2) - Integer(1), names=('z',)); (z,) = _._
↪first_ngens(1)
>>> c = Integer(2) * z + Integer(1)
>>> c * Zmod(Integer(2)).zero()
Traceback (most recent call last):
...
RuntimeError: Aborted
```

resultant(*other*)

Return the resultant of `self` and `other`, which must lie in the same polynomial ring.

INPUT:

- `other` – a polynomial

OUTPUT: an element of the base ring of the polynomial ring

EXAMPLES:

```
sage: R.<x> = GF(19)['x']
sage: f = x^3 + x + 1;  g = x^3 - x - 1
sage: r = f.resultant(g); r
11
sage: r.parent() is GF(19)
True
```

```
>>> from sage.all import *
>>> R = GF(Integer(19))['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(3) + x + Integer(1);  g = x**Integer(3) - x - Integer(1)
>>> r = f.resultant(g); r
11
>>> r.parent() is GF(Integer(19))
True
```

The following example shows that Issue #11782 has been fixed:

```
sage: R.<x> = ZZ.quo(9)['x']
sage: f = 2*x^3 + x^2 + x;  g = 6*x^2 + 2*x + 1
sage: f.resultant(g)
5
```

```
>>> from sage.all import *
>>> R = ZZ.quo(Integer(9))['x']; (x,) = R._first_ngens(1)
>>> f = Integer(2)*x**Integer(3) + x**Integer(2) + x;  g =␣
↪Integer(6)*x**Integer(2) + Integer(2)*x + Integer(1)
>>> f.resultant(g)
5
```

reverse(*degree=None*)

Return a polynomial with the coefficients of this polynomial reversed.

If the optional argument `degree` is given, the coefficient list will be truncated or zero padded as necessary before computing the reverse.

EXAMPLES:

```
sage: R.<x> = GF(5)[]
sage: p = R([1,2,3,4]); p
4*x^3 + 3*x^2 + 2*x + 1
sage: p.reverse()
x^3 + 2*x^2 + 3*x + 4
sage: p.reverse(degree=6)
x^6 + 2*x^5 + 3*x^4 + 4*x^3
sage: p.reverse(degree=2)
x^2 + 2*x + 3

sage: R.<x> = GF(101)[]
sage: f = x^3 - x + 2; f
x^3 + 100*x + 2
sage: f.reverse()
2*x^3 + 100*x^2 + 1
sage: f.reverse() == f(1/x) * x^f.degree()
True
```

```
>>> from sage.all import *
>>> R = GF(Integer(5))['x']; (x,) = R._first_ngens(1)
>>> p = R([Integer(1),Integer(2),Integer(3),Integer(4)]); p
4*x^3 + 3*x^2 + 2*x + 1
>>> p.reverse()
x^3 + 2*x^2 + 3*x + 4
>>> p.reverse(degree=Integer(6))
x^6 + 2*x^5 + 3*x^4 + 4*x^3
>>> p.reverse(degree=Integer(2))
x^2 + 2*x + 3

>>> R = GF(Integer(101))['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(3) - x + Integer(2); f
x^3 + 100*x + 2
>>> f.reverse()
```

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```
2*x^3 + 100*x^2 + 1
>>> f.reverse() == f(Integer(1)/x) * x**f.degree()
True
```

Note that if f has zero constant coefficient, its reverse will have lower degree.

```
sage: f = x^3 + 2*x
sage: f.reverse()
2*x^2 + 1
```

```
>>> from sage.all import *
>>> f = x**Integer(3) + Integer(2)*x
>>> f.reverse()
2*x^2 + 1
```

In this case, reverse is not an involution unless we explicitly specify a degree.

```
sage: f
x^3 + 2*x
sage: f.reverse().reverse()
x^2 + 2
sage: f.reverse(5).reverse(5)
x^3 + 2*x
```

```
>>> from sage.all import *
>>> f
x^3 + 2*x
>>> f.reverse().reverse()
x^2 + 2
>>> f.reverse(Integer(5)).reverse(Integer(5))
x^3 + 2*x
```

revert_series(n)

Return a polynomial f such that `f(self(x)) = self(f(x)) = x` (mod x^n).

EXAMPLES:

```
sage: R.<t> =  GF(5)[]
sage: f = t + 2*t^2 - t^3 - 3*t^4
sage: f.revert_series(5)
3*t^4 + 4*t^3 + 3*t^2 + t

sage: f.revert_series(-1)
Traceback (most recent call last):
...
ValueError: argument n must be a nonnegative integer, got -1

sage: g = - t^3 + t^5
sage: g.revert_series(6)
Traceback (most recent call last):
...
ValueError: self must have constant coefficient 0 and a unit for coefficient
```

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```
↪t^1

sage: g = t + 2*t^2 - t^3 -3*t^4 + t^5
sage: g.revert_series(6)
Traceback (most recent call last):
...
ValueError: the integers 1 up to n=5 are required to be invertible over the␣
↪base field
```

```
>>> from sage.all import *
>>> R = GF(Integer(5))['t']; (t,) = R._first_ngens(1)
>>> f = t + Integer(2)*t**Integer(2) - t**Integer(3) -␣
↪Integer(3)*t**Integer(4)
>>> f.revert_series(Integer(5))
3*t^4 + 4*t^3 + 3*t^2 + t

>>> f.revert_series(-Integer(1))
Traceback (most recent call last):
...
ValueError: argument n must be a nonnegative integer, got -1

>>> g = - t**Integer(3) + t**Integer(5)
>>> g.revert_series(Integer(6))
Traceback (most recent call last):
...
ValueError: self must have constant coefficient 0 and a unit for coefficient␣
↪t^1

>>> g = t + Integer(2)*t**Integer(2) - t**Integer(3) -
↪Integer(3)*t**Integer(4) + t**Integer(5)
>>> g.revert_series(Integer(6))
Traceback (most recent call last):
...
ValueError: the integers 1 up to n=5 are required to be invertible over the␣
↪base field
```

small_roots(**args*, ***kwds*)

See `sage.rings.polynomial.polynomial_modn_dense_ntl.small_roots()` for the documentation of this function.

EXAMPLES:

```
sage: N = 10001
sage: K = Zmod(10001)
sage: P.<x> = PolynomialRing(K)
sage: f = x^3 + 10*x^2 + 5000*x - 222
sage: f.small_roots()
[4]
```

```
>>> from sage.all import *
>>> N = Integer(10001)
>>> K = Zmod(Integer(10001))
```

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```
>>> P = PolynomialRing(K, names=('x',)); (x,) = P._first_ngens(1)
>>> f = x**Integer(3) + Integer(10)*x**Integer(2) + Integer(5000)*x -␣
↪Integer(222)
>>> f.small_roots()
[4]
```

squarefree_decomposition()

Return the squarefree decomposition of this polynomial.

EXAMPLES:

```
sage: R.<x> = GF(5)[]
sage: ((x+1)*(x^2+1)^2*x^3).squarefree_decomposition()
(x + 1) * (x^2 + 1)^2 * x^3
```

```
>>> from sage.all import *
>>> R = GF(Integer(5))['x']; (x,) = R._first_ngens(1)
>>> ((x+Integer(1))*(x**Integer(2)+Integer(1))**Integer(2)*x**Integer(3)).
↪squarefree_decomposition()
(x + 1) * (x^2 + 1)^2 * x^3
```

sage.rings.polynomial.polynomial_zmod_flint.**make_element**(*parent*, *args*)

2.1.11 Dense univariate polynomials over $\mathbf{Z}/n\mathbf{Z}$, implemented using NTL

This implementation is generally slower than the FLINT implementation in *polynomial_zmod_flint*, so we use FLINT by default when the modulus is small enough; but NTL does not require that n be int-sized, so we use it as default when n is too large for FLINT.

Note that the classes *Polynomial_dense_modn_ntl_zz* and *Polynomial_dense_modn_ntl_ZZ* are different; the former is limited to moduli less than a certain bound, while the latter supports arbitrarily large moduli.

AUTHORS:

- Robert Bradshaw: Split off from polynomial_element_generic.py (2007-09)
- Robert Bradshaw: Major rewrite to use NTL directly (2007-09)

class sage.rings.polynomial.polynomial_modn_dense_ntl.**Polynomial_dense_mod_n**

Bases: *Polynomial*

A dense polynomial over the integers modulo n, where n is composite, with the underlying arithmetic done using NTL.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(Integers(16), implementation='NTL')
sage: f = x^3 - x + 17
sage: f^2
x^6 + 14*x^4 + 2*x^3 + x^2 + 14*x + 1

sage: loads(f.dumps()) == f
True

sage: R.<x> = PolynomialRing(Integers(100), implementation='NTL')
```

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```
sage: p = 3*x
sage: q = 7*x
sage: p + q
10*x
sage: R.<x> = PolynomialRing(Integers(8), implementation='NTL')
sage: parent(p)
Univariate Polynomial Ring in x over Ring of integers modulo 100 (using NTL)
sage: p + q
10*x
sage: R({10:-1})
7*x^10
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(16)), implementation='NTL', names=('x',));
↪ (x,) = R._first_ngens(1)
>>> f = x**Integer(3) - x + Integer(17)
>>> f**Integer(2)
x^6 + 14*x^4 + 2*x^3 + x^2 + 14*x + 1

>>> loads(f.dumps()) == f
True

>>> R = PolynomialRing(Integers(Integer(100)), implementation='NTL', names=('x',
↪)); (x,) = R._first_ngens(1)
>>> p = Integer(3)*x
>>> q = Integer(7)*x
>>> p + q
10*x
>>> R = PolynomialRing(Integers(Integer(8)), implementation='NTL', names=('x',));
↪(x,) = R._first_ngens(1)
>>> parent(p)
Univariate Polynomial Ring in x over Ring of integers modulo 100 (using NTL)
>>> p + q
10*x
>>> R({Integer(10):-Integer(1)})
7*x^10
```

compose_mod(*other*, *modulus*)

Compute $f(g) \pmod{h}$.

To be precise about the order fo compostion, given `self`, `other` and `modulus` as $f(x)$, $g(x)$ and $h(x)$ compute $f(g(x)) \bmod h(x)$.

INPUT:

- `other` – a polynomial $g(x)$
- `modulus` – a polynomial $h(x)$

EXAMPLES:

```
sage: R.<x> = GF(2**127 - 1)[]
sage: f = R.random_element()
sage: g = R.random_element()
```

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```
sage: g.compose_mod(g, f) == g(g) % f
True

sage: R.<x> = GF(163)[]
sage: f = R([i for i in range(100)])
sage: g = R([i**2 for i in range(100)])
sage: h = 1 + x + x**5
sage: f.compose_mod(g, h)
82*x^4 + 56*x^3 + 45*x^2 + 60*x + 127
sage: f.compose_mod(g, h) == f(g) % h
True
```

```
>>> from sage.all import *
>>> R = GF(Integer(2)**Integer(127) - Integer(1))['x']; (x,) = R._first_
↪ngens(1)
>>> f = R.random_element()
>>> g = R.random_element()
>>> g.compose_mod(g, f) == g(g) % f
True

>>> R = GF(Integer(163))['x']; (x,) = R._first_ngens(1)
>>> f = R([i for i in range(Integer(100))])
>>> g = R([i**Integer(2) for i in range(Integer(100))])
>>> h = Integer(1) + x + x**Integer(5)
>>> f.compose_mod(g, h)
82*x^4 + 56*x^3 + 45*x^2 + 60*x + 127
>>> f.compose_mod(g, h) == f(g) % h
True
```

AUTHORS:

- Giacomo Pope (2024-08) initial implementation

degree (*gen=None*)

Return the degree of this polynomial.

The zero polynomial has degree -1.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(Integers(100), implementation='NTL')
sage: (x^3 + 3*x - 17).degree()
3
sage: R.zero().degree()
-1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(100)), implementation='NTL', names=('x
↪',)); (x,) = R._first_ngens(1)
>>> (x**Integer(3) + Integer(3)*x - Integer(17)).degree()
3
>>> R.zero().degree()
-1
```

int_list()

list(*copy=True*)

Return a new copy of the list of the underlying elements of `self`.

EXAMPLES:

```
sage: _.<x> = PolynomialRing(Integers(100), implementation='NTL')
sage: f = x^3 + 3*x - 17
sage: f.list()
[83, 3, 0, 1]
```

```
>>> from sage.all import *
>>> _ = PolynomialRing(Integers(Integer(100)), implementation='NTL', names=('x
↪',)); (x,) = _._first_ngens(1)
>>> f = x**Integer(3) + Integer(3)*x - Integer(17)
>>> f.list()
[83, 3, 0, 1]
```

minpoly_mod(*other*)

Compute the minimal polynomial of this polynomial modulo another polynomial in the same ring.

ALGORITHM:

NTL's `MinPolyMod()`, which uses Shoup's algorithm [Sho1999].

EXAMPLES:

```
sage: R.<x> = PolynomialRing(GF(101), implementation='NTL')
sage: f = x^17 + x^2 - 1
sage: (x^2).minpoly_mod(f)
x^17 + 100*x^2 + 2*x + 100
```

```
>>> from sage.all import *
>>> R = PolynomialRing(GF(Integer(101)), implementation='NTL', names=('x',));
↪(x,) = R._first_ngens(1)
>>> f = x**Integer(17) + x**Integer(2) - Integer(1)
>>> (x**Integer(2)).minpoly_mod(f)
x^17 + 100*x^2 + 2*x + 100
```

modular_composition(*other*, *modulus*)

Compute $f(g) \pmod{h}$.

To be precise about the order fo compostion, given `self`, `other` and `modulus` as $f(x)$, $g(x)$ and $h(x)$ compute $f(g(x)) \bmod h(x)$.

INPUT:

- `other` – a polynomial $g(x)$
- `modulus` – a polynomial $h(x)$

EXAMPLES:

```
sage: R.<x> = GF(2**127 - 1)[]
sage: f = R.random_element()
sage: g = R.random_element()
```

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```
sage: g.compose_mod(g, f) == g(g) % f
True

sage: R.<x> = GF(163)[]
sage: f = R([i for i in range(100)])
sage: g = R([i**2 for i in range(100)])
sage: h = 1 + x + x**5
sage: f.compose_mod(g, h)
82*x^4 + 56*x^3 + 45*x^2 + 60*x + 127
sage: f.compose_mod(g, h) == f(g) % h
True
```

```
>>> from sage.all import *
>>> R = GF(Integer(2)**Integer(127) - Integer(1))['x']; (x,) = R._first_
↪ngens(1)
>>> f = R.random_element()
>>> g = R.random_element()
>>> g.compose_mod(g, f) == g(g) % f
True

>>> R = GF(Integer(163))['x']; (x,) = R._first_ngens(1)
>>> f = R([i for i in range(Integer(100))])
>>> g = R([i**Integer(2) for i in range(Integer(100))])
>>> h = Integer(1) + x + x**Integer(5)
>>> f.compose_mod(g, h)
82*x^4 + 56*x^3 + 45*x^2 + 60*x + 127
>>> f.compose_mod(g, h) == f(g) % h
True
```

AUTHORS:

- Giacomo Pope (2024-08) initial implementation

ntl_ZZ_pX()

Return underlying NTL representation of this polynomial. Additional "bonus" functionality is available through this function.

> **Warning**
>
> You must call `ntl.set_modulus(ntl.ZZ(n))` before doing arithmetic with this object!

ntl_set_directly(*v*)

Set the value of this polynomial directly from a vector or string.

Polynomials over the integers modulo n are stored internally using NTL's `ZZ_pX` class. Use this function to set the value of this polynomial using the NTL constructor, which is potentially *very* fast. The input v is either a vector of ints or a string of the form `[ n1 n2 n3 ... ]` where the ni are integers and there are no commas between them. The optimal input format is the string format, since that's what NTL uses by default.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(Integers(100), implementation='NTL')
sage: from sage.rings.polynomial.polynomial_modn_dense_ntl import Polynomial_
↪dense_mod_n as poly_modn_dense
sage: poly_modn_dense(R, ([1,-2,3]))
3*x^2 + 98*x + 1
sage: f = poly_modn_dense(R, 0)
sage: f.ntl_set_directly([1,-2,3])
sage: f
3*x^2 + 98*x + 1
sage: f.ntl_set_directly('[1 -2 3 4]')
sage: f
4*x^3 + 3*x^2 + 98*x + 1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(100)), implementation='NTL', names=('x
↪',)); (x,) = R._first_ngens(1)
>>> from sage.rings.polynomial.polynomial_modn_dense_ntl import Polynomial_
↪dense_mod_n as poly_modn_dense
>>> poly_modn_dense(R, ([Integer(1),-Integer(2),Integer(3)]))
3*x^2 + 98*x + 1
>>> f = poly_modn_dense(R, Integer(0))
>>> f.ntl_set_directly([Integer(1),-Integer(2),Integer(3)])
>>> f
3*x^2 + 98*x + 1
>>> f.ntl_set_directly('[1 -2 3 4]')
>>> f
4*x^3 + 3*x^2 + 98*x + 1
```

quo_rem(*right*)

Return a tuple `(quotient, remainder)` where `self = quotient*other + remainder`.

shift(*n*)

Return this polynomial multiplied by the power x^n. If n is negative, terms below x^n will be discarded. Does not change this polynomial.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(Integers(12345678901234567890), implementation=
↪'NTL')
sage: p = x^2 + 2*x + 4
sage: p.shift(0)
 x^2 + 2*x + 4
sage: p.shift(-1)
 x + 2
sage: p.shift(-5)
 0
sage: p.shift(2)
 x^4 + 2*x^3 + 4*x^2
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(12345678901234567890)),
↪implementation='NTL', names=('x',)); (x,) = R._first_ngens(1)
>>> p = x**Integer(2) + Integer(2)*x + Integer(4)
```

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```
>>> p.shift(Integer(0))
 x^2 + 2*x + 4
>>> p.shift(-Integer(1))
 x + 2
>>> p.shift(-Integer(5))
 0
>>> p.shift(Integer(2))
 x^4 + 2*x^3 + 4*x^2
```

AUTHOR:

- David Harvey (2006-08-06)

small_roots(**args*, ***kwds*)

See *sage.rings.polynomial.polynomial_modn_dense_ntl.small_roots()* for the documentation of this function.

EXAMPLES:

```
sage: N = 10001
sage: K = Zmod(10001)
sage: P.<x> = PolynomialRing(K, implementation='NTL')
sage: f = x^3 + 10*x^2 + 5000*x - 222
sage: f.small_roots()
[4]
```

```
>>> from sage.all import *
>>> N = Integer(10001)
>>> K = Zmod(Integer(10001))
>>> P = PolynomialRing(K, implementation='NTL', names=('x',)); (x,) = P._
↪first_ngens(1)
>>> f = x**Integer(3) + Integer(10)*x**Integer(2) + Integer(5000)*x -␣
↪Integer(222)
>>> f.small_roots()
[4]
```

class sage.rings.polynomial.polynomial_modn_dense_ntl.**Polynomial_dense_mod_p**

Bases: *Polynomial_dense_mod_n*

A dense polynomial over the integers modulo p, where p is prime.

discriminant()

EXAMPLES:

```
sage: _.<x> = PolynomialRing(GF(19), implementation='NTL')
sage: f = x^3 + 3*x - 17
sage: f.discriminant()
12
```

```
>>> from sage.all import *
>>> _ = PolynomialRing(GF(Integer(19)), implementation='NTL', names=('x',));␣
↪(x,) = _._first_ngens(1)
>>> f = x**Integer(3) + Integer(3)*x - Integer(17)
```

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```
>>> f.discriminant()
12
```

gcd(*right*)

Return the greatest common divisor of this polynomial and `other`, as a monic polynomial.

INPUT:

- `other` – a polynomial defined over the same ring as `self`

EXAMPLES:

```
sage: R.<x> = PolynomialRing(GF(3), implementation="NTL")
sage: f, g = x + 2, x^2 - 1
sage: f.gcd(g)
x + 2
```

```
>>> from sage.all import *
>>> R = PolynomialRing(GF(Integer(3)), implementation="NTL", names=('x',));
↪(x,) = R._first_ngens(1)
>>> f, g = x + Integer(2), x**Integer(2) - Integer(1)
>>> f.gcd(g)
x + 2
```

resultant(*other*)

Return the resultant of `self` and `other`, which must lie in the same polynomial ring.

INPUT:

- `other` – a polynomial

OUTPUT: an element of the base ring of the polynomial ring

EXAMPLES:

```
sage: R.<x> = PolynomialRing(GF(19), implementation='NTL')
sage: f = x^3 + x + 1;  g = x^3 - x - 1
sage: r = f.resultant(g); r
11
sage: r.parent() is GF(19)
True
```

```
>>> from sage.all import *
>>> R = PolynomialRing(GF(Integer(19)), implementation='NTL', names=('x',));
↪(x,) = R._first_ngens(1)
>>> f = x**Integer(3) + x + Integer(1);  g = x**Integer(3) - x - Integer(1)
>>> r = f.resultant(g); r
11
>>> r.parent() is GF(Integer(19))
True
```

xgcd(*other*)

Compute the extended gcd of this element and `other`.

INPUT:

- `other` – an element in the same polynomial ring

OUTPUT:

A tuple `(r,s,t)` of elements in the polynomial ring such that `r = s*self + t*other`.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(GF(3), implementation='NTL')
sage: x.xgcd(x)
(x, 0, 1)
sage: (x^2 - 1).xgcd(x - 1)
(x + 2, 0, 1)
sage: R.zero().xgcd(R.one())
(1, 0, 1)
sage: (x^3 - 1).xgcd((x - 1)^2)
(x^2 + x + 1, 0, 1)
sage: ((x - 1)*(x + 1)).xgcd(x*(x - 1))
(x + 2, 1, 2)
```

```
>>> from sage.all import *
>>> R = PolynomialRing(GF(Integer(3)), implementation='NTL', names=('x',));
↪(x,) = R._first_ngens(1)
>>> x.xgcd(x)
(x, 0, 1)
>>> (x**Integer(2) - Integer(1)).xgcd(x - Integer(1))
(x + 2, 0, 1)
>>> R.zero().xgcd(R.one())
(1, 0, 1)
>>> (x**Integer(3) - Integer(1)).xgcd((x - Integer(1))**Integer(2))
(x^2 + x + 1, 0, 1)
>>> ((x - Integer(1))*(x + Integer(1))).xgcd(x*(x - Integer(1)))
(x + 2, 1, 2)
```

class sage.rings.polynomial.polynomial_modn_dense_ntl.**Polynomial_dense_modn_ntl_ZZ**

Bases: *Polynomial_dense_mod_n*

degree()

EXAMPLES:

```
sage: R.<x> = PolynomialRing(Integers(14^34), implementation='NTL')
sage: f = x^4 - x - 1
sage: f.degree()
4
sage: f = 14^43*x + 1
sage: f.degree()
0
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(14)**Integer(34)), implementation='NTL
↪', names=('x',)); (x,) = R._first_ngens(1)
>>> f = x**Integer(4) - x - Integer(1)
>>> f.degree()
4
>>> f = Integer(14)**Integer(43)*x + Integer(1)
```

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```
>>> f.degree()
0
```

is_gen()

list(*copy=True*)

quo_rem(*right*)

Return q and r, with the degree of r less than the degree of `right`, such that $q\cdot$ `right` $+r =$ `self`.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(Integers(10^30), implementation='NTL')
sage: f = x^5+1; g = (x+1)^2
sage: q, r = f.quo_rem(g)
sage: q
x^3 + 999999999999999999999999999998*x^2 + 3*x +␣
↪999999999999999999999999999996
sage: r
5*x + 5
sage: q*g + r
x^5 + 1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(10)**Integer(30)), implementation='NTL
↪', names=('x',)); (x,) = R._first_ngens(1)
>>> f = x**Integer(5)+Integer(1); g = (x+Integer(1))**Integer(2)
>>> q, r = f.quo_rem(g)
>>> q
x^3 + 999999999999999999999999999998*x^2 + 3*x +␣
↪999999999999999999999999999996
>>> r
5*x + 5
>>> q*g + r
x^5 + 1
```

reverse(*degree=None*)

Return the reverse of the input polynomial thought as a polynomial of degree `degree`.

If f is a degree-d polynomial, its reverse is $x^d f(1/x)$.

INPUT:

- `degree` – `None` or integer; if specified, truncate or zero pad the list of coefficients to this degree before reversing it

EXAMPLES:

```
sage: R.<x> = PolynomialRing(Integers(12^29), implementation='NTL')
sage: f = x^4 + 2*x + 5
sage: f.reverse()
5*x^4 + 2*x^3 + 1
sage: f = x^3 + x
sage: f.reverse()
x^2 + 1
```

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```
sage: f.reverse(1)
1
sage: f.reverse(5)
x^4 + x^2
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(12)**Integer(29)), implementation='NTL
↪', names=('x',)); (x,) = R._first_ngens(1)
>>> f = x**Integer(4) + Integer(2)*x + Integer(5)
>>> f.reverse()
5*x^4 + 2*x^3 + 1
>>> f = x**Integer(3) + x
>>> f.reverse()
x^2 + 1
>>> f.reverse(Integer(1))
1
>>> f.reverse(Integer(5))
x^4 + x^2
```

shift(n)

Shift `self` to left by n, which is multiplication by x^n, truncating if n is negative.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(Integers(12^30), implementation='NTL')
sage: f = x^7 + x + 1
sage: f.shift(1)
x^8 + x^2 + x
sage: f.shift(-1)
x^6 + 1
sage: f.shift(10).shift(-10) == f
True
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(12)**Integer(30)), implementation='NTL
↪', names=('x',)); (x,) = R._first_ngens(1)
>>> f = x**Integer(7) + x + Integer(1)
>>> f.shift(Integer(1))
x^8 + x^2 + x
>>> f.shift(-Integer(1))
x^6 + 1
>>> f.shift(Integer(10)).shift(-Integer(10)) == f
True
```

truncate(n)

Return this polynomial mod x^n.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(Integers(15^30), implementation='NTL')
sage: f = sum(x^n for n in range(10)); f
x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1
```

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```
sage: f.truncate(6)
x^5 + x^4 + x^3 + x^2 + x + 1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(15)**Integer(30)), implementation='NTL
↪', names=('x',)); (x,) = R._first_ngens(1)
>>> f = sum(x**n for n in range(Integer(10))); f
x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1
>>> f.truncate(Integer(6))
x^5 + x^4 + x^3 + x^2 + x + 1
```

valuation()

Return the valuation of `self`, that is, the power of the lowest nonzero monomial of `self`.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(Integers(10^50), implementation='NTL')
sage: x.valuation()
1
sage: f = x - 3; f.valuation()
0
sage: f = x^99; f.valuation()
99
sage: f = x - x; f.valuation()
+Infinity
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(10)**Integer(50)), implementation='NTL
↪', names=('x',)); (x,) = R._first_ngens(1)
>>> x.valuation()
1
>>> f = x - Integer(3); f.valuation()
0
>>> f = x**Integer(99); f.valuation()
99
>>> f = x - x; f.valuation()
+Infinity
```

class sage.rings.polynomial.polynomial_modn_dense_ntl.**Polynomial_dense_modn_ntl_zz**

Bases: *Polynomial_dense_mod_n*

Polynomial on $\mathbf{Z}/n\mathbf{Z}$ implemented via NTL.

add(*_right*)

sub(*_right*)

lmul(*c*)

rmul(*c*)

mul(*_right*)

_mul_trunc_(*right*, *n*)

Return the product of `self` and `right` truncated to the given length n

EXAMPLES:

```
sage: R.<x> = PolynomialRing(Integers(100), implementation="NTL")
sage: f = x - 2
sage: g = x^2 - 8*x + 16
sage: f*g
x^3 + 90*x^2 + 32*x + 68
sage: f._mul_trunc_(g, 42)
x^3 + 90*x^2 + 32*x + 68
sage: f._mul_trunc_(g, 3)
90*x^2 + 32*x + 68
sage: f._mul_trunc_(g, 2)
32*x + 68
sage: f._mul_trunc_(g, 1)
68
sage: f._mul_trunc_(g, 0)
0
sage: f = x^2 - 8*x + 16
sage: f._mul_trunc_(f, 2)
44*x + 56
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(100)), implementation="NTL", names=('x
↪',)); (x,) = R._first_ngens(1)
>>> f = x - Integer(2)
>>> g = x**Integer(2) - Integer(8)*x + Integer(16)
>>> f*g
x^3 + 90*x^2 + 32*x + 68
>>> f._mul_trunc_(g, Integer(42))
x^3 + 90*x^2 + 32*x + 68
>>> f._mul_trunc_(g, Integer(3))
90*x^2 + 32*x + 68
>>> f._mul_trunc_(g, Integer(2))
32*x + 68
>>> f._mul_trunc_(g, Integer(1))
68
>>> f._mul_trunc_(g, Integer(0))
0
>>> f = x**Integer(2) - Integer(8)*x + Integer(16)
>>> f._mul_trunc_(f, Integer(2))
44*x + 56
```

degree()

EXAMPLES:

```
sage: R.<x> = PolynomialRing(Integers(77), implementation='NTL')
sage: f = x^4 - x - 1
sage: f.degree()
4
sage: f = 77*x + 1
```

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```
sage: f.degree()
0
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(77)), implementation='NTL', names=('x
↪',)); (x,) = R._first_ngens(1)
>>> f = x**Integer(4) - x - Integer(1)
>>> f.degree()
4
>>> f = Integer(77)*x + Integer(1)
>>> f.degree()
0
```

int_list()

Return the coefficients of `self` as efficiently as possible as a list of python ints.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(Integers(100), implementation='NTL')
sage: from sage.rings.polynomial.polynomial_modn_dense_ntl import Polynomial_
↪dense_mod_n as poly_modn_dense
sage: f = poly_modn_dense(R,[5,0,0,1])
sage: f.int_list()
[5, 0, 0, 1]
sage: [type(a) for a in f.int_list()]
[<... 'int'>, <... 'int'>, <... 'int'>, <... 'int'>]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(100)), implementation='NTL', names=('x
↪',)); (x,) = R._first_ngens(1)
>>> from sage.rings.polynomial.polynomial_modn_dense_ntl import Polynomial_
↪dense_mod_n as poly_modn_dense
>>> f = poly_modn_dense(R,[Integer(5),Integer(0),Integer(0),Integer(1)])
>>> f.int_list()
[5, 0, 0, 1]
>>> [type(a) for a in f.int_list()]
[<... 'int'>, <... 'int'>, <... 'int'>, <... 'int'>]
```

is_gen()

ntl_set_directly(*v*)

quo_rem(*right*)

Return q and r, with the degree of r less than the degree of `right`, such that $q\cdot$ `right` $+ r =$ `self`.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(Integers(125), implementation='NTL')
sage: f = x^5+1; g = (x+1)^2
sage: q, r = f.quo_rem(g)
sage: q
x^3 + 123*x^2 + 3*x + 121
sage: r
```

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```
5*x + 5
sage: q*g + r
x^5 + 1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(125)), implementation='NTL', names=('x
˓→',)); (x,) = R._first_ngens(1)
>>> f = x**Integer(5)+Integer(1); g = (x+Integer(1))**Integer(2)
>>> q, r = f.quo_rem(g)
>>> q
x^3 + 123*x^2 + 3*x + 121
>>> r
5*x + 5
>>> q*g + r
x^5 + 1
```

reverse(*degree=None*)

Return the reverse of the input polynomial thought as a polynomial of degree `degree`.

If f is a degree-d polynomial, its reverse is $x^d f(1/x)$.

INPUT:

- `degree` – `None` or integer; if specified, truncate or zero pad the list of coefficients to this degree before reversing it

EXAMPLES:

```
sage: R.<x> = PolynomialRing(Integers(77), implementation='NTL')
sage: f = x^4 - x - 1
sage: f.reverse()
76*x^4 + 76*x^3 + 1
sage: f.reverse(2)
76*x^2 + 76*x
sage: f.reverse(5)
76*x^5 + 76*x^4 + x
sage: g = x^3 - x
sage: g.reverse()
76*x^2 + 1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(77)), implementation='NTL', names=('x
˓→',)); (x,) = R._first_ngens(1)
>>> f = x**Integer(4) - x - Integer(1)
>>> f.reverse()
76*x^4 + 76*x^3 + 1
>>> f.reverse(Integer(2))
76*x^2 + 76*x
>>> f.reverse(Integer(5))
76*x^5 + 76*x^4 + x
>>> g = x**Integer(3) - x
>>> g.reverse()
76*x^2 + 1
```

shift(*n*)

Shift `self` to left by n, which is multiplication by x^n, truncating if n is negative.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(Integers(77), implementation='NTL')
sage: f = x^7 + x + 1
sage: f.shift(1)
x^8 + x^2 + x
sage: f.shift(-1)
x^6 + 1
sage: f.shift(10).shift(-10) == f
True
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(77)), implementation='NTL', names=('x
↪',)); (x,) = R._first_ngens(1)
>>> f = x**Integer(7) + x + Integer(1)
>>> f.shift(Integer(1))
x^8 + x^2 + x
>>> f.shift(-Integer(1))
x^6 + 1
>>> f.shift(Integer(10)).shift(-Integer(10)) == f
True
```

truncate(*n*)

Return this polynomial mod x^n.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(Integers(77), implementation='NTL')
sage: f = sum(x^n for n in range(10)); f
x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1
sage: f.truncate(6)
x^5 + x^4 + x^3 + x^2 + x + 1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(77)), implementation='NTL', names=('x
↪',)); (x,) = R._first_ngens(1)
>>> f = sum(x**n for n in range(Integer(10))); f
x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1
>>> f.truncate(Integer(6))
x^5 + x^4 + x^3 + x^2 + x + 1
```

valuation()

Return the valuation of `self`, that is, the power of the lowest nonzero monomial of `self`.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(Integers(10), implementation='NTL')
sage: x.valuation()
1
sage: f = x-3; f.valuation()
0
```

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```
sage: f = x^99; f.valuation()
99
sage: f = x-x; f.valuation()
+Infinity
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(10)), implementation='NTL', names=('x
↪',)); (x,) = R._first_ngens(1)
>>> x.valuation()
1
>>> f = x-Integer(3); f.valuation()
0
>>> f = x**Integer(99); f.valuation()
99
>>> f = x-x; f.valuation()
+Infinity
```

sage.rings.polynomial.polynomial_modn_dense_ntl.**make_element**(*parent*, *args*)

sage.rings.polynomial.polynomial_modn_dense_ntl.**small_roots**(*self*, *X=None*, *beta=1.0*, *epsilon=None*, ***kwds*)

Let N be the characteristic of the base ring this polynomial is defined over: `N = self.base_ring().characteristic()`. This method returns small roots of this polynomial modulo some factor b of N with the constraint that $b >= N^\beta$. Small in this context means that if x is a root of f modulo b then $|x| < X$. This X is either provided by the user or the maximum X is chosen such that this algorithm terminates in polynomial time. If X is chosen automatically it is $X = ceil(1/2N^{\beta^2/\delta-\epsilon})$. The algorithm may also return some roots which are larger than X. 'This algorithm' in this context means Coppersmith's algorithm for finding small roots using the LLL algorithm. The implementation of this algorithm follows Alexander May's PhD thesis referenced below.

INPUT:

- `X` – an absolute bound for the root (default: see above)
- `beta` – compute a root mod b where b is a factor of N and $b \geq N^\beta$ (default: 1.0, so $b = N$.)
- `epsilon` – the parameter ϵ described above. (default: $\beta/8$)
- `**kwds` – passed through to method `Matrix_integer_dense.LLL()`

EXAMPLES:

First consider a small example:

```
sage: N = 10001
sage: K = Zmod(10001)
sage: P.<x> = PolynomialRing(K, implementation='NTL')
sage: f = x^3 + 10*x^2 + 5000*x - 222
```

```
>>> from sage.all import *
>>> N = Integer(10001)
>>> K = Zmod(Integer(10001))
>>> P = PolynomialRing(K, implementation='NTL', names=('x',)); (x,) = P._first_
↪ngens(1)
>>> f = x**Integer(3) + Integer(10)*x**Integer(2) + Integer(5000)*x - Integer(222)
```

This polynomial has no roots without modular reduction (i.e. over $\mathbf{Z}$):

```
sage: f.change_ring(ZZ).roots()
[]
```

```
>>> from sage.all import *
>>> f.change_ring(ZZ).roots()
[]
```

To compute its roots we need to factor the modulus N and use the Chinese remainder theorem:

```
sage: p, q = N.prime_divisors()
sage: f.change_ring(GF(p)).roots()
[(4, 1)]
sage: f.change_ring(GF(q)).roots()
[(4, 1)]

sage: crt(4, 4, p, q)
4
```

```
>>> from sage.all import *
>>> p, q = N.prime_divisors()
>>> f.change_ring(GF(p)).roots()
[(4, 1)]
>>> f.change_ring(GF(q)).roots()
[(4, 1)]

>>> crt(Integer(4), Integer(4), p, q)
4
```

This root is quite small compared to N, so we can attempt to recover it without factoring N using Coppersmith's small root method:

```
sage: f.small_roots()
[4]
```

```
>>> from sage.all import *
>>> f.small_roots()
[4]
```

An application of this method is to consider RSA. We are using 512-bit RSA with public exponent $e = 3$ to encrypt a 56-bit DES key. Because it would be easy to attack this setting if no padding was used we pad the key K with 1s to get a large number:

```
sage: Nbits, Kbits = 512, 56
sage: e = 3
```

```
>>> from sage.all import *
>>> Nbits, Kbits = Integer(512), Integer(56)
>>> e = Integer(3)
```

We choose two primes of size 256-bit each:

```
sage: p = 2^256 + 2^8 + 2^5 + 2^3 + 1
sage: q = 2^256 + 2^8 + 2^5 + 2^3 + 2^2 + 1
sage: N = p*q
sage: ZmodN = Zmod( N )
```

```
>>> from sage.all import *
>>> p = Integer(2)**Integer(256) + Integer(2)**Integer(8) +␣
↪Integer(2)**Integer(5) + Integer(2)**Integer(3) + Integer(1)
>>> q = Integer(2)**Integer(256) + Integer(2)**Integer(8) +␣
↪Integer(2)**Integer(5) + Integer(2)**Integer(3) + Integer(2)**Integer(2) +␣
↪Integer(1)
>>> N = p*q
>>> ZmodN = Zmod( N )
```

We choose a random key:

```
sage: K = ZZ.random_element(0, 2^Kbits)
```

```
>>> from sage.all import *
>>> K = ZZ.random_element(Integer(0), Integer(2)**Kbits)
```

and pad it with $512 - 56 = 456$ 1s:

```
sage: Kdigits = K.digits(2)
sage: M = [0]*Kbits + [1]*(Nbits-Kbits)
sage: for i in range(len(Kdigits)): M[i] = Kdigits[i]

sage: M = ZZ(M, 2)
```

```
>>> from sage.all import *
>>> Kdigits = K.digits(Integer(2))
>>> M = [Integer(0)]*Kbits + [Integer(1)]*(Nbits-Kbits)
>>> for i in range(len(Kdigits)): M[i] = Kdigits[i]

>>> M = ZZ(M, Integer(2))
```

Now we encrypt the resulting message:

```
sage: C = ZmodN(M)^e
```

```
>>> from sage.all import *
>>> C = ZmodN(M)**e
```

To recover K we consider the following polynomial modulo N:

```
sage: P.<x> = PolynomialRing(ZmodN, implementation='NTL')
sage: f = (2^Nbits - 2^Kbits + x)^e - C
```

```
>>> from sage.all import *
>>> P = PolynomialRing(ZmodN, implementation='NTL', names=('x',)); (x,) = P._
↪first_ngens(1)
>>> f = (Integer(2)**Nbits - Integer(2)**Kbits + x)**e - C
```

and recover its small roots:

```
sage: Kbar = f.small_roots()[0]
sage: K == Kbar
True
```

```
>>> from sage.all import *
>>> Kbar = f.small_roots()[Integer(0)]
>>> K == Kbar
True
```

The same algorithm can be used to factor $N = pq$ if partial knowledge about q is available. This example is from the Magma handbook:

First, we set up p, q and N:

```
sage: length = 512
sage: hidden = 110
sage: p = next_prime(2^int(round(length/2)))
sage: q = next_prime(round(pi.n()*p))                                    #␣
↪needs sage.symbolic
sage: N = p*q                                                            #␣
↪needs sage.symbolic
```

```
>>> from sage.all import *
>>> length = Integer(512)
>>> hidden = Integer(110)
>>> p = next_prime(Integer(2)**int(round(length/Integer(2))))
>>> q = next_prime(round(pi.n()*p))                                     #␣
↪needs sage.symbolic
>>> N = p*q                                                             #␣
↪needs sage.symbolic
```

Now we disturb the low 110 bits of q:

```
sage: qbar = q + ZZ.random_element(0, 2^hidden - 1)                      #␣
↪needs sage.symbolic
```

```
>>> from sage.all import *
>>> qbar = q + ZZ.random_element(Integer(0), Integer(2)**hidden - Integer(1))    ␣
↪                        # needs sage.symbolic
```

And try to recover q from it:

```
sage: F.<x> = PolynomialRing(Zmod(N), implementation='NTL')              #␣
↪needs sage.symbolic
sage: f = x - qbar                                                       #␣
↪needs sage.symbolic
```

```
>>> from sage.all import *
>>> F = PolynomialRing(Zmod(N), implementation='NTL', names=('x',)); (x,) = F._
↪first_ngens(1)# needs sage.symbolic
>>> f = x - qbar                                                        #␣
↪needs sage.symbolic
```

We know that the error is $\leq 2^{\text{hidden}} - 1$ and that the modulus we are looking for is $\geq \sqrt{N}$:

```
sage: from sage.misc.verbose import set_verbose
sage: set_verbose(2)
sage: d = f.small_roots(X=2^hidden-1, beta=0.5)[0]  # time random              #␣
↪needs sage.symbolic
verbose 2 (<module>) m = 4
verbose 2 (<module>) t = 4
verbose 2 (<module>) X = 1298074214633706907132624082305023
verbose 1 (<module>) LLL of 8x8 matrix (algorithm fpLLL:wrapper)
verbose 1 (<module>) LLL finished (time = 0.006998)
sage: q == qbar - d                                                             #␣
↪needs sage.symbolic
True
```

```
>>> from sage.all import *
>>> from sage.misc.verbose import set_verbose
>>> set_verbose(Integer(2))
>>> d = f.small_roots(X=Integer(2)**hidden-Integer(1), beta=RealNumber('0.5
↪'))[Integer(0)]  # time random              # needs sage.symbolic
verbose 2 (<module>) m = 4
verbose 2 (<module>) t = 4
verbose 2 (<module>) X = 1298074214633706907132624082305023
verbose 1 (<module>) LLL of 8x8 matrix (algorithm fpLLL:wrapper)
verbose 1 (<module>) LLL finished (time = 0.006998)
>>> q == qbar - d                                                             #␣
↪needs sage.symbolic
True
```

REFERENCES:

Don Coppersmith. *Finding a small root of a univariate modular equation.* In Advances in Cryptology, EuroCrypt 1996, volume 1070 of Lecture Notes in Computer Science, p. 155–165. Springer, 1996. http://cr.yp.to/bib/2001/coppersmith.pdf

Alexander May. *New RSA Vulnerabilities Using Lattice Reduction Methods.* PhD thesis, University of Paderborn, 2003. http://www.cs.uni-paderborn.de/uploads/tx_sibibtex/bp.pdf

2.1.12 Dense univariate polynomials over $\mathbf{R}$, implemented using MPFR

class sage.rings.polynomial.polynomial_real_mpfr_dense.**PolynomialRealDense**

Bases: *Polynomial*

change_ring(*R*)

EXAMPLES:

```
sage: from sage.rings.polynomial.polynomial_real_mpfr_dense import␣
↪PolynomialRealDense
sage: f = PolynomialRealDense(RR['x'], [-2, 0, 1.5])
sage: f.change_ring(QQ)
3/2*x^2 - 2
sage: f.change_ring(RealField(10))
1.5*x^2 - 2.0
sage: f.change_ring(RealField(100))
1.5000000000000000000000000000*x^2 - 2.0000000000000000000000000000
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polynomial_real_mpfr_dense import␣
↪PolynomialRealDense
>>> f = PolynomialRealDense(RR['x'], [-Integer(2), Integer(0), RealNumber('1.5
↪')])
>>> f.change_ring(QQ)
3/2*x^2 - 2
>>> f.change_ring(RealField(Integer(10)))
1.5*x^2 - 2.0
>>> f.change_ring(RealField(Integer(100)))
1.5000000000000000000000000000*x^2 - 2.0000000000000000000000000000
```

degree()

Return the degree of the polynomial.

EXAMPLES:

```
sage: from sage.rings.polynomial.polynomial_real_mpfr_dense import␣
↪PolynomialRealDense
sage: f = PolynomialRealDense(RR['x'], [1, 2, 3]); f
3.00000000000000*x^2 + 2.00000000000000*x + 1.00000000000000
sage: f.degree()
2
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polynomial_real_mpfr_dense import␣
↪PolynomialRealDense
>>> f = PolynomialRealDense(RR['x'], [Integer(1), Integer(2), Integer(3)]); f
3.00000000000000*x^2 + 2.00000000000000*x + 1.00000000000000
>>> f.degree()
2
```

integral()

EXAMPLES:

```
sage: from sage.rings.polynomial.polynomial_real_mpfr_dense import␣
↪PolynomialRealDense
sage: f = PolynomialRealDense(RR['x'], [3, pi, 1])                           #␣
↪needs sage.symbolic
sage: f.integral()                                                           #␣
↪needs sage.symbolic
0.333333333333333*x^3 + 1.57079632679490*x^2 + 3.00000000000000*x
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polynomial_real_mpfr_dense import␣
↪PolynomialRealDense
>>> f = PolynomialRealDense(RR['x'], [Integer(3), pi, Integer(1)])           ␣
↪              # needs sage.symbolic
>>> f.integral()                                                           #␣
↪needs sage.symbolic
0.333333333333333*x^3 + 1.57079632679490*x^2 + 3.00000000000000*x
```

list(*copy=True*)

EXAMPLES:

```
sage: from sage.rings.polynomial.polynomial_real_mpfr_dense import␣
↪PolynomialRealDense
sage: f = PolynomialRealDense(RR['x'], [1, 0, -2]); f
-2.00000000000000*x^2 + 1.00000000000000
sage: f.list()
[1.00000000000000, 0.000000000000000, -2.00000000000000]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polynomial_real_mpfr_dense import␣
↪PolynomialRealDense
>>> f = PolynomialRealDense(RR['x'], [Integer(1), Integer(0), -Integer(2)]); f
-2.00000000000000*x^2 + 1.00000000000000
>>> f.list()
[1.00000000000000, 0.000000000000000, -2.00000000000000]
```

quo_rem(*other*)

Return the quotient with remainder of `self` by `other`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polynomial_real_mpfr_dense import␣
↪PolynomialRealDense
sage: f = PolynomialRealDense(RR['x'], [-2, 0, 1])
sage: g = PolynomialRealDense(RR['x'], [5, 1])
sage: q, r = f.quo_rem(g)
sage: q
x - 5.00000000000000
sage: r
23.0000000000000
sage: q*g + r == f
True
sage: fg = f*g
sage: fg.quo_rem(f)
(x + 5.00000000000000, 0)
sage: fg.quo_rem(g)
(x^2 - 2.00000000000000, 0)

sage: # needs sage.symbolic
sage: f = PolynomialRealDense(RR['x'], range(5))
sage: g = PolynomialRealDense(RR['x'], [pi,3000,4])
sage: q, r = f.quo_rem(g)
sage: g*q + r == f
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polynomial_real_mpfr_dense import␣
↪PolynomialRealDense
>>> f = PolynomialRealDense(RR['x'], [-Integer(2), Integer(0), Integer(1)])
>>> g = PolynomialRealDense(RR['x'], [Integer(5), Integer(1)])
>>> q, r = f.quo_rem(g)
```

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```
>>> q
x - 5.00000000000000
>>> r
23.0000000000000
>>> q*g + r == f
True
>>> fg = f*g
>>> fg.quo_rem(f)
(x + 5.00000000000000, 0)
>>> fg.quo_rem(g)
(x^2 - 2.00000000000000, 0)

>>> # needs sage.symbolic
>>> f = PolynomialRealDense(RR['x'], range(Integer(5)))
>>> g = PolynomialRealDense(RR['x'], [pi,Integer(3000),Integer(4)])
>>> q, r = f.quo_rem(g)
>>> g*q + r == f
True
```

reverse(*degree=None*)

Return reverse of the input polynomial thought as a polynomial of degree `degree`.

If f is a degree-d polynomial, its reverse is $x^d f(1/x)$.

INPUT:

- `degree` – `None` or an integer; if specified, truncate or zero pad the list of coefficients to this degree before reversing it

EXAMPLES:

```
sage: # needs sage.symbolic
sage: f = RR['x']([-3, pi, 0, 1])
sage: f.reverse()
-3.00000000000000*x^3 + 3.14159265358979*x^2 + 1.00000000000000
sage: f.reverse(2)
-3.00000000000000*x^2 + 3.14159265358979*x
sage: f.reverse(5)
-3.00000000000000*x^5 + 3.14159265358979*x^4 + x^2
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> f = RR['x']([-Integer(3), pi, Integer(0), Integer(1)])
>>> f.reverse()
-3.00000000000000*x^3 + 3.14159265358979*x^2 + 1.00000000000000
>>> f.reverse(Integer(2))
-3.00000000000000*x^2 + 3.14159265358979*x
>>> f.reverse(Integer(5))
-3.00000000000000*x^5 + 3.14159265358979*x^4 + x^2
```

shift(*n*)

Return this polynomial multiplied by the power x^n. If n is negative, terms below x^n will be discarded. Does not change this polynomial.

EXAMPLES:

```
sage: from sage.rings.polynomial.polynomial_real_mpfr_dense import
↪PolynomialRealDense
sage: f = PolynomialRealDense(RR['x'], [1, 2, 3]); f
3.00000000000000*x^2 + 2.00000000000000*x + 1.00000000000000
sage: f.shift(10)
3.00000000000000*x^12 + 2.00000000000000*x^11 + x^10
sage: f.shift(-1)
3.00000000000000*x + 2.00000000000000
sage: f.shift(-10)
0
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polynomial_real_mpfr_dense import
↪PolynomialRealDense
>>> f = PolynomialRealDense(RR['x'], [Integer(1), Integer(2), Integer(3)]); f
3.00000000000000*x^2 + 2.00000000000000*x + 1.00000000000000
>>> f.shift(Integer(10))
3.00000000000000*x^12 + 2.00000000000000*x^11 + x^10
>>> f.shift(-Integer(1))
3.00000000000000*x + 2.00000000000000
>>> f.shift(-Integer(10))
0
```

truncate(*n*)

Return the polynomial of degree $< n$ which is equivalent to `self` modulo x^n.

EXAMPLES:

```
sage: from sage.rings.polynomial.polynomial_real_mpfr_dense import
↪PolynomialRealDense
sage: f = PolynomialRealDense(RealField(10)['x'], [1, 2, 4, 8])
sage: f.truncate(3)
4.0*x^2 + 2.0*x + 1.0
sage: f.truncate(100)
8.0*x^3 + 4.0*x^2 + 2.0*x + 1.0
sage: f.truncate(1)
1.0
sage: f.truncate(0)
0
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polynomial_real_mpfr_dense import
↪PolynomialRealDense
>>> f = PolynomialRealDense(RealField(Integer(10))['x'], [Integer(1),
↪Integer(2), Integer(4), Integer(8)])
>>> f.truncate(Integer(3))
4.0*x^2 + 2.0*x + 1.0
>>> f.truncate(Integer(100))
8.0*x^3 + 4.0*x^2 + 2.0*x + 1.0
>>> f.truncate(Integer(1))
1.0
>>> f.truncate(Integer(0))
0
```

truncate_abs(*bound*)

Truncate all high order coefficients below `bound`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polynomial_real_mpfr_dense import␣
↪PolynomialRealDense
sage: f = PolynomialRealDense(RealField(10)['x'], [10^-k for k in range(10)])
sage: f
1.0e-9*x^9 + 1.0e-8*x^8 + 1.0e-7*x^7 + 1.0e-6*x^6 + 0.000010*x^5
 + 0.00010*x^4 + 0.0010*x^3 + 0.010*x^2 + 0.10*x + 1.0
sage: f.truncate_abs(0.5e-6)
1.0e-6*x^6 + 0.000010*x^5 + 0.00010*x^4 + 0.0010*x^3 + 0.010*x^2 + 0.10*x + 1.
↪0
sage: f.truncate_abs(10.0)
0
sage: f.truncate_abs(1e-100) == f
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polynomial_real_mpfr_dense import␣
↪PolynomialRealDense
>>> f = PolynomialRealDense(RealField(Integer(10))['x'], [Integer(10)**-k for␣
↪k in range(Integer(10))])
>>> f
1.0e-9*x^9 + 1.0e-8*x^8 + 1.0e-7*x^7 + 1.0e-6*x^6 + 0.000010*x^5
 + 0.00010*x^4 + 0.0010*x^3 + 0.010*x^2 + 0.10*x + 1.0
>>> f.truncate_abs(RealNumber('0.5e-6'))
1.0e-6*x^6 + 0.000010*x^5 + 0.00010*x^4 + 0.0010*x^3 + 0.010*x^2 + 0.10*x + 1.
↪0
>>> f.truncate_abs(RealNumber('10.0'))
0
>>> f.truncate_abs(RealNumber('1e-100')) == f
True
```

`sage.rings.polynomial.polynomial_real_mpfr_dense.`**make_PolynomialRealDense**(*parent*, *data*)

EXAMPLES:

```
sage: from sage.rings.polynomial.polynomial_real_mpfr_dense import make_
↪PolynomialRealDense
sage: make_PolynomialRealDense(RR['x'], [1,2,3])
3.00000000000000*x^2 + 2.00000000000000*x + 1.00000000000000
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polynomial_real_mpfr_dense import make_
↪PolynomialRealDense
>>> make_PolynomialRealDense(RR['x'], [Integer(1),Integer(2),Integer(3)])
3.00000000000000*x^2 + 2.00000000000000*x + 1.00000000000000
```

2.1.13 Polynomial Interfaces to Singular

AUTHORS:

- Martin Albrecht <malb@informatik.uni-bremen.de> (2006-04-21)
- Robert Bradshaw: Re-factor to avoid multiple inheritance vs. Cython (2007-09)
- Syed Ahmad Lavasani: Added function field to _singular_init_ (2011-12-16); Added non-prime finite fields to _singular_init_ (2012-1-22)

class sage.rings.polynomial.polynomial_singular_interface.**PolynomialRing_singular_repr**

Bases: object

Implement methods to convert polynomial rings to Singular.

This class is a base class for all univariate and multivariate polynomial rings which support conversion from and to Singular rings.

class sage.rings.polynomial.polynomial_singular_interface.**Polynomial_singular_repr**

Bases: object

Implement coercion of polynomials to Singular polynomials.

This class is a base class for all (univariate and multivariate) polynomial classes which support conversion from and to Singular polynomials.

Due to the incompatibility of Python extension classes and multiple inheritance, this just defers to module-level functions.

sage.rings.polynomial.polynomial_singular_interface.**can_convert_to_singular**(*R*)

Return True if this ring's base field or ring can be represented in Singular, and the polynomial ring has at least one generator.

The following base rings are supported: finite fields, rationals, number fields, and real and complex fields.

EXAMPLES:

```
sage: from sage.rings.polynomial.polynomial_singular_interface import can_convert_
↪to_singular
sage: can_convert_to_singular(PolynomialRing(QQ, names=['x']))
True
sage: can_convert_to_singular(PolynomialRing(ZZ, names=['x']))
True

sage: can_convert_to_singular(PolynomialRing(QQ, names=[]))
False
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polynomial_singular_interface import can_convert_
↪to_singular
>>> can_convert_to_singular(PolynomialRing(QQ, names=['x']))
True
>>> can_convert_to_singular(PolynomialRing(ZZ, names=['x']))
True

>>> can_convert_to_singular(PolynomialRing(QQ, names=[]))
False
```

2.1.14 Base class for generic p-adic polynomials

This provides common functionality for all p-adic polynomials, such as printing and factoring.

AUTHORS:

- Jeroen Demeyer (2013-11-22): initial version, split off from other files, made Polynomial_padic the common base class for all p-adic polynomials.

class sage.rings.polynomial.padics.polynomial_padic.**Polynomial_padic**(*parent*, *x=None*, *check=True*, *is_gen=False*, *construct=False*)

Bases: *Polynomial*

content()

Compute the content of this polynomial.

OUTPUT:

If this is the zero polynomial, return the constant coefficient. Otherwise, since the content is only defined up to a unit, return the content as π^k with maximal precision where k is the minimal valuation of any of the coefficients.

EXAMPLES:

```
sage: # needs sage.libs.ntl
sage: K = Zp(13,7)
sage: R.<t> = K[]
sage: f = 13^7*t^3 + K(169,4)*t - 13^4
sage: f.content()
13^2 + O(13^9)
sage: R(0).content()
0
sage: f = R(K(0,3)); f
O(13^3)
sage: f.content()
O(13^3)

sage: # needs sage.libs.ntl
sage: P.<x> = ZZ[]
sage: f = x + 2
sage: f.content()
1
sage: fp = f.change_ring(pAdicRing(2, 10))
sage: fp
(1 + O(2^10))*x + 2 + O(2^11)
sage: fp.content()
1 + O(2^10)
sage: (2*fp).content()
2 + O(2^11)
```

```
>>> from sage.all import *
>>> # needs sage.libs.ntl
>>> K = Zp(Integer(13),Integer(7))
>>> R = K['t']; (t,) = R._first_ngens(1)
```

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```
>>> f = Integer(13)**Integer(7)*t**Integer(3) + K(Integer(169),Integer(4))*t -
↪ Integer(13)**Integer(4)
>>> f.content()
13^2 + O(13^9)
>>> R(Integer(0)).content()
0
>>> f = R(K(Integer(0),Integer(3))); f
O(13^3)
>>> f.content()
O(13^3)

>>> # needs sage.libs.ntl
>>> P = ZZ['x']; (x,) = P._first_ngens(1)
>>> f = x + Integer(2)
>>> f.content()
1
>>> fp = f.change_ring(pAdicRing(Integer(2), Integer(10)))
>>> fp
(1 + O(2^10))*x + 2 + O(2^11)
>>> fp.content()
1 + O(2^10)
>>> (Integer(2)*fp).content()
2 + O(2^11)
```

Over a field it would be sufficient to return only zero or one, as the content is only defined up to multiplication with a unit. However, we return π^k where k is the minimal valuation of any coefficient:

```
sage: # needs sage.libs.ntl
sage: K = Qp(13,7)
sage: R.<t> = K[]
sage: f = 13^7*t^3 + K(169,4)*t - 13^-4
sage: f.content()
13^-4 + O(13^3)
sage: f = R.zero()
sage: f.content()
0
sage: f = R(K(0,3))
sage: f.content()
O(13^3)
sage: f = 13*t^3 + K(0,1)*t
sage: f.content()
13 + O(13^8)
```

```
>>> from sage.all import *
>>> # needs sage.libs.ntl
>>> K = Qp(Integer(13),Integer(7))
>>> R = K['t']; (t,) = R._first_ngens(1)
>>> f = Integer(13)**Integer(7)*t**Integer(3) + K(Integer(169),Integer(4))*t -
↪ Integer(13)**-Integer(4)
>>> f.content()
13^-4 + O(13^3)
>>> f = R.zero()
```

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```
>>> f.content()
0
>>> f = R(K(Integer(0),Integer(3)))
>>> f.content()
O(13^3)
>>> f = Integer(13)*t**Integer(3) + K(Integer(0),Integer(1))*t
>>> f.content()
13 + O(13^8)
```

factor()

Return the factorization of this polynomial.

EXAMPLES:

```
sage: # needs sage.libs.ntl
sage: R.<t> = PolynomialRing(Qp(3, 3, print_mode='terse', print_pos=False))
sage: pol = t^8 - 1
sage: for p,e in pol.factor():
....:     print("{} {}".format(e, p))
1 (1 + O(3^3))*t + 1 + O(3^3)
1 (1 + O(3^3))*t - 1 + O(3^3)
1 (1 + O(3^3))*t^2 + (5 + O(3^3))*t - 1 + O(3^3)
1 (1 + O(3^3))*t^2 + (-5 + O(3^3))*t - 1 + O(3^3)
1 (1 + O(3^3))*t^2 + O(3^3)*t + 1 + O(3^3)
sage: R.<t> = PolynomialRing(Qp(5, 6, print_mode='terse', print_pos=False))
sage: pol = 100 * (5*t - 1) * (t - 5); pol
(500 + O(5^9))*t^2 + (-2600 + O(5^8))*t + 500 + O(5^9)
sage: pol.factor()
(500 + O(5^9)) * ((1 + O(5^5))*t - 1/5 + O(5^5)) * ((1 + O(5^6))*t - 5 + O(5^
↪6))
sage: pol.factor().value()
(500 + O(5^8))*t^2 + (-2600 + O(5^8))*t + 500 + O(5^8)
```

```
>>> from sage.all import *
>>> # needs sage.libs.ntl
>>> R = PolynomialRing(Qp(Integer(3), Integer(3), print_mode='terse', print_
↪pos=False), names=('t',)); (t,) = R._first_ngens(1)
>>> pol = t**Integer(8) - Integer(1)
>>> for p,e in pol.factor():
...     print("{} {}".format(e, p))
1 (1 + O(3^3))*t + 1 + O(3^3)
1 (1 + O(3^3))*t - 1 + O(3^3)
1 (1 + O(3^3))*t^2 + (5 + O(3^3))*t - 1 + O(3^3)
1 (1 + O(3^3))*t^2 + (-5 + O(3^3))*t - 1 + O(3^3)
1 (1 + O(3^3))*t^2 + O(3^3)*t + 1 + O(3^3)
>>> R = PolynomialRing(Qp(Integer(5), Integer(6), print_mode='terse', print_
↪pos=False), names=('t',)); (t,) = R._first_ngens(1)
>>> pol = Integer(100) * (Integer(5)*t - Integer(1)) * (t - Integer(5)); pol
(500 + O(5^9))*t^2 + (-2600 + O(5^8))*t + 500 + O(5^9)
>>> pol.factor()
(500 + O(5^9)) * ((1 + O(5^5))*t - 1/5 + O(5^5)) * ((1 + O(5^6))*t - 5 + O(5^
↪6))
```

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```
>>> pol.factor().value()
(500 + O(5^8))*t^2 + (-2600 + O(5^8))*t + 500 + O(5^8)
```

The same factorization over $\mathbf{Z}_p$. In this case, the "unit" part is a p-adic unit and the power of p is considered to be a factor:

```
sage: # needs sage.libs.ntl
sage: R.<t> = PolynomialRing(Zp(5, 6, print_mode='terse', print_pos=False))
sage: pol = 100 * (5*t - 1) * (t - 5); pol
(500 + O(5^9))*t^2 + (-2600 + O(5^8))*t + 500 + O(5^9)
sage: pol.factor()
(4 + O(5^6)) * (5 + O(5^7))^2 * ((1 + O(5^6))*t - 5 + O(5^6)) * ((5 + O(5^
↪6))*t - 1 + O(5^6))
sage: pol.factor().value()
(500 + O(5^8))*t^2 + (-2600 + O(5^8))*t + 500 + O(5^8)
```

```
>>> from sage.all import *
>>> # needs sage.libs.ntl
>>> R = PolynomialRing(Zp(Integer(5), Integer(6), print_mode='terse', print_
↪pos=False), names=('t',)); (t,) = R._first_ngens(1)
>>> pol = Integer(100) * (Integer(5)*t - Integer(1)) * (t - Integer(5)); pol
(500 + O(5^9))*t^2 + (-2600 + O(5^8))*t + 500 + O(5^9)
>>> pol.factor()
(4 + O(5^6)) * (5 + O(5^7))^2 * ((1 + O(5^6))*t - 5 + O(5^6)) * ((5 + O(5^
↪6))*t - 1 + O(5^6))
>>> pol.factor().value()
(500 + O(5^8))*t^2 + (-2600 + O(5^8))*t + 500 + O(5^8)
```

In the following example, the discriminant is zero, so the p-adic factorization is not well defined:

```
sage: factor(t^2)                                                          #␣
↪needs sage.libs.ntl
Traceback (most recent call last):
...
PrecisionError: p-adic factorization not well-defined since
the discriminant is zero up to the requestion p-adic precision
```

```
>>> from sage.all import *
>>> factor(t**Integer(2))                                                  ␣
↪       # needs sage.libs.ntl
Traceback (most recent call last):
...
PrecisionError: p-adic factorization not well-defined since
the discriminant is zero up to the requestion p-adic precision
```

An example of factoring a constant polynomial (see Issue #26669):

```
sage: R.<x> = Qp(5)[]                                                      #␣
↪needs sage.libs.ntl
sage: R(2).factor()                                                        #␣
↪needs sage.libs.ntl
2 + O(5^20)
```

```
>>> from sage.all import *
>>> R = Qp(Integer(5))['x']; (x,) = R._first_ngens(1)# needs sage.libs.ntl
>>> R(Integer(2)).factor()                                                      ␣
↪     # needs sage.libs.ntl
2 + O(5^20)
```

More examples over $\mathbf{Z}_p$:

```
sage: R.<w> = PolynomialRing(Zp(5, prec=6, type='capped-abs', print_mode='val-
↪unit'))
sage: f = w^5 - 1
sage: f.factor()                                                               #␣
↪needs sage.libs.ntl
((1 + O(5^6))*w + 3124 + O(5^6))
* ((1 + O(5^6))*w^4 + (12501 + O(5^6))*w^3 + (9376 + O(5^6))*w^2
     + (6251 + O(5^6))*w + 3126 + O(5^6))
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Zp(Integer(5), prec=Integer(6), type='capped-abs',␣
↪print_mode='val-unit'), names=('w',)); (w,) = R._first_ngens(1)
>>> f = w**Integer(5) - Integer(1)
>>> f.factor()                                                               #␣
↪needs sage.libs.ntl
((1 + O(5^6))*w + 3124 + O(5^6))
* ((1 + O(5^6))*w^4 + (12501 + O(5^6))*w^3 + (9376 + O(5^6))*w^2
     + (6251 + O(5^6))*w + 3126 + O(5^6))
```

See Issue #4038:

```
sage: # needs sage.libs.ntl sage.schemes
sage: E = EllipticCurve('37a1')
sage: K = Qp(7,10)
sage: EK = E.base_extend(K)
sage: g = EK.division_polynomial_0(3)
sage: g.factor()
(3 + O(7^10))
* ((1 + O(7^10))*x
     + 1 + 2*7 + 4*7^2 + 2*7^3 + 5*7^4 + 7^5 + 5*7^6 + 3*7^7 + 5*7^8 + 3*7^9␣
↪+ O(7^10))
* ((1 + O(7^10))*x^3
     + (6 + 4*7 + 2*7^2 + 4*7^3 + 7^4 + 5*7^5
          + 7^6 + 3*7^7 + 7^8 + 3*7^9 + O(7^10))*x^2
     + (6 + 3*7 + 5*7^2 + 2*7^4 + 7^5 + 7^6 + 2*7^8 + 3*7^9 + O(7^10))*x
     + 2 + 5*7 + 4*7^2 + 2*7^3 + 6*7^4 + 3*7^5 + 7^6 + 4*7^7 + O(7^10))
```

```
>>> from sage.all import *
>>> # needs sage.libs.ntl sage.schemes
>>> E = EllipticCurve('37a1')
>>> K = Qp(Integer(7),Integer(10))
>>> EK = E.base_extend(K)
>>> g = EK.division_polynomial_0(Integer(3))
>>> g.factor()
(3 + O(7^10))
```

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```
* ((1 + O(7^10))*x
    + 1 + 2*7 + 4*7^2 + 2*7^3 + 5*7^4 + 7^5 + 5*7^6 + 3*7^7 + 5*7^8 + 3*7^9␣
↪+ O(7^10))
* ((1 + O(7^10))*x^3
    + (6 + 4*7 + 2*7^2 + 4*7^3 + 7^4 + 5*7^5
         + 7^6 + 3*7^7 + 7^8 + 3*7^9 + O(7^10))*x^2
    + (6 + 3*7 + 5*7^2 + 2*7^4 + 7^5 + 7^6 + 2*7^8 + 3*7^9 + O(7^10))*x
    + 2 + 5*7 + 4*7^2 + 2*7^3 + 6*7^4 + 3*7^5 + 7^6 + 4*7^7 + O(7^10))
```

root_field(*names*, *check_irreducible=True*, ***kwds*)

Return the p-adic extension field generated by the roots of the irreducible polynomial `self`.

INPUT:

- `names` – name of the generator of the extension
- `check_irreducible` – check whether the polynomial is irreducible
- `kwds` – see `sage.rings.padics.padic_generic.pAdicGeneric.extension()`

EXAMPLES:

```
sage: R.<x> = Qp(3,5,print_mode='digits')[]                                 #␣
↪needs sage.libs.ntl
sage: f = x^2 - 3                                                           #␣
↪needs sage.libs.ntl
sage: f.root_field('x')                                                     #␣
↪needs sage.libs.ntl
3-adic Eisenstein Extension Field in x defined by x^2 - 3
```

```
>>> from sage.all import *
>>> R = Qp(Integer(3),Integer(5),print_mode='digits')['x']; (x,) = R._first_
↪ngens(1)# needs sage.libs.ntl
>>> f = x**Integer(2) - Integer(3)                                          ␣
↪                # needs sage.libs.ntl
>>> f.root_field('x')                                                      #␣
↪needs sage.libs.ntl
3-adic Eisenstein Extension Field in x defined by x^2 - 3
```

```
sage: R.<x> = Qp(5,5,print_mode='digits')[]                                 #␣
↪needs sage.libs.ntl
sage: f = x^2 - 3                                                           #␣
↪needs sage.libs.ntl
sage: f.root_field('x', print_mode='bars')                                  #␣
↪needs sage.libs.ntl
5-adic Unramified Extension Field in x defined by x^2 - 3
```

```
>>> from sage.all import *
>>> R = Qp(Integer(5),Integer(5),print_mode='digits')['x']; (x,) = R._first_
↪ngens(1)# needs sage.libs.ntl
>>> f = x**Integer(2) - Integer(3)                                          ␣
↪                # needs sage.libs.ntl
>>> f.root_field('x', print_mode='bars')                                   #␣
```

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```
↪needs sage.libs.ntl
5-adic Unramified Extension Field in x defined by x^2 - 3
```

```
sage: R.<x> = Qp(11,5,print_mode='digits')[]                              #␣
↪needs sage.libs.ntl
sage: f = x^2 - 3                                                         #␣
↪needs sage.libs.ntl
sage: f.root_field('x', print_mode='bars')                                #␣
↪needs sage.libs.ntl
Traceback (most recent call last):
...
ValueError: polynomial must be irreducible
```

```
>>> from sage.all import *
>>> R = Qp(Integer(11),Integer(5),print_mode='digits')['x']; (x,) = R._first_
↪ngens(1)# needs sage.libs.ntl
>>> f = x**Integer(2) - Integer(3)                                         ␣
↪               # needs sage.libs.ntl
>>> f.root_field('x', print_mode='bars')                                #␣
↪needs sage.libs.ntl
Traceback (most recent call last):
...
ValueError: polynomial must be irreducible
```

2.1.15 p-adic Capped Relative Dense Polynomials

class sage.rings.polynomial.padics.polynomial_padic_capped_relative_dense.**Polynomial_padic_capped_rel**

Bases: *Polynomial_generic_cdv*, *Polynomial_padic*

degree(*secure=False*)

Return the degree of `self`.

INPUT:

- `secure` – boolean (default: `False`)

If `secure` is `True` and the degree of this polynomial is not determined (because the leading coefficient is indistinguishable from 0), an error is raised.

If `secure` is `False`, the returned value is the largest n so that the coefficient of x^n does not compare equal to 0.

EXAMPLES:

```
sage: K = Qp(3,10)
sage: R.<T> = K[]
sage: f = T + 2; f
(1 + O(3^10))*T + 2 + O(3^10)
sage: f.degree()
1
sage: (f-T).degree()
0
sage: (f-T).degree(secure=True)
Traceback (most recent call last):
...
PrecisionError: the leading coefficient is indistinguishable from 0

sage: x = O(3^5)
sage: li = [3^i * x for i in range(0,5)]; li
[O(3^5), O(3^6), O(3^7), O(3^8), O(3^9)]
sage: f = R(li); f
O(3^9)*T^4 + O(3^8)*T^3 + O(3^7)*T^2 + O(3^6)*T + O(3^5)
sage: f.degree()
-1
sage: f.degree(secure=True)
Traceback (most recent call last):
...
PrecisionError: the leading coefficient is indistinguishable from 0
```

```
>>> from sage.all import *
>>> K = Qp(Integer(3),Integer(10))
>>> R = K['T']; (T,) = R._first_ngens(1)
>>> f = T + Integer(2); f
(1 + O(3^10))*T + 2 + O(3^10)
>>> f.degree()
1
>>> (f-T).degree()
0
>>> (f-T).degree(secure=True)
Traceback (most recent call last):
...
PrecisionError: the leading coefficient is indistinguishable from 0

>>> x = O(Integer(3)**Integer(5))
>>> li = [Integer(3)**i * x for i in range(Integer(0),Integer(5))]; li
[O(3^5), O(3^6), O(3^7), O(3^8), O(3^9)]
>>> f = R(li); f
O(3^9)*T^4 + O(3^8)*T^3 + O(3^7)*T^2 + O(3^6)*T + O(3^5)
>>> f.degree()
-1
>>> f.degree(secure=True)
Traceback (most recent call last):
...
```

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```
PrecisionError: the leading coefficient is indistinguishable from 0
```

disc()

factor_mod()

Return the factorization of `self` modulo p.

is_eisenstein(*secure=False*)

Return `True` if this polynomial is an Eisenstein polynomial.

EXAMPLES:

```
sage: K = Qp(5)
sage: R.<t> = K[]
sage: f = 5 + 5*t + t^4
sage: f.is_eisenstein()
True
```

```
>>> from sage.all import *
>>> K = Qp(Integer(5))
>>> R = K['t']; (t,) = R._first_ngens(1)
>>> f = Integer(5) + Integer(5)*t + t**Integer(4)
>>> f.is_eisenstein()
True
```

AUTHOR:

- Xavier Caruso (2013-03)

lift()

Return an integer polynomial congruent to this one modulo the precision of each coefficient.

> **Note**
>
> The lift that is returned will not necessarily be the same for polynomials with the same coefficients (i.e. same values and precisions): it will depend on how the polynomials are created.

EXAMPLES:

```
sage: K = Qp(13,7)
sage: R.<t> = K[]
sage: a = 13^7*t^3 + K(169,4)*t - 13^4
sage: a.lift()
62748517*t^3 + 169*t - 28561
```

```
>>> from sage.all import *
>>> K = Qp(Integer(13),Integer(7))
>>> R = K['t']; (t,) = R._first_ngens(1)
>>> a = Integer(13)**Integer(7)*t**Integer(3) + K(Integer(169),Integer(4))*t -
↪ Integer(13)**Integer(4)
>>> a.lift()
62748517*t^3 + 169*t - 28561
```

list(*copy=True*)

Return a list of coefficients of `self`.

Note

The length of the list returned may be greater than expected since it includes any leading zeros that have finite absolute precision.

EXAMPLES:

```
sage: K = Qp(13,7)
sage: R.<t> = K[]
sage: a = 2*t^3 + 169*t - 1
sage: a
(2 + O(13^7))*t^3 + (13^2 + O(13^9))*t + 12 + 12*13 + 12*13^2 + 12*13^3 +␣
↪12*13^4 + 12*13^5 + 12*13^6 + O(13^7)
sage: a.list()
[12 + 12*13 + 12*13^2 + 12*13^3 + 12*13^4 + 12*13^5 + 12*13^6 + O(13^7),
 13^2 + O(13^9),
 0,
 2 + O(13^7)]
```

```
>>> from sage.all import *
>>> K = Qp(Integer(13),Integer(7))
>>> R = K['t']; (t,) = R._first_ngens(1)
>>> a = Integer(2)*t**Integer(3) + Integer(169)*t - Integer(1)
>>> a
(2 + O(13^7))*t^3 + (13^2 + O(13^9))*t + 12 + 12*13 + 12*13^2 + 12*13^3 +␣
↪12*13^4 + 12*13^5 + 12*13^6 + O(13^7)
>>> a.list()
[12 + 12*13 + 12*13^2 + 12*13^3 + 12*13^4 + 12*13^5 + 12*13^6 + O(13^7),
 13^2 + O(13^9),
 0,
 2 + O(13^7)]
```

lshift_coeffs(*shift*, *no_list=False*)

Return a new polynomials whose coefficients are multiplied by p^shift.

EXAMPLES:

```
sage: K = Qp(13, 4)
sage: R.<t> = K[]
sage: a = t + 52
sage: a.lshift_coeffs(3)
(13^3 + O(13^7))*t + 4*13^4 + O(13^8)
```

```
>>> from sage.all import *
>>> K = Qp(Integer(13), Integer(4))
>>> R = K['t']; (t,) = R._first_ngens(1)
>>> a = t + Integer(52)
>>> a.lshift_coeffs(Integer(3))
(13^3 + O(13^7))*t + 4*13^4 + O(13^8)
```

newton_polygon()

Return the Newton polygon of this polynomial.

> **Note**
>
> If some coefficients have not enough precision an error is raised.

OUTPUT: a NewtonPolygon

EXAMPLES:

```
sage: K = Qp(2, prec=5)
sage: P.<x> = K[]
sage: f = x^4 + 2^3*x^3 + 2^13*x^2 + 2^21*x + 2^37
sage: f.newton_polygon()                                                    #␣
↪needs sage.geometry.polyhedron
Finite Newton polygon with 4 vertices: (0, 37), (1, 21), (3, 3), (4, 0)

sage: K = Qp(5)
sage: R.<t> = K[]
sage: f = 5 + 3*t + t^4 + 25*t^10
sage: f.newton_polygon()                                                    #␣
↪needs sage.geometry.polyhedron
Finite Newton polygon with 4 vertices: (0, 1), (1, 0), (4, 0), (10, 2)
```

```
>>> from sage.all import *
>>> K = Qp(Integer(2), prec=Integer(5))
>>> P = K['x']; (x,) = P._first_ngens(1)
>>> f = x**Integer(4) + Integer(2)**Integer(3)*x**Integer(3) +␣
↪Integer(2)**Integer(13)*x**Integer(2) + Integer(2)**Integer(21)*x +␣
↪Integer(2)**Integer(37)
>>> f.newton_polygon()                                                    #␣
↪needs sage.geometry.polyhedron
Finite Newton polygon with 4 vertices: (0, 37), (1, 21), (3, 3), (4, 0)

>>> K = Qp(Integer(5))
>>> R = K['t']; (t,) = R._first_ngens(1)
>>> f = Integer(5) + Integer(3)*t + t**Integer(4) + Integer(25)*t**Integer(10)
>>> f.newton_polygon()                                                    #␣
↪needs sage.geometry.polyhedron
Finite Newton polygon with 4 vertices: (0, 1), (1, 0), (4, 0), (10, 2)
```

Here is an example where the computation fails because precision is not sufficient:

```
sage: g = f + K(0,0)*t^4; g
(5^2 + O(5^22))*t^10 + O(5^0)*t^4 + (3 + O(5^20))*t + 5 + O(5^21)
sage: g.newton_polygon()                                                    #␣
↪needs sage.geometry.polyhedron
Traceback (most recent call last):
...
PrecisionError: The coefficient of t^4 has not enough precision
```

```
>>> from sage.all import *
>>> g = f + K(Integer(0),Integer(0))*t**Integer(4); g
(5^2 + O(5^22))*t^10 + O(5^0)*t^4 + (3 + O(5^20))*t + 5 + O(5^21)
>>> g.newton_polygon()                                                  #␣
↪needs sage.geometry.polyhedron
Traceback (most recent call last):
...
PrecisionError: The coefficient of t^4 has not enough precision
```

AUTHOR:

- Xavier Caruso (2013-03-20)

newton_slopes(*repetition=True*)

Return a list of the Newton slopes of this polynomial.

These are the valuations of the roots of this polynomial.

If `repetition` is `True`, each slope is repeated a number of times equal to its multiplicity. Otherwise it appears only one time.

INPUT:

- `repetition` – boolean (default: `True`)

OUTPUT: list of rationals

EXAMPLES:

```
sage: K = Qp(5)
sage: R.<t> = K[]
sage: f = 5 + 3*t + t^4 + 25*t^10
sage: f.newton_polygon()                                                  #␣
↪needs sage.geometry.polyhedron
Finite Newton polygon with 4 vertices: (0, 1), (1, 0), (4, 0),
(10, 2)
sage: f.newton_slopes()                                                   #␣
↪needs sage.geometry.polyhedron
[1, 0, 0, 0, -1/3, -1/3, -1/3, -1/3, -1/3, -1/3]

sage: f.newton_slopes(repetition=False)                                   #␣
↪needs sage.geometry.polyhedron
[1, 0, -1/3]
```

```
>>> from sage.all import *
>>> K = Qp(Integer(5))
>>> R = K['t']; (t,) = R._first_ngens(1)
>>> f = Integer(5) + Integer(3)*t + t**Integer(4) + Integer(25)*t**Integer(10)
>>> f.newton_polygon()                                                  #␣
↪needs sage.geometry.polyhedron
Finite Newton polygon with 4 vertices: (0, 1), (1, 0), (4, 0),
(10, 2)
>>> f.newton_slopes()                                                   #␣
↪needs sage.geometry.polyhedron
[1, 0, 0, 0, -1/3, -1/3, -1/3, -1/3, -1/3, -1/3]
```

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```
>>> f.newton_slopes(repetition=False)                                          #␣
↪needs sage.geometry.polyhedron
[1, 0, -1/3]
```

AUTHOR:

- Xavier Caruso (2013-03-20)

prec_degree()

Return the largest n so that precision information is stored about the coefficient of x^n.

Always greater than or equal to degree.

EXAMPLES:

```
sage: K = Qp(3,10)
sage: R.<T> = K[]
sage: f = T + 2; f
(1 + O(3^10))*T + 2 + O(3^10)
sage: f.prec_degree()
1
```

```
>>> from sage.all import *
>>> K = Qp(Integer(3),Integer(10))
>>> R = K['T']; (T,) = R._first_ngens(1)
>>> f = T + Integer(2); f
(1 + O(3^10))*T + 2 + O(3^10)
>>> f.prec_degree()
1
```

precision_absolute(*n=None*)

Return absolute precision information about `self`.

INPUT:

- `self` – a p-adic polynomial
- `n` – `None` or integer (default: `None`)

OUTPUT:

If `n` is `None`, returns a list of absolute precisions of coefficients. Otherwise, returns the absolute precision of the coefficient of x^n.

EXAMPLES:

```
sage: K = Qp(3,10)
sage: R.<T> = K[]
sage: f = T + 2; f
(1 + O(3^10))*T + 2 + O(3^10)
sage: f.precision_absolute()
[10, 10]
```

```
>>> from sage.all import *
>>> K = Qp(Integer(3),Integer(10))
>>> R = K['T']; (T,) = R._first_ngens(1)
```

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```
>>> f = T + Integer(2); f
(1 + O(3^10))*T + 2 + O(3^10)
>>> f.precision_absolute()
[10, 10]
```

precision_relative(*n=None*)

Return relative precision information about `self`.

INPUT:

- `self` – a p-adic polynomial
- `n` – `None` or integer (default: `None`)

OUTPUT:

If `n` is `None`, returns a list of relative precisions of coefficients. Otherwise, returns the relative precision of the coefficient of x^n.

EXAMPLES:

```
sage: K = Qp(3,10)
sage: R.<T> = K[]
sage: f = T + 2; f
(1 + O(3^10))*T + 2 + O(3^10)
sage: f.precision_relative()
[10, 10]
```

```
>>> from sage.all import *
>>> K = Qp(Integer(3),Integer(10))
>>> R = K['T']; (T,) = R._first_ngens(1)
>>> f = T + Integer(2); f
(1 + O(3^10))*T + 2 + O(3^10)
>>> f.precision_relative()
[10, 10]
```

quo_rem(*right*, *secure=False*)

Return the quotient and remainder in division of `self` by `right`.

EXAMPLES:

```
sage: K = Qp(3,10)
sage: R.<T> = K[]
sage: f = T + 2
sage: g = T**4 + 3*T+22
sage: g.quo_rem(f)
((1 + O(3^10))*T^3 + (1 + 2*3 + 2*3^2 + 2*3^3 + 2*3^4 + 2*3^5 + 2*3^6 + 2*3^7␣
↪+ 2*3^8 + 2*3^9 + O(3^10))*T^2 + (1 + 3 + O(3^10))*T + 1 + 3 + 2*3^2 + 2*3^
↪3 + 2*3^4 + 2*3^5 + 2*3^6 + 2*3^7 + 2*3^8 + 2*3^9 + O(3^10),
 2 + 3 + 3^3 + O(3^10))
```

```
>>> from sage.all import *
>>> K = Qp(Integer(3),Integer(10))
>>> R = K['T']; (T,) = R._first_ngens(1)
>>> f = T + Integer(2)
```

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```
>>> g = T**Integer(4) + Integer(3)*T+Integer(22)
>>> g.quo_rem(f)
((1 + O(3^10))*T^3 + (1 + 2*3 + 2*3^2 + 2*3^3 + 2*3^4 + 2*3^5 + 2*3^6 + 2*3^7␣
↪+ 2*3^8 + 2*3^9 + O(3^10))*T^2 + (1 + 3 + O(3^10))*T + 1 + 3 + 2*3^2 + 2*3^
↪3 + 2*3^4 + 2*3^5 + 2*3^6 + 2*3^7 + 2*3^8 + 2*3^9 + O(3^10),
 2 + 3 + 3^3 + O(3^10))
```

rescale(*a*)

Return $f(a \cdot x)$.

Todo

Need to write this function for integer polynomials before this works.

EXAMPLES:

```
sage: K = Zp(13, 5)
sage: R.<t> = K[]
sage: f = t^3 + K(13, 3) * t
sage: f.rescale(2)  # not implemented
```

```
>>> from sage.all import *
>>> K = Zp(Integer(13), Integer(5))
>>> R = K['t']; (t,) = R._first_ngens(1)
>>> f = t**Integer(3) + K(Integer(13), Integer(3)) * t
>>> f.rescale(Integer(2))  # not implemented
```

reverse(*degree=None*)

Return the reverse of the input polynomial, thought as a polynomial of degree `degree`.

If f is a degree-d polynomial, its reverse is $x^d f(1/x)$.

INPUT:

- `degree` – `None` or integer; if specified, truncate or zero pad the list of coefficients to this degree before reversing it

EXAMPLES:

```
sage: K = Qp(13,7)
sage: R.<t> = K[]
sage: f = t^3 + 4*t; f
(1 + O(13^7))*t^3 + (4 + O(13^7))*t
sage: f.reverse()
0*t^3 + (4 + O(13^7))*t^2 + 1 + O(13^7)
sage: f.reverse(3)
0*t^3 + (4 + O(13^7))*t^2 + 1 + O(13^7)
sage: f.reverse(2)
0*t^2 + (4 + O(13^7))*t
sage: f.reverse(4)
0*t^4 + (4 + O(13^7))*t^3 + (1 + O(13^7))*t
sage: f.reverse(6)
0*t^6 + (4 + O(13^7))*t^5 + (1 + O(13^7))*t^3
```

```
>>> from sage.all import *
>>> K = Qp(Integer(13),Integer(7))
>>> R = K['t']; (t,) = R._first_ngens(1)
>>> f = t**Integer(3) + Integer(4)*t; f
(1 + O(13^7))*t^3 + (4 + O(13^7))*t
>>> f.reverse()
0*t^3 + (4 + O(13^7))*t^2 + 1 + O(13^7)
>>> f.reverse(Integer(3))
0*t^3 + (4 + O(13^7))*t^2 + 1 + O(13^7)
>>> f.reverse(Integer(2))
0*t^2 + (4 + O(13^7))*t
>>> f.reverse(Integer(4))
0*t^4 + (4 + O(13^7))*t^3 + (1 + O(13^7))*t
>>> f.reverse(Integer(6))
0*t^6 + (4 + O(13^7))*t^5 + (1 + O(13^7))*t^3
```

rshift_coeffs(*shift*, *no_list=False*)

Return a new polynomial whose coefficients are p-adically shifted to the right by `shift`.

> **Note**
>
> Type `Qp(5)(0).__rshift__?` for more information.

EXAMPLES:

```
sage: K = Zp(13, 4)
sage: R.<t> = K[]
sage: a = t^2 + K(13,3)*t + 169; a
(1 + O(13^4))*t^2 + (13 + O(13^3))*t + 13^2 + O(13^6)
sage: b = a.rshift_coeffs(1); b
O(13^3)*t^2 + (1 + O(13^2))*t + 13 + O(13^5)
sage: b.list()
[13 + O(13^5), 1 + O(13^2), O(13^3)]
sage: b = a.rshift_coeffs(2); b
O(13^2)*t^2 + O(13)*t + 1 + O(13^4)
sage: b.list()
[1 + O(13^4), O(13), O(13^2)]
```

```
>>> from sage.all import *
>>> K = Zp(Integer(13), Integer(4))
>>> R = K['t']; (t,) = R._first_ngens(1)
>>> a = t**Integer(2) + K(Integer(13),Integer(3))*t + Integer(169); a
(1 + O(13^4))*t^2 + (13 + O(13^3))*t + 13^2 + O(13^6)
>>> b = a.rshift_coeffs(Integer(1)); b
O(13^3)*t^2 + (1 + O(13^2))*t + 13 + O(13^5)
>>> b.list()
[13 + O(13^5), 1 + O(13^2), O(13^3)]
>>> b = a.rshift_coeffs(Integer(2)); b
O(13^2)*t^2 + O(13)*t + 1 + O(13^4)
>>> b.list()
[1 + O(13^4), O(13), O(13^2)]
```

valuation(*val_of_var=None*)

Return the valuation of `self`.

INPUT:

- `self` – a p-adic polynomial
- `val_of_var` – `None` or a rational (default: `None`)

OUTPUT:

If `val_of_var` is `None`, returns the largest power of the variable dividing `self`. Otherwise, returns the valuation of `self` where the variable is assigned valuation `val_of_var`

EXAMPLES:

```
sage: K = Qp(3,10)
sage: R.<T> = K[]
sage: f = T + 2; f
(1 + O(3^10))*T + 2 + O(3^10)
sage: f.valuation()
0
```

```
>>> from sage.all import *
>>> K = Qp(Integer(3),Integer(10))
>>> R = K['T']; (T,) = R._first_ngens(1)
>>> f = T + Integer(2); f
(1 + O(3^10))*T + 2 + O(3^10)
>>> f.valuation()
0
```

valuation_of_coefficient(*n=None*)

Return valuation information about `self`'s coefficients.

INPUT:

- `self` – a p-adic polynomial
- `n` – `None` or integer (default: `None`)

OUTPUT:

If `n` is `None`, returns a list of valuations of coefficients. Otherwise, returns the valuation of the coefficient of x^n.

EXAMPLES:

```
sage: K = Qp(3,10)
sage: R.<T> = K[]
sage: f = T + 2; f
(1 + O(3^10))*T + 2 + O(3^10)
sage: f.valuation_of_coefficient(1)
0
```

```
>>> from sage.all import *
>>> K = Qp(Integer(3),Integer(10))
>>> R = K['T']; (T,) = R._first_ngens(1)
>>> f = T + Integer(2); f
```

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```
(1 + O(3^10))*T + 2 + O(3^10)
>>> f.valuation_of_coefficient(Integer(1))
0
```

`sage.rings.polynomial.padics.polynomial_padic_capped_relative_dense.`**`make_padic_poly`**(*parent*, *x*, *version*)

2.1.16 p-adic Flat Polynomials

class `sage.rings.polynomial.padics.polynomial_padic_flat.`**`Polynomial_padic_flat`**(*parent*, *x=None*, *check=True*, *is_gen=False*, *construct=False*, *absprec=None*)

Bases: *Polynomial_generic_dense*, *Polynomial_padic*

2.1.17 Univariate Polynomials over GF(p^e) via NTL's ZZ_pEX

AUTHOR:

- Yann Laigle-Chapuy (2010-01) initial implementation
- Lorenz Panny (2023-01): `minpoly_mod()`
- Giacomo Pope (2023-08): `reverse()`, `inverse_series_trunc()`

class `sage.rings.polynomial.polynomial_zz_pex.`**`Polynomial_ZZ_pEX`**

Bases: *Polynomial_template*

Univariate Polynomials over $\mathbf{F}_{p^n}$ via NTL's `ZZ_pEX`.

EXAMPLES:

```
sage: K.<a> = GF(next_prime(2**60)**3)
sage: R.<x> = PolynomialRing(K, implementation='NTL')
sage: (x^3 + a*x^2 + 1) * (x + a)
x^4 + 2*a*x^3 + a^2*x^2 + x + a
```

```
>>> from sage.all import *
>>> K = GF(next_prime(Integer(2)**Integer(60))**Integer(3), names=('a',)); (a,) =␣
↪K._first_ngens(1)
>>> R = PolynomialRing(K, implementation='NTL', names=('x',)); (x,) = R._first_
↪ngens(1)
>>> (x**Integer(3) + a*x**Integer(2) + Integer(1)) * (x + a)
x^4 + 2*a*x^3 + a^2*x^2 + x + a
```

`compose_mod`(*other*, *modulus*)

Compute $f(g)$ mod h.

To be precise about the order fo compostion, given `self`, `other` and `modulus` as $f(x)$, $g(x)$ and $h(x)$ compute $f(g(x))$ mod $h(x)$.

INPUT:

- `other` – a polynomial $g(x)$
- `modulus` – a polynomial $h(x)$

EXAMPLES:

```
sage: R.<x> = GF(3**6)[]
sage: f = R.random_element()
sage: g = R.random_element()
sage: g.compose_mod(g, f) == g(g) % f
True

sage: F.<z3> = GF(3**6)
sage: R.<x> = F[]
sage: f = 2*z3^2*x^2 + (z3 + 1)*x + z3^2 + 2
sage: g = (z3^2 + 2*z3)*x^2 + (2*z3 + 2)*x + 2*z3^2 + z3 + 2
sage: h = (2*z3 + 2)*x^2 + (2*z3^2 + 1)*x + 2*z3^2 + z3 + 2
sage: f.compose_mod(g, h)
(z3^5 + z3^4 + z3^3 + z3^2 + z3)*x + z3^5 + z3^3 + 2*z3 + 2
sage: f.compose_mod(g, h) == f(g) % h
True
```

```
>>> from sage.all import *
>>> R = GF(Integer(3)**Integer(6))['x']; (x,) = R._first_ngens(1)
>>> f = R.random_element()
>>> g = R.random_element()
>>> g.compose_mod(g, f) == g(g) % f
True

>>> F = GF(Integer(3)**Integer(6), names=('z3',)); (z3,) = F._first_ngens(1)
>>> R = F['x']; (x,) = R._first_ngens(1)
>>> f = Integer(2)*z3**Integer(2)*x**Integer(2) + (z3 + Integer(1))*x +␣
↪z3**Integer(2) + Integer(2)
>>> g = (z3**Integer(2) + Integer(2)*z3)*x**Integer(2) + (Integer(2)*z3 +␣
↪Integer(2))*x + Integer(2)*z3**Integer(2) + z3 + Integer(2)
>>> h = (Integer(2)*z3 + Integer(2))*x**Integer(2) +␣
↪(Integer(2)*z3**Integer(2) + Integer(1))*x + Integer(2)*z3**Integer(2) + z3␣
↪+ Integer(2)
>>> f.compose_mod(g, h)
(z3^5 + z3^4 + z3^3 + z3^2 + z3)*x + z3^5 + z3^3 + 2*z3 + 2
>>> f.compose_mod(g, h) == f(g) % h
True
```

AUTHORS:

- Giacomo Pope (2024-08) initial implementation

inverse_series_trunc(*prec*)

Compute and return the inverse of `self` modulo x^{prec}.

The constant term of `self` must be invertible.

EXAMPLES:

```
sage: R.<x> = GF(101^2)[]
sage: z2 =  R.base_ring().gen()
sage: f = (3*z2 + 57)*x^3 + (13*z2 + 94)*x^2 + (7*z2 + 2)*x + 66*z2 + 15
sage: f.inverse_series_trunc(1)
51*z2 + 92
sage: f.inverse_series_trunc(2)
(30*z2 + 30)*x + 51*z2 + 92
sage: f.inverse_series_trunc(3)
(42*z2 + 94)*x^2 + (30*z2 + 30)*x + 51*z2 + 92
sage: f.inverse_series_trunc(4)
(99*z2 + 96)*x^3 + (42*z2 + 94)*x^2 + (30*z2 + 30)*x + 51*z2 + 92
```

```
>>> from sage.all import *
>>> R = GF(Integer(101)**Integer(2))['x']; (x,) = R._first_ngens(1)
>>> z2 =  R.base_ring().gen()
>>> f = (Integer(3)*z2 + Integer(57))*x**Integer(3) + (Integer(13)*z2 +
↪Integer(94))*x**Integer(2) + (Integer(7)*z2 + Integer(2))*x +
↪Integer(66)*z2 + Integer(15)
>>> f.inverse_series_trunc(Integer(1))
51*z2 + 92
>>> f.inverse_series_trunc(Integer(2))
(30*z2 + 30)*x + 51*z2 + 92
>>> f.inverse_series_trunc(Integer(3))
(42*z2 + 94)*x^2 + (30*z2 + 30)*x + 51*z2 + 92
>>> f.inverse_series_trunc(Integer(4))
(99*z2 + 96)*x^3 + (42*z2 + 94)*x^2 + (30*z2 + 30)*x + 51*z2 + 92
```

is_irreducible(*algorithm='fast_when_false'*, *iter=1*)

Return `True` precisely when `self` is irreducible over its base ring.

INPUT:

- `algorithm` – string (default: `'fast_when_false'`); there are 3 available algorithms: `'fast_when_true'`, `'fast_when_false'`, and `'probabilistic'`
- `iter` – (default: 1) if the algorithm is `'probabilistic'`, defines the number of iterations. The error probability is bounded by $q^{-\text{iter}}$ for polynomials in $\mathbf{F}_q[x]$.

EXAMPLES:

```
sage: K.<a> = GF(next_prime(2**60)**3)
sage: R.<x> = PolynomialRing(K, implementation='NTL')
sage: P = x^3 + (2-a)*x + 1
sage: P.is_irreducible(algorithm='fast_when_false')
True
sage: P.is_irreducible(algorithm='fast_when_true')
True
sage: P.is_irreducible(algorithm='probabilistic')
True
sage: Q = (x^2+a)*(x+a^3)
sage: Q.is_irreducible(algorithm='fast_when_false')
False
sage: Q.is_irreducible(algorithm='fast_when_true')
False
```

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```
sage: Q.is_irreducible(algorithm='probabilistic')
False
```

```
>>> from sage.all import *
>>> K = GF(next_prime(Integer(2)**Integer(60))**Integer(3), names=('a',)); (a,
↪) = K._first_ngens(1)
>>> R = PolynomialRing(K, implementation='NTL', names=('x',)); (x,) = R._
↪first_ngens(1)
>>> P = x**Integer(3) + (Integer(2)-a)*x + Integer(1)
>>> P.is_irreducible(algorithm='fast_when_false')
True
>>> P.is_irreducible(algorithm='fast_when_true')
True
>>> P.is_irreducible(algorithm='probabilistic')
True
>>> Q = (x**Integer(2)+a)*(x+a**Integer(3))
>>> Q.is_irreducible(algorithm='fast_when_false')
False
>>> Q.is_irreducible(algorithm='fast_when_true')
False
>>> Q.is_irreducible(algorithm='probabilistic')
False
```

list(*copy=True*)

Return the list of coefficients.

EXAMPLES:

```
sage: K.<a> = GF(5^3)
sage: P = PolynomialRing(K, 'x')
sage: f = P.random_element(100)
sage: f.list() == [f[i] for i in range(f.degree()+1)]
True
sage: P.0.list()
[0, 1]
```

```
>>> from sage.all import *
>>> K = GF(Integer(5)**Integer(3), names=('a',)); (a,) = K._first_ngens(1)
>>> P = PolynomialRing(K, 'x')
>>> f = P.random_element(Integer(100))
>>> f.list() == [f[i] for i in range(f.degree()+Integer(1))]
True
>>> P.gen(0).list()
[0, 1]
```

minpoly_mod(*other*)

Compute the minimal polynomial of this polynomial modulo another polynomial in the same ring.

ALGORITHM:

NTL's `MinPolyMod()`, which uses Shoup's algorithm [Sho1999].

EXAMPLES:

```
sage: R.<x> = GF(101^2)[]
sage: f = x^17 + x^2 - 1
sage: (x^2).minpoly_mod(f)
x^17 + 100*x^2 + 2*x + 100
```

```
>>> from sage.all import *
>>> R = GF(Integer(101)**Integer(2))['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(17) + x**Integer(2) - Integer(1)
>>> (x**Integer(2)).minpoly_mod(f)
x^17 + 100*x^2 + 2*x + 100
```

modular_composition(*other*, *modulus*)

Compute $f(g)$ mod h.

To be precise about the order fo compostion, given `self`, `other` and `modulus` as $f(x)$, $g(x)$ and $h(x)$ compute $f(g(x))$ mod $h(x)$.

INPUT:

- `other` – a polynomial $g(x)$
- `modulus` – a polynomial $h(x)$

EXAMPLES:

```
sage: R.<x> = GF(3**6)[]
sage: f = R.random_element()
sage: g = R.random_element()
sage: g.compose_mod(g, f) == g(g) % f
True

sage: F.<z3> = GF(3**6)
sage: R.<x> = F[]
sage: f = 2*z3^2*x^2 + (z3 + 1)*x + z3^2 + 2
sage: g = (z3^2 + 2*z3)*x^2 + (2*z3 + 2)*x + 2*z3^2 + z3 + 2
sage: h = (2*z3 + 2)*x^2 + (2*z3^2 + 1)*x + 2*z3^2 + z3 + 2
sage: f.compose_mod(g, h)
(z3^5 + z3^4 + z3^3 + z3^2 + z3)*x + z3^5 + z3^3 + 2*z3 + 2
sage: f.compose_mod(g, h) == f(g) % h
True
```

```
>>> from sage.all import *
>>> R = GF(Integer(3)**Integer(6))['x']; (x,) = R._first_ngens(1)
>>> f = R.random_element()
>>> g = R.random_element()
>>> g.compose_mod(g, f) == g(g) % f
True

>>> F = GF(Integer(3)**Integer(6), names=('z3',)); (z3,) = F._first_ngens(1)
>>> R = F['x']; (x,) = R._first_ngens(1)
>>> f = Integer(2)*z3**Integer(2)*x**Integer(2) + (z3 + Integer(1))*x +
↪z3**Integer(2) + Integer(2)
>>> g = (z3**Integer(2) + Integer(2)*z3)*x**Integer(2) + (Integer(2)*z3 +
↪Integer(2))*x + Integer(2)*z3**Integer(2) + z3 + Integer(2)
>>> h = (Integer(2)*z3 + Integer(2))*x**Integer(2) +
```

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```
↪(Integer(2)*z3**Integer(2) + Integer(1))*x + Integer(2)*z3**Integer(2) + z3␣
↪+ Integer(2)
>>> f.compose_mod(g, h)
(z3^5 + z3^4 + z3^3 + z3^2 + z3)*x + z3^5 + z3^3 + 2*z3 + 2
>>> f.compose_mod(g, h) == f(g) % h
True
```

AUTHORS:

- Giacomo Pope (2024-08) initial implementation

resultant(*other*)

Return the resultant of `self` and `other`, which must lie in the same polynomial ring.

INPUT:

- `other` – a polynomial

OUTPUT: an element of the base ring of the polynomial ring

EXAMPLES:

```
sage: K.<a> = GF(next_prime(2**60)**3)
sage: R.<x> = PolynomialRing(K, implementation='NTL')
sage: f = (x-a)*(x-a**2)*(x+1)
sage: g = (x-a**3)*(x-a**4)*(x+a)
sage: r = f.resultant(g)
sage: r == prod(u - v for (u,eu) in f.roots() for (v,ev) in g.roots())
True
```

```
>>> from sage.all import *
>>> K = GF(next_prime(Integer(2)**Integer(60))**Integer(3), names=('a',)); (a,
↪) = K._first_ngens(1)
>>> R = PolynomialRing(K, implementation='NTL', names=('x',)); (x,) = R._
↪first_ngens(1)
>>> f = (x-a)*(x-a**Integer(2))*(x+Integer(1))
>>> g = (x-a**Integer(3))*(x-a**Integer(4))*(x+a)
>>> r = f.resultant(g)
>>> r == prod(u - v for (u,eu) in f.roots() for (v,ev) in g.roots())
True
```

reverse(*degree=None*)

Return the polynomial obtained by reversing the coefficients of this polynomial. If degree is set then this function behaves as if this polynomial has degree `degree`.

EXAMPLES:

```
sage: R.<x> = GF(101^2)[]
sage: f = x^13 + 11*x^10 + 32*x^6 + 4
sage: f.reverse()
4*x^13 + 32*x^7 + 11*x^3 + 1
sage: f.reverse(degree=15)
4*x^15 + 32*x^9 + 11*x^5 + x^2
sage: f.reverse(degree=2)
4*x^2
```

```
>>> from sage.all import *
>>> R = GF(Integer(101)**Integer(2))['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(13) + Integer(11)*x**Integer(10) +␣
↪Integer(32)*x**Integer(6) + Integer(4)
>>> f.reverse()
4*x^13 + 32*x^7 + 11*x^3 + 1
>>> f.reverse(degree=Integer(15))
4*x^15 + 32*x^9 + 11*x^5 + x^2
>>> f.reverse(degree=Integer(2))
4*x^2
```

shift(*n*)

EXAMPLES:

```
sage: K.<a> = GF(next_prime(2**60)**3)
sage: R.<x> = PolynomialRing(K, implementation='NTL')
sage: f = x^3 + x^2 + 1
sage: f.shift(1)
x^4 + x^3 + x
sage: f.shift(-1)
x^2 + x
```

```
>>> from sage.all import *
>>> K = GF(next_prime(Integer(2)**Integer(60))**Integer(3), names=('a',)); (a,
↪) = K._first_ngens(1)
>>> R = PolynomialRing(K, implementation='NTL', names=('x',)); (x,) = R._
↪first_ngens(1)
>>> f = x**Integer(3) + x**Integer(2) + Integer(1)
>>> f.shift(Integer(1))
x^4 + x^3 + x
>>> f.shift(-Integer(1))
x^2 + x
```

class sage.rings.polynomial.polynomial_zz_pex.**Polynomial_template**

Bases: *Polynomial*

Template for interfacing to external C / C++ libraries for implementations of polynomials.

AUTHORS:

- Robert Bradshaw (2008-10): original idea for templating
- Martin Albrecht (2008-10): initial implementation

This file implements a simple templating engine for linking univariate polynomials to their C/C++ library implementations. It requires a 'linkage' file which implements the `celement_` functions (see `sage.libs.ntl.ntl_GF2X_linkage` for an example). Both parts are then plugged together by inclusion of the linkage file when inheriting from this class. See *sage.rings.polynomial.polynomial_gf2x* for an example.

We illustrate the generic glueing using univariate polynomials over $\mathrm{GF}(2)$.

Note

Implementations using this template MUST implement coercion from base ring elements and `get_unsafe()`. See *Polynomial_GF2X* for an example.

degree()

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: x.degree()
1
sage: P(1).degree()
0
sage: P(0).degree()
-1
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> x.degree()
1
>>> P(Integer(1)).degree()
0
>>> P(Integer(0)).degree()
-1
```

gcd(*other*)

Return the greatest common divisor of `self` and `other`.

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: f = x*(x+1)
sage: f.gcd(x+1)
x + 1
sage: f.gcd(x^2)
x
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> f = x*(x+Integer(1))
>>> f.gcd(x+Integer(1))
x + 1
>>> f.gcd(x**Integer(2))
x
```

get_cparent()

is_gen()

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: x.is_gen()
True
sage: (x+1).is_gen()
False
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
```

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```
>>> x.is_gen()
True
>>> (x+Integer(1)).is_gen()
False
```

is_one()

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: P(1).is_one()
True
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> P(Integer(1)).is_one()
True
```

is_zero()

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: x.is_zero()
False
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> x.is_zero()
False
```

list(*copy=True*)

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: x.list()
[0, 1]
sage: list(x)
[0, 1]
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> x.list()
[0, 1]
>>> list(x)
[0, 1]
```

quo_rem(*right*)

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: f = x^2 + x + 1
sage: f.quo_rem(x + 1)
(x, 1)
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> f = x**Integer(2) + x + Integer(1)
>>> f.quo_rem(x + Integer(1))
(x, 1)
```

shift(*n*)

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: f = x^3 + x^2 + 1
sage: f.shift(1)
x^4 + x^3 + x
sage: f.shift(-1)
x^2 + x
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x']; (x,) = P._first_ngens(1)
>>> f = x**Integer(3) + x**Integer(2) + Integer(1)
>>> f.shift(Integer(1))
x^4 + x^3 + x
>>> f.shift(-Integer(1))
x^2 + x
```

truncate(*n*)

Return this polynomial mod x^n.

EXAMPLES:

```
sage: R.<x> =GF(2)[]
sage: f = sum(x^n for n in range(10)); f
x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1
sage: f.truncate(6)
x^5 + x^4 + x^3 + x^2 + x + 1
```

```
>>> from sage.all import *
>>> R = GF(Integer(2))['x']; (x,) = R._first_ngens(1)
>>> f = sum(x**n for n in range(Integer(10))); f
x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1
>>> f.truncate(Integer(6))
x^5 + x^4 + x^3 + x^2 + x + 1
```

If the precision is higher than the degree of the polynomial then the polynomial itself is returned:

```
sage: f.truncate(10) is f
True
```

```
>>> from sage.all import *
>>> f.truncate(Integer(10)) is f
True
```

If the precision is negative, the zero polynomial is returned:

```
sage: f.truncate(-1)
0
```

```
>>> from sage.all import *
>>> f.truncate(-Integer(1))
0
```

xgcd(*other*)

Compute extended gcd of `self` and `other`.

EXAMPLES:

```
sage: P.<x> = GF(7)[]
sage: f = x*(x+1)
sage: f.xgcd(x+1)
(x + 1, 0, 1)
sage: f.xgcd(x^2)
(x, 1, 6)
```

```
>>> from sage.all import *
>>> P = GF(Integer(7))['x']; (x,) = P._first_ngens(1)
>>> f = x*(x+Integer(1))
>>> f.xgcd(x+Integer(1))
(x + 1, 0, 1)
>>> f.xgcd(x**Integer(2))
(x, 1, 6)
```

sage.rings.polynomial.polynomial_zz_pex.**make_element**(*parent*, *args*)

2.1.18 Isolate Real Roots of Real Polynomials

AUTHOR:

- Carl Witty (2007-09-19): initial version

This is an implementation of real root isolation. That is, given a polynomial with exact real coefficients, we compute isolating intervals for the real roots of the polynomial. (Polynomials with integer, rational, or algebraic real coefficients are supported.)

We convert the polynomials into the Bernstein basis, and then use de Casteljau's algorithm and Descartes' rule of signs on the Bernstein basis polynomial (using interval arithmetic) to locate the roots. The algorithm is similar to that in "A Descartes Algorithm for Polynomials with Bit-Stream Coefficients", by Eigenwillig, Kettner, Krandick, Mehlhorn, Schmitt, and Wolpert, but has three crucial optimizations over the algorithm in that paper:

- Precision reduction: at certain points in the computation, we discard the low-order bits of the coefficients, widening the intervals.
- Degree reduction: at certain points in the computation, we find lower-degree polynomials that are approximately equal to our high-degree polynomial over the region of interest.
- When the intervals are too wide to continue (either because of a too-low initial precision, or because of precision or degree reduction), and we need to restart with higher precision, we recall which regions have already been proven not to have any roots and do not examine them again.

The best description of the algorithms used (other than this source code itself) is in the slides for my Sage Days 4 talk, currently available from https://wiki.sagemath.org/days4schedule .

exception sage.rings.polynomial.real_roots.**PrecisionError**

Bases: ValueError

sage.rings.polynomial.real_roots.**bernstein_down**(*d1*, *d2*, *s*)

Given polynomial degrees d1 and d2 (where d1 < d2), and a number of samples s, computes a matrix bd.

If you have a Bernstein polynomial of formal degree d2, and select s of its coefficients (according to subsample_vec), and multiply the resulting vector by bd, then you get the coefficients of a Bernstein polynomial of formal degree d1, where this second polynomial is a good approximation to the first polynomial over the region of the Bernstein basis.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: bernstein_down(3, 8, 5)
[ 612/245 -348/245   -37/49  338/245 -172/245]
[-724/441   132/49  395/441 -290/147  452/441]
[ 452/441 -290/147  395/441   132/49 -724/441]
[-172/245  338/245   -37/49 -348/245  612/245]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> bernstein_down(Integer(3), Integer(8), Integer(5))
[ 612/245 -348/245   -37/49  338/245 -172/245]
[-724/441   132/49  395/441 -290/147  452/441]
[ 452/441 -290/147  395/441   132/49 -724/441]
[-172/245  338/245   -37/49 -348/245  612/245]
```

sage.rings.polynomial.real_roots.**bernstein_expand**(*c*, *d2*)

Given an integer vector representing a Bernstein polynomial p, and a degree d2, compute the representation of p as a Bernstein polynomial of formal degree d2.

This is similar to multiplying by the result of bernstein_up, but should be faster for large d2 (this has about the same number of multiplies, but in this version all the multiplies are by single machine words).

This returns a pair consisting of the expanded polynomial, and the maximum error E. (So if an element of the returned polynomial is a, and the true value of that coefficient is b, then a <= b < a + E.)

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: c = vector(ZZ, [1000, 2000, -3000])
sage: bernstein_expand(c, 3)
((1000, 1666, 333, -3000), 1)
sage: bernstein_expand(c, 4)
((1000, 1500, 1000, -500, -3000), 1)
sage: bernstein_expand(c, 20)
((1000, 1100, 1168, 1205, 1210, 1184, 1126, 1036, 915, 763, 578, 363, 115, -164, -
↪474, -816, -1190, -1595, -2032, -2500, -3000), 1)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> c = vector(ZZ, [Integer(1000), Integer(2000), -Integer(3000)])
>>> bernstein_expand(c, Integer(3))
((1000, 1666, 333, -3000), 1)
```

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```
>>> bernstein_expand(c, Integer(4))
((1000, 1500, 1000, -500, -3000), 1)
>>> bernstein_expand(c, Integer(20))
((1000, 1100, 1168, 1205, 1210, 1184, 1126, 1036, 915, 763, 578, 363, 115, -164, -
↪474, -816, -1190, -1595, -2032, -2500, -3000), 1)
```

class sage.rings.polynomial.real_roots.**bernstein_polynomial_factory**

Bases: object

An abstract base class for bernstein_polynomial factories. That is, elements of subclasses represent Bernstein polynomials (exactly), and are responsible for creating interval_bernstein_polynomial_integer approximations at arbitrary precision.

Supports four methods, coeffs_bitsize(), bernstein_polynomial(), lsign(), and usign(). The coeffs_bitsize() method gives an integer approximation to the log2 of the max of the absolute values of the Bernstein coefficients. The bernstein_polynomial(scale_log2) method gives an approximation where the maximum coefficient has approximately coeffs_bitsize() - scale_log2 bits. The lsign() and usign() methods give the (exact) sign of the first and last coefficient, respectively.

lsign()

Return the sign of the first coefficient of this Bernstein polynomial.

usign()

Return the sign of the last coefficient of this Bernstein polynomial.

class sage.rings.polynomial.real_roots.**bernstein_polynomial_factory_ar**(*poly*, *neg*)

Bases: *bernstein_polynomial_factory*

This class holds an exact Bernstein polynomial (represented as a list of algebraic real coefficients), and returns arbitrarily-precise interval approximations of this polynomial on demand.

bernstein_polynomial(*scale_log2*)

Compute an interval_bernstein_polynomial_integer that approximates this polynomial, using the given scale_log2. (Smaller scale_log2 values give more accurate approximations.)

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: x = polygen(AA)
sage: p = (x - 1) * (x - sqrt(AA(2))) * (x - 2)
sage: bpf = bernstein_polynomial_factory_ar(p, False)
sage: print(bpf.bernstein_polynomial(0))
degree 3 IBP with 2-bit coefficients
sage: bpf.bernstein_polynomial(-20)
<IBP: ((-2965821, 2181961, -1542880, 1048576) + [0 .. 1)) * 2^-20>
sage: bpf = bernstein_polynomial_factory_ar(p, True)
sage: bpf.bernstein_polynomial(-20)
<IBP: ((-2965821, -2181962, -1542880, -1048576) + [0 .. 1)) * 2^-20>
sage: p = x^2 - 1
sage: bpf = bernstein_polynomial_factory_ar(p, False)
sage: bpf.bernstein_polynomial(-10)
<IBP: ((-1024, 0, 1024) + [0 .. 1)) * 2^-10>
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
```

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```
>>> x = polygen(AA)
>>> p = (x - Integer(1)) * (x - sqrt(AA(Integer(2)))) * (x - Integer(2))
>>> bpf = bernstein_polynomial_factory_ar(p, False)
>>> print(bpf.bernstein_polynomial(Integer(0)))
degree 3 IBP with 2-bit coefficients
>>> bpf.bernstein_polynomial(-Integer(20))
<IBP: ((-2965821, 2181961, -1542880, 1048576) + [0 .. 1)) * 2^-20>
>>> bpf = bernstein_polynomial_factory_ar(p, True)
>>> bpf.bernstein_polynomial(-Integer(20))
<IBP: ((-2965821, -2181962, -1542880, -1048576) + [0 .. 1)) * 2^-20>
>>> p = x**Integer(2) - Integer(1)
>>> bpf = bernstein_polynomial_factory_ar(p, False)
>>> bpf.bernstein_polynomial(-Integer(10))
<IBP: ((-1024, 0, 1024) + [0 .. 1)) * 2^-10>
```

coeffs_bitsize()

Compute the approximate log2 of the maximum of the absolute values of the coefficients.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: x = polygen(AA)
sage: p = (x - 1) * (x - sqrt(AA(2))) * (x - 2)
sage: bernstein_polynomial_factory_ar(p, False).coeffs_bitsize()
1
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> x = polygen(AA)
>>> p = (x - Integer(1)) * (x - sqrt(AA(Integer(2)))) * (x - Integer(2))
>>> bernstein_polynomial_factory_ar(p, False).coeffs_bitsize()
1
```

class sage.rings.polynomial.real_roots.**bernstein_polynomial_factory_intlist**(*coeffs*)

Bases: *bernstein_polynomial_factory*

This class holds an exact Bernstein polynomial (represented as a list of integer coefficients), and returns arbitrarily-precise interval approximations of this polynomial on demand.

bernstein_polynomial(*scale_log2*)

Compute an interval_bernstein_polynomial_integer that approximates this polynomial, using the given scale_log2. (Smaller scale_log2 values give more accurate approximations.)

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: bpf = bernstein_polynomial_factory_intlist([10, -20, 30, -40])
sage: print(bpf.bernstein_polynomial(0))
degree 3 IBP with 6-bit coefficients
sage: bpf.bernstein_polynomial(20)
<IBP: ((0, -1, 0, -1) + [0 .. 1)) * 2^20; lsign 1>
sage: bpf.bernstein_polynomial(0)
<IBP: (10, -20, 30, -40) + [0 .. 1)>
```

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```
sage: bpf.bernstein_polynomial(-20)
<IBP: ((10485760, -20971520, 31457280, -41943040) + [0 .. 1)) * 2^-20>
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> bpf = bernstein_polynomial_factory_intlist([Integer(10), -Integer(20),␣
↪Integer(30), -Integer(40)])
>>> print(bpf.bernstein_polynomial(Integer(0)))
degree 3 IBP with 6-bit coefficients
>>> bpf.bernstein_polynomial(Integer(20))
<IBP: ((0, -1, 0, -1) + [0 .. 1)) * 2^20; lsign 1>
>>> bpf.bernstein_polynomial(Integer(0))
<IBP: (10, -20, 30, -40) + [0 .. 1)>
>>> bpf.bernstein_polynomial(-Integer(20))
<IBP: ((10485760, -20971520, 31457280, -41943040) + [0 .. 1)) * 2^-20>
```

coeffs_bitsize()

Compute the approximate log2 of the maximum of the absolute values of the coefficients.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: bernstein_polynomial_factory_intlist([1, 2, 3, -60000]).coeffs_bitsize()
16
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> bernstein_polynomial_factory_intlist([Integer(1), Integer(2), Integer(3),␣
↪-Integer(60000)]).coeffs_bitsize()
16
```

class sage.rings.polynomial.real_roots.**bernstein_polynomial_factory_ratlist**(*coeffs*)

Bases: *bernstein_polynomial_factory*

This class holds an exact Bernstein polynomial (represented as a list of rational coefficients), and returns arbitrarily-precise interval approximations of this polynomial on demand.

bernstein_polynomial(*scale_log2*)

Compute an interval_bernstein_polynomial_integer that approximates this polynomial, using the given scale_log2. (Smaller scale_log2 values give more accurate approximations.)

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: bpf = bernstein_polynomial_factory_ratlist([1/3, -22/7, 193/71, -140/
↪99])
sage: print(bpf.bernstein_polynomial(0))
degree 3 IBP with 3-bit coefficients
sage: bpf.bernstein_polynomial(20)
<IBP: ((0, -1, 0, -1) + [0 .. 1)) * 2^20; lsign 1>
sage: bpf.bernstein_polynomial(0)
<IBP: (0, -4, 2, -2) + [0 .. 1); lsign 1>
sage: bpf.bernstein_polynomial(-20)
<IBP: ((349525, -3295525, 2850354, -1482835) + [0 .. 1)) * 2^-20>
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> bpf = bernstein_polynomial_factory_ratlist([Integer(1)/Integer(3), -
↪Integer(22)/Integer(7), Integer(193)/Integer(71), -Integer(140)/
↪Integer(99)])
>>> print(bpf.bernstein_polynomial(Integer(0)))
degree 3 IBP with 3-bit coefficients
>>> bpf.bernstein_polynomial(Integer(20))
<IBP: ((0, -1, 0, -1) + [0 .. 1)) * 2^20; lsign 1>
>>> bpf.bernstein_polynomial(Integer(0))
<IBP: (0, -4, 2, -2) + [0 .. 1); lsign 1>
>>> bpf.bernstein_polynomial(-Integer(20))
<IBP: ((349525, -3295525, 2850354, -1482835) + [0 .. 1)) * 2^-20>
```

coeffs_bitsize()

Compute the approximate log2 of the maximum of the absolute values of the coefficients.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: bernstein_polynomial_factory_ratlist([1, 2, 3, -60000]).coeffs_bitsize()
15
sage: bernstein_polynomial_factory_ratlist([65535/65536]).coeffs_bitsize()
-1
sage: bernstein_polynomial_factory_ratlist([65536/65535]).coeffs_bitsize()
1
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> bernstein_polynomial_factory_ratlist([Integer(1), Integer(2), Integer(3),␣
↪-Integer(60000)]).coeffs_bitsize()
15
>>> bernstein_polynomial_factory_ratlist([Integer(65535)/Integer(65536)]).
↪coeffs_bitsize()
-1
>>> bernstein_polynomial_factory_ratlist([Integer(65536)/Integer(65535)]).
↪coeffs_bitsize()
1
```

sage.rings.polynomial.real_roots.**bernstein_up**(*d1*, *d2*, *s=None*)

Given polynomial degrees d1 and d2, where d1 < d2, compute a matrix bu.

If you have a Bernstein polynomial of formal degree d1, and multiply its coefficient vector by bu, then the result is the coefficient vector of the same polynomial represented as a Bernstein polynomial of formal degree d2.

If s is not None, then it represents a number of samples; then the product only gives s of the coefficients of the new Bernstein polynomial, selected according to subsample_vec.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: bernstein_down(3, 7, 4)
[  12/5     -4      3   -2/5]
[-13/15   16/3     -4   8/15]
```

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```
[  8/15     -4   16/3 -13/15]
[  -2/5      3     -4   12/5]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> bernstein_down(Integer(3), Integer(7), Integer(4))
[  12/5     -4      3   -2/5]
[-13/15   16/3     -4   8/15]
[  8/15     -4   16/3 -13/15]
[  -2/5      3     -4   12/5]
```

sage.rings.polynomial.real_roots.**bitsize_doctest**(*n*)

sage.rings.polynomial.real_roots.**cl_maximum_root**(*cl*)

Given a polynomial represented by a list of its coefficients (as RealIntervalFieldElements), compute an upper bound on its largest real root.

Uses two algorithms of Akritas, Strzebo'nski, and Vigklas, and picks the better result.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: cl_maximum_root([RIF(-1), RIF(0), RIF(1)])
1.00000000000000
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> cl_maximum_root([RIF(-Integer(1)), RIF(Integer(0)), RIF(Integer(1))])
1.00000000000000
```

sage.rings.polynomial.real_roots.**cl_maximum_root_first_lambda**(*cl*)

Given a polynomial represented by a list of its coefficients (as RealIntervalFieldElements), compute an upper bound on its largest real root.

Uses the first-lambda algorithm from "Implementations of a New Theorem for Computing Bounds for Positive Roots of Polynomials", by Akritas, Strzebo'nski, and Vigklas.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: cl_maximum_root_first_lambda([RIF(-1), RIF(0), RIF(1)])
1.00000000000000
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> cl_maximum_root_first_lambda([RIF(-Integer(1)), RIF(Integer(0)),␣
↪RIF(Integer(1))])
1.00000000000000
```

sage.rings.polynomial.real_roots.**cl_maximum_root_local_max**(*cl*)

Given a polynomial represented by a list of its coefficients (as RealIntervalFieldElements), compute an upper bound on its largest real root.

Uses the local-max algorithm from "Implementations of a New Theorem for Computing Bounds for Positive Roots of Polynomials", by Akritas, Strzebo'nski, and Vigklas.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: cl_maximum_root_local_max([RIF(-1), RIF(0), RIF(1)])
1.41421356237310
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> cl_maximum_root_local_max([RIF(-Integer(1)), RIF(Integer(0)),␣
↪RIF(Integer(1))])
1.41421356237310
```

class sage.rings.polynomial.real_roots.**context**

Bases: object

A simple context class, which is passed through parts of the real root isolation algorithm to avoid global variables.

Holds logging information, a random number generator, and the target machine wordsize.

get_be_log()

get_dc_log()

sage.rings.polynomial.real_roots.**de_casteljau_doublevec**(*c*, *x*)

Given a polynomial in Bernstein form with floating-point coefficients over the region [0 .. 1], and a split point x, use de Casteljau's algorithm to give polynomials in Bernstein form over [0 .. x] and [x .. 1].

This function will work for an arbitrary rational split point x, as long as 0 < x < 1; but it has a specialized code path for x==1/2.

INPUT:

- c – vector of coefficients of polynomial in Bernstein form
- x – rational splitting point; 0 < x < 1

OUTPUT:

- c1 – coefficients of polynomial over range [0 .. x]
- c2 – coefficients of polynomial over range [x .. 1]
- err_inc – number of half-ulps by which error intervals widened

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: c = vector(RDF, [0.7, 0, 0, 0, 0, 0])
sage: de_casteljau_doublevec(c, 1/2)
((0.7, 0.35, 0.175, 0.0875, 0.04375, 0.021875), (0.021875, 0.0, 0.0, 0.0, 0.0, 0.
↪0), 5)
sage: de_casteljau_doublevec(c, 1/3)  # rel tol
((0.7, 0.4666666666666667, 0.3111111111111117, 0.20740740740740746, 0.
↪13827160493827165, 0.09218106995884777), (0.09218106995884777, 0.0, 0.0, 0.0, 0.
↪0, 0.0), 15)
sage: de_casteljau_doublevec(c, 7/22)  # rel tol
((0.7, 0.4772727272727273, 0.3254132231404959, 0.22187265214124724, 0.
↪15127680827812312, 0.10314327837144759), (0.10314327837144759, 0.0, 0.0, 0.0, 0.
↪0, 0.0), 15)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> c = vector(RDF, [RealNumber('0.7'), Integer(0), Integer(0), Integer(0),␣
↪Integer(0), Integer(0)])
>>> de_casteljau_doublevec(c, Integer(1)/Integer(2))
((0.7, 0.35, 0.175, 0.0875, 0.04375, 0.021875), (0.021875, 0.0, 0.0, 0.0, 0.0, 0.
↪0), 5)
>>> de_casteljau_doublevec(c, Integer(1)/Integer(3))  # rel tol
((0.7, 0.4666666666666667, 0.31111111111111117, 0.20740740740740746, 0.
↪13827160493827165, 0.09218106995884777), (0.09218106995884777, 0.0, 0.0, 0.0, 0.
↪0, 0.0), 15)
>>> de_casteljau_doublevec(c, Integer(7)/Integer(22))  # rel tol
((0.7, 0.4772727272727273, 0.3254132231404959, 0.22187265214124724, 0.
↪15127680827812312, 0.10314327837144759), (0.10314327837144759, 0.0, 0.0, 0.0, 0.
↪0, 0.0), 15)
```

sage.rings.polynomial.real_roots.**de_casteljau_intvec**(*c*, *c_bitsize*, *x*, *use_ints*)

Given a polynomial in Bernstein form with integer coefficients over the region [0 .. 1], and a split point x, use de Casteljau's algorithm to give polynomials in Bernstein form over [0 .. x] and [x .. 1].

This function will work for an arbitrary rational split point x, as long as 0 < x < 1; but it has specialized code paths that make some values of x faster than others. If x == a/(a + b), there are special efficient cases for a==1, b==1, a+b fits in a machine word, a+b is a power of 2, a fits in a machine word, b fits in a machine word. The most efficient case is x==1/2.

Given split points x == a/(a + b) and y == c/(c + d), where min(a, b) and min(c, d) fit in the same number of machine words and a+b and c+d are both powers of two, then x and y should be equally fast split points.

If use_ints is nonzero, then instead of checking whether numerators and denominators fit in machine words, we check whether they fit in ints (32 bits, even on 64-bit machines). This slows things down, but allows for identical results across machines.

INPUT:

- `c` – vector of coefficients of polynomial in Bernstein form
- `c_bitsize` – approximate size of coefficients in c (in bits)
- `x` – rational splitting point; 0 < x < 1

OUTPUT:

- `c1` – coefficients of polynomial over range [0 .. x]
- `c2` – coefficients of polynomial over range [x .. 1]
- `err_inc` – amount by which error intervals widened

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: c = vector(ZZ, [1048576, 0, 0, 0, 0, 0])
sage: de_casteljau_intvec(c, 20, 1/2, 1)
((1048576, 524288, 262144, 131072, 65536, 32768), (32768, 0, 0, 0, 0, 0), 1)
sage: de_casteljau_intvec(c, 20, 1/3, 1)
((1048576, 699050, 466033, 310689, 207126, 138084), (138084, 0, 0, 0, 0, 0), 1)
sage: de_casteljau_intvec(c, 20, 7/22, 1)
((1048576, 714938, 487457, 332357, 226607, 154505), (154505, 0, 0, 0, 0, 0), 1)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> c = vector(ZZ, [Integer(1048576), Integer(0), Integer(0), Integer(0),␣
↪Integer(0), Integer(0)])
>>> de_casteljau_intvec(c, Integer(20), Integer(1)/Integer(2), Integer(1))
((1048576, 524288, 262144, 131072, 65536, 32768), (32768, 0, 0, 0, 0, 0), 1)
>>> de_casteljau_intvec(c, Integer(20), Integer(1)/Integer(3), Integer(1))
((1048576, 699050, 466033, 310689, 207126, 138084), (138084, 0, 0, 0, 0, 0), 1)
>>> de_casteljau_intvec(c, Integer(20), Integer(7)/Integer(22), Integer(1))
((1048576, 714938, 487457, 332357, 226607, 154505), (154505, 0, 0, 0, 0, 0), 1)
```

sage.rings.polynomial.real_roots.**degree_reduction_next_size**(*n*)

Given n (a polynomial degree), returns either a smaller integer or `None`. This defines the sequence of degrees followed by our degree reduction implementation.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: degree_reduction_next_size(1000)
30
sage: degree_reduction_next_size(20)
15
sage: degree_reduction_next_size(3)
2
sage: degree_reduction_next_size(2) is None
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> degree_reduction_next_size(Integer(1000))
30
>>> degree_reduction_next_size(Integer(20))
15
>>> degree_reduction_next_size(Integer(3))
2
>>> degree_reduction_next_size(Integer(2)) is None
True
```

sage.rings.polynomial.real_roots.**dprod_imatrow_vec**(*m*, *v*, *k*)

Compute the dot product of row k of the matrix m with the vector v (that is, compute one element of the product m*v).

If v has more elements than m has columns, then elements of v are selected using subsample_vec.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: m = matrix(3, range(9))
sage: dprod_imatrow_vec(m, vector(ZZ, [1, 0, 0, 0]), 1)
0
sage: dprod_imatrow_vec(m, vector(ZZ, [0, 1, 0, 0]), 1)
3
sage: dprod_imatrow_vec(m, vector(ZZ, [0, 0, 1, 0]), 1)
4
```

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```
sage: dprod_imatrow_vec(m, vector(ZZ, [0, 0, 0, 1]), 1)
5
sage: dprod_imatrow_vec(m, vector(ZZ, [1, 0, 0]), 1)
3
sage: dprod_imatrow_vec(m, vector(ZZ, [0, 1, 0]), 1)
4
sage: dprod_imatrow_vec(m, vector(ZZ, [0, 0, 1]), 1)
5
sage: dprod_imatrow_vec(m, vector(ZZ, [1, 2, 3]), 1)
26
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> m = matrix(Integer(3), range(Integer(9)))
>>> dprod_imatrow_vec(m, vector(ZZ, [Integer(1), Integer(0), Integer(0),␣
↪Integer(0)]), Integer(1))
0
>>> dprod_imatrow_vec(m, vector(ZZ, [Integer(0), Integer(1), Integer(0),␣
↪Integer(0)]), Integer(1))
3
>>> dprod_imatrow_vec(m, vector(ZZ, [Integer(0), Integer(0), Integer(1),␣
↪Integer(0)]), Integer(1))
4
>>> dprod_imatrow_vec(m, vector(ZZ, [Integer(0), Integer(0), Integer(0),␣
↪Integer(1)]), Integer(1))
5
>>> dprod_imatrow_vec(m, vector(ZZ, [Integer(1), Integer(0), Integer(0)]),␣
↪Integer(1))
3
>>> dprod_imatrow_vec(m, vector(ZZ, [Integer(0), Integer(1), Integer(0)]),␣
↪Integer(1))
4
>>> dprod_imatrow_vec(m, vector(ZZ, [Integer(0), Integer(0), Integer(1)]),␣
↪Integer(1))
5
>>> dprod_imatrow_vec(m, vector(ZZ, [Integer(1), Integer(2), Integer(3)]),␣
↪Integer(1))
26
```

sage.rings.polynomial.real_roots.**get_realfield_rndu**(*n*)

A simple cache for RealField fields (with rounding set to round-to-positive-infinity).

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: get_realfield_rndu(20)
Real Field with 20 bits of precision and rounding RNDU
sage: get_realfield_rndu(53)
Real Field with 53 bits of precision and rounding RNDU
sage: get_realfield_rndu(20)
Real Field with 20 bits of precision and rounding RNDU
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> get_realfield_rndu(Integer(20))
Real Field with 20 bits of precision and rounding RNDU
>>> get_realfield_rndu(Integer(53))
Real Field with 53 bits of precision and rounding RNDU
>>> get_realfield_rndu(Integer(20))
Real Field with 20 bits of precision and rounding RNDU
```

class sage.rings.polynomial.real_roots.**interval_bernstein_polynomial**

Bases: object

An interval_bernstein_polynomial is an approximation to an exact polynomial. This approximation is in the form of a Bernstein polynomial (a polynomial given as coefficients over a Bernstein basis) with interval coefficients.

The Bernstein basis of degree n over the region [a .. b] is the set of polynomials

$$\binom{n}{k}(x-a)^k(b-x)^{n-k}/(b-a)^n$$

for $0 \leq k \leq n$.

A degree-n interval Bernstein polynomial P with its region [a .. b] can represent an exact polynomial p in two different ways: it can "contain" the polynomial or it can "bound" the polynomial.

We say that P contains p if, when p is represented as a degree-n Bernstein polynomial over [a .. b], its coefficients are contained in the corresponding interval coefficients of P. For instance, [0.9 .. 1.1]*x^2 (which is a degree-2 interval Bernstein polynomial over [0 .. 1]) contains x^2.

We say that P bounds p if, for all a <= x <= b, there exists a polynomial p' contained in P such that p(x) == p'(x). For instance, [0 .. 1]*x is a degree-1 interval Bernstein polynomial which bounds x^2 over [0 .. 1].

If P contains p, then P bounds p; but the converse is not necessarily true. In particular, if n < m, it is possible for a degree-n interval Bernstein polynomial to bound a degree-m polynomial; but it cannot contain the polynomial.

In the case where P bounds p, we maintain extra information, the "slope error". We say that P (over [a .. b]) bounds p with a slope error of E (where E is an interval) if there is a polynomial p' contained in P such that the derivative of (p - p') is bounded by E in the range [a .. b]. If P bounds p with a slope error of 0 then P contains p.

(Note that "contains" and "bounds" are not standard terminology; I just made them up.)

Interval Bernstein polynomials are useful in finding real roots because of the following properties:

- Given an exact real polynomial p, we can compute an interval Bernstein polynomial over an arbitrary region containing p.
- Given an interval Bernstein polynomial P over [a .. c], where a < b < c, we can compute interval Bernstein polynomials P1 over [a .. b] and P2 over [b .. c], where P1 and P2 contain (or bound) all polynomials that P contains (or bounds).
- Given a degree-n interval Bernstein polynomial P over [a .. b], and m < n, we can compute a degree-m interval Bernstein polynomial P' over [a .. b] that bounds all polynomials that P bounds.
- It is sometimes possible to prove that no polynomial bounded by P over [a .. b] has any roots in [a .. b]. (Roughly, this is possible when no polynomial contained by P has any complex roots near the line segment [a .. b], where "near" is defined relative to the length b-a.)
- It is sometimes possible to prove that every polynomial bounded by P over [a .. b] with slope error E has exactly one root in [a .. b]. (Roughly, this is possible when every polynomial contained by P over [a .. b] has exactly one root in [a .. b], there are no other complex roots near the line segment [a .. b], and every

polynomial contained in P has a derivative which is bounded away from zero over [a .. b] by an amount which is large relative to E.)

- Starting from a sufficiently precise interval Bernstein polynomial, it is always possible to split it into polynomials which provably have 0 or 1 roots (as long as your original polynomial has no multiple real roots).

So a rough outline of a family of algorithms would be:

- Given a polynomial p, compute a region [a .. b] in which any real roots must lie.
- Compute an interval Bernstein polynomial P containing p over [a .. b].
- Keep splitting P until you have isolated all the roots. Optionally, reduce the degree or the precision of the interval Bernstein polynomials at intermediate stages (to reduce computation time). If this seems not to be working, go back and try again with higher precision.

Obviously, there are many details to be worked out to turn this into a full algorithm, like:

- What initial precision is selected for computing P?
- How do you decide when to reduce the degree of intermediate polynomials?
- How do you decide when to reduce the precision of intermediate polynomials?
- How do you decide where to split the interval Bernstein polynomial regions?
- How do you decide when to give up and start over with higher precision?

Each set of answers to these questions gives a different algorithm (potentially with very different performance characteristics), but all of them can use this `interval_bernstein_polynomial` class as their basic building block.

To save computation time, all coefficients in an `interval_bernstein_polynomial` share the same interval width. (There is one exception: when creating an `interval_bernstein_polynomial`, the first and last coefficients can be marked as "known positive" or "known negative". This has some of the same effect as having a (potentially) smaller interval width for these two coefficients, although it does not affect de Casteljau splitting.) To allow for widely varying coefficient magnitudes, all coefficients in an interval_bernstein_polynomial are scaled by 2^n (where n may be positive, negative, or zero).

There are two representations for interval_bernstein_polynomials, integer and floating-point. These are the two subclasses of this class; `interval_bernstein_polynomial` itself is an abstract class.

`interval_bernstein_polynomial` and its subclasses are not expected to be used outside this file.

region()

region_width()

try_rand_split(*ctx*, *logging_note*)

Compute a random split point r (using the random number generator embedded in ctx). We require 1/4 <= r < 3/4 (to ensure that recursive algorithms make progress).

Then, try doing a de Casteljau split of this polynomial at r, resulting in polynomials p1 and p2. If we see that the sign of this polynomial is determined at r, then return (p1, p2, r); otherwise, return None.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: bp = mk_ibpi([50, 20, -90, -70, 200], error=5)
sage: bp1, bp2, _ = bp.try_rand_split(mk_context(), None)
sage: bp1
<IBP: (50, 29, -27, -56, -11) + [0 .. 6) over [0 .. 43/64]>
sage: bp2
```

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```
<IBP: (-11, 10, 49, 111, 200) + [0 .. 6) over [43/64 .. 1]>
sage: bp1, bp2, _ = bp.try_rand_split(mk_context(seed=42), None)
sage: bp1
<IBP: (50, 32, -11, -41, -29) + [0 .. 6) over [0 .. 583/1024]>
sage: bp2
<IBP: (-29, -20, 13, 83, 200) + [0 .. 6) over [583/1024 .. 1]>
sage: bp = mk_ibpf([0.5, 0.2, -0.9, -0.7, 0.99], neg_err=-0.1, pos_err=0.01)
sage: bp1, bp2, _ = bp.try_rand_split(mk_context(), None)
sage: bp1  # rel tol
<IBP: (0.5, 0.2984375, -0.2642578125, -0.5511661529541015, -0.
↪3145806974172592) + [-0.10000000000000069 .. 0.010000000000000677] over [0 .
↪. 43/64]>
sage: bp2  # rel tol
<IBP: (-0.3145806974172592, -0.19903896331787108, 0.04135986328125002, 0.
↪43546875, 0.99) + [-0.10000000000000069 .. 0.010000000000000677] over [43/
↪64 .. 1]>
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> bp = mk_ibpi([Integer(50), Integer(20), -Integer(90), -Integer(70),␣
↪Integer(200)], error=Integer(5))
>>> bp1, bp2, _ = bp.try_rand_split(mk_context(), None)
>>> bp1
<IBP: (50, 29, -27, -56, -11) + [0 .. 6) over [0 .. 43/64]>
>>> bp2
<IBP: (-11, 10, 49, 111, 200) + [0 .. 6) over [43/64 .. 1]>
>>> bp1, bp2, _ = bp.try_rand_split(mk_context(seed=Integer(42)), None)
>>> bp1
<IBP: (50, 32, -11, -41, -29) + [0 .. 6) over [0 .. 583/1024]>
>>> bp2
<IBP: (-29, -20, 13, 83, 200) + [0 .. 6) over [583/1024 .. 1]>
>>> bp = mk_ibpf([RealNumber('0.5'), RealNumber('0.2'), -RealNumber('0.9'), -
↪RealNumber('0.7'), RealNumber('0.99')], neg_err=-RealNumber('0.1'), pos_
↪err=RealNumber('0.01'))
>>> bp1, bp2, _ = bp.try_rand_split(mk_context(), None)
>>> bp1  # rel tol
<IBP: (0.5, 0.2984375, -0.2642578125, -0.5511661529541015, -0.
↪3145806974172592) + [-0.10000000000000069 .. 0.010000000000000677] over [0 .
↪. 43/64]>
>>> bp2  # rel tol
<IBP: (-0.3145806974172592, -0.19903896331787108, 0.04135986328125002, 0.
↪43546875, 0.99) + [-0.10000000000000069 .. 0.010000000000000677] over [43/
↪64 .. 1]>
```

try_split(*ctx*, *logging_note*)

Try doing a de Casteljau split of this polynomial at 1/2, resulting in polynomials p1 and p2. If we see that the sign of this polynomial is determined at 1/2, then return (p1, p2, 1/2); otherwise, return None.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: bp = mk_ibpi([50, 20, -90, -70, 200], error=5)
```

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```
sage: bp1, bp2, _ = bp.try_split(mk_context(), None)
sage: bp1
<IBP: (50, 35, 0, -29, -31) + [0 .. 6) over [0 .. 1/2]>
sage: bp2
<IBP: (-31, -33, -8, 65, 200) + [0 .. 6) over [1/2 .. 1]>
sage: bp = mk_ibpf([0.5, 0.2, -0.9, -0.7, 0.99], neg_err=-0.1, pos_err=0.01)
sage: bp1, bp2, _ = bp.try_split(mk_context(), None)
sage: bp1
<IBP: (0.5, 0.35, 0.0, -0.2875, -0.369375) + [-0.10000000000000023 .. 0.
→010000000000000226] over [0 .. 1/2]>
sage: bp2
<IBP: (-0.369375, -0.45125, -0.3275, 0.14500000000000002, 0.99) + [-0.
→10000000000000023 .. 0.010000000000000226] over [1/2 .. 1]>
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> bp = mk_ibpi([Integer(50), Integer(20), -Integer(90), -Integer(70),
→Integer(200)], error=Integer(5))
>>> bp1, bp2, _ = bp.try_split(mk_context(), None)
>>> bp1
<IBP: (50, 35, 0, -29, -31) + [0 .. 6) over [0 .. 1/2]>
>>> bp2
<IBP: (-31, -33, -8, 65, 200) + [0 .. 6) over [1/2 .. 1]>
>>> bp = mk_ibpf([RealNumber('0.5'), RealNumber('0.2'), -RealNumber('0.9'), -
→RealNumber('0.7'), RealNumber('0.99')], neg_err=-RealNumber('0.1'), pos_
→err=RealNumber('0.01'))
>>> bp1, bp2, _ = bp.try_split(mk_context(), None)
>>> bp1
<IBP: (0.5, 0.35, 0.0, -0.2875, -0.369375) + [-0.10000000000000023 .. 0.
→010000000000000226] over [0 .. 1/2]>
>>> bp2
<IBP: (-0.369375, -0.45125, -0.3275, 0.14500000000000002, 0.99) + [-0.
→10000000000000023 .. 0.010000000000000226] over [1/2 .. 1]>
```

variations()

Consider a polynomial (written in either the normal power basis or the Bernstein basis). Take its list of coefficients, omitting zeroes. Count the number of positions in the list where the sign of one coefficient is opposite the sign of the next coefficient.

This count is the number of sign variations of the polynomial. According to Descartes' rule of signs, the number of real roots of the polynomial (counted with multiplicity) in a certain interval is always less than or equal to the number of sign variations, and the difference is always even. (If the polynomial is written in the power basis, the region is the positive reals; if the polynomial is written in the Bernstein basis over a particular region, then we count roots in that region.)

In particular, a polynomial with no sign variations has no real roots in the region, and a polynomial with one sign variation has one real root in the region.

In an interval Bernstein polynomial, we do not necessarily know the signs of the coefficients (if some of the coefficient intervals contain zero), so the polynomials contained by this interval polynomial may not all have the same number of sign variations. However, we can compute a range of possible numbers of sign variations.

This function returns the range, as a 2-tuple of integers.

class sage.rings.polynomial.real_roots.**interval_bernstein_polynomial_float**

Bases: *interval_bernstein_polynomial*

This is the subclass of *interval_bernstein_polynomial* where polynomial coefficients are represented using floating-point numbers.

In the floating-point representation, each coefficient is represented as an IEEE double-precision float A, and the (shared) lower and upper interval widths E1 and E2. These represent the coefficients (A+E1)*2^n <= c <= (A+E2)*2^n.

Note that we always have E1 <= 0 <= E2. Also, each floating-point coefficient has absolute value less than one.

(Note that *mk_ibpf()* is a simple helper function for creating elements of *interval_bernstein_polynomial_float* in doctests.)

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: bp = mk_ibpf([0.1, 0.2, 0.3], pos_err=0.5); print(bp)
degree 2 IBP with floating-point coefficients
sage: bp
<IBP: (0.1, 0.2, 0.3) + [0.0 .. 0.5]>
sage: bp.variations()
(0, 0)
sage: bp = mk_ibpf([-0.3, -0.1, 0.1, -0.1, -0.3, -0.1],                    #␣
↪needs sage.symbolic
....:              lower=1, upper=5/4, usign=1, pos_err=0.2,
....:              scale_log2=-3, level=2, slope_err=RIF(pi)); print(bp)
degree 5 IBP with floating-point coefficients
sage: bp                                                                    #␣
↪needs sage.symbolic
<IBP: ((-0.3, -0.1, 0.1, -0.1, -0.3, -0.1) + [0.0 .. 0.2]) * 2^-3
      over [1 .. 5/4]; usign 1; level 2; slope_err 3.141592653589794?>
sage: bp.variations()                                                       #␣
↪needs sage.symbolic
(3, 3)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> bp = mk_ibpf([RealNumber('0.1'), RealNumber('0.2'), RealNumber('0.3')], pos_
↪err=RealNumber('0.5')); print(bp)
degree 2 IBP with floating-point coefficients
>>> bp
<IBP: (0.1, 0.2, 0.3) + [0.0 .. 0.5]>
>>> bp.variations()
(0, 0)
>>> bp = mk_ibpf([-RealNumber('0.3'), -RealNumber('0.1'), RealNumber('0.1'), -
↪RealNumber('0.1'), -RealNumber('0.3'), -RealNumber('0.1')],                  ␣
↪    # needs sage.symbolic
...              lower=Integer(1), upper=Integer(5)/Integer(4), usign=Integer(1),␣
↪pos_err=RealNumber('0.2'),
...              scale_log2=-Integer(3), level=Integer(2), slope_err=RIF(pi));␣
↪print(bp)
degree 5 IBP with floating-point coefficients
>>> bp                                                                      #␣
```

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```
↪needs sage.symbolic
<IBP: ((-0.3, -0.1, 0.1, -0.1, -0.3, -0.1) + [0.0 .. 0.2]) * 2^-3
      over [1 .. 5/4]; usign 1; level 2; slope_err 3.141592653589794?>
>>> bp.variations()                                                    #␣
↪needs sage.symbolic
(3, 3)
```

as_float()

de_casteljau(*ctx*, *mid*, *msign=0*)

Uses de Casteljau's algorithm to compute the representation of this polynomial in a Bernstein basis over new regions.

INPUT:

- `mid` – where to split the Bernstein basis region; `0 < mid < 1`
- `msign` – default 0 (unknown); the sign of this polynomial at `mid`

OUTPUT:

- `bp1`, `bp2` – the new interval Bernstein polynomials
- `ok` – boolean; `True` if the sign of the original polynomial at `mid` is known

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: ctx = mk_context()
sage: bp = mk_ibpf([0.5, 0.2, -0.9, -0.7, 0.99], neg_err=-0.1, pos_err=0.01)
sage: bp1, bp2, ok = bp.de_casteljau(ctx, 1/2)
sage: bp1
<IBP: (0.5, 0.35, 0.0, -0.2875, -0.369375) + [-0.10000000000000023 .. 0.
↪010000000000000226] over [0 .. 1/2]>
sage: bp2
<IBP: (-0.369375, -0.45125, -0.3275, 0.14500000000000002, 0.99) + [-0.
↪10000000000000023 .. 0.010000000000000226] over [1/2 .. 1]>
sage: bp1, bp2, ok = bp.de_casteljau(ctx, 2/3)
sage: bp1 # rel tol 2e-16
<IBP: (0.5, 0.30000000000000004, -0.2555555555555555, -0.5444444444444444, -0.
↪32172839506172846) + [-0.10000000000000069 .. 0.010000000000000677] over [0␣
↪.. 2/3]>
sage: bp2  # rel tol 3e-15
<IBP: (-0.32172839506172846, -0.21037037037037046, 0.028888888888888797, 0.
↪4266666666666666, 0.99) + [-0.10000000000000069 .. 0.010000000000000677]␣
↪over [2/3 .. 1]>
sage: bp1, bp2, ok = bp.de_casteljau(ctx, 7/39)
sage: bp1  # rel tol
<IBP: (0.5, 0.4461538461538461, 0.36653517422748183, 0.27328680523946786, 0.
↪1765692706232836) + [-0.10000000000000069 .. 0.010000000000000677] over [0 .
↪. 7/39]>
sage: bp2  # rel tol
<IBP: (0.1765692706232836, -0.26556803047927313, -0.7802038132807364, -0.
↪3966666666666666, 0.99) + [-0.10000000000000069 .. 0.010000000000000677]␣
↪over [7/39 .. 1]>
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> ctx = mk_context()
>>> bp = mk_ibpf([RealNumber('0.5'), RealNumber('0.2'), -RealNumber('0.9'), -
↪RealNumber('0.7'), RealNumber('0.99')], neg_err=-RealNumber('0.1'), pos_
↪err=RealNumber('0.01'))
>>> bp1, bp2, ok = bp.de_casteljau(ctx, Integer(1)/Integer(2))
>>> bp1
<IBP: (0.5, 0.35, 0.0, -0.2875, -0.369375) + [-0.10000000000000023 .. 0.
↪010000000000000226] over [0 .. 1/2]>
>>> bp2
<IBP: (-0.369375, -0.45125, -0.3275, 0.14500000000000002, 0.99) + [-0.
↪10000000000000023 .. 0.010000000000000226] over [1/2 .. 1]>
>>> bp1, bp2, ok = bp.de_casteljau(ctx, Integer(2)/Integer(3))
>>> bp1 # rel tol 2e-16
<IBP: (0.5, 0.30000000000000004, -0.2555555555555555, -0.5444444444444444, -0.
↪32172839506172846) + [-0.10000000000000069 .. 0.010000000000000677] over [0␣
↪.. 2/3]>
>>> bp2  # rel tol 3e-15
<IBP: (-0.32172839506172846, -0.21037037037037046, 0.028888888888888797, 0.
↪4266666666666666, 0.99) + [-0.10000000000000069 .. 0.010000000000000677]␣
↪over [2/3 .. 1]>
>>> bp1, bp2, ok = bp.de_casteljau(ctx, Integer(7)/Integer(39))
>>> bp1  # rel tol
<IBP: (0.5, 0.4461538461538461, 0.36653517422748183, 0.27328680523946786, 0.
↪1765692706232836) + [-0.10000000000000069 .. 0.010000000000000677] over [0 .
↪. 7/39]>
>>> bp2  # rel tol
<IBP: (0.1765692706232836, -0.26556803047927313, -0.7802038132807364, -0.
↪3966666666666666, 0.99) + [-0.10000000000000069 .. 0.010000000000000677]␣
↪over [7/39 .. 1]>
```

get_msb_bit()

Return an approximation of the log2 of the maximum of the absolute values of the coefficients, as an integer.

slope_range()

Compute a bound on the derivative of this polynomial, over its region.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: bp = mk_ibpf([0.5, 0.2, -0.9, -0.7, 0.99], neg_err=-0.1, pos_err=0.01)
sage: bp.slope_range().str(style='brackets')
'[-4.8400000000000017 .. 7.2000000000000011]'
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> bp = mk_ibpf([RealNumber('0.5'), RealNumber('0.2'), -RealNumber('0.9'), -
↪RealNumber('0.7'), RealNumber('0.99')], neg_err=-RealNumber('0.1'), pos_
↪err=RealNumber('0.01'))
>>> bp.slope_range().str(style='brackets')
'[-4.8400000000000017 .. 7.2000000000000011]'
```

class sage.rings.polynomial.real_roots.**interval_bernstein_polynomial_integer**

Bases: `interval_bernstein_polynomial`

This is the subclass of `interval_bernstein_polynomial` where polynomial coefficients are represented using integers.

In this integer representation, each coefficient is represented by a GMP arbitrary-precision integer A, and a (shared) interval width E (which is a machine integer). These represent the coefficients A*2^n <= c < (A+E)*2^n.

(Note that `mk_ibpi is a simple helper()` function for creating elements of `interval_bernstein_polynomial_integer` in doctests.)

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: bp = mk_ibpi([1, 2, 3], error=5); print(bp)
degree 2 IBP with 2-bit coefficients
sage: bp
<IBP: (1, 2, 3) + [0 .. 5)>
sage: bp.variations()
(0, 0)
sage: bp = mk_ibpi([-3, -1, 1, -1, -3, -1], lower=1, upper=5/4, usign=1,        #␣
↪needs sage.symbolic
....:              error=2, scale_log2=-3, level=2, slope_err=RIF(pi)); print(bp)
degree 5 IBP with 2-bit coefficients
sage: bp                                                                         #␣
↪needs sage.symbolic
<IBP: ((-3, -1, 1, -1, -3, -1) + [0 .. 2)) * 2^-3 over [1 .. 5/4]; usign 1;
      level 2; slope_err 3.141592653589794?>
sage: bp.variations()                                                            #␣
↪needs sage.symbolic
(3, 3)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> bp = mk_ibpi([Integer(1), Integer(2), Integer(3)], error=Integer(5));␣
↪print(bp)
degree 2 IBP with 2-bit coefficients
>>> bp
<IBP: (1, 2, 3) + [0 .. 5)>
>>> bp.variations()
(0, 0)
>>> bp = mk_ibpi([-Integer(3), -Integer(1), Integer(1), -Integer(1), -Integer(3),␣
↪-Integer(1)], lower=Integer(1), upper=Integer(5)/Integer(4), usign=Integer(1), ␣
↪      # needs sage.symbolic
...               error=Integer(2), scale_log2=-Integer(3), level=Integer(2),␣
↪slope_err=RIF(pi)); print(bp)
degree 5 IBP with 2-bit coefficients
>>> bp                                                                         #␣
↪needs sage.symbolic
<IBP: ((-3, -1, 1, -1, -3, -1) + [0 .. 2)) * 2^-3 over [1 .. 5/4]; usign 1;
      level 2; slope_err 3.141592653589794?>
>>> bp.variations()                                                            #␣
↪needs sage.symbolic
(3, 3)
```

as_float()

Compute an interval_bernstein_polynomial_float which contains (or bounds) all the polynomials this interval polynomial contains (or bounds).

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: bp = mk_ibpi([50, 20, -90, -70, 200], error=5)
sage: print(bp.as_float())
degree 4 IBP with floating-point coefficients
sage: bp.as_float()
<IBP: ((0.1953125, 0.078125, -0.3515625, -0.2734375, 0.78125) + [-1.
→1275702593849246e-16 .. 0.01953125000000017]) * 2^8>
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> bp = mk_ibpi([Integer(50), Integer(20), -Integer(90), -Integer(70),␣
→Integer(200)], error=Integer(5))
>>> print(bp.as_float())
degree 4 IBP with floating-point coefficients
>>> bp.as_float()
<IBP: ((0.1953125, 0.078125, -0.3515625, -0.2734375, 0.78125) + [-1.
→1275702593849246e-16 .. 0.01953125000000017]) * 2^8>
```

de_casteljau(*ctx*, *mid*, *msign=0*)

Uses de Casteljau's algorithm to compute the representation of this polynomial in a Bernstein basis over new regions.

INPUT:

- `mid` – where to split the Bernstein basis region; 0 < mid < 1
- `msign` – default 0 (unknown); the sign of this polynomial at mid

OUTPUT:

- `bp1`, `bp2` – the new interval Bernstein polynomials
- `ok` – boolean; `True` if the sign of the original polynomial at mid is known

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: bp = mk_ibpi([50, 20, -90, -70, 200], error=5)
sage: ctx = mk_context()
sage: bp1, bp2, ok = bp.de_casteljau(ctx, 1/2)
sage: bp1
<IBP: (50, 35, 0, -29, -31) + [0 .. 6) over [0 .. 1/2]>
sage: bp2
<IBP: (-31, -33, -8, 65, 200) + [0 .. 6) over [1/2 .. 1]>
sage: bp1, bp2, ok = bp.de_casteljau(ctx, 2/3)
sage: bp1
<IBP: (50, 30, -26, -55, -13) + [0 .. 6) over [0 .. 2/3]>
sage: bp2
<IBP: (-13, 8, 47, 110, 200) + [0 .. 6) over [2/3 .. 1]>
sage: bp1, bp2, ok = bp.de_casteljau(ctx, 7/39)
sage: bp1
```

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```
<IBP: (50, 44, 36, 27, 17) + [0 .. 6) over [0 .. 7/39]>
sage: bp2
<IBP: (17, -26, -75, -22, 200) + [0 .. 6) over [7/39 .. 1]>
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> bp = mk_ibpi([Integer(50), Integer(20), -Integer(90), -Integer(70),␣
↪Integer(200)], error=Integer(5))
>>> ctx = mk_context()
>>> bp1, bp2, ok = bp.de_casteljau(ctx, Integer(1)/Integer(2))
>>> bp1
<IBP: (50, 35, 0, -29, -31) + [0 .. 6) over [0 .. 1/2]>
>>> bp2
<IBP: (-31, -33, -8, 65, 200) + [0 .. 6) over [1/2 .. 1]>
>>> bp1, bp2, ok = bp.de_casteljau(ctx, Integer(2)/Integer(3))
>>> bp1
<IBP: (50, 30, -26, -55, -13) + [0 .. 6) over [0 .. 2/3]>
>>> bp2
<IBP: (-13, 8, 47, 110, 200) + [0 .. 6) over [2/3 .. 1]>
>>> bp1, bp2, ok = bp.de_casteljau(ctx, Integer(7)/Integer(39))
>>> bp1
<IBP: (50, 44, 36, 27, 17) + [0 .. 6) over [0 .. 7/39]>
>>> bp2
<IBP: (17, -26, -75, -22, 200) + [0 .. 6) over [7/39 .. 1]>
```

down_degree(*ctx*, *max_err*, *exp_err_shift*)

Compute an interval_bernstein_polynomial_integer which bounds all the polynomials this interval polynomial bounds, but is of lesser degree.

During the computation, we find an "expected error" expected_err, which is the error inherent in our approach (this depends on the degrees involved, and is proportional to the error of the current polynomial).

We require that the error of the new interval polynomial be bounded both by max_err, and by expected_err << exp_err_shift. If we find such a polynomial p, then we return a pair of p and some debugging/logging information. Otherwise, we return the pair (None, None).

If the resulting polynomial would have error more than 2^17, then it is downscaled before returning.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: bp = mk_ibpi([0, 100, 400, 903], error=2)
sage: ctx = mk_context()
sage: bp
<IBP: (0, 100, 400, 903) + [0 .. 2)>
sage: dbp, _ = bp.down_degree(ctx, 10, 32)
sage: dbp
<IBP: (-1, 148, 901) + [0 .. 4); level 1; slope_err 0.?e2>
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> bp = mk_ibpi([Integer(0), Integer(100), Integer(400), Integer(903)],␣
↪error=Integer(2))
```

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```
>>> ctx = mk_context()
>>> bp
<IBP: (0, 100, 400, 903) + [0 .. 2)>
>>> dbp, _ = bp.down_degree(ctx, Integer(10), Integer(32))
>>> dbp
<IBP: (-1, 148, 901) + [0 .. 4); level 1; slope_err 0.?e2>
```

down_degree_iter(*ctx*, *max_scale*)

Compute a degree-reduced version of this interval polynomial, by iterating down_degree.

We stop when degree reduction would give a polynomial which is too inaccurate, meaning that either we think the current polynomial may have more roots in its region than the degree of the reduced polynomial, or that the least significant accurate bit in the result (on the absolute scale) would be larger than 1 << max_scale.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: bp = mk_ibpi([0, 100, 400, 903, 1600, 2500], error=2)
sage: ctx = mk_context()
sage: bp
<IBP: (0, 100, 400, 903, 1600, 2500) + [0 .. 2)>
sage: rbp = bp.down_degree_iter(ctx, 6)
sage: rbp
<IBP: (-4, 249, 2497) + [0 .. 9); level 2; slope_err 0.?e3>
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> bp = mk_ibpi([Integer(0), Integer(100), Integer(400), Integer(903),␣
↪Integer(1600), Integer(2500)], error=Integer(2))
>>> ctx = mk_context()
>>> bp
<IBP: (0, 100, 400, 903, 1600, 2500) + [0 .. 2)>
>>> rbp = bp.down_degree_iter(ctx, Integer(6))
>>> rbp
<IBP: (-4, 249, 2497) + [0 .. 9); level 2; slope_err 0.?e3>
```

downscale(*bits*)

Compute an interval_bernstein_polynomial_integer which contains (or bounds) all the polynomials this interval polynomial contains (or bounds), but uses "bits" fewer bits.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: bp = mk_ibpi([0, 100, 400, 903], error=2)
sage: bp.downscale(5)
<IBP: ((0, 3, 12, 28) + [0 .. 1)) * 2^5>
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> bp = mk_ibpi([Integer(0), Integer(100), Integer(400), Integer(903)],␣
↪error=Integer(2))
>>> bp.downscale(Integer(5))
<IBP: ((0, 3, 12, 28) + [0 .. 1)) * 2^5>
```

get_msb_bit()

Return an approximation of the log2 of the maximum of the absolute values of the coefficients, as an integer.

slope_range()

Compute a bound on the derivative of this polynomial, over its region.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: bp = mk_ibpi([0, 100, 400, 903], error=2)
sage: bp.slope_range().str(style='brackets')
'[294.00000000000000 .. 1515.0000000000000]'
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> bp = mk_ibpi([Integer(0), Integer(100), Integer(400), Integer(903)],␣
↪error=Integer(2))
>>> bp.slope_range().str(style='brackets')
'[294.00000000000000 .. 1515.0000000000000]'
```

sage.rings.polynomial.real_roots.**intvec_to_doublevec**(*b*, *err*)

Given a vector of integers A = [a1, …, an], and an integer error bound E, returns a vector of floating-point numbers B = [b1, …, bn], lower and upper error bounds F1 and F2, and a scaling factor d, such that

$$(bk + F1) * 2^d \leq ak$$

and

$$ak + E \leq (bk + F2) * 2^d$$

If bj is the element of B with largest absolute value, then 0.5 <= abs(bj) < 1.0 .

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: intvec_to_doublevec(vector(ZZ, [1, 2, 3, 4, 5]), 3)
((0.125, 0.25, 0.375, 0.5, 0.625), -1.1275702593849246e-16, 0.37500000000000017,␣
↪3)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> intvec_to_doublevec(vector(ZZ, [Integer(1), Integer(2), Integer(3),␣
↪Integer(4), Integer(5)]), Integer(3))
((0.125, 0.25, 0.375, 0.5, 0.625), -1.1275702593849246e-16, 0.37500000000000017,␣
↪3)
```

class sage.rings.polynomial.real_roots.**island**

Bases: object

This implements the island portion of my ocean-island root isolation algorithm. See the documentation for class ocean, for more information on the overall algorithm.

Island root refinement starts with a Bernstein polynomial whose region is the whole island (or perhaps slightly more than the island in certain cases). There are two subalgorithms; one when looking at a Bernstein polynomial covering a whole island (so we know that there are gaps on the left and right), and one when looking at a Bernstein polynomial covering the left segment of an island (so we know that there is a gap on the left, but the right is in the middle of an island). An important invariant of the left-segment subalgorithm over the region [l .. r] is that it always finds a gap [r0 .. r] ending at its right endpoint.

Ignoring degree reduction, downscaling (precision reduction), and failures to split, the algorithm is roughly:

Whole island:

1. If the island definitely has exactly one root, then return.
2. Split the island in (approximately) half.
3. If both halves definitely have no roots, then remove this island from its doubly-linked list (merging its left and right gaps) and return.
4. If either half definitely has no roots, then discard that half and call the whole-island algorithm with the other half, then return.
5. If both halves may have roots, then call the left-segment algorithm on the left half.
6. We now know that there is a gap immediately to the left of the right half, so call the whole-island algorithm on the right half, then return.

Left segment:

1. Split the left segment in (approximately) half.
2. If both halves definitely have no roots, then extend the left gap over the segment and return.
3. If the left half definitely has no roots, then extend the left gap over this half and call the left-segment algorithm on the right half, then return.
4. If the right half definitely has no roots, then split the island in two, creating a new gap. Call the whole-island algorithm on the left half, then return.
5. Both halves may have roots. Call the left-segment algorithm on the left half.
6. We now know that there is a gap immediately to the left of the right half, so call the left-segment algorithm on the right half, then return.

Degree reduction complicates this picture only slightly. Basically, we use heuristics to decide when degree reduction might be likely to succeed and be helpful; whenever this is the case, we attempt degree reduction.

Precision reduction and split failure add more complications. The algorithm maintains a stack of different-precision representations of the interval Bernstein polynomial. The base of the stack is at the highest (currently known) precision; each stack entry has approximately half the precision of the entry below it. When we do a split, we pop off the top of the stack, split it, then push whichever half we're interested in back on the stack (so the different Bernstein polynomials may be over different regions). When we push a polynomial onto the stack, we may heuristically decide to push further lower-precision versions of the same polynomial onto the stack.

In the algorithm above, whenever we say "split in (approximately) half", we attempt to split the top-of-stack polynomial using try_split() and try_rand_split(). However, these will fail if the sign of the polynomial at the chosen split point is unknown (if the polynomial is not known to high enough precision, or if the chosen split point actually happens to be a root of the polynomial). If this fails, then we discard the top-of-stack polynomial, and try again with the next polynomial down (which has approximately twice the precision). This next polynomial may not be over the same region; if not, we split it using de Casteljau's algorithm to get a polynomial over (approximately) the same region first.

If we run out of higher-precision polynomials (if we empty out the entire stack), then we give up on root refinement for this island. The ocean class will notice this, provide the island with a higher-precision polynomial, and restart root refinement. Basically the only information kept in that case is the lower and upper bounds on the island. Since these are updated whenever we discover a "half" (of an island or a segment) that definitely contains no roots, we never need to re-examine these gaps. (We could keep more information. For example, we could keep a record of split points that succeeded and failed. However, a split point that failed at lower precision is likely to succeed at higher precision, so it's not worth avoiding. It could be useful to select split points that are known to succeed, but starting from a new Bernstein polynomial over a slightly different region, hitting such split points would require de Casteljau splits with non-power-of-two denominators, which are much much slower.)

bp_done(*bp*)

Examine the given Bernstein polynomial to see if it is known to have exactly one root in its region. (In addition, we require that the polynomial region not include 0 or 1. This makes things work if the user gives explicit bounds to real_roots(), where the lower or upper bound is a root of the polynomial. real_roots() deals with this by explicitly detecting it, dividing out the appropriate linear polynomial, and adding the root to the returned list of roots; but then if the island considers itself "done" with a region including 0 or 1, the returned root regions can overlap with each other.)

done(*ctx*)

Check to see if the island is known to contain zero roots or is known to contain one root.

has_root()

Assuming that the island is done (has either 0 or 1 roots), reports whether the island has a root.

less_bits(*ancestors*, *bp*)

Heuristically push lower-precision polynomials on the polynomial stack. See the class documentation for class island for more information.

more_bits(*ctx*, *ancestors*, *bp*, *rightmost*)

Find a Bernstein polynomial on the "ancestors" stack with more precision than bp; if it is over a different region, then shrink its region to (approximately) match that of bp. (If this is rightmost – if bp covers the whole island – then we only require that the new region cover the whole island fairly tightly; if this is not rightmost, then the new region will have exactly the same right boundary as bp, although the left boundary may vary slightly.)

refine(*ctx*)

Attempts to shrink and/or split this island into sub-island that each definitely contain exactly one root.

refine_recurse(*ctx*, *bp*, *ancestors*, *history*, *rightmost*)

This implements the root isolation algorithm described in the class documentation for class island. This is the implementation of both the whole-island and the left-segment algorithms; if the flag rightmost is True, then it is the whole-island algorithm, otherwise the left-segment algorithm.

The precision-reduction stack is (ancestors + [bp]); that is, the top-of-stack is maintained separately.

reset_root_width(*target_width*)

Modify the criteria for this island to require that it is not "done" until its width is less than or equal to target_width.

shrink_bp(*ctx*)

If the island's Bernstein polynomial covers a region much larger than the island itself (in particular, if either the island's left gap or right gap are totally contained in the polynomial's region) then shrink the polynomial down to cover the island more tightly.

class `sage.rings.polynomial.real_roots.`**linear_map**(*lower*, *upper*)

Bases: `object`

A simple class to map linearly between original coordinates (ranging from [lower .. upper]) and ocean coordinates (ranging from [0 .. 1]).

from_ocean(*region*)

to_ocean(*region*)

sage.rings.polynomial.real_roots.**max_abs_doublevec**(*c*)

Given a floating-point vector, return the maximum of the absolute values of its elements.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: max_abs_doublevec(vector(RDF, [0.1, -0.767, 0.3, 0.693]))
0.767
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> max_abs_doublevec(vector(RDF, [RealNumber('0.1'), -RealNumber('0.767'),␣
↪RealNumber('0.3'), RealNumber('0.693')]))
0.767
```

sage.rings.polynomial.real_roots.**max_bitsize_intvec_doctest**(*b*)

sage.rings.polynomial.real_roots.**maximum_root_first_lambda**(*p*)

Given a polynomial with real coefficients, computes an upper bound on its largest real root.

This is using the first-lambda algorithm from "Implementations of a New Theorem for Computing Bounds for Positive Roots of Polynomials", by Akritas, Strzebo'nski, and Vigklas.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: x = polygen(ZZ)
sage: maximum_root_first_lambda((x-1)*(x-2)*(x-3))
6.00000000000001
sage: maximum_root_first_lambda((x+1)*(x+2)*(x+3))
0.000000000000000
sage: maximum_root_first_lambda(x^2 - 1)
1.00000000000000
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> x = polygen(ZZ)
>>> maximum_root_first_lambda((x-Integer(1))*(x-Integer(2))*(x-Integer(3)))
6.00000000000001
>>> maximum_root_first_lambda((x+Integer(1))*(x+Integer(2))*(x+Integer(3)))
0.000000000000000
>>> maximum_root_first_lambda(x**Integer(2) - Integer(1))
1.00000000000000
```

sage.rings.polynomial.real_roots.**maximum_root_local_max**(*p*)

Given a polynomial with real coefficients, computes an upper bound on its largest real root, using the local-max algorithm from "Implementations of a New Theorem for Computing Bounds for Positive Roots of Polynomials", by Akritas, Strzebo'nski, and Vigklas.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: x = polygen(ZZ)
sage: maximum_root_local_max((x-1)*(x-2)*(x-3))
12.0000000000001
```

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```
sage: maximum_root_local_max((x+1)*(x+2)*(x+3))
0.000000000000000
sage: maximum_root_local_max(x^2 - 1)
1.41421356237310
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> x = polygen(ZZ)
>>> maximum_root_local_max((x-Integer(1))*(x-Integer(2))*(x-Integer(3)))
12.0000000000001
>>> maximum_root_local_max((x+Integer(1))*(x+Integer(2))*(x+Integer(3)))
0.000000000000000
>>> maximum_root_local_max(x**Integer(2) - Integer(1))
1.41421356237310
```

sage.rings.polynomial.real_roots.**min_max_delta_intvec**(*a*, *b*)

Given two integer vectors a and b (of equal, nonzero length), return a pair of the minimum and maximum values taken on by a[i] - b[i].

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: a = vector(ZZ, [10, -30])
sage: b = vector(ZZ, [15, -60])
sage: min_max_delta_intvec(a, b)
(30, -5)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> a = vector(ZZ, [Integer(10), -Integer(30)])
>>> b = vector(ZZ, [Integer(15), -Integer(60)])
>>> min_max_delta_intvec(a, b)
(30, -5)
```

sage.rings.polynomial.real_roots.**min_max_diff_doublevec**(*c*)

Given a floating-point vector b = (b0, …, bn), compute the minimum and maximum values of b_{j+1} - b_j.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: min_max_diff_doublevec(vector(RDF, [1, 7, -2]))
(-9.0, 6.0)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> min_max_diff_doublevec(vector(RDF, [Integer(1), Integer(7), -Integer(2)]))
(-9.0, 6.0)
```

sage.rings.polynomial.real_roots.**min_max_diff_intvec**(*b*)

Given an integer vector b = (b0, …, bn), compute the minimum and maximum values of b_{j+1} - b_j.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: min_max_diff_intvec(vector(ZZ, [1, 7, -2]))
(-9, 6)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> min_max_diff_intvec(vector(ZZ, [Integer(1), Integer(7), -Integer(2)]))
(-9, 6)
```

sage.rings.polynomial.real_roots.**mk_context**(*do_logging=False*, *seed=0*, *wordsize=32*)

A simple wrapper for creating context objects with coercions, defaults, etc.

For use in doctests.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: mk_context(do_logging=True, seed=3, wordsize=64)
root isolation context: seed=3; do_logging=True; wordsize=64
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> mk_context(do_logging=True, seed=Integer(3), wordsize=Integer(64))
root isolation context: seed=3; do_logging=True; wordsize=64
```

sage.rings.polynomial.real_roots.**mk_ibpf**(*coeffs*, *lower=0*, *upper=1*, *lsign=0*, *usign=0*, *neg_err=0*, *pos_err=0*, *scale_log2=0*, *level=0*, *slope_err=None*)

A simple wrapper for creating *interval_bernstein_polynomial_float* objects with coercions, defaults, etc.

For use in doctests.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: print(mk_ibpf([0.5, 0.2, -0.9, -0.7, 0.99], pos_err=0.1, neg_err=-0.01))
degree 4 IBP with floating-point coefficients
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> print(mk_ibpf([RealNumber('0.5'), RealNumber('0.2'), -RealNumber('0.9'), -
↪RealNumber('0.7'), RealNumber('0.99')], pos_err=RealNumber('0.1'), neg_err=-
↪RealNumber('0.01')))
degree 4 IBP with floating-point coefficients
```

sage.rings.polynomial.real_roots.**mk_ibpi**(*coeffs*, *lower=0*, *upper=1*, *lsign=0*, *usign=0*, *error=1*, *scale_log2=0*, *level=0*, *slope_err=None*)

A simple wrapper for creating interval_bernstein_polynomial_integer objects with coercions, defaults, etc.

For use in doctests.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: print(mk_ibpi([50, 20, -90, -70, 200], error=5))
degree 4 IBP with 8-bit coefficients
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> print(mk_ibpi([Integer(50), Integer(20), -Integer(90), -Integer(70),␣
↪Integer(200)], error=Integer(5)))
degree 4 IBP with 8-bit coefficients
```

class sage.rings.polynomial.real_roots.**ocean**

Bases: object

Given the tools we've defined so far, there are many possible root isolation algorithms that differ on where to select split points, what precision to work at when, and when to attempt degree reduction.

Here we implement one particular algorithm, which I call the ocean-island algorithm. We start with an interval Bernstein polynomial defined over the region [0 .. 1]. This region is the "ocean". Using de Casteljau's algorithm and Descartes' rule of signs, we divide this region into subregions which may contain roots, and subregions which are guaranteed not to contain roots. Subregions which may contain roots are "islands"; subregions known not to contain roots are "gaps".

All the real root isolation work happens in class island. See the documentation of that class for more information.

An island can be told to refine itself until it contains only a single root. This may not succeed, if the island's interval Bernstein polynomial does not have enough precision. The ocean basically loops, refining each of its islands, then increasing the precision of islands which did not succeed in isolating a single root; until all islands are done.

Increasing the precision of unsuccessful islands is done in a single pass using split_for_target(); this means it is possible to share work among multiple islands.

all_done()

Return True iff all islands are known to contain exactly one root.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([1/3, -22/
↪7, 193/71, -140/99]), lmap)
sage: oc.all_done()
False
sage: oc.find_roots()
sage: oc.all_done()
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([Integer(1)/
↪Integer(3), -Integer(22)/Integer(7), Integer(193)/Integer(71), -
↪Integer(140)/Integer(99)]), lmap)
>>> oc.all_done()
False
>>> oc.find_roots()
>>> oc.all_done()
True
```

approx_bp(*scale_log2*)

Return an approximation to our Bernstein polynomial with the given scale_log2.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([1/3, -22/
↪7, 193/71, -140/99]), lmap)
sage: oc.approx_bp(0)
<IBP: (0, -4, 2, -2) + [0 .. 1); lsign 1>
sage: oc.approx_bp(-20)
<IBP: ((349525, -3295525, 2850354, -1482835) + [0 .. 1)) * 2^-20>
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([Integer(1)/
↪Integer(3), -Integer(22)/Integer(7), Integer(193)/Integer(71), -
↪Integer(140)/Integer(99)]), lmap)
>>> oc.approx_bp(Integer(0))
<IBP: (0, -4, 2, -2) + [0 .. 1); lsign 1>
>>> oc.approx_bp(-Integer(20))
<IBP: ((349525, -3295525, 2850354, -1482835) + [0 .. 1)) * 2^-20>
```

find_roots()

Isolate all roots in this ocean.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([1/3, -22/
↪7, 193/71, -140/99]), lmap)
sage: oc
ocean with precision 120 and 1 island(s)
sage: oc.find_roots()
sage: oc
ocean with precision 120 and 3 island(s)
sage: oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([1, 0, -
↪1111/2, 0, 11108889/14, 0, 0, 0, 0, -1]), lmap)
sage: oc.find_roots()
sage: oc
ocean with precision 240 and 3 island(s)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([Integer(1)/
↪Integer(3), -Integer(22)/Integer(7), Integer(193)/Integer(71), -
↪Integer(140)/Integer(99)]), lmap)
>>> oc
ocean with precision 120 and 1 island(s)
>>> oc.find_roots()
>>> oc
ocean with precision 120 and 3 island(s)
>>> oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([Integer(1),
↪ Integer(0), -Integer(1111)/Integer(2), Integer(0), Integer(11108889)/
```

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```
↪Integer(14), Integer(0), Integer(0), Integer(0), Integer(0), -Integer(1)]),␣
↪lmap)
>>> oc.find_roots()
>>> oc
ocean with precision 240 and 3 island(s)
```

increase_precision()

Increase the precision of the interval Bernstein polynomial held by any islands which are not done. (In normal use, calls to this function are separated by calls to self.refine_all().)

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([1/3, -22/
↪7, 193/71, -140/99]), lmap)
sage: oc
ocean with precision 120 and 1 island(s)
sage: oc.increase_precision()
sage: oc.increase_precision()
sage: oc.increase_precision()
sage: oc
ocean with precision 960 and 1 island(s)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([Integer(1)/
↪Integer(3), -Integer(22)/Integer(7), Integer(193)/Integer(71), -
↪Integer(140)/Integer(99)]), lmap)
>>> oc
ocean with precision 120 and 1 island(s)
>>> oc.increase_precision()
>>> oc.increase_precision()
>>> oc.increase_precision()
>>> oc
ocean with precision 960 and 1 island(s)
```

refine_all()

Refine all islands which are not done (which are not known to contain exactly one root).

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([1/3, -22/
↪7, 193/71, -140/99]), lmap)
sage: oc
ocean with precision 120 and 1 island(s)
sage: oc.refine_all()
sage: oc
ocean with precision 120 and 3 island(s)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
```

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```
>>> oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([Integer(1)/
↪Integer(3), -Integer(22)/Integer(7), Integer(193)/Integer(71), -
↪Integer(140)/Integer(99)]), lmap)
>>> oc
ocean with precision 120 and 1 island(s)
>>> oc.refine_all()
>>> oc
ocean with precision 120 and 3 island(s)
```

reset_root_width (*isle_num*, *target_width*)

Require that the isle_num island have a width at most target_width.

If this is followed by a call to find_roots(), then the corresponding root will be refined to the specified width.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([-1, -1,␣
↪1]), lmap)
sage: oc.find_roots()
sage: oc.roots()
[(1/2, 3/4)]
sage: oc.reset_root_width(0, 1/2^200)
sage: oc.find_roots()
sage: oc
ocean with precision 240 and 1 island(s)
sage: RR(RealIntervalField(300)(oc.roots()[0]).absolute_diameter()).log2()
-232.668979560890
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([-
↪Integer(1), -Integer(1), Integer(1)]), lmap)
>>> oc.find_roots()
>>> oc.roots()
[(1/2, 3/4)]
>>> oc.reset_root_width(Integer(0), Integer(1)/Integer(2)**Integer(200))
>>> oc.find_roots()
>>> oc
ocean with precision 240 and 1 island(s)
>>> RR(RealIntervalField(Integer(300))(oc.roots()[Integer(0)]).absolute_
↪diameter()).log2()
-232.668979560890
```

roots ()

Return the locations of all islands in this ocean. (If run after find_roots(), this is the location of all roots in the ocean.)

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([1/3, -22/
↪7, 193/71, -140/99]), lmap)
```

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```
sage: oc.find_roots()
sage: oc.roots()
[(1/32, 1/16), (1/2, 5/8), (3/4, 7/8)]
sage: oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([1, 0, -
↪1111/2, 0, 11108889/14, 0, 0, 0, 0, -1]), lmap)
sage: oc.find_roots()
sage: oc.roots()
[(95761241267509487747625/9671406556917033397649408, 191522482605387719863145/
↪19342813113834066795298816), (1496269395904347376805/
↪151115727451828646838272, 374067366568272936175/37778931862957161709568),␣
↪(31/32, 63/64)]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([Integer(1)/
↪Integer(3), -Integer(22)/Integer(7), Integer(193)/Integer(71), -
↪Integer(140)/Integer(99)]), lmap)
>>> oc.find_roots()
>>> oc.roots()
[(1/32, 1/16), (1/2, 5/8), (3/4, 7/8)]
>>> oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([Integer(1),
↪ Integer(0), -Integer(1111)/Integer(2), Integer(0), Integer(11108889)/
↪Integer(14), Integer(0), Integer(0), Integer(0), Integer(0), -Integer(1)]),␣
↪lmap)
>>> oc.find_roots()
>>> oc.roots()
[(95761241267509487747625/9671406556917033397649408, 191522482605387719863145/
↪19342813113834066795298816), (1496269395904347376805/
↪151115727451828646838272, 374067366568272936175/37778931862957161709568),␣
↪(31/32, 63/64)]
```

`sage.rings.polynomial.real_roots.`**`precompute_degree_reduction_cache`**`(`*n*`)`

Compute and cache the matrices used for degree reduction, starting from degree n.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: precompute_degree_reduction_cache(5)
sage: dr_cache[5]
(
   [121/126    8/63    -1/9   -2/63  11/126   -2/63]
   [   -3/7   37/42   16/21    1/21    -3/7     1/6]
   [    1/6    -3/7    1/21   16/21   37/42    -3/7]
3, [  -2/63  11/126   -2/63    -1/9    8/63 121/126], 2,

([121  16 -14  -4  11  -4]
[-54 111  96   6 -54  21]
[ 21 -54   6  96 111 -54]
[ -4  11  -4 -14  16 121], 126)
)
```

```
>>> from sage.all import *
```

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```
>>> from sage.rings.polynomial.real_roots import *
>>> precompute_degree_reduction_cache(Integer(5))
>>> dr_cache[Integer(5)]
(
   [121/126    8/63    -1/9   -2/63  11/126   -2/63]
   [   -3/7   37/42   16/21    1/21    -3/7     1/6]
   [    1/6    -3/7    1/21   16/21   37/42    -3/7]
3, [  -2/63  11/126   -2/63    -1/9    8/63 121/126], 2,
<BLANKLINE>
([121  16 -14  -4  11  -4]
[-54 111  96   6 -54  21]
[ 21 -54   6  96 111 -54]
[ -4  11  -4 -14  16 121], 126)
)
```

sage.rings.polynomial.real_roots.**pseudoinverse**(*m*)

sage.rings.polynomial.real_roots.**rational_root_bounds**(*p*)

Given a polynomial p with real coefficients, computes rationals a and b, such that for every real root r of p, a < r < b. We try to find rationals which bound the roots somewhat tightly, yet are simple (have small numerators and denominators).

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: x = polygen(ZZ)
sage: rational_root_bounds((x-1)*(x-2)*(x-3))
(0, 7)
sage: rational_root_bounds(x^2)
(-1/2, 1/2)
sage: rational_root_bounds(x*(x+1))
(-3/2, 1/2)
sage: rational_root_bounds((x+2)*(x-3))
(-3, 6)
sage: rational_root_bounds(x^995 * (x^2 - 9999) - 1)
(-100, 1000/7)
sage: rational_root_bounds(x^995 * (x^2 - 9999) + 1)
(-142, 213/2)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> x = polygen(ZZ)
>>> rational_root_bounds((x-Integer(1))*(x-Integer(2))*(x-Integer(3)))
(0, 7)
>>> rational_root_bounds(x**Integer(2))
(-1/2, 1/2)
>>> rational_root_bounds(x*(x+Integer(1)))
(-3/2, 1/2)
>>> rational_root_bounds((x+Integer(2))*(x-Integer(3)))
(-3, 6)
>>> rational_root_bounds(x**Integer(995) * (x**Integer(2) - Integer(9999)) -␣
↪Integer(1))
```

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```
(-100, 1000/7)
>>> rational_root_bounds(x**Integer(995) * (x**Integer(2) - Integer(9999)) +
↪Integer(1))
(-142, 213/2)
```

If we can see that the polynomial has no real roots, return None.
sage: rational_root_bounds(x^2 + 7) is None True

`sage.rings.polynomial.real_roots.`**`real_roots`**(*p*, *bounds=None*, *seed=None*, *skip_squarefree=False*, *do_logging=False*, *wordsize=32*, *retval='rational'*, *strategy=None*, *max_diameter=None*)

Compute the real roots of a given polynomial with exact coefficients (integer, rational, and algebraic real coefficients are supported).

This returns a list of pairs of a root and its multiplicity.

The root itself can be returned in one of three different ways. If retval=='rational', then it is returned as a pair of rationals that define a region that includes exactly one root. If retval=='interval', then it is returned as a RealIntervalFieldElement that includes exactly one root. If retval=='algebraic_real', then it is returned as an AlgebraicReal. In the former two cases, all the intervals are disjoint.

An alternate high-level algorithm can be used by selecting strategy='warp'. This affects the conversion into Bernstein polynomial form, but still uses the same ocean-island algorithm as the default algorithm. The 'warp' algorithm performs the conversion into Bernstein polynomial form much more quickly, but performs the rest of the computation slightly slower in some benchmarks. The 'warp' algorithm is particularly likely to be helpful for low-degree polynomials.

Part of the algorithm is randomized; the seed parameter gives a seed for the random number generator. (By default, the same seed is used for every call, so that results are repeatable.) The random seed may affect the running time, or the exact intervals returned, but the results are correct regardless of the seed used.

The bounds parameter lets you find roots in some proper subinterval of the reals; it takes a pair of a rational lower and upper bound and only roots within this bound will be found. Currently, specifying bounds does not work if you select strategy='warp', or if you use a polynomial with algebraic real coefficients.

By default, the algorithm will do a squarefree decomposition to get squarefree polynomials. The skip_squarefree parameter lets you skip this step. (If this step is skipped, and the polynomial has a repeated real root, then the algorithm will loop forever! However, repeated non-real roots are not a problem.)

For integer and rational coefficients, the squarefree decomposition is very fast, but it may be slow for algebraic reals. (It may trigger exact computation, so it might be arbitrarily slow. The only other way that this algorithm might trigger exact computation on algebraic real coefficients is that it checks the constant term of the input polynomial for equality with zero.)

Part of the algorithm works (approximately) by splitting numbers into word-size pieces (that is, pieces that fit into a machine word). For portability, this defaults to always selecting pieces suitable for a 32-bit machine; the wordsize parameter lets you make choices suitable for a 64-bit machine instead. (This affects the running time, and the exact intervals returned, but the results are correct on both 32- and 64-bit machines even if the wordsize is chosen "wrong".)

The precision of the results can be improved (at the expense of time, of course) by specifying the max_diameter parameter. If specified, this sets the maximum diameter() of the intervals returned. (Sage defines diameter() to be the relative diameter for intervals that do not contain 0, and the absolute diameter for intervals containing 0.) This directly affects the results in rational or interval return mode; in algebraic_real mode, it increases the precision of the intervals passed to the algebraic number package, which may speed up some operations on that algebraic real.

Some logging can be enabled with do_logging=True. If logging is enabled, then the normal values are not returned; instead, a pair of the internal context object and a list of all the roots in their internal form is returned.

ALGORITHM: We convert the polynomial into the Bernstein basis, and then use de Casteljau's algorithm and Descartes' rule of signs (using interval arithmetic) to locate the roots.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: x = polygen(ZZ)
sage: real_roots(x^3 - x^2 - x - 1)
[((7/4, 19/8), 1)]
sage: real_roots((x-1)*(x-2)*(x-3)*(x-5)*(x-8)*(x-13)*(x-21)*(x-34))
[((11/16, 33/32), 1), ((11/8, 33/16), 1), ((11/4, 55/16), 1), ((77/16, 165/32),␣
↪1), ((11/2, 33/4), 1), ((11, 55/4), 1), ((165/8, 341/16), 1), ((22, 44), 1)]
sage: real_roots(x^5 * (x^2 - 9999)^2 - 1)
[((-29274496381311/9007199254740992, 419601125186091/2251799813685248), 1),␣
↪((2126658450145849453951061654415153249597/
↪21267647932558653966460912964485513216,␣
↪4253316902721330018853696359533061621799/
↪42535295865117307932921825928971026432), 1),␣
↪((1063329226287740282451317352558954186101/
↪10633823966279326983230456482242756608, 531664614358685696701445201630854654353/
↪5316911983139663491615228241121378304), 1)]
sage: real_roots(x^5 * (x^2 - 9999)^2 - 1, seed=42)
[((-123196838480289/18014398509481984, 293964743458749/9007199254740992), 1),␣
↪((8307259573979551907841696381986376143/83076749736557242056487941267521536,␣
↪16614519150981033789137940378745325503/166153499473114484112975882535043072),␣
↪1), ((519203723562592617581015249797434335/5192296858534827628530496329220096,␣
↪60443268924081068060312183/6044629098073145873530880), 1)]
sage: real_roots(x^5 * (x^2 - 9999)^2 - 1, wordsize=64)
[((-62866503803202151050003/19342813113834066795298816, 901086554512564177624143/
↪4835703278458516698824704), 1), ((544424563237337315214990987922809050101157/
↪5444517870735015415413993718908291383296,␣
↪1088849127096660194637118845654929064385439/
↪1088903574147003083082798743781658276659 2), 1),␣
↪((272212281929661439711063928866060007142141/
↪2722258935367507707706996859454145691648,␣
↪136106141275823501959100399337685485662633/
↪1361129467683753853853498429727072845824), 1)]
sage: real_roots(x)
[((-47/256, 81/512), 1)]
sage: real_roots(x * (x-1))
[((-47/256, 81/512), 1), ((1/2, 1201/1024), 1)]
sage: real_roots(x-1)
[((209/256, 593/512), 1)]
sage: real_roots(x*(x-1)*(x-2), bounds=(0, 2))
[((0, 0), 1), ((81/128, 337/256), 1), ((2, 2), 1)]
sage: real_roots(x*(x-1)*(x-2), bounds=(0, 2), retval='algebraic_real')
[(0, 1), (1, 1), (2, 1)]
sage: v = 2^40
sage: real_roots((x^2-1)^2 * (x^2 - (v+1)/v))
[(((-12855504354077768210885019021174120740504020581912910106032833/12855504354071922204335696738729300820177623950262342682411008, -642775217703888410544250951058
```

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```
↪7059395588605840418680645585479/64277521770359611021678483693646504100888119751З
↪1171341205504), 1), ((-1125899906842725/1125899906842624, -562949953421275/56294
↪9953421312), 2), ((621654045512233302694227810183526039346434035867603307617722О
↪440998294021880493573365З/621654045512233302694227810183526050125570188496684646
↪80057997111644937126566671941632, 388533778445145814183892381364703787178704153Я
↪340705594199885610069035709862106085785/3885337784451458141838923813647037813284
↪813678104279042503624819477808570410416996352), 2), ((509258994083853105745586ОО
↪183704583974906376779892204678713082380416982642672696544969781Я/50925899408362Я
↪521567111422102344540262867098416484062659035112338595324940834176545849344, 257
↪11008708155536421770038042348240136257704305733983563630791/25711008708143844408
↪671393477458601640355247900524685364822016), 1)]
sage: real_roots(x^2 - 2)
[((-3/2, -1), 1), ((1, 3/2), 1)]
sage: real_roots(x^2 - 2, retval='interval')
[(-2.?, 1), (2.?, 1)]
sage: real_roots(x^2 - 2, max_diameter=1/2^30)
[((-22506280506048041472675379598886543645348790970912519198456805737131269246430Я
↪53365310109/1591434356511317254897223194069826688321459682551512695809484726058Я
↪1039044010680170577Я2, -450125610120960829453507591977730865244489723094211820ЗЯ
↪053065599548946985601579935498343/318286871302263450979444638813965337664291936Я
↪10302539161896945211622078088021360341155Я4), 1), ((450125610120960829453507591Я
↪7773086524448972309421182031053065599548946985601579935498343/318286871302263450
↪979444638813965337664291936510302539161896945211622078088021360341155Я4, 2250628
↪0506048041472675379598886543645348790970912519198456805737131269246430553365310Я
↪09/1591434356511317254897223194069826688321459682551512695809484726058110390440Я
↪068017057792), 1)]
sage: real_roots(x^2 - 2, retval='interval', max_diameter=1/2^500)
[(-1.41421356237309504880168872420969807856967187537694807317667973799073247846210
↪7038850387534327641572735013846230912297024924836055850737212644121497099935831Я
↪1322266592750559275579995050115278206057147010955997160597027453459686201472851Я
↪418640889198609552?, 1), (1.414213562373095048801688724209698078569671875376948О
↪7317667973799073247846210703885038753432764157273501384623091229702492483605585О
↪737212644121497099935831413222665927505592755799950501152782060571470109559971ЯО
↪597027453459686201472851741864088919860955Я?, 1)]
sage: ar_rts = real_roots(x^2 - 2, retval='algebraic_real'); ar_rts
[(-1.414213562373095?, 1), (1.414213562373095?, 1)]
sage: ar_rts[0][0]^2 - 2 == 0
True
sage: v = 2^40
sage: real_roots((x-1) * (x-(v+1)/v), retval='interval')
[(1.000000000000?, 1), (1.000000000001?, 1)]
sage: v = 2^60
sage: real_roots((x-1) * (x-(v+1)/v), retval='interval')
[(1.000000000000000000?, 1), (1.000000000000000001?, 1)]
sage: real_roots((x-1) * (x-2), strategy='warp')
[((499/525, 1173/875), 1), ((337/175, 849/175), 1)]
sage: real_roots((x+3)*(x+1)*x*(x-1)*(x-2), strategy='warp')
[((-1713/335, -689/335), 1), ((-2067/2029, -689/1359), 1), ((0, 0), 1), ((499/525,
↪ 1173/875), 1), ((337/175, 849/175), 1)]
sage: real_roots((x+3)*(x+1)*x*(x-1)*(x-2), strategy='warp', retval='algebraic_
↪real')
[(-3.000000000000000?, 1), (-1.000000000000000?, 1), (0, 1), (1.000000000000000?,␣
```

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```
↪1), (2.000000000000000?, 1)]
sage: ar_rts = real_roots(x-1, retval='algebraic_real')
sage: ar_rts[0][0] == 1
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> x = polygen(ZZ)
>>> real_roots(x**Integer(3) - x**Integer(2) - x - Integer(1))
[((7/4, 19/8), 1)]
>>> real_roots((x-Integer(1))*(x-Integer(2))*(x-Integer(3))*(x-Integer(5))*(x-
↪Integer(8))*(x-Integer(13))*(x-Integer(21))*(x-Integer(34)))
[((11/16, 33/32), 1), ((11/8, 33/16), 1), ((11/4, 55/16), 1), ((77/16, 165/32),␣
↪1), ((11/2, 33/4), 1), ((11, 55/4), 1), ((165/8, 341/16), 1), ((22, 44), 1)]
>>> real_roots(x**Integer(5) * (x**Integer(2) - Integer(9999))**Integer(2) -␣
↪Integer(1))
[((-29274496381311/9007199254740992, 419601125186091/2251799813685248), 1),␣
↪((2126658450145849453951061654415153249597/
↪21267647932558653966460912964485513216,␣
↪4253316902721330018853696359533061621799/
↪42535295865117307932921825928971026432), 1),␣
↪((1063329226287740282451317352558954186101/
↪10633823966279326983230456482242756608, 531664614358685696701445201630854654353/
↪5316911983139663491615228241121378304), 1)]
>>> real_roots(x**Integer(5) * (x**Integer(2) - Integer(9999))**Integer(2) -␣
↪Integer(1), seed=Integer(42))
[((-123196838480289/18014398509481984, 293964743458749/9007199254740992), 1),␣
↪((8307259573979551907841696381986376143/83076749736557242056487941267521536,␣
↪16614519150981033789137940378745325503/166153499473114484112975882535043072),␣
↪1), ((519203723562592617581015249797434335/5192296858534827628530496329220096,␣
↪60443268924081068060312183/6044629098073145873530880), 1)]
>>> real_roots(x**Integer(5) * (x**Integer(2) - Integer(9999))**Integer(2) -␣
↪Integer(1), wordsize=Integer(64))
[((-62866503803202151050003/19342813113834066795298816, 901086554512564177624143/
↪4835703278458516698824704), 1), ((544424563237337315214990987922809050101157/
↪5444517870735015415413993718908291383296,␣
↪1088849127096660194637118845654929064385439/
↪10889035741470030830827987437816582766592), 1),␣
↪((272212281929661439711063928866060007142141/
↪2722258935367507707706996859454145691648,␣
↪136106141275823501959100399337685485662633/
↪1361129467683753853853498429727072845824), 1)]
>>> real_roots(x)
[((-47/256, 81/512), 1)]
>>> real_roots(x * (x-Integer(1)))
[((-47/256, 81/512), 1), ((1/2, 1201/1024), 1)]
>>> real_roots(x-Integer(1))
[((209/256, 593/512), 1)]
>>> real_roots(x*(x-Integer(1))*(x-Integer(2)), bounds=(Integer(0), Integer(2)))
[((0, 0), 1), ((81/128, 337/256), 1), ((2, 2), 1)]
>>> real_roots(x*(x-Integer(1))*(x-Integer(2)), bounds=(Integer(0), Integer(2)),␣
↪retval='algebraic_real')
```

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```
[(0, 1), (1, 1), (2, 1)]
>>> v = Integer(2)**Integer(40)
>>> real_roots((x**Integer(2)-Integer(1))**Integer(2) * (x**Integer(2) -␣
↪(v+Integer(1))/v))
[((-12855504354077768210885019021174120740504020581912910106032833/12855504354071922204335696738729300820177623950262342682411008, -6427752177038884105442509510587059395588605840418680645585479/64277521770359611021678483693646504100888119751311713412055004), 1), ((-1125899906842725/1125899906842624, -562949953421275/562949953421312), 2), ((62165404551223330269422781018352603934643403586760330761772204409982940218804935733653/6216540455122333026942278101835260501255701884966846468005799711164493712656667194163 2, 38853377844514581418389238136470378717870415393407055941998856100690357098621060857 85/38853377844514581418389238136470378132848136781042790425036248194778085704104169963 52), 2), ((509258994083853105745586001837045839749063767798922046787130823804169826426726965449697819/509258994083621521567111422102344540262867098416484062659035112338595324940834176545849344, 257110087081555364217700380423482401362577043057339835636307 91/2571100870814384440867139347745860164035524790052468536482201 6), 1)]
>>> real_roots(x**Integer(2) - Integer(2))
[((-3/2, -1), 1), ((1, 3/2), 1)]
>>> real_roots(x**Integer(2) - Integer(2), retval='interval')
[(-2.?, 1), (2.?, 1)]
>>> real_roots(x**Integer(2) - Integer(2), max_diameter=Integer(1)/
↪Integer(2)**Integer(30))
[((-22506280506048041472675379598886543645348790970912519198456805737131269246430553365310109/1591434356511317254897223194069826688321459682551512695809484726058110390440106801705779 2, -4501256101209608294535075919777308652444897230942118203105306559954894698560157993549834 3/3182868713022634509794446388139653376642919365103025391618969452116220780880213603411558 4), 1), ((45012561012096082945350759197773086524448972309421182031053065599548946985601579935498343/31828687130226345097944463881396533766429193651030253916189694521162207808802136034115584, 225062805060480414726753795988865436453487909709125191984568057371312692464305533653101 09/159143435651131725489722319406982668832145968255151269580948472605811039044010680170577 92), 1)]
>>> real_roots(x**Integer(2) - Integer(2), retval='interval', max_
↪diameter=Integer(1)/Integer(2)**Integer(500))
[(-1.414213562373095048801688724209698078569671875376948073176679737990732478462107038850387534327641572735013846230912297024924836055850737212644121497099935831413222665927505592755799950501152782060571470109559971605970274534596862014728517418640889198609552?, 1), (1.414213562373095048801688724209698078569671875376948073176679737990732478462107038850387534327641572735013846230912297024924836055850737212644121497099935831413222665927505592755799950501152782060571470109559971605970274534596862014728517418640889198609552?, 1)]
>>> ar_rts = real_roots(x**Integer(2) - Integer(2), retval='algebraic_real'); ar_
↪rts
[(-1.414213562373095?, 1), (1.414213562373095?, 1)]
>>> ar_rts[Integer(0)][Integer(0)]**Integer(2) - Integer(2) == Integer(0)
True
>>> v = Integer(2)**Integer(40)
>>> real_roots((x-Integer(1)) * (x-(v+Integer(1))/v), retval='interval')
[(1.000000000000?, 1), (1.000000000001?, 1)]
>>> v = Integer(2)**Integer(60)
>>> real_roots((x-Integer(1)) * (x-(v+Integer(1))/v), retval='interval')
```

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```
[(1.000000000000000000?, 1), (1.000000000000000001?, 1)]
>>> real_roots((x-Integer(1)) * (x-Integer(2)), strategy='warp')
[((499/525, 1173/875), 1), ((337/175, 849/175), 1)]
>>> real_roots((x+Integer(3))*(x+Integer(1))*x*(x-Integer(1))*(x-Integer(2)),␣
→strategy='warp')
[((-1713/335, -689/335), 1), ((-2067/2029, -689/1359), 1), ((0, 0), 1), ((499/525,
→ 1173/875), 1), ((337/175, 849/175), 1)]
>>> real_roots((x+Integer(3))*(x+Integer(1))*x*(x-Integer(1))*(x-Integer(2)),␣
→strategy='warp', retval='algebraic_real')
[(-3.000000000000000?, 1), (-1.000000000000000?, 1), (0, 1), (1.000000000000000?,␣
→1), (2.000000000000000?, 1)]
>>> ar_rts = real_roots(x-Integer(1), retval='algebraic_real')
>>> ar_rts[Integer(0)][Integer(0)] == Integer(1)
True
```

If the polynomial has no real roots, we get an empty list.

```
sage: (x^2 + 1).real_root_intervals()
[]
```

```
>>> from sage.all import *
>>> (x**Integer(2) + Integer(1)).real_root_intervals()
[]
```

We can compute Conway's constant (see http://mathworld.wolfram.com/ConwaysConstant.html) to arbitrary precision.

```
sage: p = x^71 - x^69 - 2*x^68 - x^67 + 2*x^66 + 2*x^65 + x^64 - x^63 - x^62 - x^
→61 - x^60 - x^59 + 2*x^58 + 5*x^57 + 3*x^56 - 2*x^55 - 10*x^54 - 3*x^53 - 2*x^
→52 + 6*x^51 + 6*x^50 + x^49 + 9*x^48 - 3*x^47 - 7*x^46 - 8*x^45 - 8*x^44 + 10*x^
→43 + 6*x^42 + 8*x^41 - 5*x^40 - 12*x^39 + 7*x^38 - 7*x^37 + 7*x^36 + x^35 - 3*x^
→34 + 10*x^33 + x^32 - 6*x^31 - 2*x^30 - 10*x^29 - 3*x^28 + 2*x^27 + 9*x^26 -␣
→3*x^25 + 14*x^24 - 8*x^23 - 7*x^21 + 9*x^20 + 3*x^19 - 4*x^18 - 10*x^17 - 7*x^
→16 + 12*x^15 + 7*x^14 + 2*x^13 - 12*x^12 - 4*x^11 - 2*x^10 + 5*x^9 + x^7 - 7*x^
→6 + 7*x^5 - 4*x^4 + 12*x^3 - 6*x^2 + 3*x - 6
sage: cc = real_roots(p, retval='algebraic_real')[2][0] # long time
sage: RealField(180)(cc)                                # long time
1.3035772690342963912570991121525518907307025046594049
```

```
>>> from sage.all import *
>>> p = x**Integer(71) - x**Integer(69) - Integer(2)*x**Integer(68) -␣
→x**Integer(67) + Integer(2)*x**Integer(66) + Integer(2)*x**Integer(65) +␣
→x**Integer(64) - x**Integer(63) - x**Integer(62) - x**Integer(61) -␣
→x**Integer(60) - x**Integer(59) + Integer(2)*x**Integer(58) +␣
→Integer(5)*x**Integer(57) + Integer(3)*x**Integer(56) -␣
→Integer(2)*x**Integer(55) - Integer(10)*x**Integer(54) -␣
→Integer(3)*x**Integer(53) - Integer(2)*x**Integer(52) +␣
→Integer(6)*x**Integer(51) + Integer(6)*x**Integer(50) + x**Integer(49) +␣
→Integer(9)*x**Integer(48) - Integer(3)*x**Integer(47) -␣
→Integer(7)*x**Integer(46) - Integer(8)*x**Integer(45) -␣
→Integer(8)*x**Integer(44) + Integer(10)*x**Integer(43) +␣
→Integer(6)*x**Integer(42) + Integer(8)*x**Integer(41) -␣
```

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```
↪Integer(5)*x**Integer(40) - Integer(12)*x**Integer(39) +␣
↪Integer(7)*x**Integer(38) - Integer(7)*x**Integer(37) +␣
↪Integer(7)*x**Integer(36) + x**Integer(35) - Integer(3)*x**Integer(34) +␣
↪Integer(10)*x**Integer(33) + x**Integer(32) - Integer(6)*x**Integer(31) -␣
↪Integer(2)*x**Integer(30) - Integer(10)*x**Integer(29) -␣
↪Integer(3)*x**Integer(28) + Integer(2)*x**Integer(27) +␣
↪Integer(9)*x**Integer(26) - Integer(3)*x**Integer(25) +␣
↪Integer(14)*x**Integer(24) - Integer(8)*x**Integer(23) -␣
↪Integer(7)*x**Integer(21) + Integer(9)*x**Integer(20) +␣
↪Integer(3)*x**Integer(19) - Integer(4)*x**Integer(18) -␣
↪Integer(10)*x**Integer(17) - Integer(7)*x**Integer(16) +␣
↪Integer(12)*x**Integer(15) + Integer(7)*x**Integer(14) +␣
↪Integer(2)*x**Integer(13) - Integer(12)*x**Integer(12) -␣
↪Integer(4)*x**Integer(11) - Integer(2)*x**Integer(10) +␣
↪Integer(5)*x**Integer(9) + x**Integer(7) - Integer(7)*x**Integer(6) +␣
↪Integer(7)*x**Integer(5) - Integer(4)*x**Integer(4) + Integer(12)*x**Integer(3)␣
↪- Integer(6)*x**Integer(2) + Integer(3)*x - Integer(6)
>>> cc = real_roots(p, retval='algebraic_real')[Integer(2)][Integer(0)] # long␣
↪time
>>> RealField(Integer(180))(cc)                          # long time
1.3035772690342963912570991121525518907307025046594049
```

Now we play with algebraic real coefficients.

```
sage: x = polygen(AA)
sage: p = (x - 1) * (x - sqrt(AA(2))) * (x - 2)
sage: real_roots(p)
[((499/525, 2171/1925), 1), ((1173/875, 2521/1575), 1), ((337/175, 849/175), 1)]
sage: ar_rts = real_roots(p, retval='algebraic_real'); ar_rts
[(1.000000000000000?, 1), (1.414213562373095?, 1), (2.000000000000000?, 1)]
sage: ar_rts[1][0]^2 == 2
True
sage: ar_rts = real_roots(x*(x-1), retval='algebraic_real')
sage: ar_rts[0][0] == 0
True
sage: p2 = p * (p - 1/100); p2
x^6 - 8.82842712474619?*x^5 + 31.97056274847714?*x^4 - 60.77955262170047?*x^3 +␣
↪63.98526763257801?*x^2 - 35.37613490585595?*x + 8.028284271247462?
sage: real_roots(p2, retval='interval')
[(1.00?, 1), (1.1?, 1), (1.38?, 1), (1.5?, 1), (2.00?, 1), (2.1?, 1)]
sage: p = (x - 1) * (x - sqrt(AA(2)))^2 * (x - 2)^3 * sqrt(AA(3))
sage: real_roots(p, retval='interval')
[(1.000000000000000?, 1), (1.414213562373095?, 2), (2.000000000000000?, 3)]
```

```
>>> from sage.all import *
>>> x = polygen(AA)
>>> p = (x - Integer(1)) * (x - sqrt(AA(Integer(2)))) * (x - Integer(2))
>>> real_roots(p)
[((499/525, 2171/1925), 1), ((1173/875, 2521/1575), 1), ((337/175, 849/175), 1)]
>>> ar_rts = real_roots(p, retval='algebraic_real'); ar_rts
[(1.000000000000000?, 1), (1.414213562373095?, 1), (2.000000000000000?, 1)]
>>> ar_rts[Integer(1)][Integer(0)]**Integer(2) == Integer(2)
```

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```
True
>>> ar_rts = real_roots(x*(x-Integer(1)), retval='algebraic_real')
>>> ar_rts[Integer(0)][Integer(0)] == Integer(0)
True
>>> p2 = p * (p - Integer(1)/Integer(100)); p2
x^6 - 8.82842712474619?*x^5 + 31.97056274847714?*x^4 - 60.77955262170047?*x^3 +␣
↪63.98526763257801?*x^2 - 35.37613490585595?*x + 8.028284271247462?
>>> real_roots(p2, retval='interval')
[(1.00?, 1), (1.1?, 1), (1.38?, 1), (1.5?, 1), (2.00?, 1), (2.1?, 1)]
>>> p = (x - Integer(1)) * (x - sqrt(AA(Integer(2))))**Integer(2) * (x -␣
↪Integer(2))**Integer(3) * sqrt(AA(Integer(3)))
>>> real_roots(p, retval='interval')
[(1.000000000000000?, 1), (1.414213562373095?, 2), (2.000000000000000?, 3)]
```

sage.rings.polynomial.real_roots.**relative_bounds**(*a*, *b*)

INPUT:

- `(al, ah)` – pair of rationals
- `(bl, bh)` – pair of rationals

OUTPUT:

- `(cl, ch)` – pair of rationals

Computes the linear transformation that maps (al, ah) to (0, 1); then applies this transformation to (bl, bh) and returns the result.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: relative_bounds((1/7, 1/4), (1/6, 1/5))
(2/9, 8/15)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> relative_bounds((Integer(1)/Integer(7), Integer(1)/Integer(4)), (Integer(1)/
↪Integer(6), Integer(1)/Integer(5)))
(2/9, 8/15)
```

sage.rings.polynomial.real_roots.**reverse_intvec**(*c*)

Given a vector of integers, reverse the vector (like the reverse() method on lists).

Modifies the input vector; has no return value.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: v = vector(ZZ, [1, 2, 3, 4]); v
(1, 2, 3, 4)
sage: reverse_intvec(v)
sage: v
(4, 3, 2, 1)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> v = vector(ZZ, [Integer(1), Integer(2), Integer(3), Integer(4)]); v
(1, 2, 3, 4)
>>> reverse_intvec(v)
>>> v
(4, 3, 2, 1)
```

sage.rings.polynomial.real_roots.**root_bounds**(*p*)

Given a polynomial with real coefficients, computes a lower and upper bound on its real roots. Uses algorithms of Akritas, Strzebo'nski, and Vigklas.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: x = polygen(ZZ)
sage: root_bounds((x-1)*(x-2)*(x-3))
(0.545454545454545, 6.00000000000001)
sage: root_bounds(x^2)
(0.000000000000000, 0.000000000000000)
sage: root_bounds(x*(x+1))
(-1.00000000000000, 0.000000000000000)
sage: root_bounds((x+2)*(x-3))
(-2.44948974278317, 3.46410161513776)
sage: root_bounds(x^995 * (x^2 - 9999) - 1)
(-99.9949998749937, 141.414284992713)
sage: root_bounds(x^995 * (x^2 - 9999) + 1)
(-141.414284992712, 99.9949998749938)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> x = polygen(ZZ)
>>> root_bounds((x-Integer(1))*(x-Integer(2))*(x-Integer(3)))
(0.545454545454545, 6.00000000000001)
>>> root_bounds(x**Integer(2))
(0.000000000000000, 0.000000000000000)
>>> root_bounds(x*(x+Integer(1)))
(-1.00000000000000, 0.000000000000000)
>>> root_bounds((x+Integer(2))*(x-Integer(3)))
(-2.44948974278317, 3.46410161513776)
>>> root_bounds(x**Integer(995) * (x**Integer(2) - Integer(9999)) - Integer(1))
(-99.9949998749937, 141.414284992713)
>>> root_bounds(x**Integer(995) * (x**Integer(2) - Integer(9999)) + Integer(1))
(-141.414284992712, 99.9949998749938)
```

If we can see that the polynomial has no real roots, return None.

```
sage: root_bounds(x^2 + 1) is None
True
```

```
>>> from sage.all import *
>>> root_bounds(x**Integer(2) + Integer(1)) is None
True
```

class sage.rings.polynomial.real_roots.**rr_gap**

Bases: object

A simple class representing the gaps between islands, in my ocean-island root isolation algorithm. Named "rr_gap" for "real roots gap", because "gap" seemed too short and generic.

region()

sage.rings.polynomial.real_roots.**scale_intvec_var**(*c*, *k*)

Given a vector of integers c of length n+1, and a rational k == kn / kd, multiplies each element c[i] by (kd^i)*(kn^(n-i)).

Modifies the input vector; has no return value.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: v = vector(ZZ, [1, 1, 1, 1])
sage: scale_intvec_var(v, 3/4)
sage: v
(64, 48, 36, 27)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> v = vector(ZZ, [Integer(1), Integer(1), Integer(1), Integer(1)])
>>> scale_intvec_var(v, Integer(3)/Integer(4))
>>> v
(64, 48, 36, 27)
```

sage.rings.polynomial.real_roots.**split_for_targets**(*ctx*, *bp*, *target_list*, *precise=False*)

Given an interval Bernstein polynomial over a particular region (assumed to be a (not necessarily proper) subregion of [0 .. 1]), and a list of targets, uses de Casteljau's method to compute representations of the Bernstein polynomial over each target. Uses degree reduction as often as possible while maintaining the requested precision.

Each target is of the form (lgap, ugap, b). Suppose lgap.region() is (l1, l2), and ugap.region() is (u1, u2). Then we will compute an interval Bernstein polynomial over a region [l .. u], where l1 <= l <= l2 and u1 <= u <= u2. (split_for_targets() is free to select arbitrary region endpoints within these bounds; it picks endpoints which make the computation easier.) The third component of the target, b, is the maximum allowed scale_log2 of the result; this is used to decide when degree reduction is allowed.

The pair (l1, l2) can be replaced by None, meaning [-infinity .. 0]; or, (u1, u2) can be replaced by None, meaning [1 .. infinity].

There is another constraint on the region endpoints selected by split_for_targets() for a target ((l1, l2), (u1, u2), b). We set a size goal g, such that (u - l) <= g * (u1 - l2). Normally g is 256/255, but if precise is True, then g is 65536/65535.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: bp = mk_ibpi([1000000, -2000000, 3000000, -4000000, -5000000, -6000000])
sage: ctx = mk_context()
sage: bps = split_for_targets(ctx, bp, [(rr_gap(1/1234567893, 1/1234567892, 1),␣
↪rr_gap(1/1234567891, 1/1234567890, 1), 12), (rr_gap(1/3, 1/2, -1), rr_gap(2/3,␣
↪3/4, -1), 6)])
sage: bps[0]
<IBP: (999992, 999992, 999992) + [0 .. 15) over [8613397477114467984778830327/
```

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```
↪10633823966279326983230456482242756608 ..␣
↪5919081680259343948138365274959382947 87/
↪7307508186654514591018424163581415098279662714 88]; level 2; slope_err 0.?e12>
sage: bps[1]
<IBP: (-1562500, -1875001, -2222223, -2592593, -2969137, -3337450) + [0 .. 4)␣
↪over [1/2 .. 2863311531/4294967296]>
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> bp = mk_ibpi([Integer(1000000), -Integer(2000000), Integer(3000000), -
↪Integer(4000000), -Integer(5000000), -Integer(6000000)])
>>> ctx = mk_context()
>>> bps = split_for_targets(ctx, bp, [(rr_gap(Integer(1)/Integer(1234567893),␣
↪Integer(1)/Integer(1234567892), Integer(1)), rr_gap(Integer(1)/
↪Integer(1234567891), Integer(1)/Integer(1234567890), Integer(1)), Integer(12)),␣
↪(rr_gap(Integer(1)/Integer(3), Integer(1)/Integer(2), -Integer(1)), rr_
↪gap(Integer(2)/Integer(3), Integer(3)/Integer(4), -Integer(1)), Integer(6))])
>>> bps[Integer(0)]
<IBP: (999992, 999992, 999992) + [0 .. 15) over [8613397477114467984778830327/
↪10633823966279326983230456482242756608 ..␣
↪591908168025934394813836527495938294787/
↪730750818665451459101842416358141509827966271488]; level 2; slope_err 0.?e12>
>>> bps[Integer(1)]
<IBP: (-1562500, -1875001, -2222223, -2592593, -2969137, -3337450) + [0 .. 4)␣
↪over [1/2 .. 2863311531/4294967296]>
```

sage.rings.polynomial.real_roots.**subsample_vec_doctest**(*a*, *slen*, *llen*)

sage.rings.polynomial.real_roots.**taylor_shift1_intvec**(*c*)

Given a vector of integers c of length d+1, representing the coefficients of a degree-d polynomial p, modify the vector to perform a Taylor shift by 1 (that is, p becomes p(x+1)).

This is the straightforward algorithm, which is not asymptotically optimal.

Modifies the input vector; has no return value.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: x = polygen(ZZ)
sage: p = (x-1)*(x-2)*(x-3)
sage: v = vector(ZZ, p.list())
sage: p, v
(x^3 - 6*x^2 + 11*x - 6, (-6, 11, -6, 1))
sage: taylor_shift1_intvec(v)
sage: p(x+1), v
(x^3 - 3*x^2 + 2*x, (0, 2, -3, 1))
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> x = polygen(ZZ)
>>> p = (x-Integer(1))*(x-Integer(2))*(x-Integer(3))
>>> v = vector(ZZ, p.list())
```

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```
>>> p, v
(x^3 - 6*x^2 + 11*x - 6, (-6, 11, -6, 1))
>>> taylor_shift1_intvec(v)
>>> p(x+Integer(1)), v
(x^3 - 3*x^2 + 2*x, (0, 2, -3, 1))
```

sage.rings.polynomial.real_roots.**to_bernstein**(*p*, *low=0*, *high=1*, *degree=None*)

Given a polynomial p with integer coefficients, and rational bounds low and high, compute the exact rational Bernstein coefficients of p over the region [low .. high]. The optional parameter degree can be used to give a formal degree higher than the actual degree.

The return value is a pair (c, scale); c represents the same polynomial as p*scale. (If you only care about the roots of the polynomial, then of course scale can be ignored.)

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: x = polygen(ZZ)
sage: to_bernstein(x)
([0, 1], 1)
sage: to_bernstein(x, degree=5)
([0, 1/5, 2/5, 3/5, 4/5, 1], 1)
sage: to_bernstein(x^3 + x^2 - x - 1, low=-3, high=3)
([-16, 24, -32, 32], 1)
sage: to_bernstein(x^3 + x^2 - x - 1, low=3, high=22/7)
([296352, 310464, 325206, 340605], 9261)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> x = polygen(ZZ)
>>> to_bernstein(x)
([0, 1], 1)
>>> to_bernstein(x, degree=Integer(5))
([0, 1/5, 2/5, 3/5, 4/5, 1], 1)
>>> to_bernstein(x**Integer(3) + x**Integer(2) - x - Integer(1), low=-Integer(3),
↪high=Integer(3))
([-16, 24, -32, 32], 1)
>>> to_bernstein(x**Integer(3) + x**Integer(2) - x - Integer(1), low=Integer(3),
↪high=Integer(22)/Integer(7))
([296352, 310464, 325206, 340605], 9261)
```

sage.rings.polynomial.real_roots.**to_bernstein_warp**(*p*)

Given a polynomial p with rational coefficients, compute the exact rational Bernstein coefficients of p(x/(x+1)).

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: x = polygen(ZZ)
sage: to_bernstein_warp(1 + x + x^2 + x^3 + x^4 + x^5)
[1, 1/5, 1/10, 1/10, 1/5, 1]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
```

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```
>>> x = polygen(ZZ)
>>> to_bernstein_warp(Integer(1) + x + x**Integer(2) + x**Integer(3) +
↪x**Integer(4) + x**Integer(5))
[1, 1/5, 1/10, 1/10, 1/5, 1]
```

class sage.rings.polynomial.real_roots.**warp_map**(*neg*)

Bases: object

A class to map between original coordinates and ocean coordinates. If neg is False, then the original->ocean transform is x -> x/(x+1), and the ocean->original transform is x/(1-x); this maps between [0 .. infinity] and [0 .. 1]. If neg is True, then the original->ocean transform is x -> -x/(1-x), and the ocean->original transform is the same thing: -x/(1-x). This maps between [0 .. -infinity] and [0 .. 1].

from_ocean(*region*)

to_ocean(*region*)

sage.rings.polynomial.real_roots.**wordsize_rational**(*a*, *b*, *wordsize*)

Given rationals a and b, select a de Casteljau split point r between a and b.

An attempt is made to select an efficient split point (according to the criteria mentioned in the documentation for de_casteljau_intvec), with a bias towards split points near a.

In full detail:

This takes as input two rationals, a and b, such that 0<=a<=1, 0<=b<=1, and a!=b. This returns rational r, such that a<=r<=b or b<=r<=a. The denominator of r is a power of 2. Let m be min(r, 1-r), nm be numerator(m), and dml be log2(denominator(m)). The return value r is taken from the first of the following classes to have any members between a and b (except that if a <= 1/8, or 7/8 <= a, then class 2 is preferred to class 1).

1. dml < wordsize
2. bitsize(nm) <= wordsize
3. bitsize(nm) <= 2*wordsize
4. bitsize(nm) <= 3*wordsize

…

k. bitsize(nm) <= (k-1)*wordsize

From the first class to have members between a and b, r is chosen as the element of the class which is closest to a.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: wordsize_rational(1/5, 1/7, 32)
429496729/2147483648
sage: wordsize_rational(1/7, 1/5, 32)
306783379/2147483648
sage: wordsize_rational(1/5, 1/7, 64)
1844674407370955161/9223372036854775808
sage: wordsize_rational(1/7, 1/5, 64)
658812288346769701/4611686018427387904
sage: wordsize_rational(1/17, 1/19, 32)
252645135/4294967296
sage: wordsize_rational(1/17, 1/19, 64)
1085102592571150095/18446744073709551616
```

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```
sage: wordsize_rational(1/1234567890, 1/1234567891, 32)
933866427/1152921504606846976
sage: wordsize_rational(1/1234567890, 1/1234567891, 64)
4010925763784056541/4951760157141521099596496896
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.real_roots import *
>>> wordsize_rational(Integer(1)/Integer(5), Integer(1)/Integer(7), Integer(32))
429496729/2147483648
>>> wordsize_rational(Integer(1)/Integer(7), Integer(1)/Integer(5), Integer(32))
306783379/2147483648
>>> wordsize_rational(Integer(1)/Integer(5), Integer(1)/Integer(7), Integer(64))
1844674407370955161/9223372036854775808
>>> wordsize_rational(Integer(1)/Integer(7), Integer(1)/Integer(5), Integer(64))
658812288346769701/4611686018427387904
>>> wordsize_rational(Integer(1)/Integer(17), Integer(1)/Integer(19), Integer(32))
252645135/4294967296
>>> wordsize_rational(Integer(1)/Integer(17), Integer(1)/Integer(19), Integer(64))
1085102592571150095/18446744073709551616
>>> wordsize_rational(Integer(1)/Integer(1234567890), Integer(1)/
→Integer(1234567891), Integer(32))
933866427/1152921504606846976
>>> wordsize_rational(Integer(1)/Integer(1234567890), Integer(1)/
→Integer(1234567891), Integer(64))
4010925763784056541/4951760157141521099596496896
```

2.1.19 Isolate Complex Roots of Polynomials

AUTHOR:

- Carl Witty (2007-11-18): initial version

This is an implementation of complex root isolation. That is, given a polynomial with exact complex coefficients, we compute isolating intervals for the complex roots of the polynomial. (Polynomials with integer, rational, Gaussian rational, or algebraic coefficients are supported.)

We use a simple algorithm. First, we compute a squarefree decomposition of the input polynomial; the resulting polynomials have no multiple roots. Then, we find the roots numerically, using NumPy (at low precision) or Pari (at high precision). Then, we verify the roots using interval arithmetic.

EXAMPLES:

```
sage: x = polygen(ZZ)
sage: (x^5 - x - 1).roots(ring=CIF)
[(1.167303978261419?, 1),
 (-0.764884433600585? - 0.352471546031727?*I, 1),
 (-0.764884433600585? + 0.352471546031727?*I, 1),
 (0.181232444469876? - 1.083954101317711?*I, 1),
 (0.181232444469876? + 1.083954101317711?*I, 1)]
```

```
>>> from sage.all import *
>>> x = polygen(ZZ)
>>> (x**Integer(5) - x - Integer(1)).roots(ring=CIF)
```

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```
[(1.167303978261419?, 1),
 (-0.764884433600585? - 0.352471546031727?*I, 1),
 (-0.764884433600585? + 0.352471546031727?*I, 1),
 (0.181232444469876? - 1.083954101317711?*I, 1),
 (0.181232444469876? + 1.083954101317711?*I, 1)]
```

sage.rings.polynomial.complex_roots.**complex_roots**(*p*, *skip_squarefree=False*, *retval='interval'*, *min_prec=0*)

Compute the complex roots of a given polynomial with exact coefficients (integer, rational, Gaussian rational, and algebraic coefficients are supported). Returns a list of pairs of a root and its multiplicity.

Roots are returned as a ComplexIntervalFieldElement; each interval includes exactly one root, and the intervals are disjoint.

By default, the algorithm will do a squarefree decomposition to get squarefree polynomials. The skip_squarefree parameter lets you skip this step. (If this step is skipped, and the polynomial has a repeated root, then the algorithm will loop forever!)

You can specify retval='interval' (the default) to get roots as complex intervals. The other options are retval='algebraic' to get elements of QQbar, or retval='algebraic_real' to get only the real roots, and to get them as elements of AA.

EXAMPLES:

```
sage: from sage.rings.polynomial.complex_roots import complex_roots
sage: x = polygen(ZZ)
sage: complex_roots(x^5 - x - 1)
[(1.167303978261419?, 1),
 (-0.764884433600585? - 0.352471546031727?*I, 1),
 (-0.764884433600585? + 0.352471546031727?*I, 1),
 (0.181232444469876? - 1.083954101317711?*I, 1),
 (0.181232444469876? + 1.083954101317711?*I, 1)]
sage: v = complex_roots(x^2 + 27*x + 181)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.complex_roots import complex_roots
>>> x = polygen(ZZ)
>>> complex_roots(x**Integer(5) - x - Integer(1))
[(1.167303978261419?, 1),
 (-0.764884433600585? - 0.352471546031727?*I, 1),
 (-0.764884433600585? + 0.352471546031727?*I, 1),
 (0.181232444469876? - 1.083954101317711?*I, 1),
 (0.181232444469876? + 1.083954101317711?*I, 1)]
>>> v = complex_roots(x**Integer(2) + Integer(27)*x + Integer(181))
```

Unfortunately due to numerical noise there can be a small imaginary part to each root depending on CPU, compiler, etc, and that affects the printing order. So we verify the real part of each root and check that the imaginary part is small in both cases:

```
sage: v  # random
[(-14.61803398874990?..., 1), (-12.3819660112501...? + 0.?e-27*I, 1)]
sage: sorted((v[0][0].real(),v[1][0].real()))
[-14.61803398874989?, -12.3819660112501...?]
sage: v[0][0].imag().upper() < 1e25
```

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```
True
sage: v[1][0].imag().upper() < 1e25
True

sage: K.<im> = QuadraticField(-1)
sage: eps = 1/2^100
sage: x = polygen(K)
sage: p = (x-1)*(x-1-eps)*(x-1+eps)*(x-1-eps*im)*(x-1+eps*im)
```

```
>>> from sage.all import *
>>> v  # random
[(-14.61803398874990?..., 1), (-12.3819660112501...? + 0.?e-27*I, 1)]
>>> sorted((v[Integer(0)][Integer(0)].real(),v[Integer(1)][Integer(0)].real()))
[-14.61803398874989?, -12.3819660112501...?]
>>> v[Integer(0)][Integer(0)].imag().upper() < RealNumber('1e25')
True
>>> v[Integer(1)][Integer(0)].imag().upper() < RealNumber('1e25')
True

>>> K = QuadraticField(-Integer(1), names=('im',)); (im,) = K._first_ngens(1)
>>> eps = Integer(1)/Integer(2)**Integer(100)
>>> x = polygen(K)
>>> p = (x-Integer(1))*(x-Integer(1)-eps)*(x-Integer(1)+eps)*(x-Integer(1)-
↪eps*im)*(x-Integer(1)+eps*im)
```

This polynomial actually has all-real coefficients, and is very, very close to (x-1)^5:

```
sage: [RR(QQ(a)) for a in list(p - (x-1)^5)]
[3.87259191484932e-121, -3.87259191484932e-121]
sage: rts = complex_roots(p)
sage: [ComplexIntervalField(10)(rt[0] - 1) for rt in rts]
[-7.8887?e-31, 0, 7.8887?e-31, -7.8887?e-31*I, 7.8887?e-31*I]
```

```
>>> from sage.all import *
>>> [RR(QQ(a)) for a in list(p - (x-Integer(1))**Integer(5))]
[3.87259191484932e-121, -3.87259191484932e-121]
>>> rts = complex_roots(p)
>>> [ComplexIntervalField(Integer(10))(rt[Integer(0)] - Integer(1)) for rt in rts]
[-7.8887?e-31, 0, 7.8887?e-31, -7.8887?e-31*I, 7.8887?e-31*I]
```

We can get roots either as intervals, or as elements of QQbar or AA.

```
sage: p = (x^2 + x - 1)
sage: p = p * p(x*im)
sage: p
-x^4 + (im - 1)*x^3 + im*x^2 + (-im - 1)*x + 1
```

```
>>> from sage.all import *
>>> p = (x**Integer(2) + x - Integer(1))
>>> p = p * p(x*im)
>>> p
-x^4 + (im - 1)*x^3 + im*x^2 + (-im - 1)*x + 1
```

Two of the roots have a zero real component; two have a zero imaginary component. These zero components will be found slightly inaccurately, and the exact values returned are very sensitive to the (non-portable) results of NumPy. So we post-process the roots for printing, to get predictable doctest results.

```
sage: def tiny(x):
....:     return x.contains_zero() and x.absolute_diameter() <  1e-14
sage: def smash(x):
....:     x = CIF(x[0]) # discard multiplicity
....:     if tiny(x.imag()): return x.real()
....:     if tiny(x.real()): return CIF(0, x.imag())
sage: rts = complex_roots(p); type(rts[0][0]), sorted(map(smash, rts))
(<class 'sage.rings.complex_interval.ComplexIntervalFieldElement'>,
 [-1.618033988749895?, -0.618033988749895?*I,
  1.618033988749895?*I, 0.618033988749895?])
sage: rts = complex_roots(p, retval='algebraic'); type(rts[0][0]),␣
↪sorted(map(smash, rts))
(<class 'sage.rings.qqbar.AlgebraicNumber'>,
 [-1.618033988749895?, -0.618033988749895?*I,
  1.618033988749895?*I, 0.618033988749895?])
sage: rts = complex_roots(p, retval='algebraic_real'); type(rts[0][0]), rts
(<class 'sage.rings.qqbar.AlgebraicReal'>,
 [(-1.618033988749895?, 1), (0.618033988749895?, 1)])
```

```
>>> from sage.all import *
>>> def tiny(x):
...     return x.contains_zero() and x.absolute_diameter() <  RealNumber('1e-14')
>>> def smash(x):
...     x = CIF(x[Integer(0)]) # discard multiplicity
...     if tiny(x.imag()): return x.real()
...     if tiny(x.real()): return CIF(Integer(0), x.imag())
>>> rts = complex_roots(p); type(rts[Integer(0)][Integer(0)]), sorted(map(smash,␣
↪rts))
(<class 'sage.rings.complex_interval.ComplexIntervalFieldElement'>,
 [-1.618033988749895?, -0.618033988749895?*I,
  1.618033988749895?*I, 0.618033988749895?])
>>> rts = complex_roots(p, retval='algebraic'); type(rts[Integer(0)][Integer(0)]),
↪ sorted(map(smash, rts))
(<class 'sage.rings.qqbar.AlgebraicNumber'>,
 [-1.618033988749895?, -0.618033988749895?*I,
  1.618033988749895?*I, 0.618033988749895?])
>>> rts = complex_roots(p, retval='algebraic_real');␣
↪type(rts[Integer(0)][Integer(0)]), rts
(<class 'sage.rings.qqbar.AlgebraicReal'>,
 [(-1.618033988749895?, 1), (0.618033988749895?, 1)])
```

`sage.rings.polynomial.complex_roots.`**`interval_roots`**`(`*p*, *rts*, *prec*`)`

We are given a squarefree polynomial p, a list of estimated roots, and a precision.

We attempt to verify that the estimated roots are in fact distinct roots of the polynomial, using interval arithmetic of precision `prec`. If we succeed, we return a list of intervals bounding the roots; if we fail, we return `None`.

EXAMPLES:

```
sage: x = polygen(ZZ)
```

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```
sage: p = x^3 - 1
sage: rts = [CC.zeta(3)^i for i in range(0, 3)]
sage: from sage.rings.polynomial.complex_roots import interval_roots
sage: interval_roots(p, rts, 53)
[1, -0.500000000000000? + 0.866025403784439?*I,
    -0.500000000000000? - 0.866025403784439?*I]
sage: interval_roots(p, rts, 200)
[1, -0.500000000000000000000000000000000000000000000000000000000000?
      + 0.866025403784438646763723170752936183471402626905190314027904?*I,
    -0.500000000000000000000000000000000000000000000000000000000000?
      - 0.866025403784438646763723170752936183471402626905190314027904?*I]
```

```
>>> from sage.all import *
>>> x = polygen(ZZ)
>>> p = x**Integer(3) - Integer(1)
>>> rts = [CC.zeta(Integer(3))**i for i in range(Integer(0), Integer(3))]
>>> from sage.rings.polynomial.complex_roots import interval_roots
>>> interval_roots(p, rts, Integer(53))
[1, -0.500000000000000? + 0.866025403784439?*I,
    -0.500000000000000? - 0.866025403784439?*I]
>>> interval_roots(p, rts, Integer(200))
[1, -0.500000000000000000000000000000000000000000000000000000000000?
      + 0.866025403784438646763723170752936183471402626905190314027904?*I,
    -0.500000000000000000000000000000000000000000000000000000000000?
      - 0.866025403784438646763723170752936183471402626905190314027904?*I]
```

sage.rings.polynomial.complex_roots.**intervals_disjoint**(*intvs*)

Given a list of complex intervals, check whether they are pairwise disjoint.

EXAMPLES:

```
sage: from sage.rings.polynomial.complex_roots import intervals_disjoint
sage: a = CIF(RIF(0, 3), 0)
sage: b = CIF(0, RIF(1, 3))
sage: c = CIF(RIF(1, 2), RIF(1, 2))
sage: d = CIF(RIF(2, 3), RIF(2, 3))
sage: intervals_disjoint([a,b,c,d])
False
sage: d2 = CIF(RIF(2, 3), RIF(2.001, 3))
sage: intervals_disjoint([a,b,c,d2])
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.complex_roots import intervals_disjoint
>>> a = CIF(RIF(Integer(0), Integer(3)), Integer(0))
>>> b = CIF(Integer(0), RIF(Integer(1), Integer(3)))
>>> c = CIF(RIF(Integer(1), Integer(2)), RIF(Integer(1), Integer(2)))
>>> d = CIF(RIF(Integer(2), Integer(3)), RIF(Integer(2), Integer(3)))
>>> intervals_disjoint([a,b,c,d])
False
>>> d2 = CIF(RIF(Integer(2), Integer(3)), RIF(RealNumber('2.001'), Integer(3)))
>>> intervals_disjoint([a,b,c,d2])
```

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```
True
```

2.1.20 Refine polynomial roots using Newton–Raphson

This is an implementation of the Newton–Raphson algorithm to approximate roots of complex polynomials. The implementation is based on interval arithmetic

AUTHORS:

- Carl Witty (2007-11-18): initial version

`sage.rings.polynomial.refine_root.`**`refine_root`**(*ip*, *ipd*, *irt*, *fld*)

We are given a polynomial and its derivative (with complex interval coefficients), an estimated root, and a complex interval field to use in computations. We use interval arithmetic to refine the root and prove that we have in fact isolated a unique root.

If we succeed, we return the isolated root; if we fail, we return None.

EXAMPLES:

```
sage: # needs sage.symbolic
sage: from sage.rings.polynomial.refine_root import refine_root
sage: x = polygen(ZZ)
sage: p = x^9 - 1
sage: ip = CIF['x'](p); ip
x^9 - 1
sage: ipd = CIF['x'](p.derivative()); ipd
9*x^8
sage: irt = CIF(CC(cos(2*pi/9), sin(2*pi/9))); irt
0.76604444311897802? + 0.64278760968653926?*I
sage: ip(irt)
0.?e-14 + 0.?e-14*I
sage: ipd(irt)
6.89439998807080? - 5.78508848717885?*I
sage: refine_root(ip, ipd, irt, CIF)
0.766044443118978? + 0.642787609686540?*I
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> from sage.rings.polynomial.refine_root import refine_root
>>> x = polygen(ZZ)
>>> p = x**Integer(9) - Integer(1)
>>> ip = CIF['x'](p); ip
x^9 - 1
>>> ipd = CIF['x'](p.derivative()); ipd
9*x^8
>>> irt = CIF(CC(cos(Integer(2)*pi/Integer(9)), sin(Integer(2)*pi/Integer(9))));
↪irt
0.76604444311897802? + 0.64278760968653926?*I
>>> ip(irt)
0.?e-14 + 0.?e-14*I
>>> ipd(irt)
6.89439998807080? - 5.78508848717885?*I
```

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```
>>> refine_root(ip, ipd, irt, CIF)
0.766044443118978? + 0.642787609686540?*I
```

2.1.21 Ideals in Univariate Polynomial Rings

AUTHORS:

- David Roe (2009-12-14) – initial version.

class sage.rings.polynomial.ideal.**Ideal_1poly_field**(*ring*, *gens*, *coerce=True*, ***kwds*)

Bases: Ideal_pid

An ideal in a univariate polynomial ring over a field.

change_ring(*R*)

Coerce an ideal into a new ring.

EXAMPLES:

```
sage: R.<q> = QQ[]
sage: I = R.ideal([q^2 + q - 1])
sage: I.change_ring(RR['q'])                                          #␣
↪needs sage.rings.real_mpfr
Principal ideal (q^2 + q - 1.00000000000000) of
 Univariate Polynomial Ring in q over Real Field with 53 bits of precision
```

```
>>> from sage.all import *
>>> R = QQ['q']; (q,) = R._first_ngens(1)
>>> I = R.ideal([q**Integer(2) + q - Integer(1)])
>>> I.change_ring(RR['q'])                                          #␣
↪needs sage.rings.real_mpfr
Principal ideal (q^2 + q - 1.00000000000000) of
 Univariate Polynomial Ring in q over Real Field with 53 bits of precision
```

groebner_basis(*algorithm=None*)

Return a Gröbner basis for this ideal.

The Gröbner basis has 1 element, namely the generator of the ideal. This trivial method exists for compatibility with multi-variate polynomial rings.

INPUT:

- algorithm – ignored

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: I = R.ideal([x^2 - 1, x^3 - 1])
sage: G = I.groebner_basis(); G
[x - 1]
sage: type(G)
<class 'sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_
↪generic'>
sage: list(G)
[x - 1]
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> I = R.ideal([x**Integer(2) - Integer(1), x**Integer(3) - Integer(1)])
>>> G = I.groebner_basis(); G
[x - 1]
>>> type(G)
<class 'sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_
↪generic'>
>>> list(G)
[x - 1]
```

residue_class_degree()

Return the degree of the generator of this ideal.

This function is included for compatibility with ideals in rings of integers of number fields.

EXAMPLES:

```
sage: R.<t> = GF(5)[]
sage: P = R.ideal(t^4 + t + 1)
sage: P.residue_class_degree()
4
```

```
>>> from sage.all import *
>>> R = GF(Integer(5))['t']; (t,) = R._first_ngens(1)
>>> P = R.ideal(t**Integer(4) + t + Integer(1))
>>> P.residue_class_degree()
4
```

residue_field(*names=None*, *check=True*)

If this ideal is $P \subset F_p[t]$, return the quotient $F_p[t]/P$.

EXAMPLES:

```
sage: R.<t> = GF(17)[]; P = R.ideal(t^3 + 2*t + 9)
sage: k.<a> = P.residue_field(); k                                          #␣
↪needs sage.rings.finite_rings
Residue field in a of Principal ideal (t^3 + 2*t + 9) of
 Univariate Polynomial Ring in t over Finite Field of size 17
```

```
>>> from sage.all import *
>>> R = GF(Integer(17))['t']; (t,) = R._first_ngens(1); P = R.
↪ideal(t**Integer(3) + Integer(2)*t + Integer(9))
>>> k = P.residue_field(names=('a',)); (a,) = k._first_ngens(1); k          ␣
↪                               # needs sage.rings.finite_rings
Residue field in a of Principal ideal (t^3 + 2*t + 9) of
 Univariate Polynomial Ring in t over Finite Field of size 17
```

2.1.22 Quotients of Univariate Polynomial Rings

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: S = R.quotient(x**3 - 3*x + 1, 'alpha')
sage: S.gen()**2 in S
True
sage: x in S
True
sage: S.gen() in R
False
sage: 1 in S
True
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(3) - Integer(3)*x + Integer(1), 'alpha')
>>> S.gen()**Integer(2) in S
True
>>> x in S
True
>>> S.gen() in R
False
>>> Integer(1) in S
True
```

class sage.rings.polynomial.polynomial_quotient_ring.**PolynomialQuotientRingFactory**

Bases: UniqueFactory

Create a quotient of a polynomial ring.

INPUT:

- ring – a univariate polynomial ring
- polynomial – an element of ring with a unit leading coefficient
- names – (optional) name for the variable

OUTPUT: creates the quotient ring R/I, where R is the ring and I is the principal ideal generated by polynomial.

EXAMPLES:

We create the quotient ring $\mathbf{Z}[x]/(x^3 + 7)$, and demonstrate many basic functions with it:

```
sage: Z = IntegerRing()
sage: R = PolynomialRing(Z, 'x'); x = R.gen()
sage: S = R.quotient(x^3 + 7, 'a'); a = S.gen()
sage: S
Univariate Quotient Polynomial Ring in a
 over Integer Ring with modulus x^3 + 7
sage: a^3
-7
sage: S.is_field()
False
sage: a in S
True
sage: x in S
True
```

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```
sage: a in R
False
sage: S.polynomial_ring()
Univariate Polynomial Ring in x over Integer Ring
sage: S.modulus()
x^3 + 7
sage: S.degree()
3
```

```
>>> from sage.all import *
>>> Z = IntegerRing()
>>> R = PolynomialRing(Z, 'x'); x = R.gen()
>>> S = R.quotient(x**Integer(3) + Integer(7), 'a'); a = S.gen()
>>> S
Univariate Quotient Polynomial Ring in a
 over Integer Ring with modulus x^3 + 7
>>> a**Integer(3)
-7
>>> S.is_field()
False
>>> a in S
True
>>> x in S
True
>>> a in R
False
>>> S.polynomial_ring()
Univariate Polynomial Ring in x over Integer Ring
>>> S.modulus()
x^3 + 7
>>> S.degree()
3
```

We create the "iterated" polynomial ring quotient

$$R = (\mathbf{F}_2[y]/(y^2 + y + 1))[x]/(x^3 - 5).$$

```
sage: # needs sage.libs.ntl
sage: A.<y> = PolynomialRing(GF(2)); A
Univariate Polynomial Ring in y over Finite Field of size 2 (using GF2X)
sage: B = A.quotient(y^2 + y + 1, 'y2'); B
Univariate Quotient Polynomial Ring in y2 over Finite Field of size 2
 with modulus y^2 + y + 1
sage: C = PolynomialRing(B, 'x'); x = C.gen(); C
Univariate Polynomial Ring in x
 over Univariate Quotient Polynomial Ring in y2
  over Finite Field of size 2 with modulus y^2 + y + 1
sage: R = C.quotient(x^3 - 5); R
Univariate Quotient Polynomial Ring in xbar
 over Univariate Quotient Polynomial Ring in y2
  over Finite Field of size 2 with modulus y^2 + y + 1
  with modulus x^3 + 1
```

```
>>> from sage.all import *
>>> # needs sage.libs.ntl
>>> A = PolynomialRing(GF(Integer(2)), names=('y',)); (y,) = A._first_ngens(1); A
Univariate Polynomial Ring in y over Finite Field of size 2 (using GF2X)
>>> B = A.quotient(y**Integer(2) + y + Integer(1), 'y2'); B
Univariate Quotient Polynomial Ring in y2 over Finite Field of size 2
 with modulus y^2 + y + 1
>>> C = PolynomialRing(B, 'x'); x = C.gen(); C
Univariate Polynomial Ring in x
 over Univariate Quotient Polynomial Ring in y2
  over Finite Field of size 2 with modulus y^2 + y + 1
>>> R = C.quotient(x**Integer(3) - Integer(5)); R
Univariate Quotient Polynomial Ring in xbar
 over Univariate Quotient Polynomial Ring in y2
  over Finite Field of size 2 with modulus y^2 + y + 1
  with modulus x^3 + 1
```

Next we create a number field, but viewed as a quotient of a polynomial ring over **Q**:

```
sage: R = PolynomialRing(RationalField(), 'x'); x = R.gen()
sage: S = R.quotient(x^3 + 2*x - 5, 'a'); S
Univariate Quotient Polynomial Ring in a over Rational Field
 with modulus x^3 + 2*x - 5
sage: S.is_field()
True
sage: S.degree()
3
```

```
>>> from sage.all import *
>>> R = PolynomialRing(RationalField(), 'x'); x = R.gen()
>>> S = R.quotient(x**Integer(3) + Integer(2)*x - Integer(5), 'a'); S
Univariate Quotient Polynomial Ring in a over Rational Field
 with modulus x^3 + 2*x - 5
>>> S.is_field()
True
>>> S.degree()
3
```

There are conversion functions for easily going back and forth between quotients of polynomial rings over **Q** and number fields:

```
sage: K = S.number_field(); K                                              #␣
↪needs sage.rings.number_field
Number Field in a with defining polynomial x^3 + 2*x - 5
sage: K.polynomial_quotient_ring()                                         #␣
↪needs sage.rings.number_field
Univariate Quotient Polynomial Ring in a
 over Rational Field with modulus x^3 + 2*x - 5
```

```
>>> from sage.all import *
>>> K = S.number_field(); K                                              #␣
↪needs sage.rings.number_field
Number Field in a with defining polynomial x^3 + 2*x - 5
```

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```
>>> K.polynomial_quotient_ring()                                                #␣
↪needs sage.rings.number_field
Univariate Quotient Polynomial Ring in a
 over Rational Field with modulus x^3 + 2*x - 5
```

The leading coefficient must be a unit (but need not be 1).

```
sage: R = PolynomialRing(Integers(9), 'x'); x = R.gen()
sage: S = R.quotient(2*x^4 + 2*x^3 + x + 2, 'a')
sage: S = R.quotient(3*x^4 + 2*x^3 + x + 2, 'a')
Traceback (most recent call last):
...
TypeError: polynomial must have unit leading coefficient
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(9)), 'x'); x = R.gen()
>>> S = R.quotient(Integer(2)*x**Integer(4) + Integer(2)*x**Integer(3) + x +␣
↪Integer(2), 'a')
>>> S = R.quotient(Integer(3)*x**Integer(4) + Integer(2)*x**Integer(3) + x +␣
↪Integer(2), 'a')
Traceback (most recent call last):
...
TypeError: polynomial must have unit leading coefficient
```

Another example:

```
sage: R.<x> = PolynomialRing(IntegerRing())
sage: f = x^2 + 1
sage: R.quotient(f)
Univariate Quotient Polynomial Ring in xbar over Integer Ring with modulus x^2 + 1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(IntegerRing(), names=('x',)); (x,) = R._first_ngens(1)
>>> f = x**Integer(2) + Integer(1)
>>> R.quotient(f)
Univariate Quotient Polynomial Ring in xbar over Integer Ring with modulus x^2 + 1
```

This shows that the issue at Issue #5482 is solved:

```
sage: R.<x> = PolynomialRing(QQ)
sage: f = x^2 - 1
sage: R.quotient_by_principal_ideal(f)
Univariate Quotient Polynomial Ring in xbar over Rational Field with modulus x^2 -
↪ 1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = x**Integer(2) - Integer(1)
>>> R.quotient_by_principal_ideal(f)
Univariate Quotient Polynomial Ring in xbar over Rational Field with modulus x^2 -
↪ 1
```

create_key(*ring*, *polynomial*, *names=None*)

Return a unique description of the quotient ring specified by the arguments.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: PolynomialQuotientRing.create_key(R, x + 1)
(Univariate Polynomial Ring in x over Rational Field, x + 1, ('xbar',))
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> PolynomialQuotientRing.create_key(R, x + Integer(1))
(Univariate Polynomial Ring in x over Rational Field, x + 1, ('xbar',))
```

create_object(*version*, *key*)

Return the quotient ring specified by `key`.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: PolynomialQuotientRing.create_object((8, 0, 0),
....:                                      (R, x^2 - 1, ('xbar')))
Univariate Quotient Polynomial Ring in xbar over Rational Field with modulus␣
→x^2 - 1
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> PolynomialQuotientRing.create_object((Integer(8), Integer(0), Integer(0)),
...                                      (R, x**Integer(2) - Integer(1), (
→'xbar')))
Univariate Quotient Polynomial Ring in xbar over Rational Field with modulus␣
→x^2 - 1
```

class sage.rings.polynomial.polynomial_quotient_ring.**PolynomialQuotientRing_coercion**

Bases: `DefaultConvertMap_unique`

A coercion map from a `PolynomialQuotientRing` to a `PolynomialQuotientRing` that restricts to the coercion map on the underlying ring of constants.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: S.<x> = QQ[]
sage: f = S.quo(x^2 + 1).coerce_map_from(R.quo(x^2 + 1)); f
Coercion map:
  From: Univariate Quotient Polynomial Ring in xbar over Integer Ring
        with modulus x^2 + 1
  To:   Univariate Quotient Polynomial Ring in xbar over Rational Field
        with modulus x^2 + 1
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> S = QQ['x']; (x,) = S._first_ngens(1)
>>> f = S.quo(x**Integer(2) + Integer(1)).coerce_map_from(R.quo(x**Integer(2) +␣
```

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```
↪Integer(1))); f
Coercion map:
  From: Univariate Quotient Polynomial Ring in xbar over Integer Ring
        with modulus x^2 + 1
  To:   Univariate Quotient Polynomial Ring in xbar over Rational Field
        with modulus x^2 + 1
```

is_injective()

Return whether this coercion is injective.

EXAMPLES:

If the modulus of the domain and the codomain is the same and the leading coefficient is a unit in the domain, then the map is injective if the underlying map on the constants is:

```
sage: R.<x> = ZZ[]
sage: S.<x> = QQ[]
sage: f = S.quo(x^2 + 1).coerce_map_from(R.quo(x^2 + 1))
sage: f.is_injective()
True
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> S = QQ['x']; (x,) = S._first_ngens(1)
>>> f = S.quo(x**Integer(2) + Integer(1)).coerce_map_from(R.quo(x**Integer(2)␣
↪+ Integer(1)))
>>> f.is_injective()
True
```

is_surjective()

Return whether this coercion is surjective.

EXAMPLES:

If the underlying map on constants is surjective, then this coercion is surjective since the modulus of the codomain divides the modulus of the domain:

```
sage: R.<x> = ZZ[]
sage: f = R.quo(x).coerce_map_from(R.quo(x^2))
sage: f.is_surjective()
True
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> f = R.quo(x).coerce_map_from(R.quo(x**Integer(2)))
>>> f.is_surjective()
True
```

If the modulus of the domain and the codomain is the same, then the map is surjective iff the underlying map on the constants is:

```
sage: # needs sage.rings.padics
sage: A.<a> = ZqCA(9)
sage: R.<x> = A[]
```

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```
sage: S.<x> = A.fraction_field()[]
sage: f = S.quo(x^2 + 2).coerce_map_from(R.quo(x^2 + 2))
sage: f.is_surjective()
False
```

```
>>> from sage.all import *
>>> # needs sage.rings.padics
>>> A = ZqCA(Integer(9), names=('a',)); (a,) = A._first_ngens(1)
>>> R = A['x']; (x,) = R._first_ngens(1)
>>> S = A.fraction_field()['x']; (x,) = S._first_ngens(1)
>>> f = S.quo(x**Integer(2) + Integer(2)).coerce_map_from(R.quo(x**Integer(2)
↪+ Integer(2)))
>>> f.is_surjective()
False
```

class sage.rings.polynomial.polynomial_quotient_ring.**PolynomialQuotientRing_domain**(*ring*, *polynomial*, *name=None*, *category=None*)

Bases: *PolynomialQuotientRing_generic*, CommutativeRing

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ)
sage: S.<xbar> = R.quotient(x^2 + 1)
sage: S
Univariate Quotient Polynomial Ring in xbar
 over Integer Ring with modulus x^2 + 1
sage: loads(S.dumps()) == S
True
sage: loads(xbar.dumps()) == xbar
True
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(2) + Integer(1), names=('xbar',)); (xbar,) = S._
↪first_ngens(1)
>>> S
Univariate Quotient Polynomial Ring in xbar
 over Integer Ring with modulus x^2 + 1
>>> loads(S.dumps()) == S
True
>>> loads(xbar.dumps()) == xbar
True
```

field_extension(*names*)

Take a polynomial quotient ring, and return a tuple with three elements: the NumberField defined by the same polynomial quotient ring, a homomorphism from its parent to the NumberField sending the generators to one another, and the inverse isomorphism.

OUTPUT:

- field
- homomorphism from `self` to field
- homomorphism from field to `self`

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x> = PolynomialRing(Rationals())
sage: S.<alpha> = R.quotient(x^3 - 2)
sage: F.<b>, f, g = S.field_extension()
sage: F
Number Field in b with defining polynomial x^3 - 2
sage: a = F.gen()
sage: f(alpha)
b
sage: g(a)
alpha
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = PolynomialRing(Rationals(), names=('x',)); (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(3) - Integer(2), names=('alpha',)); (alpha,) =
↪S._first_ngens(1)
>>> F, f, g  = S.field_extension(names=('b',)); (b,) = F._first_ngens(1)
>>> F
Number Field in b with defining polynomial x^3 - 2
>>> a = F.gen()
>>> f(alpha)
b
>>> g(a)
alpha
```

Note that the parent ring must be an integral domain:

```
sage: R.<x> = GF(25, 'f25')['x']                                          #
↪needs sage.rings.finite_rings
sage: S.<a> = R.quo(x^3 - 2)                                              #
↪needs sage.rings.finite_rings
sage: F, g, h = S.field_extension('b')                                    #
↪needs sage.rings.finite_rings
Traceback (most recent call last):
...
AttributeError: 'PolynomialQuotientRing_generic_with_category' object has no
↪attribute 'field_extension'...
```

```
>>> from sage.all import *
>>> R = GF(Integer(25), 'f25')['x']; (x,) = R._first_ngens(1)# needs sage.
↪rings.finite_rings
>>> S = R.quo(x**Integer(3) - Integer(2), names=('a',)); (a,) = S._first_
↪ngens(1)# needs sage.rings.finite_rings
>>> F, g, h = S.field_extension('b')                                    #
```

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```
↪needs sage.rings.finite_rings
Traceback (most recent call last):
...
AttributeError: 'PolynomialQuotientRing_generic_with_category' object has no␣
↪attribute 'field_extension'...
```

Over a finite field, the corresponding field extension is not a number field:

```
sage: # needs sage.modules sage.rings.finite_rings
sage: R.<x> = GF(25, 'a')['x']
sage: S.<a> = R.quo(x^3 + 2*x + 1)
sage: F, g, h = S.field_extension('b')
sage: h(F.0^2 + 3)
a^2 + 3
sage: g(x^2 + 2)
b^2 + 2
```

```
>>> from sage.all import *
>>> # needs sage.modules sage.rings.finite_rings
>>> R = GF(Integer(25), 'a')['x']; (x,) = R._first_ngens(1)
>>> S = R.quo(x**Integer(3) + Integer(2)*x + Integer(1), names=('a',)); (a,)␣
↪= S._first_ngens(1)
>>> F, g, h = S.field_extension('b')
>>> h(F.gen(0)**Integer(2) + Integer(3))
a^2 + 3
>>> g(x**Integer(2) + Integer(2))
b^2 + 2
```

We do an example involving a relative number field:

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQ['x']
sage: K.<a> = NumberField(x^3 - 2)
sage: S.<X> = K['X']
sage: Q.<b> = S.quo(X^3 + 2*X + 1)
sage: Q.field_extension('b')
(Number Field in b with defining polynomial X^3 + 2*X + 1 over its base field,
↪ ...
  Defn: b |--> b, Relative number field morphism:
  From: Number Field in b with defining polynomial X^3 + 2*X + 1 over its␣
↪base field
  To:   Univariate Quotient Polynomial Ring in b over Number Field in a with␣
↪defining polynomial x^3 - 2 with modulus X^3 + 2*X + 1
  Defn: b |--> b
        a |--> a)
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> K = NumberField(x**Integer(3) - Integer(2), names=('a',)); (a,) = K._
↪first_ngens(1)
>>> S = K['X']; (X,) = S._first_ngens(1)
```

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```
>>> Q = S.quo(X**Integer(3) + Integer(2)*X + Integer(1), names=('b',)); (b,)␣
↪= Q._first_ngens(1)
>>> Q.field_extension('b')
(Number Field in b with defining polynomial X^3 + 2*X + 1 over its base field,
↪ ...
  Defn: b |--> b, Relative number field morphism:
  From: Number Field in b with defining polynomial X^3 + 2*X + 1 over its␣
↪base field
  To:   Univariate Quotient Polynomial Ring in b over Number Field in a with␣
↪defining polynomial x^3 - 2 with modulus X^3 + 2*X + 1
  Defn: b |--> b
        a |--> a)
```

We slightly change the example above so it works.

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQ['x']
sage: K.<a> = NumberField(x^3 - 2)
sage: S.<X> = K['X']
sage: f = (X+a)^3 + 2*(X+a) + 1
sage: f
X^3 + 3*a*X^2 + (3*a^2 + 2)*X + 2*a + 3
sage: Q.<z> = S.quo(f)
sage: F.<w>, g, h = Q.field_extension()
sage: c = g(z)
sage: f(c)
0
sage: h(g(z))
z
sage: g(h(w))
w
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> K = NumberField(x**Integer(3) - Integer(2), names=('a',)); (a,) = K._
↪first_ngens(1)
>>> S = K['X']; (X,) = S._first_ngens(1)
>>> f = (X+a)**Integer(3) + Integer(2)*(X+a) + Integer(1)
>>> f
X^3 + 3*a*X^2 + (3*a^2 + 2)*X + 2*a + 3
>>> Q = S.quo(f, names=('z',)); (z,) = Q._first_ngens(1)
>>> F, g, h  = Q.field_extension(names=('w',)); (w,) = F._first_ngens(1)
>>> c = g(z)
>>> f(c)
0
>>> h(g(z))
z
>>> g(h(w))
w
```

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class sage.rings.polynomial.polynomial_quotient_ring.**PolynomialQuotientRing_field**(*ring*, *polynomial*, *name=None*, *category=None*)

Bases: *PolynomialQuotientRing_domain*, Field

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x> = PolynomialRing(QQ)
sage: S.<xbar> = R.quotient(x^2 + 1)
sage: S
Univariate Quotient Polynomial Ring in xbar over Rational Field
 with modulus x^2 + 1
sage: loads(S.dumps()) == S
True
sage: loads(xbar.dumps()) == xbar
True
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(2) + Integer(1), names=('xbar',)); (xbar,) = S._
↪first_ngens(1)
>>> S
Univariate Quotient Polynomial Ring in xbar over Rational Field
 with modulus x^2 + 1
>>> loads(S.dumps()) == S
True
>>> loads(xbar.dumps()) == xbar
True
```

base_field()

Alias for base_ring(), when we're defined over a field.

complex_embeddings(*prec=53*)

Return all homomorphisms of this ring into the approximate complex field with precision prec.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: f = x^5 + x + 17
sage: k = R.quotient(f)
sage: v = k.complex_embeddings(100)
sage: [phi(k.0^2) for phi in v]
[2.9757207403766761469671194565,
 -2.4088994371613850098316292196 + 1.9025410530350528612407363802*I,
```

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```
 -2.4088994371613850098316292196 - 1.9025410530350528612407363802*I,
 0.92103906697304693634806949137 - 3.0755331188457794473265418086*I,
 0.92103906697304693634806949137 + 3.0755331188457794473265418086*I]
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = x**Integer(5) + x + Integer(17)
>>> k = R.quotient(f)
>>> v = k.complex_embeddings(Integer(100))
>>> [phi(k.gen(0)**Integer(2)) for phi in v]
[2.9757207403766761469671194565,
 -2.4088994371613850098316292196 + 1.9025410530350528612407363802*I,
 -2.4088994371613850098316292196 - 1.9025410530350528612407363802*I,
 0.92103906697304693634806949137 - 3.0755331188457794473265418086*I,
 0.92103906697304693634806949137 + 3.0755331188457794473265418086*I]
```

class sage.rings.polynomial.polynomial_quotient_ring.**PolynomialQuotientRing_generic**(*ring*, *polynomial*, *name=None*, *category=None*)

Bases: QuotientRing_generic

Quotient of a univariate polynomial ring by an ideal.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(Integers(8)); R
Univariate Polynomial Ring in x over Ring of integers modulo 8
sage: S.<xbar> = R.quotient(x^2 + 1); S
Univariate Quotient Polynomial Ring in xbar over Ring of integers modulo 8
 with modulus x^2 + 1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(8)), names=('x',)); (x,) = R._first_
↪ngens(1); R
Univariate Polynomial Ring in x over Ring of integers modulo 8
>>> S = R.quotient(x**Integer(2) + Integer(1), names=('xbar',)); (xbar,) = S._
↪first_ngens(1); S
Univariate Quotient Polynomial Ring in xbar over Ring of integers modulo 8
 with modulus x^2 + 1
```

We demonstrate object persistence.

```
sage: loads(S.dumps()) == S
True
sage: loads(xbar.dumps()) == xbar
True
```

```
>>> from sage.all import *
>>> loads(S.dumps()) == S
True
>>> loads(xbar.dumps()) == xbar
True
```

We create some sample homomorphisms;

```
sage: R.<x> = PolynomialRing(ZZ)
sage: S = R.quo(x^2 - 4)
sage: f = S.hom([2])
sage: f
Ring morphism:
  From: Univariate Quotient Polynomial Ring in xbar over Integer Ring
        with modulus x^2 - 4
  To:   Integer Ring
  Defn: xbar |--> 2
sage: f(x)
2
sage: f(x^2 - 4)
0
sage: f(x^2)
4
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> S = R.quo(x**Integer(2) - Integer(4))
>>> f = S.hom([Integer(2)])
>>> f
Ring morphism:
  From: Univariate Quotient Polynomial Ring in xbar over Integer Ring
        with modulus x^2 - 4
  To:   Integer Ring
  Defn: xbar |--> 2
>>> f(x)
2
>>> f(x**Integer(2) - Integer(4))
0
>>> f(x**Integer(2))
4
```

Element

alias of *PolynomialQuotientRingElement*

S_class_group(*S*, *proof=True*)

If `self` is an étale algebra D over a number field K (i.e. a quotient of $K[x]$ by a squarefree polynomial) and S is a finite set of places of K, return a list of generators of the S-class group of D.

NOTE:

Since the `ideal` function behaves differently over number fields than over polynomial quotient rings (the quotient does not even know its ring of integers), we return a set of pairs `(gen, order)`, where `gen` is a tuple of generators of an ideal I and `order` is the order of I in the S-class group.

INPUT:

- `S` – set of primes of the coefficient ring
- `proof` – if `False`, assume the GRH in computing the class group

OUTPUT:

A list of generators of the S-class group, in the form `(gen, order)`, where `gen` is a tuple of elements generating a fractional ideal I and `order` is the order of I in the S-class group.

EXAMPLES:

A trivial algebra over $\mathbf{Q}(\sqrt{-5})$ has the same class group as its base:

```
sage: # needs sage.rings.number_field
sage: K.<a> = QuadraticField(-5)
sage: R.<x> = K[]
sage: S.<xbar> = R.quotient(x)
sage: S.S_class_group([])
[((2, -a + 1), 2)]
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> K = QuadraticField(-Integer(5), names=('a',)); (a,) = K._first_ngens(1)
>>> R = K['x']; (x,) = R._first_ngens(1)
>>> S = R.quotient(x, names=('xbar',)); (xbar,) = S._first_ngens(1)
>>> S.S_class_group([])
[((2, -a + 1), 2)]
```

When we include the prime $(2, -a + 1)$, the S-class group becomes trivial:

```
sage: S.S_class_group([K.ideal(2, -a+1)])                                    #␣
↪needs sage.rings.number_field
[]
```

```
>>> from sage.all import *
>>> S.S_class_group([K.ideal(Integer(2), -a+Integer(1))])                     ␣
↪               # needs sage.rings.number_field
[]
```

Here is an example where the base and the extension both contribute to the class group:

```
sage: # needs sage.rings.number_field
sage: K.<a> = QuadraticField(-5)
sage: K.class_group()
Class group of order 2 with structure C2 of Number Field in a
 with defining polynomial x^2 + 5 with a = 2.236067977499790?*I
sage: R.<x> = K[]
sage: S.<xbar> = R.quotient(x^2 + 23)
sage: S.S_class_group([])
[((2, a + 1, -1/2*xbar + 3/2, 1/2*a*xbar - 1/2*a + 1), 6)]
sage: S.S_class_group([K.ideal(3, a-1)])
[]
sage: S.S_class_group([K.ideal(2, a+1)])
[]
sage: S.S_class_group([K.ideal(a)])
[((2, a + 1, -1/2*xbar + 3/2, 1/2*a*xbar - 1/2*a + 1), 6)]
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> K = QuadraticField(-Integer(5), names=('a',)); (a,) = K._first_ngens(1)
>>> K.class_group()
Class group of order 2 with structure C2 of Number Field in a
 with defining polynomial x^2 + 5 with a = 2.236067977499790?*I
>>> R = K['x']; (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(2) + Integer(23), names=('xbar',)); (xbar,) = S.
↪_first_ngens(1)
>>> S.S_class_group([])
[((2, a + 1, -1/2*xbar + 3/2, 1/2*a*xbar - 1/2*a + 1), 6)]
>>> S.S_class_group([K.ideal(Integer(3), a-Integer(1))])
[]
>>> S.S_class_group([K.ideal(Integer(2), a+Integer(1))])
[]
>>> S.S_class_group([K.ideal(a)])
[((2, a + 1, -1/2*xbar + 3/2, 1/2*a*xbar - 1/2*a + 1), 6)]
```

Now we take an example over a nontrivial base with two factors, each contributing to the class group:

```
sage: # needs sage.rings.number_field
sage: K.<a> = QuadraticField(-5)
sage: R.<x> = K[]
sage: S.<xbar> = R.quotient((x^2 + 23) * (x^2 + 31))
sage: S.S_class_group([])            # not tested
[((1/4*xbar^2 + 31/4,
   (-1/8*a + 1/8)*xbar^2 - 31/8*a + 31/8,
   1/16*xbar^3 + 1/16*xbar^2 + 31/16*xbar + 31/16,
   -1/16*a*xbar^3 + (1/16*a + 1/8)*xbar^2 - 31/16*a*xbar + 31/16*a + 31/8),
  6),
 ((-1/4*xbar^2 - 23/4,
   (1/8*a - 1/8)*xbar^2 + 23/8*a - 23/8,
   -1/16*xbar^3 - 1/16*xbar^2 - 23/16*xbar - 23/16,
   1/16*a*xbar^3 + (-1/16*a - 1/8)*xbar^2 + 23/16*a*xbar - 23/16*a - 23/8),
  6),
 ((-5/4*xbar^2 - 115/4,
   1/4*a*xbar^2 + 23/4*a,
   -1/16*xbar^3 - 7/16*xbar^2 - 23/16*xbar - 161/16,
   1/16*a*xbar^3 - 1/16*a*xbar^2 + 23/16*a*xbar - 23/16*a),
  2)]
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> K = QuadraticField(-Integer(5), names=('a',)); (a,) = K._first_ngens(1)
>>> R = K['x']; (x,) = R._first_ngens(1)
>>> S = R.quotient((x**Integer(2) + Integer(23)) * (x**Integer(2) +␣
↪Integer(31)), names=('xbar',)); (xbar,) = S._first_ngens(1)
>>> S.S_class_group([])            # not tested
[((1/4*xbar^2 + 31/4,
   (-1/8*a + 1/8)*xbar^2 - 31/8*a + 31/8,
   1/16*xbar^3 + 1/16*xbar^2 + 31/16*xbar + 31/16,
   -1/16*a*xbar^3 + (1/16*a + 1/8)*xbar^2 - 31/16*a*xbar + 31/16*a + 31/8),
  6),
```

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```
((-1/4*xbar^2 - 23/4,
  (1/8*a - 1/8)*xbar^2 + 23/8*a - 23/8,
  -1/16*xbar^3 - 1/16*xbar^2 - 23/16*xbar - 23/16,
  1/16*a*xbar^3 + (-1/16*a - 1/8)*xbar^2 + 23/16*a*xbar - 23/16*a - 23/8),
 6),
((-5/4*xbar^2 - 115/4,
  1/4*a*xbar^2 + 23/4*a,
  -1/16*xbar^3 - 7/16*xbar^2 - 23/16*xbar - 161/16,
  1/16*a*xbar^3 - 1/16*a*xbar^2 + 23/16*a*xbar - 23/16*a),
 2)]
```

By using the ideal (a), we cut the part of the class group coming from $x^2 + 31$ from 12 to 2, i.e. we lose a generator of order 6 (this was fixed in Issue #14489):

```
sage: S.S_class_group([K.ideal(a)])  # representation varies        # not␣
↪tested, needs sage.rings.number_field
[((1/4*xbar^2 + 31/4, (-1/8*a + 1/8)*xbar^2 - 31/8*a + 31/8,
   1/16*xbar^3 + 1/16*xbar^2 + 31/16*xbar + 31/16,
   -1/16*a*xbar^3 + (1/16*a + 1/8)*xbar^2 - 31/16*a*xbar + 31/16*a + 31/8),
  6),
 ((-1/4*xbar^2 - 23/4, (1/8*a - 1/8)*xbar^2 + 23/8*a - 23/8,
   -1/16*xbar^3 - 1/16*xbar^2 - 23/16*xbar - 23/16,
   1/16*a*xbar^3 + (-1/16*a - 1/8)*xbar^2 + 23/16*a*xbar - 23/16*a - 23/8),
  2)]
```

```
>>> from sage.all import *
>>> S.S_class_group([K.ideal(a)])  # representation varies        # not␣
↪tested, needs sage.rings.number_field
[((1/4*xbar^2 + 31/4, (-1/8*a + 1/8)*xbar^2 - 31/8*a + 31/8,
   1/16*xbar^3 + 1/16*xbar^2 + 31/16*xbar + 31/16,
   -1/16*a*xbar^3 + (1/16*a + 1/8)*xbar^2 - 31/16*a*xbar + 31/16*a + 31/8),
  6),
 ((-1/4*xbar^2 - 23/4, (1/8*a - 1/8)*xbar^2 + 23/8*a - 23/8,
   -1/16*xbar^3 - 1/16*xbar^2 - 23/16*xbar - 23/16,
   1/16*a*xbar^3 + (-1/16*a - 1/8)*xbar^2 + 23/16*a*xbar - 23/16*a - 23/8),
  2)]
```

Note that all the returned values live where we expect them to:

```
sage: # needs sage.rings.number_field
sage: CG = S.S_class_group([])
sage: type(CG[0][0][1])
<class 'sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_
↪generic_with_category.element_class'>
sage: type(CG[0][1])
<class 'sage.rings.integer.Integer'>
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> CG = S.S_class_group([])
>>> type(CG[Integer(0)][Integer(0)][Integer(1)])
<class 'sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_
```

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```
↪generic_with_category.element_class'>
>>> type(CG[Integer(0)][Integer(1)])
<class 'sage.rings.integer.Integer'>
```

S_units(*S*, *proof=True*)

If `self` is an étale algebra D over a number field K (i.e. a quotient of $K[x]$ by a squarefree polynomial) and S is a finite set of places of K, return a list of generators of the group of S-units of D.

INPUT:

- `S` – set of primes of the base field
- `proof` – if `False`, assume the GRH in computing the class group

OUTPUT:

A list of generators of the S-unit group, in the form `(gen, order)`, where `gen` is a unit of order `order`.

EXAMPLES:

```
sage: K.<a> = QuadraticField(-3)                                              #␣
↪needs sage.rings.number_field
sage: K.unit_group()                                                          #␣
↪needs sage.rings.number_field
Unit group with structure C6 of Number Field in a
 with defining polynomial x^2 + 3 with a = 1.732050807568878?*I

sage: # needs sage.rings.number_field
sage: x = polygen(ZZ, 'x')
sage: K.<a> = QQ['x'].quotient(x^2 + 3)
sage: u, o = K.S_units([])[0]; o
6
sage: 2*u - 1 in {a, -a}
True
sage: u^6
1
sage: u^3
-1
sage: 2*u^2 + 1 in {a, -a}
True
```

```
>>> from sage.all import *
>>> K = QuadraticField(-Integer(3), names=('a',)); (a,) = K._first_ngens(1)#␣
↪needs sage.rings.number_field
>>> K.unit_group()                                                          #␣
↪needs sage.rings.number_field
Unit group with structure C6 of Number Field in a
 with defining polynomial x^2 + 3 with a = 1.732050807568878?*I

>>> # needs sage.rings.number_field
>>> x = polygen(ZZ, 'x')
>>> K = QQ['x'].quotient(x**Integer(2) + Integer(3), names=('a',)); (a,) = K._
↪first_ngens(1)
>>> u, o = K.S_units([])[Integer(0)]; o
6
```

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```
>>> Integer(2)*u - Integer(1) in {a, -a}
True
>>> u**Integer(6)
1
>>> u**Integer(3)
-1
>>> Integer(2)*u**Integer(2) + Integer(1) in {a, -a}
True
```

```
sage: # needs sage.rings.number_field
sage: K.<a> = QuadraticField(-3)
sage: y = polygen(K)
sage: L.<b> = K['y'].quotient(y^3 + 5); L
Univariate Quotient Polynomial Ring in b over Number Field in a
 with defining polynomial x^2 + 3 with a = 1.732050807568878?*I
 with modulus y^3 + 5
sage: [u for u, o in L.S_units([]) if o is Infinity]
[(-1/3*a - 1)*b^2 - 4/3*a*b - 4/3*a + 3,
 (-1/3*a - 1)*b^2 + (2/3*a - 2)*b + 13/6*a - 1/2]
sage: [u for u, o in L.S_units([K.ideal(1/2*a - 3/2)])
....:  if o is Infinity]
[(-1/6*a - 1/2)*b^2 + (1/3*a - 1)*b + 4/3*a,
 (-1/3*a - 1)*b^2 - 4/3*a*b - 4/3*a + 3,
 (-1/3*a - 1)*b^2 + (2/3*a - 2)*b + 13/6*a - 1/2]
sage: [u for u, o in L.S_units([K.ideal(2)]) if o is Infinity]
[(1/2*a - 1/2)*b^2 + (a + 1)*b + 3,
 (1/6*a + 1/2)*b^2 + (-1/3*a + 1)*b - 5/6*a - 1/2,
 1/3*a*b^2 + (1/3*a + 1)*b - 1/6*a + 3/2,
 (-1/3*a - 1)*b^2 - 4/3*a*b - 4/3*a + 3,
 (-1/3*a - 1)*b^2 + (2/3*a - 2)*b + 13/6*a - 1/2]
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> K = QuadraticField(-Integer(3), names=('a',)); (a,) = K._first_ngens(1)
>>> y = polygen(K)
>>> L = K['y'].quotient(y**Integer(3) + Integer(5), names=('b',)); (b,) = L._
↪first_ngens(1); L
Univariate Quotient Polynomial Ring in b over Number Field in a
 with defining polynomial x^2 + 3 with a = 1.732050807568878?*I
 with modulus y^3 + 5
>>> [u for u, o in L.S_units([]) if o is Infinity]
[(-1/3*a - 1)*b^2 - 4/3*a*b - 4/3*a + 3,
 (-1/3*a - 1)*b^2 + (2/3*a - 2)*b + 13/6*a - 1/2]
>>> [u for u, o in L.S_units([K.ideal(Integer(1)/Integer(2)*a - Integer(3)/
↪Integer(2))])
...  if o is Infinity]
[(-1/6*a - 1/2)*b^2 + (1/3*a - 1)*b + 4/3*a,
 (-1/3*a - 1)*b^2 - 4/3*a*b - 4/3*a + 3,
 (-1/3*a - 1)*b^2 + (2/3*a - 2)*b + 13/6*a - 1/2]
>>> [u for u, o in L.S_units([K.ideal(Integer(2))]) if o is Infinity]
[(1/2*a - 1/2)*b^2 + (a + 1)*b + 3,
```

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```
 (1/6*a + 1/2)*b^2 + (-1/3*a + 1)*b - 5/6*a - 1/2,
 1/3*a*b^2 + (1/3*a + 1)*b - 1/6*a + 3/2,
 (-1/3*a - 1)*b^2 - 4/3*a*b - 4/3*a + 3,
 (-1/3*a - 1)*b^2 + (2/3*a - 2)*b + 13/6*a - 1/2]
```

Note that all the returned values live where we expect them to:

```
sage: # needs sage.rings.number_field
sage: U = L.S_units([])
sage: type(U[0][0])
<class 'sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_
↪field_with_category.element_class'>
sage: type(U[0][1])
<class 'sage.rings.integer.Integer'>
sage: type(U[1][1])
<class 'sage.rings.infinity.PlusInfinity'>
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> U = L.S_units([])
>>> type(U[Integer(0)][Integer(0)])
<class 'sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_
↪field_with_category.element_class'>
>>> type(U[Integer(0)][Integer(1)])
<class 'sage.rings.integer.Integer'>
>>> type(U[Integer(1)][Integer(1)])
<class 'sage.rings.infinity.PlusInfinity'>
```

ambient()

base_ring()

Return the base ring of the polynomial ring, of which this ring is a quotient.

EXAMPLES:

The base ring of $\mathbf{Z}[z]/(z^3 + z^2 + z + 1)$ is $\mathbf{Z}$.

```
sage: R.<z> = PolynomialRing(ZZ)
sage: S.<beta> = R.quo(z^3 + z^2 + z + 1)
sage: S.base_ring()
Integer Ring
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('z',)); (z,) = R._first_ngens(1)
>>> S = R.quo(z**Integer(3) + z**Integer(2) + z + Integer(1), names=('beta',
↪)); (beta,) = S._first_ngens(1)
>>> S.base_ring()
Integer Ring
```

Next we make a polynomial quotient ring over S and ask for its base ring.

```
sage: T.<t> = PolynomialRing(S)
sage: W = T.quotient(t^99 + 99)
```

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```
sage: W.base_ring()
Univariate Quotient Polynomial Ring in beta
 over Integer Ring with modulus z^3 + z^2 + z + 1
```

```
>>> from sage.all import *
>>> T = PolynomialRing(S, names=('t',)); (t,) = T._first_ngens(1)
>>> W = T.quotient(t**Integer(99) + Integer(99))
>>> W.base_ring()
Univariate Quotient Polynomial Ring in beta
 over Integer Ring with modulus z^3 + z^2 + z + 1
```

cardinality()

Return the number of elements of this quotient ring.

`order` is an alias of `cardinality`.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: R.quo(1).cardinality()
1
sage: R.quo(x^3 - 2).cardinality()
+Infinity

sage: R.quo(1).order()
1
sage: R.quo(x^3 - 2).order()
+Infinity
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> R.quo(Integer(1)).cardinality()
1
>>> R.quo(x**Integer(3) - Integer(2)).cardinality()
+Infinity

>>> R.quo(Integer(1)).order()
1
>>> R.quo(x**Integer(3) - Integer(2)).order()
+Infinity
```

```
sage: # needs sage.rings.finite_rings
sage: R.<x> = GF(9, 'a')[]
sage: R.quo(2*x^3 + x + 1).cardinality()
729
sage: GF(9, 'a').extension(2*x^3 + x + 1).cardinality()
729
sage: R.quo(2).cardinality()
1
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
```

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```
>>> R = GF(Integer(9), 'a')['x']; (x,) = R._first_ngens(1)
>>> R.quo(Integer(2)*x**Integer(3) + x + Integer(1)).cardinality()
729
>>> GF(Integer(9), 'a').extension(Integer(2)*x**Integer(3) + x + Integer(1)).
↪cardinality()
729
>>> R.quo(Integer(2)).cardinality()
1
```

characteristic()

Return the characteristic of this quotient ring.

This is always the same as the characteristic of the base ring.

EXAMPLES:

```
sage: R.<z> = PolynomialRing(ZZ)
sage: S.<a> = R.quo(z - 19)
sage: S.characteristic()
0
sage: R.<x> = PolynomialRing(GF(9, 'a'))                                    #␣
↪needs sage.rings.finite_rings
sage: S = R.quotient(x^3 + 1)                                               #␣
↪needs sage.rings.finite_rings
sage: S.characteristic()                                                    #␣
↪needs sage.rings.finite_rings
3
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('z',)); (z,) = R._first_ngens(1)
>>> S = R.quo(z - Integer(19), names=('a',)); (a,) = S._first_ngens(1)
>>> S.characteristic()
0
>>> R = PolynomialRing(GF(Integer(9), 'a'), names=('x',)); (x,) = R._first_
↪ngens(1)# needs sage.rings.finite_rings
>>> S = R.quotient(x**Integer(3) + Integer(1))                              ␣
↪              # needs sage.rings.finite_rings
>>> S.characteristic()                                                    #␣
↪needs sage.rings.finite_rings
3
```

class_group(*proof=True*)

If `self` is a quotient ring of a polynomial ring over a number field K, by a polynomial of nonzero discriminant, return a list of generators of the class group.

NOTE:

Since the `ideal` function behaves differently over number fields than over polynomial quotient rings (the quotient does not even know its ring of integers), we return a set of pairs `(gen, order)`, where `gen` is a tuple of generators of an ideal I and `order` is the order of I in the class group.

INPUT:

- `proof` – if `False`, assume the GRH in computing the class group

OUTPUT:

A list of pairs `(gen, order)`, where `gen` is a tuple of elements generating a fractional ideal and `order` is the order of I in the class group.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: K.<a> = QuadraticField(-3)
sage: K.class_group()
Class group of order 1 of Number Field in a
 with defining polynomial x^2 + 3 with a = 1.732050807568878?*I
sage: x = polygen(QQ, 'x')
sage: K.<a> = QQ['x'].quotient(x^2 + 3)
sage: K.class_group()
[]
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> K = QuadraticField(-Integer(3), names=('a',)); (a,) = K._first_ngens(1)
>>> K.class_group()
Class group of order 1 of Number Field in a
 with defining polynomial x^2 + 3 with a = 1.732050807568878?*I
>>> x = polygen(QQ, 'x')
>>> K = QQ['x'].quotient(x**Integer(2) + Integer(3), names=('a',)); (a,) = K._
↪first_ngens(1)
>>> K.class_group()
[]
```

A trivial algebra over $\mathbf{Q}(\sqrt{-5})$ has the same class group as its base:

```
sage: # needs sage.rings.number_field
sage: K.<a> = QuadraticField(-5)
sage: R.<x> = K[]
sage: S.<xbar> = R.quotient(x)
sage: S.class_group()
[((2, -a + 1), 2)]
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> K = QuadraticField(-Integer(5), names=('a',)); (a,) = K._first_ngens(1)
>>> R = K['x']; (x,) = R._first_ngens(1)
>>> S = R.quotient(x, names=('xbar',)); (xbar,) = S._first_ngens(1)
>>> S.class_group()
[((2, -a + 1), 2)]
```

The same algebra constructed in a different way:

```
sage: x = polygen(ZZ, 'x')
sage: K.<a> = QQ['x'].quotient(x^2 + 5)
sage: K.class_group(())                                                  #␣
↪needs sage.rings.number_field
[((2, a + 1), 2)]
```

```
>>> from sage.all import *
>>> x = polygen(ZZ, 'x')
```

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```
>>> K = QQ['x'].quotient(x**Integer(2) + Integer(5), names=('a',)); (a,) = K._
↪first_ngens(1)
>>> K.class_group(())                                                          #␣
↪needs sage.rings.number_field
[((2, a + 1), 2)]
```

Here is an example where the base and the extension both contribute to the class group:

```
sage: # needs sage.rings.number_field
sage: K.<a> = QuadraticField(-5)
sage: K.class_group()
Class group of order 2 with structure C2 of Number Field in a
 with defining polynomial x^2 + 5 with a = 2.236067977499790?*I
sage: R.<x> = K[]
sage: S.<xbar> = R.quotient(x^2 + 23)
sage: S.class_group()
[((2, a + 1, -1/2*xbar + 3/2, 1/2*a*xbar - 1/2*a + 1), 6)]
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> K = QuadraticField(-Integer(5), names=('a',)); (a,) = K._first_ngens(1)
>>> K.class_group()
Class group of order 2 with structure C2 of Number Field in a
 with defining polynomial x^2 + 5 with a = 2.236067977499790?*I
>>> R = K['x']; (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(2) + Integer(23), names=('xbar',)); (xbar,) = S.
↪_first_ngens(1)
>>> S.class_group()
[((2, a + 1, -1/2*xbar + 3/2, 1/2*a*xbar - 1/2*a + 1), 6)]
```

Here is an example of a product of number fields, both of which contribute to the class group:

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: S.<xbar> = R.quotient((x^2 + 23) * (x^2 + 47))
sage: S.class_group()
[((1/12*xbar^2 + 47/12,
   1/48*xbar^3 - 1/48*xbar^2 + 47/48*xbar - 47/48),
  3),
 ((-1/12*xbar^2 - 23/12,
   -1/48*xbar^3 - 1/48*xbar^2 - 23/48*xbar - 23/48),
  5)]
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> S = R.quotient((x**Integer(2) + Integer(23)) * (x**Integer(2) +␣
↪Integer(47)), names=('xbar',)); (xbar,) = S._first_ngens(1)
>>> S.class_group()
[((1/12*xbar^2 + 47/12,
   1/48*xbar^3 - 1/48*xbar^2 + 47/48*xbar - 47/48),
  3),
```

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```
((-1/12*xbar^2 - 23/12,
  -1/48*xbar^3 - 1/48*xbar^2 - 23/48*xbar - 23/48),
 5)]
```

Now we take an example over a nontrivial base with two factors, each contributing to the class group:

```
sage: # needs sage.rings.number_field
sage: K.<a> = QuadraticField(-5)
sage: R.<x> = K[]
sage: S.<xbar> = R.quotient((x^2 + 23) * (x^2 + 31))
sage: S.class_group()                  # not tested
[((1/4*xbar^2 + 31/4,
   (-1/8*a + 1/8)*xbar^2 - 31/8*a + 31/8,
   1/16*xbar^3 + 1/16*xbar^2 + 31/16*xbar + 31/16,
   -1/16*a*xbar^3 + (1/16*a + 1/8)*xbar^2 - 31/16*a*xbar + 31/16*a + 31/8),
  6),
 ((-1/4*xbar^2 - 23/4,
   (1/8*a - 1/8)*xbar^2 + 23/8*a - 23/8,
   -1/16*xbar^3 - 1/16*xbar^2 - 23/16*xbar - 23/16,
   1/16*a*xbar^3 + (-1/16*a - 1/8)*xbar^2 + 23/16*a*xbar - 23/16*a - 23/8),
  6),
 ((-5/4*xbar^2 - 115/4,
   1/4*a*xbar^2 + 23/4*a,
   -1/16*xbar^3 - 7/16*xbar^2 - 23/16*xbar - 161/16,
   1/16*a*xbar^3 - 1/16*a*xbar^2 + 23/16*a*xbar - 23/16*a),
  2)]
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> K = QuadraticField(-Integer(5), names=('a',)); (a,) = K._first_ngens(1)
>>> R = K['x']; (x,) = R._first_ngens(1)
>>> S = R.quotient((x**Integer(2) + Integer(23)) * (x**Integer(2) +␣
↪Integer(31)), names=('xbar',)); (xbar,) = S._first_ngens(1)
>>> S.class_group()                  # not tested
[((1/4*xbar^2 + 31/4,
   (-1/8*a + 1/8)*xbar^2 - 31/8*a + 31/8,
   1/16*xbar^3 + 1/16*xbar^2 + 31/16*xbar + 31/16,
   -1/16*a*xbar^3 + (1/16*a + 1/8)*xbar^2 - 31/16*a*xbar + 31/16*a + 31/8),
  6),
 ((-1/4*xbar^2 - 23/4,
   (1/8*a - 1/8)*xbar^2 + 23/8*a - 23/8,
   -1/16*xbar^3 - 1/16*xbar^2 - 23/16*xbar - 23/16,
   1/16*a*xbar^3 + (-1/16*a - 1/8)*xbar^2 + 23/16*a*xbar - 23/16*a - 23/8),
  6),
 ((-5/4*xbar^2 - 115/4,
   1/4*a*xbar^2 + 23/4*a,
   -1/16*xbar^3 - 7/16*xbar^2 - 23/16*xbar - 161/16,
   1/16*a*xbar^3 - 1/16*a*xbar^2 + 23/16*a*xbar - 23/16*a),
  2)]
```

Note that all the returned values live where we expect them to:

```
sage: # needs sage.rings.number_field
sage: CG = S.class_group()
sage: type(CG[0][0][1])
<class 'sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_
↪generic_with_category.element_class'>
sage: type(CG[0][1])
<class 'sage.rings.integer.Integer'>
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> CG = S.class_group()
>>> type(CG[Integer(0)][Integer(0)][Integer(1)])
<class 'sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_
↪generic_with_category.element_class'>
>>> type(CG[Integer(0)][Integer(1)])
<class 'sage.rings.integer.Integer'>
```

construction()

Functorial construction of `self`.

EXAMPLES:

```
sage: P.<t> = ZZ[]
sage: Q = P.quo(5 + t^2)
sage: F, R = Q.construction()
sage: F(R) == Q
True
sage: P.<t> = GF(3)[]
sage: Q = P.quo([2 + t^2])
sage: F, R = Q.construction()
sage: F(R) == Q
True
```

```
>>> from sage.all import *
>>> P = ZZ['t']; (t,) = P._first_ngens(1)
>>> Q = P.quo(Integer(5) + t**Integer(2))
>>> F, R = Q.construction()
>>> F(R) == Q
True
>>> P = GF(Integer(3))['t']; (t,) = P._first_ngens(1)
>>> Q = P.quo([Integer(2) + t**Integer(2)])
>>> F, R = Q.construction()
>>> F(R) == Q
True
```

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cover_ring()

Return the polynomial ring of which this ring is the quotient.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: S = R.quotient(x^2 - 2)
sage: S.polynomial_ring()
Univariate Polynomial Ring in x over Rational Field
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(2) - Integer(2))
>>> S.polynomial_ring()
Univariate Polynomial Ring in x over Rational Field
```

degree()

Return the degree of this quotient ring. The degree is the degree of the polynomial that we quotiented out by.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(GF(3))
sage: S = R.quotient(x^2005 + 1)
sage: S.degree()
2005
```

```
>>> from sage.all import *
>>> R = PolynomialRing(GF(Integer(3)), names=('x',)); (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(2005) + Integer(1))
>>> S.degree()
2005
```

discriminant(*v=None*)

Return the discriminant of this ring over the base ring. This is by definition the discriminant of the polynomial that we quotiented out by.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: S = R.quotient(x^3 + x^2 + x + 1)
sage: S.discriminant()
-16
sage: S = R.quotient((x + 1) * (x + 1))
sage: S.discriminant()
0
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(3) + x**Integer(2) + x + Integer(1))
>>> S.discriminant()
-16
>>> S = R.quotient((x + Integer(1)) * (x + Integer(1)))
>>> S.discriminant()
0
```

The discriminant of the quotient polynomial ring need not equal the discriminant of the corresponding number field, since the discriminant of a number field is by definition the discriminant of the ring of integers of the number field:

```
sage: S = R.quotient(x^2 - 8)
sage: S.number_field().discriminant()                                    #␣
↪needs sage.rings.number_field
8
sage: S.discriminant()
32
```

```
>>> from sage.all import *
>>> S = R.quotient(x**Integer(2) - Integer(8))
>>> S.number_field().discriminant()                                     #␣
↪needs sage.rings.number_field
8
>>> S.discriminant()
32
```

gen(*n=0*)

Return the generator of this quotient ring. This is the equivalence class of the image of the generator of the polynomial ring.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: S = R.quotient(x^2 - 8, 'gamma')
sage: S.gen()
gamma
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(2) - Integer(8), 'gamma')
>>> S.gen()
gamma
```

is_field(*proof=True*)

Return whether or not this quotient ring is a field.

EXAMPLES:

```
sage: R.<z> = PolynomialRing(ZZ)
sage: S = R.quo(z^2 - 2)
sage: S.is_field()
False
sage: R.<x> = PolynomialRing(QQ)
sage: S = R.quotient(x^2 - 2)
sage: S.is_field()
True
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('z',)); (z,) = R._first_ngens(1)
>>> S = R.quo(z**Integer(2) - Integer(2))
>>> S.is_field()
False
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(2) - Integer(2))
```

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```
>>> S.is_field()
True
```

If proof is `True`, requires the `is_irreducible` method of the modulus to be implemented:

```
sage: # needs sage.rings.padics
sage: R1.<x> = Qp(2)[]
sage: F1 = R1.quotient_ring(x^2 + x + 1)
sage: R2.<x> = F1[]
sage: F2 = R2.quotient_ring(x^2 + x + 1)
sage: F2.is_field()
Traceback (most recent call last):
...
NotImplementedError: cannot rewrite Univariate Quotient Polynomial Ring in
 xbar over 2-adic Field with capped relative precision 20 with modulus
 (1 + O(2^20))*x^2 + (1 + O(2^20))*x + 1 + O(2^20) as an isomorphic ring
sage: F2.is_field(proof = False)
False
```

```
>>> from sage.all import *
>>> # needs sage.rings.padics
>>> R1 = Qp(Integer(2))['x']; (x,) = R1._first_ngens(1)
>>> F1 = R1.quotient_ring(x**Integer(2) + x + Integer(1))
>>> R2 = F1['x']; (x,) = R2._first_ngens(1)
>>> F2 = R2.quotient_ring(x**Integer(2) + x + Integer(1))
>>> F2.is_field()
Traceback (most recent call last):
...
NotImplementedError: cannot rewrite Univariate Quotient Polynomial Ring in
 xbar over 2-adic Field with capped relative precision 20 with modulus
 (1 + O(2^20))*x^2 + (1 + O(2^20))*x + 1 + O(2^20) as an isomorphic ring
>>> F2.is_field(proof = False)
False
```

is_finite()

Return whether or not this quotient ring is finite.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: R.quo(1).is_finite()
True
sage: R.quo(x^3 - 2).is_finite()
False
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> R.quo(Integer(1)).is_finite()
True
>>> R.quo(x**Integer(3) - Integer(2)).is_finite()
False
```

```
sage: R.<x> = GF(9, 'a')[]                                               #␣
↪needs sage.rings.finite_rings
sage: R.quo(2*x^3 + x + 1).is_finite()                                   #␣
↪needs sage.rings.finite_rings
True
sage: R.quo(2).is_finite()                                               #␣
↪needs sage.rings.finite_rings
True
```

```
>>> from sage.all import *
>>> R = GF(Integer(9), 'a')['x']; (x,) = R._first_ngens(1)# needs sage.rings.
↪finite_rings
>>> R.quo(Integer(2)*x**Integer(3) + x + Integer(1)).is_finite()          ␣
↪                         # needs sage.rings.finite_rings
True
>>> R.quo(Integer(2)).is_finite()                                         ␣
↪     # needs sage.rings.finite_rings
True
```

```
sage: P.<v> = GF(2)[]
sage: P.quotient(v^2 - v).is_finite()
True
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['v']; (v,) = P._first_ngens(1)
>>> P.quotient(v**Integer(2) - v).is_finite()
True
```

is_integral_domain(*proof=True*)

Return whether or not this quotient ring is an integral domain.

EXAMPLES:

```
sage: R.<z> = PolynomialRing(ZZ)

sage: S = R.quotient(z^2 - z)
sage: S.is_integral_domain()
False
sage: T = R.quotient(z^2 + 1)
sage: T.is_integral_domain()
True
sage: U = R.quotient(-1)
sage: U.is_integral_domain()
False

sage: # needs sage.libs.singular
sage: R2.<y> = PolynomialRing(R)
sage: S2 = R2.quotient(z^2 - y^3)
sage: S2.is_integral_domain()
True
sage: S3 = R2.quotient(z^2 - 2*y*z + y^2)
sage: S3.is_integral_domain()
```

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```
False

sage: R.<z> = PolynomialRing(ZZ.quotient(4))
sage: S = R.quotient(z - 1)
sage: S.is_integral_domain()
False
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('z',)); (z,) = R._first_ngens(1)

>>> S = R.quotient(z**Integer(2) - z)
>>> S.is_integral_domain()
False
>>> T = R.quotient(z**Integer(2) + Integer(1))
>>> T.is_integral_domain()
True
>>> U = R.quotient(-Integer(1))
>>> U.is_integral_domain()
False

>>> # needs sage.libs.singular
>>> R2 = PolynomialRing(R, names=('y',)); (y,) = R2._first_ngens(1)
>>> S2 = R2.quotient(z**Integer(2) - y**Integer(3))
>>> S2.is_integral_domain()
True
>>> S3 = R2.quotient(z**Integer(2) - Integer(2)*y*z + y**Integer(2))
>>> S3.is_integral_domain()
False

>>> R = PolynomialRing(ZZ.quotient(Integer(4)), names=('z',)); (z,) = R._
↪first_ngens(1)
>>> S = R.quotient(z - Integer(1))
>>> S.is_integral_domain()
False
```

krull_dimension()

Return the Krull dimension.

This is the Krull dimension of the base ring, unless the quotient is zero.

EXAMPLES:

```
sage: x = polygen(ZZ, 'x')
sage: R = PolynomialRing(ZZ, 'x').quotient(x**6 - 1)
sage: R.krull_dimension()
1
sage: R = PolynomialRing(ZZ, 'x').quotient(1)
sage: R.krull_dimension()
-1
```

```
>>> from sage.all import *
>>> x = polygen(ZZ, 'x')
```

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```
>>> R = PolynomialRing(ZZ, 'x').quotient(x**Integer(6) - Integer(1))
>>> R.krull_dimension()
1
>>> R = PolynomialRing(ZZ, 'x').quotient(Integer(1))
>>> R.krull_dimension()
-1
```

lift(*x*)

Return an element of the ambient ring mapping to the given argument.

EXAMPLES:

```
sage: P.<x> = QQ[]
sage: Q = P.quotient(x^2 + 2)
sage: Q.lift(Q.0^3)
-2*x
sage: Q(-2*x)
-2*xbar
sage: Q.0^3
-2*xbar
```

```
>>> from sage.all import *
>>> P = QQ['x']; (x,) = P._first_ngens(1)
>>> Q = P.quotient(x**Integer(2) + Integer(2))
>>> Q.lift(Q.gen(0)**Integer(3))
-2*x
>>> Q(-Integer(2)*x)
-2*xbar
>>> Q.gen(0)**Integer(3)
-2*xbar
```

modulus()

Return the polynomial modulus of this quotient ring.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(GF(3))
sage: S = R.quotient(x^2 - 2)
sage: S.modulus()
x^2 + 1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(GF(Integer(3)), names=('x',)); (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(2) - Integer(2))
>>> S.modulus()
x^2 + 1
```

ngens()

Return the number of generators of this quotient ring over the base ring. This function always returns 1.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: S.<y> = PolynomialRing(R)
sage: T.<z> = S.quotient(y + x)
sage: T
Univariate Quotient Polynomial Ring in z over
 Univariate Polynomial Ring in x over Rational Field with modulus y + x
sage: T.ngens()
1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> S = PolynomialRing(R, names=('y',)); (y,) = S._first_ngens(1)
>>> T = S.quotient(y + x, names=('z',)); (z,) = T._first_ngens(1)
>>> T
Univariate Quotient Polynomial Ring in z over
 Univariate Polynomial Ring in x over Rational Field with modulus y + x
>>> T.ngens()
1
```

number_field()

Return the number field isomorphic to this quotient polynomial ring, if possible.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x> = PolynomialRing(QQ)
sage: S.<alpha> = R.quotient(x^29 - 17*x - 1)
sage: K = S.number_field(); K
Number Field in alpha with defining polynomial x^29 - 17*x - 1
sage: alpha = K.gen()
sage: alpha^29
17*alpha + 1
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(29) - Integer(17)*x - Integer(1), names=('alpha
→',)); (alpha,) = S._first_ngens(1)
>>> K = S.number_field(); K
Number Field in alpha with defining polynomial x^29 - 17*x - 1
>>> alpha = K.gen()
>>> alpha**Integer(29)
17*alpha + 1
```

order()

Return the number of elements of this quotient ring.

`order` is an alias of `cardinality`.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: R.quo(1).cardinality()
1
```

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```
sage: R.quo(x^3 - 2).cardinality()
+Infinity

sage: R.quo(1).order()
1
sage: R.quo(x^3 - 2).order()
+Infinity
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> R.quo(Integer(1)).cardinality()
1
>>> R.quo(x**Integer(3) - Integer(2)).cardinality()
+Infinity

>>> R.quo(Integer(1)).order()
1
>>> R.quo(x**Integer(3) - Integer(2)).order()
+Infinity
```

```
sage: # needs sage.rings.finite_rings
sage: R.<x> = GF(9, 'a')[]
sage: R.quo(2*x^3 + x + 1).cardinality()
729
sage: GF(9, 'a').extension(2*x^3 + x + 1).cardinality()
729
sage: R.quo(2).cardinality()
1
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> R = GF(Integer(9), 'a')['x']; (x,) = R._first_ngens(1)
>>> R.quo(Integer(2)*x**Integer(3) + x + Integer(1)).cardinality()
729
>>> GF(Integer(9), 'a').extension(Integer(2)*x**Integer(3) + x + Integer(1)).
↪cardinality()
729
>>> R.quo(Integer(2)).cardinality()
1
```

polynomial_ring()

Return the polynomial ring of which this ring is the quotient.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: S = R.quotient(x^2 - 2)
sage: S.polynomial_ring()
Univariate Polynomial Ring in x over Rational Field
```

```
>>> from sage.all import *
```

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```
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(2) - Integer(2))
>>> S.polynomial_ring()
Univariate Polynomial Ring in x over Rational Field
```

random_element (*degree=None*, **args*, ***kwds*)

Return a random element of this quotient ring.

INPUT:

- `degree` – (optional) argument; either an integer for fixing the degree, or a tuple of the minimum and maximum degree. By default the degree is n - 1 with n the degree of the polynomial ring. Note that the degree of the polynomial is fixed before the modulo calculation. So when $degree$ is bigger than the degree of the polynomial ring, the degree of the returned polynomial would be lower than $degree$.
- `*args`, `**kwds` – arguments for randomization that are passed on to the `random_element` method of the polynomial ring, and from there to the base ring

OUTPUT: element of this quotient ring

EXAMPLES:

```
sage: # needs sage.modules sage.rings.finite_rings
sage: F1.<a> = GF(2^7)
sage: P1.<x> = F1[]
sage: F2 = F1.extension(x^2 + x + 1, 'u')
sage: F2.random_element().parent() is F2
True
```

```
>>> from sage.all import *
>>> # needs sage.modules sage.rings.finite_rings
>>> F1 = GF(Integer(2)**Integer(7), names=('a',)); (a,) = F1._first_ngens(1)
>>> P1 = F1['x']; (x,) = P1._first_ngens(1)
>>> F2 = F1.extension(x**Integer(2) + x + Integer(1), 'u')
>>> F2.random_element().parent() is F2
True
```

retract (*x*)

Return the coercion of x into this polynomial quotient ring.

The rings that coerce into the quotient ring canonically are:

- this ring
- any canonically isomorphic ring
- anything that coerces into the ring of which this is the quotient

selmer_generators (*S*, *m*, *proof=True*)

If `self` is an étale algebra D over a number field K (i.e. a quotient of $K[x]$ by a squarefree polynomial) and S is a finite set of places of K, compute the Selmer group $D(S, m)$. This is the subgroup of $D^*/(D^*)^m$ consisting of elements a such that $D(\sqrt[m]{a})/D$ is unramified at all primes of D lying above a place outside of S.

INPUT:

- `S` – set of primes of the coefficient ring (which is a number field)

- `m` – positive integer
- `proof` – if `False`, assume the GRH in computing the class group

OUTPUT:

A list of generators of $D(S, m)$.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: K.<a> = QuadraticField(-5)
sage: R.<x> = K[]
sage: D.<T> = R.quotient(x)
sage: D.selmer_generators((), 2)
[-1, 2]
sage: D.selmer_generators([K.ideal(2, -a + 1)], 2)
[2, -1]
sage: D.selmer_generators([K.ideal(2, -a + 1), K.ideal(3, a + 1)], 2)
[2, a + 1, -1]
sage: D.selmer_generators((K.ideal(2, -a + 1), K.ideal(3, a + 1)), 4)
[2, a + 1, -1]
sage: D.selmer_generators([K.ideal(2, -a + 1)], 3)
[2]
sage: D.selmer_generators([K.ideal(2, -a + 1), K.ideal(3, a + 1)], 3)
[2, a + 1]
sage: D.selmer_generators([K.ideal(2, -a + 1),
....:                       K.ideal(3, a + 1),
....:                       K.ideal(a)], 3)
[2, a + 1, a]
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> K = QuadraticField(-Integer(5), names=('a',)); (a,) = K._first_ngens(1)
>>> R = K['x']; (x,) = R._first_ngens(1)
>>> D = R.quotient(x, names=('T',)); (T,) = D._first_ngens(1)
>>> D.selmer_generators((), Integer(2))
[-1, 2]
>>> D.selmer_generators([K.ideal(Integer(2), -a + Integer(1))], Integer(2))
[2, -1]
>>> D.selmer_generators([K.ideal(Integer(2), -a + Integer(1)), K.
↪ideal(Integer(3), a + Integer(1))], Integer(2))
[2, a + 1, -1]
>>> D.selmer_generators((K.ideal(Integer(2), -a + Integer(1)), K.
↪ideal(Integer(3), a + Integer(1))), Integer(4))
[2, a + 1, -1]
>>> D.selmer_generators([K.ideal(Integer(2), -a + Integer(1))], Integer(3))
[2]
>>> D.selmer_generators([K.ideal(Integer(2), -a + Integer(1)), K.
↪ideal(Integer(3), a + Integer(1))], Integer(3))
[2, a + 1]
>>> D.selmer_generators([K.ideal(Integer(2), -a + Integer(1)),
...                       K.ideal(Integer(3), a + Integer(1)),
...                       K.ideal(a)], Integer(3))
[2, a + 1, a]
```

selmer_group(*S*, *m*, *proof=True*)

If `self` is an étale algebra D over a number field K (i.e. a quotient of $K[x]$ by a squarefree polynomial) and S is a finite set of places of K, compute the Selmer group $D(S,m)$. This is the subgroup of $D^*/(D^*)^m$ consisting of elements a such that $D(\sqrt[m]{a})/D$ is unramified at all primes of D lying above a place outside of S.

INPUT:

- `S` – set of primes of the coefficient ring (which is a number field)
- `m` – positive integer
- `proof` – if `False`, assume the GRH in computing the class group

OUTPUT:

A list of generators of $D(S,m)$.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: K.<a> = QuadraticField(-5)
sage: R.<x> = K[]
sage: D.<T> = R.quotient(x)
sage: D.selmer_generators((), 2)
[-1, 2]
sage: D.selmer_generators([K.ideal(2, -a + 1)], 2)
[2, -1]
sage: D.selmer_generators([K.ideal(2, -a + 1), K.ideal(3, a + 1)], 2)
[2, a + 1, -1]
sage: D.selmer_generators((K.ideal(2, -a + 1), K.ideal(3, a + 1)), 4)
[2, a + 1, -1]
sage: D.selmer_generators([K.ideal(2, -a + 1)], 3)
[2]
sage: D.selmer_generators([K.ideal(2, -a + 1), K.ideal(3, a + 1)], 3)
[2, a + 1]
sage: D.selmer_generators([K.ideal(2, -a + 1),
....:                      K.ideal(3, a + 1),
....:                      K.ideal(a)], 3)
[2, a + 1, a]
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> K = QuadraticField(-Integer(5), names=('a',)); (a,) = K._first_ngens(1)
>>> R = K['x']; (x,) = R._first_ngens(1)
>>> D = R.quotient(x, names=('T',)); (T,) = D._first_ngens(1)
>>> D.selmer_generators((), Integer(2))
[-1, 2]
>>> D.selmer_generators([K.ideal(Integer(2), -a + Integer(1))], Integer(2))
[2, -1]
>>> D.selmer_generators([K.ideal(Integer(2), -a + Integer(1)), K.
↪ideal(Integer(3), a + Integer(1))], Integer(2))
[2, a + 1, -1]
>>> D.selmer_generators((K.ideal(Integer(2), -a + Integer(1)), K.
↪ideal(Integer(3), a + Integer(1))), Integer(4))
[2, a + 1, -1]
```

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```
>>> D.selmer_generators([K.ideal(Integer(2), -a + Integer(1))], Integer(3))
[2]
>>> D.selmer_generators([K.ideal(Integer(2), -a + Integer(1)), K.
↪ideal(Integer(3), a + Integer(1))], Integer(3))
[2, a + 1]
>>> D.selmer_generators([K.ideal(Integer(2), -a + Integer(1)),
...                      K.ideal(Integer(3), a + Integer(1)),
...                      K.ideal(a)], Integer(3))
[2, a + 1, a]
```

units(*proof=True*)

If this quotient ring is over a number field K, by a polynomial of nonzero discriminant, returns a list of generators of the units.

INPUT:

- `proof` – if `False`, assume the GRH in computing the class group

OUTPUT:

A list of generators of the unit group, in the form `(gen, order)`, where `gen` is a unit of order `order`.

EXAMPLES:

```
sage: K.<a> = QuadraticField(-3)                                          #␣
↪needs sage.rings.number_field
sage: K.unit_group()                                                      #␣
↪needs sage.rings.number_field
Unit group with structure C6 of
 Number Field in a with defining polynomial x^2 + 3 with a = 1.
↪732050807568878?*I

sage: # needs sage.rings.number_field
sage: x = polygen(ZZ, 'x')
sage: K.<a> = QQ['x'].quotient(x^2 + 3)
sage: u = K.units()[0][0]
sage: 2*u - 1 in {a, -a}
True
sage: u^6
1
sage: u^3
-1
sage: 2*u^2 + 1 in {a, -a}
True
sage: x = polygen(ZZ, 'x')
sage: K.<a> = QQ['x'].quotient(x^2 + 5)
sage: K.units(())
[(-1, 2)]
```

```
>>> from sage.all import *
>>> K = QuadraticField(-Integer(3), names=('a',)); (a,) = K._first_ngens(1)#␣
↪needs sage.rings.number_field
>>> K.unit_group()                                                      #␣
↪needs sage.rings.number_field
```

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```
Unit group with structure C6 of
 Number Field in a with defining polynomial x^2 + 3 with a = 1.
↪732050807568878?*I

>>> # needs sage.rings.number_field
>>> x = polygen(ZZ, 'x')
>>> K = QQ['x'].quotient(x**Integer(2) + Integer(3), names=('a',)); (a,) = K._
↪first_ngens(1)
>>> u = K.units()[Integer(0)][Integer(0)]
>>> Integer(2)*u - Integer(1) in {a, -a}
True
>>> u**Integer(6)
1
>>> u**Integer(3)
-1
>>> Integer(2)*u**Integer(2) + Integer(1) in {a, -a}
True
>>> x = polygen(ZZ, 'x')
>>> K = QQ['x'].quotient(x**Integer(2) + Integer(5), names=('a',)); (a,) = K._
↪first_ngens(1)
>>> K.units(())
[(-1, 2)]
```

```
sage: # needs sage.rings.number_field
sage: K.<a> = QuadraticField(-3)
sage: y = polygen(K)
sage: L.<b> = K['y'].quotient(y^3 + 5); L
Univariate Quotient Polynomial Ring in b over Number Field in a
 with defining polynomial x^2 + 3 with a = 1.732050807568878?*I
 with modulus y^3 + 5
sage: [u for u, o in L.units() if o is Infinity]
[(-1/3*a - 1)*b^2 - 4/3*a*b - 4/3*a + 3,
 (-1/3*a - 1)*b^2 + (2/3*a - 2)*b + 13/6*a - 1/2]
sage: L.<b> = K.extension(y^3 + 5)
sage: L.unit_group()
Unit group with structure C6 x Z x Z of
 Number Field in b with defining polynomial x^3 + 5 over its base field
sage: L.unit_group().gens()     # abstract generators
(u0, u1, u2)
sage: L.unit_group().gens_values()[1:]
[(-1/3*a - 1)*b^2 - 4/3*a*b - 4/3*a + 3,
 (-1/3*a - 1)*b^2 + (2/3*a - 2)*b + 13/6*a - 1/2]
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> K = QuadraticField(-Integer(3), names=('a',)); (a,) = K._first_ngens(1)
>>> y = polygen(K)
>>> L = K['y'].quotient(y**Integer(3) + Integer(5), names=('b',)); (b,) = L._
↪first_ngens(1); L
Univariate Quotient Polynomial Ring in b over Number Field in a
 with defining polynomial x^2 + 3 with a = 1.732050807568878?*I
```

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```
 with modulus y^3 + 5
>>> [u for u, o in L.units() if o is Infinity]
[(-1/3*a - 1)*b^2 - 4/3*a*b - 4/3*a + 3,
 (-1/3*a - 1)*b^2 + (2/3*a - 2)*b + 13/6*a - 1/2]
>>> L = K.extension(y**Integer(3) + Integer(5), names=('b',)); (b,) = L._
↪first_ngens(1)
>>> L.unit_group()
Unit group with structure C6 x Z x Z of
 Number Field in b with defining polynomial x^3 + 5 over its base field
>>> L.unit_group().gens()     # abstract generators
(u0, u1, u2)
>>> L.unit_group().gens_values()[Integer(1):]
[(-1/3*a - 1)*b^2 - 4/3*a*b - 4/3*a + 3,
 (-1/3*a - 1)*b^2 + (2/3*a - 2)*b + 13/6*a - 1/2]
```

Note that all the returned values live where we expect them to:

```
sage: # needs sage.rings.number_field
sage: L.<b> = K['y'].quotient(y^3 + 5)
sage: U = L.units()
sage: type(U[0][0])
<class 'sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_
↪field_with_category.element_class'>
sage: type(U[0][1])
<class 'sage.rings.integer.Integer'>
sage: type(U[1][1])
<class 'sage.rings.infinity.PlusInfinity'>
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> L = K['y'].quotient(y**Integer(3) + Integer(5), names=('b',)); (b,) = L._
↪first_ngens(1)
>>> U = L.units()
>>> type(U[Integer(0)][Integer(0)])
<class 'sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_
↪field_with_category.element_class'>
>>> type(U[Integer(0)][Integer(1)])
<class 'sage.rings.integer.Integer'>
>>> type(U[Integer(1)][Integer(1)])
<class 'sage.rings.infinity.PlusInfinity'>
```

sage.rings.polynomial.polynomial_quotient_ring.**is_PolynomialQuotientRing**(x)

2.1.23 Elements of Quotients of Univariate Polynomial Rings

EXAMPLES: We create a quotient of a univariate polynomial ring over **Z**.

```
sage: R.<x> = ZZ[]
sage: S.<a> = R.quotient(x^3 + 3*x - 1)
sage: 2 * a^3
-6*a + 2
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(3) + Integer(3)*x - Integer(1), names=('a',)); (a,) = S.
↪_first_ngens(1)
>>> Integer(2) * a**Integer(3)
-6*a + 2
```

Next we make a univariate polynomial ring over $\mathbf{Z}[x]/(x^3 + 3x - 1)$.

```
sage: S1.<y> = S[]
```

```
>>> from sage.all import *
>>> S1 = S['y']; (y,) = S1._first_ngens(1)
```

And, we quotient out that by $y^2 + a$.

```
sage: T.<z> = S1.quotient(y^2 + a)
```

```
>>> from sage.all import *
>>> T = S1.quotient(y**Integer(2) + a, names=('z',)); (z,) = T._first_ngens(1)
```

In the quotient z^2 is $-a$.

```
sage: z^2
-a
```

```
>>> from sage.all import *
>>> z**Integer(2)
-a
```

And since $a^3 = -3x + 1$, we have:

```
sage: z^6
3*a - 1
```

```
>>> from sage.all import *
>>> z**Integer(6)
3*a - 1
```

```
sage: R.<x> = PolynomialRing(Integers(9))
sage: S.<a> = R.quotient(x^4 + 2*x^3 + x + 2)
sage: a^100
7*a^3 + 8*a + 7
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(9)), names=('x',)); (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(4) + Integer(2)*x**Integer(3) + x + Integer(2), names=(
↪'a',)); (a,) = S._first_ngens(1)
>>> a**Integer(100)
7*a^3 + 8*a + 7
```

```
sage: R.<x> = PolynomialRing(QQ)
sage: S.<a> = R.quotient(x^3 - 2)
sage: a
a
sage: a^3
2
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(3) - Integer(2), names=('a',)); (a,) = S._first_ngens(1)
>>> a
a
>>> a**Integer(3)
2
```

For the purposes of comparison in Sage the quotient element a^3 is equal to x^3. This is because when the comparison is performed, the right element is coerced into the parent of the left element, and x^3 coerces to a^3.

```
sage: a == x
True
sage: a^3 == x^3
True
sage: x^3
x^3
sage: S(x^3)
2
```

```
>>> from sage.all import *
>>> a == x
True
>>> a**Integer(3) == x**Integer(3)
True
>>> x**Integer(3)
x^3
>>> S(x**Integer(3))
2
```

AUTHORS:

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class sage.rings.polynomial.polynomial_quotient_ring_element.**PolynomialQuotientRingElement**(*parent*, *polynomial*, *check=True*)

Bases: *Polynomial_singular_repr*, CommutativeRingElement

Element of a quotient of a polynomial ring.

EXAMPLES:

```
sage: P.<x> = QQ[]
sage: Q.<xi> = P.quo([(x^2 + 1)])
sage: xi^2
-1
sage: singular(xi)                                                    #␣
↪needs sage.libs.singular
xi
sage: (singular(xi)*singular(xi)).NF('std(0)')                        #␣
↪needs sage.libs.singular
-1
```

```
>>> from sage.all import *
>>> P = QQ['x']; (x,) = P._first_ngens(1)
>>> Q = P.quo([(x**Integer(2) + Integer(1))], names=('xi',)); (xi,) = Q._first_
↪ngens(1)
>>> xi**Integer(2)
-1
>>> singular(xi)                                                      #␣
↪needs sage.libs.singular
xi
>>> (singular(xi)*singular(xi)).NF('std(0)')                          #␣
↪needs sage.libs.singular
-1
```

charpoly(*var*)

The characteristic polynomial of this element, which is by definition the characteristic polynomial of right multiplication by this element.

INPUT:

- `var` – string; the variable name

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: S.<a> = R.quo(x^3 -389*x^2 + 2*x - 5)
sage: a.charpoly('X')                                                 #␣
↪needs sage.modules
X^3 - 389*X^2 + 2*X - 5
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> S = R.quo(x**Integer(3) -Integer(389)*x**Integer(2) + Integer(2)*x -␣
↪Integer(5), names=('a',)); (a,) = S._first_ngens(1)
>>> a.charpoly('X')                                                   #␣
↪needs sage.modules
X^3 - 389*X^2 + 2*X - 5
```

fcp(*var='x'*)

Return the factorization of the characteristic polynomial of this element.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: S.<a> = R.quotient(x^3 -389*x^2 + 2*x - 5)
sage: a.fcp('x')                                                        #␣
↪needs sage.modules
x^3 - 389*x^2 + 2*x - 5
sage: S(1).fcp('y')                                                     #␣
↪needs sage.modules
(y - 1)^3
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(3) -Integer(389)*x**Integer(2) + Integer(2)*x -␣
↪Integer(5), names=('a',)); (a,) = S._first_ngens(1)
>>> a.fcp('x')                                                        #␣
↪needs sage.modules
x^3 - 389*x^2 + 2*x - 5
>>> S(Integer(1)).fcp('y')                                            ␣
↪     # needs sage.modules
(y - 1)^3
```

field_extension(*names*)

Given a polynomial with base ring a quotient ring, return a 3-tuple: a number field defined by the same polynomial, a homomorphism from its parent to the number field sending the generators to one another, and the inverse isomorphism.

INPUT:

- `names` – name of generator of output field

OUTPUT:

- field
- homomorphism from `self` to field
- homomorphism from field to `self`

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x> = PolynomialRing(QQ)
sage: S.<alpha> = R.quotient(x^3 - 2)
sage: F.<a>, f, g = alpha.field_extension()
sage: F
Number Field in a with defining polynomial x^3 - 2
sage: a = F.gen()
sage: f(alpha)
a
sage: g(a)
alpha
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(3) - Integer(2), names=('alpha',)); (alpha,) =␣
```

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```
↪S._first_ngens(1)
>>> F, f, g  = alpha.field_extension(names=('a',)); (a,) = F._first_ngens(1)
>>> F
Number Field in a with defining polynomial x^3 - 2
>>> a = F.gen()
>>> f(alpha)
a
>>> g(a)
alpha
```

Over a finite field, the corresponding field extension is not a number field:

```
sage: # needs sage.rings.finite_rings
sage: R.<x> = GF(25,'b')['x']
sage: S.<a> = R.quo(x^3 + 2*x + 1)
sage: F.<b>, g, h = a.field_extension()
sage: h(b^2 + 3)
a^2 + 3
sage: g(x^2 + 2)
b^2 + 2
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> R = GF(Integer(25),'b')['x']; (x,) = R._first_ngens(1)
>>> S = R.quo(x**Integer(3) + Integer(2)*x + Integer(1), names=('a',)); (a,)␣
↪= S._first_ngens(1)
>>> F, g, h  = a.field_extension(names=('b',)); (b,) = F._first_ngens(1)
>>> h(b**Integer(2) + Integer(3))
a^2 + 3
>>> g(x**Integer(2) + Integer(2))
b^2 + 2
```

We do an example involving a relative number field:

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQ['x']
sage: K.<a> = NumberField(x^3 - 2)
sage: S.<X> = K['X']
sage: Q.<b> = S.quo(X^3 + 2*X + 1)
sage: F, g, h = b.field_extension('c')
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> K = NumberField(x**Integer(3) - Integer(2), names=('a',)); (a,) = K._
↪first_ngens(1)
>>> S = K['X']; (X,) = S._first_ngens(1)
>>> Q = S.quo(X**Integer(3) + Integer(2)*X + Integer(1), names=('b',)); (b,)␣
↪= Q._first_ngens(1)
>>> F, g, h = b.field_extension('c')
```

Another more awkward example:

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQ['x']
sage: K.<a> = NumberField(x^3 - 2)
sage: S.<X> = K['X']
sage: f = (X+a)^3 + 2*(X+a) + 1
sage: f
X^3 + 3*a*X^2 + (3*a^2 + 2)*X + 2*a + 3
sage: Q.<z> = S.quo(f)
sage: F.<w>, g, h = z.field_extension()
sage: c = g(z)
sage: f(c)
0
sage: h(g(z))
z
sage: g(h(w))
w
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> K = NumberField(x**Integer(3) - Integer(2), names=('a',)); (a,) = K._
→first_ngens(1)
>>> S = K['X']; (X,) = S._first_ngens(1)
>>> f = (X+a)**Integer(3) + Integer(2)*(X+a) + Integer(1)
>>> f
X^3 + 3*a*X^2 + (3*a^2 + 2)*X + 2*a + 3
>>> Q = S.quo(f, names=('z',)); (z,) = Q._first_ngens(1)
>>> F, g, h  = z.field_extension(names=('w',)); (w,) = F._first_ngens(1)
>>> c = g(z)
>>> f(c)
0
>>> h(g(z))
z
>>> g(h(w))
w
```

AUTHORS:

- Craig Citro (2006-08-06)
- William Stein (2006-08-06)

is_unit()

Return `True` if `self` is invertible.

> **Warning**
>
> Only implemented when the base ring is a field.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: S.<y> = R.quotient(x^2 + 2*x + 1)
```

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```
sage: (2*y).is_unit()
True
sage: (y + 1).is_unit()
False
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(2) + Integer(2)*x + Integer(1), names=('y',));␣
↪(y,) = S._first_ngens(1)
>>> (Integer(2)*y).is_unit()
True
>>> (y + Integer(1)).is_unit()
False
```

lift()

Return lift of this polynomial quotient ring element to the unique equivalent polynomial of degree less than the modulus.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: S.<a> = R.quotient(x^3 - 2)
sage: b = a^2 - 3
sage: b
a^2 - 3
sage: b.lift()
x^2 - 3
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(3) - Integer(2), names=('a',)); (a,) = S._first_
↪ngens(1)
>>> b = a**Integer(2) - Integer(3)
>>> b
a^2 - 3
>>> b.lift()
x^2 - 3
```

list(*copy=True*)

Return list of the elements of `self`, of length the same as the degree of the quotient polynomial ring.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: S.<a> = R.quotient(x^3 + 2*x - 5)
sage: a^10
-134*a^2 - 35*a + 300
sage: (a^10).list()
[300, -35, -134]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
```

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```
>>> S = R.quotient(x**Integer(3) + Integer(2)*x - Integer(5), names=('a',));␣
↪(a,) = S._first_ngens(1)
>>> a**Integer(10)
-134*a^2 - 35*a + 300
>>> (a**Integer(10)).list()
[300, -35, -134]
```

matrix()

The matrix of right multiplication by this element on the power basis for the quotient ring.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: S.<a> = R.quotient(x^3 + 2*x - 5)
sage: a.matrix()                                                        #␣
↪needs sage.modules
[ 0  1  0]
[ 0  0  1]
[ 5 -2  0]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(3) + Integer(2)*x - Integer(5), names=('a',));␣
↪(a,) = S._first_ngens(1)
>>> a.matrix()                                                        #␣
↪needs sage.modules
[ 0  1  0]
[ 0  0  1]
[ 5 -2  0]
```

minpoly()

The minimal polynomial of this element, which is by definition the minimal polynomial of the *matrix()* of this element.

ALGORITHM: Use *minpoly_mod()* if possible, otherwise compute the minimal polynomial of the *matrix()*.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: S.<a> = R.quotient(x^3 + 2*x - 5)
sage: (a + 123).minpoly()                                               #␣
↪needs sage.modules
x^3 - 369*x^2 + 45389*x - 1861118
sage: (a + 123).matrix().minpoly()                                      #␣
↪needs sage.modules
x^3 - 369*x^2 + 45389*x - 1861118
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(3) + Integer(2)*x - Integer(5), names=('a',));␣
↪(a,) = S._first_ngens(1)
>>> (a + Integer(123)).minpoly()                                        ␣
```

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```
↪     # needs sage.modules
x^3 - 369*x^2 + 45389*x - 1861118
>>> (a + Integer(123)).matrix().minpoly()
↪     # needs sage.modules
x^3 - 369*x^2 + 45389*x - 1861118
```

One useful application of this function is to compute a minimal polynomial of a finite-field element over an intermediate extension, rather than the absolute minimal polynomial over the prime field:

```
sage: # needs sage.rings.finite_rings
sage: F2.<i> = GF((431,2), modulus=[1,0,1])
sage: F6.<u> = F2.extension(3)
sage: (u + 1).minpoly()                                                     #
↪needs sage.modules
x^6 + 425*x^5 + 19*x^4 + 125*x^3 + 189*x^2 + 239*x + 302
sage: ext = F6.over(F2)                                                     #
↪needs sage.modules
sage: ext(u + 1).minpoly()   # indirect doctest                             #
↪needs sage.modules # random
x^3 + (396*i + 428)*x^2 + (80*i + 39)*x + 9*i + 178
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> F2 = GF((Integer(431),Integer(2)), modulus=[Integer(1),Integer(0),
↪Integer(1)], names=('i',)); (i,) = F2._first_ngens(1)
>>> F6 = F2.extension(Integer(3), names=('u',)); (u,) = F6._first_ngens(1)
>>> (u + Integer(1)).minpoly()
↪     # needs sage.modules
x^6 + 425*x^5 + 19*x^4 + 125*x^3 + 189*x^2 + 239*x + 302
>>> ext = F6.over(F2)                                                     #
↪needs sage.modules
>>> ext(u + Integer(1)).minpoly()  # indirect doctest
↪     # needs sage.modules # random
x^3 + (396*i + 428)*x^2 + (80*i + 39)*x + 9*i + 178
```

norm()

The norm of this element, which is the determinant of the matrix of right multiplication by this element.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: S.<a> = R.quotient(x^3 - 389*x^2 + 2*x - 5)
sage: a.norm()                                                              #
↪needs sage.modules
5
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(3) - Integer(389)*x**Integer(2) + Integer(2)*x -
↪ Integer(5), names=('a',)); (a,) = S._first_ngens(1)
>>> a.norm()                                                              #
↪needs sage.modules
```

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```
5
```

rational_reconstruction(**args*, ***kwargs*)

Compute a rational reconstruction of this polynomial quotient ring element to its cover ring.

This method is a thin convenience wrapper around *Polynomial.rational_reconstruction()*.

EXAMPLES:

```
sage: # needs sage.rings.finite_rings
sage: R.<x> = GF(65537)[]
sage: m = (x^11 + 25345*x^10 + 10956*x^9 + 13873*x^8 + 23962*x^7
....:      + 17496*x^6 + 30348*x^5 + 7440*x^4 + 65438*x^3 + 7676*x^2
....:      + 54266*x + 47805)
sage: f = (20437*x^10 + 62630*x^9 + 63241*x^8 + 12820*x^7 + 42171*x^6
....:      + 63091*x^5 + 15288*x^4 + 32516*x^3 + 2181*x^2 + 45236*x + 2447)
sage: f_mod_m = R.quotient(m)(f)
sage: f_mod_m.rational_reconstruction()
(51388*x^5 + 29141*x^4 + 59341*x^3 + 7034*x^2 + 14152*x + 23746,
 x^5 + 15208*x^4 + 19504*x^3 + 20457*x^2 + 11180*x + 28352)
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> R = GF(Integer(65537))['x']; (x,) = R._first_ngens(1)
>>> m = (x**Integer(11) + Integer(25345)*x**Integer(10) +␣
↪Integer(10956)*x**Integer(9) + Integer(13873)*x**Integer(8) +␣
↪Integer(23962)*x**Integer(7)
...      + Integer(17496)*x**Integer(6) + Integer(30348)*x**Integer(5) +␣
↪Integer(7440)*x**Integer(4) + Integer(65438)*x**Integer(3) +␣
↪Integer(7676)*x**Integer(2)
...      + Integer(54266)*x + Integer(47805))
>>> f = (Integer(20437)*x**Integer(10) + Integer(62630)*x**Integer(9) +␣
↪Integer(63241)*x**Integer(8) + Integer(12820)*x**Integer(7) +␣
↪Integer(42171)*x**Integer(6)
...      + Integer(63091)*x**Integer(5) + Integer(15288)*x**Integer(4) +␣
↪Integer(32516)*x**Integer(3) + Integer(2181)*x**Integer(2) +␣
↪Integer(45236)*x + Integer(2447))
>>> f_mod_m = R.quotient(m)(f)
>>> f_mod_m.rational_reconstruction()
(51388*x^5 + 29141*x^4 + 59341*x^3 + 7034*x^2 + 14152*x + 23746,
 x^5 + 15208*x^4 + 19504*x^3 + 20457*x^2 + 11180*x + 28352)
```

trace()

The trace of this element, which is the trace of the matrix of right multiplication by this element.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: S.<a> = R.quotient(x^3 - 389*x^2 + 2*x - 5)
sage: a.trace()                                                          #␣
↪needs sage.modules
389
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> S = R.quotient(x**Integer(3) - Integer(389)*x**Integer(2) + Integer(2)*x -
↪ Integer(5), names=('a',)); (a,) = S._first_ngens(1)
>>> a.trace()                                                             #␣
↪needs sage.modules
389
```

2.1.24 Polynomial Compilers

AUTHORS:

- Tom Boothby, initial design & implementation
- Robert Bradshaw, bug fixes / suggested & assisted with significant design improvements

class sage.rings.polynomial.polynomial_compiled.**CompiledPolynomialFunction**

Bases: object

Build a reasonably optimized directed acyclic graph representation for a given polynomial. A CompiledPolynomialFunction is callable from python, though it is a little faster to call the eval function from pyrex.

This class is not intended to be called by a user, rather, it is intended to improve the performance of immutable polynomial objects.

Todo

- Recursive calling
- Faster casting of coefficients / argument
- Multivariate polynomials
- Cython implementation of Pippenger's Algorithm that doesn't depend heavily upon dicts.
- Computation of parameter sequence suggested by Pippenger
- Univariate exponentiation can use Brauer's method to improve extremely sparse polynomials of very high degree

class sage.rings.polynomial.polynomial_compiled.**abc_pd**

Bases: *binary_pd*

class sage.rings.polynomial.polynomial_compiled.**add_pd**

Bases: *binary_pd*

class sage.rings.polynomial.polynomial_compiled.**binary_pd**

Bases: *generic_pd*

class sage.rings.polynomial.polynomial_compiled.**coeff_pd**

Bases: *generic_pd*

class sage.rings.polynomial.polynomial_compiled.**dummy_pd**

Bases: *generic_pd*

class sage.rings.polynomial.polynomial_compiled.**generic_pd**

Bases: object

class sage.rings.polynomial.polynomial_compiled.**mul_pd**

Bases: *binary_pd*

class sage.rings.polynomial.polynomial_compiled.**pow_pd**

Bases: *unary_pd*

class sage.rings.polynomial.polynomial_compiled.**sqr_pd**

Bases: *unary_pd*

class sage.rings.polynomial.polynomial_compiled.**unary_pd**

Bases: *generic_pd*

class sage.rings.polynomial.polynomial_compiled.**univar_pd**

Bases: *generic_pd*

class sage.rings.polynomial.polynomial_compiled.**var_pd**

Bases: *generic_pd*

2.1.25 Polynomial multiplication by Kronecker substitution

2.1.26 Integer-valued polynomial rings

AUTHORS:

- Frédéric Chapoton (2023-03): Initial version

class sage.rings.polynomial.integer_valued_polynomials.**IntegerValuedPolynomialRing**(R)

Bases: UniqueRepresentation, Parent

The integer-valued polynomial ring over a base ring R.

Integer-valued polynomial rings are commutative and associative algebras, with a basis indexed by nonnegative integers.

There are two natural bases, made of the sequence $\binom{x}{n}$ for $n \geq 0$ (the *binomial basis*) and of the other sequence $\binom{x+n}{n}$ for $n \geq 0$ (the *shifted basis*).

These two bases are available as follows:

```
sage: B = IntegerValuedPolynomialRing(QQ).Binomial()
sage: S = IntegerValuedPolynomialRing(QQ).Shifted()
```

```
>>> from sage.all import *
>>> B = IntegerValuedPolynomialRing(QQ).Binomial()
>>> S = IntegerValuedPolynomialRing(QQ).Shifted()
```

or by using the shortcuts:

```
sage: B = IntegerValuedPolynomialRing(QQ).B()
sage: S = IntegerValuedPolynomialRing(QQ).S()
```

```
>>> from sage.all import *
>>> B = IntegerValuedPolynomialRing(QQ).B()
>>> S = IntegerValuedPolynomialRing(QQ).S()
```

There is a conversion formula between the two bases:

$$\binom{x}{i} = \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \binom{x+k}{k}$$

with inverse:

$$\binom{x+i}{i} = \sum_{k=0}^{i} \binom{i}{k} \binom{x}{k}.$$

REFERENCES:

- Wikipedia article Integer-valued polynomial

B

alias of *Binomial*

class Bases(*parent_with_realization*)

Bases: `Category_realization_of_parent`

class ElementMethods

Bases: `object`

content()

Return the content of `self`.

This is the gcd of the coefficients.

EXAMPLES:

```
sage: F = IntegerValuedPolynomialRing(ZZ).S()
sage: B = F.basis()
sage: (3*B[4]+6*B[7]).content()
3
```

```
>>> from sage.all import *
>>> F = IntegerValuedPolynomialRing(ZZ).S()
>>> B = F.basis()
>>> (Integer(3)*B[Integer(4)]+Integer(6)*B[Integer(7)]).content()
3
```

polynomial()

Convert to a polynomial in x.

EXAMPLES:

```
sage: F = IntegerValuedPolynomialRing(ZZ).S()
sage: B = F.gen()
sage: (B+1).polynomial()
x + 2

sage: F = IntegerValuedPolynomialRing(ZZ).B()
sage: B = F.gen()
sage: (B+1).polynomial()
x + 1
```

```
>>> from sage.all import *
>>> F = IntegerValuedPolynomialRing(ZZ).S()
>>> B = F.gen()
>>> (B+Integer(1)).polynomial()
x + 2
```

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```
>>> F = IntegerValuedPolynomialRing(ZZ).B()
>>> B = F.gen()
>>> (B+Integer(1)).polynomial()
x + 1
```

shift(*j=1*)

Shift all indices by j.

INPUT:

- j – integer (default: 1)

In the binomial basis, the shift by 1 corresponds to a summation operator from 0 to x.

EXAMPLES:

```
sage: F = IntegerValuedPolynomialRing(ZZ).B()
sage: B = F.gen()
sage: (B+1).shift()
B[1] + B[2]
sage: (B+1).shift(3)
B[3] + B[4]
```

```
>>> from sage.all import *
>>> F = IntegerValuedPolynomialRing(ZZ).B()
>>> B = F.gen()
>>> (B+Integer(1)).shift()
B[1] + B[2]
>>> (B+Integer(1)).shift(Integer(3))
B[3] + B[4]
```

sum_of_coefficients()

Return the sum of coefficients.

In the shifted basis, this is the evaluation at $x = 0$.

EXAMPLES:

```
sage: F = IntegerValuedPolynomialRing(ZZ).S()
sage: B = F.basis()
sage: (B[2]*B[4]).sum_of_coefficients()
1
```

```
>>> from sage.all import *
>>> F = IntegerValuedPolynomialRing(ZZ).S()
>>> B = F.basis()
>>> (B[Integer(2)]*B[Integer(4)]).sum_of_coefficients()
1
```

class ParentMethods

Bases: object

algebra_generators()

Return the generators of this algebra.

EXAMPLES:

```
sage: A = IntegerValuedPolynomialRing(ZZ).S(); A
Integer-Valued Polynomial Ring over Integer Ring
in the shifted basis
sage: A.algebra_generators()
Family (S[1],)
```

```
>>> from sage.all import *
>>> A = IntegerValuedPolynomialRing(ZZ).S(); A
Integer-Valued Polynomial Ring over Integer Ring
in the shifted basis
>>> A.algebra_generators()
Family (S[1],)
```

degree_on_basis(*m*)

Return the degree of the basis element indexed by `m`.

EXAMPLES:

```
sage: A = IntegerValuedPolynomialRing(QQ).S()
sage: A.degree_on_basis(4)
4
```

```
>>> from sage.all import *
>>> A = IntegerValuedPolynomialRing(QQ).S()
>>> A.degree_on_basis(Integer(4))
4
```

from_polynomial(*p*)

Convert a polynomial into the ring of integer-valued polynomials.

This raises a `ValueError` if this is not possible.

INPUT:

- `p` – a polynomial in one variable

EXAMPLES:

```
sage: A = IntegerValuedPolynomialRing(ZZ).S()
sage: S = A.basis()
sage: S[5].polynomial()
1/120*x^5 + 1/8*x^4 + 17/24*x^3 + 15/8*x^2 + 137/60*x + 1
sage: A.from_polynomial(_)
S[5]
sage: x = polygen(QQ, 'x')
sage: A.from_polynomial(x)
-S[0] + S[1]

sage: A = IntegerValuedPolynomialRing(ZZ).B()
sage: B = A.basis()
sage: B[5].polynomial()
1/120*x^5 - 1/12*x^4 + 7/24*x^3 - 5/12*x^2 + 1/5*x
sage: A.from_polynomial(_)
B[5]
sage: x = polygen(QQ, 'x')
```

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```
sage: A.from_polynomial(x)
B[1]
```

```
>>> from sage.all import *
>>> A = IntegerValuedPolynomialRing(ZZ).S()
>>> S = A.basis()
>>> S[Integer(5)].polynomial()
1/120*x^5 + 1/8*x^4 + 17/24*x^3 + 15/8*x^2 + 137/60*x + 1
>>> A.from_polynomial(_)
S[5]
>>> x = polygen(QQ, 'x')
>>> A.from_polynomial(x)
-S[0] + S[1]

>>> A = IntegerValuedPolynomialRing(ZZ).B()
>>> B = A.basis()
>>> B[Integer(5)].polynomial()
1/120*x^5 - 1/12*x^4 + 7/24*x^3 - 5/12*x^2 + 1/5*x
>>> A.from_polynomial(_)
B[5]
>>> x = polygen(QQ, 'x')
>>> A.from_polynomial(x)
B[1]
```

gen (*i=0*)

Return the generator of this algebra.

The optional argument is ignored.

EXAMPLES:

```
sage: F = IntegerValuedPolynomialRing(ZZ).B()
sage: F.gen()
B[1]
```

```
>>> from sage.all import *
>>> F = IntegerValuedPolynomialRing(ZZ).B()
>>> F.gen()
B[1]
```

gens ()

Return the generators of this algebra.

EXAMPLES:

```
sage: A = IntegerValuedPolynomialRing(ZZ).S(); A
Integer-Valued Polynomial Ring over Integer Ring
in the shifted basis
sage: A.algebra_generators()
Family (S[1],)
```

```
>>> from sage.all import *
>>> A = IntegerValuedPolynomialRing(ZZ).S(); A
```

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```
Integer-Valued Polynomial Ring over Integer Ring
in the shifted basis
>>> A.algebra_generators()
Family (S[1],)
```

one_basis()

Return the number 0, which index the unit of this algebra.

EXAMPLES:

```
sage: A = IntegerValuedPolynomialRing(QQ).S()
sage: A.one_basis()
0
sage: A.one()
S[0]
```

```
>>> from sage.all import *
>>> A = IntegerValuedPolynomialRing(QQ).S()
>>> A.one_basis()
0
>>> A.one()
S[0]
```

super_categories()

Return the super-categories of `self`.

EXAMPLES:

```
sage: A = IntegerValuedPolynomialRing(QQ); A
Integer-Valued Polynomial Ring over Rational Field
sage: C = A.Bases(); C
Category of bases of Integer-Valued Polynomial Ring
over Rational Field
sage: C.super_categories()
[Category of realizations of Integer-Valued Polynomial Ring
 over Rational Field,
 Join of Category of algebras with basis over Rational Field and
 Category of filtered algebras over Rational Field and
 Category of commutative algebras over Rational Field and
 Category of realizations of unital magmas]
```

```
>>> from sage.all import *
>>> A = IntegerValuedPolynomialRing(QQ); A
Integer-Valued Polynomial Ring over Rational Field
>>> C = A.Bases(); C
Category of bases of Integer-Valued Polynomial Ring
over Rational Field
>>> C.super_categories()
[Category of realizations of Integer-Valued Polynomial Ring
 over Rational Field,
 Join of Category of algebras with basis over Rational Field and
 Category of filtered algebras over Rational Field and
```

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```
Category of commutative algebras over Rational Field and
Category of realizations of unital magmas]
```

class Binomial(A)

Bases: CombinatorialFreeModule, BindableClass

The integer-valued polynomial ring in the binomial basis.

The basis used here is given by $B[i] = \binom{x}{i}$ for $i \in \mathbf{N}$.

Assuming $n_1 \leq n_2$, the product of two monomials $B[n_1] \cdot B[n_2]$ is given by the sum

$$\sum_{k=0}^{n_1} \binom{n_1}{k} \binom{n_1 + n_2 - k}{n_1} B[n_1 + n_2 - k].$$

The product of two monomials is therefore a positive linear combination of monomials.

EXAMPLES:

```
sage: F = IntegerValuedPolynomialRing(QQ).B(); F
Integer-Valued Polynomial Ring over Rational Field
in the binomial basis

sage: F.gen()
B[1]

sage: S = IntegerValuedPolynomialRing(ZZ).B(); S
Integer-Valued Polynomial Ring over Integer Ring
in the binomial basis
sage: S.base_ring()
Integer Ring

sage: G = IntegerValuedPolynomialRing(S).B(); G
Integer-Valued Polynomial Ring over Integer-Valued Polynomial Ring
over Integer Ring in the binomial basis in the binomial basis
sage: G.base_ring()
Integer-Valued Polynomial Ring over Integer Ring
in the binomial basis
```

```
>>> from sage.all import *
>>> F = IntegerValuedPolynomialRing(QQ).B(); F
Integer-Valued Polynomial Ring over Rational Field
in the binomial basis

>>> F.gen()
B[1]

>>> S = IntegerValuedPolynomialRing(ZZ).B(); S
Integer-Valued Polynomial Ring over Integer Ring
in the binomial basis
>>> S.base_ring()
Integer Ring

>>> G = IntegerValuedPolynomialRing(S).B(); G
```

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```
Integer-Valued Polynomial Ring over Integer-Valued Polynomial Ring
over Integer Ring in the binomial basis in the binomial basis
>>> G.base_ring()
Integer-Valued Polynomial Ring over Integer Ring
in the binomial basis
```

Integer-valued polynomial rings commute with their base ring:

```
sage: K = IntegerValuedPolynomialRing(QQ).B()
sage: a = K.gen()
sage: K.is_commutative()
True
sage: L = IntegerValuedPolynomialRing(K).B()
sage: c = L.gen()
sage: L.is_commutative()
True
sage: s = a * c^3; s
B[1]*B[1] + 6*B[1]*B[2] + 6*B[1]*B[3]
sage: parent(s)
Integer-Valued Polynomial Ring over Integer-Valued Polynomial
Ring over Rational Field in the binomial basis in the binomial basis
```

```
>>> from sage.all import *
>>> K = IntegerValuedPolynomialRing(QQ).B()
>>> a = K.gen()
>>> K.is_commutative()
True
>>> L = IntegerValuedPolynomialRing(K).B()
>>> c = L.gen()
>>> L.is_commutative()
True
>>> s = a * c**Integer(3); s
B[1]*B[1] + 6*B[1]*B[2] + 6*B[1]*B[3]
>>> parent(s)
Integer-Valued Polynomial Ring over Integer-Valued Polynomial
Ring over Rational Field in the binomial basis in the binomial basis
```

Integer-valued polynomial rings are commutative:

```
sage: c^3 * a == c * a * c * c
True
```

```
>>> from sage.all import *
>>> c**Integer(3) * a == c * a * c * c
True
```

We can also manipulate elements in the basis:

```
sage: F = IntegerValuedPolynomialRing(QQ).B()
sage: B = F.basis()
sage: B[2] * B[3]
3*B[3] + 12*B[4] + 10*B[5]
```

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```
sage: 1 - B[2] * B[2] / 2
B[0] - 1/2*B[2] - 3*B[3] - 3*B[4]
```

```
>>> from sage.all import *
>>> F = IntegerValuedPolynomialRing(QQ).B()
>>> B = F.basis()
>>> B[Integer(2)] * B[Integer(3)]
3*B[3] + 12*B[4] + 10*B[5]
>>> Integer(1) - B[Integer(2)] * B[Integer(2)] / Integer(2)
B[0] - 1/2*B[2] - 3*B[3] - 3*B[4]
```

and coerce elements from our base field:

```
sage: F(4/3)
4/3*B[0]
```

```
>>> from sage.all import *
>>> F(Integer(4)/Integer(3))
4/3*B[0]
```

class Element

Bases: IndexedFreeModuleElement

variable_shift(*k=1*)

Return the image by the shift of variables.

On polynomials, the action is the shift on variables $x \mapsto x + k$.

INPUT:

- k – integer (default: 1)

EXAMPLES:

```
sage: A = IntegerValuedPolynomialRing(ZZ).B()
sage: B = A.basis()
sage: B[5].variable_shift()
B[4] + B[5]
sage: B[5].variable_shift(-1)
-B[0] + B[1] - B[2] + B[3] - B[4] + B[5]
```

```
>>> from sage.all import *
>>> A = IntegerValuedPolynomialRing(ZZ).B()
>>> B = A.basis()
>>> B[Integer(5)].variable_shift()
B[4] + B[5]
>>> B[Integer(5)].variable_shift(-Integer(1))
-B[0] + B[1] - B[2] + B[3] - B[4] + B[5]
```

product_on_basis(*n1*, *n2*)

Return the product of basis elements n1 and n2.

INPUT:

- n1, n2 – integers

EXAMPLES:

```
sage: A = IntegerValuedPolynomialRing(QQ).B()
sage: A.product_on_basis(0, 1)
B[1]
sage: A.product_on_basis(1, 2)
2*B[2] + 3*B[3]
```

```
>>> from sage.all import *
>>> A = IntegerValuedPolynomialRing(QQ).B()
>>> A.product_on_basis(Integer(0), Integer(1))
B[1]
>>> A.product_on_basis(Integer(1), Integer(2))
2*B[2] + 3*B[3]
```

S

alias of *Shifted*

class Shifted(*A*)

Bases: CombinatorialFreeModule, BindableClass

The integer-valued polynomial ring in the shifted basis.

The basis used here is given by $S[i] = \binom{i+x}{i}$ for $i \in \mathbf{N}$.

Assuming $n_1 \leq n_2$, the product of two monomials $S[n_1] \cdot S[n_2]$ is given by the sum

$$\sum_{k=0}^{n_1} (-1)^k \binom{n_1}{k} \binom{n_1+n_2-k}{n_1} S[n_1+n_2-k].$$

EXAMPLES:

```
sage: F = IntegerValuedPolynomialRing(QQ).S(); F
Integer-Valued Polynomial Ring over Rational Field
in the shifted basis

sage: F.gen()
S[1]

sage: S = IntegerValuedPolynomialRing(ZZ).S(); S
Integer-Valued Polynomial Ring over Integer Ring
in the shifted basis
sage: S.base_ring()
Integer Ring

sage: G = IntegerValuedPolynomialRing(S).S(); G
Integer-Valued Polynomial Ring over Integer-Valued Polynomial
Ring over Integer Ring in the shifted basis in the shifted basis
sage: G.base_ring()
Integer-Valued Polynomial Ring over Integer Ring
in the shifted basis
```

```
>>> from sage.all import *
>>> F = IntegerValuedPolynomialRing(QQ).S(); F
Integer-Valued Polynomial Ring over Rational Field
in the shifted basis
```

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```
>>> F.gen()
S[1]

>>> S = IntegerValuedPolynomialRing(ZZ).S(); S
Integer-Valued Polynomial Ring over Integer Ring
in the shifted basis
>>> S.base_ring()
Integer Ring

>>> G = IntegerValuedPolynomialRing(S).S(); G
Integer-Valued Polynomial Ring over Integer-Valued Polynomial
Ring over Integer Ring in the shifted basis in the shifted basis
>>> G.base_ring()
Integer-Valued Polynomial Ring over Integer Ring
in the shifted basis
```

Integer-valued polynomial rings commute with their base ring:

```
sage: K = IntegerValuedPolynomialRing(QQ).S()
sage: a = K.gen()
sage: K.is_commutative()
True
sage: L = IntegerValuedPolynomialRing(K).S()
sage: c = L.gen()
sage: L.is_commutative()
True
sage: s = a * c^3; s
S[1]*S[1] + (-6*S[1])*S[2] + 6*S[1]*S[3]
sage: parent(s)
Integer-Valued Polynomial Ring over Integer-Valued Polynomial
Ring over Rational Field in the shifted basis in the shifted basis
```

```
>>> from sage.all import *
>>> K = IntegerValuedPolynomialRing(QQ).S()
>>> a = K.gen()
>>> K.is_commutative()
True
>>> L = IntegerValuedPolynomialRing(K).S()
>>> c = L.gen()
>>> L.is_commutative()
True
>>> s = a * c**Integer(3); s
S[1]*S[1] + (-6*S[1])*S[2] + 6*S[1]*S[3]
>>> parent(s)
Integer-Valued Polynomial Ring over Integer-Valued Polynomial
Ring over Rational Field in the shifted basis in the shifted basis
```

Integer-valued polynomial rings are commutative:

```
sage: c^3 * a == c * a * c * c
True
```

```
>>> from sage.all import *
>>> c**Integer(3) * a == c * a * c * c
True
```

We can also manipulate elements in the basis and coerce elements from our base field:

```
sage: F = IntegerValuedPolynomialRing(QQ).S()
sage: S = F.basis()
sage: S[2] * S[3]
3*S[3] - 12*S[4] + 10*S[5]
sage: 1 - S[2] * S[2] / 2
S[0] - 1/2*S[2] + 3*S[3] - 3*S[4]
```

```
>>> from sage.all import *
>>> F = IntegerValuedPolynomialRing(QQ).S()
>>> S = F.basis()
>>> S[Integer(2)] * S[Integer(3)]
3*S[3] - 12*S[4] + 10*S[5]
>>> Integer(1) - S[Integer(2)] * S[Integer(2)] / Integer(2)
S[0] - 1/2*S[2] + 3*S[3] - 3*S[4]
```

class Element

Bases: `IndexedFreeModuleElement`

delta()

Return the image by the difference operator Δ.

The operator Δ is defined on polynomials by

$$f \mapsto f(x+1) - f(x).$$

EXAMPLES:

```
sage: F = IntegerValuedPolynomialRing(ZZ).S()
sage: S = F.basis()
sage: S[5].delta()
S[0] + S[1] + S[2] + S[3] + S[4]
```

```
>>> from sage.all import *
>>> F = IntegerValuedPolynomialRing(ZZ).S()
>>> S = F.basis()
>>> S[Integer(5)].delta()
S[0] + S[1] + S[2] + S[3] + S[4]
```

derivative_at_minus_one()

Return the derivative at -1.

This is sometimes useful when -1 is a root.

See also

umbra()

EXAMPLES:

```
sage: F = IntegerValuedPolynomialRing(ZZ).S()
sage: B = F.gen()
sage: (B+1).derivative_at_minus_one()
1
```

```
>>> from sage.all import *
>>> F = IntegerValuedPolynomialRing(ZZ).S()
>>> B = F.gen()
>>> (B+Integer(1)).derivative_at_minus_one()
1
```

fraction()

Return the generating series of values as a fraction.

In the case of Ehrhart polynomials, this is known as the Ehrhart series.

See also

h_vector(), *h_polynomial()*

EXAMPLES:

```
sage: A = IntegerValuedPolynomialRing(ZZ).S()
sage: ex = A.monomial(4)
sage: f = ex.fraction();f
1/(-t^5 + 5*t^4 - 10*t^3 + 10*t^2 - 5*t + 1)

sage: F = LazyPowerSeriesRing(QQ, 't')
sage: F(f)
1 + 5*t + 15*t^2 + 35*t^3 + 70*t^4 + 126*t^5 + 210*t^6 + O(t^7)

sage: poly = ex.polynomial()
sage: [poly(i) for i in range(6)]
[1, 5, 15, 35, 70, 126]

sage: y = polygen(QQ, 'y')
sage: penta = A.from_polynomial(7/2*y^2 + 7/2*y + 1)
sage: penta.fraction()
(t^2 + 5*t + 1)/(-t^3 + 3*t^2 - 3*t + 1)
```

```
>>> from sage.all import *
>>> A = IntegerValuedPolynomialRing(ZZ).S()
>>> ex = A.monomial(Integer(4))
>>> f = ex.fraction();f
1/(-t^5 + 5*t^4 - 10*t^3 + 10*t^2 - 5*t + 1)

>>> F = LazyPowerSeriesRing(QQ, 't')
>>> F(f)
1 + 5*t + 15*t^2 + 35*t^3 + 70*t^4 + 126*t^5 + 210*t^6 + O(t^7)

>>> poly = ex.polynomial()
>>> [poly(i) for i in range(Integer(6))]
```

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```
[1, 5, 15, 35, 70, 126]

>>> y = polygen(QQ, 'y')
>>> penta = A.from_polynomial(Integer(7)/Integer(2)*y**Integer(2) +
↪Integer(7)/Integer(2)*y + Integer(1))
>>> penta.fraction()
(t^2 + 5*t + 1)/(-t^3 + 3*t^2 - 3*t + 1)
```

h_polynomial()

Return the h-vector as a polynomial.

See also

h_vector(), *fraction()*

EXAMPLES:

```
sage: x = polygen(QQ,'x')
sage: A = IntegerValuedPolynomialRing(ZZ).S()
sage: ex = A.from_polynomial((1+x)**3)
sage: ex.h_polynomial()
z^2 + 4*z + 1
```

```
>>> from sage.all import *
>>> x = polygen(QQ,'x')
>>> A = IntegerValuedPolynomialRing(ZZ).S()
>>> ex = A.from_polynomial((Integer(1)+x)**Integer(3))
>>> ex.h_polynomial()
z^2 + 4*z + 1
```

h_vector()

Return the numerator of the generating series of values.

If `self` is an Ehrhart polynomial, this is the h-vector.

See also

h_polynomial(), *fraction()*

EXAMPLES:

```
sage: x = polygen(QQ,'x')
sage: A = IntegerValuedPolynomialRing(ZZ).S()
sage: ex = A.from_polynomial((1+x)**3)
sage: ex.h_vector()
(0, 1, 4, 1)
```

```
>>> from sage.all import *
>>> x = polygen(QQ,'x')
>>> A = IntegerValuedPolynomialRing(ZZ).S()
```

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```
>>> ex = A.from_polynomial((Integer(1)+x)**Integer(3))
>>> ex.h_vector()
(0, 1, 4, 1)
```

umbra()

Return the Bernoulli umbra.

This is the derivative at -1 of the shift by one.

This is related to Bernoulli numbers.

See also

derivative_at_minus_one()

EXAMPLES:

```
sage: F = IntegerValuedPolynomialRing(ZZ).S()
sage: B = F.gen()
sage: (B+1).umbra()
3/2
```

```
>>> from sage.all import *
>>> F = IntegerValuedPolynomialRing(ZZ).S()
>>> B = F.gen()
>>> (B+Integer(1)).umbra()
3/2
```

variable_shift(*k=1*)

Return the image by the shift of variables.

On polynomials, the action is the shift on variables $x \mapsto x + k$.

INPUT:

- k – integer (default: 1)

EXAMPLES:

```
sage: A = IntegerValuedPolynomialRing(ZZ).S()
sage: S = A.basis()
sage: S[5].variable_shift()
S[0] + S[1] + S[2] + S[3] + S[4] + S[5]

sage: S[5].variable_shift(-1)
-S[4] + S[5]
```

```
>>> from sage.all import *
>>> A = IntegerValuedPolynomialRing(ZZ).S()
>>> S = A.basis()
>>> S[Integer(5)].variable_shift()
S[0] + S[1] + S[2] + S[3] + S[4] + S[5]

>>> S[Integer(5)].variable_shift(-Integer(1))
-S[4] + S[5]
```

from_h_vector(*h*)

Convert from some h-vector.

INPUT:

- h – tuple or vector

See also

Element.h_vector()

EXAMPLES:

```
sage: A = IntegerValuedPolynomialRing(ZZ).S()
sage: S = A.basis()
sage: ex = S[2]+S[4]
sage: A.from_h_vector(ex.h_vector())
S[2] + S[4]
```

```
>>> from sage.all import *
>>> A = IntegerValuedPolynomialRing(ZZ).S()
>>> S = A.basis()
>>> ex = S[Integer(2)]+S[Integer(4)]
>>> A.from_h_vector(ex.h_vector())
S[2] + S[4]
```

product_on_basis(*n1*, *n2*)

Return the product of basis elements n1 and n2.

INPUT:

- n1, n2 – integers

EXAMPLES:

```
sage: A = IntegerValuedPolynomialRing(QQ).S()
sage: A.product_on_basis(0, 1)
S[1]
sage: A.product_on_basis(1, 2)
-2*S[2] + 3*S[3]
```

```
>>> from sage.all import *
>>> A = IntegerValuedPolynomialRing(QQ).S()
>>> A.product_on_basis(Integer(0), Integer(1))
S[1]
>>> A.product_on_basis(Integer(1), Integer(2))
-2*S[2] + 3*S[3]
```

a_realization()

Return a default realization.

The Binomial realization is chosen.

EXAMPLES:

```
sage: IntegerValuedPolynomialRing(QQ).a_realization()
Integer-Valued Polynomial Ring over Rational Field
in the binomial basis
```

```
>>> from sage.all import *
>>> IntegerValuedPolynomialRing(QQ).a_realization()
Integer-Valued Polynomial Ring over Rational Field
in the binomial basis
```

2.1.27 Quantum-valued polynomial rings

This provides a q-analogue of the `IntegerValuedPolynomialRing`.

AUTHORS:

- Frédéric Chapoton (2024-03): Initial version

class `sage.rings.polynomial.q_integer_valued_polynomials.`**`QuantumValuedPolynomialRing`**`(R, q)`

Bases: `UniqueRepresentation`, `Parent`

The quantum-valued polynomial ring over a base ring.

Quantum-valued polynomial rings are commutative and associative algebras, with a basis indexed by nonnegative integers.

The elements are polynomials in one variable x with coefficients in the field of rational functions in q, such that the value at every nonnegative q-integer is a polynomial in q.

This algebra is endowed with two bases, named `B` or `Binomial` and `S` or `Shifted`.

INPUT:

- `R` – commutative ring
- `q` – optional variable

There are two possible input formats:

- If the argument `q` is not given, then the ring `R` is taken as a base ring and the ring of Laurent polynomials in q over `R` is built and used.
- If the argument `q` is given, then it should belong to the ring `R` and be invertible in this ring.

EXAMPLES:

```
sage: F = QuantumValuedPolynomialRing(QQ).S(); F
Quantum-Valued Polynomial Ring over Rational Field
in the shifted basis

sage: F.gen()
S[1]

sage: S = QuantumValuedPolynomialRing(ZZ); S
Quantum-Valued Polynomial Ring over Integer Ring
sage: S.base_ring()
Univariate Laurent Polynomial Ring in q over Integer Ring
```

```
>>> from sage.all import *
>>> F = QuantumValuedPolynomialRing(QQ).S(); F
Quantum-Valued Polynomial Ring over Rational Field
in the shifted basis
```

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```
>>> F.gen()
S[1]

>>> S = QuantumValuedPolynomialRing(ZZ); S
Quantum-Valued Polynomial Ring over Integer Ring
>>> S.base_ring()
Univariate Laurent Polynomial Ring in q over Integer Ring
```

Quantum-valued polynomial rings commute with their base ring:

```
sage: K = QuantumValuedPolynomialRing(QQ).S()
sage: a = K.gen()
sage: c = K.monomial(2)
```

```
>>> from sage.all import *
>>> K = QuantumValuedPolynomialRing(QQ).S()
>>> a = K.gen()
>>> c = K.monomial(Integer(2))
```

Quantum-valued polynomial rings are commutative:

```
sage: c^3 * a == c * a * c * c
True
```

```
>>> from sage.all import *
>>> c**Integer(3) * a == c * a * c * c
True
```

We can also manipulate elements in the basis and coerce elements from our base field:

```
sage: F = QuantumValuedPolynomialRing(QQ).S()
sage: B = F.basis()
sage: B[2] * B[3]
(q^-5+q^-4+q^-3)*S[3] - (q^-6+2*q^-5+3*q^-4+3*q^-3+2*q^-2+q^-1)*S[4]
+ (q^-6+q^-5+2*q^-4+2*q^-3+2*q^-2+q^-1+1)*S[5]
sage: 1 - B[2] * B[2] / 2
S[0] - (1/2*q^-3)*S[2] + (1/2*q^-4+q^-3+q^-2+1/2*q^-1)*S[3]
- (1/2*q^-4+1/2*q^-3+q^-2+1/2*q^-1+1/2)*S[4]
```

```
>>> from sage.all import *
>>> F = QuantumValuedPolynomialRing(QQ).S()
>>> B = F.basis()
>>> B[Integer(2)] * B[Integer(3)]
(q^-5+q^-4+q^-3)*S[3] - (q^-6+2*q^-5+3*q^-4+3*q^-3+2*q^-2+q^-1)*S[4]
+ (q^-6+q^-5+2*q^-4+2*q^-3+2*q^-2+q^-1+1)*S[5]
>>> Integer(1) - B[Integer(2)] * B[Integer(2)] / Integer(2)
S[0] - (1/2*q^-3)*S[2] + (1/2*q^-4+q^-3+q^-2+1/2*q^-1)*S[3]
- (1/2*q^-4+1/2*q^-3+q^-2+1/2*q^-1+1/2)*S[4]
```

B

alias of *Binomial*

class Bases(*parent_with_realization*)

Bases: `Category_realization_of_parent`

class ElementMethods

Bases: `object`

polynomial()

Convert to a polynomial in x.

EXAMPLES:

```
sage: F = QuantumValuedPolynomialRing(ZZ).S()
sage: S = F.gen()
sage: (S+1).polynomial()
q*x + 2

sage: F = QuantumValuedPolynomialRing(ZZ).B()
sage: B = F.gen()
sage: (B+1).polynomial()
x + 1
```

```
>>> from sage.all import *
>>> F = QuantumValuedPolynomialRing(ZZ).S()
>>> S = F.gen()
>>> (S+Integer(1)).polynomial()
q*x + 2

>>> F = QuantumValuedPolynomialRing(ZZ).B()
>>> B = F.gen()
>>> (B+Integer(1)).polynomial()
x + 1
```

shift(*j=1*)

Shift all indices by j.

INPUT:

- j – integer (default 1)

In the binomial basis, the shift by 1 corresponds to a summation operator from 0 to x.

EXAMPLES:

```
sage: F = QuantumValuedPolynomialRing(ZZ).S()
sage: B = F.gen()
sage: (B+1).shift()
S[1] + S[2]
```

```
>>> from sage.all import *
>>> F = QuantumValuedPolynomialRing(ZZ).S()
>>> B = F.gen()
>>> (B+Integer(1)).shift()
S[1] + S[2]
```

sum_of_coefficients()

Return the sum of coefficients.

In the shifted basis, this is the evaluation at $x = 0$.

EXAMPLES:

```
sage: F = QuantumValuedPolynomialRing(ZZ).S()
sage: B = F.basis()
sage: (B[2]*B[4]).sum_of_coefficients()
1
```

```
>>> from sage.all import *
>>> F = QuantumValuedPolynomialRing(ZZ).S()
>>> B = F.basis()
>>> (B[Integer(2)]*B[Integer(4)]).sum_of_coefficients()
1
```

class ParentMethods

Bases: `object`

algebra_generators()

Return the generators of this algebra.

EXAMPLES:

```
sage: A = QuantumValuedPolynomialRing(ZZ).S(); A
Quantum-Valued Polynomial Ring over Integer Ring
in the shifted basis
sage: A.algebra_generators()
Family (S[1],)
```

```
>>> from sage.all import *
>>> A = QuantumValuedPolynomialRing(ZZ).S(); A
Quantum-Valued Polynomial Ring over Integer Ring
in the shifted basis
>>> A.algebra_generators()
Family (S[1],)
```

degree_on_basis(*m*)

Return the degree of the basis element indexed by `m`.

EXAMPLES:

```
sage: A = QuantumValuedPolynomialRing(QQ).S()
sage: A.degree_on_basis(4)
4
```

```
>>> from sage.all import *
>>> A = QuantumValuedPolynomialRing(QQ).S()
>>> A.degree_on_basis(Integer(4))
4
```

from_polynomial(*p*)

Convert a polynomial into the ring of quantum-valued polynomials.

This raises a `ValueError` if this is not possible.

INPUT:

- `p` – a polynomial in `x` with coefficients in `QQ(q)`

EXAMPLES:

```
sage: A = QuantumValuedPolynomialRing(ZZ).S()
sage: S = A.basis()
sage: A.from_polynomial((S[1]).polynomial())
S[1]
sage: A.from_polynomial((S[2]+2*S[3]).polynomial())
S[2] + 2*S[3]

sage: A = QuantumValuedPolynomialRing(ZZ).B()
sage: B = A.basis()
sage: A.from_polynomial((B[1]).polynomial())
B[1]
sage: A.from_polynomial((B[2]+2*B[3]).polynomial())
B[2] + 2*B[3]
```

```
>>> from sage.all import *
>>> A = QuantumValuedPolynomialRing(ZZ).S()
>>> S = A.basis()
>>> A.from_polynomial((S[Integer(1)]).polynomial())
S[1]
>>> A.from_polynomial((S[Integer(2)]+Integer(2)*S[Integer(3)]).
→polynomial())
S[2] + 2*S[3]

>>> A = QuantumValuedPolynomialRing(ZZ).B()
>>> B = A.basis()
>>> A.from_polynomial((B[Integer(1)]).polynomial())
B[1]
>>> A.from_polynomial((B[Integer(2)]+Integer(2)*B[Integer(3)]).
→polynomial())
B[2] + 2*B[3]
```

gen (*i=0*)

Return the generator of the algebra.

The optional argument is ignored.

EXAMPLES:

```
sage: F = QuantumValuedPolynomialRing(ZZ).S()
sage: F.gen()
S[1]
```

```
>>> from sage.all import *
>>> F = QuantumValuedPolynomialRing(ZZ).S()
>>> F.gen()
S[1]
```

gens ()

Return the generators of this algebra.

EXAMPLES:

```
sage: A = QuantumValuedPolynomialRing(ZZ).S(); A
Quantum-Valued Polynomial Ring over Integer Ring
in the shifted basis
sage: A.algebra_generators()
Family (S[1],)
```

```
>>> from sage.all import *
>>> A = QuantumValuedPolynomialRing(ZZ).S(); A
Quantum-Valued Polynomial Ring over Integer Ring
in the shifted basis
>>> A.algebra_generators()
Family (S[1],)
```

ground_ring()

Return the ring of coefficients.

This ring is not supposed to contain the variable q.

EXAMPLES:

```
sage: A = QuantumValuedPolynomialRing(QQ).S()
sage: A.ground_ring()
Rational Field
```

```
>>> from sage.all import *
>>> A = QuantumValuedPolynomialRing(QQ).S()
>>> A.ground_ring()
Rational Field
```

one_basis()

Return the number 0, which index the unit of this algebra.

EXAMPLES:

```
sage: A = QuantumValuedPolynomialRing(QQ).S()
sage: A.one_basis()
0
sage: A.one()
S[0]
```

```
>>> from sage.all import *
>>> A = QuantumValuedPolynomialRing(QQ).S()
>>> A.one_basis()
0
>>> A.one()
S[0]
```

q()

Return the variable q.

EXAMPLES:

```
sage: A = QuantumValuedPolynomialRing(QQ).S()
sage: A.q()
q
```

```
>>> from sage.all import *
>>> A = QuantumValuedPolynomialRing(QQ).S()
>>> A.q()
q
```

super_categories()

Return the super-categories of `self`.

EXAMPLES:

```
sage: A = QuantumValuedPolynomialRing(QQ); A
Quantum-Valued Polynomial Ring over Rational Field
sage: C = A.Bases(); C
Category of bases of Quantum-Valued Polynomial Ring
over Rational Field
sage: C.super_categories()
[Category of realizations of Quantum-Valued Polynomial Ring
 over Rational Field,
 Join of Category of algebras with basis
 over Univariate Laurent Polynomial Ring in q over Rational Field and
 Category of filtered algebras
 over Univariate Laurent Polynomial Ring in q over Rational Field and
 Category of commutative algebras
 over Univariate Laurent Polynomial Ring in q over Rational Field and
 Category of realizations of unital magmas]
```

```
>>> from sage.all import *
>>> A = QuantumValuedPolynomialRing(QQ); A
Quantum-Valued Polynomial Ring over Rational Field
>>> C = A.Bases(); C
Category of bases of Quantum-Valued Polynomial Ring
over Rational Field
>>> C.super_categories()
[Category of realizations of Quantum-Valued Polynomial Ring
 over Rational Field,
 Join of Category of algebras with basis
 over Univariate Laurent Polynomial Ring in q over Rational Field and
 Category of filtered algebras
 over Univariate Laurent Polynomial Ring in q over Rational Field and
 Category of commutative algebras
 over Univariate Laurent Polynomial Ring in q over Rational Field and
 Category of realizations of unital magmas]
```

class Binomial(A)

Bases: `CombinatorialFreeModule`, `BindableClass`

The quantum-valued polynomial ring in the binomial basis.

The basis used here is given by $B[i] = \begin{bmatrix} x \\ i \end{bmatrix}_q$ for $i \in \mathbf{N}$.

Assuming $n_1 \leq n_2$, the product of two monomials $B[n_1] \cdot B[n_2]$ is given by the sum

$$\sum_{k=0}^{n_1} q^{(k-n_1)(k-n_2)} \begin{bmatrix} n_1 \\ k \end{bmatrix}_q \begin{bmatrix} n_1+n_2-k \\ n_1 \end{bmatrix}_q B[n_1+n_2-k].$$

class Element

Bases: `IndexedFreeModuleElement`

variable_shift(*k=1*)

Return the image by the shift of variables.

On polynomials, the action for $k = 1$ is the shift on variables $x \mapsto 1 + qx$.

This implementation follows formula (5.5) in [HaHo2017].

INPUT:

- k – nonnegative integer (default: 1)

EXAMPLES:

```
sage: A = QuantumValuedPolynomialRing(ZZ).B()
sage: B = A.basis()
sage: B[5].variable_shift()
B[4] + q^5*B[5]
```

```
>>> from sage.all import *
>>> A = QuantumValuedPolynomialRing(ZZ).B()
>>> B = A.basis()
>>> B[Integer(5)].variable_shift()
B[4] + q^5*B[5]
```

product_on_basis(*n1*, *n2*)

Return the product of basis elements `n1` and `n2`.

INPUT:

- `n1`, `n2` – integers

The formula is taken from Theorem 3.4 in [HaHo2017].

EXAMPLES:

```
sage: A = QuantumValuedPolynomialRing(QQ).B()
sage: A.product_on_basis(0, 1)
B[1]
```

```
>>> from sage.all import *
>>> A = QuantumValuedPolynomialRing(QQ).B()
>>> A.product_on_basis(Integer(0), Integer(1))
B[1]
```

S

alias of *Shifted*

class Shifted(*A*)

Bases: `CombinatorialFreeModule`, `BindableClass`

The quantum-valued polynomial ring in the shifted basis.

The basis used here is given by $S[i] = \begin{bmatrix} i+x \\ i \end{bmatrix}_q$ for $i \in \mathbf{N}$.

Assuming $n_1 \leq n_2$, the product of two monomials $S[n_1] \cdot S[n_2]$ is given by the sum

$$\sum_{k=0}^{n_1} (-1)^k q^{\binom{k}{2} - n_1 * n_2} \begin{bmatrix} n_1 \\ k \end{bmatrix}_q \begin{bmatrix} n_1 + n_2 - k \\ n_1 \end{bmatrix}_q S[n_1 + n_2 - k].$$

class Element

Bases: `IndexedFreeModuleElement`

derivative_at_minus_one()

Return the 'derivative' at -1.

See also

umbra()

EXAMPLES:

```
sage: F = QuantumValuedPolynomialRing(ZZ).S()
sage: B = F.gen()
sage: (B+1).derivative_at_minus_one()
1
```

```
>>> from sage.all import *
>>> F = QuantumValuedPolynomialRing(ZZ).S()
>>> B = F.gen()
>>> (B+Integer(1)).derivative_at_minus_one()
1
```

fraction()

Return the generating series of values as a fraction.

See also

h_vector(), *h_polynomial()*

EXAMPLES:

```
sage: A = QuantumValuedPolynomialRing(QQ).S()
sage: ex = A.basis()[4]
sage: ex.fraction().factor()
(-1) * (t - 1)^-1 * (q*t - 1)^-1 * (q^2*t - 1)^-1 * (q^3*t - 1)^-1 *␣
↪(q^4*t - 1)^-1

sage: q = polygen(QQ,'q')
sage: x = polygen(q.parent(), 'x')
sage: ex = A.from_polynomial((1+q*x)**3)
sage: ex.fraction().factor()
(t - 1)^-1 * (q*t - 1)^-1 * (q^2*t - 1)^-1 * (q^3*t - 1)^-1 * (q^3*t^2␣
↪+ 2*q^2*t + 2*q*t + 1)
```

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```
sage: ex.fraction().numerator()
q^3*t^2 + 2*q^2*t + 2*q*t + 1
```

```
>>> from sage.all import *
>>> A = QuantumValuedPolynomialRing(QQ).S()
>>> ex = A.basis()[Integer(4)]
>>> ex.fraction().factor()
(-1) * (t - 1)^-1 * (q*t - 1)^-1 * (q^2*t - 1)^-1 * (q^3*t - 1)^-1 *␣
↪(q^4*t - 1)^-1

>>> q = polygen(QQ,'q')
>>> x = polygen(q.parent(), 'x')
>>> ex = A.from_polynomial((Integer(1)+q*x)**Integer(3))
>>> ex.fraction().factor()
(t - 1)^-1 * (q*t - 1)^-1 * (q^2*t - 1)^-1 * (q^3*t - 1)^-1 * (q^3*t^2␣
↪+ 2*q^2*t + 2*q*t + 1)
>>> ex.fraction().numerator()
q^3*t^2 + 2*q^2*t + 2*q*t + 1
```

h_polynomial()

Return the h-vector as a polynomial.

See also

h_vector(), *fraction()*

EXAMPLES:

```
sage: A = QuantumValuedPolynomialRing(ZZ).S()
sage: q = polygen(ZZ,'q')
sage: x = polygen(q.parent(),'x')
sage: ex = A.from_polynomial((1+q*x)**3)
sage: ex.h_polynomial()
z^3 + (2*q + 2*q^2)*z^2 + q^3*z
```

```
>>> from sage.all import *
>>> A = QuantumValuedPolynomialRing(ZZ).S()
>>> q = polygen(ZZ,'q')
>>> x = polygen(q.parent(),'x')
>>> ex = A.from_polynomial((Integer(1)+q*x)**Integer(3))
>>> ex.h_polynomial()
z^3 + (2*q + 2*q^2)*z^2 + q^3*z
```

h_vector()

Return the numerator of the generating series of values.

If `self` is an Ehrhart polynomial, this is the h-vector.

See also

h_polynomial(), *fraction()*

EXAMPLES:

```
sage: A = QuantumValuedPolynomialRing(ZZ).S()
sage: ex = A.basis()[4]
sage: ex.h_vector()
(0, 0, 0, 0, 1)

sage: q = polygen(QQ,'q')
sage: x = polygen(q.parent(),'x')
sage: ex = A.from_polynomial((1+q*x)**3)
sage: ex.h_vector()
(0, q^3, 2*q + 2*q^2, 1)
```

```
>>> from sage.all import *
>>> A = QuantumValuedPolynomialRing(ZZ).S()
>>> ex = A.basis()[Integer(4)]
>>> ex.h_vector()
(0, 0, 0, 0, 1)

>>> q = polygen(QQ,'q')
>>> x = polygen(q.parent(),'x')
>>> ex = A.from_polynomial((Integer(1)+q*x)**Integer(3))
>>> ex.h_vector()
(0, q^3, 2*q + 2*q^2, 1)
```

umbra()

Return the Bernoulli umbra.

This is the derivative at -1 of the shift by one.

This is related to Carlitz's q-Bernoulli numbers.

See also

derivative_at_minus_one()

EXAMPLES:

```
sage: F = QuantumValuedPolynomialRing(ZZ).S()
sage: B = F.gen()
sage: (B+1).umbra()
(q + 2)/(q + 1)
```

```
>>> from sage.all import *
>>> F = QuantumValuedPolynomialRing(ZZ).S()
>>> B = F.gen()
>>> (B+Integer(1)).umbra()
(q + 2)/(q + 1)
```

variable_shift(*k=1*)

Return the image by the shift on variables.

The shift is the substitution operator

$$x \mapsto qx + 1.$$

INPUT:

- k – integer (default: 1)

EXAMPLES:

```
sage: A = QuantumValuedPolynomialRing(ZZ).S()
sage: S = A.basis()
sage: S[5].variable_shift()
S[0] + q*S[1] + q^2*S[2] + q^3*S[3] + q^4*S[4] + q^5*S[5]

sage: S[5].variable_shift(-1)
-(q^-5)*S[4] + (q^-5)*S[5]
```

```
>>> from sage.all import *
>>> A = QuantumValuedPolynomialRing(ZZ).S()
>>> S = A.basis()
>>> S[Integer(5)].variable_shift()
S[0] + q*S[1] + q^2*S[2] + q^3*S[3] + q^4*S[4] + q^5*S[5]

>>> S[Integer(5)].variable_shift(-Integer(1))
-(q^-5)*S[4] + (q^-5)*S[5]
```

from_h_vector(*hv*)

Convert from some h-vector.

EXAMPLES:

```
sage: A = QuantumValuedPolynomialRing(ZZ).S()
sage: B = A.basis()
sage: ex = B[2] + B[3]
sage: A.from_h_vector(ex.h_vector())
S[2] + S[3]

sage: q = A.base_ring().gen()
sage: ex = B[2] + q*B[3]
sage: A.from_h_vector(ex.h_vector())
S[2] + q*S[3]
```

```
>>> from sage.all import *
>>> A = QuantumValuedPolynomialRing(ZZ).S()
>>> B = A.basis()
>>> ex = B[Integer(2)] + B[Integer(3)]
>>> A.from_h_vector(ex.h_vector())
S[2] + S[3]

>>> q = A.base_ring().gen()
>>> ex = B[Integer(2)] + q*B[Integer(3)]
>>> A.from_h_vector(ex.h_vector())
S[2] + q*S[3]
```

product_on_basis(*n1*, *n2*)

Return the product of basis elements `n1` and `n2`.

INPUT:

- `n1`, `n2` – integers

EXAMPLES:

```
sage: A = QuantumValuedPolynomialRing(QQ).S()
sage: A.product_on_basis(0, 1)
S[1]
```

```
>>> from sage.all import *
>>> A = QuantumValuedPolynomialRing(QQ).S()
>>> A.product_on_basis(Integer(0), Integer(1))
S[1]
```

a_realization()

Return a default realization.

The Shifted realization is chosen.

EXAMPLES:

```
sage: QuantumValuedPolynomialRing(QQ).a_realization()
Quantum-Valued Polynomial Ring over Rational Field
in the shifted basis
```

```
>>> from sage.all import *
>>> QuantumValuedPolynomialRing(QQ).a_realization()
Quantum-Valued Polynomial Ring over Rational Field
in the shifted basis
```

sage.rings.polynomial.q_integer_valued_polynomials.**q_binomial_x**(*m*, *n*, *q=None*)

Return a q-analogue of `binomial(m + x, n)`.

When evaluated at the q-integer $[k]_q$, this gives the usual q-binomial coefficient $[m+k, n]_q$.

INPUT:

- `m` and `n` – positive integers
- `q` – optional variable

EXAMPLES:

```
sage: from sage.combinat.q_analogues import q_int
sage: from sage.rings.polynomial.q_integer_valued_polynomials import q_binomial_x,
↪ q_int_x
sage: q_binomial_x(4,2)(0) == q_binomial(4,2)
True
sage: q_binomial_x(3,2)(1) == q_binomial(4,2)
True
sage: q_binomial_x(3,1) == q_int_x(4)
True
sage: q_binomial_x(2,0).parent()
Univariate Polynomial Ring in x over Fraction Field of
Univariate Polynomial Ring in q over Integer Ring
```

```
>>> from sage.all import *
>>> from sage.combinat.q_analogues import q_int
>>> from sage.rings.polynomial.q_integer_valued_polynomials import q_binomial_x,␣
```

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```
↪q_int_x
>>> q_binomial_x(Integer(4),Integer(2))(Integer(0)) == q_binomial(Integer(4),
↪Integer(2))
True
>>> q_binomial_x(Integer(3),Integer(2))(Integer(1)) == q_binomial(Integer(4),
↪Integer(2))
True
>>> q_binomial_x(Integer(3),Integer(1)) == q_int_x(Integer(4))
True
>>> q_binomial_x(Integer(2),Integer(0)).parent()
Univariate Polynomial Ring in x over Fraction Field of
Univariate Polynomial Ring in q over Integer Ring
```

sage.rings.polynomial.q_integer_valued_polynomials.**q_int_x**(*n*, *q=None*)

Return the interpolating polynomial of q-integers.

INPUT:

- n – a positive integer
- q – optional variable

EXAMPLES:

```
sage: from sage.rings.polynomial.q_integer_valued_polynomials import q_int_x
sage: q_int_x(3)
q^2*x + q + 1
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.q_integer_valued_polynomials import q_int_x
>>> q_int_x(Integer(3))
q^2*x + q + 1
```

2.2 Generic Convolution

Asymptotically fast convolution of lists over any commutative ring in which the multiply-by-two map is injective. (More precisely, if $x \in R$, and $x = 2^k * y$ for some $k \geq 0$, we require that $R(x/2^k)$ returns y.)

The main function to be exported is *convolution()*.

EXAMPLES:

```
sage: convolution([1, 2, 3, 4, 5], [6, 7])
[6, 19, 32, 45, 58, 35]
```

```
>>> from sage.all import *
>>> convolution([Integer(1), Integer(2), Integer(3), Integer(4), Integer(5)],␣
↪[Integer(6), Integer(7)])
[6, 19, 32, 45, 58, 35]
```

The convolution function is reasonably fast, even though it is written in pure Python. For example, the following takes less than a second:

```
sage: v = convolution(list(range(1000)), list(range(1000)))
```

```
>>> from sage.all import *
>>> v = convolution(list(range(Integer(1000))), list(range(Integer(1000))))
```

ALGORITHM:

Converts the problem to multiplication in the ring $S[x]/(x^M - 1)$, where $S = R[y]/(y^K + 1)$ (where R is the original base ring). Performs FFT with respect to the roots of unity $1, y, y^2, \ldots, y^{2K-1}$ in S. The FFT/IFFT are accomplished with just additions and subtractions and rotating python lists. (I think this algorithm is essentially due to Schonhage, not completely sure.) The pointwise multiplications are handled recursively, switching to a classical algorithm at some point.

Complexity is O(n log(n) log(log(n))) additions/subtractions in R and O(n log(n)) multiplications in R.

AUTHORS:

- David Harvey (2007-07): first implementation
- William Stein: editing the docstrings for inclusion in Sage.

`sage.rings.polynomial.convolution.`**`convolution`**`(`*L1*, *L2*`)`

Return convolution of non-empty lists L1 and L2.

L1 and L2 may have arbitrary lengths.

EXAMPLES:

```
sage: convolution([1, 2, 3], [4, 5, 6, 7])
[4, 13, 28, 34, 32, 21]
```

```
>>> from sage.all import *
>>> convolution([Integer(1), Integer(2), Integer(3)], [Integer(4), Integer(5),␣
↪Integer(6), Integer(7)])
[4, 13, 28, 34, 32, 21]
```

2.3 Fast calculation of cyclotomic polynomials

This module provides a function *`cyclotomic_coeffs()`*, which calculates the coefficients of cyclotomic polynomials. This is not intended to be invoked directly by the user, but it is called by the method *`cyclotomic_polynomial()`* method of univariate polynomial ring objects and the top-level `cyclotomic_polynomial()` function.

`sage.rings.polynomial.cyclotomic.`**`bateman_bound`**`(`*nn*`)`

Reference:

Bateman, P. T.; Pomerance, C.; Vaughan, R. C. *On the size of the coefficients of the cyclotomic polynomial.*

EXAMPLES:

```
sage: from sage.rings.polynomial.cyclotomic import bateman_bound
sage: bateman_bound(2**8 * 1234567893377)                                    #␣
↪needs sage.libs.pari
66944986927
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.cyclotomic import bateman_bound
>>> bateman_bound(Integer(2)**Integer(8) * Integer(1234567893377))              ␣
```

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```
↪                          # needs sage.libs.pari
66944986927
```

sage.rings.polynomial.cyclotomic.**cyclotomic_coeffs**(*nn*, *sparse=None*)

Return the coefficients of the n-th cyclotomic polynomial by using the formula

$$\Phi_n(x) = \prod_{d|n} (1 - x^{n/d})^{\mu(d)}$$

where $\mu(d)$ is the Möbius function that is 1 if d has an even number of distinct prime divisors, -1 if it has an odd number of distinct prime divisors, and 0 if d is not squarefree.

Multiplications and divisions by polynomials of the form $1 - x^n$ can be done very quickly in a single pass.

If `sparse` is `True`, the result is returned as a dictionary of the nonzero entries, otherwise the result is returned as a list of python ints.

EXAMPLES:

```
sage: from sage.rings.polynomial.cyclotomic import cyclotomic_coeffs
sage: cyclotomic_coeffs(30)
[1, 1, 0, -1, -1, -1, 0, 1, 1]
sage: cyclotomic_coeffs(10^5)
{0: 1, 10000: -1, 20000: 1, 30000: -1, 40000: 1}
sage: R = QQ['x']
sage: R(cyclotomic_coeffs(30))
x^8 + x^7 - x^5 - x^4 - x^3 + x + 1
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.cyclotomic import cyclotomic_coeffs
>>> cyclotomic_coeffs(Integer(30))
[1, 1, 0, -1, -1, -1, 0, 1, 1]
>>> cyclotomic_coeffs(Integer(10)**Integer(5))
{0: 1, 10000: -1, 20000: 1, 30000: -1, 40000: 1}
>>> R = QQ['x']
>>> R(cyclotomic_coeffs(Integer(30)))
x^8 + x^7 - x^5 - x^4 - x^3 + x + 1
```

Check that it has the right degree:

```
sage: euler_phi(30)                                                     #␣
↪needs sage.libs.pari
8
sage: R(cyclotomic_coeffs(14)).factor()                                 #␣
↪needs sage.libs.pari
x^6 - x^5 + x^4 - x^3 + x^2 - x + 1
```

```
>>> from sage.all import *
>>> euler_phi(Integer(30))                                               ␣
↪     # needs sage.libs.pari
8
>>> R(cyclotomic_coeffs(Integer(14))).factor()                           ␣
↪     # needs sage.libs.pari
x^6 - x^5 + x^4 - x^3 + x^2 - x + 1
```

The coefficients are not always +/-1:

```
sage: cyclotomic_coeffs(105)
[1, 1, 1, 0, 0, -1, -1, -2, -1, -1, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, -1,
 0, -1, 0, -1, 0, -1, 0, -1, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, -1, -1, -2,
 -1, -1, 0, 0, 1, 1, 1]
```

```
>>> from sage.all import *
>>> cyclotomic_coeffs(Integer(105))
[1, 1, 1, 0, 0, -1, -1, -2, -1, -1, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, -1,
 0, -1, 0, -1, 0, -1, 0, -1, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, -1, -1, -2,
 -1, -1, 0, 0, 1, 1, 1]
```

In fact the height is not bounded by any polynomial in n (Erdos), although takes a while just to exceed linear:

```
sage: v = cyclotomic_coeffs(1181895)
sage: max(v)
14102773
```

```
>>> from sage.all import *
>>> v = cyclotomic_coeffs(Integer(1181895))
>>> max(v)
14102773
```

The polynomial is a palindrome for any n:

```
sage: n = ZZ.random_element(50000)
sage: v = cyclotomic_coeffs(n, sparse=False)
sage: v == list(reversed(v))
True
```

```
>>> from sage.all import *
>>> n = ZZ.random_element(Integer(50000))
>>> v = cyclotomic_coeffs(n, sparse=False)
>>> v == list(reversed(v))
True
```

AUTHORS:

- Robert Bradshaw (2007-10-27): initial version (inspired by work of Andrew Arnold and Michael Monagan)

REFERENCE:

- http://www.cecm.sfu.ca/~ada26/cyclotomic/

sage.rings.polynomial.cyclotomic.**cyclotomic_value**(*n*, *x*)

Return the value of the n-th cyclotomic polynomial evaluated at x.

INPUT:

- `n` – an Integer, specifying which cyclotomic polynomial is to be evaluated
- `x` – an element of a ring

OUTPUT: the value of the cyclotomic polynomial Φ_n at x

ALGORITHM:

- Reduce to the case that n is squarefree: use the identity

$$\Phi_n(x) = \Phi_q(x^{n/q})$$

where q is the radical of n.

- Use the identity

$$\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(n/d)},$$

where μ is the Möbius function.

- Handles the case that $x^d = 1$ for some d, but not the case that $x^d - 1$ is non-invertible: in this case polynomial evaluation is used instead.

EXAMPLES:

```
sage: cyclotomic_value(51, 3)
1282860140677441
sage: cyclotomic_polynomial(51)(3)
1282860140677441
```

```
>>> from sage.all import *
>>> cyclotomic_value(Integer(51), Integer(3))
1282860140677441
>>> cyclotomic_polynomial(Integer(51))(Integer(3))
1282860140677441
```

It works for non-integral values as well:

```
sage: cyclotomic_value(144, 4/3)
79148745433504023621920372161/79766443076872509863361
sage: cyclotomic_polynomial(144)(4/3)
79148745433504023621920372161/79766443076872509863361
```

```
>>> from sage.all import *
>>> cyclotomic_value(Integer(144), Integer(4)/Integer(3))
79148745433504023621920372161/79766443076872509863361
>>> cyclotomic_polynomial(Integer(144))(Integer(4)/Integer(3))
79148745433504023621920372161/79766443076872509863361
```

CHAPTER

THREE

MULTIVARIATE POLYNOMIALS

3.1 Multivariate Polynomials and Polynomial Rings

Sage implements multivariate polynomial rings through several backends. The most generic implementation uses the classes *sage.rings.polynomial.polydict.PolyDict* and *sage.rings.polynomial.polydict.ETuple* to construct a dictionary with exponent tuples as keys and coefficients as values.

Additionally, specialized and optimized implementations over many specific coefficient rings are implemented via a shared library interface to SINGULAR; and polynomials in the boolean polynomial ring

$$\mathbf{F}_2[x_1, ..., x_n]/\langle x_1^2 + x_1, ..., x_n^2 + x_n \rangle.$$

are implemented using the PolyBoRi library (cf. `sage.rings.polynomial.pbori`).

3.1.1 Term orders

Sage supports the following term orders:

Lexicographic (lex)

$x^a < x^b$ if and only if there exists $1 \le i \le n$ such that $a_1 = b_1, \ldots, a_{i-1} = b_{i-1}, a_i < b_i$. This term order is called 'lp' in Singular.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ, 3, order='lex')
sage: x > y
True
sage: x > y^2
True
sage: x > 1
True
sage: x^1*y^2 > y^3*z^4
True
sage: x^3*y^2*z^4 < x^3*y^2*z^1
False
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, Integer(3), order='lex', names=('x', 'y', 'z',)); (x,␣
↪y, z,) = P._first_ngens(3)
>>> x > y
True
>>> x > y**Integer(2)
```

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```
True
>>> x > Integer(1)
True
>>> x**Integer(1)*y**Integer(2) > y**Integer(3)*z**Integer(4)
True
>>> x**Integer(3)*y**Integer(2)*z**Integer(4) <␣
↪x**Integer(3)*y**Integer(2)*z**Integer(1)
False
```

Degree reverse lexicographic (degrevlex)

Let $\deg(x^a) = a_1 + a_2 + \cdots + a_n$, then $x^a < x^b$ if and only if $\deg(x^a) < \deg(x^b)$ or $\deg(x^a) = \deg(x^b)$ and there exists $1 \le i \le n$ such that $a_n = b_n, \ldots, a_{i+1} = b_{i+1}, a_i > b_i$. This term order is called 'dp' in Singular.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ, 3, order='degrevlex')
sage: x > y
True
sage: x > y^2*z
False
sage: x > 1
True
sage: x^1*y^5*z^2 > x^4*y^1*z^3
True
sage: x^2*y*z^2 > x*y^3*z
False
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, Integer(3), order='degrevlex', names=('x', 'y', 'z',));
↪ (x, y, z,) = P._first_ngens(3)
>>> x > y
True
>>> x > y**Integer(2)*z
False
>>> x > Integer(1)
True
>>> x**Integer(1)*y**Integer(5)*z**Integer(2) >␣
↪x**Integer(4)*y**Integer(1)*z**Integer(3)
True
>>> x**Integer(2)*y*z**Integer(2) > x*y**Integer(3)*z
False
```

Degree lexicographic (deglex)

Let $\deg(x^a) = a_1 + a_2 + \cdots + a_n$, then $x^a < x^b$ if and only if $\deg(x^a) < \deg(x^b)$ or $\deg(x^a) = \deg(x^b)$ and there exists $1 \le i \le n$ such that $a_1 = b_1, \ldots, a_{i-1} = b_{i-1}, a_i < b_i$. This term order is called 'Dp' in Singular.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ, 3, order='deglex')
sage: x > y
True
sage: x > y^2*z
False
```

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```
sage: x > 1
True
sage: x^1*y^2*z^3 > x^3*y^2*z^0
True
sage: x^2*y*z^2 > x*y^3*z
True
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, Integer(3), order='deglex', names=('x', 'y', 'z',));␣
↪(x, y, z,) = P._first_ngens(3)
>>> x > y
True
>>> x > y**Integer(2)*z
False
>>> x > Integer(1)
True
>>> x**Integer(1)*y**Integer(2)*z**Integer(3) >␣
↪x**Integer(3)*y**Integer(2)*z**Integer(0)
True
>>> x**Integer(2)*y*z**Integer(2) > x*y**Integer(3)*z
True
```

Inverse lexicographic (invlex)

$x^a < x^b$ if and only if there exists $1 \le i \le n$ such that $a_n = b_n, \ldots, a_{i+1} = b_{i+1}, a_i < b_i$. This order is called 'rp' in Singular.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ, 3, order='invlex')
sage: x > y
False
sage: y > x^2
True
sage: x > 1
True
sage: x*y > z
False
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, Integer(3), order='invlex', names=('x', 'y', 'z',));␣
↪(x, y, z,) = P._first_ngens(3)
>>> x > y
False
>>> y > x**Integer(2)
True
>>> x > Integer(1)
True
>>> x*y > z
False
```

This term order only makes sense in a non-commutative setting because if P is the ring $k[x_1, \ldots, x_n]$ and term order 'invlex' then it is equivalent to the ring $k[x_n, \ldots, x_1]$ with term order 'lex'.

Negative lexicographic (neglex)

$x^a < x^b$ if and only if there exists $1 \le i \le n$ such that $a_1 = b_1, \ldots, a_{i-1} = b_{i-1}, a_i > b_i$. This term order is called 'ls' in Singular.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ, 3, order='neglex')
sage: x > y
False
sage: x > 1
False
sage: x^1*y^2 > y^3*z^4
False
sage: x^3*y^2*z^4 < x^3*y^2*z^1
True
sage: x*y > z
False
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, Integer(3), order='neglex', names=('x', 'y', 'z',));
↪(x, y, z,) = P._first_ngens(3)
>>> x > y
False
>>> x > Integer(1)
False
>>> x**Integer(1)*y**Integer(2) > y**Integer(3)*z**Integer(4)
False
>>> x**Integer(3)*y**Integer(2)*z**Integer(4) <
↪x**Integer(3)*y**Integer(2)*z**Integer(1)
True
>>> x*y > z
False
```

Negative degree reverse lexicographic (negdegrevlex)

Let $\deg(x^a) = a_1 + a_2 + \cdots + a_n$, then $x^a < x^b$ if and only if $\deg(x^a) > \deg(x^b)$ or $\deg(x^a) = \deg(x^b)$ and there exists $1 \le i \le n$ such that $a_n = b_n, \ldots, a_{i+1} = b_{i+1}, a_i > b_i$. This term order is called 'ds' in Singular.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ, 3, order='negdegrevlex')
sage: x > y
True
sage: x > x^2
True
sage: x > 1
False
sage: x^1*y^2 > y^3*z^4
True
sage: x^2*y*z^2 > x*y^3*z
False
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, Integer(3), order='negdegrevlex', names=('x', 'y', 'z',
↪)); (x, y, z,) = P._first_ngens(3)
```

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```
>>> x > y
True
>>> x > x**Integer(2)
True
>>> x > Integer(1)
False
>>> x**Integer(1)*y**Integer(2) > y**Integer(3)*z**Integer(4)
True
>>> x**Integer(2)*y*z**Integer(2) > x*y**Integer(3)*z
False
```

Negative degree lexicographic (negdeglex)

Let $\deg(x^a) = a_1 + a_2 + \cdots + a_n$, then $x^a < x^b$ if and only if $\deg(x^a) > \deg(x^b)$ or $\deg(x^a) = \deg(x^b)$ and there exists $1 \le i \le n$ such that $a_1 = b_1, \ldots, a_{i-1} = b_{i-1}, a_i < b_i$. This term order is called 'Ds' in Singular.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ, 3, order='negdeglex')
sage: x > y
True
sage: x > x^2
True
sage: x > 1
False
sage: x^1*y^2 > y^3*z^4
True
sage: x^2*y*z^2 > x*y^3*z
True
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, Integer(3), order='negdeglex', names=('x', 'y', 'z',));
↪ (x, y, z,) = P._first_ngens(3)
>>> x > y
True
>>> x > x**Integer(2)
True
>>> x > Integer(1)
False
>>> x**Integer(1)*y**Integer(2) > y**Integer(3)*z**Integer(4)
True
>>> x**Integer(2)*y*z**Integer(2) > x*y**Integer(3)*z
True
```

Weighted degree reverse lexicographic (wdegrevlex), positive integral weights

Let $\deg_w(x^a) = a_1 w_1 + a_2 w_2 + \cdots + a_n w_n$ with weights w, then $x^a < x^b$ if and only if $\deg_w(x^a) < \deg_w(x^b)$ or $\deg_w(x^a) = \deg_w(x^b)$ and there exists $1 \le i \le n$ such that $a_n = b_n, \ldots, a_{i+1} = b_{i+1}, a_i > b_i$. This term order is called 'wp' in Singular.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ, 3, order=TermOrder('wdegrevlex',(1,2,3)))
sage: x > y
False
```

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```
sage: x > x^2
False
sage: x > 1
True
sage: x^1*y^2 > x^2*z
True
sage: y*z > x^3*y
False
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, Integer(3), order=TermOrder('wdegrevlex',(Integer(1),
↪Integer(2),Integer(3))), names=('x', 'y', 'z',)); (x, y, z,) = P._first_ngens(3)
>>> x > y
False
>>> x > x**Integer(2)
False
>>> x > Integer(1)
True
>>> x**Integer(1)*y**Integer(2) > x**Integer(2)*z
True
>>> y*z > x**Integer(3)*y
False
```

Weighted degree lexicographic (wdeglex), positive integral weights

Let $\deg_w(x^a) = a_1w_1 + a_2w_2 + \cdots + a_nw_n$ with weights w, then $x^a < x^b$ if and only if $\deg_w(x^a) < \deg_w(x^b)$ or $\deg_w(x^a) = \deg_w(x^b)$ and there exists $1 \leq i \leq n$ such that $a_1 = b_1, \ldots, a_{i-1} = b_{i-1}, a_i < b_i$. This term order is called 'Wp' in Singular.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ, 3, order=TermOrder('wdeglex',(1,2,3)))
sage: x > y
False
sage: x > x^2
False
sage: x > 1
True
sage: x^1*y^2 > x^2*z
False
sage: y*z > x^3*y
False
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, Integer(3), order=TermOrder('wdeglex',(Integer(1),
↪Integer(2),Integer(3))), names=('x', 'y', 'z',)); (x, y, z,) = P._first_ngens(3)
>>> x > y
False
>>> x > x**Integer(2)
False
>>> x > Integer(1)
True
>>> x**Integer(1)*y**Integer(2) > x**Integer(2)*z
```

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```
False
>>> y*z > x**Integer(3)*y
False
```

Negative weighted degree reverse lexicographic (negwdegrevlex), positive integral weights

Let $\deg_w(x^a) = a_1w_1 + a_2w_2 + \dots + a_nw_n$ with weights w, then $x^a < x^b$ if and only if $\deg_w(x^a) > \deg_w(x^b)$ or $\deg_w(x^a) = \deg_w(x^b)$ and there exists $1 \le i \le n$ such that $a_n = b_n, \dots, a_{i+1} = b_{i+1}, a_i > b_i$. This term order is called 'ws' in Singular.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ, 3, order=TermOrder('negwdegrevlex',(1,2,3)))
sage: x > y
True
sage: x > x^2
True
sage: x > 1
False
sage: x^1*y^2 > x^2*z
True
sage: y*z > x^3*y
False
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, Integer(3), order=TermOrder('negwdegrevlex',
↪(Integer(1),Integer(2),Integer(3))), names=('x', 'y', 'z',)); (x, y, z,) = P._
↪first_ngens(3)
>>> x > y
True
>>> x > x**Integer(2)
True
>>> x > Integer(1)
False
>>> x**Integer(1)*y**Integer(2) > x**Integer(2)*z
True
>>> y*z > x**Integer(3)*y
False
```

Degree negative lexicographic (degneglex)

Let $\deg(x^a) = a_1 + a_2 + \dots + a_n$, then $x^a < x^b$ if and only if $\deg(x^a) < \deg(x^b)$ or $\deg(x^a) = \deg(x^b)$ and there exists $1 \le i \le n$ such that $a_1 = b_1, \dots, a_{i-1} = b_{i-1}, a_i > b_i$. This term order is called 'dp_asc' in PolyBoRi. Singular has the extra weight vector ordering `(a(1:n),ls)` for this purpose.

EXAMPLES:

```
sage: t = TermOrder('degneglex')
sage: P.<x,y,z> = PolynomialRing(QQ, order=t)
sage: x*y > y*z # indirect doctest
False
sage: x*y > x
True
```

```
>>> from sage.all import *
>>> t = TermOrder('degneglex')
>>> P = PolynomialRing(QQ, order=t, names=('x', 'y', 'z',)); (x, y, z,) = P._
↪first_ngens(3)
>>> x*y > y*z # indirect doctest
False
>>> x*y > x
True
```

Negative weighted degree lexicographic (negwdeglex), positive integral weights

Let $\deg_w(x^a) = a_1 w_1 + a_2 w_2 + \cdots + a_n w_n$ with weights w, then $x^a < x^b$ if and only if $\deg_w(x^a) > \deg_w(x^b)$ or $\deg_w(x^a) = \deg_w(x^b)$ and there exists $1 \le i \le n$ such that $a_1 = b_1, \ldots, a_{i-1} = b_{i-1}, a_i < b_i$. This term order is called 'Ws' in Singular.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ, 3, order=TermOrder('negwdeglex',(1,2,3)))
sage: x > y
True
sage: x > x^2
True
sage: x > 1
False
sage: x^1*y^2 > x^2*z
False
sage: y*z > x^3*y
False
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, Integer(3), order=TermOrder('negwdeglex',(Integer(1),
↪Integer(2),Integer(3))), names=('x', 'y', 'z',)); (x, y, z,) = P._first_ngens(3)
>>> x > y
True
>>> x > x**Integer(2)
True
>>> x > Integer(1)
False
>>> x**Integer(1)*y**Integer(2) > x**Integer(2)*z
False
>>> y*z > x**Integer(3)*y
False
```

Of these, only 'degrevlex', 'deglex', 'degneglex', 'wdegrevlex', 'wdeglex', 'invlex' and 'lex' are global orders.

Sage also supports matrix term order. Given a square matrix A,

$x^a <_A x^b$ if and only if $Aa < Ab$

where $<$ is the lexicographic term order.

EXAMPLES:

```
sage: # needs sage.modules
sage: m = matrix(2, [2,3,0,1]); m
[2 3]
```

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```
[0 1]
sage: T = TermOrder(m); T
Matrix term order with matrix
[2 3]
[0 1]
sage: P.<a,b> = PolynomialRing(QQ, 2, order=T)
sage: P
Multivariate Polynomial Ring in a, b over Rational Field
sage: a > b
False
sage: a^3 < b^2
True
sage: S = TermOrder('M(2,3,0,1)')
sage: T == S
True
```

```
>>> from sage.all import *
>>> # needs sage.modules
>>> m = matrix(Integer(2), [Integer(2),Integer(3),Integer(0),Integer(1)]); m
[2 3]
[0 1]
>>> T = TermOrder(m); T
Matrix term order with matrix
[2 3]
[0 1]
>>> P = PolynomialRing(QQ, Integer(2), order=T, names=('a', 'b',)); (a, b,) = P._
↪first_ngens(2)
>>> P
Multivariate Polynomial Ring in a, b over Rational Field
>>> a > b
False
>>> a**Integer(3) < b**Integer(2)
True
>>> S = TermOrder('M(2,3,0,1)')
>>> T == S
True
```

Additionally all these monomial orders may be combined to product or block orders, defined as:

Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_m)$ be two ordered sets of variables, $<_1$ a monomial order on $k[x]$ and $<_2$ a monomial order on $k[y]$.

The product order (or block order) $< := (<_1, <_2)$ on $k[x, y]$ is defined as: $x^a y^b < x^A y^B$ if and only if $x^a <_1 x^A$ or ($x^a = x^A$ and $y^b <_2 y^B$).

These block orders are constructed in Sage by giving a comma separated list of monomial orders with the length of each block attached to them.

EXAMPLES:

As an example, consider constructing a block order where the first four variables are compared using the degree reverse lexicographical order while the last two variables in the second block are compared using negative lexicographical order.

```
sage: P.<a,b,c,d,e,f> = PolynomialRing(QQ, 6, order='degrevlex(4),neglex(2)')
sage: a > c^4
False
sage: a > e^4
True
sage: e > f^2
False
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, Integer(6), order='degrevlex(4),neglex(2)', names=('a', 'b
↪', 'c', 'd', 'e', 'f',)); (a, b, c, d, e, f,) = P._first_ngens(6)
>>> a > c**Integer(4)
False
>>> a > e**Integer(4)
True
>>> e > f**Integer(2)
False
```

The same result can be achieved by:

```
sage: T1 = TermOrder('degrevlex',4)
sage: T2 = TermOrder('neglex',2)
sage: T = T1 + T2
sage: P.<a,b,c,d,e,f> = PolynomialRing(QQ, 6, order=T)
sage: a > c^4
False
sage: a > e^4
True
```

```
>>> from sage.all import *
>>> T1 = TermOrder('degrevlex',Integer(4))
>>> T2 = TermOrder('neglex',Integer(2))
>>> T = T1 + T2
>>> P = PolynomialRing(QQ, Integer(6), order=T, names=('a', 'b', 'c', 'd', 'e', 'f',
↪)); (a, b, c, d, e, f,) = P._first_ngens(6)
>>> a > c**Integer(4)
False
>>> a > e**Integer(4)
True
```

If any other unsupported term order is given the provided string can be forced to be passed through as is to Singular, Macaulay2, and Magma. This ensures that it is for example possible to calculate a Groebner basis with respect to some term order Singular supports but Sage doesn't:

```
sage: T = TermOrder("royalorder")
Traceback (most recent call last):
...
ValueError: unknown term order 'royalorder'
sage: T = TermOrder("royalorder", force=True)
sage: T
royalorder term order
sage: T.singular_str()
'royalorder'
```

```
>>> from sage.all import *
>>> T = TermOrder("royalorder")
Traceback (most recent call last):
...
ValueError: unknown term order 'royalorder'
>>> T = TermOrder("royalorder", force=True)
>>> T
royalorder term order
>>> T.singular_str()
'royalorder'
```

AUTHORS:

- David Joyner and William Stein: initial version of multi_polynomial_ring
- Kiran S. Kedlaya: added macaulay2 interface
- Martin Albrecht: implemented native term orders, refactoring
- Kwankyu Lee: implemented matrix and weighted degree term orders
- Simon King (2011-06-06): added termorder_from_singular

class sage.rings.polynomial.term_order.**TermOrder**(*name='lex'*, *n=0*, *force=False*)

Bases: SageObject

A term order.

See sage.rings.polynomial.term_order for details on supported term orders.

blocks()

Return the term order blocks of self.

NOTE:

This method has been added in Issue #11316. There used to be an *attribute* of the same name and the same content. So, it is a backward incompatible syntax change.

EXAMPLES:

```
sage: t = TermOrder('deglex',2) + TermOrder('lex',2)
sage: t.blocks()
(Degree lexicographic term order, Lexicographic term order)
```

```
>>> from sage.all import *
>>> t = TermOrder('deglex',Integer(2)) + TermOrder('lex',Integer(2))
>>> t.blocks()
(Degree lexicographic term order, Lexicographic term order)
```

property greater_tuple

The default greater_tuple method for this term order.

EXAMPLES:

```
sage: O = TermOrder()
sage: O.greater_tuple.__func__ is O.greater_tuple_lex.__func__
True
sage: O = TermOrder('deglex')
```

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```
sage: O.greater_tuple.__func__ is O.greater_tuple_deglex.__func__
True
```

```
>>> from sage.all import *
>>> O = TermOrder()
>>> O.greater_tuple.__func__ is O.greater_tuple_lex.__func__
True
>>> O = TermOrder('deglex')
>>> O.greater_tuple.__func__ is O.greater_tuple_deglex.__func__
True
```

greater_tuple_block (f, g)

Return the greater exponent tuple with respect to the block order as specified when constructing this element.

This method is called by the lm/lc/lt methods of `MPolynomial_polydict`.

INPUT:

- `f` – exponent tuple
- `g` – exponent tuple

EXAMPLES:

```
sage: P.<a,b,c,d,e,f> = PolynomialRing(QQbar, 6,                              #␣
↪needs sage.rings.number_field
....:                                  order='degrevlex(3),degrevlex(3)')
sage: f = a + c^4; f.lm()   # indirect doctest                               #␣
↪needs sage.rings.number_field
c^4
sage: g = a + e^4; g.lm()                                                     #␣
↪needs sage.rings.number_field
a
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQbar, Integer(6),                             # needs␣
↪sage.rings.number_field
...                                  order='degrevlex(3),degrevlex(3)',␣
↪names=('a', 'b', 'c', 'd', 'e', 'f',)); (a, b, c, d, e, f,) = P._first_
↪ngens(6)
>>> f = a + c**Integer(4); f.lm()   # indirect doctest                        ␣
↪       # needs sage.rings.number_field
c^4
>>> g = a + e**Integer(4); g.lm()                                             ␣
↪       # needs sage.rings.number_field
a
```

greater_tuple_deglex (f, g)

Return the greater exponent tuple with respect to the total degree lexicographical term order.

INPUT:

- `f` – exponent tuple
- `g` – exponent tuple

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQbar, 3, order='deglex')                    #␣
↪needs sage.rings.number_field
sage: f = x + y; f.lm()   # indirect doctest                                   #␣
↪needs sage.rings.number_field
x
sage: f = x + y^2*z; f.lm()                                                    #␣
↪needs sage.rings.number_field
y^2*z
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQbar, Integer(3), order='deglex', names=('x', 'y', 'z
↪',)); (x, y, z,) = P._first_ngens(3)# needs sage.rings.number_field
>>> f = x + y; f.lm()   # indirect doctest                                   #␣
↪needs sage.rings.number_field
x
>>> f = x + y**Integer(2)*z; f.lm()                                            ␣
↪       # needs sage.rings.number_field
y^2*z
```

This method is called by the lm/lc/lt methods of `MPolynomial_polydict`.

greater_tuple_degneglex(*f*, *g*)

Return the greater exponent tuple with respect to the degree negative lexicographical term order.

INPUT:

- `f` – exponent tuple
- `g` – exponent tuple

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQbar, 3, order='degneglex')                 #␣
↪needs sage.rings.number_field
sage: f = x + y; f.lm()   # indirect doctest                                   #␣
↪needs sage.rings.number_field
y
sage: f = x + y^2*z; f.lm()                                                    #␣
↪needs sage.rings.number_field
y^2*z
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQbar, Integer(3), order='degneglex', names=('x', 'y',
↪'z',)); (x, y, z,) = P._first_ngens(3)# needs sage.rings.number_field
>>> f = x + y; f.lm()   # indirect doctest                                   #␣
↪needs sage.rings.number_field
y
>>> f = x + y**Integer(2)*z; f.lm()                                            ␣
↪       # needs sage.rings.number_field
y^2*z
```

This method is called by the lm/lc/lt methods of `MPolynomial_polydict`.

greater_tuple_degrevlex(*f*, *g*)

Return the greater exponent tuple with respect to the total degree reversed lexicographical term order.

INPUT:

- f – exponent tuple
- g – exponent tuple

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQbar, 3, order='degrevlex')                  #␣
↪needs sage.rings.number_field
sage: f = x + y; f.lm()   # indirect doctest                                   #␣
↪needs sage.rings.number_field
x
sage: f = x + y^2*z; f.lm()                                                    #␣
↪needs sage.rings.number_field
y^2*z
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQbar, Integer(3), order='degrevlex', names=('x', 'y',
↪'z',)); (x, y, z,) = P._first_ngens(3)# needs sage.rings.number_field
>>> f = x + y; f.lm()   # indirect doctest                                   #␣
↪needs sage.rings.number_field
x
>>> f = x + y**Integer(2)*z; f.lm()                                           ␣
↪       # needs sage.rings.number_field
y^2*z
```

This method is called by the lm/lc/lt methods of MPolynomial_polydict.

greater_tuple_invlex(*f*, *g*)

Return the greater exponent tuple with respect to the inversed lexicographical term order.

INPUT:

- f – exponent tuple
- g – exponent tuple

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQbar, 3, order='invlex')                     #␣
↪needs sage.rings.number_field
sage: f = x + y; f.lm()   # indirect doctest                                   #␣
↪needs sage.rings.number_field
y
sage: f = y + x^2; f.lm()                                                      #␣
↪needs sage.rings.number_field
y
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQbar, Integer(3), order='invlex', names=('x', 'y', 'z
↪',)); (x, y, z,) = P._first_ngens(3)# needs sage.rings.number_field
>>> f = x + y; f.lm()   # indirect doctest                                   #␣
↪needs sage.rings.number_field
```

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```
y
>>> f = y + x**Integer(2); f.lm()                                                    ␣
↪       # needs sage.rings.number_field
y
```

This method is called by the lm/lc/lt methods of `MPolynomial_polydict`.

greater_tuple_lex(*f*, *g*)

Return the greater exponent tuple with respect to the lexicographical term order.

INPUT:

- `f` – exponent tuple
- `g` – exponent tuple

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQbar, 3, order='lex')                            #␣
↪needs sage.rings.number_field
sage: f = x + y^2; f.lm()   # indirect doctest                                     #␣
↪needs sage.rings.number_field
x
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQbar, Integer(3), order='lex', names=('x', 'y', 'z',
↪)); (x, y, z,) = P._first_ngens(3)# needs sage.rings.number_field
>>> f = x + y**Integer(2); f.lm()   # indirect doctest                               ␣
↪       # needs sage.rings.number_field
x
```

This method is called by the lm/lc/lt methods of `MPolynomial_polydict`.

greater_tuple_matrix(*f*, *g*)

Return the greater exponent tuple with respect to the matrix term order.

INPUT:

- `f` – exponent tuple
- `g` – exponent tuple

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQbar, 2, order='m(1,3,1,0)')                       #␣
↪needs sage.rings.number_field
sage: y > x^2   # indirect doctest                                                 #␣
↪needs sage.rings.number_field
True
sage: y > x^3                                                                      #␣
↪needs sage.rings.number_field
False
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQbar, Integer(2), order='m(1,3,1,0)', names=('x', 'y',
↪)); (x, y,) = P._first_ngens(2)# needs sage.rings.number_field
```

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```
>>> y > x**Integer(2)  # indirect doctest                                   ␣
↪       # needs sage.rings.number_field
True
>>> y > x**Integer(3)                                                       ␣
↪       # needs sage.rings.number_field
False
```

greater_tuple_negdeglex(*f*, *g*)

Return the greater exponent tuple with respect to the negative degree lexicographical term order.

INPUT:

- `f` – exponent tuple
- `g` – exponent tuple

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: P.<x,y,z> = PolynomialRing(QQbar, 3, order='negdeglex')
sage: f = x + y; f.lm() # indirect doctest
x
sage: f = x + x^2; f.lm()
x
sage: f = x^2*y*z^2 + x*y^3*z; f.lm()
x^2*y*z^2
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> P = PolynomialRing(QQbar, Integer(3), order='negdeglex', names=('x', 'y',
↪'z',)); (x, y, z,) = P._first_ngens(3)
>>> f = x + y; f.lm() # indirect doctest
x
>>> f = x + x**Integer(2); f.lm()
x
>>> f = x**Integer(2)*y*z**Integer(2) + x*y**Integer(3)*z; f.lm()
x^2*y*z^2
```

This method is called by the lm/lc/lt methods of `MPolynomial_polydict`.

greater_tuple_negdegrevlex(*f*, *g*)

Return the greater exponent tuple with respect to the negative degree reverse lexicographical term order.

INPUT:

- `f` – exponent tuple
- `g` – exponent tuple

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: P.<x,y,z> = PolynomialRing(QQbar, 3, order='negdegrevlex')
sage: f = x + y; f.lm() # indirect doctest
x
sage: f = x + x^2; f.lm()
```

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```
x
sage: f = x^2*y*z^2 + x*y^3*z; f.lm()
x*y^3*z
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> P = PolynomialRing(QQbar, Integer(3), order='negdegrevlex', names=('x', 'y
↪', 'z',)); (x, y, z,) = P._first_ngens(3)
>>> f = x + y; f.lm() # indirect doctest
x
>>> f = x + x**Integer(2); f.lm()
x
>>> f = x**Integer(2)*y*z**Integer(2) + x*y**Integer(3)*z; f.lm()
x*y^3*z
```

This method is called by the lm/lc/lt methods of `MPolynomial_polydict`.

greater_tuple_neglex(*f*, *g*)

Return the greater exponent tuple with respect to the negative lexicographical term order.

This method is called by the lm/lc/lt methods of `MPolynomial_polydict`.

INPUT:

- `f` – exponent tuple
- `g` – exponent tuple

EXAMPLES:

```
sage: P.<a,b,c,d,e,f> = PolynomialRing(QQbar, 6,                              #␣
↪needs sage.rings.number_field
....:                                   order='degrevlex(3),degrevlex(3)')
sage: f = a + c^4; f.lm()   # indirect doctest                                 #␣
↪needs sage.rings.number_field
c^4
sage: g = a + e^4; g.lm()                                                      #␣
↪needs sage.rings.number_field
a
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQbar, Integer(6),                           # needs␣
↪sage.rings.number_field
...                                 order='degrevlex(3),degrevlex(3)',␣
↪names=('a', 'b', 'c', 'd', 'e', 'f',)); (a, b, c, d, e, f,) = P._first_
↪ngens(6)
>>> f = a + c**Integer(4); f.lm()   # indirect doctest                        ␣
↪       # needs sage.rings.number_field
c^4
>>> g = a + e**Integer(4); g.lm()                                             ␣
↪       # needs sage.rings.number_field
a
```

greater_tuple_negwdeglex(*f*, *g*)

Return the greater exponent tuple with respect to the negative weighted degree lexicographical term order.

INPUT:

- `f` – exponent tuple
- `g` – exponent tuple

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: t = TermOrder('negwdeglex',(1,2,3))
sage: P.<x,y,z> = PolynomialRing(QQbar, 3, order=t)
sage: f = x + y; f.lm() # indirect doctest
x
sage: f = x + x^2; f.lm()
x
sage: f = x^3 + z; f.lm()
x^3
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> t = TermOrder('negwdeglex',(Integer(1),Integer(2),Integer(3)))
>>> P = PolynomialRing(QQbar, Integer(3), order=t, names=('x', 'y', 'z',));
↪(x, y, z,) = P._first_ngens(3)
>>> f = x + y; f.lm() # indirect doctest
x
>>> f = x + x**Integer(2); f.lm()
x
>>> f = x**Integer(3) + z; f.lm()
x^3
```

This method is called by the lm/lc/lt methods of `MPolynomial_polydict`.

greater_tuple_negwdegrevlex(f, g)

Return the greater exponent tuple with respect to the negative weighted degree reverse lexicographical term order.

INPUT:

- `f` – exponent tuple
- `g` – exponent tuple

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: t = TermOrder('negwdegrevlex',(1,2,3))
sage: P.<x,y,z> = PolynomialRing(QQbar, 3, order=t)
sage: f = x + y; f.lm() # indirect doctest
x
sage: f = x + x^2; f.lm()
x
sage: f = x^3 + z; f.lm()
x^3
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> t = TermOrder('negwdegrevlex',(Integer(1),Integer(2),Integer(3)))
```

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```
>>> P = PolynomialRing(QQbar, Integer(3), order=t, names=('x', 'y', 'z',));␣
↪(x, y, z,) = P._first_ngens(3)
>>> f = x + y; f.lm() # indirect doctest
x
>>> f = x + x**Integer(2); f.lm()
x
>>> f = x**Integer(3) + z; f.lm()
x^3
```

This method is called by the lm/lc/lt methods of `MPolynomial_polydict`.

greater_tuple_wdeglex(*f*, *g*)

Return the greater exponent tuple with respect to the weighted degree lexicographical term order.

INPUT:

- `f` – exponent tuple
- `g` – exponent tuple

EXAMPLES:

```
sage: t = TermOrder('wdeglex',(1,2,3))
sage: P.<x,y,z> = PolynomialRing(QQbar, 3, order=t)                         #␣
↪needs sage.rings.number_field
sage: f = x + y; f.lm()   # indirect doctest                                 #␣
↪needs sage.rings.number_field
y
sage: f = x*y + z; f.lm()                                                    #␣
↪needs sage.rings.number_field
x*y
```

```
>>> from sage.all import *
>>> t = TermOrder('wdeglex',(Integer(1),Integer(2),Integer(3)))
>>> P = PolynomialRing(QQbar, Integer(3), order=t, names=('x', 'y', 'z',));␣
↪(x, y, z,) = P._first_ngens(3)# needs sage.rings.number_field
>>> f = x + y; f.lm()   # indirect doctest                                 #␣
↪needs sage.rings.number_field
y
>>> f = x*y + z; f.lm()                                                    #␣
↪needs sage.rings.number_field
x*y
```

This method is called by the lm/lc/lt methods of `MPolynomial_polydict`.

greater_tuple_wdegrevlex(*f*, *g*)

Return the greater exponent tuple with respect to the weighted degree reverse lexicographical term order.

INPUT:

- `f` – exponent tuple
- `g` – exponent tuple

EXAMPLES:

```
sage: t = TermOrder('wdegrevlex',(1,2,3))
sage: P.<x,y,z> = PolynomialRing(QQbar, 3, order=t)                      #␣
↪needs sage.rings.number_field
sage: f = x + y; f.lm()   # indirect doctest                             #␣
↪needs sage.rings.number_field
y
sage: f = x + y^2*z; f.lm()                                              #␣
↪needs sage.rings.number_field
y^2*z
```

```
>>> from sage.all import *
>>> t = TermOrder('wdegrevlex',(Integer(1),Integer(2),Integer(3)))
>>> P = PolynomialRing(QQbar, Integer(3), order=t, names=('x', 'y', 'z',));␣
↪(x, y, z,) = P._first_ngens(3)# needs sage.rings.number_field
>>> f = x + y; f.lm()   # indirect doctest                             #␣
↪needs sage.rings.number_field
y
>>> f = x + y**Integer(2)*z; f.lm()                                      ␣
↪       # needs sage.rings.number_field
y^2*z
```

This method is called by the lm/lc/lt methods of `MPolynomial_polydict`.

is_block_order()

Return `True` if `self` is a block term order.

EXAMPLES:

```
sage: t = TermOrder('deglex',2) + TermOrder('lex',2)
sage: t.is_block_order()
True
```

```
>>> from sage.all import *
>>> t = TermOrder('deglex',Integer(2)) + TermOrder('lex',Integer(2))
>>> t.is_block_order()
True
```

is_global()

Return `True` if this term order is definitely global. Return false otherwise, which includes unknown term orders.

EXAMPLES:

```
sage: T = TermOrder('lex')
sage: T.is_global()
True
sage: T = TermOrder('degrevlex', 3) + TermOrder('degrevlex', 3)
sage: T.is_global()
True
sage: T = TermOrder('degrevlex', 3) + TermOrder('negdegrevlex', 3)
sage: T.is_global()
False
sage: T = TermOrder('degneglex', 3)
sage: T.is_global()
```

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```
True
sage: T = TermOrder('invlex', 3)
sage: T.is_global()
True
```

```
>>> from sage.all import *
>>> T = TermOrder('lex')
>>> T.is_global()
True
>>> T = TermOrder('degrevlex', Integer(3)) + TermOrder('degrevlex',␣
↪Integer(3))
>>> T.is_global()
True
>>> T = TermOrder('degrevlex', Integer(3)) + TermOrder('negdegrevlex',␣
↪Integer(3))
>>> T.is_global()
False
>>> T = TermOrder('degneglex', Integer(3))
>>> T.is_global()
True
>>> T = TermOrder('invlex', Integer(3))
>>> T.is_global()
True
```

is_local()

Return `True` if this term order is definitely local. Return false otherwise, which includes unknown term orders.

EXAMPLES:

```
sage: T = TermOrder('lex')
sage: T.is_local()
False
sage: T = TermOrder('negdeglex', 3) + TermOrder('negdegrevlex', 3)
sage: T.is_local()
True
sage: T = TermOrder('degrevlex', 3) + TermOrder('negdegrevlex', 3)
sage: T.is_local()
False
```

```
>>> from sage.all import *
>>> T = TermOrder('lex')
>>> T.is_local()
False
>>> T = TermOrder('negdeglex', Integer(3)) + TermOrder('negdegrevlex',␣
↪Integer(3))
>>> T.is_local()
True
>>> T = TermOrder('degrevlex', Integer(3)) + TermOrder('negdegrevlex',␣
↪Integer(3))
>>> T.is_local()
False
```

is_weighted_degree_order()

Return True if self is a weighted degree term order.

EXAMPLES:

```
sage: t = TermOrder('wdeglex',(2,3))
sage: t.is_weighted_degree_order()
True
```

```
>>> from sage.all import *
>>> t = TermOrder('wdeglex',(Integer(2),Integer(3)))
>>> t.is_weighted_degree_order()
True
```

macaulay2_str()

Return a Macaulay2 representation of self.

Used to convert polynomial rings to their Macaulay2 representation.

EXAMPLES:

```
sage: P = PolynomialRing(GF(127), 8, names='x', order='degrevlex(3),lex(5)')
sage: T = P.term_order()
sage: T.macaulay2_str()
'{GRevLex => 3,Lex => 5}'
sage: P._macaulay2_().options()['MonomialOrder']     # optional - macaulay2
{MonomialSize => 16  }
{GRevLex => {1, 1, 1}}
{Lex => 5            }
{Position => Up      }
```

```
>>> from sage.all import *
>>> P = PolynomialRing(GF(Integer(127)), Integer(8), names='x', order=
↪'degrevlex(3),lex(5)')
>>> T = P.term_order()
>>> T.macaulay2_str()
'{GRevLex => 3,Lex => 5}'
>>> P._macaulay2_().options()['MonomialOrder']     # optional - macaulay2
{MonomialSize => 16  }
{GRevLex => {1, 1, 1}}
{Lex => 5            }
{Position => Up      }
```

magma_str()

Return a MAGMA representation of self.

Used to convert polynomial rings to their MAGMA representation.

EXAMPLES:

```
sage: P = PolynomialRing(GF(127), 10, names='x', order='degrevlex')
sage: magma(P)                                        # optional - magma
Polynomial ring of rank 10 over GF(127)
Order: Graded Reverse Lexicographical
Variables: x0, x1, x2, x3, x4, x5, x6, x7, x8, x9
```

```
>>> from sage.all import *
>>> P = PolynomialRing(GF(Integer(127)), Integer(10), names='x', order=
↪'degrevlex')
>>> magma(P)                                          # optional - magma
Polynomial ring of rank 10 over GF(127)
Order: Graded Reverse Lexicographical
Variables: x0, x1, x2, x3, x4, x5, x6, x7, x8, x9
```

```
sage: T = P.term_order()
sage: T.magma_str()
'"grevlex"'
```

```
>>> from sage.all import *
>>> T = P.term_order()
>>> T.magma_str()
'"grevlex"'
```

matrix()

Return the matrix defining matrix term order.

EXAMPLES:

```
sage: t = TermOrder("M(1,2,0,1)")                                          #␣
↪needs sage.modules
sage: t.matrix()                                                           #␣
↪needs sage.modules
[1 2]
[0 1]
```

```
>>> from sage.all import *
>>> t = TermOrder("M(1,2,0,1)")                                          #␣
↪needs sage.modules
>>> t.matrix()                                                           #␣
↪needs sage.modules
[1 2]
[0 1]
```

name()

EXAMPLES:

```
sage: TermOrder('lex').name()
'lex'
```

```
>>> from sage.all import *
>>> TermOrder('lex').name()
'lex'
```

singular_moreblocks()

Return a the number of additional blocks SINGULAR needs to allocate for handling non-native orderings like `degneglex`.

EXAMPLES:

```
sage: P = PolynomialRing(GF(127), 10, names='x',
....:                    order='lex(3),deglex(5),lex(2)')
sage: T = P.term_order()
sage: T.singular_moreblocks()
0
sage: P = PolynomialRing(GF(127), 10, names='x',
....:                    order='lex(3),degneglex(5),lex(2)')
sage: T = P.term_order()
sage: T.singular_moreblocks()
1
sage: P = PolynomialRing(GF(127), 10, names='x',
....:                    order='degneglex(5),degneglex(5)')
sage: T = P.term_order()
sage: T.singular_moreblocks()
2
```

```
>>> from sage.all import *
>>> P = PolynomialRing(GF(Integer(127)), Integer(10), names='x',
...                    order='lex(3),deglex(5),lex(2)')
>>> T = P.term_order()
>>> T.singular_moreblocks()
0
>>> P = PolynomialRing(GF(Integer(127)), Integer(10), names='x',
...                    order='lex(3),degneglex(5),lex(2)')
>>> T = P.term_order()
>>> T.singular_moreblocks()
1
>>> P = PolynomialRing(GF(Integer(127)), Integer(10), names='x',
...                    order='degneglex(5),degneglex(5)')
>>> T = P.term_order()
>>> T.singular_moreblocks()
2
```

singular_str()

Return a SINGULAR representation of `self`.

Used to convert polynomial rings to their SINGULAR representation.

EXAMPLES:

```
sage: P = PolynomialRing(GF(127), 10, names='x',
....:                    order='lex(3),deglex(5),lex(2)')
sage: T = P.term_order()
sage: T.singular_str()
'(lp(3),Dp(5),lp(2))'
sage: P._singular_()                                                    #␣
↪needs sage.libs.singular
polynomial ring, over a field, global ordering
// coefficients: ZZ/127...
// number of vars : 10
//        block   1 : ordering lp
//                  : names    x0 x1 x2
//        block   2 : ordering Dp
```

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```
//                    : names     x3 x4 x5 x6 x7
//         block   3 : ordering lp
//                    : names     x8 x9
//         block   4 : ordering C
```

```
>>> from sage.all import *
>>> P = PolynomialRing(GF(Integer(127)), Integer(10), names='x',
...                        order='lex(3),deglex(5),lex(2)')
>>> T = P.term_order()
>>> T.singular_str()
'(lp(3),Dp(5),lp(2))'
>>> P._singular_()                                                          #␣
↪needs sage.libs.singular
polynomial ring, over a field, global ordering
// coefficients: ZZ/127...
// number of vars : 10
//         block   1 : ordering lp
//                    : names     x0 x1 x2
//         block   2 : ordering Dp
//                    : names     x3 x4 x5 x6 x7
//         block   3 : ordering lp
//                    : names     x8 x9
//         block   4 : ordering C
```

The `degneglex` ordering is somehow special, it looks like a block ordering in SINGULAR:

```
sage: T = TermOrder("degneglex", 2)
sage: P = PolynomialRing(QQ,2, names='x', order=T)
sage: T = P.term_order()
sage: T.singular_str()
'(a(1:2),ls(2))'

sage: T = TermOrder("degneglex", 2) + TermOrder("degneglex", 2)
sage: P = PolynomialRing(QQ,4, names='x', order=T)
sage: T = P.term_order()
sage: T.singular_str()
'(a(1:2),ls(2),a(1:2),ls(2))'
sage: P._singular_()                                                        #␣
↪needs sage.libs.singular
polynomial ring, over a field, global ordering
// coefficients: QQ...
// number of vars : 4
//         block   1 : ordering a
//                    : names     x0 x1
//                    : weights    1  1
//         block   2 : ordering ls
//                    : names     x0 x1
//         block   3 : ordering a
//                    : names     x2 x3
//                    : weights    1  1
//         block   4 : ordering ls
//                    : names     x2 x3
```

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```
//        block   5 : ordering C
```

```
>>> from sage.all import *
>>> T = TermOrder("degneglex", Integer(2))
>>> P = PolynomialRing(QQ,Integer(2), names='x', order=T)
>>> T = P.term_order()
>>> T.singular_str()
'(a(1:2),ls(2))'

>>> T = TermOrder("degneglex", Integer(2)) + TermOrder("degneglex",␣
↪Integer(2))
>>> P = PolynomialRing(QQ,Integer(4), names='x', order=T)
>>> T = P.term_order()
>>> T.singular_str()
'(a(1:2),ls(2),a(1:2),ls(2))'
>>> P._singular_()                                                  #␣
↪needs sage.libs.singular
polynomial ring, over a field, global ordering
// coefficients: QQ...
// number of vars : 4
//        block   1 : ordering a
//                  : names    x0 x1
//                  : weights   1  1
//        block   2 : ordering ls
//                  : names    x0 x1
//        block   3 : ordering a
//                  : names    x2 x3
//                  : weights   1  1
//        block   4 : ordering ls
//                  : names    x2 x3
//        block   5 : ordering C
```

The position of the `ordering C` block can be controlled by setting `_singular_ringorder_column` attribute to an integer:

```
sage: T = TermOrder("degneglex", 2) + TermOrder("degneglex", 2)
sage: T._singular_ringorder_column = 0
sage: P = PolynomialRing(QQ, 4, names='x', order=T)
sage: P._singular_()                                                  #␣
↪needs sage.libs.singular
polynomial ring, over a field, global ordering
// coefficients: QQ...
// number of vars : 4
//        block   1 : ordering C
//        block   2 : ordering a
//                  : names    x0 x1
//                  : weights   1  1
//        block   3 : ordering ls
//                  : names    x0 x1
//        block   4 : ordering a
//                  : names    x2 x3
//                  : weights   1  1
```

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```
//         block   5 : ordering ls
//                   : names    x2 x3

sage: T._singular_ringorder_column = 1
sage: P = PolynomialRing(QQ, 4, names='y', order=T)
sage: P._singular_()                                                        #␣
↪needs sage.libs.singular
polynomial ring, over a field, global ordering
// coefficients: QQ...
// number of vars : 4
//         block   1 : ordering c
//         block   2 : ordering a
//                   : names    y0 y1
//                   : weights   1  1
//         block   3 : ordering ls
//                   : names    y0 y1
//         block   4 : ordering a
//                   : names    y2 y3
//                   : weights   1  1
//         block   5 : ordering ls
//                   : names    y2 y3

sage: T._singular_ringorder_column = 2
sage: P = PolynomialRing(QQ, 4, names='z', order=T)
sage: P._singular_()                                                        #␣
↪needs sage.libs.singular
polynomial ring, over a field, global ordering
// coefficients: QQ...
// number of vars : 4
//         block   1 : ordering a
//                   : names    z0 z1
//                   : weights   1  1
//         block   2 : ordering C
//         block   3 : ordering ls
//                   : names    z0 z1
//         block   4 : ordering a
//                   : names    z2 z3
//                   : weights   1  1
//         block   5 : ordering ls
//                   : names    z2 z3
```

```
>>> from sage.all import *
>>> T = TermOrder("degneglex", Integer(2)) + TermOrder("degneglex",␣
↪Integer(2))
>>> T._singular_ringorder_column = Integer(0)
>>> P = PolynomialRing(QQ, Integer(4), names='x', order=T)
>>> P._singular_()                                                      #␣
↪needs sage.libs.singular
polynomial ring, over a field, global ordering
// coefficients: QQ...
// number of vars : 4
//         block   1 : ordering C
```

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```
//        block   2 : ordering a
//                  : names    x0 x1
//                  : weights   1  1
//        block   3 : ordering ls
//                  : names    x0 x1
//        block   4 : ordering a
//                  : names    x2 x3
//                  : weights   1  1
//        block   5 : ordering ls
//                  : names    x2 x3

>>> T._singular_ringorder_column = Integer(1)
>>> P = PolynomialRing(QQ, Integer(4), names='y', order=T)
>>> P._singular_()                                                      #␣
↪needs sage.libs.singular
polynomial ring, over a field, global ordering
// coefficients: QQ...
// number of vars : 4
//        block   1 : ordering c
//        block   2 : ordering a
//                  : names    y0 y1
//                  : weights   1  1
//        block   3 : ordering ls
//                  : names    y0 y1
//        block   4 : ordering a
//                  : names    y2 y3
//                  : weights   1  1
//        block   5 : ordering ls
//                  : names    y2 y3

>>> T._singular_ringorder_column = Integer(2)
>>> P = PolynomialRing(QQ, Integer(4), names='z', order=T)
>>> P._singular_()                                                      #␣
↪needs sage.libs.singular
polynomial ring, over a field, global ordering
// coefficients: QQ...
// number of vars : 4
//        block   1 : ordering a
//                  : names    z0 z1
//                  : weights   1  1
//        block   2 : ordering C
//        block   3 : ordering ls
//                  : names    z0 z1
//        block   4 : ordering a
//                  : names    z2 z3
//                  : weights   1  1
//        block   5 : ordering ls
//                  : names    z2 z3
```

property sortkey

The default `sortkey` method for this term order.

EXAMPLES:

```
sage: O = TermOrder()
sage: O.sortkey.__func__ is O.sortkey_lex.__func__
True
sage: O = TermOrder('deglex')
sage: O.sortkey.__func__ is O.sortkey_deglex.__func__
True
```

```
>>> from sage.all import *
>>> O = TermOrder()
>>> O.sortkey.__func__ is O.sortkey_lex.__func__
True
>>> O = TermOrder('deglex')
>>> O.sortkey.__func__ is O.sortkey_deglex.__func__
True
```

sortkey_block(*f*)

Return the sortkey of an exponent tuple with respect to the block order as specified when constructing this element.

INPUT:

- f – exponent tuple

EXAMPLES:

```
sage: P.<a,b,c,d,e,f> = PolynomialRing(QQbar, 6,                              #␣
↪needs sage.rings.number_field
....:                                order='degrevlex(3),degrevlex(3)')
sage: a > c^4   # indirect doctest                                         #␣
↪needs sage.rings.number_field
False
sage: a > e^4                                                              #␣
↪needs sage.rings.number_field
True
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQbar, Integer(6),                              # needs␣
↪sage.rings.number_field
...                                order='degrevlex(3),degrevlex(3)',␣
↪names=('a', 'b', 'c', 'd', 'e', 'f',)); (a, b, c, d, e, f,) = P._first_
↪ngens(6)
>>> a > c**Integer(4)   # indirect doctest                                  ␣
↪       # needs sage.rings.number_field
False
>>> a > e**Integer(4)                                                       ␣
↪       # needs sage.rings.number_field
True
```

sortkey_deglex(*f*)

Return the sortkey of an exponent tuple with respect to the degree lexicographical term order.

INPUT:

- f – exponent tuple

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQbar, 2, order='deglex')                   #␣
↪needs sage.rings.number_field
sage: x > y^2   # indirect doctest                                          #␣
↪needs sage.rings.number_field
False
sage: x > 1                                                                  #␣
↪needs sage.rings.number_field
True
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQbar, Integer(2), order='deglex', names=('x', 'y',));␣
↪(x, y,) = P._first_ngens(2)# needs sage.rings.number_field
>>> x > y**Integer(2)   # indirect doctest                                    ␣
↪       # needs sage.rings.number_field
False
>>> x > Integer(1)                                                            ␣
↪      # needs sage.rings.number_field
True
```

sortkey_degneglex(*f*)

Return the sortkey of an exponent tuple with respect to the degree negative lexicographical term order.

INPUT:

- f – exponent tuple

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQbar, 3, order='degneglex')              #␣
↪needs sage.rings.number_field
sage: x*y > y*z   # indirect doctest                                        #␣
↪needs sage.rings.number_field
False
sage: x*y > x                                                                #␣
↪needs sage.rings.number_field
True
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQbar, Integer(3), order='degneglex', names=('x', 'y',
↪'z',)); (x, y, z,) = P._first_ngens(3)# needs sage.rings.number_field
>>> x*y > y*z   # indirect doctest                                          #␣
↪needs sage.rings.number_field
False
>>> x*y > x                                                                  #␣
↪needs sage.rings.number_field
True
```

sortkey_degrevlex(*f*)

Return the sortkey of an exponent tuple with respect to the degree reversed lexicographical term order.

INPUT:

- f – exponent tuple

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQbar, 2, order='degrevlex')                #␣
↪needs sage.rings.number_field
sage: x > y^2   # indirect doctest                                        #␣
↪needs sage.rings.number_field
False
sage: x > 1                                                               #␣
↪needs sage.rings.number_field
True
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQbar, Integer(2), order='degrevlex', names=('x', 'y',
↪)); (x, y,) = P._first_ngens(2)# needs sage.rings.number_field
>>> x > y**Integer(2)   # indirect doctest                                  ␣
↪       # needs sage.rings.number_field
False
>>> x > Integer(1)                                                          ␣
↪      # needs sage.rings.number_field
True
```

sortkey_invlex(*f*)

Return the sortkey of an exponent tuple with respect to the inversed lexicographical term order.

INPUT:

- `f` – exponent tuple

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQbar, 2, order='invlex')                   #␣
↪needs sage.rings.number_field
sage: x > y^2   # indirect doctest                                        #␣
↪needs sage.rings.number_field
False
sage: x > 1                                                               #␣
↪needs sage.rings.number_field
True
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQbar, Integer(2), order='invlex', names=('x', 'y',));␣
↪(x, y,) = P._first_ngens(2)# needs sage.rings.number_field
>>> x > y**Integer(2)   # indirect doctest                                  ␣
↪       # needs sage.rings.number_field
False
>>> x > Integer(1)                                                          ␣
↪      # needs sage.rings.number_field
True
```

sortkey_lex(*f*)

Return the sortkey of an exponent tuple with respect to the lexicographical term order.

INPUT:

- `f` – exponent tuple

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQbar, 2, order='lex')                        #␣
↪needs sage.rings.number_field
sage: x > y^2   # indirect doctest                                           #␣
↪needs sage.rings.number_field
True
sage: x > 1                                                                  #␣
↪needs sage.rings.number_field
True
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQbar, Integer(2), order='lex', names=('x', 'y',)); (x,
↪ y,) = P._first_ngens(2)# needs sage.rings.number_field
>>> x > y**Integer(2)   # indirect doctest                                    ␣
↪       # needs sage.rings.number_field
True
>>> x > Integer(1)                                                            ␣
↪      # needs sage.rings.number_field
True
```

sortkey_matrix(*f*)

Return the sortkey of an exponent tuple with respect to the matrix term order.

INPUT:

- `f` – exponent tuple

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQbar, 2, order='m(1,3,1,0)')                 #␣
↪needs sage.rings.number_field
sage: y > x^2   # indirect doctest                                           #␣
↪needs sage.rings.number_field
True
sage: y > x^3                                                                #␣
↪needs sage.rings.number_field
False
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQbar, Integer(2), order='m(1,3,1,0)', names=('x', 'y',
↪)); (x, y,) = P._first_ngens(2)# needs sage.rings.number_field
>>> y > x**Integer(2)   # indirect doctest                                    ␣
↪       # needs sage.rings.number_field
True
>>> y > x**Integer(3)                                                         ␣
↪       # needs sage.rings.number_field
False
```

sortkey_negdeglex(*f*)

Return the sortkey of an exponent tuple with respect to the negative degree lexicographical term order.

INPUT:

- `f` – exponent tuple

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQbar, 2, order='negdeglex')                    #␣
↪needs sage.rings.number_field
sage: x > y^2   # indirect doctest                                             #␣
↪needs sage.rings.number_field
True
sage: x > 1                                                                    #␣
↪needs sage.rings.number_field
False
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQbar, Integer(2), order='negdeglex', names=('x', 'y',
↪)); (x, y,) = P._first_ngens(2)# needs sage.rings.number_field
>>> x > y**Integer(2)   # indirect doctest                                     ␣
↪       # needs sage.rings.number_field
True
>>> x > Integer(1)                                                             ␣
↪      # needs sage.rings.number_field
False
```

sortkey_negdegrevlex(*f*)

Return the sortkey of an exponent tuple with respect to the negative degree reverse lexicographical term order.

INPUT:

- f – exponent tuple

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQbar, 2, order='negdegrevlex')                 #␣
↪needs sage.rings.number_field
sage: x > y^2   # indirect doctest                                             #␣
↪needs sage.rings.number_field
True
sage: x > 1                                                                    #␣
↪needs sage.rings.number_field
False
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQbar, Integer(2), order='negdegrevlex', names=('x', 'y
↪',)); (x, y,) = P._first_ngens(2)# needs sage.rings.number_field
>>> x > y**Integer(2)   # indirect doctest                                     ␣
↪       # needs sage.rings.number_field
True
>>> x > Integer(1)                                                             ␣
↪      # needs sage.rings.number_field
False
```

sortkey_neglex(*f*)

Return the sortkey of an exponent tuple with respect to the negative lexicographical term order.

INPUT:

- f – exponent tuple

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQbar, 2, order='neglex')                    #␣
↪needs sage.rings.number_field
sage: x > y^2   # indirect doctest                                          #␣
↪needs sage.rings.number_field
False
sage: x > 1                                                                 #␣
↪needs sage.rings.number_field
False
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQbar, Integer(2), order='neglex', names=('x', 'y',));␣
↪(x, y,) = P._first_ngens(2)# needs sage.rings.number_field
>>> x > y**Integer(2)   # indirect doctest                                   ␣
↪        # needs sage.rings.number_field
False
>>> x > Integer(1)                                                           ␣
↪       # needs sage.rings.number_field
False
```

sortkey_negwdeglex(*f*)

Return the sortkey of an exponent tuple with respect to the negative weighted degree lexicographical term order.

INPUT:

- `f` – exponent tuple

EXAMPLES:

```
sage: t = TermOrder('negwdeglex',(3,2))
sage: P.<x,y> = PolynomialRing(QQbar, 2, order=t)                           #␣
↪needs sage.rings.number_field
sage: x > y^2   # indirect doctest                                          #␣
↪needs sage.rings.number_field
True
sage: x^2 > y^3                                                             #␣
↪needs sage.rings.number_field
True
```

```
>>> from sage.all import *
>>> t = TermOrder('negwdeglex',(Integer(3),Integer(2)))
>>> P = PolynomialRing(QQbar, Integer(2), order=t, names=('x', 'y',)); (x, y,
↪) = P._first_ngens(2)# needs sage.rings.number_field
>>> x > y**Integer(2)   # indirect doctest                                   ␣
↪        # needs sage.rings.number_field
True
>>> x**Integer(2) > y**Integer(3)                                            ␣
↪                  # needs sage.rings.number_field
True
```

sortkey_negwdegrevlex(*f*)

Return the sortkey of an exponent tuple with respect to the negative weighted degree reverse lexicographical term order.

INPUT:

- f – exponent tuple

EXAMPLES:

```
sage: t = TermOrder('negwdegrevlex',(3,2))
sage: P.<x,y> = PolynomialRing(QQbar, 2, order=t)                        #␣
↪needs sage.rings.number_field
sage: x > y^2  # indirect doctest                                         #␣
↪needs sage.rings.number_field
True
sage: x^2 > y^3                                                           #␣
↪needs sage.rings.number_field
True
```

```
>>> from sage.all import *
>>> t = TermOrder('negwdegrevlex',(Integer(3),Integer(2)))
>>> P = PolynomialRing(QQbar, Integer(2), order=t, names=('x', 'y',)); (x, y,
↪) = P._first_ngens(2)# needs sage.rings.number_field
>>> x > y**Integer(2)   # indirect doctest                                  ␣
↪      # needs sage.rings.number_field
True
>>> x**Integer(2) > y**Integer(3)                                           ␣
↪                    # needs sage.rings.number_field
True
```

sortkey_wdeglex(*f*)

Return the sortkey of an exponent tuple with respect to the weighted degree lexicographical term order.

INPUT:

- f – exponent tuple

EXAMPLES:

```
sage: t = TermOrder('wdeglex',(3,2))
sage: P.<x,y> = PolynomialRing(QQbar, 2, order=t)                        #␣
↪needs sage.rings.number_field
sage: x > y^2  # indirect doctest                                         #␣
↪needs sage.rings.number_field
False
sage: x > y                                                               #␣
↪needs sage.rings.number_field
True
```

```
>>> from sage.all import *
>>> t = TermOrder('wdeglex',(Integer(3),Integer(2)))
>>> P = PolynomialRing(QQbar, Integer(2), order=t, names=('x', 'y',)); (x, y,
↪) = P._first_ngens(2)# needs sage.rings.number_field
>>> x > y**Integer(2)   # indirect doctest                                  ␣
↪      # needs sage.rings.number_field
False
>>> x > y                                                               #␣
↪needs sage.rings.number_field
True
```

sortkey_wdegrevlex(*f*)

Return the sortkey of an exponent tuple with respect to the weighted degree reverse lexicographical term order.

INPUT:

- f – exponent tuple

EXAMPLES:

```
sage: t = TermOrder('wdegrevlex',(3,2))
sage: P.<x,y> = PolynomialRing(QQbar, 2, order=t)                              #␣
↪needs sage.rings.number_field
sage: x > y^2   # indirect doctest                                              #␣
↪needs sage.rings.number_field
False
sage: x^2 > y^3                                                                 #␣
↪needs sage.rings.number_field
True
```

```
>>> from sage.all import *
>>> t = TermOrder('wdegrevlex',(Integer(3),Integer(2)))
>>> P = PolynomialRing(QQbar, Integer(2), order=t, names=('x', 'y',)); (x, y,
↪) = P._first_ngens(2)# needs sage.rings.number_field
>>> x > y**Integer(2)   # indirect doctest                                      ␣
↪      # needs sage.rings.number_field
False
>>> x**Integer(2) > y**Integer(3)                                               ␣
↪                # needs sage.rings.number_field
True
```

tuple_weight(*f*)

Return the weight of tuple f.

INPUT:

- f – exponent tuple

EXAMPLES:

```
sage: t = TermOrder('wdeglex',(1,2,3))
sage: P.<a,b,c> = PolynomialRing(QQbar, order=t)                                #␣
↪needs sage.rings.number_field
sage: P.term_order().tuple_weight([3,2,1])                                      #␣
↪needs sage.rings.number_field
10
```

```
>>> from sage.all import *
>>> t = TermOrder('wdeglex',(Integer(1),Integer(2),Integer(3)))
>>> P = PolynomialRing(QQbar, order=t, names=('a', 'b', 'c',)); (a, b, c,) =␣
↪P._first_ngens(3)# needs sage.rings.number_field
>>> P.term_order().tuple_weight([Integer(3),Integer(2),Integer(1)])             ␣
↪                          # needs sage.rings.number_field
10
```

weights()

Return the weights for weighted term orders.

EXAMPLES:

```
sage: t = TermOrder('wdeglex',(2,3))
sage: t.weights()
(2, 3)
```

```
>>> from sage.all import *
>>> t = TermOrder('wdeglex',(Integer(2),Integer(3)))
>>> t.weights()
(2, 3)
```

sage.rings.polynomial.term_order.**termorder_from_singular**(S)

Return the Sage term order of the basering in the given Singular interface.

INPUT:

An instance of the Singular interface.

EXAMPLES:

```
sage: from sage.rings.polynomial.term_order import termorder_from_singular
sage: singular.eval('ring r1 = (9,x),(a,b,c,d,e,f),(M((1,2,3,0)),wp(2,3),lp)')   #␣
↪needs sage.libs.singular
''
sage: termorder_from_singular(singular)                                           #␣
↪needs sage.libs.singular
Block term order with blocks:
(Matrix term order with matrix
   [1 2]
   [3 0],
 Weighted degree reverse lexicographic term order with weights (2, 3),
 Lexicographic term order of length 2)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.term_order import termorder_from_singular
>>> singular.eval('ring r1 = (9,x),(a,b,c,d,e,f),(M((1,2,3,0)),wp(2,3),lp)')   #␣
↪needs sage.libs.singular
''
>>> termorder_from_singular(singular)                                           #␣
↪needs sage.libs.singular
Block term order with blocks:
(Matrix term order with matrix
   [1 2]
   [3 0],
 Weighted degree reverse lexicographic term order with weights (2, 3),
 Lexicographic term order of length 2)
```

A term order in Singular also involves information on orders for modules. This information is reflected in `_singular_ringorder_column` attribute of the term order.

```
sage: # needs sage.libs.singular
sage: singular.ring(0, '(x,y,z,w)', '(C,dp(2),lp(2))')
```

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(continued from previous page)

```
polynomial ring, over a field, global ordering
// coefficients: QQ...
// number of vars : 4
//        block   1 : ordering C
//        block   2 : ordering dp
//                  : names    x y
//        block   3 : ordering lp
//                  : names    z w
sage: T = termorder_from_singular(singular)
sage: T
Block term order with blocks:
(Degree reverse lexicographic term order of length 2,
 Lexicographic term order of length 2)
sage: T._singular_ringorder_column
0

sage: # needs sage.libs.singular
sage: singular.ring(0, '(x,y,z,w)', '(c,dp(2),lp(2))')
polynomial ring, over a field, global ordering
// coefficients: QQ...
// number of vars : 4
//        block   1 : ordering c
//        block   2 : ordering dp
//                  : names    x y
//        block   3 : ordering lp
//                  : names    z w
sage: T = termorder_from_singular(singular)
sage: T
Block term order with blocks:
(Degree reverse lexicographic term order of length 2,
 Lexicographic term order of length 2)
sage: T._singular_ringorder_column
1
```

```
>>> from sage.all import *
>>> # needs sage.libs.singular
>>> singular.ring(Integer(0), '(x,y,z,w)', '(C,dp(2),lp(2))')
polynomial ring, over a field, global ordering
// coefficients: QQ...
// number of vars : 4
//        block   1 : ordering C
//        block   2 : ordering dp
//                  : names    x y
//        block   3 : ordering lp
//                  : names    z w
>>> T = termorder_from_singular(singular)
>>> T
Block term order with blocks:
(Degree reverse lexicographic term order of length 2,
 Lexicographic term order of length 2)
>>> T._singular_ringorder_column
0
```

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```
>>> # needs sage.libs.singular
>>> singular.ring(Integer(0), '(x,y,z,w)', '(c,dp(2),lp(2))')
polynomial ring, over a field, global ordering
// coefficients: QQ...
// number of vars : 4
//        block   1 : ordering c
//        block   2 : ordering dp
//                  : names    x y
//        block   3 : ordering lp
//                  : names    z w
>>> T = termorder_from_singular(singular)
>>> T
Block term order with blocks:
(Degree reverse lexicographic term order of length 2,
 Lexicographic term order of length 2)
>>> T._singular_ringorder_column
1
```

3.1.2 Base class for multivariate polynomial rings

class sage.rings.polynomial.multi_polynomial_ring_base.**BooleanPolynomialRing_base**

Bases: *MPolynomialRing_base*

Abstract base class for *BooleanPolynomialRing*.

This class is defined for the purpose of `isinstance` tests. It should not be instantiated.

EXAMPLES:

```
sage: from sage.rings.polynomial.multi_polynomial_ring_base import␣
↪BooleanPolynomialRing_base
sage: R.<x, y, z> = BooleanPolynomialRing()                                  #␣
↪needs sage.rings.polynomial.pbori
sage: isinstance(R, BooleanPolynomialRing_base)                              #␣
↪needs sage.rings.polynomial.pbori
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.multi_polynomial_ring_base import␣
↪BooleanPolynomialRing_base
>>> R = BooleanPolynomialRing(names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)# needs sage.rings.polynomial.pbori
>>> isinstance(R, BooleanPolynomialRing_base)                              #␣
↪needs sage.rings.polynomial.pbori
True
```

By design, there is only one direct implementation subclass:

```
sage: len(BooleanPolynomialRing_base.__subclasses__()) <= 1
True
```

```
>>> from sage.all import *
>>> len(BooleanPolynomialRing_base.__subclasses__()) <= Integer(1)
True
```

class sage.rings.polynomial.multi_polynomial_ring_base.**MPolynomialRing_base**

Bases: CommutativeRing

Create a polynomial ring in several variables over a commutative ring.

EXAMPLES:

```
sage: R.<x,y> = ZZ['x,y']; R
Multivariate Polynomial Ring in x, y over Integer Ring
sage: cat = Rings().Commutative()
sage: class CR(Parent):
....:     def __init__(self):
....:         Parent.__init__(self, self, category=cat)
....:     def __call__(self, x):
....:         return None
sage: cr = CR()
sage: cr.is_commutative()
True
sage: cr['x,y']
Multivariate Polynomial Ring in x, y over
<__main__.CR_with_category object at ...>
```

```
>>> from sage.all import *
>>> R = ZZ['x,y']; (x, y,) = R._first_ngens(2); R
Multivariate Polynomial Ring in x, y over Integer Ring
>>> cat = Rings().Commutative()
>>> class CR(Parent):
...     def __init__(self):
...         Parent.__init__(self, self, category=cat)
...     def __call__(self, x):
...         return None
>>> cr = CR()
>>> cr.is_commutative()
True
>>> cr['x,y']
Multivariate Polynomial Ring in x, y over
<__main__.CR_with_category object at ...>
```

change_ring(*base_ring=None*, *names=None*, *order=None*)

Return a new multivariate polynomial ring which is isomorphic to `self`, but has a different ordering given by the parameter `order` or names given by the parameter `names`.

INPUT:

- `base_ring` – a base ring
- `names` – variable names
- `order` – a term order

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(GF(127), 3, order='lex')
sage: x > y^2
True
sage: Q.<x,y,z> = P.change_ring(order='degrevlex')
sage: x > y^2
False
```

```
>>> from sage.all import *
>>> P = PolynomialRing(GF(Integer(127)), Integer(3), order='lex', names=('x',
↪'y', 'z',)); (x, y, z,) = P._first_ngens(3)
>>> x > y**Integer(2)
True
>>> Q = P.change_ring(order='degrevlex', names=('x', 'y', 'z',)); (x, y, z,)␣
↪= Q._first_ngens(3)
>>> x > y**Integer(2)
False
```

characteristic()

Return the characteristic of this polynomial ring.

EXAMPLES:

```
sage: R = PolynomialRing(QQ, 'x', 3)
sage: R.characteristic()
0
sage: R = PolynomialRing(GF(7), 'x', 20)
sage: R.characteristic()
7
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, 'x', Integer(3))
>>> R.characteristic()
0
>>> R = PolynomialRing(GF(Integer(7)), 'x', Integer(20))
>>> R.characteristic()
7
```

completion(*names=None*, *prec=20*, *extras={}*, ***kwds*)

Return the completion of `self` with respect to the ideal generated by the variable(s) `names`.

INPUT:

- `names` – (optional) variable or list/tuple of variables (given either as elements of the polynomial ring or as strings); the default is all variables of `self`
- `prec` – default precision of resulting power series ring, possibly infinite
- `extras` – passed as keywords to `PowerSeriesRing` or `LazyPowerSeriesRing`; can also be keyword arguments

EXAMPLES:

```
sage: P.<x,y,z,w> = PolynomialRing(ZZ)
sage: P.completion('w')
Power Series Ring in w over Multivariate Polynomial Ring in
```

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```
 x, y, z over Integer Ring
sage: P.completion((w,x,y))
Multivariate Power Series Ring in w, x, y over
 Univariate Polynomial Ring in z over Integer Ring
sage: Q.<w,x,y,z> = P.completion(); Q
Multivariate Power Series Ring in w, x, y, z over Integer Ring

sage: H = PolynomialRing(PolynomialRing(ZZ,3,'z'),4,'f'); H
Multivariate Polynomial Ring in f0, f1, f2, f3 over
 Multivariate Polynomial Ring in z0, z1, z2 over Integer Ring

sage: H.completion(H.gens())
Multivariate Power Series Ring in f0, f1, f2, f3 over
 Multivariate Polynomial Ring in z0, z1, z2 over Integer Ring

sage: H.completion(H.gens()[2])
Power Series Ring in f2 over
 Multivariate Polynomial Ring in f0, f1, f3 over
 Multivariate Polynomial Ring in z0, z1, z2 over Integer Ring

sage: P.<x,y,z,w> = PolynomialRing(ZZ)
sage: P.completion(prec=oo)                                             #␣
↪needs sage.combinat
Multivariate Lazy Taylor Series Ring in x, y, z, w over Integer Ring
sage: P.completion((w,x,y), prec=oo)                                    #␣
↪needs sage.combinat
Multivariate Lazy Taylor Series Ring in w, x, y over
 Univariate Polynomial Ring in z over Integer Ring
```

```
>>> from sage.all import *
>>> P = PolynomialRing(ZZ, names=('x', 'y', 'z', 'w',)); (x, y, z, w,) = P._
↪first_ngens(4)
>>> P.completion('w')
Power Series Ring in w over Multivariate Polynomial Ring in
 x, y, z over Integer Ring
>>> P.completion((w,x,y))
Multivariate Power Series Ring in w, x, y over
 Univariate Polynomial Ring in z over Integer Ring
>>> Q = P.completion(names=('w', 'x', 'y', 'z',)); (w, x, y, z,) = Q._first_
↪ngens(4); Q
Multivariate Power Series Ring in w, x, y, z over Integer Ring

>>> H = PolynomialRing(PolynomialRing(ZZ,Integer(3),'z'),Integer(4),'f'); H
Multivariate Polynomial Ring in f0, f1, f2, f3 over
 Multivariate Polynomial Ring in z0, z1, z2 over Integer Ring

>>> H.completion(H.gens())
Multivariate Power Series Ring in f0, f1, f2, f3 over
 Multivariate Polynomial Ring in z0, z1, z2 over Integer Ring

>>> H.completion(H.gens()[Integer(2)])
Power Series Ring in f2 over
```

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```
 Multivariate Polynomial Ring in f0, f1, f3 over
 Multivariate Polynomial Ring in z0, z1, z2 over Integer Ring

>>> P = PolynomialRing(ZZ, names=('x', 'y', 'z', 'w',)); (x, y, z, w,) = P._
↪first_ngens(4)
>>> P.completion(prec=oo)                                                    #␣
↪needs sage.combinat
Multivariate Lazy Taylor Series Ring in x, y, z, w over Integer Ring
>>> P.completion((w,x,y), prec=oo)                                           #␣
↪needs sage.combinat
Multivariate Lazy Taylor Series Ring in w, x, y over
 Univariate Polynomial Ring in z over Integer Ring
```

construction()

Return a functor `F` and base ring `R` such that `F(R) == self`.

EXAMPLES:

```
sage: S = ZZ['x,y']
sage: F, R = S.construction(); R
Integer Ring
sage: F
MPoly[x,y]
sage: F(R) == S
True
sage: F(R) == ZZ['x']['y']
False
```

```
>>> from sage.all import *
>>> S = ZZ['x,y']
>>> F, R = S.construction(); R
Integer Ring
>>> F
MPoly[x,y]
>>> F(R) == S
True
>>> F(R) == ZZ['x']['y']
False
```

flattening_morphism()

Return the flattening morphism of this polynomial ring.

EXAMPLES:

```
sage: QQ['a','b']['x','y'].flattening_morphism()
Flattening morphism:
  From: Multivariate Polynomial Ring in x, y
        over Multivariate Polynomial Ring in a, b over Rational Field
  To:   Multivariate Polynomial Ring in a, b, x, y over Rational Field

sage: QQ['x,y'].flattening_morphism()
Identity endomorphism of
 Multivariate Polynomial Ring in x, y over Rational Field
```

```
>>> from sage.all import *
>>> QQ['a','b']['x','y'].flattening_morphism()
Flattening morphism:
  From: Multivariate Polynomial Ring in x, y
        over Multivariate Polynomial Ring in a, b over Rational Field
  To:   Multivariate Polynomial Ring in a, b, x, y over Rational Field

>>> QQ['x,y'].flattening_morphism()
Identity endomorphism of
 Multivariate Polynomial Ring in x, y over Rational Field
```

gen(*n=0*)

interpolation(*bound*, **args*)

Create a polynomial with specified evaluations.

CALL FORMATS:

This function can be called in two ways:

1. `interpolation(bound, points, values)`
2. `interpolation(bound, function)`

INPUT:

- `bound` – either an integer bounding the total degree or a list/tuple of integers bounding the degree of the variables
- `points` – list/tuple containing the evaluation points
- `values` – list/tuple containing the desired values at `points`
- `function` – evaluable function in n variables, where n is the number of variables of the polynomial ring

OUTPUT:

1. A polynomial respecting the bounds and having `values` as values when evaluated at `points`.
2. A polynomial respecting the bounds and having the same values as `function` at exactly so many points so that the polynomial is unique.

EXAMPLES:

```
sage: def F(a, b, c):
....:     return a^3*b + b + c^2 + 25
....:
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: R.interpolation(4, F)                                                    #␣
↪needs sage.modules
x^3*y + z^2 + y + 25

sage: def F(a, b, c):
....:     return a^3*b + b + c^2 + 25
....:
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: R.interpolation([3,1,2], F)                                              #␣
↪needs sage.modules
x^3*y + z^2 + y + 25
```

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```
sage: def F(a, b, c):
....:     return a^3*b + b + c^2 + 25
....:
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: points = [(5,1,1),(7,2,2),(8,5,-1),(2,5,3),(1,4,0),(5,9,0),
....: (2,7,0),(1,10,13),(0,0,1),(-1,1,0),(2,5,3),(1,1,1),(7,4,11),
....: (12,1,9),(1,1,3),(4,-1,2),(0,1,5),(5,1,3),(3,1,-2),(2,11,3),
....: (4,12,19),(3,1,1),(5,2,-3),(12,1,1),(2,3,4)]
sage: R.interpolation([3,1,2], points, [F(*x) for x in points])          #␣
↪needs sage.modules
x^3*y + z^2 + y + 25
```

```
>>> from sage.all import *
>>> def F(a, b, c):
...     return a**Integer(3)*b + b + c**Integer(2) + Integer(25)
....:
>>> R = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> R.interpolation(Integer(4), F)                                            ␣
↪     # needs sage.modules
x^3*y + z^2 + y + 25

>>> def F(a, b, c):
...     return a**Integer(3)*b + b + c**Integer(2) + Integer(25)
....:
>>> R = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> R.interpolation([Integer(3),Integer(1),Integer(2)], F)                    ␣
↪                    # needs sage.modules
x^3*y + z^2 + y + 25

>>> def F(a, b, c):
...     return a**Integer(3)*b + b + c**Integer(2) + Integer(25)
....:
>>> R = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> points = [(Integer(5),Integer(1),Integer(1)),(Integer(7),Integer(2),
↪Integer(2)),(Integer(8),Integer(5),-Integer(1)),(Integer(2),Integer(5),
↪Integer(3)),(Integer(1),Integer(4),Integer(0)),(Integer(5),Integer(9),
↪Integer(0)),
...  (Integer(2),Integer(7),Integer(0)),(Integer(1),Integer(10),Integer(13)),
↪(Integer(0),Integer(0),Integer(1)),(-Integer(1),Integer(1),Integer(0)),
↪(Integer(2),Integer(5),Integer(3)),(Integer(1),Integer(1),Integer(1)),
↪(Integer(7),Integer(4),Integer(11)),
...  (Integer(12),Integer(1),Integer(9)),(Integer(1),Integer(1),Integer(3)),
↪(Integer(4),-Integer(1),Integer(2)),(Integer(0),Integer(1),Integer(5)),
↪(Integer(5),Integer(1),Integer(3)),(Integer(3),Integer(1),-Integer(2)),
↪(Integer(2),Integer(11),Integer(3)),
...  (Integer(4),Integer(12),Integer(19)),(Integer(3),Integer(1),Integer(1)),
↪(Integer(5),Integer(2),-Integer(3)),(Integer(12),Integer(1),Integer(1)),
↪(Integer(2),Integer(3),Integer(4))]
```

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```
>>> R.interpolation([Integer(3),Integer(1),Integer(2)], points, [F(*x) for x␣
↪in points])             # needs sage.modules
x^3*y + z^2 + y + 25
```

ALGORITHM:

Solves a linear system of equations with the linear algebra module. If the points are not specified, it samples exactly as many points as needed for a unique solution.

Note

It will only run if the base ring is a field, even though it might work otherwise as well. If your base ring is an integral domain, let it run over the fraction field.

Also, if the solution is not unique, it spits out one solution, without any notice that there are more.

For interpolation in the univariate case use *lagrange_polynomial()*.

Warning

If you don't provide point/value pairs but just a function, it will only use as many points as needed for a unique solution with the given bounds. In particular it will *not* notice or check whether the result yields the correct evaluation for other points as well. So if you give wrong bounds, you will get a wrong answer without any warning.

```
sage: def F(a, b, c):
....:     return a^3*b + b + c^2 + 25
....:
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: R.interpolation(3, F)                                              #␣
↪needs sage.modules
1/2*x^3 + x*y + z^2 - 1/2*x + y + 25
```

```
>>> from sage.all import *
>>> def F(a, b, c):
...     return a**Integer(3)*b + b + c**Integer(2) + Integer(25)
....:
>>> R = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> R.interpolation(Integer(3), F)                                        ␣
↪    # needs sage.modules
1/2*x^3 + x*y + z^2 - 1/2*x + y + 25
```

See also

lagrange_polynomial()

irrelevant_ideal()

Return the irrelevant ideal of this multivariate polynomial ring.

This is the ideal generated by all of the indeterminate generators of this ring.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: R.irrelevant_ideal()
Ideal (x, y, z) of Multivariate Polynomial Ring in x, y, z over
Rational Field
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> R.irrelevant_ideal()
Ideal (x, y, z) of Multivariate Polynomial Ring in x, y, z over
Rational Field
```

is_exact()

Test whether this multivariate polynomial ring is defined over an exact base ring.

EXAMPLES:

```
sage: PolynomialRing(QQ, 2, 'x').is_exact()
True
sage: PolynomialRing(RDF, 2, 'x').is_exact()
False
```

```
>>> from sage.all import *
>>> PolynomialRing(QQ, Integer(2), 'x').is_exact()
True
>>> PolynomialRing(RDF, Integer(2), 'x').is_exact()
False
```

is_field(*proof=True*)

Test whether this multivariate polynomial ring is a field.

A polynomial ring is a field when there are no variable and the base ring is a field.

EXAMPLES:

```
sage: PolynomialRing(QQ, 'x', 2).is_field()
False
sage: PolynomialRing(QQ, 'x', 0).is_field()
True
sage: PolynomialRing(ZZ, 'x', 0).is_field()
False
sage: PolynomialRing(Zmod(1), names=['x','y']).is_finite()
True
```

```
>>> from sage.all import *
>>> PolynomialRing(QQ, 'x', Integer(2)).is_field()
False
>>> PolynomialRing(QQ, 'x', Integer(0)).is_field()
True
>>> PolynomialRing(ZZ, 'x', Integer(0)).is_field()
False
>>> PolynomialRing(Zmod(Integer(1)), names=['x','y']).is_finite()
True
```

is_integral_domain(*proof=True*)

EXAMPLES:

```
sage: ZZ['x,y'].is_integral_domain()
True
sage: Integers(8)['x,y'].is_integral_domain()
False
```

```
>>> from sage.all import *
>>> ZZ['x,y'].is_integral_domain()
True
>>> Integers(Integer(8))['x,y'].is_integral_domain()
False
```

is_noetherian()

EXAMPLES:

```
sage: ZZ['x,y'].is_noetherian()
True
sage: Integers(8)['x,y'].is_noetherian()
True
```

```
>>> from sage.all import *
>>> ZZ['x,y'].is_noetherian()
True
>>> Integers(Integer(8))['x,y'].is_noetherian()
True
```

krull_dimension()

macaulay_resultant(**args*, ***kwds*)

Return the Macaulay resultant.

This computes the resultant of universal polynomials as well as polynomials with constant coefficients. This is a project done in sage days 55. It is based on the implementation in Maple by Manfred Minimair, which in turn is based on the references listed below. It calculates the Macaulay resultant for a list of polynomials, up to sign!

REFERENCES:

- [CLO2005]
- [Can1990]
- [Mac1916]

AUTHORS:

- Hao Chen, Solomon Vishkautsan (7-2014)

INPUT:

- `args` – list of n homogeneous polynomials in n variables works when `args[0]` is the list of polynomials, or `args` is itself the list of polynomials

kwds:

- `sparse` – boolean (default: `False`); if `True`, the function creates sparse matrices

OUTPUT: the Macaulay resultant, an element of the base ring of `self`

Todo

Working with sparse matrices should usually give faster results, but with the current implementation it actually works slower. There should be a way to improve performance with regards to this.

EXAMPLES:

The number of polynomials has to match the number of variables:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: R.macaulay_resultant([y, x + z])                                      #␣
↪needs sage.modules
Traceback (most recent call last):
...
TypeError: number of polynomials(= 2) must equal number of variables (= 3)
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) =␣
↪R._first_ngens(3)
>>> R.macaulay_resultant([y, x + z])                                      #␣
↪needs sage.modules
Traceback (most recent call last):
...
TypeError: number of polynomials(= 2) must equal number of variables (= 3)
```

The polynomials need to be all homogeneous:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: R.macaulay_resultant([y, x + z, z + x^3])                             #␣
↪needs sage.modules
Traceback (most recent call last):
...
TypeError: resultant for non-homogeneous polynomials is not supported
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) =␣
↪R._first_ngens(3)
>>> R.macaulay_resultant([y, x + z, z + x**Integer(3)])                    ␣
↪       # needs sage.modules
Traceback (most recent call last):
...
TypeError: resultant for non-homogeneous polynomials is not supported
```

All polynomials must be in the same ring:

```
sage: S.<x,y> = PolynomialRing(QQ, 2)
sage: R.<x,y,z> = PolynomialRing(QQ,3)
sage: S.macaulay_resultant([y, z+x])                                        #␣
↪needs sage.modules
Traceback (most recent call last):
```

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```
...
TypeError: not all inputs are polynomials in the calling ring
```

```
>>> from sage.all import *
>>> S = PolynomialRing(QQ, Integer(2), names=('x', 'y',)); (x, y,) = S._first_
↪ngens(2)
>>> R = PolynomialRing(QQ,Integer(3), names=('x', 'y', 'z',)); (x, y, z,) = R.
↪_first_ngens(3)
>>> S.macaulay_resultant([y, z+x])                                          #␣
↪needs sage.modules
Traceback (most recent call last):
...
TypeError: not all inputs are polynomials in the calling ring
```

The following example recreates Proposition 2.10 in Ch.3 in [CLO2005]:

```
sage: K.<x,y> = PolynomialRing(ZZ, 2)
sage: flist, R = K._macaulay_resultant_universal_polynomials([1,1,2])
sage: R.macaulay_resultant(flist)                                           #␣
↪needs sage.modules
u2^2*u4^2*u6 - 2*u1*u2*u4*u5*u6 + u1^2*u5^2*u6 - u2^2*u3*u4*u7 +
u1*u2*u3*u5*u7 + u0*u2*u4*u5*u7 - u0*u1*u5^2*u7 + u1*u2*u3*u4*u8 -
u0*u2*u4^2*u8 - u1^2*u3*u5*u8 + u0*u1*u4*u5*u8 + u2^2*u3^2*u9 -
2*u0*u2*u3*u5*u9 + u0^2*u5^2*u9 - u1*u2*u3^2*u10 +
u0*u2*u3*u4*u10 + u0*u1*u3*u5*u10 - u0^2*u4*u5*u10 +
u1^2*u3^2*u11 - 2*u0*u1*u3*u4*u11 + u0^2*u4^2*u11
```

```
>>> from sage.all import *
>>> K = PolynomialRing(ZZ, Integer(2), names=('x', 'y',)); (x, y,) = K._first_
↪ngens(2)
>>> flist, R = K._macaulay_resultant_universal_polynomials([Integer(1),
↪Integer(1),Integer(2)])
>>> R.macaulay_resultant(flist)                                             #␣
↪needs sage.modules
u2^2*u4^2*u6 - 2*u1*u2*u4*u5*u6 + u1^2*u5^2*u6 - u2^2*u3*u4*u7 +
u1*u2*u3*u5*u7 + u0*u2*u4*u5*u7 - u0*u1*u5^2*u7 + u1*u2*u3*u4*u8 -
u0*u2*u4^2*u8 - u1^2*u3*u5*u8 + u0*u1*u4*u5*u8 + u2^2*u3^2*u9 -
2*u0*u2*u3*u5*u9 + u0^2*u5^2*u9 - u1*u2*u3^2*u10 +
u0*u2*u3*u4*u10 + u0*u1*u3*u5*u10 - u0^2*u4*u5*u10 +
u1^2*u3^2*u11 - 2*u0*u1*u3*u4*u11 + u0^2*u4^2*u11
```

The following example degenerates into the determinant of a 3×3 matrix:

```
sage: K.<x,y> = PolynomialRing(ZZ, 2)
sage: flist,R = K._macaulay_resultant_universal_polynomials([1,1,1])
sage: R.macaulay_resultant(flist)                                           #␣
↪needs sage.modules
-u2*u4*u6 + u1*u5*u6 + u2*u3*u7 - u0*u5*u7 - u1*u3*u8 + u0*u4*u8
```

```
>>> from sage.all import *
>>> K = PolynomialRing(ZZ, Integer(2), names=('x', 'y',)); (x, y,) = K._first_
↪ngens(2)
```

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```
>>> flist,R = K._macaulay_resultant_universal_polynomials([Integer(1),
↪Integer(1),Integer(1)])
>>> R.macaulay_resultant(flist)                                          #␣
↪needs sage.modules
-u2*u4*u6 + u1*u5*u6 + u2*u3*u7 - u0*u5*u7 - u1*u3*u8 + u0*u4*u8
```

The following example is by Patrick Ingram (arXiv 1310.4114):

```
sage: U = PolynomialRing(ZZ,'y',2); y0,y1 = U.gens()
sage: R = PolynomialRing(U,'x',3); x0,x1,x2 = R.gens()
sage: f0 = y0*x2^2 - x0^2 + 2*x1*x2
sage: f1 = y1*x2^2 - x1^2 + 2*x0*x2
sage: f2 = x0*x1 - x2^2
sage: flist = [f0,f1,f2]
sage: R.macaulay_resultant([f0,f1,f2])                                   #␣
↪needs sage.modules
y0^2*y1^2 - 4*y0^3 - 4*y1^3 + 18*y0*y1 - 27
```

```
>>> from sage.all import *
>>> U = PolynomialRing(ZZ,'y',Integer(2)); y0,y1 = U.gens()
>>> R = PolynomialRing(U,'x',Integer(3)); x0,x1,x2 = R.gens()
>>> f0 = y0*x2**Integer(2) - x0**Integer(2) + Integer(2)*x1*x2
>>> f1 = y1*x2**Integer(2) - x1**Integer(2) + Integer(2)*x0*x2
>>> f2 = x0*x1 - x2**Integer(2)
>>> flist = [f0,f1,f2]
>>> R.macaulay_resultant([f0,f1,f2])                                     #␣
↪needs sage.modules
y0^2*y1^2 - 4*y0^3 - 4*y1^3 + 18*y0*y1 - 27
```

A simple example with constant rational coefficients:

```
sage: R.<x,y,z,w> = PolynomialRing(QQ, 4)
sage: R.macaulay_resultant([w, z, y, x])                                 #␣
↪needs sage.modules
1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(4), names=('x', 'y', 'z', 'w',)); (x, y, z,
↪ w,) = R._first_ngens(4)
>>> R.macaulay_resultant([w, z, y, x])                                   #␣
↪needs sage.modules
1
```

An example where the resultant vanishes:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: R.macaulay_resultant([x + y, y^2, x])                              #␣
↪needs sage.modules
0
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) =␣
```

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```
↪R._first_ngens(3)
>>> R.macaulay_resultant([x + y, y**Integer(2), x])
↪       # needs sage.modules
0
```

An example of bad reduction at a prime $p = 5$:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: R.macaulay_resultant([y, x^3 + 25*y^2*x, 5*z])                           #
↪needs sage.libs.pari sage.modules
125
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) =
↪R._first_ngens(3)
>>> R.macaulay_resultant([y, x**Integer(3) + Integer(25)*y**Integer(2)*x,
↪Integer(5)*z])                         # needs sage.libs.pari sage.modules
125
```

The input can given as an unpacked list of polynomials:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: R.macaulay_resultant(y, x^3 + 25*y^2*x, 5*z)                             #
↪needs sage.libs.pari sage.modules
125
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) =
↪R._first_ngens(3)
>>> R.macaulay_resultant(y, x**Integer(3) + Integer(25)*y**Integer(2)*x,
↪Integer(5)*z)                          # needs sage.libs.pari sage.modules
125
```

An example when the coefficients live in a finite field:

```
sage: F = FiniteField(11)
sage: R.<x,y,z,w> = PolynomialRing(F, 4)
sage: R.macaulay_resultant([z, x^3, 5*y, w])                                   #
↪needs sage.modules sage.rings.finite_rings
4
```

```
>>> from sage.all import *
>>> F = FiniteField(Integer(11))
>>> R = PolynomialRing(F, Integer(4), names=('x', 'y', 'z', 'w',)); (x, y, z,
↪w,) = R._first_ngens(4)
>>> R.macaulay_resultant([z, x**Integer(3), Integer(5)*y, w])
↪               # needs sage.modules sage.rings.finite_rings
4
```

Example when the denominator in the algorithm vanishes(in this case the resultant is the constant term of the quotient of char polynomials of numerator/denominator):

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: R.macaulay_resultant([y, x + z, z^2])                                     #␣
↪needs sage.libs.pari sage.modules
-1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) =␣
↪R._first_ngens(3)
>>> R.macaulay_resultant([y, x + z, z**Integer(2)])                            ␣
↪      # needs sage.libs.pari sage.modules
-1
```

When there are only 2 polynomials, the Macaulay resultant degenerates to the traditional resultant:

```
sage: R.<x> = PolynomialRing(QQ, 1)
sage: f =  x^2 + 1; g = x^5 + 1
sage: fh = f.homogenize()
sage: gh = g.homogenize()
sage: RH = fh.parent()
sage: f.resultant(g) == RH.macaulay_resultant([fh, gh])                         #␣
↪needs sage.modules
True
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(1), names=('x',)); (x,) = R._first_ngens(1)
>>> f =  x**Integer(2) + Integer(1); g = x**Integer(5) + Integer(1)
>>> fh = f.homogenize()
>>> gh = g.homogenize()
>>> RH = fh.parent()
>>> f.resultant(g) == RH.macaulay_resultant([fh, gh])                         #␣
↪needs sage.modules
True
```

monomial(**exponents*)

Return the monomial with given exponents.

EXAMPLES:

```
sage: R.<x,y,z> = PolynomialRing(ZZ, 3)
sage: R.monomial(1,1,1)
x*y*z
sage: e=(1,2,3)
sage: R.monomial(*e)
x*y^2*z^3
sage: m = R.monomial(1,2,3)
sage: R.monomial(*m.degrees()) == m
True
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) =␣
↪R._first_ngens(3)
>>> R.monomial(Integer(1),Integer(1),Integer(1))
```

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```
x*y*z
>>> e=(Integer(1),Integer(2),Integer(3))
>>> R.monomial(*e)
x*y^2*z^3
>>> m = R.monomial(Integer(1),Integer(2),Integer(3))
>>> R.monomial(*m.degrees()) == m
True
```

We also allow to specify the exponents in a single tuple:

```
sage: R.monomial(e)
x*y^2*z^3
```

```
>>> from sage.all import *
>>> R.monomial(e)
x*y^2*z^3
```

monomials_of_degree(*degree*)

Return a list of all monomials of the given total degree in this multivariate polynomial ring.

EXAMPLES:

```
sage: # needs sage.combinat
sage: R.<x,y,z> = ZZ[]
sage: mons = R.monomials_of_degree(2)
sage: mons
[z^2, y*z, x*z, y^2, x*y, x^2]
sage: P = PolynomialRing(QQ, 3, 'x, y, z', order=TermOrder('wdeglex', [1, 2,␣
↪1]))
sage: P.monomials_of_degree(2)
[z^2, y, x*z, x^2]
sage: P = PolynomialRing(QQ, 3, 'x, y, z', order='lex')
sage: P.monomials_of_degree(3)
[z^3, y*z^2, y^2*z, y^3, x*z^2, x*y*z, x*y^2, x^2*z, x^2*y, x^3]
sage: P = PolynomialRing(QQ, 3, 'x, y, z', order='invlex')
sage: P.monomials_of_degree(3)
[x^3, x^2*y, x*y^2, y^3, x^2*z, x*y*z, y^2*z, x*z^2, y*z^2, z^3]
```

```
>>> from sage.all import *
>>> # needs sage.combinat
>>> R = ZZ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> mons = R.monomials_of_degree(Integer(2))
>>> mons
[z^2, y*z, x*z, y^2, x*y, x^2]
>>> P = PolynomialRing(QQ, Integer(3), 'x, y, z', order=TermOrder('wdeglex',␣
↪[Integer(1), Integer(2), Integer(1)]))
>>> P.monomials_of_degree(Integer(2))
[z^2, y, x*z, x^2]
>>> P = PolynomialRing(QQ, Integer(3), 'x, y, z', order='lex')
>>> P.monomials_of_degree(Integer(3))
[z^3, y*z^2, y^2*z, y^3, x*z^2, x*y*z, x*y^2, x^2*z, x^2*y, x^3]
>>> P = PolynomialRing(QQ, Integer(3), 'x, y, z', order='invlex')
```

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```
>>> P.monomials_of_degree(Integer(3))
[x^3, x^2*y, x*y^2, y^3, x^2*z, x*y*z, y^2*z, x*z^2, y*z^2, z^3]
```

The number of such monomials equals $\binom{n+k-1}{k}$ where n is the number of variables and k the degree:

```
sage: len(mons) == binomial(3 + 2 - 1, 2)                                    #␣
↪needs sage.combinat
True
```

```
>>> from sage.all import *
>>> len(mons) == binomial(Integer(3) + Integer(2) - Integer(1), Integer(2))  ␣
↪                                  # needs sage.combinat
True
```

ngens()

random_element(*degree=2*, *terms=None*, *choose_degree=False*, **args*, ***kwargs*)

Return a random polynomial of at most degree d and at most t terms.

First monomials are chosen uniformly random from the set of all possible monomials of degree up to d (inclusive). This means that it is more likely that a monomial of degree d appears than a monomial of degree $d-1$ because the former class is bigger.

Exactly t *distinct* monomials are chosen this way and each one gets a random coefficient (possibly zero) from the base ring assigned.

The returned polynomial is the sum of this list of terms.

INPUT:

- `degree` – maximal degree (likely to be reached) (default: 2)
- `terms` – number of terms requested (default: 5). If more terms are requested than exist, then this parameter is silently reduced to the maximum number of available terms.
- `choose_degree` – choose degrees of monomials randomly first rather than monomials uniformly random
- `**kwargs` – passed to the random element generator of the base ring

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: f = P.random_element(2, 5)
sage: f.degree() <= 2
True
sage: f.parent() is P
True
sage: len(list(f)) <= 5
True

sage: f = P.random_element(2, 5, choose_degree=True)
sage: f.degree() <= 2
True
sage: f.parent() is P
True
```

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```
sage: len(list(f)) <= 5
True
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = P._first_
↪ngens(3)
>>> f = P.random_element(Integer(2), Integer(5))
>>> f.degree() <= Integer(2)
True
>>> f.parent() is P
True
>>> len(list(f)) <= Integer(5)
True

>>> f = P.random_element(Integer(2), Integer(5), choose_degree=True)
>>> f.degree() <= Integer(2)
True
>>> f.parent() is P
True
>>> len(list(f)) <= Integer(5)
True
```

Stacked rings:

```
sage: R = QQ['x,y']
sage: S = R['t,u']
sage: f = S._random_nonzero_element(degree=2, terms=1)
sage: len(list(f))
1
sage: f.degree() <= 2
True
sage: f.parent() is S
True
```

```
>>> from sage.all import *
>>> R = QQ['x,y']
>>> S = R['t,u']
>>> f = S._random_nonzero_element(degree=Integer(2), terms=Integer(1))
>>> len(list(f))
1
>>> f.degree() <= Integer(2)
True
>>> f.parent() is S
True
```

Default values apply if no degree and/or number of terms is provided:

```
sage: # needs sage.modules
sage: M = random_matrix(QQ['x,y,z'], 2, 2)
sage: all(a.degree() <= 2 for a in M.list())
True
sage: all(len(list(a)) <= 5 for a in M.list())
```

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```
True
sage: M = random_matrix(QQ['x,y,z'], 2, 2, terms=1, degree=2)
sage: all(a.degree() <= 2 for a in M.list())
True
sage: all(len(list(a)) <= 1 for a in M.list())
True

sage: P.random_element(0, 1) in QQ
True

sage: P.random_element(2, 0)
0

sage: R.<x> = PolynomialRing(Integers(3), 1)
sage: f = R.random_element()
sage: f.degree() <= 2
True
sage: len(list(f)) <= 3
True
```

```
>>> from sage.all import *
>>> # needs sage.modules
>>> M = random_matrix(QQ['x,y,z'], Integer(2), Integer(2))
>>> all(a.degree() <= Integer(2) for a in M.list())
True
>>> all(len(list(a)) <= Integer(5) for a in M.list())
True
>>> M = random_matrix(QQ['x,y,z'], Integer(2), Integer(2), terms=Integer(1),␣
↪degree=Integer(2))
>>> all(a.degree() <= Integer(2) for a in M.list())
True
>>> all(len(list(a)) <= Integer(1) for a in M.list())
True

>>> P.random_element(Integer(0), Integer(1)) in QQ
True

>>> P.random_element(Integer(2), Integer(0))
0

>>> R = PolynomialRing(Integers(Integer(3)), Integer(1), names=('x',)); (x,)␣
↪= R._first_ngens(1)
>>> f = R.random_element()
>>> f.degree() <= Integer(2)
True
>>> len(list(f)) <= Integer(3)
True
```

To produce a dense polynomial, pick `terms=Infinity`:

```
sage: P.<x,y,z> = GF(127)[]
sage: f = P.random_element(degree=2, terms=Infinity)
```

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```
sage: while len(list(f)) != 10:
....:     f = P.random_element(degree=2, terms=Infinity)
sage: f = P.random_element(degree=3, terms=Infinity)
sage: while len(list(f)) != 20:
....:     f = P.random_element(degree=3, terms=Infinity)
sage: f = P.random_element(degree=3, terms=Infinity, choose_degree=True)
sage: while len(list(f)) != 20:
....:     f = P.random_element(degree=3, terms=Infinity)
```

```
>>> from sage.all import *
>>> P = GF(Integer(127))['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> f = P.random_element(degree=Integer(2), terms=Infinity)
>>> while len(list(f)) != Integer(10):
...     f = P.random_element(degree=Integer(2), terms=Infinity)
>>> f = P.random_element(degree=Integer(3), terms=Infinity)
>>> while len(list(f)) != Integer(20):
...     f = P.random_element(degree=Integer(3), terms=Infinity)
>>> f = P.random_element(degree=Integer(3), terms=Infinity, choose_
↪degree=True)
>>> while len(list(f)) != Integer(20):
...     f = P.random_element(degree=Integer(3), terms=Infinity)
```

The number of terms is silently reduced to the maximum available if more terms are requested:

```
sage: P.<x,y,z> = GF(127)[]
sage: f = P.random_element(degree=2, terms=1000)
sage: len(list(f)) <= 10
True
```

```
>>> from sage.all import *
>>> P = GF(Integer(127))['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> f = P.random_element(degree=Integer(2), terms=Integer(1000))
>>> len(list(f)) <= Integer(10)
True
```

remove_var(*order=None*, **var*)

Remove a variable or sequence of variables from `self`.

If `order` is not specified, then the subring inherits the term order of the original ring, if possible.

EXAMPLES:

```
sage: P.<x,y,z,w> = PolynomialRing(ZZ)
sage: P.remove_var(z)
Multivariate Polynomial Ring in x, y, w over Integer Ring
sage: P.remove_var(z, x)
Multivariate Polynomial Ring in y, w over Integer Ring
sage: P.remove_var(y, z, x)
Univariate Polynomial Ring in w over Integer Ring
```

```
>>> from sage.all import *
>>> P = PolynomialRing(ZZ, names=('x', 'y', 'z', 'w',)); (x, y, z, w,) = P._
```

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```
↪first_ngens(4)
>>> P.remove_var(z)
Multivariate Polynomial Ring in x, y, w over Integer Ring
>>> P.remove_var(z, x)
Multivariate Polynomial Ring in y, w over Integer Ring
>>> P.remove_var(y, z, x)
Univariate Polynomial Ring in w over Integer Ring
```

Removing all variables results in the base ring:

```
sage: P.remove_var(y, z, x, w)
Integer Ring
```

```
>>> from sage.all import *
>>> P.remove_var(y, z, x, w)
Integer Ring
```

If possible, the term order is kept:

```
sage: R.<x,y,z,w> = PolynomialRing(ZZ, order='deglex')
sage: R.remove_var(y).term_order()
Degree lexicographic term order

sage: R.<x,y,z,w> = PolynomialRing(ZZ, order='lex')
sage: R.remove_var(y).term_order()
Lexicographic term order
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, order='deglex', names=('x', 'y', 'z', 'w',)); (x,␣
↪y, z, w,) = R._first_ngens(4)
>>> R.remove_var(y).term_order()
Degree lexicographic term order

>>> R = PolynomialRing(ZZ, order='lex', names=('x', 'y', 'z', 'w',)); (x, y,␣
↪z, w,) = R._first_ngens(4)
>>> R.remove_var(y).term_order()
Lexicographic term order
```

Be careful with block orders when removing variables:

```
sage: R.<x,y,z,u,v> = PolynomialRing(ZZ, order='deglex(2),lex(3)')
sage: R.remove_var(x, y, z)
Traceback (most recent call last):
...
ValueError: impossible to use the original term order (most
likely because it was a block order). Please specify the term
order for the subring
sage: R.remove_var(x,y,z, order='degrevlex')
Multivariate Polynomial Ring in u, v over Integer Ring
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, order='deglex(2),lex(3)', names=('x', 'y', 'z', 'u
```

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```
↪', 'v',)); (x, y, z, u, v,) = R._first_ngens(5)
>>> R.remove_var(x, y, z)
Traceback (most recent call last):
...
ValueError: impossible to use the original term order (most
likely because it was a block order). Please specify the term
order for the subring
>>> R.remove_var(x,y,z, order='degrevlex')
Multivariate Polynomial Ring in u, v over Integer Ring
```

repr_long()

Return structured string representation of `self`.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ, order=TermOrder('degrevlex',1)
....:                                    + TermOrder('lex',2))
sage: print(P.repr_long())
Polynomial Ring
 Base Ring : Rational Field
      Size : 3 Variables
  Block  0 : Ordering : degrevlex
             Names    : x
  Block  1 : Ordering : lex
             Names    : y, z
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, order=TermOrder('degrevlex',Integer(1))
...                                  + TermOrder('lex',Integer(2)),␣
↪names=('x', 'y', 'z',)); (x, y, z,) = P._first_ngens(3)
>>> print(P.repr_long())
Polynomial Ring
 Base Ring : Rational Field
      Size : 3 Variables
  Block  0 : Ordering : degrevlex
             Names    : x
  Block  1 : Ordering : lex
             Names    : y, z
```

some_elements()

Return a list of polynomials.

This is typically used for running generic tests.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: R.some_elements()
[x, y, x + y, x^2 + x*y, 0, 1]
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
```

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```
>>> R.some_elements()
[x, y, x + y, x^2 + x*y, 0, 1]
```

term_order()

univariate_ring(*x*)

Return a univariate polynomial ring whose base ring comprises all but one variables of `self`.

INPUT:

- `x` – a variable of `self`

EXAMPLES:

```
sage: P.<x,y,z> = QQ[]
sage: P.univariate_ring(y)
Univariate Polynomial Ring in y
 over Multivariate Polynomial Ring in x, z over Rational Field
```

```
>>> from sage.all import *
>>> P = QQ['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> P.univariate_ring(y)
Univariate Polynomial Ring in y
 over Multivariate Polynomial Ring in x, z over Rational Field
```

variable_names_recursive(*depth=None*)

Return the list of variable names of this and its base rings, as if it were a single multi-variate polynomial.

EXAMPLES:

```
sage: R = QQ['x,y']['z,w']
sage: R.variable_names_recursive()
('x', 'y', 'z', 'w')
sage: R.variable_names_recursive(3)
('y', 'z', 'w')
```

```
>>> from sage.all import *
>>> R = QQ['x,y']['z,w']
>>> R.variable_names_recursive()
('x', 'y', 'z', 'w')
>>> R.variable_names_recursive(Integer(3))
('y', 'z', 'w')
```

weyl_algebra()

Return the Weyl algebra generated from `self`.

EXAMPLES:

```
sage: R = QQ['x,y,z']
sage: W = R.weyl_algebra(); W                                              #␣
↪needs sage.modules
Differential Weyl algebra of polynomials in x, y, z over Rational Field
sage: W.polynomial_ring() == R                                             #␣
↪needs sage.modules
True
```

```
>>> from sage.all import *
>>> R = QQ['x,y,z']
>>> W = R.weyl_algebra(); W                                                   #␣
↪needs sage.modules
Differential Weyl algebra of polynomials in x, y, z over Rational Field
>>> W.polynomial_ring() == R                                                  #␣
↪needs sage.modules
True
```

sage.rings.polynomial.multi_polynomial_ring_base.**is_MPolynomialRing**(*x*)

sage.rings.polynomial.multi_polynomial_ring_base.**unpickle_MPolynomialRing_generic**(*base_ring*, *n*, *names*, *order*)

sage.rings.polynomial.multi_polynomial_ring_base.**unpickle_MPolynomialRing_generic_v1**(*base_ring*, *n*, *names*, *order*)

3.1.3 Base class for elements of multivariate polynomial rings

class sage.rings.polynomial.multi_polynomial.**MPolynomial**

Bases: CommutativePolynomial

args()

Return the names of the arguments of self, in the order they are accepted from call.

EXAMPLES:

```
sage: R.<x,y> = ZZ[]
sage: x.args()
(x, y)
```

```
>>> from sage.all import *
>>> R = ZZ['x, y']; (x, y,) = R._first_ngens(2)
>>> x.args()
(x, y)
```

change_ring(*R*)

Return this polynomial with coefficients converted to R.

INPUT:

- R – a ring or morphism; if a morphism, the coefficients are mapped to the codomain of R

OUTPUT: a new polynomial with the base ring changed to R

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: f = x^3 + 3/5*y + 1
sage: f.change_ring(GF(7))
```

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```
x^3 + 2*y + 1
sage: g = x^2 + 5*y
sage: g.change_ring(GF(5))
x^2
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = x**Integer(3) + Integer(3)/Integer(5)*y + Integer(1)
>>> f.change_ring(GF(Integer(7)))
x^3 + 2*y + 1
>>> g = x**Integer(2) + Integer(5)*y
>>> g.change_ring(GF(Integer(5)))
x^2
```

```
sage: # needs sage.rings.finite_rings
sage: R.<x,y> = GF(9,'a')[]
sage: (x+2*y).change_ring(GF(3))
x - y
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> R = GF(Integer(9),'a')['x, y']; (x, y,) = R._first_ngens(2)
>>> (x+Integer(2)*y).change_ring(GF(Integer(3)))
x - y
```

```
sage: # needs sage.rings.finite_rings
sage: F.<a> = GF(7^2)
sage: R.<x,y> = F[]
sage: f = x^2 + a^2*y^2 + a*x + a^3*y
sage: g = f.change_ring(F.frobenius_endomorphism()); g
x^2 + (-a - 2)*y^2 + (-a + 1)*x + (2*a + 2)*y
sage: g.change_ring(F.frobenius_endomorphism()) == f
True
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> F = GF(Integer(7)**Integer(2), names=('a',)); (a,) = F._first_ngens(1)
>>> R = F['x, y']; (x, y,) = R._first_ngens(2)
>>> f = x**Integer(2) + a**Integer(2)*y**Integer(2) + a*x + a**Integer(3)*y
>>> g = f.change_ring(F.frobenius_endomorphism()); g
x^2 + (-a - 2)*y^2 + (-a + 1)*x + (2*a + 2)*y
>>> g.change_ring(F.frobenius_endomorphism()) == f
True
```

```
sage: # needs sage.rings.number_field
sage: K.<z> = CyclotomicField(3)
sage: R.<x,y> = K[]
sage: f = x^2 + z*y
sage: f.change_ring(K.embeddings(CC)[1])
x^2 + (-0.500000000000000 - 0.866025403784438*I)*y
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> K = CyclotomicField(Integer(3), names=('z',)); (z,) = K._first_ngens(1)
>>> R = K['x, y']; (x, y,) = R._first_ngens(2)
>>> f = x**Integer(2) + z*y
>>> f.change_ring(K.embeddings(CC)[Integer(1)])
x^2 + (-0.500000000000000 - 0.866025403784438*I)*y
```

```
sage: # needs sage.rings.number_field
sage: K.<w> = CyclotomicField(5)
sage: R.<x,y> = K[]
sage: f = x^2 + w*y
sage: f.change_ring(K.embeddings(QQbar)[1])
x^2 + (-0.8090169943749474? + 0.5877852522924731?*I)*y
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> K = CyclotomicField(Integer(5), names=('w',)); (w,) = K._first_ngens(1)
>>> R = K['x, y']; (x, y,) = R._first_ngens(2)
>>> f = x**Integer(2) + w*y
>>> f.change_ring(K.embeddings(QQbar)[Integer(1)])
x^2 + (-0.8090169943749474? + 0.5877852522924731?*I)*y
```

coefficients()

Return the nonzero coefficients of this polynomial in a list.

The returned list is decreasingly ordered by the term ordering of `self.parent()`, i.e. the list of coefficients matches the list of monomials returned by *sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular.monomials()*.

EXAMPLES:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3, order='degrevlex')
sage: f = 23*x^6*y^7 + x^3*y+6*x^7*z
sage: f.coefficients()
[23, 6, 1]
sage: R.<x,y,z> = PolynomialRing(QQ, 3, order='lex')
sage: f = 23*x^6*y^7 + x^3*y+6*x^7*z
sage: f.coefficients()
[6, 23, 1]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(3), order='degrevlex', names=('x', 'y', 'z
↪',)); (x, y, z,) = R._first_ngens(3)
>>> f = Integer(23)*x**Integer(6)*y**Integer(7) +␣
↪x**Integer(3)*y+Integer(6)*x**Integer(7)*z
>>> f.coefficients()
[23, 6, 1]
>>> R = PolynomialRing(QQ, Integer(3), order='lex', names=('x', 'y', 'z',));␣
↪(x, y, z,) = R._first_ngens(3)
>>> f = Integer(23)*x**Integer(6)*y**Integer(7) +␣
↪x**Integer(3)*y+Integer(6)*x**Integer(7)*z
```

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```
>>> f.coefficients()
[6, 23, 1]
```

Test the same stuff with base ring **Z** – different implementation:

```
sage: R.<x,y,z> = PolynomialRing(ZZ, 3, order='degrevlex')
sage: f = 23*x^6*y^7 + x^3*y+6*x^7*z
sage: f.coefficients()
[23, 6, 1]
sage: R.<x,y,z> = PolynomialRing(ZZ, 3, order='lex')
sage: f = 23*x^6*y^7 + x^3*y+6*x^7*z
sage: f.coefficients()
[6, 23, 1]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, Integer(3), order='degrevlex', names=('x', 'y', 'z
→',)); (x, y, z,) = R._first_ngens(3)
>>> f = Integer(23)*x**Integer(6)*y**Integer(7) +␣
→x**Integer(3)*y+Integer(6)*x**Integer(7)*z
>>> f.coefficients()
[23, 6, 1]
>>> R = PolynomialRing(ZZ, Integer(3), order='lex', names=('x', 'y', 'z',));␣
→(x, y, z,) = R._first_ngens(3)
>>> f = Integer(23)*x**Integer(6)*y**Integer(7) +␣
→x**Integer(3)*y+Integer(6)*x**Integer(7)*z
>>> f.coefficients()
[6, 23, 1]
```

AUTHOR:

- Didier Deshommes

content()

Return the content of this polynomial. Here, we define content as the gcd of the coefficients in the base ring.

See also

content_ideal()

EXAMPLES:

```
sage: R.<x,y> = ZZ[]
sage: f = 4*x+6*y
sage: f.content()
2
sage: f.content().parent()
Integer Ring
```

```
>>> from sage.all import *
>>> R = ZZ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = Integer(4)*x+Integer(6)*y
>>> f.content()
```

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```
2
>>> f.content().parent()
Integer Ring
```

content_ideal()

Return the content ideal of this polynomial, defined as the ideal generated by its coefficients.

See also

content()

EXAMPLES:

```
sage: R.<x,y> = ZZ[]
sage: f = 2*x*y + 6*x - 4*y + 2
sage: f.content_ideal()
Principal ideal (2) of Integer Ring
sage: S.<z,t> = R[]
sage: g = x*z + y*t
sage: g.content_ideal()
Ideal (x, y) of Multivariate Polynomial Ring in x, y over Integer Ring
```

```
>>> from sage.all import *
>>> R = ZZ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = Integer(2)*x*y + Integer(6)*x - Integer(4)*y + Integer(2)
>>> f.content_ideal()
Principal ideal (2) of Integer Ring
>>> S = R['z, t']; (z, t,) = S._first_ngens(2)
>>> g = x*z + y*t
>>> g.content_ideal()
Ideal (x, y) of Multivariate Polynomial Ring in x, y over Integer Ring
```

denominator()

Return a denominator of `self`.

First, the lcm of the denominators of the entries of `self` is computed and returned. If this computation fails, the unit of the parent of `self` is returned.

Note that some subclasses may implement its own denominator function.

Warning

This is not the denominator of the rational function defined by `self`, which would always be 1 since `self` is a polynomial.

EXAMPLES:

First we compute the denominator of a polynomial with integer coefficients, which is of course 1.

```
sage: R.<x,y> = ZZ[]
sage: f = x^3 + 17*y + x + y
```

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```
sage: f.denominator()
1
```

```
>>> from sage.all import *
>>> R = ZZ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = x**Integer(3) + Integer(17)*y + x + y
>>> f.denominator()
1
```

Next we compute the denominator of a polynomial over a number field.

```
sage: # needs sage.rings.number_field sage.symbolic
sage: R.<x,y> = NumberField(symbolic_expression(x^2+3),'a')['x,y']
sage: f = (1/17)*x^19 + (1/6)*y - (2/3)*x + 1/3; f
1/17*x^19 - 2/3*x + 1/6*y + 1/3
sage: f.denominator()
102
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field sage.symbolic
>>> R = NumberField(symbolic_expression(x**Integer(2)+Integer(3)),'a')['x,y'];
↪ (x, y,) = R._first_ngens(2)
>>> f = (Integer(1)/Integer(17))*x**Integer(19) + (Integer(1)/Integer(6))*y -␣
↪(Integer(2)/Integer(3))*x + Integer(1)/Integer(3); f
1/17*x^19 - 2/3*x + 1/6*y + 1/3
>>> f.denominator()
102
```

Finally, we try to compute the denominator of a polynomial with coefficients in the real numbers, which is a ring whose elements do not have a denominator method.

```
sage: # needs sage.rings.real_mpfr
sage: R.<a,b,c> = RR[]
sage: f = a + b + RR('0.3'); f
a + b + 0.300000000000000
sage: f.denominator()
1.00000000000000
```

```
>>> from sage.all import *
>>> # needs sage.rings.real_mpfr
>>> R = RR['a, b, c']; (a, b, c,) = R._first_ngens(3)
>>> f = a + b + RR('0.3'); f
a + b + 0.300000000000000
>>> f.denominator()
1.00000000000000
```

Check that the denominator is an element over the base whenever the base has no denominator function. This closes Issue #9063:

```
sage: R.<a,b,c> = GF(5)[]
sage: x = R(0)
sage: x.denominator()
```

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```
1
sage: type(x.denominator())
<class 'sage.rings.finite_rings.integer_mod.IntegerMod_int'>
sage: type(a.denominator())
<class 'sage.rings.finite_rings.integer_mod.IntegerMod_int'>
sage: from sage.rings.polynomial.multi_polynomial_element import MPolynomial
sage: isinstance(a / b, MPolynomial)
False
sage: isinstance(a.numerator() / a.denominator(), MPolynomial)
True
```

```
>>> from sage.all import *
>>> R = GF(Integer(5))['a, b, c']; (a, b, c,) = R._first_ngens(3)
>>> x = R(Integer(0))
>>> x.denominator()
1
>>> type(x.denominator())
<class 'sage.rings.finite_rings.integer_mod.IntegerMod_int'>
>>> type(a.denominator())
<class 'sage.rings.finite_rings.integer_mod.IntegerMod_int'>
>>> from sage.rings.polynomial.multi_polynomial_element import MPolynomial
>>> isinstance(a / b, MPolynomial)
False
>>> isinstance(a.numerator() / a.denominator(), MPolynomial)
True
```

derivative(**args*)

The formal derivative of this polynomial, with respect to variables supplied in `args`.

Multiple variables and iteration counts may be supplied; see documentation for the global function *derivative()* for more details.

See also

`_derivative()`

EXAMPLES:

Polynomials implemented via Singular:

```
sage: # needs sage.libs.singular
sage: R.<x, y> = PolynomialRing(FiniteField(5))
sage: f = x^3*y^5 + x^7*y
sage: type(f)
<class 'sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_
↪libsingular'>
sage: f.derivative(x)
2*x^6*y - 2*x^2*y^5
sage: f.derivative(y)
x^7
```

```
>>> from sage.all import *
>>> # needs sage.libs.singular
>>> R = PolynomialRing(FiniteField(Integer(5)), names=('x', 'y',)); (x, y,) =␣
↪R._first_ngens(2)
>>> f = x**Integer(3)*y**Integer(5) + x**Integer(7)*y
>>> type(f)
<class 'sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_
↪libsingular'>
>>> f.derivative(x)
2*x^6*y - 2*x^2*y^5
>>> f.derivative(y)
x^7
```

Generic multivariate polynomials:

```
sage: R.<t> = PowerSeriesRing(QQ)
sage: S.<x, y> = PolynomialRing(R)
sage: f = (t^2 + O(t^3))*x^2*y^3 + (37*t^4 + O(t^5))*x^3
sage: type(f)
<class 'sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict'>
sage: f.derivative(x)      # with respect to x
(2*t^2 + O(t^3))*x*y^3 + (111*t^4 + O(t^5))*x^2
sage: f.derivative(y)      # with respect to y
(3*t^2 + O(t^3))*x^2*y^2
sage: f.derivative(t)      # with respect to t (recurses into base ring)
(2*t + O(t^2))*x^2*y^3 + (148*t^3 + O(t^4))*x^3
sage: f.derivative(x, y)   # with respect to x and then y
(6*t^2 + O(t^3))*x*y^2
sage: f.derivative(y, 3)   # with respect to y three times
(6*t^2 + O(t^3))*x^2
sage: f.derivative()       # can't figure out the variable
Traceback (most recent call last):
...
ValueError: must specify which variable to differentiate with respect to
```

```
>>> from sage.all import *
>>> R = PowerSeriesRing(QQ, names=('t',)); (t,) = R._first_ngens(1)
>>> S = PolynomialRing(R, names=('x', 'y',)); (x, y,) = S._first_ngens(2)
>>> f = (t**Integer(2) + O(t**Integer(3)))*x**Integer(2)*y**Integer(3) +␣
↪(Integer(37)*t**Integer(4) + O(t**Integer(5)))*x**Integer(3)
>>> type(f)
<class 'sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict'>
>>> f.derivative(x)      # with respect to x
(2*t^2 + O(t^3))*x*y^3 + (111*t^4 + O(t^5))*x^2
>>> f.derivative(y)      # with respect to y
(3*t^2 + O(t^3))*x^2*y^2
>>> f.derivative(t)      # with respect to t (recurses into base ring)
(2*t + O(t^2))*x^2*y^3 + (148*t^3 + O(t^4))*x^3
>>> f.derivative(x, y)   # with respect to x and then y
(6*t^2 + O(t^3))*x*y^2
>>> f.derivative(y, Integer(3))    # with respect to y three times
(6*t^2 + O(t^3))*x^2
>>> f.derivative()       # can't figure out the variable
```

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```
Traceback (most recent call last):
...
ValueError: must specify which variable to differentiate with respect to
```

Polynomials over the symbolic ring (just for fun....):

```
sage: # needs sage.symbolic
sage: x = var("x")
sage: S.<u, v> = PolynomialRing(SR)
sage: f = u*v*x
sage: f.derivative(x) == u*v
True
sage: f.derivative(u) == v*x
True
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> x = var("x")
>>> S = PolynomialRing(SR, names=('u', 'v',)); (u, v,) = S._first_ngens(2)
>>> f = u*v*x
>>> f.derivative(x) == u*v
True
>>> f.derivative(u) == v*x
True
```

discriminant(*variable*)

Return the discriminant of `self` with respect to the given variable.

INPUT:

- `variable` – the variable with respect to which we compute the discriminant

OUTPUT: an element of the base ring of the polynomial ring

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: f = 4*x*y^2 + 1/4*x*y*z + 3/2*x*z^2 - 1/2*z^2
sage: f.discriminant(x)                                                   #␣
↪needs sage.libs.singular
1
sage: f.discriminant(y)                                                   #␣
↪needs sage.libs.singular
-383/16*x^2*z^2 + 8*x*z^2
sage: f.discriminant(z)                                                   #␣
↪needs sage.libs.singular
-383/16*x^2*y^2 + 8*x*y^2
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> f = Integer(4)*x*y**Integer(2) + Integer(1)/Integer(4)*x*y*z + Integer(3)/
↪Integer(2)*x*z**Integer(2) - Integer(1)/Integer(2)*z**Integer(2)
>>> f.discriminant(x)                                                     #␣
↪needs sage.libs.singular
```

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```
1
>>> f.discriminant(y)                                                    #␣
↪needs sage.libs.singular
-383/16*x^2*z^2 + 8*x*z^2
>>> f.discriminant(z)                                                    #␣
↪needs sage.libs.singular
-383/16*x^2*y^2 + 8*x*y^2
```

Note that, unlike the univariate case, the result lives in the same ring as the polynomial:

```
sage: R.<x,y> = QQ[]
sage: f = x^5*y + 3*x^2*y^2 - 2*x + y - 1
sage: f.discriminant(y)                                                    #␣
↪needs sage.libs.singular
x^10 + 2*x^5 + 24*x^3 + 12*x^2 + 1
sage: f.polynomial(y).discriminant()                                       #␣
↪needs sage.libs.pari sage.modules
x^10 + 2*x^5 + 24*x^3 + 12*x^2 + 1
sage: f.discriminant(y).parent() == f.polynomial(y).discriminant().parent()  ␣
↪          # needs sage.libs.singular sage.modules
False
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = x**Integer(5)*y + Integer(3)*x**Integer(2)*y**Integer(2) -␣
↪Integer(2)*x + y - Integer(1)
>>> f.discriminant(y)                                                    #␣
↪needs sage.libs.singular
x^10 + 2*x^5 + 24*x^3 + 12*x^2 + 1
>>> f.polynomial(y).discriminant()                                       #␣
↪needs sage.libs.pari sage.modules
x^10 + 2*x^5 + 24*x^3 + 12*x^2 + 1
>>> f.discriminant(y).parent() == f.polynomial(y).discriminant().parent()    ␣
↪         # needs sage.libs.singular sage.modules
False
```

AUTHOR: Miguel Marco

gcd(*other*)

Return a greatest common divisor of this polynomial and `other`.

INPUT:

- `other` – a polynomial with the same parent as this polynomial

EXAMPLES:

```
sage: Q.<z> = Frac(QQ['z'])
sage: R.<x,y> = Q[]
sage: r = x*y - (2*z-1)/(z^2+z+1) * x + y/z
sage: p = r * (x + z*y - 1/z^2)
sage: q = r * (x*y*z + 1)
sage: gcd(p, q)
(z^3 + z^2 + z)*x*y + (-2*z^2 + z)*x + (z^2 + z + 1)*y
```

```
>>> from sage.all import *
>>> Q = Frac(QQ['z'], names=('z',)); (z,) = Q._first_ngens(1)
>>> R = Q['x, y']; (x, y,) = R._first_ngens(2)
>>> r = x*y - (Integer(2)*z-Integer(1))/(z**Integer(2)+z+Integer(1)) * x + y/z
>>> p = r * (x + z*y - Integer(1)/z**Integer(2))
>>> q = r * (x*y*z + Integer(1))
>>> gcd(p, q)
(z^3 + z^2 + z)*x*y + (-2*z^2 + z)*x + (z^2 + z + 1)*y
```

Polynomials over polynomial rings are converted to a simpler polynomial ring with all variables to compute the gcd:

```
sage: A.<z,t> = ZZ[]
sage: B.<x,y> = A[]
sage: r = x*y*z*t + 1
sage: p = r * (x - y + z - t + 1)
sage: q = r * (x*z - y*t)
sage: gcd(p, q)                                                             #␣
↪needs sage.libs.singular
z*t*x*y + 1
sage: _.parent()
Multivariate Polynomial Ring in x, y over
 Multivariate Polynomial Ring in z, t over Integer Ring
```

```
>>> from sage.all import *
>>> A = ZZ['z, t']; (z, t,) = A._first_ngens(2)
>>> B = A['x, y']; (x, y,) = B._first_ngens(2)
>>> r = x*y*z*t + Integer(1)
>>> p = r * (x - y + z - t + Integer(1))
>>> q = r * (x*z - y*t)
>>> gcd(p, q)                                                             #␣
↪needs sage.libs.singular
z*t*x*y + 1
>>> _.parent()
Multivariate Polynomial Ring in x, y over
 Multivariate Polynomial Ring in z, t over Integer Ring
```

Some multivariate polynomial rings have no gcd implementation:

```
sage: R.<x,y> = GaussianIntegers()[]                                        #␣
↪needs sage.rings.number_field
sage: x.gcd(x)
Traceback (most recent call last):
...
NotImplementedError: GCD is not implemented for multivariate polynomials over
Gaussian Integers generated by I in Number Field in I with defining␣
↪polynomial x^2 + 1 with I = 1*I
```

```
>>> from sage.all import *
>>> R = GaussianIntegers()['x, y']; (x, y,) = R._first_ngens(2)# needs sage.
↪rings.number_field
>>> x.gcd(x)
Traceback (most recent call last):
```

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```
...
NotImplementedError: GCD is not implemented for multivariate polynomials over
Gaussian Integers generated by I in Number Field in I with defining␣
↪polynomial x^2 + 1 with I = 1*I
```

gradient()

Return a list of partial derivatives of this polynomial, ordered by the variables of `self.parent()`.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(ZZ, 3)
sage: f = x*y + 1
sage: f.gradient()
[y, x, 0]
```

```
>>> from sage.all import *
>>> P = PolynomialRing(ZZ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) =␣
↪P._first_ngens(3)
>>> f = x*y + Integer(1)
>>> f.gradient()
[y, x, 0]
```

homogeneous_components()

Return the homogeneous components of this polynomial.

OUTPUT: a dictionary mapping degrees to homogeneous polynomials

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: (x^3 + 2*x*y^3 + 4*y^3 + y).homogeneous_components()
{1: y, 3: x^3 + 4*y^3, 4: 2*x*y^3}
sage: R.zero().homogeneous_components()
{}
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> (x**Integer(3) + Integer(2)*x*y**Integer(3) + Integer(4)*y**Integer(3) +␣
↪y).homogeneous_components()
{1: y, 3: x^3 + 4*y^3, 4: 2*x*y^3}
>>> R.zero().homogeneous_components()
{}
```

In case of weighted term orders, the polynomials are homogeneous with respect to the weights:

```
sage: S.<a,b,c> = PolynomialRing(ZZ, order=TermOrder('wdegrevlex', (1,2,3)))
sage: (a^6 + b^3 + b*c + a^2*c + c + a + 1).homogeneous_components()
{0: 1, 1: a, 3: c, 5: a^2*c + b*c, 6: a^6 + b^3}
```

```
>>> from sage.all import *
>>> S = PolynomialRing(ZZ, order=TermOrder('wdegrevlex', (Integer(1),
↪Integer(2),Integer(3))), names=('a', 'b', 'c',)); (a, b, c,) = S._first_
↪ngens(3)
```

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```
>>> (a**Integer(6) + b**Integer(3) + b*c + a**Integer(2)*c + c + a +␣
↪Integer(1)).homogeneous_components()
{0: 1, 1: a, 3: c, 5: a^2*c + b*c, 6: a^6 + b^3}
```

homogenize(*var='h'*)

Return the homogenization of this polynomial.

The polynomial itself is returned if it is homogeneous already. Otherwise, the monomials are multiplied with the smallest powers of `var` such that they all have the same total degree.

INPUT:

- `var` – a variable in the polynomial ring (as a string, an element of the ring, or a zero-based index in the list of variables) or a name for a new variable (default: `'h'`)

OUTPUT:

If `var` specifies a variable in the polynomial ring, then a homogeneous element in that ring is returned. Otherwise, a homogeneous element is returned in a polynomial ring with an extra last variable `var`.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: f = x^2 + y + 1 + 5*x*y^10
sage: f.homogenize()
5*x*y^10 + x^2*h^9 + y*h^10 + h^11
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = x**Integer(2) + y + Integer(1) + Integer(5)*x*y**Integer(10)
>>> f.homogenize()
5*x*y^10 + x^2*h^9 + y*h^10 + h^11
```

The parameter `var` can be used to specify the name of the variable:

```
sage: g = f.homogenize('z'); g
5*x*y^10 + x^2*z^9 + y*z^10 + z^11
sage: g.parent()
Multivariate Polynomial Ring in x, y, z over Rational Field
```

```
>>> from sage.all import *
>>> g = f.homogenize('z'); g
5*x*y^10 + x^2*z^9 + y*z^10 + z^11
>>> g.parent()
Multivariate Polynomial Ring in x, y, z over Rational Field
```

However, if the polynomial is homogeneous already, then that parameter is ignored and no extra variable is added to the polynomial ring:

```
sage: f = x^2 + y^2
sage: g = f.homogenize('z'); g
x^2 + y^2
sage: g.parent()
Multivariate Polynomial Ring in x, y over Rational Field
```

```
>>> from sage.all import *
>>> f = x**Integer(2) + y**Integer(2)
>>> g = f.homogenize('z'); g
x^2 + y^2
>>> g.parent()
Multivariate Polynomial Ring in x, y over Rational Field
```

If you want the ring of the result to be independent of whether the polynomial is homogenized, you can use `var` to use an existing variable to homogenize:

```
sage: R.<x,y,z> = QQ[]
sage: f = x^2 + y^2
sage: g = f.homogenize(z); g
x^2 + y^2
sage: g.parent()
Multivariate Polynomial Ring in x, y, z over Rational Field
sage: f = x^2 - y
sage: g = f.homogenize(z); g
x^2 - y*z
sage: g.parent()
Multivariate Polynomial Ring in x, y, z over Rational Field
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> f = x**Integer(2) + y**Integer(2)
>>> g = f.homogenize(z); g
x^2 + y^2
>>> g.parent()
Multivariate Polynomial Ring in x, y, z over Rational Field
>>> f = x**Integer(2) - y
>>> g = f.homogenize(z); g
x^2 - y*z
>>> g.parent()
Multivariate Polynomial Ring in x, y, z over Rational Field
```

The parameter `var` can also be given as a zero-based index in the list of variables:

```
sage: g = f.homogenize(2); g
x^2 - y*z
```

```
>>> from sage.all import *
>>> g = f.homogenize(Integer(2)); g
x^2 - y*z
```

If the variable specified by `var` is not present in the polynomial, then setting it to 1 yields the original polynomial:

```
sage: g(x,y,1)
x^2 - y
```

```
>>> from sage.all import *
>>> g(x,y,Integer(1))
x^2 - y
```

If it is present already, this might not be the case:

```
sage: g = f.homogenize(x); g
x^2 - x*y
sage: g(1,y,z)
-y + 1
```

```
>>> from sage.all import *
>>> g = f.homogenize(x); g
x^2 - x*y
>>> g(Integer(1),y,z)
-y + 1
```

In particular, this can be surprising in positive characteristic:

```
sage: R.<x,y> = GF(2)[]
sage: f = x + 1
sage: f.homogenize(x)
0
```

```
>>> from sage.all import *
>>> R = GF(Integer(2))['x, y']; (x, y,) = R._first_ngens(2)
>>> f = x + Integer(1)
>>> f.homogenize(x)
0
```

inverse_mod(I)

Return an inverse of `self` modulo the polynomial ideal I, namely a multivariate polynomial f such that `self * f - 1` belongs to I.

INPUT:

- `I` – an ideal of the polynomial ring in which `self` lives

OUTPUT: a multivariate polynomial representing the inverse of `f` modulo I

EXAMPLES:

```
sage: R.<x1,x2> = QQ[]
sage: I = R.ideal(x2**2 + x1 - 2, x1**2 - 1)
sage: f = x1 + 3*x2^2; g = f.inverse_mod(I); g
↪# needs sage.libs.singular
1/16*x1 + 3/16
sage: (f*g).reduce(I)
↪# needs sage.libs.singular
1
```

```
>>> from sage.all import *
>>> R = QQ['x1, x2']; (x1, x2,) = R._first_ngens(2)
>>> I = R.ideal(x2**Integer(2) + x1 - Integer(2), x1**Integer(2) - Integer(1))
>>> f = x1 + Integer(3)*x2**Integer(2); g = f.inverse_mod(I); g                 ␣
↪                 # needs sage.libs.singular
1/16*x1 + 3/16
>>> (f*g).reduce(I)                                                             #␣
```

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```
↪needs sage.libs.singular
1
```

Test a non-invertible element:

```
sage: R.<x1,x2> = QQ[]
sage: I = R.ideal(x2**2 + x1 - 2, x1**2 - 1)
sage: f = x1 + x2
sage: f.inverse_mod(I)
↪# needs sage.libs.singular
Traceback (most recent call last):
...
ArithmeticError: element is non-invertible
```

```
>>> from sage.all import *
>>> R = QQ['x1, x2']; (x1, x2,) = R._first_ngens(2)
>>> I = R.ideal(x2**Integer(2) + x1 - Integer(2), x1**Integer(2) - Integer(1))
>>> f = x1 + x2
>>> f.inverse_mod(I)                                                          #␣
↪needs sage.libs.singular
Traceback (most recent call last):
...
ArithmeticError: element is non-invertible
```

is_gen()

Return `True` if this polynomial is a generator of its parent.

EXAMPLES:

```
sage: R.<x,y> = ZZ[]
sage: x.is_gen()
True
sage: (x + y - y).is_gen()
True
sage: (x*y).is_gen()
False
sage: R.<x,y> = QQ[]
sage: x.is_gen()
True
sage: (x + y - y).is_gen()
True
sage: (x*y).is_gen()
False
```

```
>>> from sage.all import *
>>> R = ZZ['x, y']; (x, y,) = R._first_ngens(2)
>>> x.is_gen()
True
>>> (x + y - y).is_gen()
True
>>> (x*y).is_gen()
False
```

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```
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> x.is_gen()
True
>>> (x + y - y).is_gen()
True
>>> (x*y).is_gen()
False
```

is_generator(**args*, ***kwds*)

Deprecated: Use *is_gen()* instead. See Issue #38942 for details.

is_homogeneous()

Return True if self is a homogeneous polynomial.

> **Note**
>
> This is a generic implementation which is likely overridden by subclasses.

is_lorentzian(*explain=False*)

Return whether this is a Lorentzian polynomial.

INPUT:

- explain – boolean (default: False); if True return a tuple whose first element is the boolean result of the test, and the second element is a string describing the reason the test failed, or None if the test succeeded.

Lorentzian polynomials are a class of polynomials connected with the area of discrete convex analysis. A polynomial f with positive real coefficients is Lorentzian if:

- f is homogeneous;
- the support of f is M-convex
- f has degree less than 2, or if its degree is at least two, the collection of sequential partial derivatives of f which are quadratic forms have Gram matrices with at most one positive eigenvalue.

Note in particular that the zero polynomial is Lorentzian. Examples of Lorentzian polynomials include homogeneous stable polynomials, volume polynomials of convex bodies and projective varieties, and Schur polynomials after renormalizing the coefficient of each monomial x^α by $1/\alpha!$.

EXAMPLES:

Renormalized Schur polynomials are Lorentzian, but not in general if the renormalization is skipped:

```
sage: P.<x,y> = QQ[]
sage: p = (x^2 / 2) + x*y + (y^2 / 2)
sage: p.is_lorentzian()
True
sage: p = x^2 + x*y + y^2
sage: p.is_lorentzian()
False
```

```
>>> from sage.all import *
>>> P = QQ['x, y']; (x, y,) = P._first_ngens(2)
```

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```
>>> p = (x**Integer(2) / Integer(2)) + x*y + (y**Integer(2) / Integer(2))
>>> p.is_lorentzian()
True
>>> p = x**Integer(2) + x*y + y**Integer(2)
>>> p.is_lorentzian()
False
```

Homogeneous linear forms and constant polynomials with positive coefficients are Lorentzian, as well as the zero polynomial:

```
sage: p = x + 2*y
sage: p.is_lorentzian()
True
sage: p = P(5)
sage: p.is_lorentzian()
True
sage: P.zero().is_lorentzian()
True
```

```
>>> from sage.all import *
>>> p = x + Integer(2)*y
>>> p.is_lorentzian()
True
>>> p = P(Integer(5))
>>> p.is_lorentzian()
True
>>> P.zero().is_lorentzian()
True
```

Inhomogeneous polynomials and polynomials with negative coefficients are not Lorentzian:

```
sage: p = x^2 + 2*x + y^2
sage: p.is_lorentzian()
False
sage: p = 2*x^2 - y^2
sage: p.is_lorentzian()
False
```

```
>>> from sage.all import *
>>> p = x**Integer(2) + Integer(2)*x + y**Integer(2)
>>> p.is_lorentzian()
False
>>> p = Integer(2)*x**Integer(2) - y**Integer(2)
>>> p.is_lorentzian()
False
```

It is an error to check if a polynomial is Lorentzian if its base ring is not a subring of the real numbers, as the notion is not defined in this case:

```
sage: # needs sage.rings.real_mpfr
sage: Q.<z,w> = CC[]
sage: q = z^2 + w^2
```

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```
sage: q.is_lorentzian()
Traceback (most recent call last):
...
NotImplementedError: is_lorentzian only implemented for real polynomials
```

```
>>> from sage.all import *
>>> # needs sage.rings.real_mpfr
>>> Q = CC['z, w']; (z, w,) = Q._first_ngens(2)
>>> q = z**Integer(2) + w**Integer(2)
>>> q.is_lorentzian()
Traceback (most recent call last):
...
NotImplementedError: is_lorentzian only implemented for real polynomials
```

The method can give a reason for a polynomial failing to be Lorentzian:

```
sage: p = x^2 + 2*x + y^2
sage: p.is_lorentzian(explain=True)
(False, 'inhomogeneous')
```

```
>>> from sage.all import *
>>> p = x**Integer(2) + Integer(2)*x + y**Integer(2)
>>> p.is_lorentzian(explain=True)
(False, 'inhomogeneous')
```

REFERENCES:

For full definitions and related discussion, see [BrHu2019] and [HMMS2019]. The second reference gives the characterization of Lorentzian polynomials applied in this implementation explicitly.

is_nilpotent()

Return `True` if `self` is nilpotent, i.e., some power of `self` is 0.

EXAMPLES:

```
sage: R.<x,y> = QQbar[]                                                     #␣
↪needs sage.rings.number_field
sage: (x + y).is_nilpotent()                                                #␣
↪needs sage.rings.number_field
False
sage: R(0).is_nilpotent()                                                   #␣
↪needs sage.rings.number_field
True
sage: _.<x,y> = Zmod(4)[]
sage: (2*x).is_nilpotent()
True
sage: (2 + y*x).is_nilpotent()
False
sage: _.<x,y> = Zmod(36)[]
sage: (4 + 6*x).is_nilpotent()
False
sage: (6*x + 12*y + 18*x*y + 24*(x^2+y^2)).is_nilpotent()
True
```

```
>>> from sage.all import *
>>> R = QQbar['x, y']; (x, y,) = R._first_ngens(2)# needs sage.rings.number_
↪field
>>> (x + y).is_nilpotent()                                                  #␣
↪needs sage.rings.number_field
False
>>> R(Integer(0)).is_nilpotent()                                            ␣
↪     # needs sage.rings.number_field
True
>>> _ = Zmod(Integer(4))['x, y']; (x, y,) = _._first_ngens(2)
>>> (Integer(2)*x).is_nilpotent()
True
>>> (Integer(2) + y*x).is_nilpotent()
False
>>> _ = Zmod(Integer(36))['x, y']; (x, y,) = _._first_ngens(2)
>>> (Integer(4) + Integer(6)*x).is_nilpotent()
False
>>> (Integer(6)*x + Integer(12)*y + Integer(18)*x*y +␣
↪Integer(24)*(x**Integer(2)+y**Integer(2))).is_nilpotent()
True
```

is_square(*root=False*)

Test whether this polynomial is a square.

INPUT:

- `root` – if set to `True`, return a pair `(True, root)` where `root` is a square root or `(False, None)` if it is not a square.

EXAMPLES:

```
sage: R.<a,b> = QQ[]
sage: a.is_square()
False
sage: ((1+a*b^2)^2).is_square()
True
sage: ((1+a*b^2)^2).is_square(root=True)
(True, a*b^2 + 1)
```

```
>>> from sage.all import *
>>> R = QQ['a, b']; (a, b,) = R._first_ngens(2)
>>> a.is_square()
False
>>> ((Integer(1)+a*b**Integer(2))**Integer(2)).is_square()
True
>>> ((Integer(1)+a*b**Integer(2))**Integer(2)).is_square(root=True)
(True, a*b^2 + 1)
```

is_symmetric(*group=None*)

Return whether this polynomial is symmetric.

INPUT:

- `group` – (default: symmetric group) if set, test whether the polynomial is invariant with respect to the given permutation group

EXAMPLES:

```
sage: # needs sage.groups
sage: R.<x,y,z> = QQ[]
sage: p = (x+y+z)**2 - 3 * (x+y)*(x+z)*(y+z)
sage: p.is_symmetric()
True
sage: (x + y - z).is_symmetric()
False
sage: R.one().is_symmetric()
True
sage: p = (x-y)*(y-z)*(z-x)
sage: p.is_symmetric()
False
sage: p.is_symmetric(AlternatingGroup(3))
True

sage: R.<x,y> = QQ[]
sage: ((x + y)**2).is_symmetric()                                              #␣
↪needs sage.groups
True
sage: R.one().is_symmetric()                                                   #␣
↪needs sage.groups
True
sage: (x + 2*y).is_symmetric()                                                 #␣
↪needs sage.groups
False
```

```
>>> from sage.all import *
>>> # needs sage.groups
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> p = (x+y+z)**Integer(2) - Integer(3) * (x+y)*(x+z)*(y+z)
>>> p.is_symmetric()
True
>>> (x + y - z).is_symmetric()
False
>>> R.one().is_symmetric()
True
>>> p = (x-y)*(y-z)*(z-x)
>>> p.is_symmetric()
False
>>> p.is_symmetric(AlternatingGroup(Integer(3)))
True

>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> ((x + y)**Integer(2)).is_symmetric()                                       ␣
↪      # needs sage.groups
True
>>> R.one().is_symmetric()                                                   #␣
↪needs sage.groups
True
>>> (x + Integer(2)*y).is_symmetric()                                          ␣
↪      # needs sage.groups
```

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```
False
```

An example with a GAP permutation group (here the quaternions):

```
sage: R = PolynomialRing(QQ, 'x', 8)
sage: x = R.gens()
sage: p = sum(prod(x[i] for i in e)
....:         for e in [(0,1,2), (0,1,7), (0,2,7), (1,2,7),
....:                   (3,4,5), (3,4,6), (3,5,6), (4,5,6)])
sage: p.is_symmetric(libgap.TransitiveGroup(8, 5))                      #␣
↪needs sage.groups
True
sage: p = sum(prod(x[i] for i in e)
....:     for e in [(0,1,2), (0,1,7), (0,2,7), (1,2,7),
....:               (3,4,5), (3,4,6), (3,5,6)])
sage: p.is_symmetric(libgap.TransitiveGroup(8, 5))                      #␣
↪needs sage.groups
False
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, 'x', Integer(8))
>>> x = R.gens()
>>> p = sum(prod(x[i] for i in e)
...         for e in [(Integer(0),Integer(1),Integer(2)), (Integer(0),
↪Integer(1),Integer(7)), (Integer(0),Integer(2),Integer(7)), (Integer(1),
↪Integer(2),Integer(7)),
...                   (Integer(3),Integer(4),Integer(5)), (Integer(3),
↪Integer(4),Integer(6)), (Integer(3),Integer(5),Integer(6)), (Integer(4),
↪Integer(5),Integer(6))])
>>> p.is_symmetric(libgap.TransitiveGroup(Integer(8), Integer(5)))      ␣
↪               # needs sage.groups
True
>>> p = sum(prod(x[i] for i in e)
...     for e in [(Integer(0),Integer(1),Integer(2)), (Integer(0),Integer(1),
↪Integer(7)), (Integer(0),Integer(2),Integer(7)), (Integer(1),Integer(2),
↪Integer(7)),
...               (Integer(3),Integer(4),Integer(5)), (Integer(3),Integer(4),
↪Integer(6)), (Integer(3),Integer(5),Integer(6))])
>>> p.is_symmetric(libgap.TransitiveGroup(Integer(8), Integer(5)))      ␣
↪               # needs sage.groups
False
```

is_unit()

Return `True` if `self` is a unit, that is, has a multiplicative inverse.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: (x + y).is_unit()
False
sage: R(0).is_unit()
```

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```
False
sage: R(-1).is_unit()
True
sage: R(-1 + x).is_unit()
False
sage: R(2).is_unit()
True
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQbar['x, y']; (x, y,) = R._first_ngens(2)
>>> (x + y).is_unit()
False
>>> R(Integer(0)).is_unit()
False
>>> R(-Integer(1)).is_unit()
True
>>> R(-Integer(1) + x).is_unit()
False
>>> R(Integer(2)).is_unit()
True
```

Check that Issue #22454 is fixed:

```
sage: _.<x,y> = Zmod(4)[]
sage: (1 + 2*x).is_unit()
True
sage: (x*y).is_unit()
False
sage: _.<x,y> = Zmod(36)[]
sage: (7+ 6*x + 12*y - 18*x*y).is_unit()
True
```

```
>>> from sage.all import *
>>> _ = Zmod(Integer(4))['x, y']; (x, y,) = _._first_ngens(2)
>>> (Integer(1) + Integer(2)*x).is_unit()
True
>>> (x*y).is_unit()
False
>>> _ = Zmod(Integer(36))['x, y']; (x, y,) = _._first_ngens(2)
>>> (Integer(7)+ Integer(6)*x + Integer(12)*y - Integer(18)*x*y).is_unit()
True
```

iterator_exp_coeff(*as_ETuples=True*)

Iterate over `self` as pairs of ((E)Tuple, coefficient).

INPUT:

- `as_ETuples` – boolean (default: `True`); if `True`, iterate over pairs whose first element is an `ETuple`, otherwise as a tuples

EXAMPLES:

```
sage: R.<a,b,c> = QQ[]
sage: f = a*c^3 + a^2*b + 2*b^4
sage: list(f.iterator_exp_coeff())
[((0, 4, 0), 2), ((1, 0, 3), 1), ((2, 1, 0), 1)]
sage: list(f.iterator_exp_coeff(as_ETuples=False))
[((0, 4, 0), 2), ((1, 0, 3), 1), ((2, 1, 0), 1)]

sage: R.<a,b,c> = PolynomialRing(QQ, 3, order='lex')
sage: f = a*c^3 + a^2*b + 2*b^4
sage: list(f.iterator_exp_coeff())
[((2, 1, 0), 1), ((1, 0, 3), 1), ((0, 4, 0), 2)]
```

```
>>> from sage.all import *
>>> R = QQ['a, b, c']; (a, b, c,) = R._first_ngens(3)
>>> f = a*c**Integer(3) + a**Integer(2)*b + Integer(2)*b**Integer(4)
>>> list(f.iterator_exp_coeff())
[((0, 4, 0), 2), ((1, 0, 3), 1), ((2, 1, 0), 1)]
>>> list(f.iterator_exp_coeff(as_ETuples=False))
[((0, 4, 0), 2), ((1, 0, 3), 1), ((2, 1, 0), 1)]

>>> R = PolynomialRing(QQ, Integer(3), order='lex', names=('a', 'b', 'c',));
↪(a, b, c,) = R._first_ngens(3)
>>> f = a*c**Integer(3) + a**Integer(2)*b + Integer(2)*b**Integer(4)
>>> list(f.iterator_exp_coeff())
[((2, 1, 0), 1), ((1, 0, 3), 1), ((0, 4, 0), 2)]
```

jacobian_ideal()

Return the Jacobian ideal of the polynomial `self`.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: f = x^3 + y^3 + z^3
sage: f.jacobian_ideal()
Ideal (3*x^2, 3*y^2, 3*z^2) of
 Multivariate Polynomial Ring in x, y, z over Rational Field
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> f = x**Integer(3) + y**Integer(3) + z**Integer(3)
>>> f.jacobian_ideal()
Ideal (3*x^2, 3*y^2, 3*z^2) of
 Multivariate Polynomial Ring in x, y, z over Rational Field
```

leading_support(**args*, ***kwds*)

Return the maximal element of the support of `self`, according to the term order.

If the term ordering of the basis elements is not what is desired, a comparison key, `key(x)`, can be provided.

EXAMPLES:

```
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: (3*x*y^2 + 2*y*z^3 + y^4 + 4*x*y*z).leading_support()
(0, 4, 0)
```

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```
sage: R.<x,y,z> = PolynomialRing(QQ, order='lex')
sage: (3*x*y^2 + 2*y*z^3 + y^4 + 4*x*y*z).leading_support()
(1, 2, 0)
sage: R.<x,y,z> = PolynomialRing(QQ, order='invlex')
sage: (3*x*y^2 + 2*y*z^3 + y^4 + 4*x*y*z).leading_support()
(0, 1, 3)
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> (Integer(3)*x*y**Integer(2) + Integer(2)*y*z**Integer(3) + y**Integer(4)␣
↪+ Integer(4)*x*y*z).leading_support()
(0, 4, 0)
>>> R = PolynomialRing(QQ, order='lex', names=('x', 'y', 'z',)); (x, y, z,) =␣
↪R._first_ngens(3)
>>> (Integer(3)*x*y**Integer(2) + Integer(2)*y*z**Integer(3) + y**Integer(4)␣
↪+ Integer(4)*x*y*z).leading_support()
(1, 2, 0)
>>> R = PolynomialRing(QQ, order='invlex', names=('x', 'y', 'z',)); (x, y, z,
↪) = R._first_ngens(3)
>>> (Integer(3)*x*y**Integer(2) + Integer(2)*y*z**Integer(3) + y**Integer(4)␣
↪+ Integer(4)*x*y*z).leading_support()
(0, 1, 3)
```

lift(*I*)

Given an ideal $I = (f_1, \ldots, f_r)$ that contains `self`, find $s_1, \ldots, s_r$ such that `self` $= s_1 f_1 + ... + s_r f_r$.

EXAMPLES:

```
sage: # needs sage.rings.real_mpfr
sage: A.<x,y> = PolynomialRing(CC, 2, order='degrevlex')
sage: I = A.ideal([x^10 + x^9*y^2, y^8 - x^2*y^7 ])
sage: f = x*y^13 + y^12
sage: M = f.lift(I); M                                                  #␣
↪needs sage.libs.singular
[y^7, x^7*y^2 + x^8 + x^5*y^3 + x^6*y + x^3*y^4 + x^4*y^2 + x*y^5 + x^2*y^3 +␣
↪y^4]
sage: sum(map(mul, zip(M, I.gens()))) == f                              #␣
↪needs sage.libs.singular
True
```

```
>>> from sage.all import *
>>> # needs sage.rings.real_mpfr
>>> A = PolynomialRing(CC, Integer(2), order='degrevlex', names=('x', 'y',));␣
↪(x, y,) = A._first_ngens(2)
>>> I = A.ideal([x**Integer(10) + x**Integer(9)*y**Integer(2), y**Integer(8) -
↪ x**Integer(2)*y**Integer(7) ])
>>> f = x*y**Integer(13) + y**Integer(12)
>>> M = f.lift(I); M                                                  #␣
↪needs sage.libs.singular
[y^7, x^7*y^2 + x^8 + x^5*y^3 + x^6*y + x^3*y^4 + x^4*y^2 + x*y^5 + x^2*y^3 +␣
↪y^4]
```

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```
>>> sum(map(mul, zip(M, I.gens()))) == f                                        #␣
↪needs sage.libs.singular
True
```

macaulay_resultant(**args*)

This is an implementation of the Macaulay resultant. It computes the resultant of universal polynomials as well as polynomials with constant coefficients. This is a project done in sage days 55. It's based on the implementation in Maple by Manfred Minimair, which in turn is based on the references [CLO], [Can], [Mac]. It calculates the Macaulay resultant for a list of Polynomials, up to sign!

AUTHORS:

- Hao Chen, Solomon Vishkautsan (7-2014)

INPUT:

- `args` – list of $n - 1$ homogeneous polynomials in n variables; works when `args[0]` is the list of polynomials, or `args` is itself the list of polynomials

OUTPUT: the Macaulay resultant

EXAMPLES:

The number of polynomials has to match the number of variables:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: y.macaulay_resultant(x + z)                                                #␣
↪needs sage.modules
Traceback (most recent call last):
...
TypeError: number of polynomials(= 2) must equal number of variables (= 3)
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) =␣
↪R._first_ngens(3)
>>> y.macaulay_resultant(x + z)                                                #␣
↪needs sage.modules
Traceback (most recent call last):
...
TypeError: number of polynomials(= 2) must equal number of variables (= 3)
```

The polynomials need to be all homogeneous:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: y.macaulay_resultant([x + z, z + x^3])                                     #␣
↪needs sage.modules
Traceback (most recent call last):
...
TypeError: resultant for non-homogeneous polynomials is not supported
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) =␣
↪R._first_ngens(3)
>>> y.macaulay_resultant([x + z, z + x**Integer(3)])                             ␣
↪      # needs sage.modules
```

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```
Traceback (most recent call last):
...
TypeError: resultant for non-homogeneous polynomials is not supported
```

All polynomials must be in the same ring:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: S.<x,y> = PolynomialRing(QQ, 2)
sage: y.macaulay_resultant(z + x, z)                                    #␣
↪needs sage.modules
Traceback (most recent call last):
...
TypeError: not all inputs are polynomials in the calling ring
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) =␣
↪R._first_ngens(3)
>>> S = PolynomialRing(QQ, Integer(2), names=('x', 'y',)); (x, y,) = S._first_
↪ngens(2)
>>> y.macaulay_resultant(z + x, z)                                     #␣
↪needs sage.modules
Traceback (most recent call last):
...
TypeError: not all inputs are polynomials in the calling ring
```

The following example recreates Proposition 2.10 in Ch.3 of Using Algebraic Geometry:

```
sage: K.<x,y> = PolynomialRing(ZZ, 2)
sage: flist, R = K._macaulay_resultant_universal_polynomials([1,1,2])
sage: flist[0].macaulay_resultant(flist[1:])                            #␣
↪needs sage.modules
u2^2*u4^2*u6 - 2*u1*u2*u4*u5*u6 + u1^2*u5^2*u6 - u2^2*u3*u4*u7 +␣
↪u1*u2*u3*u5*u7
+ u0*u2*u4*u5*u7 - u0*u1*u5^2*u7 + u1*u2*u3*u4*u8 - u0*u2*u4^2*u8 - u1^
↪2*u3*u5*u8
+ u0*u1*u4*u5*u8 + u2^2*u3^2*u9 - 2*u0*u2*u3*u5*u9 + u0^2*u5^2*u9
- u1*u2*u3^2*u10 + u0*u2*u3*u4*u10 + u0*u1*u3*u5*u10 - u0^2*u4*u5*u10
+ u1^2*u3^2*u11 - 2*u0*u1*u3*u4*u11 + u0^2*u4^2*u11
```

```
>>> from sage.all import *
>>> K = PolynomialRing(ZZ, Integer(2), names=('x', 'y',)); (x, y,) = K._first_
↪ngens(2)
>>> flist, R = K._macaulay_resultant_universal_polynomials([Integer(1),
↪Integer(1),Integer(2)])
>>> flist[Integer(0)].macaulay_resultant(flist[Integer(1):])            ␣
↪               # needs sage.modules
u2^2*u4^2*u6 - 2*u1*u2*u4*u5*u6 + u1^2*u5^2*u6 - u2^2*u3*u4*u7 +␣
↪u1*u2*u3*u5*u7
+ u0*u2*u4*u5*u7 - u0*u1*u5^2*u7 + u1*u2*u3*u4*u8 - u0*u2*u4^2*u8 - u1^
↪2*u3*u5*u8
+ u0*u1*u4*u5*u8 + u2^2*u3^2*u9 - 2*u0*u2*u3*u5*u9 + u0^2*u5^2*u9
- u1*u2*u3^2*u10 + u0*u2*u3*u4*u10 + u0*u1*u3*u5*u10 - u0^2*u4*u5*u10
```

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```
+ u1^2*u3^2*u11 - 2*u0*u1*u3*u4*u11 + u0^2*u4^2*u11
```

The following example degenerates into the determinant of a 3×3 matrix:

```
sage: K.<x,y> = PolynomialRing(ZZ, 2)
sage: flist, R = K._macaulay_resultant_universal_polynomials([1,1,1])
sage: flist[0].macaulay_resultant(flist[1:])                                   #␣
↪needs sage.modules
-u2*u4*u6 + u1*u5*u6 + u2*u3*u7 - u0*u5*u7 - u1*u3*u8 + u0*u4*u8
```

```
>>> from sage.all import *
>>> K = PolynomialRing(ZZ, Integer(2), names=('x', 'y',)); (x, y,) = K._first_
↪ngens(2)
>>> flist, R = K._macaulay_resultant_universal_polynomials([Integer(1),
↪Integer(1),Integer(1)])
>>> flist[Integer(0)].macaulay_resultant(flist[Integer(1):])                   ␣
↪              # needs sage.modules
-u2*u4*u6 + u1*u5*u6 + u2*u3*u7 - u0*u5*u7 - u1*u3*u8 + u0*u4*u8
```

The following example is by Patrick Ingram (arXiv 1310.4114):

```
sage: U = PolynomialRing(ZZ,'y',2); y0,y1 = U.gens()
sage: R = PolynomialRing(U,'x',3); x0,x1,x2 = R.gens()
sage: f0 = y0*x2^2 - x0^2 + 2*x1*x2
sage: f1 = y1*x2^2 - x1^2 + 2*x0*x2
sage: f2 = x0*x1 - x2^2
sage: f0.macaulay_resultant(f1, f2)                                            #␣
↪needs sage.modules
y0^2*y1^2 - 4*y0^3 - 4*y1^3 + 18*y0*y1 - 27
```

```
>>> from sage.all import *
>>> U = PolynomialRing(ZZ,'y',Integer(2)); y0,y1 = U.gens()
>>> R = PolynomialRing(U,'x',Integer(3)); x0,x1,x2 = R.gens()
>>> f0 = y0*x2**Integer(2) - x0**Integer(2) + Integer(2)*x1*x2
>>> f1 = y1*x2**Integer(2) - x1**Integer(2) + Integer(2)*x0*x2
>>> f2 = x0*x1 - x2**Integer(2)
>>> f0.macaulay_resultant(f1, f2)                                            #␣
↪needs sage.modules
y0^2*y1^2 - 4*y0^3 - 4*y1^3 + 18*y0*y1 - 27
```

a simple example with constant rational coefficients:

```
sage: R.<x,y,z,w> = PolynomialRing(QQ, 4)
sage: w.macaulay_resultant([z, y, x])                                          #␣
↪needs sage.modules
1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(4), names=('x', 'y', 'z', 'w',)); (x, y, z,
↪ w,) = R._first_ngens(4)
>>> w.macaulay_resultant([z, y, x])                                          #␣
```

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```
↪needs sage.modules
1
```

an example where the resultant vanishes:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: (x + y).macaulay_resultant([y^2, x])                                    #␣
↪needs sage.modules
0
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) =␣
↪R._first_ngens(3)
>>> (x + y).macaulay_resultant([y**Integer(2), x])                            ␣
↪      # needs sage.modules
0
```

an example of bad reduction at a prime `p = 5`:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: y.macaulay_resultant([x^3 + 25*y^2*x, 5*z])                             #␣
↪needs sage.libs.pari sage.modules
125
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) =␣
↪R._first_ngens(3)
>>> y.macaulay_resultant([x**Integer(3) + Integer(25)*y**Integer(2)*x,␣
↪Integer(5)*z])                           # needs sage.libs.pari sage.modules
125
```

The input can given as an unpacked list of polynomials:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: y.macaulay_resultant(x^3 + 25*y^2*x, 5*z)                               #␣
↪needs sage.libs.pari sage.modules
125
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) =␣
↪R._first_ngens(3)
>>> y.macaulay_resultant(x**Integer(3) + Integer(25)*y**Integer(2)*x,␣
↪Integer(5)*z)                            # needs sage.libs.pari sage.
↪modules
125
```

an example when the coefficients live in a finite field:

```
sage: F = FiniteField(11)
sage: R.<x,y,z,w> = PolynomialRing(F, 4)
sage: z.macaulay_resultant([x^3, 5*y, w])                                     #␣
```

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```
↪needs sage.modules sage.rings.finite_rings
4
```

```
>>> from sage.all import *
>>> F = FiniteField(Integer(11))
>>> R = PolynomialRing(F, Integer(4), names=('x', 'y', 'z', 'w',)); (x, y, z,␣
↪w,) = R._first_ngens(4)
>>> z.macaulay_resultant([x**Integer(3), Integer(5)*y, w])                  ␣
↪                # needs sage.modules sage.rings.finite_rings
4
```

example when the denominator in the algorithm vanishes(in this case the resultant is the constant term of the quotient of char polynomials of numerator/denominator):

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: y.macaulay_resultant([x + z, z^2])                                   #␣
↪needs sage.libs.pari sage.modules
-1
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) =␣
↪R._first_ngens(3)
>>> y.macaulay_resultant([x + z, z**Integer(2)])                            ␣
↪      # needs sage.libs.pari sage.modules
-1
```

When there are only 2 polynomials, the Macaulay resultant degenerates to the traditional resultant:

```
sage: R.<x> = PolynomialRing(QQ, 1)
sage: f = x^2 + 1; g = x^5 + 1
sage: fh = f.homogenize()
sage: gh = g.homogenize()
sage: RH = fh.parent()
sage: f.resultant(g) == fh.macaulay_resultant(gh)                           #␣
↪needs sage.modules
True
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(1), names=('x',)); (x,) = R._first_ngens(1)
>>> f = x**Integer(2) + Integer(1); g = x**Integer(5) + Integer(1)
>>> fh = f.homogenize()
>>> gh = g.homogenize()
>>> RH = fh.parent()
>>> f.resultant(g) == fh.macaulay_resultant(gh)                            #␣
↪needs sage.modules
True
```

map_coefficients(*f*, *new_base_ring=None*)

Return the polynomial obtained by applying `f` to the nonzero coefficients of `self`.

If `f` is a `sage.categories.map.Map`, then the resulting polynomial will be defined over the codomain of `f`. Otherwise, the resulting polynomial will be over the same ring as `self`. Set `new_base_ring` to override this behaviour.

INPUT:

- `f` – a callable that will be applied to the coefficients of `self`
- `new_base_ring` – (optional) if given, the resulting polynomial will be defined over this ring

EXAMPLES:

```
sage: k.<a> = GF(9); R.<x,y> = k[];  f = x*a + 2*x^3*y*a + a                #␣
↪needs sage.rings.finite_rings
sage: f.map_coefficients(lambda a: a + 1)                                   #␣
↪needs sage.rings.finite_rings
(-a + 1)*x^3*y + (a + 1)*x + (a + 1)
```

```
>>> from sage.all import *
>>> k = GF(Integer(9), names=('a',)); (a,) = k._first_ngens(1); R = k['x, y'];
↪ (x, y,) = R._first_ngens(2);  f = x*a + Integer(2)*x**Integer(3)*y*a + a  ␣
↪              # needs sage.rings.finite_rings
>>> f.map_coefficients(lambda a: a + Integer(1))                              ␣
↪      # needs sage.rings.finite_rings
(-a + 1)*x^3*y + (a + 1)*x + (a + 1)
```

Examples with different base ring:

```
sage: # needs sage.rings.finite_rings
sage: R.<r> = GF(9); S.<s> = GF(81)
sage: h = Hom(R,S)[0]; h
Ring morphism:
  From: Finite Field in r of size 3^2
  To:   Finite Field in s of size 3^4
  Defn: r |--> 2*s^3 + 2*s^2 + 1
sage: T.<X,Y> = R[]
sage: f = r*X + Y
sage: g = f.map_coefficients(h); g
(-s^3 - s^2 + 1)*X + Y
sage: g.parent()
Multivariate Polynomial Ring in X, Y over Finite Field in s of size 3^4
sage: h = lambda x: x.trace()
sage: g = f.map_coefficients(h); g
X - Y
sage: g.parent()
Multivariate Polynomial Ring in X, Y over Finite Field in r of size 3^2
sage: g = f.map_coefficients(h, new_base_ring=GF(3)); g
X - Y
sage: g.parent()
Multivariate Polynomial Ring in X, Y over Finite Field of size 3
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> R = GF(Integer(9), names=('r',)); (r,) = R._first_ngens(1); S =␣
↪GF(Integer(81), names=('s',)); (s,) = S._first_ngens(1)
>>> h = Hom(R,S)[Integer(0)]; h
Ring morphism:
  From: Finite Field in r of size 3^2
  To:   Finite Field in s of size 3^4
```

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```
  Defn: r |--> 2*s^3 + 2*s^2 + 1
>>> T = R['X, Y']; (X, Y,) = T._first_ngens(2)
>>> f = r*X + Y
>>> g = f.map_coefficients(h); g
(-s^3 - s^2 + 1)*X + Y
>>> g.parent()
Multivariate Polynomial Ring in X, Y over Finite Field in s of size 3^4
>>> h = lambda x: x.trace()
>>> g = f.map_coefficients(h); g
X - Y
>>> g.parent()
Multivariate Polynomial Ring in X, Y over Finite Field in r of size 3^2
>>> g = f.map_coefficients(h, new_base_ring=GF(Integer(3))); g
X - Y
>>> g.parent()
Multivariate Polynomial Ring in X, Y over Finite Field of size 3
```

newton_polytope()

Return the Newton polytope of this polynomial.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: f = 1 + x*y + x^3 + y^3
sage: P = f.newton_polytope(); P                                            #␣
↪needs sage.geometry.polyhedron
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
sage: P.is_simple()                                                         #␣
↪needs sage.geometry.polyhedron
True
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = Integer(1) + x*y + x**Integer(3) + y**Integer(3)
>>> P = f.newton_polytope(); P                                            #␣
↪needs sage.geometry.polyhedron
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
>>> P.is_simple()                                                         #␣
↪needs sage.geometry.polyhedron
True
```

nth_root(n)

Return a n-th root of this element.

If there is no such root, a `ValueError` is raised.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: a = 32 * (x*y + 1)^5 * (x+y+z)^5
sage: a.nth_root(5)
2*x^2*y + 2*x*y^2 + 2*x*y*z + 2*x + 2*y + 2*z
sage: b = x + 2*y + 3*z
```

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```
sage: b.nth_root(42)
Traceback (most recent call last):
...
ValueError: not a 42nd power

sage: R.<x,y> = QQ[]
sage: S.<z,t> = R[]
sage: T.<u,v> = S[]
sage: p = (1 + x*u + y + v) * (1 + z*t)
sage: (p**3).nth_root(3)
(x*z*t + x)*u + (z*t + 1)*v + (y + 1)*z*t + y + 1
sage: (p**3).nth_root(3).parent() is p.parent()
True
sage: ((1+x+z+t)**2).nth_root(3)
Traceback (most recent call last):
...
ValueError: not a 3rd power
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> a = Integer(32) * (x*y + Integer(1))**Integer(5) * (x+y+z)**Integer(5)
>>> a.nth_root(Integer(5))
2*x^2*y + 2*x*y^2 + 2*x*y*z + 2*x + 2*y + 2*z
>>> b = x + Integer(2)*y + Integer(3)*z
>>> b.nth_root(Integer(42))
Traceback (most recent call last):
...
ValueError: not a 42nd power

>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> S = R['z, t']; (z, t,) = S._first_ngens(2)
>>> T = S['u, v']; (u, v,) = T._first_ngens(2)
>>> p = (Integer(1) + x*u + y + v) * (Integer(1) + z*t)
>>> (p**Integer(3)).nth_root(Integer(3))
(x*z*t + x)*u + (z*t + 1)*v + (y + 1)*z*t + y + 1
>>> (p**Integer(3)).nth_root(Integer(3)).parent() is p.parent()
True
>>> ((Integer(1)+x+z+t)**Integer(2)).nth_root(Integer(3))
Traceback (most recent call last):
...
ValueError: not a 3rd power
```

numerator()

Return a numerator of `self`, computed as `self * self.denominator()`.

Note that some subclasses may implement its own numerator function.

> **Warning**
>
> This is not the numerator of the rational function defined by `self`, which would always be `self` since `self` is a polynomial.

EXAMPLES:

First we compute the numerator of a polynomial with integer coefficients, which is of course `self`.

```
sage: R.<x, y> = ZZ[]
sage: f = x^3 + 17*x + y + 1
sage: f.numerator()
x^3 + 17*x + y + 1
sage: f == f.numerator()
True
```

```
>>> from sage.all import *
>>> R = ZZ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = x**Integer(3) + Integer(17)*x + y + Integer(1)
>>> f.numerator()
x^3 + 17*x + y + 1
>>> f == f.numerator()
True
```

Next we compute the numerator of a polynomial over a number field.

```
sage: # needs sage.rings.number_field sage.symbolic
sage: R.<x,y> = NumberField(symbolic_expression(x^2+3), 'a')['x,y']
sage: f = (1/17)*y^19 - (2/3)*x + 1/3; f
1/17*y^19 - 2/3*x + 1/3
sage: f.numerator()
3*y^19 - 34*x + 17
sage: f == f.numerator()
False
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field sage.symbolic
>>> R = NumberField(symbolic_expression(x**Integer(2)+Integer(3)), 'a')['x,y
↪']; (x, y,) = R._first_ngens(2)
>>> f = (Integer(1)/Integer(17))*y**Integer(19) - (Integer(2)/Integer(3))*x +␣
↪Integer(1)/Integer(3); f
1/17*y^19 - 2/3*x + 1/3
>>> f.numerator()
3*y^19 - 34*x + 17
>>> f == f.numerator()
False
```

We try to compute the numerator of a polynomial with coefficients in the finite field of 3 elements.

```
sage: K.<x,y,z> = GF(3)['x, y, z']
sage: f = 2*x*z + 2*z^2 + 2*y + 1; f
-x*z - z^2 - y + 1
sage: f.numerator()
-x*z - z^2 - y + 1
```

```
>>> from sage.all import *
>>> K = GF(Integer(3))['x, y, z']; (x, y, z,) = K._first_ngens(3)
>>> f = Integer(2)*x*z + Integer(2)*z**Integer(2) + Integer(2)*y + Integer(1);
```

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```
↪ f
-x*z - z^2 - y + 1
>>> f.numerator()
-x*z - z^2 - y + 1
```

We check that the computation the numerator and denominator are valid.

```
sage: # needs sage.rings.number_field sage.symbolic
sage: K = NumberField(symbolic_expression('x^3+2'), 'a')['x']['s,t']
sage: f = K.random_element()
sage: f.numerator() / f.denominator() == f
True
sage: R = RR['x,y,z']
sage: f = R.random_element()
sage: f.numerator() / f.denominator() == f
True
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field sage.symbolic
>>> K = NumberField(symbolic_expression('x^3+2'), 'a')['x']['s,t']
>>> f = K.random_element()
>>> f.numerator() / f.denominator() == f
True
>>> R = RR['x,y,z']
>>> f = R.random_element()
>>> f.numerator() / f.denominator() == f
True
```

polynomial(*var*)

Let `var` be one of the variables of the parent of `self`. This returns `self` viewed as a univariate polynomial in `var` over the polynomial ring generated by all the other variables of the parent.

EXAMPLES:

```
sage: R.<x,w,z> = QQ[]
sage: f = x^3 + 3*w*x + w^5 + (17*w^3)*x + z^5
sage: f.polynomial(x)
x^3 + (17*w^3 + 3*w)*x + w^5 + z^5
sage: parent(f.polynomial(x))
Univariate Polynomial Ring in x
 over Multivariate Polynomial Ring in w, z over Rational Field

sage: f.polynomial(w)
w^5 + 17*x*w^3 + 3*x*w + z^5 + x^3
sage: f.polynomial(z)
z^5 + w^5 + 17*x*w^3 + x^3 + 3*x*w
sage: R.<x,w,z,k> = ZZ[]
sage: f = x^3 + 3*w*x + w^5 + (17*w^3)*x + z^5 +x*w*z*k + 5
sage: f.polynomial(x)
x^3 + (17*w^3 + w*z*k + 3*w)*x + w^5 + z^5 + 5
sage: f.polynomial(w)
w^5 + 17*x*w^3 + (x*z*k + 3*x)*w + z^5 + x^3 + 5
```

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```
sage: f.polynomial(z)
z^5 + x*w*k*z + w^5 + 17*x*w^3 + x^3 + 3*x*w + 5
sage: f.polynomial(k)
x*w*z*k + w^5 + z^5 + 17*x*w^3 + x^3 + 3*x*w + 5
sage: R.<x,y> = GF(5)[]
sage: f = x^2 + x + y
sage: f.polynomial(x)
x^2 + x + y
sage: f.polynomial(y)
y + x^2 + x
```

```
>>> from sage.all import *
>>> R = QQ['x, w, z']; (x, w, z,) = R._first_ngens(3)
>>> f = x**Integer(3) + Integer(3)*w*x + w**Integer(5) +␣
↪(Integer(17)*w**Integer(3))*x + z**Integer(5)
>>> f.polynomial(x)
x^3 + (17*w^3 + 3*w)*x + w^5 + z^5
>>> parent(f.polynomial(x))
Univariate Polynomial Ring in x
 over Multivariate Polynomial Ring in w, z over Rational Field

>>> f.polynomial(w)
w^5 + 17*x*w^3 + 3*x*w + z^5 + x^3
>>> f.polynomial(z)
z^5 + w^5 + 17*x*w^3 + x^3 + 3*x*w
>>> R = ZZ['x, w, z, k']; (x, w, z, k,) = R._first_ngens(4)
>>> f = x**Integer(3) + Integer(3)*w*x + w**Integer(5) +␣
↪(Integer(17)*w**Integer(3))*x + z**Integer(5) +x*w*z*k + Integer(5)
>>> f.polynomial(x)
x^3 + (17*w^3 + w*z*k + 3*w)*x + w^5 + z^5 + 5
>>> f.polynomial(w)
w^5 + 17*x*w^3 + (x*z*k + 3*x)*w + z^5 + x^3 + 5
>>> f.polynomial(z)
z^5 + x*w*k*z + w^5 + 17*x*w^3 + x^3 + 3*x*w + 5
>>> f.polynomial(k)
x*w*z*k + w^5 + z^5 + 17*x*w^3 + x^3 + 3*x*w + 5
>>> R = GF(Integer(5))['x, y']; (x, y,) = R._first_ngens(2)
>>> f = x**Integer(2) + x + y
>>> f.polynomial(x)
x^2 + x + y
>>> f.polynomial(y)
y + x^2 + x
```

reduced_form(***kwds*)

Return a reduced form of this polynomial.

The algorithm is from Stoll and Cremona's "On the Reduction Theory of Binary Forms" [CS2003]. This takes a two variable homogeneous polynomial and finds a reduced form. This is a $SL(2,\mathbf{Z})$-equivalent binary form whose covariant in the upper half plane is in the fundamental domain. If the polynomial has multiple roots, they are removed and the algorithm is applied to the portion without multiple roots.

This reduction should also minimize the sum of the squares of the coefficients, but this is not always the case. By default the coefficient minimizing algorithm in [HS2018] is applied. The coefficients can be minimized either with respect to the sum of their squares or the maximum of their global heights.

A portion of the algorithm uses Newton's method to find a solution to a system of equations. If Newton's method fails to converge to a point in the upper half plane, the function will use the less precise z_0 covariant from the Q_0 form as defined on page 7 of [CS2003]. Additionally, if this polynomial has a root with multiplicity at least half the total degree of the polynomial, then we must also use the z_0 covariant. See [CS2003] for details.

Note that, if the covariant is within `error_limit` of the boundary but outside the fundamental domain, our function will erroneously move it to within the fundamental domain, hence our conjugation will be off by 1. If you don't want this to happen, decrease your `error_limit` and increase your precision.

Implemented by Rebecca Lauren Miller as part of GSOC 2016. Smallest coefficients added by Ben Hutz July 2018.

INPUT: keyword arguments:

- `prec` – integer (default: 300); sets the precision
- `return_conjugation` – boolean (default: `True`); whether to return element of $SL(2, \mathbf{Z})$
- `error_limit` – sets the error tolerance (default: 0.000001)
- `smallest_coeffs` – boolean (default: `True`); whether to find the model with smallest coefficients
- `norm_type` – either `'norm'` or `'height'`; what type of norm to use for smallest coefficients
- `emb` – (optional) embedding of based field into `CC`

OUTPUT:

- a polynomial (reduced binary form)
- a matrix (element of $SL(2, \mathbf{Z})$)

Todo

When Newton's Method doesn't converge to a root in the upper half plane. Now we just return z_0. It would be better to modify and find the unique root in the upper half plane.

EXAMPLES:

```
sage: R.<x,h> = PolynomialRing(QQ)
sage: f = 19*x^8 - 262*x^7*h + 1507*x^6*h^2 - 4784*x^5*h^3 + 9202*x^4*h^4\
....: -10962*x^3*h^5 + 7844*x^2*h^6 - 3040*x*h^7 + 475*h^8
sage: f.reduced_form(prec=200, smallest_coeffs=False)                    #␣
↪needs sage.modules sage.rings.complex_interval_field
(
-x^8 - 2*x^7*h + 7*x^6*h^2 + 16*x^5*h^3 + 2*x^4*h^4 - 2*x^3*h^5 + 4*x^2*h^6 -␣
↪5*h^8,

[ 1 -2]
[ 1 -1]
)
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'h',)); (x, h,) = R._first_ngens(2)
>>> f = Integer(19)*x**Integer(8) - Integer(262)*x**Integer(7)*h +␣
```

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```
↪Integer(1507)*x**Integer(6)*h**Integer(2) -␣
↪Integer(4784)*x**Integer(5)*h**Integer(3) +␣
↪Integer(9202)*x**Integer(4)*h**Integer(4)-
↪Integer(10962)*x**Integer(3)*h**Integer(5) +␣
↪Integer(7844)*x**Integer(2)*h**Integer(6) - Integer(3040)*x*h**Integer(7) +␣
↪Integer(475)*h**Integer(8)
>>> f.reduced_form(prec=Integer(200), smallest_coeffs=False)                   ␣
↪     # needs sage.modules sage.rings.complex_interval_field
(
-x^8 - 2*x^7*h + 7*x^6*h^2 + 16*x^5*h^3 + 2*x^4*h^4 - 2*x^3*h^5 + 4*x^2*h^6 -␣
↪5*h^8,
<BLANKLINE>
[ 1 -2]
[ 1 -1]
)
```

An example where the multiplicity is too high:

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: f = x^3 + 378666*x^2*y - 12444444*x*y^2 + 1234567890*y^3
sage: j = f * (x-545*y)^9
sage: j.reduced_form(prec=200, smallest_coeffs=False)                         #␣
↪needs sage.modules sage.rings.complex_interval_field
Traceback (most recent call last):
...
ValueError: cannot have a root with multiplicity >= 12/2
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> f = x**Integer(3) + Integer(378666)*x**Integer(2)*y -␣
↪Integer(12444444)*x*y**Integer(2) + Integer(1234567890)*y**Integer(3)
>>> j = f * (x-Integer(545)*y)**Integer(9)
>>> j.reduced_form(prec=Integer(200), smallest_coeffs=False)                   ␣
↪     # needs sage.modules sage.rings.complex_interval_field
Traceback (most recent call last):
...
ValueError: cannot have a root with multiplicity >= 12/2
```

An example where Newton's Method does not find the right root:

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: F = x^6 + 3*x^5*y - 8*x^4*y^2 - 2*x^3*y^3 - 44*x^2*y^4 - 8*x*y^5
sage: F.reduced_form(smallest_coeffs=False, prec=400)                         #␣
↪needs sage.modules sage.rings.complex_interval_field
Traceback (most recent call last):
...
ArithmeticError: Newton's method converged to z not in the upper half plane
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> F = x**Integer(6) + Integer(3)*x**Integer(5)*y -␣
↪Integer(8)*x**Integer(4)*y**Integer(2) -␣
```

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```
↪Integer(2)*x**Integer(3)*y**Integer(3) -␣
↪Integer(44)*x**Integer(2)*y**Integer(4) - Integer(8)*x*y**Integer(5)
>>> F.reduced_form(smallest_coeffs=False, prec=Integer(400))                  ␣
↪     # needs sage.modules sage.rings.complex_interval_field
Traceback (most recent call last):
...
ArithmeticError: Newton's method converged to z not in the upper half plane
```

An example with covariant on the boundary, therefore a non-unique form:

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: F = 5*x^2*y - 5*x*y^2 - 30*y^3
sage: F.reduced_form(smallest_coeffs=False)                                   #␣
↪needs sage.modules sage.rings.complex_interval_field
(
                            [1 1]
5*x^2*y + 5*x*y^2 - 30*y^3, [0 1]
)
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> F = Integer(5)*x**Integer(2)*y - Integer(5)*x*y**Integer(2) -␣
↪Integer(30)*y**Integer(3)
>>> F.reduced_form(smallest_coeffs=False)                                   #␣
↪needs sage.modules sage.rings.complex_interval_field
(
                            [1 1]
5*x^2*y + 5*x*y^2 - 30*y^3, [0 1]
)
```

An example where precision needs to be increased:

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: F = (-16*x^7 - 114*x^6*y - 345*x^5*y^2 - 599*x^4*y^3
....:      - 666*x^3*y^4 - 481*x^2*y^5 - 207*x*y^6 - 40*y^7)
sage: F.reduced_form(prec=50, smallest_coeffs=False)                          #␣
↪needs sage.modules sage.rings.complex_interval_field
Traceback (most recent call last):
...
ValueError: accuracy of Newton's root not within tolerance(0.000012... > 1e-
↪06),
increase precision
sage: F.reduced_form(prec=100, smallest_coeffs=False)                         #␣
↪needs sage.modules sage.rings.complex_interval_field
(
                                                          [-1 -1]
-x^5*y^2 - 24*x^3*y^4 - 3*x^2*y^5 - 2*x*y^6 + 16*y^7, [ 1  0]
)
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> F = (-Integer(16)*x**Integer(7) - Integer(114)*x**Integer(6)*y -␣
```

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```
↪Integer(345)*x**Integer(5)*y**Integer(2) -␣
↪Integer(599)*x**Integer(4)*y**Integer(3)
...       - Integer(666)*x**Integer(3)*y**Integer(4) -␣
↪Integer(481)*x**Integer(2)*y**Integer(5) - Integer(207)*x*y**Integer(6) -␣
↪Integer(40)*y**Integer(7))
>>> F.reduced_form(prec=Integer(50), smallest_coeffs=False)                    ␣
↪     # needs sage.modules sage.rings.complex_interval_field
Traceback (most recent call last):
...
ValueError: accuracy of Newton's root not within tolerance(0.000012... > 1e-
↪06),
increase precision
>>> F.reduced_form(prec=Integer(100), smallest_coeffs=False)                   ␣
↪     # needs sage.modules sage.rings.complex_interval_field
(
                                                    [-1 -1]
-x^5*y^2 - 24*x^3*y^4 - 3*x^2*y^5 - 2*x*y^6 + 16*y^7, [ 1  0]
)
```

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: F = - 8*x^4 - 3933*x^3*y - 725085*x^2*y^2 - 59411592*x*y^3 -␣
↪1825511633*y^4
sage: F.reduced_form(return_conjugation=False)                                #␣
↪needs sage.modules sage.rings.complex_interval_field
x^4 + 9*x^3*y - 3*x*y^3 - 8*y^4
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> F = - Integer(8)*x**Integer(4) - Integer(3933)*x**Integer(3)*y -␣
↪Integer(725085)*x**Integer(2)*y**Integer(2) -␣
↪Integer(59411592)*x*y**Integer(3) - Integer(1825511633)*y**Integer(4)
>>> F.reduced_form(return_conjugation=False)                                #␣
↪needs sage.modules sage.rings.complex_interval_field
x^4 + 9*x^3*y - 3*x*y^3 - 8*y^4
```

```
sage: R.<x,y> = QQ[]
sage: F = -2*x^3 + 2*x^2*y + 3*x*y^2 + 127*y^3
sage: F.reduced_form()                                                        #␣
↪needs sage.modules sage.rings.complex_interval_field
(
                                      [1 4]
-2*x^3 - 22*x^2*y - 77*x*y^2 + 43*y^3, [0 1]
)
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> F = -Integer(2)*x**Integer(3) + Integer(2)*x**Integer(2)*y +␣
↪Integer(3)*x*y**Integer(2) + Integer(127)*y**Integer(3)
>>> F.reduced_form()                                                        #␣
↪needs sage.modules sage.rings.complex_interval_field
(
```

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```
                                        [1 4]
-2*x^3 - 22*x^2*y - 77*x*y^2 + 43*y^3, [0 1]
)
```

```
sage: R.<x,y> = QQ[]
sage: F = -2*x^3 + 2*x^2*y + 3*x*y^2 + 127*y^3
sage: F.reduced_form(norm_type='height')                                      #␣
↪needs sage.modules sage.rings.complex_interval_field
(
                                         [5 4]
-58*x^3 - 47*x^2*y + 52*x*y^2 + 43*y^3, [1 1]
)
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> F = -Integer(2)*x**Integer(3) + Integer(2)*x**Integer(2)*y +␣
↪Integer(3)*x*y**Integer(2) + Integer(127)*y**Integer(3)
>>> F.reduced_form(norm_type='height')                                      #␣
↪needs sage.modules sage.rings.complex_interval_field
(
                                         [5 4]
-58*x^3 - 47*x^2*y + 52*x*y^2 + 43*y^3, [1 1]
)
```

```
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: F = x^4 + x^3*y*z + y^2*z
sage: F.reduced_form()                                                        #␣
↪needs sage.modules sage.rings.complex_interval_field
Traceback (most recent call last):
...
ValueError: (=x^3*y*z + x^4 + y^2*z) must have two variables
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> F = x**Integer(4) + x**Integer(3)*y*z + y**Integer(2)*z
>>> F.reduced_form()                                                        #␣
↪needs sage.modules sage.rings.complex_interval_field
Traceback (most recent call last):
...
ValueError: (=x^3*y*z + x^4 + y^2*z) must have two variables
```

```
sage: R.<x,y> = PolynomialRing(ZZ)
sage: F = - 8*x^6 - 3933*x^3*y - 725085*x^2*y^2 - 59411592*x*y^3 - 99*y^6
sage: F.reduced_form(return_conjugation=False)                                #␣
↪needs sage.modules sage.rings.complex_interval_field
Traceback (most recent call last):
...
ValueError: (=-8*x^6 - 99*y^6 - 3933*x^3*y - 725085*x^2*y^2 -
59411592*x*y^3) must be homogeneous
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> F = - Integer(8)*x**Integer(6) - Integer(3933)*x**Integer(3)*y -␣
↪Integer(725085)*x**Integer(2)*y**Integer(2) -␣
↪Integer(59411592)*x*y**Integer(3) - Integer(99)*y**Integer(6)
>>> F.reduced_form(return_conjugation=False)                          #␣
↪needs sage.modules sage.rings.complex_interval_field
Traceback (most recent call last):
...
ValueError: (=-8*x^6 - 99*y^6 - 3933*x^3*y - 725085*x^2*y^2 -
59411592*x*y^3) must be homogeneous
```

```
sage: R.<x,y> = PolynomialRing(RR)
sage: F = (217.992172373276*x^3 + 96023.1505442490*x^2*y
....:      + 1.40987971253579e7*x*y^2 + 6.90016027113216e8*y^3)
sage: F.reduced_form(smallest_coeffs=False)  # tol 1e-8               #␣
↪needs sage.modules sage.rings.complex_interval_field
(
-39.5673942565918*x^3 + 111.874026298523*x^2*y
 + 231.052762985229*x*y^2 - 138.380829811096*y^3,

[-147 -148]
[   1    1]
)
```

```
>>> from sage.all import *
>>> R = PolynomialRing(RR, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> F = (RealNumber('217.992172373276')*x**Integer(3) + RealNumber('96023.
↪1505442490')*x**Integer(2)*y
...      + RealNumber('1.40987971253579e7')*x*y**Integer(2) + RealNumber('6.
↪90016027113216e8')*y**Integer(3))
>>> F.reduced_form(smallest_coeffs=False)  # tol 1e-8                 #␣
↪needs sage.modules sage.rings.complex_interval_field
(
-39.5673942565918*x^3 + 111.874026298523*x^2*y
 + 231.052762985229*x*y^2 - 138.380829811096*y^3,
<BLANKLINE>
[-147 -148]
[   1    1]
)
```

```
sage: R.<x,y> = PolynomialRing(CC)                                    #␣
↪needs sage.rings.real_mpfr
sage: F = ((0.759099196558145 + 0.845425869641446*CC.0)*x^3           #␣
↪needs sage.rings.real_mpfr
....:      + (84.8317207268542 + 93.8840848648033*CC.0)*x^2*y
....:      + (3159.07040755858 + 3475.33037377779*CC.0)*x*y^2
....:      + (39202.5965389079 + 42882.5139724962*CC.0)*y^3)
sage: F.reduced_form(smallest_coeffs=False)  # tol 1e-11              #␣
↪needs sage.modules sage.rings.complex_interval_field sage.rings.real_mpfr
(
(-0.759099196558145 - 0.845425869641446*I)*x^3
```

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```
+ (-0.571709908900118 - 0.0418133346027929*I)*x^2*y
+ (0.856525964330103 - 0.0721403997649759*I)*x*y^2
+ (-0.965531044130330 + 0.754252314465703*I)*y^3,

[-1 37]
[ 0 -1]
)
```

```
>>> from sage.all import *
>>> R = PolynomialRing(CC, names=('x', 'y',)); (x, y,) = R._first_ngens(2)#␣
↪needs sage.rings.real_mpfr
>>> F = ((RealNumber('0.759099196558145') + RealNumber('0.845425869641446
↪')*CC.gen(0))*x**Integer(3)                    # needs sage.rings.real_mpfr
...       + (RealNumber('84.8317207268542') + RealNumber('93.8840848648033
↪')*CC.gen(0))*x**Integer(2)*y
...       + (RealNumber('3159.07040755858') + RealNumber('3475.33037377779
↪')*CC.gen(0))*x*y**Integer(2)
...       + (RealNumber('39202.5965389079') + RealNumber('42882.5139724962
↪')*CC.gen(0))*y**Integer(3))
>>> F.reduced_form(smallest_coeffs=False)  # tol 1e-11                    #␣
↪needs sage.modules sage.rings.complex_interval_field sage.rings.real_mpfr
(
(-0.759099196558145 - 0.845425869641446*I)*x^3
+ (-0.571709908900118 - 0.0418133346027929*I)*x^2*y
+ (0.856525964330103 - 0.0721403997649759*I)*x*y^2
+ (-0.965531044130330 + 0.754252314465703*I)*y^3,
<BLANKLINE>
[-1 37]
[ 0 -1]
)
```

specialization(*D=None*, *phi=None*)

Specialization of this polynomial.

Given a family of polynomials defined over a polynomial ring. A specialization is a particular member of that family. The specialization can be specified either by a dictionary or a `SpecializationMorphism`.

INPUT:

- `D` – dictionary (optional)
- `phi` – `SpecializationMorphism` (optional)

OUTPUT: a new polynomial

EXAMPLES:

```
sage: R.<c> = PolynomialRing(QQ)
sage: S.<x,y> = PolynomialRing(R)
sage: F = x^2 + c*y^2
sage: F.specialization({c:2})
x^2 + 2*y^2
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('c',)); (c,) = R._first_ngens(1)
>>> S = PolynomialRing(R, names=('x', 'y',)); (x, y,) = S._first_ngens(2)
>>> F = x**Integer(2) + c*y**Integer(2)
>>> F.specialization({c:Integer(2)})
x^2 + 2*y^2
```

```
sage: S.<a,b> = PolynomialRing(QQ)
sage: P.<x,y,z> = PolynomialRing(S)
sage: RR.<c,d> = PolynomialRing(P)
sage: f = a*x^2 + b*y^3 + c*y^2 - b*a*d + d^2 - a*c*b*z^2
sage: f.specialization({a:2, z:4, d:2})
(y^2 - 32*b)*c + b*y^3 + 2*x^2 - 4*b + 4
```

```
>>> from sage.all import *
>>> S = PolynomialRing(QQ, names=('a', 'b',)); (a, b,) = S._first_ngens(2)
>>> P = PolynomialRing(S, names=('x', 'y', 'z',)); (x, y, z,) = P._first_
↪ngens(3)
>>> RR = PolynomialRing(P, names=('c', 'd',)); (c, d,) = RR._first_ngens(2)
>>> f = a*x**Integer(2) + b*y**Integer(3) + c*y**Integer(2) - b*a*d +␣
↪d**Integer(2) - a*c*b*z**Integer(2)
>>> f.specialization({a:Integer(2), z:Integer(4), d:Integer(2)})
(y^2 - 32*b)*c + b*y^3 + 2*x^2 - 4*b + 4
```

Check that we preserve multi- versus uni-variate:

```
sage: R.<l> = PolynomialRing(QQ, 1)
sage: S.<k> = PolynomialRing(R)
sage: K.<a, b, c> = PolynomialRing(S)
sage: F = a*k^2 + b*l + c^2
sage: F.specialization({b:56, c:5}).parent()
Univariate Polynomial Ring in a over Univariate Polynomial Ring in k
over Multivariate Polynomial Ring in l over Rational Field
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(1), names=('l',)); (l,) = R._first_ngens(1)
>>> S = PolynomialRing(R, names=('k',)); (k,) = S._first_ngens(1)
>>> K = PolynomialRing(S, names=('a', 'b', 'c',)); (a, b, c,) = K._first_
↪ngens(3)
>>> F = a*k**Integer(2) + b*l + c**Integer(2)
>>> F.specialization({b:Integer(56), c:Integer(5)}).parent()
Univariate Polynomial Ring in a over Univariate Polynomial Ring in k
over Multivariate Polynomial Ring in l over Rational Field
```

subresultants(*other*, *variable=None*)

Return the nonzero subresultant polynomials of `self` and `other`.

INPUT:

- `other` – a polynomial

OUTPUT: list of polynomials in the same ring as `self`

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: p = (y^2 + 6)*(x - 1) - y*(x^2 + 1)
sage: q = (x^2 + 6)*(y - 1) - x*(y^2 + 1)
sage: p.subresultants(q, y)
[2*x^6 - 22*x^5 + 102*x^4 - 274*x^3 + 488*x^2 - 552*x + 288,
 -x^3 - x^2*y + 6*x^2 + 5*x*y - 11*x - 6*y + 6]
sage: p.subresultants(q, x)
[2*y^6 - 22*y^5 + 102*y^4 - 274*y^3 + 488*y^2 - 552*y + 288,
 x*y^2 + y^3 - 5*x*y - 6*y^2 + 6*x + 11*y - 6]
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> p = (y**Integer(2) + Integer(6))*(x - Integer(1)) - y*(x**Integer(2) +␣
↪Integer(1))
>>> q = (x**Integer(2) + Integer(6))*(y - Integer(1)) - x*(y**Integer(2) +␣
↪Integer(1))
>>> p.subresultants(q, y)
[2*x^6 - 22*x^5 + 102*x^4 - 274*x^3 + 488*x^2 - 552*x + 288,
 -x^3 - x^2*y + 6*x^2 + 5*x*y - 11*x - 6*y + 6]
>>> p.subresultants(q, x)
[2*y^6 - 22*y^5 + 102*y^4 - 274*y^3 + 488*y^2 - 552*y + 288,
 x*y^2 + y^3 - 5*x*y - 6*y^2 + 6*x + 11*y - 6]
```

sylvester_matrix(*right*, *variable=None*)

Given two nonzero polynomials `self` and `right`, return the Sylvester matrix of the polynomials with respect to a given variable.

Note that the Sylvester matrix is not defined if one of the polynomials is zero.

INPUT:

- `self`, `right` – multivariate polynomials
- `variable` – (optional) compute the Sylvester matrix with respect to this variable. If `variable` is not provided, the first variable of the polynomial ring is used.

OUTPUT: the Sylvester matrix of `self` and `right`

EXAMPLES:

```
sage: R.<x, y> = PolynomialRing(ZZ)
sage: f = (y + 1)*x + 3*x**2
sage: g = (y + 2)*x + 4*x**2
sage: M = f.sylvester_matrix(g, x)                                    #␣
↪needs sage.modules
sage: M                                                               #␣
↪needs sage.modules
[    3 y + 1     0     0]
[    0     3 y + 1     0]
[    4 y + 2     0     0]
[    0     4 y + 2     0]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> f = (y + Integer(1))*x + Integer(3)*x**Integer(2)
```

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```
>>> g = (y + Integer(2))*x + Integer(4)*x**Integer(2)
>>> M = f.sylvester_matrix(g, x)                                           #␣
↪needs sage.modules
>>> M                                                                      #␣
↪needs sage.modules
[    3 y + 1      0      0]
[    0      3 y + 1      0]
[    4 y + 2      0      0]
[    0      4 y + 2      0]
```

If the polynomials share a non-constant common factor then the determinant of the Sylvester matrix will be zero:

```
sage: M.determinant()                                                      #␣
↪needs sage.modules
0

sage: f.sylvester_matrix(1 + g, x).determinant()                           #␣
↪needs sage.modules
y^2 - y + 7
```

```
>>> from sage.all import *
>>> M.determinant()                                                        #␣
↪needs sage.modules
0

>>> f.sylvester_matrix(Integer(1) + g, x).determinant()                    ␣
↪     # needs sage.modules
y^2 - y + 7
```

If both polynomials are of positive degree with respect to `variable`, the determinant of the Sylvester matrix is the resultant:

```
sage: f = R.random_element(4) or (x^2 * y^2)
sage: g = R.random_element(4) or (x^2 * y^2)
sage: f.sylvester_matrix(g, x).determinant() == f.resultant(g, x)          #␣
↪needs sage.libs.singular sage.modules
True
```

```
>>> from sage.all import *
>>> f = R.random_element(Integer(4)) or (x**Integer(2) * y**Integer(2))
>>> g = R.random_element(Integer(4)) or (x**Integer(2) * y**Integer(2))
>>> f.sylvester_matrix(g, x).determinant() == f.resultant(g, x)            #␣
↪needs sage.libs.singular sage.modules
True
```

trailing_support(**args, **kwds*)

Return the minimal element of the support of `self`, according to the term order.

If the term ordering of the basis elements is not what is desired, a comparison key, `key(x)`, can be provided.

EXAMPLES:

```
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: (3*x*y^2 + 2*y*z^3 + y^4 + 4*x*y*z).trailing_support()
(1, 1, 1)
sage: R.<x,y,z> = PolynomialRing(QQ, order='lex')
sage: (3*x*y^2 + 2*y*z^3 + y^4 + 4*x*y*z).trailing_support()
(0, 1, 3)
sage: R.<x,y,z> = PolynomialRing(QQ, order='invlex')
sage: (3*x*y^2 + 2*y*z^3 + y^4 + 4*x*y*z).trailing_support()
(1, 2, 0)
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> (Integer(3)*x*y**Integer(2) + Integer(2)*y*z**Integer(3) + y**Integer(4)␣
↪+ Integer(4)*x*y*z).trailing_support()
(1, 1, 1)
>>> R = PolynomialRing(QQ, order='lex', names=('x', 'y', 'z',)); (x, y, z,) =␣
↪R._first_ngens(3)
>>> (Integer(3)*x*y**Integer(2) + Integer(2)*y*z**Integer(3) + y**Integer(4)␣
↪+ Integer(4)*x*y*z).trailing_support()
(0, 1, 3)
>>> R = PolynomialRing(QQ, order='invlex', names=('x', 'y', 'z',)); (x, y, z,
↪) = R._first_ngens(3)
>>> (Integer(3)*x*y**Integer(2) + Integer(2)*y*z**Integer(3) + y**Integer(4)␣
↪+ Integer(4)*x*y*z).trailing_support()
(1, 2, 0)
```

truncate(*var*, *n*)

Return a new multivariate polynomial obtained from `self` by deleting all terms that involve the given variable to a power at least `n`.

weighted_degree(**weights*)

Return the weighted degree of `self`, which is the maximum weighted degree of all monomials in `self`; the weighted degree of a monomial is the sum of all powers of the variables in the monomial, each power multiplied with its respective weight in `weights`.

This method is given for convenience. It is faster to use polynomial rings with weighted term orders and the standard `degree` function.

INPUT:

- `weights` – either individual numbers, an iterable or a dictionary, specifying the weights of each variable. If it is a dictionary, it maps each variable of `self` to its weight. If it is a sequence of individual numbers or a tuple, the weights are specified in the order of the generators as given by `self.parent().gens()`.

EXAMPLES:

```
sage: R.<x,y,z> = GF(7)[]
sage: p = x^3 + y + x*z^2
sage: p.weighted_degree({z:0, x:1, y:2})
3
sage: p.weighted_degree(1, 2, 0)
3
sage: p.weighted_degree((1, 4, 2))
5
```

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```
sage: p.weighted_degree((1, 4, 1))
4
sage: p.weighted_degree(2**64, 2**50, 2**128)
680564733841876926945195958937245974528
sage: q = R.random_element(100, 20)
sage: q.weighted_degree(1, 1, 1) == q.total_degree()
True
```

```
>>> from sage.all import *
>>> R = GF(Integer(7))['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> p = x**Integer(3) + y + x*z**Integer(2)
>>> p.weighted_degree({z:Integer(0), x:Integer(1), y:Integer(2)})
3
>>> p.weighted_degree(Integer(1), Integer(2), Integer(0))
3
>>> p.weighted_degree((Integer(1), Integer(4), Integer(2)))
5
>>> p.weighted_degree((Integer(1), Integer(4), Integer(1)))
4
>>> p.weighted_degree(Integer(2)**Integer(64), Integer(2)**Integer(50),␣
↪Integer(2)**Integer(128))
680564733841876926945195958937245974528
>>> q = R.random_element(Integer(100), Integer(20))
>>> q.weighted_degree(Integer(1), Integer(1), Integer(1)) == q.total_degree()
True
```

You may also work with negative weights

```
sage: p.weighted_degree(-1, -2, -1)
-2
```

```
>>> from sage.all import *
>>> p.weighted_degree(-Integer(1), -Integer(2), -Integer(1))
-2
```

Note that only integer weights are allowed

```
sage: p.weighted_degree(x, 1, 1)
Traceback (most recent call last):
...
TypeError: unable to convert non-constant polynomial x to Integer Ring
sage: p.weighted_degree(2/1, 1, 1)
6
```

```
>>> from sage.all import *
>>> p.weighted_degree(x, Integer(1), Integer(1))
Traceback (most recent call last):
...
TypeError: unable to convert non-constant polynomial x to Integer Ring
>>> p.weighted_degree(Integer(2)/Integer(1), Integer(1), Integer(1))
6
```

The `weighted_degree()` coincides with the `degree()` of a weighted polynomial ring, but the latter is faster.

```
sage: K = PolynomialRing(QQ, 'x,y', order=TermOrder('wdegrevlex', (2,3)))
sage: p = K.random_element(10)
sage: p.degree() == p.weighted_degree(2,3)
True
```

```
>>> from sage.all import *
>>> K = PolynomialRing(QQ, 'x,y', order=TermOrder('wdegrevlex', (Integer(2),
↪Integer(3))))
>>> p = K.random_element(Integer(10))
>>> p.degree() == p.weighted_degree(Integer(2),Integer(3))
True
```

class sage.rings.polynomial.multi_polynomial.**MPolynomial_libsingular**

Bases: `MPolynomial`

Abstract base class for `MPolynomial_libsingular`.

This class is defined for the purpose of `isinstance()` tests. It should not be instantiated.

EXAMPLES:

```
sage: from sage.rings.polynomial.multi_polynomial import MPolynomial_libsingular
sage: R1.<x> = QQ[]
sage: isinstance(x, MPolynomial_libsingular)
False
sage: R2.<y,z> = QQ[]
sage: isinstance(y, MPolynomial_libsingular)                                 #␣
↪needs sage.libs.singular
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.multi_polynomial import MPolynomial_libsingular
>>> R1 = QQ['x']; (x,) = R1._first_ngens(1)
>>> isinstance(x, MPolynomial_libsingular)
False
>>> R2 = QQ['y, z']; (y, z,) = R2._first_ngens(2)
>>> isinstance(y, MPolynomial_libsingular)                                 #␣
↪needs sage.libs.singular
True
```

By design, there is a unique direct subclass:

```
sage: len(sage.rings.polynomial.multi_polynomial.MPolynomial_libsingular.__
↪subclasses__()) <= 1
True
```

```
>>> from sage.all import *
>>> len(sage.rings.polynomial.multi_polynomial.MPolynomial_libsingular.__
↪subclasses__()) <= Integer(1)
True
```

sage.rings.polynomial.multi_polynomial.**is_MPolynomial**(*x*)

3.1.4 Multivariate Polynomial Rings over Generic Rings

Sage implements multivariate polynomial rings through several backends. This generic implementation uses the classes `PolyDict` and `ETuple` to construct a dictionary with exponent tuples as keys and coefficients as values.

AUTHORS:

- David Joyner and William Stein
- Kiran S. Kedlaya (2006-02-12): added Macaulay2 analogues of Singular features
- Martin Albrecht (2006-04-21): reorganize class hierarchy for singular rep
- Martin Albrecht (2007-04-20): reorganized class hierarchy to support Pyrex implementations
- Robert Bradshaw (2007-08-15): Coercions from rings in a subset of the variables.

EXAMPLES:

We construct the Frobenius morphism on $\mathbf{F}_5[x, y, z]$ over $\mathbf{F}_5$:

```
sage: R.<x,y,z> = GF(5)[]
sage: frob = R.hom([x^5, y^5, z^5])
sage: frob(x^2 + 2*y - z^4)
-z^20 + x^10 + 2*y^5
sage: frob((x + 2*y)^3)                                                    #␣
↪needs sage.rings.finite_rings
x^15 + x^10*y^5 + 2*x^5*y^10 - 2*y^15
sage: (x^5 + 2*y^5)^3                                                      #␣
↪needs sage.rings.finite_rings
x^15 + x^10*y^5 + 2*x^5*y^10 - 2*y^15
```

```
>>> from sage.all import *
>>> R = GF(Integer(5))['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> frob = R.hom([x**Integer(5), y**Integer(5), z**Integer(5)])
>>> frob(x**Integer(2) + Integer(2)*y - z**Integer(4))
-z^20 + x^10 + 2*y^5
>>> frob((x + Integer(2)*y)**Integer(3))                                   ␣
↪              # needs sage.rings.finite_rings
x^15 + x^10*y^5 + 2*x^5*y^10 - 2*y^15
>>> (x**Integer(5) + Integer(2)*y**Integer(5))**Integer(3)                 ␣
↪                          # needs sage.rings.finite_rings
x^15 + x^10*y^5 + 2*x^5*y^10 - 2*y^15
```

We make a polynomial ring in one variable over a polynomial ring in two variables:

```
sage: R.<x, y> = PolynomialRing(QQ, 2)
sage: S.<t> = PowerSeriesRing(R)
sage: t*(x+y)
(x + y)*t
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(2), names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> S = PowerSeriesRing(R, names=('t',)); (t,) = S._first_ngens(1)
>>> t*(x+y)
(x + y)*t
```

class sage.rings.polynomial.multi_polynomial_ring.**MPolynomialRing_macaulay2_repr**

Bases: object

A mixin class for polynomial rings that support conversion to Macaulay2.

class sage.rings.polynomial.multi_polynomial_ring.**MPolynomialRing_polydict**(*base_ring*, *n*, *names*, *order*)

Bases: *MPolynomialRing_macaulay2_repr*, *PolynomialRing_singular_repr*, *MPolynomialRing_base*

Multivariable polynomial ring.

EXAMPLES:

```
sage: R = PolynomialRing(Integers(12), 'x', 5); R
Multivariate Polynomial Ring in x0, x1, x2, x3, x4 over Ring of integers modulo 12
sage: loads(R.dumps()) == R
True
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Integers(Integer(12)), 'x', Integer(5)); R
Multivariate Polynomial Ring in x0, x1, x2, x3, x4 over Ring of integers modulo 12
>>> loads(R.dumps()) == R
True
```

Element_hidden

alias of *MPolynomial_polydict*

monomial_all_divisors(*t*)

Return a list of all monomials that divide t, coefficients are ignored.

INPUT:

- t – a monomial

OUTPUT: list of monomials

EXAMPLES:

```
sage: from sage.rings.polynomial.multi_polynomial_ring import MPolynomialRing_
↪polydict_domain
sage: P.<x,y,z> = MPolynomialRing_polydict_domain(QQ,3, order='degrevlex')
sage: P.monomial_all_divisors(x^2*z^3)
[x, x^2, z, x*z, x^2*z, z^2, x*z^2, x^2*z^2, z^3, x*z^3, x^2*z^3]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.multi_polynomial_ring import MPolynomialRing_
↪polydict_domain
>>> P = MPolynomialRing_polydict_domain(QQ,Integer(3), order='degrevlex',␣
↪names=('x', 'y', 'z',)); (x, y, z,) = P._first_ngens(3)
>>> P.monomial_all_divisors(x**Integer(2)*z**Integer(3))
[x, x^2, z, x*z, x^2*z, z^2, x*z^2, x^2*z^2, z^3, x*z^3, x^2*z^3]
```

ALGORITHM: addwithcarry idea by Toon Segers

monomial_divides(*a*, *b*)

Return False if a does not divide b and True otherwise.

INPUT:

- a – monomial
- b – monomial

OUTPUT: boolean

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(ZZ,3, order='degrevlex')
sage: P.monomial_divides(x*y*z, x^3*y^2*z^4)
True
sage: P.monomial_divides(x^3*y^2*z^4, x*y*z)
False
```

```
>>> from sage.all import *
>>> P = PolynomialRing(ZZ,Integer(3), order='degrevlex', names=('x', 'y', 'z',
↪)); (x, y, z,) = P._first_ngens(3)
>>> P.monomial_divides(x*y*z, x**Integer(3)*y**Integer(2)*z**Integer(4))
True
>>> P.monomial_divides(x**Integer(3)*y**Integer(2)*z**Integer(4), x*y*z)
False
```

monomial_lcm(f, g)

LCM for monomials. Coefficients are ignored.

INPUT:

- f – monomial
- g – monomial

OUTPUT: monomial

EXAMPLES:

```
sage: from sage.rings.polynomial.multi_polynomial_ring import MPolynomialRing_
↪polydict_domain
sage: P.<x,y,z> = MPolynomialRing_polydict_domain(QQ,3, order='degrevlex')
sage: P.monomial_lcm(3/2*x*y, x)
x*y
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.multi_polynomial_ring import MPolynomialRing_
↪polydict_domain
>>> P = MPolynomialRing_polydict_domain(QQ,Integer(3), order='degrevlex',␣
↪names=('x', 'y', 'z',)); (x, y, z,) = P._first_ngens(3)
>>> P.monomial_lcm(Integer(3)/Integer(2)*x*y, x)
x*y
```

monomial_pairwise_prime(h, g)

Return True if h and g are pairwise prime.

Both are treated as monomials.

INPUT:

- h – monomial

- g – monomial

OUTPUT: boolean

EXAMPLES:

```
sage: from sage.rings.polynomial.multi_polynomial_ring import MPolynomialRing_
→polydict_domain
sage: P.<x,y,z> = MPolynomialRing_polydict_domain(QQ,3, order='degrevlex')
sage: P.monomial_pairwise_prime(x^2*z^3, y^4)
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.multi_polynomial_ring import MPolynomialRing_
→polydict_domain
>>> P = MPolynomialRing_polydict_domain(QQ,Integer(3), order='degrevlex',␣
→names=('x', 'y', 'z',)); (x, y, z,) = P._first_ngens(3)
>>> P.monomial_pairwise_prime(x**Integer(2)*z**Integer(3), y**Integer(4))
True
```

```
sage: P.monomial_pairwise_prime(1/2*x^3*y^2, 3/4*y^3)
False
```

```
>>> from sage.all import *
>>> P.monomial_pairwise_prime(Integer(1)/
→Integer(2)*x**Integer(3)*y**Integer(2), Integer(3)/Integer(4)*y**Integer(3))
False
```

monomial_quotient(*f*, *g*, *coeff=False*)

Return f/g, where both f and g are treated as monomials.

Coefficients are ignored by default.

INPUT:

- f – monomial
- g – monomial
- coeff – divide coefficients as well (default: False)

OUTPUT: monomial

EXAMPLES:

```
sage: from sage.rings.polynomial.multi_polynomial_ring import MPolynomialRing_
→polydict_domain
sage: P.<x,y,z> = MPolynomialRing_polydict_domain(QQ, 3, order='degrevlex')
sage: P.monomial_quotient(3/2*x*y, x)
y
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.multi_polynomial_ring import MPolynomialRing_
→polydict_domain
>>> P = MPolynomialRing_polydict_domain(QQ, Integer(3), order='degrevlex',␣
→names=('x', 'y', 'z',)); (x, y, z,) = P._first_ngens(3)
```

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```
>>> P.monomial_quotient(Integer(3)/Integer(2)*x*y, x)
y
```

```
sage: P.monomial_quotient(3/2*x*y, 2*x, coeff=True)
3/4*y
```

```
>>> from sage.all import *
>>> P.monomial_quotient(Integer(3)/Integer(2)*x*y, Integer(2)*x, coeff=True)
3/4*y
```

Note

Assumes that the head term of f is a multiple of the head term of g and return the multiplicant m. If this rule is violated, funny things may happen.

monomial_reduce(*f*, *G*)

Try to find a `g` in `G` where `g.lm()` divides `f`.

If found, `(flt,g)` is returned, `(0,0)` otherwise, where `flt` is `f/g.lm()`. It is assumed that `G` is iterable and contains ONLY elements in this ring.

INPUT:

- `f` – monomial
- `G` – list/set of mpolynomials

EXAMPLES:

```
sage: from sage.rings.polynomial.multi_polynomial_ring import MPolynomialRing_
↪polydict_domain
sage: P.<x,y,z>=MPolynomialRing_polydict_domain(QQ,3, order='degrevlex')
sage: f = x*y^2
sage: G = [3/2*x^3 + y^2 + 1/2, 1/4*x*y + 2/7, P(1/2)]
sage: P.monomial_reduce(f,G)
(y, 1/4*x*y + 2/7)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.multi_polynomial_ring import MPolynomialRing_
↪polydict_domain
>>> P = MPolynomialRing_polydict_domain(QQ,Integer(3), order='degrevlex',␣
↪names=('x', 'y', 'z',)); (x, y, z,) = P._first_ngens(3)
>>> f = x*y**Integer(2)
>>> G = [Integer(3)/Integer(2)*x**Integer(3) + y**Integer(2) + Integer(1)/
↪Integer(2), Integer(1)/Integer(4)*x*y + Integer(2)/Integer(7), P(Integer(1)/
↪Integer(2))]
>>> P.monomial_reduce(f,G)
(y, 1/4*x*y + 2/7)
```

```
sage: from sage.rings.polynomial.multi_polynomial_ring import MPolynomialRing_
↪polydict_domain
sage: P.<x,y,z> = MPolynomialRing_polydict_domain(Zmod(23432),3, order=
```

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```
↪'degrevlex')
sage: f = x*y^2
sage: G = [3*x^3 + y^2 + 2, 4*x*y + 7, P(2)]
sage: P.monomial_reduce(f,G)
(y, 4*x*y + 7)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.multi_polynomial_ring import MPolynomialRing_
↪polydict_domain
>>> P = MPolynomialRing_polydict_domain(Zmod(Integer(23432)),Integer(3),␣
↪order='degrevlex', names=('x', 'y', 'z',)); (x, y, z,) = P._first_ngens(3)
>>> f = x*y**Integer(2)
>>> G = [Integer(3)*x**Integer(3) + y**Integer(2) + Integer(2),␣
↪Integer(4)*x*y + Integer(7), P(Integer(2))]
>>> P.monomial_reduce(f,G)
(y, 4*x*y + 7)
```

sum(*terms*)

Return a sum of elements of this multipolynomial ring.

This is method is much faster than the Python builtin *sum()*.

EXAMPLES:

```
sage: R = QQ['x']
sage: S = R['y, z']
sage: x = R.gen()
sage: y, z = S.gens()
sage: S.sum([x*y, 2*x^2*z - 2*x*y, 1 + y + z])
(-x + 1)*y + (2*x^2 + 1)*z + 1
```

```
>>> from sage.all import *
>>> R = QQ['x']
>>> S = R['y, z']
>>> x = R.gen()
>>> y, z = S.gens()
>>> S.sum([x*y, Integer(2)*x**Integer(2)*z - Integer(2)*x*y, Integer(1) + y +␣
↪z])
(-x + 1)*y + (2*x^2 + 1)*z + 1
```

Comparison with builtin *sum()*:

```
sage: sum([x*y, 2*x^2*z - 2*x*y, 1 + y + z])
(-x + 1)*y + (2*x^2 + 1)*z + 1
```

```
>>> from sage.all import *
>>> sum([x*y, Integer(2)*x**Integer(2)*z - Integer(2)*x*y, Integer(1) + y +␣
↪z])
(-x + 1)*y + (2*x^2 + 1)*z + 1
```

class sage.rings.polynomial.multi_polynomial_ring.**MPolynomialRing_polydict_domain**(*base_ring*, *n*, *names*, *order*)

Bases: `MPolynomialRing_polydict`

is_field(*proof=True*)

3.1.5 Generic Multivariate Polynomials

AUTHORS:

- David Joyner: first version
- William Stein: use dict's instead of lists
- Martin Albrecht malb@informatik.uni-bremen.de: some functions added
- William Stein (2006-02-11): added better __div__ behavior.
- Kiran S. Kedlaya (2006-02-12): added Macaulay2 analogues of some Singular features
- William Stein (2006-04-19): added e.g., `f[1,3]` to get coeff of xy^3; added examples of the new `R.x,y = PolynomialRing(QQ,2)` notation.
- Martin Albrecht: improved singular coercions (restructured class hierarchy) and added ETuples
- Robert Bradshaw (2007-08-14): added support for coercion of polynomials in a subset of variables (including multi-level univariate rings)
- Joel B. Mohler (2008-03): Refactored interactions with ETuples.

EXAMPLES:

We verify Lagrange's four squares identity:

```
sage: R.<a0,a1,a2,a3,b0,b1,b2,b3> = QQbar[]                                   #␣
↪needs sage.rings.number_field
sage: ((a0^2 + a1^2 + a2^2 + a3^2) * (b0^2 + b1^2 + b2^2 + b3^2) ==           #␣
↪needs sage.rings.number_field
....:  (a0*b0 - a1*b1 - a2*b2 - a3*b3)^2 + (a0*b1 + a1*b0 + a2*b3 - a3*b2)^2
....:  + (a0*b2 - a1*b3 + a2*b0 + a3*b1)^2 + (a0*b3 + a1*b2 - a2*b1 + a3*b0)^2)
True
```

```
>>> from sage.all import *
>>> R = QQbar['a0, a1, a2, a3, b0, b1, b2, b3']; (a0, a1, a2, a3, b0, b1, b2, b3,) =␣
↪R._first_ngens(8)# needs sage.rings.number_field
>>> ((a0**Integer(2) + a1**Integer(2) + a2**Integer(2) + a3**Integer(2)) *␣
↪(b0**Integer(2) + b1**Integer(2) + b2**Integer(2) + b3**Integer(2)) ==       ␣
↪   # needs sage.rings.number_field
...  (a0*b0 - a1*b1 - a2*b2 - a3*b3)**Integer(2) + (a0*b1 + a1*b0 + a2*b3 -␣
↪a3*b2)**Integer(2)
...  + (a0*b2 - a1*b3 + a2*b0 + a3*b1)**Integer(2) + (a0*b3 + a1*b2 - a2*b1 +␣
↪a3*b0)**Integer(2))
True
```

class sage.rings.polynomial.multi_polynomial_element.**MPolynomial_element**(*parent*, *x*)

Bases: `MPolynomial`

Generic multivariate polynomial.

This implementation is based on the `PolyDict`.

Todo

As mentioned in their docstring, *PolyDict* objects never clear zeros. In all arithmetic operations on *MPolynomial_element* there is an additional call to the method `remove_zeros` to clear them. This is not ideal because of the presence of inexact zeros, see Issue #35174.

element()

hamming_weight()

Return the number of nonzero coefficients of this polynomial.

This is also called weight, *hamming_weight()* or sparsity.

EXAMPLES:

```
sage: # needs sage.rings.real_mpfr
sage: R.<x, y> = CC[]
sage: f = x^3 - y
sage: f.number_of_terms()
2
sage: R(0).number_of_terms()
0
sage: f = (x+y)^100
sage: f.number_of_terms()
101
```

```
>>> from sage.all import *
>>> # needs sage.rings.real_mpfr
>>> R = CC['x, y']; (x, y,) = R._first_ngens(2)
>>> f = x**Integer(3) - y
>>> f.number_of_terms()
2
>>> R(Integer(0)).number_of_terms()
0
>>> f = (x+y)**Integer(100)
>>> f.number_of_terms()
101
```

The method *hamming_weight()* is an alias:

```
sage: f.hamming_weight()                                                    #␣
↪needs sage.rings.real_mpfr
101
```

```
>>> from sage.all import *
>>> f.hamming_weight()                                                    #␣
↪needs sage.rings.real_mpfr
101
```

number_of_terms()

Return the number of nonzero coefficients of this polynomial.

This is also called weight, *hamming_weight()* or sparsity.

EXAMPLES:

```
sage: # needs sage.rings.real_mpfr
sage: R.<x, y> = CC[]
sage: f = x^3 - y
sage: f.number_of_terms()
2
sage: R(0).number_of_terms()
0
sage: f = (x+y)^100
sage: f.number_of_terms()
101
```

```
>>> from sage.all import *
>>> # needs sage.rings.real_mpfr
>>> R = CC['x, y']; (x, y,) = R._first_ngens(2)
>>> f = x**Integer(3) - y
>>> f.number_of_terms()
2
>>> R(Integer(0)).number_of_terms()
0
>>> f = (x+y)**Integer(100)
>>> f.number_of_terms()
101
```

The method `hamming_weight()` is an alias:

```
sage: f.hamming_weight()                                                    #␣
↪needs sage.rings.real_mpfr
101
```

```
>>> from sage.all import *
>>> f.hamming_weight()                                                    #␣
↪needs sage.rings.real_mpfr
101
```

class sage.rings.polynomial.multi_polynomial_element.**MPolynomial_polydict**(*parent*, *x*)

Bases: `Polynomial_singular_repr`, `MPolynomial_element`

Multivariate polynomials implemented in pure python using polydicts.

coefficient(*degrees*)

Return the coefficient of the variables with the degrees specified in the python dictionary `degrees`. Mathematically, this is the coefficient in the base ring adjoined by the variables of this ring not listed in `degrees`. However, the result has the same parent as this polynomial.

This function contrasts with the function `monomial_coefficient` which returns the coefficient in the base ring of a monomial.

INPUT:

- `degrees` – can be any of:
  - a dictionary of degree restrictions
  - a list of degree restrictions (with `None` in the unrestricted variables)
  - a monomial (very fast, but not as flexible)

OUTPUT: element of the parent of `self`

See also

For coefficients of specific monomials, look at *monomial_coefficient()*.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x, y> = QQbar[]
sage: f = 2 * x * y
sage: c = f.coefficient({x: 1, y: 1}); c
2
sage: c.parent()
Multivariate Polynomial Ring in x, y over Algebraic Field
sage: c in PolynomialRing(QQbar, 2, names=['x', 'y'])
True
sage: f = y^2 - x^9 - 7*x + 5*x*y
sage: f.coefficient({y: 1})
5*x
sage: f.coefficient({y: 0})
-x^9 + (-7)*x
sage: f.coefficient({x: 0, y: 0})
0
sage: f = (1+y+y^2) * (1+x+x^2)
sage: f.coefficient({x: 0})
y^2 + y + 1
sage: f.coefficient([0, None])
y^2 + y + 1
sage: f.coefficient(x)
y^2 + y + 1
sage: # Be aware that this may not be what you think!
sage: # The physical appearance of the variable x is deceiving --␣
↪particularly if the exponent would be a variable.
sage: f.coefficient(x^0) # outputs the full polynomial
x^2*y^2 + x^2*y + x*y^2 + x^2 + x*y + y^2 + x + y + 1
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQbar['x, y']; (x, y,) = R._first_ngens(2)
>>> f = Integer(2) * x * y
>>> c = f.coefficient({x: Integer(1), y: Integer(1)}); c
2
>>> c.parent()
Multivariate Polynomial Ring in x, y over Algebraic Field
>>> c in PolynomialRing(QQbar, Integer(2), names=['x', 'y'])
True
>>> f = y**Integer(2) - x**Integer(9) - Integer(7)*x + Integer(5)*x*y
>>> f.coefficient({y: Integer(1)})
5*x
>>> f.coefficient({y: Integer(0)})
-x^9 + (-7)*x
```

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```
>>> f.coefficient({x: Integer(0), y: Integer(0)})
0
>>> f = (Integer(1)+y+y**Integer(2)) * (Integer(1)+x+x**Integer(2))
>>> f.coefficient({x: Integer(0)})
y^2 + y + 1
>>> f.coefficient([Integer(0), None])
y^2 + y + 1
>>> f.coefficient(x)
y^2 + y + 1
>>> # Be aware that this may not be what you think!
>>> # The physical appearance of the variable x is deceiving -- particularly␣
↪if the exponent would be a variable.
>>> f.coefficient(x**Integer(0)) # outputs the full polynomial
x^2*y^2 + x^2*y + x*y^2 + x^2 + x*y + y^2 + x + y + 1
```

```
sage: # needs sage.rings.real_mpfr
sage: R.<x,y> = RR[]
sage: f = x*y + 5
sage: c = f.coefficient({x: 0, y: 0}); c
5.00000000000000
sage: parent(c)
Multivariate Polynomial Ring in x, y over Real Field with 53 bits of precision
```

```
>>> from sage.all import *
>>> # needs sage.rings.real_mpfr
>>> R = RR['x, y']; (x, y,) = R._first_ngens(2)
>>> f = x*y + Integer(5)
>>> c = f.coefficient({x: Integer(0), y: Integer(0)}); c
5.00000000000000
>>> parent(c)
Multivariate Polynomial Ring in x, y over Real Field with 53 bits of precision
```

AUTHORS:

- Joel B. Mohler (2007-10-31)

constant_coefficient()

Return the constant coefficient of this multivariate polynomial.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: f = 3*x^2 - 2*y + 7*x^2*y^2 + 5
sage: f.constant_coefficient()
5
sage: f = 3*x^2
sage: f.constant_coefficient()
0
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQbar['x, y']; (x, y,) = R._first_ngens(2)
```

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```
>>> f = Integer(3)*x**Integer(2) - Integer(2)*y +␣
↪Integer(7)*x**Integer(2)*y**Integer(2) + Integer(5)
>>> f.constant_coefficient()
5
>>> f = Integer(3)*x**Integer(2)
>>> f.constant_coefficient()
0
```

degree(*x=None*, *std_grading=False*)

Return the degree of `self` in `x`, where `x` must be one of the generators for the parent of `self`.

INPUT:

- `x` – multivariate polynomial (a generator of the parent of `self`). If `x` is not specified (or is None), return the total degree, which is the maximum degree of any monomial. Note that a weighted term ordering alters the grading of the generators of the ring; see the tests below. To avoid this behavior, set the optional argument `std_grading=True`.

OUTPUT: integer

EXAMPLES:

```
sage: R.<x,y> = RR[]
sage: f = y^2 - x^9 - x
sage: f.degree(x)
9
sage: f.degree(y)
2
sage: (y^10*x - 7*x^2*y^5 + 5*x^3).degree(x)
3
sage: (y^10*x - 7*x^2*y^5 + 5*x^3).degree(y)
10
```

```
>>> from sage.all import *
>>> R = RR['x, y']; (x, y,) = R._first_ngens(2)
>>> f = y**Integer(2) - x**Integer(9) - x
>>> f.degree(x)
9
>>> f.degree(y)
2
>>> (y**Integer(10)*x - Integer(7)*x**Integer(2)*y**Integer(5) +␣
↪Integer(5)*x**Integer(3)).degree(x)
3
>>> (y**Integer(10)*x - Integer(7)*x**Integer(2)*y**Integer(5) +␣
↪Integer(5)*x**Integer(3)).degree(y)
10
```

Note that total degree takes into account if we are working in a polynomial ring with a weighted term order.

```
sage: R = PolynomialRing(QQ, 'x,y', order=TermOrder('wdeglex',(2,3)))
sage: x,y = R.gens()
sage: x.degree()
2
sage: y.degree()
```

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```
3
sage: x.degree(y), x.degree(x), y.degree(x), y.degree(y)
(0, 1, 0, 1)
sage: f = x^2*y + x*y^2
sage: f.degree(x)
2
sage: f.degree(y)
2
sage: f.degree()
8
sage: f.degree(std_grading=True)
3
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, 'x,y', order=TermOrder('wdeglex',(Integer(2),
↪Integer(3))))
>>> x,y = R.gens()
>>> x.degree()
2
>>> y.degree()
3
>>> x.degree(y), x.degree(x), y.degree(x), y.degree(y)
(0, 1, 0, 1)
>>> f = x**Integer(2)*y + x*y**Integer(2)
>>> f.degree(x)
2
>>> f.degree(y)
2
>>> f.degree()
8
>>> f.degree(std_grading=True)
3
```

Note that if `x` is not a generator of the parent of `self`, for example if it is a generator of a polynomial algebra which maps naturally to this one, then it is converted to an element of this algebra. (This fixes the problem reported in Issue #17366.)

```
sage: x, y = ZZ['x','y'].gens()
sage: GF(3037000453)['x','y'].gen(0).degree(x)                              #␣
↪needs sage.rings.finite_rings
1

sage: x0, y0 = QQ['x','y'].gens()
sage: GF(3037000453)['x','y'].gen(0).degree(x0)                             #␣
↪needs sage.rings.finite_rings
Traceback (most recent call last):
...
TypeError: argument is not coercible to the parent

sage: GF(3037000453)['x','y'].gen(0).degree(x^2)                            #␣
↪needs sage.rings.finite_rings
Traceback (most recent call last):
```

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```
...
TypeError: argument is not a generator
```

```
>>> from sage.all import *
>>> x, y = ZZ['x','y'].gens()
>>> GF(Integer(3037000453))['x','y'].gen(Integer(0)).degree(x)                   ␣
↪              # needs sage.rings.finite_rings
1

>>> x0, y0 = QQ['x','y'].gens()
>>> GF(Integer(3037000453))['x','y'].gen(Integer(0)).degree(x0)                  ␣
↪              # needs sage.rings.finite_rings
Traceback (most recent call last):
...
TypeError: argument is not coercible to the parent

>>> GF(Integer(3037000453))['x','y'].gen(Integer(0)).degree(x**Integer(2))       ␣
↪                          # needs sage.rings.finite_rings
Traceback (most recent call last):
...
TypeError: argument is not a generator
```

degrees()

Return a tuple (precisely - an `ETuple`) with the degree of each variable in this polynomial. The list of degrees is, of course, ordered by the order of the generators.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x,y,z> = PolynomialRing(QQbar)
sage: f = 3*x^2 - 2*y + 7*x^2*y^2 + 5
sage: f.degrees()
(2, 2, 0)
sage: f = x^2 + z^2
sage: f.degrees()
(2, 0, 2)
sage: f.total_degree()  # this simply illustrates that total degree is not␣
↪the sum of the degrees
2
sage: R.<x,y,z,u> = PolynomialRing(QQbar)
sage: f = (1-x) * (1+y+z+x^3)^5
sage: f.degrees()
(16, 5, 5, 0)
sage: R(0).degrees()
(0, 0, 0, 0)
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = PolynomialRing(QQbar, names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> f = Integer(3)*x**Integer(2) - Integer(2)*y +␣
↪Integer(7)*x**Integer(2)*y**Integer(2) + Integer(5)
```

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```
>>> f.degrees()
(2, 2, 0)
>>> f = x**Integer(2) + z**Integer(2)
>>> f.degrees()
(2, 0, 2)
>>> f.total_degree()  # this simply illustrates that total degree is not the␣
↪sum of the degrees
2
>>> R = PolynomialRing(QQbar, names=('x', 'y', 'z', 'u',)); (x, y, z, u,) = R.
↪_first_ngens(4)
>>> f = (Integer(1)-x) * (Integer(1)+y+z+x**Integer(3))**Integer(5)
>>> f.degrees()
(16, 5, 5, 0)
>>> R(Integer(0)).degrees()
(0, 0, 0, 0)
```

dict(*copy=None*)

Return underlying dictionary with keys the exponents and values the coefficients of this polynomial.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x,y,z> = PolynomialRing(QQbar, order='lex')
sage: f = (x^1*y^5*z^2 + x^2*z + x^4*y^1*z^3)
sage: f.monomial_coefficients()
{(1, 5, 2): 1, (2, 0, 1): 1, (4, 1, 3): 1}
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = PolynomialRing(QQbar, order='lex', names=('x', 'y', 'z',)); (x, y, z,
↪) = R._first_ngens(3)
>>> f = (x**Integer(1)*y**Integer(5)*z**Integer(2) + x**Integer(2)*z +␣
↪x**Integer(4)*y**Integer(1)*z**Integer(3))
>>> f.monomial_coefficients()
{(1, 5, 2): 1, (2, 0, 1): 1, (4, 1, 3): 1}
```

`dict` is an alias:

```
sage: f.dict()                                                               #␣
↪needs sage.rings.number_field
{(1, 5, 2): 1, (2, 0, 1): 1, (4, 1, 3): 1}
```

```
>>> from sage.all import *
>>> f.dict()                                                               #␣
↪needs sage.rings.number_field
{(1, 5, 2): 1, (2, 0, 1): 1, (4, 1, 3): 1}
```

exponents(*as_ETuples=True*)

Return the exponents of the monomials appearing in `self`.

INPUT:

- `as_ETuples` – (default: `True`) return the list of exponents as a list of ETuples

OUTPUT: the list of exponents as a list of ETuples or tuples

EXAMPLES:

```
sage: R.<a,b,c> = PolynomialRing(QQbar, 3)                                      #␣
↪needs sage.rings.number_field
sage: f = a^3 + b + 2*b^2                                                       #␣
↪needs sage.rings.number_field
sage: f.exponents()                                                             #␣
↪needs sage.rings.number_field
[(3, 0, 0), (0, 2, 0), (0, 1, 0)]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQbar, Integer(3), names=('a', 'b', 'c',)); (a, b, c,)␣
↪= R._first_ngens(3)# needs sage.rings.number_field
>>> f = a**Integer(3) + b + Integer(2)*b**Integer(2)                            ␣
↪                           # needs sage.rings.number_field
>>> f.exponents()                                                              #␣
↪needs sage.rings.number_field
[(3, 0, 0), (0, 2, 0), (0, 1, 0)]
```

By default the list of exponents is a list of ETuples:

```
sage: type(f.exponents()[0])                                                    #␣
↪needs sage.rings.number_field
<class 'sage.rings.polynomial.polydict.ETuple'>
sage: type(f.exponents(as_ETuples=False)[0])                                    #␣
↪needs sage.rings.number_field
<... 'tuple'>
```

```
>>> from sage.all import *
>>> type(f.exponents()[Integer(0)])                                             ␣
↪     # needs sage.rings.number_field
<class 'sage.rings.polynomial.polydict.ETuple'>
>>> type(f.exponents(as_ETuples=False)[Integer(0)])                             ␣
↪     # needs sage.rings.number_field
<... 'tuple'>
```

factor(*proof=None*)

Compute the irreducible factorization of this polynomial.

INPUT:

- `proof` – insist on provably correct results (default: `True` unless explicitly disabled for the `'polynomial'` subsystem with `sage.structure.proof.proof.WithProof`.)

global_height(*prec=None*)

Return the (projective) global height of the polynomial.

This returns the absolute logarithmic height of the coefficients thought of as a projective point.

INPUT:

- `prec` – desired floating point precision (default: default `RealField` precision)

OUTPUT: a real number

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQbar, 2)                                    #␣
↪needs sage.rings.number_field
sage: f = QQbar(i)*x^2 + 3*x*y                                              #␣
↪needs sage.rings.number_field
sage: f.global_height()                                                     #␣
↪needs sage.rings.number_field
1.09861228866811
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQbar, Integer(2), names=('x', 'y',)); (x, y,) = R._
↪first_ngens(2)# needs sage.rings.number_field
>>> f = QQbar(i)*x**Integer(2) + Integer(3)*x*y                             ␣
↪               # needs sage.rings.number_field
>>> f.global_height()                                                     #␣
↪needs sage.rings.number_field
1.09861228866811
```

Scaling should not change the result:

```
sage: # needs sage.rings.number_field sage.symbolic
sage: R.<x, y> = PolynomialRing(QQbar, 2)
sage: f = 1/25*x^2 + 25/3*x + 1 + QQbar(sqrt(2))*y^2
sage: f.global_height()
6.43775164973640
sage: g = 100 * f
sage: g.global_height()
6.43775164973640
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field sage.symbolic
>>> R = PolynomialRing(QQbar, Integer(2), names=('x', 'y',)); (x, y,) = R._
↪first_ngens(2)
>>> f = Integer(1)/Integer(25)*x**Integer(2) + Integer(25)/Integer(3)*x +␣
↪Integer(1) + QQbar(sqrt(Integer(2)))*y**Integer(2)
>>> f.global_height()
6.43775164973640
>>> g = Integer(100) * f
>>> g.global_height()
6.43775164973640
```

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: K.<k> = NumberField(x^2 + 1)
sage: Q.<q,r> = PolynomialRing(K, implementation='generic')
sage: f = 12 * q
sage: f.global_height()
0.000000000000000
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> K = NumberField(x**Integer(2) + Integer(1), names=('k',)); (k,) = K._
```

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```
↪first_ngens(1)
>>> Q = PolynomialRing(K, implementation='generic', names=('q', 'r',)); (q, r,
↪) = Q._first_ngens(2)
>>> f = Integer(12) * q
>>> f.global_height()
0.000000000000000
```

```
sage: R.<x,y> = PolynomialRing(QQ, implementation='generic')
sage: f = 1/123*x*y + 12
sage: f.global_height(prec=2)                                           #␣
↪needs sage.symbolic
8.0
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, implementation='generic', names=('x', 'y',)); (x,␣
↪y,) = R._first_ngens(2)
>>> f = Integer(1)/Integer(123)*x*y + Integer(12)
>>> f.global_height(prec=Integer(2))                                    ␣
↪     # needs sage.symbolic
8.0
```

```
sage: R.<x,y> = PolynomialRing(QQ, implementation='generic')
sage: f = 0*x*y
sage: f.global_height()                                                 #␣
↪needs sage.rings.real_mpfr
0.000000000000000
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, implementation='generic', names=('x', 'y',)); (x,␣
↪y,) = R._first_ngens(2)
>>> f = Integer(0)*x*y
>>> f.global_height()                                                 #␣
↪needs sage.rings.real_mpfr
0.000000000000000
```

integral(*var=None*)

Integrate `self` with respect to variable `var`.

Note

The integral is always chosen so the constant term is 0.

If `var` is not one of the generators of this ring, `integral(var)` is called recursively on each coefficient of this polynomial.

EXAMPLES:

On polynomials with rational coefficients:

```
sage: x, y = PolynomialRing(QQ, 'x, y').gens()
sage: ex = x*y + x - y
```

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```
sage: it = ex.integral(x); it
1/2*x^2*y + 1/2*x^2 - x*y
sage: it.parent() == x.parent()
True

sage: R = ZZ['x']['y, z']
sage: y, z = R.gens()
sage: R.an_element().integral(y).parent()
Multivariate Polynomial Ring in y, z
 over Univariate Polynomial Ring in x over Rational Field
```

```
>>> from sage.all import *
>>> x, y = PolynomialRing(QQ, 'x, y').gens()
>>> ex = x*y + x - y
>>> it = ex.integral(x); it
1/2*x^2*y + 1/2*x^2 - x*y
>>> it.parent() == x.parent()
True

>>> R = ZZ['x']['y, z']
>>> y, z = R.gens()
>>> R.an_element().integral(y).parent()
Multivariate Polynomial Ring in y, z
 over Univariate Polynomial Ring in x over Rational Field
```

On polynomials with coefficients in power series:

```
sage: # needs sage.rings.number_field
sage: R.<t> = PowerSeriesRing(QQbar)
sage: S.<x, y> = PolynomialRing(R)
sage: f = (t^2 + O(t^3))*x^2*y^3 + (37*t^4 + O(t^5))*x^3
sage: f.parent()
Multivariate Polynomial Ring in x, y
 over Power Series Ring in t over Algebraic Field
sage: f.integral(x)    # with respect to x
(1/3*t^2 + O(t^3))*x^3*y^3 + (37/4*t^4 + O(t^5))*x^4
sage: f.integral(x).parent()
Multivariate Polynomial Ring in x, y
 over Power Series Ring in t over Algebraic Field
sage: f.integral(y)    # with respect to y
(1/4*t^2 + O(t^3))*x^2*y^4 + (37*t^4 + O(t^5))*x^3*y
sage: f.integral(t)    # with respect to t (recurses into base ring)
(1/3*t^3 + O(t^4))*x^2*y^3 + (37/5*t^5 + O(t^6))*x^3
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = PowerSeriesRing(QQbar, names=('t',)); (t,) = R._first_ngens(1)
>>> S = PolynomialRing(R, names=('x', 'y',)); (x, y,) = S._first_ngens(2)
>>> f = (t**Integer(2) + O(t**Integer(3)))*x**Integer(2)*y**Integer(3) +␣
↪(Integer(37)*t**Integer(4) + O(t**Integer(5)))*x**Integer(3)
>>> f.parent()
Multivariate Polynomial Ring in x, y
```

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```
 over Power Series Ring in t over Algebraic Field
>>> f.integral(x)   # with respect to x
(1/3*t^2 + O(t^3))*x^3*y^3 + (37/4*t^4 + O(t^5))*x^4
>>> f.integral(x).parent()
Multivariate Polynomial Ring in x, y
 over Power Series Ring in t over Algebraic Field
>>> f.integral(y)   # with respect to y
(1/4*t^2 + O(t^3))*x^2*y^4 + (37*t^4 + O(t^5))*x^3*y
>>> f.integral(t)   # with respect to t (recurses into base ring)
(1/3*t^3 + O(t^4))*x^2*y^3 + (37/5*t^5 + O(t^6))*x^3
```

inverse_of_unit()

Return the inverse of a unit in a ring.

is_constant()

Return `True` if `self` is a constant and `False` otherwise.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: f = 3*x^2 - 2*y + 7*x^2*y^2 + 5
sage: f.is_constant()
False
sage: g = 10*x^0
sage: g.is_constant()
True
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQbar['x, y']; (x, y,) = R._first_ngens(2)
>>> f = Integer(3)*x**Integer(2) - Integer(2)*y +␣
↪Integer(7)*x**Integer(2)*y**Integer(2) + Integer(5)
>>> f.is_constant()
False
>>> g = Integer(10)*x**Integer(0)
>>> g.is_constant()
True
```

is_gen()

Return `True` if `self` is a generator of its parent.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: x.is_gen()
True
sage: (x + y - y).is_gen()
True
sage: (x*y).is_gen()
False
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQbar['x, y']; (x, y,) = R._first_ngens(2)
>>> x.is_gen()
True
>>> (x + y - y).is_gen()
True
>>> (x*y).is_gen()
False
```

is_generator(**args, **kwds*)

Deprecated: Use `is_gen()` instead. See Issue #38942 for details.

is_homogeneous()

Return `True` if `self` is a homogeneous polynomial.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: (x + y).is_homogeneous()
True
sage: (x.parent()(0)).is_homogeneous()
True
sage: (x + y^2).is_homogeneous()
False
sage: (x^2 + y^2).is_homogeneous()
True
sage: (x^2 + y^2*x).is_homogeneous()
False
sage: (x^2*y + y^2*x).is_homogeneous()
True
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQbar['x, y']; (x, y,) = R._first_ngens(2)
>>> (x + y).is_homogeneous()
True
>>> (x.parent()(Integer(0))).is_homogeneous()
True
>>> (x + y**Integer(2)).is_homogeneous()
False
>>> (x**Integer(2) + y**Integer(2)).is_homogeneous()
True
>>> (x**Integer(2) + y**Integer(2)*x).is_homogeneous()
False
>>> (x**Integer(2)*y + y**Integer(2)*x).is_homogeneous()
True
```

The weight of the parent ring is respected:

```
sage: term_order = TermOrder("wdegrevlex", [1, 3])
sage: R.<x, y> = PolynomialRing(Qp(5), order=term_order)
```

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```
sage: (x + y).is_homogeneous()
False
sage: (x^3 + y).is_homogeneous()
True
```

```
>>> from sage.all import *
>>> term_order = TermOrder("wdegrevlex", [Integer(1), Integer(3)])
>>> R = PolynomialRing(Qp(Integer(5)), order=term_order, names=('x', 'y',));␣
↪(x, y,) = R._first_ngens(2)
>>> (x + y).is_homogeneous()
False
>>> (x**Integer(3) + y).is_homogeneous()
True
```

is_monomial()

Return `True` if `self` is a monomial, which we define to be a product of generators with coefficient 1.

Use *is_term()* to allow the coefficient to not be 1.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: x.is_monomial()
True
sage: (x + 2*y).is_monomial()
False
sage: (2*x).is_monomial()
False
sage: (x*y).is_monomial()
True
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQbar['x, y']; (x, y,) = R._first_ngens(2)
>>> x.is_monomial()
True
>>> (x + Integer(2)*y).is_monomial()
False
>>> (Integer(2)*x).is_monomial()
False
>>> (x*y).is_monomial()
True
```

To allow a non-1 leading coefficient, use *is_term()*:

```
sage: (2*x*y).is_term()                                                 #␣
↪needs sage.rings.number_field
True
sage: (2*x*y).is_monomial()                                             #␣
↪needs sage.rings.number_field
False
```

```
>>> from sage.all import *
>>> (Integer(2)*x*y).is_term()                                        ␣
↪     # needs sage.rings.number_field
True
>>> (Integer(2)*x*y).is_monomial()                                    ␣
↪     # needs sage.rings.number_field
False
```

is_term()

Return `True` if `self` is a term, which we define to be a product of generators times some coefficient, which need not be 1.

Use *is_monomial()* to require that the coefficient be 1.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: x.is_term()
True
sage: (x + 2*y).is_term()
False
sage: (2*x).is_term()
True
sage: (7*x^5*y).is_term()
True
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQbar['x, y']; (x, y,) = R._first_ngens(2)
>>> x.is_term()
True
>>> (x + Integer(2)*y).is_term()
False
>>> (Integer(2)*x).is_term()
True
>>> (Integer(7)*x**Integer(5)*y).is_term()
True
```

To require leading coefficient 1, use *is_monomial()*:

```
sage: (2*x*y).is_monomial()                                          #␣
↪needs sage.rings.number_field
False
sage: (2*x*y).is_term()                                              #␣
↪needs sage.rings.number_field
True
```

```
>>> from sage.all import *
>>> (Integer(2)*x*y).is_monomial()                                    ␣
↪     # needs sage.rings.number_field
False
>>> (Integer(2)*x*y).is_term()                                        ␣
```

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```
↪      # needs sage.rings.number_field
True
```

is_univariate()

Return `True` if this multivariate polynomial is univariate and `False` otherwise.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: f = 3*x^2 - 2*y + 7*x^2*y^2 + 5
sage: f.is_univariate()
False
sage: g = f.subs({x: 10}); g
700*y^2 + (-2)*y + 305
sage: g.is_univariate()
True
sage: f = x^0
sage: f.is_univariate()
True
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQbar['x, y']; (x, y,) = R._first_ngens(2)
>>> f = Integer(3)*x**Integer(2) - Integer(2)*y +␣
↪Integer(7)*x**Integer(2)*y**Integer(2) + Integer(5)
>>> f.is_univariate()
False
>>> g = f.subs({x: Integer(10)}); g
700*y^2 + (-2)*y + 305
>>> g.is_univariate()
True
>>> f = x**Integer(0)
>>> f.is_univariate()
True
```

iterator_exp_coeff(*as_ETuples=True*)

Iterate over `self` as pairs of ((E)Tuple, coefficient).

INPUT:

- `as_ETuples` – boolean (default: `True`); if `True` iterate over pairs whose first element is an ETuple, otherwise as a tuples

EXAMPLES:

```
sage: R.<x,y,z> = PolynomialRing(QQbar, order='lex')                     #␣
↪needs sage.rings.number_field
sage: f = (x^1*y^5*z^2 + x^2*z + x^4*y^1*z^3)                            #␣
↪needs sage.rings.number_field
sage: list(f.iterator_exp_coeff())                                        #␣
↪needs sage.rings.number_field
[((4, 1, 3), 1), ((2, 0, 1), 1), ((1, 5, 2), 1)]
```

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```
sage: R.<x,y,z> = PolynomialRing(QQbar, order='deglex')                        #␣
↪needs sage.rings.number_field
sage: f = (x^1*y^5*z^2 + x^2*z + x^4*y^1*z^3)                                   #␣
↪needs sage.rings.number_field
sage: list(f.iterator_exp_coeff(as_ETuples=False))                             #␣
↪needs sage.rings.number_field
[((4, 1, 3), 1), ((1, 5, 2), 1), ((2, 0, 1), 1)]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQbar, order='lex', names=('x', 'y', 'z',)); (x, y, z,
↪) = R._first_ngens(3)# needs sage.rings.number_field
>>> f = (x**Integer(1)*y**Integer(5)*z**Integer(2) + x**Integer(2)*z +␣
↪x**Integer(4)*y**Integer(1)*z**Integer(3))                                   #␣
↪needs sage.rings.number_field
>>> list(f.iterator_exp_coeff())                                              #␣
↪needs sage.rings.number_field
[((4, 1, 3), 1), ((2, 0, 1), 1), ((1, 5, 2), 1)]

>>> R = PolynomialRing(QQbar, order='deglex', names=('x', 'y', 'z',)); (x, y,␣
↪z,) = R._first_ngens(3)# needs sage.rings.number_field
>>> f = (x**Integer(1)*y**Integer(5)*z**Integer(2) + x**Integer(2)*z +␣
↪x**Integer(4)*y**Integer(1)*z**Integer(3))                                   #␣
↪needs sage.rings.number_field
>>> list(f.iterator_exp_coeff(as_ETuples=False))                              #␣
↪needs sage.rings.number_field
[((4, 1, 3), 1), ((1, 5, 2), 1), ((2, 0, 1), 1)]
```

lc()

Return the leading coefficient of `self`, i.e., `self.coefficient(self.lm())`.

EXAMPLES:

```
sage: R.<x,y,z> = QQbar[]                                                      #␣
↪needs sage.rings.number_field
sage: f = 3*x^2 - y^2 - x*y                                                    #␣
↪needs sage.rings.number_field
sage: f.lc()                                                                   #␣
↪needs sage.rings.number_field
3
```

```
>>> from sage.all import *
>>> R = QQbar['x, y, z']; (x, y, z,) = R._first_ngens(3)# needs sage.rings.
↪number_field
>>> f = Integer(3)*x**Integer(2) - y**Integer(2) - x*y                         ␣
↪                          # needs sage.rings.number_field
>>> f.lc()                                                                    #␣
↪needs sage.rings.number_field
3
```

lift(I)

Given an ideal $I = (f_1, ..., f_r)$ and some g (= `self`) in I, find $s_1, ..., s_r$ such that $g = s_1 f_1 + ... + s_r f_r$.

ALGORITHM: Use Singular.

EXAMPLES:

```
sage: # needs sage.rings.real_mpfr
sage: A.<x,y> = PolynomialRing(CC, 2, order='degrevlex')
sage: I = A.ideal([x^10 + x^9*y^2, y^8 - x^2*y^7])
sage: f = x*y^13 + y^12
sage: M = f.lift(I); M                                                  #␣
↪needs sage.libs.singular
[y^7, x^7*y^2 + x^8 + x^5*y^3 + x^6*y + x^3*y^4 + x^4*y^2 + x*y^5 + x^2*y^3 +␣
↪y^4]
sage: sum(map(mul, zip(M, I.gens()))) == f                              #␣
↪needs sage.libs.singular
True
```

```
>>> from sage.all import *
>>> # needs sage.rings.real_mpfr
>>> A = PolynomialRing(CC, Integer(2), order='degrevlex', names=('x', 'y',));␣
↪(x, y,) = A._first_ngens(2)
>>> I = A.ideal([x**Integer(10) + x**Integer(9)*y**Integer(2), y**Integer(8) -
↪ x**Integer(2)*y**Integer(7)])
>>> f = x*y**Integer(13) + y**Integer(12)
>>> M = f.lift(I); M                                                    #␣
↪needs sage.libs.singular
[y^7, x^7*y^2 + x^8 + x^5*y^3 + x^6*y + x^3*y^4 + x^4*y^2 + x*y^5 + x^2*y^3 +␣
↪y^4]
>>> sum(map(mul, zip(M, I.gens()))) == f                                #␣
↪needs sage.libs.singular
True
```

lm()

Return the lead monomial of `self` with respect to the term order of `self.parent()`.

EXAMPLES:

```
sage: R.<x,y,z> = PolynomialRing(GF(7), 3, order='lex')
sage: (x^1*y^2 + y^3*z^4).lm()
x*y^2
sage: (x^3*y^2*z^4 + x^3*y^2*z^1).lm()
x^3*y^2*z^4
```

```
>>> from sage.all import *
>>> R = PolynomialRing(GF(Integer(7)), Integer(3), order='lex', names=('x', 'y
↪', 'z',)); (x, y, z,) = R._first_ngens(3)
>>> (x**Integer(1)*y**Integer(2) + y**Integer(3)*z**Integer(4)).lm()
x*y^2
>>> (x**Integer(3)*y**Integer(2)*z**Integer(4) +␣
↪x**Integer(3)*y**Integer(2)*z**Integer(1)).lm()
x^3*y^2*z^4
```

```
sage: # needs sage.rings.real_mpfr
sage: R.<x,y,z> = PolynomialRing(CC, 3, order='deglex')
sage: (x^1*y^2*z^3 + x^3*y^2*z^0).lm()
x*y^2*z^3
```

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```
sage: (x^1*y^2*z^4 + x^1*y^1*z^5).lm()
x*y^2*z^4
```

```
>>> from sage.all import *
>>> # needs sage.rings.real_mpfr
>>> R = PolynomialRing(CC, Integer(3), order='deglex', names=('x', 'y', 'z',
↪)); (x, y, z,) = R._first_ngens(3)
>>> (x**Integer(1)*y**Integer(2)*z**Integer(3) +␣
↪x**Integer(3)*y**Integer(2)*z**Integer(0)).lm()
x*y^2*z^3
>>> (x**Integer(1)*y**Integer(2)*z**Integer(4) +␣
↪x**Integer(1)*y**Integer(1)*z**Integer(5)).lm()
x*y^2*z^4
```

```
sage: # needs sage.rings.number_field
sage: R.<x,y,z> = PolynomialRing(QQbar, 3, order='degrevlex')
sage: (x^1*y^5*z^2 + x^4*y^1*z^3).lm()
x*y^5*z^2
sage: (x^4*y^7*z^1 + x^4*y^2*z^3).lm()
x^4*y^7*z
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = PolynomialRing(QQbar, Integer(3), order='degrevlex', names=('x', 'y',
↪'z',)); (x, y, z,) = R._first_ngens(3)
>>> (x**Integer(1)*y**Integer(5)*z**Integer(2) +␣
↪x**Integer(4)*y**Integer(1)*z**Integer(3)).lm()
x*y^5*z^2
>>> (x**Integer(4)*y**Integer(7)*z**Integer(1) +␣
↪x**Integer(4)*y**Integer(2)*z**Integer(3)).lm()
x^4*y^7*z
```

local_height(*v*, *prec=None*)

Return the maximum of the local height of the coefficients of this polynomial.

INPUT:

- v – a prime or prime ideal of the base ring
- prec – desired floating point precision (default: default RealField precision)

OUTPUT: a real number

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQ, implementation='generic')
sage: f = 1/1331*x^2 + 1/4000*y
sage: f.local_height(1331)                                              #␣
↪needs sage.rings.real_mpfr
7.19368581839511
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, implementation='generic', names=('x', 'y',)); (x,␣
```

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```
↪y,) = R._first_ngens(2)
>>> f = Integer(1)/Integer(1331)*x**Integer(2) + Integer(1)/Integer(4000)*y
>>> f.local_height(Integer(1331))                                              ␣
↪     # needs sage.rings.real_mpfr
7.19368581839511
```

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: K.<k> = NumberField(x^2 - 5)
sage: T.<t,w> = PolynomialRing(K, implementation='generic')
sage: I = K.ideal(3)
sage: f = 1/3*t*w + 3
sage: f.local_height(I)                                                       #␣
↪needs sage.symbolic
1.09861228866811
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> K = NumberField(x**Integer(2) - Integer(5), names=('k',)); (k,) = K._
↪first_ngens(1)
>>> T = PolynomialRing(K, implementation='generic', names=('t', 'w',)); (t, w,
↪) = T._first_ngens(2)
>>> I = K.ideal(Integer(3))
>>> f = Integer(1)/Integer(3)*t*w + Integer(3)
>>> f.local_height(I)                                                       #␣
↪needs sage.symbolic
1.09861228866811
```

```
sage: R.<x,y> = PolynomialRing(QQ, implementation='generic')
sage: f = 1/2*x*y + 2
sage: f.local_height(2, prec=2)                                               #␣
↪needs sage.rings.real_mpfr
0.75
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, implementation='generic', names=('x', 'y',)); (x,␣
↪y,) = R._first_ngens(2)
>>> f = Integer(1)/Integer(2)*x*y + Integer(2)
>>> f.local_height(Integer(2), prec=Integer(2))                                ␣
↪              # needs sage.rings.real_mpfr
0.75
```

local_height_arch(*i*, *prec=None*)

Return the maximum of the local height at the `i`-th infinite place of the coefficients of this polynomial.

INPUT:

- `i` – integer
- `prec` – desired floating point precision (default: default `RealField` precision)

OUTPUT: a real number

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQ, implementation='generic')
sage: f = 210*x*y
sage: f.local_height_arch(0)                                                #␣
↪needs sage.rings.real_mpfr
5.34710753071747
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, implementation='generic', names=('x', 'y',)); (x,␣
↪y,) = R._first_ngens(2)
>>> f = Integer(210)*x*y
>>> f.local_height_arch(Integer(0))                                          ␣
↪      # needs sage.rings.real_mpfr
5.34710753071747
```

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: K.<k> = NumberField(x^2 - 5)
sage: T.<t,w> = PolynomialRing(K, implementation='generic')
sage: f = 1/2*t*w + 3
sage: f.local_height_arch(1, prec=52)
1.09861228866811
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> K = NumberField(x**Integer(2) - Integer(5), names=('k',)); (k,) = K._
↪first_ngens(1)
>>> T = PolynomialRing(K, implementation='generic', names=('t', 'w',)); (t, w,
↪) = T._first_ngens(2)
>>> f = Integer(1)/Integer(2)*t*w + Integer(3)
>>> f.local_height_arch(Integer(1), prec=Integer(52))
1.09861228866811
```

```
sage: R.<x,y> = PolynomialRing(QQ, implementation='generic')
sage: f = 1/2*x*y + 3
sage: f.local_height_arch(0, prec=2)                                        #␣
↪needs sage.rings.real_mpfr
1.0
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, implementation='generic', names=('x', 'y',)); (x,␣
↪y,) = R._first_ngens(2)
>>> f = Integer(1)/Integer(2)*x*y + Integer(3)
>>> f.local_height_arch(Integer(0), prec=Integer(2))                         ␣
↪               # needs sage.rings.real_mpfr
1.0
```

lt()

Return the leading term of `self` i.e., `self.lc()*self.lm()`. The notion of "leading term" depends on the ordering defined in the parent ring.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x,y,z> = PolynomialRing(QQbar)
sage: f = 3*x^2 - y^2 - x*y
sage: f.lt()
3*x^2
sage: R.<x,y,z> = PolynomialRing(QQbar, order='invlex')
sage: f = 3*x^2 - y^2 - x*y
sage: f.lt()
-y^2
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = PolynomialRing(QQbar, names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> f = Integer(3)*x**Integer(2) - y**Integer(2) - x*y
>>> f.lt()
3*x^2
>>> R = PolynomialRing(QQbar, order='invlex', names=('x', 'y', 'z',)); (x, y,␣
↪z,) = R._first_ngens(3)
>>> f = Integer(3)*x**Integer(2) - y**Integer(2) - x*y
>>> f.lt()
-y^2
```

monomial_coefficient(*mon*)

Return the coefficient in the base ring of the monomial `mon` in `self`, where `mon` must have the same parent as `self`.

This function contrasts with the function `coefficient` which returns the coefficient of a monomial viewing this polynomial in a polynomial ring over a base ring having fewer variables.

INPUT:

- `mon` – a monomial

OUTPUT: coefficient in base ring

See also

For coefficients in a base ring of fewer variables, look at *coefficient()*.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: f = 2 * x * y
sage: c = f.monomial_coefficient(x*y); c
2
sage: c.parent()
Algebraic Field
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
```

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```
>>> R = QQbar['x, y']; (x, y,) = R._first_ngens(2)
>>> f = Integer(2) * x * y
>>> c = f.monomial_coefficient(x*y); c
2
>>> c.parent()
Algebraic Field
```

```
sage: # needs sage.rings.number_field
sage: f = y^2 + y^2*x - x^9 - 7*x + 5*x*y
sage: f.monomial_coefficient(y^2)
1
sage: f.monomial_coefficient(x*y)
5
sage: f.monomial_coefficient(x^9)
-1
sage: f.monomial_coefficient(x^10)
0
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> f = y**Integer(2) + y**Integer(2)*x - x**Integer(9) - Integer(7)*x +␣
↪Integer(5)*x*y
>>> f.monomial_coefficient(y**Integer(2))
1
>>> f.monomial_coefficient(x*y)
5
>>> f.monomial_coefficient(x**Integer(9))
-1
>>> f.monomial_coefficient(x**Integer(10))
0
```

```
sage: # needs sage.rings.number_field
sage: a = polygen(ZZ, 'a')
sage: K.<a> = NumberField(a^2 + a + 1)
sage: P.<x,y> = K[]
sage: f = (a*x - 1) * ((a+1)*y - 1); f
-x*y + (-a)*x + (-a - 1)*y + 1
sage: f.monomial_coefficient(x)
-a
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> a = polygen(ZZ, 'a')
>>> K = NumberField(a**Integer(2) + a + Integer(1), names=('a',)); (a,) = K._
↪first_ngens(1)
>>> P = K['x, y']; (x, y,) = P._first_ngens(2)
>>> f = (a*x - Integer(1)) * ((a+Integer(1))*y - Integer(1)); f
-x*y + (-a)*x + (-a - 1)*y + 1
>>> f.monomial_coefficient(x)
-a
```

monomial_coefficients(*copy=None*)

Return underlying dictionary with keys the exponents and values the coefficients of this polynomial.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x,y,z> = PolynomialRing(QQbar, order='lex')
sage: f = (x^1*y^5*z^2 + x^2*z + x^4*y^1*z^3)
sage: f.monomial_coefficients()
{(1, 5, 2): 1, (2, 0, 1): 1, (4, 1, 3): 1}
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = PolynomialRing(QQbar, order='lex', names=('x', 'y', 'z',)); (x, y, z,
↪) = R._first_ngens(3)
>>> f = (x**Integer(1)*y**Integer(5)*z**Integer(2) + x**Integer(2)*z +␣
↪x**Integer(4)*y**Integer(1)*z**Integer(3))
>>> f.monomial_coefficients()
{(1, 5, 2): 1, (2, 0, 1): 1, (4, 1, 3): 1}
```

`dict` is an alias:

```
sage: f.dict()                                                          #␣
↪needs sage.rings.number_field
{(1, 5, 2): 1, (2, 0, 1): 1, (4, 1, 3): 1}
```

```
>>> from sage.all import *
>>> f.dict()                                                          #␣
↪needs sage.rings.number_field
{(1, 5, 2): 1, (2, 0, 1): 1, (4, 1, 3): 1}
```

monomials()

Return the list of monomials in `self`. The returned list is decreasingly ordered by the term ordering of `self.parent()`.

OUTPUT: list of *MPolynomial* instances, representing monomials

EXAMPLES:

```
sage: R.<x,y> = QQbar[]                                                 #␣
↪needs sage.rings.number_field
sage: f = 3*x^2 - 2*y + 7*x^2*y^2 + 5                                   #␣
↪needs sage.rings.number_field
sage: f.monomials()                                                     #␣
↪needs sage.rings.number_field
[x^2*y^2, x^2, y, 1]
```

```
>>> from sage.all import *
>>> R = QQbar['x, y']; (x, y,) = R._first_ngens(2)# needs sage.rings.number_
↪field
>>> f = Integer(3)*x**Integer(2) - Integer(2)*y +␣
↪Integer(7)*x**Integer(2)*y**Integer(2) + Integer(5)                    ␣
↪               # needs sage.rings.number_field
>>> f.monomials()                                                     #␣
```

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```
↪needs sage.rings.number_field
[x^2*y^2, x^2, y, 1]
```

```
sage: # needs sage.rings.number_field
sage: R.<fx,fy,gx,gy> = QQbar[]
sage: F = (fx*gy - fy*gx)^3; F
-fy^3*gx^3 + 3*fx*fy^2*gx^2*gy + (-3)*fx^2*fy*gx*gy^2 + fx^3*gy^3
sage: F.monomials()
[fy^3*gx^3, fx*fy^2*gx^2*gy, fx^2*fy*gx*gy^2, fx^3*gy^3]
sage: F.coefficients()
[-1, 3, -3, 1]
sage: sum(map(mul, zip(F.coefficients(), F.monomials()))) == F
True
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQbar['fx, fy, gx, gy']; (fx, fy, gx, gy,) = R._first_ngens(4)
>>> F = (fx*gy - fy*gx)**Integer(3); F
-fy^3*gx^3 + 3*fx*fy^2*gx^2*gy + (-3)*fx^2*fy*gx*gy^2 + fx^3*gy^3
>>> F.monomials()
[fy^3*gx^3, fx*fy^2*gx^2*gy, fx^2*fy*gx*gy^2, fx^3*gy^3]
>>> F.coefficients()
[-1, 3, -3, 1]
>>> sum(map(mul, zip(F.coefficients(), F.monomials()))) == F
True
```

nvariables()

Return the number of variables in this polynomial.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: f = 3*x^2 - 2*y + 7*x^2*y^2 + 5
sage: f.nvariables()
2
sage: g = f.subs({x: 10}); g
700*y^2 + (-2)*y + 305
sage: g.nvariables()
1
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQbar['x, y']; (x, y,) = R._first_ngens(2)
>>> f = Integer(3)*x**Integer(2) - Integer(2)*y +␣
↪Integer(7)*x**Integer(2)*y**Integer(2) + Integer(5)
>>> f.nvariables()
2
>>> g = f.subs({x: Integer(10)}); g
700*y^2 + (-2)*y + 305
>>> g.nvariables()
1
```

quo_rem(*right*)

Return quotient and remainder of `self` and `right`.

EXAMPLES:

```
sage: R.<x,y> = CC[]                                                    #␣
↪needs sage.rings.real_mpfr
sage: f = y*x^2 + x + 1                                                 #␣
↪needs sage.rings.real_mpfr
sage: f.quo_rem(x)                                                      #␣
↪needs sage.libs.singular sage.rings.real_mpfr
(x*y + 1.00000000000000, 1.00000000000000)

sage: R = QQ['a','b']['x','y','z']
sage: p1 = R('a + (1+2*b)*x*y + (3-a^2)*z')
sage: p2 = R('x-1')
sage: p1.quo_rem(p2)                                                    #␣
↪needs sage.libs.singular
((2*b + 1)*y, (2*b + 1)*y + (-a^2 + 3)*z + a)

sage: R.<x,y> = Qp(5)[]                                                 #␣
↪needs sage.rings.padics
sage: x.quo_rem(y)                                                      #␣
↪needs sage.libs.singular sage.rings.padics
Traceback (most recent call last):
...
TypeError: no conversion of this ring to a Singular ring defined
```

```
>>> from sage.all import *
>>> R = CC['x, y']; (x, y,) = R._first_ngens(2)# needs sage.rings.real_mpfr
>>> f = y*x**Integer(2) + x + Integer(1)                                  ␣
↪                # needs sage.rings.real_mpfr
>>> f.quo_rem(x)                                                      #␣
↪needs sage.libs.singular sage.rings.real_mpfr
(x*y + 1.00000000000000, 1.00000000000000)

>>> R = QQ['a','b']['x','y','z']
>>> p1 = R('a + (1+2*b)*x*y + (3-a^2)*z')
>>> p2 = R('x-1')
>>> p1.quo_rem(p2)                                                    #␣
↪needs sage.libs.singular
((2*b + 1)*y, (2*b + 1)*y + (-a^2 + 3)*z + a)

>>> R = Qp(Integer(5))['x, y']; (x, y,) = R._first_ngens(2)# needs sage.rings.
↪padics
>>> x.quo_rem(y)                                                      #␣
↪needs sage.libs.singular sage.rings.padics
Traceback (most recent call last):
...
TypeError: no conversion of this ring to a Singular ring defined
```

ALGORITHM: Use Singular.

reduce(I)

Reduce this polynomial by the polynomials in I.

INPUT:

- `I` – list of polynomials or an ideal

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: P.<x,y,z> = QQbar[]
sage: f1 = -2 * x^2 + x^3
sage: f2 = -2 * y + x * y
sage: f3 = -x^2 + y^2
sage: F = Ideal([f1, f2, f3])
sage: g = x*y - 3*x*y^2
sage: g.reduce(F)                                                          #␣
↪needs sage.libs.singular
(-6)*y^2 + 2*y
sage: g.reduce(F.gens())                                                   #␣
↪needs sage.libs.singular
(-6)*y^2 + 2*y
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> P = QQbar['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> f1 = -Integer(2) * x**Integer(2) + x**Integer(3)
>>> f2 = -Integer(2) * y + x * y
>>> f3 = -x**Integer(2) + y**Integer(2)
>>> F = Ideal([f1, f2, f3])
>>> g = x*y - Integer(3)*x*y**Integer(2)
>>> g.reduce(F)                                                          #␣
↪needs sage.libs.singular
(-6)*y^2 + 2*y
>>> g.reduce(F.gens())                                                   #␣
↪needs sage.libs.singular
(-6)*y^2 + 2*y
```

```
sage: f = 3*x                                                              #␣
↪needs sage.rings.number_field
sage: f.reduce([2*x, y])                                                   #␣
↪needs sage.rings.number_field
0
```

```
>>> from sage.all import *
>>> f = Integer(3)*x                                                       ␣
↪     # needs sage.rings.number_field
>>> f.reduce([Integer(2)*x, y])                                            ␣
↪     # needs sage.rings.number_field
0
```

```
sage: # needs sage.rings.number_field
sage: k.<w> = CyclotomicField(3)
sage: A.<y9,y12,y13,y15> = PolynomialRing(k)
sage: J = [y9 + y12]
sage: f = y9 - y12; f.reduce(J)
```

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```
-2*y12
sage: f = y13*y15; f.reduce(J)
y13*y15
sage: f = y13*y15 + y9 - y12; f.reduce(J)
y13*y15 - 2*y12
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> k = CyclotomicField(Integer(3), names=('w',)); (w,) = k._first_ngens(1)
>>> A = PolynomialRing(k, names=('y9', 'y12', 'y13', 'y15',)); (y9, y12, y13,␣
↪y15,) = A._first_ngens(4)
>>> J = [y9 + y12]
>>> f = y9 - y12; f.reduce(J)
-2*y12
>>> f = y13*y15; f.reduce(J)
y13*y15
>>> f = y13*y15 + y9 - y12; f.reduce(J)
y13*y15 - 2*y12
```

Make sure the remainder returns the correct type, fixing Issue #13903:

```
sage: R.<y1,y2> = PolynomialRing(Qp(5), 2, order='lex')                    #␣
↪needs sage.rings.padics
sage: G = [y1^2 + y2^2, y1*y2 + y2^2, y2^3]                                #␣
↪needs sage.rings.padics
sage: type((y2^3).reduce(G))                                               #␣
↪needs sage.rings.padics
<class 'sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict'>
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Qp(Integer(5)), Integer(2), order='lex', names=('y1',
↪'y2',)); (y1, y2,) = R._first_ngens(2)# needs sage.rings.padics
>>> G = [y1**Integer(2) + y2**Integer(2), y1*y2 + y2**Integer(2),␣
↪y2**Integer(3)]                                        # needs sage.rings.padics
>>> type((y2**Integer(3)).reduce(G))                                       ␣
↪      # needs sage.rings.padics
<class 'sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict'>
```

resultant(*other*, *variable=None*)

Compute the resultant of `self` and `other` with respect to `variable`.

If a second argument is not provided, the first variable of `self.parent()` is chosen.

For inexact rings or rings not available in Singular, this computes the determinant of the Sylvester matrix.

INPUT:

- `other` – polynomial in `self.parent()`
- `variable` – (optional) variable (of type polynomial) in `self.parent()`

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQ, 2)
sage: a = x + y
```

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```
sage: b = x^3 - y^3
sage: a.resultant(b)                                                      #␣
↪needs sage.libs.singular
-2*y^3
sage: a.resultant(b, y)                                                   #␣
↪needs sage.libs.singular
2*x^3
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, Integer(2), names=('x', 'y',)); (x, y,) = P._first_
↪ngens(2)
>>> a = x + y
>>> b = x**Integer(3) - y**Integer(3)
>>> a.resultant(b)                                                      #␣
↪needs sage.libs.singular
-2*y^3
>>> a.resultant(b, y)                                                   #␣
↪needs sage.libs.singular
2*x^3
```

subresultants(*other*, *variable=None*)

Return the nonzero subresultant polynomials of `self` and `other`.

INPUT:

- `other` – a polynomial

OUTPUT: list of polynomials in the same ring as `self`

EXAMPLES:

```
sage: # needs sage.libs.singular sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: p = (y^2 + 6)*(x - 1) - y*(x^2 + 1)
sage: q = (x^2 + 6)*(y - 1) - x*(y^2 + 1)
sage: p.subresultants(q, y)
[2*x^6 + (-22)*x^5 + 102*x^4 + (-274)*x^3 + 488*x^2 + (-552)*x + 288,
 -x^3 - x^2*y + 6*x^2 + 5*x*y + (-11)*x + (-6)*y + 6]
sage: p.subresultants(q, x)
[2*y^6 + (-22)*y^5 + 102*y^4 + (-274)*y^3 + 488*y^2 + (-552)*y + 288,
 x*y^2 + y^3 + (-5)*x*y + (-6)*y^2 + 6*x + 11*y - 6]
```

```
>>> from sage.all import *
>>> # needs sage.libs.singular sage.rings.number_field
>>> R = QQbar['x, y']; (x, y,) = R._first_ngens(2)
>>> p = (y**Integer(2) + Integer(6))*(x - Integer(1)) - y*(x**Integer(2) +␣
↪Integer(1))
>>> q = (x**Integer(2) + Integer(6))*(y - Integer(1)) - x*(y**Integer(2) +␣
↪Integer(1))
>>> p.subresultants(q, y)
[2*x^6 + (-22)*x^5 + 102*x^4 + (-274)*x^3 + 488*x^2 + (-552)*x + 288,
 -x^3 - x^2*y + 6*x^2 + 5*x*y + (-11)*x + (-6)*y + 6]
>>> p.subresultants(q, x)
```

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```
[2*y^6 + (-22)*y^5 + 102*y^4 + (-274)*y^3 + 488*y^2 + (-552)*y + 288,
 x*y^2 + y^3 + (-5)*x*y + (-6)*y^2 + 6*x + 11*y - 6]
```

subs(*fixed=None*, ***kwds*)

Fix some given variables in a given multivariate polynomial and return the changed multivariate polynomials. The polynomial itself is not affected. The variable, value pairs for fixing are to be provided as a dictionary of the form `{variable: value}`.

This is a special case of evaluating the polynomial with some of the variables constants and the others the original variables.

INPUT:

- `fixed` – (optional) dictionary of inputs
- `**kwds` – named parameters

OUTPUT: new `MPolynomial`

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: f = x^2 + y + x^2*y^2 + 5
sage: f((5, y))
25*y^2 + y + 30
sage: f.subs({x: 5})
25*y^2 + y + 30
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQbar['x, y']; (x, y,) = R._first_ngens(2)
>>> f = x**Integer(2) + y + x**Integer(2)*y**Integer(2) + Integer(5)
>>> f((Integer(5), y))
25*y^2 + y + 30
>>> f.subs({x: Integer(5)})
25*y^2 + y + 30
```

total_degree()

Return the total degree of `self`, which is the maximum degree of any monomial in `self`.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x,y,z> = QQbar[]
sage: f = 2*x*y^3*z^2
sage: f.total_degree()
6
sage: f = 4*x^2*y^2*z^3
sage: f.total_degree()
7
sage: f = 99*x^6*y^3*z^9
sage: f.total_degree()
18
sage: f = x*y^3*z^6 + 3*x^2
```

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```
sage: f.total_degree()
10
sage: f = z^3 + 8*x^4*y^5*z
sage: f.total_degree()
10
sage: f = z^9 + 10*x^4 + y^8*x^2
sage: f.total_degree()
10
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQbar['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> f = Integer(2)*x*y**Integer(3)*z**Integer(2)
>>> f.total_degree()
6
>>> f = Integer(4)*x**Integer(2)*y**Integer(2)*z**Integer(3)
>>> f.total_degree()
7
>>> f = Integer(99)*x**Integer(6)*y**Integer(3)*z**Integer(9)
>>> f.total_degree()
18
>>> f = x*y**Integer(3)*z**Integer(6) + Integer(3)*x**Integer(2)
>>> f.total_degree()
10
>>> f = z**Integer(3) + Integer(8)*x**Integer(4)*y**Integer(5)*z
>>> f.total_degree()
10
>>> f = z**Integer(9) + Integer(10)*x**Integer(4) +␣
↪y**Integer(8)*x**Integer(2)
>>> f.total_degree()
10
```

univariate_polynomial(*R=None*)

Return a univariate polynomial associated to this multivariate polynomial.

INPUT:

- `R` – (default: `None`) *PolynomialRing*

If this polynomial is not in at most one variable, then a `ValueError` exception is raised. This is checked using the method *is_univariate()*. The new *Polynomial* is over the same base ring as the given *MPolynomial*.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: f = 3*x^2 - 2*y + 7*x^2*y^2 + 5
sage: f.univariate_polynomial()
Traceback (most recent call last):
...
TypeError: polynomial must involve at most one variable
sage: g = f.subs({x: 10}); g
700*y^2 + (-2)*y + 305
```

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```
sage: g.univariate_polynomial()
700*y^2 - 2*y + 305
sage: g.univariate_polynomial(PolynomialRing(QQ, 'z'))
700*z^2 - 2*z + 305
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQbar['x, y']; (x, y,) = R._first_ngens(2)
>>> f = Integer(3)*x**Integer(2) - Integer(2)*y +␣
↪Integer(7)*x**Integer(2)*y**Integer(2) + Integer(5)
>>> f.univariate_polynomial()
Traceback (most recent call last):
...
TypeError: polynomial must involve at most one variable
>>> g = f.subs({x: Integer(10)}); g
700*y^2 + (-2)*y + 305
>>> g.univariate_polynomial()
700*y^2 - 2*y + 305
>>> g.univariate_polynomial(PolynomialRing(QQ, 'z'))
700*z^2 - 2*z + 305
```

variable(i)

Return the i-th variable occurring in this polynomial.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: f = 3*x^2 - 2*y + 7*x^2*y^2 + 5
sage: f.variable(0)
x
sage: f.variable(1)
y
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQbar['x, y']; (x, y,) = R._first_ngens(2)
>>> f = Integer(3)*x**Integer(2) - Integer(2)*y +␣
↪Integer(7)*x**Integer(2)*y**Integer(2) + Integer(5)
>>> f.variable(Integer(0))
x
>>> f.variable(Integer(1))
y
```

variables()

Return the tuple of variables occurring in this polynomial.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: f = 3*x^2 - 2*y + 7*x^2*y^2 + 5
```

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```
sage: f.variables()
(x, y)
sage: g = f.subs({x: 10}); g
700*y^2 + (-2)*y + 305
sage: g.variables()
(y,)
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQbar['x, y']; (x, y,) = R._first_ngens(2)
>>> f = Integer(3)*x**Integer(2) - Integer(2)*y +␣
↪Integer(7)*x**Integer(2)*y**Integer(2) + Integer(5)
>>> f.variables()
(x, y)
>>> g = f.subs({x: Integer(10)}); g
700*y^2 + (-2)*y + 305
>>> g.variables()
(y,)
```

sage.rings.polynomial.multi_polynomial_element.**degree_lowest_rational_function**(*r*, *x*)

Return the difference of valuations of `r` with respect to variable `x`.

INPUT:

- `r` – a multivariate rational function
- `x` – a multivariate polynomial ring generator

OUTPUT: integer; the difference $val_x(p) - val_x(q)$ where $r = p/q$

Note

This function should be made a method of the `FractionFieldElement` class.

EXAMPLES:

```
sage: R1 = PolynomialRing(FiniteField(5), 3, names=["a", "b", "c"])
sage: F = FractionField(R1)
sage: a,b,c = R1.gens()
sage: f = 3*a*b^2*c^3 + 4*a*b*c
sage: g = a^2*b*c^2 + 2*a^2*b^4*c^7
```

```
>>> from sage.all import *
>>> R1 = PolynomialRing(FiniteField(Integer(5)), Integer(3), names=["a", "b", "c
↪"])
>>> F = FractionField(R1)
>>> a,b,c = R1.gens()
>>> f = Integer(3)*a*b**Integer(2)*c**Integer(3) + Integer(4)*a*b*c
>>> g = a**Integer(2)*b*c**Integer(2) +␣
↪Integer(2)*a**Integer(2)*b**Integer(4)*c**Integer(7)
```

Consider the quotient $f/g = \frac{4+3bc^2}{ac+2ab^3c^6}$ (note the cancellation).

```
sage: # needs sage.rings.finite_rings
sage: r = f/g; r
(-2*b*c^2 - 1)/(2*a*b^3*c^6 + a*c)
sage: degree_lowest_rational_function(r, a)
-1
sage: degree_lowest_rational_function(r, b)
0
sage: degree_lowest_rational_function(r, c)
-1
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> r = f/g; r
(-2*b*c^2 - 1)/(2*a*b^3*c^6 + a*c)
>>> degree_lowest_rational_function(r, a)
-1
>>> degree_lowest_rational_function(r, b)
0
>>> degree_lowest_rational_function(r, c)
-1
```

3.1.6 Ideals in multivariate polynomial rings

Sage has a powerful system to compute with multivariate polynomial rings. Most algorithms dealing with these ideals are centered on the computation of *Groebner bases*. Sage mainly uses Singular to implement this functionality. Singular is widely regarded as the best open-source system for Groebner basis calculation in multivariate polynomial rings over fields.

EXAMPLES:

We compute a Groebner basis for some given ideal. The type returned by the `groebner_basis` method is `PolynomialSequence`, i.e., it is not a *MPolynomialIdeal*:

```
sage: x,y,z = QQ['x,y,z'].gens()
sage: I = ideal(x^5 + y^4 + z^3 - 1,  x^3 + y^3 + z^2 - 1)
sage: B = I.groebner_basis()
sage: type(B)
<class 'sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic'>
```

```
>>> from sage.all import *
>>> x,y,z = QQ['x,y,z'].gens()
>>> I = ideal(x**Integer(5) + y**Integer(4) + z**Integer(3) - Integer(1), ␣
↪x**Integer(3) + y**Integer(3) + z**Integer(2) - Integer(1))
>>> B = I.groebner_basis()
>>> type(B)
<class 'sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic'>
```

Groebner bases can be used to solve the ideal membership problem:

```
sage: f,g,h = B
sage: (2*x*f + g).reduce(B)
0

sage: (2*x*f + g) in I
```

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```
True

sage: (2*x*f + 2*z*h + y^3).reduce(B)
y^3

sage: (2*x*f + 2*z*h + y^3) in I
False
```

```
>>> from sage.all import *
>>> f,g,h = B
>>> (Integer(2)*x*f + g).reduce(B)
0

>>> (Integer(2)*x*f + g) in I
True

>>> (Integer(2)*x*f + Integer(2)*z*h + y**Integer(3)).reduce(B)
y^3

>>> (Integer(2)*x*f + Integer(2)*z*h + y**Integer(3)) in I
False
```

We compute a Groebner basis for Cyclic 6, which is a standard benchmark and test ideal.

```
sage: R.<x,y,z,t,u,v> = QQ['x,y,z,t,u,v']
sage: I = sage.rings.ideal.Cyclic(R,6)
sage: B = I.groebner_basis()
sage: len(B)
45
```

```
>>> from sage.all import *
>>> R = QQ['x,y,z,t,u,v']; (x, y, z, t, u, v,) = R._first_ngens(6)
>>> I = sage.rings.ideal.Cyclic(R,Integer(6))
>>> B = I.groebner_basis()
>>> len(B)
45
```

We compute in a quotient of a polynomial ring over $\mathbf{Z}/17\mathbf{Z}$:

```
sage: R.<x,y> = ZZ[]
sage: S.<a,b> = R.quotient((x^2 + y^2, 17))
sage: S
Quotient of Multivariate Polynomial Ring in x, y over Integer Ring
by the ideal (x^2 + y^2, 17)

sage: a^2 + b^2 == 0
True
sage: a^3 - b^2
-a*b^2 - b^2
```

```
>>> from sage.all import *
>>> R = ZZ['x, y']; (x, y,) = R._first_ngens(2)
```

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```
>>> S = R.quotient((x**Integer(2) + y**Integer(2), Integer(17)), names=('a', 'b',));␣
↪(a, b,) = S._first_ngens(2)
>>> S
Quotient of Multivariate Polynomial Ring in x, y over Integer Ring
by the ideal (x^2 + y^2, 17)

>>> a**Integer(2) + b**Integer(2) == Integer(0)
True
>>> a**Integer(3) - b**Integer(2)
-a*b^2 - b^2
```

Note that the result of a computation is not necessarily reduced:

```
sage: (a+b)^17
a*b^16 + b^17
sage: S(17) == 0
True
```

```
>>> from sage.all import *
>>> (a+b)**Integer(17)
a*b^16 + b^17
>>> S(Integer(17)) == Integer(0)
True
```

Or we can work with $\mathbf{Z}/17\mathbf{Z}$ directly:

```
sage: R.<x,y> = Zmod(17)[]
sage: S.<a,b> = R.quotient((x^2 + y^2,))
sage: S
Quotient of Multivariate Polynomial Ring in x, y over Ring of
integers modulo 17 by the ideal (x^2 + y^2)

sage: a^2 + b^2 == 0
True
sage: a^3 - b^2 == -a*b^2 - b^2 == 16*a*b^2 + 16*b^2
True
sage: (a+b)^17
a*b^16 + b^17
sage: S(17) == 0
True
```

```
>>> from sage.all import *
>>> R = Zmod(Integer(17))['x, y']; (x, y,) = R._first_ngens(2)
>>> S = R.quotient((x**Integer(2) + y**Integer(2),), names=('a', 'b',)); (a, b,) = S._
↪first_ngens(2)
>>> S
Quotient of Multivariate Polynomial Ring in x, y over Ring of
integers modulo 17 by the ideal (x^2 + y^2)

>>> a**Integer(2) + b**Integer(2) == Integer(0)
True
>>> a**Integer(3) - b**Integer(2) == -a*b**Integer(2) - b**Integer(2) ==␣
```

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```
↪Integer(16)*a*b**Integer(2) + Integer(16)*b**Integer(2)
True
>>> (a+b)**Integer(17)
a*b^16 + b^17
>>> S(Integer(17)) == Integer(0)
True
```

Working with a polynomial ring over **Z**:

```
sage: R.<x,y,z,w> = ZZ[]
sage: I = ideal(x^2 + y^2 - z^2 - w^2, x-y)
sage: J = I^2
sage: J.groebner_basis()
[4*y^4 - 4*y^2*z^2 + z^4 - 4*y^2*w^2 + 2*z^2*w^2 + w^4,
 2*x*y^2 - 2*y^3 - x*z^2 + y*z^2 - x*w^2 + y*w^2,
 x^2 - 2*x*y + y^2]

sage: y^2 - 2*x*y + x^2 in J
True
sage: 0 in J
True
```

```
>>> from sage.all import *
>>> R = ZZ['x, y, z, w']; (x, y, z, w,) = R._first_ngens(4)
>>> I = ideal(x**Integer(2) + y**Integer(2) - z**Integer(2) - w**Integer(2), x-y)
>>> J = I**Integer(2)
>>> J.groebner_basis()
[4*y^4 - 4*y^2*z^2 + z^4 - 4*y^2*w^2 + 2*z^2*w^2 + w^4,
 2*x*y^2 - 2*y^3 - x*z^2 + y*z^2 - x*w^2 + y*w^2,
 x^2 - 2*x*y + y^2]

>>> y**Integer(2) - Integer(2)*x*y + x**Integer(2) in J
True
>>> Integer(0) in J
True
```

We do a Groebner basis computation over a number field:

```
sage: K.<zeta> = CyclotomicField(3)
sage: R.<x,y,z> = K[]; R
Multivariate Polynomial Ring in x, y, z over Cyclotomic Field of order 3 and degree 2

sage: i = ideal(x - zeta*y + 1, x^3 - zeta*y^3); i
Ideal (x + (-zeta)*y + 1, x^3 + (-zeta)*y^3) of Multivariate
Polynomial Ring in x, y, z over Cyclotomic Field of order 3 and degree 2

sage: i.groebner_basis()
[y^3 + (2*zeta + 1)*y^2 + (zeta - 1)*y + (-1/3*zeta - 2/3), x + (-zeta)*y + 1]

sage: S = R.quotient(i); S
Quotient of Multivariate Polynomial Ring in x, y, z over
Cyclotomic Field of order 3 and degree 2 by the ideal (x +
```

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```
(-zeta)*y + 1, x^3 + (-zeta)*y^3)

sage: S.0  - zeta*S.1
-1
sage: S.0^3 - zeta*S.1^3
0
```

```
>>> from sage.all import *
>>> K = CyclotomicField(Integer(3), names=('zeta',)); (zeta,) = K._first_ngens(1)
>>> R = K['x, y, z']; (x, y, z,) = R._first_ngens(3); R
Multivariate Polynomial Ring in x, y, z over Cyclotomic Field of order 3 and degree 2

>>> i = ideal(x - zeta*y + Integer(1), x**Integer(3) - zeta*y**Integer(3)); i
Ideal (x + (-zeta)*y + 1, x^3 + (-zeta)*y^3) of Multivariate
Polynomial Ring in x, y, z over Cyclotomic Field of order 3 and degree 2

>>> i.groebner_basis()
[y^3 + (2*zeta + 1)*y^2 + (zeta - 1)*y + (-1/3*zeta - 2/3), x + (-zeta)*y + 1]

>>> S = R.quotient(i); S
Quotient of Multivariate Polynomial Ring in x, y, z over
Cyclotomic Field of order 3 and degree 2 by the ideal (x +
(-zeta)*y + 1, x^3 + (-zeta)*y^3)

>>> S.gen(0)  - zeta*S.gen(1)
-1
>>> S.gen(0)**Integer(3) - zeta*S.gen(1)**Integer(3)
0
```

Two examples from the Mathematica documentation (done in Sage):

We compute a Groebner basis:

```
sage: R.<x,y> = PolynomialRing(QQ, order='lex')
sage: ideal(x^2 - 2*y^2, x*y - 3).groebner_basis()
[x - 2/3*y^3, y^4 - 9/2]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, order='lex', names=('x', 'y',)); (x, y,) = R._
↪first_ngens(2)
>>> ideal(x**Integer(2) - Integer(2)*y**Integer(2), x*y - Integer(3)).
↪groebner_basis()
[x - 2/3*y^3, y^4 - 9/2]
```

We show that three polynomials have no common root:

```
sage: R.<x,y> = QQ[]
sage: ideal(x+y, x^2 - 1, y^2 - 2*x).groebner_basis()
[1]
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
```

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```
>>> ideal(x+y, x**Integer(2) - Integer(1), y**Integer(2) - Integer(2)*x).
↪groebner_basis()
[1]
```

The next example shows how we can use Groebner bases over **Z** to find the primes modulo which a system of equations has a solution, when the system has no solutions over the rationals.

We first form a certain ideal I in $\mathbf{Z}[x, y, z]$, and note that the Groebner basis of I over $\mathbf{Q}$ contains 1, so there are no solutions over $\mathbf{Q}$ or an algebraic closure of it (this is not surprising as there are 4 equations in 3 unknowns).

```
sage: P.<x,y,z> = PolynomialRing(ZZ,order='lex')
sage: I = ideal(-y^2 - 3*y + z^2 + 3, -2*y*z + z^2 + 2*z + 1,
....:           x*z + y*z + z^2, -3*x*y + 2*y*z + 6*z^2)
sage: I.change_ring(P.change_ring(QQ)).groebner_basis()
[1]
```

```
>>> from sage.all import *
>>> P = PolynomialRing(ZZ,order='lex', names=('x', 'y', 'z',)); (x, y, z,) =
↪P._first_ngens(3)
>>> I = ideal(-y**Integer(2) - Integer(3)*y + z**Integer(2) + Integer(3), -
↪Integer(2)*y*z + z**Integer(2) + Integer(2)*z + Integer(1),
...           x*z + y*z + z**Integer(2), -Integer(3)*x*y + Integer(2)*y*z +
↪Integer(6)*z**Integer(2))
>>> I.change_ring(P.change_ring(QQ)).groebner_basis()
[1]
```

However, when we compute the Groebner basis of I (defined over **Z**), we note that there is a certain integer in the ideal which is not 1.

```
sage: I.groebner_basis()
[x + y + 57119*z + 4, y^2 + 3*y + 17220, y*z + ...,
 2*y + 158864, z^2 + 17223, 2*z + 41856, 164878]
```

```
>>> from sage.all import *
>>> I.groebner_basis()
[x + y + 57119*z + 4, y^2 + 3*y + 17220, y*z + ...,
 2*y + 158864, z^2 + 17223, 2*z + 41856, 164878]
```

Now for each prime p dividing this integer 164878, the Groebner basis of I modulo p will be non-trivial and will thus give a solution of the original system modulo p.

```
sage: factor(164878)
2 * 7 * 11777

sage: I.change_ring(P.change_ring(GF(2))).groebner_basis()
↪   # needs sage.rings.finite_rings
[x + y + z, y^2 + y, y*z + y, z^2 + 1]
sage: I.change_ring(P.change_ring(GF(7))).groebner_basis()
↪   # needs sage.rings.finite_rings
[x - 1, y + 3, z - 2]
sage: I.change_ring(P.change_ring(GF(11777))).groebner_basis()
```

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```
↪   # needs sage.rings.finite_rings
[x + 5633, y - 3007, z - 2626]
```

```
>>> from sage.all import *
>>> factor(Integer(164878))
2 * 7 * 11777

>>> I.change_ring(P.change_ring(GF(Integer(2)))).groebner_basis()                    ␣
↪          # needs sage.rings.finite_rings
[x + y + z, y^2 + y, y*z + y, z^2 + 1]
>>> I.change_ring(P.change_ring(GF(Integer(7)))).groebner_basis()                    ␣
↪          # needs sage.rings.finite_rings
[x - 1, y + 3, z - 2]
>>> I.change_ring(P.change_ring(GF(Integer(11777)))).groebner_basis()                ␣
↪          # needs sage.rings.finite_rings
[x + 5633, y - 3007, z - 2626]
```

The Groebner basis modulo any product of the prime factors is also non-trivial:

```
sage: I.change_ring(P.change_ring(IntegerModRing(2 * 7))).groebner_basis()
[x + ..., y^2 + 3*y, y*z + 7*y + 6, 2*y + 6, z^2 + 3, 2*z + 10]
```

```
>>> from sage.all import *
>>> I.change_ring(P.change_ring(IntegerModRing(Integer(2) * Integer(7)))).
↪groebner_basis()
[x + ..., y^2 + 3*y, y*z + 7*y + 6, 2*y + 6, z^2 + 3, 2*z + 10]
```

Modulo any other prime the Groebner basis is trivial so there are no other solutions. For example:

```
sage: I.change_ring(P.change_ring(GF(3))).groebner_basis()                           ␣
↪   # needs sage.rings.finite_rings
[1]
```

```
>>> from sage.all import *
>>> I.change_ring(P.change_ring(GF(Integer(3)))).groebner_basis()                    ␣
↪          # needs sage.rings.finite_rings
[1]
```

Note

Sage distinguishes between lists or sequences of polynomials and ideals. Thus an ideal is not identified with a particular set of generators. For sequences of multivariate polynomials see *sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic*.

AUTHORS:

- William Stein: initial version
- Kiran S. Kedlaya (2006-02-12): added Macaulay2 analogues of some Singular features
- Martin Albrecht (2007,2008): refactoring, many Singular related functions, added plot()
- Martin Albrecht (2009): added Groebner basis over rings functionality from Singular 3.1

- John Perry (2012): bug fixing equality & containment of ideals

class sage.rings.polynomial.multi_polynomial_ideal.**MPolynomialIdeal**(*ring*, *gens*, *coerce=True*)

Bases: *MPolynomialIdeal_singular_repr*, *MPolynomialIdeal_macaulay2_repr*, *MPolynomialIdeal_magma_repr*, Ideal_generic

Create an ideal in a multivariate polynomial ring.

INPUT:

- ring – the ring the ideal is defined in
- gens – list of generators for the ideal
- coerce – whether to coerce elements to the ring ring

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(IntegerRing(), 2, order='lex')
sage: R.ideal([x, y])
Ideal (x, y) of Multivariate Polynomial Ring in x, y over Integer Ring
sage: R.<x0,x1> = GF(3)[]
sage: R.ideal([x0^2, x1^3])
Ideal (x0^2, x1^3) of Multivariate Polynomial Ring in x0, x1 over Finite Field of␣
↪size 3
```

```
>>> from sage.all import *
>>> R = PolynomialRing(IntegerRing(), Integer(2), order='lex', names=('x', 'y',));
↪ (x, y,) = R._first_ngens(2)
>>> R.ideal([x, y])
Ideal (x, y) of Multivariate Polynomial Ring in x, y over Integer Ring
>>> R = GF(Integer(3))['x0, x1']; (x0, x1,) = R._first_ngens(2)
>>> R.ideal([x0**Integer(2), x1**Integer(3)])
Ideal (x0^2, x1^3) of Multivariate Polynomial Ring in x0, x1 over Finite Field of␣
↪size 3
```

property basis

Shortcut to gens().

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQ,2)
sage: I = Ideal([x, y + 1])
sage: I.basis
[x, y + 1]
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ,Integer(2), names=('x', 'y',)); (x, y,) = P._first_
↪ngens(2)
>>> I = Ideal([x, y + Integer(1)])
>>> I.basis
[x, y + 1]
```

change_ring(*P*)

Return the ideal I in P spanned by the generators $g_1, ..., g_n$ of self as returned by self.gens().

INPUT:

- `P` – a multivariate polynomial ring

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ,3,order='lex')
sage: I = sage.rings.ideal.Cyclic(P)
sage: I
Ideal (x + y + z, x*y + x*z + y*z, x*y*z - 1) of
Multivariate Polynomial Ring in x, y, z over Rational Field
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ,Integer(3),order='lex', names=('x', 'y', 'z',)); (x,
↪ y, z,) = P._first_ngens(3)
>>> I = sage.rings.ideal.Cyclic(P)
>>> I
Ideal (x + y + z, x*y + x*z + y*z, x*y*z - 1) of
Multivariate Polynomial Ring in x, y, z over Rational Field
```

```
sage: I.groebner_basis()
[x + y + z, y^2 + y*z + z^2, z^3 - 1]
```

```
>>> from sage.all import *
>>> I.groebner_basis()
[x + y + z, y^2 + y*z + z^2, z^3 - 1]
```

```
sage: Q.<x,y,z> = P.change_ring(order='degrevlex'); Q
Multivariate Polynomial Ring in x, y, z over Rational Field
sage: Q.term_order()
Degree reverse lexicographic term order
```

```
>>> from sage.all import *
>>> Q = P.change_ring(order='degrevlex', names=('x', 'y', 'z',)); (x, y, z,)␣
↪= Q._first_ngens(3); Q
Multivariate Polynomial Ring in x, y, z over Rational Field
>>> Q.term_order()
Degree reverse lexicographic term order
```

```
sage: J = I.change_ring(Q); J
Ideal (x + y + z, x*y + x*z + y*z, x*y*z - 1) of
Multivariate Polynomial Ring in x, y, z over Rational Field
```

```
>>> from sage.all import *
>>> J = I.change_ring(Q); J
Ideal (x + y + z, x*y + x*z + y*z, x*y*z - 1) of
Multivariate Polynomial Ring in x, y, z over Rational Field
```

```
sage: J.groebner_basis()
[z^3 - 1, y^2 + y*z + z^2, x + y + z]
```

```
>>> from sage.all import *
>>> J.groebner_basis()
[z^3 - 1, y^2 + y*z + z^2, x + y + z]
```

degree_of_semi_regularity()

Return the degree of semi-regularity of this ideal under the assumption that it is semi-regular.

Let $\{f_1, ..., f_m\} \subset K[x_1, ..., x_n]$ be homogeneous polynomials of degrees $d_1, ..., d_m$ respectively. This sequence is semi-regular if:

- $\{f_1, ..., f_m\} \neq K[x_1, ..., x_n]$
- for all $1 \leq i \leq m$ and $g \in K[x_1, \ldots, x_n]$: $deg(g \cdot pi) < D$ and $g \cdot f_i \in < f_1, \ldots, f_{i-1} >$ implies that $g \in < f_1, ..., f_{i-1} >$ where D is the degree of regularity.

This notion can be extended to affine polynomials by considering their homogeneous components of highest degree.

The degree of regularity of a semi-regular sequence $f_1, ..., f_m$ of respective degrees $d_1, ..., d_m$ is given by the index of the first nonpositive coefficient of:

$\sum c_k z^k = \frac{\prod (1-z^{d_i})}{(1-z)^n}$

EXAMPLES:

We consider a homogeneous example:

```
sage: n = 8
sage: K = GF(127)
sage: P = PolynomialRing(K, n, 'x')
sage: s = [K.random_element() for _ in range(n)]
sage: L = []
sage: for i in range(2 * n):
....:     f = P.random_element(degree=2, terms=binomial(n, 2))
....:     f -= f(*s)
....:     L.append(f.homogenize())
sage: I = Ideal(L)
sage: I.degree_of_semi_regularity()
4
```

```
>>> from sage.all import *
>>> n = Integer(8)
>>> K = GF(Integer(127))
>>> P = PolynomialRing(K, n, 'x')
>>> s = [K.random_element() for _ in range(n)]
>>> L = []
>>> for i in range(Integer(2) * n):
...     f = P.random_element(degree=Integer(2), terms=binomial(n, Integer(2)))
...     f -= f(*s)
...     L.append(f.homogenize())
>>> I = Ideal(L)
>>> I.degree_of_semi_regularity()
4
```

From this, we expect a Groebner basis computation to reach at most degree 4. For homogeneous systems this is equivalent to the largest degree in the Groebner basis:

```
sage: max(f.degree() for f in I.groebner_basis())
4
```

```
>>> from sage.all import *
>>> max(f.degree() for f in I.groebner_basis())
4
```

We increase the number of polynomials and observe a decrease the degree of regularity:

```
sage: for i in range(2 * n):
....:     f = P.random_element(degree=2, terms=binomial(n, 2))
....:     f -= f(*s)
....:     L.append(f.homogenize())
sage: I = Ideal(L)
sage: I.degree_of_semi_regularity()
3

sage: max(f.degree() for f in I.groebner_basis())
3
```

```
>>> from sage.all import *
>>> for i in range(Integer(2) * n):
...     f = P.random_element(degree=Integer(2), terms=binomial(n, Integer(2)))
...     f -= f(*s)
...     L.append(f.homogenize())
>>> I = Ideal(L)
>>> I.degree_of_semi_regularity()
3

>>> max(f.degree() for f in I.groebner_basis())
3
```

The degree of regularity approaches 2 for quadratic systems as the number of polynomials approaches n^2:

```
sage: for i in range((n-4) * n):
....:     f = P.random_element(degree=2, terms=binomial(n, 2))
....:     f -= f(*s)
....:     L.append(f.homogenize())
sage: I = Ideal(L)
sage: I.degree_of_semi_regularity()
2

sage: max(f.degree() for f in I.groebner_basis())
2
```

```
>>> from sage.all import *
>>> for i in range((n-Integer(4)) * n):
...     f = P.random_element(degree=Integer(2), terms=binomial(n, Integer(2)))
...     f -= f(*s)
...     L.append(f.homogenize())
>>> I = Ideal(L)
>>> I.degree_of_semi_regularity()
2

>>> max(f.degree() for f in I.groebner_basis())
2
```

Note

It is unknown whether semi-regular sequences exist. However, it is expected that random systems are semi-regular sequences. For more details about semi-regular sequences see [BFS2004].

gens()

Return a set of generators / a basis of this ideal.

This is usually the set of generators provided during object creation.

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQ,2)
sage: I = Ideal([x, y + 1]); I
Ideal (x, y + 1) of Multivariate Polynomial Ring in x, y over Rational Field
sage: I.gens()
[x, y + 1]
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ,Integer(2), names=('x', 'y',)); (x, y,) = P._first_
↪ngens(2)
>>> I = Ideal([x, y + Integer(1)]); I
Ideal (x, y + 1) of Multivariate Polynomial Ring in x, y over Rational Field
>>> I.gens()
[x, y + 1]
```

groebner_basis(*algorithm=''*, *deg_bound=None*, *mult_bound=None*, *prot=False*, **args*, ***kwds*)

Return the reduced Groebner basis of this ideal.

A Groebner basis $g_1, ..., g_n$ for an ideal I is a generating set such that $< LM(g_i) >= LM(I)$, i.e., the leading monomial ideal of I is spanned by the leading terms of $g_1, ..., g_n$. Groebner bases are the key concept in computational ideal theory in multivariate polynomial rings which allows a variety of problems to be solved.

Additionally, a *reduced* Groebner basis G is a unique representation for the ideal $< G >$ with respect to the chosen monomial ordering.

INPUT:

- `algorithm` – determines the algorithm to use, see below for available algorithms
- `deg_bound` – only compute to degree `deg_bound`, that is, ignore all S-polynomials of higher degree. (default: `None`)
- `mult_bound` – the computation is stopped if the ideal is zero-dimensional in a ring with local ordering and its multiplicity is lower than `mult_bound`. Singular only. (default: `None`)
- `prot` – if set to `True` the computation protocol of the underlying implementation is printed. If an algorithm from the `singular:` or `magma:` family is used, `prot` may also be `sage` in which case the output is parsed and printed in a common format where the amount of information printed can be controlled via calls to `set_verbose()`.
- `*args` – additional parameters passed to the respective implementations
- `**kwds` – additional keyword parameters passed to the respective implementations

ALGORITHMS:

`''`

autoselect (default)

'singular:groebner'
Singular's `groebner` command

'singular:std'
Singular's `std` command

'singular:stdhilb'
Singular's `stdhib` command

'singular:stdfglm'
Singular's `stdfglm` command

'singular:slimgb'
Singular's `slimgb` command

'libsingular:groebner'
libSingular's `groebner` command

'libsingular:std'
libSingular's `std` command

'libsingular:slimgb'
libSingular's `slimgb` command

'libsingular:stdhilb'
libSingular's `stdhib` command

'libsingular:stdfglm'
libSingular's `stdfglm` command

'toy:buchberger'
Sage's toy/educational buchberger without Buchberger criteria

'toy:buchberger2'
Sage's toy/educational buchberger with Buchberger criteria

'toy:d_basis'
Sage's toy/educational algorithm for computation over PIDs

'macaulay2:gb'
Macaulay2's `gb` command (if available)

'macaulay2:f4'
Macaulay2's `GroebnerBasis` command with the strategy "F4" (if available)

'macaulay2:mgb'
Macaulay2's `GroebnerBasis` command with the strategy "MGB" (if available)

'msolve'
optional package msolve (degrevlex order)

'magma:GroebnerBasis'
Magma's `Groebnerbasis` command (if available)

'ginv:TQ', 'ginv:TQBlockHigh', 'ginv:TQBlockLow' and 'ginv:TQDegree'
One of GINV's implementations (if available)

'giac:gbasis'
Giac's `gbasis` command (if available)

If only a system is given - e.g. `'magma'` - the default algorithm is chosen for that system.

Note

The Singular and libSingular versions of the respective algorithms are identical, but the former calls an external Singular process while the latter calls a C function, and thus the calling overhead is smaller. However, the libSingular interface does not support pretty printing of computation protocols.

EXAMPLES:

Consider Katsura-3 over **Q** with lexicographical term ordering. We compute the reduced Groebner basis using every available implementation and check their equality.

```
sage: P.<a,b,c> = PolynomialRing(QQ,3, order='lex')
sage: I = sage.rings.ideal.Katsura(P,3)  # regenerate to prevent caching
sage: I.groebner_basis()
[a - 60*c^3 + 158/7*c^2 + 8/7*c - 1, b + 30*c^3 - 79/7*c^2 + 3/7*c, c^4 - 10/
↪21*c^3 + 1/84*c^2 + 1/84*c]
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ,Integer(3), order='lex', names=('a', 'b', 'c',));␣
↪(a, b, c,) = P._first_ngens(3)
>>> I = sage.rings.ideal.Katsura(P,Integer(3))  # regenerate to prevent␣
↪caching
>>> I.groebner_basis()
[a - 60*c^3 + 158/7*c^2 + 8/7*c - 1, b + 30*c^3 - 79/7*c^2 + 3/7*c, c^4 - 10/
↪21*c^3 + 1/84*c^2 + 1/84*c]
```

```
sage: I = sage.rings.ideal.Katsura(P,3)  # regenerate to prevent caching
sage: I.groebner_basis('libsingular:groebner')
[a - 60*c^3 + 158/7*c^2 + 8/7*c - 1, b + 30*c^3 - 79/7*c^2 + 3/7*c, c^4 - 10/
↪21*c^3 + 1/84*c^2 + 1/84*c]
```

```
>>> from sage.all import *
>>> I = sage.rings.ideal.Katsura(P,Integer(3))  # regenerate to prevent␣
↪caching
>>> I.groebner_basis('libsingular:groebner')
[a - 60*c^3 + 158/7*c^2 + 8/7*c - 1, b + 30*c^3 - 79/7*c^2 + 3/7*c, c^4 - 10/
↪21*c^3 + 1/84*c^2 + 1/84*c]
```

```
sage: I = sage.rings.ideal.Katsura(P,3)  # regenerate to prevent caching
sage: I.groebner_basis('libsingular:std')
[a - 60*c^3 + 158/7*c^2 + 8/7*c - 1, b + 30*c^3 - 79/7*c^2 + 3/7*c, c^4 - 10/
↪21*c^3 + 1/84*c^2 + 1/84*c]
```

```
>>> from sage.all import *
>>> I = sage.rings.ideal.Katsura(P,Integer(3))  # regenerate to prevent␣
↪caching
>>> I.groebner_basis('libsingular:std')
[a - 60*c^3 + 158/7*c^2 + 8/7*c - 1, b + 30*c^3 - 79/7*c^2 + 3/7*c, c^4 - 10/
↪21*c^3 + 1/84*c^2 + 1/84*c]
```

```
sage: I = sage.rings.ideal.Katsura(P,3)  # regenerate to prevent caching
sage: I.groebner_basis('libsingular:stdhilb')
[a - 60*c^3 + 158/7*c^2 + 8/7*c - 1, b + 30*c^3 - 79/7*c^2 + 3/7*c, c^4 - 10/
↪21*c^3 + 1/84*c^2 + 1/84*c]
```

```
>>> from sage.all import *
>>> I = sage.rings.ideal.Katsura(P,Integer(3))  # regenerate to prevent␣
↪caching
>>> I.groebner_basis('libsingular:stdhilb')
[a - 60*c^3 + 158/7*c^2 + 8/7*c - 1, b + 30*c^3 - 79/7*c^2 + 3/7*c, c^4 - 10/
↪21*c^3 + 1/84*c^2 + 1/84*c]
```

```
sage: I = sage.rings.ideal.Katsura(P,3)  # regenerate to prevent caching
sage: I.groebner_basis('libsingular:stdfglm')
[a - 60*c^3 + 158/7*c^2 + 8/7*c - 1, b + 30*c^3 - 79/7*c^2 + 3/7*c, c^4 - 10/
↪21*c^3 + 1/84*c^2 + 1/84*c]
```

```
>>> from sage.all import *
>>> I = sage.rings.ideal.Katsura(P,Integer(3))  # regenerate to prevent␣
↪caching
>>> I.groebner_basis('libsingular:stdfglm')
[a - 60*c^3 + 158/7*c^2 + 8/7*c - 1, b + 30*c^3 - 79/7*c^2 + 3/7*c, c^4 - 10/
↪21*c^3 + 1/84*c^2 + 1/84*c]
```

```
sage: I = sage.rings.ideal.Katsura(P,3)  # regenerate to prevent caching
sage: I.groebner_basis('libsingular:slimgb')
[a - 60*c^3 + 158/7*c^2 + 8/7*c - 1, b + 30*c^3 - 79/7*c^2 + 3/7*c, c^4 - 10/
↪21*c^3 + 1/84*c^2 + 1/84*c]
```

```
>>> from sage.all import *
>>> I = sage.rings.ideal.Katsura(P,Integer(3))  # regenerate to prevent␣
↪caching
>>> I.groebner_basis('libsingular:slimgb')
[a - 60*c^3 + 158/7*c^2 + 8/7*c - 1, b + 30*c^3 - 79/7*c^2 + 3/7*c, c^4 - 10/
↪21*c^3 + 1/84*c^2 + 1/84*c]
```

Although Giac does support lexicographical ordering, we use degree reverse lexicographical ordering here, in order to test against Issue #21884:

```
sage: # needs sage.libs.giac
sage: I = sage.rings.ideal.Katsura(P,3)  # regenerate to prevent caching
sage: J = I.change_ring(P.change_ring(order='degrevlex'))
sage: gb = J.groebner_basis('giac')  # random
sage: gb
[c^3 - 79/210*c^2 + 1/30*b + 1/70*c, b^2 - 3/5*c^2 - 1/5*b + 1/5*c, b*c + 6/
↪5*c^2 - 1/10*b - 2/5*c, a + 2*b + 2*c - 1]
sage: J.groebner_basis.set_cache(gb)
sage: ideal(J.transformed_basis()).change_ring(P).interreduced_basis()  #␣
↪testing issue #21884
...[a - 60*c^3 + 158/7*c^2 + 8/7*c - 1, b + 30*c^3 - 79/7*c^2 + 3/7*c, c^4 -␣
↪10/21*c^3 + 1/84*c^2 + 1/84*c]
```

```
>>> from sage.all import *
>>> # needs sage.libs.giac
>>> I = sage.rings.ideal.Katsura(P,Integer(3))  # regenerate to prevent␣
↪caching
>>> J = I.change_ring(P.change_ring(order='degrevlex'))
>>> gb = J.groebner_basis('giac')  # random
>>> gb
[c^3 - 79/210*c^2 + 1/30*b + 1/70*c, b^2 - 3/5*c^2 - 1/5*b + 1/5*c, b*c + 6/
↪5*c^2 - 1/10*b - 2/5*c, a + 2*b + 2*c - 1]
>>> J.groebner_basis.set_cache(gb)
>>> ideal(J.transformed_basis()).change_ring(P).interreduced_basis()  #␣
↪testing issue #21884
...[a - 60*c^3 + 158/7*c^2 + 8/7*c - 1, b + 30*c^3 - 79/7*c^2 + 3/7*c, c^4 -␣
↪10/21*c^3 + 1/84*c^2 + 1/84*c]
```

Giac's gbasis over **Q** can benefit from a probabilistic lifting and multi threaded operations:

```
sage: # needs sage.libs.giac
sage: A9 = PolynomialRing(QQ, 9, 'x')
sage: I9 = sage.rings.ideal.Katsura(A9)
sage: print("possible output from giac", flush=True); I9.groebner_basis("giac
↪", proba_epsilon=1e-7)  # long time (3s)
possible output...
Polynomial Sequence with 143 Polynomials in 9 Variables
```

```
>>> from sage.all import *
>>> # needs sage.libs.giac
>>> A9 = PolynomialRing(QQ, Integer(9), 'x')
>>> I9 = sage.rings.ideal.Katsura(A9)
>>> print("possible output from giac", flush=True); I9.groebner_basis("giac",␣
↪proba_epsilon=RealNumber('1e-7'))  # long time (3s)
possible output...
Polynomial Sequence with 143 Polynomials in 9 Variables
```

The list of available Giac options is provided at `sage.libs.giac.groebner_basis()`.

Note that `toy:buchberger` does not return the reduced Groebner basis,

```
sage: I = sage.rings.ideal.Katsura(P,3)  # regenerate to prevent caching
sage: gb = I.groebner_basis('toy:buchberger')
sage: gb.is_groebner()
True
sage: gb == gb.reduced()
False
```

```
>>> from sage.all import *
>>> I = sage.rings.ideal.Katsura(P,Integer(3))  # regenerate to prevent␣
↪caching
>>> gb = I.groebner_basis('toy:buchberger')
>>> gb.is_groebner()
True
>>> gb == gb.reduced()
False
```

but that `toy:buchberger2` does.

```
sage: I = sage.rings.ideal.Katsura(P,3)  # regenerate to prevent caching
sage: gb = I.groebner_basis('toy:buchberger2'); gb
[a - 60*c^3 + 158/7*c^2 + 8/7*c - 1, b + 30*c^3 - 79/7*c^2 + 3/7*c, c^4 - 10/
↪21*c^3 + 1/84*c^2 + 1/84*c]
sage: gb == gb.reduced()
True
```

```
>>> from sage.all import *
>>> I = sage.rings.ideal.Katsura(P,Integer(3))  # regenerate to prevent␣
↪caching
>>> gb = I.groebner_basis('toy:buchberger2'); gb
[a - 60*c^3 + 158/7*c^2 + 8/7*c - 1, b + 30*c^3 - 79/7*c^2 + 3/7*c, c^4 - 10/
↪21*c^3 + 1/84*c^2 + 1/84*c]
>>> gb == gb.reduced()
True
```

Here we use Macaulay2 with three different strategies over a finite field.

```
sage: # optional - macaulay2
sage: R.<a,b,c> = PolynomialRing(GF(101), 3)
sage: I = sage.rings.ideal.Katsura(R,3)  # regenerate to prevent caching
sage: I.groebner_basis('macaulay2:gb')
[c^3 + 28*c^2 - 37*b + 13*c, b^2 - 41*c^2 + 20*b - 20*c,
 b*c - 19*c^2 + 10*b + 40*c, a + 2*b + 2*c - 1]
sage: I = sage.rings.ideal.Katsura(R,3)  # regenerate to prevent caching
sage: I.groebner_basis('macaulay2:f4')
[c^3 + 28*c^2 - 37*b + 13*c, b^2 - 41*c^2 + 20*b - 20*c,
 b*c - 19*c^2 + 10*b + 40*c, a + 2*b + 2*c - 1]
sage: I = sage.rings.ideal.Katsura(R,3)  # regenerate to prevent caching
sage: I.groebner_basis('macaulay2:mgb')
[c^3 + 28*c^2 - 37*b + 13*c, b^2 - 41*c^2 + 20*b - 20*c,
 b*c - 19*c^2 + 10*b + 40*c, a + 2*b + 2*c - 1]
```

```
>>> from sage.all import *
>>> # optional - macaulay2
>>> R = PolynomialRing(GF(Integer(101)), Integer(3), names=('a', 'b', 'c',));␣
↪(a, b, c,) = R._first_ngens(3)
>>> I = sage.rings.ideal.Katsura(R,Integer(3))  # regenerate to prevent␣
↪caching
>>> I.groebner_basis('macaulay2:gb')
[c^3 + 28*c^2 - 37*b + 13*c, b^2 - 41*c^2 + 20*b - 20*c,
 b*c - 19*c^2 + 10*b + 40*c, a + 2*b + 2*c - 1]
>>> I = sage.rings.ideal.Katsura(R,Integer(3))  # regenerate to prevent␣
↪caching
>>> I.groebner_basis('macaulay2:f4')
[c^3 + 28*c^2 - 37*b + 13*c, b^2 - 41*c^2 + 20*b - 20*c,
 b*c - 19*c^2 + 10*b + 40*c, a + 2*b + 2*c - 1]
>>> I = sage.rings.ideal.Katsura(R,Integer(3))  # regenerate to prevent␣
↪caching
>>> I.groebner_basis('macaulay2:mgb')
[c^3 + 28*c^2 - 37*b + 13*c, b^2 - 41*c^2 + 20*b - 20*c,
```

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```
 b*c - 19*c^2 + 10*b + 40*c, a + 2*b + 2*c - 1]
```

Over prime fields of small characteristic, we can also use the optional package msolve:

```
sage: R.<a,b,c> = PolynomialRing(GF(101), 3)
sage: I = sage.rings.ideal.Katsura(R,3)  # regenerate to prevent caching
sage: I.groebner_basis('msolve')                     # optional - msolve
[a + 2*b + 2*c - 1, b*c - 19*c^2 + 10*b + 40*c,
 b^2 - 41*c^2 + 20*b - 20*c, c^3 + 28*c^2 - 37*b + 13*c]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(GF(Integer(101)), Integer(3), names=('a', 'b', 'c',));␣
↪(a, b, c,) = R._first_ngens(3)
>>> I = sage.rings.ideal.Katsura(R,Integer(3))  # regenerate to prevent␣
↪caching
>>> I.groebner_basis('msolve')                     # optional - msolve
[a + 2*b + 2*c - 1, b*c - 19*c^2 + 10*b + 40*c,
 b^2 - 41*c^2 + 20*b - 20*c, c^3 + 28*c^2 - 37*b + 13*c]
```

```
sage: I = sage.rings.ideal.Katsura(P,3)  # regenerate to prevent caching
sage: I.groebner_basis('magma:GroebnerBasis')        # optional - magma
[a - 60*c^3 + 158/7*c^2 + 8/7*c - 1,
 b + 30*c^3 - 79/7*c^2 + 3/7*c, c^4 - 10/21*c^3 + 1/84*c^2 + 1/84*c]
```

```
>>> from sage.all import *
>>> I = sage.rings.ideal.Katsura(P,Integer(3))  # regenerate to prevent␣
↪caching
>>> I.groebner_basis('magma:GroebnerBasis')        # optional - magma
[a - 60*c^3 + 158/7*c^2 + 8/7*c - 1,
 b + 30*c^3 - 79/7*c^2 + 3/7*c, c^4 - 10/21*c^3 + 1/84*c^2 + 1/84*c]
```

Singular and libSingular can compute Groebner basis with degree restrictions.

```
sage: R.<x,y> = QQ[]
sage: I = R*[x^3 + y^2, x^2*y + 1]
sage: I.groebner_basis(algorithm='singular')
[x^3 + y^2, x^2*y + 1, y^3 - x]
sage: I.groebner_basis(algorithm='singular', deg_bound=2)
[x^3 + y^2, x^2*y + 1]
sage: I.groebner_basis()
[x^3 + y^2, x^2*y + 1, y^3 - x]
sage: I.groebner_basis(deg_bound=2)
[x^3 + y^2, x^2*y + 1]
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> I = R*[x**Integer(3) + y**Integer(2), x**Integer(2)*y + Integer(1)]
>>> I.groebner_basis(algorithm='singular')
[x^3 + y^2, x^2*y + 1, y^3 - x]
>>> I.groebner_basis(algorithm='singular', deg_bound=Integer(2))
[x^3 + y^2, x^2*y + 1]
```

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```
>>> I.groebner_basis()
[x^3 + y^2, x^2*y + 1, y^3 - x]
>>> I.groebner_basis(deg_bound=Integer(2))
[x^3 + y^2, x^2*y + 1]
```

A protocol is printed, if the verbosity level is at least 2, or if the argument `prot` is provided. Historically, the protocol did not appear during doctests, so, we skip the examples with protocol output.

```
sage: from sage.misc.verbose import set_verbose
sage: set_verbose(2)
sage: I = R*[x^3+y^2,x^2*y+1]
sage: I.groebner_basis()  # not tested
std in (QQ),(x,y),(dp(2),C)
[...:2]3ss4s6
(S:2)--
product criterion:1 chain criterion:0
[x^3 + y^2, x^2*y + 1, y^3 - x]
sage: I.groebner_basis(prot=False)
std in (QQ),(x,y),(dp(2),C)
[...:2]3ss4s6
(S:2)--
product criterion:1 chain criterion:0
[x^3 + y^2, x^2*y + 1, y^3 - x]
sage: set_verbose(0)
sage: I.groebner_basis(prot=True)  # not tested
std in (QQ),(x,y),(dp(2),C)
[...:2]3ss4s6
(S:2)--
product criterion:1 chain criterion:0
[x^3 + y^2, x^2*y + 1, y^3 - x]
```

```
>>> from sage.all import *
>>> from sage.misc.verbose import set_verbose
>>> set_verbose(Integer(2))
>>> I = R*[x**Integer(3)+y**Integer(2),x**Integer(2)*y+Integer(1)]
>>> I.groebner_basis()  # not tested
std in (QQ),(x,y),(dp(2),C)
[...:2]3ss4s6
(S:2)--
product criterion:1 chain criterion:0
[x^3 + y^2, x^2*y + 1, y^3 - x]
>>> I.groebner_basis(prot=False)
std in (QQ),(x,y),(dp(2),C)
[...:2]3ss4s6
(S:2)--
product criterion:1 chain criterion:0
[x^3 + y^2, x^2*y + 1, y^3 - x]
>>> set_verbose(Integer(0))
>>> I.groebner_basis(prot=True)  # not tested
std in (QQ),(x,y),(dp(2),C)
[...:2]3ss4s6
(S:2)--
```

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```
product criterion:1 chain criterion:0
[x^3 + y^2, x^2*y + 1, y^3 - x]
```

The list of available options is provided at `LibSingularOptions`.

Note that Groebner bases over $\mathbf{Z}$ can also be computed.

```
sage: P.<a,b,c> = PolynomialRing(ZZ,3)
sage: I = P * (a + 2*b + 2*c - 1, a^2 - a + 2*b^2 + 2*c^2, 2*a*b + 2*b*c - b)
sage: I.groebner_basis()
[b^3 + b*c^2 + 12*c^3 + b^2 + b*c - 4*c^2,
 2*b*c^2 - 6*c^3 - b^2 - b*c + 2*c^2,
 42*c^3 + b^2 + 2*b*c - 14*c^2 + b,
 2*b^2 + 6*b*c + 6*c^2 - b - 2*c,
 10*b*c + 12*c^2 - b - 4*c,
 a + 2*b + 2*c - 1]
```

```
>>> from sage.all import *
>>> P = PolynomialRing(ZZ,Integer(3), names=('a', 'b', 'c',)); (a, b, c,) = P.
↪_first_ngens(3)
>>> I = P * (a + Integer(2)*b + Integer(2)*c - Integer(1), a**Integer(2) - a␣
↪+ Integer(2)*b**Integer(2) + Integer(2)*c**Integer(2), Integer(2)*a*b +␣
↪Integer(2)*b*c - b)
>>> I.groebner_basis()
[b^3 + b*c^2 + 12*c^3 + b^2 + b*c - 4*c^2,
 2*b*c^2 - 6*c^3 - b^2 - b*c + 2*c^2,
 42*c^3 + b^2 + 2*b*c - 14*c^2 + b,
 2*b^2 + 6*b*c + 6*c^2 - b - 2*c,
 10*b*c + 12*c^2 - b - 4*c,
 a + 2*b + 2*c - 1]
```

```
sage: I.groebner_basis('macaulay2')                # optional - macaulay2
[b^3 + b*c^2 + 12*c^3 + b^2 + b*c - 4*c^2,
 2*b*c^2 - 6*c^3 + b^2 + 5*b*c + 8*c^2 - b - 2*c,
 42*c^3 + b^2 + 2*b*c - 14*c^2 + b,
 2*b^2 - 4*b*c - 6*c^2 + 2*c, 10*b*c + 12*c^2 - b - 4*c,
 a + 2*b + 2*c - 1]
```

```
>>> from sage.all import *
>>> I.groebner_basis('macaulay2')                # optional - macaulay2
[b^3 + b*c^2 + 12*c^3 + b^2 + b*c - 4*c^2,
 2*b*c^2 - 6*c^3 + b^2 + 5*b*c + 8*c^2 - b - 2*c,
 42*c^3 + b^2 + 2*b*c - 14*c^2 + b,
 2*b^2 - 4*b*c - 6*c^2 + 2*c, 10*b*c + 12*c^2 - b - 4*c,
 a + 2*b + 2*c - 1]
```

Groebner bases over $\mathbf{Z}/n\mathbf{Z}$ are also supported:

```
sage: P.<a,b,c> = PolynomialRing(Zmod(1000), 3)
sage: I = P * (a + 2*b + 2*c - 1, a^2 - a + 2*b^2 + 2*c^2, 2*a*b + 2*b*c - b)
sage: I.groebner_basis()
[b*c^2 + 732*b*c + 808*b,
```

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```
 2*c^3 + 884*b*c + 666*c^2 + 320*b,
 b^2 + 438*b*c + 281*b,
 5*b*c + 156*c^2 + 112*b + 948*c,
 50*c^2 + 600*b + 650*c,
 a + 2*b + 2*c + 999,
 125*b]
```

```
>>> from sage.all import *
>>> P = PolynomialRing(Zmod(Integer(1000)), Integer(3), names=('a', 'b', 'c',
↪)); (a, b, c,) = P._first_ngens(3)
>>> I = P * (a + Integer(2)*b + Integer(2)*c - Integer(1), a**Integer(2) - a␣
↪+ Integer(2)*b**Integer(2) + Integer(2)*c**Integer(2), Integer(2)*a*b +␣
↪Integer(2)*b*c - b)
>>> I.groebner_basis()
[b*c^2 + 732*b*c + 808*b,
 2*c^3 + 884*b*c + 666*c^2 + 320*b,
 b^2 + 438*b*c + 281*b,
 5*b*c + 156*c^2 + 112*b + 948*c,
 50*c^2 + 600*b + 650*c,
 a + 2*b + 2*c + 999,
 125*b]
```

```
sage: R.<x,y,z> = PolynomialRing(Zmod(2233497349584))
sage: I = R.ideal([z*(x-3*y), 3^2*x^2-y*z, z^2+y^2])
sage: I.groebner_basis()
[2*z^4, y*z^2 + 81*z^3, 248166372176*z^3, 9*x^2 + 2233497349583*y*z, y^2 + z^
↪2, x*z + 2233497349581*y*z, 248166372176*y*z]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Zmod(Integer(2233497349584)), names=('x', 'y', 'z',));␣
↪(x, y, z,) = R._first_ngens(3)
>>> I = R.ideal([z*(x-Integer(3)*y), Integer(3)**Integer(2)*x**Integer(2)-y*z,
↪ z**Integer(2)+y**Integer(2)])
>>> I.groebner_basis()
[2*z^4, y*z^2 + 81*z^3, 248166372176*z^3, 9*x^2 + 2233497349583*y*z, y^2 + z^
↪2, x*z + 2233497349581*y*z, 248166372176*y*z]
```

Sage also supports local orderings:

```
sage: P.<x,y,z> = PolynomialRing(QQ, 3, order='negdegrevlex')
sage: I = P * (  x*y*z + z^5, 2*x^2 + y^3 + z^7, 3*z^5 + y^5 )
sage: I.groebner_basis()
[x^2 + 1/2*y^3, x*y*z + z^5, y^5 + 3*z^5, y^4*z - 2*x*z^5, z^6]
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, Integer(3), order='negdegrevlex', names=('x', 'y',
↪'z',)); (x, y, z,) = P._first_ngens(3)
>>> I = P * (  x*y*z + z**Integer(5), Integer(2)*x**Integer(2) +␣
↪y**Integer(3) + z**Integer(7), Integer(3)*z**Integer(5) + y**Integer(5) )
>>> I.groebner_basis()
[x^2 + 1/2*y^3, x*y*z + z^5, y^5 + 3*z^5, y^4*z - 2*x*z^5, z^6]
```

We can represent every element in the ideal as a combination of the generators using the `lift()` method:

```
sage: P.<x,y,z> = PolynomialRing(QQ, 3)
sage: I = P * ( x*y*z + z^5, 2*x^2 + y^3 + z^7, 3*z^5 + y^5 )
sage: J = Ideal(I.groebner_basis())
sage: f = sum(P.random_element(terms=2)*f for f in I.gens())
sage: f                          # random
1/2*y^2*z^7 - 1/4*y*z^8 + 2*x*z^5 + 95*z^6 + 1/2*y^5 - 1/4*y^4*z + x^2*y^2 +
→3/2*x^2*y*z + 95*x*y*z^2
sage: f.lift(I.gens())           # random
[2*x + 95*z, 1/2*y^2 - 1/4*y*z, 0]
sage: l = f.lift(J.gens()); l  # random
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1/2*y^2 + 1/4*y*z, 1/2*y^2*z^2 - 1/
→4*y*z^3 + 2*x + 95*z]
sage: sum(map(mul, zip(l,J.gens()))) == f
True
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) =
→P._first_ngens(3)
>>> I = P * ( x*y*z + z**Integer(5), Integer(2)*x**Integer(2) + y**Integer(3)
→+ z**Integer(7), Integer(3)*z**Integer(5) + y**Integer(5) )
>>> J = Ideal(I.groebner_basis())
>>> f = sum(P.random_element(terms=Integer(2))*f for f in I.gens())
>>> f                          # random
1/2*y^2*z^7 - 1/4*y*z^8 + 2*x*z^5 + 95*z^6 + 1/2*y^5 - 1/4*y^4*z + x^2*y^2 +
→3/2*x^2*y*z + 95*x*y*z^2
>>> f.lift(I.gens())           # random
[2*x + 95*z, 1/2*y^2 - 1/4*y*z, 0]
>>> l = f.lift(J.gens()); l  # random
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1/2*y^2 + 1/4*y*z, 1/2*y^2*z^2 - 1/
→4*y*z^3 + 2*x + 95*z]
>>> sum(map(mul, zip(l,J.gens()))) == f
True
```

Groebner bases over fraction fields of polynomial rings are also supported:

```
sage: P.<t> = QQ[]
sage: F = Frac(P)
sage: R.<X,Y,Z> = F[]
sage: I = Ideal([f + P.random_element() for f in sage.rings.ideal.Katsura(R).
→gens()])
sage: I.groebner_basis().ideal() == I
True
```

```
>>> from sage.all import *
>>> P = QQ['t']; (t,) = P._first_ngens(1)
>>> F = Frac(P)
>>> R = F['X, Y, Z']; (X, Y, Z,) = R._first_ngens(3)
>>> I = Ideal([f + P.random_element() for f in sage.rings.ideal.Katsura(R).
→gens()])
>>> I.groebner_basis().ideal() == I
True
```

In cases where a characteristic cannot be determined, we use a toy implementation of Buchberger's algorithm (see Issue #6581):

```
sage: R.<a,b> = QQ[]; I = R.ideal(a^2+b^2-1)
sage: Q = QuotientRing(R,I); K = Frac(Q)
sage: R2.<x,y> = K[]; J = R2.ideal([(a^2+b^2)*x + y, x+y])
sage: J.groebner_basis()
verbose 0 (...: multi_polynomial_ideal.py, groebner_basis) Warning: falling␣
↪back to very slow toy implementation.
[x + y]
```

```
>>> from sage.all import *
>>> R = QQ['a, b']; (a, b,) = R._first_ngens(2); I = R.
↪ideal(a**Integer(2)+b**Integer(2)-Integer(1))
>>> Q = QuotientRing(R,I); K = Frac(Q)
>>> R2 = K['x, y']; (x, y,) = R2._first_ngens(2); J = R2.
↪ideal([(a**Integer(2)+b**Integer(2))*x + y, x+y])
>>> J.groebner_basis()
verbose 0 (...: multi_polynomial_ideal.py, groebner_basis) Warning: falling␣
↪back to very slow toy implementation.
[x + y]
```

ALGORITHM:

Uses Singular, one of the other systems listed above (if available), or a toy implementation.

groebner_cover()

Compute the Gröbner cover of the ideal, over a field with parameters.

The Groebner cover is a partition of the space of parameters, such that the Groebner basis in each part is given by the same expression.

EXAMPLES:

```
sage: F = PolynomialRing(QQ,'a').fraction_field()
sage: F.inject_variables()
Defining a
sage: R.<x,y,z> = F[]
sage: I = R.ideal([-x+3*y+z-5,2*x+a*z+4,4*x-3*z-1/a])
sage: I.groebner_cover()
{Quasi-affine subscheme X - Y of Affine Space of dimension 1 over Rational␣
↪Field,
   where X is defined by:
     0
   and Y is defined by:
     2*a^2 + 3*a: [(2*a^2 + 3*a)*z + (8*a + 1),
                   (12*a^2 + 18*a)*y + (-20*a^2 - 35*a - 2), (4*a + 6)*x +␣
↪11],
 Quasi-affine subscheme X - Y of Affine Space of dimension 1 over Rational␣
↪Field,
   where X is defined by:
     ...
   and Y is defined by:
     1: [1],
 Quasi-affine subscheme X - Y of Affine Space of dimension 1 over Rational␣
```

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```
↪Field,
   where X is defined by:
     ...
   and Y is defined by:
     1: [1]}
```

```
>>> from sage.all import *
>>> F = PolynomialRing(QQ,'a').fraction_field()
>>> F.inject_variables()
Defining a
>>> R = F['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> I = R.ideal([-x+Integer(3)*y+z-Integer(5),Integer(2)*x+a*z+Integer(4),
↪Integer(4)*x-Integer(3)*z-Integer(1)/a])
>>> I.groebner_cover()
{Quasi-affine subscheme X - Y of Affine Space of dimension 1 over Rational␣
↪Field,
   where X is defined by:
     0
   and Y is defined by:
     2*a^2 + 3*a: [(2*a^2 + 3*a)*z + (8*a + 1),
                  (12*a^2 + 18*a)*y + (-20*a^2 - 35*a - 2), (4*a + 6)*x +␣
↪11],
 Quasi-affine subscheme X - Y of Affine Space of dimension 1 over Rational␣
↪Field,
   where X is defined by:
     ...
   and Y is defined by:
     1: [1],
 Quasi-affine subscheme X - Y of Affine Space of dimension 1 over Rational␣
↪Field,
   where X is defined by:
     ...
   and Y is defined by:
     1: [1]}
```

groebner_fan(*is_groebner_basis=False*, *symmetry=None*, *verbose=False*)

Return the Groebner fan of this ideal.

The base ring must be $\mathbf{Q}$ or a finite field $\mathbf{F}_p$ of with $p \leq 32749$.

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQ)
sage: i = ideal(x^2 - y^2 + 1)
sage: g = i.groebner_fan()
sage: g.reduced_groebner_bases()
[[x^2 - y^2 + 1], [-x^2 + y^2 - 1]]
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = P._first_ngens(2)
>>> i = ideal(x**Integer(2) - y**Integer(2) + Integer(1))
>>> g = i.groebner_fan()
```

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```
>>> g.reduced_groebner_bases()
[[x^2 - y^2 + 1], [-x^2 + y^2 - 1]]
```

INPUT:

- `is_groebner_basis` – boolean (default: `False`); if `True`, then `I.gens()` must be a Groebner basis with respect to the standard degree lexicographic term order
- `symmetry` – (default: `None`) if not `None`, describes symmetries of the ideal
- `verbose` – (default: `False`) if `True`, printout useful info during computations

homogenize(*var='h'*)

Return homogeneous ideal spanned by the homogeneous polynomials generated by homogenizing the generators of this ideal.

INPUT:

- `h` – variable name or variable in cover ring (default: `'h'`)

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(GF(2))
sage: I = Ideal([x^2*y + z + 1, x + y^2 + 1]); I
Ideal (x^2*y + z + 1, y^2 + x + 1) of Multivariate
Polynomial Ring in x, y, z over Finite Field of size 2
```

```
>>> from sage.all import *
>>> P = PolynomialRing(GF(Integer(2)), names=('x', 'y', 'z',)); (x, y, z,) =␣
↪P._first_ngens(3)
>>> I = Ideal([x**Integer(2)*y + z + Integer(1), x + y**Integer(2) +␣
↪Integer(1)]); I
Ideal (x^2*y + z + 1, y^2 + x + 1) of Multivariate
Polynomial Ring in x, y, z over Finite Field of size 2
```

```
sage: I.homogenize()
Ideal (x^2*y + z*h^2 + h^3, y^2 + x*h + h^2) of
Multivariate Polynomial Ring in x, y, z, h over Finite
Field of size 2
```

```
>>> from sage.all import *
>>> I.homogenize()
Ideal (x^2*y + z*h^2 + h^3, y^2 + x*h + h^2) of
Multivariate Polynomial Ring in x, y, z, h over Finite
Field of size 2
```

```
sage: I.homogenize(y)
Ideal (x^2*y + y^3 + y^2*z, x*y) of Multivariate
Polynomial Ring in x, y, z over Finite Field of size 2
```

```
>>> from sage.all import *
>>> I.homogenize(y)
Ideal (x^2*y + y^3 + y^2*z, x*y) of Multivariate
Polynomial Ring in x, y, z over Finite Field of size 2
```

```
sage: I = Ideal([x^2*y + z^3 + y^2*x, x + y^2 + 1])
sage: I.homogenize()
Ideal (x^2*y + x*y^2 + z^3, y^2 + x*h + h^2) of
Multivariate Polynomial Ring in x, y, z, h over Finite
Field of size 2
```

```
>>> from sage.all import *
>>> I = Ideal([x**Integer(2)*y + z**Integer(3) + y**Integer(2)*x, x +␣
↪y**Integer(2) + Integer(1)])
>>> I.homogenize()
Ideal (x^2*y + x*y^2 + z^3, y^2 + x*h + h^2) of
Multivariate Polynomial Ring in x, y, z, h over Finite
Field of size 2
```

is_homogeneous()

Return `True` if this ideal is spanned by homogeneous polynomials, i.e., if it is a homogeneous ideal.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ,3)
sage: I = sage.rings.ideal.Katsura(P)
sage: I
Ideal (x + 2*y + 2*z - 1, x^2 + 2*y^2 + 2*z^2 - x, 2*x*y +
2*y*z - y) of Multivariate Polynomial Ring in x, y, z over
Rational Field
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ,Integer(3), names=('x', 'y', 'z',)); (x, y, z,) = P.
↪_first_ngens(3)
>>> I = sage.rings.ideal.Katsura(P)
>>> I
Ideal (x + 2*y + 2*z - 1, x^2 + 2*y^2 + 2*z^2 - x, 2*x*y +
2*y*z - y) of Multivariate Polynomial Ring in x, y, z over
Rational Field
```

```
sage: I.is_homogeneous()
False
```

```
>>> from sage.all import *
>>> I.is_homogeneous()
False
```

```
sage: J = I.homogenize()
sage: J
Ideal (x + 2*y + 2*z - h, x^2 + 2*y^2 + 2*z^2 - x*h, 2*x*y
+ 2*y*z - y*h) of Multivariate Polynomial Ring in x, y, z,
h over Rational Field
```

```
>>> from sage.all import *
>>> J = I.homogenize()
>>> J
```

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```
Ideal (x + 2*y + 2*z - h, x^2 + 2*y^2 + 2*z^2 - x*h, 2*x*y
+ 2*y*z - y*h) of Multivariate Polynomial Ring in x, y, z,
h over Rational Field
```

```
sage: J.is_homogeneous()
True
```

```
>>> from sage.all import *
>>> J.is_homogeneous()
True
```

plot(**args*, ***kwds*)

Plot the real zero locus of this principal ideal.

INPUT:

- `self` – a principal ideal in 2 variables
- `algorithm` – set this to 'surf' if you want 'surf' to plot the ideal (default: `None`)
- `*args` – (optional) tuples `(variable, minimum, maximum)` for plotting dimensions
- `**kwds` – optional keyword arguments passed on to `implicit_plot`

EXAMPLES:

Implicit plotting in 2-d:

```
sage: R.<x,y> = PolynomialRing(QQ, 2)
sage: I = R.ideal([y^3 - x^2])
sage: I.plot()                                  # cusp                          #␣
↪needs sage.plot
Graphics object consisting of 1 graphics primitive
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(2), names=('x', 'y',)); (x, y,) = R._first_
↪ngens(2)
>>> I = R.ideal([y**Integer(3) - x**Integer(2)])
>>> I.plot()                                # cusp                          #␣
↪needs sage.plot
Graphics object consisting of 1 graphics primitive
```

```
sage: I = R.ideal([y^2 - x^2 - 1])
sage: I.plot((x,-3, 3), (y, -2, 2))     # hyperbola                     #␣
↪needs sage.plot
Graphics object consisting of 1 graphics primitive
```

```
>>> from sage.all import *
>>> I = R.ideal([y**Integer(2) - x**Integer(2) - Integer(1)])
>>> I.plot((x,-Integer(3), Integer(3)), (y, -Integer(2), Integer(2)))     #␣
↪hyperbola                              # needs sage.plot
Graphics object consisting of 1 graphics primitive
```

```
sage: I = R.ideal([y^2 + x^2*(1/4) - 1])
sage: I.plot()                          # ellipse                          #␣
↪needs sage.plot
Graphics object consisting of 1 graphics primitive
```

```
>>> from sage.all import *
>>> I = R.ideal([y**Integer(2) + x**Integer(2)*(Integer(1)/Integer(4)) -␣
↪Integer(1)])
>>> I.plot()                          # ellipse                          #␣
↪needs sage.plot
Graphics object consisting of 1 graphics primitive
```

```
sage: I = R.ideal([y^2-(x^2-1)*(x-2)])
sage: I.plot()                          # elliptic curve                   #␣
↪needs sage.plot
Graphics object consisting of 1 graphics primitive
```

```
>>> from sage.all import *
>>> I = R.ideal([y**Integer(2)-(x**Integer(2)-Integer(1))*(x-Integer(2))])
>>> I.plot()                          # elliptic curve                   #␣
↪needs sage.plot
Graphics object consisting of 1 graphics primitive
```

```
sage: f = ((x+3)^3 + 2*(x+3)^2 - y^2)*(x^3 - y^2)*((x-3)^3-2*(x-3)^2-y^2)
sage: I = R.ideal(f)
sage: I.plot()                          # the Singular logo                #␣
↪needs sage.plot
Graphics object consisting of 1 graphics primitive
```

```
>>> from sage.all import *
>>> f = ((x+Integer(3))**Integer(3) + Integer(2)*(x+Integer(3))**Integer(2) -␣
↪y**Integer(2))*(x**Integer(3) - y**Integer(2))*((x-Integer(3))**Integer(3)-
↪Integer(2)*(x-Integer(3))**Integer(2)-y**Integer(2))
>>> I = R.ideal(f)
>>> I.plot()                          # the Singular logo                #␣
↪needs sage.plot
Graphics object consisting of 1 graphics primitive
```

```
sage: R.<x,y> = PolynomialRing(QQ, 2)
sage: I = R.ideal([x - 1])
sage: I.plot((y, -2, 2))                # vertical line                    #␣
↪needs sage.plot
Graphics object consisting of 1 graphics primitive
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(2), names=('x', 'y',)); (x, y,) = R._first_
↪ngens(2)
>>> I = R.ideal([x - Integer(1)])
>>> I.plot((y, -Integer(2), Integer(2)))                # vertical line       ␣
↪                # needs sage.plot
Graphics object consisting of 1 graphics primitive
```

```
sage: I = R.ideal([-x^2*y + 1])
sage: I.plot()                          # blow up                          #␣
↪needs sage.plot
Graphics object consisting of 1 graphics primitive
```

```
>>> from sage.all import *
>>> I = R.ideal([-x**Integer(2)*y + Integer(1)])
>>> I.plot()                          # blow up                          #␣
↪needs sage.plot
Graphics object consisting of 1 graphics primitive
```

random_element(*degree*, *compute_gb=False*, **args*, ***kwds*)

Return a random element in this ideal as $r = \sum h_i \cdot f_i$.

INPUT:

- `compute_gb` – if `True` then a Gröbner basis is computed first and f_i are the elements in the Gröbner basis. Otherwise whatever basis is returned by `self.gens()` is used.
- `*args` and `**kwds` are passed to `R.random_element()` with `R = self.ring()`.

EXAMPLES:

We compute a uniformly random element up to the provided degree.

```
sage: P.<x,y,z> = GF(127)[]
sage: I = sage.rings.ideal.Katsura(P)
sage: f = I.random_element(degree=4, compute_gb=True, terms=infinity)
sage: f.degree() <= 4
True
sage: len(list(f)) <= 35
True
```

```
>>> from sage.all import *
>>> P = GF(Integer(127))['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> I = sage.rings.ideal.Katsura(P)
>>> f = I.random_element(degree=Integer(4), compute_gb=True, terms=infinity)
>>> f.degree() <= Integer(4)
True
>>> len(list(f)) <= Integer(35)
True
```

Note that sampling uniformly at random from the ideal at some large enough degree is equivalent to computing a Gröbner basis. We give an example showing how to compute a Gröbner basis if we can sample uniformly at random from an ideal:

```
sage: n = 3; d = 4
sage: P = PolynomialRing(GF(127), n, 'x')
sage: I = sage.rings.ideal.Cyclic(P)
```

```
>>> from sage.all import *
>>> n = Integer(3); d = Integer(4)
>>> P = PolynomialRing(GF(Integer(127)), n, 'x')
>>> I = sage.rings.ideal.Cyclic(P)
```

1. We sample n^d uniformly random elements in the ideal:

```
sage: F = Sequence(I.random_element(degree=d, compute_gb=True,
....:                               terms=infinity)
....:                for _ in range(n^d))
```

```
>>> from sage.all import *
>>> F = Sequence(I.random_element(degree=d, compute_gb=True,
...                               terms=infinity)
...                for _ in range(n**d))
```

2. We linearize and compute the echelon form:

```
sage: A, v = F.coefficients_monomials()
sage: A.echelonize()
```

```
>>> from sage.all import *
>>> A, v = F.coefficients_monomials()
>>> A.echelonize()
```

3. The result is the desired Gröbner basis:

```
sage: G = Sequence((A * v).list())
sage: G.is_groebner()
True
sage: Ideal(G) == I
True
```

```
>>> from sage.all import *
>>> G = Sequence((A * v).list())
>>> G.is_groebner()
True
>>> Ideal(G) == I
True
```

We return some element in the ideal with no guarantee on the distribution:

```
sage: # needs sage.rings.finite_rings
sage: P = PolynomialRing(GF(127), 10, 'x')
sage: I = sage.rings.ideal.Katsura(P)
sage: f = I.random_element(degree=3)
sage: f   # random
-25*x0^2*x1 + 14*x1^3 + 57*x0*x1*x2 + ... + 19*x7*x9 + 40*x8*x9 + 49*x1
sage: f.degree()
3
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> P = PolynomialRing(GF(Integer(127)), Integer(10), 'x')
>>> I = sage.rings.ideal.Katsura(P)
>>> f = I.random_element(degree=Integer(3))
>>> f   # random
```

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```
-25*x0^2*x1 + 14*x1^3 + 57*x0*x1*x2 + ... + 19*x7*x9 + 40*x8*x9 + 49*x1
>>> f.degree()
3
```

We show that the default method does not sample uniformly at random from the ideal:

```
sage: # needs sage.rings.finite_rings
sage: P.<x,y,z> = GF(127)[]
sage: G = Sequence([x + 7, y - 2, z + 110])
sage: I = Ideal([sum(P.random_element() * g for g in G)
....:              for _ in range(4)])
sage: all(I.random_element(degree=1) == 0 for _ in range(100))
True
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> P = GF(Integer(127))['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> G = Sequence([x + Integer(7), y - Integer(2), z + Integer(110)])
>>> I = Ideal([sum(P.random_element() * g for g in G)
...              for _ in range(Integer(4))])
>>> all(I.random_element(degree=Integer(1)) == Integer(0) for _ in␣
→range(Integer(100)))
True
```

If `degree` equals the degree of the generators, a random linear combination of the generators is returned:

```
sage: P.<x,y> = QQ[]
sage: I = P.ideal([x^2,y^2])
sage: set_random_seed(5)
sage: I.random_element(degree=2)
-2*x^2 + 2*y^2
```

```
>>> from sage.all import *
>>> P = QQ['x, y']; (x, y,) = P._first_ngens(2)
>>> I = P.ideal([x**Integer(2),y**Integer(2)])
>>> set_random_seed(Integer(5))
>>> I.random_element(degree=Integer(2))
-2*x^2 + 2*y^2
```

reduce (*f*)

Reduce an element modulo the reduced Groebner basis for this ideal. This returns 0 if and only if the element is in this ideal. In any case, this reduction is unique up to monomial orders.

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQ, 2)
sage: I = (x^3 + y, y) * R
sage: I.reduce(y)
0
sage: I.reduce(x^3)
0
sage: I.reduce(x - y)
```

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```
x

sage: I = (y^2 - (x^3 + x)) * R
sage: I.reduce(x^3)
y^2 - x
sage: I.reduce(x^6)
y^4 - 2*x*y^2 + x^2
sage: (y^2 - x)^2
y^4 - 2*x*y^2 + x^2
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(2), names=('x', 'y',)); (x, y,) = R._first_
↪ngens(2)
>>> I = (x**Integer(3) + y, y) * R
>>> I.reduce(y)
0
>>> I.reduce(x**Integer(3))
0
>>> I.reduce(x - y)
x

>>> I = (y**Integer(2) - (x**Integer(3) + x)) * R
>>> I.reduce(x**Integer(3))
y^2 - x
>>> I.reduce(x**Integer(6))
y^4 - 2*x*y^2 + x^2
>>> (y**Integer(2) - x)**Integer(2)
y^4 - 2*x*y^2 + x^2
```

Note

Requires computation of a Groebner basis, which can be a very expensive operation.

subs(*in_dict=None*, ***kwds*)

Substitute variables.

This method substitutes some variables in the polynomials that generate the ideal with given values. Variables that are not specified in the input remain unchanged.

INPUT:

- `in_dict` – (optional) dictionary of inputs
- `**kwds` – named parameters

OUTPUT:

A new ideal with modified generators. If possible, in the same polynomial ring. Raises a `TypeError` if no common polynomial ring of the substituted generators can be found.

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(ZZ, 2, 'xy')
sage: I = R.ideal(x^5 + y^5, x^2 + y + x^2*y^2 + 5); I
```

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```
Ideal (x^5 + y^5, x^2*y^2 + x^2 + y + 5)
 of Multivariate Polynomial Ring in x, y over Integer Ring
sage: I.subs(x=y)
Ideal (2*y^5, y^4 + y^2 + y + 5)
 of Multivariate Polynomial Ring in x, y over Integer Ring
sage: I.subs({x: y})    # same substitution but with dictionary
Ideal (2*y^5, y^4 + y^2 + y + 5)
 of Multivariate Polynomial Ring in x, y over Integer Ring
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, Integer(2), 'xy', names=('x', 'y',)); (x, y,) = R._
↪first_ngens(2)
>>> I = R.ideal(x**Integer(5) + y**Integer(5), x**Integer(2) + y +␣
↪x**Integer(2)*y**Integer(2) + Integer(5)); I
Ideal (x^5 + y^5, x^2*y^2 + x^2 + y + 5)
 of Multivariate Polynomial Ring in x, y over Integer Ring
>>> I.subs(x=y)
Ideal (2*y^5, y^4 + y^2 + y + 5)
 of Multivariate Polynomial Ring in x, y over Integer Ring
>>> I.subs({x: y})    # same substitution but with dictionary
Ideal (2*y^5, y^4 + y^2 + y + 5)
 of Multivariate Polynomial Ring in x, y over Integer Ring
```

The new ideal can be in a different ring:

```
sage: R.<a,b> = PolynomialRing(QQ, 2)
sage: S.<x,y> = PolynomialRing(QQ, 2)
sage: I = R.ideal(a^2 + b^2 + a - b + 2); I
Ideal (a^2 + b^2 + a - b + 2)
 of Multivariate Polynomial Ring in a, b over Rational Field
sage: I.subs(a=x, b=y)
Ideal (x^2 + y^2 + x - y + 2)
 of Multivariate Polynomial Ring in x, y over Rational Field
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(2), names=('a', 'b',)); (a, b,) = R._first_
↪ngens(2)
>>> S = PolynomialRing(QQ, Integer(2), names=('x', 'y',)); (x, y,) = S._first_
↪ngens(2)
>>> I = R.ideal(a**Integer(2) + b**Integer(2) + a - b + Integer(2)); I
Ideal (a^2 + b^2 + a - b + 2)
 of Multivariate Polynomial Ring in a, b over Rational Field
>>> I.subs(a=x, b=y)
Ideal (x^2 + y^2 + x - y + 2)
 of Multivariate Polynomial Ring in x, y over Rational Field
```

The resulting ring need not be a multivariate polynomial ring:

```
sage: T.<t> = PolynomialRing(QQ)
sage: I.subs(a=t, b=t)
Principal ideal (t^2 + 1) of Univariate Polynomial Ring in t over Rational␣
↪Field
```

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```
sage: var("z")                                                                  #␣
↪needs sage.symbolic
z
sage: I.subs(a=z, b=z)                                                          #␣
↪needs sage.symbolic
Principal ideal (2*z^2 + 2) of Symbolic Ring
```

```
>>> from sage.all import *
>>> T = PolynomialRing(QQ, names=('t',)); (t,) = T._first_ngens(1)
>>> I.subs(a=t, b=t)
Principal ideal (t^2 + 1) of Univariate Polynomial Ring in t over Rational␣
↪Field
>>> var("z")                                                                  #␣
↪needs sage.symbolic
z
>>> I.subs(a=z, b=z)                                                          #␣
↪needs sage.symbolic
Principal ideal (2*z^2 + 2) of Symbolic Ring
```

Variables that are not substituted remain unchanged:

```
sage: R.<x,y> = PolynomialRing(QQ, 2)
sage: I = R.ideal(x^2 + y^2 + x - y + 2); I
Ideal (x^2 + y^2 + x - y + 2)
 of Multivariate Polynomial Ring in x, y over Rational Field
sage: I.subs(x=1)
Ideal (y^2 - y + 4) of Multivariate Polynomial Ring in x, y over Rational␣
↪Field
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(2), names=('x', 'y',)); (x, y,) = R._first_
↪ngens(2)
>>> I = R.ideal(x**Integer(2) + y**Integer(2) + x - y + Integer(2)); I
Ideal (x^2 + y^2 + x - y + 2)
 of Multivariate Polynomial Ring in x, y over Rational Field
>>> I.subs(x=Integer(1))
Ideal (y^2 - y + 4) of Multivariate Polynomial Ring in x, y over Rational␣
↪Field
```

weil_restriction()

Compute the Weil restriction of this ideal over some extension field. If the field is a finite field, then this computes the Weil restriction to the prime subfield.

A Weil restriction of scalars - denoted $Res_{L/k}$ - is a functor which, for any finite extension of fields L/k and any algebraic variety X over L, produces another corresponding variety $Res_{L/k}(X)$, defined over k. It is useful for reducing questions about varieties over large fields to questions about more complicated varieties over smaller fields.

This function does not compute this Weil restriction directly but computes on generating sets of polynomial ideals:

Let d be the degree of the field extension L/k, let a a generator of L/k and p the minimal polynomial of L/k. Denote this ideal by I.

Specifically, this function first maps each variable x to its representation over k: $\sum_{i=0}^{d-1} a^i x_i$. Then each generator of I is evaluated over these representations and reduced modulo the minimal polynomial p. The result is interpreted as a univariate polynomial in a and its coefficients are the new generators of the returned ideal.

If the input and the output ideals are radical, this is equivalent to the statement about algebraic varieties above.

OUTPUT: `MPolynomialIdeal`

EXAMPLES:

```
sage: # needs sage.rings.finite_rings
sage: k.<a> = GF(2^2)
sage: P.<x,y> = PolynomialRing(k, 2)
sage: I = Ideal([x*y + 1, a*x + 1])
sage: I.variety()
[{y: a, x: a + 1}]
sage: J = I.weil_restriction()
sage: J
Ideal (x0*y0 + x1*y1 + 1, x1*y0 + x0*y1 + x1*y1, x1 + 1, x0 + x1) of
 Multivariate Polynomial Ring in x0, x1, y0, y1 over Finite Field of size 2
sage: J += sage.rings.ideal.FieldIdeal(J.ring())  # ensure radical ideal
sage: J.variety()
[{y1: 1, y0: 0, x1: 1, x0: 1}]
sage: J.weil_restriction()  # returns J
Ideal (x0*y0 + x1*y1 + 1, x1*y0 + x0*y1 + x1*y1, x1 + 1, x0 + x1,
 x0^2 + x0, x1^2 + x1, y0^2 + y0, y1^2 + y1) of Multivariate
 Polynomial Ring in x0, x1, y0, y1 over Finite Field of size 2

sage: # needs sage.rings.finite_rings
sage: k.<a> = GF(3^5)
sage: P.<x,y,z> = PolynomialRing(k)
sage: I = sage.rings.ideal.Katsura(P)
sage: I.dimension()
0
sage: I.variety()
 [{z: 0, y: 0, x: 1}]
sage: J = I.weil_restriction(); J
Ideal (x0 - y0 - z0 - 1,
       x1 - y1 - z1, x2 - y2 - z2, x3 - y3 - z3, x4 - y4 - z4,
       x0^2 + x2*x3 + x1*x4 - y0^2 - y2*y3 - y1*y4 - z0^2 - z2*z3 - z1*z4 -␣
↪x0,
       -x0*x1 - x2*x3 - x3^2 - x1*x4 + x2*x4 + y0*y1 + y2*y3
            + y3^2 + y1*y4 - y2*y4 + z0*z1 + z2*z3 + z3^2 + z1*z4 - z2*z4 -␣
↪x1,
       x1^2 - x0*x2 + x3^2 - x2*x4 + x3*x4 - y1^2 + y0*y2
             - y3^2 + y2*y4 - y3*y4 - z1^2 + z0*z2 - z3^2 + z2*z4 - z3*z4 -␣
↪x2,
       -x1*x2 - x0*x3 - x3*x4 - x4^2
            + y1*y2 + y0*y3 + y3*y4 + y4^2 + z1*z2 + z0*z3 + z3*z4 + z4^2 -␣
↪x3,
       x2^2 - x1*x3 - x0*x4 + x4^2 - y2^2
                     + y1*y3 + y0*y4 - y4^2 - z2^2 + z1*z3 + z0*z4 - z4^2 -␣
↪x4,
       -x0*y0 + x4*y1 + x3*y2 + x2*y3
```

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```
                         + x1*y4 - y0*z0 + y4*z1 + y3*z2 + y2*z3 + y1*z4 -␣
↪y0,
      -x1*y0 - x0*y1 - x4*y1 - x3*y2 + x4*y2 - x2*y3 + x3*y3
                         - x1*y4 + x2*y4 - y1*z0 - y0*z1 - y4*z1 - y3*z2
                                 + y4*z2 - y2*z3 + y3*z3 - y1*z4 + y2*z4 -␣
↪y1,
      -x2*y0 - x1*y1 - x0*y2 - x4*y2 - x3*y3 + x4*y3 - x2*y4 + x3*y4
         - y2*z0 - y1*z1 - y0*z2 - y4*z2 - y3*z3 + y4*z3 - y2*z4 + y3*z4 -␣
↪y2,
      -x3*y0 - x2*y1 - x1*y2 - x0*y3 - x4*y3 - x3*y4 + x4*y4
                 - y3*z0 - y2*z1 - y1*z2 - y0*z3 - y4*z3 - y3*z4 + y4*z4 -␣
↪y3,
      -x4*y0 - x3*y1 - x2*y2 - x1*y3 - x0*y4 - x4*y4
                         - y4*z0 - y3*z1 - y2*z2 - y1*z3 - y0*z4 - y4*z4 -␣
↪y4)
 of Multivariate Polynomial Ring in x0, x1, x2, x3, x4, y0, y1, y2, y3, y4,
 z0, z1, z2, z3, z4 over Finite Field of size 3
sage: J += sage.rings.ideal.FieldIdeal(J.ring())  # ensure radical ideal
sage: J.variety()
[{z4: 0,
  z3: 0,
  z2: 0,
  z1: 0,
  z0: 0,
  y4: 0,
  y3: 0,
  y2: 0,
  y1: 0,
  y0: 0,
  x4: 0,
  x3: 0,
  x2: 0,
  x1: 0,
  x0: 1}]
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> k = GF(Integer(2)**Integer(2), names=('a',)); (a,) = k._first_ngens(1)
>>> P = PolynomialRing(k, Integer(2), names=('x', 'y',)); (x, y,) = P._first_
↪ngens(2)
>>> I = Ideal([x*y + Integer(1), a*x + Integer(1)])
>>> I.variety()
[{y: a, x: a + 1}]
>>> J = I.weil_restriction()
>>> J
Ideal (x0*y0 + x1*y1 + 1, x1*y0 + x0*y1 + x1*y1, x1 + 1, x0 + x1) of
 Multivariate Polynomial Ring in x0, x1, y0, y1 over Finite Field of size 2
>>> J += sage.rings.ideal.FieldIdeal(J.ring())  # ensure radical ideal
>>> J.variety()
[{y1: 1, y0: 0, x1: 1, x0: 1}]
>>> J.weil_restriction()  # returns J
Ideal (x0*y0 + x1*y1 + 1, x1*y0 + x0*y1 + x1*y1, x1 + 1, x0 + x1,
```

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```
 x0^2 + x0, x1^2 + x1, y0^2 + y0, y1^2 + y1) of Multivariate
 Polynomial Ring in x0, x1, y0, y1 over Finite Field of size 2

>>> # needs sage.rings.finite_rings
>>> k = GF(Integer(3)**Integer(5), names=('a',)); (a,) = k._first_ngens(1)
>>> P = PolynomialRing(k, names=('x', 'y', 'z',)); (x, y, z,) = P._first_
↪ngens(3)
>>> I = sage.rings.ideal.Katsura(P)
>>> I.dimension()
0
>>> I.variety()
 [{z: 0, y: 0, x: 1}]
>>> J = I.weil_restriction(); J
Ideal (x0 - y0 - z0 - 1,
       x1 - y1 - z1, x2 - y2 - z2, x3 - y3 - z3, x4 - y4 - z4,
       x0^2 + x2*x3 + x1*x4 - y0^2 - y2*y3 - y1*y4 - z0^2 - z2*z3 - z1*z4 -␣
↪x0,
       -x0*x1 - x2*x3 - x3^2 - x1*x4 + x2*x4 + y0*y1 + y2*y3
            + y3^2 + y1*y4 - y2*y4 + z0*z1 + z2*z3 + z3^2 + z1*z4 - z2*z4 -␣
↪x1,
       x1^2 - x0*x2 + x3^2 - x2*x4 + x3*x4 - y1^2 + y0*y2
             - y3^2 + y2*y4 - y3*y4 - z1^2 + z0*z2 - z3^2 + z2*z4 - z3*z4 -␣
↪x2,
       -x1*x2 - x0*x3 - x3*x4 - x4^2
            + y1*y2 + y0*y3 + y3*y4 + y4^2 + z1*z2 + z0*z3 + z3*z4 + z4^2 -␣
↪x3,
       x2^2 - x1*x3 - x0*x4 + x4^2 - y2^2
                     + y1*y3 + y0*y4 - y4^2 - z2^2 + z1*z3 + z0*z4 - z4^2 -␣
↪x4,
       -x0*y0 + x4*y1 + x3*y2 + x2*y3
                          + x1*y4 - y0*z0 + y4*z1 + y3*z2 + y2*z3 + y1*z4 -␣
↪y0,
       -x1*y0 - x0*y1 - x4*y1 - x3*y2 + x4*y2 - x2*y3 + x3*y3
                          - x1*y4 + x2*y4 - y1*z0 - y0*z1 - y4*z1 - y3*z2
                                  + y4*z2 - y2*z3 + y3*z3 - y1*z4 + y2*z4 -␣
↪y1,
       -x2*y0 - x1*y1 - x0*y2 - x4*y2 - x3*y3 + x4*y3 - x2*y4 + x3*y4
          - y2*z0 - y1*z1 - y0*z2 - y4*z2 - y3*z3 + y4*z3 - y2*z4 + y3*z4 -␣
↪y2,
       -x3*y0 - x2*y1 - x1*y2 - x0*y3 - x4*y3 - x3*y4 + x4*y4
                  - y3*z0 - y2*z1 - y1*z2 - y0*z3 - y4*z3 - y3*z4 + y4*z4 -␣
↪y3,
       -x4*y0 - x3*y1 - x2*y2 - x1*y3 - x0*y4 - x4*y4
                          - y4*z0 - y3*z1 - y2*z2 - y1*z3 - y0*z4 - y4*z4 -␣
↪y4)
 of Multivariate Polynomial Ring in x0, x1, x2, x3, x4, y0, y1, y2, y3, y4,
 z0, z1, z2, z3, z4 over Finite Field of size 3
>>> J += sage.rings.ideal.FieldIdeal(J.ring())  # ensure radical ideal
>>> J.variety()
[{z4: 0,
  z3: 0,
  z2: 0,
```

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```
  z1: 0,
  z0: 0,
  y4: 0,
  y3: 0,
  y2: 0,
  y1: 0,
  y0: 0,
  x4: 0,
  x3: 0,
  x2: 0,
  x1: 0,
  x0: 1}]
```

Weil restrictions are often used to study elliptic curves over extension fields so we give a simple example involving those:

```
sage: K.<a> = QuadraticField(1/3)                                        #␣
↪needs sage.rings.number_field
sage: E = EllipticCurve(K, [1,2,3,4,5])                                  #␣
↪needs sage.rings.number_field
```

```
>>> from sage.all import *
>>> K = QuadraticField(Integer(1)/Integer(3), names=('a',)); (a,) = K._first_
↪ngens(1)# needs sage.rings.number_field
>>> E = EllipticCurve(K, [Integer(1),Integer(2),Integer(3),Integer(4),
↪Integer(5)])                                       # needs sage.rings.number_
↪field
```

We pick a point on `E`:

```
sage: p = E.lift_x(1); p                                                 #␣
↪needs sage.rings.number_field
(1 : -6 : 1)

sage: I = E.defining_ideal(); I                                          #␣
↪needs sage.rings.number_field
Ideal (-x^3 - 2*x^2*z + x*y*z + y^2*z - 4*x*z^2 + 3*y*z^2 - 5*z^3)
 of Multivariate Polynomial Ring in x, y, z
  over Number Field in a with defining polynomial x^2 - 1/3
   with a = 0.5773502691896258?
```

```
>>> from sage.all import *
>>> p = E.lift_x(Integer(1)); p                                          ␣
↪     # needs sage.rings.number_field
(1 : -6 : 1)

>>> I = E.defining_ideal(); I                                          #␣
↪needs sage.rings.number_field
Ideal (-x^3 - 2*x^2*z + x*y*z + y^2*z - 4*x*z^2 + 3*y*z^2 - 5*z^3)
 of Multivariate Polynomial Ring in x, y, z
  over Number Field in a with defining polynomial x^2 - 1/3
   with a = 0.5773502691896258?
```

Of course, the point `p` is a root of all generators of `I`:

```
sage: I.subs(x=1, y=2, z=1)                                                    #␣
↪needs sage.rings.number_field
Ideal (0) of Multivariate Polynomial Ring in x, y, z
 over Number Field in a with defining polynomial x^2 - 1/3
  with a = 0.5773502691896258?
```

```
>>> from sage.all import *
>>> I.subs(x=Integer(1), y=Integer(2), z=Integer(1))                            ␣
↪                        # needs sage.rings.number_field
Ideal (0) of Multivariate Polynomial Ring in x, y, z
 over Number Field in a with defining polynomial x^2 - 1/3
  with a = 0.5773502691896258?
```

`I` is also radical:

```
sage: I.radical() == I                                                         #␣
↪needs sage.rings.number_field
True
```

```
>>> from sage.all import *
>>> I.radical() == I                                                          #␣
↪needs sage.rings.number_field
True
```

So we compute its Weil restriction:

```
sage: J = I.weil_restriction(); J                                              #␣
↪needs sage.rings.number_field
Ideal (-x0^3 - x0*x1^2 - 2*x0^2*z0 - 2/3*x1^2*z0 + x0*y0*z0 + y0^2*z0
         + 1/3*x1*y1*z0 + 1/3*y1^2*z0 - 4*x0*z0^2 + 3*y0*z0^2 - 5*z0^3
         - 4/3*x0*x1*z1 + 1/3*x1*y0*z1 + 1/3*x0*y1*z1 + 2/3*y0*y1*z1
         - 8/3*x1*z0*z1 + 2*y1*z0*z1 - 4/3*x0*z1^2 + y0*z1^2 - 5*z0*z1^2,
       -3*x0^2*x1 - 1/3*x1^3 - 4*x0*x1*z0 + x1*y0*z0 + x0*y1*z0
         + 2*y0*y1*z0 - 4*x1*z0^2 + 3*y1*z0^2 - 2*x0^2*z1 - 2/3*x1^2*z1
         + x0*y0*z1 + y0^2*z1 + 1/3*x1*y1*z1 + 1/3*y1^2*z1 - 8*x0*z0*z1
         + 6*y0*z0*z1 - 15*z0^2*z1 - 4/3*x1*z1^2 + y1*z1^2 - 5/3*z1^3)
of Multivariate Polynomial Ring in x0, x1, y0, y1, z0, z1 over Rational Field
```

```
>>> from sage.all import *
>>> J = I.weil_restriction(); J                                               #␣
↪needs sage.rings.number_field
Ideal (-x0^3 - x0*x1^2 - 2*x0^2*z0 - 2/3*x1^2*z0 + x0*y0*z0 + y0^2*z0
         + 1/3*x1*y1*z0 + 1/3*y1^2*z0 - 4*x0*z0^2 + 3*y0*z0^2 - 5*z0^3
         - 4/3*x0*x1*z1 + 1/3*x1*y0*z1 + 1/3*x0*y1*z1 + 2/3*y0*y1*z1
         - 8/3*x1*z0*z1 + 2*y1*z0*z1 - 4/3*x0*z1^2 + y0*z1^2 - 5*z0*z1^2,
       -3*x0^2*x1 - 1/3*x1^3 - 4*x0*x1*z0 + x1*y0*z0 + x0*y1*z0
         + 2*y0*y1*z0 - 4*x1*z0^2 + 3*y1*z0^2 - 2*x0^2*z1 - 2/3*x1^2*z1
         + x0*y0*z1 + y0^2*z1 + 1/3*x1*y1*z1 + 1/3*y1^2*z1 - 8*x0*z0*z1
         + 6*y0*z0*z1 - 15*z0^2*z1 - 4/3*x1*z1^2 + y1*z1^2 - 5/3*z1^3)
of Multivariate Polynomial Ring in x0, x1, y0, y1, z0, z1 over Rational Field
```

We can check that the point `p` is still a root of all generators of `J`:

```
sage: J.subs(x0=1, y0=2, z0=1, x1=0, y1=0, z1=0)                              #␣
↪needs sage.rings.number_field
Ideal (0, 0) of Multivariate Polynomial Ring in x0, x1, y0, y1, z0, z1
 over Rational Field
```

```
>>> from sage.all import *
>>> J.subs(x0=Integer(1), y0=Integer(2), z0=Integer(1), x1=Integer(0),␣
↪y1=Integer(0), z1=Integer(0))                              # needs sage.rings.
↪number_field
Ideal (0, 0) of Multivariate Polynomial Ring in x0, x1, y0, y1, z0, z1
 over Rational Field
```

Example for relative number fields:

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: K.<w> = NumberField(x^5 - 2)
sage: R.<x> = K[]
sage: L.<v> = K.extension(x^2 + 1)
sage: S.<x,y> = L[]
sage: I = S.ideal([y^2 - x^3 - 1])
sage: I.weil_restriction()
Ideal (-x0^3 + 3*x0*x1^2 + y0^2 - y1^2 - 1, -3*x0^2*x1 + x1^3 + 2*y0*y1) of
 Multivariate Polynomial Ring in x0, x1, y0, y1
  over Number Field in w with defining polynomial x^5 - 2
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> K = NumberField(x**Integer(5) - Integer(2), names=('w',)); (w,) = K._
↪first_ngens(1)
>>> R = K['x']; (x,) = R._first_ngens(1)
>>> L = K.extension(x**Integer(2) + Integer(1), names=('v',)); (v,) = L._
↪first_ngens(1)
>>> S = L['x, y']; (x, y,) = S._first_ngens(2)
>>> I = S.ideal([y**Integer(2) - x**Integer(3) - Integer(1)])
>>> I.weil_restriction()
Ideal (-x0^3 + 3*x0*x1^2 + y0^2 - y1^2 - 1, -3*x0^2*x1 + x1^3 + 2*y0*y1) of
 Multivariate Polynomial Ring in x0, x1, y0, y1
  over Number Field in w with defining polynomial x^5 - 2
```

Note

Based on a Singular implementation by Michael Brickenstein

class sage.rings.polynomial.multi_polynomial_ideal.**MPolynomialIdeal_macaulay2_repr**

Bases: `object`

An ideal in a multivariate polynomial ring, which has an underlying Macaulay2 ring associated to it.

EXAMPLES:

```
sage: R.<x,y,z,w> = PolynomialRing(ZZ, 4)
sage: I = ideal(x*y-z^2, y^2-w^2)
sage: I
Ideal (x*y - z^2, y^2 - w^2) of Multivariate Polynomial Ring in x, y, z, w over␣
↪Integer Ring
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, Integer(4), names=('x', 'y', 'z', 'w',)); (x, y, z, w,
↪) = R._first_ngens(4)
>>> I = ideal(x*y-z**Integer(2), y**Integer(2)-w**Integer(2))
>>> I
Ideal (x*y - z^2, y^2 - w^2) of Multivariate Polynomial Ring in x, y, z, w over␣
↪Integer Ring
```

class sage.rings.polynomial.multi_polynomial_ideal.**MPolynomialIdeal_magma_repr**

Bases: object

class sage.rings.polynomial.multi_polynomial_ideal.**MPolynomialIdeal_quotient**(*ring*, *gens*, *coerce=True*)

Bases: QuotientRingIdeal_generic, *MPolynomialIdeal*

An ideal in a quotient of a multivariate polynomial ring.

EXAMPLES:

```
sage: Q.<x,y,z,w> = QQ['x,y,z,w'].quotient(['x*y-z^2', 'y^2-w^2'])
sage: I = ideal(x + y^2 + z - 1)
sage: I
Ideal (w^2 + x + z - 1) of Quotient
 of Multivariate Polynomial Ring in x, y, z, w over Rational Field
 by the ideal (x*y - z^2, y^2 - w^2)
```

```
>>> from sage.all import *
>>> Q = QQ['x,y,z,w'].quotient(['x*y-z^2', 'y^2-w^2'], names=('x', 'y', 'z', 'w',
↪)); (x, y, z, w,) = Q._first_ngens(4)
>>> I = ideal(x + y**Integer(2) + z - Integer(1))
>>> I
Ideal (w^2 + x + z - 1) of Quotient
 of Multivariate Polynomial Ring in x, y, z, w over Rational Field
 by the ideal (x*y - z^2, y^2 - w^2)
```

reduce(*f*)

Reduce an element modulo a Gröbner basis for this ideal. This returns 0 if and only if the element is in this ideal. In any case, this reduction is unique up to monomial orders.

EXAMPLES:

```
sage: R.<T,U,V,W,X,Y,Z> = PolynomialRing(QQ, order='lex')
sage: I = R.ideal([T^2 + U^2 - 1, V^2 + W^2 - 1, X^2 + Y^2 + Z^2 - 1])
sage: Q.<t,u,v,w,x,y,z> = R.quotient(I)
sage: J = Q.ideal([u*v - x, u*w - y, t - z])
sage: J.reduce(t^2 - z^2)
0
```

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```
sage: J.reduce(u^2)
-z^2 + 1
sage: t^2 - z^2 in J
True
sage: u^2 in J
False
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, order='lex', names=('T', 'U', 'V', 'W', 'X', 'Y',
↪'Z',)); (T, U, V, W, X, Y, Z,) = R._first_ngens(7)
>>> I = R.ideal([T**Integer(2) + U**Integer(2) - Integer(1), V**Integer(2) +␣
↪W**Integer(2) - Integer(1), X**Integer(2) + Y**Integer(2) + Z**Integer(2) -␣
↪Integer(1)])
>>> Q = R.quotient(I, names=('t', 'u', 'v', 'w', 'x', 'y', 'z',)); (t, u, v,␣
↪w, x, y, z,) = Q._first_ngens(7)
>>> J = Q.ideal([u*v - x, u*w - y, t - z])
>>> J.reduce(t**Integer(2) - z**Integer(2))
0
>>> J.reduce(u**Integer(2))
-z^2 + 1
>>> t**Integer(2) - z**Integer(2) in J
True
>>> u**Integer(2) in J
False
```

class sage.rings.polynomial.multi_polynomial_ideal.**MPolynomialIdeal_singular_base_repr**

Bases: object

syzygy_module()

Compute the first syzygy (i.e., the module of relations of the given generators) of the ideal.

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: f = 2*x^2 + y
sage: g = y
sage: h = 2*f + g
sage: I = Ideal([f,g,h])
sage: M = I.syzygy_module(); M
[       -2        -1         1]
[       -y 2*x^2 + y         0]
sage: G = vector(I.gens())
sage: M*G
(0, 0)
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> f = Integer(2)*x**Integer(2) + y
>>> g = y
>>> h = Integer(2)*f + g
>>> I = Ideal([f,g,h])
>>> M = I.syzygy_module(); M
```

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```
[        -2         -1          1]
[        -y 2*x^2 + y          0]
>>> G = vector(I.gens())
>>> M*G
(0, 0)
```

ALGORITHM: Uses Singular's syz command

class sage.rings.polynomial.multi_polynomial_ideal.**MPolynomialIdeal_singular_repr**

Bases: *MPolynomialIdeal_singular_base_repr*

An ideal in a multivariate polynomial ring, which has an underlying Singular ring associated to it.

associated_primes(*algorithm='sy'*)

Return a list of the associated primes of primary ideals of which the intersection is I = `self`.

An ideal Q is called primary if it is a proper ideal of the ring R and if whenever $ab \in Q$ and $a \notin Q$ then $b^n \in Q$ for some $n \in \mathbf{Z}$.

If Q is a primary ideal of the ring R, then the radical ideal P of Q, i.e. $P = \{a \in R, a^n \in Q\}$ for some $n \in \mathbf{Z}$, is called the *associated prime* of Q.

If I is a proper ideal of the ring R then there exists a decomposition in primary ideals Q_i such that

- their intersection is I
- none of the Q_i contains the intersection of the rest, and
- the associated prime ideals of Q_i are pairwise different.

This method returns the associated primes of the Q_i.

INPUT:

- `algorithm` – string; one of
  - `'sy'` – (default) use the Shimoyama-Yokoyama algorithm
  - `'gtz'` – use the Gianni-Trager-Zacharias algorithm

OUTPUT: list of associated primes

EXAMPLES:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3, order='lex')
sage: p = z^2 + 1; q = z^3 + 2
sage: I = (p*q^2, y - z^2) * R
sage: pd = I.associated_primes(); sorted(pd, key=str)
[Ideal (z^2 + 1, y + 1) of Multivariate Polynomial Ring in x, y, z over
↪Rational Field,
 Ideal (z^3 + 2, y - z^2) of Multivariate Polynomial Ring in x, y, z over
↪Rational Field]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(3), order='lex', names=('x', 'y', 'z',));
↪(x, y, z,) = R._first_ngens(3)
>>> p = z**Integer(2) + Integer(1); q = z**Integer(3) + Integer(2)
>>> I = (p*q**Integer(2), y - z**Integer(2)) * R
>>> pd = I.associated_primes(); sorted(pd, key=str)
```

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```
[Ideal (z^2 + 1, y + 1) of Multivariate Polynomial Ring in x, y, z over␣
↪Rational Field,
 Ideal (z^3 + 2, y - z^2) of Multivariate Polynomial Ring in x, y, z over␣
↪Rational Field]
```

ALGORITHM:

Uses Singular.

REFERENCES:

- Thomas Becker and Volker Weispfenning. Groebner Bases - A Computational Approach To Commutative Algebra. Springer, New York 1993.

basis_is_groebner(*singular=None*)

Return `True` if the generators of this ideal (`self.gens()`) form a Groebner basis.

Let I be the set of generators of this ideal. The check is performed by trying to lift $Syz(LM(I))$ to $Syz(I)$ as I forms a Groebner basis if and only if for every element S in $Syz(LM(I))$:

$$S * G = \sum_{i=0}^{m} h_i g_i - - - - >_G 0.$$

ALGORITHM:

Uses Singular.

EXAMPLES:

```
sage: # needs sage.rings.finite_rings
sage: R.<a,b,c,d,e,f,g,h,i,j> = PolynomialRing(GF(127), 10)
sage: I = sage.rings.ideal.Cyclic(R, 4)
sage: I.basis_is_groebner()
False
sage: I2 = Ideal(I.groebner_basis())
sage: I2.basis_is_groebner()
True
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> R = PolynomialRing(GF(Integer(127)), Integer(10), names=('a', 'b', 'c', 'd
↪', 'e', 'f', 'g', 'h', 'i', 'j',)); (a, b, c, d, e, f, g, h, i, j,) = R._
↪first_ngens(10)
>>> I = sage.rings.ideal.Cyclic(R, Integer(4))
>>> I.basis_is_groebner()
False
>>> I2 = Ideal(I.groebner_basis())
>>> I2.basis_is_groebner()
True
```

A more complicated example:

```
sage: R.<U6,U5,U4,U3,U2, u6,u5,u4,u3,u2, h> = PolynomialRing(GF(7583))       #␣
↪needs sage.rings.finite_rings
sage: l = [u6 + u5 + u4 + u3 + u2 - 3791*h,                                  #␣
```

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```
↪needs sage.rings.finite_rings
....:       U6 + U5 + U4 + U3 + U2 - 3791*h,
....:       U2*u2 - h^2, U3*u3 - h^2, U4*u4 - h^2,
....:       U5*u4 + U5*u3 + U4*u3 + U5*u2 + U4*u2 + U3*u2 - 3791*U5*h
....:         - 3791*U4*h - 3791*U3*h - 3791*U2*h - 2842*h^2,
....:       U4*u5 + U3*u5 + U2*u5 + U3*u4 + U2*u4 + U2*u3 - 3791*u5*h
....:         - 3791*u4*h - 3791*u3*h - 3791*u2*h - 2842*h^2,
....:       U5*u5 - h^2, U4*U2*u3 + U5*U3*u2 + U4*U3*u2 + U3^2*u2 -␣
↪3791*U5*U3*h
....:         - 3791*U4*U3*h - 3791*U3^2*h - 3791*U5*U2*h- 3791*U4*U2*h +␣
↪U3*U2*h
....:         - 3791*U2^2*h - 3791*U4*u3*h - 3791*U4*u2*h - 3791*U3*u2*h
....:         - 2843*U5*h^2 + 1897*U4*h^2 - 946*U3*h^2 - 947*U2*h^2 + 2370*h^3,
....:       U3*u5*u4 + U2*u5*u4 + U3*u4^2 + U2*u4^2 + U2*u4*u3 - 3791*u5*u4*h
....:         - 3791*u4^2*h - 3791*u4*u3*h - 3791*u4*u2*h + u5*h^2 - 2842*u4*h^
↪2,
....:       U2*u5*u4*u3 + U2*u4^2*u3 + U2*u4*u3^2 - 3791*u5*u4*u3*h
....:         - 3791*u4^2*u3*h - 3791*u4*u3^2*h - 3791*u4*u3*u2*h + u5*u4*h^2
....:         + u4^2*h^2 + u5*u3*h^2 - 2842*u4*u3*h^2,
....:       U5^2*U4*u3 + U5*U4^2*u3 + U5^2*U4*u2 + U5*U4^2*u2 + U5^2*U3*u2
....:         + 2*U5*U4*U3*u2 + U5*U3^2*u2 - 3791*U5^2*U4*h - 3791*U5*U4^2*h
....:         - 3791*U5^2*U3*h + U5*U4*U3*h - 3791*U5*U3^2*h - 3791*U5^2*U2*h
....:         + U5*U4*U2*h+ U5*U3*U2*h - 3791*U5*U2^2*h - 3791*U5*U3*u2*h
....:         - 2842*U5^2*h^2 + 1897*U5*U4*h^2 - U4^2*h^2 - 947*U5*U3*h^2
....:         - U4*U3*h^2 - 948*U5*U2*h^2 - U4*U2*h^2 - 1422*U5*h^3 +␣
↪3791*U4*h^3,
....:       u5*u4*u3*u2*h + u4^2*u3*u2*h + u4*u3^2*u2*h + u4*u3*u2^2*h
....:         + 2*u5*u4*u3*h^2 + 2*u4^2*u3*h^2 + 2*u4*u3^2*h^2 + 2*u5*u4*u2*h^2
....:         + 2*u4^2*u2*h^2 + 2*u5*u3*u2*h^2 + 1899*u4*u3*u2*h^2,
....:       U5^2*U4*U3*u2 + U5*U4^2*U3*u2 + U5*U4*U3^2*u2 - 3791*U5^2*U4*U3*h
....:         - 3791*U5*U4^2*U3*h - 3791*U5*U4*U3^2*h - 3791*U5*U4*U3*U2*h
....:         + 3791*U5*U4*U3*u2*h + U5^2*U4*h^2 + U5*U4^2*h^2 + U5^2*U3*h^2
....:         - U4^2*U3*h^2 - U5*U3^2*h^2 - U4*U3^2*h^2 - U5*U4*U2*h^2 -␣
↪U5*U3*U2*h^2
....:         - U4*U3*U2*h^2 + 3791*U5*U4*h^3 + 3791*U5*U3*h^3 + 3791*U4*U3*h^
↪3,
....:       u4^2*u3*u2*h^2 + 1515*u5*u3^2*u2*h^2 + u4*u3^2*u2*h^2
....:         + 1515*u5*u4*u2^2*h^2 + 1515*u5*u3*u2^2*h^2 + u4*u3*u2^2*h^2
....:         + 1521*u5*u4*u3*h^3 - 3028*u4^2*u3*h^3 - 3028*u4*u3^2*h^3
....:         + 1521*u5*u4*u2*h^3 - 3028*u4^2*u2*h^3 + 1521*u5*u3*u2*h^3
....:         + 3420*u4*u3*u2*h^3,
....:       U5^2*U4*U3*U2*h + U5*U4^2*U3*U2*h + U5*U4*U3^2*U2*h + U5*U4*U3*U2^
↪2*h
....:         + 2*U5^2*U4*U3*h^2 + 2*U5*U4^2*U3*h^2 + 2*U5*U4*U3^2*h^2
....:         + 2*U5^2*U4*U2*h^2 + 2*U5*U4^2*U2*h^2 + 2*U5^2*U3*U2*h^2
....:         - 2*U4^2*U3*U2*h^2 - 2*U5*U3^2*U2*h^2 - 2*U4*U3^2*U2*h^2
....:         - 2*U5*U4*U2^2*h^2 - 2*U5*U3*U2^2*h^2 - 2*U4*U3*U2^2*h^2
....:         - U5*U4*U3*h^3 - U5*U4*U2*h^3 - U5*U3*U2*h^3 - U4*U3*U2*h^3]

sage: Ideal(l).basis_is_groebner()                                      #␣
↪needs sage.rings.finite_rings
False
```

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```
sage: gb = Ideal(l).groebner_basis()                                          #␣
↪needs sage.rings.finite_rings
sage: Ideal(gb).basis_is_groebner()                                           #␣
↪needs sage.rings.finite_rings
True
```

```
>>> from sage.all import *
>>> R = PolynomialRing(GF(Integer(7583)), names=('U6', 'U5', 'U4', 'U3', 'U2',
↪ 'u6', 'u5', 'u4', 'u3', 'u2', 'h',)); (U6, U5, U4, U3, U2, u6, u5, u4, u3,␣
↪u2, h,) = R._first_ngens(11)# needs sage.rings.finite_rings
>>> l = [u6 + u5 + u4 + u3 + u2 - Integer(3791)*h,                             ␣
↪     # needs sage.rings.finite_rings
...       U6 + U5 + U4 + U3 + U2 - Integer(3791)*h,
...       U2*u2 - h**Integer(2), U3*u3 - h**Integer(2), U4*u4 - h**Integer(2),
...       U5*u4 + U5*u3 + U4*u3 + U5*u2 + U4*u2 + U3*u2 - Integer(3791)*U5*h
...         - Integer(3791)*U4*h - Integer(3791)*U3*h - Integer(3791)*U2*h -␣
↪Integer(2842)*h**Integer(2),
...       U4*u5 + U3*u5 + U2*u5 + U3*u4 + U2*u4 + U2*u3 - Integer(3791)*u5*h
...         - Integer(3791)*u4*h - Integer(3791)*u3*h - Integer(3791)*u2*h -␣
↪Integer(2842)*h**Integer(2),
...       U5*u5 - h**Integer(2), U4*U2*u3 + U5*U3*u2 + U4*U3*u2 +␣
↪U3**Integer(2)*u2 - Integer(3791)*U5*U3*h
...         - Integer(3791)*U4*U3*h - Integer(3791)*U3**Integer(2)*h -␣
↪Integer(3791)*U5*U2*h- Integer(3791)*U4*U2*h + U3*U2*h
...         - Integer(3791)*U2**Integer(2)*h - Integer(3791)*U4*u3*h -␣
↪Integer(3791)*U4*u2*h - Integer(3791)*U3*u2*h
...         - Integer(2843)*U5*h**Integer(2) + Integer(1897)*U4*h**Integer(2) -
↪ Integer(946)*U3*h**Integer(2) - Integer(947)*U2*h**Integer(2) +␣
↪Integer(2370)*h**Integer(3),
...       U3*u5*u4 + U2*u5*u4 + U3*u4**Integer(2) + U2*u4**Integer(2) +␣
↪U2*u4*u3 - Integer(3791)*u5*u4*h
...         - Integer(3791)*u4**Integer(2)*h - Integer(3791)*u4*u3*h -␣
↪Integer(3791)*u4*u2*h + u5*h**Integer(2) - Integer(2842)*u4*h**Integer(2),
...       U2*u5*u4*u3 + U2*u4**Integer(2)*u3 + U2*u4*u3**Integer(2) -␣
↪Integer(3791)*u5*u4*u3*h
...         - Integer(3791)*u4**Integer(2)*u3*h -␣
↪Integer(3791)*u4*u3**Integer(2)*h - Integer(3791)*u4*u3*u2*h +␣
↪u5*u4*h**Integer(2)
...         + u4**Integer(2)*h**Integer(2) + u5*u3*h**Integer(2) -␣
↪Integer(2842)*u4*u3*h**Integer(2),
...       U5**Integer(2)*U4*u3 + U5*U4**Integer(2)*u3 + U5**Integer(2)*U4*u2 +␣
↪U5*U4**Integer(2)*u2 + U5**Integer(2)*U3*u2
...         + Integer(2)*U5*U4*U3*u2 + U5*U3**Integer(2)*u2 -␣
↪Integer(3791)*U5**Integer(2)*U4*h - Integer(3791)*U5*U4**Integer(2)*h
...         - Integer(3791)*U5**Integer(2)*U3*h + U5*U4*U3*h -␣
↪Integer(3791)*U5*U3**Integer(2)*h - Integer(3791)*U5**Integer(2)*U2*h
...         + U5*U4*U2*h+ U5*U3*U2*h - Integer(3791)*U5*U2**Integer(2)*h -␣
↪Integer(3791)*U5*U3*u2*h
...         - Integer(2842)*U5**Integer(2)*h**Integer(2) +␣
↪Integer(1897)*U5*U4*h**Integer(2) - U4**Integer(2)*h**Integer(2) -␣
↪Integer(947)*U5*U3*h**Integer(2)
...         - U4*U3*h**Integer(2) - Integer(948)*U5*U2*h**Integer(2) -␣
```

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```
↪U4*U2*h**Integer(2) - Integer(1422)*U5*h**Integer(3) +␣
↪Integer(3791)*U4*h**Integer(3),
...      u5*u4*u3*u2*h + u4**Integer(2)*u3*u2*h + u4*u3**Integer(2)*u2*h +␣
↪u4*u3*u2**Integer(2)*h
...        + Integer(2)*u5*u4*u3*h**Integer(2) +␣
↪Integer(2)*u4**Integer(2)*u3*h**Integer(2) +␣
↪Integer(2)*u4*u3**Integer(2)*h**Integer(2) +␣
↪Integer(2)*u5*u4*u2*h**Integer(2)
...        + Integer(2)*u4**Integer(2)*u2*h**Integer(2) +␣
↪Integer(2)*u5*u3*u2*h**Integer(2) + Integer(1899)*u4*u3*u2*h**Integer(2),
...      U5**Integer(2)*U4*U3*u2 + U5*U4**Integer(2)*U3*u2 +␣
↪U5*U4*U3**Integer(2)*u2 - Integer(3791)*U5**Integer(2)*U4*U3*h
...        - Integer(3791)*U5*U4**Integer(2)*U3*h -␣
↪Integer(3791)*U5*U4*U3**Integer(2)*h - Integer(3791)*U5*U4*U3*U2*h
...        + Integer(3791)*U5*U4*U3*u2*h + U5**Integer(2)*U4*h**Integer(2) +␣
↪U5*U4**Integer(2)*h**Integer(2) + U5**Integer(2)*U3*h**Integer(2)
...        - U4**Integer(2)*U3*h**Integer(2) -␣
↪U5*U3**Integer(2)*h**Integer(2) - U4*U3**Integer(2)*h**Integer(2) -␣
↪U5*U4*U2*h**Integer(2) - U5*U3*U2*h**Integer(2)
...        - U4*U3*U2*h**Integer(2) + Integer(3791)*U5*U4*h**Integer(3) +␣
↪Integer(3791)*U5*U3*h**Integer(3) + Integer(3791)*U4*U3*h**Integer(3),
...      u4**Integer(2)*u3*u2*h**Integer(2) +␣
↪Integer(1515)*u5*u3**Integer(2)*u2*h**Integer(2) +␣
↪u4*u3**Integer(2)*u2*h**Integer(2)
...        + Integer(1515)*u5*u4*u2**Integer(2)*h**Integer(2) +␣
↪Integer(1515)*u5*u3*u2**Integer(2)*h**Integer(2) +␣
↪u4*u3*u2**Integer(2)*h**Integer(2)
...        + Integer(1521)*u5*u4*u3*h**Integer(3) -␣
↪Integer(3028)*u4**Integer(2)*u3*h**Integer(3) -␣
↪Integer(3028)*u4*u3**Integer(2)*h**Integer(3)
...        + Integer(1521)*u5*u4*u2*h**Integer(3) -␣
↪Integer(3028)*u4**Integer(2)*u2*h**Integer(3) +␣
↪Integer(1521)*u5*u3*u2*h**Integer(3)
...        + Integer(3420)*u4*u3*u2*h**Integer(3),
...      U5**Integer(2)*U4*U3*U2*h + U5*U4**Integer(2)*U3*U2*h +␣
↪U5*U4*U3**Integer(2)*U2*h + U5*U4*U3*U2**Integer(2)*h
...        + Integer(2)*U5**Integer(2)*U4*U3*h**Integer(2) +␣
↪Integer(2)*U5*U4**Integer(2)*U3*h**Integer(2) +␣
↪Integer(2)*U5*U4*U3**Integer(2)*h**Integer(2)
...        + Integer(2)*U5**Integer(2)*U4*U2*h**Integer(2) +␣
↪Integer(2)*U5*U4**Integer(2)*U2*h**Integer(2) +␣
↪Integer(2)*U5**Integer(2)*U3*U2*h**Integer(2)
...        - Integer(2)*U4**Integer(2)*U3*U2*h**Integer(2) -␣
↪Integer(2)*U5*U3**Integer(2)*U2*h**Integer(2) -␣
↪Integer(2)*U4*U3**Integer(2)*U2*h**Integer(2)
...        - Integer(2)*U5*U4*U2**Integer(2)*h**Integer(2) -␣
↪Integer(2)*U5*U3*U2**Integer(2)*h**Integer(2) -␣
↪Integer(2)*U4*U3*U2**Integer(2)*h**Integer(2)
...        - U5*U4*U3*h**Integer(3) - U5*U4*U2*h**Integer(3) -␣
↪U5*U3*U2*h**Integer(3) - U4*U3*U2*h**Integer(3)]

>>> Ideal(l).basis_is_groebner()                                          #␣
```

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```
↪needs sage.rings.finite_rings
False
>>> gb = Ideal(l).groebner_basis()                                            #␣
↪needs sage.rings.finite_rings
>>> Ideal(gb).basis_is_groebner()                                             #␣
↪needs sage.rings.finite_rings
True
```

Note

From the Singular Manual for the reduce function we use in this method: 'The result may have no meaning if the second argument (`self`) is not a standard basis'. I (malb) believe this refers to the mathematical fact that the results may have no meaning if `self` is no standard basis, i.e., Singular doesn't 'add' any additional 'nonsense' to the result. So we may actually use reduce to determine if `self` is a Groebner basis.

complete_primary_decomposition()

A decorator that creates a cached version of an instance method of a class.

Note

For proper behavior, the method must be a pure function (no side effects). Arguments to the method must be hashable or transformed into something hashable using `key` or they must define `sage.structure.sage_object.SageObject._cache_key()`.

EXAMPLES:

```
sage: class Foo():
....:     @cached_method
....:     def f(self, t, x=2):
....:         print('computing')
....:         return t**x
sage: a = Foo()
```

```
>>> from sage.all import *
>>> class Foo():
...     @cached_method
...     def f(self, t, x=Integer(2)):
...         print('computing')
...         return t**x
>>> a = Foo()
```

The example shows that the actual computation takes place only once, and that the result is identical for equivalent input:

```
sage: res = a.f(3, 2); res
computing
9
sage: a.f(t = 3, x = 2) is res
True
```

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```
sage: a.f(3) is res
True
```

```
>>> from sage.all import *
>>> res = a.f(Integer(3), Integer(2)); res
computing
9
>>> a.f(t = Integer(3), x = Integer(2)) is res
True
>>> a.f(Integer(3)) is res
True
```

Note, however, that accessing the attribute directly will call `__get__()`, and returns a `CachedMethodCaller` or `CachedMethodCallerNoArgs`.

```
sage: P.<a,b,c,d> = QQ[]
sage: I = P*[a,b]
sage: type(I.__class__.gens)
<class 'sage.misc.cachefunc.CachedMethodCallerNoArgs'>
sage: type(I.__class__.__dict__["gens"])
<class 'sage.misc.cachefunc.CachedMethod'>
```

```
>>> from sage.all import *
>>> P = QQ['a, b, c, d']; (a, b, c, d,) = P._first_ngens(4)
>>> I = P*[a,b]
>>> type(I.__class__.gens)
<class 'sage.misc.cachefunc.CachedMethodCallerNoArgs'>
>>> type(I.__class__.__dict__["gens"])
<class 'sage.misc.cachefunc.CachedMethod'>
```

The parameter `key` can be used to pass a function which creates a custom cache key for inputs. In the following example, this parameter is used to ignore the `algorithm` keyword for caching:

```
sage: class A():
....:     def _f_normalize(self, x, algorithm): return x
....:     @cached_method(key=_f_normalize)
....:     def f(self, x, algorithm='default'): return x
sage: a = A()
sage: a.f(1, algorithm='default') is a.f(1) is a.f(1, algorithm='algorithm')
True
```

```
>>> from sage.all import *
>>> class A():
...     def _f_normalize(self, x, algorithm): return x
...     @cached_method(key=_f_normalize)
...     def f(self, x, algorithm='default'): return x
>>> a = A()
>>> a.f(Integer(1), algorithm='default') is a.f(Integer(1)) is a.f(Integer(1),
↪ algorithm='algorithm')
True
```

The parameter `do_pickle` can be used to enable pickling of the cache. Usually the cache is not stored when pickling:

```
sage: class A():
....:     @cached_method
....:     def f(self, x): return None
sage: import __main__
sage: __main__.A = A
sage: a = A()
sage: a.f(1)
sage: len(a.f.cache)
1
sage: b = loads(dumps(a))
sage: len(b.f.cache)
0
```

```
>>> from sage.all import *
>>> class A():
...     @cached_method
...     def f(self, x): return None
>>> import __main__
>>> __main__.A = A
>>> a = A()
>>> a.f(Integer(1))
>>> len(a.f.cache)
1
>>> b = loads(dumps(a))
>>> len(b.f.cache)
0
```

When `do_pickle` is set, the pickle contains the contents of the cache:

```
sage: class A():
....:     @cached_method(do_pickle=True)
....:     def f(self, x): return None
sage: __main__.A = A
sage: a = A()
sage: a.f(1)
sage: len(a.f.cache)
1
sage: b = loads(dumps(a))
sage: len(b.f.cache)
1
```

```
>>> from sage.all import *
>>> class A():
...     @cached_method(do_pickle=True)
...     def f(self, x): return None
>>> __main__.A = A
>>> a = A()
>>> a.f(Integer(1))
>>> len(a.f.cache)
1
>>> b = loads(dumps(a))
```

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```
>>> len(b.f.cache)
1
```

Cached methods cannot be copied like usual methods, see Issue #12603. Copying them can lead to very surprising results:

```
sage: class A:
....:     @cached_method
....:     def f(self):
....:         return 1
sage: class B:
....:     g=A.f
....:     def f(self):
....:         return 2

sage: b=B()
sage: b.f()
2
sage: b.g()
1
sage: b.f()
1
```

```
>>> from sage.all import *
>>> class A:
...     @cached_method
...     def f(self):
...         return Integer(1)
>>> class B:
...     g=A.f
...     def f(self):
...         return Integer(2)

>>> b=B()
>>> b.f()
2
>>> b.g()
1
>>> b.f()
1
```

dimension(*singular=None*)

The dimension of the ring modulo this ideal.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(GF(32003), order='degrevlex')              #␣
↪needs sage.rings.finite_rings
sage: I = ideal(x^2 - y, x^3)                                                  #␣
↪needs sage.rings.finite_rings
sage: I.dimension()                                                            #␣
↪needs sage.rings.finite_rings
1
```

```
>>> from sage.all import *
>>> P = PolynomialRing(GF(Integer(32003)), order='degrevlex', names=('x', 'y',
↪ 'z',)); (x, y, z,) = P._first_ngens(3)# needs sage.rings.finite_rings
>>> I = ideal(x**Integer(2) - y, x**Integer(3))                              ␣
↪                # needs sage.rings.finite_rings
>>> I.dimension()                                                           #␣
↪needs sage.rings.finite_rings
1
```

If the ideal is the total ring, the dimension is -1 by convention.

ALGORITHM:

For principal ideals, Theorem 3.5.1 of [Ger2008] is used. Otherwise Singular is used, unless the characteristic is too large. This requires computation of a Groebner basis, which can be very expensive.

For polynomials over a finite field of order too large for Singular, this falls back on a toy implementation of Buchberger to compute the Groebner basis, then uses the algorithm described in Chapter 9, Section 1 of Cox, Little, and O'Shea's "Ideals, Varieties, and Algorithms".

EXAMPLES:

```
sage: # needs sage.rings.finite_rings
sage: R.<x,y> = PolynomialRing(GF(2147483659^2), order='lex')
sage: I = R.ideal([x*y, x*y + 1])
sage: I.dimension()
verbose 0 (...: multi_polynomial_ideal.py, dimension) Warning: falling back␣
↪to very slow toy implementation.
-1
sage: I=ideal([x*(x*y+1), y*(x*y+1)])
sage: I.dimension()
verbose 0 (...: multi_polynomial_ideal.py, dimension) Warning: falling back␣
↪to very slow toy implementation.
1
sage: I = R.ideal([x^3*y, x*y^2])
sage: I.dimension()
verbose 0 (...: multi_polynomial_ideal.py, dimension) Warning: falling back␣
↪to very slow toy implementation.
1
sage: R.<x,y> = PolynomialRing(GF(2147483659^2), order='lex')
sage: I = R.ideal(0)
sage: I.dimension()
2
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> R = PolynomialRing(GF(Integer(2147483659)**Integer(2)), order='lex',␣
↪names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> I = R.ideal([x*y, x*y + Integer(1)])
>>> I.dimension()
verbose 0 (...: multi_polynomial_ideal.py, dimension) Warning: falling back␣
↪to very slow toy implementation.
-1
>>> I=ideal([x*(x*y+Integer(1)), y*(x*y+Integer(1))])
>>> I.dimension()
```

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```
verbose 0 (...: multi_polynomial_ideal.py, dimension) Warning: falling back␣
↪to very slow toy implementation.
1
>>> I = R.ideal([x**Integer(3)*y, x*y**Integer(2)])
>>> I.dimension()
verbose 0 (...: multi_polynomial_ideal.py, dimension) Warning: falling back␣
↪to very slow toy implementation.
1
>>> R = PolynomialRing(GF(Integer(2147483659)**Integer(2)), order='lex',␣
↪names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> I = R.ideal(Integer(0))
>>> I.dimension()
2
```

elimination_ideal(*variables*, *algorithm=None*, **args*, ***kwds*)

Return the elimination ideal of this ideal with respect to the variables given in `variables`.

INPUT:

- `variables` – list or tuple of variables in `self.ring()`
- `algorithm` – determines the algorithm to use, see below for available algorithms

ALGORITHMS:

- `'libsingular:eliminate'` – libSingular's `eliminate` command (default)
- `'giac:eliminate'` – Giac's `eliminate` command (if available)

If only a system is given - e.g. 'giac' - the default algorithm is chosen for that system.

EXAMPLES:

```
sage: R.<x,y,t,s,z> = PolynomialRing(QQ,5)
sage: I = R * [x - t, y - t^2, z - t^3, s - x + y^3]
sage: J = I.elimination_ideal([t, s]); J
Ideal (y^2 - x*z, x*y - z, x^2 - y)
 of Multivariate Polynomial Ring in x, y, t, s, z over Rational Field
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ,Integer(5), names=('x', 'y', 't', 's', 'z',)); (x,␣
↪y, t, s, z,) = R._first_ngens(5)
>>> I = R * [x - t, y - t**Integer(2), z - t**Integer(3), s - x +␣
↪y**Integer(3)]
>>> J = I.elimination_ideal([t, s]); J
Ideal (y^2 - x*z, x*y - z, x^2 - y)
 of Multivariate Polynomial Ring in x, y, t, s, z over Rational Field
```

You can use Giac to compute the elimination ideal:

```
sage: # needs sage.libs.giac
sage: print("possible output from giac", flush=True); I.elimination_ideal([t,␣
↪s], algorithm='giac') == J
possible output...
True
```

```
>>> from sage.all import *
>>> # needs sage.libs.giac
>>> print("possible output from giac", flush=True); I.elimination_ideal([t,␣
↪s], algorithm='giac') == J
possible output...
True
```

The list of available Giac options is provided at `sage.libs.giac.groebner_basis()`.

ALGORITHM:

Uses Singular, or Giac (if available).

> **Note**
>
> Requires computation of a Groebner basis, which can be a very expensive operation.

free_resolution(**args, **kwds*)

Return a free resolution of `self`.

For input options, see `FreeResolution`.

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: f = 2*x^2 + y
sage: g = y
sage: h = 2*f + g
sage: I = R.ideal([f,g,h])
sage: res = I.free_resolution()
sage: res
S^1 <-- S^2 <-- S^1 <-- 0
sage: ascii_art(res.chain_complex())
                            [-x^2]
            [   y x^2]      [    y]
 0 <-- C_0 <---------- C_1 <------- C_2 <-- 0

sage: q = ZZ['q'].fraction_field().gen()
sage: R.<x,y,z> = q.parent()[]
sage: I = R.ideal([x^2+y^2+z^2, x*y*z^4])
sage: I.free_resolution()
Traceback (most recent call last):
...
NotImplementedError: the ring must be a polynomial ring using Singular
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> f = Integer(2)*x**Integer(2) + y
>>> g = y
>>> h = Integer(2)*f + g
>>> I = R.ideal([f,g,h])
>>> res = I.free_resolution()
>>> res
```

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```
S^1 <-- S^2 <-- S^1 <-- 0
>>> ascii_art(res.chain_complex())
                                [-x^2]
             [  y x^2]          [   y]
 0 <-- C_0 <---------- C_1 <------- C_2 <-- 0

>>> q = ZZ['q'].fraction_field().gen()
>>> R = q.parent()['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> I = R.ideal([x**Integer(2)+y**Integer(2)+z**Integer(2),␣
↪x*y*z**Integer(4)])
>>> I.free_resolution()
Traceback (most recent call last):
...
NotImplementedError: the ring must be a polynomial ring using Singular
```

genus()

A decorator that creates a cached version of an instance method of a class.

> **Note**
>
> For proper behavior, the method must be a pure function (no side effects). Arguments to the method must be hashable or transformed into something hashable using `key` or they must define `sage.structure.sage_object.SageObject._cache_key()`.

EXAMPLES:

```
sage: class Foo():
....:     @cached_method
....:     def f(self, t, x=2):
....:         print('computing')
....:         return t**x
sage: a = Foo()
```

```
>>> from sage.all import *
>>> class Foo():
...     @cached_method
...     def f(self, t, x=Integer(2)):
...         print('computing')
...         return t**x
>>> a = Foo()
```

The example shows that the actual computation takes place only once, and that the result is identical for equivalent input:

```
sage: res = a.f(3, 2); res
computing
9
sage: a.f(t = 3, x = 2) is res
True
sage: a.f(3) is res
True
```

```
>>> from sage.all import *
>>> res = a.f(Integer(3), Integer(2)); res
computing
9
>>> a.f(t = Integer(3), x = Integer(2)) is res
True
>>> a.f(Integer(3)) is res
True
```

Note, however, that accessing the attribute directly will call `__get__()`, and returns a `CachedMethodCaller` or `CachedMethodCallerNoArgs`.

```
sage: P.<a,b,c,d> = QQ[]
sage: I = P*[a,b]
sage: type(I.__class__.gens)
<class 'sage.misc.cachefunc.CachedMethodCallerNoArgs'>
sage: type(I.__class__.__dict__["gens"])
<class 'sage.misc.cachefunc.CachedMethod'>
```

```
>>> from sage.all import *
>>> P = QQ['a, b, c, d']; (a, b, c, d,) = P._first_ngens(4)
>>> I = P*[a,b]
>>> type(I.__class__.gens)
<class 'sage.misc.cachefunc.CachedMethodCallerNoArgs'>
>>> type(I.__class__.__dict__["gens"])
<class 'sage.misc.cachefunc.CachedMethod'>
```

The parameter `key` can be used to pass a function which creates a custom cache key for inputs. In the following example, this parameter is used to ignore the `algorithm` keyword for caching:

```
sage: class A():
....:     def _f_normalize(self, x, algorithm): return x
....:     @cached_method(key=_f_normalize)
....:     def f(self, x, algorithm='default'): return x
sage: a = A()
sage: a.f(1, algorithm='default') is a.f(1) is a.f(1, algorithm='algorithm')
True
```

```
>>> from sage.all import *
>>> class A():
...     def _f_normalize(self, x, algorithm): return x
...     @cached_method(key=_f_normalize)
...     def f(self, x, algorithm='default'): return x
>>> a = A()
>>> a.f(Integer(1), algorithm='default') is a.f(Integer(1)) is a.f(Integer(1),
↪ algorithm='algorithm')
True
```

The parameter `do_pickle` can be used to enable pickling of the cache. Usually the cache is not stored when pickling:

```
sage: class A():
....:     @cached_method
```

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```
....:     def f(self, x): return None
sage: import __main__
sage: __main__.A = A
sage: a = A()
sage: a.f(1)
sage: len(a.f.cache)
1
sage: b = loads(dumps(a))
sage: len(b.f.cache)
0
```

```
>>> from sage.all import *
>>> class A():
...     @cached_method
...     def f(self, x): return None
>>> import __main__
>>> __main__.A = A
>>> a = A()
>>> a.f(Integer(1))
>>> len(a.f.cache)
1
>>> b = loads(dumps(a))
>>> len(b.f.cache)
0
```

When `do_pickle` is set, the pickle contains the contents of the cache:

```
sage: class A():
....:     @cached_method(do_pickle=True)
....:     def f(self, x): return None
sage: __main__.A = A
sage: a = A()
sage: a.f(1)
sage: len(a.f.cache)
1
sage: b = loads(dumps(a))
sage: len(b.f.cache)
1
```

```
>>> from sage.all import *
>>> class A():
...     @cached_method(do_pickle=True)
...     def f(self, x): return None
>>> __main__.A = A
>>> a = A()
>>> a.f(Integer(1))
>>> len(a.f.cache)
1
>>> b = loads(dumps(a))
>>> len(b.f.cache)
1
```

Cached methods cannot be copied like usual methods, see Issue #12603. Copying them can lead to very surprising results:

```
sage: class A:
....:     @cached_method
....:     def f(self):
....:         return 1
sage: class B:
....:     g=A.f
....:     def f(self):
....:         return 2

sage: b=B()
sage: b.f()
2
sage: b.g()
1
sage: b.f()
1
```

```
>>> from sage.all import *
>>> class A:
...     @cached_method
...     def f(self):
...         return Integer(1)
>>> class B:
...     g=A.f
...     def f(self):
...         return Integer(2)

>>> b=B()
>>> b.f()
2
>>> b.g()
1
>>> b.f()
1
```

graded_free_resolution(**args*, ***kwds*)

Return a graded free resolution of `self`.

For input options, see `GradedFreeResolution`.

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: f = 2*x^2 + y^2
sage: g = y^2
sage: h = 2*f + g
sage: I = R.ideal([f,g,h])
sage: I.graded_free_resolution(shifts=[1])
S(-1) <-- S(-3)⊕S(-3) <-- S(-5) <-- 0

sage: f = 2*x^2 + y
```

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```
sage: g = y
sage: h = 2*f + g
sage: I = R.ideal([f,g,h])
sage: I.graded_free_resolution(degrees=[1,2])
S(0) <-- S(-2)⊕S(-2) <-- S(-4) <-- 0

sage: q = ZZ['q'].fraction_field().gen()
sage: R.<x,y,z> = q.parent()[]
sage: I = R.ideal([x^2+y^2+z^2, x*y*z^4])
sage: I.graded_free_resolution()
Traceback (most recent call last):
...
NotImplementedError: the ring must be a polynomial ring using Singular
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> f = Integer(2)*x**Integer(2) + y**Integer(2)
>>> g = y**Integer(2)
>>> h = Integer(2)*f + g
>>> I = R.ideal([f,g,h])
>>> I.graded_free_resolution(shifts=[Integer(1)])
S(-1) <-- S(-3)⊕S(-3) <-- S(-5) <-- 0

>>> f = Integer(2)*x**Integer(2) + y
>>> g = y
>>> h = Integer(2)*f + g
>>> I = R.ideal([f,g,h])
>>> I.graded_free_resolution(degrees=[Integer(1),Integer(2)])
S(0) <-- S(-2)⊕S(-2) <-- S(-4) <-- 0

>>> q = ZZ['q'].fraction_field().gen()
>>> R = q.parent()['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> I = R.ideal([x**Integer(2)+y**Integer(2)+z**Integer(2),␣
↪x*y*z**Integer(4)])
>>> I.graded_free_resolution()
Traceback (most recent call last):
...
NotImplementedError: the ring must be a polynomial ring using Singular
```

hilbert_numerator(*grading=None*, *algorithm='sage'*)

Return the Hilbert numerator of this ideal.

INPUT:

- `grading` – (optional) a list or tuple of integers
- `algorithm` – (default: `'sage'`) must be either `'sage'` or `'singular'`

Let I (which is `self`) be a homogeneous ideal and $R = \bigoplus_d R_d$ (which is `self.ring()`) be a graded commutative algebra over a field K. Then the *Hilbert function* is defined as $H(d) = \dim_K R_d$ and the *Hilbert series* of I is defined as the formal power series $HS(t) = \sum_{d=0}^{\infty} H(d)t^d$.

This power series can be expressed as $HS(t) = Q(t)/(1-t)^n$ where $Q(t)$ is a polynomial over $\mathbf{Z}$ and n the number of variables in R. This method returns $Q(t)$, the numerator; hence the name, `hilbert_numerator`. An optional `grading` can be given, in which case the graded (or weighted) Hilbert numerator is given.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: I = Ideal([x^3*y^2 + 3*x^2*y^2*z + y^3*z^2 + z^5])
sage: I.hilbert_numerator()                                                 #␣
↪needs sage.libs.flint
-t^5 + 1
sage: R.<a,b> = PolynomialRing(QQ)
sage: J = R.ideal([a^2*b, a*b^2])
sage: J.hilbert_numerator()                                                 #␣
↪needs sage.libs.flint
t^4 - 2*t^3 + 1
sage: J.hilbert_numerator(grading=(10,3))                                   #␣
↪needs sage.libs.flint
t^26 - t^23 - t^16 + 1
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = P._first_
↪ngens(3)
>>> I = Ideal([x**Integer(3)*y**Integer(2) +␣
↪Integer(3)*x**Integer(2)*y**Integer(2)*z + y**Integer(3)*z**Integer(2) +␣
↪z**Integer(5)])
>>> I.hilbert_numerator()                                                 #␣
↪needs sage.libs.flint
-t^5 + 1
>>> R = PolynomialRing(QQ, names=('a', 'b',)); (a, b,) = R._first_ngens(2)
>>> J = R.ideal([a**Integer(2)*b, a*b**Integer(2)])
>>> J.hilbert_numerator()                                                 #␣
↪needs sage.libs.flint
t^4 - 2*t^3 + 1
>>> J.hilbert_numerator(grading=(Integer(10),Integer(3)))                   ␣
↪               # needs sage.libs.flint
t^26 - t^23 - t^16 + 1
```

hilbert_polynomial(*algorithm='sage'*)

Return the Hilbert polynomial of this ideal.

INPUT:

- `algorithm` – (default: `'sage'`) must be either `'sage'` or `'singular'`

Let I (which is `self`) be a homogeneous ideal and $R = \bigoplus_d R_d$ (which is `self.ring()`) be a graded commutative algebra over a field K. The *Hilbert polynomial* is the unique polynomial $HP(t)$ with rational coefficients such that $HP(d) = \dim_K R_d$ for all but finitely many positive integers d.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: I = Ideal([x^3*y^2 + 3*x^2*y^2*z + y^3*z^2 + z^5])
sage: I.hilbert_polynomial()                                                #␣
↪needs sage.libs.flint
5*t - 5
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = P._first_
```

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```
↪ngens(3)
>>> I = Ideal([x**Integer(3)*y**Integer(2) +␣
↪Integer(3)*x**Integer(2)*y**Integer(2)*z + y**Integer(3)*z**Integer(2) +␣
↪z**Integer(5)])
>>> I.hilbert_polynomial()                                              #␣
↪needs sage.libs.flint
5*t - 5
```

Of course, the Hilbert polynomial of a zero-dimensional ideal is zero:

```
sage: J0 = Ideal([x^3*y^2 + 3*x^2*y^2*z + y^3*z^2 + z^5,
....:             y^3 - 2*x*z^2 + x*y, x^4 + x*y - y*z^2])
sage: J = P*[m.lm() for m in J0.groebner_basis()]
sage: J.dimension()
0
sage: J.hilbert_polynomial()                                            #␣
↪needs sage.libs.flint
0
```

```
>>> from sage.all import *
>>> J0 = Ideal([x**Integer(3)*y**Integer(2) +␣
↪Integer(3)*x**Integer(2)*y**Integer(2)*z + y**Integer(3)*z**Integer(2) +␣
↪z**Integer(5),
...             y**Integer(3) - Integer(2)*x*z**Integer(2) + x*y,␣
↪x**Integer(4) + x*y - y*z**Integer(2)])
>>> J = P*[m.lm() for m in J0.groebner_basis()]
>>> J.dimension()
0
>>> J.hilbert_polynomial()                                              #␣
↪needs sage.libs.flint
0
```

It is possible to request a computation using the Singular library:

```
sage: I.hilbert_polynomial(algorithm='singular') == I.hilbert_polynomial()   #␣
↪needs sage.libs.flint
True
sage: J.hilbert_polynomial(algorithm='singular') == J.hilbert_polynomial()   #␣
↪needs sage.libs.flint
True
```

```
>>> from sage.all import *
>>> I.hilbert_polynomial(algorithm='singular') == I.hilbert_polynomial()   #␣
↪needs sage.libs.flint
True
>>> J.hilbert_polynomial(algorithm='singular') == J.hilbert_polynomial()   #␣
↪needs sage.libs.flint
True
```

Here is a bigger examples:

```
sage: n = 4; m = 11; P = PolynomialRing(QQ, n * m, "x"); x = P.gens()
sage: M = Matrix(n, x)
sage: Minors = P.ideal(M.minors(2))
sage: hp = Minors.hilbert_polynomial(); hp                                  #␣
↪needs sage.libs.flint
1/21772800*t^13 + 61/21772800*t^12 + 1661/21772800*t^11
 + 26681/21772800*t^10 + 93841/7257600*t^9 + 685421/7257600*t^8
 + 1524809/3110400*t^7 + 39780323/21772800*t^6 + 6638071/1360800*t^5
 + 12509761/1360800*t^4 + 2689031/226800*t^3 + 1494509/151200*t^2
 + 12001/2520*t + 1
```

```
>>> from sage.all import *
>>> n = Integer(4); m = Integer(11); P = PolynomialRing(QQ, n * m, "x"); x =␣
↪P.gens()
>>> M = Matrix(n, x)
>>> Minors = P.ideal(M.minors(Integer(2)))
>>> hp = Minors.hilbert_polynomial(); hp                                  #␣
↪needs sage.libs.flint
1/21772800*t^13 + 61/21772800*t^12 + 1661/21772800*t^11
 + 26681/21772800*t^10 + 93841/7257600*t^9 + 685421/7257600*t^8
 + 1524809/3110400*t^7 + 39780323/21772800*t^6 + 6638071/1360800*t^5
 + 12509761/1360800*t^4 + 2689031/226800*t^3 + 1494509/151200*t^2
 + 12001/2520*t + 1
```

Because Singular uses 32-bit integers, the above example would fail with Singular. We don't test it here, as it has a side-effect on other tests that is not understood yet (see Issue #26300):

```
sage: Minors.hilbert_polynomial(algorithm='singular')        # not tested
Traceback (most recent call last):
...
RuntimeError: error in Singular function call 'hilbPoly':
int overflow in hilb 1
error occurred in or before poly.lib::hilbPoly line 58: `   intvec v=hilb(I,
↪2);`
expected intvec-expression. type 'help intvec;'
```

```
>>> from sage.all import *
>>> Minors.hilbert_polynomial(algorithm='singular')        # not tested
Traceback (most recent call last):
...
RuntimeError: error in Singular function call 'hilbPoly':
int overflow in hilb 1
error occurred in or before poly.lib::hilbPoly line 58: `   intvec v=hilb(I,
↪2);`
expected intvec-expression. type 'help intvec;'
```

Note that in this example, the Hilbert polynomial gives the coefficients of the Hilbert-Poincaré series in all degrees:

```
sage: P = PowerSeriesRing(QQ, 't', default_prec=50)
sage: hs = Minors.hilbert_series()                                  #␣
↪needs sage.libs.flint
sage: list(P(hs.numerator()) / P(hs.denominator())) == [hp(t=k)          #␣
```

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```
↪needs sage.libs.flint
....:                                                   for k in range(50)]
True
```

```
>>> from sage.all import *
>>> P = PowerSeriesRing(QQ, 't', default_prec=Integer(50))
>>> hs = Minors.hilbert_series()                                              #␣
↪needs sage.libs.flint
>>> list(P(hs.numerator()) / P(hs.denominator())) == [hp(t=k)                 #␣
↪needs sage.libs.flint
...                                                   for k in␣
↪range(Integer(50))]
True
```

hilbert_series(*grading=None*, *algorithm='sage'*)

Return the Hilbert series of this ideal.

INPUT:

- `grading` – (optional) a list or tuple of integers
- `algorithm` – (default: `'sage'`) must be either `'sage'` or `'singular'`

Let I (which is `self`) be a homogeneous ideal and $R = \bigoplus_d R_d$ (which is `self.ring()`) be a graded commutative algebra over a field K. Then the *Hilbert function* is defined as $H(d) = \dim_K R_d$ and the *Hilbert series* of I is defined as the formal power series $HS(t) = \sum_{d=0}^{\infty} H(d)t^d$.

This power series can be expressed as $HS(t) = Q(t)/(1-t)^n$ where $Q(t)$ is a polynomial over $\mathbf{Z}$ and n the number of variables in R. This method returns $Q(t)/(1-t)^n$, normalised so that the leading monomial of the numerator is positive.

An optional `grading` can be given, in which case the graded (or weighted) Hilbert series is given.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: I = Ideal([x^3*y^2 + 3*x^2*y^2*z + y^3*z^2 + z^5])
sage: I.hilbert_series()                                                       #␣
↪needs sage.libs.flint
(t^4 + t^3 + t^2 + t + 1)/(t^2 - 2*t + 1)
sage: R.<a,b> = PolynomialRing(QQ)
sage: J = R.ideal([a^2*b, a*b^2])
sage: J.hilbert_series()                                                       #␣
↪needs sage.libs.flint
(t^3 - t^2 - t - 1)/(t - 1)
sage: J.hilbert_series(grading=(10,3))                                         #␣
↪needs sage.libs.flint
(t^25 + t^24 + t^23 - t^15 - t^14 - t^13 - t^12 - t^11
 - t^10 - t^9 - t^8 - t^7 - t^6 - t^5 - t^4 - t^3 - t^2
 - t - 1)/(t^12 + t^11 + t^10 - t^2 - t - 1)

sage: K = R.ideal([a^2*b^3, a*b^4 + a^3*b^2])
sage: K.hilbert_series(grading=[1,2])                                          #␣
↪needs sage.libs.flint
(t^11 + t^8 - t^6 - t^5 - t^4 - t^3 - t^2 - t - 1)/(t^2 - 1)
```

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```
sage: K.hilbert_series(grading=[2,1])                                         #␣
↪needs sage.libs.flint
(2*t^7 - t^6 - t^4 - t^2 - 1)/(t - 1)
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = P._first_
↪ngens(3)
>>> I = Ideal([x**Integer(3)*y**Integer(2) +␣
↪Integer(3)*x**Integer(2)*y**Integer(2)*z + y**Integer(3)*z**Integer(2) +␣
↪z**Integer(5)])
>>> I.hilbert_series()                                                       #␣
↪needs sage.libs.flint
(t^4 + t^3 + t^2 + t + 1)/(t^2 - 2*t + 1)
>>> R = PolynomialRing(QQ, names=('a', 'b',)); (a, b,) = R._first_ngens(2)
>>> J = R.ideal([a**Integer(2)*b, a*b**Integer(2)])
>>> J.hilbert_series()                                                       #␣
↪needs sage.libs.flint
(t^3 - t^2 - t - 1)/(t - 1)
>>> J.hilbert_series(grading=(Integer(10),Integer(3)))                          ␣
↪               # needs sage.libs.flint
(t^25 + t^24 + t^23 - t^15 - t^14 - t^13 - t^12 - t^11
 - t^10 - t^9 - t^8 - t^7 - t^6 - t^5 - t^4 - t^3 - t^2
 - t - 1)/(t^12 + t^11 + t^10 - t^2 - t - 1)

>>> K = R.ideal([a**Integer(2)*b**Integer(3), a*b**Integer(4) +␣
↪a**Integer(3)*b**Integer(2)])
>>> K.hilbert_series(grading=[Integer(1),Integer(2)])                           ␣
↪               # needs sage.libs.flint
(t^11 + t^8 - t^6 - t^5 - t^4 - t^3 - t^2 - t - 1)/(t^2 - 1)
>>> K.hilbert_series(grading=[Integer(2),Integer(1)])                           ␣
↪               # needs sage.libs.flint
(2*t^7 - t^6 - t^4 - t^2 - 1)/(t - 1)
```

integral_closure(*p=0*, *r=True*, *singular=None*)

Let I = `self`.

Return the integral closure of $I, ..., I^p$, where sI is an ideal in the polynomial ring $R = k[x(1), ...x(n)]$. If p is not given, or $p = 0$, compute the closure of all powers up to the maximum degree in t occurring in the closure of $R[It]$ (so this is the last power whose closure is not just the sum/product of the smaller). If r is given and `r is True`, `I.integral_closure()` starts with a check whether I is already a radical ideal.

INPUT:

- `p` – powers of I (default: 0)
- `r` – check whether `self` is a radical ideal first (default: `True`)

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: I = ideal([x^2, x*y^4, y^5])
sage: I.integral_closure()
[x^2, x*y^4, y^5, x*y^3]
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> I = ideal([x**Integer(2), x*y**Integer(4), y**Integer(5)])
>>> I.integral_closure()
[x^2, x*y^4, y^5, x*y^3]
```

ALGORITHM:

Uses libSINGULAR.

interreduced_basis()

If this ideal is spanned by $(f_1, ..., f_n)$, return $(g_1, ..., g_s)$ such that:

- $(f_1, ..., f_n) = (g_1, ..., g_s)$
- $LT(g_i) \neq LT(g_j)$ for all $i \neq j$
- $LT(g_i)$ does not divide m for all monomials m of $\{g_1, ..., g_{i-1}, g_{i+1}, ..., g_s\}$
- $LC(g_i) = 1$ for all i if the coefficient ring is a field.

EXAMPLES:

```
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: I = Ideal([z*x + y^3, z + y^3, z + x*y])
sage: I.interreduced_basis()
[y^3 + z, x*y + z, x*z - z]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> I = Ideal([z*x + y**Integer(3), z + y**Integer(3), z + x*y])
>>> I.interreduced_basis()
[y^3 + z, x*y + z, x*z - z]
```

Note that tail reduction for local orderings is not well-defined:

```
sage: R.<x,y,z> = PolynomialRing(QQ,order='negdegrevlex')
sage: I = Ideal([z*x + y^3, z + y^3, z + x*y])
sage: I.interreduced_basis()
[z + x*y, x*y - y^3, x^2*y - y^3]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ,order='negdegrevlex', names=('x', 'y', 'z',)); (x,␣
↪y, z,) = R._first_ngens(3)
>>> I = Ideal([z*x + y**Integer(3), z + y**Integer(3), z + x*y])
>>> I.interreduced_basis()
[z + x*y, x*y - y^3, x^2*y - y^3]
```

A fixed error with nonstandard base fields:

```
sage: R.<t> = QQ['t']
sage: K.<x,y> = R.fraction_field()['x,y']
sage: I = t*x * K
sage: I.interreduced_basis()
[x]
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> K = R.fraction_field()['x,y']; (x, y,) = K._first_ngens(2)
>>> I = t*x * K
>>> I.interreduced_basis()
[x]
```

The interreduced basis of 0 is 0:

```
sage: P.<x,y,z> = GF(2)[]
sage: Ideal(P(0)).interreduced_basis()
[0]
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> Ideal(P(Integer(0))).interreduced_basis()
[0]
```

ALGORITHM:

Uses Singular's `interred` command or *sage.rings.polynomial.toy_buchberger.inter_reduction()* if conversion to Singular fails.

intersection(**others*)

Return the intersection of the arguments with this ideal.

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQ, 2, order='lex')
sage: I = x*R
sage: J = y*R
sage: I.intersection(J)
Ideal (x*y) of Multivariate Polynomial Ring in x, y over Rational Field
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(2), order='lex', names=('x', 'y',)); (x, y,
↪) = R._first_ngens(2)
>>> I = x*R
>>> J = y*R
>>> I.intersection(J)
Ideal (x*y) of Multivariate Polynomial Ring in x, y over Rational Field
```

The following simple example illustrates that the product need not equal the intersection.

```
sage: I = (x^2, y) * R
sage: J = (y^2, x) * R
sage: K = I.intersection(J); K
Ideal (y^2, x*y, x^2) of Multivariate Polynomial Ring in x, y over Rational␣
↪Field
sage: IJ = I*J; IJ
Ideal (x^2*y^2, x^3, y^3, x*y)
 of Multivariate Polynomial Ring in x, y over Rational Field
sage: IJ == K
False
```

```
>>> from sage.all import *
>>> I = (x**Integer(2), y) * R
>>> J = (y**Integer(2), x) * R
>>> K = I.intersection(J); K
Ideal (y^2, x*y, x^2) of Multivariate Polynomial Ring in x, y over Rational␣
↪Field
>>> IJ = I*J; IJ
Ideal (x^2*y^2, x^3, y^3, x*y)
 of Multivariate Polynomial Ring in x, y over Rational Field
>>> IJ == K
False
```

Intersection of several ideals:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3, order='lex')
sage: I1 = x * R
sage: I2 = y * R
sage: I3 = (x, y) * R
sage: I4 = (x^2 + x*y*z, y^2 - z^3*y, z^3 + y^5*x*z) * R
sage: I1.intersection(I2, I3, I4).groebner_basis()
[x^2*y + x*y*z^4, x*y^2 - x*y*z^3, x*y*z^20 - x*y*z^3]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(3), order='lex', names=('x', 'y', 'z',));␣
↪(x, y, z,) = R._first_ngens(3)
>>> I1 = x * R
>>> I2 = y * R
>>> I3 = (x, y) * R
>>> I4 = (x**Integer(2) + x*y*z, y**Integer(2) - z**Integer(3)*y,␣
↪z**Integer(3) + y**Integer(5)*x*z) * R
>>> I1.intersection(I2, I3, I4).groebner_basis()
[x^2*y + x*y*z^4, x*y^2 - x*y*z^3, x*y*z^20 - x*y*z^3]
```

The ideals must share the same ring:

```
sage: R2.<x,y> = PolynomialRing(QQ, 2, order='lex')
sage: R3.<x,y,z> = PolynomialRing(QQ, 3, order='lex')
sage: I2 = x*R2
sage: I3 = x*R3
sage: I2.intersection(I3)
Traceback (most recent call last):
...
TypeError: Intersection is only available for ideals of the same ring.
```

```
>>> from sage.all import *
>>> R2 = PolynomialRing(QQ, Integer(2), order='lex', names=('x', 'y',)); (x,␣
↪y,) = R2._first_ngens(2)
>>> R3 = PolynomialRing(QQ, Integer(3), order='lex', names=('x', 'y', 'z',));␣
↪(x, y, z,) = R3._first_ngens(3)
>>> I2 = x*R2
>>> I3 = x*R3
>>> I2.intersection(I3)
Traceback (most recent call last):
```

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```
...
TypeError: Intersection is only available for ideals of the same ring.
```

is_prime(***kwds*)

Return `True` if this ideal is prime.

INPUT:

- keyword arguments are passed on to `complete_primary_decomposition()`; in this way you can specify the algorithm to use.

EXAMPLES:

```
sage: R.<x, y> = PolynomialRing(QQ, 2)
sage: I = (x^2 - y^2 - 1) * R
sage: I.is_prime()
True
sage: (I^2).is_prime()
False

sage: J = (x^2 - y^2) * R
sage: J.is_prime()
False
sage: (J^3).is_prime()
False

sage: (I * J).is_prime()
False
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(2), names=('x', 'y',)); (x, y,) = R._first_
↪ngens(2)
>>> I = (x**Integer(2) - y**Integer(2) - Integer(1)) * R
>>> I.is_prime()
True
>>> (I**Integer(2)).is_prime()
False

>>> J = (x**Integer(2) - y**Integer(2)) * R
>>> J.is_prime()
False
>>> (J**Integer(3)).is_prime()
False

>>> (I * J).is_prime()
False
```

The following is Issue #5982. Note that the quotient ring is not recognized as being a field at this time, so the fraction field is not the quotient ring itself:

```
sage: Q = R.quotient(I); Q
Quotient of Multivariate Polynomial Ring in x, y over Rational Field
 by the ideal (x^2 - y^2 - 1)
```

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```
sage: Q.fraction_field()
Fraction Field of
 Quotient of Multivariate Polynomial Ring in x, y over Rational Field
  by the ideal (x^2 - y^2 - 1)
```

```
>>> from sage.all import *
>>> Q = R.quotient(I); Q
Quotient of Multivariate Polynomial Ring in x, y over Rational Field
 by the ideal (x^2 - y^2 - 1)
>>> Q.fraction_field()
Fraction Field of
 Quotient of Multivariate Polynomial Ring in x, y over Rational Field
  by the ideal (x^2 - y^2 - 1)
```

minimal_associated_primes()

OUTPUT: list of prime ideals

EXAMPLES:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3, 'xyz')
sage: p = z^2 + 1; q = z^3 + 2
sage: I = (p*q^2, y - z^2) * R
sage: sorted(I.minimal_associated_primes(), key=str)
[Ideal (z^2 + 1, -z^2 + y)
  of Multivariate Polynomial Ring in x, y, z over Rational Field,
 Ideal (z^3 + 2, -z^2 + y)
  of Multivariate Polynomial Ring in x, y, z over Rational Field]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(3), 'xyz', names=('x', 'y', 'z',)); (x, y,␣
↪z,) = R._first_ngens(3)
>>> p = z**Integer(2) + Integer(1); q = z**Integer(3) + Integer(2)
>>> I = (p*q**Integer(2), y - z**Integer(2)) * R
>>> sorted(I.minimal_associated_primes(), key=str)
[Ideal (z^2 + 1, -z^2 + y)
  of Multivariate Polynomial Ring in x, y, z over Rational Field,
 Ideal (z^3 + 2, -z^2 + y)
  of Multivariate Polynomial Ring in x, y, z over Rational Field]
```

ALGORITHM:

Uses Singular.

normal_basis(*degree=None*, *algorithm='libsingular'*, *singular=None*)

Return a vector space basis of the quotient ring of this ideal.

INPUT:

- `degree` – integer (default: `None`)
- `algorithm` – string (default: `'libsingular'`); if not the default, this will use the `kbase()` or `weightKB()` command from Singular
- `singular` – the singular interpreter to use when `algorithm` is not `'libsingular'` (default: the default instance)

OUTPUT:

Monomials in the basis. If `degree` is given, only the monomials of the given degree are returned.

EXAMPLES:

```
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: I = R.ideal(x^2 + y^2 + z^2 - 4, x^2 + 2*y^2 - 5, x*z - 1)
sage: I.normal_basis()
[y*z^2, z^2, y*z, z, x*y, y, x, 1]
sage: I.normal_basis(algorithm='singular')
[y*z^2, z^2, y*z, z, x*y, y, x, 1]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> I = R.ideal(x**Integer(2) + y**Integer(2) + z**Integer(2) - Integer(4),␣
↪x**Integer(2) + Integer(2)*y**Integer(2) - Integer(5), x*z - Integer(1))
>>> I.normal_basis()
[y*z^2, z^2, y*z, z, x*y, y, x, 1]
>>> I.normal_basis(algorithm='singular')
[y*z^2, z^2, y*z, z, x*y, y, x, 1]
```

The result can be restricted to monomials of a chosen degree, which is particularly useful when the quotient ring is not finite-dimensional as a vector space.

```
sage: J = R.ideal(x^2 + y^2 + z^2 - 4, x^2 + 2*y^2 - 5)
sage: J.dimension()
1
sage: [J.normal_basis(d) for d in (0..3)]
[[1], [z, y, x], [z^2, y*z, x*z, x*y], [z^3, y*z^2, x*z^2, x*y*z]]
sage: [J.normal_basis(d, algorithm='singular') for d in (0..3)]
[[1], [z, y, x], [z^2, y*z, x*z, x*y], [z^3, y*z^2, x*z^2, x*y*z]]
```

```
>>> from sage.all import *
>>> J = R.ideal(x**Integer(2) + y**Integer(2) + z**Integer(2) - Integer(4),␣
↪x**Integer(2) + Integer(2)*y**Integer(2) - Integer(5))
>>> J.dimension()
1
>>> [J.normal_basis(d) for d in (ellipsis_iter(Integer(0),Ellipsis,
↪Integer(3)))]
[[1], [z, y, x], [z^2, y*z, x*z, x*y], [z^3, y*z^2, x*z^2, x*y*z]]
>>> [J.normal_basis(d, algorithm='singular') for d in (ellipsis_
↪iter(Integer(0),Ellipsis,Integer(3)))]
[[1], [z, y, x], [z^2, y*z, x*z, x*y], [z^3, y*z^2, x*z^2, x*y*z]]
```

In case of a polynomial ring with a weighted term order, the degree of the monomials is taken with respect to the weights.

```
sage: T = TermOrder('wdegrevlex', (1, 2, 3))
sage: R.<x,y,z> = PolynomialRing(QQ, order=T)
sage: B = R.ideal(x*y^2 + x^5, z*y + x^3*y).normal_basis(9); B
[x^2*y^2*z, x^3*z^2, x*y*z^2, z^3]
sage: all(f.degree() == 9 for f in B)
True
```

```
>>> from sage.all import *
>>> T = TermOrder('wdegrevlex', (Integer(1), Integer(2), Integer(3)))
>>> R = PolynomialRing(QQ, order=T, names=('x', 'y', 'z',)); (x, y, z,) = R._
↪first_ngens(3)
>>> B = R.ideal(x*y**Integer(2) + x**Integer(5), z*y + x**Integer(3)*y).
↪normal_basis(Integer(9)); B
[x^2*y^2*z, x^3*z^2, x*y*z^2, z^3]
>>> all(f.degree() == Integer(9) for f in B)
True
```

plot(*singular=None*)

If you somehow manage to install surf, perhaps you can use this function to implicitly plot the real zero locus of this ideal (if principal).

INPUT:

- `self` – must be a principal ideal in 2 or 3 vars over $\mathbf{Q}$

EXAMPLES:

Implicit plotting in 2-d:

```
sage: R.<x,y> = PolynomialRing(QQ,2)
sage: I = R.ideal([y^3 - x^2])
sage: I.plot()        # cusp
Graphics object consisting of 1 graphics primitive
sage: I = R.ideal([y^2 - x^2 - 1])
sage: I.plot()        # hyperbola
Graphics object consisting of 1 graphics primitive
sage: I = R.ideal([y^2 + x^2*(1/4) - 1])
sage: I.plot()        # ellipse
Graphics object consisting of 1 graphics primitive
sage: I = R.ideal([y^2-(x^2-1)*(x-2)])
sage: I.plot()        # elliptic curve
Graphics object consisting of 1 graphics primitive
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ,Integer(2), names=('x', 'y',)); (x, y,) = R._first_
↪ngens(2)
>>> I = R.ideal([y**Integer(3) - x**Integer(2)])
>>> I.plot()        # cusp
Graphics object consisting of 1 graphics primitive
>>> I = R.ideal([y**Integer(2) - x**Integer(2) - Integer(1)])
>>> I.plot()        # hyperbola
Graphics object consisting of 1 graphics primitive
>>> I = R.ideal([y**Integer(2) + x**Integer(2)*(Integer(1)/Integer(4)) -␣
↪Integer(1)])
>>> I.plot()        # ellipse
Graphics object consisting of 1 graphics primitive
>>> I = R.ideal([y**Integer(2)-(x**Integer(2)-Integer(1))*(x-Integer(2))])
>>> I.plot()        # elliptic curve
Graphics object consisting of 1 graphics primitive
```

Implicit plotting in 3-d:

```
sage: R.<x,y,z> = PolynomialRing(QQ,3)
sage: I = R.ideal([y^2 + x^2*(1/4) - z])
sage: I.plot()          # a cone; optional - surf
sage: I = R.ideal([y^2 + z^2*(1/4) - x])
sage: I.plot()          # same code, from a different angle; optional - surf
sage: I = R.ideal([x^2*y^2+x^2*z^2+y^2*z^2-16*x*y*z])
sage: I.plot()          # Steiner surface; optional - surf
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ,Integer(3), names=('x', 'y', 'z',)); (x, y, z,) = R.
↪_first_ngens(3)
>>> I = R.ideal([y**Integer(2) + x**Integer(2)*(Integer(1)/Integer(4)) - z])
>>> I.plot()          # a cone; optional - surf
>>> I = R.ideal([y**Integer(2) + z**Integer(2)*(Integer(1)/Integer(4)) - x])
>>> I.plot()          # same code, from a different angle; optional - surf
>>> I = R.ideal([x**Integer(2)*y**Integer(2)+x**Integer(2)*z**Integer(2)+y**In
↪teger(2)*z**Integer(2)-Integer(16)*x*y*z])
>>> I.plot()          # Steiner surface; optional - surf
```

AUTHORS:

- David Joyner (2006-02-12)

primary_decomposition(*algorithm='sy'*)

Return a list of primary ideals such that their intersection is `self`.

An ideal Q is called primary if it is a proper ideal of the ring R, and if whenever $ab \in Q$ and $a \notin Q$, then $b^n \in Q$ for some $n \in \mathbf{Z}$.

If Q is a primary ideal of the ring R, then the radical ideal P of Q (i.e., the ideal consisting of all $a \in R$ with $a^n \in Q$ for some $n \in \mathbf{Z}$), is called the associated prime of Q.

If I is a proper ideal of a Noetherian ring R, then there exists a finite collection of primary ideals Q_i such that the following hold:

- the intersection of the Q_i is I;
- none of the Q_i contains the intersection of the others;
- the associated prime ideals of the Q_i are pairwise distinct.

INPUT:

- `algorithm` – string; one of
  - `'sy'` – (default) use the Shimoyama-Yokoyama algorithm
  - `'gtz'` – use the Gianni-Trager-Zacharias algorithm

OUTPUT:

- a list of primary ideals Q_i forming a primary decomposition of `self`.

EXAMPLES:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3, order='lex')
sage: p = z^2 + 1; q = z^3 + 2
sage: I = (p*q^2, y - z^2) * R
sage: pd = I.primary_decomposition(); sorted(pd, key=str)
[Ideal (z^2 + 1, y + 1)
```

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```
 of Multivariate Polynomial Ring in x, y, z over Rational Field,
Ideal (z^6 + 4*z^3 + 4, y - z^2)
 of Multivariate Polynomial Ring in x, y, z over Rational Field]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(3), order='lex', names=('x', 'y', 'z',));␣
↪(x, y, z,) = R._first_ngens(3)
>>> p = z**Integer(2) + Integer(1); q = z**Integer(3) + Integer(2)
>>> I = (p*q**Integer(2), y - z**Integer(2)) * R
>>> pd = I.primary_decomposition(); sorted(pd, key=str)
[Ideal (z^2 + 1, y + 1)
 of Multivariate Polynomial Ring in x, y, z over Rational Field,
Ideal (z^6 + 4*z^3 + 4, y - z^2)
 of Multivariate Polynomial Ring in x, y, z over Rational Field]
```

```
sage: from functools import reduce
sage: reduce(lambda Qi, Qj: Qi.intersection(Qj), pd) == I
True
```

```
>>> from sage.all import *
>>> from functools import reduce
>>> reduce(lambda Qi, Qj: Qi.intersection(Qj), pd) == I
True
```

ALGORITHM:

Uses Singular.

REFERENCES:

- Thomas Becker and Volker Weispfenning. Groebner Bases - A Computational Approach To Commutative Algebra. Springer, New York 1993.

primary_decomposition_complete()

A decorator that creates a cached version of an instance method of a class.

Note

For proper behavior, the method must be a pure function (no side effects). Arguments to the method must be hashable or transformed into something hashable using `key` or they must define `sage.structure.sage_object.SageObject._cache_key()`.

EXAMPLES:

```
sage: class Foo():
....:     @cached_method
....:     def f(self, t, x=2):
....:         print('computing')
....:         return t**x
sage: a = Foo()
```

```
>>> from sage.all import *
>>> class Foo():
...     @cached_method
...     def f(self, t, x=Integer(2)):
...         print('computing')
...         return t**x
>>> a = Foo()
```

The example shows that the actual computation takes place only once, and that the result is identical for equivalent input:

```
sage: res = a.f(3, 2); res
computing
9
sage: a.f(t = 3, x = 2) is res
True
sage: a.f(3) is res
True
```

```
>>> from sage.all import *
>>> res = a.f(Integer(3), Integer(2)); res
computing
9
>>> a.f(t = Integer(3), x = Integer(2)) is res
True
>>> a.f(Integer(3)) is res
True
```

Note, however, that accessing the attribute directly will call `__get__()`, and returns a `CachedMethodCaller` or `CachedMethodCallerNoArgs`.

```
sage: P.<a,b,c,d> = QQ[]
sage: I = P*[a,b]
sage: type(I.__class__.gens)
<class 'sage.misc.cachefunc.CachedMethodCallerNoArgs'>
sage: type(I.__class__.__dict__["gens"])
<class 'sage.misc.cachefunc.CachedMethod'>
```

```
>>> from sage.all import *
>>> P = QQ['a, b, c, d']; (a, b, c, d,) = P._first_ngens(4)
>>> I = P*[a,b]
>>> type(I.__class__.gens)
<class 'sage.misc.cachefunc.CachedMethodCallerNoArgs'>
>>> type(I.__class__.__dict__["gens"])
<class 'sage.misc.cachefunc.CachedMethod'>
```

The parameter `key` can be used to pass a function which creates a custom cache key for inputs. In the following example, this parameter is used to ignore the `algorithm` keyword for caching:

```
sage: class A():
....:     def _f_normalize(self, x, algorithm): return x
....:     @cached_method(key=_f_normalize)
....:     def f(self, x, algorithm='default'): return x
```

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```
sage: a = A()
sage: a.f(1, algorithm='default') is a.f(1) is a.f(1, algorithm='algorithm')
True
```

```
>>> from sage.all import *
>>> class A():
...     def _f_normalize(self, x, algorithm): return x
...     @cached_method(key=_f_normalize)
...     def f(self, x, algorithm='default'): return x
>>> a = A()
>>> a.f(Integer(1), algorithm='default') is a.f(Integer(1)) is a.f(Integer(1),
↪ algorithm='algorithm')
True
```

The parameter `do_pickle` can be used to enable pickling of the cache. Usually the cache is not stored when pickling:

```
sage: class A():
....:     @cached_method
....:     def f(self, x): return None
sage: import __main__
sage: __main__.A = A
sage: a = A()
sage: a.f(1)
sage: len(a.f.cache)
1
sage: b = loads(dumps(a))
sage: len(b.f.cache)
0
```

```
>>> from sage.all import *
>>> class A():
...     @cached_method
...     def f(self, x): return None
>>> import __main__
>>> __main__.A = A
>>> a = A()
>>> a.f(Integer(1))
>>> len(a.f.cache)
1
>>> b = loads(dumps(a))
>>> len(b.f.cache)
0
```

When `do_pickle` is set, the pickle contains the contents of the cache:

```
sage: class A():
....:     @cached_method(do_pickle=True)
....:     def f(self, x): return None
sage: __main__.A = A
sage: a = A()
sage: a.f(1)
```

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```
sage: len(a.f.cache)
1
sage: b = loads(dumps(a))
sage: len(b.f.cache)
1
```

```
>>> from sage.all import *
>>> class A():
...     @cached_method(do_pickle=True)
...     def f(self, x): return None
>>> __main__.A = A
>>> a = A()
>>> a.f(Integer(1))
>>> len(a.f.cache)
1
>>> b = loads(dumps(a))
>>> len(b.f.cache)
1
```

Cached methods cannot be copied like usual methods, see Issue #12603. Copying them can lead to very surprising results:

```
sage: class A:
....:     @cached_method
....:     def f(self):
....:         return 1
sage: class B:
....:     g=A.f
....:     def f(self):
....:         return 2

sage: b=B()
sage: b.f()
2
sage: b.g()
1
sage: b.f()
1
```

```
>>> from sage.all import *
>>> class A:
...     @cached_method
...     def f(self):
...         return Integer(1)
>>> class B:
...     g=A.f
...     def f(self):
...         return Integer(2)

>>> b=B()
>>> b.f()
2
```

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```
>>> b.g()
1
>>> b.f()
1
```

quotient(J)

Given ideals I = `self` and J in the same polynomial ring P, return the ideal quotient of I by J consisting of the polynomials a of P such that $\{aJ \subset I\}$.

This is also referred to as the colon ideal (I:J).

INPUT:

- `J` – multivariate polynomial ideal

EXAMPLES:

```
sage: # needs sage.rings.finite_rings
sage: R.<x,y,z> = PolynomialRing(GF(181), 3)
sage: I = Ideal([x^2 + x*y*z, y^2 - z^3*y, z^3 + y^5*x*z])
sage: J = Ideal([x])
sage: Q = I.quotient(J)
sage: y*z + x in I
False
sage: x in J
True
sage: x * (y*z + x) in I
True
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> R = PolynomialRing(GF(Integer(181)), Integer(3), names=('x', 'y', 'z',));
↪(x, y, z,) = R._first_ngens(3)
>>> I = Ideal([x**Integer(2) + x*y*z, y**Integer(2) - z**Integer(3)*y,
↪z**Integer(3) + y**Integer(5)*x*z])
>>> J = Ideal([x])
>>> Q = I.quotient(J)
>>> y*z + x in I
False
>>> x in J
True
>>> x * (y*z + x) in I
True
```

radical()

The radical of this ideal.

EXAMPLES:

This is an obviously not radical ideal:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: I = (x^2, y^3, (x*z)^4 + y^3 + 10*x^2) * R
sage: I.radical()
Ideal (y, x) of Multivariate Polynomial Ring in x, y, z over Rational Field
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) =␣
↪R._first_ngens(3)
>>> I = (x**Integer(2), y**Integer(3), (x*z)**Integer(4) + y**Integer(3) +␣
↪Integer(10)*x**Integer(2)) * R
>>> I.radical()
Ideal (y, x) of Multivariate Polynomial Ring in x, y, z over Rational Field
```

That the radical is correct is clear from the Groebner basis.

```
sage: I.groebner_basis()
[y^3, x^2]
```

```
>>> from sage.all import *
>>> I.groebner_basis()
[y^3, x^2]
```

This is the example from the Singular manual:

```
sage: p = z^2 + 1; q = z^3 + 2
sage: I = (p*q^2, y - z^2) * R
sage: I.radical()
Ideal (z^2 - y, y^2*z + y*z + 2*y + 2)
 of Multivariate Polynomial Ring in x, y, z over Rational Field
```

```
>>> from sage.all import *
>>> p = z**Integer(2) + Integer(1); q = z**Integer(3) + Integer(2)
>>> I = (p*q**Integer(2), y - z**Integer(2)) * R
>>> I.radical()
Ideal (z^2 - y, y^2*z + y*z + 2*y + 2)
 of Multivariate Polynomial Ring in x, y, z over Rational Field
```

Note

From the Singular manual: A combination of the algorithms of Krick/Logar and Kemper is used. Works also in positive characteristic (Kempers algorithm).

```
sage: # needs sage.rings.finite_rings
sage: R.<x,y,z> = PolynomialRing(GF(37), 3)
sage: p = z^2 + 1; q = z^3 + 2
sage: I = (p*q^2, y - z^2) * R
sage: I.radical()
Ideal (z^2 - y, y^2*z + y*z + 2*y + 2)
 of Multivariate Polynomial Ring in x, y, z over Finite Field of size 37
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> R = PolynomialRing(GF(Integer(37)), Integer(3), names=('x', 'y', 'z',));␣
↪(x, y, z,) = R._first_ngens(3)
>>> p = z**Integer(2) + Integer(1); q = z**Integer(3) + Integer(2)
```

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```
>>> I = (p*q**Integer(2), y - z**Integer(2)) * R
>>> I.radical()
Ideal (z^2 - y, y^2*z + y*z + 2*y + 2)
 of Multivariate Polynomial Ring in x, y, z over Finite Field of size 37
```

saturation(*other*)

Return the saturation (and saturation exponent) of the ideal `self` with respect to the ideal `other`.

INPUT:

- `other` – another ideal in the same ring

OUTPUT: a pair (ideal, integer)

EXAMPLES:

```
sage: R.<x, y, z> = QQ[]
sage: I = R.ideal(x^5*z^3, x*y*z, y*z^4)
sage: J = R.ideal(z)
sage: I.saturation(J)
(Ideal (y, x^5) of Multivariate Polynomial Ring in x, y, z over Rational␣
↪Field, 4)
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> I = R.ideal(x**Integer(5)*z**Integer(3), x*y*z, y*z**Integer(4))
>>> J = R.ideal(z)
>>> I.saturation(J)
(Ideal (y, x^5) of Multivariate Polynomial Ring in x, y, z over Rational␣
↪Field, 4)
```

syzygy_module()

Compute the first syzygy (i.e., the module of relations of the given generators) of the ideal.

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: f = 2*x^2 + y
sage: g = y
sage: h = 2*f + g
sage: I = Ideal([f,g,h])
sage: M = I.syzygy_module(); M
[       -2        -1         1]
[       -y 2*x^2 + y         0]
sage: G = vector(I.gens())
sage: M*G
(0, 0)
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> f = Integer(2)*x**Integer(2) + y
>>> g = y
>>> h = Integer(2)*f + g
>>> I = Ideal([f,g,h])
```

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```
>>> M = I.syzygy_module(); M
[       -2        -1         1]
[       -y 2*x^2 + y         0]
>>> G = vector(I.gens())
>>> M*G
(0, 0)
```

ALGORITHM:

Uses Singular's syz command.

transformed_basis(*algorithm='gwalk'*, *other_ring=None*, *singular=None*)

Return a lex or `other_ring` Groebner Basis for this ideal.

INPUT:

- `algorithm` – see below for options
- `other_ring` – only valid for `algorithm='fglm'`; if provided, conversion will be performed to this ring. Otherwise a lex Groebner basis will be returned.

ALGORITHMS:

- `'fglm'` – FGLM algorithm. The input ideal must be given with a reduced Groebner Basis of a zero-dimensional ideal
- `'gwalk'` – Groebner Walk algorithm (*default*)
- `'awalk1'` – 'first alternative' algorithm
- `'awalk2'` – 'second alternative' algorithm
- `'twalk'` – Tran algorithm
- `'fwalk'` – Fractal Walk algorithm

EXAMPLES:

```
sage: R.<x,y,z> = PolynomialRing(QQ,3)
sage: I = Ideal([y^3+x^2,x^2*y+x^2, x^3-x^2, z^4-x^2-y])
sage: I = Ideal(I.groebner_basis())
sage: S.<z,x,y> = PolynomialRing(QQ,3,order='lex')
sage: J = Ideal(I.transformed_basis('fglm',S))
sage: J
Ideal (z^4 + y^3 - y, x^2 + y^3, x*y^3 - y^3, y^4 + y^3)
 of Multivariate Polynomial Ring in z, x, y over Rational Field
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ,Integer(3), names=('x', 'y', 'z',)); (x, y, z,) = R.
↪_first_ngens(3)
>>> I = Ideal([y**Integer(3)+x**Integer(2),x**Integer(2)*y+x**Integer(2),␣
↪x**Integer(3)-x**Integer(2), z**Integer(4)-x**Integer(2)-y])
>>> I = Ideal(I.groebner_basis())
>>> S = PolynomialRing(QQ,Integer(3),order='lex', names=('z', 'x', 'y',)); (z,
↪ x, y,) = S._first_ngens(3)
>>> J = Ideal(I.transformed_basis('fglm',S))
>>> J
```

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```
Ideal (z^4 + y^3 - y, x^2 + y^3, x*y^3 - y^3, y^4 + y^3)
 of Multivariate Polynomial Ring in z, x, y over Rational Field
```

```
sage: R.<z,y,x> = PolynomialRing(GF(32003), 3, order='lex')                 #␣
↪needs sage.rings.finite_rings
sage: I = Ideal([y^3 + x*y*z + y^2*z + x*z^3, 3 + x*y + x^2*y + y^2*z])     #␣
↪needs sage.rings.finite_rings
sage: I.transformed_basis('gwalk')                                          #␣
↪needs sage.rings.finite_rings
[z*y^2 + y*x^2 + y*x + 3,
 z*x + 8297*y^8*x^2 + 8297*y^8*x + 3556*y^7 - 8297*y^6*x^4 + 15409*y^6*x^3
   - 8297*y^6*x^2 - 8297*y^5*x^5 + 15409*y^5*x^4 - 8297*y^5*x^3 + 3556*y^5*x^2
   + 3556*y^5*x + 3556*y^4*x^3 + 3556*y^4*x^2 - 10668*y^4 - 10668*y^3*x
   - 8297*y^2*x^9 - 1185*y^2*x^8 + 14224*y^2*x^7 - 1185*y^2*x^6 - 8297*y^2*x^5
   - 14223*y*x^7 - 10666*y*x^6 - 10666*y*x^5 - 14223*y*x^4 + x^5 + 2*x^4 + x^
↪3,
 y^9 - y^7*x^2 - y^7*x - y^6*x^3 - y^6*x^2 - 3*y^6 - 3*y^5*x - y^3*x^7
   - 3*y^3*x^6 - 3*y^3*x^5 - y^3*x^4 - 9*y^2*x^5 - 18*y^2*x^4 - 9*y^2*x^3
   - 27*y*x^3 - 27*y*x^2 - 27*x]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(GF(Integer(32003)), Integer(3), order='lex', names=('z
↪', 'y', 'x',)); (z, y, x,) = R._first_ngens(3)# needs sage.rings.finite_
↪rings
>>> I = Ideal([y**Integer(3) + x*y*z + y**Integer(2)*z + x*z**Integer(3),␣
↪Integer(3) + x*y + x**Integer(2)*y + y**Integer(2)*z])      # needs sage.
↪rings.finite_rings
>>> I.transformed_basis('gwalk')                                          #␣
↪needs sage.rings.finite_rings
[z*y^2 + y*x^2 + y*x + 3,
 z*x + 8297*y^8*x^2 + 8297*y^8*x + 3556*y^7 - 8297*y^6*x^4 + 15409*y^6*x^3
   - 8297*y^6*x^2 - 8297*y^5*x^5 + 15409*y^5*x^4 - 8297*y^5*x^3 + 3556*y^5*x^2
   + 3556*y^5*x + 3556*y^4*x^3 + 3556*y^4*x^2 - 10668*y^4 - 10668*y^3*x
   - 8297*y^2*x^9 - 1185*y^2*x^8 + 14224*y^2*x^7 - 1185*y^2*x^6 - 8297*y^2*x^5
   - 14223*y*x^7 - 10666*y*x^6 - 10666*y*x^5 - 14223*y*x^4 + x^5 + 2*x^4 + x^
↪3,
 y^9 - y^7*x^2 - y^7*x - y^6*x^3 - y^6*x^2 - 3*y^6 - 3*y^5*x - y^3*x^7
   - 3*y^3*x^6 - 3*y^3*x^5 - y^3*x^4 - 9*y^2*x^5 - 18*y^2*x^4 - 9*y^2*x^3
   - 27*y*x^3 - 27*y*x^2 - 27*x]
```

ALGORITHM:

Uses Singular.

triangular_decomposition(*algorithm=None*, *singular=None*)

Decompose zero-dimensional ideal `self` into triangular sets.

This requires that the given basis is reduced w.r.t. to the lexicographical monomial ordering. If the basis of `self` does not have this property, the required Groebner basis is computed implicitly.

INPUT:

- `algorithm` – string or `None` (default: `None`)

ALGORITHMS:

- `'singular:triangL'` – decomposition of `self` into triangular systems (Lazard)
- `'singular:triangLfak'` – decomposition of `self` into triangular systems plus factorization
- `'singular:triangM'` – decomposition of `self` into triangular systems (Moeller)

OUTPUT: list T of lists t such that the variety of `self` is the union of the varieties of t in L and each t is in triangular form

EXAMPLES:

```
sage: P.<e,d,c,b,a> = PolynomialRing(QQ, 5, order='lex')
sage: I = sage.rings.ideal.Cyclic(P)
sage: GB = Ideal(I.groebner_basis('libsingular:stdfglm'))
sage: GB.triangular_decomposition('singular:triangLfak')
[Ideal (a - 1, b - 1, c - 1, d^2 + 3*d + 1, e + d + 3) of Multivariate␣
↪Polynomial Ring in e, d, c, b, a over Rational Field,
 Ideal (a - 1, b - 1, c^2 + 3*c + 1, d + c + 3, e - 1) of Multivariate␣
↪Polynomial Ring in e, d, c, b, a over Rational Field,
 Ideal (a - 1, b^2 + 3*b + 1, c + b + 3, d - 1, e - 1) of Multivariate␣
↪Polynomial Ring in e, d, c, b, a over Rational Field,
 Ideal (a - 1, b^4 + b^3 + b^2 + b + 1, -c + b^2, -d + b^3, e + b^3 + b^2 + b␣
↪+ 1) of Multivariate
  Polynomial Ring in e, d, c, b, a over Rational Field,
 Ideal (a^2 + 3*a + 1, b - 1, c - 1, d - 1, e + a + 3) of Multivariate␣
↪Polynomial Ring in e, d, c, b, a over Rational Field,
 Ideal (a^2 + 3*a + 1, b + a + 3, c - 1, d - 1, e - 1) of Multivariate␣
↪Polynomial Ring in e, d, c, b, a over Rational Field,
 Ideal (a^4 - 4*a^3 + 6*a^2 + a + 1,
        -11*b^2 + 6*b*a^3 - 26*b*a^2 + 41*b*a - 4*b - 8*a^3 + 31*a^2 - 40*a -␣
↪24,
        11*c + 3*a^3 - 13*a^2 + 26*a - 2, 11*d + 3*a^3 - 13*a^2 + 26*a - 2,
        -11*e - 11*b + 6*a^3 - 26*a^2 + 41*a - 4) of Multivariate Polynomial␣
↪Ring in e, d, c, b, a over Rational Field,
 Ideal (a^4 + a^3 + a^2 + a + 1, b - a, c - a, d^2 + 3*d*a + a^2,
        e + d + 3*a) of Multivariate Polynomial Ring in e, d, c, b, a over␣
↪Rational Field,
 Ideal (a^4 + a^3 + a^2 + a + 1, b - a, c^2 + 3*c*a + a^2, d + c + 3*a,
        e - a) of Multivariate Polynomial Ring in e, d, c, b, a over Rational␣
↪Field,
 Ideal (a^4 + a^3 + a^2 + a + 1, b - 1, c + a^3 + a^2 + a + 1, -d + a^3,
        -e + a^2) of Multivariate Polynomial Ring in e, d, c, b, a over␣
↪Rational Field,
 Ideal (a^4 + a^3 + a^2 + a + 1, b^2 + 3*b*a + a^2, c + b + 3*a, d - a,
        e - a) of Multivariate Polynomial Ring in e, d, c, b, a over Rational␣
↪Field,
 Ideal (a^4 + a^3 + a^2 + a + 1, b^3 + b^2*a + b^2 + b*a^2 + b*a + b + a^3 +␣
↪a^2 + a + 1,
        c + b^2*a^3 + b^2*a^2 + b^2*a + b^2, -d + b^2*a^2 + b^2*a + b^2 + b*a^
↪2 + b*a + a^2,
        -e + b^2*a^3 - b*a^2 - b*a - b - a^2 - a) of Multivariate Polynomial␣
↪Ring in e, d, c, b, a over Rational Field,
 Ideal (a^4 + a^3 + 6*a^2 - 4*a + 1, -11*b^2 + 6*b*a^3 + 10*b*a^2 + 39*b*a +␣
↪2*b + 16*a^3 + 23*a^2 + 104*a - 24,
        11*c + 3*a^3 + 5*a^2 + 25*a + 1, 11*d + 3*a^3 + 5*a^2 + 25*a + 1, -
```

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```
↪11*e - 11*b + 6*a^3 + 10*a^2 + 39*a + 2)
      of Multivariate Polynomial Ring in e, d, c, b, a over Rational Field]

sage: R.<x1,x2> = PolynomialRing(QQ, 2, order='lex')
sage: f1 = 1/2*((x1^2 + 2*x1 - 4)*x2^2 + 2*(x1^2 + x1)*x2 + x1^2)
sage: f2 = 1/2*((x1^2 + 2*x1 + 1)*x2^2 + 2*(x1^2 + x1)*x2 - 4*x1^2)
sage: I = Ideal(f1,f2)
sage: I.triangular_decomposition()
[Ideal (x2, x1^2) of Multivariate Polynomial Ring in x1, x2 over Rational␣
↪Field,
 Ideal (x2, x1^2) of Multivariate Polynomial Ring in x1, x2 over Rational␣
↪Field,
 Ideal (x2, x1^2) of Multivariate Polynomial Ring in x1, x2 over Rational␣
↪Field,
 Ideal (x2^4 + 4*x2^3 - 6*x2^2 - 20*x2 + 5, 8*x1 - x2^3 + x2^2 + 13*x2 - 5)
  of Multivariate Polynomial Ring in x1, x2 over Rational Field]
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, Integer(5), order='lex', names=('e', 'd', 'c', 'b',
↪ 'a',)); (e, d, c, b, a,) = P._first_ngens(5)
>>> I = sage.rings.ideal.Cyclic(P)
>>> GB = Ideal(I.groebner_basis('libsingular:stdfglm'))
>>> GB.triangular_decomposition('singular:triangLfak')
[Ideal (a - 1, b - 1, c - 1, d^2 + 3*d + 1, e + d + 3) of Multivariate␣
↪Polynomial Ring in e, d, c, b, a over Rational Field,
 Ideal (a - 1, b - 1, c^2 + 3*c + 1, d + c + 3, e - 1) of Multivariate␣
↪Polynomial Ring in e, d, c, b, a over Rational Field,
 Ideal (a - 1, b^2 + 3*b + 1, c + b + 3, d - 1, e - 1) of Multivariate␣
↪Polynomial Ring in e, d, c, b, a over Rational Field,
 Ideal (a - 1, b^4 + b^3 + b^2 + b + 1, -c + b^2, -d + b^3, e + b^3 + b^2 + b␣
↪+ 1) of Multivariate
  Polynomial Ring in e, d, c, b, a over Rational Field,
 Ideal (a^2 + 3*a + 1, b - 1, c - 1, d - 1, e + a + 3) of Multivariate␣
↪Polynomial Ring in e, d, c, b, a over Rational Field,
 Ideal (a^2 + 3*a + 1, b + a + 3, c - 1, d - 1, e - 1) of Multivariate␣
↪Polynomial Ring in e, d, c, b, a over Rational Field,
 Ideal (a^4 - 4*a^3 + 6*a^2 + a + 1,
        -11*b^2 + 6*b*a^3 - 26*b*a^2 + 41*b*a - 4*b - 8*a^3 + 31*a^2 - 40*a -␣
↪24,
        11*c + 3*a^3 - 13*a^2 + 26*a - 2, 11*d + 3*a^3 - 13*a^2 + 26*a - 2,
        -11*e - 11*b + 6*a^3 - 26*a^2 + 41*a - 4) of Multivariate Polynomial␣
↪Ring in e, d, c, b, a over Rational Field,
 Ideal (a^4 + a^3 + a^2 + a + 1, b - a, c - a, d^2 + 3*d*a + a^2,
        e + d + 3*a) of Multivariate Polynomial Ring in e, d, c, b, a over␣
↪Rational Field,
 Ideal (a^4 + a^3 + a^2 + a + 1, b - a, c^2 + 3*c*a + a^2, d + c + 3*a,
        e - a) of Multivariate Polynomial Ring in e, d, c, b, a over Rational␣
↪Field,
 Ideal (a^4 + a^3 + a^2 + a + 1, b - 1, c + a^3 + a^2 + a + 1, -d + a^3,
        -e + a^2) of Multivariate Polynomial Ring in e, d, c, b, a over␣
↪Rational Field,
 Ideal (a^4 + a^3 + a^2 + a + 1, b^2 + 3*b*a + a^2, c + b + 3*a, d - a,
```

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```
        e - a) of Multivariate Polynomial Ring in e, d, c, b, a over Rational␣
↪Field,
 Ideal (a^4 + a^3 + a^2 + a + 1, b^3 + b^2*a + b^2 + b*a^2 + b*a + b + a^3 +␣
↪a^2 + a + 1,
        c + b^2*a^3 + b^2*a^2 + b^2*a + b^2, -d + b^2*a^2 + b^2*a + b^2 + b*a^
↪2 + b*a + a^2,
        -e + b^2*a^3 - b*a^2 - b*a - b - a^2 - a) of Multivariate Polynomial␣
↪Ring in e, d, c, b, a over Rational Field,
 Ideal (a^4 + a^3 + 6*a^2 - 4*a + 1, -11*b^2 + 6*b*a^3 + 10*b*a^2 + 39*b*a +␣
↪2*b + 16*a^3 + 23*a^2 + 104*a - 24,
        11*c + 3*a^3 + 5*a^2 + 25*a + 1, 11*d + 3*a^3 + 5*a^2 + 25*a + 1, -
↪11*e - 11*b + 6*a^3 + 10*a^2 + 39*a + 2)
       of Multivariate Polynomial Ring in e, d, c, b, a over Rational Field]

>>> R = PolynomialRing(QQ, Integer(2), order='lex', names=('x1', 'x2',)); (x1,
↪ x2,) = R._first_ngens(2)
>>> f1 = Integer(1)/Integer(2)*((x1**Integer(2) + Integer(2)*x1 -␣
↪Integer(4))*x2**Integer(2) + Integer(2)*(x1**Integer(2) + x1)*x2 +␣
↪x1**Integer(2))
>>> f2 = Integer(1)/Integer(2)*((x1**Integer(2) + Integer(2)*x1 +␣
↪Integer(1))*x2**Integer(2) + Integer(2)*(x1**Integer(2) + x1)*x2 -␣
↪Integer(4)*x1**Integer(2))
>>> I = Ideal(f1,f2)
>>> I.triangular_decomposition()
[Ideal (x2, x1^2) of Multivariate Polynomial Ring in x1, x2 over Rational␣
↪Field,
 Ideal (x2, x1^2) of Multivariate Polynomial Ring in x1, x2 over Rational␣
↪Field,
 Ideal (x2, x1^2) of Multivariate Polynomial Ring in x1, x2 over Rational␣
↪Field,
 Ideal (x2^4 + 4*x2^3 - 6*x2^2 - 20*x2 + 5, 8*x1 - x2^3 + x2^2 + 13*x2 - 5)
  of Multivariate Polynomial Ring in x1, x2 over Rational Field]
```

variety(*ring=None*)

Return the variety of this ideal.

Given a zero-dimensional ideal I (= `self`) of a polynomial ring P whose order is lexicographic, return the variety of I as a list of dictionaries with `(variable, value)` pairs. By default, the variety of the ideal over its coefficient field K is returned; `ring` can be specified to find the variety over a different ring.

These dictionaries have cardinality equal to the number of variables in P and represent assignments of values to these variables such that all polynomials in I vanish.

If `ring` is specified, then a triangular decomposition of `self` is found over the original coefficient field K; then the triangular systems are solved using root-finding over `ring`. This is particularly useful when K is `QQ` (to allow fast symbolic computation of the triangular decomposition) and `ring` is `RR`, `AA`, `CC`, or `QQbar` (to compute the whole real or complex variety of the ideal).

Note that with `ring=RR` or `CC`, computation is done numerically and potentially inaccurately; in particular, the number of points in the real variety may be miscomputed. With `ring=AA` or `QQbar`, computation is done exactly (which may be much slower, of course).

INPUT:

- `ring` – return roots in the `ring` instead of the base ring of this ideal (default: `None`)

- `algorithm` – algorithm or implementation to use; see below for supported values
- `proof` – return a provably correct result (default: `True`)

EXAMPLES:

```
sage: # needs sage.rings.finite_rings
sage: K.<w> = GF(27)  # this example is from the MAGMA handbook
sage: P.<x, y> = PolynomialRing(K, 2, order='lex')
sage: I = Ideal([x^8 + y + 2, y^6 + x*y^5 + x^2])
sage: I = Ideal(I.groebner_basis()); I
Ideal (x - y^47 - y^45 + y^44 - y^43 + y^41 - y^39 - y^38 - y^37 - y^36
         + y^35 - y^34 - y^33 + y^32 - y^31 + y^30 + y^28 + y^27 + y^26
         + y^25 - y^23 + y^22 + y^21 - y^19 - y^18 - y^16 + y^15 + y^13
         + y^12 - y^10 + y^9 + y^8 + y^7 - y^6 + y^4 + y^3 + y^2 + y - 1,
       y^48 + y^41 - y^40 + y^37 - y^36 - y^33 + y^32 - y^29 + y^28
         - y^25 + y^24 + y^2 + y + 1)
of Multivariate Polynomial Ring in x, y over Finite Field in w of size 3^3
sage: V = I.variety();
sage: sorted(V, key=str)
[{y: w^2 + 2*w, x: 2*w + 2}, {y: w^2 + 2, x: 2*w}, {y: w^2 + w, x: 2*w + 1}]
sage: [f.subs(v)                          # check that all polynomials vanish
....:  for f in I.gens() for v in V]
[0, 0, 0, 0, 0, 0]
sage: [I.subs(v).is_zero() for v in V]   # same test, but nicer syntax
[True, True, True]
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> K = GF(Integer(27), names=('w',)); (w,) = K._first_ngens(1)# this example␣
↪is from the MAGMA handbook
>>> P = PolynomialRing(K, Integer(2), order='lex', names=('x', 'y',)); (x, y,
↪) = P._first_ngens(2)
>>> I = Ideal([x**Integer(8) + y + Integer(2), y**Integer(6) +␣
↪x*y**Integer(5) + x**Integer(2)])
>>> I = Ideal(I.groebner_basis()); I
Ideal (x - y^47 - y^45 + y^44 - y^43 + y^41 - y^39 - y^38 - y^37 - y^36
         + y^35 - y^34 - y^33 + y^32 - y^31 + y^30 + y^28 + y^27 + y^26
         + y^25 - y^23 + y^22 + y^21 - y^19 - y^18 - y^16 + y^15 + y^13
         + y^12 - y^10 + y^9 + y^8 + y^7 - y^6 + y^4 + y^3 + y^2 + y - 1,
       y^48 + y^41 - y^40 + y^37 - y^36 - y^33 + y^32 - y^29 + y^28
         - y^25 + y^24 + y^2 + y + 1)
of Multivariate Polynomial Ring in x, y over Finite Field in w of size 3^3
>>> V = I.variety();
>>> sorted(V, key=str)
[{y: w^2 + 2*w, x: 2*w + 2}, {y: w^2 + 2, x: 2*w}, {y: w^2 + w, x: 2*w + 1}]
>>> [f.subs(v)                          # check that all polynomials vanish
...  for f in I.gens() for v in V]
[0, 0, 0, 0, 0, 0]
>>> [I.subs(v).is_zero() for v in V]   # same test, but nicer syntax
[True, True, True]
```

However, we only account for solutions in the ground field and not in the algebraic closure:

```
sage: I.vector_space_dimension()                                                #␣
↪needs sage.rings.finite_rings
48
```

```
>>> from sage.all import *
>>> I.vector_space_dimension()                                              #␣
↪needs sage.rings.finite_rings
48
```

Here we compute the points of intersection of a hyperbola and a circle, in several fields:

```
sage: K.<x, y> = PolynomialRing(QQ, 2, order='lex')
sage: I = Ideal([x*y - 1, (x-2)^2 + (y-1)^2 - 1])
sage: I = Ideal(I.groebner_basis()); I
Ideal (x + y^3 - 2*y^2 + 4*y - 4, y^4 - 2*y^3 + 4*y^2 - 4*y + 1)
 of Multivariate Polynomial Ring in x, y over Rational Field
```

```
>>> from sage.all import *
>>> K = PolynomialRing(QQ, Integer(2), order='lex', names=('x', 'y',)); (x, y,
↪) = K._first_ngens(2)
>>> I = Ideal([x*y - Integer(1), (x-Integer(2))**Integer(2) + (y-
↪Integer(1))**Integer(2) - Integer(1)])
>>> I = Ideal(I.groebner_basis()); I
Ideal (x + y^3 - 2*y^2 + 4*y - 4, y^4 - 2*y^3 + 4*y^2 - 4*y + 1)
 of Multivariate Polynomial Ring in x, y over Rational Field
```

These two curves have one rational intersection:

```
sage: I.variety()
[{y: 1, x: 1}]
```

```
>>> from sage.all import *
>>> I.variety()
[{y: 1, x: 1}]
```

There are two real intersections:

```
sage: sorted(I.variety(ring=RR), key=str)
[{y: 0.361103080528647, x: 2.76929235423863},
 {y: 1.00000000000000, x: 1.00000000000000}]
sage: I.variety(ring=AA)                                                        #␣
↪needs sage.rings.number_field
[{y: 1, x: 1},
 {y: 0.3611030805286474?, x: 2.769292354238632?}]
```

```
>>> from sage.all import *
>>> sorted(I.variety(ring=RR), key=str)
[{y: 0.361103080528647, x: 2.76929235423863},
 {y: 1.00000000000000, x: 1.00000000000000}]
>>> I.variety(ring=AA)                                                      #␣
↪needs sage.rings.number_field
[{y: 1, x: 1},
 {y: 0.3611030805286474?, x: 2.769292354238632?}]
```

and a total of four intersections:

```
sage: sorted(I.variety(ring=CC), key=str)
[{y: 0.31944845973567... + 1.6331702409152...*I,
  x: 0.11535382288068... - 0.58974280502220...*I},
 {y: 0.31944845973567... - 1.6331702409152...*I,
  x: 0.11535382288068... + 0.58974280502220...*I},
 {y: 0.36110308052864..., x: 2.7692923542386...},
 {y: 1.00000000000000, x: 1.00000000000000}]
sage: sorted(I.variety(ring=QQbar), key=str)                              #␣
↪needs sage.rings.number_field
[{y: 0.3194484597356763? + 1.633170240915238?*I,
  x: 0.1153538228806842? - 0.5897428050222055?*I},
 {y: 0.3194484597356763? - 1.633170240915238?*I,
  x: 0.1153538228806842? + 0.5897428050222055?*I},
 {y: 0.3611030805286474?, x: 2.769292354238632?},
 {y: 1, x: 1}]
```

```
>>> from sage.all import *
>>> sorted(I.variety(ring=CC), key=str)
[{y: 0.31944845973567... + 1.6331702409152...*I,
  x: 0.11535382288068... - 0.58974280502220...*I},
 {y: 0.31944845973567... - 1.6331702409152...*I,
  x: 0.11535382288068... + 0.58974280502220...*I},
 {y: 0.36110308052864..., x: 2.7692923542386...},
 {y: 1.00000000000000, x: 1.00000000000000}]
>>> sorted(I.variety(ring=QQbar), key=str)                              #␣
↪needs sage.rings.number_field
[{y: 0.3194484597356763? + 1.633170240915238?*I,
  x: 0.1153538228806842? - 0.5897428050222055?*I},
 {y: 0.3194484597356763? - 1.633170240915238?*I,
  x: 0.1153538228806842? + 0.5897428050222055?*I},
 {y: 0.3611030805286474?, x: 2.769292354238632?},
 {y: 1, x: 1}]
```

We can also use the optional package msolve to compute the variety. See `msolve` for more information.

```
sage: I.variety(RBF, algorithm='msolve', proof=False)    # optional - msolve
[{y: [0.361103080528647 +/- 4.53e-16], x: [2.76929235423863 +/- 2.08e-15]},
 {y: 1.000000000000000, x: 1.000000000000000}]
```

```
>>> from sage.all import *
>>> I.variety(RBF, algorithm='msolve', proof=False)    # optional - msolve
[{y: [0.361103080528647 +/- 4.53e-16], x: [2.76929235423863 +/- 2.08e-15]},
 {y: 1.000000000000000, x: 1.000000000000000}]
```

Computation over floating point numbers may compute only a partial solution, or even none at all. Notice that x values are missing from the following variety:

```
sage: R.<x,y> = CC[]
sage: I = ideal([x^2+y^2-1,x*y-1])
sage: sorted(I.variety(), key=str)
verbose 0 (...: multi_polynomial_ideal.py, variety) Warning: computations in␣
```

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```
↪the complex field are inexact; variety may be computed partially or␣
↪incorrectly.
verbose 0 (...: multi_polynomial_ideal.py, variety) Warning: falling back to␣
↪very slow toy implementation.
[{y: -0.86602540378443... + 0.500000000000000*I},
 {y: -0.86602540378443... - 0.500000000000000*I},
 {y: 0.86602540378443... + 0.500000000000000*I},
 {y: 0.86602540378443... - 0.500000000000000*I}]
```

```
>>> from sage.all import *
>>> R = CC['x, y']; (x, y,) = R._first_ngens(2)
>>> I = ideal([x**Integer(2)+y**Integer(2)-Integer(1),x*y-Integer(1)])
>>> sorted(I.variety(), key=str)
verbose 0 (...: multi_polynomial_ideal.py, variety) Warning: computations in␣
↪the complex field are inexact; variety may be computed partially or␣
↪incorrectly.
verbose 0 (...: multi_polynomial_ideal.py, variety) Warning: falling back to␣
↪very slow toy implementation.
[{y: -0.86602540378443... + 0.500000000000000*I},
 {y: -0.86602540378443... - 0.500000000000000*I},
 {y: 0.86602540378443... + 0.500000000000000*I},
 {y: 0.86602540378443... - 0.500000000000000*I}]
```

This is due to precision error, which causes the computation of an intermediate Groebner basis to fail.

If the ground field's characteristic is too large for Singular, we resort to a toy implementation:

```
sage: # needs sage.rings.finite_rings
sage: R.<x,y> = PolynomialRing(GF(2147483659^3), order='lex')
sage: I = ideal([x^3 - 2*y^2, 3*x + y^4])
sage: I.variety()
verbose 0 (...: multi_polynomial_ideal.py, groebner_basis) Warning: falling␣
↪back to very slow toy implementation.
verbose 0 (...: multi_polynomial_ideal.py, dimension) Warning: falling back␣
↪to very slow toy implementation.
verbose 0 (...: multi_polynomial_ideal.py, variety) Warning: falling back to␣
↪very slow toy implementation.
[{y: 0, x: 0}]
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> R = PolynomialRing(GF(Integer(2147483659)**Integer(3)), order='lex',␣
↪names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> I = ideal([x**Integer(3) - Integer(2)*y**Integer(2), Integer(3)*x +␣
↪y**Integer(4)])
>>> I.variety()
verbose 0 (...: multi_polynomial_ideal.py, groebner_basis) Warning: falling␣
↪back to very slow toy implementation.
verbose 0 (...: multi_polynomial_ideal.py, dimension) Warning: falling back␣
↪to very slow toy implementation.
verbose 0 (...: multi_polynomial_ideal.py, variety) Warning: falling back to␣
↪very slow toy implementation.
```

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```
[{y: 0, x: 0}]
```

The dictionary expressing the variety will be indexed by generators of the polynomial ring after changing to the target field. But the mapping will also accept generators of the original ring, or even generator names as strings, when provided as keys:

```
sage: # needs sage.rings.number_field
sage: K.<x,y> = QQ[]
sage: I = ideal([x^2 + 2*y - 5, x + y + 3])
sage: v = I.variety(AA)[0]; v[x], v[y]
(4.464101615137755?, -7.464101615137755?)
sage: list(v)[0].parent()
Multivariate Polynomial Ring in x, y over Algebraic Real Field
sage: v[x]
4.464101615137755?
sage: v["y"]
-7.464101615137755?
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> K = QQ['x, y']; (x, y,) = K._first_ngens(2)
>>> I = ideal([x**Integer(2) + Integer(2)*y - Integer(5), x + y + Integer(3)])
>>> v = I.variety(AA)[Integer(0)]; v[x], v[y]
(4.464101615137755?, -7.464101615137755?)
>>> list(v)[Integer(0)].parent()
Multivariate Polynomial Ring in x, y over Algebraic Real Field
>>> v[x]
4.464101615137755?
>>> v["y"]
-7.464101615137755?
```

`msolve` also works over finite fields:

```
sage: R.<x, y> = PolynomialRing(GF(536870909), 2, order='lex')             #␣
↪needs sage.rings.finite_rings
sage: I = Ideal([x^2 - 1, y^2 - 1])                                          #␣
↪needs sage.rings.finite_rings
sage: sorted(I.variety(algorithm='msolve',          # optional - msolve,␣
↪needs sage.rings.finite_rings
....:                  proof=False),
....:        key=str)
[{y: 1, x: 1},
 {y: 1, x: 536870908},
 {y: 536870908, x: 1},
 {y: 536870908, x: 536870908}]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(GF(Integer(536870909)), Integer(2), order='lex',␣
↪names=('x', 'y',)); (x, y,) = R._first_ngens(2)# needs sage.rings.finite_
↪rings
>>> I = Ideal([x**Integer(2) - Integer(1), y**Integer(2) - Integer(1)])      ␣
↪                                  # needs sage.rings.finite_rings
```

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```
>>> sorted(I.variety(algorithm='msolve',          # optional - msolve, needs␣
↪sage.rings.finite_rings
...                     proof=False),
...         key=str)
[{y: 1, x: 1},
 {y: 1, x: 536870908},
 {y: 536870908, x: 1},
 {y: 536870908, x: 536870908}]
```

but may fail in small characteristic, especially with ideals of high degree with respect to the characteristic:

```
sage: R.<x, y> = PolynomialRing(GF(3), 2, order='lex')
sage: I = Ideal([x^2 - 1, y^2 - 1])
sage: I.variety(algorithm='msolve', proof=False)     # optional - msolve
Traceback (most recent call last):
...
NotImplementedError: characteristic 3 too small
```

```
>>> from sage.all import *
>>> R = PolynomialRing(GF(Integer(3)), Integer(2), order='lex', names=('x', 'y
↪',)); (x, y,) = R._first_ngens(2)
>>> I = Ideal([x**Integer(2) - Integer(1), y**Integer(2) - Integer(1)])
>>> I.variety(algorithm='msolve', proof=False)     # optional - msolve
Traceback (most recent call last):
...
NotImplementedError: characteristic 3 too small
```

ALGORITHM:

- With `algorithm` = `'triangular_decomposition'` (default), uses triangular decomposition, via Singular if possible, falling back on a toy implementation otherwise.
- With `algorithm` = `'msolve'`, uses the optional package msolve. Note that msolve uses heuristics and therefore requires setting the `proof` flag to `False`. See *msolve* for more information.

vector_space_dimension()

Return the vector space dimension of the ring modulo this ideal. If the ideal is not zero-dimensional, a `TypeError` is raised.

ALGORITHM:

Uses Singular.

EXAMPLES:

```
sage: R.<u,v> = PolynomialRing(QQ)
sage: g = u^4 + v^4 + u^3 + v^3
sage: I = ideal(g) + ideal(g.gradient())
sage: I.dimension()
0
sage: I.vector_space_dimension()
4
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('u', 'v',)); (u, v,) = R._first_ngens(2)
```

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```
>>> g = u**Integer(4) + v**Integer(4) + u**Integer(3) + v**Integer(3)
>>> I = ideal(g) + ideal(g.gradient())
>>> I.dimension()
0
>>> I.vector_space_dimension()
4
```

When the ideal is not zero-dimensional, we return infinity:

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: I = R.ideal(x)
sage: I.dimension()
1
sage: I.vector_space_dimension()
+Infinity
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> I = R.ideal(x)
>>> I.dimension()
1
>>> I.vector_space_dimension()
+Infinity
```

Due to integer overflow, the result is correct only modulo `2^32`, see Issue #8586:

```
sage: P.<x,y,z> = PolynomialRing(GF(32003), 3)                                    #␣
↪needs sage.rings.finite_rings
sage: sage.rings.ideal.FieldIdeal(P).vector_space_dimension()         # known␣
↪bug, needs sage.rings.finite_rings
32777216864027
```

```
>>> from sage.all import *
>>> P = PolynomialRing(GF(Integer(32003)), Integer(3), names=('x', 'y', 'z',
↪)); (x, y, z,) = P._first_ngens(3)# needs sage.rings.finite_rings
>>> sage.rings.ideal.FieldIdeal(P).vector_space_dimension()        # known bug,
↪ needs sage.rings.finite_rings
32777216864027
```

class `sage.rings.polynomial.multi_polynomial_ideal.`**`NCPolynomialIdeal`**(*ring*, *gens*, *coerce=True*, *side='left'*)

Bases: *`MPolynomialIdeal_singular_repr`*, `Ideal_nc`

Create a non-commutative polynomial ideal.

INPUT:

- `ring` – the g-algebra to which this ideal belongs
- `gens` – the generators of this ideal
- `coerce` – boolean (default: `True`); generators are coerced into the ring before creating the ideal
- `side` – (optional) string; either `'left'` (default) or `'twosided'`. Defines whether this ideal is left or two-sided.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H = A.g_algebra({y*x: x*y-z, z*x: x*z+2*x, z*y: y*z-2*y})
sage: H.inject_variables()
Defining x, y, z
sage: I = H.ideal([y^2, x^2, z^2 - H.one()],              # indirect doctest
....:             coerce=False)
sage: I  # random
Left Ideal (y^2, x^2, z^2 - 1) of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
 nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - z}
sage: sorted(I.gens(), key=str)
[x^2, y^2, z^2 - 1]
sage: H.ideal([y^2, x^2, z^2 - H.one()], side='twosided')  # random
Twosided Ideal (y^2, x^2, z^2 - 1) of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
 nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - z}
sage: sorted(H.ideal([y^2, x^2, z^2 - H.one()], side='twosided').gens(),
....:        key=str)
[x^2, y^2, z^2 - 1]
sage: H.ideal([y^2, x^2, z^2 - H.one()], side='right')
Traceback (most recent call last):
...
ValueError: Only left and two-sided ideals are allowed.
```

```
>>> from sage.all import *
>>> # needs sage.combinat sage.modules
>>> A = FreeAlgebra(QQ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) = A._
↪first_ngens(3)
>>> H = A.g_algebra({y*x: x*y-z, z*x: x*z+Integer(2)*x, z*y: y*z-Integer(2)*y})
>>> H.inject_variables()
Defining x, y, z
>>> I = H.ideal([y**Integer(2), x**Integer(2), z**Integer(2) - H.one()],
↪   # indirect doctest
...             coerce=False)
>>> I  # random
Left Ideal (y^2, x^2, z^2 - 1) of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
 nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - z}
>>> sorted(I.gens(), key=str)
[x^2, y^2, z^2 - 1]
>>> H.ideal([y**Integer(2), x**Integer(2), z**Integer(2) - H.one()], side=
↪'twosided')  # random
Twosided Ideal (y^2, x^2, z^2 - 1) of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
 nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - z}
>>> sorted(H.ideal([y**Integer(2), x**Integer(2), z**Integer(2) - H.one()], side=
↪'twosided').gens(),
...        key=str)
[x^2, y^2, z^2 - 1]
>>> H.ideal([y**Integer(2), x**Integer(2), z**Integer(2) - H.one()], side='right')
```

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```
Traceback (most recent call last):
...
ValueError: Only left and two-sided ideals are allowed.
```

elimination_ideal(*variables*)

Return the elimination ideal of this ideal with respect to the variables given in `variables`.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H = A.g_algebra({y*x: x*y-z, z*x: x*z+2*x, z*y: y*z-2*y})
sage: H.inject_variables()
Defining x, y, z
sage: I = H.ideal([y^2, x^2, z^2 - H.one()], coerce=False)
sage: I.elimination_ideal([x, z])
Left Ideal (y^2) of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
 nc-relations: {...}
sage: J = I.twostd(); J
Twosided Ideal (z^2 - 1, y*z - y, x*z + x, y^2, 2*x*y - z - 1, x^2) of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
 nc-relations: {...}
sage: J.elimination_ideal([x, z])
Twosided Ideal (y^2) of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
 nc-relations: {...}
```

```
>>> from sage.all import *
>>> # needs sage.combinat sage.modules
>>> A = FreeAlgebra(QQ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) = A._
→first_ngens(3)
>>> H = A.g_algebra({y*x: x*y-z, z*x: x*z+Integer(2)*x, z*y: y*z-Integer(2)*y}
→)
>>> H.inject_variables()
Defining x, y, z
>>> I = H.ideal([y**Integer(2), x**Integer(2), z**Integer(2) - H.one()],␣
→coerce=False)
>>> I.elimination_ideal([x, z])
Left Ideal (y^2) of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
 nc-relations: {...}
>>> J = I.twostd(); J
Twosided Ideal (z^2 - 1, y*z - y, x*z + x, y^2, 2*x*y - z - 1, x^2) of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
 nc-relations: {...}
>>> J.elimination_ideal([x, z])
Twosided Ideal (y^2) of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
 nc-relations: {...}
```

ALGORITHM: Uses Singular's `eliminate` command

reduce(*p*)

Reduce an element modulo a Groebner basis for this ideal.

It returns 0 if and only if the element is in this ideal. In any case, this reduction is unique up to monomial orders.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H.<x,y,z> = A.g_algebra({y*x: x*y-z, z*x: x*z+2*x, z*y: y*z-2*y})
sage: I = H.ideal([y^2, x^2, z^2 - H.one()],
....:             coerce=False, side='twosided')
sage: Q = H.quotient(I); Q  #random
Quotient of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
 nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - z}
 by the ideal (y^2, x^2, z^2 - 1)
sage: Q.2^2 == Q.one()   # indirect doctest
True
```

```
>>> from sage.all import *
>>> # needs sage.combinat sage.modules
>>> A = FreeAlgebra(QQ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) = A._
↪first_ngens(3)
>>> H = A.g_algebra({y*x: x*y-z, z*x: x*z+Integer(2)*x, z*y: y*z-Integer(2)*y}
↪, names=('x', 'y', 'z',)); (x, y, z,) = H._first_ngens(3)
>>> I = H.ideal([y**Integer(2), x**Integer(2), z**Integer(2) - H.one()],
...             coerce=False, side='twosided')
>>> Q = H.quotient(I); Q  #random
Quotient of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
 nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - z}
 by the ideal (y^2, x^2, z^2 - 1)
>>> Q.gen(2)**Integer(2) == Q.one()   # indirect doctest
True
```

Here, we see that the relation that we just found in the quotient is actually a consequence of the given relations:

```
sage: H.2^2 - H.one() in I.std().gens()                                      #␣
↪needs sage.combinat sage.modules
True
```

```
>>> from sage.all import *
>>> H.gen(2)**Integer(2) - H.one() in I.std().gens()                          ␣
↪           # needs sage.combinat sage.modules
True
```

Here is the corresponding direct test:

```
sage: I.reduce(z^2)                                                          #␣
↪needs sage.combinat sage.modules
1
```

```
>>> from sage.all import *
>>> I.reduce(z**Integer(2))                                                    ␣
↪       # needs sage.combinat sage.modules
1
```

res(*length*)

Compute the resolution up to a given length of the ideal.

NOTE:

Only left syzygies can be computed. So, even if the ideal is two-sided, then the resolution is only one-sided. In that case, a warning is printed.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H = A.g_algebra({y*x: x*y-z, z*x: x*z+2*x, z*y: y*z-2*y})
sage: H.inject_variables()
Defining x, y, z
sage: I = H.ideal([y^2, x^2, z^2-H.one()], coerce=False)
sage: I.res(3)
<Resolution>
```

```
>>> from sage.all import *
>>> # needs sage.combinat sage.modules
>>> A = FreeAlgebra(QQ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) = A._
↪first_ngens(3)
>>> H = A.g_algebra({y*x: x*y-z, z*x: x*z+Integer(2)*x, z*y: y*z-Integer(2)*y}
↪)
>>> H.inject_variables()
Defining x, y, z
>>> I = H.ideal([y**Integer(2), x**Integer(2), z**Integer(2)-H.one()],␣
↪coerce=False)
>>> I.res(Integer(3))
<Resolution>
```

std()

Compute a GB of the ideal. It is two-sided if and only if the ideal is two-sided.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H = A.g_algebra({y*x: x*y-z, z*x: x*z+2*x, z*y: y*z-2*y})
sage: H.inject_variables()
Defining x, y, z
sage: I = H.ideal([y^2, x^2, z^2 - H.one()], coerce=False)
sage: I.std()  #random
Left Ideal (z^2 - 1, y*z - y, x*z + x, y^2, 2*x*y - z - 1, x^2) of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
 nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - z}
sage: sorted(I.std().gens(), key=str)
[2*x*y - z - 1, x*z + x, x^2, y*z - y, y^2, z^2 - 1]
```

```
>>> from sage.all import *
>>> # needs sage.combinat sage.modules
>>> A = FreeAlgebra(QQ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) = A._
↪first_ngens(3)
>>> H = A.g_algebra({y*x: x*y-z, z*x: x*z+Integer(2)*x, z*y: y*z-Integer(2)*y}
↪)
>>> H.inject_variables()
Defining x, y, z
>>> I = H.ideal([y**Integer(2), x**Integer(2), z**Integer(2) - H.one()],
↪coerce=False)
>>> I.std()  #random
Left Ideal (z^2 - 1, y*z - y, x*z + x, y^2, 2*x*y - z - 1, x^2) of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
 nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - z}
>>> sorted(I.std().gens(), key=str)
[2*x*y - z - 1, x*z + x, x^2, y*z - y, y^2, z^2 - 1]
```

If the ideal is a left ideal, then *std()* returns a left Groebner basis. But if it is a two-sided ideal, then the output of *std()* and *twostd()* coincide:

```
sage: # needs sage.combinat sage.modules
sage: JL = H.ideal([x^3, y^3, z^3 - 4*z])
sage: JL  #random
Left Ideal (x^3, y^3, z^3 - 4*z) of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
 nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - z}
sage: sorted(JL.gens(), key=str)
[x^3, y^3, z^3 - 4*z]
sage: JL.std()  # random
Left Ideal (z^3 - 4*z, y*z^2 - 2*y*z,
            x*z^2 + 2*x*z, 2*x*y*z - z^2 - 2*z, y^3, x^3) of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
 nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - z}
sage: sorted(JL.std().gens(), key=str)
[2*x*y*z - z^2 - 2*z, x*z^2 + 2*x*z, x^3, y*z^2 - 2*y*z, y^3, z^3 - 4*z]
sage: JT = H.ideal([x^3, y^3, z^3 - 4*z], side='twosided')
sage: JT  #random
Twosided Ideal (x^3, y^3, z^3 - 4*z) of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
 nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - z}
sage: sorted(JT.gens(), key=str)
[x^3, y^3, z^3 - 4*z]
sage: JT.std()  #random
Twosided Ideal (z^3 - 4*z, y*z^2 - 2*y*z, x*z^2 + 2*x*z,
                y^2*z - 2*y^2, 2*x*y*z - z^2 - 2*z, x^2*z + 2*x^2,
                y^3, x*y^2 - y*z, x^2*y - x*z - 2*x, x^3) of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
 nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - z}
sage: sorted(JT.std().gens(), key=str)
[2*x*y*z - z^2 - 2*z, x*y^2 - y*z, x*z^2 + 2*x*z, x^2*y - x*z - 2*x,
 x^2*z + 2*x^2, x^3, y*z^2 - 2*y*z, y^2*z - 2*y^2, y^3, z^3 - 4*z]
sage: JT.std() == JL.twostd()
True
```

```
>>> from sage.all import *
>>> # needs sage.combinat sage.modules
>>> JL = H.ideal([x**Integer(3), y**Integer(3), z**Integer(3) - Integer(4)*z])
>>> JL  #random
Left Ideal (x^3, y^3, z^3 - 4*z) of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
 nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - z}
>>> sorted(JL.gens(), key=str)
[x^3, y^3, z^3 - 4*z]
>>> JL.std()  # random
Left Ideal (z^3 - 4*z, y*z^2 - 2*y*z,
            x*z^2 + 2*x*z, 2*x*y*z - z^2 - 2*z, y^3, x^3) of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
 nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - z}
>>> sorted(JL.std().gens(), key=str)
[2*x*y*z - z^2 - 2*z, x*z^2 + 2*x*z, x^3, y*z^2 - 2*y*z, y^3, z^3 - 4*z]
>>> JT = H.ideal([x**Integer(3), y**Integer(3), z**Integer(3) - Integer(4)*z],
↪ side='twosided')
>>> JT  #random
Twosided Ideal (x^3, y^3, z^3 - 4*z) of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
 nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - z}
>>> sorted(JT.gens(), key=str)
[x^3, y^3, z^3 - 4*z]
>>> JT.std()  #random
Twosided Ideal (z^3 - 4*z, y*z^2 - 2*y*z, x*z^2 + 2*x*z,
                y^2*z - 2*y^2, 2*x*y*z - z^2 - 2*z, x^2*z + 2*x^2,
                y^3, x*y^2 - y*z, x^2*y - x*z - 2*x, x^3) of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
 nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - z}
>>> sorted(JT.std().gens(), key=str)
[2*x*y*z - z^2 - 2*z, x*y^2 - y*z, x*z^2 + 2*x*z, x^2*y - x*z - 2*x,
 x^2*z + 2*x^2, x^3, y*z^2 - 2*y*z, y^2*z - 2*y^2, y^3, z^3 - 4*z]
>>> JT.std() == JL.twostd()
True
```

ALGORITHM: Uses Singular's `std` command

syzygy_module()

Compute the first syzygy (i.e., the module of relations of the given generators) of the ideal.

NOTE:

Only left syzygies can be computed. So, even if the ideal is two-sided, then the syzygies are only one-sided. In that case, a warning is printed.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H = A.g_algebra({y*x: x*y-z, z*x: x*z+2*x, z*y: y*z-2*y})
sage: H.inject_variables()
Defining x, y, z
sage: I = H.ideal([y^2, x^2, z^2-H.one()], coerce=False)
sage: G = vector(I.gens()); G
```

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```
d...: UserWarning: You are constructing a free module
over a noncommutative ring. Sage does not have a concept
of left/right and both sided modules, so be careful.
It's also not guaranteed that all multiplications are
done from the right side.
d...: UserWarning: You are constructing a free module
over a noncommutative ring. Sage does not have a concept
of left/right and both sided modules, so be careful.
It's also not guaranteed that all multiplications are
done from the right side.
(y^2, x^2, z^2 - 1)
sage: M = I.syzygy_module(); M
[                                                                             -z^
↪2 - 8*z - 15                                                                   ␣
↪                        0                                                      ␣
↪                                   y^2]
[                                                                               ␣
↪          0                                                                    ␣
↪         -z^2 + 8*z - 15                                                       ␣
↪                                  x^2]
[                                                                  x^2*z^2 + 8*x^
↪2*z + 15*x^2                                                                   -
↪y^2*z^2 + 8*y^2*z - 15*y^2                                                     ␣
↪                  -4*x*y*z + 2*z^2 + 2*z]
[                 x^2*y*z^2 + 9*x^2*y*z - 6*x*z^3 + 20*x^2*y - 72*x*z^2 -␣
↪282*x*z - 360*x                                                                ␣
↪ -y^3*z^2 + 7*y^3*z - 12*y^3                                                   ␣
↪                                  6*y*z^2]
[                                                                  x^3*z^2 + 7*x^
↪3*z + 12*x^3                    -x*y^2*z^2 + 9*x*y^2*z - 4*y*z^3 - 20*x*y^2 +␣
↪52*y*z^2 - 224*y*z + 320*y                                                     ␣
↪                              -6*x*z^2]
[  x^2*y^2*z + 4*x^2*y^2 - 8*x*y*z^2 - 48*x*y*z + 12*z^3 - 64*x*y + 108*z^2 +␣
↪312*z + 288                                                                    ␣
↪          -y^4*z + 4*y^4                                                       ␣
↪                                    0]
[                                                      2*x^3*y*z + 8*x^3*y + 9*x^
↪2*z + 27*x^2                                    -2*x*y^3*z + 8*x*y^3 - 12*y^
↪2*z^2 + 99*y^2*z - 195*y^2                                                     ␣
↪               -36*x*y*z + 24*z^2 + 18*z]
[                                                                              x^
↪4*z + 4*x^4    -x^2*y^2*z + 4*x^2*y^2 - 4*x*y*z^2 + 32*x*y*z - 6*z^3 -␣
↪64*x*y + 66*z^2 - 240*z + 288                                                  ␣
↪                                        0]
[x^3*y^2*z + 4*x^3*y^2 + 18*x^2*y*z - 36*x*z^3 + 66*x^2*y - 432*x*z^2 -␣
↪1656*x*z - 2052*x                                      -x*y^4*z + 4*x*y^4 -␣
↪8*y^3*z^2 + 62*y^3*z - 114*y^3                                                 ␣
↪                           48*y*z^2 - 36*y*z]

sage: M*G                                                                      #␣
↪needs sage.combinat sage.modules
(0, 0, 0, 0, 0, 0, 0, 0, 0)
```

```
>>> from sage.all import *
>>> # needs sage.combinat sage.modules
>>> A = FreeAlgebra(QQ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) = A._
↪first_ngens(3)
>>> H = A.g_algebra({y*x: x*y-z, z*x: x*z+Integer(2)*x, z*y: y*z-Integer(2)*y}
↪)
>>> H.inject_variables()
Defining x, y, z
>>> I = H.ideal([y**Integer(2), x**Integer(2), z**Integer(2)-H.one()],
↪coerce=False)
>>> G = vector(I.gens()); G
d...: UserWarning: You are constructing a free module
over a noncommutative ring. Sage does not have a concept
of left/right and both sided modules, so be careful.
It's also not guaranteed that all multiplications are
done from the right side.
d...: UserWarning: You are constructing a free module
over a noncommutative ring. Sage does not have a concept
of left/right and both sided modules, so be careful.
It's also not guaranteed that all multiplications are
done from the right side.
(y^2, x^2, z^2 - 1)
>>> M = I.syzygy_module(); M
[                                                                         -z^
↪2 - 8*z - 15
↪                         0
↪                                    y^2]
[
↪          0
↪         -z^2 + 8*z - 15
↪                                   x^2]
[                                                              x^2*z^2 + 8*x^
↪2*z + 15*x^2                                                                -
↪y^2*z^2 + 8*y^2*z - 15*y^2
↪                  -4*x*y*z + 2*z^2 + 2*z]
[                 x^2*y*z^2 + 9*x^2*y*z - 6*x*z^3 + 20*x^2*y - 72*x*z^2 -
↪282*x*z - 360*x
↪ -y^3*z^2 + 7*y^3*z - 12*y^3
↪                                   6*y*z^2]
[                                                              x^3*z^2 + 7*x^
↪3*z + 12*x^3                   -x*y^2*z^2 + 9*x*y^2*z - 4*y*z^3 - 20*x*y^2 +
↪52*y*z^2 - 224*y*z + 320*y
↪                               -6*x*z^2]
[  x^2*y^2*z + 4*x^2*y^2 - 8*x*y*z^2 - 48*x*y*z + 12*z^3 - 64*x*y + 108*z^2 +
↪312*z + 288
↪          -y^4*z + 4*y^4
↪                                     0]
[                                                    2*x^3*y*z + 8*x^3*y + 9*x^
↪2*z + 27*x^2                                      -2*x*y^3*z + 8*x*y^3 - 12*y^
↪2*z^2 + 99*y^2*z - 195*y^2
↪               -36*x*y*z + 24*z^2 + 18*z]
[                                                                            x^
↪4*z + 4*x^4    -x^2*y^2*z + 4*x^2*y^2 - 4*x*y*z^2 + 32*x*y*z - 6*z^3 -
```

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```
↪64*x*y + 66*z^2 - 240*z + 288                                                    ␣
↪                                                    0]
[x^3*y^2*z + 4*x^3*y^2 + 18*x^2*y*z - 36*x*z^3 + 66*x^2*y - 432*x*z^2 -␣
↪1656*x*z - 2052*x                                       -x*y^4*z + 4*x*y^4 -␣
↪8*y^3*z^2 + 62*y^3*z - 114*y^3                                                   ␣
↪                              48*y*z^2 - 36*y*z]

>>> M*G                                                                       #␣
↪needs sage.combinat sage.modules
(0, 0, 0, 0, 0, 0, 0, 0, 0)
```

ALGORITHM: Uses Singular's `syz` command

twostd()

Compute a two-sided GB of the ideal (even if it is a left ideal).

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H = A.g_algebra({y*x: x*y-z, z*x: x*z+2*x, z*y: y*z-2*y})
sage: H.inject_variables()
Defining x, y, z
sage: I = H.ideal([y^2, x^2, z^2 - H.one()], coerce=False)
sage: I.twostd()  #random
Twosided Ideal (z^2 - 1, y*z - y, x*z + x, y^2, 2*x*y - z - 1, x^2) of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field...
sage: sorted(I.twostd().gens(), key=str)
[2*x*y - z - 1, x*z + x, x^2, y*z - y, y^2, z^2 - 1]
```

```
>>> from sage.all import *
>>> # needs sage.combinat sage.modules
>>> A = FreeAlgebra(QQ, Integer(3), names=('x', 'y', 'z',)); (x, y, z,) = A._
↪first_ngens(3)
>>> H = A.g_algebra({y*x: x*y-z, z*x: x*z+Integer(2)*x, z*y: y*z-Integer(2)*y}
↪)
>>> H.inject_variables()
Defining x, y, z
>>> I = H.ideal([y**Integer(2), x**Integer(2), z**Integer(2) - H.one()],␣
↪coerce=False)
>>> I.twostd()  #random
Twosided Ideal (z^2 - 1, y*z - y, x*z + x, y^2, 2*x*y - z - 1, x^2) of
 Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field...
>>> sorted(I.twostd().gens(), key=str)
[2*x*y - z - 1, x*z + x, x^2, y*z - y, y^2, z^2 - 1]
```

ALGORITHM: Uses Singular's `twostd` command

class sage.rings.polynomial.multi_polynomial_ideal.**RequireField**(f)

Bases: `MethodDecorator`

Decorator which throws an exception if a computation over a coefficient ring which is not a field is attempted.

Note

This decorator is used automatically internally so the user does not need to use it manually.

sage.rings.polynomial.multi_polynomial_ideal.**is_MPolynomialIdeal**(*x*)

Return True if the provided argument x is an ideal in a multivariate polynomial ring.

INPUT:

- x – an arbitrary object

EXAMPLES:

```
sage: from sage.rings.polynomial.multi_polynomial_ideal import is_MPolynomialIdeal
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: I = [x + 2*y + 2*z - 1, x^2 + 2*y^2 + 2*z^2 - x, 2*x*y + 2*y*z - y]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.multi_polynomial_ideal import is_MPolynomialIdeal
>>> P = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = P._first_ngens(3)
>>> I = [x + Integer(2)*y + Integer(2)*z - Integer(1), x**Integer(2) +␣
↪Integer(2)*y**Integer(2) + Integer(2)*z**Integer(2) - x, Integer(2)*x*y +␣
↪Integer(2)*y*z - y]
```

Sage distinguishes between a list of generators for an ideal and the ideal itself. This distinction is inconsistent with Singular but matches Magma's behavior.

```
sage: is_MPolynomialIdeal(I)
doctest:warning...
DeprecationWarning: The function is_MPolynomialIdeal is deprecated;
use 'isinstance(..., MPolynomialIdeal)' instead.
See https://github.com/sagemath/sage/issues/38266 for details.
False
```

```
>>> from sage.all import *
>>> is_MPolynomialIdeal(I)
doctest:warning...
DeprecationWarning: The function is_MPolynomialIdeal is deprecated;
use 'isinstance(..., MPolynomialIdeal)' instead.
See https://github.com/sagemath/sage/issues/38266 for details.
False
```

```
sage: I = Ideal(I)
sage: is_MPolynomialIdeal(I)
True
```

```
>>> from sage.all import *
>>> I = Ideal(I)
>>> is_MPolynomialIdeal(I)
True
```

sage.rings.polynomial.multi_polynomial_ideal.**require_field**

alias of *RequireField*

3.1.7 Polynomial Sequences

We call a finite list of polynomials a `Polynomial Sequence`.

Polynomial sequences in Sage can optionally be viewed as consisting of various parts or sub-sequences. These kind of polynomial sequences which naturally split into parts arise naturally for example in algebraic cryptanalysis of symmetric cryptographic primitives. The most prominent examples of these systems are: the small scale variants of the AES [CMR2005] (cf. `sage.crypto.mq.sr.SR()`) and Flurry/Curry [BPW2006]. By default, a polynomial sequence has exactly one part.

AUTHORS:

- Martin Albrecht (2007ff): initial version
- Martin Albrecht (2009): refactoring, clean-up, new functions
- Martin Albrecht (2011): refactoring, moved to sage.rings.polynomial
- Alex Raichev (2011-06): added algebraic_dependence()
- Charles Bouillaguet (2013-1): added solve()

EXAMPLES:

As an example consider a small scale variant of the AES:

```
sage: sr = mq.SR(2, 1, 2, 4, gf2=True, polybori=True)                    #␣
↪needs sage.rings.polynomial.pbori
sage: sr                                                                  #␣
↪needs sage.rings.polynomial.pbori
SR(2,1,2,4)
```

```
>>> from sage.all import *
>>> sr = mq.SR(Integer(2), Integer(1), Integer(2), Integer(4), gf2=True,␣
↪polybori=True)                                   # needs sage.rings.polynomial.pbori
>>> sr                                                                   #␣
↪needs sage.rings.polynomial.pbori
SR(2,1,2,4)
```

We can construct a polynomial sequence for a random plaintext-ciphertext pair and study it:

```
sage: set_random_seed(1)
sage: while True:  # workaround (see :issue:`31891`)                      #␣
↪needs sage.rings.polynomial.pbori
....:     try:
....:         F, s = sr.polynomial_system()
....:         break
....:     except ZeroDivisionError:
....:         pass
sage: F                                                                   #␣
↪needs sage.rings.polynomial.pbori
Polynomial Sequence with 112 Polynomials in 64 Variables

sage: r2 = F.part(2); r2                                                  #␣
↪needs sage.rings.polynomial.pbori
(w200 + k100 + x100 + x102 + x103,
 w201 + k101 + x100 + x101 + x103 + 1,
 w202 + k102 + x100 + x101 + x102 + 1,
```

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```
w203 + k103 + x101 + x102 + x103,
w210 + k110 + x110 + x112 + x113,
w211 + k111 + x110 + x111 + x113 + 1,
w212 + k112 + x110 + x111 + x112 + 1,
w213 + k113 + x111 + x112 + x113,
x100*w100 + x100*w103 + x101*w102 + x102*w101 + x103*w100,
x100*w100 + x100*w101 + x101*w100 + x101*w103 + x102*w102 + x103*w101,
x100*w101 + x100*w102 + x101*w100 + x101*w101 + x102*w100 + x102*w103 + x103*w102,
x100*w100 + x100*w102 + x100*w103 + x101*w100 + x101*w101 + x102*w102 + x103*w100 +␣
↪x100,
x100*w101 + x100*w103 + x101*w101 + x101*w102 + x102*w100 + x102*w103 + x103*w101 +␣
↪x101,
x100*w100 + x100*w102 + x101*w100 + x101*w102 + x101*w103 + x102*w100 + x102*w101 +␣
↪x103*w102 + x102,
x100*w101 + x100*w102 + x101*w100 + x101*w103 + x102*w101 + x103*w103 + x103,
x100*w100 + x100*w101 + x100*w103 + x101*w101 + x102*w100 + x102*w102 + x103*w100 +␣
↪w100,
x100*w102 + x101*w100 + x101*w101 + x101*w103 + x102*w101 + x103*w100 + x103*w102 +␣
↪w101,
x100*w100 + x100*w101 + x100*w102 + x101*w102 + x102*w100 + x102*w101 + x102*w103 +␣
↪x103*w101 + w102,
x100*w101 + x101*w100 + x101*w102 + x102*w100 + x103*w101 + x103*w103 + w103,
x100*w102 + x101*w101 + x102*w100 + x103*w103 + 1,
x110*w110 + x110*w113 + x111*w112 + x112*w111 + x113*w110,
x110*w110 + x110*w111 + x111*w110 + x111*w113 + x112*w112 + x113*w111,
x110*w111 + x110*w112 + x111*w110 + x111*w111 + x112*w110 + x112*w113 + x113*w112,
x110*w110 + x110*w112 + x110*w113 + x111*w110 + x111*w111 + x112*w112 + x113*w110 +␣
↪x110,
x110*w111 + x110*w113 + x111*w111 + x111*w112 + x112*w110 + x112*w113 + x113*w111 +␣
↪x111,
x110*w110 + x110*w112 + x111*w110 + x111*w112 + x111*w113 + x112*w110 + x112*w111 +␣
↪x113*w112 + x112,
x110*w111 + x110*w112 + x111*w110 + x111*w113 + x112*w111 + x113*w113 + x113,
x110*w110 + x110*w111 + x110*w113 + x111*w111 + x112*w110 + x112*w112 + x113*w110 +␣
↪w110,
x110*w112 + x111*w110 + x111*w111 + x111*w113 + x112*w111 + x113*w110 + x113*w112 +␣
↪w111,
x110*w110 + x110*w111 + x110*w112 + x111*w112 + x112*w110 + x112*w111 + x112*w113 +␣
↪x113*w111 + w112,
x110*w111 + x111*w110 + x111*w112 + x112*w110 + x113*w111 + x113*w113 + w113,
x110*w112 + x111*w111 + x112*w110 + x113*w113 + 1)
```

```
>>> from sage.all import *
>>> set_random_seed(Integer(1))
>>> while True:  # workaround (see :issue:`31891`)                           #␣
↪needs sage.rings.polynomial.pbori
...     try:
...         F, s = sr.polynomial_system()
...         break
...     except ZeroDivisionError:
...         pass
>>> F                                                                         #␣
```

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```
↪needs sage.rings.polynomial.pbori
Polynomial Sequence with 112 Polynomials in 64 Variables

>>> r2 = F.part(Integer(2)); r2                                                         ␣
↪     # needs sage.rings.polynomial.pbori
(w200 + k100 + x100 + x102 + x103,
 w201 + k101 + x100 + x101 + x103 + 1,
 w202 + k102 + x100 + x101 + x102 + 1,
 w203 + k103 + x101 + x102 + x103,
 w210 + k110 + x110 + x112 + x113,
 w211 + k111 + x110 + x111 + x113 + 1,
 w212 + k112 + x110 + x111 + x112 + 1,
 w213 + k113 + x111 + x112 + x113,
 x100*w100 + x100*w103 + x101*w102 + x102*w101 + x103*w100,
 x100*w100 + x100*w101 + x101*w100 + x101*w103 + x102*w102 + x103*w101,
 x100*w101 + x100*w102 + x101*w100 + x101*w101 + x102*w100 + x102*w103 + x103*w102,
 x100*w100 + x100*w102 + x100*w103 + x101*w100 + x101*w101 + x102*w102 + x103*w100 +␣
↪x100,
 x100*w101 + x100*w103 + x101*w101 + x101*w102 + x102*w100 + x102*w103 + x103*w101 +␣
↪x101,
 x100*w100 + x100*w102 + x101*w100 + x101*w102 + x101*w103 + x102*w100 + x102*w101 +␣
↪x103*w102 + x102,
 x100*w101 + x100*w102 + x101*w100 + x101*w103 + x102*w101 + x103*w103 + x103,
 x100*w100 + x100*w101 + x100*w103 + x101*w101 + x102*w100 + x102*w102 + x103*w100 +␣
↪w100,
 x100*w102 + x101*w100 + x101*w101 + x101*w103 + x102*w101 + x103*w100 + x103*w102 +␣
↪w101,
 x100*w100 + x100*w101 + x100*w102 + x101*w102 + x102*w100 + x102*w101 + x102*w103 +␣
↪x103*w101 + w102,
 x100*w101 + x101*w100 + x101*w102 + x102*w100 + x103*w101 + x103*w103 + w103,
 x100*w102 + x101*w101 + x102*w100 + x103*w103 + 1,
 x110*w110 + x110*w113 + x111*w112 + x112*w111 + x113*w110,
 x110*w110 + x110*w111 + x111*w110 + x111*w113 + x112*w112 + x113*w111,
 x110*w111 + x110*w112 + x111*w110 + x111*w111 + x112*w110 + x112*w113 + x113*w112,
 x110*w110 + x110*w112 + x110*w113 + x111*w110 + x111*w111 + x112*w112 + x113*w110 +␣
↪x110,
 x110*w111 + x110*w113 + x111*w111 + x111*w112 + x112*w110 + x112*w113 + x113*w111 +␣
↪x111,
 x110*w110 + x110*w112 + x111*w110 + x111*w112 + x111*w113 + x112*w110 + x112*w111 +␣
↪x113*w112 + x112,
 x110*w111 + x110*w112 + x111*w110 + x111*w113 + x112*w111 + x113*w113 + x113,
 x110*w110 + x110*w111 + x110*w113 + x111*w111 + x112*w110 + x112*w112 + x113*w110 +␣
↪w110,
 x110*w112 + x111*w110 + x111*w111 + x111*w113 + x112*w111 + x113*w110 + x113*w112 +␣
↪w111,
 x110*w110 + x110*w111 + x110*w112 + x111*w112 + x112*w110 + x112*w111 + x112*w113 +␣
↪x113*w111 + w112,
 x110*w111 + x111*w110 + x111*w112 + x112*w110 + x113*w111 + x113*w113 + w113,
 x110*w112 + x111*w111 + x112*w110 + x113*w113 + 1)
```

We separate the system in independent subsystems:

```
sage: C = Sequence(r2).connected_components(); C                          #␣
↪needs sage.rings.polynomial.pbori
[[w200 + k100 + x100 + x102 + x103,
  w201 + k101 + x100 + x101 + x103 + 1,
  w202 + k102 + x100 + x101 + x102 + 1,
  w203 + k103 + x101 + x102 + x103,
  x100*w100 + x100*w103 + x101*w102 + x102*w101 + x103*w100,
  x100*w100 + x100*w101 + x101*w100 + x101*w103 + x102*w102 + x103*w101,
  x100*w101 + x100*w102 + x101*w100 + x101*w101 + x102*w100 + x102*w103 + x103*w102,
  x100*w100 + x100*w102 + x100*w103 + x101*w100 + x101*w101 + x102*w102 + x103*w100 +␣
↪x100,
  x100*w101 + x100*w103 + x101*w101 + x101*w102 + x102*w100 + x102*w103 + x103*w101 +␣
↪x101,
  x100*w100 + x100*w102 + x101*w100 + x101*w102 + x101*w103 + x102*w100 + x102*w101 +␣
↪x103*w102 + x102,
  x100*w101 + x100*w102 + x101*w100 + x101*w103 + x102*w101 + x103*w103 + x103,
  x100*w100 + x100*w101 + x100*w103 + x101*w101 + x102*w100 + x102*w102 + x103*w100 +␣
↪w100,
  x100*w102 + x101*w100 + x101*w101 + x101*w103 + x102*w101 + x103*w100 + x103*w102 +␣
↪w101,
  x100*w100 + x100*w101 + x100*w102 + x101*w102 + x102*w100 + x102*w101 + x102*w103 +␣
↪x103*w101 + w102,
  x100*w101 + x101*w100 + x101*w102 + x102*w100 + x103*w101 + x103*w103 + w103,
  x100*w102 + x101*w101 + x102*w100 + x103*w103 + 1],
 [w210 + k110 + x110 + x112 + x113,
  w211 + k111 + x110 + x111 + x113 + 1,
  w212 + k112 + x110 + x111 + x112 + 1,
  w213 + k113 + x111 + x112 + x113,
  x110*w110 + x110*w113 + x111*w112 + x112*w111 + x113*w110,
  x110*w110 + x110*w111 + x111*w110 + x111*w113 + x112*w112 + x113*w111,
  x110*w111 + x110*w112 + x111*w110 + x111*w111 + x112*w110 + x112*w113 + x113*w112,
  x110*w110 + x110*w112 + x110*w113 + x111*w110 + x111*w111 + x112*w112 + x113*w110 +␣
↪x110,
  x110*w111 + x110*w113 + x111*w111 + x111*w112 + x112*w110 + x112*w113 + x113*w111 +␣
↪x111,
  x110*w110 + x110*w112 + x111*w110 + x111*w112 + x111*w113 + x112*w110 + x112*w111 +␣
↪x113*w112 + x112,
  x110*w111 + x110*w112 + x111*w110 + x111*w113 + x112*w111 + x113*w113 + x113,
  x110*w110 + x110*w111 + x110*w113 + x111*w111 + x112*w110 + x112*w112 + x113*w110 +␣
↪w110,
  x110*w112 + x111*w110 + x111*w111 + x111*w113 + x112*w111 + x113*w110 + x113*w112 +␣
↪w111,
  x110*w110 + x110*w111 + x110*w112 + x111*w112 + x112*w110 + x112*w111 + x112*w113 +␣
↪x113*w111 + w112,
  x110*w111 + x111*w110 + x111*w112 + x112*w110 + x113*w111 + x113*w113 + w113,
  x110*w112 + x111*w111 + x112*w110 + x113*w113 + 1]]
sage: C[0].groebner_basis()                                               #␣
↪needs sage.rings.polynomial.pbori
Polynomial Sequence with 30 Polynomials in 16 Variables
```

```
>>> from sage.all import *
>>> C = Sequence(r2).connected_components(); C                          #␣
```

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```
↪needs sage.rings.polynomial.pbori
[[w200 + k100 + x100 + x102 + x103,
  w201 + k101 + x100 + x101 + x103 + 1,
  w202 + k102 + x100 + x101 + x102 + 1,
  w203 + k103 + x101 + x102 + x103,
  x100*w100 + x100*w103 + x101*w102 + x102*w101 + x103*w100,
  x100*w100 + x100*w101 + x101*w100 + x101*w103 + x102*w102 + x103*w101,
  x100*w101 + x100*w102 + x101*w100 + x101*w101 + x102*w100 + x102*w103 + x103*w102,
  x100*w100 + x100*w102 + x100*w103 + x101*w100 + x101*w101 + x102*w102 + x103*w100 +␣
↪x100,
  x100*w101 + x100*w103 + x101*w101 + x101*w102 + x102*w100 + x102*w103 + x103*w101 +␣
↪x101,
  x100*w100 + x100*w102 + x101*w100 + x101*w102 + x101*w103 + x102*w100 + x102*w101 +␣
↪x103*w102 + x102,
  x100*w101 + x100*w102 + x101*w100 + x101*w103 + x102*w101 + x103*w103 + x103,
  x100*w100 + x100*w101 + x100*w103 + x101*w101 + x102*w100 + x102*w102 + x103*w100 +␣
↪w100,
  x100*w102 + x101*w100 + x101*w101 + x101*w103 + x102*w101 + x103*w100 + x103*w102 +␣
↪w101,
  x100*w100 + x100*w101 + x100*w102 + x101*w102 + x102*w100 + x102*w101 + x102*w103 +␣
↪x103*w101 + w102,
  x100*w101 + x101*w100 + x101*w102 + x102*w100 + x103*w101 + x103*w103 + w103,
  x100*w102 + x101*w101 + x102*w100 + x103*w103 + 1],
 [w210 + k110 + x110 + x112 + x113,
  w211 + k111 + x110 + x111 + x113 + 1,
  w212 + k112 + x110 + x111 + x112 + 1,
  w213 + k113 + x111 + x112 + x113,
  x110*w110 + x110*w113 + x111*w112 + x112*w111 + x113*w110,
  x110*w110 + x110*w111 + x111*w110 + x111*w113 + x112*w112 + x113*w111,
  x110*w111 + x110*w112 + x111*w110 + x111*w111 + x112*w110 + x112*w113 + x113*w112,
  x110*w110 + x110*w112 + x110*w113 + x111*w110 + x111*w111 + x112*w112 + x113*w110 +␣
↪x110,
  x110*w111 + x110*w113 + x111*w111 + x111*w112 + x112*w110 + x112*w113 + x113*w111 +␣
↪x111,
  x110*w110 + x110*w112 + x111*w110 + x111*w112 + x111*w113 + x112*w110 + x112*w111 +␣
↪x113*w112 + x112,
  x110*w111 + x110*w112 + x111*w110 + x111*w113 + x112*w111 + x113*w113 + x113,
  x110*w110 + x110*w111 + x110*w113 + x111*w111 + x112*w110 + x112*w112 + x113*w110 +␣
↪w110,
  x110*w112 + x111*w110 + x111*w111 + x111*w113 + x112*w111 + x113*w110 + x113*w112 +␣
↪w111,
  x110*w110 + x110*w111 + x110*w112 + x111*w112 + x112*w110 + x112*w111 + x112*w113 +␣
↪x113*w111 + w112,
  x110*w111 + x111*w110 + x111*w112 + x112*w110 + x113*w111 + x113*w113 + w113,
  x110*w112 + x111*w111 + x112*w110 + x113*w113 + 1]]
>>> C[Integer(0)].groebner_basis()                                          ␣
↪      # needs sage.rings.polynomial.pbori
Polynomial Sequence with 30 Polynomials in 16 Variables
```

and compute the coefficient matrix:

```
sage: A,v = Sequence(r2).coefficients_monomials()                             #␣
```

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```
↪needs sage.rings.polynomial.pbori
sage: A.rank()                                                          #␣
↪needs sage.rings.polynomial.pbori
32
```

```
>>> from sage.all import *
>>> A,v = Sequence(r2).coefficients_monomials()                        #␣
↪needs sage.rings.polynomial.pbori
>>> A.rank()                                                            #␣
↪needs sage.rings.polynomial.pbori
32
```

Using these building blocks we can implement a simple XL algorithm easily:

```
sage: sr = mq.SR(1,1,1,4, gf2=True, polybori=True, order='lex')         #␣
↪needs sage.rings.polynomial.pbori
sage: while True:  # workaround (see :issue:`31891`)                     #␣
↪needs sage.rings.polynomial.pbori
....:     try:
....:         F, s = sr.polynomial_system()
....:         break
....:     except ZeroDivisionError:
....:         pass

sage: # needs sage.rings.polynomial.pbori
sage: monomials = [a*b for a in F.variables() for b in F.variables() if a < b]
sage: len(monomials)
190
sage: F2 = Sequence(map(mul, cartesian_product_iterator((monomials, F))))
sage: A, v = F2.coefficients_monomials(sparse=False)
sage: A.echelonize()
sage: A
6840 x 4474 dense matrix over Finite Field of size 2...
sage: A.rank()
4056
sage: A[4055] * v
k001*k003
```

```
>>> from sage.all import *
>>> sr = mq.SR(Integer(1),Integer(1),Integer(1),Integer(4), gf2=True, polybori=True,␣
↪order='lex')                    # needs sage.rings.polynomial.pbori
>>> while True:  # workaround (see :issue:`31891`)                     #␣
↪needs sage.rings.polynomial.pbori
...     try:
...         F, s = sr.polynomial_system()
...         break
...     except ZeroDivisionError:
...         pass

>>> # needs sage.rings.polynomial.pbori
>>> monomials = [a*b for a in F.variables() for b in F.variables() if a < b]
>>> len(monomials)
```

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```
190
>>> F2 = Sequence(map(mul, cartesian_product_iterator((monomials, F))))
>>> A, v = F2.coefficients_monomials(sparse=False)
>>> A.echelonize()
>>> A
6840 x 4474 dense matrix over Finite Field of size 2...
>>> A.rank()
4056
>>> A[Integer(4055)] * v
k001*k003
```

Note

In many other computer algebra systems (cf. Singular) this class would be called `Ideal` but an ideal is a very distinct object from its generators and thus this is not an ideal in Sage.

Classes

`sage.rings.polynomial.multi_polynomial_sequence.`**`PolynomialSequence`**(*arg1*, *arg2=None*, *immutable=False*, *cr=False*, *cr_str=None*)

Construct a new polynomial sequence object.

INPUT:

- `arg1` – a multivariate polynomial ring, an ideal or a matrix
- `arg2` – an iterable object of parts or polynomials (default: `None`)
  - `immutable` – if `True` the sequence is immutable (default: `False`)
  - `cr`, `cr_str` – see `Sequence()`

EXAMPLES:

```
sage: P.<a,b,c,d> = PolynomialRing(GF(127), 4)
sage: I = sage.rings.ideal.Katsura(P)                                        #␣
↪needs sage.libs.singular
```

```
>>> from sage.all import *
>>> P = PolynomialRing(GF(Integer(127)), Integer(4), names=('a', 'b', 'c', 'd',));
↪ (a, b, c, d,) = P._first_ngens(4)
>>> I = sage.rings.ideal.Katsura(P)                                         #␣
↪needs sage.libs.singular
```

If a list of tuples is provided, those form the parts:

```
sage: F = Sequence([I.gens(),I.gens()], I.ring()); F  # indirect doctest      #␣
↪needs sage.libs.singular
[a + 2*b + 2*c + 2*d - 1,
 a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
 2*a*b + 2*b*c + 2*c*d - b,
 b^2 + 2*a*c + 2*b*d - c,
```

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```
 a + 2*b + 2*c + 2*d - 1,
 a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
 2*a*b + 2*b*c + 2*c*d - b,
 b^2 + 2*a*c + 2*b*d - c]
sage: F.nparts()                                                      #␣
↪needs sage.libs.singular
2
```

```
>>> from sage.all import *
>>> F = Sequence([I.gens(),I.gens()], I.ring()); F  # indirect doctest     #␣
↪needs sage.libs.singular
[a + 2*b + 2*c + 2*d - 1,
 a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
 2*a*b + 2*b*c + 2*c*d - b,
 b^2 + 2*a*c + 2*b*d - c,
 a + 2*b + 2*c + 2*d - 1,
 a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
 2*a*b + 2*b*c + 2*c*d - b,
 b^2 + 2*a*c + 2*b*d - c]
>>> F.nparts()                                                        #␣
↪needs sage.libs.singular
2
```

If an ideal is provided, the generators are used:

```
sage: Sequence(I)                                                     #␣
↪needs sage.libs.singular
[a + 2*b + 2*c + 2*d - 1,
 a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
 2*a*b + 2*b*c + 2*c*d - b,
 b^2 + 2*a*c + 2*b*d - c]
```

```
>>> from sage.all import *
>>> Sequence(I)                                                       #␣
↪needs sage.libs.singular
[a + 2*b + 2*c + 2*d - 1,
 a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
 2*a*b + 2*b*c + 2*c*d - b,
 b^2 + 2*a*c + 2*b*d - c]
```

If a list of polynomials is provided, the system has only one part:

```
sage: F = Sequence(I.gens(), I.ring()); F                             #␣
↪needs sage.libs.singular
[a + 2*b + 2*c + 2*d - 1,
 a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
 2*a*b + 2*b*c + 2*c*d - b,
 b^2 + 2*a*c + 2*b*d - c]
 sage: F.nparts()                                                     #␣
↪needs sage.libs.singular
 1
```

```
>>> from sage.all import *
>>> F = Sequence(I.gens(), I.ring()); F                                    #␣
↪needs sage.libs.singular
[a + 2*b + 2*c + 2*d - 1,
 a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
 2*a*b + 2*b*c + 2*c*d - b,
 b^2 + 2*a*c + 2*b*d - c]
>>> F.nparts()                                                            #␣
↪needs sage.libs.singular
1
```

We test that the ring is inferred correctly:

```
sage: P.<x,y,z> = GF(2)[]
sage: from sage.rings.polynomial.multi_polynomial_sequence import␣
↪PolynomialSequence
sage: PolynomialSequence([1,x,y]).ring()
Multivariate Polynomial Ring in x, y, z over Finite Field of size 2

sage: PolynomialSequence([[1,x,y], [0]]).ring()
Multivariate Polynomial Ring in x, y, z over Finite Field of size 2
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> from sage.rings.polynomial.multi_polynomial_sequence import PolynomialSequence
>>> PolynomialSequence([Integer(1),x,y]).ring()
Multivariate Polynomial Ring in x, y, z over Finite Field of size 2

>>> PolynomialSequence([[Integer(1),x,y], [Integer(0)]]).ring()
Multivariate Polynomial Ring in x, y, z over Finite Field of size 2
```

class sage.rings.polynomial.multi_polynomial_sequence.**PolynomialSequence_generic**(*parts*, *ring*, *immutable=False*, *cr=False*, *cr_str=None*)

Bases: `Sequence_generic`

Construct a new system of multivariate polynomials.

INPUT:

- `parts` – a list of lists with polynomials
- `ring` – a multivariate polynomial ring
- `immutable` – if `True` the sequence is immutable (default: `False`)
- `cr`, `cr_str` – see `Sequence()`

EXAMPLES:

```
sage: P.<a,b,c,d> = PolynomialRing(GF(127), 4)
sage: I = sage.rings.ideal.Katsura(P)                                    #␣
↪needs sage.libs.singular
```

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```
sage: Sequence([I.gens()], I.ring())  # indirect doctest                    #␣
↪needs sage.libs.singular
[a + 2*b + 2*c + 2*d - 1, a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
 2*a*b + 2*b*c + 2*c*d - b, b^2 + 2*a*c + 2*b*d - c]
```

```
>>> from sage.all import *
>>> P = PolynomialRing(GF(Integer(127)), Integer(4), names=('a', 'b', 'c', 'd',));
↪ (a, b, c, d,) = P._first_ngens(4)
>>> I = sage.rings.ideal.Katsura(P)                                         # needs␣
↪sage.libs.singular

>>> Sequence([I.gens()], I.ring())  # indirect doctest                      # needs␣
↪sage.libs.singular
[a + 2*b + 2*c + 2*d - 1, a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
 2*a*b + 2*b*c + 2*c*d - b, b^2 + 2*a*c + 2*b*d - c]
```

If an ideal is provided, the generators are used.:

```
sage: Sequence(I)                                                           #␣
↪needs sage.libs.singular
[a + 2*b + 2*c + 2*d - 1, a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
 2*a*b + 2*b*c + 2*c*d - b, b^2 + 2*a*c + 2*b*d - c]
```

```
>>> from sage.all import *
>>> Sequence(I)                                                             # needs␣
↪sage.libs.singular
[a + 2*b + 2*c + 2*d - 1, a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
 2*a*b + 2*b*c + 2*c*d - b, b^2 + 2*a*c + 2*b*d - c]
```

If a list of polynomials is provided, the system has only one part.:

```
sage: Sequence(I.gens(), I.ring())                                          #␣
↪needs sage.libs.singular
[a + 2*b + 2*c + 2*d - 1, a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
 2*a*b + 2*b*c + 2*c*d - b, b^2 + 2*a*c + 2*b*d - c]
```

```
>>> from sage.all import *
>>> Sequence(I.gens(), I.ring())                                            # needs␣
↪sage.libs.singular
[a + 2*b + 2*c + 2*d - 1, a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
 2*a*b + 2*b*c + 2*c*d - b, b^2 + 2*a*c + 2*b*d - c]
```

algebraic_dependence()

Return the ideal of annihilating polynomials for the polynomials in `self`, if those polynomials are algebraically dependent. Otherwise, return the zero ideal.

OUTPUT:

If the polynomials $f_1, \ldots, f_r$ in `self` are algebraically dependent, then the output is the ideal $\{F \in K[T_1, \ldots, T_r] : F(f_1, \ldots, f_r) = 0\}$ of annihilating polynomials of $f_1, \ldots, f_r$. Here K is the coefficient ring of polynomial ring of $f_1, \ldots, f_r$ and $T_1, \ldots, T_r$ are new indeterminates. If $f_1, \ldots, f_r$ are algebraically independent, then the output is the zero ideal in $K[T_1, \ldots, T_r]$.

EXAMPLES:

```
sage: # needs sage.libs.singular
sage: R.<x,y> = PolynomialRing(QQ)
sage: S = Sequence([x, x*y])
sage: I = S.algebraic_dependence(); I
Ideal (0) of Multivariate Polynomial Ring in T0, T1 over Rational Field
```

```
>>> from sage.all import *
>>> # needs sage.libs.singular
>>> R = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> S = Sequence([x, x*y])
>>> I = S.algebraic_dependence(); I
Ideal (0) of Multivariate Polynomial Ring in T0, T1 over Rational Field
```

```
sage: # needs sage.libs.singular
sage: R.<x,y> = PolynomialRing(QQ)
sage: S = Sequence([x, (x^2 + y^2 - 1)^2, x*y - 2])
sage: I = S.algebraic_dependence(); I
Ideal (16 + 32*T2 - 8*T0^2 + 24*T2^2 - 8*T0^2*T2 + 8*T2^3 + 9*T0^4 - 2*T0^
↪2*T2^2
        + T2^4 - T0^4*T1 + 8*T0^4*T2 - 2*T0^6 + 2*T0^4*T2^2 + T0^8)
 of Multivariate Polynomial Ring in T0, T1, T2 over Rational Field
sage: [F(S) for F in I.gens()]
[0]
```

```
>>> from sage.all import *
>>> # needs sage.libs.singular
>>> R = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> S = Sequence([x, (x**Integer(2) + y**Integer(2) - Integer(1))**Integer(2),
↪ x*y - Integer(2)])
>>> I = S.algebraic_dependence(); I
Ideal (16 + 32*T2 - 8*T0^2 + 24*T2^2 - 8*T0^2*T2 + 8*T2^3 + 9*T0^4 - 2*T0^
↪2*T2^2
        + T2^4 - T0^4*T1 + 8*T0^4*T2 - 2*T0^6 + 2*T0^4*T2^2 + T0^8)
 of Multivariate Polynomial Ring in T0, T1, T2 over Rational Field
>>> [F(S) for F in I.gens()]
[0]
```

```
sage: # needs sage.libs.singular
sage: R.<x,y> = PolynomialRing(GF(7))
sage: S = Sequence([x, (x^2 + y^2 - 1)^2, x*y - 2])
sage: I = S.algebraic_dependence(); I
Ideal (2 - 3*T2 - T0^2 + 3*T2^2 - T0^2*T2 + T2^3 + 2*T0^4 - 2*T0^2*T2^2
       + T2^4 - T0^4*T1 + T0^4*T2 - 2*T0^6 + 2*T0^4*T2^2 + T0^8)
 of Multivariate Polynomial Ring in T0, T1, T2 over Finite Field of size 7
sage: [F(S) for F in I.gens()]
[0]
```

```
>>> from sage.all import *
>>> # needs sage.libs.singular
>>> R = PolynomialRing(GF(Integer(7)), names=('x', 'y',)); (x, y,) = R._first_
```

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```
↪ngens(2)
>>> S = Sequence([x, (x**Integer(2) + y**Integer(2) - Integer(1))**Integer(2),
↪ x*y - Integer(2)])
>>> I = S.algebraic_dependence(); I
Ideal (2 - 3*T2 - T0^2 + 3*T2^2 - T0^2*T2 + T2^3 + 2*T0^4 - 2*T0^2*T2^2
       + T2^4 - T0^4*T1 + T0^4*T2 - 2*T0^6 + 2*T0^4*T2^2 + T0^8)
 of Multivariate Polynomial Ring in T0, T1, T2 over Finite Field of size 7
>>> [F(S) for F in I.gens()]
[0]
```

Note

This function's code also works for sequences of polynomials from a univariate polynomial ring, but i don't know where in the Sage codebase to put it to use it to that effect.

AUTHORS:

- Alex Raichev (2011-06-22)

coefficient_matrix(*sparse=True*)

Return tuple `(A,v)` where `A` is the coefficient matrix of this system and `v` the matching monomial vector.

Thus value of `A[i,j]` corresponds the coefficient of the monomial `v[j]` in the `i`-th polynomial in this system.

Monomials are order w.r.t. the term ordering of `self.ring()` in reverse order, i.e. such that the smallest entry comes last.

INPUT:

- `sparse` – construct a sparse matrix (default: `True`)

EXAMPLES:

```
sage: # needs sage.libs.singular
sage: P.<a,b,c,d> = PolynomialRing(GF(127), 4)
sage: I = sage.rings.ideal.Katsura(P)
sage: I.gens()
[a + 2*b + 2*c + 2*d - 1,
 a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
 2*a*b + 2*b*c + 2*c*d - b,
 b^2 + 2*a*c + 2*b*d - c]
sage: F = Sequence(I)
sage: A,v = F.coefficient_matrix()
doctest:warning...
DeprecationWarning: the function coefficient_matrix is deprecated; use
↪coefficients_monomials instead
See https://github.com/sagemath/sage/issues/37035 for details.
sage: A
[  0   0   0   0   0   0   0   0   0   1   2   2   2 126]
[  1   0   2   0   0   2   0   0   2 126   0   0   0   0]
[  0   2   0   0   2   0   0   2   0   0 126   0   0   0]
[  0   0   1   2   0   0   2   0   0   0   0 126   0   0]
sage: v
[a^2]
```

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```
[a*b]
[b^2]
[a*c]
[b*c]
[c^2]
[b*d]
[c*d]
[d^2]
[  a]
[  b]
[  c]
[  d]
[  1]
sage: A*v
[        a + 2*b + 2*c + 2*d - 1]
[a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a]
[      2*a*b + 2*b*c + 2*c*d - b]
[        b^2 + 2*a*c + 2*b*d - c]
```

```
>>> from sage.all import *
>>> # needs sage.libs.singular
>>> P = PolynomialRing(GF(Integer(127)), Integer(4), names=('a', 'b', 'c', 'd
↪',)); (a, b, c, d,) = P._first_ngens(4)
>>> I = sage.rings.ideal.Katsura(P)
>>> I.gens()
[a + 2*b + 2*c + 2*d - 1,
 a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
 2*a*b + 2*b*c + 2*c*d - b,
 b^2 + 2*a*c + 2*b*d - c]
>>> F = Sequence(I)
>>> A,v = F.coefficient_matrix()
doctest:warning...
DeprecationWarning: the function coefficient_matrix is deprecated; use
↪coefficients_monomials instead
See https://github.com/sagemath/sage/issues/37035 for details.
>>> A
[  0   0   0   0   0   0   0   0   0   1   2   2   2 126]
[  1   0   2   0   0   2   0   0   2 126   0   0   0   0]
[  0   2   0   0   2   0   0   2   0   0 126   0   0   0]
[  0   0   1   2   0   0   2   0   0   0   0 126   0   0]
>>> v
[a^2]
[a*b]
[b^2]
[a*c]
[b*c]
[c^2]
[b*d]
[c*d]
[d^2]
[  a]
[  b]
```

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```
[   c]
[   d]
[   1]
>>> A*v
[         a + 2*b + 2*c + 2*d - 1]
[a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a]
[      2*a*b + 2*b*c + 2*c*d - b]
[         b^2 + 2*a*c + 2*b*d - c]
```

coefficients_monomials(*order=None*, *sparse=True*)

Return the matrix of coefficients `A` and the matching vector of monomials `v`, such that `A*v == vector(self)`.

Thus value of `A[i,j]` corresponds the coefficient of the monomial `v[j]` in the `i`-th polynomial in this system.

Monomials are ordered w.r.t. the term ordering of `order` if given; otherwise, they are ordered w.r.t. `self.ring()` in reverse order, i.e., such that the smallest entry comes last.

INPUT:

- `sparse` – construct a sparse matrix (default: `True`)
- `order` – list or tuple specifying the order of monomials (default: `None`)

EXAMPLES:

```
sage: # needs sage.libs.singular
sage: P.<a,b,c,d> = PolynomialRing(GF(127), 4)
sage: I = sage.rings.ideal.Katsura(P)
sage: I.gens()
[a + 2*b + 2*c + 2*d - 1,
 a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
 2*a*b + 2*b*c + 2*c*d - b,
 b^2 + 2*a*c + 2*b*d - c]
sage: F = Sequence(I)
sage: A,v = F.coefficients_monomials()
sage: A
[  0   0   0   0   0   0   0   0   0   1   2   2   2 126]
[  1   0   2   0   0   2   0   0   2 126   0   0   0   0]
[  0   2   0   0   2   0   0   2   0   0 126   0   0   0]
[  0   0   1   2   0   0   2   0   0   0   0 126   0   0]
sage: v
(a^2, a*b, b^2, a*c, b*c, c^2, b*d, c*d, d^2, a, b, c, d, 1)
sage: A*v
(a + 2*b + 2*c + 2*d - 1, a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
 2*a*b + 2*b*c + 2*c*d - b, b^2 + 2*a*c + 2*b*d - c)
```

```
>>> from sage.all import *
>>> # needs sage.libs.singular
>>> P = PolynomialRing(GF(Integer(127)), Integer(4), names=('a', 'b', 'c', 'd
↪',)); (a, b, c, d,) = P._first_ngens(4)
>>> I = sage.rings.ideal.Katsura(P)
>>> I.gens()
[a + 2*b + 2*c + 2*d - 1,
 a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
```

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```
 2*a*b + 2*b*c + 2*c*d - b,
 b^2 + 2*a*c + 2*b*d - c]
>>> F = Sequence(I)
>>> A,v = F.coefficients_monomials()
>>> A
[  0   0   0   0   0   0   0   0   0   1   2   2   2 126]
[  1   0   2   0   0   2   0   0   2 126   0   0   0   0]
[  0   2   0   0   2   0   0   2   0   0 126   0   0   0]
[  0   0   1   2   0   0   2   0   0   0   0 126   0   0]
>>> v
(a^2, a*b, b^2, a*c, b*c, c^2, b*d, c*d, d^2, a, b, c, d, 1)
>>> A*v
(a + 2*b + 2*c + 2*d - 1, a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
 2*a*b + 2*b*c + 2*c*d - b, b^2 + 2*a*c + 2*b*d - c)
```

connected_components()

Split the polynomial system in systems which do not share any variables.

EXAMPLES:

As an example consider one part of AES, which naturally splits into four subsystems which are independent:

```
sage: # needs sage.rings.polynomial.pbori
sage: sr = mq.SR(2, 4, 4, 8, gf2=True, polybori=True)
sage: while True:  # workaround (see :issue:`31891`)
....:     try:
....:         F, s = sr.polynomial_system()
....:         break
....:     except ZeroDivisionError:
....:         pass
sage: Fz = Sequence(F.part(2))
sage: Fz.connected_components()
[Polynomial Sequence with 128 Polynomials in 128 Variables,
 Polynomial Sequence with 128 Polynomials in 128 Variables,
 Polynomial Sequence with 128 Polynomials in 128 Variables,
 Polynomial Sequence with 128 Polynomials in 128 Variables]
```

```
>>> from sage.all import *
>>> # needs sage.rings.polynomial.pbori
>>> sr = mq.SR(Integer(2), Integer(4), Integer(4), Integer(8), gf2=True,␣
↪polybori=True)
>>> while True:  # workaround (see :issue:`31891`)
...     try:
...         F, s = sr.polynomial_system()
...         break
...     except ZeroDivisionError:
...         pass
>>> Fz = Sequence(F.part(Integer(2)))
>>> Fz.connected_components()
[Polynomial Sequence with 128 Polynomials in 128 Variables,
 Polynomial Sequence with 128 Polynomials in 128 Variables,
 Polynomial Sequence with 128 Polynomials in 128 Variables,
 Polynomial Sequence with 128 Polynomials in 128 Variables]
```

connection_graph()

Return the graph which has the variables of this system as vertices and edges between two variables if they appear in the same polynomial.

EXAMPLES:

```
sage: # needs sage.rings.polynomial.pbori
sage: B.<x,y,z> = BooleanPolynomialRing()
sage: F = Sequence([x*y + y + 1, z + 1])
sage: G = F.connection_graph(); G
Graph on 3 vertices
sage: G.is_connected()
False
sage: F = Sequence([x])
sage: F.connection_graph()
Graph on 1 vertex
```

```
>>> from sage.all import *
>>> # needs sage.rings.polynomial.pbori
>>> B = BooleanPolynomialRing(names=('x', 'y', 'z',)); (x, y, z,) = B._first_
↪ngens(3)
>>> F = Sequence([x*y + y + Integer(1), z + Integer(1)])
>>> G = F.connection_graph(); G
Graph on 3 vertices
>>> G.is_connected()
False
>>> F = Sequence([x])
>>> F.connection_graph()
Graph on 1 vertex
```

groebner_basis(**args*, ***kwargs*)

Compute and return a Groebner basis for the ideal spanned by the polynomials in this system.

INPUT:

- `args` – list of arguments passed to `MPolynomialIdeal.groebner_basis` call
- `kwargs` – dictionary of arguments passed to `MPolynomialIdeal.groebner_basis` call

EXAMPLES:

```
sage: # needs sage.rings.polynomial.pbori
sage: sr = mq.SR(allow_zero_inversions=True)
sage: F, s = sr.polynomial_system()
sage: gb = F.groebner_basis()
sage: Ideal(gb).basis_is_groebner()
True
```

```
>>> from sage.all import *
>>> # needs sage.rings.polynomial.pbori
>>> sr = mq.SR(allow_zero_inversions=True)
>>> F, s = sr.polynomial_system()
>>> gb = F.groebner_basis()
>>> Ideal(gb).basis_is_groebner()
True
```

ideal()

Return ideal spanned by the elements of this system.

EXAMPLES:

```
sage: # needs sage.rings.polynomial.pbori
sage: sr = mq.SR(allow_zero_inversions=True)
sage: F, s = sr.polynomial_system()
sage: P = F.ring()
sage: I = F.ideal()
sage: J = I.elimination_ideal(P.gens()[4:-4])
sage: J <= I
True
sage: set(J.gens().variables()).issubset(P.gens()[:4] + P.gens()[-4:])
True
```

```
>>> from sage.all import *
>>> # needs sage.rings.polynomial.pbori
>>> sr = mq.SR(allow_zero_inversions=True)
>>> F, s = sr.polynomial_system()
>>> P = F.ring()
>>> I = F.ideal()
>>> J = I.elimination_ideal(P.gens()[Integer(4):-Integer(4)])
>>> J <= I
True
>>> set(J.gens().variables()).issubset(P.gens()[:Integer(4)] + P.gens()[-
↪Integer(4):])
True
```

is_groebner(*singular=Singular*)

Return `True` if the generators of this ideal (`self.gens()`) form a Groebner basis.

Let I be the set of generators of this ideal. The check is performed by trying to lift $Syz(LM(I))$ to $Syz(I)$ as I forms a Groebner basis if and only if for every element S in $Syz(LM(I))$:

$$S \star G = \sum_{i=0}^{m} h_i g_i \longrightarrow_G 0.$$

EXAMPLES:

```
sage: # needs sage.libs.singular
sage: R.<a,b,c,d,e,f,g,h,i,j> = PolynomialRing(GF(127), 10)
sage: I = sage.rings.ideal.Cyclic(R, 4)
sage: I.basis.is_groebner()
False
sage: I2 = Ideal(I.groebner_basis())
sage: I2.basis.is_groebner()
True
```

```
>>> from sage.all import *
>>> # needs sage.libs.singular
>>> R = PolynomialRing(GF(Integer(127)), Integer(10), names=('a', 'b', 'c', 'd
↪', 'e', 'f', 'g', 'h', 'i', 'j',)); (a, b, c, d, e, f, g, h, i, j,) = R._
↪first_ngens(10)
```

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```
>>> I = sage.rings.ideal.Cyclic(R, Integer(4))
>>> I.basis.is_groebner()
False
>>> I2 = Ideal(I.groebner_basis())
>>> I2.basis.is_groebner()
True
```

maximal_degree()

Return the maximal degree of any polynomial in this sequence.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(GF(7))
sage: F = Sequence([x*y + x, x])
sage: F.maximal_degree()
2
sage: P.<x,y,z> = PolynomialRing(GF(7))
sage: F = Sequence([], universe=P)
sage: F.maximal_degree()
-1
```

```
>>> from sage.all import *
>>> P = PolynomialRing(GF(Integer(7)), names=('x', 'y', 'z',)); (x, y, z,) =␣
↪P._first_ngens(3)
>>> F = Sequence([x*y + x, x])
>>> F.maximal_degree()
2
>>> P = PolynomialRing(GF(Integer(7)), names=('x', 'y', 'z',)); (x, y, z,) =␣
↪P._first_ngens(3)
>>> F = Sequence([], universe=P)
>>> F.maximal_degree()
-1
```

monomials()

Return an unordered tuple of monomials in this polynomial system.

EXAMPLES:

```
sage: sr = mq.SR(allow_zero_inversions=True)                                 #␣
↪needs sage.rings.polynomial.pbori
sage: F,s = sr.polynomial_system()                                           #␣
↪needs sage.rings.polynomial.pbori
sage: len(F.monomials())                                                     #␣
↪needs sage.rings.polynomial.pbori
49
```

```
>>> from sage.all import *
>>> sr = mq.SR(allow_zero_inversions=True)                                 #␣
↪needs sage.rings.polynomial.pbori
>>> F,s = sr.polynomial_system()                                           #␣
↪needs sage.rings.polynomial.pbori
>>> len(F.monomials())                                                     #␣
```

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```
↪needs sage.rings.polynomial.pbori
49
```

nmonomials()

Return the number of monomials present in this system.

EXAMPLES:

```
sage: sr = mq.SR(allow_zero_inversions=True)                                  #␣
↪needs sage.rings.polynomial.pbori
sage: F,s = sr.polynomial_system()                                            #␣
↪needs sage.rings.polynomial.pbori
sage: F.nmonomials()                                                          #␣
↪needs sage.rings.polynomial.pbori
49
```

```
>>> from sage.all import *
>>> sr = mq.SR(allow_zero_inversions=True)                                    #␣
↪needs sage.rings.polynomial.pbori
>>> F,s = sr.polynomial_system()                                              #␣
↪needs sage.rings.polynomial.pbori
>>> F.nmonomials()                                                            #␣
↪needs sage.rings.polynomial.pbori
49
```

nparts()

Return number of parts of this system.

EXAMPLES:

```
sage: sr = mq.SR(allow_zero_inversions=True)                                  #␣
↪needs sage.rings.polynomial.pbori
sage: F, s = sr.polynomial_system()                                           #␣
↪needs sage.rings.polynomial.pbori
sage: F.nparts()                                                              #␣
↪needs sage.rings.polynomial.pbori
4
```

```
>>> from sage.all import *
>>> sr = mq.SR(allow_zero_inversions=True)                                    #␣
↪needs sage.rings.polynomial.pbori
>>> F, s = sr.polynomial_system()                                             #␣
↪needs sage.rings.polynomial.pbori
>>> F.nparts()                                                                #␣
↪needs sage.rings.polynomial.pbori
4
```

nvariables()

Return number of variables present in this system.

EXAMPLES:

```
sage: sr = mq.SR(allow_zero_inversions=True)                                    #␣
↪needs sage.rings.polynomial.pbori
sage: F,s = sr.polynomial_system()                                              #␣
↪needs sage.rings.polynomial.pbori
sage: F.nvariables()                                                            #␣
↪needs sage.rings.polynomial.pbori
20
```

```
>>> from sage.all import *
>>> sr = mq.SR(allow_zero_inversions=True)                                    #␣
↪needs sage.rings.polynomial.pbori
>>> F,s = sr.polynomial_system()                                              #␣
↪needs sage.rings.polynomial.pbori
>>> F.nvariables()                                                            #␣
↪needs sage.rings.polynomial.pbori
20
```

part(*i*)

Return i-th part of this system.

EXAMPLES:

```
sage: # needs sage.rings.polynomial.pbori
sage: sr = mq.SR(allow_zero_inversions=True)
sage: F, s = sr.polynomial_system()
sage: R0 = F.part(1)
sage: R0
(k000^2 + k001, k001^2 + k002, k002^2 + k003, k003^2 + k000)
```

```
>>> from sage.all import *
>>> # needs sage.rings.polynomial.pbori
>>> sr = mq.SR(allow_zero_inversions=True)
>>> F, s = sr.polynomial_system()
>>> R0 = F.part(Integer(1))
>>> R0
(k000^2 + k001, k001^2 + k002, k002^2 + k003, k003^2 + k000)
```

parts()

Return a tuple of parts of this system.

EXAMPLES:

```
sage: # needs sage.rings.polynomial.pbori
sage: sr = mq.SR(allow_zero_inversions=True)
sage: F, s = sr.polynomial_system()
sage: l = F.parts()
sage: len(l)
4
```

```
>>> from sage.all import *
>>> # needs sage.rings.polynomial.pbori
>>> sr = mq.SR(allow_zero_inversions=True)
>>> F, s = sr.polynomial_system()
```

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```
>>> l = F.parts()
>>> len(l)
4
```

reduced()

If this sequence is $(f_1, ..., f_n)$ then this method returns $(g_1, ..., g_s)$ such that:

- $(f_1, ..., f_n) = (g_1, ..., g_s)$
- $LT(g_i) \neq LT(g_j)$ for all $i \neq j$
- $LT(g_i)$ does not divide m for all monomials m of $\{g_1, ..., g_{i-1}, g_{i+1}, ..., g_s\}$
- $LC(g_i) = 1$ for all i if the coefficient ring is a field.

EXAMPLES:

```
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: F = Sequence([z*x+y^3,z+y^3,z+x*y])
sage: F.reduced()
[y^3 + z, x*y + z, x*z - z]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> F = Sequence([z*x+y**Integer(3),z+y**Integer(3),z+x*y])
>>> F.reduced()
[y^3 + z, x*y + z, x*z - z]
```

Note that tail reduction for local orderings is not well-defined:

```
sage: R.<x,y,z> = PolynomialRing(QQ,order='negdegrevlex')
sage: F = Sequence([z*x+y^3,z+y^3,z+x*y])
sage: F.reduced()
[z + x*y, x*y - y^3, x^2*y - y^3]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ,order='negdegrevlex', names=('x', 'y', 'z',)); (x,␣
↪y, z,) = R._first_ngens(3)
>>> F = Sequence([z*x+y**Integer(3),z+y**Integer(3),z+x*y])
>>> F.reduced()
[z + x*y, x*y - y^3, x^2*y - y^3]
```

A fixed error with nonstandard base fields:

```
sage: R.<t>=QQ['t']
sage: K.<x,y>=R.fraction_field()['x,y']
sage: I=t*x*K
sage: I.basis.reduced()
[x]
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> K = R.fraction_field()['x,y']; (x, y,) = K._first_ngens(2)
```

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```
>>> I=t*x*K
>>> I.basis.reduced()
[x]
```

The interreduced basis of 0 is 0:

```
sage: P.<x,y,z> = GF(2)[]
sage: Sequence([P(0)]).reduced()
[0]
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> Sequence([P(Integer(0))]).reduced()
[0]
```

Leading coefficients are reduced to 1:

```
sage: P.<x,y> = QQ[]
sage: Sequence([2*x,y]).reduced()
[x, y]

sage: P.<x,y> = CC[]                                                        #␣
↪needs sage.rings.real_mpfr
sage: Sequence([2*x,y]).reduced()
[x, y]
```

```
>>> from sage.all import *
>>> P = QQ['x, y']; (x, y,) = P._first_ngens(2)
>>> Sequence([Integer(2)*x,y]).reduced()
[x, y]

>>> P = CC['x, y']; (x, y,) = P._first_ngens(2)# needs sage.rings.real_mpfr
>>> Sequence([Integer(2)*x,y]).reduced()
[x, y]
```

ALGORITHM:

Uses Singular's interred command or `sage.rings.polynomial.toy_buchberger.inter_reduction()` if conversion to Singular fails.

ring()

Return the polynomial ring all elements live in.

EXAMPLES:

```
sage: sr = mq.SR(allow_zero_inversions=True, gf2=True, order='block')       #␣
↪needs sage.rings.polynomial.pbori
sage: F, s = sr.polynomial_system()                                         #␣
↪needs sage.rings.polynomial.pbori
sage: print(F.ring().repr_long())                                           #␣
↪needs sage.rings.polynomial.pbori
Polynomial Ring
 Base Ring : Finite Field of size 2
```

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```
      Size : 20 Variables
  Block  0 : Ordering : deglex
             Names    : k100, k101, k102, k103, x100, x101, x102, x103, w100,␣
↪w101, w102, w103, s000, s001, s002, s003
  Block  1 : Ordering : deglex
             Names    : k000, k001, k002, k003
```

```
>>> from sage.all import *
>>> sr = mq.SR(allow_zero_inversions=True, gf2=True, order='block')      #␣
↪needs sage.rings.polynomial.pbori
>>> F, s = sr.polynomial_system()                                          #␣
↪needs sage.rings.polynomial.pbori
>>> print(F.ring().repr_long())                                            #␣
↪needs sage.rings.polynomial.pbori
Polynomial Ring
 Base Ring : Finite Field of size 2
      Size : 20 Variables
  Block  0 : Ordering : deglex
             Names    : k100, k101, k102, k103, x100, x101, x102, x103, w100,␣
↪w101, w102, w103, s000, s001, s002, s003
  Block  1 : Ordering : deglex
             Names    : k000, k001, k002, k003
```

subs (**args*, ***kwargs*)

Substitute variables for every polynomial in this system and return a new system. See `MPolynomial.subs()` for calling convention.

INPUT:

- `args` – arguments to be passed to `MPolynomial.subs()`
- `kwargs` – keyword arguments to be passed to `MPolynomial.subs()`

EXAMPLES:

```
sage: sr = mq.SR(allow_zero_inversions=True)                                 #␣
↪needs sage.rings.polynomial.pbori
sage: F, s = sr.polynomial_system(); F                                       #␣
↪needs sage.rings.polynomial.pbori
Polynomial Sequence with 40 Polynomials in 20 Variables
sage: F = F.subs(s); F                                                       #␣
↪needs sage.rings.polynomial.pbori
Polynomial Sequence with 40 Polynomials in 16 Variables
```

```
>>> from sage.all import *
>>> sr = mq.SR(allow_zero_inversions=True)                                 #␣
↪needs sage.rings.polynomial.pbori
>>> F, s = sr.polynomial_system(); F                                       #␣
↪needs sage.rings.polynomial.pbori
Polynomial Sequence with 40 Polynomials in 20 Variables
>>> F = F.subs(s); F                                                       #␣
↪needs sage.rings.polynomial.pbori
Polynomial Sequence with 40 Polynomials in 16 Variables
```

universe()

Return the polynomial ring all elements live in.

EXAMPLES:

```
sage: sr = mq.SR(allow_zero_inversions=True, gf2=True, order='block')        #␣
↪needs sage.rings.polynomial.pbori
sage: F, s = sr.polynomial_system()                                          #␣
↪needs sage.rings.polynomial.pbori
sage: print(F.ring().repr_long())                                            #␣
↪needs sage.rings.polynomial.pbori
Polynomial Ring
 Base Ring : Finite Field of size 2
      Size : 20 Variables
  Block  0 : Ordering : deglex
             Names    : k100, k101, k102, k103, x100, x101, x102, x103, w100,␣
↪w101, w102, w103, s000, s001, s002, s003
  Block  1 : Ordering : deglex
             Names    : k000, k001, k002, k003
```

```
>>> from sage.all import *
>>> sr = mq.SR(allow_zero_inversions=True, gf2=True, order='block')        #␣
↪needs sage.rings.polynomial.pbori
>>> F, s = sr.polynomial_system()                                          #␣
↪needs sage.rings.polynomial.pbori
>>> print(F.ring().repr_long())                                            #␣
↪needs sage.rings.polynomial.pbori
Polynomial Ring
 Base Ring : Finite Field of size 2
      Size : 20 Variables
  Block  0 : Ordering : deglex
             Names    : k100, k101, k102, k103, x100, x101, x102, x103, w100,␣
↪w101, w102, w103, s000, s001, s002, s003
  Block  1 : Ordering : deglex
             Names    : k000, k001, k002, k003
```

variables()

Return all variables present in this system. This tuple may or may not be equal to the generators of the ring of this system.

EXAMPLES:

```
sage: sr = mq.SR(allow_zero_inversions=True)                                 #␣
↪needs sage.rings.polynomial.pbori
sage: F,s = sr.polynomial_system()                                           #␣
↪needs sage.rings.polynomial.pbori
sage: F.variables()[:10]                                                     #␣
↪needs sage.rings.polynomial.pbori
(k003, k002, k001, k000, s003, s002, s001, s000, w103, w102)
```

```
>>> from sage.all import *
>>> sr = mq.SR(allow_zero_inversions=True)                                 #␣
↪needs sage.rings.polynomial.pbori
>>> F,s = sr.polynomial_system()                                           #␣
```

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```
↪needs sage.rings.polynomial.pbori
>>> F.variables()[:Integer(10)]                                                   ␣
↪     # needs sage.rings.polynomial.pbori
(k003, k002, k001, k000, s003, s002, s001, s000, w103, w102)
```

class sage.rings.polynomial.multi_polynomial_sequence.**PolynomialSequence_gf2**(*parts*, *ring*, *immutable=False*, *cr=False*, *cr_str=None*)

Bases: `PolynomialSequence_generic`

Polynomial Sequences over $\mathbf{F}_2$.

coefficients_monomials(*order=None*, *sparse=True*)

Return the matrix of coefficients `A` and the matching vector of monomials `v`, such that `A*v == vector(self)`.

Thus value of `A[i,j]` corresponds the coefficient of the monomial `v[j]` in the `i`-th polynomial in this system.

Monomials are ordered w.r.t. the term ordering of `order` if given; otherwise, they are ordered w.r.t. `self.ring()` in reverse order, i.e., such that the smallest entry comes last.

INPUT:

- `sparse` – construct a sparse matrix (default: `True`)
- `order` – list or tuple specifying the order of monomials (default: `None`)

EXAMPLES:

```
sage: # needs sage.rings.polynomial.pbori
sage: B.<x,y,z> = BooleanPolynomialRing()
sage: F = Sequence([x*y + y + 1, z + 1])
sage: A, v = F.coefficients_monomials()
sage: A
[1 1 0 1]
[0 0 1 1]
sage: v
(x*y, y, z, 1)
sage: A*v
(x*y + y + 1, z + 1)
```

```
>>> from sage.all import *
>>> # needs sage.rings.polynomial.pbori
>>> B = BooleanPolynomialRing(names=('x', 'y', 'z',)); (x, y, z,) = B._first_
↪ngens(3)
>>> F = Sequence([x*y + y + Integer(1), z + Integer(1)])
>>> A, v = F.coefficients_monomials()
>>> A
[1 1 0 1]
[0 0 1 1]
>>> v
(x*y, y, z, 1)
>>> A*v
(x*y + y + 1, z + 1)
```

eliminate_linear_variables(*maxlength=+Infinity*, *skip=None*, *return_reductors=False*, *use_polybori=False*)

Return a new system where linear leading variables are eliminated if the tail of the polynomial has length at most `maxlength`.

INPUT:

- `maxlength` – an optional upper bound on the number of monomials by which a variable is replaced. If `maxlength==+Infinity` then no condition is checked. (default: +Infinity).
- `skip` – an optional callable to skip eliminations. It must accept two parameters and return either `True` or `False`. The two parameters are the leading term and the tail of a polynomial (default: `None`).
- `return_reductors` – if `True` the list of polynomials with linear leading terms which were used for reduction is also returned (default: `False`).
- `use_polybori` – if `True` then `polybori.ll.eliminate` is called. While this is typically faster than what is implemented here, it is less flexible (`skip` is not supported) and may increase the degree (default: `False`)

OUTPUT:

With `return_reductors=True`, a pair of sequences of boolean polynomials are returned, along with the promises that:

1. The union of the two sequences spans the same boolean ideal as the argument of the method
2. The second sequence only contains linear polynomials, and it forms a reduced groebner basis (they all have pairwise distinct leading variables, and the leading variable of a polynomial does not occur anywhere in other polynomials).
3. The leading variables of the second sequence do not occur anywhere in the first sequence (these variables have been eliminated).

With `return_reductors=False`, only the first sequence is returned.

EXAMPLES:

```
sage: # needs sage.rings.polynomial.pbori
sage: B.<a,b,c,d> = BooleanPolynomialRing()
sage: F = Sequence([c + d + b + 1, a + c + d, a*b + c, b*c*d + c])
sage: F.eliminate_linear_variables() # everything vanishes
[]
sage: F.eliminate_linear_variables(maxlength=2)
[b + c + d + 1, b*c + b*d + c, b*c*d + c]
sage: F.eliminate_linear_variables(skip=lambda lm,tail: str(lm)=='a')
[a + c + d, a*c + a*d + a + c, c*d + c]
```

```
>>> from sage.all import *
>>> # needs sage.rings.polynomial.pbori
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd',)); (a, b, c, d,) = B.
↪_first_ngens(4)
>>> F = Sequence([c + d + b + Integer(1), a + c + d, a*b + c, b*c*d + c])
>>> F.eliminate_linear_variables() # everything vanishes
[]
>>> F.eliminate_linear_variables(maxlength=Integer(2))
[b + c + d + 1, b*c + b*d + c, b*c*d + c]
>>> F.eliminate_linear_variables(skip=lambda lm,tail: str(lm)=='a')
[a + c + d, a*c + a*d + a + c, c*d + c]
```

The list of reductors can be requested by setting `return_reductors` to `True`:

```
sage: # needs sage.rings.polynomial.pbori
sage: B.<a,b,c,d> = BooleanPolynomialRing()
sage: F = Sequence([a + b + d, a + b + c])
sage: F, R = F.eliminate_linear_variables(return_reductors=True)
sage: F
[]
sage: R
[a + b + d, c + d]
```

```
>>> from sage.all import *
>>> # needs sage.rings.polynomial.pbori
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd',)); (a, b, c, d,) = B.
↪_first_ngens(4)
>>> F = Sequence([a + b + d, a + b + c])
>>> F, R = F.eliminate_linear_variables(return_reductors=True)
>>> F
[]
>>> R
[a + b + d, c + d]
```

If the input system is detected to be inconsistent then `[1]` is returned, and the list of reductors is empty:

```
sage: # needs sage.rings.polynomial.pbori
sage: R.<x,y,z> = BooleanPolynomialRing()
sage: S = Sequence([x*y*z + x*y + z*y + x*z, x + y + z + 1, x + y + z])
sage: S.eliminate_linear_variables()
[1]
sage: R.<x,y,z> = BooleanPolynomialRing()
sage: S = Sequence([x*y*z + x*y + z*y + x*z, x + y + z + 1, x + y + z])
sage: S.eliminate_linear_variables(return_reductors=True)
([1], [])
```

```
>>> from sage.all import *
>>> # needs sage.rings.polynomial.pbori
>>> R = BooleanPolynomialRing(names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> S = Sequence([x*y*z + x*y + z*y + x*z, x + y + z + Integer(1), x + y + z])
>>> S.eliminate_linear_variables()
[1]
>>> R = BooleanPolynomialRing(names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> S = Sequence([x*y*z + x*y + z*y + x*z, x + y + z + Integer(1), x + y + z])
>>> S.eliminate_linear_variables(return_reductors=True)
([1], [])
```

> **Note**
>
> This is called "massaging" in [BCJ2007].

reduced()

If this sequence is $f_1, ..., f_n$, return $g_1, ..., g_s$ such that:

- $(f_1, ..., f_n) = (g_1, ..., g_s)$
- $LT(g_i) \neq LT(g_j)$ for all $i \neq j$
- $LT(g_i)$ does not divide m for all monomials m of $g_1, ..., g_{i-1}, g_{i+1}, ..., g_s$

EXAMPLES:

```
sage: # needs sage.rings.polynomial.pbori
sage: sr = mq.SR(1, 1, 1, 4, gf2=True, polybori=True)
sage: while True:  # workaround (see :issue:`31891`)
....:     try:
....:         F, s = sr.polynomial_system()
....:         break
....:     except ZeroDivisionError:
....:         pass
sage: g = F.reduced()
sage: len(g) == len(set(gi.lt() for gi in g))
True
sage: for i in range(len(g)):
....:     for j in range(len(g)):
....:         if i == j:
....:             continue
....:         for t in list(g[j]):
....:             assert g[i].lt() not in t.divisors()
```

```
>>> from sage.all import *
>>> # needs sage.rings.polynomial.pbori
>>> sr = mq.SR(Integer(1), Integer(1), Integer(1), Integer(4), gf2=True,␣
↪polybori=True)
>>> while True:  # workaround (see :issue:`31891`)
...     try:
...         F, s = sr.polynomial_system()
...         break
...     except ZeroDivisionError:
...         pass
>>> g = F.reduced()
>>> len(g) == len(set(gi.lt() for gi in g))
True
>>> for i in range(len(g)):
...     for j in range(len(g)):
...         if i == j:
...             continue
...         for t in list(g[j]):
...             assert g[i].lt() not in t.divisors()
```

solve(*algorithm='polybori'*, *n=1*, *eliminate_linear_variables=True*, *verbose=False*, ***kwds*)

Find solutions of this boolean polynomial system.

This function provide a unified interface to several algorithms dedicated to solving systems of boolean equations. Depending on the particular nature of the system, some might be much faster than some others.

INPUT:

- `self` – a sequence of boolean polynomials

- `algorithm` – the method to use. Possible values are `'polybori'`, `'sat'` and `'exhaustive_search'`. (default: `'polybori'`, since it is always available)
- `n` – (default: 1) number of solutions to return. If `n == +Infinity` then all solutions are returned. If $n < \infty$ then n solutions are returned if the equations have at least n solutions. Otherwise, all the solutions are returned.
- `eliminate_linear_variables` – whether to eliminate variables that appear linearly. This reduces the number of variables (makes solving faster a priori), but is likely to make the equations denser (may make solving slower depending on the method).
- `verbose` – boolean (default: `False`); whether to display progress and (potentially) useful information while the computation runs

EXAMPLES:

Without argument, a single arbitrary solution is returned:

```
sage: # needs sage.rings.polynomial.pbori
sage: R.<x,y,z> = BooleanPolynomialRing()
sage: S = Sequence([x*y + z, y*z + x, x + y + z + 1])
sage: sol = S.solve()
sage: sol
[{z: 0, y: 1, x: 0}]
```

```
>>> from sage.all import *
>>> # needs sage.rings.polynomial.pbori
>>> R = BooleanPolynomialRing(names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> S = Sequence([x*y + z, y*z + x, x + y + z + Integer(1)])
>>> sol = S.solve()
>>> sol
[{z: 0, y: 1, x: 0}]
```

We check that it is actually a solution:

```
sage: S.subs(sol[0])                                                    #␣
↪needs sage.rings.polynomial.pbori
[0, 0, 0]
```

```
>>> from sage.all import *
>>> S.subs(sol[Integer(0)])                                             ␣
↪    # needs sage.rings.polynomial.pbori
[0, 0, 0]
```

We obtain all solutions:

```
sage: sols = S.solve(n=Infinity)                                        #␣
↪needs sage.rings.polynomial.pbori
sage: sols                                                              #␣
↪needs sage.rings.polynomial.pbori
[{z: 0, y: 1, x: 0}, {z: 1, y: 1, x: 1}]
sage: [S.subs(x) for x in sols]                                         #␣
↪needs sage.rings.polynomial.pbori
[[0, 0, 0], [0, 0, 0]]
```

```
>>> from sage.all import *
>>> sols = S.solve(n=Infinity)                                              #␣
↪needs sage.rings.polynomial.pbori
>>> sols                                                                    #␣
↪needs sage.rings.polynomial.pbori
[{z: 0, y: 1, x: 0}, {z: 1, y: 1, x: 1}]
>>> [S.subs(x) for x in sols]                                               #␣
↪needs sage.rings.polynomial.pbori
[[0, 0, 0], [0, 0, 0]]
```

We can force the use of exhaustive search if the optional package `FES` is present:

```
sage: sol = S.solve(algorithm='exhaustive_search')  # optional - fes        #␣
↪needs sage.rings.polynomial.pbori
sage: sol                                           # optional - fes        #␣
↪needs sage.rings.polynomial.pbori
[{z: 1, y: 1, x: 1}]
sage: S.subs(sol[0])                                # optional - fes        #␣
↪needs sage.rings.polynomial.pbori
[0, 0, 0]
```

```
>>> from sage.all import *
>>> sol = S.solve(algorithm='exhaustive_search')  # optional - fes        #␣
↪needs sage.rings.polynomial.pbori
>>> sol                                           # optional - fes        #␣
↪needs sage.rings.polynomial.pbori
[{z: 1, y: 1, x: 1}]
>>> S.subs(sol[Integer(0)])                                # optional - fes ␣
↪      # needs sage.rings.polynomial.pbori
[0, 0, 0]
```

And we may use SAT-solvers if they are available:

```
sage: sol = S.solve(algorithm='sat')        # optional - pycryptosat        #␣
↪needs sage.rings.polynomial.pbori
sage: sol                                   # optional - pycryptosat        #␣
↪needs sage.rings.polynomial.pbori
[{z: 0, y: 1, x: 0}]
sage: S.subs(sol[0])                                                        #␣
↪needs sage.rings.polynomial.pbori
[0, 0, 0]
```

```
>>> from sage.all import *
>>> sol = S.solve(algorithm='sat')        # optional - pycryptosat        #␣
↪needs sage.rings.polynomial.pbori
>>> sol                                   # optional - pycryptosat        #␣
↪needs sage.rings.polynomial.pbori
[{z: 0, y: 1, x: 0}]
>>> S.subs(sol[Integer(0)])                                                 ␣
↪      # needs sage.rings.polynomial.pbori
[0, 0, 0]
```

class sage.rings.polynomial.multi_polynomial_sequence.**PolynomialSequence_gf2e**(*parts*, *ring*, *immutable=False*, *cr=False*, *cr_str=None*)

Bases: *PolynomialSequence_generic*

PolynomialSequence over $\mathbf{F}_{2^e}$, i.e extensions over $\mathbf{F}_(2)$.

weil_restriction()

Project this polynomial system to $\mathbf{F}_2$.

That is, compute the Weil restriction of scalars for the variety corresponding to this polynomial system and express it as a polynomial system over $\mathbf{F}_2$.

EXAMPLES:

```
sage: # needs sage.rings.finite_rings
sage: k.<a> = GF(2^2)
sage: P.<x,y> = PolynomialRing(k, 2)
sage: a = P.base_ring().gen()
sage: F = Sequence([x*y + 1, a*x + 1], P)
sage: F2 = F.weil_restriction()
sage: F2
[x0*y0 + x1*y1 + 1, x1*y0 + x0*y1 + x1*y1, x1 + 1, x0 + x1, x0^2 + x0,
 x1^2 + x1, y0^2 + y0, y1^2 + y1]
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> k = GF(Integer(2)**Integer(2), names=('a',)); (a,) = k._first_ngens(1)
>>> P = PolynomialRing(k, Integer(2), names=('x', 'y',)); (x, y,) = P._first_
↪ngens(2)
>>> a = P.base_ring().gen()
>>> F = Sequence([x*y + Integer(1), a*x + Integer(1)], P)
>>> F2 = F.weil_restriction()
>>> F2
[x0*y0 + x1*y1 + 1, x1*y0 + x0*y1 + x1*y1, x1 + 1, x0 + x1, x0^2 + x0,
 x1^2 + x1, y0^2 + y0, y1^2 + y1]
```

Another bigger example for a small scale AES:

```
sage: # needs sage.rings.polynomial.pbori
sage: sr = mq.SR(1, 1, 1, 4, gf2=False)
sage: while True:   # workaround (see :issue:`31891`)
....:     try:
....:         F, s = sr.polynomial_system()
....:         break
....:     except ZeroDivisionError:
....:         pass
sage: F
Polynomial Sequence with 40 Polynomials in 20 Variables
sage: F2 = F.weil_restriction(); F2
Polynomial Sequence with 240 Polynomials in 80 Variables
```

```
>>> from sage.all import *
>>> # needs sage.rings.polynomial.pbori
>>> sr = mq.SR(Integer(1), Integer(1), Integer(1), Integer(4), gf2=False)
>>> while True:  # workaround (see :issue:`31891`)
...     try:
...         F, s = sr.polynomial_system()
...         break
...     except ZeroDivisionError:
...         pass
>>> F
Polynomial Sequence with 40 Polynomials in 20 Variables
>>> F2 = F.weil_restriction(); F2
Polynomial Sequence with 240 Polynomials in 80 Variables
```

sage.rings.polynomial.multi_polynomial_sequence.**is_PolynomialSequence**(F)

Return `True` if `F` is a `PolynomialSequence`.

INPUT:

- `F` – anything

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQ)
sage: I = [[x^2 + y^2], [x^2 - y^2]]
sage: F = Sequence(I, P); F
[x^2 + y^2, x^2 - y^2]

sage: from sage.rings.polynomial.multi_polynomial_sequence import␣
↪PolynomialSequence_generic
sage: isinstance(F, PolynomialSequence_generic)
True
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = P._first_ngens(2)
>>> I = [[x**Integer(2) + y**Integer(2)], [x**Integer(2) - y**Integer(2)]]
>>> F = Sequence(I, P); F
[x^2 + y^2, x^2 - y^2]

>>> from sage.rings.polynomial.multi_polynomial_sequence import␣
↪PolynomialSequence_generic
>>> isinstance(F, PolynomialSequence_generic)
True
```

3.1.8 Multivariate Polynomials via libSINGULAR

This module implements specialized and optimized implementations for multivariate polynomials over many coefficient rings, via a shared library interface to SINGULAR. In particular, the following coefficient rings are supported by this implementation:

- the rational numbers $\mathbf{Q}$,
- the ring of integers $\mathbf{Z}$,
- $\mathbf{Z}/n\mathbf{Z}$ for any integer n,

- finite fields $\mathbf{F}_{p^n}$ for p prime and $n > 0$,
- and absolute number fields $\mathbf{Q}(a)$.

EXAMPLES:

We show how to construct various multivariate polynomial rings:

```
sage: P.<x,y,z> = QQ[]
sage: P
Multivariate Polynomial Ring in x, y, z over Rational Field

sage: f = 27/113 * x^2 + y*z + 1/2; f
27/113*x^2 + y*z + 1/2

sage: P.term_order()
Degree reverse lexicographic term order

sage: P = PolynomialRing(GF(127), 3, names='abc', order='lex'); P
Multivariate Polynomial Ring in a, b, c over Finite Field of size 127
sage: a,b,c = P.gens()
sage: f = 57 * a^2*b + 43 * c + 1; f
57*a^2*b + 43*c + 1
sage: P.term_order()
Lexicographic term order

sage: z = QQ['z'].0
sage: K.<s> = NumberField(z^2 - 2)                                      #␣
↪needs sage.rings.number_field
sage: P.<x,y> = PolynomialRing(K, 2)                                    #␣
↪needs sage.rings.number_field
sage: 1/2*s*x^2 + 3/4*s                                                 #␣
↪needs sage.rings.number_field
(1/2*s)*x^2 + (3/4*s)

sage: P.<x,y,z> = ZZ[]; P
Multivariate Polynomial Ring in x, y, z over Integer Ring

sage: P.<x,y,z> = Zmod(2^10)[]; P
Multivariate Polynomial Ring in x, y, z over Ring of integers modulo 1024

sage: P.<x,y,z> = Zmod(3^10)[]; P
Multivariate Polynomial Ring in x, y, z over Ring of integers modulo 59049

sage: P.<x,y,z> = Zmod(2^100)[]; P
Multivariate Polynomial Ring in x, y, z over
 Ring of integers modulo 1267650600228229401496703205376

sage: P.<x,y,z> = Zmod(2521352)[]; P
Multivariate Polynomial Ring in x, y, z over Ring of integers modulo 2521352
sage: type(P)
<class 'sage.rings.polynomial.multi_polynomial_libsingular.MPolynomialRing_libsingular
↪'>

sage: P.<x,y,z> = Zmod(25213521351515232)[]; P
```

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```
Multivariate Polynomial Ring in x, y, z over Ring of integers modulo 25213521351515232
sage: type(P)
<class 'sage.rings.polynomial.multi_polynomial_libsingular.MPolynomialRing_libsingular
↪'>
```

```
>>> from sage.all import *
>>> P = QQ['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> P
Multivariate Polynomial Ring in x, y, z over Rational Field

>>> f = Integer(27)/Integer(113) * x**Integer(2) + y*z + Integer(1)/Integer(2); f
27/113*x^2 + y*z + 1/2

>>> P.term_order()
Degree reverse lexicographic term order

>>> P = PolynomialRing(GF(Integer(127)), Integer(3), names='abc', order='lex'); P
Multivariate Polynomial Ring in a, b, c over Finite Field of size 127
>>> a,b,c = P.gens()
>>> f = Integer(57) * a**Integer(2)*b + Integer(43) * c + Integer(1); f
57*a^2*b + 43*c + 1
>>> P.term_order()
Lexicographic term order

>>> z = QQ['z'].gen(0)
>>> K = NumberField(z**Integer(2) - Integer(2), names=('s',)); (s,) = K._first_
↪ngens(1)# needs sage.rings.number_field
>>> P = PolynomialRing(K, Integer(2), names=('x', 'y',)); (x, y,) = P._first_ngens(2)
↪# needs sage.rings.number_field
>>> Integer(1)/Integer(2)*s*x**Integer(2) + Integer(3)/Integer(4)*s                  ␣
↪                                         # needs sage.rings.number_field
(1/2*s)*x^2 + (3/4*s)

>>> P = ZZ['x, y, z']; (x, y, z,) = P._first_ngens(3); P
Multivariate Polynomial Ring in x, y, z over Integer Ring

>>> P = Zmod(Integer(2)**Integer(10))['x, y, z']; (x, y, z,) = P._first_ngens(3); P
Multivariate Polynomial Ring in x, y, z over Ring of integers modulo 1024

>>> P = Zmod(Integer(3)**Integer(10))['x, y, z']; (x, y, z,) = P._first_ngens(3); P
Multivariate Polynomial Ring in x, y, z over Ring of integers modulo 59049

>>> P = Zmod(Integer(2)**Integer(100))['x, y, z']; (x, y, z,) = P._first_ngens(3); P
Multivariate Polynomial Ring in x, y, z over
 Ring of integers modulo 1267650600228229401496703205376

>>> P = Zmod(Integer(2521352))['x, y, z']; (x, y, z,) = P._first_ngens(3); P
Multivariate Polynomial Ring in x, y, z over Ring of integers modulo 2521352
>>> type(P)
<class 'sage.rings.polynomial.multi_polynomial_libsingular.MPolynomialRing_libsingular
↪'>
```

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```
>>> P = Zmod(Integer(25213521351515232))['x, y, z']; (x, y, z,) = P._first_ngens(3); P
Multivariate Polynomial Ring in x, y, z over Ring of integers modulo 25213521351515232
>>> type(P)
<class 'sage.rings.polynomial.multi_polynomial_libsingular.MPolynomialRing_libsingular
↪'>
```

We construct the Frobenius morphism on $\mathbf{F}_5[x, y, z]$ over $\mathbf{F}_5$:

```
sage: R.<x,y,z> = PolynomialRing(GF(5), 3)
sage: frob = R.hom([x^5, y^5, z^5])
sage: frob(x^2 + 2*y - z^4)
-z^20 + x^10 + 2*y^5
sage: frob((x + 2*y)^3)                                                         #␣
↪needs sage.rings.finite_rings
x^15 + x^10*y^5 + 2*x^5*y^10 - 2*y^15
sage: (x^5 + 2*y^5)^3                                                           #␣
↪needs sage.rings.finite_rings
x^15 + x^10*y^5 + 2*x^5*y^10 - 2*y^15
```

```
>>> from sage.all import *
>>> R = PolynomialRing(GF(Integer(5)), Integer(3), names=('x', 'y', 'z',)); (x, y, z,
↪) = R._first_ngens(3)
>>> frob = R.hom([x**Integer(5), y**Integer(5), z**Integer(5)])
>>> frob(x**Integer(2) + Integer(2)*y - z**Integer(4))
-z^20 + x^10 + 2*y^5
>>> frob((x + Integer(2)*y)**Integer(3))                                         ␣
↪             # needs sage.rings.finite_rings
x^15 + x^10*y^5 + 2*x^5*y^10 - 2*y^15
>>> (x**Integer(5) + Integer(2)*y**Integer(5))**Integer(3)                       ␣
↪                         # needs sage.rings.finite_rings
x^15 + x^10*y^5 + 2*x^5*y^10 - 2*y^15
```

We make a polynomial ring in one variable over a polynomial ring in two variables:

```
sage: R.<x, y> = PolynomialRing(QQ, 2)
sage: S.<t> = PowerSeriesRing(R)
sage: t*(x+y)
(x + y)*t
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(2), names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> S = PowerSeriesRing(R, names=('t',)); (t,) = S._first_ngens(1)
>>> t*(x+y)
(x + y)*t
```

Todo

Implement Real, Complex coefficient rings via libSINGULAR

AUTHORS:

- Martin Albrecht (2007-01): initial implementation

- Joel Mohler (2008-01): misc improvements, polishing
- Martin Albrecht (2008-08): added $\mathbf{Q}(a)$ and $\mathbf{Z}$ support
- Simon King (2009-04): improved coercion
- Martin Albrecht (2009-05): added $\mathbf{Z}/n\mathbf{Z}$ support, refactoring
- Martin Albrecht (2009-06): refactored the code to allow better re-use
- Simon King (2011-03): use a faster way of conversion from the base ring.
- Volker Braun (2011-06): major cleanup, refcount singular rings, bugfixes.

class sage.rings.polynomial.multi_polynomial_libsingular.**MPolynomialRing_libsingular**

Bases: *MPolynomialRing_base*

Construct a multivariate polynomial ring subject to the following conditions:

INPUT:

- **base_ring – base ring (must be either GF(q), ZZ, ZZ/nZZ,**
  QQ or absolute number field)
- n – number of variables (must be at least 1)
- names – names of ring variables, may be string of list/tuple
- order – term order (default: degrevlex)

EXAMPLES:

```
sage: P.<x,y,z> = QQ[]; P
Multivariate Polynomial Ring in x, y, z over Rational Field
sage: f = 27/113 * x^2 + y*z + 1/2; f
27/113*x^2 + y*z + 1/2
sage: P.term_order()
Degree reverse lexicographic term order

sage: P = PolynomialRing(GF(127), 3, names='abc', order='lex'); P
Multivariate Polynomial Ring in a, b, c over Finite Field of size 127
sage: a,b,c = P.gens()
sage: f = 57 * a^2*b + 43 * c + 1; f
57*a^2*b + 43*c + 1
sage: P.term_order()
Lexicographic term order

sage: z = QQ['z'].0
sage: K.<s> = NumberField(z^2 - 2)                                          #␣
↪needs sage.rings.number_field
sage: P.<x,y> = PolynomialRing(K, 2)                                        #␣
↪needs sage.rings.number_field
sage: 1/2*s*x^2 + 3/4*s                                                     #␣
↪needs sage.rings.number_field
(1/2*s)*x^2 + (3/4*s)

sage: P.<x,y,z> = ZZ[]; P
Multivariate Polynomial Ring in x, y, z over Integer Ring

sage: P.<x,y,z> = Zmod(2^10)[]; P
```

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```
Multivariate Polynomial Ring in x, y, z over Ring of integers modulo 1024

sage: P.<x,y,z> = Zmod(3^10)[]; P
Multivariate Polynomial Ring in x, y, z over Ring of integers modulo 59049

sage: P.<x,y,z> = Zmod(2^100)[]; P
Multivariate Polynomial Ring in x, y, z over
 Ring of integers modulo 1267650600228229401496703205376

sage: P.<x,y,z> = Zmod(2521352)[]; P
Multivariate Polynomial Ring in x, y, z over Ring of integers modulo 2521352
sage: type(P)
<class 'sage.rings.polynomial.multi_polynomial_libsingular.MPolynomialRing_
→libsingular'>

sage: P.<x,y,z> = Zmod(25213521351515232)[]; P
Multivariate Polynomial Ring in x, y, z over
 Ring of integers modulo 25213521351515232
sage: type(P)
<class 'sage.rings.polynomial.multi_polynomial_libsingular.MPolynomialRing_
→libsingular'>

sage: P.<x,y,z> = PolynomialRing(Integers(2^32), order='lex')
sage: P(2^32-1)
-1
```

```
>>> from sage.all import *
>>> P = QQ['x, y, z']; (x, y, z,) = P._first_ngens(3); P
Multivariate Polynomial Ring in x, y, z over Rational Field
>>> f = Integer(27)/Integer(113) * x**Integer(2) + y*z + Integer(1)/Integer(2); f
27/113*x^2 + y*z + 1/2
>>> P.term_order()
Degree reverse lexicographic term order

>>> P = PolynomialRing(GF(Integer(127)), Integer(3), names='abc', order='lex'); P
Multivariate Polynomial Ring in a, b, c over Finite Field of size 127
>>> a,b,c = P.gens()
>>> f = Integer(57) * a**Integer(2)*b + Integer(43) * c + Integer(1); f
57*a^2*b + 43*c + 1
>>> P.term_order()
Lexicographic term order

>>> z = QQ['z'].gen(0)
>>> K = NumberField(z**Integer(2) - Integer(2), names=('s',)); (s,) = K._first_
→ngens(1)# needs sage.rings.number_field
>>> P = PolynomialRing(K, Integer(2), names=('x', 'y',)); (x, y,) = P._first_
→ngens(2)# needs sage.rings.number_field
>>> Integer(1)/Integer(2)*s*x**Integer(2) + Integer(3)/Integer(4)*s                    ␣
→                                    # needs sage.rings.number_field
(1/2*s)*x^2 + (3/4*s)

>>> P = ZZ['x, y, z']; (x, y, z,) = P._first_ngens(3); P
```

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```
Multivariate Polynomial Ring in x, y, z over Integer Ring

>>> P = Zmod(Integer(2)**Integer(10))['x, y, z']; (x, y, z,) = P._first_ngens(3);␣
↪P
Multivariate Polynomial Ring in x, y, z over Ring of integers modulo 1024

>>> P = Zmod(Integer(3)**Integer(10))['x, y, z']; (x, y, z,) = P._first_ngens(3);␣
↪P
Multivariate Polynomial Ring in x, y, z over Ring of integers modulo 59049

>>> P = Zmod(Integer(2)**Integer(100))['x, y, z']; (x, y, z,) = P._first_ngens(3);
↪ P
Multivariate Polynomial Ring in x, y, z over
 Ring of integers modulo 1267650600228229401496703205376

>>> P = Zmod(Integer(2521352))['x, y, z']; (x, y, z,) = P._first_ngens(3); P
Multivariate Polynomial Ring in x, y, z over Ring of integers modulo 2521352
>>> type(P)
<class 'sage.rings.polynomial.multi_polynomial_libsingular.MPolynomialRing_
↪libsingular'>

>>> P = Zmod(Integer(25213521351515232))['x, y, z']; (x, y, z,) = P._first_
↪ngens(3); P
Multivariate Polynomial Ring in x, y, z over
 Ring of integers modulo 25213521351515232
>>> type(P)
<class 'sage.rings.polynomial.multi_polynomial_libsingular.MPolynomialRing_
↪libsingular'>

>>> P = PolynomialRing(Integers(Integer(2)**Integer(32)), order='lex', names=('x',
↪ 'y', 'z',)); (x, y, z,) = P._first_ngens(3)
>>> P(Integer(2)**Integer(32)-Integer(1))
-1
```

Element

alias of *MPolynomial_libsingular*

gen(*n=0*)

Return the n-th generator of this multivariate polynomial ring.

INPUT:

- n – nonnegative integer

EXAMPLES:

```
sage: P.<x,y,z> = QQ[]
sage: P.gen(), P.gen(1)
(x, y)

sage: P = PolynomialRing(GF(127), 1000, 'x')
sage: P.gen(500)
x500
```

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```
sage: P.<SAGE,SINGULAR> = QQ[] # weird names
sage: P.gen(1)
SINGULAR
```

```
>>> from sage.all import *
>>> P = QQ['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> P.gen(), P.gen(Integer(1))
(x, y)

>>> P = PolynomialRing(GF(Integer(127)), Integer(1000), 'x')
>>> P.gen(Integer(500))
x500

>>> P = QQ['SAGE, SINGULAR']; (SAGE, SINGULAR,) = P._first_ngens(2)# weird␣
↪names
>>> P.gen(Integer(1))
SINGULAR
```

ideal(**gens*, ***kwds*)

Create an ideal in this polynomial ring.

INPUT:

- `*gens` – list or tuple of generators (or several input arguments)
- `coerce` – boolean (default: `True`); this must be a keyword argument. Only set it to `False` if you are certain that each generator is already in the ring.

EXAMPLES:

```
sage: P.<x,y,z> = QQ[]
sage: sage.rings.ideal.Katsura(P)                                          #␣
↪needs sage.rings.function_field
Ideal (x + 2*y + 2*z - 1, x^2 + 2*y^2 + 2*z^2 - x, 2*x*y + 2*y*z - y)
 of Multivariate Polynomial Ring in x, y, z over Rational Field

sage: P.ideal([x + 2*y + 2*z-1, 2*x*y + 2*y*z-y, x^2 + 2*y^2 + 2*z^2-x])
Ideal (x + 2*y + 2*z - 1, 2*x*y + 2*y*z - y, x^2 + 2*y^2 + 2*z^2 - x)
 of Multivariate Polynomial Ring in x, y, z over Rational Field
```

```
>>> from sage.all import *
>>> P = QQ['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> sage.rings.ideal.Katsura(P)                                        #␣
↪needs sage.rings.function_field
Ideal (x + 2*y + 2*z - 1, x^2 + 2*y^2 + 2*z^2 - x, 2*x*y + 2*y*z - y)
 of Multivariate Polynomial Ring in x, y, z over Rational Field

>>> P.ideal([x + Integer(2)*y + Integer(2)*z-Integer(1), Integer(2)*x*y +␣
↪Integer(2)*y*z-y, x**Integer(2) + Integer(2)*y**Integer(2) +␣
↪Integer(2)*z**Integer(2)-x])
Ideal (x + 2*y + 2*z - 1, 2*x*y + 2*y*z - y, x^2 + 2*y^2 + 2*z^2 - x)
 of Multivariate Polynomial Ring in x, y, z over Rational Field
```

monomial_all_divisors(*t*)

Return a list of all monomials that divide `t`.

Coefficients are ignored.

INPUT:

- `t` – a monomial

OUTPUT: list of monomials

EXAMPLES:

```
sage: P.<x,y,z> = QQ[]
sage: P.monomial_all_divisors(x^2*z^3)
[x, x^2, z, x*z, x^2*z, z^2, x*z^2, x^2*z^2, z^3, x*z^3, x^2*z^3]
```

```
>>> from sage.all import *
>>> P = QQ['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> P.monomial_all_divisors(x**Integer(2)*z**Integer(3))
[x, x^2, z, x*z, x^2*z, z^2, x*z^2, x^2*z^2, z^3, x*z^3, x^2*z^3]
```

ALGORITHM: addwithcarry idea by Toon Segers

monomial_divides(a, b)

Return `False` if a does not divide b and `True` otherwise.

Coefficients are ignored.

INPUT:

- `a` – monomial
- `b` – monomial

EXAMPLES:

```
sage: P.<x,y,z> = QQ[]
sage: P.monomial_divides(x*y*z, x^3*y^2*z^4)
True
sage: P.monomial_divides(x^3*y^2*z^4, x*y*z)
False
```

```
>>> from sage.all import *
>>> P = QQ['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> P.monomial_divides(x*y*z, x**Integer(3)*y**Integer(2)*z**Integer(4))
True
>>> P.monomial_divides(x**Integer(3)*y**Integer(2)*z**Integer(4), x*y*z)
False
```

monomial_lcm(f, g)

LCM for monomials. Coefficients are ignored.

INPUT:

- `f` – monomial
- `g` – monomial

EXAMPLES:

```
sage: P.<x,y,z> = QQ[]
sage: P.monomial_lcm(3/2*x*y,x)
x*y
```

```
>>> from sage.all import *
>>> P = QQ['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> P.monomial_lcm(Integer(3)/Integer(2)*x*y,x)
x*y
```

monomial_pairwise_prime(*g*, *h*)

Return `True` if `h` and `g` are pairwise prime. Both are treated as monomials.

Coefficients are ignored.

INPUT:

- `h` – monomial
- `g` – monomial

EXAMPLES:

```
sage: P.<x,y,z> = QQ[]
sage: P.monomial_pairwise_prime(x^2*z^3, y^4)
True

sage: P.monomial_pairwise_prime(1/2*x^3*y^2, 3/4*y^3)
False
```

```
>>> from sage.all import *
>>> P = QQ['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> P.monomial_pairwise_prime(x**Integer(2)*z**Integer(3), y**Integer(4))
True

>>> P.monomial_pairwise_prime(Integer(1)/
↪Integer(2)*x**Integer(3)*y**Integer(2), Integer(3)/Integer(4)*y**Integer(3))
False
```

monomial_quotient(*f*, *g*, *coeff=False*)

Return `f/g`, where both `f` and`` `g` are treated as monomials.

Coefficients are ignored by default.

INPUT:

- `f` – monomial
- `g` – monomial
- `coeff` – divide coefficients as well (default: `False`)

EXAMPLES:

```
sage: P.<x,y,z> = QQ[]
sage: P.monomial_quotient(3/2*x*y,x)
y
```

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```
sage: P.monomial_quotient(3/2*x*y,x,coeff=True)
3/2*y
```

```
>>> from sage.all import *
>>> P = QQ['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> P.monomial_quotient(Integer(3)/Integer(2)*x*y,x)
y

>>> P.monomial_quotient(Integer(3)/Integer(2)*x*y,x,coeff=True)
3/2*y
```

Note, that **Z** behaves different if `coeff=True`:

```
sage: P.monomial_quotient(2*x,3*x)
1

sage: P.<x,y> = PolynomialRing(ZZ)
sage: P.monomial_quotient(2*x,3*x,coeff=True)
Traceback (most recent call last):
...
ArithmeticError: Cannot divide these coefficients.
```

```
>>> from sage.all import *
>>> P.monomial_quotient(Integer(2)*x,Integer(3)*x)
1

>>> P = PolynomialRing(ZZ, names=('x', 'y',)); (x, y,) = P._first_ngens(2)
>>> P.monomial_quotient(Integer(2)*x,Integer(3)*x,coeff=True)
Traceback (most recent call last):
...
ArithmeticError: Cannot divide these coefficients.
```

Warning

Assumes that the head term of f is a multiple of the head term of g and return the multiplicant m. If this rule is violated, funny things may happen.

monomial_reduce(f, G)

Try to find a `g` in `G` where `g.lm()` divides `f`. If found `(flt,g)` is returned, `(0,0)` otherwise, where `flt` is `f/g.lm()`.

It is assumed that `G` is iterable and contains *only* elements in this polynomial ring.

Coefficients are ignored.

INPUT:

- `f` – monomial
- `G` – list/set of mpolynomials

EXAMPLES:

```
sage: P.<x,y,z> = QQ[]
sage: f = x*y^2
sage: G = [ 3/2*x^3 + y^2 + 1/2, 1/4*x*y + 2/7, 1/2  ]
sage: P.monomial_reduce(f,G)
(y, 1/4*x*y + 2/7)
```

```
>>> from sage.all import *
>>> P = QQ['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> f = x*y**Integer(2)
>>> G = [ Integer(3)/Integer(2)*x**Integer(3) + y**Integer(2) + Integer(1)/
↪Integer(2), Integer(1)/Integer(4)*x*y + Integer(2)/Integer(7), Integer(1)/
↪Integer(2)  ]
>>> P.monomial_reduce(f,G)
(y, 1/4*x*y + 2/7)
```

ngens()

Return the number of variables in this multivariate polynomial ring.

EXAMPLES:

```
sage: P.<x,y> = QQ[]
sage: P.ngens()
2

sage: k.<a> = GF(2^16)                                                  #␣
↪needs sage.rings.finite_rings
sage: P = PolynomialRing(k, 1000, 'x')                                  #␣
↪needs sage.rings.finite_rings
sage: P.ngens()                                                         #␣
↪needs sage.rings.finite_rings
1000
```

```
>>> from sage.all import *
>>> P = QQ['x, y']; (x, y,) = P._first_ngens(2)
>>> P.ngens()
2

>>> k = GF(Integer(2)**Integer(16), names=('a',)); (a,) = k._first_ngens(1)#␣
↪needs sage.rings.finite_rings
>>> P = PolynomialRing(k, Integer(1000), 'x')                             ␣
↪     # needs sage.rings.finite_rings
>>> P.ngens()                                                           #␣
↪needs sage.rings.finite_rings
1000
```

class sage.rings.polynomial.multi_polynomial_libsingular.**MPolynomial_libsingular**

Bases: *MPolynomial_libsingular*

A multivariate polynomial implemented using libSINGULAR.

add_m_mul_q(*m*, *q*)

Return `self + m*q`, where `m` must be a monomial and `q` a polynomial.

INPUT:

- m – a monomial
- q – a polynomial

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ,3)
sage: x.add_m_mul_q(y,z)
y*z + x
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ,Integer(3), names=('x', 'y', 'z',)); (x, y, z,) = P.
↪_first_ngens(3)
>>> x.add_m_mul_q(y,z)
y*z + x
```

coefficient(*degrees*)

Return the coefficient of the variables with the degrees specified in the python dictionary degrees. Mathematically, this is the coefficient in the base ring adjoined by the variables of this ring not listed in degrees. However, the result has the same parent as this polynomial.

This function contrasts with the function monomial_coefficient which returns the coefficient in the base ring of a monomial.

INPUT:

- degrees – can be any of: - a dictionary of degree restrictions - a list of degree restrictions (with None in the unrestricted variables) - a monomial (very fast, but not as flexible)

OUTPUT: element of the parent of this element

Note

For coefficients of specific monomials, look at *monomial_coefficient()*.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: f=x*y+y+5
sage: f.coefficient({x:0,y:1})
1
sage: f.coefficient({x:0})
y + 5
sage: f=(1+y+y^2)*(1+x+x^2)
sage: f.coefficient({x:0})
y^2 + y + 1
sage: f.coefficient([0,None])
y^2 + y + 1
sage: f.coefficient(x)
y^2 + y + 1
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> f=x*y+y+Integer(5)
>>> f.coefficient({x:Integer(0),y:Integer(1)})
```

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```
1
>>> f.coefficient({x:Integer(0)})
y + 5
>>> f=(Integer(1)+y+y**Integer(2))*(Integer(1)+x+x**Integer(2))
>>> f.coefficient({x:Integer(0)})
y^2 + y + 1
>>> f.coefficient([Integer(0),None])
y^2 + y + 1
>>> f.coefficient(x)
y^2 + y + 1
```

Note that exponents have all variables specified:

```
sage: x.coefficient(x.exponents()[0])
1
sage: f.coefficient([1,0])
1
sage: f.coefficient({x:1,y:0})
1
```

```
>>> from sage.all import *
>>> x.coefficient(x.exponents()[Integer(0)])
1
>>> f.coefficient([Integer(1),Integer(0)])
1
>>> f.coefficient({x:Integer(1),y:Integer(0)})
1
```

Be aware that this may not be what you think! The physical appearance of the variable x is deceiving – particularly if the exponent would be a variable.

```
sage: f.coefficient(x^0) # outputs the full polynomial
x^2*y^2 + x^2*y + x*y^2 + x^2 + x*y + y^2 + x + y + 1
sage: R.<x,y> = GF(389)[]
sage: f = x*y + 5
sage: c = f.coefficient({x:0, y:0}); c
5
sage: parent(c)
Multivariate Polynomial Ring in x, y over Finite Field of size 389
```

```
>>> from sage.all import *
>>> f.coefficient(x**Integer(0)) # outputs the full polynomial
x^2*y^2 + x^2*y + x*y^2 + x^2 + x*y + y^2 + x + y + 1
>>> R = GF(Integer(389))['x, y']; (x, y,) = R._first_ngens(2)
>>> f = x*y + Integer(5)
>>> c = f.coefficient({x:Integer(0), y:Integer(0)}); c
5
>>> parent(c)
Multivariate Polynomial Ring in x, y over Finite Field of size 389
```

AUTHOR:

- Joel B. Mohler (2007.10.31)

coefficients()

Return the nonzero coefficients of this polynomial in a list. The returned list is decreasingly ordered by the term ordering of the parent.

EXAMPLES:

```
sage: R.<x,y,z> = PolynomialRing(QQ, order='degrevlex')
sage: f=23*x^6*y^7 + x^3*y+6*x^7*z
sage: f.coefficients()
[23, 6, 1]

sage: R.<x,y,z> = PolynomialRing(QQ, order='lex')
sage: f=23*x^6*y^7 + x^3*y+6*x^7*z
sage: f.coefficients()
[6, 23, 1]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, order='degrevlex', names=('x', 'y', 'z',)); (x, y,␣
↪z,) = R._first_ngens(3)
>>> f=Integer(23)*x**Integer(6)*y**Integer(7) +␣
↪x**Integer(3)*y+Integer(6)*x**Integer(7)*z
>>> f.coefficients()
[23, 6, 1]

>>> R = PolynomialRing(QQ, order='lex', names=('x', 'y', 'z',)); (x, y, z,) =␣
↪R._first_ngens(3)
>>> f=Integer(23)*x**Integer(6)*y**Integer(7) +␣
↪x**Integer(3)*y+Integer(6)*x**Integer(7)*z
>>> f.coefficients()
[6, 23, 1]
```

AUTHOR:

- Didier Deshommes

constant_coefficient()

Return the constant coefficient of this multivariate polynomial.

EXAMPLES:

```
sage: P.<x, y> = QQ[]
sage: f = 3*x^2 - 2*y + 7*x^2*y^2 + 5
sage: f.constant_coefficient()
5
sage: f = 3*x^2
sage: f.constant_coefficient()
0
```

```
>>> from sage.all import *
>>> P = QQ['x, y']; (x, y,) = P._first_ngens(2)
>>> f = Integer(3)*x**Integer(2) - Integer(2)*y +␣
↪Integer(7)*x**Integer(2)*y**Integer(2) + Integer(5)
>>> f.constant_coefficient()
5
>>> f = Integer(3)*x**Integer(2)
```

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```
>>> f.constant_coefficient()
0
```

degree(*x=None*, *std_grading=False*)

Return the degree of this polynomial.

INPUT:

- `x` – (default: `None`) a generator of the parent ring

OUTPUT:

If `x` is `None`, return the total degree of `self`. Note that this result is affected by the weighting given to the generators of the parent ring. Otherwise, if `x` is (or is coercible to) a generator of the parent ring, the output is the maximum degree of `x` in `self`. This is not affected by the weighting of the generators.

EXAMPLES:

```
sage: R.<x, y> = QQ[]
sage: f = y^2 - x^9 - x
sage: f.degree(x)
9
sage: f.degree(y)
2
sage: (y^10*x - 7*x^2*y^5 + 5*x^3).degree(x)
3
sage: (y^10*x - 7*x^2*y^5 + 5*x^3).degree(y)
10
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = y**Integer(2) - x**Integer(9) - x
>>> f.degree(x)
9
>>> f.degree(y)
2
>>> (y**Integer(10)*x - Integer(7)*x**Integer(2)*y**Integer(5) +␣
↪Integer(5)*x**Integer(3)).degree(x)
3
>>> (y**Integer(10)*x - Integer(7)*x**Integer(2)*y**Integer(5) +␣
↪Integer(5)*x**Integer(3)).degree(y)
10
```

When the generators have a grading (weighting) then the total degree respects this, but the degree for a given generator is unaffected:

```
sage: T = TermOrder("wdegrevlex", (2, 3))
sage: R.<x, y> = PolynomialRing(QQ, order=T)
sage: f = x^2 * y + y^4
sage: f.degree()
12
sage: f.degree(x)
2
sage: f.degree(y)
4
```

```
>>> from sage.all import *
>>> T = TermOrder("wdegrevlex", (Integer(2), Integer(3)))
>>> R = PolynomialRing(QQ, order=T, names=('x', 'y',)); (x, y,) = R._first_
↪ngens(2)
>>> f = x**Integer(2) * y + y**Integer(4)
>>> f.degree()
12
>>> f.degree(x)
2
>>> f.degree(y)
4
```

The term ordering of the parent ring determines the grading of the generators.

```
sage: T = TermOrder('wdegrevlex', (1,2,3,4))
sage: R = PolynomialRing(QQ, 'x', 12, order=T+T+T)
sage: [x.degree() for x in R.gens()]
[1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4]
```

```
>>> from sage.all import *
>>> T = TermOrder('wdegrevlex', (Integer(1),Integer(2),Integer(3),Integer(4)))
>>> R = PolynomialRing(QQ, 'x', Integer(12), order=T+T+T)
>>> [x.degree() for x in R.gens()]
[1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4]
```

A matrix term ordering determines the grading of the generators by the first row of the matrix.

```
sage: m = matrix(3, [3,2,1,1,1,0,1,0,0])
sage: m
[3 2 1]
[1 1 0]
[1 0 0]
sage: R.<x,y,z> = PolynomialRing(QQ, order=TermOrder(m))
sage: x.degree(), y.degree(), z.degree()
(3, 2, 1)
sage: f = x^3*y + x*z^4
sage: f.degree()
11
```

```
>>> from sage.all import *
>>> m = matrix(Integer(3), [Integer(3),Integer(2),Integer(1),Integer(1),
↪Integer(1),Integer(0),Integer(1),Integer(0),Integer(0)])
>>> m
[3 2 1]
[1 1 0]
[1 0 0]
>>> R = PolynomialRing(QQ, order=TermOrder(m), names=('x', 'y', 'z',)); (x, y,
↪ z,) = R._first_ngens(3)
>>> x.degree(), y.degree(), z.degree()
(3, 2, 1)
>>> f = x**Integer(3)*y + x*z**Integer(4)
>>> f.degree()
11
```

If the first row contains zero, the grading becomes the standard one.

```
sage: m = matrix(3, [3,0,1,1,1,0,1,0,0])
sage: m
[3 0 1]
[1 1 0]
[1 0 0]
sage: R.<x,y,z> = PolynomialRing(QQ, order=TermOrder(m))
sage: x.degree(), y.degree(), z.degree()
(1, 1, 1)
sage: f = x^3*y + x*z^4
sage: f.degree()
5
```

```
>>> from sage.all import *
>>> m = matrix(Integer(3), [Integer(3),Integer(0),Integer(1),Integer(1),
↪Integer(1),Integer(0),Integer(1),Integer(0),Integer(0)])
>>> m
[3 0 1]
[1 1 0]
[1 0 0]
>>> R = PolynomialRing(QQ, order=TermOrder(m), names=('x', 'y', 'z',)); (x, y,
↪ z,) = R._first_ngens(3)
>>> x.degree(), y.degree(), z.degree()
(1, 1, 1)
>>> f = x**Integer(3)*y + x*z**Integer(4)
>>> f.degree()
5
```

To get the degree with the standard grading regardless of the term ordering of the parent ring, use `std_grading=True`.

```
sage: f.degree(std_grading=True)
5
```

```
>>> from sage.all import *
>>> f.degree(std_grading=True)
5
```

degrees()

Return a tuple with the maximal degree of each variable in this polynomial. The list of degrees is ordered by the order of the generators.

EXAMPLES:

```
sage: R.<y0,y1,y2> = PolynomialRing(QQ,3)
sage: q = 3*y0*y1*y1*y2; q
3*y0*y1^2*y2
sage: q.degrees()
(1, 2, 1)
sage: (q + y0^5).degrees()
(5, 2, 1)
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ,Integer(3), names=('y0', 'y1', 'y2',)); (y0, y1, y2,
↪) = R._first_ngens(3)
>>> q = Integer(3)*y0*y1*y1*y2; q
3*y0*y1^2*y2
>>> q.degrees()
(1, 2, 1)
>>> (q + y0**Integer(5)).degrees()
(5, 2, 1)
```

dict(*copy=None*)

Return a dictionary representing `self`. This dictionary is in the same format as the generic MPolynomial: The dictionary consists of `ETuple:coefficient` pairs.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: f = 2*x*y^3*z^2 + 1/7*x^2 + 2/3
sage: f.monomial_coefficients()
{(0, 0, 0): 2/3, (1, 3, 2): 2, (2, 0, 0): 1/7}
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> f = Integer(2)*x*y**Integer(3)*z**Integer(2) + Integer(1)/
↪Integer(7)*x**Integer(2) + Integer(2)/Integer(3)
>>> f.monomial_coefficients()
{(0, 0, 0): 2/3, (1, 3, 2): 2, (2, 0, 0): 1/7}
```

`dict` is an alias:

```
sage: f.dict()
{(0, 0, 0): 2/3, (1, 3, 2): 2, (2, 0, 0): 1/7}
```

```
>>> from sage.all import *
>>> f.dict()
{(0, 0, 0): 2/3, (1, 3, 2): 2, (2, 0, 0): 1/7}
```

divides(*other*)

Return `True` if this polynomial divides `other`.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: p = 3*x*y + 2*y*z + x*z
sage: q = x + y + z + 1
sage: r = p * q
sage: p.divides(r)
True
sage: q.divides(p)
False
sage: r.divides(0)
True
sage: R.zero().divides(r)
```

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```
False
sage: R.zero().divides(0)
True
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> p = Integer(3)*x*y + Integer(2)*y*z + x*z
>>> q = x + y + z + Integer(1)
>>> r = p * q
>>> p.divides(r)
True
>>> q.divides(p)
False
>>> r.divides(Integer(0))
True
>>> R.zero().divides(r)
False
>>> R.zero().divides(Integer(0))
True
```

exponents(*as_ETuples=True*)

Return the exponents of the monomials appearing in this polynomial.

INPUT:

- `as_ETuples` – boolean (default: `True`); if `True` returns the result as a list of ETuples, otherwise returns a list of tuples

EXAMPLES:

```
sage: R.<a,b,c> = QQ[]
sage: f = a^3 + b + 2*b^2
sage: f.exponents()
[(3, 0, 0), (0, 2, 0), (0, 1, 0)]
sage: f.exponents(as_ETuples=False)
[(3, 0, 0), (0, 2, 0), (0, 1, 0)]
```

```
>>> from sage.all import *
>>> R = QQ['a, b, c']; (a, b, c,) = R._first_ngens(3)
>>> f = a**Integer(3) + b + Integer(2)*b**Integer(2)
>>> f.exponents()
[(3, 0, 0), (0, 2, 0), (0, 1, 0)]
>>> f.exponents(as_ETuples=False)
[(3, 0, 0), (0, 2, 0), (0, 1, 0)]
```

factor(*proof=None*)

Return the factorization of this polynomial.

INPUT:

- `proof` – ignored

EXAMPLES:

```
sage: R.<x, y> = QQ[]
sage: f = (x^3 + 2*y^2*x) * (x^2 + x + 1); f
x^5 + 2*x^3*y^2 + x^4 + 2*x^2*y^2 + x^3 + 2*x*y^2
sage: F = f.factor(); F
x * (x^2 + x + 1) * (x^2 + 2*y^2)
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = (x**Integer(3) + Integer(2)*y**Integer(2)*x) * (x**Integer(2) + x +␣
↪Integer(1)); f
x^5 + 2*x^3*y^2 + x^4 + 2*x^2*y^2 + x^3 + 2*x*y^2
>>> F = f.factor(); F
x * (x^2 + x + 1) * (x^2 + 2*y^2)
```

Next we factor the same polynomial, but over the finite field of order 3.:

```
sage: R.<x, y> = GF(3)[]
sage: f = (x^3 + 2*y^2*x) * (x^2 + x + 1); f
x^5 - x^3*y^2 + x^4 - x^2*y^2 + x^3 - x*y^2
sage: F = f.factor()
sage: F  # order is somewhat random
(-1) * x * (-x + y) * (x + y) * (x - 1)^2
```

```
>>> from sage.all import *
>>> R = GF(Integer(3))['x, y']; (x, y,) = R._first_ngens(2)
>>> f = (x**Integer(3) + Integer(2)*y**Integer(2)*x) * (x**Integer(2) + x +␣
↪Integer(1)); f
x^5 - x^3*y^2 + x^4 - x^2*y^2 + x^3 - x*y^2
>>> F = f.factor()
>>> F  # order is somewhat random
(-1) * x * (-x + y) * (x + y) * (x - 1)^2
```

Next we factor a polynomial, but over a finite field of order 9.:

```
sage: # needs sage.rings.finite_rings
sage: K.<a> = GF(3^2)
sage: R.<x, y> = K[]
sage: f = (x^3 + 2*a*y^2*x) * (x^2 + x + 1); f
x^5 + (-a)*x^3*y^2 + x^4 + (-a)*x^2*y^2 + x^3 + (-a)*x*y^2
sage: F = f.factor(); F
((-a)) * x * (x - 1)^2 * ((-a + 1)*x^2 + y^2)
sage: f - F
0
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> K = GF(Integer(3)**Integer(2), names=('a',)); (a,) = K._first_ngens(1)
>>> R = K['x, y']; (x, y,) = R._first_ngens(2)
>>> f = (x**Integer(3) + Integer(2)*a*y**Integer(2)*x) * (x**Integer(2) + x +␣
↪Integer(1)); f
x^5 + (-a)*x^3*y^2 + x^4 + (-a)*x^2*y^2 + x^3 + (-a)*x*y^2
>>> F = f.factor(); F
```

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```
((-a)) * x * (x - 1)^2 * ((-a + 1)*x^2 + y^2)
>>> f - F
0
```

Next we factor a polynomial over a number field.:

```
sage: # needs sage.rings.number_field
sage: p = polygen(ZZ, 'p')
sage: K.<s> = NumberField(p^3 - 2)
sage: KXY.<x,y> = K[]
sage: factor(x^3 - 2*y^3)
(x + (-s)*y) * (x^2 + s*x*y + (s^2)*y^2)
sage: k = (x^3-2*y^3)^5*(x+s*y)^2*(2/3 + s^2)
sage: k.factor()
((s^2 + 2/3)) * (x + s*y)^2 * (x + (-s)*y)^5 * (x^2 + s*x*y + (s^2)*y^2)^5
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> p = polygen(ZZ, 'p')
>>> K = NumberField(p**Integer(3) - Integer(2), names=('s',)); (s,) = K._
↪first_ngens(1)
>>> KXY = K['x, y']; (x, y,) = KXY._first_ngens(2)
>>> factor(x**Integer(3) - Integer(2)*y**Integer(3))
(x + (-s)*y) * (x^2 + s*x*y + (s^2)*y^2)
>>> k = (x**Integer(3)-
↪Integer(2)*y**Integer(3))**Integer(5)*(x+s*y)**Integer(2)*(Integer(2)/
↪Integer(3) + s**Integer(2))
>>> k.factor()
((s^2 + 2/3)) * (x + s*y)^2 * (x + (-s)*y)^5 * (x^2 + s*x*y + (s^2)*y^2)^5
```

This shows that issue Issue #2780 is fixed, i.e. that the unit part of the factorization is set correctly:

```
sage: # needs sage.rings.number_field
sage: x = polygen(ZZ, 'x')
sage: K.<a> = NumberField(x^2 + 1)
sage: R.<y, z> = PolynomialRing(K)
sage: f = 2*y^2 + 2*z^2
sage: F = f.factor(); F.unit()
2
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> x = polygen(ZZ, 'x')
>>> K = NumberField(x**Integer(2) + Integer(1), names=('a',)); (a,) = K._
↪first_ngens(1)
>>> R = PolynomialRing(K, names=('y', 'z',)); (y, z,) = R._first_ngens(2)
>>> f = Integer(2)*y**Integer(2) + Integer(2)*z**Integer(2)
>>> F = f.factor(); F.unit()
2
```

Another example:

```
sage: R.<x,y,z> = GF(32003)[]                                               #␣
↪needs sage.rings.finite_rings
sage: f = 9*(x-1)^2*(y+z)                                                   #␣
↪needs sage.rings.finite_rings
sage: f.factor()                                                            #␣
↪needs sage.rings.finite_rings
(9) * (y + z) * (x - 1)^2

sage: R.<x,w,v,u> = QQ['x','w','v','u']
sage: p = (4*v^4*u^2 - 16*v^2*u^4 + 16*u^6 - 4*v^4*u + 8*v^2*u^3 + v^4)
sage: p.factor()
(-2*v^2*u + 4*u^3 + v^2)^2
sage: R.<a,b,c,d> = QQ[]
sage: f =  (-2) * (a - d) * (-a + b) * (b - d) * (a - c) * (b - c) * (c - d)
sage: F = f.factor(); F
(-2) * (-a + c) * (-a + b) * (-b + c) * (c - d) * (b - d) * (a - d)
sage: F[0][0]
-a + c
sage: F.unit()
-2
```

```
>>> from sage.all import *
>>> R = GF(Integer(32003))['x, y, z']; (x, y, z,) = R._first_ngens(3)# needs␣
↪sage.rings.finite_rings
>>> f = Integer(9)*(x-Integer(1))**Integer(2)*(y+z)                         ␣
↪                           # needs sage.rings.finite_rings
>>> f.factor()                                                            #␣
↪needs sage.rings.finite_rings
(9) * (y + z) * (x - 1)^2

>>> R = QQ['x','w','v','u']; (x, w, v, u,) = R._first_ngens(4)
>>> p = (Integer(4)*v**Integer(4)*u**Integer(2) -␣
↪Integer(16)*v**Integer(2)*u**Integer(4) + Integer(16)*u**Integer(6) -␣
↪Integer(4)*v**Integer(4)*u + Integer(8)*v**Integer(2)*u**Integer(3) +␣
↪v**Integer(4))
>>> p.factor()
(-2*v^2*u + 4*u^3 + v^2)^2
>>> R = QQ['a, b, c, d']; (a, b, c, d,) = R._first_ngens(4)
>>> f =  (-Integer(2)) * (a - d) * (-a + b) * (b - d) * (a - c) * (b - c) *␣
↪(c - d)
>>> F = f.factor(); F
(-2) * (-a + c) * (-a + b) * (-b + c) * (c - d) * (b - d) * (a - d)
>>> F[Integer(0)][Integer(0)]
-a + c
>>> F.unit()
-2
```

Constant elements are factorized in the base rings.

```
sage: P.<x,y> = ZZ[]
sage: P(2^3*7).factor()
2^3 * 7
sage: P.<x,y> = GF(2)[]
```

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```
sage: P(1).factor()
1
```

```
>>> from sage.all import *
>>> P = ZZ['x, y']; (x, y,) = P._first_ngens(2)
>>> P(Integer(2)**Integer(3)*Integer(7)).factor()
2^3 * 7
>>> P = GF(Integer(2))['x, y']; (x, y,) = P._first_ngens(2)
>>> P(Integer(1)).factor()
1
```

Factorization for finite prime fields with characteristic $> 2^{29}$ is not supported

```
sage: q = 1073741789
sage: T.<aa, bb> = PolynomialRing(GF(q))                                    #␣
↪needs sage.rings.finite_rings
sage: f = aa^2 + 12124343*bb*aa + 32434598*bb^2                             #␣
↪needs sage.rings.finite_rings
sage: f.factor()                                                            #␣
↪needs sage.rings.finite_rings
Traceback (most recent call last):
...
NotImplementedError: Factorization of multivariate polynomials
over prime fields with characteristic > 2^29 is not implemented.
```

```
>>> from sage.all import *
>>> q = Integer(1073741789)
>>> T = PolynomialRing(GF(q), names=('aa', 'bb',)); (aa, bb,) = T._first_
↪ngens(2)# needs sage.rings.finite_rings
>>> f = aa**Integer(2) + Integer(12124343)*bb*aa +␣
↪Integer(32434598)*bb**Integer(2)                               # needs sage.
↪rings.finite_rings
>>> f.factor()                                                            #␣
↪needs sage.rings.finite_rings
Traceback (most recent call last):
...
NotImplementedError: Factorization of multivariate polynomials
over prime fields with characteristic > 2^29 is not implemented.
```

Factorization over the integers is now supported, see Issue #17840:

```
sage: P.<x,y> = PolynomialRing(ZZ)
sage: f = 12 * (3*x*y + 4) * (5*x - 2) * (2*y + 7)^2
sage: f.factor()
2^2 * 3 * (2*y + 7)^2 * (5*x - 2) * (3*x*y + 4)
sage: g = -12 * (x^2 - y^2)
sage: g.factor()
(-1) * 2^2 * 3 * (x - y) * (x + y)
sage: factor(-4*x*y - 2*x + 2*y + 1)
(-1) * (2*y + 1) * (2*x - 1)
```

```
>>> from sage.all import *
>>> P = PolynomialRing(ZZ, names=('x', 'y',)); (x, y,) = P._first_ngens(2)
>>> f = Integer(12) * (Integer(3)*x*y + Integer(4)) * (Integer(5)*x -␣
↪Integer(2)) * (Integer(2)*y + Integer(7))**Integer(2)
>>> f.factor()
2^2 * 3 * (2*y + 7)^2 * (5*x - 2) * (3*x*y + 4)
>>> g = -Integer(12) * (x**Integer(2) - y**Integer(2))
>>> g.factor()
(-1) * 2^2 * 3 * (x - y) * (x + y)
>>> factor(-Integer(4)*x*y - Integer(2)*x + Integer(2)*y + Integer(1))
(-1) * (2*y + 1) * (2*x - 1)
```

Factorization over non-integral domains is not supported

```
sage: R.<x,y> = PolynomialRing(Zmod(4))
sage: f = (2*x + 1) * (x^2 + x + 1)
sage: f.factor()
Traceback (most recent call last):
...
NotImplementedError: Factorization of multivariate polynomials
over Ring of integers modulo 4 is not implemented.
```

```
>>> from sage.all import *
>>> R = PolynomialRing(Zmod(Integer(4)), names=('x', 'y',)); (x, y,) = R._
↪first_ngens(2)
>>> f = (Integer(2)*x + Integer(1)) * (x**Integer(2) + x + Integer(1))
>>> f.factor()
Traceback (most recent call last):
...
NotImplementedError: Factorization of multivariate polynomials
over Ring of integers modulo 4 is not implemented.
```

gcd(*right*, *algorithm=None*, ***kwds*)

Return the greatest common divisor of `self` and `right`.

INPUT:

- `right` – polynomial
- `algorithm` - `'ezgcd'` – EZGCD algorithm - `'modular'` – multi-modular algorithm (default)
- `**kwds` – ignored

EXAMPLES:

```
sage: P.<x,y,z> = QQ[]
sage: f = (x*y*z)^6 - 1
sage: g = (x*y*z)^4 - 1
sage: f.gcd(g)
x^2*y^2*z^2 - 1
sage: GCD([x^3 - 3*x + 2, x^4 - 1, x^6 -1])
x - 1

sage: R.<x,y> = QQ[]
sage: f = (x^3 + 2*y^2*x)^2
```

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```
sage: g = x^2*y^2
sage: f.gcd(g)
x^2
```

```
>>> from sage.all import *
>>> P = QQ['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> f = (x*y*z)**Integer(6) - Integer(1)
>>> g = (x*y*z)**Integer(4) - Integer(1)
>>> f.gcd(g)
x^2*y^2*z^2 - 1
>>> GCD([x**Integer(3) - Integer(3)*x + Integer(2), x**Integer(4) -
↪Integer(1), x**Integer(6) -Integer(1)])
x - 1

>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = (x**Integer(3) + Integer(2)*y**Integer(2)*x)**Integer(2)
>>> g = x**Integer(2)*y**Integer(2)
>>> f.gcd(g)
x^2
```

We compute a gcd over a finite field:

```
sage: # needs sage.rings.finite_rings
sage: F.<u> = GF(31^2)
sage: R.<x,y,z> = F[]
sage: p = x^3 + (1+u)*y^3 + z^3
sage: q = p^3 * (x - y + z*u)
sage: gcd(p,q)
x^3 + (u + 1)*y^3 + z^3
sage: gcd(p,q)  # yes, twice -- tests that singular ring is properly set.
x^3 + (u + 1)*y^3 + z^3
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> F = GF(Integer(31)**Integer(2), names=('u',)); (u,) = F._first_ngens(1)
>>> R = F['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> p = x**Integer(3) + (Integer(1)+u)*y**Integer(3) + z**Integer(3)
>>> q = p**Integer(3) * (x - y + z*u)
>>> gcd(p,q)
x^3 + (u + 1)*y^3 + z^3
>>> gcd(p,q)  # yes, twice -- tests that singular ring is properly set.
x^3 + (u + 1)*y^3 + z^3
```

We compute a gcd over a number field:

```
sage: # needs sage.rings.number_field
sage: x = polygen(QQ)
sage: F.<u> = NumberField(x^3 - 2)
sage: R.<x,y,z> = F[]
sage: p = x^3 + (1+u)*y^3 + z^3
sage: q = p^3 * (x - y + z*u)
sage: gcd(p,q)
```

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```
x^3 + (u + 1)*y^3 + z^3
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> x = polygen(QQ)
>>> F = NumberField(x**Integer(3) - Integer(2), names=('u',)); (u,) = F._
↪first_ngens(1)
>>> R = F['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> p = x**Integer(3) + (Integer(1)+u)*y**Integer(3) + z**Integer(3)
>>> q = p**Integer(3) * (x - y + z*u)
>>> gcd(p,q)
x^3 + (u + 1)*y^3 + z^3
```

global_height (*prec=None*)

Return the (projective) global height of the polynomial.

This returns the absolute logarithmic height of the coefficients thought of as a projective point.

INPUT:

- `prec` – desired floating point precision (default: default `RealField` precision)

OUTPUT: a real number

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: f = 3*x^3 + 2*x*y^2
sage: exp(f.global_height())                                            #␣
↪needs sage.symbolic
3.00000000000000
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> f = Integer(3)*x**Integer(3) + Integer(2)*x*y**Integer(2)
>>> exp(f.global_height())                                             #␣
↪needs sage.symbolic
3.00000000000000
```

```
sage: # needs sage.rings.number_field
sage: K.<k> = CyclotomicField(3)
sage: R.<x,y> = PolynomialRing(K, sparse=True)
sage: f = k*x*y + 1
sage: exp(f.global_height())
1.00000000000000
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> K = CyclotomicField(Integer(3), names=('k',)); (k,) = K._first_ngens(1)
>>> R = PolynomialRing(K, sparse=True, names=('x', 'y',)); (x, y,) = R._first_
↪ngens(2)
>>> f = k*x*y + Integer(1)
>>> exp(f.global_height())
1.00000000000000
```

Scaling should not change the result:

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: f = 1/25*x^2 + 25/3*x*y + y^2
sage: f.global_height()                                                    #␣
↪needs sage.symbolic
6.43775164973640
sage: g = 100 * f
sage: g.global_height()                                                    #␣
↪needs sage.symbolic
6.43775164973640
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> f = Integer(1)/Integer(25)*x**Integer(2) + Integer(25)/Integer(3)*x*y +␣
↪y**Integer(2)
>>> f.global_height()                                                    #␣
↪needs sage.symbolic
6.43775164973640
>>> g = Integer(100) * f
>>> g.global_height()                                                    #␣
↪needs sage.symbolic
6.43775164973640
```

```
sage: R.<x> = PolynomialRing(QQ)
sage: K.<k> = NumberField(x^2 + 5)
sage: T.<t,w> = PolynomialRing(K)
sage: f = 1/1331 * t^2 + 5 * w + 7
sage: f.global_height()                                                    #␣
↪needs sage.symbolic
9.13959596745043
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> K = NumberField(x**Integer(2) + Integer(5), names=('k',)); (k,) = K._
↪first_ngens(1)
>>> T = PolynomialRing(K, names=('t', 'w',)); (t, w,) = T._first_ngens(2)
>>> f = Integer(1)/Integer(1331) * t**Integer(2) + Integer(5) * w + Integer(7)
>>> f.global_height()                                                    #␣
↪needs sage.symbolic
9.13959596745043
```

```
sage: R.<x,y> = QQ[]
sage: f = 1/123*x*y + 12
sage: f.global_height(prec=2)                                              #␣
↪needs sage.symbolic
8.0
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = Integer(1)/Integer(123)*x*y + Integer(12)
>>> f.global_height(prec=Integer(2))                                      ␣
```

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```
↪     # needs sage.symbolic
8.0
```

```
sage: R.<x,y> = QQ[]
sage: f = 0*x*y
sage: f.global_height()
0.000000000000000
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = Integer(0)*x*y
>>> f.global_height()
0.000000000000000
```

gradient()

Return a list of partial derivatives of this polynomial, ordered by the variables of the parent.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ,3)
sage: f= x*y + 1
sage: f.gradient()
[y, x, 0]
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ,Integer(3), names=('x', 'y', 'z',)); (x, y, z,) = P.
↪_first_ngens(3)
>>> f= x*y + Integer(1)
>>> f.gradient()
[y, x, 0]
```

hamming_weight()

Return the number of nonzero coefficients of this polynomial.

This is also called weight, *hamming_weight()* or sparsity.

EXAMPLES:

```
sage: R.<x, y> = ZZ[]
sage: f = x^3 - y
sage: f.number_of_terms()
2
sage: R(0).number_of_terms()
0
sage: f = (x+y)^100
sage: f.number_of_terms()
101
```

```
>>> from sage.all import *
>>> R = ZZ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = x**Integer(3) - y
>>> f.number_of_terms()
```

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```
2
>>> R(Integer(0)).number_of_terms()
0
>>> f = (x+y)**Integer(100)
>>> f.number_of_terms()
101
```

The method `hamming_weight()` is an alias:

```
sage: f.hamming_weight()
101
```

```
>>> from sage.all import *
>>> f.hamming_weight()
101
```

in_subalgebra(*J*, *algorithm='algebra_containment'*, *certificate=False*)

Return whether this polynomial is contained in the subalgebra generated by `J`

INPUT:

- `J` – list of elements of the parent polynomial ring
- `algorithm` – can be `'algebra_containment'` (the default), `'inSubring'`, or `'groebner'`
  - `'algebra_containment'`: use Singular's `algebra_containment` function, https://www.singular.uni-kl.de/Manual/4-2-1/sing_1247.htm#SEC1328. The Singular documentation suggests that this is frequently faster than the next option.
  - `'inSubring'`: use Singular's `inSubring` function, https://www.singular.uni-kl.de/Manual/4-2-0/sing_1240.htm#SEC1321.
  - `'groebner'`: use the algorithm described in Singular's documentation, but within Sage: if the subalgebra generators are $y_1, \ldots, y_m$, then create a new polynomial algebra with the old generators along with new ones: $z_1, \ldots, z_m$. Create the ideal $(z_1 - y_1, \ldots, z_m - y_m)$, and reduce the polynomial modulo this ideal. The polynomial is contained in the subalgebra if and only if the remainder involves only the new variables z_i.
- `certificate` – boolean (default: `False`) or string; if `True` and `algorithm='groebner'`, return the polynomial generated by the algorithm if such a polynomial exists, otherwise return `None`; if a nonempty string, then this is used as the name of the variables in the returned polynomial, otherwise the name will be `'newgens'`

EXAMPLES:

```
sage: P.<x,y,z> = QQ[]
sage: J = [x^2 + y^2, x^2 + z^2]
sage: (y^2).in_subalgebra(J)
False
sage: a = (x^2 + y^2) * (x^2 + z^2)
sage: a.in_subalgebra(J, algorithm='inSubring')
True
sage: (a^2).in_subalgebra(J, algorithm='groebner')
True
sage: (a^2).in_subalgebra(J, algorithm='groebner', certificate='x')
x0^2*x1^2
```

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```
sage: (a + a^2).in_subalgebra(J)
True
```

```
>>> from sage.all import *
>>> P = QQ['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> J = [x**Integer(2) + y**Integer(2), x**Integer(2) + z**Integer(2)]
>>> (y**Integer(2)).in_subalgebra(J)
False
>>> a = (x**Integer(2) + y**Integer(2)) * (x**Integer(2) + z**Integer(2))
>>> a.in_subalgebra(J, algorithm='inSubring')
True
>>> (a**Integer(2)).in_subalgebra(J, algorithm='groebner')
True
>>> (a**Integer(2)).in_subalgebra(J, algorithm='groebner', certificate='x')
x0^2*x1^2
>>> (a + a**Integer(2)).in_subalgebra(J)
True
```

integral(*var*)

Integrate this polynomial with respect to the provided variable.

One requires that **Q** is contained in the ring.

INPUT:

- `variable` – the integral is taken with respect to variable

EXAMPLES:

```
sage: R.<x, y> = PolynomialRing(QQ, 2)
sage: f = 3*x^3*y^2 + 5*y^2 + 3*x + 2
sage: f.integral(x)
3/4*x^4*y^2 + 5*x*y^2 + 3/2*x^2 + 2*x
sage: f.integral(y)
x^3*y^3 + 5/3*y^3 + 3*x*y + 2*y
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(2), names=('x', 'y',)); (x, y,) = R._first_
↪ngens(2)
>>> f = Integer(3)*x**Integer(3)*y**Integer(2) + Integer(5)*y**Integer(2) +␣
↪Integer(3)*x + Integer(2)
>>> f.integral(x)
3/4*x^4*y^2 + 5*x*y^2 + 3/2*x^2 + 2*x
>>> f.integral(y)
x^3*y^3 + 5/3*y^3 + 3*x*y + 2*y
```

Check that Issue #15896 is solved:

```
sage: s = x+y
sage: s.integral(x)+x
1/2*x^2 + x*y + x
sage: s.integral(x)*s
1/2*x^3 + 3/2*x^2*y + x*y^2
```

```
>>> from sage.all import *
>>> s = x+y
>>> s.integral(x)+x
1/2*x^2 + x*y + x
>>> s.integral(x)*s
1/2*x^3 + 3/2*x^2*y + x*y^2
```

inverse_of_unit()

Return the inverse of this polynomial if it is a unit.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: x.inverse_of_unit()
Traceback (most recent call last):
...
ArithmeticError: Element is not a unit.

sage: R(1/2).inverse_of_unit()
2
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> x.inverse_of_unit()
Traceback (most recent call last):
...
ArithmeticError: Element is not a unit.

>>> R(Integer(1)/Integer(2)).inverse_of_unit()
2
```

is_constant()

Return `True` if this polynomial is constant.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(GF(127))
sage: x.is_constant()
False
sage: P(1).is_constant()
True
```

```
>>> from sage.all import *
>>> P = PolynomialRing(GF(Integer(127)), names=('x', 'y', 'z',)); (x, y, z,)
↪= P._first_ngens(3)
>>> x.is_constant()
False
>>> P(Integer(1)).is_constant()
True
```

is_homogeneous()

Return `True` if this polynomial is homogeneous.

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(RationalField(), 2)
sage: (x+y).is_homogeneous()
True
sage: (x.parent()(0)).is_homogeneous()
True
sage: (x+y^2).is_homogeneous()
False
sage: (x^2 + y^2).is_homogeneous()
True
sage: (x^2 + y^2*x).is_homogeneous()
False
sage: (x^2*y + y^2*x).is_homogeneous()
True
```

```
>>> from sage.all import *
>>> P = PolynomialRing(RationalField(), Integer(2), names=('x', 'y',)); (x, y,
↪) = P._first_ngens(2)
>>> (x+y).is_homogeneous()
True
>>> (x.parent()(Integer(0))).is_homogeneous()
True
>>> (x+y**Integer(2)).is_homogeneous()
False
>>> (x**Integer(2) + y**Integer(2)).is_homogeneous()
True
>>> (x**Integer(2) + y**Integer(2)*x).is_homogeneous()
False
>>> (x**Integer(2)*y + y**Integer(2)*x).is_homogeneous()
True
```

is_monomial()

Return `True` if this polynomial is a monomial. A monomial is defined to be a product of generators with coefficient 1.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: x.is_monomial()
True
sage: (2*x).is_monomial()
False
sage: (x*y).is_monomial()
True
sage: (x*y + x).is_monomial()
False
sage: P(2).is_monomial()
False
sage: P.zero().is_monomial()
False
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = P._first_
↪ngens(3)
```

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```
>>> x.is_monomial()
True
>>> (Integer(2)*x).is_monomial()
False
>>> (x*y).is_monomial()
True
>>> (x*y + x).is_monomial()
False
>>> P(Integer(2)).is_monomial()
False
>>> P.zero().is_monomial()
False
```

is_squarefree()

Return `True` if this polynomial is square free.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: f= x^2 + 2*x*y + 1/2*z
sage: f.is_squarefree()
True
sage: h = f^2
sage: h.is_squarefree()
False
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = P._first_
↪ngens(3)
>>> f= x**Integer(2) + Integer(2)*x*y + Integer(1)/Integer(2)*z
>>> f.is_squarefree()
True
>>> h = f**Integer(2)
>>> h.is_squarefree()
False
```

is_term()

Return `True` if `self` is a term, which we define to be a product of generators times some coefficient, which need not be 1.

Use *is_monomial()* to require that the coefficient be 1.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: x.is_term()
True
sage: (2*x).is_term()
True
sage: (x*y).is_term()
True
sage: (x*y + x).is_term()
False
```

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```
sage: P(2).is_term()
True
sage: P.zero().is_term()
False
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = P._first_
↪ngens(3)
>>> x.is_term()
True
>>> (Integer(2)*x).is_term()
True
>>> (x*y).is_term()
True
>>> (x*y + x).is_term()
False
>>> P(Integer(2)).is_term()
True
>>> P.zero().is_term()
False
```

is_univariate()

Return `True` if `self` is a univariate polynomial, that is if `self` contains only one variable.

EXAMPLES:

```
sage: P.<x,y,z> = GF(2)[]
sage: f = x^2 + 1
sage: f.is_univariate()
True
sage: f = y*x^2 + 1
sage: f.is_univariate()
False
sage: f = P(0)
sage: f.is_univariate()
True
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> f = x**Integer(2) + Integer(1)
>>> f.is_univariate()
True
>>> f = y*x**Integer(2) + Integer(1)
>>> f.is_univariate()
False
>>> f = P(Integer(0))
>>> f.is_univariate()
True
```

is_zero()

Return `True` if this polynomial is zero.

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQ)
sage: x.is_zero()
False
sage: (x - x).is_zero()
True
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = P._first_ngens(2)
>>> x.is_zero()
False
>>> (x - x).is_zero()
True
```

iterator_exp_coeff(*as_ETuples=True*)

Iterate over `self` as pairs of ((E)Tuple, coefficient).

INPUT:

- `as_ETuples` – boolean (default: `True`); if `True` iterate over pairs whose first element is an ETuple, otherwise as a tuples

EXAMPLES:

```
sage: R.<a,b,c> = QQ[]
sage: f = a*c^3 + a^2*b + 2*b^4
sage: list(f.iterator_exp_coeff())
[((0, 4, 0), 2), ((1, 0, 3), 1), ((2, 1, 0), 1)]
sage: list(f.iterator_exp_coeff(as_ETuples=False))
[((0, 4, 0), 2), ((1, 0, 3), 1), ((2, 1, 0), 1)]

sage: R.<a,b,c> = PolynomialRing(QQ, 3, order='lex')
sage: f = a*c^3 + a^2*b + 2*b^4
sage: list(f.iterator_exp_coeff())
[((2, 1, 0), 1), ((1, 0, 3), 1), ((0, 4, 0), 2)]
```

```
>>> from sage.all import *
>>> R = QQ['a, b, c']; (a, b, c,) = R._first_ngens(3)
>>> f = a*c**Integer(3) + a**Integer(2)*b + Integer(2)*b**Integer(4)
>>> list(f.iterator_exp_coeff())
[((0, 4, 0), 2), ((1, 0, 3), 1), ((2, 1, 0), 1)]
>>> list(f.iterator_exp_coeff(as_ETuples=False))
[((0, 4, 0), 2), ((1, 0, 3), 1), ((2, 1, 0), 1)]

>>> R = PolynomialRing(QQ, Integer(3), order='lex', names=('a', 'b', 'c',));
→(a, b, c,) = R._first_ngens(3)
>>> f = a*c**Integer(3) + a**Integer(2)*b + Integer(2)*b**Integer(4)
>>> list(f.iterator_exp_coeff())
[((2, 1, 0), 1), ((1, 0, 3), 1), ((0, 4, 0), 2)]
```

lc()

Leading coefficient of this polynomial with respect to the term order of `self.parent()`.

EXAMPLES:

```
sage: R.<x,y,z> = PolynomialRing(GF(7), 3, order='lex')
sage: f = 3*x^1*y^2 + 2*y^3*z^4
sage: f.lc()
3

sage: f = 5*x^3*y^2*z^4 + 4*x^3*y^2*z^1
sage: f.lc()
5
```

```
>>> from sage.all import *
>>> R = PolynomialRing(GF(Integer(7)), Integer(3), order='lex', names=('x', 'y
↪', 'z',)); (x, y, z,) = R._first_ngens(3)
>>> f = Integer(3)*x**Integer(1)*y**Integer(2) +␣
↪Integer(2)*y**Integer(3)*z**Integer(4)
>>> f.lc()
3

>>> f = Integer(5)*x**Integer(3)*y**Integer(2)*z**Integer(4) +␣
↪Integer(4)*x**Integer(3)*y**Integer(2)*z**Integer(1)
>>> f.lc()
5
```

lcm(g)

Return the least common multiple of `self` and g.

EXAMPLES:

```
sage: P.<x,y,z> = QQ[]
sage: p = (x+y)*(y+z)
sage: q = (z^4+2)*(y+z)
sage: lcm(p,q)
x*y*z^4 + y^2*z^4 + x*z^5 + y*z^5 + 2*x*y + 2*y^2 + 2*x*z + 2*y*z

sage: P.<x,y,z> = ZZ[]
sage: p = 2*(x+y)*(y+z)
sage: q = 3*(z^4+2)*(y+z)
sage: lcm(p,q)
6*x*y*z^4 + 6*y^2*z^4 + 6*x*z^5 + 6*y*z^5 + 12*x*y + 12*y^2 + 12*x*z + 12*y*z

sage: # needs sage.rings.finite_rings
sage: r.<x,y> = PolynomialRing(GF(2**8, 'a'), 2)
sage: a = r.base_ring().0
sage: f = (a^2+a)*x^2*y + (a^4+a^3+a)*y + a^5
sage: f.lcm(x^4)
(a^2 + a)*x^6*y + (a^4 + a^3 + a)*x^4*y + (a^5)*x^4

sage: # needs sage.rings.number_field
sage: w = polygen(ZZ, 'w')
sage: r.<x,y> = PolynomialRing(NumberField(w^4 + 1, 'a'), 2)
sage: a = r.base_ring().0
sage: f = (a^2+a)*x^2*y + (a^4+a^3+a)*y + a^5
sage: f.lcm(x^4)
(a^2 + a)*x^6*y + (a^3 + a - 1)*x^4*y + (-a)*x^4
```

```
>>> from sage.all import *
>>> P = QQ['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> p = (x+y)*(y+z)
>>> q = (z**Integer(4)+Integer(2))*(y+z)
>>> lcm(p,q)
x*y*z^4 + y^2*z^4 + x*z^5 + y*z^5 + 2*x*y + 2*y^2 + 2*x*z + 2*y*z

>>> P = ZZ['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> p = Integer(2)*(x+y)*(y+z)
>>> q = Integer(3)*(z**Integer(4)+Integer(2))*(y+z)
>>> lcm(p,q)
6*x*y*z^4 + 6*y^2*z^4 + 6*x*z^5 + 6*y*z^5 + 12*x*y + 12*y^2 + 12*x*z + 12*y*z

>>> # needs sage.rings.finite_rings
>>> r = PolynomialRing(GF(Integer(2)**Integer(8), 'a'), Integer(2), names=('x
↪', 'y',)); (x, y,) = r._first_ngens(2)
>>> a = r.base_ring().gen(0)
>>> f = (a**Integer(2)+a)*x**Integer(2)*y + (a**Integer(4)+a**Integer(3)+a)*y␣
↪+ a**Integer(5)
>>> f.lcm(x**Integer(4))
(a^2 + a)*x^6*y + (a^4 + a^3 + a)*x^4*y + (a^5)*x^4

>>> # needs sage.rings.number_field
>>> w = polygen(ZZ, 'w')
>>> r = PolynomialRing(NumberField(w**Integer(4) + Integer(1), 'a'),␣
↪Integer(2), names=('x', 'y',)); (x, y,) = r._first_ngens(2)
>>> a = r.base_ring().gen(0)
>>> f = (a**Integer(2)+a)*x**Integer(2)*y + (a**Integer(4)+a**Integer(3)+a)*y␣
↪+ a**Integer(5)
>>> f.lcm(x**Integer(4))
(a^2 + a)*x^6*y + (a^3 + a - 1)*x^4*y + (-a)*x^4
```

lift(*I*)

Given an ideal `I = (f_1,...,f_r)` and some `g (== self)` in `I`, find `s_1,...,s_r` such that `g = s_1 f_1 + ... + s_r f_r`.

A `ValueError` exception is raised if `g (== self)` does not belong to `I`.

EXAMPLES:

```
sage: A.<x,y> = PolynomialRing(QQ,2,order='degrevlex')
sage: I = A.ideal([x^10 + x^9*y^2, y^8 - x^2*y^7 ])
sage: f = x*y^13 + y^12
sage: M = f.lift(I)
sage: M
[y^7, x^7*y^2 + x^8 + x^5*y^3 + x^6*y + x^3*y^4 + x^4*y^2 + x*y^5 + x^2*y^3 +␣
↪y^4]
sage: sum( map( mul , zip( M, I.gens() ) ) ) == f
True
```

```
>>> from sage.all import *
>>> A = PolynomialRing(QQ,Integer(2),order='degrevlex', names=('x', 'y',));␣
↪(x, y,) = A._first_ngens(2)
```

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```
>>> I = A.ideal([x**Integer(10) + x**Integer(9)*y**Integer(2), y**Integer(8) -
↪ x**Integer(2)*y**Integer(7) ])
>>> f = x*y**Integer(13) + y**Integer(12)
>>> M = f.lift(I)
>>> M
[y^7, x^7*y^2 + x^8 + x^5*y^3 + x^6*y + x^3*y^4 + x^4*y^2 + x*y^5 + x^2*y^3 +␣
↪y^4]
>>> sum( map( mul , zip( M, I.gens() ) ) ) == f
True
```

Check that Issue #13671 is fixed:

```
sage: R.<x1,x2> = QQ[]
sage: I = R.ideal(x2**2 + x1 - 2, x1**2 - 1)
sage: f = I.gen(0) + x2*I.gen(1)
sage: f.lift(I)
[1, x2]
sage: (f+1).lift(I)
Traceback (most recent call last):
...
ValueError: polynomial is not in the ideal
```

```
>>> from sage.all import *
>>> R = QQ['x1, x2']; (x1, x2,) = R._first_ngens(2)
>>> I = R.ideal(x2**Integer(2) + x1 - Integer(2), x1**Integer(2) - Integer(1))
>>> f = I.gen(Integer(0)) + x2*I.gen(Integer(1))
>>> f.lift(I)
[1, x2]
>>> (f+Integer(1)).lift(I)
Traceback (most recent call last):
...
ValueError: polynomial is not in the ideal
```

Check that we can work over the integers:

```
sage: R.<x1,x2> = ZZ[]
sage: I = R.ideal(x2**2 + x1 - 2, x1**2 - 1)
sage: f = I.gen(0) + x2*I.gen(1)
sage: f.lift(I)
[1, x2]
sage: (f+1).lift(I)
Traceback (most recent call last):
...
ValueError: polynomial is not in the ideal
sage: A.<x,y> = PolynomialRing(ZZ,2,order='degrevlex')
sage: I = A.ideal([x^10 + x^9*y^2, y^8 - x^2*y^7 ])
sage: f = x*y^13 + y^12
sage: M = f.lift(I)
sage: M
[y^7, x^7*y^2 + x^8 + x^5*y^3 + x^6*y + x^3*y^4 + x^4*y^2 + x*y^5 + x^2*y^3 +␣
↪y^4]
```

```
>>> from sage.all import *
>>> R = ZZ['x1, x2']; (x1, x2,) = R._first_ngens(2)
>>> I = R.ideal(x2**Integer(2) + x1 - Integer(2), x1**Integer(2) - Integer(1))
>>> f = I.gen(Integer(0)) + x2*I.gen(Integer(1))
>>> f.lift(I)
[1, x2]
>>> (f+Integer(1)).lift(I)
Traceback (most recent call last):
...
ValueError: polynomial is not in the ideal
>>> A = PolynomialRing(ZZ,Integer(2),order='degrevlex', names=('x', 'y',));␣
↪(x, y,) = A._first_ngens(2)
>>> I = A.ideal([x**Integer(10) + x**Integer(9)*y**Integer(2), y**Integer(8) -
↪ x**Integer(2)*y**Integer(7) ])
>>> f = x*y**Integer(13) + y**Integer(12)
>>> M = f.lift(I)
>>> M
[y^7, x^7*y^2 + x^8 + x^5*y^3 + x^6*y + x^3*y^4 + x^4*y^2 + x*y^5 + x^2*y^3 +␣
↪y^4]
```

lm()

Return the lead monomial of `self` with respect to the term order of `self.parent()`. In Sage a monomial is a product of variables in some power without a coefficient.

EXAMPLES:

```
sage: R.<x,y,z> = PolynomialRing(GF(7), 3, order='lex')
sage: f = x^1*y^2 + y^3*z^4
sage: f.lm()
x*y^2
sage: f = x^3*y^2*z^4 + x^3*y^2*z^1
sage: f.lm()
x^3*y^2*z^4

sage: R.<x,y,z>=PolynomialRing(QQ, 3, order='deglex')
sage: f = x^1*y^2*z^3 + x^3*y^2*z^0
sage: f.lm()
x*y^2*z^3
sage: f = x^1*y^2*z^4 + x^1*y^1*z^5
sage: f.lm()
x*y^2*z^4

sage: R.<x,y,z>=PolynomialRing(GF(127), 3, order='degrevlex')
sage: f = x^1*y^5*z^2 + x^4*y^1*z^3
sage: f.lm()
x*y^5*z^2
sage: f = x^4*y^7*z^1 + x^4*y^2*z^3
sage: f.lm()
x^4*y^7*z
```

```
>>> from sage.all import *
>>> R = PolynomialRing(GF(Integer(7)), Integer(3), order='lex', names=('x', 'y
↪', 'z',)); (x, y, z,) = R._first_ngens(3)
```

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```
>>> f = x**Integer(1)*y**Integer(2) + y**Integer(3)*z**Integer(4)
>>> f.lm()
x*y^2
>>> f = x**Integer(3)*y**Integer(2)*z**Integer(4) +␣
↪x**Integer(3)*y**Integer(2)*z**Integer(1)
>>> f.lm()
x^3*y^2*z^4

>>> R = PolynomialRing(QQ, Integer(3), order='deglex', names=('x', 'y', 'z',
↪)); (x, y, z,) = R._first_ngens(3)
>>> f = x**Integer(1)*y**Integer(2)*z**Integer(3) +␣
↪x**Integer(3)*y**Integer(2)*z**Integer(0)
>>> f.lm()
x*y^2*z^3
>>> f = x**Integer(1)*y**Integer(2)*z**Integer(4) +␣
↪x**Integer(1)*y**Integer(1)*z**Integer(5)
>>> f.lm()
x*y^2*z^4

>>> R = PolynomialRing(GF(Integer(127)), Integer(3), order='degrevlex',␣
↪names=('x', 'y', 'z',)); (x, y, z,) = R._first_ngens(3)
>>> f = x**Integer(1)*y**Integer(5)*z**Integer(2) +␣
↪x**Integer(4)*y**Integer(1)*z**Integer(3)
>>> f.lm()
x*y^5*z^2
>>> f = x**Integer(4)*y**Integer(7)*z**Integer(1) +␣
↪x**Integer(4)*y**Integer(2)*z**Integer(3)
>>> f.lm()
x^4*y^7*z
```

local_height(*v*, *prec=None*)

Return the maximum of the local height of the coefficients of this polynomial.

INPUT:

- v – a prime or prime ideal of the base ring
- prec – desired floating point precision (default: default RealField precision)

OUTPUT: a real number

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: f = 1/1331*x^2 + 1/4000*y^2
sage: f.local_height(1331)
7.193685818395112
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> f = Integer(1)/Integer(1331)*x**Integer(2) + Integer(1)/
↪Integer(4000)*y**Integer(2)
>>> f.local_height(Integer(1331))
7.193685818395112
```

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: K.<k> = NumberField(x^2 - 5)
sage: T.<t,w> = K[]
sage: I = K.ideal(3)
sage: f = 1/3*t*w + 3
sage: f.local_height(I)
1.09861228866811
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> K = NumberField(x**Integer(2) - Integer(5), names=('k',)); (k,) = K._
→first_ngens(1)
>>> T = K['t, w']; (t, w,) = T._first_ngens(2)
>>> I = K.ideal(Integer(3))
>>> f = Integer(1)/Integer(3)*t*w + Integer(3)
>>> f.local_height(I)
1.09861228866811
```

```
sage: R.<x,y> = QQ[]
sage: f = 1/2*x*y + 2
sage: f.local_height(2, prec=2)
0.75
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = Integer(1)/Integer(2)*x*y + Integer(2)
>>> f.local_height(Integer(2), prec=Integer(2))
0.75
```

local_height_arch(*i*, *prec=None*)

Return the maximum of the local height at the `i`-th infinite place of the coefficients of this polynomial.

INPUT:

- `i` – integer
- `prec` – desired floating point precision (default: default `RealField` precision)

OUTPUT: a real number

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: f = 210*x*y
sage: f.local_height_arch(0)
5.34710753071747
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> f = Integer(210)*x*y
>>> f.local_height_arch(Integer(0))
5.34710753071747
```

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: K.<k> = NumberField(x^2 - 5)
sage: T.<t,w> = K[]
sage: f = 1/2*t*w + 3
sage: f.local_height_arch(1, prec=52)
1.09861228866811
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> K = NumberField(x**Integer(2) - Integer(5), names=('k',)); (k,) = K._
↪first_ngens(1)
>>> T = K['t, w']; (t, w,) = T._first_ngens(2)
>>> f = Integer(1)/Integer(2)*t*w + Integer(3)
>>> f.local_height_arch(Integer(1), prec=Integer(52))
1.09861228866811
```

```
sage: R.<x,y> = QQ[]
sage: f = 1/2*x*y + 3
sage: f.local_height_arch(0, prec=2)
1.0
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = Integer(1)/Integer(2)*x*y + Integer(3)
>>> f.local_height_arch(Integer(0), prec=Integer(2))
1.0
```

lt()

Leading term of this polynomial. In Sage a term is a product of variables in some power and a coefficient.

EXAMPLES:

```
sage: R.<x,y,z> = PolynomialRing(GF(7), 3, order='lex')
sage: f = 3*x^1*y^2 + 2*y^3*z^4
sage: f.lt()
3*x*y^2

sage: f = 5*x^3*y^2*z^4 + 4*x^3*y^2*z^1
sage: f.lt()
-2*x^3*y^2*z^4
```

```
>>> from sage.all import *
>>> R = PolynomialRing(GF(Integer(7)), Integer(3), order='lex', names=('x', 'y
↪', 'z',)); (x, y, z,) = R._first_ngens(3)
>>> f = Integer(3)*x**Integer(1)*y**Integer(2) +␣
↪Integer(2)*y**Integer(3)*z**Integer(4)
>>> f.lt()
3*x*y^2

>>> f = Integer(5)*x**Integer(3)*y**Integer(2)*z**Integer(4) +␣
```

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```
↪Integer(4)*x**Integer(3)*y**Integer(2)*z**Integer(1)
>>> f.lt()
-2*x^3*y^2*z^4
```

monomial_coefficient(*mon*)

Return the coefficient in the base ring of the monomial mon in `self`, where mon must have the same parent as `self`.

This function contrasts with the function `coefficient` which returns the coefficient of a monomial viewing this polynomial in a polynomial ring over a base ring having fewer variables.

INPUT:

- `mon` – a monomial

OUTPUT: coefficient in base ring

See also

For coefficients in a base ring of fewer variables, look at `coefficient`.

EXAMPLES:

```
sage: P.<x,y> = QQ[]

The parent of the return is a member of the base ring.
sage: f = 2 * x * y
sage: c = f.monomial_coefficient(x*y); c
2
sage: c.parent()
Rational Field

sage: f = y^2 + y^2*x - x^9 - 7*x + 5*x*y
sage: f.monomial_coefficient(y^2)
1
sage: f.monomial_coefficient(x*y)
5
sage: f.monomial_coefficient(x^9)
-1
sage: f.monomial_coefficient(x^10)
0
```

```
>>> from sage.all import *
>>> P = QQ['x, y']; (x, y,) = P._first_ngens(2)

The parent of the return is a member of the base ring.
>>> f = Integer(2) * x * y
>>> c = f.monomial_coefficient(x*y); c
2
>>> c.parent()
Rational Field

>>> f = y**Integer(2) + y**Integer(2)*x - x**Integer(9) - Integer(7)*x +␣
```

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```
↪Integer(5)*x*y
>>> f.monomial_coefficient(y**Integer(2))
1
>>> f.monomial_coefficient(x*y)
5
>>> f.monomial_coefficient(x**Integer(9))
-1
>>> f.monomial_coefficient(x**Integer(10))
0
```

monomial_coefficients(*copy=None*)

Return a dictionary representing `self`. This dictionary is in the same format as the generic MPolynomial: The dictionary consists of `ETuple:coefficient` pairs.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: f = 2*x*y^3*z^2 + 1/7*x^2 + 2/3
sage: f.monomial_coefficients()
{(0, 0, 0): 2/3, (1, 3, 2): 2, (2, 0, 0): 1/7}
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> f = Integer(2)*x*y**Integer(3)*z**Integer(2) + Integer(1)/
↪Integer(7)*x**Integer(2) + Integer(2)/Integer(3)
>>> f.monomial_coefficients()
{(0, 0, 0): 2/3, (1, 3, 2): 2, (2, 0, 0): 1/7}
```

`dict` is an alias:

```
sage: f.dict()
{(0, 0, 0): 2/3, (1, 3, 2): 2, (2, 0, 0): 1/7}
```

```
>>> from sage.all import *
>>> f.dict()
{(0, 0, 0): 2/3, (1, 3, 2): 2, (2, 0, 0): 1/7}
```

monomials()

Return the list of monomials in `self`. The returned list is decreasingly ordered by the term ordering of `self.parent()`.

EXAMPLES:

```
sage: P.<x,y,z> = QQ[]
sage: f = x + 3/2*y*z^2 + 2/3
sage: f.monomials()
[y*z^2, x, 1]
sage: f = P(3/2)
sage: f.monomials()
[1]
```

```
>>> from sage.all import *
>>> P = QQ['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> f = x + Integer(3)/Integer(2)*y*z**Integer(2) + Integer(2)/Integer(3)
>>> f.monomials()
[y*z^2, x, 1]
>>> f = P(Integer(3)/Integer(2))
>>> f.monomials()
[1]
```

number_of_terms()

Return the number of nonzero coefficients of this polynomial.

This is also called weight, *hamming_weight()* or sparsity.

EXAMPLES:

```
sage: R.<x, y> = ZZ[]
sage: f = x^3 - y
sage: f.number_of_terms()
2
sage: R(0).number_of_terms()
0
sage: f = (x+y)^100
sage: f.number_of_terms()
101
```

```
>>> from sage.all import *
>>> R = ZZ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = x**Integer(3) - y
>>> f.number_of_terms()
2
>>> R(Integer(0)).number_of_terms()
0
>>> f = (x+y)**Integer(100)
>>> f.number_of_terms()
101
```

The method *hamming_weight()* is an alias:

```
sage: f.hamming_weight()
101
```

```
>>> from sage.all import *
>>> f.hamming_weight()
101
```

numerator()

Return a numerator of `self` computed as `self * self.denominator()`.

If the base_field of `self` is the Rational Field then the numerator is a polynomial whose base_ring is the Integer Ring, this is done for compatibility to the univariate case.

> ⚠ **Warning**
>
> This is not the numerator of the rational function defined by `self`, which would always be `self` since `self` is a polynomial.

EXAMPLES:

First we compute the numerator of a polynomial with integer coefficients, which is of course `self`.

```
sage: R.<x, y> = ZZ[]
sage: f = x^3 + 17*y + 1
sage: f.numerator()
x^3 + 17*y + 1
sage: f == f.numerator()
True
```

```
>>> from sage.all import *
>>> R = ZZ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = x**Integer(3) + Integer(17)*y + Integer(1)
>>> f.numerator()
x^3 + 17*y + 1
>>> f == f.numerator()
True
```

Next we compute the numerator of a polynomial with rational coefficients.

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: f = (1/17)*x^19 - (2/3)*y + 1/3; f
1/17*x^19 - 2/3*y + 1/3
sage: f.numerator()
3*x^19 - 34*y + 17
sage: f == f.numerator()
False
sage: f.numerator().base_ring()
Integer Ring
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> f = (Integer(1)/Integer(17))*x**Integer(19) - (Integer(2)/Integer(3))*y +
→Integer(1)/Integer(3); f
1/17*x^19 - 2/3*y + 1/3
>>> f.numerator()
3*x^19 - 34*y + 17
>>> f == f.numerator()
False
>>> f.numerator().base_ring()
Integer Ring
```

We check that the computation of numerator and denominator is valid.

```
sage: K=QQ['x,y']
sage: f=K.random_element()
```

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```
sage: f.numerator() / f.denominator() == f
True
```

```
>>> from sage.all import *
>>> K=QQ['x,y']
>>> f=K.random_element()
>>> f.numerator() / f.denominator() == f
True
```

The following tests against a bug fixed in Issue #11780:

```
sage: P.<foo,bar> = ZZ[]
sage: Q.<foo,bar> = QQ[]
sage: f = Q.random_element()
sage: f.numerator().parent() is P
True
```

```
>>> from sage.all import *
>>> P = ZZ['foo, bar']; (foo, bar,) = P._first_ngens(2)
>>> Q = QQ['foo, bar']; (foo, bar,) = Q._first_ngens(2)
>>> f = Q.random_element()
>>> f.numerator().parent() is P
True
```

nvariables()

Return the number variables in this polynomial.

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(GF(127))
sage: f = x*y + z
sage: f.nvariables()
3
sage: f = x + y
sage: f.nvariables()
2
```

```
>>> from sage.all import *
>>> P = PolynomialRing(GF(Integer(127)), names=('x', 'y', 'z',)); (x, y, z,)
↪= P._first_ngens(3)
>>> f = x*y + z
>>> f.nvariables()
3
>>> f = x + y
>>> f.nvariables()
2
```

quo_rem(*right*)

Return quotient and remainder of `self` and `right`.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: f = y*x^2 + x + 1
sage: f.quo_rem(x)
(x*y + 1, 1)
sage: f.quo_rem(y)
(x^2, x + 1)

sage: R.<x,y> = ZZ[]
sage: f = 2*y*x^2 + x + 1
sage: f.quo_rem(x)
(2*x*y + 1, 1)
sage: f.quo_rem(y)
(2*x^2, x + 1)
sage: f.quo_rem(3*x)
(0, 2*x^2*y + x + 1)
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = y*x**Integer(2) + x + Integer(1)
>>> f.quo_rem(x)
(x*y + 1, 1)
>>> f.quo_rem(y)
(x^2, x + 1)

>>> R = ZZ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = Integer(2)*y*x**Integer(2) + x + Integer(1)
>>> f.quo_rem(x)
(2*x*y + 1, 1)
>>> f.quo_rem(y)
(2*x^2, x + 1)
>>> f.quo_rem(Integer(3)*x)
(0, 2*x^2*y + x + 1)
```

reduce(*I*)

Return a remainder of this polynomial modulo the polynomials in `I`.

INPUT:

- `I` – an ideal or a list/set/iterable of polynomials

OUTPUT: a polynomial `r` such that:

- `self` – `r` is in the ideal generated by `I`.
- No term in `r` is divisible by any of the leading monomials of `I`.

The result `r` is canonical if:

- `I` is an ideal, and Sage can compute a Groebner basis of it.
- `I` is a list/set/iterable that is a (strong) Groebner basis for the term order of `self`. (A strong Groebner basis is such that for every leading term `t` of the ideal generated by `I`, there exists an element `g` of `I` such that the leading term of `g` divides `t`.)

The result `r` is implementation-dependent (and possibly order-dependent) otherwise. If `I` is an ideal and no Groebner basis can be computed, its list of generators `I.gens()` is used for the reduction.

EXAMPLES:

```
sage: P.<x,y,z> = QQ[]
sage: f1 = -2 * x^2 + x^3
sage: f2 = -2 * y + x* y
sage: f3 = -x^2 + y^2
sage: F = Ideal([f1,f2,f3])
sage: g = x*y - 3*x*y^2
sage: g.reduce(F)
-6*y^2 + 2*y
sage: g.reduce(F.gens())
-6*y^2 + 2*y
```

```
>>> from sage.all import *
>>> P = QQ['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> f1 = -Integer(2) * x**Integer(2) + x**Integer(3)
>>> f2 = -Integer(2) * y + x* y
>>> f3 = -x**Integer(2) + y**Integer(2)
>>> F = Ideal([f1,f2,f3])
>>> g = x*y - Integer(3)*x*y**Integer(2)
>>> g.reduce(F)
-6*y^2 + 2*y
>>> g.reduce(F.gens())
-6*y^2 + 2*y
```

Z is also supported.

```
sage: P.<x,y,z> = ZZ[]
sage: f1 = -2 * x^2 + x^3
sage: f2 = -2 * y + x* y
sage: f3 = -x^2 + y^2
sage: F = Ideal([f1,f2,f3])
sage: g = x*y - 3*x*y^2
sage: g.reduce(F)
-6*y^2 + 2*y
sage: g.reduce(F.gens())
-6*y^2 + 2*y

sage: f = 3*x
sage: f.reduce([2*x,y])
x
```

```
>>> from sage.all import *
>>> P = ZZ['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> f1 = -Integer(2) * x**Integer(2) + x**Integer(3)
>>> f2 = -Integer(2) * y + x* y
>>> f3 = -x**Integer(2) + y**Integer(2)
>>> F = Ideal([f1,f2,f3])
>>> g = x*y - Integer(3)*x*y**Integer(2)
>>> g.reduce(F)
-6*y^2 + 2*y
>>> g.reduce(F.gens())
-6*y^2 + 2*y
```

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```
>>> f = Integer(3)*x
>>> f.reduce([Integer(2)*x,y])
x
```

The reduction is not canonical when `I` is not a Groebner basis:

```
sage: A.<x,y> = QQ[]
sage: (x+y).reduce([x+y, x-y])
2*y
sage: (x+y).reduce([x-y, x+y])
0
```

```
>>> from sage.all import *
>>> A = QQ['x, y']; (x, y,) = A._first_ngens(2)
>>> (x+y).reduce([x+y, x-y])
2*y
>>> (x+y).reduce([x-y, x+y])
0
```

resultant(*other*, *variable=None*)

Compute the resultant of this polynomial and the first argument with respect to the variable given as the second argument.

If a second argument is not provide the first variable of the parent is chosen.

INPUT:

- `other` – polynomial
- `variable` – optional variable (default: `None`)

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQ,2)
sage: a = x+y
sage: b = x^3-y^3
sage: c = a.resultant(b); c
-2*y^3
sage: d = a.resultant(b,y); d
2*x^3
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ,Integer(2), names=('x', 'y',)); (x, y,) = P._first_
↪ngens(2)
>>> a = x+y
>>> b = x**Integer(3)-y**Integer(3)
>>> c = a.resultant(b); c
-2*y^3
>>> d = a.resultant(b,y); d
2*x^3
```

The SINGULAR example:

```
sage: R.<x,y,z> = PolynomialRing(GF(32003), 3)                              #␣
↪needs sage.rings.finite_rings
sage: f = 3 * (x+2)^3 + y
sage: g = x + y + z                                                         #␣
↪needs sage.rings.finite_rings
sage: f.resultant(g, x)                                                     #␣
↪needs sage.rings.finite_rings
3*y^3 + 9*y^2*z + 9*y*z^2 + 3*z^3 - 18*y^2 - 36*y*z - 18*z^2 + 35*y + 36*z -␣
↪24
```

```
>>> from sage.all import *
>>> R = PolynomialRing(GF(Integer(32003)), Integer(3), names=('x', 'y', 'z',
↪)); (x, y, z,) = R._first_ngens(3)# needs sage.rings.finite_rings
>>> f = Integer(3) * (x+Integer(2))**Integer(3) + y
>>> g = x + y + z                                                           #␣
↪needs sage.rings.finite_rings
>>> f.resultant(g, x)                                                       #␣
↪needs sage.rings.finite_rings
3*y^3 + 9*y^2*z + 9*y*z^2 + 3*z^3 - 18*y^2 - 36*y*z - 18*z^2 + 35*y + 36*z -␣
↪24
```

Resultants are also supported over the Integers:

```
sage: R.<x,y,a,b,u> = PolynomialRing(ZZ, 5, order='lex')
sage: r = (x^4*y^2 + x^2*y - y).resultant(x*y - y*a - x*b + a*b + u, x)
sage: r
y^6*a^4 - 4*y^5*a^4*b - 4*y^5*a^3*u + y^5*a^2 - y^5 + 6*y^4*a^4*b^2 + 12*y^
↪4*a^3*b*u - 4*y^4*a^2*b + 6*y^4*a^2*u^2 - 2*y^4*a*u + 4*y^4*b - 4*y^3*a^4*b^
↪3 - 12*y^3*a^3*b^2*u + 6*y^3*a^2*b^2 - 12*y^3*a^2*b*u^2 + 6*y^3*a*b*u - 4*y^
↪3*a*u^3 - 6*y^3*b^2 + y^3*u^2 + y^2*a^4*b^4 + 4*y^2*a^3*b^3*u - 4*y^2*a^2*b^
↪3 + 6*y^2*a^2*b^2*u^2 - 6*y^2*a*b^2*u + 4*y^2*a*b*u^3 + 4*y^2*b^3 - 2*y^
↪2*b*u^2 + y^2*u^4 + y*a^2*b^4 + 2*y*a*b^3*u - y*b^4 + y*b^2*u^2
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, Integer(5), order='lex', names=('x', 'y', 'a', 'b',
↪ 'u',)); (x, y, a, b, u,) = R._first_ngens(5)
>>> r = (x**Integer(4)*y**Integer(2) + x**Integer(2)*y - y).resultant(x*y -␣
↪y*a - x*b + a*b + u, x)
>>> r
y^6*a^4 - 4*y^5*a^4*b - 4*y^5*a^3*u + y^5*a^2 - y^5 + 6*y^4*a^4*b^2 + 12*y^
↪4*a^3*b*u - 4*y^4*a^2*b + 6*y^4*a^2*u^2 - 2*y^4*a*u + 4*y^4*b - 4*y^3*a^4*b^
↪3 - 12*y^3*a^3*b^2*u + 6*y^3*a^2*b^2 - 12*y^3*a^2*b*u^2 + 6*y^3*a*b*u - 4*y^
↪3*a*u^3 - 6*y^3*b^2 + y^3*u^2 + y^2*a^4*b^4 + 4*y^2*a^3*b^3*u - 4*y^2*a^2*b^
↪3 + 6*y^2*a^2*b^2*u^2 - 6*y^2*a*b^2*u + 4*y^2*a*b*u^3 + 4*y^2*b^3 - 2*y^
↪2*b*u^2 + y^2*u^4 + y*a^2*b^4 + 2*y*a*b^3*u - y*b^4 + y*b^2*u^2
```

sub_m_mul_q(*m*, *q*)

Return `self - m*q`, where `m` must be a monomial and `q` a polynomial.

INPUT:

- `m` – a monomial
- `q` – a polynomial

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ,3)
sage: x.sub_m_mul_q(y, z)
-y*z + x
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ,Integer(3), names=('x', 'y', 'z',)); (x, y, z,) = P.
↪_first_ngens(3)
>>> x.sub_m_mul_q(y, z)
-y*z + x
```

subs(*fixed=None*, ***kw*)

Fixes some given variables in a given multivariate polynomial and returns the changed multivariate polynomials. The polynomial itself is not affected. The variable,value pairs for fixing are to be provided as dictionary of the form `{variable:value}`.

This is a special case of evaluating the polynomial with some of the variables constants and the others the original variables, but should be much faster if only few variables are to be fixed.

INPUT:

- `fixed` – (optional) dict with variable:value pairs
- `**kw` – names parameters

OUTPUT: a new multivariate polynomial

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: f = x^2 + y + x^2*y^2 + 5
sage: f(5,y)
25*y^2 + y + 30
sage: f.subs({x: 5})
25*y^2 + y + 30
sage: f.subs(x=5)
25*y^2 + y + 30

sage: P.<x,y,z> = PolynomialRing(GF(2), 3)
sage: f = x + y + 1
sage: f.subs({x:y+1})
0
sage: f.subs(x=y)
1
sage: f.subs(x=x)
x + y + 1
sage: f.subs({x: z})
y + z + 1
sage: f.subs(x=z + 1)
y + z

sage: f.subs(x=1/y)
(y^2 + y + 1)/y
sage: f.subs({x: 1/y})
(y^2 + y + 1)/y
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = x**Integer(2) + y + x**Integer(2)*y**Integer(2) + Integer(5)
>>> f(Integer(5),y)
25*y^2 + y + 30
>>> f.subs({x: Integer(5)})
25*y^2 + y + 30
>>> f.subs(x=Integer(5))
25*y^2 + y + 30

>>> P = PolynomialRing(GF(Integer(2)), Integer(3), names=('x', 'y', 'z',));
↪(x, y, z,) = P._first_ngens(3)
>>> f = x + y + Integer(1)
>>> f.subs({x:y+Integer(1)})
0
>>> f.subs(x=y)
1
>>> f.subs(x=x)
x + y + 1
>>> f.subs({x: z})
y + z + 1
>>> f.subs(x=z + Integer(1))
y + z

>>> f.subs(x=Integer(1)/y)
(y^2 + y + 1)/y
>>> f.subs({x: Integer(1)/y})
(y^2 + y + 1)/y
```

The parameters are substituted in order and without side effects:

```
sage: R.<x,y>=QQ[]
sage: g=x+y
sage: g.subs({x:x+1,y:x*y})
x*y + x + 1
sage: g.subs({x:x+1}).subs({y:x*y})
x*y + x + 1
sage: g.subs({y:x*y}).subs({x:x+1})
x*y + x + y + 1
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> g=x+y
>>> g.subs({x:x+Integer(1),y:x*y})
x*y + x + 1
>>> g.subs({x:x+Integer(1)}).subs({y:x*y})
x*y + x + 1
>>> g.subs({y:x*y}).subs({x:x+Integer(1)})
x*y + x + y + 1
```

```
sage: R.<x,y> = QQ[]
sage: f = x + 2*y
```

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```
sage: f.subs(x=y,y=x)
2*x + y
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = x + Integer(2)*y
>>> f.subs(x=y,y=x)
2*x + y
```

total_degree(*std_grading=False*)

Return the total degree of `self`, which is the maximum degree of all monomials in `self`.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: f = 2*x*y^3*z^2
sage: f.total_degree()
6
sage: f = 4*x^2*y^2*z^3
sage: f.total_degree()
7
sage: f = 99*x^6*y^3*z^9
sage: f.total_degree()
18
sage: f = x*y^3*z^6+3*x^2
sage: f.total_degree()
10
sage: f = z^3+8*x^4*y^5*z
sage: f.total_degree()
10
sage: f = z^9+10*x^4+y^8*x^2
sage: f.total_degree()
10
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> f = Integer(2)*x*y**Integer(3)*z**Integer(2)
>>> f.total_degree()
6
>>> f = Integer(4)*x**Integer(2)*y**Integer(2)*z**Integer(3)
>>> f.total_degree()
7
>>> f = Integer(99)*x**Integer(6)*y**Integer(3)*z**Integer(9)
>>> f.total_degree()
18
>>> f = x*y**Integer(3)*z**Integer(6)+Integer(3)*x**Integer(2)
>>> f.total_degree()
10
>>> f = z**Integer(3)+Integer(8)*x**Integer(4)*y**Integer(5)*z
>>> f.total_degree()
10
>>> f = z**Integer(9)+Integer(10)*x**Integer(4)+y**Integer(8)*x**Integer(2)
```

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```
>>> f.total_degree()
10
```

A matrix term ordering changes the grading. To get the total degree using the standard grading, use `std_grading=True`:

```
sage: tord = TermOrder(matrix(3, [3,2,1,1,1,0,1,0,0]))
sage: tord
Matrix term order with matrix
[3 2 1]
[1 1 0]
[1 0 0]
sage: R.<x,y,z> = PolynomialRing(QQ, order=tord)
sage: f = x^2*y
sage: f.total_degree()
8
sage: f.total_degree(std_grading=True)
3
```

```
>>> from sage.all import *
>>> tord = TermOrder(matrix(Integer(3), [Integer(3),Integer(2),Integer(1),
↪Integer(1),Integer(1),Integer(0),Integer(1),Integer(0),Integer(0)]))
>>> tord
Matrix term order with matrix
[3 2 1]
[1 1 0]
[1 0 0]
>>> R = PolynomialRing(QQ, order=tord, names=('x', 'y', 'z',)); (x, y, z,) =␣
↪R._first_ngens(3)
>>> f = x**Integer(2)*y
>>> f.total_degree()
8
>>> f.total_degree(std_grading=True)
3
```

univariate_polynomial(*R=None*)

Return a univariate polynomial associated to this multivariate polynomial.

INPUT:

- `R` – (default: `None`) PolynomialRing

If this polynomial is not in at most one variable, then a `ValueError` exception is raised. This is checked using the *`is_univariate()`* method. The new Polynomial is over the same base ring as the given `MPolynomial` and in the variable `x` if no ring `R` is provided.

EXAMPLES:

```
sage: R.<x, y> = QQ[]
sage: f = 3*x^2 - 2*y + 7*x^2*y^2 + 5
sage: f.univariate_polynomial()
Traceback (most recent call last):
...
TypeError: polynomial must involve at most one variable
```

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```
sage: g = f.subs({x:10}); g
700*y^2 - 2*y + 305
sage: g.univariate_polynomial ()
700*y^2 - 2*y + 305
sage: g.univariate_polynomial(PolynomialRing(QQ,'z'))
700*z^2 - 2*z + 305
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = Integer(3)*x**Integer(2) - Integer(2)*y +␣
↪Integer(7)*x**Integer(2)*y**Integer(2) + Integer(5)
>>> f.univariate_polynomial()
Traceback (most recent call last):
...
TypeError: polynomial must involve at most one variable
>>> g = f.subs({x:Integer(10)}); g
700*y^2 - 2*y + 305
>>> g.univariate_polynomial ()
700*y^2 - 2*y + 305
>>> g.univariate_polynomial(PolynomialRing(QQ,'z'))
700*z^2 - 2*z + 305
```

Here's an example with a constant multivariate polynomial:

```
sage: g = R(1)
sage: h = g.univariate_polynomial(); h
1
sage: h.parent()
Univariate Polynomial Ring in x over Rational Field
```

```
>>> from sage.all import *
>>> g = R(Integer(1))
>>> h = g.univariate_polynomial(); h
1
>>> h.parent()
Univariate Polynomial Ring in x over Rational Field
```

variable(*i=0*)

Return the i-th variable occurring in `self`. The index i is the index in `self.variables()`.

EXAMPLES:

```
sage: P.<x,y,z> = GF(2)[]
sage: f = x*z^2 + z + 1
sage: f.variables()
(x, z)
sage: f.variable(1)
z
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> f = x*z**Integer(2) + z + Integer(1)
```

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```
>>> f.variables()
(x, z)
>>> f.variable(Integer(1))
z
```

variables()

Return a tuple of all variables occurring in `self`.

EXAMPLES:

```
sage: P.<x,y,z> = GF(2)[]
sage: f = x*z^2 + z + 1
sage: f.variables()
(x, z)
```

```
>>> from sage.all import *
>>> P = GF(Integer(2))['x, y, z']; (x, y, z,) = P._first_ngens(3)
>>> f = x*z**Integer(2) + z + Integer(1)
>>> f.variables()
(x, z)
```

sage.rings.polynomial.multi_polynomial_libsingular.**unpickle_MPolynomialRing_libsingular**(*base_ring*, *names*, *term_order*)

Inverse function for `MPolynomialRing_libsingular.__reduce__`.

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQ)
sage: loads(dumps(P)) is P # indirect doctest
True
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = P._first_ngens(2)
>>> loads(dumps(P)) is P # indirect doctest
True
```

sage.rings.polynomial.multi_polynomial_libsingular.**unpickle_MPolynomial_libsingular**(*R*, *d*)

Deserialize an `MPolynomial_libsingular` object.

INPUT:

- `R` – the base ring
- `d` – a Python dictionary as returned by `MPolynomial_libsingular.dict()`

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQ)
sage: loads(dumps(x)) == x # indirect doctest
True
```

```
>>> from sage.all import *
>>> P = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = P._first_ngens(2)
>>> loads(dumps(x)) == x # indirect doctest
True
```

3.1.9 Direct low-level access to SINGULAR's Groebner basis engine via libSINGULAR

AUTHOR:

- Martin Albrecht (2007-08-08): initial version

EXAMPLES:

```
sage: x,y,z = QQ['x,y,z'].gens()
sage: I = ideal(x^5 + y^4 + z^3 - 1,  x^3 + y^3 + z^2 - 1)
sage: I.groebner_basis('libsingular:std')
[y^6 + x*y^4 + 2*y^3*z^2 + x*z^3 + z^4 - 2*y^3 - 2*z^2 - x + 1,
 x^2*y^3 - y^4 + x^2*z^2 - z^3 - x^2 + 1, x^3 + y^3 + z^2 - 1]
```

```
>>> from sage.all import *
>>> x,y,z = QQ['x,y,z'].gens()
>>> I = ideal(x**Integer(5) + y**Integer(4) + z**Integer(3) - Integer(1), ␣
↪x**Integer(3) + y**Integer(3) + z**Integer(2) - Integer(1))
>>> I.groebner_basis('libsingular:std')
[y^6 + x*y^4 + 2*y^3*z^2 + x*z^3 + z^4 - 2*y^3 - 2*z^2 - x + 1,
 x^2*y^3 - y^4 + x^2*z^2 - z^3 - x^2 + 1, x^3 + y^3 + z^2 - 1]
```

We compute a Groebner basis for cyclic 6, which is a standard benchmark and test ideal:

```
sage: R.<x,y,z,t,u,v> = QQ['x,y,z,t,u,v']
sage: I = sage.rings.ideal.Cyclic(R,6)
sage: B = I.groebner_basis('libsingular:std')
sage: len(B)
45
```

```
>>> from sage.all import *
>>> R = QQ['x,y,z,t,u,v']; (x, y, z, t, u, v,) = R._first_ngens(6)
>>> I = sage.rings.ideal.Cyclic(R,Integer(6))
>>> B = I.groebner_basis('libsingular:std')
>>> len(B)
45
```

Two examples from the Mathematica documentation (done in Sage):

- We compute a Groebner basis:

```
sage: R.<x,y> = PolynomialRing(QQ, order='lex')
sage: ideal(x^2 - 2*y^2, x*y - 3).groebner_basis('libsingular:slimgb')
[x - 2/3*y^3, y^4 - 9/2]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, order='lex', names=('x', 'y',)); (x, y,) = R._first_
```

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```
↪ngens(2)
>>> ideal(x**Integer(2) - Integer(2)*y**Integer(2), x*y - Integer(3)).groebner_
↪basis('libsingular:slimgb')
[x - 2/3*y^3, y^4 - 9/2]
```

- We show that three polynomials have no common root:

```
sage: R.<x,y> = QQ[]
sage: ideal(x+y, x^2 - 1, y^2 - 2*x).groebner_basis('libsingular:slimgb')
[1]
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> ideal(x+y, x**Integer(2) - Integer(1), y**Integer(2) - Integer(2)*x).groebner_
↪basis('libsingular:slimgb')
[1]
```

sage.rings.polynomial.multi_polynomial_ideal_libsingular.**interred_libsingular**(*I*)

SINGULAR's `interred()` command.

INPUT:

- `I` – a Sage ideal

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(ZZ)
sage: I = ideal( x^2 - 3*y, y^3 - x*y, z^3 - x, x^4 - y*z + 1 )
sage: I.interreduced_basis()
[y*z^2 - 81*x*y - 9*y - z, z^3 - x, x^2 - 3*y, 9*y^2 - y*z + 1]

sage: P.<x,y,z> = PolynomialRing(QQ)
sage: I = ideal( x^2 - 3*y, y^3 - x*y, z^3 - x, x^4 - y*z + 1 )
sage: I.interreduced_basis()
[y*z^2 - 81*x*y - 9*y - z, z^3 - x, x^2 - 3*y, y^2 - 1/9*y*z + 1/9]
```

```
>>> from sage.all import *
>>> P = PolynomialRing(ZZ, names=('x', 'y', 'z',)); (x, y, z,) = P._first_ngens(3)
>>> I = ideal( x**Integer(2) - Integer(3)*y, y**Integer(3) - x*y, z**Integer(3) -␣
↪x, x**Integer(4) - y*z + Integer(1) )
>>> I.interreduced_basis()
[y*z^2 - 81*x*y - 9*y - z, z^3 - x, x^2 - 3*y, 9*y^2 - y*z + 1]

>>> P = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = P._first_ngens(3)
>>> I = ideal( x**Integer(2) - Integer(3)*y, y**Integer(3) - x*y, z**Integer(3) -␣
↪x, x**Integer(4) - y*z + Integer(1) )
>>> I.interreduced_basis()
[y*z^2 - 81*x*y - 9*y - z, z^3 - x, x^2 - 3*y, y^2 - 1/9*y*z + 1/9]
```

sage.rings.polynomial.multi_polynomial_ideal_libsingular.**kbase_libsingular**(*I*, *degree=None*)

SINGULAR's `kbase()` algorithm.

INPUT:

- `I` – a groebner basis of an ideal
- `degree` – integer (default: `None`); if not `None`, return only the monomials of the given degree

OUTPUT:

Computes a vector space basis (consisting of monomials) of the quotient ring by the ideal, resp. of a free module by the module, in case it is finite dimensional and if the input is a standard basis with respect to the ring ordering. If the input is not a standard basis, the leading terms of the input are used and the result may have no meaning.

With two arguments: computes the part of a vector space basis of the respective quotient with degree of the monomials equal to the second argument. Here, the quotient does not need to be finite dimensional.

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQ, order='lex')
sage: I = R.ideal(x^2-2*y^2, x*y-3)
sage: I.normal_basis()  # indirect doctest
[y^3, y^2, y, 1]
sage: J = R.ideal(x^2-2*y^2)
sage: [J.normal_basis(d) for d in (0..4)]  # indirect doctest
[[1], [y, x], [y^2, x*y], [y^3, x*y^2], [y^4, x*y^3]]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, order='lex', names=('x', 'y',)); (x, y,) = R._first_
↪ngens(2)
>>> I = R.ideal(x**Integer(2)-Integer(2)*y**Integer(2), x*y-Integer(3))
>>> I.normal_basis()  # indirect doctest
[y^3, y^2, y, 1]
>>> J = R.ideal(x**Integer(2)-Integer(2)*y**Integer(2))
>>> [J.normal_basis(d) for d in (ellipsis_iter(Integer(0),Ellipsis,Integer(4)))]
↪# indirect doctest
[[1], [y, x], [y^2, x*y], [y^3, x*y^2], [y^4, x*y^3]]
```

`sage.rings.polynomial.multi_polynomial_ideal_libsingular.`**`slimgb_libsingular`**`(I)`

SINGULAR's `slimgb()` algorithm.

INPUT:

- `I` – a Sage ideal

`sage.rings.polynomial.multi_polynomial_ideal_libsingular.`**`std_libsingular`**`(I)`

SINGULAR's `std()` algorithm.

INPUT:

- `I` – a Sage ideal

3.1.10 Solution of polynomial systems using msolve

msolve is a multivariate polynomial system solver based on Gröbner bases.

This module provide implementations of some operations on polynomial ideals based on msolve.

Note that the optional package msolve must be installed.

See also

- `sage.features.msolve`

- *sage.rings.polynomial.multi_polynomial_ideal*

`sage.rings.polynomial.msolve.`**`groebner_basis_degrevlex`**(*ideal*, *proof=True*)

Compute a degrevlex Gröbner basis using msolve

EXAMPLES:

```
sage: from sage.rings.polynomial.msolve import groebner_basis_degrevlex

sage: R.<a,b,c> = PolynomialRing(GF(101), 3, order='lex')
sage: I = sage.rings.ideal.Katsura(R,3)
sage: gb = groebner_basis_degrevlex(I); gb # optional - msolve
[a + 2*b + 2*c - 1, b*c - 19*c^2 + 10*b + 40*c,
b^2 - 41*c^2 + 20*b - 20*c, c^3 + 28*c^2 - 37*b + 13*c]
sage: gb.universe() is R # optional - msolve
False
sage: gb.universe().term_order() # optional - msolve
Degree reverse lexicographic term order
sage: ideal(gb).transformed_basis(other_ring=R) # optional - msolve
[c^4 + 38*c^3 - 6*c^2 - 6*c, 30*c^3 + 32*c^2 + b - 14*c,
a + 2*b + 2*c - 1]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.msolve import groebner_basis_degrevlex

>>> R = PolynomialRing(GF(Integer(101)), Integer(3), order='lex', names=('a', 'b',
↪ 'c',)); (a, b, c,) = R._first_ngens(3)
>>> I = sage.rings.ideal.Katsura(R,Integer(3))
>>> gb = groebner_basis_degrevlex(I); gb # optional - msolve
[a + 2*b + 2*c - 1, b*c - 19*c^2 + 10*b + 40*c,
b^2 - 41*c^2 + 20*b - 20*c, c^3 + 28*c^2 - 37*b + 13*c]
>>> gb.universe() is R # optional - msolve
False
>>> gb.universe().term_order() # optional - msolve
Degree reverse lexicographic term order
>>> ideal(gb).transformed_basis(other_ring=R) # optional - msolve
[c^4 + 38*c^3 - 6*c^2 - 6*c, 30*c^3 + 32*c^2 + b - 14*c,
a + 2*b + 2*c - 1]
```

Gröbner bases over the rationals require $proof = False$:

```
sage: R.<x, y> = PolynomialRing(QQ, 2)
sage: I = Ideal([ x*y - 1, (x-2)^2 + (y-1)^2 - 1])
sage: I.groebner_basis(algorithm='msolve') # optional - msolve
Traceback (most recent call last):
...
ValueError: msolve relies on heuristics; please use proof=False
sage: I.groebner_basis(algorithm='msolve', proof=False) # optional - msolve
[x*y - 1, x^2 + y^2 - 4*x - 2*y + 4, y^3 - 2*y^2 + x + 4*y - 4]
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, Integer(2), names=('x', 'y',)); (x, y,) = R._first_
```

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```
↪ngens(2)
>>> I = Ideal([ x*y - Integer(1), (x-Integer(2))**Integer(2) + (y-
↪Integer(1))**Integer(2) - Integer(1)])
>>> I.groebner_basis(algorithm='msolve') # optional - msolve
Traceback (most recent call last):
...
ValueError: msolve relies on heuristics; please use proof=False
>>> I.groebner_basis(algorithm='msolve', proof=False) # optional - msolve
[x*y - 1, x^2 + y^2 - 4*x - 2*y + 4, y^3 - 2*y^2 + x + 4*y - 4]
```

`sage.rings.polynomial.msolve.`**`variety`**(*ideal*, *ring*, *proof*)

Compute the variety of a zero-dimensional ideal using msolve.

Part of the initial implementation was loosely based on the example interfaces available as part of msolve, with the authors' permission.

EXAMPLES:

```
sage: from sage.rings.polynomial.msolve import variety
sage: p = 536870909
sage: R.<x, y> = PolynomialRing(GF(p), 2, order='lex')
sage: I = Ideal([ x*y - 1, (x-2)^2 + (y-1)^2 - 1])
sage: sorted(variety(I, GF(p^2), proof=False), key=str) # optional - msolve
[{x: 1, y: 1},
 {x: 254228855*z2 + 114981228, y: 232449571*z2 + 402714189},
 {x: 267525699, y: 473946006},
 {x: 282642054*z2 + 154363985, y: 304421338*z2 + 197081624}]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.msolve import variety
>>> p = Integer(536870909)
>>> R = PolynomialRing(GF(p), Integer(2), order='lex', names=('x', 'y',)); (x, y,
↪) = R._first_ngens(2)
>>> I = Ideal([ x*y - Integer(1), (x-Integer(2))**Integer(2) + (y-
↪Integer(1))**Integer(2) - Integer(1)])
>>> sorted(variety(I, GF(p**Integer(2)), proof=False), key=str) # optional -␣
↪msolve
[{x: 1, y: 1},
 {x: 254228855*z2 + 114981228, y: 232449571*z2 + 402714189},
 {x: 267525699, y: 473946006},
 {x: 282642054*z2 + 154363985, y: 304421338*z2 + 197081624}]
```

3.1.11 Generic data structures for multivariate polynomials

This module provides an implementation of a generic data structure `PolyDict` and the underlying arithmetic for multi-variate polynomial rings. It uses a sparse representation of polynomials encoded as a Python dictionary where keys are exponents and values coefficients.

`{(e1,...,er):c1,...} <-> c1*x1^e1*...*xr^er+...,`

The exponent `(e1,...,er)` in this representation is an instance of the class `ETuple`.

AUTHORS:

- William Stein

- David Joyner
- Martin Albrecht (ETuple)
- Joel B. Mohler (2008-03-17) – ETuple rewrite as sparse C array

class sage.rings.polynomial.polydict.**ETuple**

Bases: object

Representation of the exponents of a polydict monomial. If (0,0,3,0,5) is the exponent tuple of x_2^3*x_4^5 then this class only stores {2:3, 4:5} instead of the full tuple. This sparse information may be obtained by provided methods.

The index/value data is all stored in the _data C int array member variable. For the example above, the C array would contain 2,3,4,5. The indices are interlaced with the values.

This data structure is very nice to work with for some functions implemented in this class, but tricky for others. One reason that I really like the format is that it requires a single memory allocation for all of the values. A hash table would require more allocations and presumably be slower. I didn't benchmark this question (although, there is no question that this is much faster than the prior use of python dicts).

combine_to_positives(*other*)

Given a pair of ETuples (self, other), returns a triple of ETuples (a, b, c) so that self = a + b, other = a + c and b and c have all positive entries.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([-2, 1, -5, 3, 1, 0])
sage: f = ETuple([1, -3, -3, 4, 0, 2])
sage: e.combine_to_positives(f)
((-2, -3, -5, 3, 0, 0), (0, 4, 0, 0, 1, 0), (3, 0, 2, 1, 0, 2))
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple
>>> e = ETuple([-Integer(2), Integer(1), -Integer(5), Integer(3), Integer(1),␣
↪Integer(0)])
>>> f = ETuple([Integer(1), -Integer(3), -Integer(3), Integer(4), Integer(0),␣
↪Integer(2)])
>>> e.combine_to_positives(f)
((-2, -3, -5, 3, 0, 0), (0, 4, 0, 0, 1, 0), (3, 0, 2, 1, 0, 2))
```

common_nonzero_positions(*other*, *sort=False*)

Return an optionally sorted list of nonzero positions either in self or other, i.e. the only positions that need to be considered for any vector operation.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 0, 2])
sage: f = ETuple([0, 0, 1])
sage: e.common_nonzero_positions(f)
{0, 2}
sage: e.common_nonzero_positions(f, sort=True)
[0, 2]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple
>>> e = ETuple([Integer(1), Integer(0), Integer(2)])
>>> f = ETuple([Integer(0), Integer(0), Integer(1)])
>>> e.common_nonzero_positions(f)
{0, 2}
>>> e.common_nonzero_positions(f, sort=True)
[0, 2]
```

divide_by_gcd(*other*)

Return `self / gcd(self, other)`.

The entries of the result are the maximum of 0 and the difference of the corresponding entries of `self` and `other`.

divide_by_var(*pos*)

Return division of `self` by the variable with index `pos`.

If `self[pos] == 0` then a `ArithmeticError` is raised. Otherwise, an *ETuple* is returned that is zero in position `pos` and coincides with `self` in the other positions.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 2, 0, 1])
sage: e.divide_by_var(0)
(0, 2, 0, 1)
sage: e.divide_by_var(1)
(1, 1, 0, 1)
sage: e.divide_by_var(3)
(1, 2, 0, 0)
sage: e.divide_by_var(2)
Traceback (most recent call last):
...
ArithmeticError: not divisible by this variable
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple
>>> e = ETuple([Integer(1), Integer(2), Integer(0), Integer(1)])
>>> e.divide_by_var(Integer(0))
(0, 2, 0, 1)
>>> e.divide_by_var(Integer(1))
(1, 1, 0, 1)
>>> e.divide_by_var(Integer(3))
(1, 2, 0, 0)
>>> e.divide_by_var(Integer(2))
Traceback (most recent call last):
...
ArithmeticError: not divisible by this variable
```

divides(*other*)

Return whether `self` divides `other`, i.e., no entry of `self` exceeds that of `other`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: ETuple([1, 1, 0, 1, 0]).divides(ETuple([2, 2, 2, 2, 2]))
True
sage: ETuple([0, 3, 0, 1, 0]).divides(ETuple([2, 2, 2, 2, 2]))
False
sage: ETuple([0, 3, 0, 1, 0]).divides(ETuple([0, 3, 2, 2, 2]))
True
sage: ETuple([0, 0, 0, 0, 0]).divides(ETuple([2, 2, 2, 2, 2]))
True

sage: ETuple({104: 18, 256: 25, 314:78}, length=400r).divides(ETuple({104: 19,
↪ 105: 20, 106: 21}, length=400r))
False
sage: ETuple({104: 18, 256: 25, 314:78}, length=400r).divides(ETuple({104: 19,
↪ 105: 20, 106: 21, 255: 2, 256: 25, 312: 5, 314: 79, 315: 28}, length=400r))
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple
>>> ETuple([Integer(1), Integer(1), Integer(0), Integer(1), Integer(0)]).
↪divides(ETuple([Integer(2), Integer(2), Integer(2), Integer(2),␣
↪Integer(2)]))
True
>>> ETuple([Integer(0), Integer(3), Integer(0), Integer(1), Integer(0)]).
↪divides(ETuple([Integer(2), Integer(2), Integer(2), Integer(2),␣
↪Integer(2)]))
False
>>> ETuple([Integer(0), Integer(3), Integer(0), Integer(1), Integer(0)]).
↪divides(ETuple([Integer(0), Integer(3), Integer(2), Integer(2),␣
↪Integer(2)]))
True
>>> ETuple([Integer(0), Integer(0), Integer(0), Integer(0), Integer(0)]).
↪divides(ETuple([Integer(2), Integer(2), Integer(2), Integer(2),␣
↪Integer(2)]))
True

>>> ETuple({Integer(104): Integer(18), Integer(256): Integer(25),␣
↪Integer(314):Integer(78)}, length=400).divides(ETuple({Integer(104):␣
↪Integer(19), Integer(105): Integer(20), Integer(106): Integer(21)},␣
↪length=400))
False
>>> ETuple({Integer(104): Integer(18), Integer(256): Integer(25),␣
↪Integer(314):Integer(78)}, length=400).divides(ETuple({Integer(104):␣
↪Integer(19), Integer(105): Integer(20), Integer(106): Integer(21),␣
↪Integer(255): Integer(2), Integer(256): Integer(25), Integer(312):␣
↪Integer(5), Integer(314): Integer(79), Integer(315): Integer(28)},␣
↪length=400))
True
```

dotprod(*other*)

Return the dot product of this tuple by `other`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 0, 2])
sage: f = ETuple([0, 1, 1])
sage: e.dotprod(f)
2
sage: e = ETuple([1, 1, -1])
sage: f = ETuple([0, -2, 1])
sage: e.dotprod(f)
-3
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple
>>> e = ETuple([Integer(1), Integer(0), Integer(2)])
>>> f = ETuple([Integer(0), Integer(1), Integer(1)])
>>> e.dotprod(f)
2
>>> e = ETuple([Integer(1), Integer(1), -Integer(1)])
>>> f = ETuple([Integer(0), -Integer(2), Integer(1)])
>>> e.dotprod(f)
-3
```

eadd(*other*)

Return the vector addition of `self` with `other`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 0, 2])
sage: f = ETuple([0, 1, 1])
sage: e.eadd(f)
(1, 1, 3)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple
>>> e = ETuple([Integer(1), Integer(0), Integer(2)])
>>> f = ETuple([Integer(0), Integer(1), Integer(1)])
>>> e.eadd(f)
(1, 1, 3)
```

Verify that Issue #6428 has been addressed:

```
sage: # needs sage.libs.singular
sage: R.<y, z> = Frac(QQ['x'])[]
sage: type(y)
<class 'sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_
↪libsingular'>
sage: y^(2^32)
Traceback (most recent call last):
...
OverflowError: exponent overflow (...)   # 64-bit
OverflowError: Python int too large to convert to C unsigned long  # 32-bit
```

```
>>> from sage.all import *
>>> # needs sage.libs.singular
>>> R = Frac(QQ['x'])['y, z']; (y, z,) = R._first_ngens(2)
>>> type(y)
<class 'sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_
↪libsingular'>
>>> y**(Integer(2)**Integer(32))
Traceback (most recent call last):
...
OverflowError: exponent overflow (...)   # 64-bit
OverflowError: Python int too large to convert to C unsigned long  # 32-bit
```

eadd_p(*other*, *pos*)

Add other to self at position pos.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 0, 2])
sage: e.eadd_p(5, 1)
(1, 5, 2)
sage: e = ETuple([0]*7)
sage: e.eadd_p(5, 4)
(0, 0, 0, 0, 5, 0, 0)

sage: ETuple([0,1]).eadd_p(1, 0) == ETuple([1,1])
True

sage: e = ETuple([0, 1, 0])
sage: e.eadd_p(0, 0).nonzero_positions()
[1]
sage: e.eadd_p(0, 1).nonzero_positions()
[1]
sage: e.eadd_p(0, 2).nonzero_positions()
[1]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple
>>> e = ETuple([Integer(1), Integer(0), Integer(2)])
>>> e.eadd_p(Integer(5), Integer(1))
(1, 5, 2)
>>> e = ETuple([Integer(0)]*Integer(7))
>>> e.eadd_p(Integer(5), Integer(4))
(0, 0, 0, 0, 5, 0, 0)

>>> ETuple([Integer(0),Integer(1)]).eadd_p(Integer(1), Integer(0)) ==␣
↪ETuple([Integer(1),Integer(1)])
True

>>> e = ETuple([Integer(0), Integer(1), Integer(0)])
>>> e.eadd_p(Integer(0), Integer(0)).nonzero_positions()
[1]
>>> e.eadd_p(Integer(0), Integer(1)).nonzero_positions()
```

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```
[1]
>>> e.eadd_p(Integer(0), Integer(2)).nonzero_positions()
[1]
```

eadd_scaled(*other*, *scalar*)

Vector addition of `self` with `scalar * other`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 0, 2])
sage: f = ETuple([0, 1, 1])
sage: e.eadd_scaled(f, 3)
(1, 3, 5)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple
>>> e = ETuple([Integer(1), Integer(0), Integer(2)])
>>> f = ETuple([Integer(0), Integer(1), Integer(1)])
>>> e.eadd_scaled(f, Integer(3))
(1, 3, 5)
```

emax(*other*)

Vector of maximum of components of `self` and `other`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 0, 2])
sage: f = ETuple([0, 1, 1])
sage: e.emax(f)
(1, 1, 2)
sage: e = ETuple((1, 2, 3, 4))
sage: f = ETuple((4, 0, 2, 1))
sage: f.emax(e)
(4, 2, 3, 4)
sage: e = ETuple((1, -2, -2, 4))
sage: f = ETuple((4, 0, 0, 0))
sage: f.emax(e)
(4, 0, 0, 4)
sage: f.emax(e).nonzero_positions()
[0, 3]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple
>>> e = ETuple([Integer(1), Integer(0), Integer(2)])
>>> f = ETuple([Integer(0), Integer(1), Integer(1)])
>>> e.emax(f)
(1, 1, 2)
>>> e = ETuple((Integer(1), Integer(2), Integer(3), Integer(4)))
>>> f = ETuple((Integer(4), Integer(0), Integer(2), Integer(1)))
>>> f.emax(e)
```

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```
(4, 2, 3, 4)
>>> e = ETuple((Integer(1), -Integer(2), -Integer(2), Integer(4)))
>>> f = ETuple((Integer(4), Integer(0), Integer(0), Integer(0)))
>>> f.emax(e)
(4, 0, 0, 4)
>>> f.emax(e).nonzero_positions()
[0, 3]
```

emin(*other*)

Vector of minimum of components of `self` and `other`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 0, 2])
sage: f = ETuple([0, 1, 1])
sage: e.emin(f)
(0, 0, 1)
sage: e = ETuple([1, 0, -1])
sage: f = ETuple([0, -2, 1])
sage: e.emin(f)
(0, -2, -1)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple
>>> e = ETuple([Integer(1), Integer(0), Integer(2)])
>>> f = ETuple([Integer(0), Integer(1), Integer(1)])
>>> e.emin(f)
(0, 0, 1)
>>> e = ETuple([Integer(1), Integer(0), -Integer(1)])
>>> f = ETuple([Integer(0), -Integer(2), Integer(1)])
>>> e.emin(f)
(0, -2, -1)
```

emul(*factor*)

Scalar Vector multiplication of `self`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 0, 2])
sage: e.emul(2)
(2, 0, 4)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple
>>> e = ETuple([Integer(1), Integer(0), Integer(2)])
>>> e.emul(Integer(2))
(2, 0, 4)
```

escalar_div(*n*)

Divide each exponent by `n`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: ETuple([1, 0, 2]).escalar_div(2)
(0, 0, 1)
sage: ETuple([0, 3, 12]).escalar_div(3)
(0, 1, 4)

sage: ETuple([1, 5, 2]).escalar_div(0)
Traceback (most recent call last):
...
ZeroDivisionError
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple
>>> ETuple([Integer(1), Integer(0), Integer(2)]).escalar_div(Integer(2))
(0, 0, 1)
>>> ETuple([Integer(0), Integer(3), Integer(12)]).escalar_div(Integer(3))
(0, 1, 4)

>>> ETuple([Integer(1), Integer(5), Integer(2)]).escalar_div(Integer(0))
Traceback (most recent call last):
...
ZeroDivisionError
```

esub(*other*)

Vector subtraction of `self` with `other`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 0, 2])
sage: f = ETuple([0, 1, 1])
sage: e.esub(f)
(1, -1, 1)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple
>>> e = ETuple([Integer(1), Integer(0), Integer(2)])
>>> f = ETuple([Integer(0), Integer(1), Integer(1)])
>>> e.esub(f)
(1, -1, 1)
```

is_constant()

Return if all exponents are zero in the tuple.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 0, 2])
sage: e.is_constant()
False
sage: e = ETuple([0, 0])
sage: e.is_constant()
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple
>>> e = ETuple([Integer(1), Integer(0), Integer(2)])
>>> e.is_constant()
False
>>> e = ETuple([Integer(0), Integer(0)])
>>> e.is_constant()
True
```

is_multiple_of(*n*)

Test whether each entry is a multiple of `n`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple

sage: ETuple([0, 0]).is_multiple_of(3)
True
sage: ETuple([0, 3, 12, 0, 6]).is_multiple_of(3)
True
sage: ETuple([0, 0, 2]).is_multiple_of(3)
False
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple

>>> ETuple([Integer(0), Integer(0)]).is_multiple_of(Integer(3))
True
>>> ETuple([Integer(0), Integer(3), Integer(12), Integer(0), Integer(6)]).is_
↪multiple_of(Integer(3))
True
>>> ETuple([Integer(0), Integer(0), Integer(2)]).is_multiple_of(Integer(3))
False
```

nonzero_positions(*sort=False*)

Return the positions of nonzero exponents in the tuple.

INPUT:

- `sort` – boolean (default: `False`); if `True` a sorted list is returned; if `False` an unsorted list is returned

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 0, 2])
sage: e.nonzero_positions()
[0, 2]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple
>>> e = ETuple([Integer(1), Integer(0), Integer(2)])
>>> e.nonzero_positions()
[0, 2]
```

nonzero_values(*sort=True*)

Return the nonzero values of the tuple.

INPUT:

- `sort` – boolean (default: `True`); if `True` the values are sorted by their indices. Otherwise the values are returned unsorted.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([2, 0, 1])
sage: e.nonzero_values()
[2, 1]
sage: f = ETuple([0, -1, 1])
sage: f.nonzero_values(sort=True)
[-1, 1]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple
>>> e = ETuple([Integer(2), Integer(0), Integer(1)])
>>> e.nonzero_values()
[2, 1]
>>> f = ETuple([Integer(0), -Integer(1), Integer(1)])
>>> f.nonzero_values(sort=True)
[-1, 1]
```

reversed()

Return the reversed ETuple of `self`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 2, 3])
sage: e.reversed()
(3, 2, 1)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple
>>> e = ETuple([Integer(1), Integer(2), Integer(3)])
>>> e.reversed()
(3, 2, 1)
```

sparse_iter()

Iterator over the elements of `self` where the elements are returned as `(i, e)` where `i` is the position of `e` in the tuple.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 0, 2, 0, 3])
sage: list(e.sparse_iter())
[(0, 1), (2, 2), (4, 3)]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple
>>> e = ETuple([Integer(1), Integer(0), Integer(2), Integer(0), Integer(3)])
>>> list(e.sparse_iter())
[(0, 1), (2, 2), (4, 3)]
```

unweighted_degree()

Return the sum of entries.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: ETuple([1, 1, 0, 2, 0]).unweighted_degree()
4
sage: ETuple([-1, 1]).unweighted_degree()
0
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple
>>> ETuple([Integer(1), Integer(1), Integer(0), Integer(2), Integer(0)]).
↪unweighted_degree()
4
>>> ETuple([-Integer(1), Integer(1)]).unweighted_degree()
0
```

unweighted_quotient_degree(*other*)

Return the degree of `self` divided by its gcd with `other`.

It amounts to counting the nonnegative entries of `self.esub(other)`.

weighted_degree(*w*)

Return the weighted sum of entries.

INPUT:

- `w` – tuple of nonnegative integers

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 1, 0, 2, 0])
sage: e.weighted_degree((1, 2, 3, 4, 5))
11
sage: ETuple([-1, 1]).weighted_degree((1, 2))
1

sage: ETuple([1, 0]).weighted_degree((1, 2, 3))
Traceback (most recent call last):
...
ValueError: w must be of the same length as the ETuple
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple
>>> e = ETuple([Integer(1), Integer(1), Integer(0), Integer(2), Integer(0)])
>>> e.weighted_degree((Integer(1), Integer(2), Integer(3), Integer(4),␣
```

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```
↪Integer(5)))
11
>>> ETuple([-Integer(1), Integer(1)]).weighted_degree((Integer(1),␣
↪Integer(2)))
1

>>> ETuple([Integer(1), Integer(0)]).weighted_degree((Integer(1), Integer(2),␣
↪Integer(3)))
Traceback (most recent call last):
...
ValueError: w must be of the same length as the ETuple
```

weighted_quotient_degree(*other*, *w*)

Return the weighted degree of `self` divided by its gcd with `other`.

INPUT:

- `other` – an *ETuple*
- `w` – tuple of nonnegative integers

class sage.rings.polynomial.polydict.**PolyDict**

Bases: `object`

Data structure for multivariate polynomials.

A PolyDict holds a dictionary all of whose keys are *ETuple* and whose values are coefficients on which it is implicitely assumed that arithmetic operations can be performed.

No arithmetic operation on *PolyDict* clear zero coefficients as of now there is no reliable way of testing it in the most general setting, see Issue #35319. For removing zero coefficients from a *PolyDict* you can use the method *remove_zeros()* which can be parametrized by a zero test.

apply_map(*f*)

Apply the map `f` on the coefficients (inplace).

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(1, 0): 1, (1, 1): -2})
sage: f.apply_map(lambda x: x^2)
sage: f
PolyDict with representation {(1, 0): 1, (1, 1): 4}
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
>>> f = PolyDict({(Integer(1), Integer(0)): Integer(1), (Integer(1),␣
↪Integer(1)): -Integer(2)})
>>> f.apply_map(lambda x: x**Integer(2))
>>> f
PolyDict with representation {(1, 0): 1, (1, 1): 4}
```

coefficient(*mon*)

Return a polydict that defines a polynomial in 1 less number of variables that gives the coefficient of mon in this polynomial.

The coefficient is defined as follows. If f is this polynomial, then the coefficient is the sum T/mon where the sum is over terms T in f that are exactly divisible by mon.

coefficients()

Return the coefficients of `self`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: sorted(f.coefficients())
[2, 3, 4]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
>>> f = PolyDict({(Integer(2), Integer(3)): Integer(2), (Integer(1),␣
↪Integer(2)): Integer(3), (Integer(2), Integer(1)): Integer(4)})
>>> sorted(f.coefficients())
[2, 3, 4]
```

coerce_coefficients(*A*)

Coerce the coefficients in the parent `A`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 0})
sage: f
PolyDict with representation {(2, 3): 0}
sage: f.coerce_coefficients(QQ)
doctest:warning
...
DeprecationWarning: coerce_cefficients is deprecated; use apply_map instead
See https://github.com/sagemath/sage/issues/34000 for details.
sage: f
PolyDict with representation {(2, 3): 0}
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
>>> f = PolyDict({(Integer(2), Integer(3)): Integer(0)})
>>> f
PolyDict with representation {(2, 3): 0}
>>> f.coerce_coefficients(QQ)
doctest:warning
...
DeprecationWarning: coerce_cefficients is deprecated; use apply_map instead
See https://github.com/sagemath/sage/issues/34000 for details.
>>> f
PolyDict with representation {(2, 3): 0}
```

degree(*x=None*)

Return the total degree or the maximum degree in the variable `x`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.degree()
5
sage: f.degree(PolyDict({(1, 0): 1}))
2
sage: f.degree(PolyDict({(0, 1): 1}))
3
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
>>> f = PolyDict({(Integer(2), Integer(3)): Integer(2), (Integer(1),␣
↪Integer(2)): Integer(3), (Integer(2), Integer(1)): Integer(4)})
>>> f.degree()
5
>>> f.degree(PolyDict({(Integer(1), Integer(0)): Integer(1)}))
2
>>> f.degree(PolyDict({(Integer(0), Integer(1)): Integer(1)}))
3
```

derivative(*x*)

Return the derivative of `self` with respect to `x`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.derivative(PolyDict({(1, 0): 1}))
PolyDict with representation {(0, 2): 3, (1, 1): 8, (1, 3): 4}
sage: f.derivative(PolyDict({(0, 1): 1}))
PolyDict with representation {(1, 1): 6, (2, 0): 4, (2, 2): 6}

sage: PolyDict({(-1,): 1}).derivative(PolyDict({(1,): 1}))
PolyDict with representation {(-2,): -1}
sage: PolyDict({(-2,): 1}).derivative(PolyDict({(1,): 1}))
PolyDict with representation {(-3,): -2}

sage: PolyDict({}).derivative(PolyDict({(1, 1): 1}))
Traceback (most recent call last):
...
ValueError: x must be a generator
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
>>> f = PolyDict({(Integer(2), Integer(3)): Integer(2), (Integer(1),␣
↪Integer(2)): Integer(3), (Integer(2), Integer(1)): Integer(4)})
>>> f.derivative(PolyDict({(Integer(1), Integer(0)): Integer(1)}))
PolyDict with representation {(0, 2): 3, (1, 1): 8, (1, 3): 4}
>>> f.derivative(PolyDict({(Integer(0), Integer(1)): Integer(1)}))
PolyDict with representation {(1, 1): 6, (2, 0): 4, (2, 2): 6}

>>> PolyDict({(-Integer(1),): Integer(1)}).derivative(PolyDict({(Integer(1),
↪): Integer(1)}))
```

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```
PolyDict with representation {(-2,): -1}
>>> PolyDict({(-Integer(2),): Integer(1)}).derivative(PolyDict({(Integer(1),
↪): Integer(1)}))
PolyDict with representation {(-3,): -2}

>>> PolyDict({}).derivative(PolyDict({(Integer(1), Integer(1)): Integer(1)}))
Traceback (most recent call last):
...
ValueError: x must be a generator
```

derivative_i(*i*)

Return the derivative of `self` with respect to the `i`-th variable.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: PolyDict({(1, 1): 1}).derivative_i(0)
PolyDict with representation {(0, 1): 1}
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
>>> PolyDict({(Integer(1), Integer(1)): Integer(1)}).derivative_i(Integer(0))
PolyDict with representation {(0, 1): 1}
```

dict()

Return a copy of the dict that defines `self`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.dict()
{(1, 2): 3, (2, 1): 4, (2, 3): 2}
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
>>> f = PolyDict({(Integer(2), Integer(3)): Integer(2), (Integer(1),␣
↪Integer(2)): Integer(3), (Integer(2), Integer(1)): Integer(4)})
>>> f.dict()
{(1, 2): 3, (2, 1): 4, (2, 3): 2}
```

exponents()

Return the exponents of `self`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: sorted(f.exponents())
[(1, 2), (2, 1), (2, 3)]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
```

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```
>>> f = PolyDict({(Integer(2), Integer(3)): Integer(2), (Integer(1),␣
↪Integer(2)): Integer(3), (Integer(2), Integer(1)): Integer(4)})
>>> sorted(f.exponents())
[(1, 2), (2, 1), (2, 3)]
```

get(*e*, *default=None*)

Return the coefficient of the ETuple `e` if present and `default` otherwise.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict, ETuple
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.get(ETuple([1,2]))
3
sage: f.get(ETuple([1,1]), 'hello')
'hello'
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict, ETuple
>>> f = PolyDict({(Integer(2), Integer(3)): Integer(2), (Integer(1),␣
↪Integer(2)): Integer(3), (Integer(2), Integer(1)): Integer(4)})
>>> f.get(ETuple([Integer(1),Integer(2)]))
3
>>> f.get(ETuple([Integer(1),Integer(1)]), 'hello')
'hello'
```

homogenize(*var*)

Return the homogeneization of `self` by increasing the degree of the variable `var`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(0, 0): 1, (2, 1): 3, (1, 1): 5})
sage: f.homogenize(0)
PolyDict with representation {(2, 1): 8, (3, 0): 1}
sage: f.homogenize(1)
PolyDict with representation {(0, 3): 1, (1, 2): 5, (2, 1): 3}

sage: PolyDict({(0, 1): 1, (1, 1): -1}).homogenize(0)
PolyDict with representation {(1, 1): 0}
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
>>> f = PolyDict({(Integer(0), Integer(0)): Integer(1), (Integer(2),␣
↪Integer(1)): Integer(3), (Integer(1), Integer(1)): Integer(5)})
>>> f.homogenize(Integer(0))
PolyDict with representation {(2, 1): 8, (3, 0): 1}
>>> f.homogenize(Integer(1))
PolyDict with representation {(0, 3): 1, (1, 2): 5, (2, 1): 3}

>>> PolyDict({(Integer(0), Integer(1)): Integer(1), (Integer(1), Integer(1)):␣
↪-Integer(1)}).homogenize(Integer(0))
PolyDict with representation {(1, 1): 0}
```

integral(*x*)

Return the integral of `self` with respect to `x`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.integral(PolyDict({(1, 0): 1}))
PolyDict with representation {(2, 2): 3/2, (3, 1): 4/3, (3, 3): 2/3}
sage: f.integral(PolyDict({(0, 1): 1}))
PolyDict with representation {(1, 3): 1, (2, 2): 2, (2, 4): 1/2}

sage: PolyDict({(-1,): 1}).integral(PolyDict({(1,): 1}))
Traceback (most recent call last):
...
ArithmeticError: integral of monomial with exponent -1
sage: PolyDict({(-2,): 1}).integral(PolyDict({(1,): 1}))
PolyDict with representation {(-1,): -1}
sage: PolyDict({}).integral(PolyDict({(1, 1): 1}))
Traceback (most recent call last):
...
ValueError: x must be a generator
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
>>> f = PolyDict({(Integer(2), Integer(3)): Integer(2), (Integer(1),␣
↪Integer(2)): Integer(3), (Integer(2), Integer(1)): Integer(4)})
>>> f.integral(PolyDict({(Integer(1), Integer(0)): Integer(1)}))
PolyDict with representation {(2, 2): 3/2, (3, 1): 4/3, (3, 3): 2/3}
>>> f.integral(PolyDict({(Integer(0), Integer(1)): Integer(1)}))
PolyDict with representation {(1, 3): 1, (2, 2): 2, (2, 4): 1/2}

>>> PolyDict({(-Integer(1),): Integer(1)}).integral(PolyDict({(Integer(1),):␣
↪Integer(1)}))
Traceback (most recent call last):
...
ArithmeticError: integral of monomial with exponent -1
>>> PolyDict({(-Integer(2),): Integer(1)}).integral(PolyDict({(Integer(1),):␣
↪Integer(1)}))
PolyDict with representation {(-1,): -1}
>>> PolyDict({}).integral(PolyDict({(Integer(1), Integer(1)): Integer(1)}))
Traceback (most recent call last):
...
ValueError: x must be a generator
```

integral_i(*i*)

Return the derivative of `self` with respect to the `i`-th variable.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: PolyDict({(1, 1): 1}).integral_i(0)
PolyDict with representation {(2, 1): 1/2}
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
>>> PolyDict({(Integer(1), Integer(1)): Integer(1)}).integral_i(Integer(0))
PolyDict with representation {(2, 1): 1/2}
```

is_constant()

Return whether this polynomial is constant.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.is_constant()
False
sage: g = PolyDict({(0, 0): 2})
sage: g.is_constant()
True
sage: h = PolyDict({})
sage: h.is_constant()
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
>>> f = PolyDict({(Integer(2), Integer(3)): Integer(2), (Integer(1),␣
→Integer(2)): Integer(3), (Integer(2), Integer(1)): Integer(4)})
>>> f.is_constant()
False
>>> g = PolyDict({(Integer(0), Integer(0)): Integer(2)})
>>> g.is_constant()
True
>>> h = PolyDict({})
>>> h.is_constant()
True
```

is_homogeneous(*w=None*)

Return whether this polynomial is homogeneous.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: PolyDict({}).is_homogeneous()
True
sage: PolyDict({(1, 2): 1, (0, 3): -2}).is_homogeneous()
True
sage: PolyDict({(1, 0): 1, (1, 2): 3}).is_homogeneous()
False
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
>>> PolyDict({}).is_homogeneous()
True
>>> PolyDict({(Integer(1), Integer(2)): Integer(1), (Integer(0), Integer(3)):␣
→-Integer(2)}).is_homogeneous()
```

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```
True
>>> PolyDict({(Integer(1), Integer(0)): Integer(1), (Integer(1), Integer(2)):␣
↪Integer(3)}).is_homogeneous()
False
```

latex(*vars*, *atomic_exponents=True*, *atomic_coefficients=True*, *sortkey=None*)

Return a nice polynomial latex representation of this PolyDict, where the vars are substituted in.

INPUT:

- `vars` – list
- `atomic_exponents` – boolean (default: `True`)
- `atomic_coefficients` – boolean (default: `True`)

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.latex(['a', 'WW'])
'2 a^{2} WW^{3} + 4 a^{2} WW + 3 a WW^{2}'
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
>>> f = PolyDict({(Integer(2), Integer(3)): Integer(2), (Integer(1),␣
↪Integer(2)): Integer(3), (Integer(2), Integer(1)): Integer(4)})
>>> f.latex(['a', 'WW'])
'2 a^{2} WW^{3} + 4 a^{2} WW + 3 a WW^{2}'
```

lcmt(*greater_etuple*)

Provides functionality of lc, lm, and lt by calling the tuple compare function on the provided term order T.

INPUT:

- `greater_etuple` – a term order

list()

Return a list that defines `self`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: sorted(f.list())
[[2, [2, 3]], [3, [1, 2]], [4, [2, 1]]]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
>>> f = PolyDict({(Integer(2), Integer(3)): Integer(2), (Integer(1),␣
↪Integer(2)): Integer(3), (Integer(2), Integer(1)): Integer(4)})
>>> sorted(f.list())
[[2, [2, 3]], [3, [1, 2]], [4, [2, 1]]]
```

max_exp()

Return an ETuple containing the maximum exponents appearing. If there are no terms at all in the PolyDict, it returns None.

The nvars parameter is necessary because a PolyDict doesn't know it from the data it has (and an empty PolyDict offers no clues).

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.max_exp()
(2, 3)
sage: PolyDict({}).max_exp() # returns None
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
>>> f = PolyDict({(Integer(2), Integer(3)): Integer(2), (Integer(1),␣
↪Integer(2)): Integer(3), (Integer(2), Integer(1)): Integer(4)})
>>> f.max_exp()
(2, 3)
>>> PolyDict({}).max_exp() # returns None
```

min_exp()

Return an ETuple containing the minimum exponents appearing. If there are no terms at all in the PolyDict, it returns None.

The nvars parameter is necessary because a PolyDict doesn't know it from the data it has (and an empty PolyDict offers no clues).

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.min_exp()
(1, 1)
sage: PolyDict({}).min_exp() # returns None
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
>>> f = PolyDict({(Integer(2), Integer(3)): Integer(2), (Integer(1),␣
↪Integer(2)): Integer(3), (Integer(2), Integer(1)): Integer(4)})
>>> f.min_exp()
(1, 1)
>>> PolyDict({}).min_exp() # returns None
```

monomial_coefficient(*mon*)

Return the coefficient of the monomial `mon`.

INPUT:

- `mon` – a PolyDict with a single key

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2,3):2, (1,2):3, (2,1):4})
sage: f.monomial_coefficient(PolyDict({(2,1):1}).dict())
doctest:warning
...
DeprecationWarning: PolyDict.monomial_coefficient is deprecated; use PolyDict.
↪get instead
See https://github.com/sagemath/sage/issues/34000 for details.
4
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
>>> f = PolyDict({(Integer(2),Integer(3)):Integer(2), (Integer(1),
↪Integer(2)):Integer(3), (Integer(2),Integer(1)):Integer(4)})
>>> f.monomial_coefficient(PolyDict({(Integer(2),Integer(1)):Integer(1)}).
↪dict())
doctest:warning
...
DeprecationWarning: PolyDict.monomial_coefficient is deprecated; use PolyDict.
↪get instead
See https://github.com/sagemath/sage/issues/34000 for details.
4
```

poly_repr(*vars*, *atomic_exponents=True*, *atomic_coefficients=True*, *sortkey=None*)

Return a nice polynomial string representation of this PolyDict, where the vars are substituted in.

INPUT:

- `vars` – list
- `atomic_exponents` – boolean (default: `True`)
- `atomic_coefficients` – boolean (default: `True`)

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2,3):2, (1,2):3, (2,1):4})
sage: f.poly_repr(['a', 'WW'])
'2*a^2*WW^3 + 4*a^2*WW + 3*a*WW^2'
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
>>> f = PolyDict({(Integer(2),Integer(3)):Integer(2), (Integer(1),
↪Integer(2)):Integer(3), (Integer(2),Integer(1)):Integer(4)})
>>> f.poly_repr(['a', 'WW'])
'2*a^2*WW^3 + 4*a^2*WW + 3*a*WW^2'
```

We check to make sure that when we are in characteristic two, we don't put negative signs on the generators.

```
sage: Integers(2)['x, y'].gens()
(x, y)
```

```
>>> from sage.all import *
>>> Integers(Integer(2))['x, y'].gens()
(x, y)
```

We make sure that intervals are correctly represented.

```
sage: f = PolyDict({(2, 3): RIF(1/2,3/2), (1, 2): RIF(-1,1)})                #␣
↪needs sage.rings.real_interval_field
sage: f.poly_repr(['x', 'y'])                                                #␣
↪needs sage.rings.real_interval_field
'1.?*x^2*y^3 + 0.?*x*y^2'
```

```
>>> from sage.all import *
>>> f = PolyDict({(Integer(2), Integer(3)): RIF(Integer(1)/Integer(2),
↪Integer(3)/Integer(2)), (Integer(1), Integer(2)): RIF(-Integer(1),
↪Integer(1))})                    # needs sage.rings.real_interval_field
>>> f.poly_repr(['x', 'y'])                                                #␣
↪needs sage.rings.real_interval_field
'1.?*x^2*y^3 + 0.?*x*y^2'
```

polynomial_coefficient(*degrees*)

Return a polydict that defines the coefficient in the current polynomial viewed as a tower of polynomial extensions.

INPUT:

- `degrees` – list of degree restrictions; list elements are `None` if the variable in that position should be unrestricted

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.polynomial_coefficient([2, None])
PolyDict with representation {(0, 1): 4, (0, 3): 2}
sage: f = PolyDict({(0, 3): 2, (0, 2): 3, (2, 1): 4})
sage: f.polynomial_coefficient([0, None])
PolyDict with representation {(0, 2): 3, (0, 3): 2}
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
>>> f = PolyDict({(Integer(2), Integer(3)): Integer(2), (Integer(1),␣
↪Integer(2)): Integer(3), (Integer(2), Integer(1)): Integer(4)})
>>> f.polynomial_coefficient([Integer(2), None])
PolyDict with representation {(0, 1): 4, (0, 3): 2}
>>> f = PolyDict({(Integer(0), Integer(3)): Integer(2), (Integer(0),␣
↪Integer(2)): Integer(3), (Integer(2), Integer(1)): Integer(4)})
>>> f.polynomial_coefficient([Integer(0), None])
PolyDict with representation {(0, 2): 3, (0, 3): 2}
```

remove_zeros(*zero_test=None*)

Remove the entries with zero coefficients.

INPUT:

- `zero_test` – (optional) function that performs test to zero of a coefficient

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3):0})
sage: f
PolyDict with representation {(2, 3): 0}
sage: f.remove_zeros()
sage: f
PolyDict with representation {}
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
>>> f = PolyDict({(Integer(2), Integer(3)):Integer(0)})
>>> f
PolyDict with representation {(2, 3): 0}
>>> f.remove_zeros()
>>> f
PolyDict with representation {}
```

The following example shows how to remove only exact zeros from a `PolyDict` containing univariate power series:

```
sage: R.<t> = PowerSeriesRing(QQ)
sage: f = PolyDict({(1, 1): O(t), (1, 0): R.zero()})
sage: f.remove_zeros(lambda s: s.is_zero() and s.prec() is Infinity)
sage: f
PolyDict with representation {(1, 1): O(t^1)}
```

```
>>> from sage.all import *
>>> R = PowerSeriesRing(QQ, names=('t',)); (t,) = R._first_ngens(1)
>>> f = PolyDict({(Integer(1), Integer(1)): O(t), (Integer(1), Integer(0)): R.
↪zero()})
>>> f.remove_zeros(lambda s: s.is_zero() and s.prec() is Infinity)
>>> f
PolyDict with representation {(1, 1): O(t^1)}
```

rich_compare(*other*, *op*, *sortkey=None*)

Compare two $PolyDict`susingaspecifiedtermordering``sortkey`$.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: from sage.structure.richcmp import op_EQ, op_NE, op_LT
sage: p1 = PolyDict({(0,): 1})
sage: p2 = PolyDict({(0,): 2})
sage: O = TermOrder()
sage: p1.rich_compare(PolyDict({(0,): 1}), op_EQ, O.sortkey)
True
sage: p1.rich_compare(p2, op_EQ, O.sortkey)
False
sage: p1.rich_compare(p2, op_NE, O.sortkey)
True
```

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```
sage: p1.rich_compare(p2, op_LT, O.sortkey)
True

sage: p3 = PolyDict({(3, 2, 4): 1, (3, 2, 5): 2})
sage: p4 = PolyDict({(3, 2, 4): 1, (3, 2, 3): 2})
sage: p3.rich_compare(p4, op_LT, O.sortkey)
False
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
>>> from sage.structure.richcmp import op_EQ, op_NE, op_LT
>>> p1 = PolyDict({(Integer(0),): Integer(1)})
>>> p2 = PolyDict({(Integer(0),): Integer(2)})
>>> O = TermOrder()
>>> p1.rich_compare(PolyDict({(Integer(0),): Integer(1)}), op_EQ, O.sortkey)
True
>>> p1.rich_compare(p2, op_EQ, O.sortkey)
False
>>> p1.rich_compare(p2, op_NE, O.sortkey)
True
>>> p1.rich_compare(p2, op_LT, O.sortkey)
True

>>> p3 = PolyDict({(Integer(3), Integer(2), Integer(4)): Integer(1),␣
↪(Integer(3), Integer(2), Integer(5)): Integer(2)})
>>> p4 = PolyDict({(Integer(3), Integer(2), Integer(4)): Integer(1),␣
↪(Integer(3), Integer(2), Integer(3)): Integer(2)})
>>> p3.rich_compare(p4, op_LT, O.sortkey)
False
```

scalar_lmult(*s*)

Return the left scalar multiplication of `self` by `s`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict

sage: x, y = FreeMonoid(2, 'x, y').gens()  # a strange object to live in a␣
↪polydict, but non-commutative!   # needs sage.combinat
sage: f = PolyDict({(2,3):x})                                          #␣
↪needs sage.combinat
sage: f.scalar_lmult(y)                                                #␣
↪needs sage.combinat
PolyDict with representation {(2, 3): y*x}

sage: f = PolyDict({(2,3):2, (1,2):3, (2,1):4})
sage: f.scalar_lmult(-2)
PolyDict with representation {(1, 2): -6, (2, 1): -8, (2, 3): -4}
sage: f.scalar_lmult(RIF(-1,1))                                        #␣
↪needs sage.rings.real_interval_field
PolyDict with representation {(1, 2): 0.?e1, (2, 1): 0.?e1, (2, 3): 0.?e1}
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict

>>> x, y = FreeMonoid(Integer(2), 'x, y').gens()  # a strange object to live␣
↪in a polydict, but non-commutative!   # needs sage.combinat
>>> f = PolyDict({(Integer(2),Integer(3)):x})                                  ␣
↪              # needs sage.combinat
>>> f.scalar_lmult(y)                                                         #␣
↪needs sage.combinat
PolyDict with representation {(2, 3): y*x}

>>> f = PolyDict({(Integer(2),Integer(3)):Integer(2), (Integer(1),
↪Integer(2)):Integer(3), (Integer(2),Integer(1)):Integer(4)})
>>> f.scalar_lmult(-Integer(2))
PolyDict with representation {(1, 2): -6, (2, 1): -8, (2, 3): -4}
>>> f.scalar_lmult(RIF(-Integer(1),Integer(1)))                                ␣
↪              # needs sage.rings.real_interval_field
PolyDict with representation {(1, 2): 0.?e1, (2, 1): 0.?e1, (2, 3): 0.?e1}
```

scalar_rmult(*s*)

Return the right scalar multiplication of `self` by `s`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict

sage: x, y = FreeMonoid(2, 'x, y').gens()  # a strange object to live in a␣
↪polydict, but non-commutative!   # needs sage.combinat
sage: f = PolyDict({(2, 3): x})                                                 #␣
↪needs sage.combinat
sage: f.scalar_rmult(y)                                                         #␣
↪needs sage.combinat
PolyDict with representation {(2, 3): x*y}

sage: f = PolyDict({(2,3):2, (1, 2): 3, (2, 1): 4})
sage: f.scalar_rmult(-2)
PolyDict with representation {(1, 2): -6, (2, 1): -8, (2, 3): -4}
sage: f.scalar_rmult(RIF(-1,1))                                                 #␣
↪needs sage.rings.real_interval_field
PolyDict with representation {(1, 2): 0.?e1, (2, 1): 0.?e1, (2, 3): 0.?e1}
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict

>>> x, y = FreeMonoid(Integer(2), 'x, y').gens()  # a strange object to live␣
↪in a polydict, but non-commutative!   # needs sage.combinat
>>> f = PolyDict({(Integer(2), Integer(3)): x})                                ␣
↪              # needs sage.combinat
>>> f.scalar_rmult(y)                                                         #␣
↪needs sage.combinat
PolyDict with representation {(2, 3): x*y}

>>> f = PolyDict({(Integer(2),Integer(3)):Integer(2), (Integer(1),␣
```

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```
↪Integer(2)): Integer(3), (Integer(2), Integer(1)): Integer(4)})
>>> f.scalar_rmult(-Integer(2))
PolyDict with representation {(1, 2): -6, (2, 1): -8, (2, 3): -4}
>>> f.scalar_rmult(RIF(-Integer(1),Integer(1)))                              ␣
↪              # needs sage.rings.real_interval_field
PolyDict with representation {(1, 2): 0.?e1, (2, 1): 0.?e1, (2, 3): 0.?e1}
```

term_lmult(*exponent*, *s*)

Return this element multiplied by `s` on the left and with exponents shifted by `exponent`.

INPUT:

- `exponent` – a ETuple
- `s` – a scalar

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple, PolyDict

sage: x, y = FreeMonoid(2, 'x, y').gens()  # a strange object to live in a␣
↪polydict, but non-commutative!   # needs sage.combinat
sage: f = PolyDict({(2, 3): x})                                               #␣
↪needs sage.combinat
sage: f.term_lmult(ETuple((1, 2)), y)                                         #␣
↪needs sage.combinat
PolyDict with representation {(3, 5): y*x}

sage: f = PolyDict({(2,3): 2, (1,2): 3, (2,1): 4})
sage: f.term_lmult(ETuple((1, 2)), -2)
PolyDict with representation {(2, 4): -6, (3, 3): -8, (3, 5): -4}
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple, PolyDict

>>> x, y = FreeMonoid(Integer(2), 'x, y').gens()  # a strange object to live␣
↪in a polydict, but non-commutative!   # needs sage.combinat
>>> f = PolyDict({(Integer(2), Integer(3)): x})                               ␣
↪              # needs sage.combinat
>>> f.term_lmult(ETuple((Integer(1), Integer(2))), y)                         ␣
↪              # needs sage.combinat
PolyDict with representation {(3, 5): y*x}

>>> f = PolyDict({(Integer(2),Integer(3)): Integer(2), (Integer(1),
↪Integer(2)): Integer(3), (Integer(2),Integer(1)): Integer(4)})
>>> f.term_lmult(ETuple((Integer(1), Integer(2))), -Integer(2))
PolyDict with representation {(2, 4): -6, (3, 3): -8, (3, 5): -4}
```

term_rmult(*exponent*, *s*)

Return this element multiplied by `s` on the right and with exponents shifted by `exponent`.

INPUT:

- `exponent` – a ETuple
- `s` – a scalar

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple, PolyDict

sage: x, y = FreeMonoid(2, 'x, y').gens()  # a strange object to live in a␣
↪polydict, but non-commutative!   # needs sage.combinat
sage: f = PolyDict({(2, 3): x})                                                #␣
↪needs sage.combinat
sage: f.term_rmult(ETuple((1, 2)), y)                                          #␣
↪needs sage.combinat
PolyDict with representation {(3, 5): x*y}

sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.term_rmult(ETuple((1, 2)), -2)
PolyDict with representation {(2, 4): -6, (3, 3): -8, (3, 5): -4}
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import ETuple, PolyDict

>>> x, y = FreeMonoid(Integer(2), 'x, y').gens()  # a strange object to live␣
↪in a polydict, but non-commutative!   # needs sage.combinat
>>> f = PolyDict({(Integer(2), Integer(3)): x})                                ␣
↪               # needs sage.combinat
>>> f.term_rmult(ETuple((Integer(1), Integer(2))), y)                          ␣
↪               # needs sage.combinat
PolyDict with representation {(3, 5): x*y}

>>> f = PolyDict({(Integer(2), Integer(3)): Integer(2), (Integer(1),␣
↪Integer(2)): Integer(3), (Integer(2), Integer(1)): Integer(4)})
>>> f.term_rmult(ETuple((Integer(1), Integer(2))), -Integer(2))
PolyDict with representation {(2, 4): -6, (3, 3): -8, (3, 5): -4}
```

total_degree(*w=None*)

Return the total degree.

INPUT:

- `w` – (optional) a tuple of weights

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.total_degree()
5
sage: f.total_degree((3, 1))
9
sage: PolyDict({}).degree()
-1
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict
>>> f = PolyDict({(Integer(2), Integer(3)): Integer(2), (Integer(1),␣
↪Integer(2)): Integer(3), (Integer(2), Integer(1)): Integer(4)})
```

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```
>>> f.total_degree()
5
>>> f.total_degree((Integer(3), Integer(1)))
9
>>> PolyDict({}).degree()
-1
```

sage.rings.polynomial.polydict.**gen_index**(*x*)

Return the index of the variable represented by `x` or `-1` if `x` is not a monomial of degree one.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict, gen_index
sage: gen_index(PolyDict({(1, 0): 1}))
0
sage: gen_index(PolyDict({(0, 1): 1}))
1
sage: gen_index(PolyDict({}))
-1
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict, gen_index
>>> gen_index(PolyDict({(Integer(1), Integer(0)): Integer(1)}))
0
>>> gen_index(PolyDict({(Integer(0), Integer(1)): Integer(1)}))
1
>>> gen_index(PolyDict({}))
-1
```

sage.rings.polynomial.polydict.**make_ETuple**(*data*, *length*)

Ensure support for pickled data from older sage versions.

sage.rings.polynomial.polydict.**make_PolyDict**(*data*)

Ensure support for pickled data from older sage versions.

sage.rings.polynomial.polydict.**monomial_exponent**(*p*)

Return the unique exponent of `p` if it is a monomial or raise a `ValueError`.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict, monomial_exponent
sage: monomial_exponent(PolyDict({(2, 3): 1}))
(2, 3)
sage: monomial_exponent(PolyDict({(2, 3): 3}))
Traceback (most recent call last):
...
ValueError: not a monomial
sage: monomial_exponent(PolyDict({(1, 0): 1, (0, 1): 1}))
Traceback (most recent call last):
...
ValueError: not a monomial
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.polydict import PolyDict, monomial_exponent
>>> monomial_exponent(PolyDict({(Integer(2), Integer(3)): Integer(1)}))
(2, 3)
>>> monomial_exponent(PolyDict({(Integer(2), Integer(3)): Integer(3)}))
Traceback (most recent call last):
...
ValueError: not a monomial
>>> monomial_exponent(PolyDict({(Integer(1), Integer(0)): Integer(1), (Integer(0),
↪ Integer(1)): Integer(1)}))
Traceback (most recent call last):
...
ValueError: not a monomial
```

3.1.12 Compute Hilbert series of monomial ideals

This implementation was provided at Issue #26243 and is supposed to be a way out when Singular fails with an int overflow, which will regularly be the case in any example with more than 34 variables.

class sage.rings.polynomial.hilbert.**Node**

Bases: object

A node of a binary tree

It has slots for data that allow to recursively compute the first Hilbert series of a monomial ideal.

sage.rings.polynomial.hilbert.**first_hilbert_series**(*I*, *grading=None*, *return_grading=False*)

Return the first Hilbert series of the given monomial ideal.

INPUT:

- I – a monomial ideal (possibly defined in singular)
- grading – (optional) a list or tuple of integers used as degree weights
- return_grading – boolean (default: False); whether to return the grading

OUTPUT:

A univariate polynomial, namely the first Hilbert function of I, and if return_grading==True also the grading used to compute the series.

EXAMPLES:

```
sage: from sage.rings.polynomial.hilbert import first_hilbert_series

sage: # needs sage.libs.singular
sage: R = singular.ring(0,'(x,y,z)','dp')
sage: I = singular.ideal(['x^2','y^2','z^2'])
sage: first_hilbert_series(I)
-t^6 + 3*t^4 - 3*t^2 + 1
sage: first_hilbert_series(I, return_grading=True)
(-t^6 + 3*t^4 - 3*t^2 + 1, (1, 1, 1))
sage: first_hilbert_series(I, grading=(1,2,3))
-t^12 + t^10 + t^8 - t^4 - t^2 + 1
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.hilbert import first_hilbert_series

>>> # needs sage.libs.singular
>>> R = singular.ring(Integer(0),'(x,y,z)','dp')
>>> I = singular.ideal(['x^2','y^2','z^2'])
>>> first_hilbert_series(I)
-t^6 + 3*t^4 - 3*t^2 + 1
>>> first_hilbert_series(I, return_grading=True)
(-t^6 + 3*t^4 - 3*t^2 + 1, (1, 1, 1))
>>> first_hilbert_series(I, grading=(Integer(1),Integer(2),Integer(3)))
-t^12 + t^10 + t^8 - t^4 - t^2 + 1
```

sage.rings.polynomial.hilbert.**hilbert_poincare_series**(*I*, *grading=None*)

Return the Hilbert Poincaré series of the given monomial ideal.

INPUT:

- `I` – a monomial ideal (possibly defined in Singular)
- `grading` – (optional) a tuple of degree weights

EXAMPLES:

```
sage: # needs sage.libs.singular
sage: from sage.rings.polynomial.hilbert import hilbert_poincare_series
sage: R = PolynomialRing(QQ,'x',9)
sage: I = [m.lm()
....:      for m in ((matrix(R, 3, R.gens())^2).list() * R).groebner_basis()] * R
sage: hilbert_poincare_series(I)
(t^7 - 3*t^6 + 2*t^5 + 2*t^4 - 2*t^3 + 6*t^2 + 5*t + 1)/(t^4 - 4*t^3 + 6*t^2 -␣
↪4*t + 1)
sage: hilbert_poincare_series((R * R.gens())^2, grading=range(1,10))
t^9 + t^8 + t^7 + t^6 + t^5 + t^4 + t^3 + t^2 + t + 1
```

```
>>> from sage.all import *
>>> # needs sage.libs.singular
>>> from sage.rings.polynomial.hilbert import hilbert_poincare_series
>>> R = PolynomialRing(QQ,'x',Integer(9))
>>> I = [m.lm()
...      for m in ((matrix(R, Integer(3), R.gens())**Integer(2)).list() * R).
↪groebner_basis()] * R
>>> hilbert_poincare_series(I)
(t^7 - 3*t^6 + 2*t^5 + 2*t^4 - 2*t^3 + 6*t^2 + 5*t + 1)/(t^4 - 4*t^3 + 6*t^2 -␣
↪4*t + 1)
>>> hilbert_poincare_series((R * R.gens())**Integer(2), grading=range(Integer(1),
↪Integer(10)))
t^9 + t^8 + t^7 + t^6 + t^5 + t^4 + t^3 + t^2 + t + 1
```

The following example is taken from Issue #20145:

```
sage: # needs sage.libs.singular
sage: n=4; m=11; P = PolynomialRing(QQ, n*m, "x"); x = P.gens(); M = Matrix(n, x)
sage: from sage.rings.polynomial.hilbert import first_hilbert_series
sage: I = P.ideal(M.minors(2))
```

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```
sage: J = P * [m.lm() for m in I.groebner_basis()]
sage: hilbert_poincare_series(J).numerator()
120*t^3 + 135*t^2 + 30*t + 1
sage: hilbert_poincare_series(J).denominator().factor()
(t - 1)^14
```

```
>>> from sage.all import *
>>> # needs sage.libs.singular
>>> n=Integer(4); m=Integer(11); P = PolynomialRing(QQ, n*m, "x"); x = P.gens();␣
↪M = Matrix(n, x)
>>> from sage.rings.polynomial.hilbert import first_hilbert_series
>>> I = P.ideal(M.minors(Integer(2)))
>>> J = P * [m.lm() for m in I.groebner_basis()]
>>> hilbert_poincare_series(J).numerator()
120*t^3 + 135*t^2 + 30*t + 1
>>> hilbert_poincare_series(J).denominator().factor()
(t - 1)^14
```

3.1.13 Class to flatten polynomial rings over polynomial ring

For example `QQ['a','b'],['x','y']` flattens to `QQ['a','b','x','y']`.

EXAMPLES:

```
sage: R = QQ['x']['y']['s','t']['X']
sage: from sage.rings.polynomial.flatten import FlatteningMorphism
sage: phi = FlatteningMorphism(R); phi
Flattening morphism:
  From: Univariate Polynomial Ring in X
        over Multivariate Polynomial Ring in s, t
        over Univariate Polynomial Ring in y
        over Univariate Polynomial Ring in x over Rational Field
  To:   Multivariate Polynomial Ring in x, y, s, t, X over Rational Field
sage: phi('x*y*s + t*X').parent()
Multivariate Polynomial Ring in x, y, s, t, X over Rational Field
```

```
>>> from sage.all import *
>>> R = QQ['x']['y']['s','t']['X']
>>> from sage.rings.polynomial.flatten import FlatteningMorphism
>>> phi = FlatteningMorphism(R); phi
Flattening morphism:
  From: Univariate Polynomial Ring in X
        over Multivariate Polynomial Ring in s, t
        over Univariate Polynomial Ring in y
        over Univariate Polynomial Ring in x over Rational Field
  To:   Multivariate Polynomial Ring in x, y, s, t, X over Rational Field
>>> phi('x*y*s + t*X').parent()
Multivariate Polynomial Ring in x, y, s, t, X over Rational Field
```

Authors:

Vincent Delecroix, Ben Hutz (July 2016): initial implementation

class sage.rings.polynomial.flatten.**FlatteningMorphism**(*domain*)

Bases: Morphism

EXAMPLES:

```
sage: R = QQ['a','b']['x','y','z']['t1','t2']
sage: from sage.rings.polynomial.flatten import FlatteningMorphism
sage: f = FlatteningMorphism(R)
sage: f.codomain()
Multivariate Polynomial Ring in a, b, x, y, z, t1, t2 over Rational Field
sage: p = R('(a+b)*x + (a^2-b)*t2*(z+y)')
sage: p
((a^2 - b)*y + (a^2 - b)*z)*t2 + (a + b)*x
sage: f(p)
a^2*y*t2 + a^2*z*t2 - b*y*t2 - b*z*t2 + a*x + b*x
sage: f(p).parent()
Multivariate Polynomial Ring in a, b, x, y, z, t1, t2 over Rational Field
```

```
>>> from sage.all import *
>>> R = QQ['a','b']['x','y','z']['t1','t2']
>>> from sage.rings.polynomial.flatten import FlatteningMorphism
>>> f = FlatteningMorphism(R)
>>> f.codomain()
Multivariate Polynomial Ring in a, b, x, y, z, t1, t2 over Rational Field
>>> p = R('(a+b)*x + (a^2-b)*t2*(z+y)')
>>> p
((a^2 - b)*y + (a^2 - b)*z)*t2 + (a + b)*x
>>> f(p)
a^2*y*t2 + a^2*z*t2 - b*y*t2 - b*z*t2 + a*x + b*x
>>> f(p).parent()
Multivariate Polynomial Ring in a, b, x, y, z, t1, t2 over Rational Field
```

Also works when univariate polynomial ring are involved:

```
sage: R = QQ['x']['y']['s','t']['X']
sage: from sage.rings.polynomial.flatten import FlatteningMorphism
sage: f = FlatteningMorphism(R)
sage: f.codomain()
Multivariate Polynomial Ring in x, y, s, t, X over Rational Field
sage: p = R('((x^2 + 1) + (x+2)*y + x*y^3)*(s+t) + x*y*X')
sage: p
x*y*X + (x*y^3 + (x + 2)*y + x^2 + 1)*s + (x*y^3 + (x + 2)*y + x^2 + 1)*t
sage: f(p)
x*y^3*s + x*y^3*t + x^2*s + x*y*s + x^2*t + x*y*t + x*y*X + 2*y*s + 2*y*t + s + t
sage: f(p).parent()
Multivariate Polynomial Ring in x, y, s, t, X over Rational Field
```

```
>>> from sage.all import *
>>> R = QQ['x']['y']['s','t']['X']
>>> from sage.rings.polynomial.flatten import FlatteningMorphism
>>> f = FlatteningMorphism(R)
>>> f.codomain()
Multivariate Polynomial Ring in x, y, s, t, X over Rational Field
```

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```
>>> p = R('((x^2 + 1) + (x+2)*y + x*y^3)*(s+t) + x*y*X')
>>> p
x*y*X + (x*y^3 + (x + 2)*y + x^2 + 1)*s + (x*y^3 + (x + 2)*y + x^2 + 1)*t
>>> f(p)
x*y^3*s + x*y^3*t + x^2*s + x*y*s + x^2*t + x*y*t + x*y*X + 2*y*s + 2*y*t + s + t
>>> f(p).parent()
Multivariate Polynomial Ring in x, y, s, t, X over Rational Field
```

inverse()

Return the inverse of this flattening morphism.

This is the same as calling *section()*.

EXAMPLES:

```
sage: f = QQ['x,y']['u,v'].flattening_morphism()
sage: f.inverse()
Unflattening morphism:
  From: Multivariate Polynomial Ring in x, y, u, v over Rational Field
  To:   Multivariate Polynomial Ring in u, v
        over Multivariate Polynomial Ring in x, y over Rational Field
```

```
>>> from sage.all import *
>>> f = QQ['x,y']['u,v'].flattening_morphism()
>>> f.inverse()
Unflattening morphism:
  From: Multivariate Polynomial Ring in x, y, u, v over Rational Field
  To:   Multivariate Polynomial Ring in u, v
        over Multivariate Polynomial Ring in x, y over Rational Field
```

section()

Inverse of this flattening morphism.

EXAMPLES:

```
sage: R = QQ['a','b','c']['x','y','z']
sage: from sage.rings.polynomial.flatten import FlatteningMorphism
sage: h = FlatteningMorphism(R)
sage: h.section()
Unflattening morphism:
  From: Multivariate Polynomial Ring in a, b, c, x, y, z over Rational Field
  To:   Multivariate Polynomial Ring in x, y, z
        over Multivariate Polynomial Ring in a, b, c over Rational Field
```

```
>>> from sage.all import *
>>> R = QQ['a','b','c']['x','y','z']
>>> from sage.rings.polynomial.flatten import FlatteningMorphism
>>> h = FlatteningMorphism(R)
>>> h.section()
Unflattening morphism:
  From: Multivariate Polynomial Ring in a, b, c, x, y, z over Rational Field
  To:   Multivariate Polynomial Ring in x, y, z
        over Multivariate Polynomial Ring in a, b, c over Rational Field
```

```
sage: R = ZZ['a']['b']['c']
sage: from sage.rings.polynomial.flatten import FlatteningMorphism
sage: FlatteningMorphism(R).section()
Unflattening morphism:
  From: Multivariate Polynomial Ring in a, b, c over Integer Ring
  To:   Univariate Polynomial Ring in c over Univariate Polynomial Ring in b
        over Univariate Polynomial Ring in a over Integer Ring
```

```
>>> from sage.all import *
>>> R = ZZ['a']['b']['c']
>>> from sage.rings.polynomial.flatten import FlatteningMorphism
>>> FlatteningMorphism(R).section()
Unflattening morphism:
  From: Multivariate Polynomial Ring in a, b, c over Integer Ring
  To:   Univariate Polynomial Ring in c over Univariate Polynomial Ring in b
        over Univariate Polynomial Ring in a over Integer Ring
```

class sage.rings.polynomial.flatten.**FractionSpecializationMorphism**(*domain*, *D*)

Bases: Morphism

A specialization morphism for fraction fields over (stacked) polynomial rings

class sage.rings.polynomial.flatten.**SpecializationMorphism**(*domain*, *D*)

Bases: Morphism

Morphisms to specialize parameters in (stacked) polynomial rings.

EXAMPLES:

```
sage: R.<c> = PolynomialRing(QQ)
sage: S.<x,y,z> = PolynomialRing(R)
sage: D = dict({c:1})
sage: from sage.rings.polynomial.flatten import SpecializationMorphism
sage: f = SpecializationMorphism(S, D)
sage: g = f(x^2 + c*y^2 - z^2); g
x^2 + y^2 - z^2
sage: g.parent()
Multivariate Polynomial Ring in x, y, z over Rational Field
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('c',)); (c,) = R._first_ngens(1)
>>> S = PolynomialRing(R, names=('x', 'y', 'z',)); (x, y, z,) = S._first_ngens(3)
>>> D = dict({c:Integer(1)})
>>> from sage.rings.polynomial.flatten import SpecializationMorphism
>>> f = SpecializationMorphism(S, D)
>>> g = f(x**Integer(2) + c*y**Integer(2) - z**Integer(2)); g
x^2 + y^2 - z^2
>>> g.parent()
Multivariate Polynomial Ring in x, y, z over Rational Field
```

```
sage: R.<c> = PolynomialRing(QQ)
sage: S.<z> = PolynomialRing(R)
sage: from sage.rings.polynomial.flatten import SpecializationMorphism
sage: xi = SpecializationMorphism(S, {c:0}); xi
```

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```
Specialization morphism:
  From: Univariate Polynomial Ring in z
        over Univariate Polynomial Ring in c over Rational Field
  To:   Univariate Polynomial Ring in z over Rational Field
sage: xi(z^2+c)
z^2
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('c',)); (c,) = R._first_ngens(1)
>>> S = PolynomialRing(R, names=('z',)); (z,) = S._first_ngens(1)
>>> from sage.rings.polynomial.flatten import SpecializationMorphism
>>> xi = SpecializationMorphism(S, {c:Integer(0)}); xi
Specialization morphism:
  From: Univariate Polynomial Ring in z
        over Univariate Polynomial Ring in c over Rational Field
  To:   Univariate Polynomial Ring in z over Rational Field
>>> xi(z**Integer(2)+c)
z^2
```

```
sage: R1.<u,v> = PolynomialRing(QQ)
sage: R2.<a,b,c> = PolynomialRing(R1)
sage: S.<x,y,z> = PolynomialRing(R2)
sage: D = dict({a:1, b:2, x:0, u:1})
sage: from sage.rings.polynomial.flatten import SpecializationMorphism
sage: xi = SpecializationMorphism(S, D); xi
Specialization morphism:
  From: Multivariate Polynomial Ring in x, y, z
        over Multivariate Polynomial Ring in a, b, c
        over Multivariate Polynomial Ring in u, v over Rational Field
  To:   Multivariate Polynomial Ring in y, z over Univariate Polynomial Ring in c
        over Univariate Polynomial Ring in v over Rational Field
sage: xi(a*(x*z+y^2)*u+b*v*u*(x*z+y^2)*y^2*c+c*y^2*z^2)
2*v*c*y^4 + c*y^2*z^2 + y^2
```

```
>>> from sage.all import *
>>> R1 = PolynomialRing(QQ, names=('u', 'v',)); (u, v,) = R1._first_ngens(2)
>>> R2 = PolynomialRing(R1, names=('a', 'b', 'c',)); (a, b, c,) = R2._first_
↪ngens(3)
>>> S = PolynomialRing(R2, names=('x', 'y', 'z',)); (x, y, z,) = S._first_ngens(3)
>>> D = dict({a:Integer(1), b:Integer(2), x:Integer(0), u:Integer(1)})
>>> from sage.rings.polynomial.flatten import SpecializationMorphism
>>> xi = SpecializationMorphism(S, D); xi
Specialization morphism:
  From: Multivariate Polynomial Ring in x, y, z
        over Multivariate Polynomial Ring in a, b, c
        over Multivariate Polynomial Ring in u, v over Rational Field
  To:   Multivariate Polynomial Ring in y, z over Univariate Polynomial Ring in c
        over Univariate Polynomial Ring in v over Rational Field
>>> xi(a*(x*z+y**Integer(2))*u+b*v*u*(x*z+y**Integer(2))*y**Integer(2)*c+c*y**In-
teger(2)*z**Integer(2))
2*v*c*y^4 + c*y^2*z^2 + y^2
```

class sage.rings.polynomial.flatten.**UnflatteningMorphism**(*domain*, *codomain*)

Bases: Morphism

Inverses for *FlatteningMorphism*.

EXAMPLES:

```
sage: R = QQ['c','x','y','z']
sage: S = QQ['c']['x','y','z']
sage: from sage.rings.polynomial.flatten import UnflatteningMorphism
sage: f = UnflatteningMorphism(R, S)
sage: g = f(R('x^2 + c*y^2 - z^2'));g
x^2 + c*y^2 - z^2
sage: g.parent()
Multivariate Polynomial Ring in x, y, z
 over Univariate Polynomial Ring in c over Rational Field
```

```
>>> from sage.all import *
>>> R = QQ['c','x','y','z']
>>> S = QQ['c']['x','y','z']
>>> from sage.rings.polynomial.flatten import UnflatteningMorphism
>>> f = UnflatteningMorphism(R, S)
>>> g = f(R('x^2 + c*y^2 - z^2'));g
x^2 + c*y^2 - z^2
>>> g.parent()
Multivariate Polynomial Ring in x, y, z
 over Univariate Polynomial Ring in c over Rational Field
```

```
sage: R = QQ['a','b', 'x','y']
sage: S = QQ['a','b']['x','y']
sage: from sage.rings.polynomial.flatten import UnflatteningMorphism
sage: UnflatteningMorphism(R, S)
Unflattening morphism:
  From: Multivariate Polynomial Ring in a, b, x, y over Rational Field
  To:   Multivariate Polynomial Ring in x, y
        over Multivariate Polynomial Ring in a, b over Rational Field
```

```
>>> from sage.all import *
>>> R = QQ['a','b', 'x','y']
>>> S = QQ['a','b']['x','y']
>>> from sage.rings.polynomial.flatten import UnflatteningMorphism
>>> UnflatteningMorphism(R, S)
Unflattening morphism:
  From: Multivariate Polynomial Ring in a, b, x, y over Rational Field
  To:   Multivariate Polynomial Ring in x, y
        over Multivariate Polynomial Ring in a, b over Rational Field
```

3.1.14 Monomials

sage.rings.monomials.**monomials**(*v*, *n*)

Given two lists `v` and `n`, of exactly the same length, return all monomials in the elements of `v`, where variable `i` (i.e., `v[i]`) in the monomial appears to degree strictly less than `n[i]`.

INPUT:

- v – list of ring elements
- n – list of integers

EXAMPLES:

```
sage: monomials([x], [3])                                                       #␣
↪needs sage.symbolic
[1, x, x^2]
sage: R.<x,y,z> = QQ[]
sage: monomials([x,y], [5,5])
[1, y, y^2, y^3, y^4, x, x*y, x*y^2, x*y^3, x*y^4, x^2, x^2*y, x^2*y^2, x^2*y^3,␣
↪x^2*y^4, x^3, x^3*y, x^3*y^2, x^3*y^3, x^3*y^4, x^4, x^4*y, x^4*y^2, x^4*y^3, x^
↪4*y^4]
sage: monomials([x,y,z], [2,3,2])
[1, z, y, y*z, y^2, y^2*z, x, x*z, x*y, x*y*z, x*y^2, x*y^2*z]
```

```
>>> from sage.all import *
>>> monomials([x], [Integer(3)])                                               ␣
↪    # needs sage.symbolic
[1, x, x^2]
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> monomials([x,y], [Integer(5),Integer(5)])
[1, y, y^2, y^3, y^4, x, x*y, x*y^2, x*y^3, x*y^4, x^2, x^2*y, x^2*y^2, x^2*y^3,␣
↪x^2*y^4, x^3, x^3*y, x^3*y^2, x^3*y^3, x^3*y^4, x^4, x^4*y, x^4*y^2, x^4*y^3, x^
↪4*y^4]
>>> monomials([x,y,z], [Integer(2),Integer(3),Integer(2)])
[1, z, y, y*z, y^2, y^2*z, x, x*z, x*y, x*y*z, x*y^2, x*y^2*z]
```

3.2 Classical Invariant Theory

3.2.1 Classical Invariant Theory

This module lists classical invariants and covariants of homogeneous polynomials (also called algebraic forms) under the action of the special linear group. That is, we are dealing with polynomials of degree d in n variables. The special linear group $SL(n, \mathbf{C})$ acts on the variables $(x_1, \dots, x_n)$ linearly,

$$(x_1, \dots, x_n)^t \to A(x_1, \dots, x_n)^t, \qquad A \in SL(n, \mathbf{C})$$

The linear action on the variables transforms a polynomial p generally into a different polynomial gp. We can think of it as an action on the space of coefficients in p. An invariant is a polynomial in the coefficients that is invariant under this action. A covariant is a polynomial in the coefficients and the variables $(x_1, \dots, x_n)$ that is invariant under the combined action.

For example, the binary quadratic $p(x, y) = ax^2 + bxy + cy^2$ has as its invariant the discriminant $disc(p) = b^2 - 4ac$. This means that for any $SL(2, \mathbf{C})$ coordinate change

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \alpha\delta - \beta\gamma = 1$$

the discriminant is invariant, $disc\big(p(x', y')\big) = disc\big(p(x, y)\big)$.

To use this module, you should use the factory object *invariant_theory*. For example, take the quartic:

```
sage: R.<x,y> = QQ[]
sage: q = x^4 + y^4
sage: quartic = invariant_theory.binary_quartic(q);  quartic
Binary quartic with coefficients (1, 0, 0, 0, 1)
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> q = x**Integer(4) + y**Integer(4)
>>> quartic = invariant_theory.binary_quartic(q);  quartic
Binary quartic with coefficients (1, 0, 0, 0, 1)
```

One invariant of a quartic is known as the Eisenstein D-invariant. Since it is an invariant, it is a polynomial in the coefficients (which are integers in this example):

```
sage: quartic.EisensteinD()
1
```

```
>>> from sage.all import *
>>> quartic.EisensteinD()
1
```

One example of a covariant of a quartic is the so-called g-covariant (actually, the Hessian). As with all covariants, it is a polynomial in x, y and the coefficients:

```
sage: quartic.g_covariant()
-x^2*y^2
```

```
>>> from sage.all import *
>>> quartic.g_covariant()
-x^2*y^2
```

As usual, use tab completion and the online help to discover the implemented invariants and covariants.

In general, the variables of the defining polynomial cannot be guessed. For example, the zero polynomial can be thought of as a homogeneous polynomial of any degree. Also, since we also want to allow polynomial coefficients we cannot just take all variables of the polynomial ring as the variables of the form. This is why you will have to specify the variables explicitly if there is any potential ambiguity. For example:

```
sage: invariant_theory.binary_quartic(R.zero(), [x,y])
Binary quartic with coefficients (0, 0, 0, 0, 0)

sage: invariant_theory.binary_quartic(x^4, [x,y])
Binary quartic with coefficients (0, 0, 0, 0, 1)

sage: R.<x,y,t> = QQ[]
sage: invariant_theory.binary_quartic(x^4 + y^4 + t*x^2*y^2, [x,y])
Binary quartic with coefficients (1, 0, t, 0, 1)
```

```
>>> from sage.all import *
>>> invariant_theory.binary_quartic(R.zero(), [x,y])
Binary quartic with coefficients (0, 0, 0, 0, 0)

>>> invariant_theory.binary_quartic(x**Integer(4), [x,y])
```

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```
Binary quartic with coefficients (0, 0, 0, 0, 1)

>>> R = QQ['x, y, t']; (x, y, t,) = R._first_ngens(3)
>>> invariant_theory.binary_quartic(x**Integer(4) + y**Integer(4) +␣
↪t*x**Integer(2)*y**Integer(2), [x,y])
Binary quartic with coefficients (1, 0, t, 0, 1)
```

Finally, it is often convenient to use inhomogeneous polynomials where it is understood that one wants to homogenize them. This is also supported, just define the form with an inhomogeneous polynomial and specify one less variable:

```
sage: R.<x,t> = QQ[]
sage: invariant_theory.binary_quartic(x^4 + 1 + t*x^2, [x])
Binary quartic with coefficients (1, 0, t, 0, 1)
```

```
>>> from sage.all import *
>>> R = QQ['x, t']; (x, t,) = R._first_ngens(2)
>>> invariant_theory.binary_quartic(x**Integer(4) + Integer(1) + t*x**Integer(2), [x])
Binary quartic with coefficients (1, 0, t, 0, 1)
```

REFERENCES:

- Wikipedia article Glossary_of_invariant_theory

AUTHORS:

- Volker Braun (2013-01-24): initial version
- Jesper Noordsij (2018-05-18): support for binary quintics added

class `sage.rings.invariants.invariant_theory.`**`AlgebraicForm`**(*n*, *d*, *polynomial*, **args*, ***kwds*)

Bases: *FormsBase*

The base class of algebraic forms (i.e. homogeneous polynomials).

You should only instantiate the derived classes of this base class.

Derived classes must implement `coeffs()` and `scaled_coeffs()`

INPUT:

- `n` – the number of variables
- `d` – the degree of the polynomial
- `polynomial` – the polynomial
- `*args` – the variables, as a single list/tuple, multiple arguments, or `None` to use all variables of the polynomial

Derived classes must implement the same arguments for the constructor.

EXAMPLES:

```
sage: from sage.rings.invariants.invariant_theory import AlgebraicForm
sage: R.<x,y> = QQ[]
sage: p = x^2 + y^2
sage: AlgebraicForm(2, 2, p).variables()
(x, y)
sage: AlgebraicForm(2, 2, p, None).variables()
(x, y)
```

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```
sage: AlgebraicForm(3, 2, p).variables()
(x, y, None)
sage: AlgebraicForm(3, 2, p, None).variables()
(x, y, None)

sage: from sage.rings.invariants.invariant_theory import AlgebraicForm
sage: R.<x,y,s,t> = QQ[]
sage: p = s*x^2 + t*y^2
sage: AlgebraicForm(2, 2, p, [x,y]).variables()
(x, y)
sage: AlgebraicForm(2, 2, p, x,y).variables()
(x, y)

sage: AlgebraicForm(3, 2, p, [x,y,None]).variables()
(x, y, None)
sage: AlgebraicForm(3, 2, p, x,y,None).variables()
(x, y, None)

sage: AlgebraicForm(2, 1, p, [x,y]).variables()
Traceback (most recent call last):
...
ValueError: polynomial is of the wrong degree

sage: AlgebraicForm(2, 2, x^2 + y, [x,y]).variables()
Traceback (most recent call last):
...
ValueError: polynomial is not homogeneous
```

```
>>> from sage.all import *
>>> from sage.rings.invariants.invariant_theory import AlgebraicForm
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> p = x**Integer(2) + y**Integer(2)
>>> AlgebraicForm(Integer(2), Integer(2), p).variables()
(x, y)
>>> AlgebraicForm(Integer(2), Integer(2), p, None).variables()
(x, y)
>>> AlgebraicForm(Integer(3), Integer(2), p).variables()
(x, y, None)
>>> AlgebraicForm(Integer(3), Integer(2), p, None).variables()
(x, y, None)

>>> from sage.rings.invariants.invariant_theory import AlgebraicForm
>>> R = QQ['x, y, s, t']; (x, y, s, t,) = R._first_ngens(4)
>>> p = s*x**Integer(2) + t*y**Integer(2)
>>> AlgebraicForm(Integer(2), Integer(2), p, [x,y]).variables()
(x, y)
>>> AlgebraicForm(Integer(2), Integer(2), p, x,y).variables()
(x, y)

>>> AlgebraicForm(Integer(3), Integer(2), p, [x,y,None]).variables()
(x, y, None)
>>> AlgebraicForm(Integer(3), Integer(2), p, x,y,None).variables()
```

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```
(x, y, None)

>>> AlgebraicForm(Integer(2), Integer(1), p, [x,y]).variables()
Traceback (most recent call last):
...
ValueError: polynomial is of the wrong degree

>>> AlgebraicForm(Integer(2), Integer(2), x**Integer(2) + y, [x,y]).variables()
Traceback (most recent call last):
...
ValueError: polynomial is not homogeneous
```

coefficients()

Alias for `coeffs()`.

See the documentation for `coeffs()` for details.

EXAMPLES:

```
sage: R.<a,b,c,d,e,f,g, x,y,z> = QQ[]
sage: p = a*x^2 + b*y^2 + c*z^2 + d*x*y + e*x*z + f*y*z
sage: q = invariant_theory.quadratic_form(p, x,y,z)
sage: q.coefficients()
(a, b, c, d, e, f)
sage: q.coeffs()
(a, b, c, d, e, f)
```

```
>>> from sage.all import *
>>> R = QQ['a, b, c, d, e, f, g, x, y, z']; (a, b, c, d, e, f, g, x, y, z,) =␣
↪R._first_ngens(10)
>>> p = a*x**Integer(2) + b*y**Integer(2) + c*z**Integer(2) + d*x*y + e*x*z +␣
↪f*y*z
>>> q = invariant_theory.quadratic_form(p, x,y,z)
>>> q.coefficients()
(a, b, c, d, e, f)
>>> q.coeffs()
(a, b, c, d, e, f)
```

form()

Return the defining polynomial.

OUTPUT: the polynomial used to define the algebraic form

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: quartic = invariant_theory.binary_quartic(x^4 + y^4)
sage: quartic.form()
x^4 + y^4
sage: quartic.polynomial()
x^4 + y^4
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> quartic = invariant_theory.binary_quartic(x**Integer(4) + y**Integer(4))
>>> quartic.form()
x^4 + y^4
>>> quartic.polynomial()
x^4 + y^4
```

homogenized(*var='h'*)

Return form as defined by a homogeneous polynomial.

INPUT:

- `var` – either a variable name, variable index or a variable (default: `'h'`)

OUTPUT:

The same algebraic form, but defined by a homogeneous polynomial.

EXAMPLES:

```
sage: T.<t> = QQ[]
sage: quadratic = invariant_theory.binary_quadratic(t^2 + 2*t + 3)
sage: quadratic
Binary quadratic with coefficients (1, 3, 2)
sage: quadratic.homogenized()
Binary quadratic with coefficients (1, 3, 2)
sage: quadratic == quadratic.homogenized()
True
sage: quadratic.form()
t^2 + 2*t + 3
sage: quadratic.homogenized().form()
t^2 + 2*t*h + 3*h^2

sage: R.<x,y,z> = QQ[]
sage: quadratic = invariant_theory.ternary_quadratic(x^2 + 1, [x,y])
sage: quadratic.homogenized().form()
x^2 + h^2

sage: R.<x> = QQ[]
sage: quintic = invariant_theory.binary_quintic(x^4 + 1, x)
sage: quintic.homogenized().form()
x^4*h + h^5
```

```
>>> from sage.all import *
>>> T = QQ['t']; (t,) = T._first_ngens(1)
>>> quadratic = invariant_theory.binary_quadratic(t**Integer(2) +␣
↪Integer(2)*t + Integer(3))
>>> quadratic
Binary quadratic with coefficients (1, 3, 2)
>>> quadratic.homogenized()
Binary quadratic with coefficients (1, 3, 2)
>>> quadratic == quadratic.homogenized()
True
>>> quadratic.form()
```

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```
t^2 + 2*t + 3
>>> quadratic.homogenized().form()
t^2 + 2*t*h + 3*h^2

>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> quadratic = invariant_theory.ternary_quadratic(x**Integer(2) + Integer(1),
↪ [x,y])
>>> quadratic.homogenized().form()
x^2 + h^2

>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> quintic = invariant_theory.binary_quintic(x**Integer(4) + Integer(1), x)
>>> quintic.homogenized().form()
x^4*h + h^5
```

polynomial()

Return the defining polynomial.

OUTPUT: the polynomial used to define the algebraic form

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: quartic = invariant_theory.binary_quartic(x^4 + y^4)
sage: quartic.form()
x^4 + y^4
sage: quartic.polynomial()
x^4 + y^4
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> quartic = invariant_theory.binary_quartic(x**Integer(4) + y**Integer(4))
>>> quartic.form()
x^4 + y^4
>>> quartic.polynomial()
x^4 + y^4
```

transformed(g)

Return the image under a linear transformation of the variables.

INPUT:

- `g` – a $GL(n, \mathbf{C})$ matrix or a dictionary with the variables as keys. A matrix is used to define the linear transformation of homogeneous variables, a dictionary acts by substitution of the variables.

OUTPUT:

A new instance of a subclass of `AlgebraicForm` obtained by replacing the variables of the homogeneous polynomial by their image under `g`.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: cubic = invariant_theory.ternary_cubic(x^3 + 2*y^3 + 3*z^3 + 4*x*y*z)
sage: cubic.transformed({x: y, y: z, z: x}).form()
```

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```
3*x^3 + y^3 + 4*x*y*z + 2*z^3
sage: cyc = matrix([[0,1,0], [0,0,1], [1,0,0]])
sage: cubic.transformed(cyc) == cubic.transformed({x:y, y:z, z:x})
True
sage: g = matrix(QQ, [[1, 0, 0], [-1, 1, -3], [-5, -5, 16]])
sage: cubic.transformed(g)
Ternary cubic with coefficients (-356, -373, 12234, -1119, 3578, -1151,
3582, -11766, -11466, 7360)
sage: cubic.transformed(g).transformed(g.inverse()) == cubic
True
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> cubic = invariant_theory.ternary_cubic(x**Integer(3) +
↪Integer(2)*y**Integer(3) + Integer(3)*z**Integer(3) + Integer(4)*x*y*z)
>>> cubic.transformed({x: y, y: z, z: x}).form()
3*x^3 + y^3 + 4*x*y*z + 2*z^3
>>> cyc = matrix([[Integer(0),Integer(1),Integer(0)], [Integer(0),Integer(0),
↪Integer(1)], [Integer(1),Integer(0),Integer(0)]])
>>> cubic.transformed(cyc) == cubic.transformed({x:y, y:z, z:x})
True
>>> g = matrix(QQ, [[Integer(1), Integer(0), Integer(0)], [-Integer(1),
↪Integer(1), -Integer(3)], [-Integer(5), -Integer(5), Integer(16)]])
>>> cubic.transformed(g)
Ternary cubic with coefficients (-356, -373, 12234, -1119, 3578, -1151,
3582, -11766, -11466, 7360)
>>> cubic.transformed(g).transformed(g.inverse()) == cubic
True
```

class sage.rings.invariants.invariant_theory.**BinaryQuartic**(*n*, *d*, *polynomial*, **args*)

Bases: *AlgebraicForm*

Invariant theory of a binary quartic.

You should use the *invariant_theory* factory object to construct instances of this class. See *binary_quartic()* for details.

EisensteinD()

One of the Eisenstein invariants of a binary quartic.

OUTPUT: the Eisenstein D-invariant of the quartic

$$f(x) = a_0 x_1^4 + 4a_1 x_0 x_1^3 + 6a_2 x_0^2 x_1^2 + 4a_3 x_0^3 x_1 + a_4 x_0^4$$
$$\Rightarrow D(f) = a_0 a_4 + 3a_2^2 - 4a_1 a_3$$

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, x0, x1> = QQ[]
sage: f = a0*x1^4 + 4*a1*x0*x1^3 + 6*a2*x0^2*x1^2 + 4*a3*x0^3*x1 + a4*x0^4
sage: inv = invariant_theory.binary_quartic(f, x0, x1)
sage: inv.EisensteinD()
3*a2^2 - 4*a1*a3 + a0*a4
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, x0, x1']; (a0, a1, a2, a3, a4, x0, x1,) = R._
↪first_ngens(7)
>>> f = a0*x1**Integer(4) + Integer(4)*a1*x0*x1**Integer(3) +␣
↪Integer(6)*a2*x0**Integer(2)*x1**Integer(2) +␣
↪Integer(4)*a3*x0**Integer(3)*x1 + a4*x0**Integer(4)
>>> inv = invariant_theory.binary_quartic(f, x0, x1)
>>> inv.EisensteinD()
3*a2^2 - 4*a1*a3 + a0*a4
```

EisensteinE()

One of the Eisenstein invariants of a binary quartic.

OUTPUT: the Eisenstein E-invariant of the quartic

$$f(x) = a_0 x_1^4 + 4a_1 x_0 x_1^3 + 6a_2 x_0^2 x_1^2 + 4a_3 x_0^3 x_1 + a_4 x_0^4$$
$$\Rightarrow E(f) = a_0 a_3^2 + a_1^2 a_4 - a_0 a_2 a_4 - 2a_1 a_2 a_3 + a_2^3$$

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, x0, x1> = QQ[]
sage: f = a0*x1^4 + 4*a1*x0*x1^3 + 6*a2*x0^2*x1^2 + 4*a3*x0^3*x1 + a4*x0^4
sage: inv = invariant_theory.binary_quartic(f, x0, x1)
sage: inv.EisensteinE()
a2^3 - 2*a1*a2*a3 + a0*a3^2 + a1^2*a4 - a0*a2*a4
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, x0, x1']; (a0, a1, a2, a3, a4, x0, x1,) = R._
↪first_ngens(7)
>>> f = a0*x1**Integer(4) + Integer(4)*a1*x0*x1**Integer(3) +␣
↪Integer(6)*a2*x0**Integer(2)*x1**Integer(2) +␣
↪Integer(4)*a3*x0**Integer(3)*x1 + a4*x0**Integer(4)
>>> inv = invariant_theory.binary_quartic(f, x0, x1)
>>> inv.EisensteinE()
a2^3 - 2*a1*a2*a3 + a0*a3^2 + a1^2*a4 - a0*a2*a4
```

coeffs()

The coefficients of a binary quartic.

Given

$$f(x) = a_0 x_1^4 + a_1 x_0 x_1^3 + a_2 x_0^2 x_1^2 + a_3 x_0^3 x_1 + a_4 x_0^4$$

this function returns $a = (a_0, a_1, a_2, a_3, a_4)$

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, x0, x1> = QQ[]
sage: p = a0*x1^4 + a1*x1^3*x0 + a2*x1^2*x0^2 + a3*x1*x0^3 + a4*x0^4
sage: quartic = invariant_theory.binary_quartic(p, x0, x1)
sage: quartic.coeffs()
(a0, a1, a2, a3, a4)

sage: R.<a0, a1, a2, a3, a4, x> = QQ[]
```

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```
sage: p = a0 + a1*x + a2*x^2 + a3*x^3 + a4*x^4
sage: quartic = invariant_theory.binary_quartic(p, x)
sage: quartic.coeffs()
(a0, a1, a2, a3, a4)
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, x0, x1']; (a0, a1, a2, a3, a4, x0, x1,) = R._
↪first_ngens(7)
>>> p = a0*x1**Integer(4) + a1*x1**Integer(3)*x0 +␣
↪a2*x1**Integer(2)*x0**Integer(2) + a3*x1*x0**Integer(3) + a4*x0**Integer(4)
>>> quartic = invariant_theory.binary_quartic(p, x0, x1)
>>> quartic.coeffs()
(a0, a1, a2, a3, a4)

>>> R = QQ['a0, a1, a2, a3, a4, x']; (a0, a1, a2, a3, a4, x,) = R._first_
↪ngens(6)
>>> p = a0 + a1*x + a2*x**Integer(2) + a3*x**Integer(3) + a4*x**Integer(4)
>>> quartic = invariant_theory.binary_quartic(p, x)
>>> quartic.coeffs()
(a0, a1, a2, a3, a4)
```

g_covariant()

The g-covariant of a binary quartic.

OUTPUT: the g-covariant of the quartic

$$f(x) = a_0 x_1^4 + 4 a_1 x_0 x_1^3 + 6 a_2 x_0^2 x_1^2 + 4 a_3 x_0^3 x_1 + a_4 x_0^4$$
$$\Rightarrow D(f) = \frac{1}{144} \left(\frac{\partial^2 f}{\partial x \partial x} \right)$$

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, x, y> = QQ[]
sage: p = a0*x^4 + 4*a1*x^3*y + 6*a2*x^2*y^2 + 4*a3*x*y^3 + a4*y^4
sage: inv = invariant_theory.binary_quartic(p, x, y)
sage: g = inv.g_covariant();  g
a1^2*x^4 - a0*a2*x^4 + 2*a1*a2*x^3*y - 2*a0*a3*x^3*y + 3*a2^2*x^2*y^2
- 2*a1*a3*x^2*y^2 - a0*a4*x^2*y^2 + 2*a2*a3*x*y^3
- 2*a1*a4*x*y^3 + a3^2*y^4 - a2*a4*y^4

sage: inv_inhomogeneous = invariant_theory.binary_quartic(p.subs(y=1), x)
sage: inv_inhomogeneous.g_covariant()
a1^2*x^4 - a0*a2*x^4 + 2*a1*a2*x^3 - 2*a0*a3*x^3 + 3*a2^2*x^2
- 2*a1*a3*x^2 - a0*a4*x^2 + 2*a2*a3*x - 2*a1*a4*x + a3^2 - a2*a4

sage: g == 1/144 * (p.derivative(x,y)^2 - p.derivative(x,x)*p.derivative(y,y))
True
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, x, y']; (a0, a1, a2, a3, a4, x, y,) = R._
↪first_ngens(7)
>>> p = a0*x**Integer(4) + Integer(4)*a1*x**Integer(3)*y +␣
↪Integer(6)*a2*x**Integer(2)*y**Integer(2) + Integer(4)*a3*x*y**Integer(3) +␣
```

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```
↪a4*y**Integer(4)
>>> inv = invariant_theory.binary_quartic(p, x, y)
>>> g = inv.g_covariant();  g
a1^2*x^4 - a0*a2*x^4 + 2*a1*a2*x^3*y - 2*a0*a3*x^3*y + 3*a2^2*x^2*y^2
- 2*a1*a3*x^2*y^2 - a0*a4*x^2*y^2 + 2*a2*a3*x*y^3
- 2*a1*a4*x*y^3 + a3^2*y^4 - a2*a4*y^4

>>> inv_inhomogeneous = invariant_theory.binary_quartic(p.subs(y=Integer(1)),␣
↪x)
>>> inv_inhomogeneous.g_covariant()
a1^2*x^4 - a0*a2*x^4 + 2*a1*a2*x^3 - 2*a0*a3*x^3 + 3*a2^2*x^2
- 2*a1*a3*x^2 - a0*a4*x^2 + 2*a2*a3*x - 2*a1*a4*x + a3^2 - a2*a4

>>> g == Integer(1)/Integer(144) * (p.derivative(x,y)**Integer(2) - p.
↪derivative(x,x)*p.derivative(y,y))
True
```

h_covariant()

The h-covariant of a binary quartic.

OUTPUT: the h-covariant of the quartic

$$f(x) = a_0x_1^4 + 4a_1x_0x_1^3 + 6a_2x_0^2x_1^2 + 4a_3x_0^3x_1 + a_4x_0^4$$
$$\Rightarrow D(f) = \frac{1}{144}\left(\frac{\partial^2 f}{\partial x \partial x}\right)$$

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, x, y> = QQ[]
sage: p = a0*x^4 + 4*a1*x^3*y + 6*a2*x^2*y^2 + 4*a3*x*y^3 + a4*y^4
sage: inv = invariant_theory.binary_quartic(p, x, y)
sage: h = inv.h_covariant();   h
-2*a1^3*x^6 + 3*a0*a1*a2*x^6 - a0^2*a3*x^6 - 6*a1^2*a2*x^5*y + 9*a0*a2^2*x^5*y
- 2*a0*a1*a3*x^5*y - a0^2*a4*x^5*y - 10*a1^2*a3*x^4*y^2 + 15*a0*a2*a3*x^4*y^2
- 5*a0*a1*a4*x^4*y^2 + 10*a0*a3^2*x^3*y^3 - 10*a1^2*a4*x^3*y^3
+ 10*a1*a3^2*x^2*y^4 - 15*a1*a2*a4*x^2*y^4 + 5*a0*a3*a4*x^2*y^4
+ 6*a2*a3^2*x*y^5 - 9*a2^2*a4*x*y^5 + 2*a1*a3*a4*x*y^5 + a0*a4^2*x*y^5
+ 2*a3^3*y^6 - 3*a2*a3*a4*y^6 + a1*a4^2*y^6

sage: inv_inhomogeneous = invariant_theory.binary_quartic(p.subs(y=1), x)
sage: inv_inhomogeneous.h_covariant()
-2*a1^3*x^6 + 3*a0*a1*a2*x^6 - a0^2*a3*x^6 - 6*a1^2*a2*x^5 + 9*a0*a2^2*x^5
- 2*a0*a1*a3*x^5 - a0^2*a4*x^5 - 10*a1^2*a3*x^4 + 15*a0*a2*a3*x^4
- 5*a0*a1*a4*x^4 + 10*a0*a3^2*x^3 - 10*a1^2*a4*x^3 + 10*a1*a3^2*x^2
- 15*a1*a2*a4*x^2 + 5*a0*a3*a4*x^2 + 6*a2*a3^2*x - 9*a2^2*a4*x
+ 2*a1*a3*a4*x + a0*a4^2*x + 2*a3^3 - 3*a2*a3*a4 + a1*a4^2

sage: g = inv.g_covariant()
sage: h == 1/8 * (p.derivative(x)*g.derivative(y) - p.derivative(y)*g.
↪derivative(x))
True
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, x, y']; (a0, a1, a2, a3, a4, x, y,) = R._
↪first_ngens(7)
>>> p = a0*x**Integer(4) + Integer(4)*a1*x**Integer(3)*y +␣
↪Integer(6)*a2*x**Integer(2)*y**Integer(2) + Integer(4)*a3*x*y**Integer(3) +␣
↪a4*y**Integer(4)
>>> inv = invariant_theory.binary_quartic(p, x, y)
>>> h = inv.h_covariant();   h
-2*a1^3*x^6 + 3*a0*a1*a2*x^6 - a0^2*a3*x^6 - 6*a1^2*a2*x^5*y + 9*a0*a2^2*x^5*y
- 2*a0*a1*a3*x^5*y - a0^2*a4*x^5*y - 10*a1^2*a3*x^4*y^2 + 15*a0*a2*a3*x^4*y^2
- 5*a0*a1*a4*x^4*y^2 + 10*a0*a3^2*x^3*y^3 - 10*a1^2*a4*x^3*y^3
+ 10*a1*a3^2*x^2*y^4 - 15*a1*a2*a4*x^2*y^4 + 5*a0*a3*a4*x^2*y^4
+ 6*a2*a3^2*x*y^5 - 9*a2^2*a4*x*y^5 + 2*a1*a3*a4*x*y^5 + a0*a4^2*x*y^5
+ 2*a3^3*y^6 - 3*a2*a3*a4*y^6 + a1*a4^2*y^6

>>> inv_inhomogeneous = invariant_theory.binary_quartic(p.subs(y=Integer(1)),␣
↪x)
>>> inv_inhomogeneous.h_covariant()
-2*a1^3*x^6 + 3*a0*a1*a2*x^6 - a0^2*a3*x^6 - 6*a1^2*a2*x^5 + 9*a0*a2^2*x^5
- 2*a0*a1*a3*x^5 - a0^2*a4*x^5 - 10*a1^2*a3*x^4 + 15*a0*a2*a3*x^4
- 5*a0*a1*a4*x^4 + 10*a0*a3^2*x^3 - 10*a1^2*a4*x^3 + 10*a1*a3^2*x^2
- 15*a1*a2*a4*x^2 + 5*a0*a3*a4*x^2 + 6*a2*a3^2*x - 9*a2^2*a4*x
+ 2*a1*a3*a4*x + a0*a4^2*x + 2*a3^3 - 3*a2*a3*a4 + a1*a4^2

>>> g = inv.g_covariant()
>>> h == Integer(1)/Integer(8) * (p.derivative(x)*g.derivative(y) - p.
↪derivative(y)*g.derivative(x))
True
```

monomials()

List the basis monomials in the form.

OUTPUT:

A tuple of monomials. They are in the same order as *coeffs()*.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: quartic = invariant_theory.binary_quartic(x^4 + y^4)
sage: quartic.monomials()
(y^4, x*y^3, x^2*y^2, x^3*y, x^4)
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> quartic = invariant_theory.binary_quartic(x**Integer(4) + y**Integer(4))
>>> quartic.monomials()
(y^4, x*y^3, x^2*y^2, x^3*y, x^4)
```

scaled_coeffs()

The coefficients of a binary quartic.

Given

$$f(x) = a_0 x_1^4 + 4a_1 x_0 x_1^3 + 6a_2 x_0^2 x_1^2 + 4a_3 x_0^3 x_1 + a_4 x_0^4$$

this function returns $a = (a_0, a_1, a_2, a_3, a_4)$

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, x0, x1> = QQ[]
sage: quartic = a0*x1^4 + 4*a1*x1^3*x0 + 6*a2*x1^2*x0^2 + 4*a3*x1*x0^3 +␣
↪a4*x0^4
sage: inv = invariant_theory.binary_quartic(quartic, x0, x1)
sage: inv.scaled_coeffs()
(a0, a1, a2, a3, a4)

sage: R.<a0, a1, a2, a3, a4, x> = QQ[]
sage: quartic = a0 + 4*a1*x + 6*a2*x^2 + 4*a3*x^3 + a4*x^4
sage: inv = invariant_theory.binary_quartic(quartic, x)
sage: inv.scaled_coeffs()
(a0, a1, a2, a3, a4)
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, x0, x1']; (a0, a1, a2, a3, a4, x0, x1,) = R._
↪first_ngens(7)
>>> quartic = a0*x1**Integer(4) + Integer(4)*a1*x1**Integer(3)*x0 +␣
↪Integer(6)*a2*x1**Integer(2)*x0**Integer(2) +␣
↪Integer(4)*a3*x1*x0**Integer(3) + a4*x0**Integer(4)
>>> inv = invariant_theory.binary_quartic(quartic, x0, x1)
>>> inv.scaled_coeffs()
(a0, a1, a2, a3, a4)

>>> R = QQ['a0, a1, a2, a3, a4, x']; (a0, a1, a2, a3, a4, x,) = R._first_
↪ngens(6)
>>> quartic = a0 + Integer(4)*a1*x + Integer(6)*a2*x**Integer(2) +␣
↪Integer(4)*a3*x**Integer(3) + a4*x**Integer(4)
>>> inv = invariant_theory.binary_quartic(quartic, x)
>>> inv.scaled_coeffs()
(a0, a1, a2, a3, a4)
```

class `sage.rings.invariants.invariant_theory.`**`BinaryQuintic`**(*n*, *d*, *polynomial*, **args*)

Bases: *`AlgebraicForm`*

Invariant theory of a binary quintic form.

You should use the *`invariant_theory`* factory object to construct instances of this class. See *`binary_quintic()`* for details.

REFERENCES:

For a description of all invariants and covariants of a binary quintic, see section 73 of [Cle1872].

`A_invariant`()

Return the invariant A of a binary quintic.

OUTPUT: the A-invariant of the binary quintic

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 +␣
↪a5*x0^5
```

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```
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.A_invariant()
4/625*a2^2*a3^2 - 12/625*a1*a3^3 - 12/625*a2^3*a4
+ 38/625*a1*a2*a3*a4 + 6/125*a0*a3^2*a4 - 18/625*a1^2*a4^2
- 16/125*a0*a2*a4^2 + 6/125*a1*a2^2*a5 - 16/125*a1^2*a3*a5
- 2/25*a0*a2*a3*a5 + 4/5*a0*a1*a4*a5 - 2*a0^2*a5^2
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, a5, x0, x1']; (a0, a1, a2, a3, a4, a5, x0, x1,
↪) = R._first_ngens(8)
>>> p = a0*x1**Integer(5) + a1*x1**Integer(4)*x0 +␣
↪a2*x1**Integer(3)*x0**Integer(2) + a3*x1**Integer(2)*x0**Integer(3) +␣
↪a4*x1*x0**Integer(4) + a5*x0**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x0, x1)
>>> quintic.A_invariant()
4/625*a2^2*a3^2 - 12/625*a1*a3^3 - 12/625*a2^3*a4
+ 38/625*a1*a2*a3*a4 + 6/125*a0*a3^2*a4 - 18/625*a1^2*a4^2
- 16/125*a0*a2*a4^2 + 6/125*a1*a2^2*a5 - 16/125*a1^2*a3*a5
- 2/25*a0*a2*a3*a5 + 4/5*a0*a1*a4*a5 - 2*a0^2*a5^2
```

B_invariant()

Return the invariant B of a binary quintic.

OUTPUT: the B-invariant of the binary quintic

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 +␣
↪a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.B_invariant()
1/1562500*a2^4*a3^4 - 3/781250*a1*a2^2*a3^5 + 9/1562500*a1^2*a3^6
- 3/781250*a2^5*a3^2*a4 + 37/1562500*a1*a2^3*a3^3*a4
- 57/1562500*a1^2*a2*a3^4*a4 + 3/312500*a0*a2^2*a3^4*a4
...
+ 8/625*a0^2*a1^2*a4^2*a5^2 - 4/125*a0^3*a2*a4^2*a5^2 - 16/3125*a1^5*a5^3
+ 4/125*a0*a1^3*a2*a5^3 - 6/125*a0^2*a1*a2^2*a5^3
- 4/125*a0^2*a1^2*a3*a5^3 + 2/25*a0^3*a2*a3*a5^3
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, a5, x0, x1']; (a0, a1, a2, a3, a4, a5, x0, x1,
↪) = R._first_ngens(8)
>>> p = a0*x1**Integer(5) + a1*x1**Integer(4)*x0 +␣
↪a2*x1**Integer(3)*x0**Integer(2) + a3*x1**Integer(2)*x0**Integer(3) +␣
↪a4*x1*x0**Integer(4) + a5*x0**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x0, x1)
>>> quintic.B_invariant()
1/1562500*a2^4*a3^4 - 3/781250*a1*a2^2*a3^5 + 9/1562500*a1^2*a3^6
- 3/781250*a2^5*a3^2*a4 + 37/1562500*a1*a2^3*a3^3*a4
- 57/1562500*a1^2*a2*a3^4*a4 + 3/312500*a0*a2^2*a3^4*a4
...
```

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```
+ 8/625*a0^2*a1^2*a4^2*a5^2 - 4/125*a0^3*a2*a4^2*a5^2 - 16/3125*a1^5*a5^3
+ 4/125*a0*a1^3*a2*a5^3 - 6/125*a0^2*a1*a2^2*a5^3
- 4/125*a0^2*a1^2*a3*a5^3 + 2/25*a0^3*a2*a3*a5^3
```

C_invariant()

Return the invariant C of a binary quintic.

OUTPUT: the C-invariant of the binary quintic

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 +␣
↪a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.C_invariant()
-3/1953125000*a2^6*a3^6 + 27/1953125000*a1*a2^4*a3^7
- 249/7812500000*a1^2*a2^2*a3^8 - 3/78125000*a0*a2^3*a3^8
+ 3/976562500*a1^3*a3^9 + 27/156250000*a0*a1*a2*a3^9
...
+ 192/15625*a0^2*a1^3*a2^2*a3*a5^4 - 36/3125*a0^3*a1*a2^3*a3*a5^4
+ 24/15625*a0^2*a1^4*a3^2*a5^4 - 24/3125*a0^3*a1^2*a2*a3^2*a5^4
+ 6/625*a0^4*a2^2*a3^2*a5^4
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, a5, x0, x1']; (a0, a1, a2, a3, a4, a5, x0, x1,
↪) = R._first_ngens(8)
>>> p = a0*x1**Integer(5) + a1*x1**Integer(4)*x0 +␣
↪a2*x1**Integer(3)*x0**Integer(2) + a3*x1**Integer(2)*x0**Integer(3) +␣
↪a4*x1*x0**Integer(4) + a5*x0**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x0, x1)
>>> quintic.C_invariant()
-3/1953125000*a2^6*a3^6 + 27/1953125000*a1*a2^4*a3^7
- 249/7812500000*a1^2*a2^2*a3^8 - 3/78125000*a0*a2^3*a3^8
+ 3/976562500*a1^3*a3^9 + 27/156250000*a0*a1*a2*a3^9
...
+ 192/15625*a0^2*a1^3*a2^2*a3*a5^4 - 36/3125*a0^3*a1*a2^3*a3*a5^4
+ 24/15625*a0^2*a1^4*a3^2*a5^4 - 24/3125*a0^3*a1^2*a2*a3^2*a5^4
+ 6/625*a0^4*a2^2*a3^2*a5^4
```

H_covariant(*as_form=False*)

Return the covariant H of a binary quintic.

INPUT:

- `as_form` – if `as_form` is `False`, the result will be returned as polynomial (default). If it is `True` the result is returned as an object of the class *AlgebraicForm*.

OUTPUT: the H-covariant of the binary quintic as polynomial or as binary form

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 +␣
↪a5*x0^5
```

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```
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.H_covariant()
-2/25*a4^2*x0^6 + 1/5*a3*a5*x0^6 - 3/25*a3*a4*x0^5*x1
+ 3/5*a2*a5*x0^5*x1 - 3/25*a3^2*x0^4*x1^2 + 3/25*a2*a4*x0^4*x1^2
+ 6/5*a1*a5*x0^4*x1^2 - 4/25*a2*a3*x0^3*x1^3 + 14/25*a1*a4*x0^3*x1^3
+ 2*a0*a5*x0^3*x1^3 - 3/25*a2^2*x0^2*x1^4 + 3/25*a1*a3*x0^2*x1^4
+ 6/5*a0*a4*x0^2*x1^4 - 3/25*a1*a2*x0*x1^5 + 3/5*a0*a3*x0*x1^5
- 2/25*a1^2*x1^6 + 1/5*a0*a2*x1^6

sage: quintic.H_covariant(as_form=True)
Binary sextic given by ...
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, a5, x0, x1']; (a0, a1, a2, a3, a4, a5, x0, x1,
↪) = R._first_ngens(8)
>>> p = a0*x1**Integer(5) + a1*x1**Integer(4)*x0 +␣
↪a2*x1**Integer(3)*x0**Integer(2) + a3*x1**Integer(2)*x0**Integer(3) +␣
↪a4*x1*x0**Integer(4) + a5*x0**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x0, x1)
>>> quintic.H_covariant()
-2/25*a4^2*x0^6 + 1/5*a3*a5*x0^6 - 3/25*a3*a4*x0^5*x1
+ 3/5*a2*a5*x0^5*x1 - 3/25*a3^2*x0^4*x1^2 + 3/25*a2*a4*x0^4*x1^2
+ 6/5*a1*a5*x0^4*x1^2 - 4/25*a2*a3*x0^3*x1^3 + 14/25*a1*a4*x0^3*x1^3
+ 2*a0*a5*x0^3*x1^3 - 3/25*a2^2*x0^2*x1^4 + 3/25*a1*a3*x0^2*x1^4
+ 6/5*a0*a4*x0^2*x1^4 - 3/25*a1*a2*x0*x1^5 + 3/5*a0*a3*x0*x1^5
- 2/25*a1^2*x1^6 + 1/5*a0*a2*x1^6

>>> quintic.H_covariant(as_form=True)
Binary sextic given by ...
```

R_invariant()

Return the invariant R of a binary quintic.

OUTPUT: the R-invariant of the binary quintic

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 +␣
↪a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.R_invariant()
3/3906250000000*a1^2*a2^5*a3^11 - 3/976562500000*a0*a2^6*a3^11
- 51/7812500000000*a1^3*a2^3*a3^12 + 27/976562500000*a0*a1*a2^4*a3^12
+ 27/1953125000000*a1^4*a2*a3^13 - 81/1562500000000*a0*a1^2*a2^2*a3^13
...
+ 384/9765625*a0*a1^10*a5^7 - 192/390625*a0^2*a1^8*a2*a5^7
+ 192/78125*a0^3*a1^6*a2^2*a5^7 - 96/15625*a0^4*a1^4*a2^3*a5^7
+ 24/3125*a0^5*a1^2*a2^4*a5^7 - 12/3125*a0^6*a2^5*a5^7
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, a5, x0, x1']; (a0, a1, a2, a3, a4, a5, x0, x1,
```

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```
↪) = R._first_ngens(8)
>>> p = a0*x1**Integer(5) + a1*x1**Integer(4)*x0 +␣
↪a2*x1**Integer(3)*x0**Integer(2) + a3*x1**Integer(2)*x0**Integer(3) +␣
↪a4*x1*x0**Integer(4) + a5*x0**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x0, x1)
>>> quintic.R_invariant()
3/3906250000000*a1^2*a2^5*a3^11 - 3/976562500000*a0*a2^6*a3^11
- 51/7812500000000*a1^3*a2^3*a3^12 + 27/976562500000*a0*a1*a2^4*a3^12
+ 27/1953125000000*a1^4*a2*a3^13 - 81/1562500000000*a0*a1^2*a2^2*a3^13
...
+ 384/9765625*a0*a1^10*a5^7 - 192/390625*a0^2*a1^8*a2*a5^7
+ 192/78125*a0^3*a1^6*a2^2*a5^7 - 96/15625*a0^4*a1^4*a2^3*a5^7
+ 24/3125*a0^5*a1^2*a2^4*a5^7 - 12/3125*a0^6*a2^5*a5^7
```

T_covariant(*as_form=False*)

Return the covariant T of a binary quintic.

INPUT:

- `as_form` – if `as_form` is `False`, the result will be returned as polynomial (default). If it is `True` the result is returned as an object of the class *AlgebraicForm*.

OUTPUT: the T-covariant of the binary quintic as polynomial or as binary form

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 +␣
↪a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.T_covariant()
2/125*a4^3*x0^9 - 3/50*a3*a4*a5*x0^9 + 1/10*a2*a5^2*x0^9
+ 9/250*a3*a4^2*x0^8*x1 - 3/25*a3^2*a5*x0^8*x1 + 1/50*a2*a4*a5*x0^8*x1
+ 2/5*a1*a5^2*x0^8*x1 + 3/250*a3^2*a4*x0^7*x1^2 + 8/125*a2*a4^2*x0^7*x1^2
...
11/25*a0*a1*a4*x0^2*x1^7 - a0^2*a5*x0^2*x1^7 - 9/250*a1^2*a2*x0*x1^8
+ 3/25*a0*a2^2*x0*x1^8 - 1/50*a0*a1*a3*x0*x1^8 - 2/5*a0^2*a4*x0*x1^8
- 2/125*a1^3*x1^9 + 3/50*a0*a1*a2*x1^9 - 1/10*a0^2*a3*x1^9

sage: quintic.T_covariant(as_form=True)
Binary nonic given by ...
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, a5, x0, x1']; (a0, a1, a2, a3, a4, a5, x0, x1,
↪) = R._first_ngens(8)
>>> p = a0*x1**Integer(5) + a1*x1**Integer(4)*x0 +␣
↪a2*x1**Integer(3)*x0**Integer(2) + a3*x1**Integer(2)*x0**Integer(3) +␣
↪a4*x1*x0**Integer(4) + a5*x0**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x0, x1)
>>> quintic.T_covariant()
2/125*a4^3*x0^9 - 3/50*a3*a4*a5*x0^9 + 1/10*a2*a5^2*x0^9
+ 9/250*a3*a4^2*x0^8*x1 - 3/25*a3^2*a5*x0^8*x1 + 1/50*a2*a4*a5*x0^8*x1
+ 2/5*a1*a5^2*x0^8*x1 + 3/250*a3^2*a4*x0^7*x1^2 + 8/125*a2*a4^2*x0^7*x1^2
...
```

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```
11/25*a0*a1*a4*x0^2*x1^7 - a0^2*a5*x0^2*x1^7 - 9/250*a1^2*a2*x0*x1^8
+ 3/25*a0*a2^2*x0*x1^8 - 1/50*a0*a1*a3*x0*x1^8 - 2/5*a0^2*a4*x0*x1^8
- 2/125*a1^3*x1^9 + 3/50*a0*a1*a2*x1^9 - 1/10*a0^2*a3*x1^9

>>> quintic.T_covariant(as_form=True)
Binary nonic given by ...
```

alpha_covariant(*as_form=False*)

Return the covariant α of a binary quintic.

INPUT:

- `as_form` – if `as_form` is `False`, the result will be returned as polynomial (default). If it is `True` the result is returned as an object of the class *AlgebraicForm*.

OUTPUT:

The α-covariant of the binary quintic as polynomial or as binary form.

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 +␣
↪a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.alpha_covariant()
1/2500*a2^2*a3^3*x0 - 3/2500*a1*a3^4*x0 - 1/625*a2^3*a3*a4*x0
+ 3/625*a1*a2*a3^2*a4*x0 + 3/625*a0*a3^3*a4*x0 + 2/625*a1*a2^2*a4^2*x0
- 6/625*a1^2*a3*a4^2*x0 - 12/625*a0*a2*a3*a4^2*x0 + 24/625*a0*a1*a4^3*x0
...
- 12/625*a1^2*a2*a3*a5*x1 - 1/125*a0*a2^2*a3*a5*x1
+ 8/125*a0*a1*a3^2*a5*x1 + 24/625*a1^3*a4*a5*x1 - 8/125*a0*a1*a2*a4*a5*x1
- 4/25*a0^2*a3*a4*a5*x1 - 4/25*a0*a1^2*a5^2*x1 + 2/5*a0^2*a2*a5^2*x1

sage: quintic.alpha_covariant(as_form=True)
Binary monic given by ...
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, a5, x0, x1']; (a0, a1, a2, a3, a4, a5, x0, x1,
↪) = R._first_ngens(8)
>>> p = a0*x1**Integer(5) + a1*x1**Integer(4)*x0 +␣
↪a2*x1**Integer(3)*x0**Integer(2) + a3*x1**Integer(2)*x0**Integer(3) +␣
↪a4*x1*x0**Integer(4) + a5*x0**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x0, x1)
>>> quintic.alpha_covariant()
1/2500*a2^2*a3^3*x0 - 3/2500*a1*a3^4*x0 - 1/625*a2^3*a3*a4*x0
+ 3/625*a1*a2*a3^2*a4*x0 + 3/625*a0*a3^3*a4*x0 + 2/625*a1*a2^2*a4^2*x0
- 6/625*a1^2*a3*a4^2*x0 - 12/625*a0*a2*a3*a4^2*x0 + 24/625*a0*a1*a4^3*x0
...
- 12/625*a1^2*a2*a3*a5*x1 - 1/125*a0*a2^2*a3*a5*x1
+ 8/125*a0*a1*a3^2*a5*x1 + 24/625*a1^3*a4*a5*x1 - 8/125*a0*a1*a2*a4*a5*x1
- 4/25*a0^2*a3*a4*a5*x1 - 4/25*a0*a1^2*a5^2*x1 + 2/5*a0^2*a2*a5^2*x1

>>> quintic.alpha_covariant(as_form=True)
```

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```
Binary monic given by ...
```

arithmetic_invariants()

Return a set of generating arithmetic invariants of a binary quintic.

An arithmetic invariants is an invariant whose coefficients are integers for a general binary quintic. They are linear combinations of the Clebsch invariants, such that they still generate the ring of invariants.

OUTPUT: the arithmetic invariants of the binary quintic. They are given by

$$I_4 = 2^{-1} \cdot 5^4 \cdot A$$
$$I_8 = 5^5 \cdot (2^{-1} \cdot 47 \cdot A^2 - 2^2 \cdot B)$$
$$I_{12} = 5^{10} \cdot (2^{-1} \cdot 3 \cdot A^3 - 2^5 \cdot 3^{-1} \cdot C)$$
$$I_{18} = 2^8 \cdot 3^{-1} \cdot 5^{15} \cdot R$$

where A, B, C and R are the `BinaryQuintic.clebsch_invariants()`.

EXAMPLES:

```
sage: R.<x0, x1> = QQ[]
sage: p = 2*x1^5 + 4*x1^4*x0 + 5*x1^3*x0^2 + 7*x1^2*x0^3 - 11*x1*x0^4 + x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.arithmetic_invariants()
{'I12': -1156502613073152,
 'I18': -1271287234804879764275 2,
 'I4': -138016,
 'I8': 14164936192}
```

```
>>> from sage.all import *
>>> R = QQ['x0, x1']; (x0, x1,) = R._first_ngens(2)
>>> p = Integer(2)*x1**Integer(5) + Integer(4)*x1**Integer(4)*x0 +␣
↪Integer(5)*x1**Integer(3)*x0**Integer(2) +␣
↪Integer(7)*x1**Integer(2)*x0**Integer(3) - Integer(11)*x1*x0**Integer(4) +␣
↪x0**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x0, x1)
>>> quintic.arithmetic_invariants()
{'I12': -1156502613073152,
 'I18': -12712872348048797642752,
 'I4': -138016,
 'I8': 14164936192}
```

We can check that the coefficients of the invariants have no common divisor for a general quintic form:

```
sage: R.<a0,a1,a2,a3,a4,a5,x0,x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 +␣
↪a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: invs = quintic.arithmetic_invariants()
sage: [invs[x].content() for x in invs]
[1, 1, 1, 1]
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, a5, x0, x1']; (a0, a1, a2, a3, a4, a5, x0, x1,
```

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```
↪) = R._first_ngens(8)
>>> p = a0*x1**Integer(5) + a1*x1**Integer(4)*x0 +␣
↪a2*x1**Integer(3)*x0**Integer(2) + a3*x1**Integer(2)*x0**Integer(3) +␣
↪a4*x1*x0**Integer(4) + a5*x0**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x0, x1)
>>> invs = quintic.arithmetic_invariants()
>>> [invs[x].content() for x in invs]
[1, 1, 1, 1]
```

beta_covariant(*as_form=False*)

Return the covariant β of a binary quintic.

INPUT:

- `as_form` – if `as_form` is `False`, the result will be returned as polynomial (default). If it is `True` the result is returned as an object of the class *AlgebraicForm*.

OUTPUT:

The β-covariant of the binary quintic as polynomial or as binary form.

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 +␣
↪a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.beta_covariant()
-1/62500*a2^3*a3^4*x0 + 9/125000*a1*a2*a3^5*x0 - 27/125000*a0*a3^6*x0
+ 13/125000*a2^4*a3^2*a4*x0 - 31/62500*a1*a2^2*a3^3*a4*x0
- 3/62500*a1^2*a3^4*a4*x0 + 27/15625*a0*a2*a3^4*a4*x0
...
- 16/125*a0^2*a1*a3^2*a5^2*x1 - 28/625*a0*a1^3*a4*a5^2*x1
+ 6/125*a0^2*a1*a2*a4*a5^2*x1 + 8/25*a0^3*a3*a4*a5^2*x1
+ 4/25*a0^2*a1^2*a5^3*x1 - 2/5*a0^3*a2*a5^3*x1

sage: quintic.beta_covariant(as_form=True)
Binary monic given by ...
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, a5, x0, x1']; (a0, a1, a2, a3, a4, a5, x0, x1,
↪) = R._first_ngens(8)
>>> p = a0*x1**Integer(5) + a1*x1**Integer(4)*x0 +␣
↪a2*x1**Integer(3)*x0**Integer(2) + a3*x1**Integer(2)*x0**Integer(3) +␣
↪a4*x1*x0**Integer(4) + a5*x0**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x0, x1)
>>> quintic.beta_covariant()
-1/62500*a2^3*a3^4*x0 + 9/125000*a1*a2*a3^5*x0 - 27/125000*a0*a3^6*x0
+ 13/125000*a2^4*a3^2*a4*x0 - 31/62500*a1*a2^2*a3^3*a4*x0
- 3/62500*a1^2*a3^4*a4*x0 + 27/15625*a0*a2*a3^4*a4*x0
...
- 16/125*a0^2*a1*a3^2*a5^2*x1 - 28/625*a0*a1^3*a4*a5^2*x1
+ 6/125*a0^2*a1*a2*a4*a5^2*x1 + 8/25*a0^3*a3*a4*a5^2*x1
+ 4/25*a0^2*a1^2*a5^3*x1 - 2/5*a0^3*a2*a5^3*x1
```

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```
>>> quintic.beta_covariant(as_form=True)
Binary monic given by ...
```

canonical_form(*reduce_gcd=False*)

Return a canonical representative of the quintic.

Given a binary quintic f with coefficients in a field K, returns a canonical representative of the $GL(2, \bar{K})$-orbit of the quintic, where $\bar{K}$ is an algebraic closure of K. This means that two binary quintics f and g are $GL(2, \bar{K})$-equivalent if and only if their canonical forms are the same.

INPUT:

- `reduce_gcd` – if set to `True`, then a variant of this canonical form is computed where the coefficients are coprime integers. The obtained form is then unique up to multiplication by a unit. See also `binary_quintic_from_invariants()`'.

OUTPUT:

A canonical $GL(2, \bar{K})$-equivalent binary quintic.

EXAMPLES:

```
sage: R.<x0, x1> = QQ[]
sage: p = 2*x1^5 + 4*x1^4*x0 + 5*x1^3*x0^2 + 7*x1^2*x0^3 - 11*x1*x0^4 + x0^5
sage: f = invariant_theory.binary_quintic(p, x0, x1)
sage: g = matrix(QQ, [[11,5],[7,2]])
sage: gf = f.transformed(g)
sage: f.canonical_form() == gf.canonical_form()
True
sage: h = f.canonical_form(reduce_gcd=True)
sage: gcd(h.coeffs())
1
```

```
>>> from sage.all import *
>>> R = QQ['x0, x1']; (x0, x1,) = R._first_ngens(2)
>>> p = Integer(2)*x1**Integer(5) + Integer(4)*x1**Integer(4)*x0 +␣
↪Integer(5)*x1**Integer(3)*x0**Integer(2) +␣
↪Integer(7)*x1**Integer(2)*x0**Integer(3) - Integer(11)*x1*x0**Integer(4) +␣
↪x0**Integer(5)
>>> f = invariant_theory.binary_quintic(p, x0, x1)
>>> g = matrix(QQ, [[Integer(11),Integer(5)],[Integer(7),Integer(2)]])
>>> gf = f.transformed(g)
>>> f.canonical_form() == gf.canonical_form()
True
>>> h = f.canonical_form(reduce_gcd=True)
>>> gcd(h.coeffs())
1
```

clebsch_invariants(*as_tuple=False*)

Return the invariants of a binary quintic as described by Clebsch.

The following invariants are returned: A, B, C and R.

OUTPUT: the Clebsch invariants of the binary quintic

EXAMPLES:

```
sage: R.<x0, x1> = QQ[]
sage: p = 2*x1^5 + 4*x1^4*x0 + 5*x1^3*x0^2 + 7*x1^2*x0^3 - 11*x1*x0^4 + x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.clebsch_invariants()
{'A': -276032/625,
 'B': 4983526016/390625,
 'C': -247056495846408/244140625,
 'R': -148978972828696847376/30517578125}

sage: quintic.clebsch_invariants(as_tuple=True)
(-276032/625,
 4983526016/390625,
 -247056495846408/244140625,
 -148978972828696847376/30517578125)
```

```
>>> from sage.all import *
>>> R = QQ['x0, x1']; (x0, x1,) = R._first_ngens(2)
>>> p = Integer(2)*x1**Integer(5) + Integer(4)*x1**Integer(4)*x0 +␣
↪Integer(5)*x1**Integer(3)*x0**Integer(2) +␣
↪Integer(7)*x1**Integer(2)*x0**Integer(3) - Integer(11)*x1*x0**Integer(4) +␣
↪x0**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x0, x1)
>>> quintic.clebsch_invariants()
{'A': -276032/625,
 'B': 4983526016/390625,
 'C': -247056495846408/244140625,
 'R': -148978972828696847376/30517578125}

>>> quintic.clebsch_invariants(as_tuple=True)
(-276032/625,
 4983526016/390625,
 -247056495846408/244140625,
 -148978972828696847376/30517578125)
```

coeffs()

The coefficients of a binary quintic.

Given

$$f(x) = a_0 x_1^5 + a_1 x_0 x_1^4 + a_2 x_0^2 x_1^3 + a_3 x_0^3 x_1^2 + a_4 x_0^4 x_1 + a_5 x_1^5$$

this function returns $a = (a_0, a_1, a_2, a_3, a_4, a_5)$

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 +␣
↪a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.coeffs()
(a0, a1, a2, a3, a4, a5)

sage: R.<a0, a1, a2, a3, a4, a5, x> = QQ[]
sage: p = a0 + a1*x + a2*x^2 + a3*x^3 + a4*x^4 + a5*x^5
```

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```
sage: quintic = invariant_theory.binary_quintic(p, x)
sage: quintic.coeffs()
(a0, a1, a2, a3, a4, a5)
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, a5, x0, x1']; (a0, a1, a2, a3, a4, a5, x0, x1,
↪) = R._first_ngens(8)
>>> p = a0*x1**Integer(5) + a1*x1**Integer(4)*x0 +␣
↪a2*x1**Integer(3)*x0**Integer(2) + a3*x1**Integer(2)*x0**Integer(3) +␣
↪a4*x1*x0**Integer(4) + a5*x0**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x0, x1)
>>> quintic.coeffs()
(a0, a1, a2, a3, a4, a5)

>>> R = QQ['a0, a1, a2, a3, a4, a5, x']; (a0, a1, a2, a3, a4, a5, x,) = R._
↪first_ngens(7)
>>> p = a0 + a1*x + a2*x**Integer(2) + a3*x**Integer(3) + a4*x**Integer(4) +␣
↪a5*x**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x)
>>> quintic.coeffs()
(a0, a1, a2, a3, a4, a5)
```

delta_covariant(*as_form=False*)

Return the covariant δ of a binary quintic.

INPUT:

- `as_form` – if `as_form` is `False`, the result will be returned as polynomial (default). If it is `True` the result is returned as an object of the class *`AlgebraicForm`*.

OUTPUT:

The δ-covariant of the binary quintic as polynomial or as binary form.

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 +␣
↪a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.delta_covariant()
1/1562500000*a2^6*a3^7*x0 - 9/1562500000*a1*a2^4*a3^8*x0
+ 9/625000000*a1^2*a2^2*a3^9*x0 + 9/781250000*a0*a2^3*a3^9*x0
- 9/1562500000*a1^3*a3^10*x0 - 81/1562500000*a0*a1*a2*a3^10*x0
...
+ 64/3125*a0^3*a1^3*a2^2*a5^5*x1 - 12/625*a0^4*a1*a2^3*a5^5*x1
+ 16/3125*a0^3*a1^4*a3*a5^5*x1 - 16/625*a0^4*a1^2*a2*a3*a5^5*x1
+ 4/125*a0^5*a2^2*a3*a5^5*x1

sage: quintic.delta_covariant(as_form=True)
Binary monic given by ...
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, a5, x0, x1']; (a0, a1, a2, a3, a4, a5, x0, x1,
```

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```
↪) = R._first_ngens(8)
>>> p = a0*x1**Integer(5) + a1*x1**Integer(4)*x0 +␣
↪a2*x1**Integer(3)*x0**Integer(2) + a3*x1**Integer(2)*x0**Integer(3) +␣
↪a4*x1*x0**Integer(4) + a5*x0**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x0, x1)
>>> quintic.delta_covariant()
1/1562500000*a2^6*a3^7*x0 - 9/1562500000*a1*a2^4*a3^8*x0
+ 9/625000000*a1^2*a2^2*a3^9*x0 + 9/781250000*a0*a2^3*a3^9*x0
- 9/1562500000*a1^3*a3^10*x0 - 81/1562500000*a0*a1*a2*a3^10*x0
...
+ 64/3125*a0^3*a1^3*a2^2*a5^5*x1 - 12/625*a0^4*a1*a2^3*a5^5*x1
+ 16/3125*a0^3*a1^4*a3*a5^5*x1 - 16/625*a0^4*a1^2*a2*a3*a5^5*x1
+ 4/125*a0^5*a2^2*a3*a5^5*x1

>>> quintic.delta_covariant(as_form=True)
Binary monic given by ...
```

classmethod from_invariants(*invariants*, *x*, *z*, **args*, ***kwargs*)

Construct a binary quintic from its invariants.

This function constructs a binary quintic whose invariants equal the ones provided as argument up to scaling.

INPUT:

- `invariants` – list or tuple of invariants that are used to reconstruct the binary quintic

OUTPUT: a BinaryQuintic

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: from sage.rings.invariants.invariant_theory import BinaryQuintic
sage: BinaryQuintic.from_invariants([3,6,12], x, y)
Binary quintic with coefficients (0, 1, 0, 0, 1, 0)
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> from sage.rings.invariants.invariant_theory import BinaryQuintic
>>> BinaryQuintic.from_invariants([Integer(3),Integer(6),Integer(12)], x, y)
Binary quintic with coefficients (0, 1, 0, 0, 1, 0)
```

gamma_covariant(*as_form=False*)

Return the covariant γ of a binary quintic.

INPUT:

- `as_form` – if `as_form` is `False`, the result will be returned as polynomial (default). If it is `True` the result is returned as an object of the class *AlgebraicForm*.

OUTPUT:

The γ-covariant of the binary quintic as polynomial or as binary form.

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 +␣
```

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```
↪a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.gamma_covariant()
1/156250000*a2^5*a3^6*x0 - 3/62500000*a1*a2^3*a3^7*x0
+ 27/312500000*a1^2*a2*a3^8*x0 + 27/312500000*a0*a2^2*a3^8*x0
- 81/312500000*a0*a1*a3^9*x0 - 19/312500000*a2^6*a3^4*a4*x0
...
- 32/3125*a0^2*a1^3*a2^2*a5^4*x1 + 6/625*a0^3*a1*a2^3*a5^4*x1
- 8/3125*a0^2*a1^4*a3*a5^4*x1 + 8/625*a0^3*a1^2*a2*a3*a5^4*x1
- 2/125*a0^4*a2^2*a3*a5^4*x1

sage: quintic.gamma_covariant(as_form=True)
Binary monic given by ...
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, a5, x0, x1']; (a0, a1, a2, a3, a4, a5, x0, x1,
↪) = R._first_ngens(8)
>>> p = a0*x1**Integer(5) + a1*x1**Integer(4)*x0 +␣
↪a2*x1**Integer(3)*x0**Integer(2) + a3*x1**Integer(2)*x0**Integer(3) +␣
↪a4*x1*x0**Integer(4) + a5*x0**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x0, x1)
>>> quintic.gamma_covariant()
1/156250000*a2^5*a3^6*x0 - 3/62500000*a1*a2^3*a3^7*x0
+ 27/312500000*a1^2*a2*a3^8*x0 + 27/312500000*a0*a2^2*a3^8*x0
- 81/312500000*a0*a1*a3^9*x0 - 19/312500000*a2^6*a3^4*a4*x0
...
- 32/3125*a0^2*a1^3*a2^2*a5^4*x1 + 6/625*a0^3*a1*a2^3*a5^4*x1
- 8/3125*a0^2*a1^4*a3*a5^4*x1 + 8/625*a0^3*a1^2*a2*a3*a5^4*x1
- 2/125*a0^4*a2^2*a3*a5^4*x1

>>> quintic.gamma_covariant(as_form=True)
Binary monic given by ...
```

i_covariant(*as_form=False*)

Return the covariant i of a binary quintic.

INPUT:

- `as_form` – if `as_form` is `False`, the result will be returned as polynomial (default). If it is `True` the result is returned as an object of the class *AlgebraicForm*.

OUTPUT: the i-covariant of the binary quintic as polynomial or as binary form

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 +␣
↪a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.i_covariant()
3/50*a3^2*x0^2 - 4/25*a2*a4*x0^2 + 2/5*a1*a5*x0^2 + 1/25*a2*a3*x0*x1
- 6/25*a1*a4*x0*x1 + 2*a0*a5*x0*x1 + 3/50*a2^2*x1^2 - 4/25*a1*a3*x1^2
+ 2/5*a0*a4*x1^2
```

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```
sage: quintic.i_covariant(as_form=True)
Binary quadratic given by ...
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, a5, x0, x1']; (a0, a1, a2, a3, a4, a5, x0, x1,
↪) = R._first_ngens(8)
>>> p = a0*x1**Integer(5) + a1*x1**Integer(4)*x0 +␣
↪a2*x1**Integer(3)*x0**Integer(2) + a3*x1**Integer(2)*x0**Integer(3) +␣
↪a4*x1*x0**Integer(4) + a5*x0**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x0, x1)
>>> quintic.i_covariant()
3/50*a3^2*x0^2 - 4/25*a2*a4*x0^2 + 2/5*a1*a5*x0^2 + 1/25*a2*a3*x0*x1
- 6/25*a1*a4*x0*x1 + 2*a0*a5*x0*x1 + 3/50*a2^2*x1^2 - 4/25*a1*a3*x1^2
+ 2/5*a0*a4*x1^2

>>> quintic.i_covariant(as_form=True)
Binary quadratic given by ...
```

invariants(*type='clebsch'*)

Return a tuple of invariants of a binary quintic.

INPUT:

- `type` – the type of invariants to return; the default choice is to return the Clebsch invariants

OUTPUT: the invariants of the binary quintic

EXAMPLES:

```
sage: R.<x0, x1> = QQ[]
sage: p = 2*x1^5 + 4*x1^4*x0 + 5*x1^3*x0^2 + 7*x1^2*x0^3 - 11*x1*x0^4 + x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.invariants()
(-276032/625,
 4983526016/390625,
 -247056495846408/244140625,
 -148978972828696847376/30517578125)
sage: quintic.invariants('unknown')
Traceback (most recent call last):
...
ValueError: unknown type of invariants unknown for a binary quintic
```

```
>>> from sage.all import *
>>> R = QQ['x0, x1']; (x0, x1,) = R._first_ngens(2)
>>> p = Integer(2)*x1**Integer(5) + Integer(4)*x1**Integer(4)*x0 +␣
↪Integer(5)*x1**Integer(3)*x0**Integer(2) +␣
↪Integer(7)*x1**Integer(2)*x0**Integer(3) - Integer(11)*x1*x0**Integer(4) +␣
↪x0**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x0, x1)
>>> quintic.invariants()
(-276032/625,
 4983526016/390625,
 -247056495846408/244140625,
```

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```
 -148978972828696847376/30517578125)
>>> quintic.invariants('unknown')
Traceback (most recent call last):
...
ValueError: unknown type of invariants unknown for a binary quintic
```

j_covariant(*as_form=False*)

Return the covariant j of a binary quintic.

INPUT:

- `as_form` – if `as_form` is `False`, the result will be returned as polynomial (default). If it is `True` the result is returned as an object of the class *AlgebraicForm*.

OUTPUT: the j-covariant of the binary quintic as polynomial or as binary form

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 +␣
↪a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.j_covariant()
-3/500*a3^3*x0^3 + 3/125*a2*a3*a4*x0^3 - 6/125*a1*a4^2*x0^3
- 3/50*a2^2*a5*x0^3 + 3/25*a1*a3*a5*x0^3 - 3/500*a2*a3^2*x0^2*x1
+ 3/250*a2^2*a4*x0^2*x1 + 3/125*a1*a3*a4*x0^2*x1 - 6/25*a0*a4^2*x0^2*x1
- 3/25*a1*a2*a5*x0^2*x1 + 3/5*a0*a3*a5*x0^2*x1 - 3/500*a2^2*a3*x0*x1^2
+ 3/250*a1*a3^2*x0*x1^2 + 3/125*a1*a2*a4*x0*x1^2 - 3/25*a0*a3*a4*x0*x1^2
- 6/25*a1^2*a5*x0*x1^2 + 3/5*a0*a2*a5*x0*x1^2 - 3/500*a2^3*x1^3
+ 3/125*a1*a2*a3*x1^3 - 3/50*a0*a3^2*x1^3 - 6/125*a1^2*a4*x1^3
+ 3/25*a0*a2*a4*x1^3

sage: quintic.j_covariant(as_form=True)
Binary cubic given by ...
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, a5, x0, x1']; (a0, a1, a2, a3, a4, a5, x0, x1,
↪) = R._first_ngens(8)
>>> p = a0*x1**Integer(5) + a1*x1**Integer(4)*x0 +␣
↪a2*x1**Integer(3)*x0**Integer(2) + a3*x1**Integer(2)*x0**Integer(3) +␣
↪a4*x1*x0**Integer(4) + a5*x0**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x0, x1)
>>> quintic.j_covariant()
-3/500*a3^3*x0^3 + 3/125*a2*a3*a4*x0^3 - 6/125*a1*a4^2*x0^3
- 3/50*a2^2*a5*x0^3 + 3/25*a1*a3*a5*x0^3 - 3/500*a2*a3^2*x0^2*x1
+ 3/250*a2^2*a4*x0^2*x1 + 3/125*a1*a3*a4*x0^2*x1 - 6/25*a0*a4^2*x0^2*x1
- 3/25*a1*a2*a5*x0^2*x1 + 3/5*a0*a3*a5*x0^2*x1 - 3/500*a2^2*a3*x0*x1^2
+ 3/250*a1*a3^2*x0*x1^2 + 3/125*a1*a2*a4*x0*x1^2 - 3/25*a0*a3*a4*x0*x1^2
- 6/25*a1^2*a5*x0*x1^2 + 3/5*a0*a2*a5*x0*x1^2 - 3/500*a2^3*x1^3
+ 3/125*a1*a2*a3*x1^3 - 3/50*a0*a3^2*x1^3 - 6/125*a1^2*a4*x1^3
+ 3/25*a0*a2*a4*x1^3

>>> quintic.j_covariant(as_form=True)
Binary cubic given by ...
```

monomials()

List the basis monomials of the form.

This function lists a basis of monomials of the space of binary quintics of which this form is an element.

OUTPUT:

A tuple of monomials. They are in the same order as `coeffs()`.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: quintic = invariant_theory.binary_quintic(x^5 + y^5)
sage: quintic.monomials()
(y^5, x*y^4, x^2*y^3, x^3*y^2, x^4*y, x^5)
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> quintic = invariant_theory.binary_quintic(x**Integer(5) + y**Integer(5))
>>> quintic.monomials()
(y^5, x*y^4, x^2*y^3, x^3*y^2, x^4*y, x^5)
```

scaled_coeffs()

The coefficients of a binary quintic.

Given

$$f(x) = a_0 x_1^5 + 5a_1 x_0 x_1^4 + 10a_2 x_0^2 x_1^3 + 10a_3 x_0^3 x_1^2 + 5a_4 x_0^4 x_1 + a_5 x_1^5$$

this function returns $a = (a_0, a_1, a_2, a_3, a_4, a_5)$

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + 5*a1*x1^4*x0 + 10*a2*x1^3*x0^2 + 10*a3*x1^2*x0^3 +␣
↪5*a4*x1*x0^4 + a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.scaled_coeffs()
(a0, a1, a2, a3, a4, a5)

sage: R.<a0, a1, a2, a3, a4, a5, x> = QQ[]
sage: p = a0 + 5*a1*x + 10*a2*x^2 + 10*a3*x^3 + 5*a4*x^4 + a5*x^5
sage: quintic = invariant_theory.binary_quintic(p, x)
sage: quintic.scaled_coeffs()
(a0, a1, a2, a3, a4, a5)
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, a5, x0, x1']; (a0, a1, a2, a3, a4, a5, x0, x1,
↪) = R._first_ngens(8)
>>> p = a0*x1**Integer(5) + Integer(5)*a1*x1**Integer(4)*x0 +␣
↪Integer(10)*a2*x1**Integer(3)*x0**Integer(2) +␣
↪Integer(10)*a3*x1**Integer(2)*x0**Integer(3) +␣
↪Integer(5)*a4*x1*x0**Integer(4) + a5*x0**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x0, x1)
>>> quintic.scaled_coeffs()
(a0, a1, a2, a3, a4, a5)
```

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```
>>> R = QQ['a0, a1, a2, a3, a4, a5, x']; (a0, a1, a2, a3, a4, a5, x,) = R._
↪first_ngens(7)
>>> p = a0 + Integer(5)*a1*x + Integer(10)*a2*x**Integer(2) +␣
↪Integer(10)*a3*x**Integer(3) + Integer(5)*a4*x**Integer(4) +␣
↪a5*x**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x)
>>> quintic.scaled_coeffs()
(a0, a1, a2, a3, a4, a5)
```

tau_covariant(*as_form=False*)

Return the covariant τ of a binary quintic.

INPUT:

- `as_form` – if `as_form` is `False`, the result will be returned as polynomial (default). If it is `True` the result is returned as an object of the class *AlgebraicForm*.

OUTPUT:

The τ-covariant of the binary quintic as polynomial or as binary form.

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 +␣
↪a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.tau_covariant()
1/62500*a2^2*a3^4*x0^2 - 3/62500*a1*a3^5*x0^2
- 1/15625*a2^3*a3^2*a4*x0^2 + 1/6250*a1*a2*a3^3*a4*x0^2
+ 3/6250*a0*a3^4*a4*x0^2 - 1/31250*a2^4*a4^2*x0^2
...
- 2/125*a0*a1*a2^2*a4*a5*x1^2 - 4/125*a0*a1^2*a3*a4*a5*x1^2
+ 2/25*a0^2*a2*a3*a4*a5*x1^2 - 8/625*a1^4*a5^2*x1^2
+ 8/125*a0*a1^2*a2*a5^2*x1^2 - 2/25*a0^2*a2^2*a5^2*x1^2

sage: quintic.tau_covariant(as_form=True)
Binary quadratic given by ...
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, a5, x0, x1']; (a0, a1, a2, a3, a4, a5, x0, x1,
↪) = R._first_ngens(8)
>>> p = a0*x1**Integer(5) + a1*x1**Integer(4)*x0 +␣
↪a2*x1**Integer(3)*x0**Integer(2) + a3*x1**Integer(2)*x0**Integer(3) +␣
↪a4*x1*x0**Integer(4) + a5*x0**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x0, x1)
>>> quintic.tau_covariant()
1/62500*a2^2*a3^4*x0^2 - 3/62500*a1*a3^5*x0^2
- 1/15625*a2^3*a3^2*a4*x0^2 + 1/6250*a1*a2*a3^3*a4*x0^2
+ 3/6250*a0*a3^4*a4*x0^2 - 1/31250*a2^4*a4^2*x0^2
...
- 2/125*a0*a1*a2^2*a4*a5*x1^2 - 4/125*a0*a1^2*a3*a4*a5*x1^2
+ 2/25*a0^2*a2*a3*a4*a5*x1^2 - 8/625*a1^4*a5^2*x1^2
```

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```
+ 8/125*a0*a1^2*a2*a5^2*x1^2 - 2/25*a0^2*a2^2*a5^2*x1^2

>>> quintic.tau_covariant(as_form=True)
Binary quadratic given by ...
```

theta_covariant(*as_form=False*)

Return the covariant θ of a binary quintic.

INPUT:

- `as_form` – if `as_form` is `False`, the result will be returned as polynomial (default). If it is `True` the result is returned as an object of the class *AlgebraicForm*.

OUTPUT:

The θ-covariant of the binary quintic as polynomial or as binary form.

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 +␣
↪a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.theta_covariant()
-1/625000*a2^3*a3^5*x0^2 + 9/1250000*a1*a2*a3^6*x0^2
- 27/1250000*a0*a3^7*x0^2 + 3/250000*a2^4*a3^3*a4*x0^2
- 7/125000*a1*a2^2*a3^4*a4*x0^2 - 3/312500*a1^2*a3^5*a4*x0^2
...
+ 6/625*a0^2*a1*a2^2*a4*a5^2*x1^2 + 24/625*a0^2*a1^2*a3*a4*a5^2*x1^2
- 12/125*a0^3*a2*a3*a4*a5^2*x1^2 + 8/625*a0*a1^4*a5^3*x1^2
- 8/125*a0^2*a1^2*a2*a5^3*x1^2 + 2/25*a0^3*a2^2*a5^3*x1^2

sage: quintic.theta_covariant(as_form=True)
Binary quadratic given by ...
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, a5, x0, x1']; (a0, a1, a2, a3, a4, a5, x0, x1,
↪) = R._first_ngens(8)
>>> p = a0*x1**Integer(5) + a1*x1**Integer(4)*x0 +␣
↪a2*x1**Integer(3)*x0**Integer(2) + a3*x1**Integer(2)*x0**Integer(3) +␣
↪a4*x1*x0**Integer(4) + a5*x0**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x0, x1)
>>> quintic.theta_covariant()
-1/625000*a2^3*a3^5*x0^2 + 9/1250000*a1*a2*a3^6*x0^2
- 27/1250000*a0*a3^7*x0^2 + 3/250000*a2^4*a3^3*a4*x0^2
- 7/125000*a1*a2^2*a3^4*a4*x0^2 - 3/312500*a1^2*a3^5*a4*x0^2
...
+ 6/625*a0^2*a1*a2^2*a4*a5^2*x1^2 + 24/625*a0^2*a1^2*a3*a4*a5^2*x1^2
- 12/125*a0^3*a2*a3*a4*a5^2*x1^2 + 8/625*a0*a1^4*a5^3*x1^2
- 8/125*a0^2*a1^2*a2*a5^3*x1^2 + 2/25*a0^3*a2^2*a5^3*x1^2

>>> quintic.theta_covariant(as_form=True)
Binary quadratic given by ...
```

class sage.rings.invariants.invariant_theory.**FormsBase**(*n*, *homogeneous*, *ring*, *variables*)

Bases: SageObject

The common base class of *AlgebraicForm* and *SeveralAlgebraicForms*.

This is an abstract base class to provide common methods. It does not make much sense to instantiate it.

is_homogeneous()

Return whether the forms were defined by homogeneous polynomials.

OUTPUT: boolean; whether the user originally defined the form via homogeneous variables

EXAMPLES:

```
sage: R.<x,y,t> = QQ[]
sage: quartic = invariant_theory.binary_quartic(x^4 + y^4 + t*x^2*y^2, [x,y])
sage: quartic.is_homogeneous()
True
sage: quartic.form()
x^2*y^2*t + x^4 + y^4

sage: R.<x,y,t> = QQ[]
sage: quartic = invariant_theory.binary_quartic(x^4 + 1 + t*x^2, [x])
sage: quartic.is_homogeneous()
False
sage: quartic.form()
x^4 + x^2*t + 1
```

```
>>> from sage.all import *
>>> R = QQ['x, y, t']; (x, y, t,) = R._first_ngens(3)
>>> quartic = invariant_theory.binary_quartic(x**Integer(4) + y**Integer(4) +␣
↪t*x**Integer(2)*y**Integer(2), [x,y])
>>> quartic.is_homogeneous()
True
>>> quartic.form()
x^2*y^2*t + x^4 + y^4

>>> R = QQ['x, y, t']; (x, y, t,) = R._first_ngens(3)
>>> quartic = invariant_theory.binary_quartic(x**Integer(4) + Integer(1) +␣
↪t*x**Integer(2), [x])
>>> quartic.is_homogeneous()
False
>>> quartic.form()
x^4 + x^2*t + 1
```

ring()

Return the polynomial ring.

OUTPUT:

A polynomial ring. This is where the defining polynomial(s) live. Note that the polynomials may be homogeneous or inhomogeneous, depending on how the user constructed the object.

EXAMPLES:

```
sage: R.<x,y,t> = QQ[]
sage: quartic = invariant_theory.binary_quartic(x^4 + y^4 + t*x^2*y^2, [x,y])
```

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```
sage: quartic.ring()
Multivariate Polynomial Ring in x, y, t over Rational Field

sage: R.<x,y,t> = QQ[]
sage: quartic = invariant_theory.binary_quartic(x^4 + 1 + t*x^2, [x])
sage: quartic.ring()
Multivariate Polynomial Ring in x, y, t over Rational Field
```

```
>>> from sage.all import *
>>> R = QQ['x, y, t']; (x, y, t,) = R._first_ngens(3)
>>> quartic = invariant_theory.binary_quartic(x**Integer(4) + y**Integer(4) +
↪t*x**Integer(2)*y**Integer(2), [x,y])
>>> quartic.ring()
Multivariate Polynomial Ring in x, y, t over Rational Field

>>> R = QQ['x, y, t']; (x, y, t,) = R._first_ngens(3)
>>> quartic = invariant_theory.binary_quartic(x**Integer(4) + Integer(1) +
↪t*x**Integer(2), [x])
>>> quartic.ring()
Multivariate Polynomial Ring in x, y, t over Rational Field
```

variables()

Return the variables of the form.

OUTPUT:

A tuple of variables. If inhomogeneous notation is used for the defining polynomial then the last entry will be `None`.

EXAMPLES:

```
sage: R.<x,y,t> = QQ[]
sage: quartic = invariant_theory.binary_quartic(x^4 + y^4 + t*x^2*y^2, [x,y])
sage: quartic.variables()
(x, y)

sage: R.<x,y,t> = QQ[]
sage: quartic = invariant_theory.binary_quartic(x^4 + 1 + t*x^2, [x])
sage: quartic.variables()
(x, None)
```

```
>>> from sage.all import *
>>> R = QQ['x, y, t']; (x, y, t,) = R._first_ngens(3)
>>> quartic = invariant_theory.binary_quartic(x**Integer(4) + y**Integer(4) +
↪t*x**Integer(2)*y**Integer(2), [x,y])
>>> quartic.variables()
(x, y)

>>> R = QQ['x, y, t']; (x, y, t,) = R._first_ngens(3)
>>> quartic = invariant_theory.binary_quartic(x**Integer(4) + Integer(1) +
↪t*x**Integer(2), [x])
>>> quartic.variables()
(x, None)
```

class sage.rings.invariants.invariant_theory.**InvariantTheoryFactory**

Bases: object

Factory object for invariants of multilinear forms.

Use the invariant_theory object to construct algebraic forms. These can then be queried for invariant and covariants.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: invariant_theory.ternary_cubic(x^3 + y^3 + z^3)
Ternary cubic with coefficients (1, 1, 1, 0, 0, 0, 0, 0, 0, 0)

sage: invariant_theory.ternary_cubic(x^3 + y^3 + z^3).J_covariant()
x^6*y^3 - x^3*y^6 - x^6*z^3 + y^6*z^3 + x^3*z^6 - y^3*z^6
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> invariant_theory.ternary_cubic(x**Integer(3) + y**Integer(3) + z**Integer(3))
Ternary cubic with coefficients (1, 1, 1, 0, 0, 0, 0, 0, 0, 0)

>>> invariant_theory.ternary_cubic(x**Integer(3) + y**Integer(3) + z**Integer(3)).
↪J_covariant()
x^6*y^3 - x^3*y^6 - x^6*z^3 + y^6*z^3 + x^3*z^6 - y^3*z^6
```

binary_form_from_invariants(*degree*, *invariants*, *variables=None*, *as_form=True*, **args*, ***kwargs*)

Reconstruct a binary form from the values of its invariants.

INPUT:

- degree – the degree of the binary form
- invariants – list or tuple of values of the invariants of the binary form
- variables – list or tuple of two variables that are used for the resulting form (only if as_form is True). If no variables are provided, two abstract variables x and z will be used.
- as_form – boolean; if False, the function will return a tuple of coefficients of a binary form

OUTPUT:

A binary form or a tuple of its coefficients, whose invariants are equal to the given invariants up to a scaling.

EXAMPLES:

In the case of binary quadratics and cubics, the form is reconstructed based on the value of the discriminant. See also binary_quadratic_coefficients_from_invariants() and binary_cubic_coefficients_from_invariants(). These methods will always return the same result if the discriminant is nonzero:

```
sage: discriminant = 1
sage: invariant_theory.binary_form_from_invariants(2, [discriminant])
Binary quadratic with coefficients (1, -1/4, 0)
sage: invariant_theory.binary_form_from_invariants(3, [discriminant], as_
↪form=false)
(0, 1, -1, 0)
```

```
>>> from sage.all import *
>>> discriminant = Integer(1)
>>> invariant_theory.binary_form_from_invariants(Integer(2), [discriminant])
Binary quadratic with coefficients (1, -1/4, 0)
>>> invariant_theory.binary_form_from_invariants(Integer(3), [discriminant],␣
↪as_form=false)
(0, 1, -1, 0)
```

For binary cubics, there is no class implemented yet, so `as_form=True` will yield a `NotImplementedError`:

```
sage: invariant_theory.binary_form_from_invariants(3, [discriminant])
Traceback (most recent call last):
...
NotImplementedError: no class for binary cubics implemented
```

```
>>> from sage.all import *
>>> invariant_theory.binary_form_from_invariants(Integer(3), [discriminant])
Traceback (most recent call last):
...
NotImplementedError: no class for binary cubics implemented
```

For binary quintics, the three Clebsch invariants of the form should be provided to reconstruct the form. For more details about these invariants, see `clebsch_invariants()`:

```
sage: invariants = [1, 0, 0]
sage: invariant_theory.binary_form_from_invariants(5, invariants)
Binary quintic with coefficients (1, 0, 0, 0, 0, 1)
```

```
>>> from sage.all import *
>>> invariants = [Integer(1), Integer(0), Integer(0)]
>>> invariant_theory.binary_form_from_invariants(Integer(5), invariants)
Binary quintic with coefficients (1, 0, 0, 0, 0, 1)
```

An optional `scaling` argument may be provided in order to scale the resulting quintic. For more details, see `binary_quintic_coefficients_from_invariants()`:

```
sage: invariants = [3, 4, 7]
sage: invariant_theory.binary_form_from_invariants(5, invariants)
Binary quintic with coefficients (-37725479487783/1048576,
565882192316745/8388608, 0, 1033866765362693115/67108864,
12849486940936328715/268435456, -23129076493685391687/2147483648)
sage: invariant_theory.binary_form_from_invariants(5, invariants,
....:                                               scaling='normalized')
Binary quintic with coefficients (24389/892616806656,
4205/11019960576, 0, 1015/209952, -145/1296, -3/16)
sage: invariant_theory.binary_form_from_invariants(5, invariants,
....:                                                  scaling='coprime')
Binary quintic with coefficients (-2048, 3840, 0, 876960, 2724840, -613089)
```

```
>>> from sage.all import *
>>> invariants = [Integer(3), Integer(4), Integer(7)]
```

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```
>>> invariant_theory.binary_form_from_invariants(Integer(5), invariants)
Binary quintic with coefficients (-37725479487783/1048576,
565882192316745/8388608, 0, 1033866765362693115/67108864,
12849486940936328715/268435456, -23129076493685391687/2147483648)
>>> invariant_theory.binary_form_from_invariants(Integer(5), invariants,
...                                               scaling='normalized')
Binary quintic with coefficients (24389/892616806656,
4205/11019960576, 0, 1015/209952, -145/1296, -3/16)
>>> invariant_theory.binary_form_from_invariants(Integer(5), invariants,
...                                                  scaling='coprime')
Binary quintic with coefficients (-2048, 3840, 0, 876960, 2724840, -613089)
```

The invariants can also be computed using the invariants of a given binary quintic. The resulting form has the same invariants up to scaling, is $GL(2, \mathbf{Q})$-equivalent to the provided form and hence has the same canonical form (see `canonical_form()`):

```
sage: R.<x0, x1> = QQ[]
sage: p = 3*x1^5 + 6*x1^4*x0 + 3*x1^3*x0^2 + 4*x1^2*x0^3 - 5*x1*x0^4 + 4*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: invariants = quintic.clebsch_invariants(as_tuple=True)
sage: newquintic = invariant_theory.binary_form_from_invariants(
....:                     5, invariants, variables=quintic.variables())
sage: newquintic
Binary quintic with coefficients (9592267437341790539005557/244140625000000,
2149296928207625556323004064707/610351562500000000,
11149651890347700974453304786783/76293945312500000,
122650775751894638395648891202734239/47683715820312500000,
323996630945706528474286334593218447/11920928955078125000,
1504506503644608395841632538558481466127/14901161193847656250000)
sage: quintic.canonical_form() == newquintic.canonical_form()
True
```

```
>>> from sage.all import *
>>> R = QQ['x0, x1']; (x0, x1,) = R._first_ngens(2)
>>> p = Integer(3)*x1**Integer(5) + Integer(6)*x1**Integer(4)*x0 +␣
↪Integer(3)*x1**Integer(3)*x0**Integer(2) +␣
↪Integer(4)*x1**Integer(2)*x0**Integer(3) - Integer(5)*x1*x0**Integer(4) +␣
↪Integer(4)*x0**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x0, x1)
>>> invariants = quintic.clebsch_invariants(as_tuple=True)
>>> newquintic = invariant_theory.binary_form_from_invariants(
...                     Integer(5), invariants, variables=quintic.variables())
>>> newquintic
Binary quintic with coefficients (9592267437341790539005557/244140625000000,
2149296928207625556323004064707/610351562500000000,
11149651890347700974453304786783/76293945312500000,
122650775751894638395648891202734239/47683715820312500000,
323996630945706528474286334593218447/11920928955078125000,
1504506503644608395841632538558481466127/14901161193847656250000)
>>> quintic.canonical_form() == newquintic.canonical_form()
True
```

For binary forms of other degrees, no reconstruction has been implemented yet. For forms of degree 6, see

Issue #26462:

```
sage: invariant_theory.binary_form_from_invariants(6, invariants)
Traceback (most recent call last):
...
NotImplementedError: no reconstruction for binary forms of degree 6␣
↪implemented
```

```
>>> from sage.all import *
>>> invariant_theory.binary_form_from_invariants(Integer(6), invariants)
Traceback (most recent call last):
...
NotImplementedError: no reconstruction for binary forms of degree 6␣
↪implemented
```

binary_quadratic(*quadratic*, **args*)

Invariant theory of a quadratic in two variables.

INPUT:

- `quadratic` – a quadratic form
- `x`, `y` – the homogeneous variables. If `y` is `None`, the quadratic is assumed to be inhomogeneous.

REFERENCES:

- Wikipedia article Invariant_of_a_binary_form

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: invariant_theory.binary_quadratic(x^2 + y^2)
Binary quadratic with coefficients (1, 1, 0)

sage: T.<t> = QQ[]
sage: invariant_theory.binary_quadratic(t^2 + 2*t + 1, [t])
Binary quadratic with coefficients (1, 1, 2)
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> invariant_theory.binary_quadratic(x**Integer(2) + y**Integer(2))
Binary quadratic with coefficients (1, 1, 0)

>>> T = QQ['t']; (t,) = T._first_ngens(1)
>>> invariant_theory.binary_quadratic(t**Integer(2) + Integer(2)*t +␣
↪Integer(1), [t])
Binary quadratic with coefficients (1, 1, 2)
```

binary_quartic(*quartic*, **args*, ***kwds*)

Invariant theory of a quartic in two variables.

The algebra of invariants of a quartic form is generated by invariants i, j of degrees 2, 3. This ring is naturally isomorphic to the ring of modular forms of level 1, with the two generators corresponding to the Eisenstein series E_4 (see `EisensteinD()`) and E_6 (see `EisensteinE()`). The algebra of covariants is generated by these two invariants together with the form f of degree 1 and order 4, the Hessian g (see `g_covariant()`) of degree 2 and order 4, and a covariant h (see `h_covariant()`) of degree 3 and order 6. They are related by a syzygy

$$jf^3 - gf^2i + 4g^3 + h^2 = 0$$

of degree 6 and order 12.

INPUT:

- `quartic` – a quartic
- `x`, `y` – the homogeneous variables. If `y` is `None`, the quartic is assumed to be inhomogeneous.

REFERENCES:

- Wikipedia article Invariant_of_a_binary_form

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: quartic = invariant_theory.binary_quartic(x^4 + y^4)
sage: quartic
Binary quartic with coefficients (1, 0, 0, 0, 1)
sage: type(quartic)
<class 'sage.rings.invariants.invariant_theory.BinaryQuartic'>
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> quartic = invariant_theory.binary_quartic(x**Integer(4) + y**Integer(4))
>>> quartic
Binary quartic with coefficients (1, 0, 0, 0, 1)
>>> type(quartic)
<class 'sage.rings.invariants.invariant_theory.BinaryQuartic'>
```

binary_quintic(*quintic*, **args*, ***kwds*)

Create a binary quintic for computing invariants.

A binary quintic is a homogeneous polynomial of degree 5 in two variables. The algebra of invariants of a binary quintic is generated by the invariants A, B and C of respective degrees 4, 8 and 12 (see `A_invariant()`, `B_invariant()` and `C_invariant()`).

INPUT:

- `quintic` – a homogeneous polynomial of degree five in two variables or a (possibly inhomogeneous) polynomial of degree at most five in one variable.
- `*args` – the two homogeneous variables. If only one variable is given, the polynomial `quintic` is assumed to be univariate. If no variables are given, they are guessed.

REFERENCES:

- Wikipedia article Invariant_of_a_binary_form
- [Cle1872]

EXAMPLES:

If no variables are provided, they will be guessed:

```
sage: R.<x,y> = QQ[]
sage: quintic = invariant_theory.binary_quintic(x^5 + y^5)
sage: quintic
Binary quintic with coefficients (1, 0, 0, 0, 0, 1)
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> quintic = invariant_theory.binary_quintic(x**Integer(5) + y**Integer(5))
>>> quintic
Binary quintic with coefficients (1, 0, 0, 0, 0, 1)
```

If only one variable is given, the quintic is the homogenisation of the provided polynomial:

```
sage: quintic = invariant_theory.binary_quintic(x^5 + y^5, x)
sage: quintic
Binary quintic with coefficients (y^5, 0, 0, 0, 0, 1)
sage: quintic.is_homogeneous()
False
```

```
>>> from sage.all import *
>>> quintic = invariant_theory.binary_quintic(x**Integer(5) + y**Integer(5),␣
↪x)
>>> quintic
Binary quintic with coefficients (y^5, 0, 0, 0, 0, 1)
>>> quintic.is_homogeneous()
False
```

If the polynomial has three or more variables, the variables should be specified:

```
sage: R.<x,y,z> = QQ[]
sage: quintic = invariant_theory.binary_quintic(x^5 + z*y^5)
Traceback (most recent call last):
...
ValueError: need 2 or 1 variables, got (x, y, z)
sage: quintic = invariant_theory.binary_quintic(x^5 + z*y^5, x, y)
sage: quintic
Binary quintic with coefficients (z, 0, 0, 0, 0, 1)

sage: type(quintic)
<class 'sage.rings.invariants.invariant_theory.BinaryQuintic'>
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> quintic = invariant_theory.binary_quintic(x**Integer(5) + z*y**Integer(5))
Traceback (most recent call last):
...
ValueError: need 2 or 1 variables, got (x, y, z)
>>> quintic = invariant_theory.binary_quintic(x**Integer(5) + z*y**Integer(5),
↪ x, y)
>>> quintic
Binary quintic with coefficients (z, 0, 0, 0, 0, 1)

>>> type(quintic)
<class 'sage.rings.invariants.invariant_theory.BinaryQuintic'>
```

inhomogeneous_quadratic_form(*polynomial*, **args*)

Invariants of an inhomogeneous quadratic form.

INPUT:

- `polynomial` – an inhomogeneous quadratic form
- `*args` – the variables as multiple arguments, or as a single list/tuple

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: quadratic = x^2 + 2*y^2 + 3*x*y + 4*x + 5*y + 6
sage: inv3 = invariant_theory.inhomogeneous_quadratic_form(quadratic)
sage: type(inv3)
<class 'sage.rings.invariants.invariant_theory.TernaryQuadratic'>
sage: inv4 = invariant_theory.inhomogeneous_quadratic_form(x^2 + y^2 + z^2)
sage: type(inv4)
<class 'sage.rings.invariants.invariant_theory.QuadraticForm'>
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> quadratic = x**Integer(2) + Integer(2)*y**Integer(2) + Integer(3)*x*y +␣
↪Integer(4)*x + Integer(5)*y + Integer(6)
>>> inv3 = invariant_theory.inhomogeneous_quadratic_form(quadratic)
>>> type(inv3)
<class 'sage.rings.invariants.invariant_theory.TernaryQuadratic'>
>>> inv4 = invariant_theory.inhomogeneous_quadratic_form(x**Integer(2) +␣
↪y**Integer(2) + z**Integer(2))
>>> type(inv4)
<class 'sage.rings.invariants.invariant_theory.QuadraticForm'>
```

quadratic_form(*polynomial*, **args*)

Invariants of a homogeneous quadratic form.

INPUT:

- `polynomial` – a homogeneous or inhomogeneous quadratic form
- `*args` – the variables as multiple arguments, or as a single list/tuple. If the last argument is `None`, the cubic is assumed to be inhomogeneous.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: quadratic = x^2 + y^2 + z^2
sage: inv = invariant_theory.quadratic_form(quadratic)
sage: type(inv)
<class 'sage.rings.invariants.invariant_theory.TernaryQuadratic'>
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> quadratic = x**Integer(2) + y**Integer(2) + z**Integer(2)
>>> inv = invariant_theory.quadratic_form(quadratic)
>>> type(inv)
<class 'sage.rings.invariants.invariant_theory.TernaryQuadratic'>
```

If some of the ring variables are to be treated as coefficients you need to specify the polynomial variables:

```
sage: R.<x,y,z, a,b> = QQ[]
sage: quadratic = a*x^2 + b*y^2 + z^2 + 2*y*z
```

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```
sage: invariant_theory.quadratic_form(quadratic, x,y,z)
Ternary quadratic with coefficients (a, b, 1, 0, 0, 2)
sage: invariant_theory.quadratic_form(quadratic, [x,y,z])   # alternate syntax
Ternary quadratic with coefficients (a, b, 1, 0, 0, 2)
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z, a, b']; (x, y, z, a, b,) = R._first_ngens(5)
>>> quadratic = a*x**Integer(2) + b*y**Integer(2) + z**Integer(2) +␣
→Integer(2)*y*z
>>> invariant_theory.quadratic_form(quadratic, x,y,z)
Ternary quadratic with coefficients (a, b, 1, 0, 0, 2)
>>> invariant_theory.quadratic_form(quadratic, [x,y,z])   # alternate syntax
Ternary quadratic with coefficients (a, b, 1, 0, 0, 2)
```

Inhomogeneous quadratic forms (see also *inhomogeneous_quadratic_form()*) can be specified by passing None as the last variable:

```
sage: inhom = quadratic.subs(z=1)
sage: invariant_theory.quadratic_form(inhom, x,y,None)
Ternary quadratic with coefficients (a, b, 1, 0, 0, 2)
```

```
>>> from sage.all import *
>>> inhom = quadratic.subs(z=Integer(1))
>>> invariant_theory.quadratic_form(inhom, x,y,None)
Ternary quadratic with coefficients (a, b, 1, 0, 0, 2)
```

quaternary_biquadratic(*quadratic1*, *quadratic2*, **args*, ***kwds*)

Invariants of two quadratics in four variables.

INPUT:

- quadratic1, quadratic2 – two polynomials. Either homogeneous quadratic in 4 homogeneous variables, or inhomogeneous quadratic in 3 variables.
- w, x, y, z – the variables. If z is None, the quadratics are assumed to be inhomogeneous.

EXAMPLES:

```
sage: R.<w,x,y,z> = QQ[]
sage: q1 = w^2 + x^2 + y^2 + z^2
sage: q2 = w*x + y*z
sage: inv = invariant_theory.quaternary_biquadratic(q1, q2)
sage: type(inv)
<class 'sage.rings.invariants.invariant_theory.TwoQuaternaryQuadratics'>
```

```
>>> from sage.all import *
>>> R = QQ['w, x, y, z']; (w, x, y, z,) = R._first_ngens(4)
>>> q1 = w**Integer(2) + x**Integer(2) + y**Integer(2) + z**Integer(2)
>>> q2 = w*x + y*z
>>> inv = invariant_theory.quaternary_biquadratic(q1, q2)
>>> type(inv)
<class 'sage.rings.invariants.invariant_theory.TwoQuaternaryQuadratics'>
```

Distance between two spheres [Sal1958], [Sal1965]

```
sage: R.<x,y,z, a,b,c, r1,r2> = QQ[]
sage: S1 = -r1^2 + x^2 + y^2 + z^2
sage: S2 = -r2^2 + (x-a)^2 + (y-b)^2 + (z-c)^2
sage: inv = invariant_theory.quaternary_biquadratic(S1, S2, [x, y, z])
sage: inv.Delta_invariant()
-r1^2
sage: inv.Delta_prime_invariant()
-r2^2
sage: inv.Theta_invariant()
a^2 + b^2 + c^2 - 3*r1^2 - r2^2
sage: inv.Theta_prime_invariant()
a^2 + b^2 + c^2 - r1^2 - 3*r2^2
sage: inv.Phi_invariant()
2*a^2 + 2*b^2 + 2*c^2 - 3*r1^2 - 3*r2^2
sage: inv.J_covariant()
0
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z, a, b, c, r1, r2']; (x, y, z, a, b, c, r1, r2,) = R._
↪first_ngens(8)
>>> S1 = -r1**Integer(2) + x**Integer(2) + y**Integer(2) + z**Integer(2)
>>> S2 = -r2**Integer(2) + (x-a)**Integer(2) + (y-b)**Integer(2) + (z-
↪c)**Integer(2)
>>> inv = invariant_theory.quaternary_biquadratic(S1, S2, [x, y, z])
>>> inv.Delta_invariant()
-r1^2
>>> inv.Delta_prime_invariant()
-r2^2
>>> inv.Theta_invariant()
a^2 + b^2 + c^2 - 3*r1^2 - r2^2
>>> inv.Theta_prime_invariant()
a^2 + b^2 + c^2 - r1^2 - 3*r2^2
>>> inv.Phi_invariant()
2*a^2 + 2*b^2 + 2*c^2 - 3*r1^2 - 3*r2^2
>>> inv.J_covariant()
0
```

quaternary_quadratic(*quadratic*, **args*)

Invariant theory of a quadratic in four variables.

INPUT:

- `quadratic` – a quadratic form
- `w`, `x`, `y`, `z` – the homogeneous variables. If `z` is `None`, the quadratic is assumed to be inhomogeneous.

REFERENCES:

- Wikipedia article Invariant_of_a_binary_form

EXAMPLES:

```
sage: R.<w,x,y,z> = QQ[]
sage: invariant_theory.quaternary_quadratic(w^2 + x^2 + y^2 + z^2)
Quaternary quadratic with coefficients (1, 1, 1, 1, 0, 0, 0, 0, 0, 0)
```

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```
sage: R.<x,y,z> = QQ[]
sage: invariant_theory.quaternary_quadratic(1 + x^2 + y^2 + z^2)
Quaternary quadratic with coefficients (1, 1, 1, 1, 0, 0, 0, 0, 0, 0)
```

```
>>> from sage.all import *
>>> R = QQ['w, x, y, z']; (w, x, y, z,) = R._first_ngens(4)
>>> invariant_theory.quaternary_quadratic(w**Integer(2) + x**Integer(2) +␣
↪y**Integer(2) + z**Integer(2))
Quaternary quadratic with coefficients (1, 1, 1, 1, 0, 0, 0, 0, 0, 0)

>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> invariant_theory.quaternary_quadratic(Integer(1) + x**Integer(2) +␣
↪y**Integer(2) + z**Integer(2))
Quaternary quadratic with coefficients (1, 1, 1, 1, 0, 0, 0, 0, 0, 0)
```

ternary_biquadratic(*quadratic1*, *quadratic2*, **args*, ***kwds*)

Invariants of two quadratics in three variables.

INPUT:

- `quadratic1`, `quadratic2` – two polynomials. Either homogeneous quadratic in 3 homogeneous variables, or inhomogeneous quadratic in 2 variables.
- `x`, `y`, `z` – the variables. If `z` is `None`, the quadratics are assumed to be inhomogeneous.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: q1 = x^2 + y^2 + z^2
sage: q2 = x*y + y*z + x*z
sage: inv = invariant_theory.ternary_biquadratic(q1, q2)
sage: type(inv)
<class 'sage.rings.invariants.invariant_theory.TwoTernaryQuadratics'>
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> q1 = x**Integer(2) + y**Integer(2) + z**Integer(2)
>>> q2 = x*y + y*z + x*z
>>> inv = invariant_theory.ternary_biquadratic(q1, q2)
>>> type(inv)
<class 'sage.rings.invariants.invariant_theory.TwoTernaryQuadratics'>
```

Distance between two circles:

```
sage: R.<x,y, a,b, r1,r2> = QQ[]
sage: S1 = -r1^2 + x^2 + y^2
sage: S2 = -r2^2 + (x-a)^2 + (y-b)^2
sage: inv = invariant_theory.ternary_biquadratic(S1, S2, [x, y])
sage: inv.Delta_invariant()
-r1^2
sage: inv.Delta_prime_invariant()
-r2^2
sage: inv.Theta_invariant()
```

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```
a^2 + b^2 - 2*r1^2 - r2^2
sage: inv.Theta_prime_invariant()
a^2 + b^2 - r1^2 - 2*r2^2
sage: inv.F_covariant()
2*x^2*a^2 + y^2*a^2 - 2*x*a^3 + a^4 + 2*x*y*a*b - 2*y*a^2*b + x^2*b^2 +
2*y^2*b^2 - 2*x*a*b^2 + 2*a^2*b^2 - 2*y*b^3 + b^4 - 2*x^2*r1^2 - 2*y^2*r1^2 +
2*x*a*r1^2 - 2*a^2*r1^2 + 2*y*b*r1^2 - 2*b^2*r1^2 + r1^4 - 2*x^2*r2^2 -
2*y^2*r2^2 + 2*x*a*r2^2 - 2*a^2*r2^2 + 2*y*b*r2^2 - 2*b^2*r2^2 + 2*r1^2*r2^2 +
r2^4
sage: inv.J_covariant()
-8*x^2*y*a^3 + 8*x*y*a^4 + 8*x^3*a^2*b - 16*x*y^2*a^2*b - 8*x^2*a^3*b +
8*y^2*a^3*b + 16*x^2*y*a*b^2 - 8*y^3*a*b^2 + 8*x*y^2*b^3 - 8*x^2*a*b^3 +
8*y^2*a*b^3 - 8*x*y*b^4 + 8*x*y*a^2*r1^2 - 8*y*a^3*r1^2 - 8*x^2*a*b*r1^2 +
8*y^2*a*b*r1^2 + 8*x*a^2*b*r1^2 - 8*x*y*b^2*r1^2 - 8*y*a*b^2*r1^2 + 8*x*b^
↪3*r1^2 -
8*x*y*a^2*r2^2 + 8*x^2*a*b*r2^2 - 8*y^2*a*b*r2^2 + 8*x*y*b^2*r2^2
```

```
>>> from sage.all import *
>>> R = QQ['x, y, a, b, r1, r2']; (x, y, a, b, r1, r2,) = R._first_ngens(6)
>>> S1 = -r1**Integer(2) + x**Integer(2) + y**Integer(2)
>>> S2 = -r2**Integer(2) + (x-a)**Integer(2) + (y-b)**Integer(2)
>>> inv = invariant_theory.ternary_biquadratic(S1, S2, [x, y])
>>> inv.Delta_invariant()
-r1^2
>>> inv.Delta_prime_invariant()
-r2^2
>>> inv.Theta_invariant()
a^2 + b^2 - 2*r1^2 - r2^2
>>> inv.Theta_prime_invariant()
a^2 + b^2 - r1^2 - 2*r2^2
>>> inv.F_covariant()
2*x^2*a^2 + y^2*a^2 - 2*x*a^3 + a^4 + 2*x*y*a*b - 2*y*a^2*b + x^2*b^2 +
2*y^2*b^2 - 2*x*a*b^2 + 2*a^2*b^2 - 2*y*b^3 + b^4 - 2*x^2*r1^2 - 2*y^2*r1^2 +
2*x*a*r1^2 - 2*a^2*r1^2 + 2*y*b*r1^2 - 2*b^2*r1^2 + r1^4 - 2*x^2*r2^2 -
2*y^2*r2^2 + 2*x*a*r2^2 - 2*a^2*r2^2 + 2*y*b*r2^2 - 2*b^2*r2^2 + 2*r1^2*r2^2 +
r2^4
>>> inv.J_covariant()
-8*x^2*y*a^3 + 8*x*y*a^4 + 8*x^3*a^2*b - 16*x*y^2*a^2*b - 8*x^2*a^3*b +
8*y^2*a^3*b + 16*x^2*y*a*b^2 - 8*y^3*a*b^2 + 8*x*y^2*b^3 - 8*x^2*a*b^3 +
8*y^2*a*b^3 - 8*x*y*b^4 + 8*x*y*a^2*r1^2 - 8*y*a^3*r1^2 - 8*x^2*a*b*r1^2 +
8*y^2*a*b*r1^2 + 8*x*a^2*b*r1^2 - 8*x*y*b^2*r1^2 - 8*y*a*b^2*r1^2 + 8*x*b^
↪3*r1^2 -
8*x*y*a^2*r2^2 + 8*x^2*a*b*r2^2 - 8*y^2*a*b*r2^2 + 8*x*y*b^2*r2^2
```

ternary_cubic(*cubic*, **args*, ***kwds*)

Invariants of a cubic in three variables.

The algebra of invariants of a ternary cubic under $SL_3(\mathbf{C})$ is a polynomial algebra generated by two invariants S (see `S_invariant()`) and T (see `T_invariant()`) of degrees 4 and 6, called Aronhold invariants.

The ring of covariants is given as follows. The identity covariant U of a ternary cubic has degree 1 and order 3. The Hessian H (see `Hessian()`) is a covariant of ternary cubics of degree 3 and order 3. There is a covariant Θ (see `Theta_covariant()`) of ternary cubics of degree 8 and order 6 that vanishes on points

x lying on the Salmon conic of the polar of x with respect to the curve and its Hessian curve. The Brioschi covariant J (see `J_covariant()`) is the Jacobian of U, Θ, and H of degree 12, order 9. The algebra of covariants of a ternary cubic is generated over the ring of invariants by U, Θ, H, and J, with a relation

$$\begin{aligned} J^2 = & 4\Theta^3 + TU^2\Theta^2 + \Theta(-4S^3U^4 + 2STU^3H - 72S^2U^2H^2 \\ & - 18TUH^3 + 108SH^4) - 16S^4U^5H - 11S^2TU^4H^2 \\ & - 4T^2U^3H^3 + 54STU^2H^4 - 432S^2UH^5 - 27TH^6 \end{aligned}$$

REFERENCES:

- Wikipedia article Ternary_cubic

INPUT:

- `cubic` – a homogeneous cubic in 3 homogeneous variables, or an inhomogeneous cubic in 2 variables
- `x`, `y`, `z` – the variables. If `z` is `None`, the cubic is assumed to be inhomogeneous.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: cubic = invariant_theory.ternary_cubic(x^3 + y^3 + z^3)
sage: type(cubic)
<class 'sage.rings.invariants.invariant_theory.TernaryCubic'>
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> cubic = invariant_theory.ternary_cubic(x**Integer(3) + y**Integer(3) +␣
↪z**Integer(3))
>>> type(cubic)
<class 'sage.rings.invariants.invariant_theory.TernaryCubic'>
```

ternary_quadratic(*quadratic*, **args*, ***kwds*)

Invariants of a quadratic in three variables.

INPUT:

- `quadratic` – a homogeneous quadratic in 3 homogeneous variables, or an inhomogeneous quadratic in 2 variables
- `x`, `y`, `z` – the variables. If `z` is `None`, the quadratic is assumed to be inhomogeneous.

REFERENCES:

- Wikipedia article Invariant_of_a_binary_form

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: invariant_theory.ternary_quadratic(x^2 + y^2 + z^2)
Ternary quadratic with coefficients (1, 1, 1, 0, 0, 0)

sage: T.<u, v> = QQ[]
sage: invariant_theory.ternary_quadratic(1 + u^2 + v^2)
Ternary quadratic with coefficients (1, 1, 1, 0, 0, 0)

sage: quadratic = x^2 + y^2 + z^2
sage: inv = invariant_theory.ternary_quadratic(quadratic)
```

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```
sage: type(inv)
<class 'sage.rings.invariants.invariant_theory.TernaryQuadratic'>
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> invariant_theory.ternary_quadratic(x**Integer(2) + y**Integer(2) +␣
↪z**Integer(2))
Ternary quadratic with coefficients (1, 1, 1, 0, 0, 0)

>>> T = QQ['u, v']; (u, v,) = T._first_ngens(2)
>>> invariant_theory.ternary_quadratic(Integer(1) + u**Integer(2) +␣
↪v**Integer(2))
Ternary quadratic with coefficients (1, 1, 1, 0, 0, 0)

>>> quadratic = x**Integer(2) + y**Integer(2) + z**Integer(2)
>>> inv = invariant_theory.ternary_quadratic(quadratic)
>>> type(inv)
<class 'sage.rings.invariants.invariant_theory.TernaryQuadratic'>
```

class sage.rings.invariants.invariant_theory.**QuadraticForm**(*n*, *d*, *polynomial*, **args*)

Bases: *AlgebraicForm*

Invariant theory of a multivariate quadratic form.

You should use the *invariant_theory* factory object to construct instances of this class. See *quadratic_form()* for details.

as_QuadraticForm()

Convert into a QuadraticForm.

OUTPUT:

Sage has a special quadratic forms subsystem. This method converts self into this QuadraticForm representation.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: p = x^2 + y^2 + z^2 + 2*x*y + 3*x*z
sage: quadratic = invariant_theory.ternary_quadratic(p)
sage: matrix(quadratic)
[  1   1 3/2]
[  1   1   0]
[3/2   0   1]
sage: quadratic.as_QuadraticForm()
Quadratic form in 3 variables over Multivariate Polynomial
Ring in x, y, z over Rational Field with coefficients:
[ 1 2 3 ]
[ * 1 0 ]
[ * * 1 ]
sage: _.polynomial('X,Y,Z')
X^2 + 2*X*Y + Y^2 + 3*X*Z + Z^2
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> p = x**Integer(2) + y**Integer(2) + z**Integer(2) + Integer(2)*x*y +␣
↪Integer(3)*x*z
>>> quadratic = invariant_theory.ternary_quadratic(p)
>>> matrix(quadratic)
[  1    1 3/2]
[  1    1    0]
[3/2    0    1]
>>> quadratic.as_QuadraticForm()
Quadratic form in 3 variables over Multivariate Polynomial
Ring in x, y, z over Rational Field with coefficients:
[ 1 2 3 ]
[ * 1 0 ]
[ * * 1 ]
>>> _.polynomial('X,Y,Z')
X^2 + 2*X*Y + Y^2 + 3*X*Z + Z^2
```

coeffs()

The coefficients of a quadratic form.

Given

$$f(x) = \sum_{0 \le i < n} a_i x_i^2 + \sum_{0 \le j < k < n} a_{jk} x_j x_k$$

this function returns $a = (a_0, \ldots, a_n, a_{00}, a_{01}, \ldots, a_{n-1,n})$

EXAMPLES:

```
sage: R.<a,b,c,d,e,f,g, x,y,z> = QQ[]
sage: p = a*x^2 + b*y^2 + c*z^2 + d*x*y + e*x*z + f*y*z
sage: inv = invariant_theory.quadratic_form(p, x,y,z); inv
Ternary quadratic with coefficients (a, b, c, d, e, f)
sage: inv.coeffs()
(a, b, c, d, e, f)
sage: inv.scaled_coeffs()
(a, b, c, 1/2*d, 1/2*e, 1/2*f)
```

```
>>> from sage.all import *
>>> R = QQ['a, b, c, d, e, f, g, x, y, z']; (a, b, c, d, e, f, g, x, y, z,) =␣
↪R._first_ngens(10)
>>> p = a*x**Integer(2) + b*y**Integer(2) + c*z**Integer(2) + d*x*y + e*x*z +␣
↪f*y*z
>>> inv = invariant_theory.quadratic_form(p, x,y,z); inv
Ternary quadratic with coefficients (a, b, c, d, e, f)
>>> inv.coeffs()
(a, b, c, d, e, f)
>>> inv.scaled_coeffs()
(a, b, c, 1/2*d, 1/2*e, 1/2*f)
```

discriminant()

Return the discriminant of the quadratic form.

Up to an overall constant factor, this is just the determinant of the defining matrix, see `matrix()`. For a quadratic form in n variables, the overall constant is 2^{n-1} if n is odd and $(-1)^{n/2} 2^n$ if n is even.

EXAMPLES:

```
sage: R.<a,b,c, x,y> = QQ[]
sage: p = a*x^2 + b*x*y + c*y^2
sage: quadratic = invariant_theory.quadratic_form(p, x,y)
sage: quadratic.discriminant()
b^2 - 4*a*c

sage: R.<a,b,c,d,e,f,g, x,y,z> = QQ[]
sage: p = a*x^2 + b*y^2 + c*z^2 + d*x*y + e*x*z + f*y*z
sage: quadratic = invariant_theory.quadratic_form(p, x,y,z)
sage: quadratic.discriminant()
4*a*b*c - c*d^2 - b*e^2 + d*e*f - a*f^2
```

```
>>> from sage.all import *
>>> R = QQ['a, b, c, x, y']; (a, b, c, x, y,) = R._first_ngens(5)
>>> p = a*x**Integer(2) + b*x*y + c*y**Integer(2)
>>> quadratic = invariant_theory.quadratic_form(p, x,y)
>>> quadratic.discriminant()
b^2 - 4*a*c

>>> R = QQ['a, b, c, d, e, f, g, x, y, z']; (a, b, c, d, e, f, g, x, y, z,) =␣
↪R._first_ngens(10)
>>> p = a*x**Integer(2) + b*y**Integer(2) + c*z**Integer(2) + d*x*y + e*x*z +␣
↪f*y*z
>>> quadratic = invariant_theory.quadratic_form(p, x,y,z)
>>> quadratic.discriminant()
4*a*b*c - c*d^2 - b*e^2 + d*e*f - a*f^2
```

dual()

Return the dual quadratic form.

OUTPUT:

A new quadratic form (with the same number of variables) defined by the adjoint matrix.

EXAMPLES:

```
sage: R.<a,b,c,x,y,z> = QQ[]
sage: cubic = x^2+y^2+z^2
sage: quadratic = invariant_theory.ternary_quadratic(a*x^2+b*y^2+c*z^2, [x,y,
↪z])
sage: quadratic.form()
a*x^2 + b*y^2 + c*z^2
sage: quadratic.dual().form()
b*c*x^2 + a*c*y^2 + a*b*z^2

sage: R.<x,y,z, t> = QQ[]
sage: cubic = x^2+y^2+z^2
sage: quadratic = invariant_theory.ternary_quadratic(x^2+y^2+z^2 + t*x*y, [x,
↪y,z])
sage: quadratic.dual()
Ternary quadratic with coefficients (1, 1, -1/4*t^2 + 1, -t, 0, 0)

sage: R.<x,y, t> = QQ[]
```

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```
sage: quadratic = invariant_theory.ternary_quadratic(x^2+y^2+1 + t*x*y, [x,y])
sage: quadratic.dual()
Ternary quadratic with coefficients (1, 1, -1/4*t^2 + 1, -t, 0, 0)
```

```
>>> from sage.all import *
>>> R = QQ['a, b, c, x, y, z']; (a, b, c, x, y, z,) = R._first_ngens(6)
>>> cubic = x**Integer(2)+y**Integer(2)+z**Integer(2)
>>> quadratic = invariant_theory.ternary_
→quadratic(a*x**Integer(2)+b*y**Integer(2)+c*z**Integer(2), [x,y,z])
>>> quadratic.form()
a*x^2 + b*y^2 + c*z^2
>>> quadratic.dual().form()
b*c*x^2 + a*c*y^2 + a*b*z^2

>>> R = QQ['x, y, z, t']; (x, y, z, t,) = R._first_ngens(4)
>>> cubic = x**Integer(2)+y**Integer(2)+z**Integer(2)
>>> quadratic = invariant_theory.ternary_
→quadratic(x**Integer(2)+y**Integer(2)+z**Integer(2) + t*x*y, [x,y,z])
>>> quadratic.dual()
Ternary quadratic with coefficients (1, 1, -1/4*t^2 + 1, -t, 0, 0)

>>> R = QQ['x, y, t']; (x, y, t,) = R._first_ngens(3)
>>> quadratic = invariant_theory.ternary_
→quadratic(x**Integer(2)+y**Integer(2)+Integer(1) + t*x*y, [x,y])
>>> quadratic.dual()
Ternary quadratic with coefficients (1, 1, -1/4*t^2 + 1, -t, 0, 0)
```

classmethod from_invariants(*discriminant*, *x*, *z*, **args*, ***kwargs*)

Construct a binary quadratic from its discriminant.

This function constructs a binary quadratic whose discriminant equal the one provided as argument up to scaling.

INPUT:

- `discriminant` – value of the discriminant used to reconstruct the binary quadratic

OUTPUT: a QuadraticForm with 2 variables

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: from sage.rings.invariants.invariant_theory import QuadraticForm
sage: QuadraticForm.from_invariants(1, x, y)
Binary quadratic with coefficients (1, -1/4, 0)
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> from sage.rings.invariants.invariant_theory import QuadraticForm
>>> QuadraticForm.from_invariants(Integer(1), x, y)
Binary quadratic with coefficients (1, -1/4, 0)
```

invariants(*type='discriminant'*)

Return a tuple of invariants of a binary quadratic.

INPUT:

- `type` – the type of invariants to return; the default choice is to return the discriminant

OUTPUT: the invariants of the binary quadratic

EXAMPLES:

```
sage: R.<x0, x1> = QQ[]
sage: p = 2*x1^2 + 5*x0*x1 + 3*x0^2
sage: quadratic = invariant_theory.binary_quadratic(p, x0, x1)
sage: quadratic.invariants()
(1,)
sage: quadratic.invariants('unknown')
Traceback (most recent call last):
...
ValueError: unknown type of invariants unknown for a binary quadratic
```

```
>>> from sage.all import *
>>> R = QQ['x0, x1']; (x0, x1,) = R._first_ngens(2)
>>> p = Integer(2)*x1**Integer(2) + Integer(5)*x0*x1 +␣
↪Integer(3)*x0**Integer(2)
>>> quadratic = invariant_theory.binary_quadratic(p, x0, x1)
>>> quadratic.invariants()
(1,)
>>> quadratic.invariants('unknown')
Traceback (most recent call last):
...
ValueError: unknown type of invariants unknown for a binary quadratic
```

matrix()

Return the quadratic form as a symmetric matrix.

OUTPUT:

This method returns a symmetric matrix A such that the quadratic Q equals

$$Q(x, y, z, \dots) = (x, y, \dots) A (x, y, \dots)^t$$

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: quadratic = invariant_theory.ternary_quadratic(x^2+y^2+z^2+x*y)
sage: matrix(quadratic)
[  1 1/2   0]
[1/2   1   0]
[  0   0   1]
sage: quadratic._matrix_() == matrix(quadratic)
True
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> quadratic = invariant_theory.ternary_
↪quadratic(x**Integer(2)+y**Integer(2)+z**Integer(2)+x*y)
>>> matrix(quadratic)
[  1 1/2   0]
```

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```
[1/2    1    0]
[  0    0    1]
>>> quadratic._matrix_() == matrix(quadratic)
True
```

monomials()

List the basis monomials in the form.

OUTPUT:

A tuple of monomials. They are in the same order as *coeffs()*.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: quadratic = invariant_theory.quadratic_form(x^2 + y^2)
sage: quadratic.monomials()
(x^2, y^2, x*y)

sage: quadratic = invariant_theory.inhomogeneous_quadratic_form(x^2 + y^2)
sage: quadratic.monomials()
(x^2, y^2, 1, x*y, x, y)
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> quadratic = invariant_theory.quadratic_form(x**Integer(2) + y**Integer(2))
>>> quadratic.monomials()
(x^2, y^2, x*y)

>>> quadratic = invariant_theory.inhomogeneous_quadratic_form(x**Integer(2) +
→y**Integer(2))
>>> quadratic.monomials()
(x^2, y^2, 1, x*y, x, y)
```

scaled_coeffs()

The scaled coefficients of a quadratic form.

Given

$$f(x) = \sum_{0\leq i<n} a_i x_i^2 + \sum_{0\leq j<k<n} 2a_{jk}x_jx_k$$

this function returns $a = (a_0, \cdots, a_n, a_{00}, a_{01}, \ldots, a_{n-1,n})$

EXAMPLES:

```
sage: R.<a,b,c,d,e,f,g, x,y,z> = QQ[]
sage: p = a*x^2 + b*y^2 + c*z^2 + d*x*y + e*x*z + f*y*z
sage: inv = invariant_theory.quadratic_form(p, x,y,z); inv
Ternary quadratic with coefficients (a, b, c, d, e, f)
sage: inv.coeffs()
(a, b, c, d, e, f)
sage: inv.scaled_coeffs()
(a, b, c, 1/2*d, 1/2*e, 1/2*f)
```

```
>>> from sage.all import *
>>> R = QQ['a, b, c, d, e, f, g, x, y, z']; (a, b, c, d, e, f, g, x, y, z,) =␣
↪R._first_ngens(10)
>>> p = a*x**Integer(2) + b*y**Integer(2) + c*z**Integer(2) + d*x*y + e*x*z +␣
↪f*y*z
>>> inv = invariant_theory.quadratic_form(p, x,y,z); inv
Ternary quadratic with coefficients (a, b, c, d, e, f)
>>> inv.coeffs()
(a, b, c, d, e, f)
>>> inv.scaled_coeffs()
(a, b, c, 1/2*d, 1/2*e, 1/2*f)
```

class sage.rings.invariants.invariant_theory.**SeveralAlgebraicForms**(*forms*)

Bases: `FormsBase`

The base class of multiple algebraic forms (i.e. homogeneous polynomials).

You should only instantiate the derived classes of this base class.

See `AlgebraicForm` for the base class of a single algebraic form.

INPUT:

- `forms` – list/tuple/iterable of at least one `AlgebraicForm` object, all with the same number of variables. Interpreted as multiple homogeneous polynomials in a common polynomial ring.

EXAMPLES:

```
sage: from sage.rings.invariants.invariant_theory import AlgebraicForm,␣
↪SeveralAlgebraicForms
sage: R.<x,y> = QQ[]
sage: p = AlgebraicForm(2, 2, x^2, (x,y))
sage: q = AlgebraicForm(2, 2, y^2, (x,y))
sage: pq = SeveralAlgebraicForms([p, q])
```

```
>>> from sage.all import *
>>> from sage.rings.invariants.invariant_theory import AlgebraicForm,␣
↪SeveralAlgebraicForms
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> p = AlgebraicForm(Integer(2), Integer(2), x**Integer(2), (x,y))
>>> q = AlgebraicForm(Integer(2), Integer(2), y**Integer(2), (x,y))
>>> pq = SeveralAlgebraicForms([p, q])
```

get_form(*i*)

Return the i-th form.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: q1 = invariant_theory.quadratic_form(x^2 + y^2)
sage: q2 = invariant_theory.quadratic_form(x*y)
sage: from sage.rings.invariants.invariant_theory import SeveralAlgebraicForms
sage: q12 = SeveralAlgebraicForms([q1, q2])
sage: q12.get_form(0) is q1
True
sage: q12.get_form(1) is q2
```

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```
True
sage: q12[0] is q12.get_form(0)    # syntactic sugar
True
sage: q12[1] is q12.get_form(1)    # syntactic sugar
True
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> q1 = invariant_theory.quadratic_form(x**Integer(2) + y**Integer(2))
>>> q2 = invariant_theory.quadratic_form(x*y)
>>> from sage.rings.invariants.invariant_theory import SeveralAlgebraicForms
>>> q12 = SeveralAlgebraicForms([q1, q2])
>>> q12.get_form(Integer(0)) is q1
True
>>> q12.get_form(Integer(1)) is q2
True
>>> q12[Integer(0)] is q12.get_form(Integer(0))    # syntactic sugar
True
>>> q12[Integer(1)] is q12.get_form(Integer(1))    # syntactic sugar
True
```

homogenized(*var='h'*)

Return form as defined by a homogeneous polynomial.

INPUT:

- `var` – either a variable name, variable index or a variable (default: `'h'`)

OUTPUT:

The same algebraic form, but defined by a homogeneous polynomial.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: q = invariant_theory.quaternary_biquadratic(x^2 + 1, y^2 + 1, [x,y,z])
sage: q
Joint quaternary quadratic with coefficients (1, 0, 0, 1, 0, 0, 0, 0, 0, 0)
and quaternary quadratic with coefficients (0, 1, 0, 1, 0, 0, 0, 0, 0, 0)
sage: q.homogenized()
Joint quaternary quadratic with coefficients (1, 0, 0, 1, 0, 0, 0, 0, 0, 0)
and quaternary quadratic with coefficients (0, 1, 0, 1, 0, 0, 0, 0, 0, 0)
sage: type(q) is type(q.homogenized())
True
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> q = invariant_theory.quaternary_biquadratic(x**Integer(2) + Integer(1),
→y**Integer(2) + Integer(1), [x,y,z])
>>> q
Joint quaternary quadratic with coefficients (1, 0, 0, 1, 0, 0, 0, 0, 0, 0)
and quaternary quadratic with coefficients (0, 1, 0, 1, 0, 0, 0, 0, 0, 0)
>>> q.homogenized()
Joint quaternary quadratic with coefficients (1, 0, 0, 1, 0, 0, 0, 0, 0, 0)
```

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```
and quaternary quadratic with coefficients (0, 1, 0, 1, 0, 0, 0, 0, 0, 0)
>>> type(q) is type(q.homogenized())
True
```

n_forms()

Return the number of forms.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: q1 = invariant_theory.quadratic_form(x^2 + y^2)
sage: q2 = invariant_theory.quadratic_form(x*y)
sage: from sage.rings.invariants.invariant_theory import SeveralAlgebraicForms
sage: q12 = SeveralAlgebraicForms([q1, q2])
sage: q12.n_forms()
2
sage: len(q12) == q12.n_forms()     # syntactic sugar
True
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> q1 = invariant_theory.quadratic_form(x**Integer(2) + y**Integer(2))
>>> q2 = invariant_theory.quadratic_form(x*y)
>>> from sage.rings.invariants.invariant_theory import SeveralAlgebraicForms
>>> q12 = SeveralAlgebraicForms([q1, q2])
>>> q12.n_forms()
2
>>> len(q12) == q12.n_forms()     # syntactic sugar
True
```

class sage.rings.invariants.invariant_theory.**TernaryCubic**(*n*, *d*, *polynomial*, **args*)

Bases: *AlgebraicForm*

Invariant theory of a ternary cubic.

You should use the *invariant_theory* factory object to construct instances of this class. See *ternary_cubic()* for details.

Hessian()

Return the Hessian covariant.

OUTPUT:

The Hessian matrix multiplied with the conventional normalization factor $1/216$.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: cubic = invariant_theory.ternary_cubic(x^3 + y^3 + z^3)
sage: cubic.Hessian()
x*y*z

sage: R.<x,y> = QQ[]
sage: cubic = invariant_theory.ternary_cubic(x^3 + y^3 + 1)
```

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```
sage: cubic.Hessian()
x*y
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> cubic = invariant_theory.ternary_cubic(x**Integer(3) + y**Integer(3) +␣
↪z**Integer(3))
>>> cubic.Hessian()
x*y*z

>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> cubic = invariant_theory.ternary_cubic(x**Integer(3) + y**Integer(3) +␣
↪Integer(1))
>>> cubic.Hessian()
x*y
```

J_covariant()

Return the J-covariant of the ternary cubic.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: cubic = invariant_theory.ternary_cubic(x^3 + y^3 + z^3)
sage: cubic.J_covariant()
x^6*y^3 - x^3*y^6 - x^6*z^3 + y^6*z^3 + x^3*z^6 - y^3*z^6

sage: R.<x,y> = QQ[]
sage: cubic = invariant_theory.ternary_cubic(x^3 + y^3 + 1)
sage: cubic.J_covariant()
x^6*y^3 - x^3*y^6 - x^6 + y^6 + x^3 - y^3
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> cubic = invariant_theory.ternary_cubic(x**Integer(3) + y**Integer(3) +␣
↪z**Integer(3))
>>> cubic.J_covariant()
x^6*y^3 - x^3*y^6 - x^6*z^3 + y^6*z^3 + x^3*z^6 - y^3*z^6

>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> cubic = invariant_theory.ternary_cubic(x**Integer(3) + y**Integer(3) +␣
↪Integer(1))
>>> cubic.J_covariant()
x^6*y^3 - x^3*y^6 - x^6 + y^6 + x^3 - y^3
```

S_invariant()

Return the S-invariant.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: cubic = invariant_theory.ternary_cubic(x^2*y + y^3 + z^3 + x*y*z)
sage: cubic.S_invariant()
-1/1296
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> cubic = invariant_theory.ternary_cubic(x**Integer(2)*y + y**Integer(3) +␣
↪z**Integer(3) + x*y*z)
>>> cubic.S_invariant()
-1/1296
```

T_invariant()

Return the T-invariant.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: cubic = invariant_theory.ternary_cubic(x^3 + y^3 + z^3)
sage: cubic.T_invariant()
1

sage: R.<x,y,z,t> = GF(7)[]
sage: cubic = invariant_theory.ternary_cubic(x^3 + y^3 + z^3 + t*x*y*z, [x,y,
↪z])
sage: cubic.T_invariant()
-t^6 - t^3 + 1
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> cubic = invariant_theory.ternary_cubic(x**Integer(3) + y**Integer(3) +␣
↪z**Integer(3))
>>> cubic.T_invariant()
1

>>> R = GF(Integer(7))['x, y, z, t']; (x, y, z, t,) = R._first_ngens(4)
>>> cubic = invariant_theory.ternary_cubic(x**Integer(3) + y**Integer(3) +␣
↪z**Integer(3) + t*x*y*z, [x,y,z])
>>> cubic.T_invariant()
-t^6 - t^3 + 1
```

Theta_covariant()

Return the Θ covariant.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: cubic = invariant_theory.ternary_cubic(x^3 + y^3 + z^3)
sage: cubic.Theta_covariant()
-x^3*y^3 - x^3*z^3 - y^3*z^3

sage: R.<x,y> = QQ[]
sage: cubic = invariant_theory.ternary_cubic(x^3 + y^3 + 1)
sage: cubic.Theta_covariant()
-x^3*y^3 - x^3 - y^3

sage: R.<x,y,z,a30,a21,a12,a03,a20,a11,a02,a10,a01,a00> = QQ[]
sage: p = ( a30*x^3 + a21*x^2*y + a12*x*y^2 + a03*y^3 + a20*x^2*z +
```

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```
....:        a11*x*y*z + a02*y^2*z + a10*x*z^2 + a01*y*z^2 + a00*z^3 )
sage: cubic = invariant_theory.ternary_cubic(p, x,y,z)
sage: len(list(cubic.Theta_covariant()))
6952
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> cubic = invariant_theory.ternary_cubic(x**Integer(3) + y**Integer(3) +␣
↪z**Integer(3))
>>> cubic.Theta_covariant()
-x^3*y^3 - x^3*z^3 - y^3*z^3

>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> cubic = invariant_theory.ternary_cubic(x**Integer(3) + y**Integer(3) +␣
↪Integer(1))
>>> cubic.Theta_covariant()
-x^3*y^3 - x^3 - y^3

>>> R = QQ['x, y, z, a30, a21, a12, a03, a20, a11, a02, a10, a01, a00']; (x,␣
↪y, z, a30, a21, a12, a03, a20, a11, a02, a10, a01, a00,) = R._first_
↪ngens(13)
>>> p = ( a30*x**Integer(3) + a21*x**Integer(2)*y + a12*x*y**Integer(2) +␣
↪a03*y**Integer(3) + a20*x**Integer(2)*z +
...        a11*x*y*z + a02*y**Integer(2)*z + a10*x*z**Integer(2) +␣
↪a01*y*z**Integer(2) + a00*z**Integer(3) )
>>> cubic = invariant_theory.ternary_cubic(p, x,y,z)
>>> len(list(cubic.Theta_covariant()))
6952
```

coeffs()

Return the coefficients of a cubic.

Given

$$p(x,y) = a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{20}x^2 + \\ a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}$$

this function returns $a = (a_{30}, a_{03}, a_{00}, a_{21}, a_{20}, a_{12}, a_{02}, a_{10}, a_{01}, a_{11})$

EXAMPLES:

```
sage: R.<x,y,z,a30,a21,a12,a03,a20,a11,a02,a10,a01,a00> = QQ[]
sage: p = ( a30*x^3 + a21*x^2*y + a12*x*y^2 + a03*y^3 + a20*x^2*z +
....:        a11*x*y*z + a02*y^2*z + a10*x*z^2 + a01*y*z^2 + a00*z^3 )
sage: invariant_theory.ternary_cubic(p, x,y,z).coeffs()
(a30, a03, a00, a21, a20, a12, a02, a10, a01, a11)
sage: invariant_theory.ternary_cubic(p.subs(z=1), x, y).coeffs()
(a30, a03, a00, a21, a20, a12, a02, a10, a01, a11)
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z, a30, a21, a12, a03, a20, a11, a02, a10, a01, a00']; (x,␣
↪y, z, a30, a21, a12, a03, a20, a11, a02, a10, a01, a00,) = R._first_
↪ngens(13)
```

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```
>>> p = ( a30*x**Integer(3) + a21*x**Integer(2)*y + a12*x*y**Integer(2) +␣
↪a03*y**Integer(3) + a20*x**Integer(2)*z +
...       a11*x*y*z + a02*y**Integer(2)*z + a10*x*z**Integer(2) +␣
↪a01*y*z**Integer(2) + a00*z**Integer(3) )
>>> invariant_theory.ternary_cubic(p, x,y,z).coeffs()
(a30, a03, a00, a21, a20, a12, a02, a10, a01, a11)
>>> invariant_theory.ternary_cubic(p.subs(z=Integer(1)), x, y).coeffs()
(a30, a03, a00, a21, a20, a12, a02, a10, a01, a11)
```

monomials()

List the basis monomials of the form.

OUTPUT:

A tuple of monomials. They are in the same order as `coeffs()`.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: cubic = invariant_theory.ternary_cubic(x^3+y*z^2)
sage: cubic.monomials()
(x^3, y^3, z^3, x^2*y, x^2*z, x*y^2, y^2*z, x*z^2, y*z^2, x*y*z)
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> cubic = invariant_theory.ternary_cubic(x**Integer(3)+y*z**Integer(2))
>>> cubic.monomials()
(x^3, y^3, z^3, x^2*y, x^2*z, x*y^2, y^2*z, x*z^2, y*z^2, x*y*z)
```

polar_conic()

Return the polar conic of the cubic.

OUTPUT:

Given the ternary cubic $f(X, Y, Z)$, this method returns the symmetric matrix $A(x, y, z)$ defined by

$$xf_X + yf_Y + zf_Z = (X, Y, Z) \cdot A(x, y, z) \cdot (X, Y, Z)^t$$

EXAMPLES:

```
sage: R.<x,y,z,X,Y,Z,a30,a21,a12,a03,a20,a11,a02,a10,a01,a00> = QQ[]
sage: p = ( a30*x^3 + a21*x^2*y + a12*x*y^2 + a03*y^3 + a20*x^2*z +
....:       a11*x*y*z + a02*y^2*z + a10*x*z^2 + a01*y*z^2 + a00*z^3 )
sage: cubic = invariant_theory.ternary_cubic(p, x,y,z)
sage: cubic.polar_conic()
[  3*x*a30 + y*a21 + z*a20 x*a21 + y*a12 + 1/2*z*a11 x*a20 + 1/2*y*a11 +␣
↪z*a10]
[x*a21 + y*a12 + 1/2*z*a11   x*a12 + 3*y*a03 + z*a02 1/2*x*a11 + y*a02 +␣
↪z*a01]
[x*a20 + 1/2*y*a11 + z*a10 1/2*x*a11 + y*a02 + z*a01   x*a10 + y*a01 +␣
↪3*z*a00]

sage: polar_eqn = X*p.derivative(x) + Y*p.derivative(y) + Z*p.derivative(z)
sage: polar = invariant_theory.ternary_quadratic(polar_eqn, [x,y,z])
```

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```
sage: polar.matrix().subs(X=x,Y=y,Z=z) == cubic.polar_conic()
True
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z, X, Y, Z, a30, a21, a12, a03, a20, a11, a02, a10, a01, a00
↪']; (x, y, z, X, Y, Z, a30, a21, a12, a03, a20, a11, a02, a10, a01, a00,) =
↪R._first_ngens(16)
>>> p = ( a30*x**Integer(3) + a21*x**Integer(2)*y + a12*x*y**Integer(2) +
↪a03*y**Integer(3) + a20*x**Integer(2)*z +
...       a11*x*y*z + a02*y**Integer(2)*z + a10*x*z**Integer(2) +
↪a01*y*z**Integer(2) + a00*z**Integer(3) )
>>> cubic = invariant_theory.ternary_cubic(p, x,y,z)
>>> cubic.polar_conic()
[  3*x*a30 + y*a21 + z*a20 x*a21 + y*a12 + 1/2*z*a11 x*a20 + 1/2*y*a11 +
↪z*a10]
[x*a21 + y*a12 + 1/2*z*a11   x*a12 + 3*y*a03 + z*a02 1/2*x*a11 + y*a02 +
↪z*a01]
[x*a20 + 1/2*y*a11 + z*a10 1/2*x*a11 + y*a02 + z*a01   x*a10 + y*a01 +
↪3*z*a00]

>>> polar_eqn = X*p.derivative(x) + Y*p.derivative(y) + Z*p.derivative(z)
>>> polar = invariant_theory.ternary_quadratic(polar_eqn, [x,y,z])
>>> polar.matrix().subs(X=x,Y=y,Z=z) == cubic.polar_conic()
True
```

scaled_coeffs()

Return the coefficients of a cubic.

Compared to *coeffs()*, this method returns rescaled coefficients that are often used in invariant theory.

Given

$$p(x,y) = a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{20}x^2 + \\ a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}$$

this function returns $a = (a_{30}, a_{03}, a_{00}, a_{21}/3, a_{20}/3, a_{12}/3, a_{02}/3, a_{10}/3, a_{01}/3, a_{11}/6)$

EXAMPLES:

```
sage: R.<x,y,z,a30,a21,a12,a03,a20,a11,a02,a10,a01,a00> = QQ[]
sage: p = ( a30*x^3 + a21*x^2*y + a12*x*y^2 + a03*y^3 + a20*x^2*z +
....:       a11*x*y*z + a02*y^2*z + a10*x*z^2 + a01*y*z^2 + a00*z^3 )
sage: invariant_theory.ternary_cubic(p, x,y,z).scaled_coeffs()
(a30, a03, a00, 1/3*a21, 1/3*a20, 1/3*a12, 1/3*a02, 1/3*a10, 1/3*a01, 1/6*a11)
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z, a30, a21, a12, a03, a20, a11, a02, a10, a01, a00']; (x,
↪y, z, a30, a21, a12, a03, a20, a11, a02, a10, a01, a00,) = R._first_
↪ngens(13)
>>> p = ( a30*x**Integer(3) + a21*x**Integer(2)*y + a12*x*y**Integer(2) +
↪a03*y**Integer(3) + a20*x**Integer(2)*z +
...       a11*x*y*z + a02*y**Integer(2)*z + a10*x*z**Integer(2) +
↪a01*y*z**Integer(2) + a00*z**Integer(3) )
```

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```
>>> invariant_theory.ternary_cubic(p, x,y,z).scaled_coeffs()
(a30, a03, a00, 1/3*a21, 1/3*a20, 1/3*a12, 1/3*a02, 1/3*a10, 1/3*a01, 1/6*a11)
```

syzygy(*U*, *S*, *T*, *H*, *Theta*, *J*)

Return the syzygy of the cubic evaluated on the invariants and covariants.

INPUT:

- U, S, T, H, Theta, J – polynomials from the same polynomial ring.

OUTPUT:

0 if evaluated for the form, the S invariant, the T invariant, the Hessian, the Θ covariant and the J-covariant of a ternary cubic.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: monomials = (x^3, y^3, z^3, x^2*y, x^2*z, x*y^2,
....:              y^2*z, x*z^2, y*z^2, x*y*z)
sage: random_poly = sum([ randint(0,10000) * m for m in monomials ])
sage: cubic = invariant_theory.ternary_cubic(random_poly)
sage: U = cubic.form()
sage: S = cubic.S_invariant()
sage: T = cubic.T_invariant()
sage: H = cubic.Hessian()
sage: Theta = cubic.Theta_covariant()
sage: J = cubic.J_covariant()
sage: cubic.syzygy(U, S, T, H, Theta, J)
0
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> monomials = (x**Integer(3), y**Integer(3), z**Integer(3), x**Integer(2)*y,
↪ x**Integer(2)*z, x*y**Integer(2),
...              y**Integer(2)*z, x*z**Integer(2), y*z**Integer(2), x*y*z)
>>> random_poly = sum([ randint(Integer(0),Integer(10000)) * m for m in␣
↪monomials ])
>>> cubic = invariant_theory.ternary_cubic(random_poly)
>>> U = cubic.form()
>>> S = cubic.S_invariant()
>>> T = cubic.T_invariant()
>>> H = cubic.Hessian()
>>> Theta = cubic.Theta_covariant()
>>> J = cubic.J_covariant()
>>> cubic.syzygy(U, S, T, H, Theta, J)
0
```

class sage.rings.invariants.invariant_theory.**TernaryQuadratic**(*n*, *d*, *polynomial*, **args*)

Bases: *QuadraticForm*

Invariant theory of a ternary quadratic.

You should use the *invariant_theory* factory object to construct instances of this class. See *ternary_quadratic()* for details.

coeffs()

Return the coefficients of a quadratic.

Given

$$p(x, y) = a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}$$

this function returns $a = (a_{20}, a_{02}, a_{00}, a_{11}, a_{10}, a_{01})$

EXAMPLES:

```
sage: R.<x,y,z,a20,a11,a02,a10,a01,a00> = QQ[]
sage: p = ( a20*x^2 + a11*x*y + a02*y^2 +
....:       a10*x*z + a01*y*z + a00*z^2 )
sage: invariant_theory.ternary_quadratic(p, x,y,z).coeffs()
(a20, a02, a00, a11, a10, a01)
sage: invariant_theory.ternary_quadratic(p.subs(z=1), x, y).coeffs()
(a20, a02, a00, a11, a10, a01)
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z, a20, a11, a02, a10, a01, a00']; (x, y, z, a20, a11, a02,␣
↪a10, a01, a00,) = R._first_ngens(9)
>>> p = ( a20*x**Integer(2) + a11*x*y + a02*y**Integer(2) +
...       a10*x*z + a01*y*z + a00*z**Integer(2) )
>>> invariant_theory.ternary_quadratic(p, x,y,z).coeffs()
(a20, a02, a00, a11, a10, a01)
>>> invariant_theory.ternary_quadratic(p.subs(z=Integer(1)), x, y).coeffs()
(a20, a02, a00, a11, a10, a01)
```

covariant_conic(*other*)

Return the ternary quadratic covariant to `self` and `other`.

INPUT:

- `other` – another ternary quadratic

OUTPUT:

The so-called covariant conic, a ternary quadratic. It is symmetric under exchange of `self` and `other`.

EXAMPLES:

```
sage: ring.<x,y,z> = QQ[]
sage: Q = invariant_theory.ternary_quadratic(x^2 + y^2 + z^2)
sage: R = invariant_theory.ternary_quadratic(x*y + x*z + y*z)
sage: Q.covariant_conic(R)
-x*y - x*z - y*z
sage: R.covariant_conic(Q)
-x*y - x*z - y*z
```

```
>>> from sage.all import *
>>> ring = QQ['x, y, z']; (x, y, z,) = ring._first_ngens(3)
>>> Q = invariant_theory.ternary_quadratic(x**Integer(2) + y**Integer(2) +␣
↪z**Integer(2))
>>> R = invariant_theory.ternary_quadratic(x*y + x*z + y*z)
>>> Q.covariant_conic(R)
```

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```
-x*y - x*z - y*z
>>> R.covariant_conic(Q)
-x*y - x*z - y*z
```

monomials()

List the basis monomials of the form.

OUTPUT:

A tuple of monomials. They are in the same order as `coeffs()`.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: quadratic = invariant_theory.ternary_quadratic(x^2 + y*z)
sage: quadratic.monomials()
(x^2, y^2, z^2, x*y, x*z, y*z)
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> quadratic = invariant_theory.ternary_quadratic(x**Integer(2) + y*z)
>>> quadratic.monomials()
(x^2, y^2, z^2, x*y, x*z, y*z)
```

scaled_coeffs()

Return the scaled coefficients of a quadratic.

Given

$$p(x, y) = a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}$$

this function returns $a = (a_{20}, a_{02}, a_{00}, a_{11}/2, a_{10}/2, a_{01}/2,)$

EXAMPLES:

```
sage: R.<x,y,z,a20,a11,a02,a10,a01,a00> = QQ[]
sage: p = ( a20*x^2 + a11*x*y + a02*y^2 +
....:       a10*x*z + a01*y*z + a00*z^2 )
sage: invariant_theory.ternary_quadratic(p, x,y,z).scaled_coeffs()
(a20, a02, a00, 1/2*a11, 1/2*a10, 1/2*a01)
sage: invariant_theory.ternary_quadratic(p.subs(z=1), x, y).scaled_coeffs()
(a20, a02, a00, 1/2*a11, 1/2*a10, 1/2*a01)
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z, a20, a11, a02, a10, a01, a00']; (x, y, z, a20, a11, a02,␣
↪a10, a01, a00,) = R._first_ngens(9)
>>> p = ( a20*x**Integer(2) + a11*x*y + a02*y**Integer(2) +
...       a10*x*z + a01*y*z + a00*z**Integer(2) )
>>> invariant_theory.ternary_quadratic(p, x,y,z).scaled_coeffs()
(a20, a02, a00, 1/2*a11, 1/2*a10, 1/2*a01)
>>> invariant_theory.ternary_quadratic(p.subs(z=Integer(1)), x, y).scaled_
↪coeffs()
(a20, a02, a00, 1/2*a11, 1/2*a10, 1/2*a01)
```

class sage.rings.invariants.invariant_theory.**TwoAlgebraicForms**(*forms*)

Bases: *SeveralAlgebraicForms*

first()

Return the first of the two forms.

OUTPUT: the first algebraic form used in the definition

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: q0 = invariant_theory.quadratic_form(x^2 + y^2)
sage: q1 = invariant_theory.quadratic_form(x*y)
sage: from sage.rings.invariants.invariant_theory import TwoAlgebraicForms
sage: q = TwoAlgebraicForms([q0, q1])
sage: q.first() is q0
True
sage: q.get_form(0) is q0
True
sage: q.first().polynomial()
x^2 + y^2
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> q0 = invariant_theory.quadratic_form(x**Integer(2) + y**Integer(2))
>>> q1 = invariant_theory.quadratic_form(x*y)
>>> from sage.rings.invariants.invariant_theory import TwoAlgebraicForms
>>> q = TwoAlgebraicForms([q0, q1])
>>> q.first() is q0
True
>>> q.get_form(Integer(0)) is q0
True
>>> q.first().polynomial()
x^2 + y^2
```

second()

Return the second of the two forms.

OUTPUT: the second form used in the definition

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: q0 = invariant_theory.quadratic_form(x^2 + y^2)
sage: q1 = invariant_theory.quadratic_form(x*y)
sage: from sage.rings.invariants.invariant_theory import TwoAlgebraicForms
sage: q = TwoAlgebraicForms([q0, q1])
sage: q.second() is q1
True
sage: q.get_form(1) is q1
True
sage: q.second().polynomial()
x*y
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> q0 = invariant_theory.quadratic_form(x**Integer(2) + y**Integer(2))
>>> q1 = invariant_theory.quadratic_form(x*y)
>>> from sage.rings.invariants.invariant_theory import TwoAlgebraicForms
>>> q = TwoAlgebraicForms([q0, q1])
>>> q.second() is q1
True
>>> q.get_form(Integer(1)) is q1
True
>>> q.second().polynomial()
x*y
```

class sage.rings.invariants.invariant_theory.**TwoQuaternaryQuadratics**(*forms*)

Bases: *TwoAlgebraicForms*

Invariant theory of two quaternary quadratics.

You should use the *invariant_theory* factory object to construct instances of this class. See quaternary_biquadratics() for details.

REFERENCES:

- section on "Invariants and Covariants of Systems of Quadrics" in [Sal1958], [Sal1965]

Delta_invariant()

Return the Δ invariant.

EXAMPLES:

```
sage: R.<x,y,z,t,a0,a1,a2,a3,b0,b1,b2,b3,b4,b5,A0,A1,A2,A3,B0,B1,B2,B3,B4,B5>␣
↪= QQ[]
sage: p1 = a0*x^2 + a1*y^2 + a2*z^2 + a3
sage: p1 += b0*x*y + b1*x*z + b2*x + b3*y*z + b4*y + b5*z
sage: p2 = A0*x^2 + A1*y^2 + A2*z^2 + A3
sage: p2 += B0*x*y + B1*x*z + B2*x + B3*y*z + B4*y + B5*z
sage: q = invariant_theory.quaternary_biquadratic(p1, p2, [x, y, z])
sage: coeffs = det(t * q[0].matrix() + q[1].matrix()).polynomial(t).
↪coefficients(sparse=False)
sage: q.Delta_invariant() == coeffs[4]
True
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z, t, a0, a1, a2, a3, b0, b1, b2, b3, b4, b5, A0, A1, A2,␣
↪A3, B0, B1, B2, B3, B4, B5']; (x, y, z, t, a0, a1, a2, a3, b0, b1, b2, b3,␣
↪b4, b5, A0, A1, A2, A3, B0, B1, B2, B3, B4, B5,) = R._first_ngens(24)
>>> p1 = a0*x**Integer(2) + a1*y**Integer(2) + a2*z**Integer(2) + a3
>>> p1 += b0*x*y + b1*x*z + b2*x + b3*y*z + b4*y + b5*z
>>> p2 = A0*x**Integer(2) + A1*y**Integer(2) + A2*z**Integer(2) + A3
>>> p2 += B0*x*y + B1*x*z + B2*x + B3*y*z + B4*y + B5*z
>>> q = invariant_theory.quaternary_biquadratic(p1, p2, [x, y, z])
>>> coeffs = det(t * q[Integer(0)].matrix() + q[Integer(1)].matrix()).
↪polynomial(t).coefficients(sparse=False)
>>> q.Delta_invariant() == coeffs[Integer(4)]
True
```

Delta_prime_invariant()

Return the Δ' invariant.

EXAMPLES:

```
sage: R.<x,y,z,t,a0,a1,a2,a3,b0,b1,b2,b3,b4,b5,A0,A1,A2,A3,B0,B1,B2,B3,B4,B5>␣
↪= QQ[]
sage: p1 = a0*x^2 + a1*y^2 + a2*z^2 + a3
sage: p1 += b0*x*y + b1*x*z + b2*x + b3*y*z + b4*y + b5*z
sage: p2 = A0*x^2 + A1*y^2 + A2*z^2 + A3
sage: p2 += B0*x*y + B1*x*z + B2*x + B3*y*z + B4*y + B5*z
sage: q = invariant_theory.quaternary_biquadratic(p1, p2, [x, y, z])
sage: coeffs = det(t * q[0].matrix() + q[1].matrix()).polynomial(t).
↪coefficients(sparse=False)
sage: q.Delta_prime_invariant() == coeffs[0]
True
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z, t, a0, a1, a2, a3, b0, b1, b2, b3, b4, b5, A0, A1, A2,␣
↪A3, B0, B1, B2, B3, B4, B5']; (x, y, z, t, a0, a1, a2, a3, b0, b1, b2, b3,␣
↪b4, b5, A0, A1, A2, A3, B0, B1, B2, B3, B4, B5,) = R._first_ngens(24)
>>> p1 = a0*x**Integer(2) + a1*y**Integer(2) + a2*z**Integer(2) + a3
>>> p1 += b0*x*y + b1*x*z + b2*x + b3*y*z + b4*y + b5*z
>>> p2 = A0*x**Integer(2) + A1*y**Integer(2) + A2*z**Integer(2) + A3
>>> p2 += B0*x*y + B1*x*z + B2*x + B3*y*z + B4*y + B5*z
>>> q = invariant_theory.quaternary_biquadratic(p1, p2, [x, y, z])
>>> coeffs = det(t * q[Integer(0)].matrix() + q[Integer(1)].matrix()).
↪polynomial(t).coefficients(sparse=False)
>>> q.Delta_prime_invariant() == coeffs[Integer(0)]
True
```

J_covariant()

The J-covariant.

This is the Jacobian determinant of the two biquadratics, the T-covariant, and the T'-covariant with respect to the four homogeneous variables.

EXAMPLES:

```
sage: R.<w,x,y,z,a0,a1,a2,a3,A0,A1,A2,A3> = QQ[]
sage: p1 = a0*x^2 + a1*y^2 + a2*z^2 + a3*w^2
sage: p2 = A0*x^2 + A1*y^2 + A2*z^2 + A3*w^2
sage: q = invariant_theory.quaternary_biquadratic(p1, p2, [w, x, y, z])
sage: q.J_covariant().factor()
z * y * x * w * (-a1*A0 + a0*A1) * (-a2*A0 + a0*A2) * (-a2*A1 + a1*A2)
* (a3*A2 - a2*A3) * (a3*A1 - a1*A3) * (a3*A0 - a0*A3)
```

```
>>> from sage.all import *
>>> R = QQ['w, x, y, z, a0, a1, a2, a3, A0, A1, A2, A3']; (w, x, y, z, a0, a1,
↪ a2, a3, A0, A1, A2, A3,) = R._first_ngens(12)
>>> p1 = a0*x**Integer(2) + a1*y**Integer(2) + a2*z**Integer(2) +␣
↪a3*w**Integer(2)
>>> p2 = A0*x**Integer(2) + A1*y**Integer(2) + A2*z**Integer(2) +␣
↪A3*w**Integer(2)
```

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```
>>> q = invariant_theory.quaternary_biquadratic(p1, p2, [w, x, y, z])
>>> q.J_covariant().factor()
z * y * x * w * (-a1*A0 + a0*A1) * (-a2*A0 + a0*A2) * (-a2*A1 + a1*A2)
* (a3*A2 - a2*A3) * (a3*A1 - a1*A3) * (a3*A0 - a0*A3)
```

Phi_invariant()

Return the Φ' invariant.

EXAMPLES:

```
sage: R.<x,y,z,t,a0,a1,a2,a3,b0,b1,b2,b3,b4,b5,A0,A1,A2,A3,B0,B1,B2,B3,B4,B5>␣
↪= QQ[]
sage: p1 = a0*x^2 + a1*y^2 + a2*z^2 + a3
sage: p1 += b0*x*y + b1*x*z + b2*x + b3*y*z + b4*y + b5*z
sage: p2 = A0*x^2 + A1*y^2 + A2*z^2 + A3
sage: p2 += B0*x*y + B1*x*z + B2*x + B3*y*z + B4*y + B5*z
sage: q = invariant_theory.quaternary_biquadratic(p1, p2, [x, y, z])
sage: coeffs = det(t * q[0].matrix() + q[1].matrix()).polynomial(t).
↪coefficients(sparse=False)
sage: q.Phi_invariant() == coeffs[2]
True
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z, t, a0, a1, a2, a3, b0, b1, b2, b3, b4, b5, A0, A1, A2,␣
↪A3, B0, B1, B2, B3, B4, B5']; (x, y, z, t, a0, a1, a2, a3, b0, b1, b2, b3,␣
↪b4, b5, A0, A1, A2, A3, B0, B1, B2, B3, B4, B5,) = R._first_ngens(24)
>>> p1 = a0*x**Integer(2) + a1*y**Integer(2) + a2*z**Integer(2) + a3
>>> p1 += b0*x*y + b1*x*z + b2*x + b3*y*z + b4*y + b5*z
>>> p2 = A0*x**Integer(2) + A1*y**Integer(2) + A2*z**Integer(2) + A3
>>> p2 += B0*x*y + B1*x*z + B2*x + B3*y*z + B4*y + B5*z
>>> q = invariant_theory.quaternary_biquadratic(p1, p2, [x, y, z])
>>> coeffs = det(t * q[Integer(0)].matrix() + q[Integer(1)].matrix()).
↪polynomial(t).coefficients(sparse=False)
>>> q.Phi_invariant() == coeffs[Integer(2)]
True
```

T_covariant()

The T-covariant.

EXAMPLES:

```
sage: R.<x,y,z,t,a0,a1,a2,a3,b0,b1,b2,b3,b4,b5,A0,A1,A2,A3,B0,B1,B2,B3,B4,B5>␣
↪= QQ[]
sage: p1 = a0*x^2 + a1*y^2 + a2*z^2 + a3
sage: p1 += b0*x*y + b1*x*z + b2*x + b3*y*z + b4*y + b5*z
sage: p2 = A0*x^2 + A1*y^2 + A2*z^2 + A3
sage: p2 += B0*x*y + B1*x*z + B2*x + B3*y*z + B4*y + B5*z
sage: q = invariant_theory.quaternary_biquadratic(p1, p2, [x, y, z])
sage: T = invariant_theory.quaternary_quadratic(q.T_covariant(), [x,y,z]).
↪matrix()
sage: M = q[0].matrix().adjugate() + t*q[1].matrix().adjugate()
sage: M = M.adjugate().apply_map(           # long time (4s on my thinkpad␣
↪W530)
```

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```
....:           lambda m: m.coefficient(t))
sage: M == q.Delta_invariant()*T              # long time
True
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z, t, a0, a1, a2, a3, b0, b1, b2, b3, b4, b5, A0, A1, A2,␣
↪A3, B0, B1, B2, B3, B4, B5']; (x, y, z, t, a0, a1, a2, a3, b0, b1, b2, b3,␣
↪b4, b5, A0, A1, A2, A3, B0, B1, B2, B3, B4, B5,) = R._first_ngens(24)
>>> p1 = a0*x**Integer(2) + a1*y**Integer(2) + a2*z**Integer(2) + a3
>>> p1 += b0*x*y + b1*x*z + b2*x + b3*y*z + b4*y + b5*z
>>> p2 = A0*x**Integer(2) + A1*y**Integer(2) + A2*z**Integer(2) + A3
>>> p2 += B0*x*y + B1*x*z + B2*x + B3*y*z + B4*y + B5*z
>>> q = invariant_theory.quaternary_biquadratic(p1, p2, [x, y, z])
>>> T = invariant_theory.quaternary_quadratic(q.T_covariant(), [x,y,z]).
↪matrix()
>>> M = q[Integer(0)].matrix().adjugate() + t*q[Integer(1)].matrix().
↪adjugate()
>>> M = M.adjugate().apply_map(              # long time (4s on my thinkpad␣
↪W530)
...           lambda m: m.coefficient(t))
>>> M == q.Delta_invariant()*T              # long time
True
```

T_prime_covariant()

The T'-covariant.

EXAMPLES:

```
sage: R.<x,y,z,t,a0,a1,a2,a3,b0,b1,b2,b3,b4,b5,A0,A1,A2,A3,B0,B1,B2,B3,B4,B5>␣
↪= QQ[]
sage: p1 = a0*x^2 + a1*y^2 + a2*z^2 + a3
sage: p1 += b0*x*y + b1*x*z + b2*x + b3*y*z + b4*y + b5*z
sage: p2 = A0*x^2 + A1*y^2 + A2*z^2 + A3
sage: p2 += B0*x*y + B1*x*z + B2*x + B3*y*z + B4*y + B5*z
sage: q = invariant_theory.quaternary_biquadratic(p1, p2, [x, y, z])
sage: Tprime = invariant_theory.quaternary_quadratic(
....:     q.T_prime_covariant(), [x,y,z]).matrix()
sage: M = q[0].matrix().adjugate() + t*q[1].matrix().adjugate()
sage: M = M.adjugate().apply_map(                # long time (4s on my␣
↪thinkpad W530)
....:           lambda m: m.coefficient(t^2))
sage: M == q.Delta_prime_invariant() * Tprime    # long time
True
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z, t, a0, a1, a2, a3, b0, b1, b2, b3, b4, b5, A0, A1, A2,␣
↪A3, B0, B1, B2, B3, B4, B5']; (x, y, z, t, a0, a1, a2, a3, b0, b1, b2, b3,␣
↪b4, b5, A0, A1, A2, A3, B0, B1, B2, B3, B4, B5,) = R._first_ngens(24)
>>> p1 = a0*x**Integer(2) + a1*y**Integer(2) + a2*z**Integer(2) + a3
>>> p1 += b0*x*y + b1*x*z + b2*x + b3*y*z + b4*y + b5*z
>>> p2 = A0*x**Integer(2) + A1*y**Integer(2) + A2*z**Integer(2) + A3
>>> p2 += B0*x*y + B1*x*z + B2*x + B3*y*z + B4*y + B5*z
```

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```
>>> q = invariant_theory.quaternary_biquadratic(p1, p2, [x, y, z])
>>> Tprime = invariant_theory.quaternary_quadratic(
...     q.T_prime_covariant(), [x,y,z]).matrix()
>>> M = q[Integer(0)].matrix().adjugate() + t*q[Integer(1)].matrix().
↪adjugate()
>>> M = M.adjugate().apply_map(                # long time (4s on my thinkpad␣
↪W530)
...         lambda m: m.coefficient(t**Integer(2)))
>>> M == q.Delta_prime_invariant() * Tprime   # long time
True
```

Theta_invariant()

Return the Θ invariant.

EXAMPLES:

```
sage: R.<x,y,z,t,a0,a1,a2,a3,b0,b1,b2,b3,b4,b5,A0,A1,A2,A3,B0,B1,B2,B3,B4,B5>␣
↪= QQ[]
sage: p1 = a0*x^2 + a1*y^2 + a2*z^2 + a3
sage: p1 += b0*x*y + b1*x*z + b2*x + b3*y*z + b4*y + b5*z
sage: p2 = A0*x^2 + A1*y^2 + A2*z^2 + A3
sage: p2 += B0*x*y + B1*x*z + B2*x + B3*y*z + B4*y + B5*z
sage: q = invariant_theory.quaternary_biquadratic(p1, p2, [x, y, z])
sage: coeffs = det(t * q[0].matrix() + q[1].matrix()).polynomial(t).
↪coefficients(sparse=False)
sage: q.Theta_invariant() == coeffs[3]
True
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z, t, a0, a1, a2, a3, b0, b1, b2, b3, b4, b5, A0, A1, A2,␣
↪A3, B0, B1, B2, B3, B4, B5']; (x, y, z, t, a0, a1, a2, a3, b0, b1, b2, b3,␣
↪b4, b5, A0, A1, A2, A3, B0, B1, B2, B3, B4, B5,) = R._first_ngens(24)
>>> p1 = a0*x**Integer(2) + a1*y**Integer(2) + a2*z**Integer(2) + a3
>>> p1 += b0*x*y + b1*x*z + b2*x + b3*y*z + b4*y + b5*z
>>> p2 = A0*x**Integer(2) + A1*y**Integer(2) + A2*z**Integer(2) + A3
>>> p2 += B0*x*y + B1*x*z + B2*x + B3*y*z + B4*y + B5*z
>>> q = invariant_theory.quaternary_biquadratic(p1, p2, [x, y, z])
>>> coeffs = det(t * q[Integer(0)].matrix() + q[Integer(1)].matrix()).
↪polynomial(t).coefficients(sparse=False)
>>> q.Theta_invariant() == coeffs[Integer(3)]
True
```

Theta_prime_invariant()

Return the Θ' invariant.

EXAMPLES:

```
sage: R.<x,y,z,t,a0,a1,a2,a3,b0,b1,b2,b3,b4,b5,A0,A1,A2,A3,B0,B1,B2,B3,B4,B5>␣
↪= QQ[]
sage: p1 = a0*x^2 + a1*y^2 + a2*z^2 + a3
sage: p1 += b0*x*y + b1*x*z + b2*x + b3*y*z + b4*y + b5*z
sage: p2 = A0*x^2 + A1*y^2 + A2*z^2 + A3
sage: p2 += B0*x*y + B1*x*z + B2*x + B3*y*z + B4*y + B5*z
```

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```
sage: q = invariant_theory.quaternary_biquadratic(p1, p2, [x, y, z])
sage: coeffs = det(t * q[0].matrix() + q[1].matrix()).polynomial(t).
↪coefficients(sparse=False)
sage: q.Theta_prime_invariant() == coeffs[1]
True
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z, t, a0, a1, a2, a3, b0, b1, b2, b3, b4, b5, A0, A1, A2,␣
↪A3, B0, B1, B2, B3, B4, B5']; (x, y, z, t, a0, a1, a2, a3, b0, b1, b2, b3,␣
↪b4, b5, A0, A1, A2, A3, B0, B1, B2, B3, B4, B5,) = R._first_ngens(24)
>>> p1 = a0*x**Integer(2) + a1*y**Integer(2) + a2*z**Integer(2) + a3
>>> p1 += b0*x*y + b1*x*z + b2*x + b3*y*z + b4*y + b5*z
>>> p2 = A0*x**Integer(2) + A1*y**Integer(2) + A2*z**Integer(2) + A3
>>> p2 += B0*x*y + B1*x*z + B2*x + B3*y*z + B4*y + B5*z
>>> q = invariant_theory.quaternary_biquadratic(p1, p2, [x, y, z])
>>> coeffs = det(t * q[Integer(0)].matrix() + q[Integer(1)].matrix()).
↪polynomial(t).coefficients(sparse=False)
>>> q.Theta_prime_invariant() == coeffs[Integer(1)]
True
```

syzygy(*Delta*, *Theta*, *Phi*, *Theta_prime*, *Delta_prime*, *U*, *V*, *T*, *T_prime*, *J*)

Return the syzygy evaluated on the invariants and covariants.

INPUT:

- `Delta`, `Theta`, `Phi`, `Theta_prime`, `Delta_prime`, `U`, `V`, `T`, `T_prime`, `J` – polynomials from the same polynomial ring.

OUTPUT:

Zero if the `U` is the first polynomial, `V` the second polynomial, and the remaining input are the invariants and covariants of a quaternary biquadratic.

EXAMPLES:

```
sage: R.<w,x,y,z> = QQ[]
sage: monomials = [x^2, x*y, y^2, x*z, y*z, z^2, x*w, y*w, z*w, w^2]
sage: def q_rnd():  return sum(randint(-1000, 1000)*m for m in monomials)
sage: biquadratic = invariant_theory.quaternary_biquadratic(q_rnd(), q_rnd())
sage: Delta = biquadratic.Delta_invariant()
sage: Theta = biquadratic.Theta_invariant()
sage: Phi = biquadratic.Phi_invariant()
sage: Theta_prime = biquadratic.Theta_prime_invariant()
sage: Delta_prime = biquadratic.Delta_prime_invariant()
sage: U = biquadratic.first().polynomial()
sage: V  = biquadratic.second().polynomial()
sage: T = biquadratic.T_covariant()
sage: T_prime  = biquadratic.T_prime_covariant()
sage: J = biquadratic.J_covariant()
sage: biquadratic.syzygy(Delta, Theta, Phi, Theta_prime, Delta_prime, U, V, T,
↪ T_prime, J)
0
```

```
>>> from sage.all import *
>>> R = QQ['w, x, y, z']; (w, x, y, z,) = R._first_ngens(4)
>>> monomials = [x**Integer(2), x*y, y**Integer(2), x*z, y*z, z**Integer(2),␣
↪x*w, y*w, z*w, w**Integer(2)]
>>> def q_rnd():  return sum(randint(-Integer(1000), Integer(1000))*m for m␣
↪in monomials)
>>> biquadratic = invariant_theory.quaternary_biquadratic(q_rnd(), q_rnd())
>>> Delta = biquadratic.Delta_invariant()
>>> Theta = biquadratic.Theta_invariant()
>>> Phi = biquadratic.Phi_invariant()
>>> Theta_prime = biquadratic.Theta_prime_invariant()
>>> Delta_prime = biquadratic.Delta_prime_invariant()
>>> U = biquadratic.first().polynomial()
>>> V  = biquadratic.second().polynomial()
>>> T = biquadratic.T_covariant()
>>> T_prime  = biquadratic.T_prime_covariant()
>>> J = biquadratic.J_covariant()
>>> biquadratic.syzygy(Delta, Theta, Phi, Theta_prime, Delta_prime, U, V, T,␣
↪T_prime, J)
0
```

If the arguments are not the invariants and covariants then the output is some (generically nonzero) polynomial:

```
sage: biquadratic.syzygy(1, 1, 1, 1, 1, 1, 1, 1, 1, x)
-x^2 + 1
```

```
>>> from sage.all import *
>>> biquadratic.syzygy(Integer(1), Integer(1), Integer(1), Integer(1),␣
↪Integer(1), Integer(1), Integer(1), Integer(1), Integer(1), x)
-x^2 + 1
```

class sage.rings.invariants.invariant_theory.**TwoTernaryQuadratics**(*forms*)

Bases: *TwoAlgebraicForms*

Invariant theory of two ternary quadratics.

You should use the *invariant_theory* factory object to construct instances of this class. See `ternary_biquadratics()` for details.

REFERENCES:

- Section on "Invariants and Covariants of Systems of Conics", Art. 388 (a) in [Sal1954]

Delta_invariant()

Return the Δ invariant.

EXAMPLES:

```
sage: R.<a00, a01, a11, a02, a12, a22, b00, b01, b11, b02, b12, b22, y0, y1,␣
↪y2, t> = QQ[]
sage: p1 = a00*y0^2 + 2*a01*y0*y1 + a11*y1^2 + 2*a02*y0*y2 + 2*a12*y1*y2 +␣
↪a22*y2^2
sage: p2 = b00*y0^2 + 2*b01*y0*y1 + b11*y1^2 + 2*b02*y0*y2 + 2*b12*y1*y2 +␣
↪b22*y2^2
sage: q = invariant_theory.ternary_biquadratic(p1, p2, [y0, y1, y2])
```

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```
sage: coeffs = det(t * q[0].matrix() + q[1].matrix()).polynomial(t).
↪coefficients(sparse=False)
sage: q.Delta_invariant() == coeffs[3]
True
```

```
>>> from sage.all import *
>>> R = QQ['a00, a01, a11, a02, a12, a22, b00, b01, b11, b02, b12, b22, y0,␣
↪y1, y2, t']; (a00, a01, a11, a02, a12, a22, b00, b01, b11, b02, b12, b22,␣
↪y0, y1, y2, t,) = R._first_ngens(16)
>>> p1 = a00*y0**Integer(2) + Integer(2)*a01*y0*y1 + a11*y1**Integer(2) +␣
↪Integer(2)*a02*y0*y2 + Integer(2)*a12*y1*y2 + a22*y2**Integer(2)
>>> p2 = b00*y0**Integer(2) + Integer(2)*b01*y0*y1 + b11*y1**Integer(2) +␣
↪Integer(2)*b02*y0*y2 + Integer(2)*b12*y1*y2 + b22*y2**Integer(2)
>>> q = invariant_theory.ternary_biquadratic(p1, p2, [y0, y1, y2])
>>> coeffs = det(t * q[Integer(0)].matrix() + q[Integer(1)].matrix()).
↪polynomial(t).coefficients(sparse=False)
>>> q.Delta_invariant() == coeffs[Integer(3)]
True
```

Delta_prime_invariant()

Return the Δ' invariant.

EXAMPLES:

```
sage: R.<a00, a01, a11, a02, a12, a22, b00, b01, b11, b02, b12, b22, y0, y1,␣
↪y2, t> = QQ[]
sage: p1 = a00*y0^2 + 2*a01*y0*y1 + a11*y1^2 + 2*a02*y0*y2 + 2*a12*y1*y2 +␣
↪a22*y2^2
sage: p2 = b00*y0^2 + 2*b01*y0*y1 + b11*y1^2 + 2*b02*y0*y2 + 2*b12*y1*y2 +␣
↪b22*y2^2
sage: q = invariant_theory.ternary_biquadratic(p1, p2, [y0, y1, y2])
sage: coeffs = det(t * q[0].matrix() + q[1].matrix()).polynomial(t).
↪coefficients(sparse=False)
sage: q.Delta_prime_invariant() == coeffs[0]
True
```

```
>>> from sage.all import *
>>> R = QQ['a00, a01, a11, a02, a12, a22, b00, b01, b11, b02, b12, b22, y0,␣
↪y1, y2, t']; (a00, a01, a11, a02, a12, a22, b00, b01, b11, b02, b12, b22,␣
↪y0, y1, y2, t,) = R._first_ngens(16)
>>> p1 = a00*y0**Integer(2) + Integer(2)*a01*y0*y1 + a11*y1**Integer(2) +␣
↪Integer(2)*a02*y0*y2 + Integer(2)*a12*y1*y2 + a22*y2**Integer(2)
>>> p2 = b00*y0**Integer(2) + Integer(2)*b01*y0*y1 + b11*y1**Integer(2) +␣
↪Integer(2)*b02*y0*y2 + Integer(2)*b12*y1*y2 + b22*y2**Integer(2)
>>> q = invariant_theory.ternary_biquadratic(p1, p2, [y0, y1, y2])
>>> coeffs = det(t * q[Integer(0)].matrix() + q[Integer(1)].matrix()).
↪polynomial(t).coefficients(sparse=False)
>>> q.Delta_prime_invariant() == coeffs[Integer(0)]
True
```

F_covariant()

Return the F covariant.

EXAMPLES:

```
sage: R.<a00, a01, a11, a02, a12, a22, b00, b01, b11, b02, b12, b22, x, y> =␣
↪QQ[]
sage: p1 = 73*x^2 + 96*x*y - 11*y^2 + 4*x + 63*y + 57
sage: p2 = 61*x^2 - 100*x*y - 72*y^2 - 81*x + 39*y - 7
sage: q = invariant_theory.ternary_biquadratic(p1, p2, [x, y])
sage: q.F_covariant()
-32566577*x^2 + 29060637/2*x*y + 20153633/4*y^2 -
30250497/2*x - 241241273/4*y - 323820473/16
```

```
>>> from sage.all import *
>>> R = QQ['a00, a01, a11, a02, a12, a22, b00, b01, b11, b02, b12, b22, x, y
↪']; (a00, a01, a11, a02, a12, a22, b00, b01, b11, b02, b12, b22, x, y,) = R.
↪_first_ngens(14)
>>> p1 = Integer(73)*x**Integer(2) + Integer(96)*x*y -␣
↪Integer(11)*y**Integer(2) + Integer(4)*x + Integer(63)*y + Integer(57)
>>> p2 = Integer(61)*x**Integer(2) - Integer(100)*x*y -␣
↪Integer(72)*y**Integer(2) - Integer(81)*x + Integer(39)*y - Integer(7)
>>> q = invariant_theory.ternary_biquadratic(p1, p2, [x, y])
>>> q.F_covariant()
-32566577*x^2 + 29060637/2*x*y + 20153633/4*y^2 -
30250497/2*x - 241241273/4*y - 323820473/16
```

J_covariant()

Return the J covariant.

EXAMPLES:

```
sage: R.<a00, a01, a11, a02, a12, a22, b00, b01, b11, b02, b12, b22, x, y> =␣
↪QQ[]
sage: p1 = 73*x^2 + 96*x*y - 11*y^2 + 4*x + 63*y + 57
sage: p2 = 61*x^2 - 100*x*y - 72*y^2 - 81*x + 39*y - 7
sage: q = invariant_theory.ternary_biquadratic(p1, p2, [x, y])
sage: q.J_covariant()
1057324024445*x^3 + 1209531088209*x^2*y + 942116599708*x*y^2 +
984553030871*y^3 + 543715345505/2*x^2 - 3065093506021/2*x*y +
755263948570*y^2 - 1118430692650*x - 509948695327/4*y + 3369951531745/8
```

```
>>> from sage.all import *
>>> R = QQ['a00, a01, a11, a02, a12, a22, b00, b01, b11, b02, b12, b22, x, y
↪']; (a00, a01, a11, a02, a12, a22, b00, b01, b11, b02, b12, b22, x, y,) = R.
↪_first_ngens(14)
>>> p1 = Integer(73)*x**Integer(2) + Integer(96)*x*y -␣
↪Integer(11)*y**Integer(2) + Integer(4)*x + Integer(63)*y + Integer(57)
>>> p2 = Integer(61)*x**Integer(2) - Integer(100)*x*y -␣
↪Integer(72)*y**Integer(2) - Integer(81)*x + Integer(39)*y - Integer(7)
>>> q = invariant_theory.ternary_biquadratic(p1, p2, [x, y])
>>> q.J_covariant()
1057324024445*x^3 + 1209531088209*x^2*y + 942116599708*x*y^2 +
984553030871*y^3 + 543715345505/2*x^2 - 3065093506021/2*x*y +
755263948570*y^2 - 1118430692650*x - 509948695327/4*y + 3369951531745/8
```

Theta_invariant()

Return the Θ invariant.

EXAMPLES:

```
sage: R.<a00, a01, a11, a02, a12, a22, b00, b01, b11, b02, b12, b22, y0, y1,␣
↪y2, t> = QQ[]
sage: p1 = a00*y0^2 + 2*a01*y0*y1 + a11*y1^2 + 2*a02*y0*y2 + 2*a12*y1*y2 +␣
↪a22*y2^2
sage: p2 = b00*y0^2 + 2*b01*y0*y1 + b11*y1^2 + 2*b02*y0*y2 + 2*b12*y1*y2 +␣
↪b22*y2^2
sage: q = invariant_theory.ternary_biquadratic(p1, p2, [y0, y1, y2])
sage: coeffs = det(t * q[0].matrix() + q[1].matrix()).polynomial(t).
↪coefficients(sparse=False)
sage: q.Theta_invariant() == coeffs[2]
True
```

```
>>> from sage.all import *
>>> R = QQ['a00, a01, a11, a02, a12, a22, b00, b01, b11, b02, b12, b22, y0,␣
↪y1, y2, t']; (a00, a01, a11, a02, a12, a22, b00, b01, b11, b02, b12, b22,␣
↪y0, y1, y2, t,) = R._first_ngens(16)
>>> p1 = a00*y0**Integer(2) + Integer(2)*a01*y0*y1 + a11*y1**Integer(2) +␣
↪Integer(2)*a02*y0*y2 + Integer(2)*a12*y1*y2 + a22*y2**Integer(2)
>>> p2 = b00*y0**Integer(2) + Integer(2)*b01*y0*y1 + b11*y1**Integer(2) +␣
↪Integer(2)*b02*y0*y2 + Integer(2)*b12*y1*y2 + b22*y2**Integer(2)
>>> q = invariant_theory.ternary_biquadratic(p1, p2, [y0, y1, y2])
>>> coeffs = det(t * q[Integer(0)].matrix() + q[Integer(1)].matrix()).
↪polynomial(t).coefficients(sparse=False)
>>> q.Theta_invariant() == coeffs[Integer(2)]
True
```

Theta_prime_invariant()

Return the Θ' invariant.

EXAMPLES:

```
sage: R.<a00, a01, a11, a02, a12, a22, b00, b01, b11, b02, b12, b22, y0, y1,␣
↪y2, t> = QQ[]
sage: p1 = a00*y0^2 + 2*a01*y0*y1 + a11*y1^2 + 2*a02*y0*y2 + 2*a12*y1*y2 +␣
↪a22*y2^2
sage: p2 = b00*y0^2 + 2*b01*y0*y1 + b11*y1^2 + 2*b02*y0*y2 + 2*b12*y1*y2 +␣
↪b22*y2^2
sage: q = invariant_theory.ternary_biquadratic(p1, p2, [y0, y1, y2])
sage: coeffs = det(t * q[0].matrix() + q[1].matrix()).polynomial(t).
↪coefficients(sparse=False)
sage: q.Theta_prime_invariant() == coeffs[1]
True
```

```
>>> from sage.all import *
>>> R = QQ['a00, a01, a11, a02, a12, a22, b00, b01, b11, b02, b12, b22, y0,␣
↪y1, y2, t']; (a00, a01, a11, a02, a12, a22, b00, b01, b11, b02, b12, b22,␣
↪y0, y1, y2, t,) = R._first_ngens(16)
>>> p1 = a00*y0**Integer(2) + Integer(2)*a01*y0*y1 + a11*y1**Integer(2) +␣
↪Integer(2)*a02*y0*y2 + Integer(2)*a12*y1*y2 + a22*y2**Integer(2)
```

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```
>>> p2 = b00*y0**Integer(2) + Integer(2)*b01*y0*y1 + b11*y1**Integer(2) +␣
↪Integer(2)*b02*y0*y2 + Integer(2)*b12*y1*y2 + b22*y2**Integer(2)
>>> q = invariant_theory.ternary_biquadratic(p1, p2, [y0, y1, y2])
>>> coeffs = det(t * q[Integer(0)].matrix() + q[Integer(1)].matrix()).
↪polynomial(t).coefficients(sparse=False)
>>> q.Theta_prime_invariant() == coeffs[Integer(1)]
True
```

syzygy(*Delta*, *Theta*, *Theta_prime*, *Delta_prime*, *S*, *S_prime*, *F*, *J*)

Return the syzygy evaluated on the invariants and covariants.

INPUT:

- `Delta`, `Theta`, `Theta_prime`, `Delta_prime`, `S`, `S_prime`, `F`, `J` – polynomials from the same polynomial ring.

OUTPUT:

Zero if `S` is the first polynomial, `S_prime` the second polynomial, and the remaining input are the invariants and covariants of a ternary biquadratic.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: monomials = [x^2, x*y, y^2, x*z, y*z, z^2]
sage: def q_rnd():  return sum(randint(-1000, 1000)*m for m in monomials)
sage: biquadratic = invariant_theory.ternary_biquadratic(q_rnd(), q_rnd(), [x,
↪y,z])
sage: Delta = biquadratic.Delta_invariant()
sage: Theta = biquadratic.Theta_invariant()
sage: Theta_prime = biquadratic.Theta_prime_invariant()
sage: Delta_prime = biquadratic.Delta_prime_invariant()
sage: S = biquadratic.first().polynomial()
sage: S_prime  = biquadratic.second().polynomial()
sage: F = biquadratic.F_covariant()
sage: J = biquadratic.J_covariant()
sage: biquadratic.syzygy(Delta, Theta, Theta_prime, Delta_prime, S, S_prime,␣
↪F, J)
0
```

```
>>> from sage.all import *
>>> R = QQ['x, y, z']; (x, y, z,) = R._first_ngens(3)
>>> monomials = [x**Integer(2), x*y, y**Integer(2), x*z, y*z, z**Integer(2)]
>>> def q_rnd():  return sum(randint(-Integer(1000), Integer(1000))*m for m␣
↪in monomials)
>>> biquadratic = invariant_theory.ternary_biquadratic(q_rnd(), q_rnd(), [x,y,
↪z])
>>> Delta = biquadratic.Delta_invariant()
>>> Theta = biquadratic.Theta_invariant()
>>> Theta_prime = biquadratic.Theta_prime_invariant()
>>> Delta_prime = biquadratic.Delta_prime_invariant()
>>> S = biquadratic.first().polynomial()
>>> S_prime  = biquadratic.second().polynomial()
>>> F = biquadratic.F_covariant()
```

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```
>>> J = biquadratic.J_covariant()
>>> biquadratic.syzygy(Delta, Theta, Theta_prime, Delta_prime, S, S_prime, F,␣
↪J)
0
```

If the arguments are not the invariants and covariants then the output is some (generically nonzero) polynomial:

```
sage: biquadratic.syzygy(1, 1, 1, 1, 1, 1, 1, x)
1/64*x^2 + 1
```

```
>>> from sage.all import *
>>> biquadratic.syzygy(Integer(1), Integer(1), Integer(1), Integer(1),␣
↪Integer(1), Integer(1), Integer(1), x)
1/64*x^2 + 1
```

sage.rings.invariants.invariant_theory.**transvectant**(*f*, *g*, *h=1*, *scale='default'*)

Return the h-th transvectant of f and g.

INPUT:

- `f`, `g` – two homogeneous binary forms in the same polynomial ring
- `h` – the order of the transvectant; if it is not specified, the first transvectant is returned
- `scale` – the scaling factor applied to the result. Possible values are `'default'` and `'none'`. The `'default'` scaling factor is the one that appears in the output statement below, if the scaling factor is `'none'` the quotient of factorials is left out.

OUTPUT:

The h-th transvectant of the listed forms f and g:

$$(f,g)_h = \frac{(d_f-h)!\cdot(d_g-h)!}{d_f!\cdot d_g!}\left(\left(\frac{\partial}{\partial x}\frac{\partial}{\partial z'}-\frac{\partial}{\partial x'}\frac{\partial}{\partial z}\right)^h (f(x,z)\cdot g(x',z'))\right)_{(x',z')=(x,z)}$$

EXAMPLES:

```
sage: from sage.rings.invariants.invariant_theory import AlgebraicForm,␣
↪transvectant
sage: R.<x,y> = QQ[]
sage: f = AlgebraicForm(2, 5, x^5 + 5*x^4*y + 5*x*y^4 + y^5)
sage: transvectant(f, f, 4)
Binary quadratic given by 2*x^2 - 4*x*y + 2*y^2
sage: transvectant(f, f, 8)
Binary form of degree -6 given by 0
```

```
>>> from sage.all import *
>>> from sage.rings.invariants.invariant_theory import AlgebraicForm, transvectant
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = AlgebraicForm(Integer(2), Integer(5), x**Integer(5) +␣
↪Integer(5)*x**Integer(4)*y + Integer(5)*x*y**Integer(4) + y**Integer(5))
>>> transvectant(f, f, Integer(4))
Binary quadratic given by 2*x^2 - 4*x*y + 2*y^2
>>> transvectant(f, f, Integer(8))
Binary form of degree -6 given by 0
```

The default scaling will yield an error for fields of positive characteristic below $d_f!$ or $d_g!$ as the denominator of the scaling factor will not be invertible in that case. The scale argument `'none'` can be used to compute the transvectant in this case:

```
sage: # needs sage.rings.finite_rings
sage: R.<a0,a1,a2,a3,a4,a5,x0,x1> = GF(5)[]
sage: f = AlgebraicForm(2, 5, a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2
....:                         + a3*x1^2*x0^3 + a4*x1*x0^4 + a5*x0^5, x0, x1)
sage: transvectant(f, f, 4)
Traceback (most recent call last):
...
ZeroDivisionError
sage: transvectant(f, f, 4, scale='none')
Binary quadratic given by -a3^2*x0^2 + a2*a4*x0^2 + a2*a3*x0*x1
- a1*a4*x0*x1 - a2^2*x1^2 + a1*a3*x1^2
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> R = GF(Integer(5))['a0, a1, a2, a3, a4, a5, x0, x1']; (a0, a1, a2, a3, a4, a5,
↪ x0, x1,) = R._first_ngens(8)
>>> f = AlgebraicForm(Integer(2), Integer(5), a0*x1**Integer(5) +␣
↪a1*x1**Integer(4)*x0 + a2*x1**Integer(3)*x0**Integer(2)
...                         + a3*x1**Integer(2)*x0**Integer(3) +␣
↪a4*x1*x0**Integer(4) + a5*x0**Integer(5), x0, x1)
>>> transvectant(f, f, Integer(4))
Traceback (most recent call last):
...
ZeroDivisionError
>>> transvectant(f, f, Integer(4), scale='none')
Binary quadratic given by -a3^2*x0^2 + a2*a4*x0^2 + a2*a3*x0*x1
- a1*a4*x0*x1 - a2^2*x1^2 + a1*a3*x1^2
```

The additional factors that appear when `scale='none'` is used can be seen if we consider the same transvectant over the rationals and compare it to the scaled version:

```
sage: R.<a0,a1,a2,a3,a4,a5,x0,x1> = QQ[]
sage: f = AlgebraicForm(2, 5, a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2
....:                         + a3*x1^2*x0^3 + a4*x1*x0^4 + a5*x0^5, x0, x1)
sage: transvectant(f, f, 4)
Binary quadratic given by 3/50*a3^2*x0^2 - 4/25*a2*a4*x0^2
+ 2/5*a1*a5*x0^2 + 1/25*a2*a3*x0*x1 - 6/25*a1*a4*x0*x1 + 2*a0*a5*x0*x1
+ 3/50*a2^2*x1^2 - 4/25*a1*a3*x1^2 + 2/5*a0*a4*x1^2
sage: transvectant(f, f, 4, scale='none')
Binary quadratic given by 864*a3^2*x0^2 - 2304*a2*a4*x0^2
+ 5760*a1*a5*x0^2 + 576*a2*a3*x0*x1 - 3456*a1*a4*x0*x1
+ 28800*a0*a5*x0*x1 + 864*a2^2*x1^2 - 2304*a1*a3*x1^2 + 5760*a0*a4*x1^2
```

```
>>> from sage.all import *
>>> R = QQ['a0, a1, a2, a3, a4, a5, x0, x1']; (a0, a1, a2, a3, a4, a5, x0, x1,) =␣
↪R._first_ngens(8)
>>> f = AlgebraicForm(Integer(2), Integer(5), a0*x1**Integer(5) +␣
↪a1*x1**Integer(4)*x0 + a2*x1**Integer(3)*x0**Integer(2)
...                         + a3*x1**Integer(2)*x0**Integer(3) +␣
```

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```
↪a4*x1*x0**Integer(4) + a5*x0**Integer(5), x0, x1)
>>> transvectant(f, f, Integer(4))
Binary quadratic given by 3/50*a3^2*x0^2 - 4/25*a2*a4*x0^2
+ 2/5*a1*a5*x0^2 + 1/25*a2*a3*x0*x1 - 6/25*a1*a4*x0*x1 + 2*a0*a5*x0*x1
+ 3/50*a2^2*x1^2 - 4/25*a1*a3*x1^2 + 2/5*a0*a4*x1^2
>>> transvectant(f, f, Integer(4), scale='none')
Binary quadratic given by 864*a3^2*x0^2 - 2304*a2*a4*x0^2
+ 5760*a1*a5*x0^2 + 576*a2*a3*x0*x1 - 3456*a1*a4*x0*x1
+ 28800*a0*a5*x0*x1 + 864*a2^2*x1^2 - 2304*a1*a3*x1^2 + 5760*a0*a4*x1^2
```

If the forms are given as inhomogeneous polynomials, the homogenisation might fail if the polynomial ring has multiple variables. You can circumvent this by making sure the base ring of the polynomial has only one variable:

```
sage: R.<x,y> = QQ[]
sage: quintic = invariant_theory.binary_quintic(x^5 + x^3 + 2*x^2 + y^5, x)
sage: transvectant(quintic, quintic, 2)
Traceback (most recent call last):
...
ValueError: polynomial is not homogeneous
sage: R.<y> = QQ[]
sage: S.<x> = R[]
sage: quintic = invariant_theory.binary_quintic(x^5 + x^3 + 2*x^2 + y^5, x)
sage: transvectant(quintic, quintic, 2)
Binary sextic given by 1/5*x^6 + 6/5*x^5*h - 3/25*x^4*h^2
+ (50*y^5 - 8)/25*x^3*h^3 - 12/25*x^2*h^4 + (3*y^5)/5*x*h^5
+ (2*y^5)/5*h^6
```

```
>>> from sage.all import *
>>> R = QQ['x, y']; (x, y,) = R._first_ngens(2)
>>> quintic = invariant_theory.binary_quintic(x**Integer(5) + x**Integer(3) +␣
↪Integer(2)*x**Integer(2) + y**Integer(5), x)
>>> transvectant(quintic, quintic, Integer(2))
Traceback (most recent call last):
...
ValueError: polynomial is not homogeneous
>>> R = QQ['y']; (y,) = R._first_ngens(1)
>>> S = R['x']; (x,) = S._first_ngens(1)
>>> quintic = invariant_theory.binary_quintic(x**Integer(5) + x**Integer(3) +␣
↪Integer(2)*x**Integer(2) + y**Integer(5), x)
>>> transvectant(quintic, quintic, Integer(2))
Binary sextic given by 1/5*x^6 + 6/5*x^5*h - 3/25*x^4*h^2
+ (50*y^5 - 8)/25*x^3*h^3 - 12/25*x^2*h^4 + (3*y^5)/5*x*h^5
+ (2*y^5)/5*h^6
```

3.2.2 Reconstruction of Algebraic Forms

This module reconstructs algebraic forms from the values of their invariants. Given a set of (classical) invariants, it returns a form that attains this values as invariants (up to scaling).

AUTHORS:

- Jesper Noordsij (2018-06): initial version sage.rings.invariants.reconstruction.**binary_cubic_coefficients_from_invariants**(*discriminant*, *invariant_choice='default'*)

Reconstruct a binary cubic from the value of its discriminant.

INPUT:

- `discriminant` – the value of the discriminant of the binary cubic
- `invariant_choice` – the type of invariants provided. The accepted options are `'discriminant'` and `'default'`, which are the same. No other options are implemented.

OUTPUT:

A set of coefficients of a binary cubic, whose discriminant is equal to the given `discriminant` up to a scaling.

EXAMPLES:

```
sage: from sage.rings.invariants.reconstruction import binary_cubic_coefficients_
↪from_invariants
sage: coeffs = binary_cubic_coefficients_from_invariants(1)
sage: coeffs
(0, 1, -1, 0)
sage: R.<x> = QQ[]
sage: R(coeffs).discriminant()                                              #␣
↪needs sage.libs.pari
1
```

```
>>> from sage.all import *
>>> from sage.rings.invariants.reconstruction import binary_cubic_coefficients_
↪from_invariants
>>> coeffs = binary_cubic_coefficients_from_invariants(Integer(1))
>>> coeffs
(0, 1, -1, 0)
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> R(coeffs).discriminant()                                                #␣
↪needs sage.libs.pari
1
```

The two non-equivalent cubics x^3 and $x^2 * z$ with discriminant 0 can't be distinguished based on their discriminant, hence an error is raised:

```
sage: binary_cubic_coefficients_from_invariants(0)
Traceback (most recent call last):
...
ValueError: no unique reconstruction possible for binary cubics with a double root
```

```
>>> from sage.all import *
>>> binary_cubic_coefficients_from_invariants(Integer(0))
Traceback (most recent call last):
...
ValueError: no unique reconstruction possible for binary cubics with a double root
```

sage.rings.invariants.reconstruction.**binary_quadratic_coefficients_from_invariants**(*discriminant*, *invariant_choice='default'*)

Reconstruct a binary quadratic from the value of its discriminant.

INPUT:

- `discriminant` – the value of the discriminant of the binary quadratic
- `invariant_choice` – the type of invariants provided. The accepted options are `'discriminant'` and `'default'`, which are the same. No other options are implemented.

OUTPUT:

A set of coefficients of a binary quadratic, whose discriminant is equal to the given `discriminant` up to a scaling.

EXAMPLES:

```
sage: from sage.rings.invariants.reconstruction import binary_quadratic_
↪coefficients_from_invariants
sage: quadratic = invariant_theory.binary_form_from_invariants(2, [24])   #␣
↪indirect doctest
sage: quadratic
Binary quadratic with coefficients (1, -6, 0)
sage: quadratic.discriminant()
24
sage: binary_quadratic_coefficients_from_invariants(0)
(1, 0, 0)
```

```
>>> from sage.all import *
>>> from sage.rings.invariants.reconstruction import binary_quadratic_
↪coefficients_from_invariants
>>> quadratic = invariant_theory.binary_form_from_invariants(Integer(2),␣
↪[Integer(24)])   # indirect doctest
>>> quadratic
Binary quadratic with coefficients (1, -6, 0)
>>> quadratic.discriminant()
24
>>> binary_quadratic_coefficients_from_invariants(Integer(0))
(1, 0, 0)
```

sage.rings.invariants.reconstruction.**binary_quintic_coefficients_from_invariants**(*invariants*, *K=None*, *invariant_choice='default'*, *scaling='none'*)

Reconstruct a binary quintic from the values of its (Clebsch) invariants.

INPUT:

- `invariants` – list or tuple of values of the three or four invariants. The default option requires the Clebsch invariants A, B, C and R of the binary quintic.
- `K` – the field over which the quintic is defined
- `invariant_choice` – the type of invariants provided. The accepted options are `'clebsch'` and `'default'`, which are the same. No other options are implemented.
- `scaling` – how the coefficients should be scaled. The accepted values are `'none'` for no scaling, `'normalized'` to scale in such a way that the resulting coefficients are independent of the scaling of the input invariants and `'coprime'` which scales the input invariants by dividing them by their gcd.

OUTPUT:

A set of coefficients of a binary quintic, whose invariants are equal to the given `invariants` up to a scaling.

EXAMPLES:

First we check the general case, where the invariant M is nonzero:

```
sage: R.<x0, x1> = QQ[]
sage: p = 3*x1^5 + 6*x1^4*x0 + 3*x1^3*x0^2 + 4*x1^2*x0^3 - 5*x1*x0^4 + 4*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: invs = quintic.clebsch_invariants(as_tuple=True)
sage: reconstructed = invariant_theory.binary_form_from_invariants(  # indirect␣
↪doctest
....:     5, invs, variables=quintic.variables())
sage: reconstructed
Binary quintic with coefficients (9592267437341790539005557/244140625000000,
2149296928207625556323004064707/610351562500000000,
11149651890347700974453304786783/76293945312500000,
122650775751894638395648891202734239/47683715820312500000,
323996630945706528474286334593218447/11920928955078125000,
1504506503644608395841632538558481466127/14901161193847656250000)
```

```
>>> from sage.all import *
>>> R = QQ['x0, x1']; (x0, x1,) = R._first_ngens(2)
>>> p = Integer(3)*x1**Integer(5) + Integer(6)*x1**Integer(4)*x0 +␣
↪Integer(3)*x1**Integer(3)*x0**Integer(2) +␣
↪Integer(4)*x1**Integer(2)*x0**Integer(3) - Integer(5)*x1*x0**Integer(4) +␣
↪Integer(4)*x0**Integer(5)
>>> quintic = invariant_theory.binary_quintic(p, x0, x1)
>>> invs = quintic.clebsch_invariants(as_tuple=True)
>>> reconstructed = invariant_theory.binary_form_from_invariants(  # indirect␣
↪doctest
...     Integer(5), invs, variables=quintic.variables())
>>> reconstructed
Binary quintic with coefficients (9592267437341790539005557/244140625000000,
2149296928207625556323004064707/610351562500000000,
11149651890347700974453304786783/76293945312500000,
122650775751894638395648891202734239/47683715820312500000,
323996630945706528474286334593218447/11920928955078125000,
1504506503644608395841632538558481466127/14901161193847656250000)
```

We can see that the invariants of the reconstructed form match the ones of the original form by scaling the invariants B and C:

```
sage: scale = invs[0]/reconstructed.A_invariant()
sage: invs[1] == reconstructed.B_invariant()*scale^2
True
sage: invs[2] == reconstructed.C_invariant()*scale^3
True
```

```
>>> from sage.all import *
>>> scale = invs[Integer(0)]/reconstructed.A_invariant()
>>> invs[Integer(1)] == reconstructed.B_invariant()*scale**Integer(2)
True
>>> invs[Integer(2)] == reconstructed.C_invariant()*scale**Integer(3)
True
```

If we compare the form obtained by this reconstruction to the one found by letting the covariants α and β be the coordinates of the form, we find the forms are the same up to a power of the determinant of α and β:

```
sage: alpha = quintic.alpha_covariant()
sage: beta = quintic.beta_covariant()
sage: g = matrix([[alpha(x0=1,x1=0), alpha(x0=0,x1=1)],
....:             [beta(x0=1,x1=0), beta(x0=0,x1=1)]])^-1
sage: transformed = tuple([g.determinant()^-5*x
....:                     for x in quintic.transformed(g).coeffs()])
sage: transformed == reconstructed.coeffs()
True
```

```
>>> from sage.all import *
>>> alpha = quintic.alpha_covariant()
>>> beta = quintic.beta_covariant()
>>> g = matrix([[alpha(x0=Integer(1),x1=Integer(0)), alpha(x0=Integer(0),
↪x1=Integer(1))],
...             [beta(x0=Integer(1),x1=Integer(0)), beta(x0=Integer(0),
↪x1=Integer(1))]])**-Integer(1)
>>> transformed = tuple([g.determinant()**-Integer(5)*x
...                     for x in quintic.transformed(g).coeffs()])
>>> transformed == reconstructed.coeffs()
True
```

This can also be seen by computing the α covariant of the obtained form:

```
sage: reconstructed.alpha_covariant().coefficient(x1)
0
sage: reconstructed.alpha_covariant().coefficient(x0) != 0
True
```

```
>>> from sage.all import *
>>> reconstructed.alpha_covariant().coefficient(x1)
0
>>> reconstructed.alpha_covariant().coefficient(x0) != Integer(0)
True
```

If the invariant M vanishes, then the coefficients are computed in a different way:

```
sage: [A,B,C] = [3,1,2]
sage: M = 2*A*B - 3*C
sage: M
0
sage: from sage.rings.invariants.reconstruction import binary_quintic_
↪coefficients_from_invariants
sage: reconstructed = binary_quintic_coefficients_from_invariants([A,B,C])
sage: reconstructed
(-66741943359375/2097152,
 -125141143798828125/134217728,
 0,
 52793920040130615234375/34359738368,
 19797720015048980712890625/1099511627776,
 -4454487003386020660400390625/17592186044416)
sage: newform = sum([ reconstructed[i]*x0^i*x1^(5-i) for i in range(6) ])
sage: newquintic = invariant_theory.binary_quintic(newform, x0, x1)
sage: scale = 3/newquintic.A_invariant()
sage: [3, newquintic.B_invariant()*scale^2, newquintic.C_invariant()*scale^3]
[3, 1, 2]
```

```
>>> from sage.all import *
>>> [A,B,C] = [Integer(3),Integer(1),Integer(2)]
>>> M = Integer(2)*A*B - Integer(3)*C
>>> M
0
>>> from sage.rings.invariants.reconstruction import binary_quintic_coefficients_
↪from_invariants
>>> reconstructed = binary_quintic_coefficients_from_invariants([A,B,C])
>>> reconstructed
(-66741943359375/2097152,
 -125141143798828125/134217728,
 0,
 52793920040130615234375/34359738368,
 19797720015048980712890625/1099511627776,
 -4454487003386020660400390625/17592186044416)
>>> newform = sum([ reconstructed[i]*x0**i*x1**(Integer(5)-i) for i in␣
↪range(Integer(6)) ])
>>> newquintic = invariant_theory.binary_quintic(newform, x0, x1)
>>> scale = Integer(3)/newquintic.A_invariant()
>>> [Integer(3), newquintic.B_invariant()*scale**Integer(2), newquintic.C_
↪invariant()*scale**Integer(3)]
[3, 1, 2]
```

Several special cases:

```
sage: quintic = invariant_theory.binary_quintic(x0^5 - x1^5, x0, x1)
sage: invs = quintic.clebsch_invariants(as_tuple=True)
sage: binary_quintic_coefficients_from_invariants(invs)
(1, 0, 0, 0, 0, 1)
sage: quintic = invariant_theory.binary_quintic(x0*x1*(x0^3-x1^3), x0, x1)
sage: invs = quintic.clebsch_invariants(as_tuple=True)
sage: binary_quintic_coefficients_from_invariants(invs)
(0, 1, 0, 0, 1, 0)
```

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```
sage: quintic = invariant_theory.binary_quintic(x0^5 + 10*x0^3*x1^2 - 15*x0*x1^4,␣
↪x0, x1)
sage: invs = quintic.clebsch_invariants(as_tuple=True)
sage: binary_quintic_coefficients_from_invariants(invs)
(1, 0, 10, 0, -15, 0)
sage: quintic = invariant_theory.binary_quintic(x0^2*(x0^3 + x1^3), x0, x1)
sage: invs = quintic.clebsch_invariants(as_tuple=True)
sage: binary_quintic_coefficients_from_invariants(invs)
(1, 0, 0, 1, 0, 0)
sage: quintic = invariant_theory.binary_quintic(x0*(x0^4 + x1^4), x0, x1)
sage: invs = quintic.clebsch_invariants(as_tuple=True)
sage: binary_quintic_coefficients_from_invariants(invs)
(1, 0, 0, 0, 1, 0)
```

```
>>> from sage.all import *
>>> quintic = invariant_theory.binary_quintic(x0**Integer(5) - x1**Integer(5), x0,
↪ x1)
>>> invs = quintic.clebsch_invariants(as_tuple=True)
>>> binary_quintic_coefficients_from_invariants(invs)
(1, 0, 0, 0, 0, 1)
>>> quintic = invariant_theory.binary_quintic(x0*x1*(x0**Integer(3)-
↪x1**Integer(3)), x0, x1)
>>> invs = quintic.clebsch_invariants(as_tuple=True)
>>> binary_quintic_coefficients_from_invariants(invs)
(0, 1, 0, 0, 1, 0)
>>> quintic = invariant_theory.binary_quintic(x0**Integer(5) +␣
↪Integer(10)*x0**Integer(3)*x1**Integer(2) - Integer(15)*x0*x1**Integer(4), x0,␣
↪x1)
>>> invs = quintic.clebsch_invariants(as_tuple=True)
>>> binary_quintic_coefficients_from_invariants(invs)
(1, 0, 10, 0, -15, 0)
>>> quintic = invariant_theory.binary_quintic(x0**Integer(2)*(x0**Integer(3) +␣
↪x1**Integer(3)), x0, x1)
>>> invs = quintic.clebsch_invariants(as_tuple=True)
>>> binary_quintic_coefficients_from_invariants(invs)
(1, 0, 0, 1, 0, 0)
>>> quintic = invariant_theory.binary_quintic(x0*(x0**Integer(4) +␣
↪x1**Integer(4)), x0, x1)
>>> invs = quintic.clebsch_invariants(as_tuple=True)
>>> binary_quintic_coefficients_from_invariants(invs)
(1, 0, 0, 0, 1, 0)
```

For fields of characteristic 2, 3 or 5, there is no reconstruction implemented. This is part of Issue #26786.:

```
sage: binary_quintic_coefficients_from_invariants([3,1,2], K=GF(5))
Traceback (most recent call last):
...
NotImplementedError: no reconstruction of binary quintics implemented
for fields of characteristic 2, 3 or 5
```

```
>>> from sage.all import *
>>> binary_quintic_coefficients_from_invariants([Integer(3),Integer(1),
```

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```
↪Integer(2)], K=GF(Integer(5)))
Traceback (most recent call last):
...
NotImplementedError: no reconstruction of binary quintics implemented
for fields of characteristic 2, 3 or 5
```

3.3 Educational Versions of Groebner Basis Related Algorithms

3.3.1 Educational versions of Groebner basis algorithms

Following [BW1993], the original Buchberger algorithm (algorithm GROEBNER in [BW1993]) and an improved version of Buchberger's algorithm (algorithm GROEBNERNEW2 in [BW1993]) are implemented.

No attempt was made to optimize either algorithm as the emphasis of these implementations is a clean and easy presentation. To compute a Groebner basis most efficiently in Sage, use the `MPolynomialIdeal.groebner_basis()` method on multivariate polynomial objects instead.

Note

The notion of 'term' and 'monomial' in [BW1993] is swapped from the notion of those words in Sage (or the other way around, however you prefer it). In Sage a term is a monomial multiplied by a coefficient, while in [BW1993] a monomial is a term multiplied by a coefficient. Also, what is called LM (the leading monomial) in Sage is called HT (the head term) in [BW1993].

EXAMPLES:

Consider Katsura-6 with respect to a `degrevlex` ordering.

```
sage: # needs sage.libs.singular sage.rings.finite_rings
sage: from sage.rings.polynomial.toy_buchberger import *
sage: P.<a,b,c,e,f,g,h,i,j,k> = PolynomialRing(GF(32003))
sage: I = sage.rings.ideal.Katsura(P, 6)
sage: g1 = buchberger(I)
sage: g2 = buchberger_improved(I)
sage: g3 = I.groebner_basis()
```

```
>>> from sage.all import *
>>> # needs sage.libs.singular sage.rings.finite_rings
>>> from sage.rings.polynomial.toy_buchberger import *
>>> P = PolynomialRing(GF(Integer(32003)), names=('a', 'b', 'c', 'e', 'f', 'g', 'h',
↪'i', 'j', 'k',)); (a, b, c, e, f, g, h, i, j, k,) = P._first_ngens(10)
>>> I = sage.rings.ideal.Katsura(P, Integer(6))
>>> g1 = buchberger(I)
>>> g2 = buchberger_improved(I)
>>> g3 = I.groebner_basis()
```

All algorithms actually compute a Groebner basis:

```
sage: # needs sage.libs.singular sage.rings.finite_rings
sage: Ideal(g1).basis_is_groebner()
True
```

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```
sage: Ideal(g2).basis_is_groebner()
True
sage: Ideal(g3).basis_is_groebner()
True
```

```
>>> from sage.all import *
>>> # needs sage.libs.singular sage.rings.finite_rings
>>> Ideal(g1).basis_is_groebner()
True
>>> Ideal(g2).basis_is_groebner()
True
>>> Ideal(g3).basis_is_groebner()
True
```

The results are correct:

```
sage: # needs sage.libs.singular sage.rings.finite_rings
sage: Ideal(g1) == Ideal(g2) == Ideal(g3)
True
```

```
>>> from sage.all import *
>>> # needs sage.libs.singular sage.rings.finite_rings
>>> Ideal(g1) == Ideal(g2) == Ideal(g3)
True
```

If `get_verbose()` is ≥ 1, a protocol is provided:

```
sage: # needs sage.libs.singular sage.rings.finite_rings
sage: from sage.misc.verbose import set_verbose
sage: set_verbose(1)
sage: P.<a,b,c> = PolynomialRing(GF(127))
sage: I = sage.rings.ideal.Katsura(P)
// sage... ideal
sage: I
Ideal (a + 2*b + 2*c - 1, a^2 + 2*b^2 + 2*c^2 - a, 2*a*b + 2*b*c - b)
 of Multivariate Polynomial Ring in a, b, c over Finite Field of size 127
sage: buchberger(I)  # random
(a + 2*b + 2*c - 1, a^2 + 2*b^2 + 2*c^2 - a) => -2*b^2 - 6*b*c - 6*c^2 + b + 2*c
G: set([a + 2*b + 2*c - 1, 2*a*b + 2*b*c - b, a^2 + 2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c␣
↪- 6*c^2 + b + 2*c])

(a^2 + 2*b^2 + 2*c^2 - a, a + 2*b + 2*c - 1) => 0
G: set([a + 2*b + 2*c - 1, 2*a*b + 2*b*c - b, a^2 + 2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c␣
↪- 6*c^2 + b + 2*c])

(a + 2*b + 2*c - 1, 2*a*b + 2*b*c - b) => -5*b*c - 6*c^2 - 63*b + 2*c
G: set([a + 2*b + 2*c - 1, 2*a*b + 2*b*c - b, -5*b*c - 6*c^2 - 63*b + 2*c, a^2 + 2*b^
↪2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c])

(2*a*b + 2*b*c - b, a + 2*b + 2*c - 1) => 0
G: set([a + 2*b + 2*c - 1, 2*a*b + 2*b*c - b, -5*b*c - 6*c^2 - 63*b + 2*c, a^2 + 2*b^
↪2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c])
```

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```
(2*a*b + 2*b*c - b, -5*b*c - 6*c^2 - 63*b + 2*c) => -22*c^3 + 24*c^2 - 60*b - 62*c
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +␣
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

(2*a*b + 2*b*c - b, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +␣
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

(2*a*b + 2*b*c - b, a^2 + 2*b^2 + 2*c^2 - a) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +␣
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

(a + 2*b + 2*c - 1, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +␣
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

(a^2 + 2*b^2 + 2*c^2 - a, 2*a*b + 2*b*c - b) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +␣
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

(-2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +␣
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

(a + 2*b + 2*c - 1, -5*b*c - 6*c^2 - 63*b + 2*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +␣
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

(a^2 + 2*b^2 + 2*c^2 - a, -5*b*c - 6*c^2 - 63*b + 2*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +␣
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

(-5*b*c - 6*c^2 - 63*b + 2*c, -22*c^3 + 24*c^2 - 60*b - 62*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +␣
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

(a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +␣
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

(a^2 + 2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +␣
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

(-2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -22*c^3 + 24*c^2 - 60*b - 62*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +␣
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

(2*a*b + 2*b*c - b, -22*c^3 + 24*c^2 - 60*b - 62*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +␣
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])
```

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```
(a^2 + 2*b^2 + 2*c^2 - a, -22*c^3 + 24*c^2 - 60*b - 62*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

15 reductions to zero.
[a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 + 2*b^2 +
↪2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c]
```

```
>>> from sage.all import *
>>> # needs sage.libs.singular sage.rings.finite_rings
>>> from sage.misc.verbose import set_verbose
>>> set_verbose(Integer(1))
>>> P = PolynomialRing(GF(Integer(127)), names=('a', 'b', 'c',)); (a, b, c,) = P._
↪first_ngens(3)
>>> I = sage.rings.ideal.Katsura(P)
// sage... ideal
>>> I
Ideal (a + 2*b + 2*c - 1, a^2 + 2*b^2 + 2*c^2 - a, 2*a*b + 2*b*c - b)
 of Multivariate Polynomial Ring in a, b, c over Finite Field of size 127
>>> buchberger(I)  # random
(a + 2*b + 2*c - 1, a^2 + 2*b^2 + 2*c^2 - a) => -2*b^2 - 6*b*c - 6*c^2 + b + 2*c
G: set([a + 2*b + 2*c - 1, 2*a*b + 2*b*c - b, a^2 + 2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c
↪- 6*c^2 + b + 2*c])
<BLANKLINE>
(a^2 + 2*b^2 + 2*c^2 - a, a + 2*b + 2*c - 1) => 0
G: set([a + 2*b + 2*c - 1, 2*a*b + 2*b*c - b, a^2 + 2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c
↪- 6*c^2 + b + 2*c])
<BLANKLINE>
(a + 2*b + 2*c - 1, 2*a*b + 2*b*c - b) => -5*b*c - 6*c^2 - 63*b + 2*c
G: set([a + 2*b + 2*c - 1, 2*a*b + 2*b*c - b, -5*b*c - 6*c^2 - 63*b + 2*c, a^2 + 2*b^
↪2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c])
<BLANKLINE>
(2*a*b + 2*b*c - b, a + 2*b + 2*c - 1) => 0
G: set([a + 2*b + 2*c - 1, 2*a*b + 2*b*c - b, -5*b*c - 6*c^2 - 63*b + 2*c, a^2 + 2*b^
↪2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c])
<BLANKLINE>
(2*a*b + 2*b*c - b, -5*b*c - 6*c^2 - 63*b + 2*c) => -22*c^3 + 24*c^2 - 60*b - 62*c
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])
<BLANKLINE>
(2*a*b + 2*b*c - b, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])
<BLANKLINE>
(2*a*b + 2*b*c - b, a^2 + 2*b^2 + 2*c^2 - a) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])
<BLANKLINE>
(a + 2*b + 2*c - 1, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])
```

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```
<BLANKLINE>
(a^2 + 2*b^2 + 2*c^2 - a, 2*a*b + 2*b*c - b) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])
<BLANKLINE>
(-2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])
<BLANKLINE>
(a + 2*b + 2*c - 1, -5*b*c - 6*c^2 - 63*b + 2*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])
<BLANKLINE>
(a^2 + 2*b^2 + 2*c^2 - a, -5*b*c - 6*c^2 - 63*b + 2*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])
<BLANKLINE>
(-5*b*c - 6*c^2 - 63*b + 2*c, -22*c^3 + 24*c^2 - 60*b - 62*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])
<BLANKLINE>
(a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])
<BLANKLINE>
(a^2 + 2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])
<BLANKLINE>
(-2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -22*c^3 + 24*c^2 - 60*b - 62*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])
<BLANKLINE>
(2*a*b + 2*b*c - b, -22*c^3 + 24*c^2 - 60*b - 62*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])
<BLANKLINE>
(a^2 + 2*b^2 + 2*c^2 - a, -22*c^3 + 24*c^2 - 60*b - 62*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
↪2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])
<BLANKLINE>
15 reductions to zero.
[a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 + 2*b^2 +
↪2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c]
```

The original Buchberger algorithm performs 15 useless reductions to zero for this example:

```
sage: # needs sage.libs.singular sage.rings.finite_rings
sage: gb = buchberger(I)
...
15 reductions to zero.
```

```
>>> from sage.all import *
>>> # needs sage.libs.singular sage.rings.finite_rings
>>> gb = buchberger(I)
...
15 reductions to zero.
```

The 'improved' Buchberger algorithm in contrast only performs 1 reduction to zero:

```
sage: # needs sage.libs.singular sage.rings.finite_rings
sage: gb = buchberger_improved(I)
...
1 reductions to zero.
sage: sorted(gb)
[a + 2*b + 2*c - 1, b*c + 52*c^2 + 38*b + 25*c,
 b^2 - 26*c^2 - 51*b + 51*c, c^3 + 22*c^2 - 55*b + 49*c]
```

```
>>> from sage.all import *
>>> # needs sage.libs.singular sage.rings.finite_rings
>>> gb = buchberger_improved(I)
...
1 reductions to zero.
>>> sorted(gb)
[a + 2*b + 2*c - 1, b*c + 52*c^2 + 38*b + 25*c,
 b^2 - 26*c^2 - 51*b + 51*c, c^3 + 22*c^2 - 55*b + 49*c]
```

AUTHORS:

- Martin Albrecht (2007-05-24): initial version
- Marshall Hampton (2009-07-08): some doctest additions

`sage.rings.polynomial.toy_buchberger.`**`LCM`**`(`f, g`)`

`sage.rings.polynomial.toy_buchberger.`**`LM`**`(`f`)`

`sage.rings.polynomial.toy_buchberger.`**`LT`**`(`f`)`

`sage.rings.polynomial.toy_buchberger.`**`buchberger`**`(`F`)`

Compute a Groebner basis using the original version of Buchberger's algorithm as presented in [BW1993], page 214.

INPUT:

- `F` – an ideal in a multivariate polynomial ring

OUTPUT: a Groebner basis for F

Note

The verbosity of this function may be controlled with a `set_verbose()` call. Any value >=1 will result in this function printing intermediate bases.

EXAMPLES:

```
sage: from sage.rings.polynomial.toy_buchberger import buchberger
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: I = R.ideal([x^2 - z - 1, z^2 - y - 1, x*y^2 - x - 1])
sage: set_verbose(0)
sage: gb = buchberger(I)                                                    #␣
↪needs sage.libs.singular
sage: gb.is_groebner()                                                      #␣
↪needs sage.libs.singular
True
sage: gb.ideal() == I                                                       #␣
↪needs sage.libs.singular
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.toy_buchberger import buchberger
>>> R = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = R._first_ngens(3)
>>> I = R.ideal([x**Integer(2) - z - Integer(1), z**Integer(2) - y - Integer(1),␣
↪x*y**Integer(2) - x - Integer(1)])
>>> set_verbose(Integer(0))
>>> gb = buchberger(I)                                                      #␣
↪needs sage.libs.singular
>>> gb.is_groebner()                                                        #␣
↪needs sage.libs.singular
True
>>> gb.ideal() == I                                                         #␣
↪needs sage.libs.singular
True
```

sage.rings.polynomial.toy_buchberger.**buchberger_improved**(*F*)

Compute a Groebner basis using an improved version of Buchberger's algorithm as presented in [BW1993], page 232.

This variant uses the Gebauer-Moeller Installation to apply Buchberger's first and second criterion to avoid useless pairs.

INPUT:

- `F` – an ideal in a multivariate polynomial ring

OUTPUT: a Groebner basis for F

Note

The verbosity of this function may be controlled with a `set_verbose()` call. Any value `>=1` will result in this function printing intermediate Groebner bases.

EXAMPLES:

```
sage: from sage.rings.polynomial.toy_buchberger import buchberger_improved
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: set_verbose(0)
sage: sorted(buchberger_improved(R.ideal([x^4 - y - z, x*y*z - 1])))       #␣
↪needs sage.libs.singular
[x*y*z - 1, x^3 - y^2*z - y*z^2, y^3*z^2 + y^2*z^3 - x^2]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.toy_buchberger import buchberger_improved
>>> R = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = R._first_ngens(3)
>>> set_verbose(Integer(0))
>>> sorted(buchberger_improved(R.ideal([x**Integer(4) - y - z, x*y*z -␣
↪Integer(1)])))              # needs sage.libs.singular
[x*y*z - 1, x^3 - y^2*z - y*z^2, y^3*z^2 + y^2*z^3 - x^2]
```

sage.rings.polynomial.toy_buchberger.**inter_reduction**(Q)

Compute inter-reduced polynomials from a set of polynomials.

INPUT:

- `Q` – set of polynomials

OUTPUT:

if `Q` is the set $f_1, ..., f_n$, this method returns $g_1, ..., g_s$ such that:

- $(f_1, ..., f_n) = (g_1, ..., g_s)$
- $LM(g_i) \neq LM(g_j)$ for all $i \neq j$
- $LM(g_i)$ does not divide m for all monomials m of $\{g_1, ..., g_{i-1}, g_{i+1}, ..., g_s\}$
- $LC(g_i) = 1$ for all i.

EXAMPLES:

```
sage: from sage.rings.polynomial.toy_buchberger import inter_reduction
sage: inter_reduction(set())
set()
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.toy_buchberger import inter_reduction
>>> inter_reduction(set())
set()
```

```
sage: P.<x,y> = QQ[]
sage: reduced = inter_reduction(set([x^2 - 5*y^2, x^3]))                     #␣
↪needs sage.libs.singular
sage: reduced == set([x*y^2, x^2 - 5*y^2])                                   #␣
↪needs sage.libs.singular
True
sage: reduced == inter_reduction(set([2*(x^2 - 5*y^2), x^3]))                #␣
↪needs sage.libs.singular
True
```

```
>>> from sage.all import *
>>> P = QQ['x, y']; (x, y,) = P._first_ngens(2)
>>> reduced = inter_reduction(set([x**Integer(2) - Integer(5)*y**Integer(2),␣
↪x**Integer(3)]))                          # needs sage.libs.singular
>>> reduced == set([x*y**Integer(2), x**Integer(2) - Integer(5)*y**Integer(2)])  ␣
↪                                     # needs sage.libs.singular
True
>>> reduced == inter_reduction(set([Integer(2)*(x**Integer(2) -␣
```

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```
↪Integer(5)*y**Integer(2)), x**Integer(3)]))                          # needs sage.libs.
↪singular
True
```

sage.rings.polynomial.toy_buchberger.**select**(*P*)

Select a polynomial using the normal selection strategy.

INPUT:

- `P` – list of critical pairs

OUTPUT: an element of P

EXAMPLES:

```
sage: from sage.rings.polynomial.toy_buchberger import select
sage: R.<x,y,z> = PolynomialRing(QQ, order='lex')
sage: ps = [x^3 - z - 1, z^3 - y - 1, x^5 - y - 2]
sage: pairs = [[ps[i], ps[j]] for i in range(3) for j in range(i + 1, 3)]
sage: select(pairs)
[x^3 - z - 1, -y + z^3 - 1]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.toy_buchberger import select
>>> R = PolynomialRing(QQ, order='lex', names=('x', 'y', 'z',)); (x, y, z,) = R._
↪first_ngens(3)
>>> ps = [x**Integer(3) - z - Integer(1), z**Integer(3) - y - Integer(1),␣
↪x**Integer(5) - y - Integer(2)]
>>> pairs = [[ps[i], ps[j]] for i in range(Integer(3)) for j in range(i +␣
↪Integer(1), Integer(3))]
>>> select(pairs)
[x^3 - z - 1, -y + z^3 - 1]
```

sage.rings.polynomial.toy_buchberger.**spol**(*f*, *g*)

Compute the S-polynomial of f and g.

INPUT:

- `f`, `g` – polynomials

OUTPUT: the S-polynomial of f and g

EXAMPLES:

```
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: from sage.rings.polynomial.toy_buchberger import spol
sage: spol(x^2 - z - 1, z^2 - y - 1)
x^2*y - z^3 + x^2 - z^2
```

```
>>> from sage.all import *
>>> R = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = R._first_ngens(3)
>>> from sage.rings.polynomial.toy_buchberger import spol
>>> spol(x**Integer(2) - z - Integer(1), z**Integer(2) - y - Integer(1))
x^2*y - z^3 + x^2 - z^2
```

`sage.rings.polynomial.toy_buchberger.`**`update`**`(`G, B, h`)`

Update `G` using the set of critical pairs `B` and the polynomial `h` as presented in [BW1993], page 230. For this, Buchberger's first and second criterion are tested.

This function implements the Gebauer-Moeller Installation.

INPUT:

- `G` – an intermediate Groebner basis
- `B` – set of critical pairs
- `h` – a polynomial

OUTPUT: a tuple of

- an intermediate Groebner basis
- a set of critical pairs

EXAMPLES:

```
sage: from sage.rings.polynomial.toy_buchberger import update
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: set_verbose(0)
sage: update(set(), set(), x*y*z)
({x*y*z}, set())
sage: G, B = update(set(), set(), x*y*z - 1)
sage: G, B = update(G, B, x*y^2 - 1)
sage: G, B
({x*y*z - 1, x*y^2 - 1}, {(x*y^2 - 1, x*y*z - 1)})
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.toy_buchberger import update
>>> R = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = R._first_ngens(3)
>>> set_verbose(Integer(0))
>>> update(set(), set(), x*y*z)
({x*y*z}, set())
>>> G, B = update(set(), set(), x*y*z - Integer(1))
>>> G, B = update(G, B, x*y**Integer(2) - Integer(1))
>>> G, B
({x*y*z - 1, x*y^2 - 1}, {(x*y^2 - 1, x*y*z - 1)})
```

3.3.2 Educational versions of Groebner basis algorithms: triangular factorization

In this file is the implementation of two algorithms in [Laz1992].

The main algorithm is `Triangular`; a secondary algorithm, necessary for the first, is `ElimPolMin`. As per Lazard's formulation, the implementation works with any term ordering, not only lexicographic.

Lazard does not specify a few of the subalgorithms implemented as the functions

- `is_triangular`,
- `is_linearly_dependent`, and
- `linear_representation`.

The implementations are not hard, and the choice of algorithm is described with the relevant function.

No attempt was made to optimize these algorithms as the emphasis of this implementation is a clean and easy presentation.

Examples appear with the appropriate function.

AUTHORS:

- John Perry (2009-02-24): initial version, but some words of documentation were stolen shamelessly from Martin Albrecht's toy_buchberger.py.

sage.rings.polynomial.toy_variety.**coefficient_matrix**(*polys*)

Generate the matrix M whose entries are the coefficients of polys.

The entries of row i of M consist of the coefficients of polys[i].

INPUT:

- polys – list/tuple of polynomials

OUTPUT: a matrix M of the coefficients of polys

EXAMPLES:

```
sage: from sage.rings.polynomial.toy_variety import coefficient_matrix
sage: R.<x,y> = PolynomialRing(QQ)
sage: coefficient_matrix([x^2 + 1, y^2 + 1, x*y + 1])                         #␣
↪needs sage.modules
[1 0 0 1]
[0 0 1 1]
[0 1 0 1]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.toy_variety import coefficient_matrix
>>> R = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> coefficient_matrix([x**Integer(2) + Integer(1), y**Integer(2) + Integer(1),␣
↪x*y + Integer(1)])                               # needs sage.modules
[1 0 0 1]
[0 0 1 1]
[0 1 0 1]
```

Note

This function may be merged with *sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic.coefficient_matrix()* in the future.

sage.rings.polynomial.toy_variety.**elim_pol**(*B*, *n=-1*)

Find the unique monic polynomial of lowest degree and lowest variable in the ideal described by B.

For the purposes of the triangularization algorithm, it is necessary to preserve the ring, so n specifies which variable to check. By default, we check the last one, which should also be the smallest.

The algorithm may not work if you are trying to cheat: B should describe the Groebner basis of a zero-dimensional ideal. However, it is not necessary for the Groebner basis to be lexicographic.

The algorithm is taken from a 1993 paper by Lazard [Laz1992].

INPUT:

- B – list/tuple of polynomials or a multivariate polynomial ideal
- n – the variable to check (see above) (default: -1)

EXAMPLES:

```
sage: # needs sage.rings.finite_rings
sage: from sage.misc.verbose import set_verbose
sage: set_verbose(0)
sage: from sage.rings.polynomial.toy_variety import elim_pol
sage: R.<x,y,z> = PolynomialRing(GF(32003))
sage: p1 = x^2*(x-1)^3*y^2*(z-3)^3
sage: p2 = z^2 - z
sage: p3 = (x-2)^2*(y-1)^3
sage: I = R.ideal(p1,p2,p3)
sage: elim_pol(I.groebner_basis())                                    #␣
↪needs sage.libs.singular
z^2 - z
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> from sage.misc.verbose import set_verbose
>>> set_verbose(Integer(0))
>>> from sage.rings.polynomial.toy_variety import elim_pol
>>> R = PolynomialRing(GF(Integer(32003)), names=('x', 'y', 'z',)); (x, y, z,) =␣
↪R._first_ngens(3)
>>> p1 = x**Integer(2)*(x-Integer(1))**Integer(3)*y**Integer(2)*(z-
↪Integer(3))**Integer(3)
>>> p2 = z**Integer(2) - z
>>> p3 = (x-Integer(2))**Integer(2)*(y-Integer(1))**Integer(3)
>>> I = R.ideal(p1,p2,p3)
>>> elim_pol(I.groebner_basis())                                      #␣
↪needs sage.libs.singular
z^2 - z
```

sage.rings.polynomial.toy_variety.**is_linearly_dependent**(*polys*)

Decide whether the polynomials of `polys` are linearly dependent.

Here `polys` is a collection of polynomials.

The algorithm creates a matrix of coefficients of the monomials of `polys`. It computes the echelon form of the matrix, then checks whether any of the rows is the zero vector.

Essentially this relies on the fact that the monomials are linearly independent, and therefore is building a linear map from the vector space of the monomials to the canonical basis of `R^n`, where `n` is the number of distinct monomials in `polys`. There is a zero vector iff there is a linear dependence among `polys`.

The case where `polys=[]` is considered to be not linearly dependent.

INPUT:

- `polys` – list/tuple of polynomials

OUTPUT:

`True` if the elements of `polys` are linearly dependent; `False` otherwise.

EXAMPLES:

```
sage: from sage.rings.polynomial.toy_variety import is_linearly_dependent
sage: R.<x,y> = PolynomialRing(QQ)
sage: B = [x^2 + 1, y^2 + 1, x*y + 1]
```

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```
sage: p = 3*B[0] - 2*B[1] + B[2]
sage: is_linearly_dependent(B + [p])                                          #␣
↪needs sage.modules
True
sage: p = x*B[0]
sage: is_linearly_dependent(B + [p])                                          #␣
↪needs sage.modules
False
sage: is_linearly_dependent([])
False
sage: R.<x> = PolynomialRing(QQ)
sage: B = [x^147 + x^99,
....:      2*x^123 + x^75,
....:      x^147 + 2*x^123 + 2*x^75,
....:      2*x^147 + x^99 + x^75]
sage: is_linearly_dependent(B)
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.toy_variety import is_linearly_dependent
>>> R = PolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> B = [x**Integer(2) + Integer(1), y**Integer(2) + Integer(1), x*y + Integer(1)]
>>> p = Integer(3)*B[Integer(0)] - Integer(2)*B[Integer(1)] + B[Integer(2)]
>>> is_linearly_dependent(B + [p])                                          #␣
↪needs sage.modules
True
>>> p = x*B[Integer(0)]
>>> is_linearly_dependent(B + [p])                                          #␣
↪needs sage.modules
False
>>> is_linearly_dependent([])
False
>>> R = PolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> B = [x**Integer(147) + x**Integer(99),
...      Integer(2)*x**Integer(123) + x**Integer(75),
...      x**Integer(147) + Integer(2)*x**Integer(123) + Integer(2)*x**Integer(75),
...      Integer(2)*x**Integer(147) + x**Integer(99) + x**Integer(75)]
>>> is_linearly_dependent(B)
True
```

sage.rings.polynomial.toy_variety.**is_triangular**(*B*)

Check whether the basis `B` of an ideal is triangular.

That is: check whether the largest variable in `B[i]` with respect to the ordering of the base ring `R` is `R.gens()[i]`.

The algorithm is based on the definition of a triangular basis, given by Lazard in 1992 in [Laz1992].

INPUT:

- `B` – list/tuple of polynomials or a multivariate polynomial ideal

OUTPUT: `True` if the basis is triangular; `False` otherwise

EXAMPLES:

```
sage: from sage.rings.polynomial.toy_variety import is_triangular
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: p1 = x^2*y + z^2
sage: p2 = y*z + z^3
sage: p3 = y+z
sage: is_triangular(R.ideal(p1,p2,p3))
False
sage: p3 = z^2 - 3
sage: is_triangular(R.ideal(p1,p2,p3))
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.toy_variety import is_triangular
>>> R = PolynomialRing(QQ, names=('x', 'y', 'z',)); (x, y, z,) = R._first_ngens(3)
>>> p1 = x**Integer(2)*y + z**Integer(2)
>>> p2 = y*z + z**Integer(3)
>>> p3 = y+z
>>> is_triangular(R.ideal(p1,p2,p3))
False
>>> p3 = z**Integer(2) - Integer(3)
>>> is_triangular(R.ideal(p1,p2,p3))
True
```

sage.rings.polynomial.toy_variety.**linear_representation**(*p*, *polys*)

Assuming that `p` is a linear combination of `polys`, determine coefficients that describe the linear combination.

This probably does not work for any inputs except `p`, a polynomial, and `polys`, a sequence of polynomials. If `p` is not in fact a linear combination of `polys`, the function raises an exception.

The algorithm creates a matrix of coefficients of the monomials of `polys` and `p`, with the coefficients of `p` in the last row. It augments this matrix with the appropriate identity matrix, then computes the echelon form of the augmented matrix. The last row should contain zeroes in the first columns, and the last columns contain a linear dependence relation. Solving for the desired linear relation is straightforward.

INPUT:

- `p` – a polynomial
- `polys` – list/tuple of polynomials

OUTPUT:

If `n == len(polys)`, returns `[a[0],a[1],...,a[n-1]]` such that `p == a[0]*poly[0] + ... + a[n-1]*poly[n-1]`.

EXAMPLES:

```
sage: # needs sage.modules sage.rings.finite_rings
sage: from sage.rings.polynomial.toy_variety import linear_representation
sage: R.<x,y> = PolynomialRing(GF(32003))
sage: B = [x^2 + 1, y^2 + 1, x*y + 1]
sage: p = 3*B[0] - 2*B[1] + B[2]
sage: linear_representation(p, B)
[3, 32001, 1]
```

```
>>> from sage.all import *
>>> # needs sage.modules sage.rings.finite_rings
>>> from sage.rings.polynomial.toy_variety import linear_representation
>>> R = PolynomialRing(GF(Integer(32003)), names=('x', 'y',)); (x, y,) = R._first_
↪ngens(2)
>>> B = [x**Integer(2) + Integer(1), y**Integer(2) + Integer(1), x*y + Integer(1)]
>>> p = Integer(3)*B[Integer(0)] - Integer(2)*B[Integer(1)] + B[Integer(2)]
>>> linear_representation(p, B)
[3, 32001, 1]
```

sage.rings.polynomial.toy_variety.**triangular_factorization**(*B*, *n=-1*)

Compute the triangular factorization of the Groebner basis `B` of an ideal.

This will not work properly if `B` is not a Groebner basis!

The algorithm used is that described in a 1992 paper by Daniel Lazard [Laz1992]. It is not necessary for the term ordering to be lexicographic.

INPUT:

- `B` – list/tuple of polynomials or a multivariate polynomial ideal
- `n` – the recursion parameter (default: `-1`)

OUTPUT: list `T` of triangular sets `T_0`, `T_1`, etc.

EXAMPLES:

```
sage: # needs sage.rings.finite_rings
sage: from sage.misc.verbose import set_verbose
sage: set_verbose(0)
sage: from sage.rings.polynomial.toy_variety import triangular_factorization
sage: R.<x,y,z> = PolynomialRing(GF(32003))
sage: p1 = x^2*(x-1)^3*y^2*(z-3)^3
sage: p2 = z^2 - z
sage: p3 = (x-2)^2*(y-1)^3
sage: I = R.ideal(p1,p2,p3)
sage: triangular_factorization(I.groebner_basis())                             #␣
↪needs sage.libs.singular
[[x^2 - 4*x + 4, y, z - 1],
 [x^5 - 3*x^4 + 3*x^3 - x^2, y - 1, z - 1],
 [x^2 - 4*x + 4, y, z],
 [x^5 - 3*x^4 + 3*x^3 - x^2, y - 1, z]]
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> from sage.misc.verbose import set_verbose
>>> set_verbose(Integer(0))
>>> from sage.rings.polynomial.toy_variety import triangular_factorization
>>> R = PolynomialRing(GF(Integer(32003)), names=('x', 'y', 'z',)); (x, y, z,) =␣
↪R._first_ngens(3)
>>> p1 = x**Integer(2)*(x-Integer(1))**Integer(3)*y**Integer(2)*(z-
↪Integer(3))**Integer(3)
>>> p2 = z**Integer(2) - z
>>> p3 = (x-Integer(2))**Integer(2)*(y-Integer(1))**Integer(3)
>>> I = R.ideal(p1,p2,p3)
```

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```
>>> triangular_factorization(I.groebner_basis())                        #␣
↪needs sage.libs.singular
[[x^2 - 4*x + 4, y, z - 1],
 [x^5 - 3*x^4 + 3*x^3 - x^2, y - 1, z - 1],
 [x^2 - 4*x + 4, y, z],
 [x^5 - 3*x^4 + 3*x^3 - x^2, y - 1, z]]
```

3.3.3 Educational version of the d-Groebner basis algorithm over PIDs

No attempt was made to optimize this algorithm as the emphasis of this implementation is a clean and easy presentation.

Note

The notion of 'term' and 'monomial' in [BW1993] is swapped from the notion of those words in Sage (or the other way around, however you prefer it). In Sage a term is a monomial multiplied by a coefficient, while in [BW1993] a monomial is a term multiplied by a coefficient. Also, what is called LM (the leading monomial) in Sage is called HT (the head term) in [BW1993].

EXAMPLES:

```
sage: from sage.rings.polynomial.toy_d_basis import d_basis
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.toy_d_basis import d_basis
```

First, consider an example from arithmetic geometry:

```
sage: A.<x,y> = PolynomialRing(ZZ, 2)
sage: B.<X,Y> = PolynomialRing(Rationals(), 2)
sage: f = -y^2 - y + x^3 + 7*x + 1
sage: fx = f.derivative(x)
sage: fy = f.derivative(y)
sage: I = B.ideal([B(f), B(fx), B(fy)])
sage: I.groebner_basis()                                                 #␣
↪needs sage.libs.singular
[1]
```

```
>>> from sage.all import *
>>> A = PolynomialRing(ZZ, Integer(2), names=('x', 'y',)); (x, y,) = A._first_ngens(2)
>>> B = PolynomialRing(Rationals(), Integer(2), names=('X', 'Y',)); (X, Y,) = B._
↪first_ngens(2)
>>> f = -y**Integer(2) - y + x**Integer(3) + Integer(7)*x + Integer(1)
>>> fx = f.derivative(x)
>>> fy = f.derivative(y)
>>> I = B.ideal([B(f), B(fx), B(fy)])
>>> I.groebner_basis()                                                  #␣
↪needs sage.libs.singular
[1]
```

Since the output is 1, we know that there are no generic singularities.

To look at the singularities of the arithmetic surface, we need to do the corresponding computation over $\mathbf{Z}$:

```
sage: I = A.ideal([f, fx, fy])
sage: gb = d_basis(I); gb                                                   #␣
↪needs sage.libs.singular
[x - 2020, y - 11313, 22627]

sage: gb[-1].factor()                                                       #␣
↪needs sage.libs.singular
11^3 * 17
```

```
>>> from sage.all import *
>>> I = A.ideal([f, fx, fy])
>>> gb = d_basis(I); gb                                                    #␣
↪needs sage.libs.singular
[x - 2020, y - 11313, 22627]

>>> gb[-Integer(1)].factor()                                                 ␣
↪     # needs sage.libs.singular
11^3 * 17
```

This Groebner Basis gives a lot of information. First, the only fibers (over $\mathbf{Z}$) that are not smooth are at $11 = 0$, and $17 = 0$. Examining the Groebner Basis, we see that we have a simple node in both the fiber at 11 and at 17. From the factorization, we see that the node at 17 is regular on the surface (an I_1 node), but the node at 11 is not. After blowing up this non-regular point, we find that it is an I_3 node.

Another example. This one is from the Magma Handbook:

```
sage: # needs sage.libs.singular
sage: P.<x, y, z> = PolynomialRing(IntegerRing(), 3, order='lex')
sage: I = ideal(x^2 - 1, y^2 - 1, 2*x*y - z)
sage: I = Ideal(d_basis(I))
sage: x.reduce(I)
x
sage: (2*x).reduce(I)
y*z
```

```
>>> from sage.all import *
>>> # needs sage.libs.singular
>>> P = PolynomialRing(IntegerRing(), Integer(3), order='lex', names=('x', 'y', 'z',
↪)); (x, y, z,) = P._first_ngens(3)
>>> I = ideal(x**Integer(2) - Integer(1), y**Integer(2) - Integer(1), Integer(2)*x*y -
↪ z)
>>> I = Ideal(d_basis(I))
>>> x.reduce(I)
x
>>> (Integer(2)*x).reduce(I)
y*z
```

To compute modulo 4, we can add the generator 4 to our basis.:

```
sage: # needs sage.libs.singular
sage: I = ideal(x^2 - 1, y^2 - 1, 2*x*y - z, 4)
sage: gb = d_basis(I)
sage: R = P.change_ring(IntegerModRing(4))
```

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```
sage: gb = [R(f) for f in gb if R(f)]; gb
[x^2 - 1, x*z + 2*y, 2*x - y*z, y^2 - 1, z^2, 2*z]
```

```
>>> from sage.all import *
>>> # needs sage.libs.singular
>>> I = ideal(x**Integer(2) - Integer(1), y**Integer(2) - Integer(1), Integer(2)*x*y -
↪ z, Integer(4))
>>> gb = d_basis(I)
>>> R = P.change_ring(IntegerModRing(Integer(4)))
>>> gb = [R(f) for f in gb if R(f)]; gb
[x^2 - 1, x*z + 2*y, 2*x - y*z, y^2 - 1, z^2, 2*z]
```

A third example is also from the Magma Handbook.

This example shows how one can use Groebner bases over the integers to find the primes modulo which a system of equations has a solution, when the system has no solutions over the rationals.

We first form a certain ideal I in $\mathbf{Z}[x, y, z]$, and note that the Groebner basis of I over $\mathbf{Q}$ contains 1, so there are no solutions over $\mathbf{Q}$ or an algebraic closure of it (this is not surprising as there are 4 equations in 3 unknowns).

```
sage: P.<x, y, z> = PolynomialRing(IntegerRing(), 3, order='degneglex')
sage: I = ideal(x^2 - 3*y, y^3 - x*y, z^3 - x, x^4 - y*z + 1)
sage: I.change_ring(P.change_ring(RationalField())).groebner_basis()              #␣
↪needs sage.libs.singular
[1]
```

```
>>> from sage.all import *
>>> P = PolynomialRing(IntegerRing(), Integer(3), order='degneglex', names=('x', 'y',
↪'z',)); (x, y, z,) = P._first_ngens(3)
>>> I = ideal(x**Integer(2) - Integer(3)*y, y**Integer(3) - x*y, z**Integer(3) - x,␣
↪x**Integer(4) - y*z + Integer(1))
>>> I.change_ring(P.change_ring(RationalField())).groebner_basis()              #␣
↪needs sage.libs.singular
[1]
```

However, when we compute the Groebner basis of I (defined over $\mathbf{Z}$), we note that there is a certain integer in the ideal which is not 1:

```
sage: gb = d_basis(I); gb                                                         #␣
↪needs sage.libs.singular
[z ..., y ..., x ..., 282687803443]
```

```
>>> from sage.all import *
>>> gb = d_basis(I); gb                                                         #␣
↪needs sage.libs.singular
[z ..., y ..., x ..., 282687803443]
```

Now for each prime p dividing this integer 282687803443, the Groebner basis of I modulo p will be non-trivial and will thus give a solution of the original system modulo p.:

```
sage: factor(282687803443)
101 * 103 * 27173681
```

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```
sage: I.change_ring(P.change_ring(GF(101))).groebner_basis()                    #␣
↪needs sage.libs.singular
[z - 33, y + 48, x + 19]

sage: I.change_ring(P.change_ring(GF(103))).groebner_basis()                    #␣
↪needs sage.libs.singular
[z - 18, y + 8, x + 39]

sage: I.change_ring(P.change_ring(GF(27173681))).groebner_basis()               #␣
↪needs sage.libs.singular sage.rings.finite_rings
[z + 10380032, y + 3186055, x - 536027]
```

```
>>> from sage.all import *
>>> factor(Integer(282687803443))
101 * 103 * 27173681

>>> I.change_ring(P.change_ring(GF(Integer(101)))).groebner_basis()            ␣
↪    # needs sage.libs.singular
[z - 33, y + 48, x + 19]

>>> I.change_ring(P.change_ring(GF(Integer(103)))).groebner_basis()            ␣
↪    # needs sage.libs.singular
[z - 18, y + 8, x + 39]

>>> I.change_ring(P.change_ring(GF(Integer(27173681)))).groebner_basis()       ␣
↪    # needs sage.libs.singular sage.rings.finite_rings
[z + 10380032, y + 3186055, x - 536027]
```

Of course, modulo any other prime the Groebner basis is trivial so there are no other solutions. For example:

```
sage: I.change_ring(P.change_ring(GF(3))).groebner_basis()                      #␣
↪needs sage.libs.singular
[1]
```

```
>>> from sage.all import *
>>> I.change_ring(P.change_ring(GF(Integer(3)))).groebner_basis()              ␣
↪    # needs sage.libs.singular
[1]
```

AUTHOR:

- Martin Albrecht (2008-08): initial version

sage.rings.polynomial.toy_d_basis.**LC**(*f*)

sage.rings.polynomial.toy_d_basis.**LM**(*f*)

sage.rings.polynomial.toy_d_basis.**d_basis**(*F*, *strat=True*)

Return the d-basis for the Ideal `F` as defined in [BW1993].

INPUT:

- `F` – an ideal
- `strat` – use update strategy (default: `True`)

EXAMPLES:

```
sage: from sage.rings.polynomial.toy_d_basis import d_basis
sage: A.<x,y> = PolynomialRing(ZZ, 2)
sage: f = -y^2 - y + x^3 + 7*x + 1
sage: fx = f.derivative(x)
sage: fy = f.derivative(y)
sage: I = A.ideal([f,fx,fy])
sage: gb = d_basis(I); gb                                                    #␣
↪needs sage.libs.singular
[x - 2020, y - 11313, 22627]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.toy_d_basis import d_basis
>>> A = PolynomialRing(ZZ, Integer(2), names=('x', 'y',)); (x, y,) = A._first_
↪ngens(2)
>>> f = -y**Integer(2) - y + x**Integer(3) + Integer(7)*x + Integer(1)
>>> fx = f.derivative(x)
>>> fy = f.derivative(y)
>>> I = A.ideal([f,fx,fy])
>>> gb = d_basis(I); gb                                                     #␣
↪needs sage.libs.singular
[x - 2020, y - 11313, 22627]
```

`sage.rings.polynomial.toy_d_basis.`**`gpol`**`(`*g1*, *g2*`)`

Return the G-Polynomial of `g_1` and `g_2`.

Let $a_i t_i$ be $LT(g_i)$, $a = a_i * c_i + a_j * c_j$ with $a = GCD(a_i, a_j)$, and $s_i = t/t_i$ with $t = LCM(t_i, t_j)$. Then the G-Polynomial is defined as: $c_1 s_1 g_1 - c_2 s_2 g_2$.

INPUT:

- `g1` – polynomial
- `g2` – polynomial

EXAMPLES:

```
sage: from sage.rings.polynomial.toy_d_basis import gpol
sage: P.<x, y, z> = PolynomialRing(IntegerRing(), 3, order='lex')
sage: f = x^2 - 1
sage: g = 2*x*y - z
sage: gpol(f,g)
x^2*y - y
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.toy_d_basis import gpol
>>> P = PolynomialRing(IntegerRing(), Integer(3), order='lex', names=('x', 'y', 'z
↪',)); (x, y, z,) = P._first_ngens(3)
>>> f = x**Integer(2) - Integer(1)
>>> g = Integer(2)*x*y - z
>>> gpol(f,g)
x^2*y - y
```

`sage.rings.polynomial.toy_d_basis.`**`select`**`(`*P*`)`

The normal selection strategy.

INPUT:

- `P` – list of critical pairs

OUTPUT: an element of P

EXAMPLES:

```
sage: from sage.rings.polynomial.toy_d_basis import select
sage: A.<x,y> = PolynomialRing(ZZ, 2)
sage: f = -y^2 - y + x^3 + 7*x + 1
sage: fx = f.derivative(x)
sage: fy = f.derivative(y)
sage: G = [f, fx, fy]
sage: B = set((f1, f2) for f1 in G for f2 in G if f1 != f2)
sage: select(B)
(-2*y - 1, 3*x^2 + 7)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.toy_d_basis import select
>>> A = PolynomialRing(ZZ, Integer(2), names=('x', 'y',)); (x, y,) = A._first_
↪ngens(2)
>>> f = -y**Integer(2) - y + x**Integer(3) + Integer(7)*x + Integer(1)
>>> fx = f.derivative(x)
>>> fy = f.derivative(y)
>>> G = [f, fx, fy]
>>> B = set((f1, f2) for f1 in G for f2 in G if f1 != f2)
>>> select(B)
(-2*y - 1, 3*x^2 + 7)
```

sage.rings.polynomial.toy_d_basis.**spol**(*g1*, *g2*)

Return the S-Polynomial of `g_1` and `g_2`.

Let $a_i t_i$ be $LT(g_i)$, $b_i = a/a_i$ with $a = LCM(a_i, a_j)$, and $s_i = t/t_i$ with $t = LCM(t_i, t_j)$. Then the S-Polynomial is defined as: $b_1 s_1 g_1 - b_2 s_2 g_2$.

INPUT:

- `g1` – polynomial
- `g2` – polynomial

EXAMPLES:

```
sage: from sage.rings.polynomial.toy_d_basis import spol
sage: P.<x, y, z> = PolynomialRing(IntegerRing(), 3, order='lex')
sage: f = x^2 - 1
sage: g = 2*x*y - z
sage: spol(f,g)
x*z - 2*y
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.toy_d_basis import spol
>>> P = PolynomialRing(IntegerRing(), Integer(3), order='lex', names=('x', 'y', 'z
↪',)); (x, y, z,) = P._first_ngens(3)
>>> f = x**Integer(2) - Integer(1)
>>> g = Integer(2)*x*y - z
```

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```
>>> spol(f,g)
x*z - 2*y
```

sage.rings.polynomial.toy_d_basis.**update**(*G*, *B*, *h*)

Update G using the list of critical pairs B and the polynomial h as presented in [BW1993], page 230. For this, Buchberger's first and second criterion are tested.

This function uses the Gebauer-Moeller Installation.

INPUT:

- G – an intermediate Groebner basis
- B – list of critical pairs
- h – a polynomial

OUTPUT: G,B where G and B are updated

EXAMPLES:

```
sage: from sage.rings.polynomial.toy_d_basis import update
sage: A.<x,y> = PolynomialRing(ZZ, 2)
sage: G = set([3*x^2 + 7, 2*y + 1, x^3 - y^2 + 7*x - y + 1])
sage: B = set()
sage: h = x^2*y - x^2 + y - 3
sage: update(G,B,h)
({2*y + 1, 3*x^2 + 7, x^2*y - x^2 + y - 3, x^3 - y^2 + 7*x - y + 1},
 {(x^2*y - x^2 + y - 3, 2*y + 1),
  (x^2*y - x^2 + y - 3, 3*x^2 + 7),
  (x^2*y - x^2 + y - 3, x^3 - y^2 + 7*x - y + 1)})
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.toy_d_basis import update
>>> A = PolynomialRing(ZZ, Integer(2), names=('x', 'y',)); (x, y,) = A._first_
↪ngens(2)
>>> G = set([Integer(3)*x**Integer(2) + Integer(7), Integer(2)*y + Integer(1),␣
↪x**Integer(3) - y**Integer(2) + Integer(7)*x - y + Integer(1)])
>>> B = set()
>>> h = x**Integer(2)*y - x**Integer(2) + y - Integer(3)
>>> update(G,B,h)
({2*y + 1, 3*x^2 + 7, x^2*y - x^2 + y - 3, x^3 - y^2 + 7*x - y + 1},
 {(x^2*y - x^2 + y - 3, 2*y + 1),
  (x^2*y - x^2 + y - 3, 3*x^2 + 7),
  (x^2*y - x^2 + y - 3, x^3 - y^2 + 7*x - y + 1)})
```

CHAPTER

FOUR

RATIONAL FUNCTIONS

4.1 Fraction Field of Integral Domains

AUTHORS:

- William Stein (with input from David Joyner, David Kohel, and Joe Wetherell)
- Burcin Erocal
- Julian Rüth (2017-06-27): embedding into the field of fractions and its section

EXAMPLES:

Quotienting is a constructor for an element of the fraction field:

```
sage: R.<x> = QQ[]
sage: (x^2-1)/(x+1)
x - 1
sage: parent((x^2-1)/(x+1))
Fraction Field of Univariate Polynomial Ring in x over Rational Field
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> (x**Integer(2)-Integer(1))/(x+Integer(1))
x - 1
>>> parent((x**Integer(2)-Integer(1))/(x+Integer(1)))
Fraction Field of Univariate Polynomial Ring in x over Rational Field
```

The GCD is not taken (since it doesn't converge sometimes) in the inexact case:

```
sage: # needs sage.rings.real_mpfr
sage: Z.<z> = CC[]
sage: I = CC.gen()
sage: (1+I+z)/(z+0.1*I)
(z + 1.00000000000000 + I)/(z + 0.100000000000000*I)
sage: (1+I*z)/(z+1.1)
(I*z + 1.00000000000000)/(z + 1.10000000000000)
```

```
>>> from sage.all import *
>>> # needs sage.rings.real_mpfr
>>> Z = CC['z']; (z,) = Z._first_ngens(1)
>>> I = CC.gen()
>>> (Integer(1)+I+z)/(z+RealNumber('0.1')*I)
```

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```
(z + 1.00000000000000 + I)/(z + 0.100000000000000*I)
>>> (Integer(1)+I*z)/(z+RealNumber('1.1'))
(I*z + 1.00000000000000)/(z + 1.10000000000000)
```

sage.rings.fraction_field.**FractionField**(*R*, *names=None*)

Create the fraction field of the integral domain R.

INPUT:

- R – an integral domain
- names – ignored

EXAMPLES:

We create some example fraction fields:

```
sage: FractionField(IntegerRing())
Rational Field
sage: FractionField(PolynomialRing(RationalField(),'x'))
Fraction Field of Univariate Polynomial Ring in x over Rational Field
sage: FractionField(PolynomialRing(IntegerRing(),'x'))
Fraction Field of Univariate Polynomial Ring in x over Integer Ring
sage: FractionField(PolynomialRing(RationalField(),2,'x'))
Fraction Field of Multivariate Polynomial Ring in x0, x1 over Rational Field
```

```
>>> from sage.all import *
>>> FractionField(IntegerRing())
Rational Field
>>> FractionField(PolynomialRing(RationalField(),'x'))
Fraction Field of Univariate Polynomial Ring in x over Rational Field
>>> FractionField(PolynomialRing(IntegerRing(),'x'))
Fraction Field of Univariate Polynomial Ring in x over Integer Ring
>>> FractionField(PolynomialRing(RationalField(),Integer(2),'x'))
Fraction Field of Multivariate Polynomial Ring in x0, x1 over Rational Field
```

Dividing elements often implicitly creates elements of the fraction field:

```
sage: x = PolynomialRing(RationalField(), 'x').gen()
sage: f = x/(x+1)
sage: g = x**3/(x+1)
sage: f/g
1/x^2
sage: g/f
x^2
```

```
>>> from sage.all import *
>>> x = PolynomialRing(RationalField(), 'x').gen()
>>> f = x/(x+Integer(1))
>>> g = x**Integer(3)/(x+Integer(1))
>>> f/g
1/x^2
>>> g/f
x^2
```

The input must be an integral domain:

```
sage: Frac(Integers(4))
Traceback (most recent call last):
...
TypeError: R must be an integral domain
```

```
>>> from sage.all import *
>>> Frac(Integers(Integer(4)))
Traceback (most recent call last):
...
TypeError: R must be an integral domain
```

class sage.rings.fraction_field.**FractionFieldEmbedding**

Bases: DefaultConvertMap_unique

The embedding of an integral domain into its field of fractions.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: f = R.fraction_field().coerce_map_from(R); f
Coercion map:
  From: Univariate Polynomial Ring in x over Rational Field
  To:   Fraction Field of Univariate Polynomial Ring in x over Rational Field
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = R.fraction_field().coerce_map_from(R); f
Coercion map:
  From: Univariate Polynomial Ring in x over Rational Field
  To:   Fraction Field of Univariate Polynomial Ring in x over Rational Field
```

is_injective()

Return whether this map is injective.

EXAMPLES:

The map from an integral domain to its fraction field is always injective:

```
sage: R.<x> = QQ[]
sage: R.fraction_field().coerce_map_from(R).is_injective()
True
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> R.fraction_field().coerce_map_from(R).is_injective()
True
```

is_surjective()

Return whether this map is surjective.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: R.fraction_field().coerce_map_from(R).is_surjective()
False
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> R.fraction_field().coerce_map_from(R).is_surjective()
False
```

section()

Return a section of this map.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: R.fraction_field().coerce_map_from(R).section()
Section map:
  From: Fraction Field of Univariate Polynomial Ring in x over Rational Field
  To:   Univariate Polynomial Ring in x over Rational Field
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> R.fraction_field().coerce_map_from(R).section()
Section map:
  From: Fraction Field of Univariate Polynomial Ring in x over Rational Field
  To:   Univariate Polynomial Ring in x over Rational Field
```

class sage.rings.fraction_field.**FractionFieldEmbeddingSection**

Bases: `Section`

The section of the embedding of an integral domain into its field of fractions.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: f = R.fraction_field().coerce_map_from(R).section(); f
Section map:
  From: Fraction Field of Univariate Polynomial Ring in x over Rational Field
  To:   Univariate Polynomial Ring in x over Rational Field
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> f = R.fraction_field().coerce_map_from(R).section(); f
Section map:
  From: Fraction Field of Univariate Polynomial Ring in x over Rational Field
  To:   Univariate Polynomial Ring in x over Rational Field
```

class sage.rings.fraction_field.**FractionField_1poly_field**(*R*, *element_class=<class 'sage.rings.fraction_field_element.FractionFieldElement_1poly_field'>*)

Bases: *FractionField_generic*

The fraction field of a univariate polynomial ring over a field.

Many of the functions here are included for coherence with number fields.

class_number()

Here for compatibility with number fields and function fields.

EXAMPLES:

```
sage: R.<t> = GF(5)[]; K = R.fraction_field()
sage: K.class_number()
1
```

```
>>> from sage.all import *
>>> R = GF(Integer(5))['t']; (t,) = R._first_ngens(1); K = R.fraction_field()
>>> K.class_number()
1
```

function_field()

Return the isomorphic function field.

EXAMPLES:

```
sage: R.<t> = GF(5)[]
sage: K = R.fraction_field()
sage: K.function_field()
Rational function field in t over Finite Field of size 5
```

```
>>> from sage.all import *
>>> R = GF(Integer(5))['t']; (t,) = R._first_ngens(1)
>>> K = R.fraction_field()
>>> K.function_field()
Rational function field in t over Finite Field of size 5
```

See also

sage.rings.function_field.RationalFunctionField.field()

maximal_order()

Return the maximal order in this fraction field.

EXAMPLES:

```
sage: K = FractionField(GF(5)['t'])
sage: K.maximal_order()
Univariate Polynomial Ring in t over Finite Field of size 5
```

```
>>> from sage.all import *
>>> K = FractionField(GF(Integer(5))['t'])
>>> K.maximal_order()
Univariate Polynomial Ring in t over Finite Field of size 5
```

ring_of_integers()

Return the ring of integers in this fraction field.

EXAMPLES:

```
sage: K = FractionField(GF(5)['t'])
sage: K.ring_of_integers()
Univariate Polynomial Ring in t over Finite Field of size 5
```

```
>>> from sage.all import *
>>> K = FractionField(GF(Integer(5))['t'])
>>> K.ring_of_integers()
Univariate Polynomial Ring in t over Finite Field of size 5
```

class sage.rings.fraction_field.**FractionField_generic**(*R*, *element_class=<class 'sage.rings.fraction_field_element.FractionFieldElement'>*, *category=Category of quotient fields*)

Bases: Field

The fraction field of an integral domain.

base_ring()

Return the base ring of self.

This is the base ring of the ring which this fraction field is the fraction field of.

EXAMPLES:

```
sage: R = Frac(ZZ['t'])
sage: R.base_ring()
Integer Ring
```

```
>>> from sage.all import *
>>> R = Frac(ZZ['t'])
>>> R.base_ring()
Integer Ring
```

characteristic()

Return the characteristic of this fraction field.

EXAMPLES:

```
sage: R = Frac(ZZ['t'])
sage: R.base_ring()
Integer Ring
sage: R = Frac(ZZ['t']); R.characteristic()
0
sage: R = Frac(GF(5)['w']); R.characteristic()
5
```

```
>>> from sage.all import *
>>> R = Frac(ZZ['t'])
>>> R.base_ring()
Integer Ring
>>> R = Frac(ZZ['t']); R.characteristic()
0
>>> R = Frac(GF(Integer(5))['w']); R.characteristic()
5
```

construction()

EXAMPLES:

```
sage: Frac(ZZ['x']).construction()
(FractionField, Univariate Polynomial Ring in x over Integer Ring)
sage: K = Frac(GF(3)['t'])
sage: f, R = K.construction()
sage: f(R)
Fraction Field of Univariate Polynomial Ring in t
 over Finite Field of size 3
sage: f(R) == K
True
```

```
>>> from sage.all import *
>>> Frac(ZZ['x']).construction()
(FractionField, Univariate Polynomial Ring in x over Integer Ring)
>>> K = Frac(GF(Integer(3))['t'])
>>> f, R = K.construction()
>>> f(R)
Fraction Field of Univariate Polynomial Ring in t
 over Finite Field of size 3
>>> f(R) == K
True
```

gen(*i=0*)

Return the `i`-th generator of `self`.

EXAMPLES:

```
sage: R = Frac(PolynomialRing(QQ,'z',10)); R
Fraction Field of Multivariate Polynomial Ring
 in z0, z1, z2, z3, z4, z5, z6, z7, z8, z9 over Rational Field
sage: R.0
z0
sage: R.gen(3)
z3
sage: R.3
z3
```

```
>>> from sage.all import *
>>> R = Frac(PolynomialRing(QQ,'z',Integer(10))); R
Fraction Field of Multivariate Polynomial Ring
 in z0, z1, z2, z3, z4, z5, z6, z7, z8, z9 over Rational Field
>>> R.gen(0)
z0
>>> R.gen(Integer(3))
z3
>>> R.gen(3)
z3
```

is_exact()

Return if `self` is exact which is if the underlying ring is exact.

EXAMPLES:

```
sage: Frac(ZZ['x']).is_exact()
True
sage: Frac(CDF['x']).is_exact()                                    #␣
↪needs sage.rings.complex_double
False
```

```
>>> from sage.all import *
>>> Frac(ZZ['x']).is_exact()
True
>>> Frac(CDF['x']).is_exact()                                  #␣
↪needs sage.rings.complex_double
False
```

is_field(*proof=True*)

Return `True`, since the fraction field is a field.

EXAMPLES:

```
sage: Frac(ZZ).is_field()
True
```

```
>>> from sage.all import *
>>> Frac(ZZ).is_field()
True
```

is_finite()

Tells whether this fraction field is finite.

> **Note**
>
> A fraction field is finite if and only if the associated integral domain is finite.

EXAMPLES:

```
sage: Frac(QQ['a','b','c']).is_finite()
False
```

```
>>> from sage.all import *
>>> Frac(QQ['a','b','c']).is_finite()
False
```

ngens()

This is the same as for the parent object.

EXAMPLES:

```
sage: R = Frac(PolynomialRing(QQ,'z',10)); R
Fraction Field of Multivariate Polynomial Ring
 in z0, z1, z2, z3, z4, z5, z6, z7, z8, z9 over Rational Field
sage: R.ngens()
10
```

```
>>> from sage.all import *
>>> R = Frac(PolynomialRing(QQ,'z',Integer(10))); R
Fraction Field of Multivariate Polynomial Ring
 in z0, z1, z2, z3, z4, z5, z6, z7, z8, z9 over Rational Field
>>> R.ngens()
10
```

random_element(**args*, ***kwds*)

Return a random element in this fraction field.

The arguments are passed to the random generator of the underlying ring.

EXAMPLES:

```
sage: F = ZZ['x'].fraction_field()
sage: F.random_element()   # random
(2*x - 8)/(-x^2 + x)
```

```
>>> from sage.all import *
>>> F = ZZ['x'].fraction_field()
>>> F.random_element()   # random
(2*x - 8)/(-x^2 + x)
```

```
sage: f = F.random_element(degree=5)
sage: f.numerator().degree() == f.denominator().degree()
True
sage: f.denominator().degree() <= 5
True
sage: while f.numerator().degree() != 5:
....:     f = F.random_element(degree=5)
```

```
>>> from sage.all import *
>>> f = F.random_element(degree=Integer(5))
>>> f.numerator().degree() == f.denominator().degree()
True
>>> f.denominator().degree() <= Integer(5)
True
>>> while f.numerator().degree() != Integer(5):
...     f = F.random_element(degree=Integer(5))
```

ring()

Return the ring that this is the fraction field of.

EXAMPLES:

```
sage: R = Frac(QQ['x,y'])
sage: R
Fraction Field of Multivariate Polynomial Ring in x, y over Rational Field
sage: R.ring()
Multivariate Polynomial Ring in x, y over Rational Field
```

```
>>> from sage.all import *
>>> R = Frac(QQ['x,y'])
```

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```
>>> R
Fraction Field of Multivariate Polynomial Ring in x, y over Rational Field
>>> R.ring()
Multivariate Polynomial Ring in x, y over Rational Field
```

some_elements()

Return some elements in this field.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: R.fraction_field().some_elements()
[0,
 1,
 x,
 2*x,
 x/(x^2 + 2*x + 1),
 1/x^2,
 ...
 (2*x^2 + 2)/(x^2 + 2*x + 1),
 (2*x^2 + 2)/x^3,
 (2*x^2 + 2)/(x^2 - 1),
 2]
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> R.fraction_field().some_elements()
[0,
 1,
 x,
 2*x,
 x/(x^2 + 2*x + 1),
 1/x^2,
 ...
 (2*x^2 + 2)/(x^2 + 2*x + 1),
 (2*x^2 + 2)/x^3,
 (2*x^2 + 2)/(x^2 - 1),
 2]
```

sage.rings.fraction_field.**is_FractionField**(*x*)

Test whether or not x inherits from *FractionField_generic*.

EXAMPLES:

```
sage: from sage.rings.fraction_field import is_FractionField
sage: is_FractionField(Frac(ZZ['x']))
doctest:warning...
DeprecationWarning: The function is_FractionField is deprecated;
use 'isinstance(..., FractionField_generic)' instead.
See https://github.com/sagemath/sage/issues/38128 for details.
True
sage: is_FractionField(QQ)
False
```

```
>>> from sage.all import *
>>> from sage.rings.fraction_field import is_FractionField
>>> is_FractionField(Frac(ZZ['x']))
doctest:warning...
DeprecationWarning: The function is_FractionField is deprecated;
use 'isinstance(..., FractionField_generic)' instead.
See https://github.com/sagemath/sage/issues/38128 for details.
True
>>> is_FractionField(QQ)
False
```

4.2 Fraction Field Elements

AUTHORS:

- William Stein (input from David Joyner, David Kohel, and Joe Wetherell)
- Sebastian Pancratz (2010-01-06): Rewrite of addition, multiplication and derivative to use Henrici's algorithms [Hor1972]

class `sage.rings.fraction_field_element.`**`FractionFieldElement`**

Bases: `FieldElement`

EXAMPLES:

```
sage: K = FractionField(PolynomialRing(QQ, 'x'))
sage: K
Fraction Field of Univariate Polynomial Ring in x over Rational Field
sage: loads(K.dumps()) == K
True
sage: x = K.gen()
sage: f = (x^3 + x)/(17 - x^19); f
(-x^3 - x)/(x^19 - 17)
sage: loads(f.dumps()) == f
True
```

```
>>> from sage.all import *
>>> K = FractionField(PolynomialRing(QQ, 'x'))
>>> K
Fraction Field of Univariate Polynomial Ring in x over Rational Field
>>> loads(K.dumps()) == K
True
>>> x = K.gen()
>>> f = (x**Integer(3) + x)/(Integer(17) - x**Integer(19)); f
(-x^3 - x)/(x^19 - 17)
>>> loads(f.dumps()) == f
True
```

`denominator``()`

Return the denominator of `self`.

EXAMPLES:

```
sage: R.<x,y> = ZZ[]
sage: f = x/y + 1; f
(x + y)/y
sage: f.denominator()
y
```

```
>>> from sage.all import *
>>> R = ZZ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = x/y + Integer(1); f
(x + y)/y
>>> f.denominator()
y
```

is_one()

Return `True` if this element is equal to one.

EXAMPLES:

```
sage: F = ZZ['x,y'].fraction_field()
sage: x,y = F.gens()
sage: (x/x).is_one()
True
sage: (x/y).is_one()
False
```

```
>>> from sage.all import *
>>> F = ZZ['x,y'].fraction_field()
>>> x,y = F.gens()
>>> (x/x).is_one()
True
>>> (x/y).is_one()
False
```

is_square(*root=False*)

Return whether or not `self` is a perfect square.

If the optional argument `root` is `True`, then also returns a square root (or `None`, if the fraction field element is not square).

INPUT:

- `root` – whether or not to also return a square root (default: `False`)

OUTPUT:

- boolean; whether or not a square
- object (optional); an actual square root if found, and `None` otherwise

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: (1/t).is_square()
False
sage: (1/t^6).is_square()
True
```

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```
sage: ((1+t)^4/t^6).is_square()
True
sage: (4*(1+t)^4/t^6).is_square()
True
sage: (2*(1+t)^4/t^6).is_square()
False
sage: ((1+t)/t^6).is_square()
False

sage: (4*(1+t)^4/t^6).is_square(root=True)
(True, (2*t^2 + 4*t + 2)/t^3)
sage: (2*(1+t)^4/t^6).is_square(root=True)
(False, None)

sage: R.<x> = QQ[]
sage: a = 2*(x+1)^2 / (2*(x-1)^2); a
(x^2 + 2*x + 1)/(x^2 - 2*x + 1)
sage: a.is_square()
True
sage: (0/x).is_square()
True
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> (Integer(1)/t).is_square()
False
>>> (Integer(1)/t**Integer(6)).is_square()
True
>>> ((Integer(1)+t)**Integer(4)/t**Integer(6)).is_square()
True
>>> (Integer(4)*(Integer(1)+t)**Integer(4)/t**Integer(6)).is_square()
True
>>> (Integer(2)*(Integer(1)+t)**Integer(4)/t**Integer(6)).is_square()
False
>>> ((Integer(1)+t)/t**Integer(6)).is_square()
False

>>> (Integer(4)*(Integer(1)+t)**Integer(4)/t**Integer(6)).is_square(root=True)
(True, (2*t^2 + 4*t + 2)/t^3)
>>> (Integer(2)*(Integer(1)+t)**Integer(4)/t**Integer(6)).is_square(root=True)
(False, None)

>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> a = Integer(2)*(x+Integer(1))**Integer(2) / (Integer(2)*(x-
↪Integer(1))**Integer(2)); a
(x^2 + 2*x + 1)/(x^2 - 2*x + 1)
>>> a.is_square()
True
>>> (Integer(0)/x).is_square()
True
```

is_zero()

Return `True` if this element is equal to zero.

EXAMPLES:

```
sage: F = ZZ['x,y'].fraction_field()
sage: x,y = F.gens()
sage: t = F(0)/x
sage: t.is_zero()
True
sage: u = 1/x - 1/x
sage: u.is_zero()
True
sage: u.parent() is F
True
```

```
>>> from sage.all import *
>>> F = ZZ['x,y'].fraction_field()
>>> x,y = F.gens()
>>> t = F(Integer(0))/x
>>> t.is_zero()
True
>>> u = Integer(1)/x - Integer(1)/x
>>> u.is_zero()
True
>>> u.parent() is F
True
```

nth_root(*n*)

Return a `n`-th root of this element.

EXAMPLES:

```
sage: R = QQ['t'].fraction_field()
sage: t = R.gen()
sage: p = (t+1)^3 / (t^2+t-1)^3
sage: p.nth_root(3)
(t + 1)/(t^2 + t - 1)

sage: p = (t+1) / (t-1)
sage: p.nth_root(2)
Traceback (most recent call last):
...
ValueError: not a 2nd power
```

```
>>> from sage.all import *
>>> R = QQ['t'].fraction_field()
>>> t = R.gen()
>>> p = (t+Integer(1))**Integer(3) / (t**Integer(2)+t-Integer(1))**Integer(3)
>>> p.nth_root(Integer(3))
(t + 1)/(t^2 + t - 1)

>>> p = (t+Integer(1)) / (t-Integer(1))
>>> p.nth_root(Integer(2))
Traceback (most recent call last):
```

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```
...
ValueError: not a 2nd power
```

numerator()

Return the numerator of `self`.

EXAMPLES:

```
sage: R.<x,y> = ZZ[]
sage: f = x/y + 1; f
(x + y)/y
sage: f.numerator()
x + y
```

```
>>> from sage.all import *
>>> R = ZZ['x, y']; (x, y,) = R._first_ngens(2)
>>> f = x/y + Integer(1); f
(x + y)/y
>>> f.numerator()
x + y
```

reduce()

Reduce this fraction.

Divides out the gcd of the numerator and denominator. If the denominator becomes a unit, it becomes 1. Additionally, depending on the base ring, the leading coefficients of the numerator and the denominator may be normalized to 1.

Automatically called for exact rings, but because it may be numerically unstable for inexact rings it must be called manually in that case.

EXAMPLES:

```
sage: R.<x> = RealField(10)[]                                           #␣
↪needs sage.rings.real_mpfr
sage: f = (x^2+2*x+1)/(x+1); f                                          #␣
↪needs sage.rings.real_mpfr
(x^2 + 2.0*x + 1.0)/(x + 1.0)
sage: f.reduce(); f                                                     #␣
↪needs sage.rings.real_mpfr
x + 1.0
```

```
>>> from sage.all import *
>>> R = RealField(Integer(10))['x']; (x,) = R._first_ngens(1)# needs sage.
↪rings.real_mpfr
>>> f = (x**Integer(2)+Integer(2)*x+Integer(1))/(x+Integer(1)); f      ␣
↪                                   # needs sage.rings.real_mpfr
(x^2 + 2.0*x + 1.0)/(x + 1.0)
>>> f.reduce(); f                                                     #␣
↪needs sage.rings.real_mpfr
x + 1.0
```

specialization(*D=None*, *phi=None*)

Return the specialization of a fraction element of a polynomial ring.

subs(*in_dict=None*, **args*, ***kwds*)

Substitute variables in the numerator and denominator of `self`.

If a dictionary is passed, the keys are mapped to generators of the parent ring. Otherwise, the arguments are transmitted unchanged to the method `subs` of the numerator and the denominator.

EXAMPLES:

```
sage: x, y = PolynomialRing(ZZ, 2, 'xy').gens()
sage: f = x^2 + y + x^2*y^2 + 5
sage: (1/f).subs(x=5)
1/(25*y^2 + y + 30)
```

```
>>> from sage.all import *
>>> x, y = PolynomialRing(ZZ, Integer(2), 'xy').gens()
>>> f = x**Integer(2) + y + x**Integer(2)*y**Integer(2) + Integer(5)
>>> (Integer(1)/f).subs(x=Integer(5))
1/(25*y^2 + y + 30)
```

valuation(*v=None*)

Return the valuation of `self`, assuming that the numerator and denominator have valuation functions defined on them.

EXAMPLES:

```
sage: x = PolynomialRing(RationalField(),'x').gen()
sage: f = (x^3 + x)/(x^2 - 2*x^3)
sage: f
(-1/2*x^2 - 1/2)/(x^2 - 1/2*x)
sage: f.valuation()
-1
sage: f.valuation(x^2 + 1)
1
```

```
>>> from sage.all import *
>>> x = PolynomialRing(RationalField(),'x').gen()
>>> f = (x**Integer(3) + x)/(x**Integer(2) - Integer(2)*x**Integer(3))
>>> f
(-1/2*x^2 - 1/2)/(x^2 - 1/2*x)
>>> f.valuation()
-1
>>> f.valuation(x**Integer(2) + Integer(1))
1
```

class sage.rings.fraction_field_element.**FractionFieldElement_1poly_field**

Bases: *FractionFieldElement*

A fraction field element where the parent is the fraction field of a univariate polynomial ring over a field.

Many of the functions here are included for coherence with number fields.

is_integral()

Return whether this element is actually a polynomial.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: elt = (t^2 + t - 2) / (t + 2); elt # == (t + 2)*(t - 1)/(t + 2)
t - 1
sage: elt.is_integral()
True
sage: elt = (t^2 - t) / (t+2); elt # == t*(t - 1)/(t + 2)
(t^2 - t)/(t + 2)
sage: elt.is_integral()
False
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> elt = (t**Integer(2) + t - Integer(2)) / (t + Integer(2)); elt # == (t +␣
↪2)*(t - 1)/(t + 2)
t - 1
>>> elt.is_integral()
True
>>> elt = (t**Integer(2) - t) / (t+Integer(2)); elt # == t*(t - 1)/(t + 2)
(t^2 - t)/(t + 2)
>>> elt.is_integral()
False
```

reduce()

Pick a normalized representation of `self`.

In particular, for any a == b, after normalization they will have the same numerator and denominator.

EXAMPLES:

For univariate rational functions over a field, we have:

```
sage: R.<x> = QQ[]
sage: (2 + 2*x) / (4*x) # indirect doctest
(1/2*x + 1/2)/x
```

```
>>> from sage.all import *
>>> R = QQ['x']; (x,) = R._first_ngens(1)
>>> (Integer(2) + Integer(2)*x) / (Integer(4)*x) # indirect doctest
(1/2*x + 1/2)/x
```

Compare with:

```
sage: R.<x> = ZZ[]
sage: (2 + 2*x) / (4*x)
(x + 1)/(2*x)
```

```
>>> from sage.all import *
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> (Integer(2) + Integer(2)*x) / (Integer(4)*x)
(x + 1)/(2*x)
```

support()

Return a sorted list of primes dividing either the numerator or denominator of this element.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: h = (t^14 + 2*t^12 - 4*t^11 - 8*t^9 + 6*t^8 + 12*t^6 - 4*t^5
....:      - 8*t^3 + t^2 + 2)/(t^6 + 6*t^5 + 9*t^4 - 2*t^2 - 12*t - 18)
sage: h.support()                                                           #␣
↪needs sage.libs.pari
[t - 1, t + 3, t^2 + 2, t^2 + t + 1, t^4 - 2]
```

```
>>> from sage.all import *
>>> R = QQ['t']; (t,) = R._first_ngens(1)
>>> h = (t**Integer(14) + Integer(2)*t**Integer(12) -␣
↪Integer(4)*t**Integer(11) - Integer(8)*t**Integer(9) +␣
↪Integer(6)*t**Integer(8) + Integer(12)*t**Integer(6) -␣
↪Integer(4)*t**Integer(5)
...      - Integer(8)*t**Integer(3) + t**Integer(2) + Integer(2))/
↪(t**Integer(6) + Integer(6)*t**Integer(5) + Integer(9)*t**Integer(4) -␣
↪Integer(2)*t**Integer(2) - Integer(12)*t - Integer(18))
>>> h.support()                                                         #␣
↪needs sage.libs.pari
[t - 1, t + 3, t^2 + 2, t^2 + t + 1, t^4 - 2]
```

sage.rings.fraction_field_element.**is_FractionFieldElement**(*x*)

Return whether or not x is a *FractionFieldElement*.

EXAMPLES:

```
sage: from sage.rings.fraction_field_element import is_FractionFieldElement
sage: R.<x> = ZZ[]
sage: is_FractionFieldElement(x/2)
doctest:warning...
DeprecationWarning: The function is_FractionFieldElement is deprecated;
use 'isinstance(..., FractionFieldElement)' instead.
See https://github.com/sagemath/sage/issues/38128 for details.
False
sage: is_FractionFieldElement(2/x)
True
sage: is_FractionFieldElement(1/3)
False
```

```
>>> from sage.all import *
>>> from sage.rings.fraction_field_element import is_FractionFieldElement
>>> R = ZZ['x']; (x,) = R._first_ngens(1)
>>> is_FractionFieldElement(x/Integer(2))
doctest:warning...
DeprecationWarning: The function is_FractionFieldElement is deprecated;
use 'isinstance(..., FractionFieldElement)' instead.
See https://github.com/sagemath/sage/issues/38128 for details.
False
>>> is_FractionFieldElement(Integer(2)/x)
True
>>> is_FractionFieldElement(Integer(1)/Integer(3))
False
```

sage.rings.fraction_field_element.**make_element**(*parent*, *numerator*, *denominator*)

Used for unpickling `FractionFieldElement` objects (and subclasses).

EXAMPLES:

```
sage: from sage.rings.fraction_field_element import make_element
sage: R = ZZ['x,y']
sage: x,y = R.gens()
sage: F = R.fraction_field()
sage: make_element(F, 1 + x, 1 + y)
(x + 1)/(y + 1)
```

```
>>> from sage.all import *
>>> from sage.rings.fraction_field_element import make_element
>>> R = ZZ['x,y']
>>> x,y = R.gens()
>>> F = R.fraction_field()
>>> make_element(F, Integer(1) + x, Integer(1) + y)
(x + 1)/(y + 1)
```

sage.rings.fraction_field_element.**make_element_old**(*parent*, *cdict*)

Used for unpickling old `FractionFieldElement` pickles.

EXAMPLES:

```
sage: from sage.rings.fraction_field_element import make_element_old
sage: R.<x,y> = ZZ[]
sage: F = R.fraction_field()
sage: make_element_old(F, {'_FractionFieldElement__numerator': x + y,
....:                      '_FractionFieldElement__denominator': x - y})
(x + y)/(x - y)
```

```
>>> from sage.all import *
>>> from sage.rings.fraction_field_element import make_element_old
>>> R = ZZ['x, y']; (x, y,) = R._first_ngens(2)
>>> F = R.fraction_field()
>>> make_element_old(F, {'_FractionFieldElement__numerator': x + y,
...                      '_FractionFieldElement__denominator': x - y})
(x + y)/(x - y)
```

4.3 Univariate rational functions over prime fields

class sage.rings.fraction_field_FpT.**FpT**(*R*, *names=None*)

Bases: `FractionField_1poly_field`

This class represents the fraction field $\mathbf{F}_p(T)$ for $2 < p < \sqrt{2^31 - 1}$.

EXAMPLES:

```
sage: R.<T> = GF(71)[]
sage: K = FractionField(R); K
Fraction Field of Univariate Polynomial Ring in T over Finite Field of size 71
sage: 1-1/T
(T + 70)/T
```

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```
sage: parent(1-1/T) is K
True
```

```
>>> from sage.all import *
>>> R = GF(Integer(71))['T']; (T,) = R._first_ngens(1)
>>> K = FractionField(R); K
Fraction Field of Univariate Polynomial Ring in T over Finite Field of size 71
>>> Integer(1)-Integer(1)/T
(T + 70)/T
>>> parent(Integer(1)-Integer(1)/T) is K
True
```

INTEGER_LIMIT = 46341

iter(*bound=None*, *start=None*)

EXAMPLES:

```
sage: from sage.rings.fraction_field_FpT import *
sage: R.<t> = FpT(GF(5)['t'])
sage: list(R.iter(2))[350:355]
[(t^2 + t + 1)/(t + 2),
 (t^2 + t + 2)/(t + 2),
 (t^2 + t + 4)/(t + 2),
 (t^2 + 2*t + 1)/(t + 2),
 (t^2 + 2*t + 2)/(t + 2)]
```

```
>>> from sage.all import *
>>> from sage.rings.fraction_field_FpT import *
>>> R = FpT(GF(Integer(5))['t'], names=('t',)); (t,) = R._first_ngens(1)
>>> list(R.iter(Integer(2)))[Integer(350):Integer(355)]
[(t^2 + t + 1)/(t + 2),
 (t^2 + t + 2)/(t + 2),
 (t^2 + t + 4)/(t + 2),
 (t^2 + 2*t + 1)/(t + 2),
 (t^2 + 2*t + 2)/(t + 2)]
```

class sage.rings.fraction_field_FpT.**FpTElement**

Bases: FieldElement

An element of an *FpT* fraction field.

denom()

Return the denominator of this element, as an element of the polynomial ring.

EXAMPLES:

```
sage: K = GF(11)['t'].fraction_field()
sage: t = K.gen(0); a = (t + 1/t)^3 - 1
sage: a.denom()
t^3
```

```
>>> from sage.all import *
>>> K = GF(Integer(11))['t'].fraction_field()
```

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```
>>> t = K.gen(Integer(0)); a = (t + Integer(1)/t)**Integer(3) - Integer(1)
>>> a.denom()
t^3
```

denominator()

Return the denominator of this element, as an element of the polynomial ring.

EXAMPLES:

```
sage: K = GF(11)['t'].fraction_field()
sage: t = K.gen(0); a = (t + 1/t)^3 - 1
sage: a.denominator()
t^3
```

```
>>> from sage.all import *
>>> K = GF(Integer(11))['t'].fraction_field()
>>> t = K.gen(Integer(0)); a = (t + Integer(1)/t)**Integer(3) - Integer(1)
>>> a.denominator()
t^3
```

factor()

EXAMPLES:

```
sage: K = Frac(GF(5)['t'])
sage: t = K.gen()
sage: f = 2 * (t+1) * (t^2+t+1)^2 / (t-1)
sage: factor(f)
(2) * (t + 4)^-1 * (t + 1) * (t^2 + t + 1)^2
```

```
>>> from sage.all import *
>>> K = Frac(GF(Integer(5))['t'])
>>> t = K.gen()
>>> f = Integer(2) * (t+Integer(1)) *␣
↪(t**Integer(2)+t+Integer(1))**Integer(2) / (t-Integer(1))
>>> factor(f)
(2) * (t + 4)^-1 * (t + 1) * (t^2 + t + 1)^2
```

is_square()

Return `True` if this element is the square of another element of the fraction field.

EXAMPLES:

```
sage: K = GF(13)['t'].fraction_field(); t = K.gen()
sage: t.is_square()
False
sage: (1/t^2).is_square()
True
sage: K(0).is_square()
True
```

```
>>> from sage.all import *
>>> K = GF(Integer(13))['t'].fraction_field(); t = K.gen()
```

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```
>>> t.is_square()
False
>>> (Integer(1)/t**Integer(2)).is_square()
True
>>> K(Integer(0)).is_square()
True
```

next()

Iterate through all polynomials, returning the "next" polynomial after this one.

The strategy is as follows:

- We always leave the denominator monic.
- We progress through the elements with both numerator and denominator monic, and with the denominator less than the numerator. For each such, we output all the scalar multiples of it, then all of the scalar multiples of its inverse.
- So if the leading coefficient of the numerator is less than $p - 1$, we scale the numerator to increase it by 1.
- Otherwise, we consider the multiple with numerator and denominator monic.
  - If the numerator is less than the denominator (lexicographically), we return the inverse of that element.
  - If the numerator is greater than the denominator, we invert, and then increase the numerator (remaining monic) until we either get something relatively prime to the new denominator, or we reach the new denominator. In this case, we increase the denominator and set the numerator to 1.

EXAMPLES:

```
sage: from sage.rings.fraction_field_FpT import *
sage: R.<t> = FpT(GF(3)['t'])
sage: a = R(0)
sage: for _ in range(30):
....:     a = a.next()
....:     print(a)
1
2
1/t
2/t
t
2*t
1/(t + 1)
2/(t + 1)
t + 1
2*t + 2
t/(t + 1)
2*t/(t + 1)
(t + 1)/t
(2*t + 2)/t
1/(t + 2)
2/(t + 2)
t + 2
2*t + 1
```

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```
t/(t + 2)
2*t/(t + 2)
(t + 2)/t
(2*t + 1)/t
(t + 1)/(t + 2)
(2*t + 2)/(t + 2)
(t + 2)/(t + 1)
(2*t + 1)/(t + 1)
1/t^2
2/t^2
t^2
2*t^2
```

```
>>> from sage.all import *
>>> from sage.rings.fraction_field_FpT import *
>>> R = FpT(GF(Integer(3))['t'], names=('t',)); (t,) = R._first_ngens(1)
>>> a = R(Integer(0))
>>> for _ in range(Integer(30)):
...     a = a.next()
...     print(a)
1
2
1/t
2/t
t
2*t
1/(t + 1)
2/(t + 1)
t + 1
2*t + 2
t/(t + 1)
2*t/(t + 1)
(t + 1)/t
(2*t + 2)/t
1/(t + 2)
2/(t + 2)
t + 2
2*t + 1
t/(t + 2)
2*t/(t + 2)
(t + 2)/t
(2*t + 1)/t
(t + 1)/(t + 2)
(2*t + 2)/(t + 2)
(t + 2)/(t + 1)
(2*t + 1)/(t + 1)
1/t^2
2/t^2
t^2
2*t^2
```

numer()

Return the numerator of this element, as an element of the polynomial ring.

EXAMPLES:

```
sage: K = GF(11)['t'].fraction_field()
sage: t = K.gen(0); a = (t + 1/t)^3 - 1
sage: a.numer()
t^6 + 3*t^4 + 10*t^3 + 3*t^2 + 1
```

```
>>> from sage.all import *
>>> K = GF(Integer(11))['t'].fraction_field()
>>> t = K.gen(Integer(0)); a = (t + Integer(1)/t)**Integer(3) - Integer(1)
>>> a.numer()
t^6 + 3*t^4 + 10*t^3 + 3*t^2 + 1
```

numerator()

Return the numerator of this element, as an element of the polynomial ring.

EXAMPLES:

```
sage: K = GF(11)['t'].fraction_field()
sage: t = K.gen(0); a = (t + 1/t)^3 - 1
sage: a.numerator()
t^6 + 3*t^4 + 10*t^3 + 3*t^2 + 1
```

```
>>> from sage.all import *
>>> K = GF(Integer(11))['t'].fraction_field()
>>> t = K.gen(Integer(0)); a = (t + Integer(1)/t)**Integer(3) - Integer(1)
>>> a.numerator()
t^6 + 3*t^4 + 10*t^3 + 3*t^2 + 1
```

sqrt(*extend=True*, *all=False*)

Return the square root of this element.

INPUT:

- `extend` – boolean (default: `True`); if `True`, return a square root in an extension ring, if necessary. Otherwise, raise a `ValueError` if the square is not in the base ring.
- `all` – boolean (default: `False`); if `True`, return all square roots of self, instead of just one

EXAMPLES:

```
sage: from sage.rings.fraction_field_FpT import *
sage: K = GF(7)['t'].fraction_field(); t = K.gen(0)
sage: p = (t + 2)^2/(3*t^3 + 1)^4
sage: p.sqrt()
(3*t + 6)/(t^6 + 3*t^3 + 4)
sage: p.sqrt()^2 == p
True
```

```
>>> from sage.all import *
>>> from sage.rings.fraction_field_FpT import *
>>> K = GF(Integer(7))['t'].fraction_field(); t = K.gen(Integer(0))
>>> p = (t + Integer(2))**Integer(2)/(Integer(3)*t**Integer(3) +␣
```

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```
↪Integer(1))**Integer(4)
>>> p.sqrt()
(3*t + 6)/(t^6 + 3*t^3 + 4)
>>> p.sqrt()**Integer(2) == p
True
```

subs(*in_dict=None*, **args*, ***kwds*)

EXAMPLES:

```
sage: K = Frac(GF(11)['t'])
sage: t = K.gen()
sage: f = (t+1)/(t-1)
sage: f.subs(t=2)
3
sage: f.subs(X=2)
(t + 1)/(t + 10)
```

```
>>> from sage.all import *
>>> K = Frac(GF(Integer(11))['t'])
>>> t = K.gen()
>>> f = (t+Integer(1))/(t-Integer(1))
>>> f.subs(t=Integer(2))
3
>>> f.subs(X=Integer(2))
(t + 1)/(t + 10)
```

valuation(*v*)

Return the valuation of `self` at v.

EXAMPLES:

```
sage: R.<t> = GF(5)[]
sage: f = (t+1)^2 * (t^2+t+1) / (t-1)^3
sage: f.valuation(t+1)
2
sage: f.valuation(t-1)
-3
sage: f.valuation(t)
0
```

```
>>> from sage.all import *
>>> R = GF(Integer(5))['t']; (t,) = R._first_ngens(1)
>>> f = (t+Integer(1))**Integer(2) * (t**Integer(2)+t+Integer(1)) / (t-
↪Integer(1))**Integer(3)
>>> f.valuation(t+Integer(1))
2
>>> f.valuation(t-Integer(1))
-3
>>> f.valuation(t)
0
```

class sage.rings.fraction_field_FpT.**FpT_Fp_section**

Bases: `Section`

This class represents the section from GF(p)(t) back to GF(p)[t].

EXAMPLES:

```
sage: R.<t> = GF(5)[]
sage: K = R.fraction_field()
sage: f = GF(5).convert_map_from(K); f
Section map:
  From: Fraction Field of Univariate Polynomial Ring in t over Finite Field of␣
↪size 5
  To:   Finite Field of size 5
sage: type(f)
<class 'sage.rings.fraction_field_FpT.FpT_Fp_section'>
```

```
>>> from sage.all import *
>>> R = GF(Integer(5))['t']; (t,) = R._first_ngens(1)
>>> K = R.fraction_field()
>>> f = GF(Integer(5)).convert_map_from(K); f
Section map:
  From: Fraction Field of Univariate Polynomial Ring in t over Finite Field of␣
↪size 5
  To:   Finite Field of size 5
>>> type(f)
<class 'sage.rings.fraction_field_FpT.FpT_Fp_section'>
```

Warning

Comparison of `FpT_Fp_section` objects is not currently implemented. See Issue #23469.

```
sage: fprime = loads(dumps(f))
sage: fprime == f
False

sage: fprime(3) == f(3)
True
```

```
>>> from sage.all import *
>>> fprime = loads(dumps(f))
>>> fprime == f
False

>>> fprime(Integer(3)) == f(Integer(3))
True
```

class sage.rings.fraction_field_FpT.**FpT_Polyring_section**

Bases: `Section`

This class represents the section from GF(p)(t) back to GF(p)[t].

EXAMPLES:

```
sage: R.<t> = GF(5)[]
sage: K = R.fraction_field()
```

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```
sage: f = R.convert_map_from(K); f
Section map:
  From: Fraction Field of Univariate Polynomial Ring in t over Finite Field of␣
↪size 5
  To:   Univariate Polynomial Ring in t over Finite Field of size 5
sage: type(f)
<class 'sage.rings.fraction_field_FpT.FpT_Polyring_section'>
```

```
>>> from sage.all import *
>>> R = GF(Integer(5))['t']; (t,) = R._first_ngens(1)
>>> K = R.fraction_field()
>>> f = R.convert_map_from(K); f
Section map:
  From: Fraction Field of Univariate Polynomial Ring in t over Finite Field of␣
↪size 5
  To:   Univariate Polynomial Ring in t over Finite Field of size 5
>>> type(f)
<class 'sage.rings.fraction_field_FpT.FpT_Polyring_section'>
```

Warning

Comparison of `FpT_Polyring_section` objects is not currently implemented. See Issue #23469.

```
sage: fprime = loads(dumps(f))
sage: fprime == f
False

sage: fprime(1+t) == f(1+t)
True
```

```
>>> from sage.all import *
>>> fprime = loads(dumps(f))
>>> fprime == f
False

>>> fprime(Integer(1)+t) == f(Integer(1)+t)
True
```

class `sage.rings.fraction_field_FpT.`**`FpT_iter`**

Bases: `object`

Return a class that iterates over all elements of an FpT.

EXAMPLES:

```
sage: K = GF(3)['t'].fraction_field()
sage: I = K.iter(1)
sage: list(I)
[0,
 1,
 2,
```

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```
 t,
 t + 1,
 t + 2,
 2*t,
 2*t + 1,
 2*t + 2,
 1/t,
 2/t,
 (t + 1)/t,
 (t + 2)/t,
 (2*t + 1)/t,
 (2*t + 2)/t,
 1/(t + 1),
 2/(t + 1),
 t/(t + 1),
 (t + 2)/(t + 1),
 2*t/(t + 1),
 (2*t + 1)/(t + 1),
 1/(t + 2),
 2/(t + 2),
 t/(t + 2),
 (t + 1)/(t + 2),
 2*t/(t + 2),
 (2*t + 2)/(t + 2)]
```

```
>>> from sage.all import *
>>> K = GF(Integer(3))['t'].fraction_field()
>>> I = K.iter(Integer(1))
>>> list(I)
[0,
 1,
 2,
 t,
 t + 1,
 t + 2,
 2*t,
 2*t + 1,
 2*t + 2,
 1/t,
 2/t,
 (t + 1)/t,
 (t + 2)/t,
 (2*t + 1)/t,
 (2*t + 2)/t,
 1/(t + 1),
 2/(t + 1),
 t/(t + 1),
 (t + 2)/(t + 1),
 2*t/(t + 1),
 (2*t + 1)/(t + 1),
 1/(t + 2),
 2/(t + 2),
```

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```
 t/(t + 2),
 (t + 1)/(t + 2),
 2*t/(t + 2),
 (2*t + 2)/(t + 2)]
```

class sage.rings.fraction_field_FpT.**Fp_FpT_coerce**

Bases: RingHomomorphism

This class represents the coercion map from GF(p) to GF(p)(t).

EXAMPLES:

```
sage: R.<t> = GF(5)[]
sage: K = R.fraction_field()
sage: f = K.coerce_map_from(GF(5)); f
Ring morphism:
  From: Finite Field of size 5
  To:   Fraction Field of Univariate Polynomial Ring in t over Finite Field of␣
↪size 5
sage: type(f)
<class 'sage.rings.fraction_field_FpT.Fp_FpT_coerce'>
```

```
>>> from sage.all import *
>>> R = GF(Integer(5))['t']; (t,) = R._first_ngens(1)
>>> K = R.fraction_field()
>>> f = K.coerce_map_from(GF(Integer(5))); f
Ring morphism:
  From: Finite Field of size 5
  To:   Fraction Field of Univariate Polynomial Ring in t over Finite Field of␣
↪size 5
>>> type(f)
<class 'sage.rings.fraction_field_FpT.Fp_FpT_coerce'>
```

section()

Return the section of this inclusion: the partially defined map from `GF(p)(t)` back to `GF(p)`, defined on constant elements.

EXAMPLES:

```
sage: R.<t> = GF(5)[]
sage: K = R.fraction_field()
sage: f = K.coerce_map_from(GF(5))
sage: g = f.section(); g
Section map:
  From: Fraction Field of Univariate Polynomial Ring in t over Finite Field␣
↪of size 5
  To:   Finite Field of size 5
sage: t = K.gen()
sage: g(f(1,3,reduce=False))
2
sage: g(t)
Traceback (most recent call last):
...
```

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```
ValueError: not constant
sage: g(1/t)
Traceback (most recent call last):
...
ValueError: not integral
```

```
>>> from sage.all import *
>>> R = GF(Integer(5))['t']; (t,) = R._first_ngens(1)
>>> K = R.fraction_field()
>>> f = K.coerce_map_from(GF(Integer(5)))
>>> g = f.section(); g
Section map:
  From: Fraction Field of Univariate Polynomial Ring in t over Finite Field␣
↪of size 5
  To:   Finite Field of size 5
>>> t = K.gen()
>>> g(f(Integer(1),Integer(3),reduce=False))
2
>>> g(t)
Traceback (most recent call last):
...
ValueError: not constant
>>> g(Integer(1)/t)
Traceback (most recent call last):
...
ValueError: not integral
```

class sage.rings.fraction_field_FpT.**Polyring_FpT_coerce**

Bases: `RingHomomorphism`

This class represents the coercion map from GF(p)[t] to GF(p)(t).

EXAMPLES:

```
sage: R.<t> = GF(5)[]
sage: K = R.fraction_field()
sage: f = K.coerce_map_from(R); f
Ring morphism:
  From: Univariate Polynomial Ring in t over Finite Field of size 5
  To:   Fraction Field of Univariate Polynomial Ring in t over Finite Field of␣
↪size 5
sage: type(f)
<class 'sage.rings.fraction_field_FpT.Polyring_FpT_coerce'>
```

```
>>> from sage.all import *
>>> R = GF(Integer(5))['t']; (t,) = R._first_ngens(1)
>>> K = R.fraction_field()
>>> f = K.coerce_map_from(R); f
Ring morphism:
  From: Univariate Polynomial Ring in t over Finite Field of size 5
  To:   Fraction Field of Univariate Polynomial Ring in t over Finite Field of␣
↪size 5
```

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```
>>> type(f)
<class 'sage.rings.fraction_field_FpT.Polyring_FpT_coerce'>
```

section()

Return the section of this inclusion: the partially defined map from `GF(p)(t)` back to `GF(p)[t]`, defined on elements with unit denominator.

EXAMPLES:

```
sage: R.<t> = GF(5)[]
sage: K = R.fraction_field()
sage: f = K.coerce_map_from(R)
sage: g = f.section(); g
Section map:
  From: Fraction Field of Univariate Polynomial Ring in t over Finite Field␣
↪of size 5
  To:   Univariate Polynomial Ring in t over Finite Field of size 5
sage: t = K.gen()
sage: g(t)
t
sage: g(1/t)
Traceback (most recent call last):
...
ValueError: not integral
```

```
>>> from sage.all import *
>>> R = GF(Integer(5))['t']; (t,) = R._first_ngens(1)
>>> K = R.fraction_field()
>>> f = K.coerce_map_from(R)
>>> g = f.section(); g
Section map:
  From: Fraction Field of Univariate Polynomial Ring in t over Finite Field␣
↪of size 5
  To:   Univariate Polynomial Ring in t over Finite Field of size 5
>>> t = K.gen()
>>> g(t)
t
>>> g(Integer(1)/t)
Traceback (most recent call last):
...
ValueError: not integral
```

class sage.rings.fraction_field_FpT.**ZZ_FpT_coerce**

Bases: `RingHomomorphism`

This class represents the coercion map from ZZ to GF(p)(t).

EXAMPLES:

```
sage: R.<t> = GF(17)[]
sage: K = R.fraction_field()
sage: f = K.coerce_map_from(ZZ); f
Ring morphism:
```

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```
  From: Integer Ring
  To:   Fraction Field of Univariate Polynomial Ring in t over Finite Field of␣
↪size 17
sage: type(f)
<class 'sage.rings.fraction_field_FpT.ZZ_FpT_coerce'>
```

```
>>> from sage.all import *
>>> R = GF(Integer(17))['t']; (t,) = R._first_ngens(1)
>>> K = R.fraction_field()
>>> f = K.coerce_map_from(ZZ); f
Ring morphism:
  From: Integer Ring
  To:   Fraction Field of Univariate Polynomial Ring in t over Finite Field of␣
↪size 17
>>> type(f)
<class 'sage.rings.fraction_field_FpT.ZZ_FpT_coerce'>
```

section()

Return the section of this inclusion: the partially defined map from `GF(p)(t)` back to `ZZ`, defined on constant elements.

EXAMPLES:

```
sage: R.<t> = GF(5)[]
sage: K = R.fraction_field()
sage: f = K.coerce_map_from(ZZ)
sage: g = f.section(); g
Composite map:
  From: Fraction Field of Univariate Polynomial Ring in t over Finite Field␣
↪of size 5
  To:   Integer Ring
  Defn:   Section map:
          From: Fraction Field of Univariate Polynomial Ring in t over Finite␣
↪Field of size 5
          To:   Finite Field of size 5
        then
          Lifting map:
          From: Finite Field of size 5
          To:   Integer Ring
sage: t = K.gen()
sage: g(f(1,3,reduce=False))
2
sage: g(t)
Traceback (most recent call last):
...
ValueError: not constant
sage: g(1/t)
Traceback (most recent call last):
...
ValueError: not integral
```

```
>>> from sage.all import *
```

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```
>>> R = GF(Integer(5))['t']; (t,) = R._first_ngens(1)
>>> K = R.fraction_field()
>>> f = K.coerce_map_from(ZZ)
>>> g = f.section(); g
Composite map:
  From: Fraction Field of Univariate Polynomial Ring in t over Finite Field␣
↪of size 5
  To:   Integer Ring
  Defn:   Section map:
          From: Fraction Field of Univariate Polynomial Ring in t over Finite␣
↪Field of size 5
          To:   Finite Field of size 5
        then
          Lifting map:
          From: Finite Field of size 5
          To:   Integer Ring
>>> t = K.gen()
>>> g(f(Integer(1),Integer(3),reduce=False))
2
>>> g(t)
Traceback (most recent call last):
...
ValueError: not constant
>>> g(Integer(1)/t)
Traceback (most recent call last):
...
ValueError: not integral
```

`sage.rings.fraction_field_FpT.`**`unpickle_FpT_element`**(*K*, *numer*, *denom*)

Used for pickling.

CHAPTER FIVE

LAURENT POLYNOMIALS

5.1 Ring of Laurent Polynomials (base class)

If R is a commutative ring, then the ring of Laurent polynomials in n variables over R is $R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$.

AUTHORS:

- David Roe (2008-2-23): created
- David Loeffler (2009-07-10): cleaned up docstrings

class sage.rings.polynomial.laurent_polynomial_ring_base.**LaurentPolynomialRing_generic**(*R*)

Bases: CommutativeRing, Parent

Laurent polynomial ring (base class).

EXAMPLES:

This base class inherits from CommutativeRing. Since Issue #11900, it is also initialised as such:

```
sage: R.<x1,x2> = LaurentPolynomialRing(QQ)
sage: R.category()
Join of Category of unique factorization domains
    and Category of algebras with basis
        over (number fields and quotient fields and metric spaces)
    and Category of commutative algebras
        over (number fields and quotient fields and metric spaces)
    and Category of infinite sets
sage: TestSuite(R).run()
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, names=('x1', 'x2',)); (x1, x2,) = R._first_
→ngens(2)
>>> R.category()
Join of Category of unique factorization domains
    and Category of algebras with basis
        over (number fields and quotient fields and metric spaces)
    and Category of commutative algebras
        over (number fields and quotient fields and metric spaces)
    and Category of infinite sets
>>> TestSuite(R).run()
```

change_ring(*base_ring=None*, *names=None*, *sparse=False*, *order=None*)

EXAMPLES:

```
sage: R = LaurentPolynomialRing(QQ, 2, 'x')
sage: R.change_ring(ZZ)
Multivariate Laurent Polynomial Ring in x0, x1 over Integer Ring
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, Integer(2), 'x')
>>> R.change_ring(ZZ)
Multivariate Laurent Polynomial Ring in x0, x1 over Integer Ring
```

Check that the distinction between a univariate ring and a multivariate ring with one generator is preserved:

```
sage: P.<x> = LaurentPolynomialRing(QQ, 1)
sage: P
Multivariate Laurent Polynomial Ring in x over Rational Field
sage: K.<i> = CyclotomicField(4)                                              ␣
↪          # needs sage.rings.number_field
sage: P.change_ring(K)                                                        ␣
↪          # needs sage.rings.number_field
Multivariate Laurent Polynomial Ring in x over
 Cyclotomic Field of order 4 and degree 2
```

```
>>> from sage.all import *
>>> P = LaurentPolynomialRing(QQ, Integer(1), names=('x',)); (x,) = P._first_
↪ngens(1)
>>> P
Multivariate Laurent Polynomial Ring in x over Rational Field
>>> K = CyclotomicField(Integer(4), names=('i',)); (i,) = K._first_ngens(1)#␣
↪needs sage.rings.number_field
>>> P.change_ring(K)                                                          ␣
↪         # needs sage.rings.number_field
Multivariate Laurent Polynomial Ring in x over
 Cyclotomic Field of order 4 and degree 2
```

characteristic()

Return the characteristic of the base ring.

EXAMPLES:

```
sage: LaurentPolynomialRing(QQ, 2, 'x').characteristic()
0
sage: LaurentPolynomialRing(GF(3), 2, 'x').characteristic()
3
```

```
>>> from sage.all import *
>>> LaurentPolynomialRing(QQ, Integer(2), 'x').characteristic()
0
>>> LaurentPolynomialRing(GF(Integer(3)), Integer(2), 'x').characteristic()
3
```

completion(*p=None*, *prec=20*, *extras=None*)

Return the completion of `self`.

Currently only implemented for the ring of formal Laurent series. The `prec` variable controls the precision used in the Laurent series ring. If `prec` is ∞, then this returns a `LazyLaurentSeriesRing`.

EXAMPLES:

```
sage: P.<x> = LaurentPolynomialRing(QQ); P
Univariate Laurent Polynomial Ring in x over Rational Field
sage: PP = P.completion(x); PP
Laurent Series Ring in x over Rational Field
sage: f = 1 - 1/x
sage: PP(f)
-x^-1 + 1
sage: g = 1 / PP(f); g
-x - x^2 - x^3 - x^4 - x^5 - x^6 - x^7 - x^8 - x^9 - x^10 - x^11
 - x^12 - x^13 - x^14 - x^15 - x^16 - x^17 - x^18 - x^19 - x^20 + O(x^21)
sage: 1 / g
-x^-1 + 1 + O(x^19)

sage: # needs sage.combinat
sage: PP = P.completion(x, prec=oo); PP
Lazy Laurent Series Ring in x over Rational Field
sage: g = 1 / PP(f); g
-x - x^2 - x^3 + O(x^4)
sage: 1 / g == f
True
```

```
>>> from sage.all import *
>>> P = LaurentPolynomialRing(QQ, names=('x',)); (x,) = P._first_ngens(1); P
Univariate Laurent Polynomial Ring in x over Rational Field
>>> PP = P.completion(x); PP
Laurent Series Ring in x over Rational Field
>>> f = Integer(1) - Integer(1)/x
>>> PP(f)
-x^-1 + 1
>>> g = Integer(1) / PP(f); g
-x - x^2 - x^3 - x^4 - x^5 - x^6 - x^7 - x^8 - x^9 - x^10 - x^11
 - x^12 - x^13 - x^14 - x^15 - x^16 - x^17 - x^18 - x^19 - x^20 + O(x^21)
>>> Integer(1) / g
-x^-1 + 1 + O(x^19)

>>> # needs sage.combinat
>>> PP = P.completion(x, prec=oo); PP
Lazy Laurent Series Ring in x over Rational Field
>>> g = Integer(1) / PP(f); g
-x - x^2 - x^3 + O(x^4)
>>> Integer(1) / g == f
True
```

construction()

Return the construction of `self`.

EXAMPLES:

```
sage: LaurentPolynomialRing(QQ, 2, 'x,y').construction()
(LaurentPolynomialFunctor,
 Univariate Laurent Polynomial Ring in x over Rational Field)
```

```
>>> from sage.all import *
>>> LaurentPolynomialRing(QQ, Integer(2), 'x,y').construction()
(LaurentPolynomialFunctor,
 Univariate Laurent Polynomial Ring in x over Rational Field)
```

fraction_field()

The fraction field is the same as the fraction field of the polynomial ring.

EXAMPLES:

```
sage: L.<x> = LaurentPolynomialRing(QQ)
sage: L.fraction_field()
Fraction Field of Univariate Polynomial Ring in x over Rational Field
sage: (x^-1 + 2) / (x - 1)
(2*x + 1)/(x^2 - x)
```

```
>>> from sage.all import *
>>> L = LaurentPolynomialRing(QQ, names=('x',)); (x,) = L._first_ngens(1)
>>> L.fraction_field()
Fraction Field of Univariate Polynomial Ring in x over Rational Field
>>> (x**-Integer(1) + Integer(2)) / (x - Integer(1))
(2*x + 1)/(x^2 - x)
```

gen(*i=0*)

Return the i-th generator of `self`. If i is not specified, then the first generator will be returned.

EXAMPLES:

```
sage: LaurentPolynomialRing(QQ, 2, 'x').gen()
x0
sage: LaurentPolynomialRing(QQ, 2, 'x').gen(0)
x0
sage: LaurentPolynomialRing(QQ, 2, 'x').gen(1)
x1
```

```
>>> from sage.all import *
>>> LaurentPolynomialRing(QQ, Integer(2), 'x').gen()
x0
>>> LaurentPolynomialRing(QQ, Integer(2), 'x').gen(Integer(0))
x0
>>> LaurentPolynomialRing(QQ, Integer(2), 'x').gen(Integer(1))
x1
```

ideal(**args*, ***kwds*)

EXAMPLES:

```
sage: LaurentPolynomialRing(QQ, 2, 'x').ideal([1])
Ideal (1) of Multivariate Laurent Polynomial Ring in x0, x1 over Rational
↪Field
```

```
>>> from sage.all import *
>>> LaurentPolynomialRing(QQ, Integer(2), 'x').ideal([Integer(1)])
```

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```
Ideal (1) of Multivariate Laurent Polynomial Ring in x0, x1 over Rational
↪Field
```

is_exact()

Return `True` if the base ring is exact.

EXAMPLES:

```
sage: LaurentPolynomialRing(QQ, 2, 'x').is_exact()
True
sage: LaurentPolynomialRing(RDF, 2, 'x').is_exact()
False
```

```
>>> from sage.all import *
>>> LaurentPolynomialRing(QQ, Integer(2), 'x').is_exact()
True
>>> LaurentPolynomialRing(RDF, Integer(2), 'x').is_exact()
False
```

is_field(*proof=True*)

EXAMPLES:

```
sage: LaurentPolynomialRing(QQ, 2, 'x').is_field()
False
```

```
>>> from sage.all import *
>>> LaurentPolynomialRing(QQ, Integer(2), 'x').is_field()
False
```

is_finite()

EXAMPLES:

```
sage: LaurentPolynomialRing(QQ, 2, 'x').is_finite()
False
```

```
>>> from sage.all import *
>>> LaurentPolynomialRing(QQ, Integer(2), 'x').is_finite()
False
```

is_integral_domain(*proof=True*)

Return `True` if `self` is an integral domain.

EXAMPLES:

```
sage: LaurentPolynomialRing(QQ, 2, 'x').is_integral_domain()
True
```

```
>>> from sage.all import *
>>> LaurentPolynomialRing(QQ, Integer(2), 'x').is_integral_domain()
True
```

The following used to fail; see Issue #7530:

```
sage: L = LaurentPolynomialRing(ZZ, 'X')
sage: L['Y']
Univariate Polynomial Ring in Y over
 Univariate Laurent Polynomial Ring in X over Integer Ring
```

```
>>> from sage.all import *
>>> L = LaurentPolynomialRing(ZZ, 'X')
>>> L['Y']
Univariate Polynomial Ring in Y over
 Univariate Laurent Polynomial Ring in X over Integer Ring
```

is_noetherian()

Return `True` if `self` is Noetherian.

EXAMPLES:

```
sage: LaurentPolynomialRing(QQ, 2, 'x').is_noetherian()
True
```

```
>>> from sage.all import *
>>> LaurentPolynomialRing(QQ, Integer(2), 'x').is_noetherian()
True
```

krull_dimension()

EXAMPLES:

```
sage: LaurentPolynomialRing(QQ, 2, 'x').krull_dimension()
Traceback (most recent call last):
...
NotImplementedError
```

```
>>> from sage.all import *
>>> LaurentPolynomialRing(QQ, Integer(2), 'x').krull_dimension()
Traceback (most recent call last):
...
NotImplementedError
```

ngens()

Return the number of generators of `self`.

EXAMPLES:

```
sage: LaurentPolynomialRing(QQ, 2, 'x').ngens()
2
sage: LaurentPolynomialRing(QQ, 1, 'x').ngens()
1
```

```
>>> from sage.all import *
>>> LaurentPolynomialRing(QQ, Integer(2), 'x').ngens()
2
>>> LaurentPolynomialRing(QQ, Integer(1), 'x').ngens()
1
```

polynomial_ring()

Return the polynomial ring associated with `self`.

EXAMPLES:

```
sage: LaurentPolynomialRing(QQ, 2, 'x').polynomial_ring()
Multivariate Polynomial Ring in x0, x1 over Rational Field
sage: LaurentPolynomialRing(QQ, 1, 'x').polynomial_ring()
Multivariate Polynomial Ring in x over Rational Field
```

```
>>> from sage.all import *
>>> LaurentPolynomialRing(QQ, Integer(2), 'x').polynomial_ring()
Multivariate Polynomial Ring in x0, x1 over Rational Field
>>> LaurentPolynomialRing(QQ, Integer(1), 'x').polynomial_ring()
Multivariate Polynomial Ring in x over Rational Field
```

random_element(*min_valuation=-2*, *max_degree=2*, **args*, ***kwds*)

Return a random polynomial with degree at most `max_degree` and lowest valuation at least `min_valuation`.

Uses the random sampling from the base polynomial ring then divides out by a monomial to ensure correct `max_degree` and `min_valuation`.

INPUT:

- `min_valuation` – integer (default: -2); the minimal allowed valuation of the polynomial
- `max_degree` – integer (default: 2); the maximal allowed degree of the polynomial
- `*args`, `**kwds` – passed to the random element generator of the base polynomial ring and base ring itself

EXAMPLES:

```
sage: L.<x> = LaurentPolynomialRing(QQ)
sage: f = L.random_element()
sage: f.degree() <= 2
True
sage: f.valuation() >= -2
True
sage: f.parent() is L
True
```

```
>>> from sage.all import *
>>> L = LaurentPolynomialRing(QQ, names=('x',)); (x,) = L._first_ngens(1)
>>> f = L.random_element()
>>> f.degree() <= Integer(2)
True
>>> f.valuation() >= -Integer(2)
True
>>> f.parent() is L
True
```

```
sage: L = LaurentPolynomialRing(ZZ, 2, 'x')
sage: f = L.random_element(10, 20)
sage: f.degree() <= 20
```

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```
True
sage: f.valuation() >= 10
True
sage: f.parent() is L
True
```

```
>>> from sage.all import *
>>> L = LaurentPolynomialRing(ZZ, Integer(2), 'x')
>>> f = L.random_element(Integer(10), Integer(20))
>>> f.degree() <= Integer(20)
True
>>> f.valuation() >= Integer(10)
True
>>> f.parent() is L
True
```

```
sage: L = LaurentPolynomialRing(GF(13), 3, 'x')
sage: f = L.random_element(-10, -1)
sage: f.degree() <= -1
True
sage: f.valuation() >= -10
True
sage: f.parent() is L
True
```

```
>>> from sage.all import *
>>> L = LaurentPolynomialRing(GF(Integer(13)), Integer(3), 'x')
>>> f = L.random_element(-Integer(10), -Integer(1))
>>> f.degree() <= -Integer(1)
True
>>> f.valuation() >= -Integer(10)
True
>>> f.parent() is L
True
```

```
sage: L.<x, y> = LaurentPolynomialRing(RR)
sage: f = L.random_element()
sage: f.degree() <= 2
True
sage: f.valuation() >= -2
True
sage: f.parent() is L
True
```

```
>>> from sage.all import *
>>> L = LaurentPolynomialRing(RR, names=('x', 'y',)); (x, y,) = L._first_
↪ngens(2)
>>> f = L.random_element()
>>> f.degree() <= Integer(2)
True
>>> f.valuation() >= -Integer(2)
```

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```
True
>>> f.parent() is L
True
```

```
sage: L = LaurentPolynomialRing(QQbar, 5, 'x')
sage: f = L.random_element(-1, 1)
sage: f = L.random_element(-1, 1)
sage: f.degree() <= 1
True
sage: f.valuation() >= -1
True
sage: f.parent() is L
True
```

```
>>> from sage.all import *
>>> L = LaurentPolynomialRing(QQbar, Integer(5), 'x')
>>> f = L.random_element(-Integer(1), Integer(1))
>>> f = L.random_element(-Integer(1), Integer(1))
>>> f.degree() <= Integer(1)
True
>>> f.valuation() >= -Integer(1)
True
>>> f.parent() is L
True
```

remove_var(*var*)

EXAMPLES:

```
sage: R = LaurentPolynomialRing(QQ,'x,y,z')
sage: R.remove_var('x')
Multivariate Laurent Polynomial Ring in y, z over Rational Field
sage: R.remove_var('x').remove_var('y')
Univariate Laurent Polynomial Ring in z over Rational Field
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ,'x,y,z')
>>> R.remove_var('x')
Multivariate Laurent Polynomial Ring in y, z over Rational Field
>>> R.remove_var('x').remove_var('y')
Univariate Laurent Polynomial Ring in z over Rational Field
```

term_order()

Return the term order of `self`.

EXAMPLES:

```
sage: LaurentPolynomialRing(QQ, 2, 'x').term_order()
Degree reverse lexicographic term order
```

```
>>> from sage.all import *
>>> LaurentPolynomialRing(QQ, Integer(2), 'x').term_order()
Degree reverse lexicographic term order
```

variable_names_recursive(*depth=+Infinity*)

Return the list of variable names of this ring and its base rings, as if it were a single multi-variate Laurent polynomial.

INPUT:

- `depth` – integer or `Infinity`

OUTPUT: a tuple of strings

EXAMPLES:

```
sage: T = LaurentPolynomialRing(QQ, 'x')
sage: S = LaurentPolynomialRing(T, 'y')
sage: R = LaurentPolynomialRing(S, 'z')
sage: R.variable_names_recursive()
('x', 'y', 'z')
sage: R.variable_names_recursive(2)
('y', 'z')
```

```
>>> from sage.all import *
>>> T = LaurentPolynomialRing(QQ, 'x')
>>> S = LaurentPolynomialRing(T, 'y')
>>> R = LaurentPolynomialRing(S, 'z')
>>> R.variable_names_recursive()
('x', 'y', 'z')
>>> R.variable_names_recursive(Integer(2))
('y', 'z')
```

5.2 Ring of Laurent Polynomials

If R is a commutative ring, then the ring of Laurent polynomials in n variables over R is $R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$. We implement it as a quotient ring

$$R[x_1, y_1, x_2, y_2, \ldots, x_n, y_n]/(x_1 y_1 - 1, x_2 y_2 - 1, \ldots, x_n y_n - 1).$$

AUTHORS:

- David Roe (2008-2-23): created
- David Loeffler (2009-07-10): cleaned up docstrings

`sage.rings.polynomial.laurent_polynomial_ring.`**LaurentPolynomialRing**(*base_ring*, **args*, ***kwds*)

Return the globally unique univariate or multivariate Laurent polynomial ring with given properties and variable name or names.

There are four ways to call the Laurent polynomial ring constructor:

1. `LaurentPolynomialRing(base_ring, name, sparse=False)`
2. `LaurentPolynomialRing(base_ring, names, order='degrevlex')`
3. `LaurentPolynomialRing(base_ring, name, n, order='degrevlex')`
4. `LaurentPolynomialRing(base_ring, n, name, order='degrevlex')`

The optional arguments `sparse` and `order` *must* be explicitly named, and the other arguments must be given positionally.

INPUT:

- `base_ring` – a commutative ring
- `name` – string
- `names` – list or tuple of names, or a comma separated string
- `n` – positive integer
- `sparse` – boolean (default: `False`); whether or not elements are sparse
- `order` – string or `TermOrder`, e.g.,
  - `'degrevlex'` – default; degree reverse lexicographic
  - `'lex'` – lexicographic
  - `'deglex'` – degree lexicographic
  - `TermOrder('deglex',3) + TermOrder('deglex',3)` – block ordering

OUTPUT:

`LaurentPolynomialRing(base_ring, name, sparse=False)` returns a univariate Laurent polynomial ring; all other input formats return a multivariate Laurent polynomial ring.

UNIQUENESS and IMMUTABILITY: In Sage there is exactly one single-variate Laurent polynomial ring over each base ring in each choice of variable and sparseness. There is also exactly one multivariate Laurent polynomial ring over each base ring for each choice of names of variables and term order.

```
sage: R.<x,y> = LaurentPolynomialRing(QQ, 2); R                              #␣
↪needs sage.modules
Multivariate Laurent Polynomial Ring in x, y over Rational Field
sage: f = x^2 - 2*y^-2                                                       #␣
↪needs sage.modules
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, Integer(2), names=('x', 'y',)); (x, y,) = R._
↪first_ngens(2); R                                        # needs sage.modules
Multivariate Laurent Polynomial Ring in x, y over Rational Field
>>> f = x**Integer(2) - Integer(2)*y**-Integer(2)                              ␣
↪                          # needs sage.modules
```

You can't just globally change the names of those variables. This is because objects all over Sage could have pointers to that polynomial ring.

```
sage: R._assign_names(['z','w'])                                             #␣
↪needs sage.modules
Traceback (most recent call last):
...
ValueError: variable names cannot be changed after object creation.
```

```
>>> from sage.all import *
>>> R._assign_names(['z','w'])                                             #␣
↪needs sage.modules
Traceback (most recent call last):
```

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```
...
ValueError: variable names cannot be changed after object creation.
```

EXAMPLES:

1. `LaurentPolynomialRing(base_ring, name, sparse=False)`

```
sage: LaurentPolynomialRing(QQ, 'w')
Univariate Laurent Polynomial Ring in w over Rational Field
```

```
>>> from sage.all import *
>>> LaurentPolynomialRing(QQ, 'w')
Univariate Laurent Polynomial Ring in w over Rational Field
```

Use the diamond brackets notation to make the variable ready for use after you define the ring:

```
sage: R.<w> = LaurentPolynomialRing(QQ)
sage: (1 + w)^3
1 + 3*w + 3*w^2 + w^3
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, names=('w',)); (w,) = R._first_ngens(1)
>>> (Integer(1) + w)**Integer(3)
1 + 3*w + 3*w^2 + w^3
```

You must specify a name:

```
sage: LaurentPolynomialRing(QQ)
Traceback (most recent call last):
...
TypeError: you must specify the names of the variables

sage: R.<abc> = LaurentPolynomialRing(QQ, sparse=True); R
Univariate Laurent Polynomial Ring in abc over Rational Field

sage: R.<w> = LaurentPolynomialRing(PolynomialRing(GF(7),'k')); R
Univariate Laurent Polynomial Ring in w over
 Univariate Polynomial Ring in k over Finite Field of size 7
```

```
>>> from sage.all import *
>>> LaurentPolynomialRing(QQ)
Traceback (most recent call last):
...
TypeError: you must specify the names of the variables

>>> R = LaurentPolynomialRing(QQ, sparse=True, names=('abc',)); (abc,) = R._
↪first_ngens(1); R
Univariate Laurent Polynomial Ring in abc over Rational Field

>>> R = LaurentPolynomialRing(PolynomialRing(GF(Integer(7)),'k'), names=('w',
↪)); (w,) = R._first_ngens(1); R
Univariate Laurent Polynomial Ring in w over
 Univariate Polynomial Ring in k over Finite Field of size 7
```

Rings with different variables are different:

```
sage: LaurentPolynomialRing(QQ, 'x') == LaurentPolynomialRing(QQ, 'y')
False
```

```
>>> from sage.all import *
>>> LaurentPolynomialRing(QQ, 'x') == LaurentPolynomialRing(QQ, 'y')
False
```

2. `LaurentPolynomialRing(base_ring, names, order='degrevlex')`

```
sage: R = LaurentPolynomialRing(QQ, 'a,b,c'); R
↪# needs sage.modules
Multivariate Laurent Polynomial Ring in a, b, c over Rational Field

sage: S = LaurentPolynomialRing(QQ, ['a','b','c']); S
↪# needs sage.modules
Multivariate Laurent Polynomial Ring in a, b, c over Rational Field

sage: T = LaurentPolynomialRing(QQ, ('a','b','c')); T
↪# needs sage.modules
Multivariate Laurent Polynomial Ring in a, b, c over Rational Field
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, 'a,b,c'); R                           #␣
↪needs sage.modules
Multivariate Laurent Polynomial Ring in a, b, c over Rational Field

>>> S = LaurentPolynomialRing(QQ, ['a','b','c']); S                     #␣
↪needs sage.modules
Multivariate Laurent Polynomial Ring in a, b, c over Rational Field

>>> T = LaurentPolynomialRing(QQ, ('a','b','c')); T                     #␣
↪needs sage.modules
Multivariate Laurent Polynomial Ring in a, b, c over Rational Field
```

All three rings are identical.

```
sage: (R is S) and  (S is T)
↪# needs sage.modules
True
```

```
>>> from sage.all import *
>>> (R is S) and  (S is T)                                              #␣
↪needs sage.modules
True
```

There is a unique Laurent polynomial ring with each term order:

```
sage: # needs sage.modules
sage: R = LaurentPolynomialRing(QQ, 'x,y,z', order='degrevlex'); R
Multivariate Laurent Polynomial Ring in x, y, z over Rational Field
sage: S = LaurentPolynomialRing(QQ, 'x,y,z', order='invlex'); S
```

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```
Multivariate Laurent Polynomial Ring in x, y, z over Rational Field
sage: S is LaurentPolynomialRing(QQ, 'x,y,z', order='invlex')
True
sage: R == S
False
```

```
>>> from sage.all import *
>>> # needs sage.modules
>>> R = LaurentPolynomialRing(QQ, 'x,y,z', order='degrevlex'); R
Multivariate Laurent Polynomial Ring in x, y, z over Rational Field
>>> S = LaurentPolynomialRing(QQ, 'x,y,z', order='invlex'); S
Multivariate Laurent Polynomial Ring in x, y, z over Rational Field
>>> S is LaurentPolynomialRing(QQ, 'x,y,z', order='invlex')
True
>>> R == S
False
```

3. `LaurentPolynomialRing(base_ring, name, n, order='degrevlex')`

   If you specify a single name as a string and a number of variables, then variables labeled with numbers are created.

```
sage: LaurentPolynomialRing(QQ, 'x', 10)
↪# needs sage.modules
Multivariate Laurent Polynomial Ring in x0, x1, x2, x3, x4, x5, x6, x7, x8, x9
 over Rational Field

sage: LaurentPolynomialRing(GF(7), 'y', 5)
↪# needs sage.modules
Multivariate Laurent Polynomial Ring in y0, y1, y2, y3, y4
 over Finite Field of size 7

sage: LaurentPolynomialRing(QQ, 'y', 3, sparse=True)
↪# needs sage.modules
Multivariate Laurent Polynomial Ring in y0, y1, y2 over Rational Field
```

```
>>> from sage.all import *
>>> LaurentPolynomialRing(QQ, 'x', Integer(10))                              ␣
↪      # needs sage.modules
Multivariate Laurent Polynomial Ring in x0, x1, x2, x3, x4, x5, x6, x7, x8, x9
 over Rational Field

>>> LaurentPolynomialRing(GF(Integer(7)), 'y', Integer(5))                   ␣
↪               # needs sage.modules
Multivariate Laurent Polynomial Ring in y0, y1, y2, y3, y4
 over Finite Field of size 7

>>> LaurentPolynomialRing(QQ, 'y', Integer(3), sparse=True)                  ␣
↪      # needs sage.modules
Multivariate Laurent Polynomial Ring in y0, y1, y2 over Rational Field
```

By calling the `inject_variables()` method, all those variable names are available for interactive use:

```
sage: R = LaurentPolynomialRing(GF(7), 15, 'w'); R
↪# needs sage.modules
Multivariate Laurent Polynomial Ring in w0, w1, w2, w3, w4, w5, w6, w7,
 w8, w9, w10, w11, w12, w13, w14 over Finite Field of size 7
sage: R.inject_variables()
↪# needs sage.modules
Defining w0, w1, w2, w3, w4, w5, w6, w7, w8, w9, w10, w11, w12, w13, w14
sage: (w0 + 2*w8 + w13)^2
↪# needs sage.modules
w0^2 + 4*w0*w8 + 4*w8^2 + 2*w0*w13 + 4*w8*w13 + w13^2
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(GF(Integer(7)), Integer(15), 'w'); R
↪                # needs sage.modules
Multivariate Laurent Polynomial Ring in w0, w1, w2, w3, w4, w5, w6, w7,
 w8, w9, w10, w11, w12, w13, w14 over Finite Field of size 7
>>> R.inject_variables()                                              #
↪needs sage.modules
Defining w0, w1, w2, w3, w4, w5, w6, w7, w8, w9, w10, w11, w12, w13, w14
>>> (w0 + Integer(2)*w8 + w13)**Integer(2)
↪                 # needs sage.modules
w0^2 + 4*w0*w8 + 4*w8^2 + 2*w0*w13 + 4*w8*w13 + w13^2
```

class sage.rings.polynomial.laurent_polynomial_ring.**LaurentPolynomialRing_mpair**(*R*)

Bases: *LaurentPolynomialRing_generic*

EXAMPLES:

```
sage: L = LaurentPolynomialRing(QQ,2,'x')                             #
↪needs sage.modules
sage: type(L)                                                         #
↪needs sage.modules
<class
'sage.rings.polynomial.laurent_polynomial_ring.LaurentPolynomialRing_mpair_with_
↪category'>
sage: L == loads(dumps(L))                                            #
↪needs sage.modules
True
```

```
>>> from sage.all import *
>>> L = LaurentPolynomialRing(QQ,Integer(2),'x')
↪ # needs sage.modules
>>> type(L)                                                      # needs
↪sage.modules
<class
'sage.rings.polynomial.laurent_polynomial_ring.LaurentPolynomialRing_mpair_with_
↪category'>
>>> L == loads(dumps(L))                                         # needs
↪sage.modules
True
```

Element

alias of `LaurentPolynomial_mpair`

monomial(**exponents*)

Return the monomial whose exponents are given in argument.

EXAMPLES:

```
sage: # needs sage.modules
sage: L = LaurentPolynomialRing(QQ, 'x', 2)
sage: L.monomial(-3, 5)
x0^-3*x1^5
sage: L.monomial(1, 1)
x0*x1
sage: L.monomial(0, 0)
1
sage: L.monomial(-2, -3)
x0^-2*x1^-3

sage: x0, x1 = L.gens()                                                 #␣
↪needs sage.modules
sage: L.monomial(-1, 2) == x0^-1 * x1^2                                 #␣
↪needs sage.modules
True

sage: L.monomial(1, 2, 3)                                               #␣
↪needs sage.modules
Traceback (most recent call last):
...
TypeError: tuple key (1, 2, 3) must have same length as ngens (= 2)
```

```
>>> from sage.all import *
>>> # needs sage.modules
>>> L = LaurentPolynomialRing(QQ, 'x', Integer(2))
>>> L.monomial(-Integer(3), Integer(5))
x0^-3*x1^5
>>> L.monomial(Integer(1), Integer(1))
x0*x1
>>> L.monomial(Integer(0), Integer(0))
1
>>> L.monomial(-Integer(2), -Integer(3))
x0^-2*x1^-3

>>> x0, x1 = L.gens()                                                 #␣
↪needs sage.modules
>>> L.monomial(-Integer(1), Integer(2)) == x0**-Integer(1) * x1**Integer(2)  ␣
↪                                    # needs sage.modules
True

>>> L.monomial(Integer(1), Integer(2), Integer(3))                     ␣
↪                      # needs sage.modules
Traceback (most recent call last):
...
TypeError: tuple key (1, 2, 3) must have same length as ngens (= 2)
```

We also allow to specify the exponents in a single tuple:

```
sage: L.monomial((-1, 2))                                                    #␣
↪needs sage.modules
x0^-1*x1^2

sage: L.monomial((-1, 2, 3))                                                 #␣
↪needs sage.modules
Traceback (most recent call last):
...
TypeError: tuple key (-1, 2, 3) must have same length as ngens (= 2)
```

```
>>> from sage.all import *
>>> L.monomial((-Integer(1), Integer(2)))                                     ␣
↪             # needs sage.modules
x0^-1*x1^2

>>> L.monomial((-Integer(1), Integer(2), Integer(3)))                         ␣
↪                      # needs sage.modules
Traceback (most recent call last):
...
TypeError: tuple key (-1, 2, 3) must have same length as ngens (= 2)
```

class sage.rings.polynomial.laurent_polynomial_ring.**LaurentPolynomialRing_univariate**(*R*)

Bases: *LaurentPolynomialRing_generic*

EXAMPLES:

```
sage: L = LaurentPolynomialRing(QQ,'x')
sage: type(L)
<class 'sage.rings.polynomial.laurent_polynomial_ring.LaurentPolynomialRing_
↪univariate_with_category'>
sage: TestSuite(L).run()
```

```
>>> from sage.all import *
>>> L = LaurentPolynomialRing(QQ,'x')
>>> type(L)
<class 'sage.rings.polynomial.laurent_polynomial_ring.LaurentPolynomialRing_
↪univariate_with_category'>
>>> TestSuite(L).run()
```

Element

alias of *LaurentPolynomial_univariate*

monomial(*arg*)

Return the monomial with the given exponent.

sage.rings.polynomial.laurent_polynomial_ring.**from_fraction_field**(*L*, *x*)

Helper function to construct a Laurent polynomial from an element of its parent's fraction field.

INPUT:

- `L` – an instance of *LaurentPolynomialRing_generic*
- `x` – an element of the fraction field of `L`

OUTPUT:

An instance of the element class of `L`. If the denominator fails to be a unit in `L` an error is raised.

EXAMPLES:

```
sage: # needs sage.modules
sage: from sage.rings.polynomial.laurent_polynomial_ring import from_fraction_
↪field
sage: L.<x, y> = LaurentPolynomialRing(ZZ)
sage: F = L.fraction_field()
sage: xi = F(~x)
sage: from_fraction_field(L, xi) == ~x
True
```

```
>>> from sage.all import *
>>> # needs sage.modules
>>> from sage.rings.polynomial.laurent_polynomial_ring import from_fraction_field
>>> L = LaurentPolynomialRing(ZZ, names=('x', 'y',)); (x, y,) = L._first_ngens(2)
>>> F = L.fraction_field()
>>> xi = F(~x)
>>> from_fraction_field(L, xi) == ~x
True
```

sage.rings.polynomial.laurent_polynomial_ring.**is_LaurentPolynomialRing**(*R*)

Return `True` if and only if R is a Laurent polynomial ring.

EXAMPLES:

```
sage: from sage.rings.polynomial.laurent_polynomial_ring import is_
↪LaurentPolynomialRing
sage: P = PolynomialRing(QQ, 2, 'x')
sage: is_LaurentPolynomialRing(P)
doctest:warning...
DeprecationWarning: is_LaurentPolynomialRing is deprecated; use isinstance(...,
sage.rings.polynomial.laurent_polynomial_ring_base.LaurentPolynomialRing_generic)␣
↪instead
See https://github.com/sagemath/sage/issues/35229 for details.
False

sage: R = LaurentPolynomialRing(QQ,3,'x')                                          #␣
↪needs sage.modules
sage: is_LaurentPolynomialRing(R)                                                  #␣
↪needs sage.modules
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.laurent_polynomial_ring import is_
↪LaurentPolynomialRing
>>> P = PolynomialRing(QQ, Integer(2), 'x')
>>> is_LaurentPolynomialRing(P)
doctest:warning...
DeprecationWarning: is_LaurentPolynomialRing is deprecated; use isinstance(...,
sage.rings.polynomial.laurent_polynomial_ring_base.LaurentPolynomialRing_generic)␣
↪instead
See https://github.com/sagemath/sage/issues/35229 for details.
```

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```
False

>>> R = LaurentPolynomialRing(QQ,Integer(3),'x')                                    ␣
↪     # needs sage.modules
>>> is_LaurentPolynomialRing(R)                                                  #␣
↪needs sage.modules
True
```

5.3 Elements of Laurent polynomial rings

class sage.rings.polynomial.laurent_polynomial.**LaurentPolynomial**

Bases: CommutativeAlgebraElement

Base class for Laurent polynomials.

change_ring(*R*)

Return a copy of this Laurent polynomial, with coefficients in R.

EXAMPLES:

```
sage: R.<x> = LaurentPolynomialRing(QQ)
sage: a = x^2 + 3*x^3 + 5*x^-1
sage: a.change_ring(GF(3))
2*x^-1 + x^2
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> a = x**Integer(2) + Integer(3)*x**Integer(3) + Integer(5)*x**-Integer(1)
>>> a.change_ring(GF(Integer(3)))
2*x^-1 + x^2
```

Check that Issue #22277 is fixed:

```
sage: # needs sage.modules
sage: R.<x, y> = LaurentPolynomialRing(QQ)
sage: a = 2*x^2 + 3*x^3 + 4*x^-1
sage: a.change_ring(GF(3))
-x^2 + x^-1
```

```
>>> from sage.all import *
>>> # needs sage.modules
>>> R = LaurentPolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_
↪ngens(2)
>>> a = Integer(2)*x**Integer(2) + Integer(3)*x**Integer(3) + Integer(4)*x**-
↪Integer(1)
>>> a.change_ring(GF(Integer(3)))
-x^2 + x^-1
```

dict()

Abstract dict method.

EXAMPLES:

```
sage: R.<x> = LaurentPolynomialRing(ZZ)
sage: from sage.rings.polynomial.laurent_polynomial import LaurentPolynomial
sage: LaurentPolynomial.monomial_coefficients(x)
Traceback (most recent call last):
...
NotImplementedError
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> from sage.rings.polynomial.laurent_polynomial import LaurentPolynomial
>>> LaurentPolynomial.monomial_coefficients(x)
Traceback (most recent call last):
...
NotImplementedError
```

hamming_weight()

Return the hamming weight of `self`.

The hamming weight is number of nonzero coefficients and also known as the weight or sparsity.

EXAMPLES:

```
sage: R.<x> = LaurentPolynomialRing(ZZ)
sage: f = x^3 - 1
sage: f.hamming_weight()
2
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = x**Integer(3) - Integer(1)
>>> f.hamming_weight()
2
```

map_coefficients(*f*, *new_base_ring=None*)

Apply `f` to the coefficients of `self`.

If `f` is a `sage.categories.map.Map`, then the resulting polynomial will be defined over the codomain of `f`. Otherwise, the resulting polynomial will be over the same ring as `self`. Set `new_base_ring` to override this behavior.

INPUT:

- `f` – a callable that will be applied to the coefficients of `self`
- `new_base_ring` – (optional) if given, the resulting polynomial will be defined over this ring

EXAMPLES:

```
sage: # needs sage.rings.finite_rings
sage: k.<a> = GF(9)
sage: R.<x> = LaurentPolynomialRing(k)
sage: f = x*a + a
sage: f.map_coefficients(lambda a: a + 1)
(a + 1) + (a + 1)*x
sage: R.<x,y> = LaurentPolynomialRing(k, 2)                                   #␣
```

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```
↪needs sage.modules
sage: f = x*a + 2*x^3*y*a + a                                                  #␣
↪needs sage.modules
sage: f.map_coefficients(lambda a: a + 1)                                      #␣
↪needs sage.modules
(2*a + 1)*x^3*y + (a + 1)*x + a + 1
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> k = GF(Integer(9), names=('a',)); (a,) = k._first_ngens(1)
>>> R = LaurentPolynomialRing(k, names=('x',)); (x,) = R._first_ngens(1)
>>> f = x*a + a
>>> f.map_coefficients(lambda a: a + Integer(1))
(a + 1) + (a + 1)*x
>>> R = LaurentPolynomialRing(k, Integer(2), names=('x', 'y',)); (x, y,) = R._
↪first_ngens(2)# needs sage.modules
>>> f = x*a + Integer(2)*x**Integer(3)*y*a + a                                 ␣
↪               # needs sage.modules
>>> f.map_coefficients(lambda a: a + Integer(1))                               ␣
↪     # needs sage.modules
(2*a + 1)*x^3*y + (a + 1)*x + a + 1
```

Examples with different base ring:

```
sage: # needs sage.modules sage.rings.finite_rings
sage: R.<r> = GF(9); S.<s> = GF(81)
sage: h = Hom(R, S)[0]; h
Ring morphism:
  From: Finite Field in r of size 3^2
  To:   Finite Field in s of size 3^4
  Defn: r |--> 2*s^3 + 2*s^2 + 1
sage: T.<X,Y> = LaurentPolynomialRing(R, 2)
sage: f = r*X + Y
sage: g = f.map_coefficients(h); g
(2*s^3 + 2*s^2 + 1)*X + Y
sage: g.parent()
Multivariate Laurent Polynomial Ring in X, Y
 over Finite Field in s of size 3^4
sage: h = lambda x: x.trace()
sage: g = f.map_coefficients(h); g
X - Y
sage: g.parent()
Multivariate Laurent Polynomial Ring in X, Y
 over Finite Field in r of size 3^2
sage: g = f.map_coefficients(h, new_base_ring=GF(3)); g
X - Y
sage: g.parent()
Multivariate Laurent Polynomial Ring in X, Y over Finite Field of size 3
```

```
>>> from sage.all import *
>>> # needs sage.modules sage.rings.finite_rings
>>> R = GF(Integer(9), names=('r',)); (r,) = R._first_ngens(1); S =␣
```

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```
↪GF(Integer(81), names=('s',)); (s,) = S._first_ngens(1)
>>> h = Hom(R, S)[Integer(0)]; h
Ring morphism:
  From: Finite Field in r of size 3^2
  To:   Finite Field in s of size 3^4
  Defn: r |--> 2*s^3 + 2*s^2 + 1
>>> T = LaurentPolynomialRing(R, Integer(2), names=('X', 'Y',)); (X, Y,) = T._
↪first_ngens(2)
>>> f = r*X + Y
>>> g = f.map_coefficients(h); g
(2*s^3 + 2*s^2 + 1)*X + Y
>>> g.parent()
Multivariate Laurent Polynomial Ring in X, Y
 over Finite Field in s of size 3^4
>>> h = lambda x: x.trace()
>>> g = f.map_coefficients(h); g
X - Y
>>> g.parent()
Multivariate Laurent Polynomial Ring in X, Y
 over Finite Field in r of size 3^2
>>> g = f.map_coefficients(h, new_base_ring=GF(Integer(3))); g
X - Y
>>> g.parent()
Multivariate Laurent Polynomial Ring in X, Y over Finite Field of size 3
```

monomial_coefficients()

Abstract `dict` method.

EXAMPLES:

```
sage: R.<x> = LaurentPolynomialRing(ZZ)
sage: from sage.rings.polynomial.laurent_polynomial import LaurentPolynomial
sage: LaurentPolynomial.monomial_coefficients(x)
Traceback (most recent call last):
...
NotImplementedError
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> from sage.rings.polynomial.laurent_polynomial import LaurentPolynomial
>>> LaurentPolynomial.monomial_coefficients(x)
Traceback (most recent call last):
...
NotImplementedError
```

number_of_terms()

Abstract method for number of terms

EXAMPLES:

```
sage: R.<x> = LaurentPolynomialRing(ZZ)
sage: from sage.rings.polynomial.laurent_polynomial import LaurentPolynomial
sage: LaurentPolynomial.number_of_terms(x)
```

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```
Traceback (most recent call last):
...
NotImplementedError
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> from sage.rings.polynomial.laurent_polynomial import LaurentPolynomial
>>> LaurentPolynomial.number_of_terms(x)
Traceback (most recent call last):
...
NotImplementedError
```

class sage.rings.polynomial.laurent_polynomial.**LaurentPolynomial_univariate**

Bases: *LaurentPolynomial*

A univariate Laurent polynomial in the form of $t^n \cdot f$ where f is a polynomial in t.

INPUT:

- `parent` – a Laurent polynomial ring
- `f` – a polynomial (or something that can be coerced to one)
- `n` – integer (default: 0)

AUTHORS:

- Tom Boothby (2011) copied this class almost verbatim from `laurent_series_ring_element.pyx`, so most of the credit goes to William Stein, David Joyner, and Robert Bradshaw
- Travis Scrimshaw (09-2013): Cleaned-up and added a few extra methods

coefficients()

Return the nonzero coefficients of `self`.

EXAMPLES:

```
sage: R.<t> = LaurentPolynomialRing(QQ)
sage: f = -5/t^(2) + t + t^2 - 10/3*t^3
sage: f.coefficients()
[-5, 1, 1, -10/3]
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, names=('t',)); (t,) = R._first_ngens(1)
>>> f = -Integer(5)/t**(Integer(2)) + t + t**Integer(2) - Integer(10)/
↪Integer(3)*t**Integer(3)
>>> f.coefficients()
[-5, 1, 1, -10/3]
```

constant_coefficient()

Return the coefficient of the constant term of `self`.

EXAMPLES:

```
sage: R.<t> = LaurentPolynomialRing(QQ)
sage: f = 3*t^-2 - t^-1 + 3 + t^2
sage: f.constant_coefficient()
```

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```
3
sage: g = -2*t^-2 + t^-1 + 3*t
sage: g.constant_coefficient()
0
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, names=('t',)); (t,) = R._first_ngens(1)
>>> f = Integer(3)*t**-Integer(2) - t**-Integer(1) + Integer(3) +␣
↪t**Integer(2)
>>> f.constant_coefficient()
3
>>> g = -Integer(2)*t**-Integer(2) + t**-Integer(1) + Integer(3)*t
>>> g.constant_coefficient()
0
```

degree()

Return the degree of `self`.

EXAMPLES:

```
sage: R.<x> = LaurentPolynomialRing(ZZ)
sage: g = x^2 - x^4
sage: g.degree()
4
sage: g = -10/x^5 + x^2 - x^7
sage: g.degree()
7
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> g = x**Integer(2) - x**Integer(4)
>>> g.degree()
4
>>> g = -Integer(10)/x**Integer(5) + x**Integer(2) - x**Integer(7)
>>> g.degree()
7
```

The zero polynomial is defined to have degree $-\infty$:

```
sage: R.<x> = LaurentPolynomialRing(ZZ)
sage: R.zero().degree()
-Infinity
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> R.zero().degree()
-Infinity
```

derivative(**args*)

The formal derivative of this Laurent polynomial, with respect to variables supplied in args.

Multiple variables and iteration counts may be supplied. See documentation for the global *derivative()* function for more details.

See also

`_derivative()`

EXAMPLES:

```
sage: R.<x> = LaurentPolynomialRing(QQ)
sage: g = 1/x^10 - x + x^2 - x^4
sage: g.derivative()
-10*x^-11 - 1 + 2*x - 4*x^3
sage: g.derivative(x)
-10*x^-11 - 1 + 2*x - 4*x^3
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> g = Integer(1)/x**Integer(10) - x + x**Integer(2) - x**Integer(4)
>>> g.derivative()
-10*x^-11 - 1 + 2*x - 4*x^3
>>> g.derivative(x)
-10*x^-11 - 1 + 2*x - 4*x^3
```

```
sage: R.<t> = PolynomialRing(ZZ)
sage: S.<x> = LaurentPolynomialRing(R)
sage: f = 2*t/x + (3*t^2 + 6*t)*x
sage: f.derivative()
-2*t*x^-2 + (3*t^2 + 6*t)
sage: f.derivative(x)
-2*t*x^-2 + (3*t^2 + 6*t)
sage: f.derivative(t)
2*x^-1 + (6*t + 6)*x
```

```
>>> from sage.all import *
>>> R = PolynomialRing(ZZ, names=('t',)); (t,) = R._first_ngens(1)
>>> S = LaurentPolynomialRing(R, names=('x',)); (x,) = S._first_ngens(1)
>>> f = Integer(2)*t/x + (Integer(3)*t**Integer(2) + Integer(6)*t)*x
>>> f.derivative()
-2*t*x^-2 + (3*t^2 + 6*t)
>>> f.derivative(x)
-2*t*x^-2 + (3*t^2 + 6*t)
>>> f.derivative(t)
2*x^-1 + (6*t + 6)*x
```

dict()

Return a dictionary representing `self`.

EXAMPLES:

```
sage: R.<x,y> = ZZ[]
sage: Q.<t> = LaurentPolynomialRing(R)
sage: f = (x^3 + y/t^3)^3 + t^2; f
y^3*t^-9 + 3*x^3*y^2*t^-6 + 3*x^6*y*t^-3 + x^9 + t^2
sage: f.monomial_coefficients()
{-9: y^3, -6: 3*x^3*y^2, -3: 3*x^6*y, 0: x^9, 2: 1}
```

```
>>> from sage.all import *
>>> R = ZZ['x, y']; (x, y,) = R._first_ngens(2)
>>> Q = LaurentPolynomialRing(R, names=('t',)); (t,) = Q._first_ngens(1)
>>> f = (x**Integer(3) + y/t**Integer(3))**Integer(3) + t**Integer(2); f
y^3*t^-9 + 3*x^3*y^2*t^-6 + 3*x^6*y*t^-3 + x^9 + t^2
>>> f.monomial_coefficients()
{-9: y^3, -6: 3*x^3*y^2, -3: 3*x^6*y, 0: x^9, 2: 1}
```

`dict` is an alias:

```
sage: f.dict()
{-9: y^3, -6: 3*x^3*y^2, -3: 3*x^6*y, 0: x^9, 2: 1}
```

```
>>> from sage.all import *
>>> f.dict()
{-9: y^3, -6: 3*x^3*y^2, -3: 3*x^6*y, 0: x^9, 2: 1}
```

divides(*other*)

Return `True` if `self` divides `other`.

EXAMPLES:

```
sage: R.<x> = LaurentPolynomialRing(ZZ)
sage: (2*x**-1 + 1).divides(4*x**-2 - 1)
True
sage: (2*x + 1).divides(4*x**2 + 1)
False
sage: (2*x + x**-1).divides(R(0))
True
sage: R(0).divides(2*x ** -1 + 1)
False
sage: R(0).divides(R(0))
True
sage: R.<x> = LaurentPolynomialRing(Zmod(6))
sage: p = 4*x + 3*x^-1
sage: q = 5*x^2 + x + 2*x^-2
sage: p.divides(q)
False

sage: R.<x,y> = GF(2)[]
sage: S.<z> = LaurentPolynomialRing(R)
sage: p = (x+y+1) * z**-1 + x*y
sage: q = (y^2-x^2) * z**-2 + z + x-y
sage: p.divides(q), p.divides(p*q)                                          #␣
↪needs sage.libs.singular
(False, True)
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> (Integer(2)*x**-Integer(1) + Integer(1)).divides(Integer(4)*x**-
↪Integer(2) - Integer(1))
True
>>> (Integer(2)*x + Integer(1)).divides(Integer(4)*x**Integer(2) + Integer(1))
```

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```
False
>>> (Integer(2)*x + x**-Integer(1)).divides(R(Integer(0)))
True
>>> R(Integer(0)).divides(Integer(2)*x ** -Integer(1) + Integer(1))
False
>>> R(Integer(0)).divides(R(Integer(0)))
True
>>> R = LaurentPolynomialRing(Zmod(Integer(6)), names=('x',)); (x,) = R._
↪first_ngens(1)
>>> p = Integer(4)*x + Integer(3)*x**-Integer(1)
>>> q = Integer(5)*x**Integer(2) + x + Integer(2)*x**-Integer(2)
>>> p.divides(q)
False

>>> R = GF(Integer(2))['x, y']; (x, y,) = R._first_ngens(2)
>>> S = LaurentPolynomialRing(R, names=('z',)); (z,) = S._first_ngens(1)
>>> p = (x+y+Integer(1)) * z**-Integer(1) + x*y
>>> q = (y**Integer(2)-x**Integer(2)) * z**-Integer(2) + z + x-y
>>> p.divides(q), p.divides(p*q)                                          #␣
↪needs sage.libs.singular
(False, True)
```

euclidean_degree()

Return the degree of `self` as an element of an Euclidean domain.

This is the Euclidean degree of the underlying polynomial.

EXAMPLES:

```
sage: R.<x> = LaurentPolynomialRing(QQ)
sage: (x^-5 + x^2).euclidean_degree()
7

sage: R.<x> = LaurentPolynomialRing(ZZ)
sage: (x^-5 + x^2).euclidean_degree()
Traceback (most recent call last):
...
NotImplementedError
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> (x**-Integer(5) + x**Integer(2)).euclidean_degree()
7

>>> R = LaurentPolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> (x**-Integer(5) + x**Integer(2)).euclidean_degree()
Traceback (most recent call last):
...
NotImplementedError
```

exponents()

Return the exponents appearing in `self` with nonzero coefficients.

EXAMPLES:

```
sage: R.<t> = LaurentPolynomialRing(QQ)
sage: f = -5/t^(2) + t + t^2 - 10/3*t^3
sage: f.exponents()
[-2, 1, 2, 3]
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, names=('t',)); (t,) = R._first_ngens(1)
>>> f = -Integer(5)/t**(Integer(2)) + t + t**Integer(2) - Integer(10)/
↪Integer(3)*t**Integer(3)
>>> f.exponents()
[-2, 1, 2, 3]
```

factor()

Return a Laurent monomial (the unit part of the factorization) and a factored polynomial.

EXAMPLES:

```
sage: R.<t> = LaurentPolynomialRing(ZZ)
sage: f = 4*t^-7 + 3*t^3 + 2*t^4 + t^-6
sage: f.factor()                                                          #␣
↪needs sage.libs.pari
(t^-7) * (4 + t + 3*t^10 + 2*t^11)
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(ZZ, names=('t',)); (t,) = R._first_ngens(1)
>>> f = Integer(4)*t**-Integer(7) + Integer(3)*t**Integer(3) +␣
↪Integer(2)*t**Integer(4) + t**-Integer(6)
>>> f.factor()                                                          #␣
↪needs sage.libs.pari
(t^-7) * (4 + t + 3*t^10 + 2*t^11)
```

gcd(*right*)

Return the gcd of `self` with `right` where the common divisor `d` makes both `self` and `right` into polynomials with the lowest possible degree.

EXAMPLES:

```
sage: R.<t> = LaurentPolynomialRing(QQ)
sage: t.gcd(2)
1
sage: gcd(t^-2 + 1, t^-4 + 3*t^-1)
t^-4
sage: gcd((t^-2 + t)*(t + t^-1), (t^5 + t^8)*(1 + t^-2))
t^-3 + t^-1 + 1 + t^2
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, names=('t',)); (t,) = R._first_ngens(1)
>>> t.gcd(Integer(2))
1
>>> gcd(t**-Integer(2) + Integer(1), t**-Integer(4) + Integer(3)*t**-
↪Integer(1))
t^-4
>>> gcd((t**-Integer(2) + t)*(t + t**-Integer(1)), (t**Integer(5) +␣
```

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```
↪t**Integer(8))*(Integer(1) + t**-Integer(2)))
t^-3 + t^-1 + 1 + t^2
```

integral()

The formal integral of this Laurent series with 0 constant term.

EXAMPLES:

The integral may or may not be defined if the base ring is not a field.

```
sage: t = LaurentPolynomialRing(ZZ, 't').0
sage: f = 2*t^-3 + 3*t^2
sage: f.integral()
-t^-2 + t^3
```

```
>>> from sage.all import *
>>> t = LaurentPolynomialRing(ZZ, 't').gen(0)
>>> f = Integer(2)*t**-Integer(3) + Integer(3)*t**Integer(2)
>>> f.integral()
-t^-2 + t^3
```

```
sage: f = t^3
sage: f.integral()
Traceback (most recent call last):
...
ArithmeticError: coefficients of integral cannot be coerced into the base ring
```

```
>>> from sage.all import *
>>> f = t**Integer(3)
>>> f.integral()
Traceback (most recent call last):
...
ArithmeticError: coefficients of integral cannot be coerced into the base ring
```

The integral of $1/t$ is $\log(t)$, which is not given by a Laurent polynomial:

```
sage: t = LaurentPolynomialRing(ZZ,'t').0
sage: f = -1/t^3 - 31/t
sage: f.integral()
Traceback (most recent call last):
...
ArithmeticError: the integral of is not a Laurent polynomial, since t^-1 has␣
↪nonzero coefficient
```

```
>>> from sage.all import *
>>> t = LaurentPolynomialRing(ZZ,'t').gen(0)
>>> f = -Integer(1)/t**Integer(3) - Integer(31)/t
>>> f.integral()
Traceback (most recent call last):
...
ArithmeticError: the integral of is not a Laurent polynomial, since t^-1 has␣
↪nonzero coefficient
```

Another example with just one negative coefficient:

```
sage: A.<t> = LaurentPolynomialRing(QQ)
sage: f = -2*t^(-4)
sage: f.integral()
2/3*t^-3
sage: f.integral().derivative() == f
True
```

```
>>> from sage.all import *
>>> A = LaurentPolynomialRing(QQ, names=('t',)); (t,) = A._first_ngens(1)
>>> f = -Integer(2)*t**(-Integer(4))
>>> f.integral()
2/3*t^-3
>>> f.integral().derivative() == f
True
```

inverse_mod(*a*, *m*)

Invert the polynomial `a` with respect to `m`, or raise a `ValueError` if no such inverse exists.

The parameter `m` may be either a single polynomial or an ideal (for consistency with *inverse_mod()* in other rings).

ALGORITHM: Solve the system $as + mt = 1$, returning s as the inverse of a mod m.

EXAMPLES:

```
sage: S.<t> = LaurentPolynomialRing(QQ)
sage: f = inverse_mod(t^-2 + 1, t^-3 + 1); f
1/2*t^2 - 1/2*t^3 - 1/2*t^4
sage: f * (t^-2 + 1) + (1/2*t^4 + 1/2*t^3) * (t^-3 + 1)
1
```

```
>>> from sage.all import *
>>> S = LaurentPolynomialRing(QQ, names=('t',)); (t,) = S._first_ngens(1)
>>> f = inverse_mod(t**-Integer(2) + Integer(1), t**-Integer(3) + Integer(1));
↪ f
1/2*t^2 - 1/2*t^3 - 1/2*t^4
>>> f * (t**-Integer(2) + Integer(1)) + (Integer(1)/Integer(2)*t**Integer(4)␣
↪+ Integer(1)/Integer(2)*t**Integer(3)) * (t**-Integer(3) + Integer(1))
1
```

inverse_of_unit()

Return the inverse of `self` if a unit.

EXAMPLES:

```
sage: R.<t> = LaurentPolynomialRing(QQ)
sage: (t^-2).inverse_of_unit()
t^2
sage: (t + 2).inverse_of_unit()
Traceback (most recent call last):
...
ArithmeticError: element is not a unit
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, names=('t',)); (t,) = R._first_ngens(1)
>>> (t**-Integer(2)).inverse_of_unit()
t^2
>>> (t + Integer(2)).inverse_of_unit()
Traceback (most recent call last):
...
ArithmeticError: element is not a unit
```

is_constant()

Return whether this Laurent polynomial is constant.

EXAMPLES:

```
sage: R.<x> = LaurentPolynomialRing(QQ)
sage: x.is_constant()
False
sage: R.one().is_constant()
True
sage: (x^-2).is_constant()
False
sage: (x^2).is_constant()
False
sage: (x^-2 + 2).is_constant()
False
sage: R(0).is_constant()
True
sage: R(42).is_constant()
True
sage: x.is_constant()
False
sage: (1/x).is_constant()
False
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> x.is_constant()
False
>>> R.one().is_constant()
True
>>> (x**-Integer(2)).is_constant()
False
>>> (x**Integer(2)).is_constant()
False
>>> (x**-Integer(2) + Integer(2)).is_constant()
False
>>> R(Integer(0)).is_constant()
True
>>> R(Integer(42)).is_constant()
True
>>> x.is_constant()
False
>>> (Integer(1)/x).is_constant()
```

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```
False
```

is_monomial()

Return `True` if `self` is a monomial; that is, if `self` is x^n for some integer n.

EXAMPLES:

```
sage: k.<z> = LaurentPolynomialRing(QQ)
sage: z.is_monomial()
True
sage: k(1).is_monomial()
True
sage: (z+1).is_monomial()
False
sage: (z^-2909).is_monomial()
True
sage: (38*z^-2909).is_monomial()
False
```

```
>>> from sage.all import *
>>> k = LaurentPolynomialRing(QQ, names=('z',)); (z,) = k._first_ngens(1)
>>> z.is_monomial()
True
>>> k(Integer(1)).is_monomial()
True
>>> (z+Integer(1)).is_monomial()
False
>>> (z**-Integer(2909)).is_monomial()
True
>>> (Integer(38)*z**-Integer(2909)).is_monomial()
False
```

is_square(*root=False*)

Return whether this Laurent polynomial is a square.

If `root` is set to `True` then return a pair made of the boolean answer together with `None` or a square root.

EXAMPLES:

```
sage: R.<t> = LaurentPolynomialRing(QQ)

sage: R.one().is_square()
True
sage: R(2).is_square()
False

sage: t.is_square()
False
sage: (t**-2).is_square()
True
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, names=('t',)); (t,) = R._first_ngens(1)
```

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```
>>> R.one().is_square()
True
>>> R(Integer(2)).is_square()
False

>>> t.is_square()
False
>>> (t**-Integer(2)).is_square()
True
```

Usage of the `root` option:

```
sage: p = (1 + t^-1 - 2*t^3)
sage: p.is_square(root=True)
(False, None)
sage: (p**2).is_square(root=True)
(True, -t^-1 - 1 + 2*t^3)
```

```
>>> from sage.all import *
>>> p = (Integer(1) + t**-Integer(1) - Integer(2)*t**Integer(3))
>>> p.is_square(root=True)
(False, None)
>>> (p**Integer(2)).is_square(root=True)
(True, -t^-1 - 1 + 2*t^3)
```

The answer is dependent of the base ring:

```
sage: # needs sage.rings.number_field
sage: S.<u> = LaurentPolynomialRing(QQbar)
sage: (2 + 4*t + 2*t^2).is_square()
False
sage: (2 + 4*u + 2*u^2).is_square()
True
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> S = LaurentPolynomialRing(QQbar, names=('u',)); (u,) = S._first_ngens(1)
>>> (Integer(2) + Integer(4)*t + Integer(2)*t**Integer(2)).is_square()
False
>>> (Integer(2) + Integer(4)*u + Integer(2)*u**Integer(2)).is_square()
True
```

is_unit()

Return `True` if this Laurent polynomial is a unit in this ring.

EXAMPLES:

```
sage: R.<t> = LaurentPolynomialRing(QQ)
sage: (2 + t).is_unit()
False
sage: f = 2*t
```

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```
sage: f.is_unit()
True
sage: 1/f
1/2*t^-1
sage: R(0).is_unit()
False
sage: R.<s> = LaurentPolynomialRing(ZZ)
sage: g = 2*s
sage: g.is_unit()
False
sage: 1/g
1/2*s^-1
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, names=('t',)); (t,) = R._first_ngens(1)
>>> (Integer(2) + t).is_unit()
False
>>> f = Integer(2)*t
>>> f.is_unit()
True
>>> Integer(1)/f
1/2*t^-1
>>> R(Integer(0)).is_unit()
False
>>> R = LaurentPolynomialRing(ZZ, names=('s',)); (s,) = R._first_ngens(1)
>>> g = Integer(2)*s
>>> g.is_unit()
False
>>> Integer(1)/g
1/2*s^-1
```

ALGORITHM: A Laurent polynomial is a unit if and only if its “unit part” is a unit.

is_zero()

Return `1` if `self` is 0, else return `0`.

EXAMPLES:

```
sage: R.<x> = LaurentPolynomialRing(QQ)
sage: f = 1/x + x + x^2 + 3*x^4
sage: f.is_zero()
0
sage: z = 0*f
sage: z.is_zero()
1
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = Integer(1)/x + x + x**Integer(2) + Integer(3)*x**Integer(4)
>>> f.is_zero()
0
>>> z = Integer(0)*f
```

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```
>>> z.is_zero()
1
```

monomial_coefficients()

Return a dictionary representing self.

EXAMPLES:

```
sage: R.<x,y> = ZZ[]
sage: Q.<t> = LaurentPolynomialRing(R)
sage: f = (x^3 + y/t^3)^3 + t^2; f
y^3*t^-9 + 3*x^3*y^2*t^-6 + 3*x^6*y*t^-3 + x^9 + t^2
sage: f.monomial_coefficients()
{-9: y^3, -6: 3*x^3*y^2, -3: 3*x^6*y, 0: x^9, 2: 1}
```

```
>>> from sage.all import *
>>> R = ZZ['x, y']; (x, y,) = R._first_ngens(2)
>>> Q = LaurentPolynomialRing(R, names=('t',)); (t,) = Q._first_ngens(1)
>>> f = (x**Integer(3) + y/t**Integer(3))**Integer(3) + t**Integer(2); f
y^3*t^-9 + 3*x^3*y^2*t^-6 + 3*x^6*y*t^-3 + x^9 + t^2
>>> f.monomial_coefficients()
{-9: y^3, -6: 3*x^3*y^2, -3: 3*x^6*y, 0: x^9, 2: 1}
```

dict is an alias:

```
sage: f.dict()
{-9: y^3, -6: 3*x^3*y^2, -3: 3*x^6*y, 0: x^9, 2: 1}
```

```
>>> from sage.all import *
>>> f.dict()
{-9: y^3, -6: 3*x^3*y^2, -3: 3*x^6*y, 0: x^9, 2: 1}
```

monomial_reduction()

Return the decomposition as a polynomial and a power of the variable. Constructed for compatibility with the multivariate case.

OUTPUT:

A tuple (u, t^n) where u is the underlying polynomial and n is the power of the exponent shift.

EXAMPLES:

```
sage: R.<x> = LaurentPolynomialRing(QQ)
sage: f = 1/x + x^2 + 3*x^4
sage: f.monomial_reduction()
(3*x^5 + x^3 + 1, x^-1)
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = Integer(1)/x + x**Integer(2) + Integer(3)*x**Integer(4)
>>> f.monomial_reduction()
(3*x^5 + x^3 + 1, x^-1)
```

number_of_terms()

Return the number of nonzero coefficients of `self`.

Also called weight, hamming weight or sparsity.

EXAMPLES:

```
sage: R.<x> = LaurentPolynomialRing(ZZ)
sage: f = x^3 - 1
sage: f.number_of_terms()
2
sage: R(0).number_of_terms()
0
sage: f = (x+1)^100
sage: f.number_of_terms()
101
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = x**Integer(3) - Integer(1)
>>> f.number_of_terms()
2
>>> R(Integer(0)).number_of_terms()
0
>>> f = (x+Integer(1))**Integer(100)
>>> f.number_of_terms()
101
```

The method `hamming_weight()` is an alias:

```
sage: f.hamming_weight()
101
```

```
>>> from sage.all import *
>>> f.hamming_weight()
101
```

polynomial_construction()

Return the polynomial and the shift in power used to construct the Laurent polynomial $t^n u$.

OUTPUT:

A tuple `(u, n)` where `u` is the underlying polynomial and `n` is the power of the exponent shift.

EXAMPLES:

```
sage: R.<x> = LaurentPolynomialRing(QQ)
sage: f = 1/x + x^2 + 3*x^4
sage: f.polynomial_construction()
(3*x^5 + x^3 + 1, -1)
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = Integer(1)/x + x**Integer(2) + Integer(3)*x**Integer(4)
>>> f.polynomial_construction()
(3*x^5 + x^3 + 1, -1)
```

quo_rem(*other*)

Divide `self` by `other` and return a quotient `q` and a remainder `r` such that `self == q * other + r`.

EXAMPLES:

```
sage: R.<t> = LaurentPolynomialRing(QQ)
sage: (t^-3 - t^3).quo_rem(t^-1 - t)
(t^-2 + 1 + t^2, 0)
sage: (t^-2 + 3 + t).quo_rem(t^-4)
(t^2 + 3*t^4 + t^5, 0)

sage: num = t^-2 + t
sage: den = t^-2 + 1
sage: q, r = num.quo_rem(den)
sage: num == q * den + r
True
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, names=('t',)); (t,) = R._first_ngens(1)
>>> (t**-Integer(3) - t**Integer(3)).quo_rem(t**-Integer(1) - t)
(t^-2 + 1 + t^2, 0)
>>> (t**-Integer(2) + Integer(3) + t).quo_rem(t**-Integer(4))
(t^2 + 3*t^4 + t^5, 0)

>>> num = t**-Integer(2) + t
>>> den = t**-Integer(2) + Integer(1)
>>> q, r = num.quo_rem(den)
>>> num == q * den + r
True
```

residue()

Return the residue of `self`.

The residue is the coefficient of t^-1.

EXAMPLES:

```
sage: R.<t> = LaurentPolynomialRing(QQ)
sage: f = 3*t^-2 - t^-1 + 3 + t^2
sage: f.residue()
-1
sage: g = -2*t^-2 + 4 + 3*t
sage: g.residue()
0
sage: f.residue().parent()
Rational Field
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, names=('t',)); (t,) = R._first_ngens(1)
>>> f = Integer(3)*t**-Integer(2) - t**-Integer(1) + Integer(3) +␣
↪t**Integer(2)
>>> f.residue()
-1
>>> g = -Integer(2)*t**-Integer(2) + Integer(4) + Integer(3)*t
```

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```
>>> g.residue()
0
>>> f.residue().parent()
Rational Field
```

shift(k)

Return this Laurent polynomial multiplied by the power t^n. Does not change this polynomial.

EXAMPLES:

```
sage: R.<t> = LaurentPolynomialRing(QQ['y'])
sage: f = (t+t^-1)^4; f
t^-4 + 4*t^-2 + 6 + 4*t^2 + t^4
sage: f.shift(10)
t^6 + 4*t^8 + 6*t^10 + 4*t^12 + t^14
sage: f >> 10
t^-14 + 4*t^-12 + 6*t^-10 + 4*t^-8 + t^-6
sage: f << 4
1 + 4*t^2 + 6*t^4 + 4*t^6 + t^8
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ['y'], names=('t',)); (t,) = R._first_ngens(1)
>>> f = (t+t**-Integer(1))**Integer(4); f
t^-4 + 4*t^-2 + 6 + 4*t^2 + t^4
>>> f.shift(Integer(10))
t^6 + 4*t^8 + 6*t^10 + 4*t^12 + t^14
>>> f >> Integer(10)
t^-14 + 4*t^-12 + 6*t^-10 + 4*t^-8 + t^-6
>>> f << Integer(4)
1 + 4*t^2 + 6*t^4 + 4*t^6 + t^8
```

truncate(n)

Return a polynomial with degree at most $n - 1$ whose j-th coefficients agree with `self` for all $j < n$.

EXAMPLES:

```
sage: R.<x> = LaurentPolynomialRing(QQ)
sage: f = 1/x^12 + x^3 + x^5 + x^9
sage: f.truncate(10)
x^-12 + x^3 + x^5 + x^9
sage: f.truncate(5)
x^-12 + x^3
sage: f.truncate(-16)
0
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = Integer(1)/x**Integer(12) + x**Integer(3) + x**Integer(5) +␣
↪x**Integer(9)
>>> f.truncate(Integer(10))
x^-12 + x^3 + x^5 + x^9
>>> f.truncate(Integer(5))
```

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```
x^-12 + x^3
>>> f.truncate(-Integer(16))
0
```

valuation(*p=None*)

Return the valuation of `self`.

The valuation of a Laurent polynomial $t^n u$ is n plus the valuation of u.

EXAMPLES:

```
sage: R.<x> = LaurentPolynomialRing(ZZ)
sage: f = 1/x + x^2 + 3*x^4
sage: g = 1 - x + x^2 - x^4
sage: f.valuation()
-1
sage: g.valuation()
0
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(ZZ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = Integer(1)/x + x**Integer(2) + Integer(3)*x**Integer(4)
>>> g = Integer(1) - x + x**Integer(2) - x**Integer(4)
>>> f.valuation()
-1
>>> g.valuation()
0
```

variable_name()

Return the name of variable of `self` as a string.

EXAMPLES:

```
sage: R.<x> = LaurentPolynomialRing(QQ)
sage: f = 1/x + x^2 + 3*x^4
sage: f.variable_name()
'x'
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = Integer(1)/x + x**Integer(2) + Integer(3)*x**Integer(4)
>>> f.variable_name()
'x'
```

variables()

Return the tuple of variables occurring in this Laurent polynomial.

EXAMPLES:

```
sage: R.<x> = LaurentPolynomialRing(QQ)
sage: f = 1/x + x^2 + 3*x^4
sage: f.variables()
(x,)
```

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```
sage: R.one().variables()
()
```

```
>>> from sage.all import *
>>> R = LaurentPolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> f = Integer(1)/x + x**Integer(2) + Integer(3)*x**Integer(4)
>>> f.variables()
(x,)
>>> R.one().variables()
()
```

xgcd(*other*)

Extended *gcd()* for univariate Laurent polynomial rings over a field.

OUTPUT:

A triple `(g, p, q)` such that `g` is the *gcd()* of `self` ($= a$) and `other` ($= b$), and `p` and `q` are cofactors satisfying the Bezout identity

$$g = p \cdot a + q \cdot b.$$

EXAMPLES:

```
sage: S.<t> = LaurentPolynomialRing(QQ)
sage: a = t^-2 + 1
sage: b = t^-3 + 1
sage: g, p, q = a.xgcd(b); (g, p, q)
(t^-3, 1/2*t^-1 - 1/2 - 1/2*t, 1/2 + 1/2*t)
sage: g == p * a + q * b
True
sage: g == a.gcd(b)
True
sage: t.xgcd(t)
(t, 0, 1)
sage: t.xgcd(5)
(1, 0, 1/5)
```

```
>>> from sage.all import *
>>> S = LaurentPolynomialRing(QQ, names=('t',)); (t,) = S._first_ngens(1)
>>> a = t**-Integer(2) + Integer(1)
>>> b = t**-Integer(3) + Integer(1)
>>> g, p, q = a.xgcd(b); (g, p, q)
(t^-3, 1/2*t^-1 - 1/2 - 1/2*t, 1/2 + 1/2*t)
>>> g == p * a + q * b
True
>>> g == a.gcd(b)
True
>>> t.xgcd(t)
(t, 0, 1)
>>> t.xgcd(Integer(5))
(1, 0, 1/5)
```

5.4 MacMahon's Partition Analysis Omega Operator

This module implements *MacMahon's Omega Operator* [Mac1915], which takes a quotient of Laurent polynomials and removes all negative exponents in the corresponding power series.

5.4.1 Examples

In the following example, all negative exponents of μ are removed. The formula

$$\Omega_{\geq} \frac{1}{(1 - x\mu)(1 - y/\mu)} = \frac{1}{(1 - x)(1 - xy)}$$

can be calculated and verified by

```
sage: L.<mu, x, y> = LaurentPolynomialRing(ZZ)
sage: MacMahonOmega(mu, 1, [1 - x*mu, 1 - y/mu])
1 * (-x + 1)^-1 * (-x*y + 1)^-1
```

```
>>> from sage.all import *
>>> L = LaurentPolynomialRing(ZZ, names=('mu', 'x', 'y',)); (mu, x, y,) = L._first_
↪ngens(3)
>>> MacMahonOmega(mu, Integer(1), [Integer(1) - x*mu, Integer(1) - y/mu])
1 * (-x + 1)^-1 * (-x*y + 1)^-1
```

5.4.2 Various

AUTHORS:

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5.4.3 Functions

sage.rings.polynomial.omega.**MacMahonOmega**(*var*, *expression*, *denominator=None*, *op=<built-in function ge>*, *Factorization_sort=False*, *Factorization_simplify=True*)

Return Ω_{op} of `expression` with respect to `var`.

To be more precise, calculate

$$\Omega_{\mathrm{op}} \frac{n}{d_1 \dots d_n}$$

for the numerator n and the factors d_1, …, d_n of the denominator, all of which are Laurent polynomials in `var` and return a (partial) factorization of the result.

INPUT:

- `var` – a variable or a representation string of a variable
- `expression` – a `Factorization` of Laurent polynomials or, if `denominator` is specified, a Laurent polynomial interpreted as the numerator of the expression
- `denominator` – a Laurent polynomial or a `Factorization` (consisting of Laurent polynomial factors) or a tuple/list of factors (Laurent polynomials)

- `op` – (default: `operator.ge`) an operator

  At the moment only `operator.ge` is implemented.
- `Factorization_sort` (default: `False`) and `Factorization_simplify` (default: `True`) – are passed on to `sage.structure.factorization.Factorization` when creating the result

OUTPUT:

A (partial) `Factorization` of the result whose factors are Laurent polynomials

Note: The numerator of the result may not be factored.

REFERENCES:

- [Mac1915]
- [APR2001]

EXAMPLES:

```
sage: L.<mu, x, y, z, w> = LaurentPolynomialRing(ZZ)

sage: MacMahonOmega(mu, 1, [1 - x*mu, 1 - y/mu])
1 * (-x + 1)^-1 * (-x*y + 1)^-1

sage: MacMahonOmega(mu, 1, [1 - x*mu, 1 - y/mu, 1 - z/mu])
1 * (-x + 1)^-1 * (-x*y + 1)^-1 * (-x*z + 1)^-1
sage: MacMahonOmega(mu, 1, [1 - x*mu, 1 - y*mu, 1 - z/mu])
(-x*y*z + 1) * (-x + 1)^-1 * (-y + 1)^-1 * (-x*z + 1)^-1 * (-y*z + 1)^-1
sage: MacMahonOmega(mu, 1, [1 - x*mu, 1 - y/mu^2])
1 * (-x + 1)^-1 * (-x^2*y + 1)^-1
sage: MacMahonOmega(mu, 1, [1 - x*mu^2, 1 - y/mu])
(x*y + 1) * (-x + 1)^-1 * (-x*y^2 + 1)^-1

sage: MacMahonOmega(mu, 1, [1 - x*mu, 1 - y*mu, 1 - z/mu^2])
(-x^2*y*z - x*y^2*z + x*y*z + 1) *
(-x + 1)^-1 * (-y + 1)^-1 * (-x^2*z + 1)^-1 * (-y^2*z + 1)^-1

sage: MacMahonOmega(mu, 1, [1 - x*mu, 1 - y/mu^3])
1 * (-x + 1)^-1 * (-x^3*y + 1)^-1
sage: MacMahonOmega(mu, 1, [1 - x*mu, 1 - y/mu^4])
1 * (-x + 1)^-1 * (-x^4*y + 1)^-1
sage: MacMahonOmega(mu, 1, [1 - x*mu^3, 1 - y/mu])
(x*y^2 + x*y + 1) * (-x + 1)^-1 * (-x*y^3 + 1)^-1
sage: MacMahonOmega(mu, 1, [1 - x*mu^4, 1 - y/mu])
(x*y^3 + x*y^2 + x*y + 1) * (-x + 1)^-1 * (-x*y^4 + 1)^-1

sage: MacMahonOmega(mu, 1, [1 - x*mu^2, 1 - y/mu, 1 - z/mu])
(x*y*z + x*y + x*z + 1) *
(-x + 1)^-1 * (-x*y^2 + 1)^-1 * (-x*z^2 + 1)^-1
sage: MacMahonOmega(mu, 1, [1 - x*mu^2, 1 - y*mu, 1 - z/mu])
(-x*y*z^2 - x*y*z + x*z + 1) *
(-x + 1)^-1 * (-y + 1)^-1 * (-x*z^2 + 1)^-1 * (-y*z + 1)^-1
```

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```
sage: MacMahonOmega(mu, 1, [1 - x*mu, 1 - y*mu, 1 - z*mu, 1 - w/mu])
(x*y*z*w^2 + x*y*z*w - x*y*w - x*z*w - y*z*w + 1) *
(-x + 1)^-1 * (-y + 1)^-1 * (-z + 1)^-1 *
(-x*w + 1)^-1 * (-y*w + 1)^-1 * (-z*w + 1)^-1
sage: MacMahonOmega(mu, 1, [1 - x*mu, 1 - y*mu, 1 - z/mu, 1 - w/mu])
(x^2*y*z*w + x*y^2*z*w - x*y*z*w - x*y*z - x*y*w + 1) *
(-x + 1)^-1 * (-y + 1)^-1 *
(-x*z + 1)^-1 * (-x*w + 1)^-1 * (-y*z + 1)^-1 * (-y*w + 1)^-1

sage: MacMahonOmega(mu, mu^-2, [1 - x*mu, 1 - y/mu])
x^2 * (-x + 1)^-1 * (-x*y + 1)^-1
sage: MacMahonOmega(mu, mu^-1, [1 - x*mu, 1 - y/mu])
x * (-x + 1)^-1 * (-x*y + 1)^-1
sage: MacMahonOmega(mu, mu, [1 - x*mu, 1 - y/mu])
(-x*y + y + 1) * (-x + 1)^-1 * (-x*y + 1)^-1
sage: MacMahonOmega(mu, mu^2, [1 - x*mu, 1 - y/mu])
(-x*y^2 - x*y + y^2 + y + 1) * (-x + 1)^-1 * (-x*y + 1)^-1
```

```
>>> from sage.all import *
>>> L = LaurentPolynomialRing(ZZ, names=('mu', 'x', 'y', 'z', 'w',)); (mu, x, y,␣
↪z, w,) = L._first_ngens(5)

>>> MacMahonOmega(mu, Integer(1), [Integer(1) - x*mu, Integer(1) - y/mu])
1 * (-x + 1)^-1 * (-x*y + 1)^-1

>>> MacMahonOmega(mu, Integer(1), [Integer(1) - x*mu, Integer(1) - y/mu,␣
↪Integer(1) - z/mu])
1 * (-x + 1)^-1 * (-x*y + 1)^-1 * (-x*z + 1)^-1
>>> MacMahonOmega(mu, Integer(1), [Integer(1) - x*mu, Integer(1) - y*mu,␣
↪Integer(1) - z/mu])
(-x*y*z + 1) * (-x + 1)^-1 * (-y + 1)^-1 * (-x*z + 1)^-1 * (-y*z + 1)^-1
>>> MacMahonOmega(mu, Integer(1), [Integer(1) - x*mu, Integer(1) - y/
↪mu**Integer(2)])
1 * (-x + 1)^-1 * (-x^2*y + 1)^-1
>>> MacMahonOmega(mu, Integer(1), [Integer(1) - x*mu**Integer(2), Integer(1) - y/
↪mu])
(x*y + 1) * (-x + 1)^-1 * (-x*y^2 + 1)^-1

>>> MacMahonOmega(mu, Integer(1), [Integer(1) - x*mu, Integer(1) - y*mu,␣
↪Integer(1) - z/mu**Integer(2)])
(-x^2*y*z - x*y^2*z + x*y*z + 1) *
(-x + 1)^-1 * (-y + 1)^-1 * (-x^2*z + 1)^-1 * (-y^2*z + 1)^-1

>>> MacMahonOmega(mu, Integer(1), [Integer(1) - x*mu, Integer(1) - y/
↪mu**Integer(3)])
1 * (-x + 1)^-1 * (-x^3*y + 1)^-1
>>> MacMahonOmega(mu, Integer(1), [Integer(1) - x*mu, Integer(1) - y/
↪mu**Integer(4)])
1 * (-x + 1)^-1 * (-x^4*y + 1)^-1
>>> MacMahonOmega(mu, Integer(1), [Integer(1) - x*mu**Integer(3), Integer(1) - y/
↪mu])
```

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```
(x*y^2 + x*y + 1) * (-x + 1)^-1 * (-x*y^3 + 1)^-1
>>> MacMahonOmega(mu, Integer(1), [Integer(1) - x*mu**Integer(4), Integer(1) - y/
↪mu])
(x*y^3 + x*y^2 + x*y + 1) * (-x + 1)^-1 * (-x*y^4 + 1)^-1

>>> MacMahonOmega(mu, Integer(1), [Integer(1) - x*mu**Integer(2), Integer(1) - y/
↪mu, Integer(1) - z/mu])
(x*y*z + x*y + x*z + 1) *
(-x + 1)^-1 * (-x*y^2 + 1)^-1 * (-x*z^2 + 1)^-1
>>> MacMahonOmega(mu, Integer(1), [Integer(1) - x*mu**Integer(2), Integer(1) -␣
↪y*mu, Integer(1) - z/mu])
(-x*y*z^2 - x*y*z + x*z + 1) *
(-x + 1)^-1 * (-y + 1)^-1 * (-x*z^2 + 1)^-1 * (-y*z + 1)^-1

>>> MacMahonOmega(mu, Integer(1), [Integer(1) - x*mu, Integer(1) - y*mu,␣
↪Integer(1) - z*mu, Integer(1) - w/mu])
(x*y*z*w^2 + x*y*z*w - x*y*w - x*z*w - y*z*w + 1) *
(-x + 1)^-1 * (-y + 1)^-1 * (-z + 1)^-1 *
(-x*w + 1)^-1 * (-y*w + 1)^-1 * (-z*w + 1)^-1
>>> MacMahonOmega(mu, Integer(1), [Integer(1) - x*mu, Integer(1) - y*mu,␣
↪Integer(1) - z/mu, Integer(1) - w/mu])
(x^2*y*z*w + x*y^2*z*w - x*y*z*w - x*y*z - x*y*w + 1) *
(-x + 1)^-1 * (-y + 1)^-1 *
(-x*z + 1)^-1 * (-x*w + 1)^-1 * (-y*z + 1)^-1 * (-y*w + 1)^-1

>>> MacMahonOmega(mu, mu**-Integer(2), [Integer(1) - x*mu, Integer(1) - y/mu])
x^2 * (-x + 1)^-1 * (-x*y + 1)^-1
>>> MacMahonOmega(mu, mu**-Integer(1), [Integer(1) - x*mu, Integer(1) - y/mu])
x * (-x + 1)^-1 * (-x*y + 1)^-1
>>> MacMahonOmega(mu, mu, [Integer(1) - x*mu, Integer(1) - y/mu])
(-x*y + y + 1) * (-x + 1)^-1 * (-x*y + 1)^-1
>>> MacMahonOmega(mu, mu**Integer(2), [Integer(1) - x*mu, Integer(1) - y/mu])
(-x*y^2 - x*y + y^2 + y + 1) * (-x + 1)^-1 * (-x*y + 1)^-1
```

We demonstrate the different allowed input variants:

```
sage: MacMahonOmega(mu,
....:     Factorization([(mu, 2), (1 - x*mu, -1), (1 - y/mu, -1)]))
(-x*y^2 - x*y + y^2 + y + 1) * (-x + 1)^-1 * (-x*y + 1)^-1

sage: MacMahonOmega(mu, mu^2,
....:     Factorization([(1 - x*mu, 1), (1 - y/mu, 1)]))
(-x*y^2 - x*y + y^2 + y + 1) * (-x + 1)^-1 * (-x*y + 1)^-1

sage: MacMahonOmega(mu, mu^2, [1 - x*mu, 1 - y/mu])
(-x*y^2 - x*y + y^2 + y + 1) * (-x + 1)^-1 * (-x*y + 1)^-1

sage: MacMahonOmega(mu, mu^2, (1 - x*mu)*(1 - y/mu))  # not tested because not␣
↪fully implemented
(-x*y^2 - x*y + y^2 + y + 1) * (-x + 1)^-1 * (-x*y + 1)^-1

sage: MacMahonOmega(mu, mu^2 / ((1 - x*mu)*(1 - y/mu)))  # not tested because not␣
```

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```
↪fully implemented
(-x*y^2 - x*y + y^2 + y + 1) * (-x + 1)^-1 * (-x*y + 1)^-1
```

```
>>> from sage.all import *
>>> MacMahonOmega(mu,
...     Factorization([(mu, Integer(2)), (Integer(1) - x*mu, -Integer(1)),␣
↪(Integer(1) - y/mu, -Integer(1))]))
(-x*y^2 - x*y + y^2 + y + 1) * (-x + 1)^-1 * (-x*y + 1)^-1

>>> MacMahonOmega(mu, mu**Integer(2),
...     Factorization([(Integer(1) - x*mu, Integer(1)), (Integer(1) - y/mu,␣
↪Integer(1))]))
(-x*y^2 - x*y + y^2 + y + 1) * (-x + 1)^-1 * (-x*y + 1)^-1

>>> MacMahonOmega(mu, mu**Integer(2), [Integer(1) - x*mu, Integer(1) - y/mu])
(-x*y^2 - x*y + y^2 + y + 1) * (-x + 1)^-1 * (-x*y + 1)^-1

>>> MacMahonOmega(mu, mu**Integer(2), (Integer(1) - x*mu)*(Integer(1) - y/mu))  #␣
↪not tested because not fully implemented
(-x*y^2 - x*y + y^2 + y + 1) * (-x + 1)^-1 * (-x*y + 1)^-1

>>> MacMahonOmega(mu, mu**Integer(2) / ((Integer(1) - x*mu)*(Integer(1) - y/mu)))␣
↪ # not tested because not fully implemented
(-x*y^2 - x*y + y^2 + y + 1) * (-x + 1)^-1 * (-x*y + 1)^-1
```

sage.rings.polynomial.omega.**Omega_ge**(*exponents*)

Return $\Omega_{\geq}$ of the expression specified by the input.

To be more precise, calculate

$$\Omega_{\geq} \frac{\mu^a}{(1 - z_0 \mu^{e_0}) \dots (1 - z_{n-1} \mu^{e_{n-1}})}$$

and return its numerator and a factorization of its denominator. Note that $z_0, \dots, z_{n-1}$ only appear in the output, but not in the input.

INPUT:

- `a` – integer
- `exponents` – tuple of integers

OUTPUT:

A pair representing a quotient as follows: Its first component is the numerator as a Laurent polynomial, its second component a factorization of the denominator as a tuple of Laurent polynomials, where each Laurent polynomial z represents a factor $1 - z$.

The parents of these Laurent polynomials is always a Laurent polynomial ring in $z_0, \dots, z_{n-1}$ over $\mathbf{Z}$, where n is the length of `exponents`.

EXAMPLES:

```
sage: from sage.rings.polynomial.omega import Omega_ge
sage: Omega_ge(0, (1, -2))
(1, (z0, z0^2*z1))
```

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```
sage: Omega_ge(0, (1, -3))
(1, (z0, z0^3*z1))
sage: Omega_ge(0, (1, -4))
(1, (z0, z0^4*z1))

sage: Omega_ge(0, (2, -1))
(z0*z1 + 1, (z0, z0*z1^2))
sage: Omega_ge(0, (3, -1))
(z0*z1^2 + z0*z1 + 1, (z0, z0*z1^3))
sage: Omega_ge(0, (4, -1))
(z0*z1^3 + z0*z1^2 + z0*z1 + 1, (z0, z0*z1^4))

sage: Omega_ge(0, (1, 1, -2))
(-z0^2*z1*z2 - z0*z1^2*z2 + z0*z1*z2 + 1, (z0, z1, z0^2*z2, z1^2*z2))
sage: Omega_ge(0, (2, -1, -1))
(z0*z1*z2 + z0*z1 + z0*z2 + 1, (z0, z0*z1^2, z0*z2^2))
sage: Omega_ge(0, (2, 1, -1))
(-z0*z1*z2^2 - z0*z1*z2 + z0*z2 + 1, (z0, z1, z0*z2^2, z1*z2))
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.omega import Omega_ge
>>> Omega_ge(Integer(0), (Integer(1), -Integer(2)))
(1, (z0, z0^2*z1))
>>> Omega_ge(Integer(0), (Integer(1), -Integer(3)))
(1, (z0, z0^3*z1))
>>> Omega_ge(Integer(0), (Integer(1), -Integer(4)))
(1, (z0, z0^4*z1))

>>> Omega_ge(Integer(0), (Integer(2), -Integer(1)))
(z0*z1 + 1, (z0, z0*z1^2))
>>> Omega_ge(Integer(0), (Integer(3), -Integer(1)))
(z0*z1^2 + z0*z1 + 1, (z0, z0*z1^3))
>>> Omega_ge(Integer(0), (Integer(4), -Integer(1)))
(z0*z1^3 + z0*z1^2 + z0*z1 + 1, (z0, z0*z1^4))

>>> Omega_ge(Integer(0), (Integer(1), Integer(1), -Integer(2)))
(-z0^2*z1*z2 - z0*z1^2*z2 + z0*z1*z2 + 1, (z0, z1, z0^2*z2, z1^2*z2))
>>> Omega_ge(Integer(0), (Integer(2), -Integer(1), -Integer(1)))
(z0*z1*z2 + z0*z1 + z0*z2 + 1, (z0, z0*z1^2, z0*z2^2))
>>> Omega_ge(Integer(0), (Integer(2), Integer(1), -Integer(1)))
(-z0*z1*z2^2 - z0*z1*z2 + z0*z2 + 1, (z0, z1, z0*z2^2, z1*z2))
```

```
sage: Omega_ge(0, (2, -2))
(-z0*z1 + 1, (z0, z0*z1, z0*z1))
sage: Omega_ge(0, (2, -3))
(z0^2*z1 + 1, (z0, z0^3*z1^2))
sage: Omega_ge(0, (3, 1, -3))
(-z0^3*z1^3*z2^3 + 2*z0^2*z1^3*z2^2 - z0*z1^3*z2
 + z0^2*z2^2 - 2*z0*z2 + 1,
 (z0, z1, z0*z2, z0*z2, z0*z2, z1^3*z2))
```

```
>>> from sage.all import *
>>> Omega_ge(Integer(0), (Integer(2), -Integer(2)))
(-z0*z1 + 1, (z0, z0*z1, z0*z1))
>>> Omega_ge(Integer(0), (Integer(2), -Integer(3)))
(z0^2*z1 + 1, (z0, z0^3*z1^2))
>>> Omega_ge(Integer(0), (Integer(3), Integer(1), -Integer(3)))
(-z0^3*z1^3*z2^3 + 2*z0^2*z1^3*z2^2 - z0*z1^3*z2
 + z0^2*z2^2 - 2*z0*z2 + 1,
 (z0, z1, z0*z2, z0*z2, z0*z2, z1^3*z2))
```

```
sage: Omega_ge(0, (3, 6, -1))
(-z0*z1*z2^8 - z0*z1*z2^7 - z0*z1*z2^6 - z0*z1*z2^5 - z0*z1*z2^4 +
 z1*z2^5 - z0*z1*z2^3 + z1*z2^4 - z0*z1*z2^2 + z1*z2^3 -
 z0*z1*z2 + z0*z2^2 + z1*z2^2 + z0*z2 + z1*z2 + 1,
 (z0, z1, z0*z2^3, z1*z2^6))
```

```
>>> from sage.all import *
>>> Omega_ge(Integer(0), (Integer(3), Integer(6), -Integer(1)))
(-z0*z1*z2^8 - z0*z1*z2^7 - z0*z1*z2^6 - z0*z1*z2^5 - z0*z1*z2^4 +
 z1*z2^5 - z0*z1*z2^3 + z1*z2^4 - z0*z1*z2^2 + z1*z2^3 -
 z0*z1*z2 + z0*z2^2 + z1*z2^2 + z0*z2 + z1*z2 + 1,
 (z0, z1, z0*z2^3, z1*z2^6))
```

sage.rings.polynomial.omega.**homogeneous_symmetric_function**(j, x)

Return a complete homogeneous symmetric polynomial (Wikipedia article Complete_homogeneous_symmetric_polynomial).

INPUT:

- j – the degree as a nonnegative integer
- x – an iterable of variables

OUTPUT: a polynomial of the common parent of all entries of x

EXAMPLES:

```
sage: from sage.rings.polynomial.omega import homogeneous_symmetric_function
sage: P = PolynomialRing(ZZ, 'X', 3)
sage: homogeneous_symmetric_function(0, P.gens())
1
sage: homogeneous_symmetric_function(1, P.gens())
X0 + X1 + X2
sage: homogeneous_symmetric_function(2, P.gens())
X0^2 + X0*X1 + X1^2 + X0*X2 + X1*X2 + X2^2
sage: homogeneous_symmetric_function(3, P.gens())
X0^3 + X0^2*X1 + X0*X1^2 + X1^3 + X0^2*X2 +
X0*X1*X2 + X1^2*X2 + X0*X2^2 + X1*X2^2 + X2^3
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.omega import homogeneous_symmetric_function
>>> P = PolynomialRing(ZZ, 'X', Integer(3))
>>> homogeneous_symmetric_function(Integer(0), P.gens())
1
```

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```
>>> homogeneous_symmetric_function(Integer(1), P.gens())
X0 + X1 + X2
>>> homogeneous_symmetric_function(Integer(2), P.gens())
X0^2 + X0*X1 + X1^2 + X0*X2 + X1*X2 + X2^2
>>> homogeneous_symmetric_function(Integer(3), P.gens())
X0^3 + X0^2*X1 + X0*X1^2 + X1^3 + X0^2*X2 +
X0*X1*X2 + X1^2*X2 + X0*X2^2 + X1*X2^2 + X2^3
```

`sage.rings.polynomial.omega.`**`partition`**(*items*, *predicate=<class 'bool'>*)

Split `items` into two parts by the given `predicate`.

INPUT:

- `item` – an iterator
- `predicate` – a function

OUTPUT:

A pair of iterators; the first contains the elements not satisfying the `predicate`, the second the elements satisfying the `predicate`.

ALGORITHM:

Source of the code: http://nedbatchelder.com/blog/201306/filter_a_list_into_two_parts.html

EXAMPLES:

```
sage: from sage.rings.polynomial.omega import partition
sage: E, O = partition(srange(10), is_odd)
sage: tuple(E), tuple(O)
((0, 2, 4, 6, 8), (1, 3, 5, 7, 9))
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.omega import partition
>>> E, O = partition(srange(Integer(10)), is_odd)
>>> tuple(E), tuple(O)
((0, 2, 4, 6, 8), (1, 3, 5, 7, 9))
```

CHAPTER

SIX

INFINITE POLYNOMIAL RINGS

6.1 Infinite Polynomial Rings

By Infinite Polynomial Rings, we mean polynomial rings in a countably infinite number of variables. The implementation consists of a wrapper around the current *finite* polynomial rings in Sage.

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An Infinite Polynomial Ring has finitely many generators $x_*, y_*, ...$ and infinitely many variables of the form $x_0, x_1, x_2, ..., y_0, y_1, y_2, ..., ...$. We refer to the natural number n as the *index* of the variable x_n.

INPUT:

- `R` – the base ring; it has to be a commutative ring, and in some applications it must even be a field
- `names` – a finite list of generator names; generator names must be alpha-numeric
- `order` – (optional) string; the default order is `'lex'` (lexicographic). `'deglex'` is degree lexicographic, and `'degrevlex'` (degree reverse lexicographic) is possible but discouraged.

Each generator `x` produces an infinite sequence of variables `x[1], x[2], ...` which are printed on screen as `x_1, x_2, ...` and are latex typeset as x_1, x_2. Then, the Infinite Polynomial Ring is formed by polynomials in these variables.

By default, the monomials are ordered lexicographically. Alternatively, degree (reverse) lexicographic ordering is possible as well. However, we do not guarantee that the computation of Groebner bases will terminate in this case.

In either case, the variables of a Infinite Polynomial Ring X are ordered according to the following rule:

`X.gen(i)[m] > X.gen(j)[n]` if and only if `i<j or (i==j and m>n)`

We provide a 'dense' and a 'sparse' implementation. In the dense implementation, the Infinite Polynomial Ring carries a finite polynomial ring that comprises *all* variables up to the maximal index that has been used so far. This is potentially a very big ring and may also comprise many variables that are not used.

In the sparse implementation, we try to keep the underlying finite polynomial rings small, using only those variables that are really needed. By default, we use the dense implementation, since it usually is much faster.

EXAMPLES:

```
sage: X.<x,y> = InfinitePolynomialRing(ZZ, implementation='sparse')
sage: A.<alpha,beta> = InfinitePolynomialRing(QQ, order='deglex')

sage: f = x[5] + 2; f
x_5 + 2
```

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```
sage: g = 3*y[1]; g
3*y_1
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(ZZ, implementation='sparse', names=('x', 'y',)); (x, y,
↪) = X._first_ngens(2)
>>> A = InfinitePolynomialRing(QQ, order='deglex', names=('alpha', 'beta',)); (alpha,
↪beta,) = A._first_ngens(2)

>>> f = x[Integer(5)] + Integer(2); f
x_5 + 2
>>> g = Integer(3)*y[Integer(1)]; g
3*y_1
```

It has some advantages to have an underlying ring that is not univariate. Hence, we always have at least two variables:

```
sage: g._p.parent()
Multivariate Polynomial Ring in y_1, y_0 over Integer Ring

sage: f2 = alpha[5] + 2; f2
alpha_5 + 2
sage: g2 = 3*beta[1]; g2
3*beta_1
sage: A.polynomial_ring()
Multivariate Polynomial Ring in alpha_5, alpha_4, alpha_3, alpha_2, alpha_1, alpha_0,
 beta_5, beta_4, beta_3, beta_2, beta_1, beta_0 over Rational Field
```

```
>>> from sage.all import *
>>> g._p.parent()
Multivariate Polynomial Ring in y_1, y_0 over Integer Ring

>>> f2 = alpha[Integer(5)] + Integer(2); f2
alpha_5 + 2
>>> g2 = Integer(3)*beta[Integer(1)]; g2
3*beta_1
>>> A.polynomial_ring()
Multivariate Polynomial Ring in alpha_5, alpha_4, alpha_3, alpha_2, alpha_1, alpha_0,
 beta_5, beta_4, beta_3, beta_2, beta_1, beta_0 over Rational Field
```

Of course, we provide the usual polynomial arithmetic:

```
sage: f + g
x_5 + 3*y_1 + 2
sage: p = x[10]^2*(f+g); p
x_10^2*x_5 + 3*x_10^2*y_1 + 2*x_10^2
sage: p2 = alpha[10]^2*(f2+g2); p2
alpha_10^2*alpha_5 + 3*alpha_10^2*beta_1 + 2*alpha_10^2
```

```
>>> from sage.all import *
>>> f + g
x_5 + 3*y_1 + 2
>>> p = x[Integer(10)]**Integer(2)*(f+g); p
```

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```
x_10^2*x_5 + 3*x_10^2*y_1 + 2*x_10^2
>>> p2 = alpha[Integer(10)]**Integer(2)*(f2+g2); p2
alpha_10^2*alpha_5 + 3*alpha_10^2*beta_1 + 2*alpha_10^2
```

There is a permutation action on the variables, by permuting positive variable indices:

```
sage: P = Permutation(((10,1)))
sage: p^P
x_5*x_1^2 + 3*x_1^2*y_10 + 2*x_1^2
sage: p2^P
alpha_5*alpha_1^2 + 3*alpha_1^2*beta_10 + 2*alpha_1^2
```

```
>>> from sage.all import *
>>> P = Permutation(((Integer(10),Integer(1))))
>>> p**P
x_5*x_1^2 + 3*x_1^2*y_10 + 2*x_1^2
>>> p2**P
alpha_5*alpha_1^2 + 3*alpha_1^2*beta_10 + 2*alpha_1^2
```

Note that $x_0^P = x_0$, since the permutations only change *positive* variable indices.

We also implemented ideals of Infinite Polynomial Rings. Here, it is thoroughly assumed that the ideals are set-wise invariant under the permutation action. We therefore refer to these ideals as *Symmetric Ideals*. Symmetric Ideals are finitely generated modulo addition, multiplication by ring elements and permutation of variables. If the base ring is a field, one can compute Symmetric Groebner Bases:

```
sage: J = A * (alpha[1]*beta[2])
sage: J.groebner_basis()                                                    #␣
↪needs sage.combinat sage.libs.singular
[alpha_1*beta_2, alpha_2*beta_1]
```

```
>>> from sage.all import *
>>> J = A * (alpha[Integer(1)]*beta[Integer(2)])
>>> J.groebner_basis()                                                     #␣
↪needs sage.combinat sage.libs.singular
[alpha_1*beta_2, alpha_2*beta_1]
```

For more details, see `SymmetricIdeal`.

Infinite Polynomial Rings can have any commutative base ring. If the base ring of an Infinite Polynomial Ring is a (classical or infinite) Polynomial Ring, then our implementation tries to merge everything into *one* ring. The basic requirement is that the monomial orders match. In the case of two Infinite Polynomial Rings, the implementations must match. Moreover, name conflicts should be avoided. An overlap is only accepted if the order of variables can be uniquely inferred, as in the following example:

```
sage: A.<a,b,c> = InfinitePolynomialRing(ZZ)
sage: B.<b,c,d> = InfinitePolynomialRing(A)
sage: B
Infinite polynomial ring in a, b, c, d over Integer Ring
```

```
>>> from sage.all import *
>>> A = InfinitePolynomialRing(ZZ, names=('a', 'b', 'c',)); (a, b, c,) = A._first_
↪ngens(3)
```

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```
>>> B = InfinitePolynomialRing(A, names=('b', 'c', 'd',)); (b, c, d,) = B._first_
↪ngens(3)
>>> B
Infinite polynomial ring in a, b, c, d over Integer Ring
```

This is also allowed if finite polynomial rings are involved:

```
sage: A.<a_3,a_1,b_1,c_2,c_0> = ZZ[]
sage: B.<b,c,d> = InfinitePolynomialRing(A, order='degrevlex')
sage: B
Infinite polynomial ring in b, c, d over
 Multivariate Polynomial Ring in a_3, a_1 over Integer Ring
```

```
>>> from sage.all import *
>>> A = ZZ['a_3, a_1, b_1, c_2, c_0']; (a_3, a_1, b_1, c_2, c_0,) = A._first_ngens(5)
>>> B = InfinitePolynomialRing(A, order='degrevlex', names=('b', 'c', 'd',)); (b, c,␣
↪d,) = B._first_ngens(3)
>>> B
Infinite polynomial ring in b, c, d over
 Multivariate Polynomial Ring in a_3, a_1 over Integer Ring
```

It is no problem if one generator of the Infinite Polynomial Ring is called `x` and one variable of the base ring is also called `x`. This is since no *variable* of the Infinite Polynomial Ring will be called `x`. However, a problem arises if the underlying classical Polynomial Ring has a variable `x_1`, since this can be confused with a variable of the Infinite Polynomial Ring. In this case, an error will be raised:

```
sage: X.<x,y_1> = ZZ[]
sage: Y.<x,z> = InfinitePolynomialRing(X)
```

```
>>> from sage.all import *
>>> X = ZZ['x, y_1']; (x, y_1,) = X._first_ngens(2)
>>> Y = InfinitePolynomialRing(X, names=('x', 'z',)); (x, z,) = Y._first_ngens(2)
```

Note that `X` is not merged into `Y`; this is since the monomial order of `X` is 'degrevlex', but of `Y` is 'lex'.

```
sage: Y
Infinite polynomial ring in x, z over
 Multivariate Polynomial Ring in x, y_1 over Integer Ring
```

```
>>> from sage.all import *
>>> Y
Infinite polynomial ring in x, z over
 Multivariate Polynomial Ring in x, y_1 over Integer Ring
```

The variable `x` of `X` can still be interpreted in `Y`, although the first generator of `Y` is called `x` as well:

```
sage: x
x_*
sage: X('x')
x
sage: Y(X('x'))
x
```

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```
sage: Y('x')
x
```

```
>>> from sage.all import *
>>> x
x_*
>>> X('x')
x
>>> Y(X('x'))
x
>>> Y('x')
x
```

But there is only merging if the resulting monomial order is uniquely determined. This is not the case in the following examples, and thus an error is raised:

```
sage: X.<y_1,x> = ZZ[]
sage: Y.<y,z> = InfinitePolynomialRing(X)
Traceback (most recent call last):
...
CoercionException: Overlapping variables (('y', 'z'),['y_1']) are incompatible
sage: Y.<z,y> = InfinitePolynomialRing(X)
Traceback (most recent call last):
...
CoercionException: Overlapping variables (('z', 'y'),['y_1']) are incompatible
sage: X.<x_3,y_1,y_2> = PolynomialRing(ZZ, order='lex')
sage: # y_1 and y_2 would be in opposite order in an Infinite Polynomial Ring
sage: Y.<y> = InfinitePolynomialRing(X)
Traceback (most recent call last):
...
CoercionException: Overlapping variables (('y',),['y_1', 'y_2']) are incompatible
```

```
>>> from sage.all import *
>>> X = ZZ['y_1, x']; (y_1, x,) = X._first_ngens(2)
>>> Y = InfinitePolynomialRing(X, names=('y', 'z',)); (y, z,) = Y._first_ngens(2)
Traceback (most recent call last):
...
CoercionException: Overlapping variables (('y', 'z'),['y_1']) are incompatible
>>> Y = InfinitePolynomialRing(X, names=('z', 'y',)); (z, y,) = Y._first_ngens(2)
Traceback (most recent call last):
...
CoercionException: Overlapping variables (('z', 'y'),['y_1']) are incompatible
>>> X = PolynomialRing(ZZ, order='lex', names=('x_3', 'y_1', 'y_2',)); (x_3, y_1, y_2,
↪) = X._first_ngens(3)
>>> # y_1 and y_2 would be in opposite order in an Infinite Polynomial Ring
>>> Y = InfinitePolynomialRing(X, names=('y',)); (y,) = Y._first_ngens(1)
Traceback (most recent call last):
...
CoercionException: Overlapping variables (('y',),['y_1', 'y_2']) are incompatible
```

If the type of monomial orderings (e.g., 'degrevlex' versus 'lex') or if the implementations do not match, there is no simplified construction available:

```
sage: X.<x,y> = InfinitePolynomialRing(ZZ)
sage: Y.<z> = InfinitePolynomialRing(X, order='degrevlex')
sage: Y
Infinite polynomial ring in z over Infinite polynomial ring in x, y over Integer Ring
sage: Y.<z> = InfinitePolynomialRing(X, implementation='sparse')
sage: Y
Infinite polynomial ring in z over Infinite polynomial ring in x, y over Integer Ring
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(ZZ, names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> Y = InfinitePolynomialRing(X, order='degrevlex', names=('z',)); (z,) = Y._first_
↪ngens(1)
>>> Y
Infinite polynomial ring in z over Infinite polynomial ring in x, y over Integer Ring
>>> Y = InfinitePolynomialRing(X, implementation='sparse', names=('z',)); (z,) = Y._
↪first_ngens(1)
>>> Y
Infinite polynomial ring in z over Infinite polynomial ring in x, y over Integer Ring
```

class sage.rings.polynomial.infinite_polynomial_ring.**GenDictWithBasering**(*parent*, *start*)

Bases: object

A dictionary-like class that is suitable for usage in sage_eval.

This pseudo-dictionary accepts strings as index, and then walks down a chain of base rings of (infinite) polynomial rings until it finds one ring that has the given string as variable name, which is then returned.

EXAMPLES:

```
sage: R.<a,b> = InfinitePolynomialRing(ZZ)
sage: D = R.gens_dict() # indirect doctest
sage: D
GenDict of Infinite polynomial ring in a, b over Integer Ring
sage: D['a_15']
a_15
sage: type(_)
<class 'sage.rings.polynomial.infinite_polynomial_element.InfinitePolynomial_dense
↪'>
sage: sage_eval('3*a_3*b_5-1/2*a_7', D)
-1/2*a_7 + 3*a_3*b_5
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(ZZ, names=('a', 'b',)); (a, b,) = R._first_ngens(2)
>>> D = R.gens_dict() # indirect doctest
>>> D
GenDict of Infinite polynomial ring in a, b over Integer Ring
>>> D['a_15']
a_15
>>> type(_)
<class 'sage.rings.polynomial.infinite_polynomial_element.InfinitePolynomial_dense
↪'>
>>> sage_eval('3*a_3*b_5-1/2*a_7', D)
-1/2*a_7 + 3*a_3*b_5
```

next()

Return a dictionary that can be used to interpret strings in the base ring of `self`.

EXAMPLES:

```
sage: R.<a,b> = InfinitePolynomialRing(QQ['t'])
sage: D = R.gens_dict()
sage: D
GenDict of Infinite polynomial ring in a, b over Univariate Polynomial Ring␣
↪in t over Rational Field
sage: next(D)
GenDict of Univariate Polynomial Ring in t over Rational Field
sage: sage_eval('t^2', next(D))
t^2
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(QQ['t'], names=('a', 'b',)); (a, b,) = R._
↪first_ngens(2)
>>> D = R.gens_dict()
>>> D
GenDict of Infinite polynomial ring in a, b over Univariate Polynomial Ring␣
↪in t over Rational Field
>>> next(D)
GenDict of Univariate Polynomial Ring in t over Rational Field
>>> sage_eval('t^2', next(D))
t^2
```

class sage.rings.polynomial.infinite_polynomial_ring.**InfiniteGenDict**(*Gens*)

Bases: `object`

A dictionary-like class that is suitable for usage in `sage_eval`.

The generators of an Infinite Polynomial Ring are not variables. Variables of an Infinite Polynomial Ring are returned by indexing a generator. The purpose of this class is to return a variable of an Infinite Polynomial Ring, given its string representation.

EXAMPLES:

```
sage: R.<a,b> = InfinitePolynomialRing(ZZ)
sage: D = R.gens_dict() # indirect doctest
sage: D._D
[InfiniteGenDict defined by ['a', 'b'], {'1': 1}]
sage: D._D[0]['a_15']
a_15
sage: type(_)
<class 'sage.rings.polynomial.infinite_polynomial_element.InfinitePolynomial_dense
↪'>
sage: sage_eval('3*a_3*b_5-1/2*a_7', D._D[0])
-1/2*a_7 + 3*a_3*b_5
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(ZZ, names=('a', 'b',)); (a, b,) = R._first_ngens(2)
>>> D = R.gens_dict() # indirect doctest
>>> D._D
[InfiniteGenDict defined by ['a', 'b'], {'1': 1}]
```

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```
>>> D._D[Integer(0)]['a_15']
a_15
>>> type(_)
<class 'sage.rings.polynomial.infinite_polynomial_element.InfinitePolynomial_dense
↪'>
>>> sage_eval('3*a_3*b_5-1/2*a_7', D._D[Integer(0)])
-1/2*a_7 + 3*a_3*b_5
```

class sage.rings.polynomial.infinite_polynomial_ring.**InfinitePolynomialGen**(*parent*, *name*)

Bases: `SageObject`

This class provides the object which is responsible for returning variables in an infinite polynomial ring (implemented in `__getitem__()`).

EXAMPLES:

```
sage: X.<x1,x2> = InfinitePolynomialRing(RR)
sage: x1
x1_*
sage: x1[5]
x1_5
sage: x1 == loads(dumps(x1))
True
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(RR, names=('x1', 'x2',)); (x1, x2,) = X._first_
↪ngens(2)
>>> x1
x1_*
>>> x1[Integer(5)]
x1_5
>>> x1 == loads(dumps(x1))
True
```

class sage.rings.polynomial.infinite_polynomial_ring.**InfinitePolynomialRingFactory**

Bases: `UniqueFactory`

A factory for creating infinite polynomial ring elements. It makes sure that they are unique as well as handling pickling. For more details, see `UniqueFactory` and *`infinite_polynomial_ring`*.

EXAMPLES:

```
sage: A.<a> = InfinitePolynomialRing(QQ)
sage: B.<b> = InfinitePolynomialRing(A)
sage: B.construction()
[InfPoly{[a,b], "lex", "dense"}, Rational Field]
sage: R.<a,b> = InfinitePolynomialRing(QQ)
sage: R is B
True
sage: X.<x> = InfinitePolynomialRing(QQ)
sage: X2.<x> = InfinitePolynomialRing(QQ, implementation='sparse')
sage: X is X2
False
```

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```
sage: X is loads(dumps(X))
True
```

```
>>> from sage.all import *
>>> A = InfinitePolynomialRing(QQ, names=('a',)); (a,) = A._first_ngens(1)
>>> B = InfinitePolynomialRing(A, names=('b',)); (b,) = B._first_ngens(1)
>>> B.construction()
[InfPoly{[a,b], "lex", "dense"}, Rational Field]
>>> R = InfinitePolynomialRing(QQ, names=('a', 'b',)); (a, b,) = R._first_ngens(2)
>>> R is B
True
>>> X = InfinitePolynomialRing(QQ, names=('x',)); (x,) = X._first_ngens(1)
>>> X2 = InfinitePolynomialRing(QQ, implementation='sparse', names=('x',)); (x,)␣
↪= X2._first_ngens(1)
>>> X is X2
False

>>> X is loads(dumps(X))
True
```

create_key(*R*, *names=('x',)*, *order='lex'*, *implementation='dense'*)

Create a key which uniquely defines the infinite polynomial ring.

create_object(*version*, *key*)

Return the infinite polynomial ring corresponding to the key `key`.

class sage.rings.polynomial.infinite_polynomial_ring.**InfinitePolynomialRing_dense**(*R*, *names*, *order*)

Bases: *InfinitePolynomialRing_sparse*

Dense implementation of Infinite Polynomial Rings.

Compared with *InfinitePolynomialRing_sparse*, from which this class inherits, it keeps a polynomial ring that comprises all elements that have been created so far.

construction()

Return the construction of `self`.

OUTPUT:

A pair `F,R`, where `F` is a construction functor and `R` is a ring, so that `F(R) is self`.

EXAMPLES:

```
sage: R.<x,y> = InfinitePolynomialRing(GF(5))
sage: R.construction()
[InfPoly{[x,y], "lex", "dense"}, Finite Field of size 5]
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(GF(Integer(5)), names=('x', 'y',)); (x, y,) =␣
↪R._first_ngens(2)
>>> R.construction()
[InfPoly{[x,y], "lex", "dense"}, Finite Field of size 5]
```

polynomial_ring()

Return the underlying *finite* polynomial ring.

Note

The ring returned can change over time as more variables are used.

Since the rings are cached, we create here a ring with variable names that do not occur in other doc tests, so that we avoid side effects.

EXAMPLES:

```
sage: X.<xx, yy> = InfinitePolynomialRing(ZZ)
sage: X.polynomial_ring()
Multivariate Polynomial Ring in xx_0, yy_0 over Integer Ring
sage: a = yy[3]
sage: X.polynomial_ring()
Multivariate Polynomial Ring in xx_3, xx_2, xx_1, xx_0, yy_3, yy_2, yy_1, yy_0
 over Integer Ring
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(ZZ, names=('xx', 'yy',)); (xx, yy,) = X._first_
↪ngens(2)
>>> X.polynomial_ring()
Multivariate Polynomial Ring in xx_0, yy_0 over Integer Ring
>>> a = yy[Integer(3)]
>>> X.polynomial_ring()
Multivariate Polynomial Ring in xx_3, xx_2, xx_1, xx_0, yy_3, yy_2, yy_1, yy_0
 over Integer Ring
```

tensor_with_ring(R)

Return the tensor product of `self` with another ring.

INPUT:

- `R` – a ring

OUTPUT:

An infinite polynomial ring that, mathematically, can be seen as the tensor product of `self` with `R`.

NOTE:

It is required that the underlying ring of `self` coerces into `R`. Hence, the tensor product is in fact merely an extension of the base ring.

EXAMPLES:

```
sage: R.<a,b> = InfinitePolynomialRing(ZZ, implementation='sparse')
sage: R.tensor_with_ring(QQ)
Infinite polynomial ring in a, b over Rational Field
sage: R
Infinite polynomial ring in a, b over Integer Ring
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(ZZ, implementation='sparse', names=('a', 'b',
```

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```
↪)); (a, b,) = R._first_ngens(2)
>>> R.tensor_with_ring(QQ)
Infinite polynomial ring in a, b over Rational Field
>>> R
Infinite polynomial ring in a, b over Integer Ring
```

The following tests against a bug that was fixed at Issue #10468:

```
sage: R.<x,y> = InfinitePolynomialRing(QQ, implementation='sparse')
sage: R.tensor_with_ring(QQ) is R
True
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(QQ, implementation='sparse', names=('x', 'y',
↪)); (x, y,) = R._first_ngens(2)
>>> R.tensor_with_ring(QQ) is R
True
```

class sage.rings.polynomial.infinite_polynomial_ring.**InfinitePolynomialRing_sparse**(*R*, *names*, *order*)

Bases: `CommutativeRing`

Sparse implementation of Infinite Polynomial Rings.

An Infinite Polynomial Ring with generators $x_*, y_*, ...$ over a field F is a free commutative F-algebra generated by $x_0, x_1, x_2, ..., y_0, y_1, y_2, ..., ...$ and is equipped with a permutation action on the generators, namely $x_n^P = x_{P(n)}, y_n^P = y_{P(n)}, ...$ for any permutation P (note that variables of index zero are invariant under such permutation).

It is known that any permutation invariant ideal in an Infinite Polynomial Ring is finitely generated modulo the permutation action – see `SymmetricIdeal` for more details.

Usually, an instance of this class is created using `InfinitePolynomialRing` with the optional parameter `implementation='sparse'`. This takes care of uniqueness of parent structures. However, a direct construction is possible, in principle:

```
sage: X.<x,y> = InfinitePolynomialRing(QQ, implementation='sparse')
sage: Y.<x,y> = InfinitePolynomialRing(QQ, implementation='sparse')
sage: X is Y
True
sage: from sage.rings.polynomial.infinite_polynomial_ring import␣
↪InfinitePolynomialRing_sparse
sage: Z = InfinitePolynomialRing_sparse(QQ, ['x','y'], 'lex')
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, implementation='sparse', names=('x', 'y',));␣
↪(x, y,) = X._first_ngens(2)
>>> Y = InfinitePolynomialRing(QQ, implementation='sparse', names=('x', 'y',));␣
↪(x, y,) = Y._first_ngens(2)
>>> X is Y
True
```

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```
>>> from sage.rings.polynomial.infinite_polynomial_ring import␣
↪InfinitePolynomialRing_sparse
>>> Z = InfinitePolynomialRing_sparse(QQ, ['x','y'], 'lex')
```

Nevertheless, since infinite polynomial rings are supposed to be unique parent structures, they do not evaluate equal.

```
sage: Z == X
False
```

```
>>> from sage.all import *
>>> Z == X
False
```

The last parameter ('lex' in the above example) can also be 'deglex' or 'degrevlex'; this would result in an Infinite Polynomial Ring in degree lexicographic or degree reverse lexicographic order.

See *infinite_polynomial_ring* for more details.

characteristic()

Return the characteristic of the base field.

EXAMPLES:

```
sage: X.<x,y> = InfinitePolynomialRing(GF(25,'a'))                          #␣
↪needs sage.rings.finite_rings
sage: X                                                                     #␣
↪needs sage.rings.finite_rings
Infinite polynomial ring in x, y over Finite Field in a of size 5^2
sage: X.characteristic()                                                    #␣
↪needs sage.rings.finite_rings
5
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(GF(Integer(25),'a'), names=('x', 'y',)); (x, y,
↪) = X._first_ngens(2)# needs sage.rings.finite_rings
>>> X                                                                     #␣
↪needs sage.rings.finite_rings
Infinite polynomial ring in x, y over Finite Field in a of size 5^2
>>> X.characteristic()                                                    #␣
↪needs sage.rings.finite_rings
5
```

construction()

Return the construction of `self`.

OUTPUT:

A pair `F,R`, where `F` is a construction functor and `R` is a ring, so that `F(R) is self`.

EXAMPLES:

```
sage: R.<x,y> = InfinitePolynomialRing(GF(5))
sage: R.construction()
[InfPoly{[x,y], "lex", "dense"}, Finite Field of size 5]
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(GF(Integer(5)), names=('x', 'y',)); (x, y,) =␣
↪R._first_ngens(2)
>>> R.construction()
[InfPoly{[x,y], "lex", "dense"}, Finite Field of size 5]
```

gen(*i=None*)

Return the i-th 'generator' (see the description in `ngens()`) of this infinite polynomial ring.

EXAMPLES:

```
sage: X = InfinitePolynomialRing(QQ)
sage: x = X.gen()
sage: x[1]
x_1
sage: X.gen() is X.gen(0)
True
sage: XX = InfinitePolynomialRing(GF(5))
sage: XX.gen(0) is XX.gen()
True
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ)
>>> x = X.gen()
>>> x[Integer(1)]
x_1
>>> X.gen() is X.gen(Integer(0))
True
>>> XX = InfinitePolynomialRing(GF(Integer(5)))
>>> XX.gen(Integer(0)) is XX.gen()
True
```

gens_dict()

Return a dictionary-like object containing the infinitely many `{var_name:variable}` pairs.

EXAMPLES:

```
sage: R = InfinitePolynomialRing(ZZ, 'a')
sage: D = R.gens_dict()
sage: D
GenDict of Infinite polynomial ring in a over Integer Ring
sage: D['a_5']
a_5
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(ZZ, 'a')
>>> D = R.gens_dict()
>>> D
GenDict of Infinite polynomial ring in a over Integer Ring
>>> D['a_5']
a_5
```

is_field(**args, **kwds*)

Return `False` since Infinite Polynomial Rings are never fields.

Since Infinite Polynomial Rings must have at least one generator, they have infinitely many variables and thus never are fields.

EXAMPLES:

```
sage: R.<x, y> = InfinitePolynomialRing(QQ)
sage: R.is_field()
False
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_
↪ngens(2)
>>> R.is_field()
False
```

is_integral_domain(**args*, ***kwds*)

An infinite polynomial ring is an integral domain if and only if the base ring is. Arguments are passed to is_integral_domain method of base ring.

EXAMPLES:

```
sage: R.<x, y> = InfinitePolynomialRing(QQ)
sage: R.is_integral_domain()
True
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_
↪ngens(2)
>>> R.is_integral_domain()
True
```

is_noetherian()

Return `False`, since polynomial rings in infinitely many variables are never Noetherian rings.

Since Infinite Polynomial Rings must have at least one generator, they have infinitely many variables and are thus not Noetherian, as a ring.

Note

Infinite Polynomial Rings over a field F are Noetherian as $F(G)$ modules, where G is the symmetric group of the natural numbers. But this is not what the method `is_noetherian()` is answering.

key_basis()

Return the basis of `self` given by key polynomials.

EXAMPLES:

```
sage: R.<x> = InfinitePolynomialRing(GF(2))
sage: R.key_basis()                                                        #␣
↪needs sage.combinat sage.modules
Key polynomial basis over Finite Field of size 2
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(GF(Integer(2)), names=('x',)); (x,) = R._first_
↪ngens(1)
>>> R.key_basis()                                                           #␣
↪needs sage.combinat sage.modules
Key polynomial basis over Finite Field of size 2
```

krull_dimension(**args*, ***kwds*)

Return `Infinity`, since polynomial rings in infinitely many variables have infinite Krull dimension.

EXAMPLES:

```
sage: R.<x, y> = InfinitePolynomialRing(QQ)
sage: R.krull_dimension()
+Infinity
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_
↪ngens(2)
>>> R.krull_dimension()
+Infinity
```

ngens()

Return the number of generators for this ring.

Since there are countably infinitely many variables in this polynomial ring, by 'generators' we mean the number of infinite families of variables. See *infinite_polynomial_ring* for more details.

EXAMPLES:

```
sage: X.<x> = InfinitePolynomialRing(ZZ)
sage: X.ngens()
1

sage: X.<x1,x2> = InfinitePolynomialRing(QQ)
sage: X.ngens()
2
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(ZZ, names=('x',)); (x,) = X._first_ngens(1)
>>> X.ngens()
1

>>> X = InfinitePolynomialRing(QQ, names=('x1', 'x2',)); (x1, x2,) = X._first_
↪ngens(2)
>>> X.ngens()
2
```

one()

order()

Return `Infinity`, since polynomial rings have infinitely many elements.

EXAMPLES:

```
sage: R.<x> = InfinitePolynomialRing(GF(2))
sage: R.order()
+Infinity
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(GF(Integer(2)), names=('x',)); (x,) = R._first_
↪ngens(1)
>>> R.order()
+Infinity
```

tensor_with_ring(*R*)

Return the tensor product of `self` with another ring.

INPUT:

- `R` – a ring

OUTPUT:

An infinite polynomial ring that, mathematically, can be seen as the tensor product of `self` with `R`.

NOTE:

It is required that the underlying ring of `self` coerces into `R`. Hence, the tensor product is in fact merely an extension of the base ring.

EXAMPLES:

```
sage: R.<a,b> = InfinitePolynomialRing(ZZ)
sage: R.tensor_with_ring(QQ)
Infinite polynomial ring in a, b over Rational Field
sage: R
Infinite polynomial ring in a, b over Integer Ring
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(ZZ, names=('a', 'b',)); (a, b,) = R._first_
↪ngens(2)
>>> R.tensor_with_ring(QQ)
Infinite polynomial ring in a, b over Rational Field
>>> R
Infinite polynomial ring in a, b over Integer Ring
```

The following tests against a bug that was fixed at Issue #10468:

```
sage: R.<x,y> = InfinitePolynomialRing(QQ)
sage: R.tensor_with_ring(QQ) is R
True
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_
↪ngens(2)
>>> R.tensor_with_ring(QQ) is R
True
```

varname_key(*x*)

Key for comparison of variable names.

INPUT:

- `x` – string of the form `a+'_'+str(n)`, where a is the name of a generator, and n is an integer

OUTPUT: a key used to sort the variables

THEORY:

The order is defined as follows:

x<y $\iff$ the string `x.split('_')[0]` is later in the list of generator names of `self` than `y.split('_')[0]`, or (`x.split('_')[0]==y.split('_')[0]` and `int(x.split('_')[1])<int(y.split('_')[1])`)

EXAMPLES:

```
sage: X.<alpha,beta> = InfinitePolynomialRing(ZZ)
sage: X.varname_key('alpha_1')
(0, 1)
sage: X.varname_key('beta_10')
(-1, 10)
sage: X.varname_key('beta_1')
(-1, 1)
sage: X.varname_key('alpha_10')
(0, 10)
sage: X.varname_key('alpha_1')
(0, 1)
sage: X.varname_key('alpha_10')
(0, 10)
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(ZZ, names=('alpha', 'beta',)); (alpha, beta,)
↪= X._first_ngens(2)
>>> X.varname_key('alpha_1')
(0, 1)
>>> X.varname_key('beta_10')
(-1, 10)
>>> X.varname_key('beta_1')
(-1, 1)
>>> X.varname_key('alpha_10')
(0, 10)
>>> X.varname_key('alpha_1')
(0, 1)
>>> X.varname_key('alpha_10')
(0, 10)
```

6.2 Elements of Infinite Polynomial Rings

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An Infinite Polynomial Ring has generators $x_*, y_*, \ldots$, so that the variables are of the form $x_0, x_1, x_2, \ldots, y_0, y_1, y_2, \ldots, \ldots$ (see *infinite_polynomial_ring*). Using the generators, we can create elements as follows:

```
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: a = x[3]
sage: b = y[4]
sage: a
x_3
sage: b
y_4
sage: c = a*b + a^3 - 2*b^4
sage: c
x_3^3 + x_3*y_4 - 2*y_4^4
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> a = x[Integer(3)]
>>> b = y[Integer(4)]
>>> a
x_3
>>> b
y_4
>>> c = a*b + a**Integer(3) - Integer(2)*b**Integer(4)
>>> c
x_3^3 + x_3*y_4 - 2*y_4^4
```

Any Infinite Polynomial Ring `X` is equipped with a monomial ordering. We only consider monomial orderings in which:

`X.gen(i)[m] > X.gen(j)[n]` $\iff$ `i<j`, or `i==j` and `m>n`

Under this restriction, the monomial ordering can be lexicographic (default), degree lexicographic, or degree reverse lexicographic. Here, the ordering is lexicographic, and elements can be compared as usual:

```
sage: X._order
'lex'
sage: a > b
True
```

```
>>> from sage.all import *
>>> X._order
'lex'
>>> a > b
True
```

Note that, when a method is called that is not directly implemented for 'InfinitePolynomial', it is tried to call this method for the underlying *classical* polynomial. This holds, e.g., when applying the `latex` function:

```
sage: latex(c)
x_{3}^{3} + x_{3} y_{4} - 2 y_{4}^{4}
```

```
>>> from sage.all import *
>>> latex(c)
x_{3}^{3} + x_{3} y_{4} - 2 y_{4}^{4}
```

There is a permutation action on Infinite Polynomial Rings by permuting the indices of the variables:

```
sage: P = Permutation(((4,5),(2,3)))
sage: c^P
x_2^3 + x_2*y_5 - 2*y_5^4
```

```
>>> from sage.all import *
>>> P = Permutation(((Integer(4),Integer(5)),(Integer(2),Integer(3))))
>>> c**P
x_2^3 + x_2*y_5 - 2*y_5^4
```

Note that `P(0)==0`, and thus variables of index zero are invariant under the permutation action. More generally, if `P` is any callable object that accepts nonnegative integers as input and returns nonnegative integers, then `c^P` means to apply `P` to the variable indices occurring in `c`.

If you want to substitute variables you can use the standard polynomial methods, such as `subs()`:

```
sage: R.<x,y> = InfinitePolynomialRing(QQ)
sage: f = x[1] + x[1]*x[2]*x[3]
sage: f.subs({x[1]: x[0]})
x_3*x_2*x_0 + x_0
sage: g = x[0] + x[1] + y[0]
sage: g.subs({x[0]: y[0]})
x_1 + 2*y_0
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> f = x[Integer(1)] + x[Integer(1)]*x[Integer(2)]*x[Integer(3)]
>>> f.subs({x[Integer(1)]: x[Integer(0)]})
x_3*x_2*x_0 + x_0
>>> g = x[Integer(0)] + x[Integer(1)] + y[Integer(0)]
>>> g.subs({x[Integer(0)]: y[Integer(0)]})
x_1 + 2*y_0
```

class `sage.rings.polynomial.infinite_polynomial_element.`**`InfinitePolynomial`**`(A, p)`

Bases: `CommutativePolynomial`

Create an element of a Polynomial Ring with a Countably Infinite Number of Variables.

Usually, an InfinitePolynomial is obtained by using the generators of an Infinite Polynomial Ring (see *infinite_polynomial_ring*) or by conversion.

INPUT:

- `A` – an Infinite Polynomial Ring
- `p` – a *classical* polynomial that can be interpreted in `A`

ASSUMPTIONS:

In the dense implementation, it must be ensured that the argument `p` coerces into `A._P` by a name preserving conversion map.

In the sparse implementation, in the direct construction of an infinite polynomial, it is *not* tested whether the argument `p` makes sense in `A`.

EXAMPLES:

```
sage: from sage.rings.polynomial.infinite_polynomial_element import␣
↪InfinitePolynomial
sage: X.<alpha> = InfinitePolynomialRing(ZZ)
sage: P.<alpha_1,alpha_2> = ZZ[]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.infinite_polynomial_element import␣
↪InfinitePolynomial
>>> X = InfinitePolynomialRing(ZZ, names=('alpha',)); (alpha,) = X._first_ngens(1)
>>> P = ZZ['alpha_1, alpha_2']; (alpha_1, alpha_2,) = P._first_ngens(2)
```

Currently, `P` and `X._P` (the underlying polynomial ring of `X`) both have two variables:

```
sage: X._P
Multivariate Polynomial Ring in alpha_1, alpha_0 over Integer Ring
```

```
>>> from sage.all import *
>>> X._P
Multivariate Polynomial Ring in alpha_1, alpha_0 over Integer Ring
```

By default, a coercion from `P` to `X._P` would not be name preserving. However, this is taken care for; a name preserving conversion is impossible, and by consequence an error is raised:

```
sage: InfinitePolynomial(X, (alpha_1+alpha_2)^2)
Traceback (most recent call last):
...
TypeError: Could not find a mapping of the passed element to this ring.
```

```
>>> from sage.all import *
>>> InfinitePolynomial(X, (alpha_1+alpha_2)**Integer(2))
Traceback (most recent call last):
...
TypeError: Could not find a mapping of the passed element to this ring.
```

When extending the underlying polynomial ring, the construction of an infinite polynomial works:

```
sage: alpha[2]
alpha_2
sage: InfinitePolynomial(X, (alpha_1+alpha_2)^2)
alpha_2^2 + 2*alpha_2*alpha_1 + alpha_1^2
```

```
>>> from sage.all import *
>>> alpha[Integer(2)]
alpha_2
>>> InfinitePolynomial(X, (alpha_1+alpha_2)**Integer(2))
alpha_2^2 + 2*alpha_2*alpha_1 + alpha_1^2
```

In the sparse implementation, it is not checked whether the polynomial really belongs to the parent, and when it does not, the results may be unexpected due to coercions:

```
sage: Y.<alpha,beta> = InfinitePolynomialRing(GF(2), implementation='sparse')
sage: a = (alpha_1+alpha_2)^2
```

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```
sage: InfinitePolynomial(Y, a)
alpha_0^2 + beta_0^2
```

```
>>> from sage.all import *
>>> Y = InfinitePolynomialRing(GF(Integer(2)), implementation='sparse', names=(
↪'alpha', 'beta',)); (alpha, beta,) = Y._first_ngens(2)
>>> a = (alpha_1+alpha_2)**Integer(2)
>>> InfinitePolynomial(Y, a)
alpha_0^2 + beta_0^2
```

However, it is checked when doing a conversion:

```
sage: Y(a)
alpha_2^2 + alpha_1^2
```

```
>>> from sage.all import *
>>> Y(a)
alpha_2^2 + alpha_1^2
```

coefficient(*monomial*)

Return the coefficient of a monomial in this polynomial.

INPUT:

- A monomial (element of the parent of self) or
- a dictionary that describes a monomial (the keys are variables of the parent of self, the values are the corresponding exponents)

EXAMPLES:

We can get the coefficient in front of monomials:

```
sage: X.<x> = InfinitePolynomialRing(QQ)
sage: a = 2*x[0]*x[1] + x[1] + x[2]
sage: a.coefficient(x[0])
2*x_1
sage: a.coefficient(x[1])
2*x_0 + 1
sage: a.coefficient(x[2])
1
sage: a.coefficient(x[0]*x[1])
2
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x',)); (x,) = X._first_ngens(1)
>>> a = Integer(2)*x[Integer(0)]*x[Integer(1)] + x[Integer(1)] + x[Integer(2)]
>>> a.coefficient(x[Integer(0)])
2*x_1
>>> a.coefficient(x[Integer(1)])
2*x_0 + 1
>>> a.coefficient(x[Integer(2)])
1
```

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```
>>> a.coefficient(x[Integer(0)]*x[Integer(1)])
2
```

We can also pass in a dictionary:

```
sage: a.coefficient({x[0]:1, x[1]:1})
2
```

```
>>> from sage.all import *
>>> a.coefficient({x[Integer(0)]:Integer(1), x[Integer(1)]:Integer(1)})
2
```

footprint()

Leading exponents sorted by index and generator.

OUTPUT: `D`; dictionary whose keys are the occurring variable indices

`D[s]` is a list `[i_1,...,i_n]`, where `i_j` gives the exponent of `self.parent().gen(j)[s]` in the leading term of `self`.

EXAMPLES:

```
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: p = x[30]*y[1]^3*x[1]^2 + 2*x[10]*y[30]
sage: sorted(p.footprint().items())
[(1, [2, 3]), (30, [1, 0])]
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = X._first_
↪ngens(2)
>>> p = x[Integer(30)]*y[Integer(1)]**Integer(3)*x[Integer(1)]**Integer(2) +␣
↪Integer(2)*x[Integer(10)]*y[Integer(30)]
>>> sorted(p.footprint().items())
[(1, [2, 3]), (30, [1, 0])]
```

gcd(x)

Compute the greatest common divisor.

EXAMPLES:

```
sage: R.<x> = InfinitePolynomialRing(QQ)
sage: p1 = x[0] + x[1]^2
sage: gcd(p1, p1 + 3)
1
sage: gcd(p1, p1) == p1
True
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> p1 = x[Integer(0)] + x[Integer(1)]**Integer(2)
>>> gcd(p1, p1 + Integer(3))
1
>>> gcd(p1, p1) == p1
True
```

is_nilpotent()

Return `True` if `self` is nilpotent, i.e., some power of `self` is 0.

EXAMPLES:

```
sage: R.<x> = InfinitePolynomialRing(QQbar)                                  #␣
↪needs sage.rings.number_field
sage: (x[0] + x[1]).is_nilpotent()                                           #␣
↪needs sage.rings.number_field
False
sage: R(0).is_nilpotent()                                                    #␣
↪needs sage.rings.number_field
True
sage: _.<x> = InfinitePolynomialRing(Zmod(4))
sage: (2*x[0]).is_nilpotent()
True
sage: (2+x[4]*x[7]).is_nilpotent()
False
sage: _.<y> = InfinitePolynomialRing(Zmod(100))
sage: (5+2*y[0] + 10*(y[0]^2+y[1]^2)).is_nilpotent()
False
sage: (10*y[2] + 20*y[5] - 30*y[2]*y[5] + 70*(y[2]^2+y[5]^2)).is_nilpotent()
True
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(QQbar, names=('x',)); (x,) = R._first_ngens(1)
↪# needs sage.rings.number_field
>>> (x[Integer(0)] + x[Integer(1)]).is_nilpotent()                           ␣
↪              # needs sage.rings.number_field
False
>>> R(Integer(0)).is_nilpotent()                                             ␣
↪     # needs sage.rings.number_field
True
>>> _ = InfinitePolynomialRing(Zmod(Integer(4)), names=('x',)); (x,) = _._
↪first_ngens(1)
>>> (Integer(2)*x[Integer(0)]).is_nilpotent()
True
>>> (Integer(2)+x[Integer(4)]*x[Integer(7)]).is_nilpotent()
False
>>> _ = InfinitePolynomialRing(Zmod(Integer(100)), names=('y',)); (y,) = _._
↪first_ngens(1)
>>> (Integer(5)+Integer(2)*y[Integer(0)] +␣
↪Integer(10)*(y[Integer(0)]**Integer(2)+y[Integer(1)]**Integer(2))).is_
↪nilpotent()
False
>>> (Integer(10)*y[Integer(2)] + Integer(20)*y[Integer(5)] -␣
↪Integer(30)*y[Integer(2)]*y[Integer(5)] +␣
↪Integer(70)*(y[Integer(2)]**Integer(2)+y[Integer(5)]**Integer(2))).is_
↪nilpotent()
True
```

is_unit()

Answer whether `self` is a unit.

EXAMPLES:

```
sage: R1.<x,y> = InfinitePolynomialRing(ZZ)
sage: R2.<a,b> = InfinitePolynomialRing(QQ)
sage: (1 + x[2]).is_unit()
False
sage: R1(1).is_unit()
True
sage: R1(2).is_unit()
False
sage: R2(2).is_unit()
True
sage: (1 + a[2]).is_unit()
False
```

```
>>> from sage.all import *
>>> R1 = InfinitePolynomialRing(ZZ, names=('x', 'y',)); (x, y,) = R1._first_
↪ngens(2)
>>> R2 = InfinitePolynomialRing(QQ, names=('a', 'b',)); (a, b,) = R2._first_
↪ngens(2)
>>> (Integer(1) + x[Integer(2)]).is_unit()
False
>>> R1(Integer(1)).is_unit()
True
>>> R1(Integer(2)).is_unit()
False
>>> R2(Integer(2)).is_unit()
True
>>> (Integer(1) + a[Integer(2)]).is_unit()
False
```

Check that Issue #22454 is fixed:

```
sage: _.<x> = InfinitePolynomialRing(Zmod(4))
sage: (1 + 2*x[0]).is_unit()
True
sage: (x[0]*x[1]).is_unit()
False
sage: _.<x> = InfinitePolynomialRing(Zmod(900))
sage: (7+150*x[0] + 30*x[1] + 120*x[1]*x[100]).is_unit()
True
```

```
>>> from sage.all import *
>>> _ = InfinitePolynomialRing(Zmod(Integer(4)), names=('x',)); (x,) = _._
↪first_ngens(1)
>>> (Integer(1) + Integer(2)*x[Integer(0)]).is_unit()
True
>>> (x[Integer(0)]*x[Integer(1)]).is_unit()
False
>>> _ = InfinitePolynomialRing(Zmod(Integer(900)), names=('x',)); (x,) = _._
↪first_ngens(1)
>>> (Integer(7)+Integer(150)*x[Integer(0)] + Integer(30)*x[Integer(1)] +␣
↪Integer(120)*x[Integer(1)]*x[Integer(100)]).is_unit()
True
```

lc()

The coefficient of the leading term of `self`.

EXAMPLES:

```
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: p = 2*x[10]*y[30] + 3*x[10]*y[1]^3*x[1]^2
sage: p.lc()
3
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = X._first_
↪ngens(2)
>>> p = Integer(2)*x[Integer(10)]*y[Integer(30)] +␣
↪Integer(3)*x[Integer(10)]*y[Integer(1)]**Integer(3)*x[Integer(1)]**Integer(2)
>>> p.lc()
3
```

lm()

The leading monomial of `self`.

EXAMPLES:

```
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: p = 2*x[10]*y[30] + x[10]*y[1]^3*x[1]^2
sage: p.lm()
x_10*x_1^2*y_1^3
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = X._first_
↪ngens(2)
>>> p = Integer(2)*x[Integer(10)]*y[Integer(30)] +␣
↪x[Integer(10)]*y[Integer(1)]**Integer(3)*x[Integer(1)]**Integer(2)
>>> p.lm()
x_10*x_1^2*y_1^3
```

lt()

The leading term (= product of coefficient and monomial) of `self`.

EXAMPLES:

```
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: p = 2*x[10]*y[30] + 3*x[10]*y[1]^3*x[1]^2
sage: p.lt()
3*x_10*x_1^2*y_1^3
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = X._first_
↪ngens(2)
>>> p = Integer(2)*x[Integer(10)]*y[Integer(30)] +␣
↪Integer(3)*x[Integer(10)]*y[Integer(1)]**Integer(3)*x[Integer(1)]**Integer(2)
>>> p.lt()
3*x_10*x_1^2*y_1^3
```

max_index()

Return the maximal index of a variable occurring in `self`, or -1 if `self` is scalar.

EXAMPLES:

```
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: p = x[1]^2 + y[2]^2 + x[1]*x[2]*y[3] + x[1]*y[4]
sage: p.max_index()
4
sage: x[0].max_index()
0
sage: X(10).max_index()
-1
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = X._first_
→ngens(2)
>>> p = x[Integer(1)]**Integer(2) + y[Integer(2)]**Integer(2) +␣
→x[Integer(1)]*x[Integer(2)]*y[Integer(3)] + x[Integer(1)]*y[Integer(4)]
>>> p.max_index()
4
>>> x[Integer(0)].max_index()
0
>>> X(Integer(10)).max_index()
-1
```

monomial_coefficient(*mon*)

Return the base ring element that is the coefficient of `mon` in `self`.

This function contrasts with the function `coefficient()`, which returns the coefficient of a monomial viewing this polynomial in a polynomial ring over a base ring having fewer variables.

INPUT:

- `mon` – a monomial in the parent of `self`

OUTPUT: coefficient in base ring

See also

For coefficients in a base ring of fewer variables, look at `coefficient()`.

EXAMPLES:

```
sage: X.<x> = InfinitePolynomialRing(QQ)
sage: f = 2*x[0]*x[2] + 3*x[1]^2
sage: c = f.monomial_coefficient(x[1]^2); c
3
sage: c.parent()
Rational Field

sage: c = f.coefficient(x[2]); c
2*x_0
sage: c.parent()
Infinite polynomial ring in x over Rational Field
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x',)); (x,) = X._first_ngens(1)
>>> f = Integer(2)*x[Integer(0)]*x[Integer(2)] +␣
↪Integer(3)*x[Integer(1)]**Integer(2)
>>> c = f.monomial_coefficient(x[Integer(1)]**Integer(2)); c
3
>>> c.parent()
Rational Field

>>> c = f.coefficient(x[Integer(2)]); c
2*x_0
>>> c.parent()
Infinite polynomial ring in x over Rational Field
```

monomials()

Return the list of monomials in `self`.

The returned list is decreasingly ordered by the term ordering of `self.parent()`.

EXAMPLES:

```
sage: X.<x> = InfinitePolynomialRing(QQ)
sage: p = x[1]^3 + x[2] - 2*x[1]*x[3]
sage: p.monomials()
[x_3*x_1, x_2, x_1^3]

sage: X.<x> = InfinitePolynomialRing(QQ, order='deglex')
sage: p = x[1]^3 + x[2] - 2*x[1]*x[3]
sage: p.monomials()
[x_1^3, x_3*x_1, x_2]
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x',)); (x,) = X._first_ngens(1)
>>> p = x[Integer(1)]**Integer(3) + x[Integer(2)] -␣
↪Integer(2)*x[Integer(1)]*x[Integer(3)]
>>> p.monomials()
[x_3*x_1, x_2, x_1^3]

>>> X = InfinitePolynomialRing(QQ, order='deglex', names=('x',)); (x,) = X._
↪first_ngens(1)
>>> p = x[Integer(1)]**Integer(3) + x[Integer(2)] -␣
↪Integer(2)*x[Integer(1)]*x[Integer(3)]
>>> p.monomials()
[x_1^3, x_3*x_1, x_2]
```

numerator()

Return a numerator of `self`, computed as `self * self.denominator()`.

> **Warning**
>
> This is not the numerator of the rational function defined by `self`, which would always be `self` since it is a polynomial.

EXAMPLES:

```
sage: X.<x> = InfinitePolynomialRing(QQ)
sage: p = 2/3*x[1] + 4/9*x[2] - 2*x[1]*x[3]
sage: num = p.numerator(); num
-18*x_3*x_1 + 4*x_2 + 6*x_1
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x',)); (x,) = X._first_ngens(1)
>>> p = Integer(2)/Integer(3)*x[Integer(1)] + Integer(4)/
↪Integer(9)*x[Integer(2)] - Integer(2)*x[Integer(1)]*x[Integer(3)]
>>> num = p.numerator(); num
-18*x_3*x_1 + 4*x_2 + 6*x_1
```

polynomial()

Return the underlying polynomial.

EXAMPLES:

```
sage: X.<x,y> = InfinitePolynomialRing(GF(7))
sage: p = x[2]*y[1] + 3*y[0]
sage: p
x_2*y_1 + 3*y_0
sage: p.polynomial()
x_2*y_1 + 3*y_0
sage: p.polynomial().parent()
Multivariate Polynomial Ring in x_2, x_1, x_0, y_2, y_1, y_0
 over Finite Field of size 7
sage: p.parent()
Infinite polynomial ring in x, y over Finite Field of size 7
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(GF(Integer(7)), names=('x', 'y',)); (x, y,) =␣
↪X._first_ngens(2)
>>> p = x[Integer(2)]*y[Integer(1)] + Integer(3)*y[Integer(0)]
>>> p
x_2*y_1 + 3*y_0
>>> p.polynomial()
x_2*y_1 + 3*y_0
>>> p.polynomial().parent()
Multivariate Polynomial Ring in x_2, x_1, x_0, y_2, y_1, y_0
 over Finite Field of size 7
>>> p.parent()
Infinite polynomial ring in x, y over Finite Field of size 7
```

reduce(*I*, *tailreduce=False*, *report=None*)

Symmetrical reduction of `self` with respect to a symmetric ideal (or list of Infinite Polynomials).

INPUT:

- `I` – a *SymmetricIdeal* or a list of Infinite Polynomials
- `tailreduce` – boolean (default: `False`); *tail reduction* is performed if this parameter is `True`.
- `report` – object (default: `None`); if not `None`, some information on the progress of computation is printed, since reduction of huge polynomials may take a long time

OUTPUT: symmetrical reduction of `self` with respect to `I`, possibly with tail reduction

THEORY:

Reducing an element p of an Infinite Polynomial Ring X by some other element q means the following:

1. Let M and N be the leading terms of p and q.
2. Test whether there is a permutation P that does not does not diminish the variable indices occurring in N and preserves their order, so that there is some term $T \in X$ with $TN^P = M$. If there is no such permutation, return p
3. Replace p by $p - Tq^P$ and continue with step 1.

EXAMPLES:

```
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: p = y[1]^2*y[3] + y[2]*x[3]^3
sage: p.reduce([y[2]*x[1]^2])
x_3^3*y_2 + y_3*y_1^2
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = X._first_
↪ngens(2)
>>> p = y[Integer(1)]**Integer(2)*y[Integer(3)] +␣
↪y[Integer(2)]*x[Integer(3)]**Integer(3)
>>> p.reduce([y[Integer(2)]*x[Integer(1)]**Integer(2)])
x_3^3*y_2 + y_3*y_1^2
```

The preceding is correct: If a permutation turns `y[2]*x[1]^2` into a factor of the leading monomial `y[2]*x[3]^3` of `p`, then it interchanges the variable indices 1 and 2; this is not allowed in a symmetric reduction. However, reduction by `y[1]*x[2]^2` works, since one can change variable index 1 into 2 and 2 into 3:

```
sage: p.reduce([y[1]*x[2]^2])                                             #␣
↪needs sage.libs.singular
y_3*y_1^2
```

```
>>> from sage.all import *
>>> p.reduce([y[Integer(1)]*x[Integer(2)]**Integer(2)])                  ␣
↪                          # needs sage.libs.singular
y_3*y_1^2
```

The next example shows that tail reduction is not done, unless it is explicitly advised. The input can also be a Symmetric Ideal:

```
sage: I = (y[3])*X
sage: p.reduce(I)
x_3^3*y_2 + y_3*y_1^2
sage: p.reduce(I, tailreduce=True)                                        #␣
↪needs sage.libs.singular
x_3^3*y_2
```

```
>>> from sage.all import *
>>> I = (y[Integer(3)])*X
>>> p.reduce(I)
```

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```
x_3^3*y_2 + y_3*y_1^2
>>> p.reduce(I, tailreduce=True)                                          #␣
↪needs sage.libs.singular
x_3^3*y_2
```

Last, we demonstrate the `report` option:

```
sage: p = x[1]^2 + y[2]^2 + x[1]*x[2]*y[3] + x[1]*y[4]
sage: p.reduce(I, tailreduce=True, report=True)                            #␣
↪needs sage.libs.singular
:T[2]:>
>
x_1^2 + y_2^2
```

```
>>> from sage.all import *
>>> p = x[Integer(1)]**Integer(2) + y[Integer(2)]**Integer(2) +␣
↪x[Integer(1)]*x[Integer(2)]*y[Integer(3)] + x[Integer(1)]*y[Integer(4)]
>>> p.reduce(I, tailreduce=True, report=True)                              #␣
↪needs sage.libs.singular
:T[2]:>
>
x_1^2 + y_2^2
```

The output ':' means that there was one reduction of the leading monomial. 'T[2]' means that a tail reduction was performed on a polynomial with two terms. At '>', one round of the reduction process is finished (there could only be several non-trivial rounds if I was generated by more than one polynomial).

ring()

The ring which `self` belongs to.

This is the same as `self.parent()`.

EXAMPLES:

```
sage: X.<x,y> = InfinitePolynomialRing(ZZ,implementation='sparse')
sage: p = x[100]*y[1]^3*x[1]^2 + 2*x[10]*y[30]
sage: p.ring()
Infinite polynomial ring in x, y over Integer Ring
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(ZZ,implementation='sparse', names=('x', 'y',));
↪ (x, y,) = X._first_ngens(2)
>>> p = x[Integer(100)]*y[Integer(1)]**Integer(3)*x[Integer(1)]**Integer(2) +␣
↪Integer(2)*x[Integer(10)]*y[Integer(30)]
>>> p.ring()
Infinite polynomial ring in x, y over Integer Ring
```

squeezed()

Reduce the variable indices occurring in `self`.

OUTPUT:

Apply a permutation to `self` that does not change the order of the variable indices of `self` but squeezes them into the range 1,2,…

EXAMPLES:

```
sage: X.<x,y> = InfinitePolynomialRing(QQ,implementation='sparse')
sage: p = x[1]*y[100] + x[50]*y[1000]
sage: p.squeezed()
x_2*y_4 + x_1*y_3
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ,implementation='sparse', names=('x', 'y',));
↪ (x, y,) = X._first_ngens(2)
>>> p = x[Integer(1)]*y[Integer(100)] + x[Integer(50)]*y[Integer(1000)]
>>> p.squeezed()
x_2*y_4 + x_1*y_3
```

stretch(*k*)

Stretch `self` by a given factor.

INPUT:

- `k` – integer

OUTPUT: replace v_n with $v_{n \cdot k}$ for all generators v_* occurring in `self`

EXAMPLES:

```
sage: X.<x> = InfinitePolynomialRing(QQ)
sage: a = x[0] + x[1] + x[2]
sage: a.stretch(2)
x_4 + x_2 + x_0

sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: a = x[0] + x[1] + y[0]*y[1]; a
x_1 + x_0 + y_1*y_0
sage: a.stretch(2)
x_2 + x_0 + y_2*y_0
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x',)); (x,) = X._first_ngens(1)
>>> a = x[Integer(0)] + x[Integer(1)] + x[Integer(2)]
>>> a.stretch(Integer(2))
x_4 + x_2 + x_0

>>> X = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = X._first_
↪ngens(2)
>>> a = x[Integer(0)] + x[Integer(1)] + y[Integer(0)]*y[Integer(1)]; a
x_1 + x_0 + y_1*y_0
>>> a.stretch(Integer(2))
x_2 + x_0 + y_2*y_0
```

subs(*fixed=None*, ***kwargs*)

Substitute variables in `self`.

INPUT:

- `fixed` – (optional) `dict` with `{variable: value}` pairs
- `**kwargs` – named parameters

OUTPUT: the resulting substitution

EXAMPLES:

```
sage: R.<x,y> = InfinitePolynomialRing(QQ)
sage: f = x[1] + x[1]*x[2]*x[3]
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_
↪ngens(2)
>>> f = x[Integer(1)] + x[Integer(1)]*x[Integer(2)]*x[Integer(3)]
```

Passing `fixed={x[1]: x[0]}`. Note that the keys may be given using the generators of the infinite polynomial ring or as a string:

```
sage: f.subs({x[1]: x[0]})
x_3*x_2*x_0 + x_0
sage: f.subs({'x_1': x[0]})
x_3*x_2*x_0 + x_0
```

```
>>> from sage.all import *
>>> f.subs({x[Integer(1)]: x[Integer(0)]})
x_3*x_2*x_0 + x_0
>>> f.subs({'x_1': x[Integer(0)]})
x_3*x_2*x_0 + x_0
```

Passing the variables as names parameters:

```
sage: f.subs(x_1=y[1])
x_3*x_2*y_1 + y_1
sage: f.subs(x_1=y[1], x_2=2)
2*x_3*y_1 + y_1
```

```
>>> from sage.all import *
>>> f.subs(x_1=y[Integer(1)])
x_3*x_2*y_1 + y_1
>>> f.subs(x_1=y[Integer(1)], x_2=Integer(2))
2*x_3*y_1 + y_1
```

The substitution returns the original polynomial if you try to substitute a variable not present:

```
sage: g = x[0] + x[1]
sage: g.subs({y[0]: x[0]})
x_1 + x_0
```

```
>>> from sage.all import *
>>> g = x[Integer(0)] + x[Integer(1)]
>>> g.subs({y[Integer(0)]: x[Integer(0)]})
x_1 + x_0
```

The substitution can also handle matrices:

```
sage: # needs sage.modules
sage: M = matrix([[1,0], [0,2]])
```

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```
sage: N = matrix([[0,3], [4,0]])
sage: g = x[0]^2 + 3*x[1]
sage: g.subs({'x_0': M})
[3*x_1 + 1          0]
[         0 3*x_1 + 4]
sage: g.subs({x[0]: M, x[1]: N})
[ 1  9]
[12  4]
```

```
>>> from sage.all import *
>>> # needs sage.modules
>>> M = matrix([[Integer(1),Integer(0)], [Integer(0),Integer(2)]])
>>> N = matrix([[Integer(0),Integer(3)], [Integer(4),Integer(0)]])
>>> g = x[Integer(0)]**Integer(2) + Integer(3)*x[Integer(1)]
>>> g.subs({'x_0': M})
[3*x_1 + 1          0]
[         0 3*x_1 + 4]
>>> g.subs({x[Integer(0)]: M, x[Integer(1)]: N})
[ 1  9]
[12  4]
```

If you pass both `fixed` and `kwargs`, any conflicts will defer to `fixed`:

```
sage: R.<x,y> = InfinitePolynomialRing(QQ)
sage: f = x[0]
sage: f.subs({x[0]: 1})
1
sage: f.subs(x_0=5)
5
sage: f.subs({x[0]: 1}, x_0=5)
1
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_
↪ngens(2)
>>> f = x[Integer(0)]
>>> f.subs({x[Integer(0)]: Integer(1)})
1
>>> f.subs(x_0=Integer(5))
5
>>> f.subs({x[Integer(0)]: Integer(1)}, x_0=Integer(5))
1
```

symmetric_cancellation_order(*other*)

Comparison of leading terms by Symmetric Cancellation Order, $<_{sc}$.

INPUT:

- `self`, `other` – two Infinite Polynomials

ASSUMPTION:

Both Infinite Polynomials are nonzero.

OUTPUT:

`(c, sigma, w)`, where

- c = -1,0,1, or None if the leading monomial of `self` is smaller, equal, greater, or incomparable with respect to `other` in the monomial ordering of the Infinite Polynomial Ring
- sigma is a permutation witnessing `self` $<_{sc}$ `other` (resp. `self` $>_{sc}$ `other`) or is 1 if `self.lm()==other.lm()`
- w is 1 or is a term so that `w*self.lt()^sigma == other.lt()` if $c \le 0$, and `w*other.lt()^sigma == self.lt()` if $c = 1$

THEORY:

If the Symmetric Cancellation Order is a well-quasi-ordering then computation of Groebner bases always terminates. This is the case, e.g., if the monomial order is lexicographic. For that reason, lexicographic order is our default order.

EXAMPLES:

```
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: (x[2]*x[1]).symmetric_cancellation_order(x[2]^2)
(None, 1, 1)
sage: (x[2]*x[1]).symmetric_cancellation_order(x[2]*x[3]*y[1])
(-1, [2, 3, 1], y_1)
sage: (x[2]*x[1]*y[1]).symmetric_cancellation_order(x[2]*x[3]*y[1])
(None, 1, 1)
sage: (x[2]*x[1]*y[1]).symmetric_cancellation_order(x[2]*x[3]*y[2])
(-1, [2, 3, 1], 1)
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = X._first_
↪ngens(2)
>>> (x[Integer(2)]*x[Integer(1)]).symmetric_cancellation_
↪order(x[Integer(2)]**Integer(2))
(None, 1, 1)
>>> (x[Integer(2)]*x[Integer(1)]).symmetric_cancellation_
↪order(x[Integer(2)]*x[Integer(3)]*y[Integer(1)])
(-1, [2, 3, 1], y_1)
>>> (x[Integer(2)]*x[Integer(1)]*y[Integer(1)]).symmetric_cancellation_
↪order(x[Integer(2)]*x[Integer(3)]*y[Integer(1)])
(None, 1, 1)
>>> (x[Integer(2)]*x[Integer(1)]*y[Integer(1)]).symmetric_cancellation_
↪order(x[Integer(2)]*x[Integer(3)]*y[Integer(2)])
(-1, [2, 3, 1], 1)
```

tail()

The tail of `self` (this is `self` minus its leading term).

EXAMPLES:

```
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: p = 2*x[10]*y[30] + 3*x[10]*y[1]^3*x[1]^2
sage: p.tail()
2*x_10*y_30
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = X._first_
↪ngens(2)
>>> p = Integer(2)*x[Integer(10)]*y[Integer(30)] +␣
↪Integer(3)*x[Integer(10)]*y[Integer(1)]**Integer(3)*x[Integer(1)]**Integer(2)
>>> p.tail()
2*x_10*y_30
```

variables()

Return the variables occurring in `self` (tuple of elements of some polynomial ring).

EXAMPLES:

```
sage: X.<x> = InfinitePolynomialRing(QQ)
sage: p = x[1] + x[2] - 2*x[1]*x[3]
sage: p.variables()
(x_3, x_2, x_1)
sage: x[1].variables()
(x_1,)
sage: X(1).variables()
()
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x',)); (x,) = X._first_ngens(1)
>>> p = x[Integer(1)] + x[Integer(2)] - Integer(2)*x[Integer(1)]*x[Integer(3)]
>>> p.variables()
(x_3, x_2, x_1)
>>> x[Integer(1)].variables()
(x_1,)
>>> X(Integer(1)).variables()
()
```

class sage.rings.polynomial.infinite_polynomial_element.**InfinitePolynomial_dense**(A, p)

Bases: *InfinitePolynomial*

Element of a dense Polynomial Ring with a Countably Infinite Number of Variables.

INPUT:

- `A` – an Infinite Polynomial Ring in dense implementation
- `p` – a *classical* polynomial that can be interpreted in `A`

Of course, one should not directly invoke this class, but rather construct elements of `A` in the usual way.

class sage.rings.polynomial.infinite_polynomial_element.**InfinitePolynomial_sparse**(A, p)

Bases: *InfinitePolynomial*

Element of a sparse Polynomial Ring with a Countably Infinite Number of Variables.

INPUT:

- `A` – an Infinite Polynomial Ring in sparse implementation
- `p` – a *classical* polynomial that can be interpreted in `A`

Of course, one should not directly invoke this class, but rather construct elements of `A` in the usual way.

EXAMPLES:

```
sage: A.<a> = QQ[]
sage: B.<b,c> = InfinitePolynomialRing(A,implementation='sparse')
sage: p = a*b[100] + 1/2*c[4]
sage: p
a*b_100 + 1/2*c_4
sage: p.parent()
Infinite polynomial ring in b, c
 over Univariate Polynomial Ring in a over Rational Field
sage: p.polynomial().parent()
Multivariate Polynomial Ring in b_100, b_0, c_4, c_0
 over Univariate Polynomial Ring in a over Rational Field
```

```
>>> from sage.all import *
>>> A = QQ['a']; (a,) = A._first_ngens(1)
>>> B = InfinitePolynomialRing(A,implementation='sparse', names=('b', 'c',)); (b,
↪c,) = B._first_ngens(2)
>>> p = a*b[Integer(100)] + Integer(1)/Integer(2)*c[Integer(4)]
>>> p
a*b_100 + 1/2*c_4
>>> p.parent()
Infinite polynomial ring in b, c
 over Univariate Polynomial Ring in a over Rational Field
>>> p.polynomial().parent()
Multivariate Polynomial Ring in b_100, b_0, c_4, c_0
 over Univariate Polynomial Ring in a over Rational Field
```

6.3 Symmetric Ideals of Infinite Polynomial Rings

This module provides an implementation of ideals of polynomial rings in a countably infinite number of variables that are invariant under variable permutation. Such ideals are called 'Symmetric Ideals' in the rest of this document. Our implementation is based on the theory of M. Aschenbrenner and C. Hillar.

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EXAMPLES:

Here, we demonstrate that working in quotient rings of Infinite Polynomial Rings works, provided that one uses symmetric Groebner bases.

```
sage: R.<x> = InfinitePolynomialRing(QQ)
sage: I = R.ideal([x[1]*x[2] + x[3]])
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(QQ, names=('x',)); (x,) = R._first_ngens(1)
>>> I = R.ideal([x[Integer(1)]*x[Integer(2)] + x[Integer(3)]])
```

Note that I is not a symmetric Groebner basis:

```
sage: # needs sage.combinat
sage: G = R * I.groebner_basis()
sage: G
```

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```
Symmetric Ideal (x_2 - x_1, x_1^2 + x_1) of Infinite polynomial ring in x
over Rational Field
sage: Q = R.quotient(G)
sage: p = x[3]*x[1] + x[2]^2 + 3
sage: Q(p)
-2*x_1 + 3
```

```
>>> from sage.all import *
>>> # needs sage.combinat
>>> G = R * I.groebner_basis()
>>> G
Symmetric Ideal (x_2 - x_1, x_1^2 + x_1) of Infinite polynomial ring in x
over Rational Field
>>> Q = R.quotient(G)
>>> p = x[Integer(3)]*x[Integer(1)] + x[Integer(2)]**Integer(2) + Integer(3)
>>> Q(p)
-2*x_1 + 3
```

By the second generator of `G`, variable x_n is equal to x_1 for any positive integer n. By the first generator of `G`, x_1^3 is equal to x_1 in `Q`. Indeed, we have

```
sage: Q(p)*x[2] == Q(p)*x[1]*x[3]*x[5]                                         #␣
↪needs sage.combinat
True
```

```
>>> from sage.all import *
>>> Q(p)*x[Integer(2)] == Q(p)*x[Integer(1)]*x[Integer(3)]*x[Integer(5)]          ␣
↪                                       # needs sage.combinat
True
```

class `sage.rings.polynomial.symmetric_ideal.`**`SymmetricIdeal`**(*ring*, *gens*, *coerce=True*)

Bases: `Ideal_generic`

Ideal in an Infinite Polynomial Ring, invariant under permutation of variable indices.

THEORY:

An Infinite Polynomial Ring with finitely many generators $x_*, y_*, ...$ over a field F is a free commutative F-algebra generated by infinitely many 'variables' $x_0, x_1, x_2, ..., y_0, y_1, y_2, ...$. We refer to the natural number n as the *index* of the variable x_n. See more detailed description at `infinite_polynomial_ring`

Infinite Polynomial Rings are equipped with a permutation action by permuting positive variable indices, i.e., $x_n^P = x_{P(n)}, y_n^P = y_{P(n)}, ...$ for any permutation P. Note that the variables $x_0, y_0, ...$ of index zero are invariant under that action.

A *Symmetric Ideal* is an ideal in an infinite polynomial ring X that is invariant under the permutation action. In other words, if $\mathfrak{S}_\infty$ denotes the symmetric group of $1, 2, ...$, then a Symmetric Ideal is a right $X[\mathfrak{S}_\infty]$-submodule of X.

It is known by work of Aschenbrenner and Hillar [AB2007] that an Infinite Polynomial Ring X with a single generator x_* is Noetherian, in the sense that any Symmetric Ideal $I \subset X$ is finitely generated modulo addition, multiplication by elements of X, and permutation of variable indices (hence, it is a finitely generated right $X[\mathfrak{S}_\infty]$-module).

Moreover, if X is equipped with a lexicographic monomial ordering with $x_1 < x_2 < x_3...$ then there is an algorithm of Buchberger type that computes a Groebner basis G for I that allows for computation of a unique normal form, that is zero precisely for the elements of I – see [AB2008]. See `groebner_basis()` for more details.

Our implementation allows more than one generator and also provides degree lexicographic and degree reverse lexicographic monomial orderings – we do, however, not guarantee termination of the Buchberger algorithm in these cases.

EXAMPLES:

```
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: I = [x[1]*y[2]*y[1] + 2*x[1]*y[2]] * X
sage: I == loads(dumps(I))
True
sage: latex(I)
\left(x_{1} y_{2} y_{1} + 2 x_{1} y_{2}\right)\Bold{Q}[x_{\ast}, y_{\ast}][\
↪mathfrak{S}_{\infty}]
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> I = [x[Integer(1)]*y[Integer(2)]*y[Integer(1)] +␣
↪Integer(2)*x[Integer(1)]*y[Integer(2)]] * X
>>> I == loads(dumps(I))
True
>>> latex(I)
\left(x_{1} y_{2} y_{1} + 2 x_{1} y_{2}\right)\Bold{Q}[x_{\ast}, y_{\ast}][\
↪mathfrak{S}_{\infty}]
```

The default ordering is lexicographic. We now compute a Groebner basis:

```
sage: J = I.groebner_basis(); J   # about 3 seconds                         #␣
↪needs sage.combinat
[x_2*x_1*y_2 - x_2*x_1*y_1, x_1*y_2*y_1 + 2*x_1*y_2,
 x_2*y_2*y_1 + 2*x_2*y_1, x_2*x_1*y_1^2 + 2*x_2*x_1*y_1]
```

```
>>> from sage.all import *
>>> J = I.groebner_basis(); J   # about 3 seconds                         #␣
↪needs sage.combinat
[x_2*x_1*y_2 - x_2*x_1*y_1, x_1*y_2*y_1 + 2*x_1*y_2,
 x_2*y_2*y_1 + 2*x_2*y_1, x_2*x_1*y_1^2 + 2*x_2*x_1*y_1]
```

Note that even though the symmetric ideal can be generated by a single polynomial, its reduced symmetric Groebner basis comprises four elements. Ideal membership in `I` can now be tested by commuting symmetric reduction modulo `J`:

```
sage: I.reduce(J)                                                           #␣
↪needs sage.combinat
Symmetric Ideal (0) of Infinite polynomial ring in x, y over Rational Field
```

```
>>> from sage.all import *
>>> I.reduce(J)                                                           #␣
↪needs sage.combinat
Symmetric Ideal (0) of Infinite polynomial ring in x, y over Rational Field
```

The Groebner basis is not point-wise invariant under permutation:

```
sage: # needs sage.combinat
sage: P = Permutation([2, 1])
sage: J[2]
x_2*y_2*y_1 + 2*x_2*y_1
sage: J[2]^P
x_1*y_2*y_1 + 2*x_1*y_2
sage: J[2]^P in J
True
```

```
>>> from sage.all import *
>>> # needs sage.combinat
>>> P = Permutation([Integer(2), Integer(1)])
>>> J[Integer(2)]
x_2*y_2*y_1 + 2*x_2*y_1
>>> J[Integer(2)]**P
x_1*y_2*y_1 + 2*x_1*y_2
>>> J[Integer(2)]**P in J
True
```

However, any element of `J` has symmetric reduction zero even after applying a permutation. This even holds when the permutations involve higher variable indices than the ones occurring in `J`:

```
sage: [[(p^P).reduce(J) for p in J] for P in Permutations(3)]                  #␣
↪needs sage.combinat
[[0, 0, 0, 0], [0, 0, 0, 0], [0, 0, 0, 0], [0, 0, 0, 0], [0, 0, 0, 0], [0, 0, 0,␣
↪0]]
```

```
>>> from sage.all import *
>>> [[(p**P).reduce(J) for p in J] for P in Permutations(Integer(3))]           ␣
↪       # needs sage.combinat
[[0, 0, 0, 0], [0, 0, 0, 0], [0, 0, 0, 0], [0, 0, 0, 0], [0, 0, 0, 0], [0, 0, 0,␣
↪0]]
```

Since `I` is not a Groebner basis, it is no surprise that it cannot detect ideal membership:

```
sage: [p.reduce(I) for p in J]                                                 #␣
↪needs sage.combinat
[x_2*x_1*y_2 - x_2*x_1*y_1, 0, x_2*y_2*y_1 + 2*x_2*y_1, x_2*x_1*y_1^2 + 2*x_2*x_
↪1*y_1]
```

```
>>> from sage.all import *
>>> [p.reduce(I) for p in J]                                                  #␣
↪needs sage.combinat
[x_2*x_1*y_2 - x_2*x_1*y_1, 0, x_2*y_2*y_1 + 2*x_2*y_1, x_2*x_1*y_1^2 + 2*x_2*x_
↪1*y_1]
```

Note that we give no guarantee that the computation of a symmetric Groebner basis will terminate in any order different from lexicographic.

When multiplying Symmetric Ideals or raising them to some integer power, the permutation action is taken into account, so that the product is indeed the product of ideals in the mathematical sense.

```
sage: I = X * (x[1])
sage: I * I                                                              #␣
↪needs sage.combinat
Symmetric Ideal (x_1^2, x_2*x_1) of
 Infinite polynomial ring in x, y over Rational Field
sage: I^3                                                                #␣
↪needs sage.combinat
Symmetric Ideal (x_1^3, x_2*x_1^2, x_2^2*x_1, x_3*x_2*x_1) of
 Infinite polynomial ring in x, y over Rational Field
sage: I * I == X * (x[1]^2)                                              #␣
↪needs sage.combinat
False
```

```
>>> from sage.all import *
>>> I = X * (x[Integer(1)])
>>> I * I                                                              #␣
↪needs sage.combinat
Symmetric Ideal (x_1^2, x_2*x_1) of
 Infinite polynomial ring in x, y over Rational Field
>>> I**Integer(3)                                                        ␣
↪      # needs sage.combinat
Symmetric Ideal (x_1^3, x_2*x_1^2, x_2^2*x_1, x_3*x_2*x_1) of
 Infinite polynomial ring in x, y over Rational Field
>>> I * I == X * (x[Integer(1)]**Integer(2))                             ␣
↪               # needs sage.combinat
False
```

groebner_basis(*tailreduce=False*, *reduced=True*, *algorithm=None*, *report=None*, *use_full_group=False*)

Return a symmetric Groebner basis (type `Sequence`) of `self`.

INPUT:

- `tailreduce` – boolean (default: `False`); if `True`, use tail reduction in intermediate computations
- `reduced` – boolean (default: `True`); if `True`, return the reduced normalised symmetric Groebner basis
- `algorithm` – string (default: `None`); determine the algorithm (see below for available algorithms)
- `report` – object (default `None`); if not `None`, print information on the progress of computation
- `use_full_group` – boolean (default: `False`); if `True` then proceed as originally suggested by [AB2008]. Our default method should be faster; see `symmetrisation()` for more details.

The computation of symmetric Groebner bases also involves the computation of *classical* Groebner bases, i.e., of Groebner bases for ideals in polynomial rings with finitely many variables. For these computations, Sage provides the following ALGORITHMS:

‘’
: autoselect (default)

‘singular:groebner’
: Singular’s `groebner` command

‘singular:std’
: Singular’s `std` command

‘singular:stdhilb’
: Singular’s `stdhib` command

'singular:stdfglm'
Singular's `stdfglm` command

'singular:slimgb'
Singular's `slimgb` command

'libsingular:std'
libSingular's `std` command

'libsingular:slimgb'
libSingular's `slimgb` command

'toy:buchberger'
Sage's toy/educational buchberger without strategy

'toy:buchberger2'
Sage's toy/educational buchberger with strategy

'toy:d_basis'
Sage's toy/educational d_basis algorithm

'macaulay2:gb'
Macaulay2's `gb` command (if available)

'magma:GroebnerBasis'
Magma's `Groebnerbasis` command (if available)

If only a system is given - e.g. 'magma' - the default algorithm is chosen for that system.

Note

The Singular and libSingular versions of the respective algorithms are identical, but the former calls an external Singular process while the later calls a C function, i.e. the calling overhead is smaller.

EXAMPLES:

```
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: I1 = X * (x[1] + x[2], x[1]*x[2])
sage: I1.groebner_basis()                                                   #␣
↪needs sage.combinat
[x_1]
sage: I2 = X * (y[1]^2*y[3] + y[1]*x[3])
sage: I2.groebner_basis()                                                   #␣
↪needs sage.combinat
[x_1*y_2 + y_2^2*y_1, x_2*y_1 + y_2*y_1^2]
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = X._first_
↪ngens(2)
>>> I1 = X * (x[Integer(1)] + x[Integer(2)], x[Integer(1)]*x[Integer(2)])
>>> I1.groebner_basis()                                                  #␣
↪needs sage.combinat
[x_1]
>>> I2 = X * (y[Integer(1)]**Integer(2)*y[Integer(3)] +␣
↪y[Integer(1)]*x[Integer(3)])
>>> I2.groebner_basis()                                                  #␣
```

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```
↪needs sage.combinat
[x_1*y_2 + y_2^2*y_1, x_2*y_1 + y_2*y_1^2]
```

Note that a symmetric Groebner basis of a principal ideal is not necessarily formed by a single polynomial.

When using the algorithm originally suggested by Aschenbrenner and Hillar, the result is the same, but the computation takes much longer:

```
sage: I2.groebner_basis(use_full_group=True)                                    #␣
↪needs sage.combinat
[x_1*y_2 + y_2^2*y_1, x_2*y_1 + y_2*y_1^2]
```

```
>>> from sage.all import *
>>> I2.groebner_basis(use_full_group=True)                                    #␣
↪needs sage.combinat
[x_1*y_2 + y_2^2*y_1, x_2*y_1 + y_2*y_1^2]
```

Last, we demonstrate how the report on the progress of computations looks like:

```
sage: I1.groebner_basis(report=True, reduced=True)                              #␣
↪needs sage.combinat
Symmetric interreduction
[1/2] >
[2/2] :>
[1/2] >
[2/2] >
Symmetrise 2 polynomials at level 2
Apply permutations
>
>
Symmetric interreduction
[1/3] >
[2/3] >
[3/3] >
Apply permutations
>
:>
:>
Symmetric interreduction
[1/3] >
[2/3] >
[3/3] >
Symmetrisation done
Classical Groebner basis
-> 2 generators
Symmetric interreduction
[1/2] >
[2/2] >
Symmetrise 2 polynomials at level 3
Apply permutations
>
>
:>
```

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```
::>
:>
::>
Symmetric interreduction
[1/4] >
[2/4] :>
-> 0
[3/4] ::>
-> 0
[4/4] :>
-> 0
[1/1] >
Apply permutations
:>
:>
:>
Symmetric interreduction
[1/1] >
Classical Groebner basis
-> 1 generators
Symmetric interreduction
[1/1] >
Symmetrise 1 polynomials at level 4
Apply permutations
>
:>
:>
>
:>
:>
Symmetric interreduction
[1/2] >
[2/2] :>
-> 0
[1/1] >
Symmetric interreduction
[1/1] >
[x_1]
```

```
>>> from sage.all import *
>>> I1.groebner_basis(report=True, reduced=True)                    #␣
↪needs sage.combinat
Symmetric interreduction
[1/2] >
[2/2] :>
[1/2] >
[2/2] >
Symmetrise 2 polynomials at level 2
Apply permutations
>
>
Symmetric interreduction
```

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```
[1/3] >
[2/3] >
[3/3] >
Apply permutations
>
:>
:>
Symmetric interreduction
[1/3] >
[2/3] >
[3/3] >
Symmetrisation done
Classical Groebner basis
-> 2 generators
Symmetric interreduction
[1/2] >
[2/2] >
Symmetrise 2 polynomials at level 3
Apply permutations
>
>
:>
::>
:>
::>
Symmetric interreduction
[1/4] >
[2/4] :>
-> 0
[3/4] ::>
-> 0
[4/4] :>
-> 0
[1/1] >
Apply permutations
:>
:>
:>
Symmetric interreduction
[1/1] >
Classical Groebner basis
-> 1 generators
Symmetric interreduction
[1/1] >
Symmetrise 1 polynomials at level 4
Apply permutations
>
:>
:>
>
:>
:>
```

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```
Symmetric interreduction
[1/2] >
[2/2] :>
-> 0
[1/1] >
Symmetric interreduction
[1/1] >
[x_1]
```

The Aschenbrenner-Hillar algorithm is only guaranteed to work if the base ring is a field. So, we raise a `TypeError` if this is not the case:

```
sage: R.<x,y> = InfinitePolynomialRing(ZZ)
sage: I = R * [x[1] + x[2], y[1]]
sage: I.groebner_basis()                                                    #␣
↪needs sage.combinat
Traceback (most recent call last):
...
TypeError: The base ring (= Integer Ring) must be a field
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(ZZ, names=('x', 'y',)); (x, y,) = R._first_
↪ngens(2)
>>> I = R * [x[Integer(1)] + x[Integer(2)], y[Integer(1)]]
>>> I.groebner_basis()                                                    #␣
↪needs sage.combinat
Traceback (most recent call last):
...
TypeError: The base ring (= Integer Ring) must be a field
```

interreduced_basis()

A fully symmetrically reduced generating set (type `Sequence`) of `self`.

This does essentially the same as *interreduction()* with the option 'tailreduce', but it returns a `Sequence` rather than a *SymmetricIdeal*.

EXAMPLES:

```
sage: X.<x> = InfinitePolynomialRing(QQ)
sage: I = X * (x[1] + x[2], x[1]*x[2])
sage: I.interreduced_basis()                                                #␣
↪needs sage.combinat
[-x_1^2, x_2 + x_1]
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x',)); (x,) = X._first_ngens(1)
>>> I = X * (x[Integer(1)] + x[Integer(2)], x[Integer(1)]*x[Integer(2)])
>>> I.interreduced_basis()                                                #␣
↪needs sage.combinat
[-x_1^2, x_2 + x_1]
```

interreduction(*tailreduce=True*, *sorted=False*, *report=None*, *RStrat=None*)

Return symmetrically interreduced form of `self`.

INPUT:

- `tailreduce` – boolean (default: `True`); if `True`, the interreduction is also performed on the non-leading monomials.
- `sorted` – boolean (default: `False`); if `True`, it is assumed that the generators of `self` are already increasingly sorted.
- `report` – object (default `None`); if not `None`, some information on the progress of computation is printed
- `RStrat` – (*SymmetricReductionStrategy*, default `None`) A reduction strategy to which the polynomials resulting from the interreduction will be added. If `RStrat` already contains some polynomials, they will be used in the interreduction. The effect is to compute in a quotient ring.

OUTPUT:

A Symmetric Ideal J (sorted list of generators) coinciding with `self` as an ideal, so that any generator is symmetrically reduced w.r.t. the other generators. Note that the leading coefficients of the result are not necessarily 1.

EXAMPLES:

```
sage: X.<x> = InfinitePolynomialRing(QQ)
sage: I = X * (x[1] + x[2], x[1]*x[2])
sage: I.interreduction()                                                    #␣
↪needs sage.combinat
Symmetric Ideal (-x_1^2, x_2 + x_1) of
 Infinite polynomial ring in x over Rational Field
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x',)); (x,) = X._first_ngens(1)
>>> I = X * (x[Integer(1)] + x[Integer(2)], x[Integer(1)]*x[Integer(2)])
>>> I.interreduction()                                                    #␣
↪needs sage.combinat
Symmetric Ideal (-x_1^2, x_2 + x_1) of
 Infinite polynomial ring in x over Rational Field
```

Here, we show the `report` option:

```
sage: I.interreduction(report=True)                                         #␣
↪needs sage.combinat
Symmetric interreduction
[1/2]  >
[2/2] :>
[1/2]  >
[2/2] T[1]>
>
Symmetric Ideal (-x_1^2, x_2 + x_1) of
 Infinite polynomial ring in x over Rational Field
```

```
>>> from sage.all import *
>>> I.interreduction(report=True)                                         #␣
↪needs sage.combinat
Symmetric interreduction
[1/2]  >
[2/2] :>
```

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```
[1/2]  >
[2/2] T[1]>
>
Symmetric Ideal (-x_1^2, x_2 + x_1) of
 Infinite polynomial ring in x over Rational Field
```

[m/n] indicates that polynomial number m is considered and the total number of polynomials under consideration is n. '-> 0' is printed if a zero reduction occurred. The rest of the report is as described in *sage.rings.polynomial.symmetric_reduction.SymmetricReductionStrategy.reduce()*.

Last, we demonstrate the use of the optional parameter RStrat:

```
sage: from sage.rings.polynomial.symmetric_reduction import␣
↪SymmetricReductionStrategy
sage: R = SymmetricReductionStrategy(X); R
Symmetric Reduction Strategy in
 Infinite polynomial ring in x over Rational Field
sage: I.interreduction(RStrat=R)                                    #␣
↪needs sage.combinat
Symmetric Ideal (-x_1^2, x_2 + x_1) of
 Infinite polynomial ring in x over Rational Field
sage: R                                                             #␣
↪needs sage.combinat
Symmetric Reduction Strategy in
 Infinite polynomial ring in x over Rational Field, modulo
    x_1^2,
    x_2 + x_1
sage: R = SymmetricReductionStrategy(X, [x[1]^2])
sage: I.interreduction(RStrat=R)                                    #␣
↪needs sage.combinat
Symmetric Ideal (x_2 + x_1) of Infinite polynomial ring in x over Rational␣
↪Field
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.symmetric_reduction import␣
↪SymmetricReductionStrategy
>>> R = SymmetricReductionStrategy(X); R
Symmetric Reduction Strategy in
 Infinite polynomial ring in x over Rational Field
>>> I.interreduction(RStrat=R)                                     #␣
↪needs sage.combinat
Symmetric Ideal (-x_1^2, x_2 + x_1) of
 Infinite polynomial ring in x over Rational Field
>>> R                                                              #␣
↪needs sage.combinat
Symmetric Reduction Strategy in
 Infinite polynomial ring in x over Rational Field, modulo
    x_1^2,
    x_2 + x_1
>>> R = SymmetricReductionStrategy(X, [x[Integer(1)]**Integer(2)])
>>> I.interreduction(RStrat=R)                                     #␣
↪needs sage.combinat
```

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```
Symmetric Ideal (x_2 + x_1) of Infinite polynomial ring in x over Rational␣
↪Field
```

is_maximal()

Answer whether `self` is a maximal ideal.

ASSUMPTION:

`self` is defined by a symmetric Groebner basis.

NOTE:

It is not checked whether `self` is in fact a symmetric Groebner basis. A wrong answer can result if this assumption does not hold. A `NotImplementedError` is raised if the base ring is not a field, since symmetric Groebner bases are not implemented in this setting.

EXAMPLES:

```
sage: R.<x,y> = InfinitePolynomialRing(QQ)
sage: I = R.ideal([x[1] + y[2], x[2] - y[1]])
sage: I = R * I.groebner_basis(); I                                         #␣
↪needs sage.combinat
Symmetric Ideal (y_1, x_1) of
 Infinite polynomial ring in x, y over Rational Field
sage: I = R.ideal([x[1] + y[2], x[2] - y[1]])                               #␣
↪needs sage.combinat
sage: I.is_maximal()                                                        #␣
↪needs sage.combinat
False
```

```
>>> from sage.all import *
>>> R = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = R._first_
↪ngens(2)
>>> I = R.ideal([x[Integer(1)] + y[Integer(2)], x[Integer(2)] -␣
↪y[Integer(1)]])
>>> I = R * I.groebner_basis(); I                                           #␣
↪needs sage.combinat
Symmetric Ideal (y_1, x_1) of
 Infinite polynomial ring in x, y over Rational Field
>>> I = R.ideal([x[Integer(1)] + y[Integer(2)], x[Integer(2)] -␣
↪y[Integer(1)]])                               # needs sage.combinat
>>> I.is_maximal()                                                          #␣
↪needs sage.combinat
False
```

The preceding answer is wrong, since it is not the case that `I` is given by a symmetric Groebner basis:

```
sage: I = R * I.groebner_basis(); I                                         #␣
↪needs sage.combinat
Symmetric Ideal (y_1, x_1) of
 Infinite polynomial ring in x, y over Rational Field
sage: I.is_maximal()                                                        #␣
↪needs sage.combinat
True
```

```
>>> from sage.all import *
>>> I = R * I.groebner_basis(); I                                    #␣
↪needs sage.combinat
Symmetric Ideal (y_1, x_1) of
 Infinite polynomial ring in x, y over Rational Field
>>> I.is_maximal()                                                   #␣
↪needs sage.combinat
True
```

normalisation()

Return an ideal that coincides with `self`, so that all generators have leading coefficient 1.

Possibly occurring zeroes are removed from the generator list.

EXAMPLES:

```
sage: X.<x> = InfinitePolynomialRing(QQ)
sage: I = X*(1/2*x[1] + 2/3*x[2], 0, 4/5*x[1]*x[2])
sage: I.normalisation()
Symmetric Ideal (x_2 + 3/4*x_1, x_2*x_1) of
 Infinite polynomial ring in x over Rational Field
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x',)); (x,) = X._first_ngens(1)
>>> I = X*(Integer(1)/Integer(2)*x[Integer(1)] + Integer(2)/
↪Integer(3)*x[Integer(2)], Integer(0), Integer(4)/
↪Integer(5)*x[Integer(1)]*x[Integer(2)])
>>> I.normalisation()
Symmetric Ideal (x_2 + 3/4*x_1, x_2*x_1) of
 Infinite polynomial ring in x over Rational Field
```

reduce(*I*, *tailreduce=False*)

Symmetric reduction of `self` by another Symmetric Ideal or list of Infinite Polynomials, or symmetric reduction of a given Infinite Polynomial by `self`.

INPUT:

- `I` – an Infinite Polynomial, or a Symmetric Ideal or a list of Infinite Polynomials
- `tailreduce` – boolean (default `False`); if `True`, the non-leading terms will be reduced as well

OUTPUT: symmetric reduction of `self` with respect to `I`

THEORY:

Reduction of an element p of an Infinite Polynomial Ring X by some other element q means the following:

1. Let M and N be the leading terms of p and q.
2. Test whether there is a permutation P that does not does not diminish the variable indices occurring in N and preserves their order, so that there is some term $T \in X$ with $TN^P = M$. If there is no such permutation, return p
3. Replace p by $p - Tq^P$ and continue with step 1.

EXAMPLES:

```
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: I = X * (y[1]^2*y[3] + y[1]*x[3]^2)
sage: I.reduce([x[1]^2*y[2]])
Symmetric Ideal (x_3^2*y_1 + y_3*y_1^2) of
 Infinite polynomial ring in x, y over Rational Field
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = X._first_
↪ngens(2)
>>> I = X * (y[Integer(1)]**Integer(2)*y[Integer(3)] +␣
↪y[Integer(1)]*x[Integer(3)]**Integer(2))
>>> I.reduce([x[Integer(1)]**Integer(2)*y[Integer(2)]])
Symmetric Ideal (x_3^2*y_1 + y_3*y_1^2) of
 Infinite polynomial ring in x, y over Rational Field
```

The preceding is correct, since any permutation that turns `x[1]^2*y[2]` into a factor of `x[3]^2*y[2]` interchanges the variable indices 1 and 2 – which is not allowed. However, reduction by `x[2]^2*y[1]` works, since one can change variable index 1 into 2 and 2 into 3:

```
sage: I.reduce([x[2]^2*y[1]])                                          #␣
↪needs sage.combinat
Symmetric Ideal (y_3*y_1^2) of
 Infinite polynomial ring in x, y over Rational Field
```

```
>>> from sage.all import *
>>> I.reduce([x[Integer(2)]**Integer(2)*y[Integer(1)]])                ␣
↪                         # needs sage.combinat
Symmetric Ideal (y_3*y_1^2) of
 Infinite polynomial ring in x, y over Rational Field
```

The next example shows that tail reduction is not done, unless it is explicitly advised. The input can also be a symmetric ideal:

```
sage: J = (y[2]) * X
sage: I.reduce(J)
Symmetric Ideal (x_3^2*y_1 + y_3*y_1^2) of
 Infinite polynomial ring in x, y over Rational Field
sage: I.reduce(J, tailreduce=True)                                     #␣
↪needs sage.combinat
Symmetric Ideal (x_3^2*y_1) of
 Infinite polynomial ring in x, y over Rational Field
```

```
>>> from sage.all import *
>>> J = (y[Integer(2)]) * X
>>> I.reduce(J)
Symmetric Ideal (x_3^2*y_1 + y_3*y_1^2) of
 Infinite polynomial ring in x, y over Rational Field
>>> I.reduce(J, tailreduce=True)                                     #␣
↪needs sage.combinat
Symmetric Ideal (x_3^2*y_1) of
 Infinite polynomial ring in x, y over Rational Field
```

squeezed()

Reduce the variable indices occurring in `self`.

OUTPUT:

A Symmetric Ideal whose generators are the result of applying `squeezed()` to the generators of `self`.

NOTE:

The output describes the same Symmetric Ideal as `self`.

EXAMPLES:

```
sage: X.<x,y> = InfinitePolynomialRing(QQ, implementation='sparse')
sage: I = X * (x[1000]*y[100], x[50]*y[1000])
sage: I.squeezed()
Symmetric Ideal (x_2*y_1, x_1*y_2) of
 Infinite polynomial ring in x, y over Rational Field
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, implementation='sparse', names=('x', 'y',
↪)); (x, y,) = X._first_ngens(2)
>>> I = X * (x[Integer(1000)]*y[Integer(100)],␣
↪x[Integer(50)]*y[Integer(1000)])
>>> I.squeezed()
Symmetric Ideal (x_2*y_1, x_1*y_2) of
 Infinite polynomial ring in x, y over Rational Field
```

symmetric_basis()

A symmetrised generating set (type `Sequence`) of `self`.

This does essentially the same as *symmetrisation()* with the option `tailreduce`, and it returns a `Sequence` rather than a *SymmetricIdeal*.

EXAMPLES:

```
sage: X.<x> = InfinitePolynomialRing(QQ)
sage: I = X * (x[1] + x[2], x[1]*x[2])
sage: I.symmetric_basis()                                              #␣
↪needs sage.combinat
[x_2^2, x_1^2, x_2 + x_1]
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x',)); (x,) = X._first_ngens(1)
>>> I = X * (x[Integer(1)] + x[Integer(2)], x[Integer(1)]*x[Integer(2)])
>>> I.symmetric_basis()                                              #␣
↪needs sage.combinat
[x_2^2, x_1^2, x_2 + x_1]
```

symmetrisation(*N=None*, *tailreduce=False*, *report=None*, *use_full_group=False*)

Apply permutations to the generators of `self` and interreduce.

INPUT:

- `N` – integer (default: `None`); apply permutations in $Sym(N)$. If it is not given then it will be replaced by the maximal variable index occurring in the generators of `self.interreduction().squeezed()`.
- `tailreduce` – boolean (default: `False`); if `True`, perform tail reductions.

- `report` – object (default `None`); if not `None`, report on the progress of computations.
- `use_full_group` – (optional) if `True`, apply *all* elements of $Sym(N)$ to the generators of `self` (this is what [AB2008] originally suggests). The default is to apply all elementary transpositions to the generators of `self.squeezed()`, interreduce, and repeat until the result stabilises, which is often much faster than applying all of $Sym(N)$, and we are convinced that both methods yield the same result.

OUTPUT:

A symmetrically interreduced symmetric ideal with respect to which any $Sym(N)$-translate of a generator of `self` is symmetrically reducible, where by default `N` is the maximal variable index that occurs in the generators of `self.interreduction().squeezed()`.

NOTE:

If `I` is a symmetric ideal whose generators are monomials, then `I.symmetrisation()` is its reduced Groebner basis. It should be noted that without symmetrisation, monomial generators, in general, do not form a Groebner basis.

EXAMPLES:

```
sage: X.<x> = InfinitePolynomialRing(QQ)
sage: I = X * (x[1] + x[2], x[1]*x[2])
sage: I.symmetrisation()                                                    #␣
↪needs sage.combinat
Symmetric Ideal (-x_2^2, -x_1^2, x_2 + x_1) of Infinite polynomial ring in x
over Rational Field
sage: I.symmetrisation(N=3)                                                 #␣
↪needs sage.combinat
Symmetric Ideal (-x_2^2, -x_1^2, -2*x_1) of Infinite polynomial ring in x
over Rational Field
sage: I.symmetrisation(N=3, use_full_group=True)                            #␣
↪needs sage.combinat
Symmetric Ideal (-x_1^2, -2*x_1) of Infinite polynomial ring in x
over Rational Field
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x',)); (x,) = X._first_ngens(1)
>>> I = X * (x[Integer(1)] + x[Integer(2)], x[Integer(1)]*x[Integer(2)])
>>> I.symmetrisation()                                                    #␣
↪needs sage.combinat
Symmetric Ideal (-x_2^2, -x_1^2, x_2 + x_1) of Infinite polynomial ring in x
over Rational Field
>>> I.symmetrisation(N=Integer(3))                                          ␣
↪     # needs sage.combinat
Symmetric Ideal (-x_2^2, -x_1^2, -2*x_1) of Infinite polynomial ring in x
over Rational Field
>>> I.symmetrisation(N=Integer(3), use_full_group=True)                     ␣
↪     # needs sage.combinat
Symmetric Ideal (-x_1^2, -2*x_1) of Infinite polynomial ring in x
over Rational Field
```

6.4 Symmetric Reduction of Infinite Polynomials

SymmetricReductionStrategy provides a framework for efficient symmetric reduction of Infinite Polynomials, see *infinite_polynomial_element*.

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THEORY:

According to M. Aschenbrenner and C. Hillar [AB2007], Symmetric Reduction of an element p of an Infinite Polynomial Ring X by some other element q means the following:

1. Let M and N be the leading terms of p and q.
2. Test whether there is a permutation P that does not diminish the variable indices occurring in N and preserves their order, so that there is some term $T \in X$ with $TN^P = M$. If there is no such permutation, return p.
3. Replace p by $p - Tq^P$ and continue with step 1.

When reducing one polynomial p with respect to a list L of other polynomials, there usually is a choice of order on which the efficiency crucially depends. Also it helps to modify the polynomials on the list in order to simplify the basic reduction steps.

The preparation of L may be expensive. Hence, if the same list is used many times then it is reasonable to perform the preparation only once. This is the background of *SymmetricReductionStrategy*.

Our current strategy is to keep the number of terms in the polynomials as small as possible. For this, we sort L by increasing number of terms. If several elements of L allow for a reduction of p, we choose the one with the smallest number of terms. Later on, it should be possible to implement further strategies for choice.

When adding a new polynomial q to L, we first reduce q with respect to L. Then, we test heuristically whether it is possible to reduce the number of terms of the elements of L by reduction modulo q. That way, we see best chances to keep the number of terms in intermediate reduction steps relatively small.

EXAMPLES:

First, we create an infinite polynomial ring and one of its elements:

```
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: p = y[1]*y[3] + y[1]^2*x[3]
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = X._first_ngens(2)
>>> p = y[Integer(1)]*y[Integer(3)] + y[Integer(1)]**Integer(2)*x[Integer(3)]
```

We want to symmetrically reduce it by another polynomial. So, we put this other polynomial into a list and create a Symmetric Reduction Strategy object:

```
sage: from sage.rings.polynomial.symmetric_reduction import SymmetricReductionStrategy
sage: S = SymmetricReductionStrategy(X, [y[2]^2*x[1]])
sage: S
Symmetric Reduction Strategy in
 Infinite polynomial ring in x, y over Rational Field, modulo
    x_1*y_2^2
sage: S.reduce(p)
x_3*y_1^2 + y_3*y_1
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.symmetric_reduction import SymmetricReductionStrategy
>>> S = SymmetricReductionStrategy(X, [y[Integer(2)]**Integer(2)*x[Integer(1)]])
>>> S
Symmetric Reduction Strategy in
 Infinite polynomial ring in x, y over Rational Field, modulo
    x_1*y_2^2
>>> S.reduce(p)
x_3*y_1^2 + y_3*y_1
```

The preceding is correct, since any permutation that turns `y[2]^2*x[1]` into a factor of `y[1]^2*x[3]` interchanges the variable indices 1 and 2 – which is not allowed in a symmetric reduction. However, reduction by `y[1]^2*x[2]` works, since one can change variable index 1 into 2 and 2 into 3. So, we add this to `S`:

```
sage: S.add_generator(y[1]^2*x[2])
sage: S
Symmetric Reduction Strategy in
 Infinite polynomial ring in x, y over Rational Field, modulo
    x_2*y_1^2,
    x_1*y_2^2
sage: S.reduce(p)                                                        #␣
↪needs sage.combinat
y_3*y_1
```

```
>>> from sage.all import *
>>> S.add_generator(y[Integer(1)]**Integer(2)*x[Integer(2)])
>>> S
Symmetric Reduction Strategy in
 Infinite polynomial ring in x, y over Rational Field, modulo
    x_2*y_1^2,
    x_1*y_2^2
>>> S.reduce(p)                                                        #␣
↪needs sage.combinat
y_3*y_1
```

The next example shows that tail reduction is not done, unless it is explicitly advised:

```
sage: S.reduce(x[3] + 2*x[2]*y[1]^2 + 3*y[2]^2*x[1])                     #␣
↪needs sage.combinat
x_3 + 2*x_2*y_1^2 + 3*x_1*y_2^2
sage: S.tailreduce(x[3] + 2*x[2]*y[1]^2 + 3*y[2]^2*x[1])                 #␣
↪needs sage.combinat
x_3
```

```
>>> from sage.all import *
>>> S.reduce(x[Integer(3)] + Integer(2)*x[Integer(2)]*y[Integer(1)]**Integer(2) +␣
↪Integer(3)*y[Integer(2)]**Integer(2)*x[Integer(1)])
↪# needs sage.combinat
x_3 + 2*x_2*y_1^2 + 3*x_1*y_2^2
>>> S.tailreduce(x[Integer(3)] + Integer(2)*x[Integer(2)]*y[Integer(1)]**Integer(2) +␣
↪Integer(3)*y[Integer(2)]**Integer(2)*x[Integer(1)])                   #␣
↪needs sage.combinat
x_3
```

However, it is possible to ask for tailreduction already when the Symmetric Reduction Strategy is created:

```
sage: S2 = SymmetricReductionStrategy(X, [y[2]^2*x[1],y[1]^2*x[2]], tailreduce=True)
sage: S2
Symmetric Reduction Strategy in
 Infinite polynomial ring in x, y over Rational Field, modulo
    x_2*y_1^2,
    x_1*y_2^2
with tailreduction
sage: S2.reduce(x[3] + 2*x[2]*y[1]^2 + 3*y[2]^2*x[1])                          #␣
↪needs sage.combinat
x_3
```

```
>>> from sage.all import *
>>> S2 = SymmetricReductionStrategy(X, [y[Integer(2)]**Integer(2)*x[Integer(1)],
↪y[Integer(1)]**Integer(2)*x[Integer(2)]], tailreduce=True)
>>> S2
Symmetric Reduction Strategy in
 Infinite polynomial ring in x, y over Rational Field, modulo
    x_2*y_1^2,
    x_1*y_2^2
with tailreduction
>>> S2.reduce(x[Integer(3)] + Integer(2)*x[Integer(2)]*y[Integer(1)]**Integer(2) +␣
↪Integer(3)*y[Integer(2)]**Integer(2)*x[Integer(1)])                          #␣
↪needs sage.combinat
x_3
```

class sage.rings.polynomial.symmetric_reduction.**SymmetricReductionStrategy**

Bases: `object`

A framework for efficient symmetric reduction of InfinitePolynomial, see *infinite_polynomial_element*.

INPUT:

- `Parent` – an Infinite Polynomial Ring, see *infinite_polynomial_element*
- `L` – list (default: the empty list); list of elements of `Parent` with respect to which will be reduced
- `good_input` – boolean (default: `None`); if this optional parameter is true, it is assumed that each element of `L` is symmetrically reduced with respect to the previous elements of `L`

EXAMPLES:

```
sage: X.<y> = InfinitePolynomialRing(QQ)
sage: from sage.rings.polynomial.symmetric_reduction import␣
↪SymmetricReductionStrategy
sage: S = SymmetricReductionStrategy(X, [y[2]^2*y[1],y[1]^2*y[2]], good_
↪input=True)
sage: S.reduce(y[3] + 2*y[2]*y[1]^2 + 3*y[2]^2*y[1])
y_3 + 3*y_2^2*y_1 + 2*y_2*y_1^2
sage: S.tailreduce(y[3] + 2*y[2]*y[1]^2 + 3*y[2]^2*y[1])                       #␣
↪needs sage.combinat
y_3
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('y',)); (y,) = X._first_ngens(1)
```

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```
>>> from sage.rings.polynomial.symmetric_reduction import␣
↪SymmetricReductionStrategy
>>> S = SymmetricReductionStrategy(X, [y[Integer(2)]**Integer(2)*y[Integer(1)],
↪y[Integer(1)]**Integer(2)*y[Integer(2)]], good_input=True)
>>> S.reduce(y[Integer(3)] + Integer(2)*y[Integer(2)]*y[Integer(1)]**Integer(2) +␣
↪Integer(3)*y[Integer(2)]**Integer(2)*y[Integer(1)])
y_3 + 3*y_2^2*y_1 + 2*y_2*y_1^2
>>> S.tailreduce(y[Integer(3)] +␣
↪Integer(2)*y[Integer(2)]*y[Integer(1)]**Integer(2) +␣
↪Integer(3)*y[Integer(2)]**Integer(2)*y[Integer(1)])                    #␣
↪needs sage.combinat
y_3
```

add_generator(*p*, *good_input=None*)

Add another polynomial to `self`.

INPUT:

- `p` – an element of the underlying infinite polynomial ring
- `good_input` – boolean (default: `None`); if `True`, it is assumed that `p` is reduced with respect to `self`. Otherwise, this reduction will be done first (which may cost some time).

Note

Previously added polynomials may be modified. All input is prepared in view of an efficient symmetric reduction.

EXAMPLES:

```
sage: from sage.rings.polynomial.symmetric_reduction import␣
↪SymmetricReductionStrategy
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: S = SymmetricReductionStrategy(X)
sage: S
Symmetric Reduction Strategy in
 Infinite polynomial ring in x, y over Rational Field
sage: S.add_generator(y[3] + y[1]*(x[3]+x[1]))
sage: S
Symmetric Reduction Strategy in
 Infinite polynomial ring in x, y over Rational Field, modulo
    x_3*y_1 + x_1*y_1 + y_3
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.symmetric_reduction import␣
↪SymmetricReductionStrategy
>>> X = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = X._first_
↪ngens(2)
>>> S = SymmetricReductionStrategy(X)
>>> S
Symmetric Reduction Strategy in
 Infinite polynomial ring in x, y over Rational Field
```

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```
>>> S.add_generator(y[Integer(3)] +␣
↪y[Integer(1)]*(x[Integer(3)]+x[Integer(1)]))
>>> S
Symmetric Reduction Strategy in
 Infinite polynomial ring in x, y over Rational Field, modulo
    x_3*y_1 + x_1*y_1 + y_3
```

Note that the first added polynomial will be simplified when adding a suitable second polynomial:

```
sage: S.add_generator(x[2] + x[1])                                      #␣
↪needs sage.combinat
sage: S                                                                 #␣
↪needs sage.combinat
Symmetric Reduction Strategy in
 Infinite polynomial ring in x, y over Rational Field, modulo
    y_3,
    x_2 + x_1
```

```
>>> from sage.all import *
>>> S.add_generator(x[Integer(2)] + x[Integer(1)])                       ␣
↪              # needs sage.combinat
>>> S                                                                  #␣
↪needs sage.combinat
Symmetric Reduction Strategy in
 Infinite polynomial ring in x, y over Rational Field, modulo
    y_3,
    x_2 + x_1
```

By default, reduction is applied to any newly added polynomial. This can be avoided by specifying the optional parameter 'good_input':

```
sage: # needs sage.combinat
sage: S.add_generator(y[2] + y[1]*x[2])
sage: S
Symmetric Reduction Strategy in
 Infinite polynomial ring in x, y over Rational Field, modulo
    y_3,
    x_1*y_1 - y_2,
    x_2 + x_1
sage: S.reduce(x[3] + x[2])
-2*x_1
sage: S.add_generator(x[3] + x[2], good_input=True)
sage: S
Symmetric Reduction Strategy in
 Infinite polynomial ring in x, y over Rational Field, modulo
    y_3,
    x_3 + x_2,
    x_1*y_1 - y_2,
    x_2 + x_1
```

```
>>> from sage.all import *
>>> # needs sage.combinat
```

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```
>>> S.add_generator(y[Integer(2)] + y[Integer(1)]*x[Integer(2)])
>>> S
Symmetric Reduction Strategy in
 Infinite polynomial ring in x, y over Rational Field, modulo
    y_3,
    x_1*y_1 - y_2,
    x_2 + x_1
>>> S.reduce(x[Integer(3)] + x[Integer(2)])
-2*x_1
>>> S.add_generator(x[Integer(3)] + x[Integer(2)], good_input=True)
>>> S
Symmetric Reduction Strategy in
 Infinite polynomial ring in x, y over Rational Field, modulo
    y_3,
    x_3 + x_2,
    x_1*y_1 - y_2,
    x_2 + x_1
```

In the previous example, `x[3] + x[2]` is added without being reduced to zero.

gens()

Return the tuple of Infinite Polynomials modulo which `self` reduces.

EXAMPLES:

```
sage: X.<y> = InfinitePolynomialRing(QQ)
sage: from sage.rings.polynomial.symmetric_reduction import␣
↪SymmetricReductionStrategy
sage: S = SymmetricReductionStrategy(X, [y[2]^2*y[1],y[1]^2*y[2]])
sage: S
Symmetric Reduction Strategy in
 Infinite polynomial ring in y over Rational Field, modulo
    y_2*y_1^2,
    y_2^2*y_1
sage: S.gens()
(y_2*y_1^2, y_2^2*y_1)
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('y',)); (y,) = X._first_ngens(1)
>>> from sage.rings.polynomial.symmetric_reduction import␣
↪SymmetricReductionStrategy
>>> S = SymmetricReductionStrategy(X,␣
↪[y[Integer(2)]**Integer(2)*y[Integer(1)],
↪y[Integer(1)]**Integer(2)*y[Integer(2)]])
>>> S
Symmetric Reduction Strategy in
 Infinite polynomial ring in y over Rational Field, modulo
    y_2*y_1^2,
    y_2^2*y_1
>>> S.gens()
(y_2*y_1^2, y_2^2*y_1)
```

reduce(*p*, *notail=False*, *report=None*)

Symmetric reduction of an infinite polynomial.

INPUT:

- `p` – an element of the underlying infinite polynomial ring
- `notail` – boolean (default: `False`); if `True`, tail reduction is avoided (but there is no guarantee that there will be no tail reduction at all)
- `report` – object (default: `None`); if not `None`, print information on the progress of the computation

OUTPUT: reduction of `p` with respect to `self`

Note

If tail reduction shall be forced, use *tailreduce()*.

EXAMPLES:

```
sage: from sage.rings.polynomial.symmetric_reduction import␣
↪SymmetricReductionStrategy
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: S = SymmetricReductionStrategy(X, [y[3]], tailreduce=True)
sage: S.reduce(y[4]*x[1] + y[1]*x[4])
x_4*y_1
sage: S.reduce(y[4]*x[1] + y[1]*x[4], notail=True)
x_4*y_1 + x_1*y_4
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.symmetric_reduction import␣
↪SymmetricReductionStrategy
>>> X = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = X._first_
↪ngens(2)
>>> S = SymmetricReductionStrategy(X, [y[Integer(3)]], tailreduce=True)
>>> S.reduce(y[Integer(4)]*x[Integer(1)] + y[Integer(1)]*x[Integer(4)])
x_4*y_1
>>> S.reduce(y[Integer(4)]*x[Integer(1)] + y[Integer(1)]*x[Integer(4)],␣
↪notail=True)
x_4*y_1 + x_1*y_4
```

Last, we demonstrate the `report` option:

```
sage: S = SymmetricReductionStrategy(X, [x[2] + y[1],
....:                                        x[2]*y[3] + x[1]*y[2] + y[4],
....:                                        y[3] + y[2]])
sage: S
Symmetric Reduction Strategy in
 Infinite polynomial ring in x, y over Rational Field, modulo
    y_3 + y_2,
    x_2 + y_1,
    x_1*y_2 + y_4 - y_3*y_1
sage: S.reduce(x[3] + x[1]*y[3] + x[1]*y[1], report=True)
:::>
x_1*y_1 + y_4 - y_3*y_1 - y_1
```

```
>>> from sage.all import *
>>> S = SymmetricReductionStrategy(X, [x[Integer(2)] + y[Integer(1)],
...                                    x[Integer(2)]*y[Integer(3)] +␣
↪x[Integer(1)]*y[Integer(2)] + y[Integer(4)],
...                                    y[Integer(3)] + y[Integer(2)]])
>>> S
Symmetric Reduction Strategy in
 Infinite polynomial ring in x, y over Rational Field, modulo
    y_3 + y_2,
    x_2 + y_1,
    x_1*y_2 + y_4 - y_3*y_1
>>> S.reduce(x[Integer(3)] + x[Integer(1)]*y[Integer(3)] +␣
↪x[Integer(1)]*y[Integer(1)], report=True)
:::>
x_1*y_1 + y_4 - y_3*y_1 - y_1
```

Each ':' indicates that one reduction of the leading monomial was performed. Eventually, the '>' indicates that the computation is finished.

reset()

Remove all polynomials from `self`.

EXAMPLES:

```
sage: X.<y> = InfinitePolynomialRing(QQ)
sage: from sage.rings.polynomial.symmetric_reduction import␣
↪SymmetricReductionStrategy
sage: S = SymmetricReductionStrategy(X, [y[2]^2*y[1],y[1]^2*y[2]])
sage: S
Symmetric Reduction Strategy in
 Infinite polynomial ring in y over Rational Field, modulo
    y_2*y_1^2,
    y_2^2*y_1
sage: S.reset()
sage: S
Symmetric Reduction Strategy in Infinite polynomial ring in y over Rational␣
↪Field
```

```
>>> from sage.all import *
>>> X = InfinitePolynomialRing(QQ, names=('y',)); (y,) = X._first_ngens(1)
>>> from sage.rings.polynomial.symmetric_reduction import␣
↪SymmetricReductionStrategy
>>> S = SymmetricReductionStrategy(X,␣
↪[y[Integer(2)]**Integer(2)*y[Integer(1)],
↪y[Integer(1)]**Integer(2)*y[Integer(2)]])
>>> S
Symmetric Reduction Strategy in
 Infinite polynomial ring in y over Rational Field, modulo
    y_2*y_1^2,
    y_2^2*y_1
>>> S.reset()
>>> S
Symmetric Reduction Strategy in Infinite polynomial ring in y over Rational␣
↪Field
```

setgens(*L*)

Define the list of Infinite Polynomials modulo which `self` reduces.

INPUT:

- `L` – list of elements of the underlying infinite polynomial ring

> **Note**
>
> It is not tested if `L` is a good input. That method simply assigns a *copy* of `L` to the generators of `self`.

EXAMPLES:

```
sage: from sage.rings.polynomial.symmetric_reduction import SymmetricReductionStrategy
sage: X.<y> = InfinitePolynomialRing(QQ)
sage: S = SymmetricReductionStrategy(X, [y[2]^2*y[1],y[1]^2*y[2]])
sage: R = SymmetricReductionStrategy(X)
sage: R.setgens(S.gens())
sage: R
Symmetric Reduction Strategy in
 Infinite polynomial ring in y over Rational Field, modulo
    y_2*y_1^2,
    y_2^2*y_1
sage: R.gens() is S.gens()
False
sage: R.gens() == S.gens()
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.symmetric_reduction import SymmetricReductionStrategy
>>> X = InfinitePolynomialRing(QQ, names=('y',)); (y,) = X._first_ngens(1)
>>> S = SymmetricReductionStrategy(X, [y[Integer(2)]**Integer(2)*y[Integer(1)], y[Integer(1)]**Integer(2)*y[Integer(2)]])
>>> R = SymmetricReductionStrategy(X)
>>> R.setgens(S.gens())
>>> R
Symmetric Reduction Strategy in
 Infinite polynomial ring in y over Rational Field, modulo
    y_2*y_1^2,
    y_2^2*y_1
>>> R.gens() is S.gens()
False
>>> R.gens() == S.gens()
True
```

tailreduce(*p*, *report=None*)

Symmetric reduction of an infinite polynomial, with forced tail reduction.

INPUT:

- `p` – an element of the underlying infinite polynomial ring

- `report` – object (default: `None`); if not `None`, print information on the progress of the computation

OUTPUT:

Reduction (including the non-leading elements) of `p` with respect to `self`.

EXAMPLES:

```
sage: from sage.rings.polynomial.symmetric_reduction import␣
↪SymmetricReductionStrategy
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: S = SymmetricReductionStrategy(X, [y[3]])
sage: S.reduce(y[4]*x[1] + y[1]*x[4])
x_4*y_1 + x_1*y_4
sage: S.tailreduce(y[4]*x[1] + y[1]*x[4])                                    #␣
↪needs sage.combinat
x_4*y_1
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.symmetric_reduction import␣
↪SymmetricReductionStrategy
>>> X = InfinitePolynomialRing(QQ, names=('x', 'y',)); (x, y,) = X._first_
↪ngens(2)
>>> S = SymmetricReductionStrategy(X, [y[Integer(3)]])
>>> S.reduce(y[Integer(4)]*x[Integer(1)] + y[Integer(1)]*x[Integer(4)])
x_4*y_1 + x_1*y_4
>>> S.tailreduce(y[Integer(4)]*x[Integer(1)] + y[Integer(1)]*x[Integer(4)])  ␣
↪                                  # needs sage.combinat
x_4*y_1
```

Last, we demonstrate the 'report' option:

```
sage: S = SymmetricReductionStrategy(X, [x[2] + y[1],
....:                                     x[2]*x[3] + x[1]*y[2] + y[4],
....:                                     y[3] + y[2]])
sage: S
Symmetric Reduction Strategy in
 Infinite polynomial ring in x, y over Rational Field, modulo
    y_3 + y_2,
    x_2 + y_1,
    x_1*y_2 + y_4 + y_1^2
sage: S.tailreduce(x[3] + x[1]*y[3] + x[1]*y[1], report=True)                #␣
↪needs sage.combinat
T[3]:::>
T[3]:>
x_1*y_1 - y_2 + y_1^2 - y_1
```

```
>>> from sage.all import *
>>> S = SymmetricReductionStrategy(X, [x[Integer(2)] + y[Integer(1)],
...                                     x[Integer(2)]*x[Integer(3)] +␣
↪x[Integer(1)]*y[Integer(2)] + y[Integer(4)],
...                                     y[Integer(3)] + y[Integer(2)]])
>>> S
Symmetric Reduction Strategy in
 Infinite polynomial ring in x, y over Rational Field, modulo
```

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```
    y_3 + y_2,
    x_2 + y_1,
    x_1*y_2 + y_4 + y_1^2
>>> S.tailreduce(x[Integer(3)] + x[Integer(1)]*y[Integer(3)] +
↪x[Integer(1)]*y[Integer(1)], report=True)                # needs sage.
↪combinat
T[3]:::>
T[3]:>
x_1*y_1 - y_2 + y_1^2 - y_1
```

The protocol means the following.

- 'T[3]' means that we currently do tail reduction for a polynomial with three terms.
- ':::>' means that there were three reductions of leading terms.
- The tail of the result of the preceding reduction still has three terms. One reduction of leading terms was possible, and then the final result was obtained.

CHAPTER

SEVEN

TROPICAL POLYNOMIALS

7.1 Univariate Tropical Polynomials

AUTHORS:

- Verrel Rievaldo Wijaya (2024-06): initial version

EXAMPLES:

```
sage: T = TropicalSemiring(QQ, use_min=False)
sage: R.<x> = PolynomialRing(T)
sage: x.parent()
Univariate Tropical Polynomial Semiring in x over Rational Field
sage: (x + R(3)*x) * (x^2 + x)
3*x^3 + 3*x^2
sage: (x^2 + R(1)*x + R(-1))^2
0*x^4 + 1*x^3 + 2*x^2 + 0*x + (-2)
sage: (x^2 + x + R(0))^4
0*x^8 + 0*x^7 + 0*x^6 + 0*x^5 + 0*x^4 + 0*x^3 + 0*x^2 + 0*x + 0
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=False)
>>> R = PolynomialRing(T, names=('x',)); (x,) = R._first_ngens(1)
>>> x.parent()
Univariate Tropical Polynomial Semiring in x over Rational Field
>>> (x + R(Integer(3))*x) * (x**Integer(2) + x)
3*x^3 + 3*x^2
>>> (x**Integer(2) + R(Integer(1))*x + R(-Integer(1)))**Integer(2)
0*x^4 + 1*x^3 + 2*x^2 + 0*x + (-2)
>>> (x**Integer(2) + x + R(Integer(0)))**Integer(4)
0*x^8 + 0*x^7 + 0*x^6 + 0*x^5 + 0*x^4 + 0*x^3 + 0*x^2 + 0*x + 0
```

REFERENCES:

- [Bru2014]
- [Fil2017]

class sage.rings.semirings.tropical_polynomial.**TropicalPolynomial**(*parent*, *x=None*, *check=True*, *is_gen=False*, *construct=False*)

Bases: *Polynomial_generic_sparse*

A univariate tropical polynomial.

A tropical polynomial is a polynomial whose coefficients come from a tropical semiring. Tropical polynomial induces a function which is piecewise linear, where each segment of the function has a non-negative integer slope. Tropical roots (zeros) of polynomial $P(x)$ is defined as all points x_0 for which the graph of $P(x)$ change its slope. The difference in the slopes of the two pieces adjacent to this root gives the order of the root.

The tropical polynomials are implemented with a sparse format by using a `dict` whose keys are the exponent and values the corresponding coefficients. Each coefficient is a tropical number.

EXAMPLES:

We construct a tropical polynomial semiring by defining a base tropical semiring and then inputting it to *PolynomialRing*:

```
sage: T = TropicalSemiring(QQ, use_min=False)
sage: R.<x> = PolynomialRing(T); R
Univariate Tropical Polynomial Semiring in x over Rational Field
sage: R.0
0*x
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=False)
>>> R = PolynomialRing(T, names=('x',)); (x,) = R._first_ngens(1); R
Univariate Tropical Polynomial Semiring in x over Rational Field
>>> R.gen(0)
0*x
```

One way to construct an element is to provide a list or tuple of coefficients. Another way to define an element is to write a polynomial equation with each coefficient converted to the semiring:

```
sage: p1 = R([1,4,None,0]); p1
0*x^3 + 4*x + 1
sage: p2 = R(1)*x^2 + R(2)*x + R(3); p2
1*x^2 + 2*x + 3
```

```
>>> from sage.all import *
>>> p1 = R([Integer(1),Integer(4),None,Integer(0)]); p1
0*x^3 + 4*x + 1
>>> p2 = R(Integer(1))*x**Integer(2) + R(Integer(2))*x + R(Integer(3)); p2
1*x^2 + 2*x + 3
```

We can do some basic arithmetic operations for these tropical polynomials. Remember that numbers given have to be in the tropical semiring. If not, then it will raise an error:

```
sage: p1 + p2
0*x^3 + 1*x^2 + 4*x + 3
sage: p1 * p2
1*x^5 + 2*x^4 + 5*x^3 + 6*x^2 + 7*x + 4
sage: 2 * p1
Traceback (most recent call last):
...
ArithmeticError: cannot negate any non-infinite element
sage: T(2) * p1
2*x^3 + 6*x + 3
sage: p1(3)
Traceback (most recent call last):
```

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```
...
TypeError: no common canonical parent for objects with parents:
'Tropical semiring over Rational Field' and 'Integer Ring'
sage: p1(T(3))
9
```

```
>>> from sage.all import *
>>> p1 + p2
0*x^3 + 1*x^2 + 4*x + 3
>>> p1 * p2
1*x^5 + 2*x^4 + 5*x^3 + 6*x^2 + 7*x + 4
>>> Integer(2) * p1
Traceback (most recent call last):
...
ArithmeticError: cannot negate any non-infinite element
>>> T(Integer(2)) * p1
2*x^3 + 6*x + 3
>>> p1(Integer(3))
Traceback (most recent call last):
...
TypeError: no common canonical parent for objects with parents:
'Tropical semiring over Rational Field' and 'Integer Ring'
>>> p1(T(Integer(3)))
9
```

We can find all tropical roots of a tropical polynomial, counted with their multiplicities:

```
sage: p1.roots()
[-3, 2, 2]
sage: p2.roots()
[1, 1]
```

```
>>> from sage.all import *
>>> p1.roots()
[-3, 2, 2]
>>> p2.roots()
[1, 1]
```

Even though every tropical polynomials have tropical roots, this does not necessarily means it can be factored into its linear factors:

```
sage: p1.factor()
(0) * (0*x^3 + 4*x + 1)
sage: p2.factor()
(1) * (0*x + 1)^2
```

```
>>> from sage.all import *
>>> p1.factor()
(0) * (0*x^3 + 4*x + 1)
>>> p2.factor()
(1) * (0*x + 1)^2
```

Every tropical polynomial $p(x)$ have a corresponding unique tropical polynomial $\bar{p}(x)$ with the same roots that can be factored. We call $\bar{p}(x)$ the tropical polynomial split form of $p(x)$:

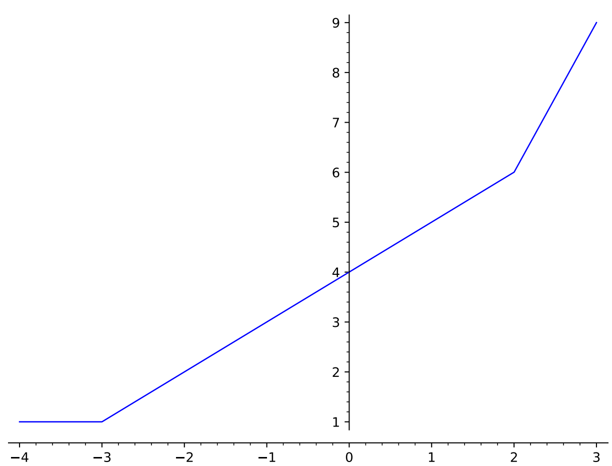
```
sage: p1.split_form()
0*x^3 + 2*x^2 + 4*x + 1
sage: p2.split_form()
1*x^2 + 2*x + 3
```

```
>>> from sage.all import *
>>> p1.split_form()
0*x^3 + 2*x^2 + 4*x + 1
>>> p2.split_form()
1*x^2 + 2*x + 3
```

Every tropical polynomial induce a piecewise linear function that can be invoked in the following way:

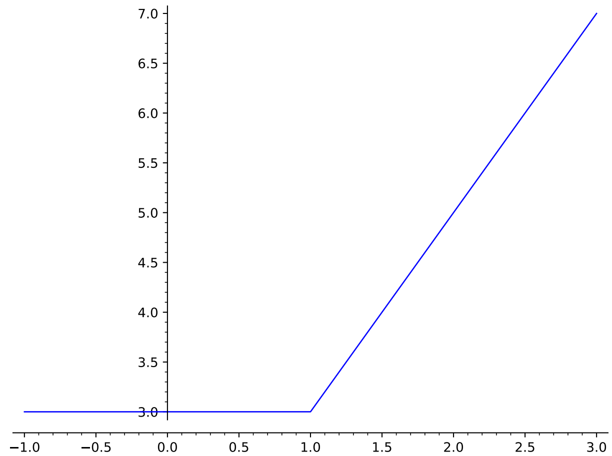
```
sage: p1.piecewise_function()
piecewise(x|-->1 on (-oo, -3], x|-->x + 4 on (-3, 2), x|-->3*x on [2, +oo); x)
sage: p1.plot()
Graphics object consisting of 1 graphics primitive
```

```
>>> from sage.all import *
>>> p1.piecewise_function()
piecewise(x|-->1 on (-oo, -3], x|-->x + 4 on (-3, 2), x|-->3*x on [2, +oo); x)
>>> p1.plot()
Graphics object consisting of 1 graphics primitive
```




```
sage: p2.piecewise_function()
piecewise(x|-->3 on (-oo, 1], x|-->2*x + 1 on (1, +oo); x)
sage: p2.plot()
Graphics object consisting of 1 graphics primitive
```

```
>>> from sage.all import *
>>> p2.piecewise_function()
piecewise(x|-->3 on (-oo, 1], x|-->2*x + 1 on (1, +oo); x)
>>> p2.plot()
Graphics object consisting of 1 graphics primitive
```



factor()

Return the factorization of `self` into its tropical linear factors.

Note that the factor $x - x_0$ in classical algebra gets transformed to the factor $x + x_0$, since the root of the tropical polynomial $x + x_0$ is x_0 and not $-x_0$. However, not every tropical polynomial can be factored.

OUTPUT: :class:'Factorization'

EXAMPLES:

```
sage: T = TropicalSemiring(QQ, use_min=True)
sage: R.<x> = PolynomialRing(T)
sage: p1 = R([6,3,1,0]); p1
0*x^3 + 1*x^2 + 3*x + 6
sage: factor(p1)
(0) * (0*x + 1) * (0*x + 2) * (0*x + 3)
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=True)
>>> R = PolynomialRing(T, names=('x',)); (x,) = R._first_ngens(1)
>>> p1 = R([Integer(6),Integer(3),Integer(1),Integer(0)]); p1
0*x^3 + 1*x^2 + 3*x + 6
>>> factor(p1)
(0) * (0*x + 1) * (0*x + 2) * (0*x + 3)
```

Such factorization is not always possible:

```
sage: p2 = R([4,4,2]); p2
2*x^2 + 4*x + 4
sage: p2.factor()
(2) * (0*x^2 + 2*x + 2)
```

```
>>> from sage.all import *
>>> p2 = R([Integer(4),Integer(4),Integer(2)]); p2
2*x^2 + 4*x + 4
>>> p2.factor()
(2) * (0*x^2 + 2*x + 2)
```

piecewise_function()

Return the tropical polynomial function of `self`.

The function is a piecewise linear function with the domains are divided by the roots. First we convert each term of polynomial to its corresponding linear function. Next, we must determine which term achieves the minimum (maximum) at each interval.

OUTPUT: a piecewise function

EXAMPLES:

```
sage: T = TropicalSemiring(QQ, use_min=False)
sage: R.<x> = PolynomialRing(T)
sage: p1 = R([4,2,1,3]); p1
3*x^3 + 1*x^2 + 2*x + 4
sage: p1.piecewise_function()
piecewise(x|-->4 on (-oo, 1/3], x|-->3*x + 3 on (1/3, +oo); x)
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=False)
>>> R = PolynomialRing(T, names=('x',)); (x,) = R._first_ngens(1)
>>> p1 = R([Integer(4),Integer(2),Integer(1),Integer(3)]); p1
3*x^3 + 1*x^2 + 2*x + 4
>>> p1.piecewise_function()
piecewise(x|-->4 on (-oo, 1/3], x|-->3*x + 3 on (1/3, +oo); x)
```

A constant tropical polynomial will result in a constant function:

```
sage: p2 = R(3)
sage: p2.piecewise_function()
3
```

```
>>> from sage.all import *
>>> p2 = R(Integer(3))
>>> p2.piecewise_function()
3
```

A monomial will resulted in a linear function:

```
sage: p3 = R(1)*x^3
sage: p3.piecewise_function()
3*x + 1
```

```
>>> from sage.all import *
>>> p3 = R(Integer(1))*x**Integer(3)
>>> p3.piecewise_function()
3*x + 1
```

plot(*xmin=None*, *xmax=None*)

Return the plot of `self`, which is the tropical polynomial function we get from `self.piecewise_function()`.

INPUT:

- `xmin` – (optional) real number
- `xmax` – (optional) real number

OUTPUT:

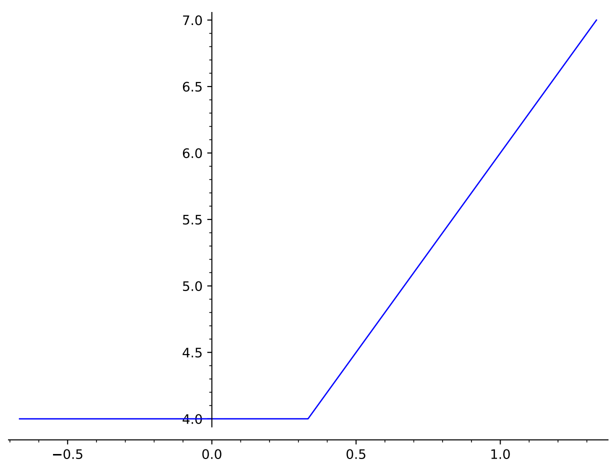
If `xmin` and `xmax` is given, then it return a plot of piecewise linear function of `self` with the axes start from `xmin` to `xmax`. Otherwise, the domain will start from the the minimum root of `self` minus 1 to maximum root of `self` plus 1. If the function of `self` is constant or linear, then the default domain will be $[-1, 1]$.

EXAMPLES:

If the tropical semiring use a max-plus algebra, then the graph will be of piecewise linear convex function:

```
sage: T = TropicalSemiring(QQ, use_min=False)
sage: R.<x> = PolynomialRing(T)
sage: p1 = R([4,2,1,3]); p1
3*x^3 + 1*x^2 + 2*x + 4
sage: p1.roots()
[1/3, 1/3, 1/3]
sage: p1.plot()
Graphics object consisting of 1 graphics primitive
```

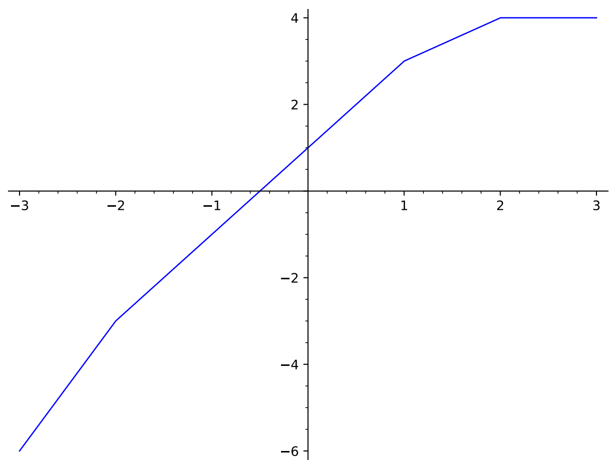
```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=False)
>>> R = PolynomialRing(T, names=('x',)); (x,) = R._first_ngens(1)
>>> p1 = R([Integer(4),Integer(2),Integer(1),Integer(3)]); p1
3*x^3 + 1*x^2 + 2*x + 4
>>> p1.roots()
[1/3, 1/3, 1/3]
>>> p1.plot()
Graphics object consisting of 1 graphics primitive
```



A different result will be obtained if the tropical semiring employs a min-plus algebra. Rather, a graph of the piecewise linear concave function will be obtained:

```
sage: T = TropicalSemiring(QQ, use_min=True)
sage: R.<x> = PolynomialRing(T)
sage: p1 = R([4,2,1,3])
sage: p1.roots()
[-2, 1, 2]
sage: p1.plot()
Graphics object consisting of 1 graphics primitive
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=True)
>>> R = PolynomialRing(T, names=('x',)); (x,) = R._first_ngens(1)
>>> p1 = R([Integer(4),Integer(2),Integer(1),Integer(3)])
>>> p1.roots()
[-2, 1, 2]
>>> p1.plot()
Graphics object consisting of 1 graphics primitive
```



roots()

Return the list of all tropical roots of `self`, counted with multiplicity.

OUTPUT: a list of tropical numbers

ALGORITHM:

For each pair of monomials in the polynomial, we find the point where their values are equal. This is the same as solving the equation $c_1 + a_1 \cdot x = c_2 + a_2 \cdot x$ for x, where (c_1, a_1) and (c_2, a_2) are the coefficients and exponents of monomials.

The solution to this equation is $x = (c_2 - c_1)/(a_1 - a_2)$. We substitute this x to each monomials in polynomial and check if the maximum (minimum) is achieved by the previous two monomials. If it is, then x is the root of tropical polynomial. In this case, the order of the root at x is the maximum of $|i - j|$ for all possible pairs i, j which realise this maximum (minimum).

EXAMPLES:

```
sage: T = TropicalSemiring(QQ, use_min=True)
sage: R.<x> = PolynomialRing(T)
sage: p1 = R([5,4,1,0,2,4,3]); p1
3*x^6 + 4*x^5 + 2*x^4 + 0*x^3 + 1*x^2 + 4*x + 5
sage: p1.roots()
[-1, -1, -1, 1, 2, 2]
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=True)
>>> R = PolynomialRing(T, names=('x',)); (x,) = R._first_ngens(1)
>>> p1 = R([Integer(5),Integer(4),Integer(1),Integer(0),Integer(2),Integer(4),
↪Integer(3)]); p1
```

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```
3*x^6 + 4*x^5 + 2*x^4 + 0*x^3 + 1*x^2 + 4*x + 5
>>> p1.roots()
[-1, -1, -1, 1, 2, 2]
```

There will be no tropical root for constant polynomial. Additionaly, for a monomial, the tropical root is assumed to be the additive identity of its base tropical semiring:

```
sage: p2 = R(2)
sage: p2.roots()
[]
sage: p3 = x^3
sage: p3.roots()
[+infinity, +infinity, +infinity]
```

```
>>> from sage.all import *
>>> p2 = R(Integer(2))
>>> p2.roots()
[]
>>> p3 = x**Integer(3)
>>> p3.roots()
[+infinity, +infinity, +infinity]
```

split_form()

Return the tropical polynomial which has the same roots as `self` but which can be reduced to its linear factors.

If a tropical polynomial has roots at $x_1, x_2, \ldots, x_n$, then its split form is the tropical product of linear terms of the form $(x + x_i)$ for all $i = 1, 2, \ldots, n$.

OUTPUT: *TropicalPolynomial*

EXAMPLES:

```
sage: T = TropicalSemiring(QQ, use_min=True)
sage: R.<x> = PolynomialRing(T)
sage: p1 = R([5,4,1,0,2,4,3]); p1
3*x^6 + 4*x^5 + 2*x^4 + 0*x^3 + 1*x^2 + 4*x + 5
sage: p1.split_form()
3*x^6 + 2*x^5 + 1*x^4 + 0*x^3 + 1*x^2 + 3*x + 5
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=True)
>>> R = PolynomialRing(T, names=('x',)); (x,) = R._first_ngens(1)
>>> p1 = R([Integer(5),Integer(4),Integer(1),Integer(0),Integer(2),Integer(4),
↪Integer(3)]); p1
3*x^6 + 4*x^5 + 2*x^4 + 0*x^3 + 1*x^2 + 4*x + 5
>>> p1.split_form()
3*x^6 + 2*x^5 + 1*x^4 + 0*x^3 + 1*x^2 + 3*x + 5
```

```
sage: T = TropicalSemiring(QQ, use_min=False)
sage: R.<x> = PolynomialRing(T)
sage: p1 = R([5,4,1,0,2,4,3])
```

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```
sage: p1.split_form()
3*x^6 + 4*x^5 + 21/5*x^4 + 22/5*x^3 + 23/5*x^2 + 24/5*x + 5
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=False)
>>> R = PolynomialRing(T, names=('x',)); (x,) = R._first_ngens(1)
>>> p1 = R([Integer(5),Integer(4),Integer(1),Integer(0),Integer(2),Integer(4),
↪Integer(3)])
>>> p1.split_form()
3*x^6 + 4*x^5 + 21/5*x^4 + 22/5*x^3 + 23/5*x^2 + 24/5*x + 5
```

class sage.rings.semirings.tropical_polynomial.**TropicalPolynomialSemiring**(*base_semiring*, *names*)

Bases: UniqueRepresentation, Parent

The semiring of univariate tropical polynomials.

This is the commutative semiring consisting of finite linear combinations of tropical monomials under (tropical) addition and multiplication with the identity element being $+\infty$ (or $-\infty$ depending on whether the base tropical semiring is using min-plus or max-plus algebra).

EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R.<x> = PolynomialRing(T)
sage: f = T(1)*x
sage: g = T(2)*x
sage: f + g
1*x
sage: f * g
3*x^2
sage: f + R.zero() == f
True
sage: f * R.zero() == R.zero()
True
sage: f * R.one() == f
True
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('x',)); (x,) = R._first_ngens(1)
>>> f = T(Integer(1))*x
>>> g = T(Integer(2))*x
>>> f + g
1*x
>>> f * g
3*x^2
>>> f + R.zero() == f
True
>>> f * R.zero() == R.zero()
True
>>> f * R.one() == f
True
```

Element

alias of *TropicalPolynomial*

gen (*n=0*)

Return the indeterminate generator of `self`.

EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R.<abc> = PolynomialRing(T)
sage: R.gen()
0*abc
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('abc',)); (abc,) = R._first_ngens(1)
>>> R.gen()
0*abc
```

gens ()

Return the generators for `self`.

EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R.<abc> = PolynomialRing(T)
sage: R.gens()
(0*abc,)
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('abc',)); (abc,) = R._first_ngens(1)
>>> R.gens()
(0*abc,)
```

interpolation (*points*)

Return a tropical polynomial with its function is a linear interpolation of point in `points` if possible.

If there is only one point, then it will give a constant polynomial. Because we are using linear interpolation, each point is actually a root of the resulted tropical polynomial.

INPUT:

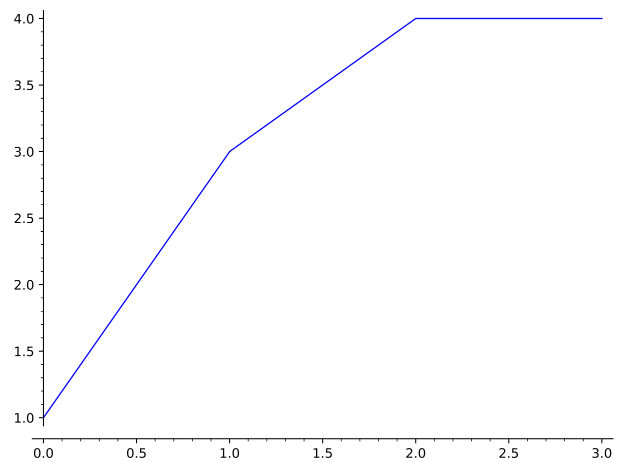
- `points` – a list of tuples `(x, y)`

OUTPUT: *TropicalPolynomial*

EXAMPLES:

```
sage: T = TropicalSemiring(QQ, use_min=True)
sage: R = PolynomialRing(T, 'x')
sage: points = [(-2,-3),(1,3),(2,4)]
sage: p1 = R.interpolation(points); p1
1*x^2 + 2*x + 4
sage: p1.plot()
Graphics object consisting of 1 graphics primitive
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=True)
>>> R = PolynomialRing(T, 'x')
>>> points = [(-Integer(2),-Integer(3)),(Integer(1),Integer(3)),(Integer(2),
↪Integer(4))]
>>> p1 = R.interpolation(points); p1
1*x^2 + 2*x + 4
>>> p1.plot()
Graphics object consisting of 1 graphics primitive
```




```
sage: T = TropicalSemiring(QQ, use_min=False)
sage: R = PolynomialRing(T,'x')
sage: points = [(0,0),(1,1),(2,4)]
sage: p1 = R.interpolation(points); p1
(-2)*x^3 + (-1)*x^2 + 0*x + 0
sage: p1.plot()
Graphics object consisting of 1 graphics primitive
```

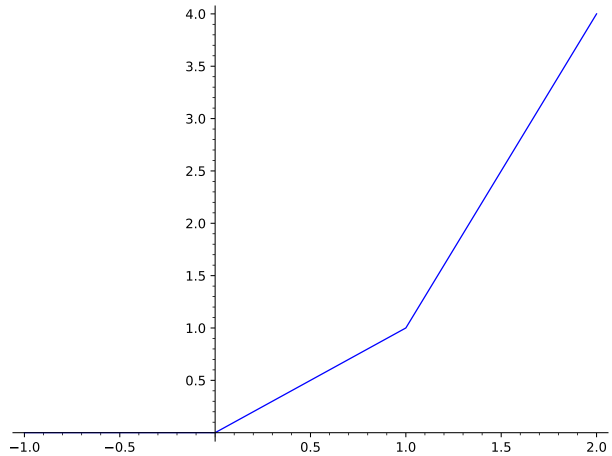
```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=False)
>>> R = PolynomialRing(T,'x')
>>> points = [(Integer(0),Integer(0)),(Integer(1),Integer(1)),(Integer(2),
↪Integer(4))]
>>> p1 = R.interpolation(points); p1
(-2)*x^3 + (-1)*x^2 + 0*x + 0
>>> p1.plot()
Graphics object consisting of 1 graphics primitive
```

is_sparse()

Return `True` to indicate that the objects are sparse polynomials.

EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R = PolynomialRing(T, 'x')
sage: R.is_sparse()
True
```




```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, 'x')
>>> R.is_sparse()
True
```

ngens()

Return the number of generators of `self`, which is 1 since it is a univariate polynomial ring.

EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R = PolynomialRing(T, 'x')
sage: R.ngens()
1
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, 'x')
>>> R.ngens()
1
```

one()

Return the multiplicative identity of `self`.

EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R = PolynomialRing(T, 'x')
sage: R.one()
0
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, 'x')
>>> R.one()
0
```

random_element(*degree=(-1, 2)*, *monic=False*, **args*, ***kwds*)

Return a random tropical polynomial of given degrees (bounds).

OUTPUT: *TropicalPolynomial*

See also:

sage.rings.polynomial.polynomial_ring.PolynomialRing_generic.random_element()

EXAMPLES:

Tropical polynomial over an integer with each coefficient bounded between `x` and `y`:

```
sage: T = TropicalSemiring(ZZ)
sage: R = PolynomialRing(T, 'x')
sage: f = R.random_element(8, x=3, y=10)
sage: f.degree()
8
sage: f.parent() is R
True
sage: all(a >= T(3) for a in f.coefficients())
True
sage: all(a < T(10) for a in f.coefficients())
True
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(ZZ)
>>> R = PolynomialRing(T, 'x')
>>> f = R.random_element(Integer(8), x=Integer(3), y=Integer(10))
>>> f.degree()
8
>>> f.parent() is R
True
>>> all(a >= T(Integer(3)) for a in f.coefficients())
True
>>> all(a < T(Integer(10)) for a in f.coefficients())
True
```

If a tuple of two integers is provided for the `degree` argument, a polynomial is selected with degrees within that range:

```
sage: all(R.random_element(degree=(1, 5)).degree() in range(1, 6) for _ in␣
↪range(10^3))
True
```

```
>>> from sage.all import *
>>> all(R.random_element(degree=(Integer(1), Integer(5))).degree() in␣
↪range(Integer(1), Integer(6)) for _ in range(Integer(10)**Integer(3)))
True
```

Note that the zero polynomial ($\pm\infty$) has degree -1. To include it, set the minimum degree to -1:

```
sage: while R.random_element(degree=(-1,2), x=-1, y=1) != R.zero():
....:     pass
```

```
>>> from sage.all import *
>>> while R.random_element(degree=(-Integer(1),Integer(2)), x=-Integer(1),␣
→y=Integer(1)) != R.zero():
...     pass
```

Monic polynomials are chosen among all monic polynomials with degree between the given `degree` argument:

```
sage: all(R.random_element(degree=(-1, 2), monic=True).is_monic() for _ in␣
→range(10^3))
True
```

```
>>> from sage.all import *
>>> all(R.random_element(degree=(-Integer(1), Integer(2)), monic=True).is_
→monic() for _ in range(Integer(10)**Integer(3)))
True
```

zero()

Return the additive identity of `self`.

EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R = PolynomialRing(T, 'x')
sage: R.zero()
+infinity
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, 'x')
>>> R.zero()
+infinity
```

7.2 Multivariate Tropical Polynomials

AUTHORS:

- Verrel Rievaldo Wijaya (2024-06): initial version

EXAMPLES:

```
sage: T = TropicalSemiring(QQ, use_min=True)
sage: R.<x,y,z> = PolynomialRing(T)
sage: z.parent()
Multivariate Tropical Polynomial Semiring in x, y, z over Rational Field
sage: R(2)*x + R(-1)*x + R(5)*y + R(-3)
(-1)*x + 5*y + (-3)
sage: (x+y+z)^2
0*x^2 + 0*x*y + 0*y^2 + 0*x*z + 0*y*z + 0*z^2
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=True)
>>> R = PolynomialRing(T, names=('x', 'y', 'z',)); (x, y, z,) = R._first_ngens(3)
>>> z.parent()
Multivariate Tropical Polynomial Semiring in x, y, z over Rational Field
>>> R(Integer(2))*x + R(-Integer(1))*x + R(Integer(5))*y + R(-Integer(3))
(-1)*x + 5*y + (-3)
>>> (x+y+z)**Integer(2)
0*x^2 + 0*x*y + 0*y^2 + 0*x*z + 0*y*z + 0*z^2
```

REFERENCES:

- [Bru2014]
- [Fil2017]

class sage.rings.semirings.tropical_mpolynomial.**TropicalMPolynomial**(*parent*, *x*)

Bases: *MPolynomial_polydict*

A multivariate tropical polynomial.

Let $x_1, x_2, \ldots, x_n$ be indeterminants. A tropical monomial is any product of these variables, possibly including repetitions: $x_1^{i_1} \cdots x_n^{i_n}$ where $i_j \in \{0, 1, \ldots\}$, for all $j \in \{1, \ldots, n\}$. A multivariate tropical polynomial is a finite linear combination of tropical monomials, $p(x_1, \ldots, x_n) = \sum_{i=1}^{n} c_i x_1^{i_1} \cdots x_n^{i_n}$.

In classical arithmetic, we can rewrite the general form of a tropical monomial: $x_1^{i_1} \cdots x_n^{i_n} \mapsto i_1 x_1 + \cdots + i_n x_n$. Thus, the tropical polynomial can be viewed as the minimum (maximum) of a finite collection of linear functions.

EXAMPLES:

Construct a multivariate tropical polynomial semiring in two variables:

```
sage: T = TropicalSemiring(QQ, use_min=False)
sage: R.<a,b> = PolynomialRing(T); R
Multivariate Tropical Polynomial Semiring in a, b over Rational Field
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=False)
>>> R = PolynomialRing(T, names=('a', 'b',)); (a, b,) = R._first_ngens(2); R
Multivariate Tropical Polynomial Semiring in a, b over Rational Field
```

Define some multivariate tropical polynomials:

```
sage: p1 = R(3)*a*b + a + R(-1)*b; p1
3*a*b + 0*a + (-1)*b
sage: p2 = R(1)*a + R(1)*b + R(1)*a*b; p2
1*a*b + 1*a + 1*b
```

```
>>> from sage.all import *
>>> p1 = R(Integer(3))*a*b + a + R(-Integer(1))*b; p1
3*a*b + 0*a + (-1)*b
>>> p2 = R(Integer(1))*a + R(Integer(1))*b + R(Integer(1))*a*b; p2
1*a*b + 1*a + 1*b
```

Some basic arithmetic operations for multivariate tropical polynomials:

```
sage: p1 + p2
3*a*b + 1*a + 1*b
sage: p1 * p2
4*a^2*b^2 + 4*a^2*b + 4*a*b^2 + 1*a^2 + 1*a*b + 0*b^2
sage: T(2) * p1
5*a*b + 2*a + 1*b
sage: p1(T(1),T(2))
6
```

```
>>> from sage.all import *
>>> p1 + p2
3*a*b + 1*a + 1*b
>>> p1 * p2
4*a^2*b^2 + 4*a^2*b + 4*a*b^2 + 1*a^2 + 1*a*b + 0*b^2
>>> T(Integer(2)) * p1
5*a*b + 2*a + 1*b
>>> p1(T(Integer(1)),T(Integer(2)))
6
```

Let us look at the different result for tropical curve and 3d graph of tropical polynomial in two variables when different algebra is used. First for the min-plus algebra:

```
sage: T = TropicalSemiring(QQ, use_min=True)
sage: R.<a,b> = PolynomialRing(T)
sage: p1 = R(3)*a*b + a + R(-1)*b
sage: p1.tropical_variety()
Tropical curve of 3*a*b + 0*a + (-1)*b
sage: p1.tropical_variety().plot()
Graphics object consisting of 3 graphics primitives
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=True)
>>> R = PolynomialRing(T, names=('a', 'b',)); (a, b,) = R._first_ngens(2)
>>> p1 = R(Integer(3))*a*b + a + R(-Integer(1))*b
>>> p1.tropical_variety()
Tropical curve of 3*a*b + 0*a + (-1)*b
>>> p1.tropical_variety().plot()
Graphics object consisting of 3 graphics primitives
```

Tropical polynomial in two variables will induce a function in three dimension that consists of a number of surfaces:

```
sage: p1.plot3d()
Graphics3d Object
```

```
>>> from sage.all import *
>>> p1.plot3d()
Graphics3d Object
```

If we use a max-plus algebra, we will get a slightly different result:

```
sage: T = TropicalSemiring(QQ, use_min=False)
sage: R.<a,b> = PolynomialRing(T)
```

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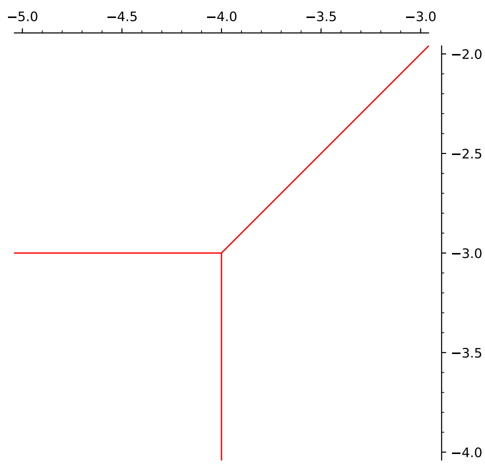
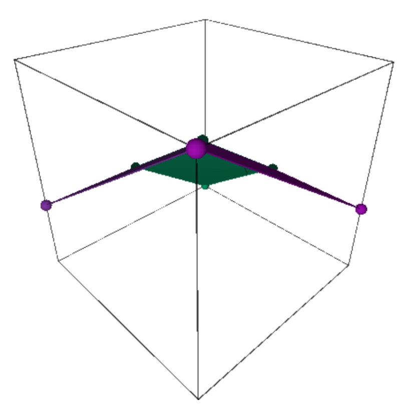

−5.0
−4.5
−4.0
−3.5
−3.0
−2.0
−2.5
−3.0
−3.5
−4.0



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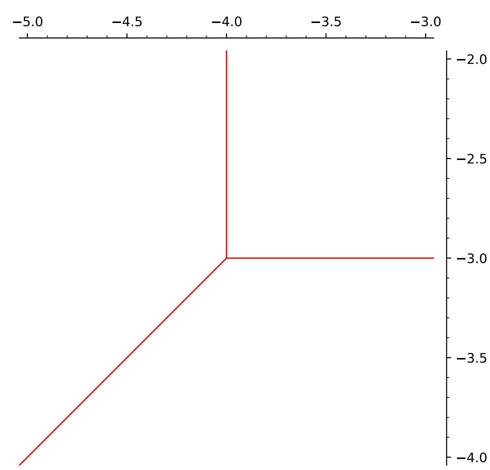
```
sage: p1 = R(3)*a*b + a + R(-1)*b
sage: p1.tropical_variety()
Tropical curve of 3*a*b + 0*a + (-1)*b
sage: p1.tropical_variety().plot()
Graphics object consisting of 3 graphics primitives
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=False)
>>> R = PolynomialRing(T, names=('a', 'b',)); (a, b,) = R._first_ngens(2)
>>> p1 = R(Integer(3))*a*b + a + R(-Integer(1))*b
>>> p1.tropical_variety()
Tropical curve of 3*a*b + 0*a + (-1)*b
>>> p1.tropical_variety().plot()
Graphics object consisting of 3 graphics primitives
```



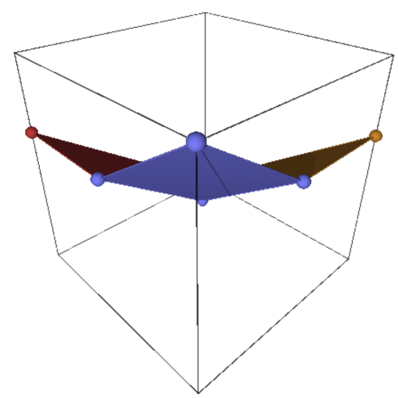

```
sage: p1.plot3d()
Graphics3d Object
```

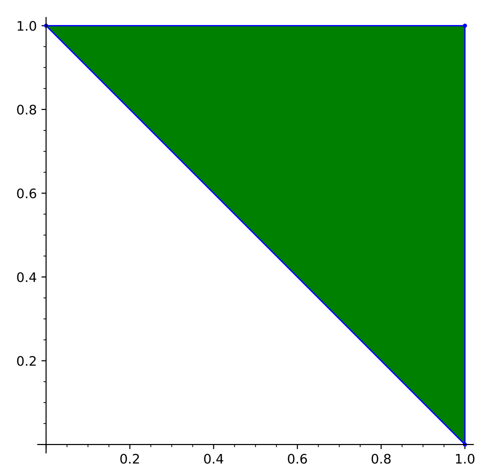
```
>>> from sage.all import *
>>> p1.plot3d()
Graphics3d Object
```

Another way to represent tropical curve is through dual subdivision, which is a subdivision of Newton polytope of tropical polynomial:

```
sage: p1.newton_polytope()
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
sage: p1.dual_subdivision()
Polyhedral complex with 1 maximal cell
```

```
>>> from sage.all import *
>>> p1.newton_polytope()
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
>>> p1.dual_subdivision()
Polyhedral complex with 1 maximal cell
```




1.0
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1.0

dual_subdivision()

Return the dual subdivision of `self`.

Dual subdivision refers to a specific decomposition of the Newton polytope of a tropical polynomial. The term "dual" is used in the sense that the combinatorial structure of the tropical variety is reflected in the dual subdivision. Specifically, vertices of the dual subdivision correspond to the intersection of multiple components. Edges of the dual subdivision correspond to the individual components.

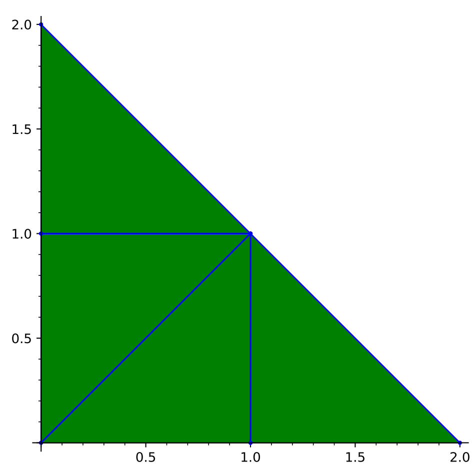
OUTPUT: `PolyhedralComplex`

EXAMPLES:

Dual subdivision of a tropical curve:

```
sage: T = TropicalSemiring(QQ, use_min=False)
sage: R.<x,y> = PolynomialRing(T)
sage: p1 = R(3) + R(2)*x + R(2)*y + R(3)*x*y + x^2 + y^2
sage: pc = p1.dual_subdivision(); pc
Polyhedral complex with 4 maximal cells
sage: [p.Vrepresentation() for p in pc.maximal_cells_sorted()]
[(A vertex at (0, 0), A vertex at (0, 1), A vertex at (1, 1)),
 (A vertex at (0, 0), A vertex at (1, 0), A vertex at (1, 1)),
 (A vertex at (0, 1), A vertex at (0, 2), A vertex at (1, 1)),
 (A vertex at (1, 0), A vertex at (1, 1), A vertex at (2, 0))]
```

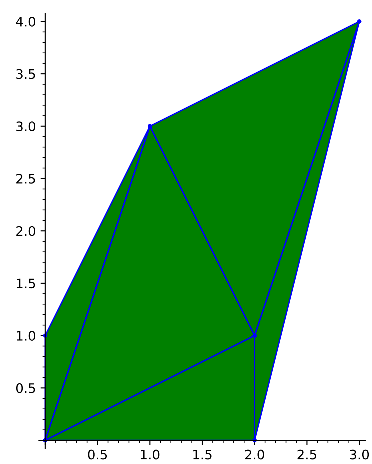
```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=False)
>>> R = PolynomialRing(T, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> p1 = R(Integer(3)) + R(Integer(2))*x + R(Integer(2))*y +␣
↪R(Integer(3))*x*y + x**Integer(2) + y**Integer(2)
>>> pc = p1.dual_subdivision(); pc
Polyhedral complex with 4 maximal cells
>>> [p.Vrepresentation() for p in pc.maximal_cells_sorted()]
[(A vertex at (0, 0), A vertex at (0, 1), A vertex at (1, 1)),
 (A vertex at (0, 0), A vertex at (1, 0), A vertex at (1, 1)),
 (A vertex at (0, 1), A vertex at (0, 2), A vertex at (1, 1)),
 (A vertex at (1, 0), A vertex at (1, 1), A vertex at (2, 0))]
```



A subdivision of a pentagonal Newton polytope:

```
sage: p2 = R(3) + x^2 + R(-2)*y + R(1/2)*x^2*y + R(2)*x*y^3 + R(-1)*x^3*y^4
sage: pc = p2.dual_subdivision(); pc
Polyhedral complex with 5 maximal cells
sage: [p.Vrepresentation() for p in pc.maximal_cells_sorted()]
[(A vertex at (0, 0), A vertex at (0, 1), A vertex at (1, 3)),
 (A vertex at (0, 0), A vertex at (1, 3), A vertex at (2, 1)),
 (A vertex at (0, 0), A vertex at (2, 0), A vertex at (2, 1)),
 (A vertex at (1, 3), A vertex at (2, 1), A vertex at (3, 4)),
 (A vertex at (2, 0), A vertex at (2, 1), A vertex at (3, 4))]
```

```
>>> from sage.all import *
>>> p2 = R(Integer(3)) + x**Integer(2) + R(-Integer(2))*y + R(Integer(1)/
↪Integer(2))*x**Integer(2)*y + R(Integer(2))*x*y**Integer(3) + R(-
↪Integer(1))*x**Integer(3)*y**Integer(4)
>>> pc = p2.dual_subdivision(); pc
Polyhedral complex with 5 maximal cells
>>> [p.Vrepresentation() for p in pc.maximal_cells_sorted()]
[(A vertex at (0, 0), A vertex at (0, 1), A vertex at (1, 3)),
 (A vertex at (0, 0), A vertex at (1, 3), A vertex at (2, 1)),
 (A vertex at (0, 0), A vertex at (2, 0), A vertex at (2, 1)),
 (A vertex at (1, 3), A vertex at (2, 1), A vertex at (3, 4)),
 (A vertex at (2, 0), A vertex at (2, 1), A vertex at (3, 4))]
```



A subdivision with many faces, not all of which are triangles:

```
sage: T = TropicalSemiring(QQ)
sage: R.<x,y> = PolynomialRing(T)
sage: p3 = (R(8) + R(4)*x + R(2)*y + R(1)*x^2 + x*y + R(1)*y^2
....:       + R(2)*x^3 + x^2*y + x*y^2 + R(4)*y^3 + R(8)*x^4
....:       + R(4)*x^3*y + x^2*y^2 + R(2)*x*y^3 + y^4)
sage: pc = p3.dual_subdivision(); pc
Polyhedral complex with 10 maximal cells
sage: [p.Vrepresentation() for p in pc.maximal_cells_sorted()]
[(A vertex at (0, 0), A vertex at (0, 1), A vertex at (1, 0)),
 (A vertex at (0, 1), A vertex at (0, 2), A vertex at (1, 1)),
 (A vertex at (0, 1), A vertex at (1, 0), A vertex at (2, 0)),
 (A vertex at (0, 1), A vertex at (1, 1), A vertex at (2, 0)),
```

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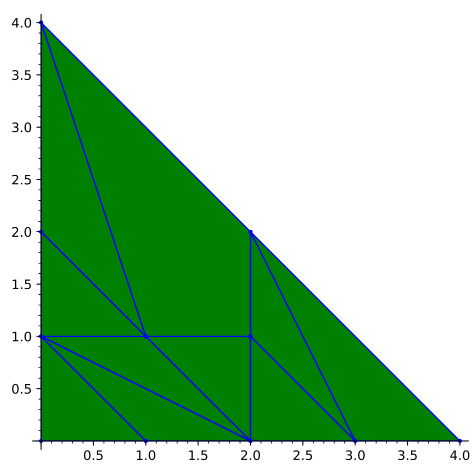
```
 (A vertex at (0, 2), A vertex at (0, 4), A vertex at (1, 1)),
 (A vertex at (0, 4),
  A vertex at (1, 1),
  A vertex at (2, 1),
  A vertex at (2, 2)),
 (A vertex at (1, 1), A vertex at (2, 0), A vertex at (2, 1)),
 (A vertex at (2, 0), A vertex at (2, 1), A vertex at (3, 0)),
 (A vertex at (2, 1), A vertex at (2, 2), A vertex at (3, 0)),
 (A vertex at (2, 2), A vertex at (3, 0), A vertex at (4, 0))]
```

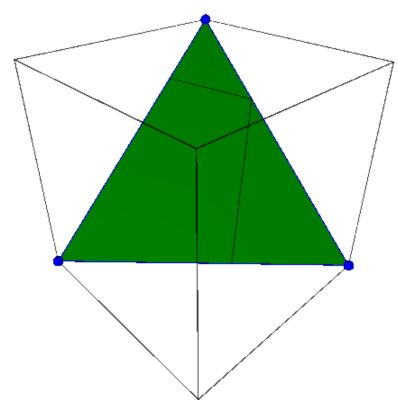
```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> p3 = (R(Integer(8)) + R(Integer(4))*x + R(Integer(2))*y +
↪R(Integer(1))*x**Integer(2) + x*y + R(Integer(1))*y**Integer(2)
...       + R(Integer(2))*x**Integer(3) + x**Integer(2)*y + x*y**Integer(2) +
↪R(Integer(4))*y**Integer(3) + R(Integer(8))*x**Integer(4)
...       + R(Integer(4))*x**Integer(3)*y + x**Integer(2)*y**Integer(2) +
↪R(Integer(2))*x*y**Integer(3) + y**Integer(4))
>>> pc = p3.dual_subdivision(); pc
Polyhedral complex with 10 maximal cells
>>> [p.Vrepresentation() for p in pc.maximal_cells_sorted()]
[(A vertex at (0, 0), A vertex at (0, 1), A vertex at (1, 0)),
 (A vertex at (0, 1), A vertex at (0, 2), A vertex at (1, 1)),
 (A vertex at (0, 1), A vertex at (1, 0), A vertex at (2, 0)),
 (A vertex at (0, 1), A vertex at (1, 1), A vertex at (2, 0)),
 (A vertex at (0, 2), A vertex at (0, 4), A vertex at (1, 1)),
 (A vertex at (0, 4),
  A vertex at (1, 1),
  A vertex at (2, 1),
  A vertex at (2, 2)),
 (A vertex at (1, 1), A vertex at (2, 0), A vertex at (2, 1)),
 (A vertex at (2, 0), A vertex at (2, 1), A vertex at (3, 0)),
 (A vertex at (2, 1), A vertex at (2, 2), A vertex at (3, 0)),
 (A vertex at (2, 2), A vertex at (3, 0), A vertex at (4, 0))]
```



Dual subdivision of a tropical surface:

```
sage: T = TropicalSemiring(QQ)
sage: R.<x,y,z> = PolynomialRing(T)
sage: p1 = x + y + z + x^2 + R(1)
sage: pc = p1.dual_subdivision(); pc
Polyhedral complex with 7 maximal cells
sage: [p.Vrepresentation() for p in pc.maximal_cells_sorted()]
[(A vertex at (0, 0, 0), A vertex at (0, 0, 1), A vertex at (0, 1, 0)),
 (A vertex at (0, 0, 0), A vertex at (0, 0, 1), A vertex at (1, 0, 0)),
 (A vertex at (0, 0, 0), A vertex at (0, 1, 0), A vertex at (1, 0, 0)),
 (A vertex at (0, 0, 1), A vertex at (0, 1, 0), A vertex at (1, 0, 0)),
 (A vertex at (0, 0, 1), A vertex at (0, 1, 0), A vertex at (2, 0, 0)),
 (A vertex at (0, 0, 1), A vertex at (1, 0, 0), A vertex at (2, 0, 0)),
 (A vertex at (0, 1, 0), A vertex at (1, 0, 0), A vertex at (2, 0, 0))]
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> p1 = x + y + z + x**Integer(2) + R(Integer(1))
>>> pc = p1.dual_subdivision(); pc
Polyhedral complex with 7 maximal cells
>>> [p.Vrepresentation() for p in pc.maximal_cells_sorted()]
[(A vertex at (0, 0, 0), A vertex at (0, 0, 1), A vertex at (0, 1, 0)),
 (A vertex at (0, 0, 0), A vertex at (0, 0, 1), A vertex at (1, 0, 0)),
 (A vertex at (0, 0, 0), A vertex at (0, 1, 0), A vertex at (1, 0, 0)),
 (A vertex at (0, 0, 1), A vertex at (0, 1, 0), A vertex at (1, 0, 0)),
 (A vertex at (0, 0, 1), A vertex at (0, 1, 0), A vertex at (2, 0, 0)),
 (A vertex at (0, 0, 1), A vertex at (1, 0, 0), A vertex at (2, 0, 0)),
 (A vertex at (0, 1, 0), A vertex at (1, 0, 0), A vertex at (2, 0, 0))]
```



Dual subdivision of a tropical hypersurface:

```
sage: T = TropicalSemiring(QQ)
sage: R.<a,b,c,d> = PolynomialRing(T)
sage: p1 = R(2)*a*b + R(3)*a*c + R(-1)*c^2 + R(-1/3)*a*d
sage: pc = p1.dual_subdivision(); pc
Polyhedral complex with 4 maximal cells
```

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```
sage: [p.Vrepresentation() for p in pc.maximal_cells_sorted()]
[(A vertex at (0, 0, 2, 0),
  A vertex at (1, 0, 0, 1),
  A vertex at (1, 0, 1, 0)),
 (A vertex at (0, 0, 2, 0),
  A vertex at (1, 0, 0, 1),
  A vertex at (1, 1, 0, 0)),
 (A vertex at (0, 0, 2, 0),
  A vertex at (1, 0, 1, 0),
  A vertex at (1, 1, 0, 0)),
 (A vertex at (1, 0, 0, 1),
  A vertex at (1, 0, 1, 0),
  A vertex at (1, 1, 0, 0))]
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('a', 'b', 'c', 'd',)); (a, b, c, d,) = R._
↪first_ngens(4)
>>> p1 = R(Integer(2))*a*b + R(Integer(3))*a*c + R(-Integer(1))*c**Integer(2)␣
↪+ R(-Integer(1)/Integer(3))*a*d
>>> pc = p1.dual_subdivision(); pc
Polyhedral complex with 4 maximal cells
>>> [p.Vrepresentation() for p in pc.maximal_cells_sorted()]
[(A vertex at (0, 0, 2, 0),
  A vertex at (1, 0, 0, 1),
  A vertex at (1, 0, 1, 0)),
 (A vertex at (0, 0, 2, 0),
  A vertex at (1, 0, 0, 1),
  A vertex at (1, 1, 0, 0)),
 (A vertex at (0, 0, 2, 0),
  A vertex at (1, 0, 1, 0),
  A vertex at (1, 1, 0, 0)),
 (A vertex at (1, 0, 0, 1),
  A vertex at (1, 0, 1, 0),
  A vertex at (1, 1, 0, 0))]
```

newton_polytope()

Return the Newton polytope of `self`.

The Newton polytope is the convex hull of all the points corresponding to the exponents of the monomials of tropical polynomial.

OUTPUT: `Polyhedron()`

EXAMPLES:

A Newton polytope for a two-variable tropical polynomial:

```
sage: T = TropicalSemiring(QQ)
sage: R.<x,y> = PolynomialRing(T)
sage: p1 = x + y
sage: p1.newton_polytope()
A 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 2 vertices
sage: p1.newton_polytope().Vrepresentation()
```

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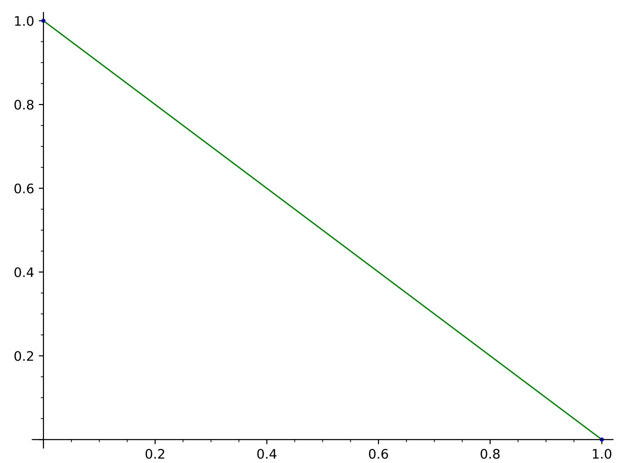
```
(A vertex at (0, 1), A vertex at (1, 0))
sage: p1.newton_polytope().Hrepresentation()
(An equation (1, 1) x - 1 == 0,
 An inequality (0, -1) x + 1 >= 0,
 An inequality (0, 1) x + 0 >= 0)
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> p1 = x + y
>>> p1.newton_polytope()
A 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 2 vertices
>>> p1.newton_polytope().Vrepresentation()
(A vertex at (0, 1), A vertex at (1, 0))
>>> p1.newton_polytope().Hrepresentation()
(An equation (1, 1) x - 1 == 0,
 An inequality (0, -1) x + 1 >= 0,
 An inequality (0, 1) x + 0 >= 0)
```



A Newton polytope in three dimension:

```
sage: T = TropicalSemiring(QQ)
sage: R.<x,y,z> = PolynomialRing(T)
sage: p1 = x^2 + x*y*z + x + y + z + R(0)
sage: p1.newton_polytope()
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 5 vertices
sage: p1.newton_polytope().Vrepresentation()
(A vertex at (0, 0, 0),
A vertex at (0, 0, 1),
A vertex at (0, 1, 0),
A vertex at (2, 0, 0),
A vertex at (1, 1, 1))
sage: p1.newton_polytope().Hrepresentation()
(An inequality (0, 1, 0) x + 0 >= 0,
 An inequality (0, 0, 1) x + 0 >= 0,
 An inequality (1, 0, 0) x + 0 >= 0,
 An inequality (1, -1, -1) x + 1 >= 0,
```

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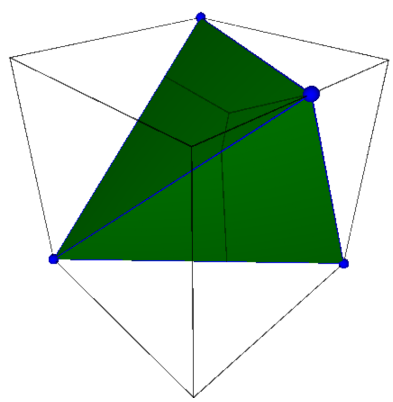
```
 An inequality (-1, -2, 1) x + 2 >= 0,
 An inequality (-1, 1, -2) x + 2 >= 0)
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> p1 = x**Integer(2) + x*y*z + x + y + z + R(Integer(0))
>>> p1.newton_polytope()
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 5 vertices
>>> p1.newton_polytope().Vrepresentation()
(A vertex at (0, 0, 0),
A vertex at (0, 0, 1),
A vertex at (0, 1, 0),
A vertex at (2, 0, 0),
A vertex at (1, 1, 1))
>>> p1.newton_polytope().Hrepresentation()
(An inequality (0, 1, 0) x + 0 >= 0,
 An inequality (0, 0, 1) x + 0 >= 0,
 An inequality (1, 0, 0) x + 0 >= 0,
 An inequality (1, -1, -1) x + 1 >= 0,
 An inequality (-1, -2, 1) x + 2 >= 0,
 An inequality (-1, 1, -2) x + 2 >= 0)
```



plot3d(*color='random'*)

Return the 3d plot of `self`.

Only implemented for tropical polynomial in two variables. The x-y axes for this 3d plot is the same as the x-y axes of the corresponding tropical curve.

OUTPUT: Graphics3d Object

EXAMPLES:

A simple tropical polynomial that consist of only one surface:

```
sage: T = TropicalSemiring(QQ, use_min=False)
sage: R.<x,y> = PolynomialRing(T)
```

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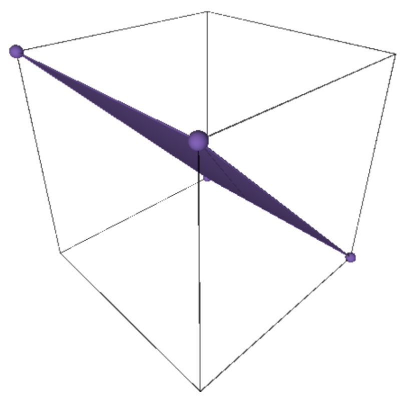
```
sage: p1 = x^2
sage: p1.plot3d()
Graphics3d Object
```

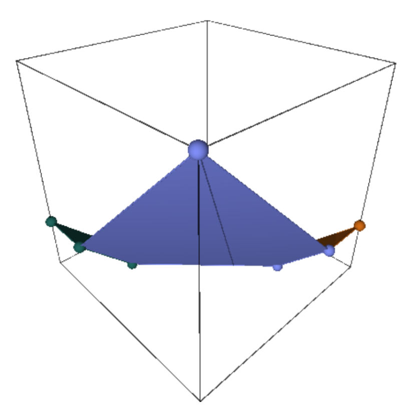
```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=False)
>>> R = PolynomialRing(T, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> p1 = x**Integer(2)
>>> p1.plot3d()
Graphics3d Object
```



Tropical polynomials often have graphs that represent a combination of multiple surfaces:

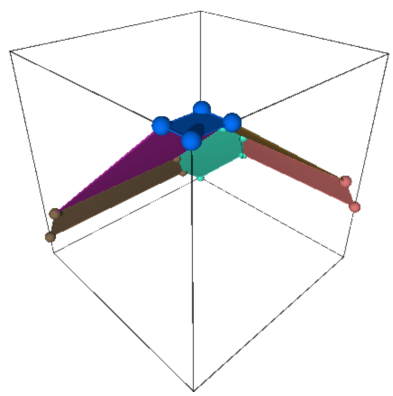
```
sage: p2 = R(3) + R(2)*x + R(2)*y + R(3)*x*y
sage: p2.plot3d()
Graphics3d Object
```

```
>>> from sage.all import *
>>> p2 = R(Integer(3)) + R(Integer(2))*x + R(Integer(2))*y + R(Integer(3))*x*y
>>> p2.plot3d()
Graphics3d Object
```



```
sage: T = TropicalSemiring(QQ)
sage: R.<x,y> = PolynomialRing(T)
sage: p3 = R(2)*x^2 + x*y + R(2)*y^2 + x + R(-1)*y + R(3)
sage: p3.plot3d()
Graphics3d Object
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> p3 = R(Integer(2))*x**Integer(2) + x*y + R(Integer(2))*y**Integer(2) + x␣
↪+ R(-Integer(1))*y + R(Integer(3))
>>> p3.plot3d()
Graphics3d Object
```



subs (*fixed=None*, ***kwds*)

Fix some given variables in `self` and return the changed tropical multivariate polynomials.

See also

sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict.subs()

EXAMPLES:

```
sage: T = TropicalSemiring(QQ, use_min=False)
sage: R.<x,y> = PolynomialRing(T)
sage: p1 = x^2 + y + R(3)
sage: p1((R(4),y))
0*y + 8
sage: p1.subs({x: 4})
0*y + 8
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=False)
>>> R = PolynomialRing(T, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> p1 = x**Integer(2) + y + R(Integer(3))
>>> p1((R(Integer(4)),y))
```

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```
0*y + 8
>>> p1.subs({x: Integer(4)})
0*y + 8
```

tropical_variety()

Return tropical roots of `self`.

In the multivariate case, the roots can be represented by a tropical variety. In two dimensions, this is known as a tropical curve. For dimensions higher than two, it is referred to as a tropical hypersurface.

OUTPUT: *sage.rings.semirings.tropical_variety.TropicalVariety*

EXAMPLES:

Tropical curve for tropical polynomials in two variables:

```
sage: T = TropicalSemiring(QQ, use_min=False)
sage: R.<x,y> = PolynomialRing(T)
sage: p1 = x + y + R(0); p1
0*x + 0*y + 0
sage: p1.tropical_variety()
Tropical curve of 0*x + 0*y + 0
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=False)
>>> R = PolynomialRing(T, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> p1 = x + y + R(Integer(0)); p1
0*x + 0*y + 0
>>> p1.tropical_variety()
Tropical curve of 0*x + 0*y + 0
```

Tropical hypersurface for tropical polynomials in more than two variables:

```
sage: T = TropicalSemiring(QQ)
sage: R.<x,y,z> = PolynomialRing(T)
sage: p1 = R(1)*x*y + R(-1/2)*x*z + R(4)*z^2; p1
1*x*y + (-1/2)*x*z + 4*z^2
sage: p1.tropical_variety()
Tropical surface of 1*x*y + (-1/2)*x*z + 4*z^2
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> p1 = R(Integer(1))*x*y + R(-Integer(1)/Integer(2))*x*z +␣
↪R(Integer(4))*z**Integer(2); p1
1*x*y + (-1/2)*x*z + 4*z^2
>>> p1.tropical_variety()
Tropical surface of 1*x*y + (-1/2)*x*z + 4*z^2
```

class sage.rings.semirings.tropical_mpolynomial.**TropicalMPolynomialSemiring**(*base_semiring*, *n*, *names*, *order*)

Bases: `UniqueRepresentation`, `Parent`

The semiring of tropical polynomials in multiple variables.

This is the commutative semiring consisting of all finite linear combinations of tropical monomials under (tropical) addition and multiplication with coefficients in a tropical semiring.

EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R.<x,y> = PolynomialRing(T)
sage: f = T(1)*x + T(-1)*y
sage: g = T(2)*x + T(-2)*y
sage: f + g
1*x + (-2)*y
sage: f * g
3*x^2 + (-1)*x*y + (-3)*y^2
sage: f + R.zero() == f
True
sage: f * R.zero() == R.zero()
True
sage: f * R.one() == f
True
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> f = T(Integer(1))*x + T(-Integer(1))*y
>>> g = T(Integer(2))*x + T(-Integer(2))*y
>>> f + g
1*x + (-2)*y
>>> f * g
3*x^2 + (-1)*x*y + (-3)*y^2
>>> f + R.zero() == f
True
>>> f * R.zero() == R.zero()
True
>>> f * R.one() == f
True
```

Element

alias of *TropicalMPolynomial*

gen(*n=0*)

Return the n-th generator of self.

EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R.<a,b,c> = PolynomialRing(T)
sage: R.gen()
0*a
sage: R.gen(2)
0*c
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
```

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```
>>> R = PolynomialRing(T, names=('a', 'b', 'c',)); (a, b, c,) = R._first_
↪ngens(3)
>>> R.gen()
0*a
>>> R.gen(Integer(2))
0*c
```

gens()

Return the generators of `self`.

EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R = PolynomialRing(T, 5, 'x')
sage: R.gens()
(0*x0, 0*x1, 0*x2, 0*x3, 0*x4)
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, Integer(5), 'x')
>>> R.gens()
(0*x0, 0*x1, 0*x2, 0*x3, 0*x4)
```

ngens()

Return the number of generators of `self`.

EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R = PolynomialRing(T, 10, 'z')
sage: R.ngens()
10
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, Integer(10), 'z')
>>> R.ngens()
10
```

one()

Return the multiplicative identity of `self`.

EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R = PolynomialRing(T, 'x,y')
sage: R.one()
0
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, 'x,y')
```

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```
>>> R.one()
0
```

random_element(*degree=2*, *terms=None*, *choose_degree=False*, **args*, ***kwargs*)

Return a random multivariate tropical polynomial from `self`.

OUTPUT: `TropicalMPolynomial`

See also

`sage.rings.polynomial.multi_polynomial_ring_base.MPolynomialRing_base.random_element()`

EXAMPLES:

A random polynomial of at most degree d and at most t terms:

```
sage: T = TropicalSemiring(QQ)
sage: R.<a,b,c> = PolynomialRing(T)
sage: f = R.random_element(2, 5)
sage: f.degree() <= 2
True
sage: f.parent() is R
True
sage: len(list(f)) <= 5
True
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('a', 'b', 'c',)); (a, b, c,) = R._first_
↪ngens(3)
>>> f = R.random_element(Integer(2), Integer(5))
>>> f.degree() <= Integer(2)
True
>>> f.parent() is R
True
>>> len(list(f)) <= Integer(5)
True
```

Choose degrees of monomials randomly first rather than monomials uniformly random:

```
sage: f = R.random_element(3, 6, choose_degree=True)
sage: f.degree() <= 3
True
sage: f.parent() is R
True
sage: len(list(f)) <= 6
True
```

```
>>> from sage.all import *
>>> f = R.random_element(Integer(3), Integer(6), choose_degree=True)
>>> f.degree() <= Integer(3)
```

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```
True
>>> f.parent() is R
True
>>> len(list(f)) <= Integer(6)
True
```

term_order()

Return the defined term order of `self`.

EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R.<x,y,z> = PolynomialRing(T)
sage: R.term_order()
Degree reverse lexicographic term order
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> R.term_order()
Degree reverse lexicographic term order
```

zero()

Return the additive identity of `self`.

EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R = PolynomialRing(T, 'x,y')
sage: R.zero()
+infinity
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, 'x,y')
>>> R.zero()
+infinity
```

7.3 Tropical Varieties

A tropical variety is a piecewise-linear geometric object derived from a classical algebraic variety by using tropical mathematics, where the tropical semiring replaces the usual arithmetic operations.

AUTHORS:

- Verrel Rievaldo Wijaya (2024-06): initial version

REFERENCES:

- [Bru2014]
- [Mac2015]
- [Fil2017]

class sage.rings.semirings.tropical_variety.**TropicalCurve**(*poly*)

Bases: *TropicalVariety*

A tropical curve in $\mathbf{R}^2$.

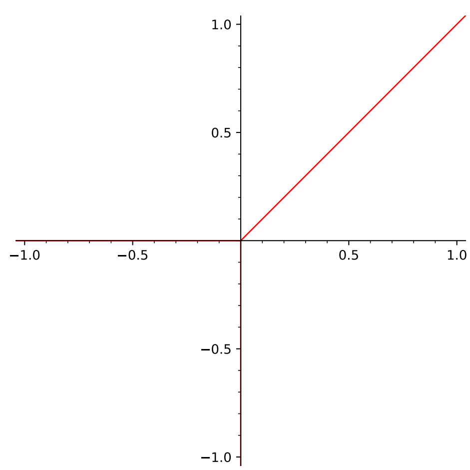
The tropical curve consists of line segments and half-lines, which we call edges. These edges are connected in such a way that they form a piecewise linear graph embedded in the plane. These edges meet at a vertices, where the balancing condition is satisfied. This balancing condition ensures that the sum of the outgoing slopes at each vertex is zero, reflecting the equilibrium.

EXAMPLES:

We define some tropical curves:

```
sage: T = TropicalSemiring(QQ, use_min=False)
sage: R.<x,y> = PolynomialRing(T)
sage: p1 = x + y + R(0)
sage: tv1 = p1.tropical_variety(); tv1
Tropical curve of 0*x + 0*y + 0
sage: tv1.components()
[[(t1, t1), [t1 >= 0], 1], [(0, t1), [t1 <= 0], 1], [(t1, 0), [t1 <= 0], 1]]
sage: tv1.plot()
Graphics object consisting of 3 graphics primitives
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=False)
>>> R = PolynomialRing(T, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> p1 = x + y + R(Integer(0))
>>> tv1 = p1.tropical_variety(); tv1
Tropical curve of 0*x + 0*y + 0
>>> tv1.components()
[[(t1, t1), [t1 >= 0], 1], [(0, t1), [t1 <= 0], 1], [(t1, 0), [t1 <= 0], 1]]
>>> tv1.plot()
Graphics object consisting of 3 graphics primitives
```




```
sage: p2 = R(-2)*x^2 + R(-1)*x + R(1/2)*y + R(1/6)
sage: tv2 = p2.tropical_variety()
sage: tv2.components()
[[(1/2*t1 + 5/4, t1), [(-1/3) <= t1], 1],
```

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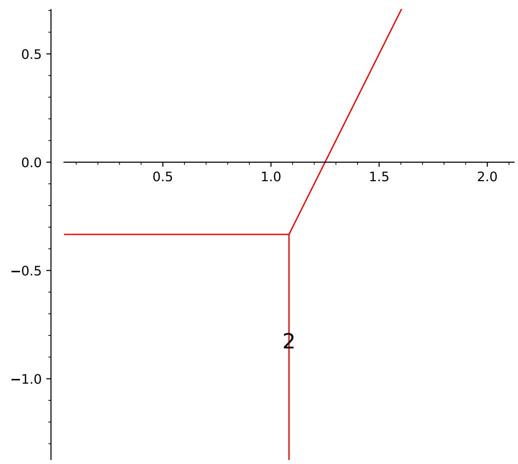
```
 [(13/12, t1), [t1 <= (-1/3)], 2],
 [(t1, -1/3), [t1 <= (13/12)], 1]]
sage: tv2.plot()
Graphics object consisting of 4 graphics primitives
```

```
>>> from sage.all import *
>>> p2 = R(-Integer(2))*x**Integer(2) + R(-Integer(1))*x + R(Integer(1)/
↪Integer(2))*y + R(Integer(1)/Integer(6))
>>> tv2 = p2.tropical_variety()
>>> tv2.components()
[[(1/2*t1 + 5/4, t1), [(-1/3) <= t1], 1],
 [(13/12, t1), [t1 <= (-1/3)], 2],
 [(t1, -1/3), [t1 <= (13/12)], 1]]
>>> tv2.plot()
Graphics object consisting of 4 graphics primitives
```



When two tropical polynomials are multiplied, the tropical curve of the resulting polynomial is the union of the tropical curves of the original polynomials:

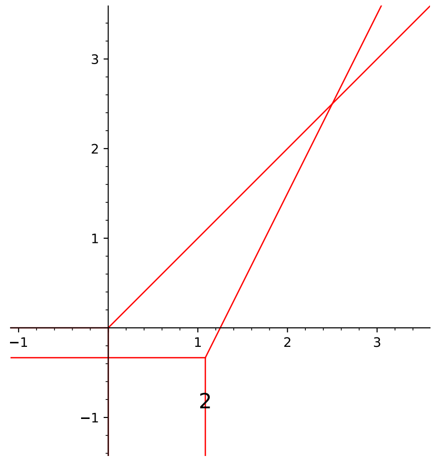
```
sage: p3 = p1 * p2; p3
(-2)*x^3 + (-2)*x^2*y + (-1)*x^2 + 1/2*x*y + 1/2*y^2 + 1/6*x + 1/2*y + 1/6
sage: tv3 = p3.tropical_variety()
sage: tv3.plot()
Graphics object consisting of 11 graphics primitives
```

```
>>> from sage.all import *
>>> p3 = p1 * p2; p3
(-2)*x^3 + (-2)*x^2*y + (-1)*x^2 + 1/2*x*y + 1/2*y^2 + 1/6*x + 1/2*y + 1/6
>>> tv3 = p3.tropical_variety()
>>> tv3.plot()
Graphics object consisting of 11 graphics primitives
```

contribution()

Return the contribution of `self`.

The contribution of a simple curve C is defined as the product of the normalized areas of all triangles in the corresponding dual subdivision. We just multiply positive integers attached to the trivalent vertices. The



contribution of a trivalent vertex equals $w_1w_2|\det(v_1,v_2)|$, with w_i are the weights of the adjacent edges and v_i are their weight vectors. That formula is independent of the choice made because of the balancing condition $w_1v_1 + w_2v_2 + w_3v_3 = 0$.

EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R.<x,y> = PolynomialRing(T)
sage: p1 = R(2)*x^2 + x*y + R(2)*y^2 + x + R(-1)*y + R(3)
sage: p1.tropical_variety().contribution()
1
sage: p2 = R(-1/3)*x^2 + R(1)*x*y + R(1)*y^2 + R(-1/3)*x + R(1/3)
sage: p2.tropical_variety().contribution()
16
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> p1 = R(Integer(2))*x**Integer(2) + x*y + R(Integer(2))*y**Integer(2) + x␣
↪+ R(-Integer(1))*y + R(Integer(3))
>>> p1.tropical_variety().contribution()
1
>>> p2 = R(-Integer(1)/Integer(3))*x**Integer(2) + R(Integer(1))*x*y +␣
↪R(Integer(1))*y**Integer(2) + R(-Integer(1)/Integer(3))*x + R(Integer(1)/
↪Integer(3))
>>> p2.tropical_variety().contribution()
16
```

genus()

Return the genus of `self`.

Let $t(C)$ be the number of trivalent vertices, and let $r(C)$ be the number of unbounded edges of C. The genus of simple tropical curve C is defined by the formula:

$$g(C) = \frac{1}{2}t(C) - \frac{1}{2}r(C) + 1.$$

EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R.<x,y> = PolynomialRing(T)
sage: p1 = x^2 + y^2 + x*y
sage: p1.tropical_variety().genus()
1
sage: p2 = R(2)*x^2 + x*y + R(2)*y^2 + x + R(-1)*y + R(3)
sage: p2.tropical_variety().genus()
0
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> p1 = x**Integer(2) + y**Integer(2) + x*y
>>> p1.tropical_variety().genus()
1
>>> p2 = R(Integer(2))*x**Integer(2) + x*y + R(Integer(2))*y**Integer(2) + x␣
↪+ R(-Integer(1))*y + R(Integer(3))
>>> p2.tropical_variety().genus()
0
```

is_simple()

Return `True` if `self` is simple and `False` otherwise.

A tropical curve C is called simple if each vertex is either trivalent or is locally the intersection of two line segments. Equivalently, C is simple if the corresponding subdivision consists only of triangles and parallelograms.

EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R.<x,y> = PolynomialRing(T)
sage: p1 = R(0) + x + y + x*y + x^2*y + x*y^2
sage: p1.tropical_variety().is_simple()
False
sage: p2 = R(2)*x^2 + x*y + R(2)*y^2 + x + R(-1)*y + R(3)
sage: p2.tropical_variety().is_simple()
True
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> p1 = R(Integer(0)) + x + y + x*y + x**Integer(2)*y + x*y**Integer(2)
>>> p1.tropical_variety().is_simple()
False
>>> p2 = R(Integer(2))*x**Integer(2) + x*y + R(Integer(2))*y**Integer(2) + x␣
↪+ R(-Integer(1))*y + R(Integer(3))
>>> p2.tropical_variety().is_simple()
True
```

is_smooth()

Return `True` if `self` is smooth and `False` otherwise.

Suppose C is a tropical curve of degree d. A tropical curve C is smooth if the dual subdivision of C consists of d^2 triangles each having unit area $1/2$. This is equivalent with C having d^2 vertices. These vertices are necessarily trivalent (has three adjacent edges).

EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R.<x,y> = PolynomialRing(T)
sage: p1 = x^2 + x + R(1)
sage: p1.tropical_variety().is_smooth()
False
sage: p2 = R(2)*x^2 + x*y + R(2)*y^2 + x + R(-1)*y + R(3)
sage: p2.tropical_variety().is_smooth()
True
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> p1 = x**Integer(2) + x + R(Integer(1))
>>> p1.tropical_variety().is_smooth()
False
>>> p2 = R(Integer(2))*x**Integer(2) + x*y + R(Integer(2))*y**Integer(2) + x␣
↪+ R(-Integer(1))*y + R(Integer(3))
>>> p2.tropical_variety().is_smooth()
True
```

plot()

Return the plot of `self`.

Generates a visual representation of the tropical curve in cartesian coordinates. The plot shows piecewise-linear segments representing each components. The axes are centered around the vertices.

OUTPUT:

A Graphics object. The weight of the component will be written if it is greater or equal than 2. The weight is written near the vertex.

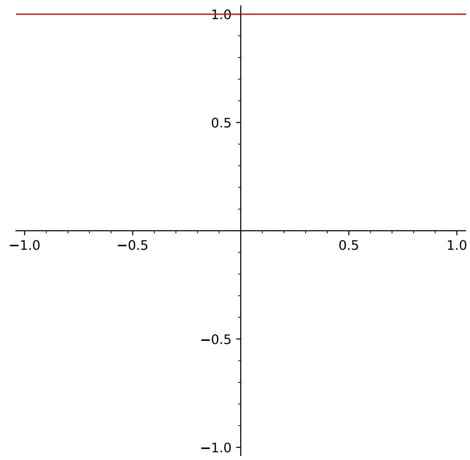
EXAMPLES:

A polynomial with only two terms will give one straight line:

```
sage: T = TropicalSemiring(QQ)
sage: R.<x,y> = PolynomialRing(T)
sage: (y+R(1)).tropical_variety().components()
[[(t1, 1), [-Infinity < t1, t1 < +Infinity], 1]]
sage: (y+R(1)).tropical_variety().plot()
Graphics object consisting of 1 graphics primitive
```

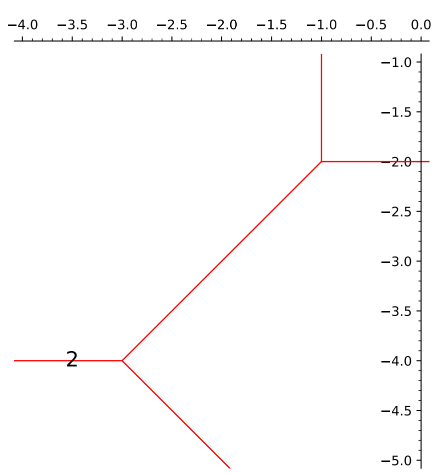
```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> (y+R(Integer(1))).tropical_variety().components()
[[(t1, 1), [-Infinity < t1, t1 < +Infinity], 1]]
>>> (y+R(Integer(1))).tropical_variety().plot()
Graphics object consisting of 1 graphics primitive
```

An intriguing and fascinating tropical curve can be obtained with a more complex tropical polynomial:




```
sage: p1 = R(1) + R(2)*x + R(3)*y + R(6)*x*y + R(10)*x*y^2
sage: p1.tropical_variety().components()
[[(-1, t1), [-2 <= t1], 1],
[(t1, -2), [-1 <= t1], 1],
[(t1 + 1, t1), [-4 <= t1, t1 <= -2], 1],
[(t1, -4), [t1 <= -3], 2],
[(-t1 - 7, t1), [t1 <= -4], 1]]
sage: p1.tropical_variety().plot()
Graphics object consisting of 6 graphics primitives
```

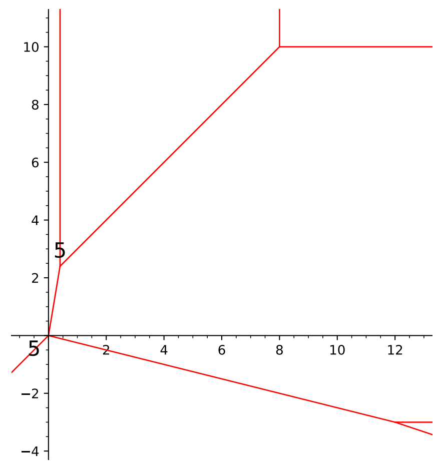
```
>>> from sage.all import *
>>> p1 = R(Integer(1)) + R(Integer(2))*x + R(Integer(3))*y +␣
↪R(Integer(6))*x*y + R(Integer(10))*x*y**Integer(2)
>>> p1.tropical_variety().components()
[[(-1, t1), [-2 <= t1], 1],
[(t1, -2), [-1 <= t1], 1],
[(t1 + 1, t1), [-4 <= t1, t1 <= -2], 1],
[(t1, -4), [t1 <= -3], 2],
[(-t1 - 7, t1), [t1 <= -4], 1]]
>>> p1.tropical_variety().plot()
Graphics object consisting of 6 graphics primitives
```



Another tropical polynomial with numerous components, resulting in a more intricate structure:

```
sage: p2 = (x^6 + R(4)*x^4*y^2 + R(2)*x^3*y^3 + R(3)*x^2*y^4
....:       + x*y^5 + R(7)*x^2 + R(5)*x*y + R(3)*y^2 + R(2)*x
....:       + y + R(10))
sage: p2.tropical_variety().plot() # long time
Graphics object consisting of 11 graphics primitives
```

```
>>> from sage.all import *
>>> p2 = (x**Integer(6) + R(Integer(4))*x**Integer(4)*y**Integer(2) +␣
↪R(Integer(2))*x**Integer(3)*y**Integer(3) +␣
↪R(Integer(3))*x**Integer(2)*y**Integer(4)
...       + x*y**Integer(5) + R(Integer(7))*x**Integer(2) + R(Integer(5))*x*y␣
↪+ R(Integer(3))*y**Integer(2) + R(Integer(2))*x
...       + y + R(Integer(10)))
>>> p2.tropical_variety().plot() # long time
Graphics object consisting of 11 graphics primitives
```



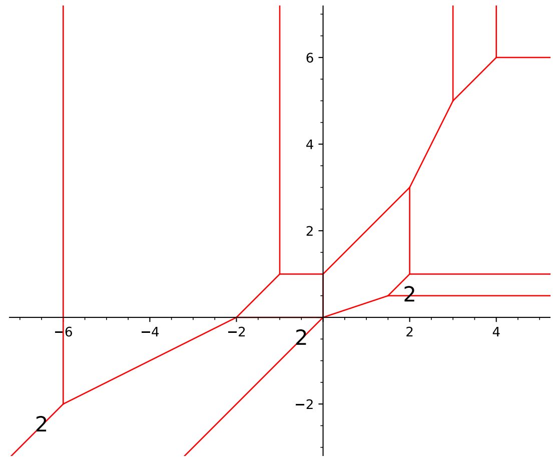

```
sage: p3 = (R(8) + R(4)*x + R(2)*y + R(1)*x^2 + x*y + R(1)*y^2
....:       + R(2)*x^3 + x^2*y + x*y^2 + R(4)*y^3 + R(8)*x^4
....:       + R(4)*x^3*y + x^2*y^2 + R(2)*x*y^3 + y^4)
sage: p3.tropical_variety().plot() # long time
Graphics object consisting of 23 graphics primitives
```

```
>>> from sage.all import *
>>> p3 = (R(Integer(8)) + R(Integer(4))*x + R(Integer(2))*y +␣
↪R(Integer(1))*x**Integer(2) + x*y + R(Integer(1))*y**Integer(2)
...       + R(Integer(2))*x**Integer(3) + x**Integer(2)*y + x*y**Integer(2) +␣
↪R(Integer(4))*y**Integer(3) + R(Integer(8))*x**Integer(4)
...       + R(Integer(4))*x**Integer(3)*y + x**Integer(2)*y**Integer(2) +␣
↪R(Integer(2))*x*y**Integer(3) + y**Integer(4))
>>> p3.tropical_variety().plot() # long time
Graphics object consisting of 23 graphics primitives
```

vertices()

Return all vertices of `self`, which is the point where three or more edges intersect.

OUTPUT: a set of (x, y) points



EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R.<x,y> = PolynomialRing(T)
sage: p1 = x + y
sage: p1.tropical_variety().vertices()
set()
sage: p2 = R(-2)*x^2 + R(-1)*x + R(1/2)*y + R(1/6)
sage: p2.tropical_variety().vertices()
{(1, -1/2), (7/6, -1/3)}
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> p1 = x + y
>>> p1.tropical_variety().vertices()
set()
>>> p2 = R(-Integer(2))*x**Integer(2) + R(-Integer(1))*x + R(Integer(1)/
↪Integer(2))*y + R(Integer(1)/Integer(6))
>>> p2.tropical_variety().vertices()
{(1, -1/2), (7/6, -1/3)}
```

weight_vectors()

Return the weight vectors for all vertices of `self`.

Weight vectors are a list of vectors associated with each vertex of the curve. Each vector corresponds to an edge emanating from that vertex and points in the direction of the edge.

Suppose v is a vertex adjacent to the edges $e_1, \ldots, e_k$ with respective weights $w_1, \ldots, w_k$. Every edge e_i is contained in a line (component) defined by an equation. Therefore, there exists a unique integer vector $v_i = (\alpha, \beta)$ in the direction of e_i such that $\gcd(\alpha, \beta) = 1$. Then, each vertex v yield the vectors $w_1 v_1, \ldots, w_k v_k$. These vectors will satisfy the following balancing condition: $\sum_{i=1}^{k} w_i v_i = 0$.

OUTPUT:

A dictionary where the keys represent the vertices, and the values are lists of vectors.

EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R.<x,y> = PolynomialRing(T)
sage: p1 = R(-2)*x^2 + R(-1)*x + R(1/2)*y + R(1/6)
sage: p1.tropical_variety().weight_vectors()
{(1, -1/2): [(0, 1), (-1, -2), (1, 1)],
 (7/6, -1/3): [(-1, -1), (0, 1), (1, 0)]}

sage: p2 = R(2)*x^2 + x*y + R(2)*y^2 + x + R(-1)*y + R(3)
sage: p2.tropical_variety().weight_vectors()
{(-2, 0): [(-1, -1), (0, 1), (1, 0)],
 (-1, -3): [(-1, -1), (0, 1), (1, 0)],
 (-1, 0): [(-1, 0), (0, -1), (1, 1)],
 (3, 4): [(-1, -1), (0, 1), (1, 0)]}
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> p1 = R(-Integer(2))*x**Integer(2) + R(-Integer(1))*x + R(Integer(1)/
↪Integer(2))*y + R(Integer(1)/Integer(6))
>>> p1.tropical_variety().weight_vectors()
{(1, -1/2): [(0, 1), (-1, -2), (1, 1)],
 (7/6, -1/3): [(-1, -1), (0, 1), (1, 0)]}

>>> p2 = R(Integer(2))*x**Integer(2) + x*y + R(Integer(2))*y**Integer(2) + x␣
↪+ R(-Integer(1))*y + R(Integer(3))
>>> p2.tropical_variety().weight_vectors()
{(-2, 0): [(-1, -1), (0, 1), (1, 0)],
 (-1, -3): [(-1, -1), (0, 1), (1, 0)],
 (-1, 0): [(-1, 0), (0, -1), (1, 1)],
 (3, 4): [(-1, -1), (0, 1), (1, 0)]}
```

class sage.rings.semirings.tropical_variety.**TropicalSurface**(*poly*)

Bases: *TropicalVariety*

A tropical surface in $\mathbf{R}^3$.

The tropical surface consists of planar regions and facets, which we can call cells. These cells are connected in such a way that they form a piecewise linear structure embedded in three-dimensional space. These cells meet along edges, where the balancing condition is satisfied. This balancing condition ensures that the sum of the outgoing normal vectors at each edge is zero, reflecting the equilibrium.

EXAMPLES:

```
sage: T = TropicalSemiring(QQ, use_min=False)
sage: R.<x,y,z> = PolynomialRing(T)
sage: p1 = x + y + z + R(0)
sage: tv = p1.tropical_variety(); tv
Tropical surface of 0*x + 0*y + 0*z + 0
sage: tv.components()
[[(t1, t1, t2), [t2 <= t1, 0 <= t1], 1],
[(t1, t2, t1), [max(0, t2) <= t1], 1],
[(0, t1, t2), [t2 <= 0, t1 <= 0], 1],
[(t1, t2, t2), [max(0, t1) <= t2], 1],
[(t1, 0, t2), [t2 <= 0, t1 <= 0], 1],
```

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```
[(t1, t2, 0), [t1 <= 0, t2 <= 0], 1]]
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=False)
>>> R = PolynomialRing(T, names=('x', 'y', 'z',)); (x, y, z,) = R._first_ngens(3)
>>> p1 = x + y + z + R(Integer(0))
>>> tv = p1.tropical_variety(); tv
Tropical surface of 0*x + 0*y + 0*z + 0
>>> tv.components()
[[(t1, t1, t2), [t2 <= t1, 0 <= t1], 1],
[(t1, t2, t1), [max(0, t2) <= t1], 1],
[(0, t1, t2), [t2 <= 0, t1 <= 0], 1],
[(t1, t2, t2), [max(0, t1) <= t2], 1],
[(t1, 0, t2), [t2 <= 0, t1 <= 0], 1],
[(t1, t2, 0), [t1 <= 0, t2 <= 0], 1]]
```

plot(*color='random'*)

Return the plot of `self` by constructing a polyhedron from vertices in `self.polygon_vertices()`.

INPUT:

- `color` – string or tuple that represent a color (default: `random`); `random` means each polygon will be assigned a different color. If instead a specific `color` is provided, then all polygon will be given the same color.

OUTPUT: Graphics3d Object

EXAMPLES:

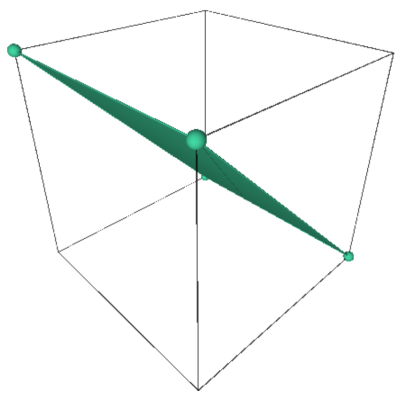
A tropical surface that consist of only one cell:

```
sage: T = TropicalSemiring(QQ)
sage: R.<x,y,z> = PolynomialRing(T)
sage: p1 = x + z
sage: tv = p1.tropical_variety()
sage: tv.plot()
Graphics3d Object
```

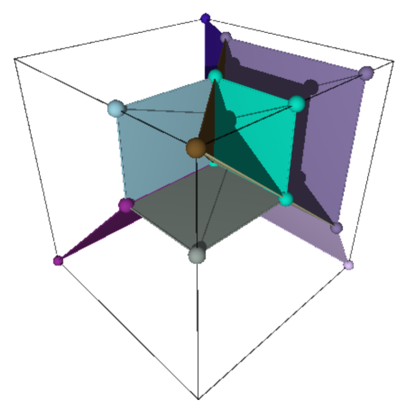
```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> p1 = x + z
>>> tv = p1.tropical_variety()
>>> tv.plot()
Graphics3d Object
```

A tropical surface with multiple cells that exhibit complex and intriguing geometric structures:

```
sage: p2 = x^2 + x + y + z + R(1)
sage: tv = p2.tropical_variety()
sage: tv.plot() # long time
Graphics3d Object
```



```
>>> from sage.all import *
>>> p2 = x**Integer(2) + x + y + z + R(Integer(1))
>>> tv = p2.tropical_variety()
>>> tv.plot() # long time
Graphics3d Object
```



class sage.rings.semirings.tropical_variety.**TropicalVariety**(*poly*)

Bases: UniqueRepresentation, SageObject

A tropical variety in $\mathbf{R}^n$.

A tropical variety is defined as a corner locus of tropical polynomial function. This means it consist of all points in $\mathbf{R}^n$ for which the minimum (maximum) of the function is attained at least twice.

We represent the tropical variety as a list of lists, where the inner list consist of three parts. The first one is a parametric equations for tropical roots. The second one is the condition for parameters. The third one is the order of the corresponding component.

INPUT:

- poly – a TropicalMPolynomial

ALGORITHM:

We need to determine a corner locus of this tropical polynomial function, which is all points $(x_1, x_2, \ldots, x_n)$ for which the maximum (minimum) is obtained at least twice. First, we convert each monomial to its corresponding linear function. Then for each two monomials of polynomial, we find the points where their values are equal. Since we attempt to solve the equality of two equations in n variables, the solution set will be described by $n-1$ parameters.

Next, we need to check if the value of previous two monomials at the points in solution set is really the maximum (minimum) of function. We do this by solving the inequality of the previous monomial with all other monomials in the polynomial after substituting the parameter. This will give us the condition of parameters. Each of this condition is then combined by union operator. If this final condition is not an empty set, then it represent one component of tropical root. Then we calculate the weight of this particular component by the maximum of gcd of the numbers $|i-k|$ and $|j-l|$ for all pairs (i,j) and (k,l) such that the value of on this component is given by the corresponding monomials.

EXAMPLES:

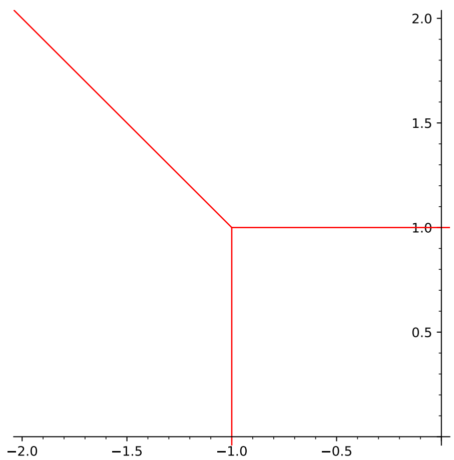
We construct a tropical variety in $\mathbf{R}^2$, where it is called a tropical curve:

```
sage: T = TropicalSemiring(QQ, use_min=False)
sage: R.<x,y> = PolynomialRing(T)
sage: p1 = R(1)*x + x*y + R(0); p1
0*x*y + 1*x + 0
sage: tv = p1.tropical_variety(); tv
Tropical curve of 0*x*y + 1*x + 0
sage: tv.components()
[[(t1, 1), [t1 >= -1], 1], [(-1, t1), [t1 <= 1], 1], [(-t1, t1), [t1 >= 1], 1]]
sage: tv.vertices()
{(-1, 1)}
sage: tv.plot()
Graphics object consisting of 3 graphics primitives
```

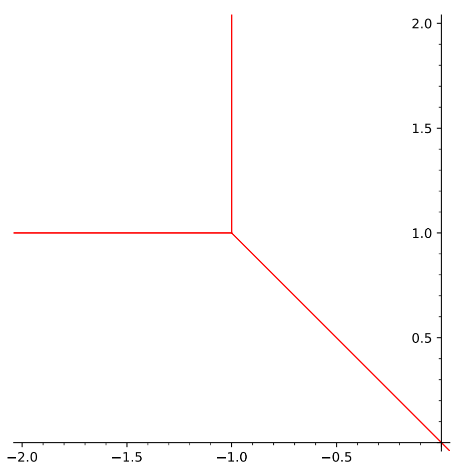
```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=False)
>>> R = PolynomialRing(T, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> p1 = R(Integer(1))*x + x*y + R(Integer(0)); p1
0*x*y + 1*x + 0
>>> tv = p1.tropical_variety(); tv
Tropical curve of 0*x*y + 1*x + 0
>>> tv.components()
[[(t1, 1), [t1 >= -1], 1], [(-1, t1), [t1 <= 1], 1], [(-t1, t1), [t1 >= 1], 1]]
>>> tv.vertices()
{(-1, 1)}
>>> tv.plot()
Graphics object consisting of 3 graphics primitives
```

A slightly different result will be obtained if we use min-plus algebra for the base tropical semiring:

```
sage: T = TropicalSemiring(QQ, use_min=True)
sage: R.<x,y> = PolynomialRing(T)
sage: p1 = R(1)*x + x*y + R(0)
sage: tv = p1.tropical_variety(); tv
Tropical curve of 0*x*y + 1*x + 0
sage: tv.components()
[[(t1, 1), [t1 <= -1], 1], [(-1, t1), [t1 >= 1], 1], [(-t1, t1), [t1 <= 1], 1]]
sage: tv.plot()
Graphics object consisting of 3 graphics primitives
```




```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=True)
>>> R = PolynomialRing(T, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> p1 = R(Integer(1))*x + x*y + R(Integer(0))
>>> tv = p1.tropical_variety(); tv
Tropical curve of 0*x*y + 1*x + 0
>>> tv.components()
[[(t1, 1), [t1 <= -1], 1], [(-1, t1), [t1 >= 1], 1], [(-t1, t1), [t1 <= 1], 1]]
>>> tv.plot()
Graphics object consisting of 3 graphics primitives
```



Tropical variety can consist of multiple components with varying orders:

```
sage: T = TropicalSemiring(QQ, use_min=False)
sage: R.<x,y> = PolynomialRing(T)
sage: p1 = R(7) + T(4)*x + y + R(4)*x*y + R(3)*y^2 + R(-3)*x^2
sage: tv = p1.tropical_variety(); tv
Tropical curve of (-3)*x^2 + 4*x*y + 3*y^2 + 4*x + 0*y + 7
sage: tv.components()
[[(3, t1), [t1 <= 0], 1],
[(-t1 + 3, t1), [0 <= t1, t1 <= 2], 1],
[(t1, 2), [t1 <= 1], 2],
```

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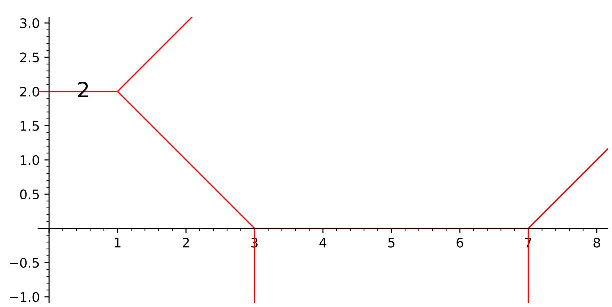
```
[(t1, 0), [3 <= t1, t1 <= 7], 1],
[(7, t1), [t1 <= 0], 1],
[(t1 - 1, t1), [2 <= t1], 1],
[(t1 + 7, t1), [0 <= t1], 1]]
sage: tv.plot()
Graphics object consisting of 8 graphics primitives
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ, use_min=False)
>>> R = PolynomialRing(T, names=('x', 'y',)); (x, y,) = R._first_ngens(2)
>>> p1 = R(Integer(7)) + T(Integer(4))*x + y + R(Integer(4))*x*y +␣
↪R(Integer(3))*y**Integer(2) + R(-Integer(3))*x**Integer(2)
>>> tv = p1.tropical_variety(); tv
Tropical curve of (-3)*x^2 + 4*x*y + 3*y^2 + 4*x + 0*y + 7
>>> tv.components()
[[(3, t1), [t1 <= 0], 1],
[(-t1 + 3, t1), [0 <= t1, t1 <= 2], 1],
[(t1, 2), [t1 <= 1], 2],
[(t1, 0), [3 <= t1, t1 <= 7], 1],
[(7, t1), [t1 <= 0], 1],
[(t1 - 1, t1), [2 <= t1], 1],
[(t1 + 7, t1), [0 <= t1], 1]]
>>> tv.plot()
Graphics object consisting of 8 graphics primitives
```



If the tropical polynomial have $n > 2$ variables, then the result will be a tropical hypersurface embedded in a real space $\mathbf{R}^n$:

```
sage: T = TropicalSemiring(QQ)
sage: R.<w,x,y,z> = PolynomialRing(T)
sage: p1 = x*y + R(-1/2)*x*z + R(4)*z^2 + w*x
sage: tv = p1.tropical_variety(); tv
Tropical hypersurface of 0*w*x + 0*x*y + (-1/2)*x*z + 4*z^2
sage: tv.components()
[[(t1, t2, t3 - 1/2, t3), [t2 - 9/2 <= t3, t3 <= t1 + 1/2, t2 - 5 <= t1], 1],
[(t1, 2*t2 - t3 + 4, t3, t2), [t3 + 1/2 <= t2, t3 <= t1], 1],
[(t1, t2, t1, t3), [max(t1 + 1/2, 1/2*t1 + 1/2*t2 - 2) <= t3], 1],
```

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```
[(t1, t2 + 9/2, t3, t2), [t2 <= min(t3 + 1/2, t1 + 1/2)], 1],
[(t1 - 1/2, t2, t3, t1), [t2 - 9/2 <= t1, t1 <= t3 + 1/2, t2 - 5 <= t3], 1],
[(2*t1 - t2 + 4, t2, t3, t1), [t1 <= min(1/2*t2 + 1/2*t3 - 2, t2 - 9/2)], 1]]
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('w', 'x', 'y', 'z',)); (w, x, y, z,) = R._first_
↪ngens(4)
>>> p1 = x*y + R(-Integer(1)/Integer(2))*x*z + R(Integer(4))*z**Integer(2) + w*x
>>> tv = p1.tropical_variety(); tv
Tropical hypersurface of 0*w*x + 0*x*y + (-1/2)*x*z + 4*z^2
>>> tv.components()
[[(t1, t2, t3 - 1/2, t3), [t2 - 9/2 <= t3, t3 <= t1 + 1/2, t2 - 5 <= t1], 1],
[(t1, 2*t2 - t3 + 4, t3, t2), [t3 + 1/2 <= t2, t3 <= t1], 1],
[(t1, t2, t1, t3), [max(t1 + 1/2, 1/2*t1 + 1/2*t2 - 2) <= t3], 1],
[(t1, t2 + 9/2, t3, t2), [t2 <= min(t3 + 1/2, t1 + 1/2)], 1],
[(t1 - 1/2, t2, t3, t1), [t2 - 9/2 <= t1, t1 <= t3 + 1/2, t2 - 5 <= t3], 1],
[(2*t1 - t2 + 4, t2, t3, t1), [t1 <= min(1/2*t2 + 1/2*t3 - 2, t2 - 9/2)], 1]]
```

components()

Return all components of `self`.

EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R.<a,x,y,z> = PolynomialRing(T)
sage: tv = (a+x+y+z).tropical_variety()
sage: tv.components()
[[(t1, t1, t2, t3), [t1 <= min(t3, t2)], 1],
 [(t1, t2, t1, t3), [t1 <= t3, t1 <= t2], 1],
 [(t1, t2, t3, t1), [t1 <= min(t3, t2)], 1],
 [(t1, t2, t2, t3), [t2 <= t3, t2 <= t1], 1],
 [(t1, t2, t3, t2), [t2 <= min(t3, t1)], 1],
 [(t1, t2, t3, t3), [t3 <= min(t1, t2)], 1]]
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('a', 'x', 'y', 'z',)); (a, x, y, z,) = R._
↪first_ngens(4)
>>> tv = (a+x+y+z).tropical_variety()
>>> tv.components()
[[(t1, t1, t2, t3), [t1 <= min(t3, t2)], 1],
 [(t1, t2, t1, t3), [t1 <= t3, t1 <= t2], 1],
 [(t1, t2, t3, t1), [t1 <= min(t3, t2)], 1],
 [(t1, t2, t2, t3), [t2 <= t3, t2 <= t1], 1],
 [(t1, t2, t3, t2), [t2 <= min(t3, t1)], 1],
 [(t1, t2, t3, t3), [t3 <= min(t1, t2)], 1]]
```

dimension()

Return the dimension of `self`.

EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R.<a,x,y,z> = PolynomialRing(T)
sage: p1 = x*y + R(-1)*x*z
sage: p1.tropical_variety().dimension()
4
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('a', 'x', 'y', 'z',)); (a, x, y, z,) = R._
↪first_ngens(4)
>>> p1 = x*y + R(-Integer(1))*x*z
>>> p1.tropical_variety().dimension()
4
```

number_of_components()

Return the number of components that make up `self`.

EXAMPLES:

```
sage: T = TropicalSemiring(QQ)
sage: R.<a,x,y,z> = PolynomialRing(T)
sage: p1 = x*y*a + x*z + y^2 + a*x + y + z
sage: p1.tropical_variety().number_of_components()
13
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('a', 'x', 'y', 'z',)); (a, x, y, z,) = R._
↪first_ngens(4)
>>> p1 = x*y*a + x*z + y**Integer(2) + a*x + y + z
>>> p1.tropical_variety().number_of_components()
13
```

weight_vectors()

Return the weight vectors for each unique intesection of components of `self`.

Weight vectors are a list of vectors associated with each unique intersection of the components of tropical variety. Each vector is a normal vector to a component with respect to the unique intersection lying within that component.

Assume `self` is a n-dimensional tropical variety. Suppose L is an intersection lying within the components $S_1, \ldots, S_k$ with respective weights $w_1, \ldots, w_k$. This L is a linear structure in $\mathbf{R}^{n-1}$ and has $n-1$ direction vectors $d_1, d_2, \ldots, d_{n-1}$. Each component $S_1, \ldots, S_k$ has a normal vector $n_1, \ldots, n_k$. Then, we scale each normal vector to an integer vector such that the greatest common divisor of its elements is 1.

The weight vector of a component S_i with respect to L can be found by calculating the cross product between direction vectors of L and normal vector n_i.These vectors will satisfy the balancing condition $\sum_{i=1}^{k} w_k v_k = 0$.

OUTPUT:

A tuple of three dictionaries. The first dictionary contains equations representing the intersections. The second dictionary contains indices of components that contains the intersection. The third dictionary contains lists of vectors.

EXAMPLES:

Weight vectors of tropical surface:

```
sage: T = TropicalSemiring(QQ)
sage: R.<x,y,z> = PolynomialRing(T)
sage: p = x^2 + R(-1)*y + z + R(1)
sage: tv = p.tropical_variety()
sage: tv.weight_vectors()
({0: ((1/2*u2, u2 + 1, u2), {u2 <= 1}),
  1: ((1/2, 2, u2), {1 <= u2}),
  2: ((1/2, u2, 1), {2 <= u2}),
  3: ((u1, 2, 1), {(1/2) <= u1})},
 {0: [0, 1, 3], 1: [0, 2, 4], 2: [1, 2, 5], 3: [3, 4, 5]},
 {0: [(1, 2, -5/2), (1, -5/2, 2), (-2, 1/2, 1/2)],
  1: [(-1, -2, 0), (0, 2, 0), (1, 0, 0)],
  2: [(1, 0, 2), (0, 0, -2), (-1, 0, 0)],
  3: [(0, 1, 1), (0, 0, -1), (0, -1, 0)]})
```

```
>>> from sage.all import *
>>> T = TropicalSemiring(QQ)
>>> R = PolynomialRing(T, names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> p = x**Integer(2) + R(-Integer(1))*y + z + R(Integer(1))
>>> tv = p.tropical_variety()
>>> tv.weight_vectors()
({0: ((1/2*u2, u2 + 1, u2), {u2 <= 1}),
  1: ((1/2, 2, u2), {1 <= u2}),
  2: ((1/2, u2, 1), {2 <= u2}),
  3: ((u1, 2, 1), {(1/2) <= u1})},
 {0: [0, 1, 3], 1: [0, 2, 4], 2: [1, 2, 5], 3: [3, 4, 5]},
 {0: [(1, 2, -5/2), (1, -5/2, 2), (-2, 1/2, 1/2)],
  1: [(-1, -2, 0), (0, 2, 0), (1, 0, 0)],
  2: [(1, 0, 2), (0, 0, -2), (-1, 0, 0)],
  3: [(0, 1, 1), (0, 0, -1), (0, -1, 0)]})
```

CHAPTER

EIGHT

BOOLEAN POLYNOMIALS

8.1 Boolean Polynomials

Elements of the quotient ring

$$\mathbf{F}_2[x_1, ..., x_n]/ < x_1^2 + x_1, ..., x_n^2 + x_n > .$$

are called boolean polynomials. Boolean polynomials arise naturally in cryptography, coding theory, formal logic, chip design and other areas. This implementation is a thin wrapper around the PolyBoRi library by Michael Brickenstein and Alexander Dreyer.

"Boolean polynomials can be modelled in a rather simple way, with both coefficients and degree per variable lying in `{0, 1}`. The ring of Boolean polynomials is, however, not a polynomial ring, but rather the quotient ring of the polynomial ring over the field with two elements modulo the field equations $x^2 = x$ for each variable x. Therefore, the usual polynomial data structures seem not to be appropriate for fast Groebner basis computations. We introduce a specialised data structure for Boolean polynomials based on zero-suppressed binary decision diagrams (ZDDs), which is capable of handling these polynomials more efficiently with respect to memory consumption and also computational speed. Furthermore, we concentrate on high-level algorithmic aspects, taking into account the new data structures as well as structural properties of Boolean polynomials." - [BD2007]

For details on the internal representation of polynomials see

> http://polybori.sourceforge.net/zdd.html

AUTHORS:

- Michael Brickenstein: PolyBoRi author
- Alexander Dreyer: PolyBoRi author
- Burcin Erocal <burcin@erocal.org>: main Sage wrapper author
- Martin Albrecht <malb@informatik.uni-bremen.de>: some contributions to the Sage wrapper
- Simon King <simon.king@uni-jena.de>: Adopt the new coercion model. Fix conversion from univariate polynomial rings. Pickling of `BooleanMonomialMonoid` (via `UniqueRepresentation`) and `BooleanMonomial`.
- Charles Bouillaguet <charles.bouillaguet@gmail.com>: minor changes to improve compatibility with MPolynomial and make the variety() function work on ideals of BooleanPolynomial's.

EXAMPLES:

Consider the ideal

$$< ab + cd + 1, ace + de, abe + ce, bc + cde + 1 > .$$

First, we compute the lexicographical Groebner basis in the polynomial ring

$$R = \mathbf{F}_2[a, b, c, d, e].$$

```
sage: P.<a,b,c,d,e> = PolynomialRing(GF(2), 5, order='lex')
sage: I1 = ideal([a*b + c*d + 1, a*c*e + d*e, a*b*e + c*e, b*c + c*d*e + 1])
sage: for f in I1.groebner_basis():
....:   f
a + c^2*d + c + d^2*e
b*c + d^3*e^2 + d^3*e + d^2*e^2 + d*e + e + 1
b*e + d*e^2 + d*e + e
c*e + d^3*e^2 + d^3*e + d^2*e^2 + d*e
d^4*e^2 + d^4*e + d^3*e + d^2*e^2 + d^2*e + d*e + e
```

```
>>> from sage.all import *
>>> P = PolynomialRing(GF(Integer(2)), Integer(5), order='lex', names=('a', 'b', 'c',
↪'d', 'e',)); (a, b, c, d, e,) = P._first_ngens(5)
>>> I1 = ideal([a*b + c*d + Integer(1), a*c*e + d*e, a*b*e + c*e, b*c + c*d*e +␣
↪Integer(1)])
>>> for f in I1.groebner_basis():
...   f
a + c^2*d + c + d^2*e
b*c + d^3*e^2 + d^3*e + d^2*e^2 + d*e + e + 1
b*e + d*e^2 + d*e + e
c*e + d^3*e^2 + d^3*e + d^2*e^2 + d*e
d^4*e^2 + d^4*e + d^3*e + d^2*e^2 + d^2*e + d*e + e
```

If one wants to solve this system over the algebraic closure of $\mathbf{F}_2$ then this Groebner basis was the one to consider. If one wants solutions over $\mathbf{F}_2$ only then one adds the field polynomials to the ideal to force the solutions in $\mathbf{F}_2$.

```
sage: J = I1 + sage.rings.ideal.FieldIdeal(P)
sage: for f in J.groebner_basis():
....:   f
a + d + 1
b + 1
c + 1
d^2 + d
e
```

```
>>> from sage.all import *
>>> J = I1 + sage.rings.ideal.FieldIdeal(P)
>>> for f in J.groebner_basis():
...   f
a + d + 1
b + 1
c + 1
d^2 + d
e
```

So the solutions over $\mathbf{F}_2$ are $\{e = 0, d = 1, c = 1, b = 1, a = 0\}$ and $\{e = 0, d = 0, c = 1, b = 1, a = 1\}$.

We can express the restriction to $\mathbf{F}_2$ by considering the quotient ring. If I is an ideal in $\mathbf{F}[x_1, ..., x_n]$ then the ideals in the quotient ring $\mathbf{F}[x_1, ..., x_n]/I$ are in one-to-one correspondence with the ideals of $\mathbf{F}[x_0, ..., x_n]$ containing I (that is, the ideals J satisfying $I \subset J \subset P$).

```
sage: Q = P.quotient( sage.rings.ideal.FieldIdeal(P) )
sage: I2 = ideal([Q(f) for f in I1.gens()])
```

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```
sage: for f in I2.groebner_basis():
....:     f
abar + dbar + 1
bbar + 1
cbar + 1
ebar
```

```
>>> from sage.all import *
>>> Q = P.quotient( sage.rings.ideal.FieldIdeal(P) )
>>> I2 = ideal([Q(f) for f in I1.gens()])
>>> for f in I2.groebner_basis():
...     f
abar + dbar + 1
bbar + 1
cbar + 1
ebar
```

This quotient ring is exactly what PolyBoRi handles well:

```
sage: B.<a,b,c,d,e> = BooleanPolynomialRing(5, order='lex')
sage: I2 = ideal([B(f) for f in I1.gens()])
sage: for f in I2.groebner_basis():
....:   f
a + d + 1
b + 1
c + 1
e
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(5), order='lex', names=('a', 'b', 'c', 'd', 'e',
↪)); (a, b, c, d, e,) = B._first_ngens(5)
>>> I2 = ideal([B(f) for f in I1.gens()])
>>> for f in I2.groebner_basis():
...   f
a + d + 1
b + 1
c + 1
e
```

Note that `d^2 + d` is not representable in `B == Q`. Also note, that PolyBoRi cannot play out its strength in such small examples, i.e. working in the polynomial ring might be faster for small examples like this.

8.1.1 Implementation specific notes

PolyBoRi comes with a Python wrapper. However this wrapper does not match Sage's style and is written using Boost. Thus Sage's wrapper is a reimplementation of Python bindings to PolyBoRi's C++ library. This interface is written in Cython like all of Sage's C/C++ library interfaces. An interface in PolyBoRi style is also provided which is effectively a reimplementation of the official Boost wrapper in Cython. This means that some functionality of the official wrapper might be missing from this wrapper and this wrapper might have bugs not present in the official Python interface.

8.1.2 Access to the original PolyBoRi interface

The re-implementation PolyBoRi's native wrapper is available to the user too:

```
sage: from sage.rings.polynomial.pbori import *
sage: declare_ring([Block('x',2),Block('y',3)],globals())
Boolean PolynomialRing in x0, x1, y0, y1, y2
sage: r
Boolean PolynomialRing in x0, x1, y0, y1, y2
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori import *
>>> declare_ring([Block('x',Integer(2)),Block('y',Integer(3))],globals())
Boolean PolynomialRing in x0, x1, y0, y1, y2
>>> r
Boolean PolynomialRing in x0, x1, y0, y1, y2
```

```
sage: [Variable(i, r) for i in range(r.ngens())]
[x(0), x(1), y(0), y(1), y(2)]
```

```
>>> from sage.all import *
>>> [Variable(i, r) for i in range(r.ngens())]
[x(0), x(1), y(0), y(1), y(2)]
```

For details on this interface see:

> http://polybori.sourceforge.net/doc/tutorial/tutorial.html.

Also, the interface provides functions for compatibility with Sage accepting convenient Sage data types which are slower than their native PolyBoRi counterparts. For instance, sets of points can be represented as tuples of tuples (Sage) or as `BooleSet` (PolyBoRi) and naturally the second option is faster.

class `sage.rings.polynomial.pbori.pbori.`**`BooleConstant`**

Bases: `object`

Construct a boolean constant (modulo 2) from integer value:

INPUT:

- `i` – integer

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import BooleConstant
sage: [BooleConstant(i) for i in range(5)]
[0, 1, 0, 1, 0]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import BooleConstant
>>> [BooleConstant(i) for i in range(Integer(5))]
[0, 1, 0, 1, 0]
```

deg()

Get degree of boolean constant.

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import BooleConstant
sage: BooleConstant(0).deg()
-1
sage: BooleConstant(1).deg()
0
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import BooleConstant
>>> BooleConstant(Integer(0)).deg()
-1
>>> BooleConstant(Integer(1)).deg()
0
```

has_constant_part()

This is true for $BooleConstant(1)$.

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import BooleConstant
sage: BooleConstant(1).has_constant_part()
True
sage: BooleConstant(0).has_constant_part()
False
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import BooleConstant
>>> BooleConstant(Integer(1)).has_constant_part()
True
>>> BooleConstant(Integer(0)).has_constant_part()
False
```

is_constant()

This is always true for in this case.

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import BooleConstant
sage: BooleConstant(1).is_constant()
True
sage: BooleConstant(0).is_constant()
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import BooleConstant
>>> BooleConstant(Integer(1)).is_constant()
True
>>> BooleConstant(Integer(0)).is_constant()
True
```

is_one()

Check whether boolean constant is one.

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import BooleConstant
sage: BooleConstant(0).is_one()
False
sage: BooleConstant(1).is_one()
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import BooleConstant
>>> BooleConstant(Integer(0)).is_one()
False
>>> BooleConstant(Integer(1)).is_one()
True
```

is_zero()

Check whether boolean constant is zero.

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import BooleConstant
sage: BooleConstant(1).is_zero()
False
sage: BooleConstant(0).is_zero()
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import BooleConstant
>>> BooleConstant(Integer(1)).is_zero()
False
>>> BooleConstant(Integer(0)).is_zero()
True
```

variables()

Get variables (return always and empty tuple).

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import BooleConstant
sage: BooleConstant(0).variables()
()
sage: BooleConstant(1).variables()
()
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import BooleConstant
>>> BooleConstant(Integer(0)).variables()
()
>>> BooleConstant(Integer(1)).variables()
()
```

class sage.rings.polynomial.pbori.pbori.**BooleSet**

Bases: `object`

Return a new set of boolean monomials. This data type is also implemented on the top of ZDDs and allows to see polynomials from a different angle. Also, it makes high-level set operations possible, which are in most cases faster than operations handling individual terms, because the complexity of the algorithms depends only on the structure of the diagrams.

Objects of type *BooleanPolynomial* can easily be converted to the type *BooleSet* by using the member function *BooleanPolynomial.set()*.

INPUT:

- `param` – either a *CCuddNavigator*, a *BooleSet* or `None`
- `ring` – boolean polynomial ring

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import BooleSet
sage: B.<a,b,c,d> = BooleanPolynomialRing(4)
sage: BS = BooleSet(a.set())
sage: BS
{{a}}

sage: BS = BooleSet((a*b + c + 1).set())
sage: BS
{{a,b}, {c}, {}}

sage: from sage.rings.polynomial.pbori.pbori import *
sage: from sage.rings.polynomial.pbori.PyPolyBoRi import Monomial
sage: BooleSet([Monomial(B)])
{{}}
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import BooleSet
>>> B = BooleanPolynomialRing(Integer(4), names=('a', 'b', 'c', 'd',)); (a, b, c,␣
↪d,) = B._first_ngens(4)
>>> BS = BooleSet(a.set())
>>> BS
{{a}}

>>> BS = BooleSet((a*b + c + Integer(1)).set())
>>> BS
{{a,b}, {c}, {}}

>>> from sage.rings.polynomial.pbori.pbori import *
>>> from sage.rings.polynomial.pbori.PyPolyBoRi import Monomial
>>> BooleSet([Monomial(B)])
{{}}
```

Note

BooleSet prints as `{}` but are not Python dictionaries.

cartesian_product(*rhs*)

Return the Cartesian product of this set and the set `rhs`.

The Cartesian product of two sets X and Y is the set of all possible ordered pairs whose first component is a member of X and whose second component is a member of Y.

$$X \times Y = \{(x, y) | x \in X \text{ and } y \in Y\}.$$

EXAMPLES:

```
sage: B = BooleanPolynomialRing(5,'x')
sage: x0,x1,x2,x3,x4 = B.gens()
sage: f = x1*x2+x2*x3
sage: s = f.set(); s
{{x1,x2}, {x2,x3}}
sage: g = x4 + 1
sage: t = g.set(); t
{{x4}, {}}
sage: s.cartesian_product(t)
{{x1,x2,x4}, {x1,x2}, {x2,x3,x4}, {x2,x3}}
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(5),'x')
>>> x0,x1,x2,x3,x4 = B.gens()
>>> f = x1*x2+x2*x3
>>> s = f.set(); s
{{x1,x2}, {x2,x3}}
>>> g = x4 + Integer(1)
>>> t = g.set(); t
{{x4}, {}}
>>> s.cartesian_product(t)
{{x1,x2,x4}, {x1,x2}, {x2,x3,x4}, {x2,x3}}
```

change (*ind*)

Swaps the presence of `x_i` in each entry of the set.

EXAMPLES:

```
sage: P.<a,b,c> = BooleanPolynomialRing()
sage: f = a+b
sage: s = f.set(); s
{{a}, {b}}
sage: s.change(0)
{{a,b}, {}}
sage: s.change(1)
{{a,b}, {}}
sage: s.change(2)
{{a,c}, {b,c}}
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(names=('a', 'b', 'c',)); (a, b, c,) = P._first_
↪ngens(3)
>>> f = a+b
>>> s = f.set(); s
{{a}, {b}}
>>> s.change(Integer(0))
{{a,b}, {}}
>>> s.change(Integer(1))
```

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```
{{a,b}, {}}
>>> s.change(Integer(2))
{{a,c}, {b,c}}
```

diff(*rhs*)

Return the set theoretic difference of this set and the set `rhs`.

The difference of two sets X and Y is defined as:

$$X\ Y = \{x|x \in X \text{ and } x \notin Y\}.$$

EXAMPLES:

```
sage: B = BooleanPolynomialRing(5,'x')
sage: x0,x1,x2,x3,x4 = B.gens()
sage: f = x1*x2+x2*x3
sage: s = f.set(); s
{{x1,x2}, {x2,x3}}
sage: g = x2*x3 + 1
sage: t = g.set(); t
{{x2,x3}, {}}
sage: s.diff(t)
{{x1,x2}}
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(5),'x')
>>> x0,x1,x2,x3,x4 = B.gens()
>>> f = x1*x2+x2*x3
>>> s = f.set(); s
{{x1,x2}, {x2,x3}}
>>> g = x2*x3 + Integer(1)
>>> t = g.set(); t
{{x2,x3}, {}}
>>> s.diff(t)
{{x1,x2}}
```

divide(*rhs*)

Divide each element of this set by the monomial `rhs` and return a new set containing the result.

EXAMPLES:

```
sage: B.<a,b,c,d,e,f> = BooleanPolynomialRing(order='lex')
sage: f = b*e + b*c*d + b
sage: s = f.set(); s
{{b,c,d}, {b,e}, {b}}
sage: s.divide(b.lm())
{{c,d}, {e}, {}}

sage: f = b*e + b*c*d + b + c
sage: s = f.set()
sage: s.divide(b.lm())
{{c,d}, {e}, {}}
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(order='lex', names=('a', 'b', 'c', 'd', 'e', 'f
↪',)); (a, b, c, d, e, f,) = B._first_ngens(6)
>>> f = b*e + b*c*d + b
>>> s = f.set(); s
{{b,c,d}, {b,e}, {b}}
>>> s.divide(b.lm())
{{c,d}, {e}, {}}

>>> f = b*e + b*c*d + b + c
>>> s = f.set()
>>> s.divide(b.lm())
{{c,d}, {e}, {}}
```

divisors_of(*m*)

Return those members which are divisors of `m`.

INPUT:

- `m` – boolean monomial

EXAMPLES:

```
sage: B = BooleanPolynomialRing(5,'x')
sage: x0,x1,x2,x3,x4 = B.gens()
sage: f = x1*x2+x2*x3
sage: s = f.set()
sage: s.divisors_of((x1*x2*x4).lead())
{{x1,x2}}
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(5),'x')
>>> x0,x1,x2,x3,x4 = B.gens()
>>> f = x1*x2+x2*x3
>>> s = f.set()
>>> s.divisors_of((x1*x2*x4).lead())
{{x1,x2}}
```

empty()

Return `True` if this set is empty.

EXAMPLES:

```
sage: B.<a,b,c,d> = BooleanPolynomialRing(4)
sage: BS = (a*b + c).set()
sage: BS.empty()
False

sage: BS = B(0).set()
sage: BS.empty()
True
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(4), names=('a', 'b', 'c', 'd',)); (a, b,
```

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```
↪ c, d,) = B._first_ngens(4)
>>> BS = (a*b + c).set()
>>> BS.empty()
False

>>> BS = B(Integer(0)).set()
>>> BS.empty()
True
```

include_divisors()

Extend this set to include all divisors of the elements already in this set and return the result as a new set.

EXAMPLES:

```
sage: B.<a,b,c,d,e,f> = BooleanPolynomialRing()
sage: f = a*d*e + a*f + b*d*e + c*d*e + 1
sage: s = f.set(); s
{{a,d,e}, {a,f}, {b,d,e}, {c,d,e}, {}}

sage: s.include_divisors()
{{a,d,e}, {a,d}, {a,e}, {a,f}, {a}, {b,d,e}, {b,d}, {b,e},
 {b}, {c,d,e}, {c,d}, {c,e}, {c}, {d,e}, {d}, {e}, {f}, {}}
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd', 'e', 'f',)); (a, b,␣
↪c, d, e, f,) = B._first_ngens(6)
>>> f = a*d*e + a*f + b*d*e + c*d*e + Integer(1)
>>> s = f.set(); s
{{a,d,e}, {a,f}, {b,d,e}, {c,d,e}, {}}

>>> s.include_divisors()
{{a,d,e}, {a,d}, {a,e}, {a,f}, {a}, {b,d,e}, {b,d}, {b,e},
 {b}, {c,d,e}, {c,d}, {c,e}, {c}, {d,e}, {d}, {e}, {f}, {}}
```

intersect(*other*)

Return the set theoretic intersection of this set and the set `rhs`.

The union of two sets X and Y is defined as:

$$X \cap Y = \{x | x \in X \text{ and } x \in Y\}.$$

EXAMPLES:

```
sage: B = BooleanPolynomialRing(5,'x')
sage: x0,x1,x2,x3,x4 = B.gens()
sage: f = x1*x2+x2*x3
sage: s = f.set(); s
{{x1,x2}, {x2,x3}}
sage: g = x2*x3 + 1
sage: t = g.set(); t
{{x2,x3}, {}}
sage: s.intersect(t)
{{x2,x3}}
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(5),'x')
>>> x0,x1,x2,x3,x4 = B.gens()
>>> f = x1*x2+x2*x3
>>> s = f.set(); s
{{x1,x2}, {x2,x3}}
>>> g = x2*x3 + Integer(1)
>>> t = g.set(); t
{{x2,x3}, {}}
>>> s.intersect(t)
{{x2,x3}}
```

minimal_elements()

Return a new set containing a divisor of all elements of this set.

EXAMPLES:

```
sage: B.<a,b,c,d,e,f> = BooleanPolynomialRing()
sage: f = a*d*e + a*f + a*b*d*e + a*c*d*e + a
sage: s = f.set(); s
{{a,b,d,e}, {a,c,d,e}, {a,d,e}, {a,f}, {a}}
sage: s.minimal_elements()
{{a}}
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd', 'e', 'f',)); (a, b,
↪c, d, e, f,) = B._first_ngens(6)
>>> f = a*d*e + a*f + a*b*d*e + a*c*d*e + a
>>> s = f.set(); s
{{a,b,d,e}, {a,c,d,e}, {a,d,e}, {a,f}, {a}}
>>> s.minimal_elements()
{{a}}
```

multiples_of(*m*)

Return those members which are multiples of m.

INPUT:

- m – boolean monomial

EXAMPLES:

```
sage: B = BooleanPolynomialRing(5,'x')
sage: x0,x1,x2,x3,x4 = B.gens()
sage: f = x1*x2+x2*x3
sage: s = f.set()
sage: s.multiples_of(x1.lm())
{{x1,x2}}
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(5),'x')
>>> x0,x1,x2,x3,x4 = B.gens()
>>> f = x1*x2+x2*x3
>>> s = f.set()
```

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```
>>> s.multiples_of(x1.lm())
{{x1,x2}}
```

n_nodes()

Return the number of nodes in the ZDD.

EXAMPLES:

```
sage: B = BooleanPolynomialRing(5,'x')
sage: x0,x1,x2,x3,x4 = B.gens()
sage: f = x1*x2+x2*x3
sage: s = f.set(); s
{{x1,x2}, {x2,x3}}
sage: s.n_nodes()
4
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(5),'x')
>>> x0,x1,x2,x3,x4 = B.gens()
>>> f = x1*x2+x2*x3
>>> s = f.set(); s
{{x1,x2}, {x2,x3}}
>>> s.n_nodes()
4
```

navigation()

Navigators provide an interface to diagram nodes, accessing their index as well as the corresponding then- and else-branches.

You should be very careful and always keep a reference to the original object, when dealing with navigators, as navigators contain only a raw pointer as data. For the same reason, it is necessary to supply the ring as argument, when constructing a set out of a navigator.

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import BooleSet
sage: B = BooleanPolynomialRing(5,'x')
sage: x0,x1,x2,x3,x4 = B.gens()
sage: f = x1*x2+x2*x3*x4+x2*x4+x3+x4+1
sage: s = f.set(); s
{{x1,x2}, {x2,x3,x4}, {x2,x4}, {x3}, {x4}, {}}

sage: nav = s.navigation()
sage: BooleSet(nav, s.ring())
{{x1,x2}, {x2,x3,x4}, {x2,x4}, {x3}, {x4}, {}}

sage: nav.value()
1

sage: nav_else = nav.else_branch()

sage: BooleSet(nav_else, s.ring())
{{x2,x3,x4}, {x2,x4}, {x3}, {x4}, {}}
```

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```
sage: nav_else.value()
2
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import BooleSet
>>> B = BooleanPolynomialRing(Integer(5),'x')
>>> x0,x1,x2,x3,x4 = B.gens()
>>> f = x1*x2+x2*x3*x4+x2*x4+x3+x4+Integer(1)
>>> s = f.set(); s
{{x1,x2}, {x2,x3,x4}, {x2,x4}, {x3}, {x4}, {}}

>>> nav = s.navigation()
>>> BooleSet(nav, s.ring())
{{x1,x2}, {x2,x3,x4}, {x2,x4}, {x3}, {x4}, {}}

>>> nav.value()
1

>>> nav_else = nav.else_branch()

>>> BooleSet(nav_else, s.ring())
{{x2,x3,x4}, {x2,x4}, {x3}, {x4}, {}}

>>> nav_else.value()
2
```

ring()

Return the parent ring.

EXAMPLES:

```
sage: B = BooleanPolynomialRing(5,'x')
sage: x0,x1,x2,x3,x4 = B.gens()
sage: f = x1*x2+x2*x3*x4+x2*x4+x3+x4+1
sage: f.set().ring() is B
True
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(5),'x')
>>> x0,x1,x2,x3,x4 = B.gens()
>>> f = x1*x2+x2*x3*x4+x2*x4+x3+x4+Integer(1)
>>> f.set().ring() is B
True
```

set()

Return `self`.

EXAMPLES:

```
sage: B.<a,b,c,d> = BooleanPolynomialRing(4)
sage: BS = (a*b + c).set()
```

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```
sage: BS.set() is BS
True
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(4), names=('a', 'b', 'c', 'd',)); (a, b,
↪ c, d,) = B._first_ngens(4)
>>> BS = (a*b + c).set()
>>> BS.set() is BS
True
```

size_double()

Return the size of this set as a floating point number.

EXAMPLES:

```
sage: B = BooleanPolynomialRing(5,'x')
sage: x0,x1,x2,x3,x4 = B.gens()
sage: f = x1*x2+x2*x3
sage: s = f.set()
sage: s.size_double()
2.0
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(5),'x')
>>> x0,x1,x2,x3,x4 = B.gens()
>>> f = x1*x2+x2*x3
>>> s = f.set()
>>> s.size_double()
2.0
```

stable_hash()

A hash value which is stable across processes.

EXAMPLES:

```
sage: B.<x,y> = BooleanPolynomialRing()
sage: x.set() is x.set()
False
sage: x.set().stable_hash() == x.set().stable_hash()
True
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(names=('x', 'y',)); (x, y,) = B._first_ngens(2)
>>> x.set() is x.set()
False
>>> x.set().stable_hash() == x.set().stable_hash()
True
```

Note

This function is part of the upstream PolyBoRi interface. In Sage all hashes are stable.

subset0(*i*)

Return a set of those elements in this set which do not contain the variable indexed by `i`.

INPUT:

- `i` – an index

EXAMPLES:

```
sage: BooleanPolynomialRing(5,'x')
Boolean PolynomialRing in x0, x1, x2, x3, x4
sage: B = BooleanPolynomialRing(5,'x')
sage: B.inject_variables()
Defining x0, x1, x2, x3, x4
sage: f = x1*x2+x2*x3
sage: s = f.set(); s
{{x1,x2}, {x2,x3}}
sage: s.subset0(1)
{{x2,x3}}
```

```
>>> from sage.all import *
>>> BooleanPolynomialRing(Integer(5),'x')
Boolean PolynomialRing in x0, x1, x2, x3, x4
>>> B = BooleanPolynomialRing(Integer(5),'x')
>>> B.inject_variables()
Defining x0, x1, x2, x3, x4
>>> f = x1*x2+x2*x3
>>> s = f.set(); s
{{x1,x2}, {x2,x3}}
>>> s.subset0(Integer(1))
{{x2,x3}}
```

subset1(*i*)

Return a set of those elements in this set which do contain the variable indexed by `i` and evaluate the variable indexed by `i` to 1.

INPUT:

- `i` – an index

EXAMPLES:

```
sage: BooleanPolynomialRing(5,'x')
Boolean PolynomialRing in x0, x1, x2, x3, x4
sage: B = BooleanPolynomialRing(5,'x')
sage: B.inject_variables()
Defining x0, x1, x2, x3, x4
sage: f = x1*x2+x2*x3
sage: s = f.set(); s
{{x1,x2}, {x2,x3}}
sage: s.subset1(1)
{{x2}}
```

```
>>> from sage.all import *
>>> BooleanPolynomialRing(Integer(5),'x')
Boolean PolynomialRing in x0, x1, x2, x3, x4
```

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```
>>> B = BooleanPolynomialRing(Integer(5),'x')
>>> B.inject_variables()
Defining x0, x1, x2, x3, x4
>>> f = x1*x2+x2*x3
>>> s = f.set(); s
{{x1,x2}, {x2,x3}}
>>> s.subset1(Integer(1))
{{x2}}
```

union(*rhs*)

Return the set theoretic union of this set and the set `rhs`.

The union of two sets X and Y is defined as:

$$X \cup Y = \{x | x \in X \text{ or } x \in Y\}.$$

EXAMPLES:

```
sage: B = BooleanPolynomialRing(5,'x')
sage: x0,x1,x2,x3,x4 = B.gens()
sage: f = x1*x2+x2*x3
sage: s = f.set(); s
{{x1,x2}, {x2,x3}}
sage: g = x2*x3 + 1
sage: t = g.set(); t
{{x2,x3}, {}}
sage: s.union(t)
{{x1,x2}, {x2,x3}, {}}
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(5),'x')
>>> x0,x1,x2,x3,x4 = B.gens()
>>> f = x1*x2+x2*x3
>>> s = f.set(); s
{{x1,x2}, {x2,x3}}
>>> g = x2*x3 + Integer(1)
>>> t = g.set(); t
{{x2,x3}, {}}
>>> s.union(t)
{{x1,x2}, {x2,x3}, {}}
```

vars()

Return the variables in this set as a monomial.

EXAMPLES:

```
sage: B.<a,b,c,d,e,f> = BooleanPolynomialRing(order='lex')
sage: f = a + b*e + d*f + e + 1
sage: s = f.set()
sage: s
{{a}, {b,e}, {d,f}, {e}, {}}
sage: s.vars()
a*b*d*e*f
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(order='lex', names=('a', 'b', 'c', 'd', 'e', 'f
↪',)); (a, b, c, d, e, f,) = B._first_ngens(6)
>>> f = a + b*e + d*f + e + Integer(1)
>>> s = f.set()
>>> s
{{a}, {b,e}, {d,f}, {e}, {}}
>>> s.vars()
a*b*d*e*f
```

class sage.rings.polynomial.pbori.pbori.**BooleSetIterator**

Bases: object

Helper class to iterate over boolean sets.

class sage.rings.polynomial.pbori.pbori.**BooleanMonomial**

Bases: MonoidElement

Construct a boolean monomial.

INPUT:

- parent – parent monoid this element lives in

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import BooleanMonomialMonoid,␣
↪BooleanMonomial
sage: P.<x,y,z> = BooleanPolynomialRing(3)
sage: M = BooleanMonomialMonoid(P)
sage: BooleanMonomial(M)
1
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import BooleanMonomialMonoid,␣
↪BooleanMonomial
>>> P = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,) = P.
↪_first_ngens(3)
>>> M = BooleanMonomialMonoid(P)
>>> BooleanMonomial(M)
1
```

Note

Use the BooleanMonomialMonoid__call__() method and not this constructor to construct these objects.

deg()

Return degree of this monomial.

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import BooleanMonomialMonoid
sage: P.<x,y,z> = BooleanPolynomialRing(3)
sage: M = BooleanMonomialMonoid(P)
```

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```
sage: M(x*y).deg()
2

sage: M(x*x*y*z).deg()
3
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import BooleanMonomialMonoid
>>> P = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= P._first_ngens(3)
>>> M = BooleanMonomialMonoid(P)
>>> M(x*y).deg()
2

>>> M(x*x*y*z).deg()
3
```

Note

This function is part of the upstream PolyBoRi interface.

degree(*x=None*)

Return the degree of this monomial in `x`, where `x` must be one of the generators of the polynomial ring.

INPUT:

- `x` – boolean multivariate polynomial (a generator of the polynomial ring). If `x` is not specified (or is `None`), return the total degree of this monomial.

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import BooleanMonomialMonoid
sage: P.<x,y,z> = BooleanPolynomialRing(3)
sage: M = BooleanMonomialMonoid(P)
sage: M(x*y).degree()
2
sage: M(x*y).degree(x)
1
sage: M(x*y).degree(z)
0
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import BooleanMonomialMonoid
>>> P = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= P._first_ngens(3)
>>> M = BooleanMonomialMonoid(P)
>>> M(x*y).degree()
2
>>> M(x*y).degree(x)
1
>>> M(x*y).degree(z)
0
```

divisors()

Return a set of boolean monomials with all divisors of this monomial.

EXAMPLES:

```
sage: B.<x,y,z> = BooleanPolynomialRing(3)
sage: f = x*y
sage: m = f.lm()
sage: m.divisors()
{{x,y}, {x}, {y}, {}}
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= B._first_ngens(3)
>>> f = x*y
>>> m = f.lm()
>>> m.divisors()
{{x,y}, {x}, {y}, {}}
```

gcd(*rhs*)

Return the greatest common divisor of this boolean monomial and `rhs`.

INPUT:

- `rhs` – boolean monomial

EXAMPLES:

```
sage: B.<a,b,c,d> = BooleanPolynomialRing()
sage: a,b,c,d = a.lm(), b.lm(), c.lm(), d.lm()
sage: (a*b).gcd(b*c)
b
sage: (a*b*c).gcd(d)
1
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd',)); (a, b, c, d,) = B.
↪_first_ngens(4)
>>> a,b,c,d = a.lm(), b.lm(), c.lm(), d.lm()
>>> (a*b).gcd(b*c)
b
>>> (a*b*c).gcd(d)
1
```

index()

Return the variable index of the first variable in this monomial.

EXAMPLES:

```
sage: B.<x,y,z> = BooleanPolynomialRing(3)
sage: f = x*y
sage: m = f.lm()
sage: m.index()
0
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= B._first_ngens(3)
>>> f = x*y
>>> m = f.lm()
>>> m.index()
0
```

Note

This function is part of the upstream PolyBoRi interface.

iterindex()

Return an iterator over the indices of the variables in `self`.

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import BooleanMonomialMonoid
sage: P.<x,y,z> = BooleanPolynomialRing(3)
sage: M = BooleanMonomialMonoid(P)
sage: list(M(x*z).iterindex())
[0, 2]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import BooleanMonomialMonoid
>>> P = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= P._first_ngens(3)
>>> M = BooleanMonomialMonoid(P)
>>> list(M(x*z).iterindex())
[0, 2]
```

multiples(*rhs*)

Return a set of boolean monomials with all multiples of this monomial up to the bound `rhs`.

INPUT:

- `rhs` – boolean monomial

EXAMPLES:

```
sage: B.<x,y,z> = BooleanPolynomialRing(3)
sage: f = x
sage: m = f.lm()
sage: g = x*y*z
sage: n = g.lm()
sage: m.multiples(n)
{{x,y,z}, {x,y}, {x,z}, {x}}
sage: n.multiples(m)
{{x,y,z}}
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
```

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```
↪= B._first_ngens(3)
>>> f = x
>>> m = f.lm()
>>> g = x*y*z
>>> n = g.lm()
>>> m.multiples(n)
{{x,y,z}, {x,y}, {x,z}, {x}}
>>> n.multiples(m)
{{x,y,z}}
```

> **Note**
>
> The returned set always contains `self` even if the bound `rhs` is smaller than `self`.

navigation()

Navigators provide an interface to diagram nodes, accessing their index as well as the corresponding then- and else-branches.

You should be very careful and always keep a reference to the original object, when dealing with navigators, as navigators contain only a raw pointer as data. For the same reason, it is necessary to supply the ring as argument, when constructing a set out of a navigator.

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import BooleSet
sage: B = BooleanPolynomialRing(5,'x')
sage: x0,x1,x2,x3,x4 = B.gens()
sage: f = x1*x2+x2*x3*x4+x2*x4+x3+x4+1
sage: m = f.lm(); m
x1*x2

sage: nav = m.navigation()
sage: BooleSet(nav, B)
{{x1,x2}}

sage: nav.value()
1
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import BooleSet
>>> B = BooleanPolynomialRing(Integer(5),'x')
>>> x0,x1,x2,x3,x4 = B.gens()
>>> f = x1*x2+x2*x3*x4+x2*x4+x3+x4+Integer(1)
>>> m = f.lm(); m
x1*x2

>>> nav = m.navigation()
>>> BooleSet(nav, B)
{{x1,x2}}

>>> nav.value()
```

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```
1
```

reducible_by(*rhs*)

Return `True` if `self` is reducible by `rhs`.

INPUT:

- `rhs` – boolean monomial

EXAMPLES:

```
sage: B.<x,y,z> = BooleanPolynomialRing(3)
sage: f = x*y
sage: m = f.lm()
sage: m.reducible_by((x*y).lm())
True
sage: m.reducible_by((x*z).lm())
False
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= B._first_ngens(3)
>>> f = x*y
>>> m = f.lm()
>>> m.reducible_by((x*y).lm())
True
>>> m.reducible_by((x*z).lm())
False
```

ring()

Return the corresponding boolean ring.

EXAMPLES:

```
sage: B.<a,b,c,d> = BooleanPolynomialRing(4)
sage: a.lm().ring() is B
True
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(4), names=('a', 'b', 'c', 'd',)); (a, b,
↪ c, d,) = B._first_ngens(4)
>>> a.lm().ring() is B
True
```

set()

Return a boolean set of variables in this monomials.

EXAMPLES:

```
sage: B.<x,y,z> = BooleanPolynomialRing(3)
sage: f = x*y
sage: m = f.lm()
sage: m.set()
{{x,y}}
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= B._first_ngens(3)
>>> f = x*y
>>> m = f.lm()
>>> m.set()
{{x,y}}
```

stable_hash()

A hash value which is stable across processes.

EXAMPLES:

```
sage: B.<x,y> = BooleanPolynomialRing()
sage: x.lm() is x.lm()
False
sage: x.lm().stable_hash() == x.lm().stable_hash()
True
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(names=('x', 'y',)); (x, y,) = B._first_ngens(2)
>>> x.lm() is x.lm()
False
>>> x.lm().stable_hash() == x.lm().stable_hash()
True
```

> **Note**
>
> This function is part of the upstream PolyBoRi interface. In Sage all hashes are stable.

variables()

Return a tuple of the variables in this monomial.

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import BooleanMonomialMonoid
sage: P.<x,y,z> = BooleanPolynomialRing(3)
sage: M = BooleanMonomialMonoid(P)
sage: M(x*z).variables() # indirect doctest
(x, z)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import BooleanMonomialMonoid
>>> P = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= P._first_ngens(3)
>>> M = BooleanMonomialMonoid(P)
>>> M(x*z).variables() # indirect doctest
(x, z)
```

class sage.rings.polynomial.pbori.pbori.**BooleanMonomialIterator**

Bases: `object`

An iterator over the variable indices of a monomial.

class sage.rings.polynomial.pbori.pbori.**BooleanMonomialMonoid**(*polring*)

Bases: UniqueRepresentation, Monoid_class

Construct a boolean monomial monoid given a boolean polynomial ring.

This object provides a parent for boolean monomials.

INPUT:

- polring – the polynomial ring our monomials lie in

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import BooleanMonomialMonoid
sage: P.<x,y> = BooleanPolynomialRing(2)
sage: M = BooleanMonomialMonoid(P)
sage: M
MonomialMonoid of Boolean PolynomialRing in x, y

sage: M.gens()
(x, y)
sage: type(M.gen(0))
<class 'sage.rings.polynomial.pbori.pbori.BooleanMonomial'>
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import BooleanMonomialMonoid
>>> P = BooleanPolynomialRing(Integer(2), names=('x', 'y',)); (x, y,) = P._first_
↪ngens(2)
>>> M = BooleanMonomialMonoid(P)
>>> M
MonomialMonoid of Boolean PolynomialRing in x, y

>>> M.gens()
(x, y)
>>> type(M.gen(Integer(0)))
<class 'sage.rings.polynomial.pbori.pbori.BooleanMonomial'>
```

Since Issue #9138, boolean monomial monoids are unique parents and are fit into the category framework:

```
sage: loads(dumps(M)) is M
True
sage: TestSuite(M).run()
```

```
>>> from sage.all import *
>>> loads(dumps(M)) is M
True
>>> TestSuite(M).run()
```

gen(*i=0*)

Return the i-th generator of self.

INPUT:

- i – integer

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import BooleanMonomialMonoid
sage: P.<x,y,z> = BooleanPolynomialRing(3)
sage: M = BooleanMonomialMonoid(P)
sage: M.gen(0)
x
sage: M.gen(2)
z

sage: P = BooleanPolynomialRing(1000, 'x')
sage: M = BooleanMonomialMonoid(P)
sage: M.gen(50)
x50
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import BooleanMonomialMonoid
>>> P = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= P._first_ngens(3)
>>> M = BooleanMonomialMonoid(P)
>>> M.gen(Integer(0))
x
>>> M.gen(Integer(2))
z

>>> P = BooleanPolynomialRing(Integer(1000), 'x')
>>> M = BooleanMonomialMonoid(P)
>>> M.gen(Integer(50))
x50
```

gens()

Return the tuple of generators of this monoid.

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import BooleanMonomialMonoid
sage: P.<x,y,z> = BooleanPolynomialRing(3)
sage: M = BooleanMonomialMonoid(P)
sage: M.gens()
(x, y, z)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import BooleanMonomialMonoid
>>> P = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= P._first_ngens(3)
>>> M = BooleanMonomialMonoid(P)
>>> M.gens()
(x, y, z)
```

ngens()

Return the number of variables in this monoid.

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import BooleanMonomialMonoid
sage: P = BooleanPolynomialRing(100, 'x')
sage: M = BooleanMonomialMonoid(P)
sage: M.ngens()
100
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import BooleanMonomialMonoid
>>> P = BooleanPolynomialRing(Integer(100), 'x')
>>> M = BooleanMonomialMonoid(P)
>>> M.ngens()
100
```

class sage.rings.polynomial.pbori.pbori.**BooleanMonomialVariableIterator**

Bases: object

class sage.rings.polynomial.pbori.pbori.**BooleanMulAction**

Bases: Action

class sage.rings.polynomial.pbori.pbori.**BooleanPolynomial**

Bases: *MPolynomial*

Construct a boolean polynomial object in the given boolean polynomial ring.

INPUT:

- parent – boolean polynomial ring

> **Note**
>
> Do not use this method to construct boolean polynomials, but use the appropriate `__call__` method in the parent.

constant()

Return True if this element is constant.

EXAMPLES:

```
sage: B.<x,y,z> = BooleanPolynomialRing(3)
sage: x.constant()
False
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= B._first_ngens(3)
>>> x.constant()
False
```

```
sage: B(1).constant()
True
```

```
>>> from sage.all import *
>>> B(Integer(1)).constant()
True
```

Note: This function is part of the upstream PolyBoRi interface.

constant_coefficient()

Return the constant coefficient of this boolean polynomial.

EXAMPLES:

```
sage: B.<a,b> = BooleanPolynomialRing()
sage: a.constant_coefficient()
0
sage: (a+1).constant_coefficient()
1
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(names=('a', 'b',)); (a, b,) = B._first_ngens(2)
>>> a.constant_coefficient()
0
>>> (a+Integer(1)).constant_coefficient()
1
```

deg()

Return the degree of `self`. This is usually equivalent to the total degree except for weighted term orderings which are not implemented yet.

EXAMPLES:

```
sage: P.<x,y> = BooleanPolynomialRing(2)
sage: (x+y).degree()
1
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(2), names=('x', 'y',)); (x, y,) = P._
↪first_ngens(2)
>>> (x+y).degree()
1
```

```
sage: P(1).degree()
0
```

```
>>> from sage.all import *
>>> P(Integer(1)).degree()
0
```

```
sage: (x*y + x + y + 1).degree()
2
```

```
>>> from sage.all import *
>>> (x*y + x + y + Integer(1)).degree()
2
```

> **Note**
>
> This function is part of the upstream PolyBoRi interface.

degree(*x=None*)

Return the maximal degree of this polynomial in `x`, where `x` must be one of the generators for the parent of this polynomial.

If x is not specified (or is `None`), return the total degree, which is the maximum degree of any monomial.

EXAMPLES:

```
sage: P.<x,y> = BooleanPolynomialRing(2)
sage: (x+y).degree()
1
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(2), names=('x', 'y',)); (x, y,) = P._
↪first_ngens(2)
>>> (x+y).degree()
1
```

```
sage: P(1).degree()
0
```

```
>>> from sage.all import *
>>> P(Integer(1)).degree()
0
```

```
sage: (x*y + x + y + 1).degree()
2

sage: (x*y + x + y + 1).degree(x)
1
```

```
>>> from sage.all import *
>>> (x*y + x + y + Integer(1)).degree()
2

>>> (x*y + x + y + Integer(1)).degree(x)
1
```

elength()

Return elimination length as used in the SlimGB algorithm.

EXAMPLES:

```
sage: P.<x,y> = BooleanPolynomialRing(2)
sage: x.elength()
1
sage: f = x*y + 1
sage: f.elength()
2
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(2), names=('x', 'y',)); (x, y,) = P._
↪first_ngens(2)
>>> x.elength()
1
>>> f = x*y + Integer(1)
>>> f.elength()
2
```

REFERENCES:

- Michael Brickenstein; SlimGB: Groebner Bases with Slim Polynomials http://www.mathematik.uni-kl.de/~zca/Reports_on_ca/35/paper_35_full.ps.gz

Note

This function is part of the upstream PolyBoRi interface.

first_term()

Return the first term with respect to the lexicographical term ordering.

EXAMPLES:

```
sage: B.<a,b,z> = BooleanPolynomialRing(3,order='lex')
sage: f = b*z + a + 1
sage: f.first_term()
a
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(3),order='lex', names=('a', 'b', 'z',));
↪ (a, b, z,) = B._first_ngens(3)
>>> f = b*z + a + Integer(1)
>>> f.first_term()
a
```

Note

This function is part of the upstream PolyBoRi interface.

graded_part(*deg*)

Return graded part of this boolean polynomial of degree `deg`.

INPUT:

- `deg` – a degree

EXAMPLES:

```
sage: B.<a,b,c,d> = BooleanPolynomialRing(4)
sage: f = a*b*c + c*d + a*b + 1
sage: f.graded_part(2)
a*b + c*d
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(4), names=('a', 'b', 'c', 'd',)); (a, b,
↪ c, d,) = B._first_ngens(4)
>>> f = a*b*c + c*d + a*b + Integer(1)
>>> f.graded_part(Integer(2))
a*b + c*d
```

```
sage: f.graded_part(0)
1
```

```
>>> from sage.all import *
>>> f.graded_part(Integer(0))
1
```

has_constant_part()

Return `True` if this boolean polynomial has a constant part, i.e. if `1` is a term.

EXAMPLES:

```
sage: B.<a,b,c,d> = BooleanPolynomialRing(4)
sage: f = a*b*c + c*d + a*b + 1
sage: f.has_constant_part()
True
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(4), names=('a', 'b', 'c', 'd',)); (a, b,
↪ c, d,) = B._first_ngens(4)
>>> f = a*b*c + c*d + a*b + Integer(1)
>>> f.has_constant_part()
True
```

```
sage: f = a*b*c + c*d + a*b
sage: f.has_constant_part()
False
```

```
>>> from sage.all import *
>>> f = a*b*c + c*d + a*b
>>> f.has_constant_part()
False
```

is_constant()

Check if `self` is constant.

EXAMPLES:

```
sage: P.<x,y> = BooleanPolynomialRing(2)
sage: P(1).is_constant()
True

sage: P(0).is_constant()
True

sage: x.is_constant()
False

sage: (x*y).is_constant()
False
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(2), names=('x', 'y',)); (x, y,) = P._
↪first_ngens(2)
>>> P(Integer(1)).is_constant()
True

>>> P(Integer(0)).is_constant()
True

>>> x.is_constant()
False

>>> (x*y).is_constant()
False
```

is_equal(*right*)

EXAMPLES:

```
sage: B.<a,b,z> = BooleanPolynomialRing(3)
sage: f = a*z + b + 1
sage: g = b + z
sage: f.is_equal(g)
False

sage: f.is_equal((f + 1) - 1)
True
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(3), names=('a', 'b', 'z',)); (a, b, z,)␣
↪= B._first_ngens(3)
>>> f = a*z + b + Integer(1)
>>> g = b + z
>>> f.is_equal(g)
False

>>> f.is_equal((f + Integer(1)) - Integer(1))
True
```

Note

This function is part of the upstream PolyBoRi interface.

is_homogeneous()

Return `True` if this element is a homogeneous polynomial.

EXAMPLES:

```
sage: P.<x, y> = BooleanPolynomialRing()
sage: (x+y).is_homogeneous()
True
sage: P(0).is_homogeneous()
True
sage: (x+1).is_homogeneous()
False
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(names=('x', 'y',)); (x, y,) = P._first_ngens(2)
>>> (x+y).is_homogeneous()
True
>>> P(Integer(0)).is_homogeneous()
True
>>> (x+Integer(1)).is_homogeneous()
False
```

is_one()

Check if `self` is 1.

EXAMPLES:

```
sage: P.<x,y> = BooleanPolynomialRing(2)
sage: P(1).is_one()
True

sage: P.one().is_one()
True

sage: x.is_one()
False

sage: P(0).is_one()
False
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(2), names=('x', 'y',)); (x, y,) = P._
↪first_ngens(2)
>>> P(Integer(1)).is_one()
True

>>> P.one().is_one()
True
```

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```
>>> x.is_one()
False

>>> P(Integer(0)).is_one()
False
```

is_pair()

Check if `self` has exactly two terms.

EXAMPLES:

```
sage: P.<x,y> = BooleanPolynomialRing(2)
sage: P(0).is_pair()
False

sage: x.is_pair()
False

sage: P(1).is_pair()
False

sage: (x*y).is_pair()
False

sage: (x + y).is_pair()
True

sage: (x + 1).is_pair()
True

sage: (x*y + 1).is_pair()
True

sage: (x + y + 1).is_pair()
False

sage: ((x + 1)*(y + 1)).is_pair()
False
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(2), names=('x', 'y',)); (x, y,) = P._
↪first_ngens(2)
>>> P(Integer(0)).is_pair()
False

>>> x.is_pair()
False

>>> P(Integer(1)).is_pair()
False

>>> (x*y).is_pair()
```

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```
False

>>> (x + y).is_pair()
True

>>> (x + Integer(1)).is_pair()
True

>>> (x*y + Integer(1)).is_pair()
True

>>> (x + y + Integer(1)).is_pair()
False

>>> ((x + Integer(1))*(y + Integer(1))).is_pair()
False
```

is_singleton()

Check if `self` has at most one term.

EXAMPLES:

```
sage: P.<x,y> = BooleanPolynomialRing(2)
sage: P(0).is_singleton()
True

sage: x.is_singleton()
True

sage: P(1).is_singleton()
True

sage: (x*y).is_singleton()
True

sage: (x + y).is_singleton()
False

sage: (x + 1).is_singleton()
False

sage: (x*y + 1).is_singleton()
False

sage: (x + y + 1).is_singleton()
False

sage: ((x + 1)*(y + 1)).is_singleton()
False
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(2), names=('x', 'y',)); (x, y,) = P._
```

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```
↪first_ngens(2)
>>> P(Integer(0)).is_singleton()
True

>>> x.is_singleton()
True

>>> P(Integer(1)).is_singleton()
True

>>> (x*y).is_singleton()
True

>>> (x + y).is_singleton()
False

>>> (x + Integer(1)).is_singleton()
False

>>> (x*y + Integer(1)).is_singleton()
False

>>> (x + y + Integer(1)).is_singleton()
False

>>> ((x + Integer(1))*(y + Integer(1))).is_singleton()
False
```

is_singleton_or_pair()

Check if `self` has at most two terms.

EXAMPLES:

```
sage: P.<x,y> = BooleanPolynomialRing(2)
sage: P(0).is_singleton_or_pair()
True

sage: x.is_singleton_or_pair()
True

sage: P(1).is_singleton_or_pair()
True

sage: (x*y).is_singleton_or_pair()
True

sage: (x + y).is_singleton_or_pair()
True

sage: (x + 1).is_singleton_or_pair()
True
```

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```
sage: (x*y + 1).is_singleton_or_pair()
True

sage: (x + y + 1).is_singleton_or_pair()
False

sage: ((x + 1)*(y + 1)).is_singleton_or_pair()
False
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(2), names=('x', 'y',)); (x, y,) = P._
↪first_ngens(2)
>>> P(Integer(0)).is_singleton_or_pair()
True

>>> x.is_singleton_or_pair()
True

>>> P(Integer(1)).is_singleton_or_pair()
True

>>> (x*y).is_singleton_or_pair()
True

>>> (x + y).is_singleton_or_pair()
True

>>> (x + Integer(1)).is_singleton_or_pair()
True

>>> (x*y + Integer(1)).is_singleton_or_pair()
True

>>> (x + y + Integer(1)).is_singleton_or_pair()
False

>>> ((x + Integer(1))*(y + Integer(1))).is_singleton_or_pair()
False
```

is_unit()

Check if `self` is invertible in the parent ring.

Note that this condition is equivalent to being 1 for boolean polynomials.

EXAMPLES:

```
sage: P.<x,y> = BooleanPolynomialRing(2)
sage: P.one().is_unit()
True

sage: x.is_unit()
False
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(2), names=('x', 'y',)); (x, y,) = P._
↪first_ngens(2)
>>> P.one().is_unit()
True

>>> x.is_unit()
False
```

is_univariate()

Return `True` if `self` is a univariate polynomial.

This means that `self` contains at most one variable.

EXAMPLES:

```
sage: P.<x,y,z> = BooleanPolynomialRing()
sage: f = x + 1
sage: f.is_univariate()
True
sage: f = y*x + 1
sage: f.is_univariate()
False
sage: f = P(0)
sage: f.is_univariate()
True
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(names=('x', 'y', 'z',)); (x, y, z,) = P._first_
↪ngens(3)
>>> f = x + Integer(1)
>>> f.is_univariate()
True
>>> f = y*x + Integer(1)
>>> f.is_univariate()
False
>>> f = P(Integer(0))
>>> f.is_univariate()
True
```

is_zero()

Check if `self` is zero.

EXAMPLES:

```
sage: P.<x,y> = BooleanPolynomialRing(2)
sage: P(0).is_zero()
True

sage: x.is_zero()
False

sage: P(1).is_zero()
False
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(2), names=('x', 'y',)); (x, y,) = P._
↪first_ngens(2)
>>> P(Integer(0)).is_zero()
True

>>> x.is_zero()
False

>>> P(Integer(1)).is_zero()
False
```

lead()

Return the leading monomial of boolean polynomial, with respect to to the order of parent ring.

EXAMPLES:

```
sage: P.<x,y,z> = BooleanPolynomialRing(3)
sage: (x+y+y*z).lead()
x
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= P._first_ngens(3)
>>> (x+y+y*z).lead()
x
```

```
sage: P.<x,y,z> = BooleanPolynomialRing(3, order='deglex')
sage: (x+y+y*z).lead()
y*z
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(3), order='deglex', names=('x', 'y', 'z
↪',)); (x, y, z,) = P._first_ngens(3)
>>> (x+y+y*z).lead()
y*z
```

Note

This function is part of the upstream PolyBoRi interface.

lead_deg()

Return the total degree of the leading monomial of `self`.

EXAMPLES:

```
sage: P.<x,y,z> = BooleanPolynomialRing(3)
sage: p = x + y*z
sage: p.lead_deg()
1
```

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```
sage: P.<x,y,z> = BooleanPolynomialRing(3,order='deglex')
sage: p = x + y*z
sage: p.lead_deg()
2

sage: P(0).lead_deg()
0
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= P._first_ngens(3)
>>> p = x + y*z
>>> p.lead_deg()
1

>>> P = BooleanPolynomialRing(Integer(3),order='deglex', names=('x', 'y', 'z',
↪)); (x, y, z,) = P._first_ngens(3)
>>> p = x + y*z
>>> p.lead_deg()
2

>>> P(Integer(0)).lead_deg()
0
```

Note

This function is part of the upstream PolyBoRi interface.

lead_divisors()

Return a `BooleSet` of all divisors of the leading monomial.

EXAMPLES:

```
sage: B.<a,b,z> = BooleanPolynomialRing(3)
sage: f = a*b + z + 1
sage: f.lead_divisors()
{{a,b}, {a}, {b}, {}}
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(3), names=('a', 'b', 'z',)); (a, b, z,)␣
↪= B._first_ngens(3)
>>> f = a*b + z + Integer(1)
>>> f.lead_divisors()
{{a,b}, {a}, {b}, {}}
```

Note

This function is part of the upstream PolyBoRi interface.

lex_lead()

Return the leading monomial of boolean polynomial, with respect to the lexicographical term ordering.

EXAMPLES:

```
sage: P.<x,y,z> = BooleanPolynomialRing(3)
sage: (x+y+y*z).lex_lead()
x

sage: P.<x,y,z> = BooleanPolynomialRing(3, order='deglex')
sage: (x+y+y*z).lex_lead()
x

sage: P(0).lex_lead()
0
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= P._first_ngens(3)
>>> (x+y+y*z).lex_lead()
x

>>> P = BooleanPolynomialRing(Integer(3), order='deglex', names=('x', 'y', 'z
↪',)); (x, y, z,) = P._first_ngens(3)
>>> (x+y+y*z).lex_lead()
x

>>> P(Integer(0)).lex_lead()
0
```

Note

This function is part of the upstream PolyBoRi interface.

lex_lead_deg()

Return degree of leading monomial with respect to the lexicographical ordering.

EXAMPLES:

```
sage: B.<x,y,z> = BooleanPolynomialRing(3,order='lex')
sage: f = x + y*z
sage: f
x + y*z
sage: f.lex_lead_deg()
1
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(3),order='lex', names=('x', 'y', 'z',));
↪ (x, y, z,) = B._first_ngens(3)
>>> f = x + y*z
>>> f
x + y*z
```

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```
>>> f.lex_lead_deg()
1
```

```
sage: B.<x,y,z> = BooleanPolynomialRing(3,order='deglex')
sage: f = x + y*z
sage: f
y*z + x
sage: f.lex_lead_deg()
1
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(3),order='deglex', names=('x', 'y', 'z',
↪)); (x, y, z,) = B._first_ngens(3)
>>> f = x + y*z
>>> f
y*z + x
>>> f.lex_lead_deg()
1
```

Note

This function is part of the upstream PolyBoRi interface.

lm()

Return the leading monomial of this boolean polynomial, with respect to the order of parent ring.

EXAMPLES:

```
sage: P.<x,y,z> = BooleanPolynomialRing(3)
sage: (x+y+y*z).lm()
x

sage: P.<x,y,z> = BooleanPolynomialRing(3, order='deglex')
sage: (x+y+y*z).lm()
y*z

sage: P(0).lm()
0
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= P._first_ngens(3)
>>> (x+y+y*z).lm()
x

>>> P = BooleanPolynomialRing(Integer(3), order='deglex', names=('x', 'y', 'z
↪',)); (x, y, z,) = P._first_ngens(3)
>>> (x+y+y*z).lm()
y*z
```

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```
>>> P(Integer(0)).lm()
0
```

lt()

Return the leading term of this boolean polynomial, with respect to the order of the parent ring.

Note that for boolean polynomials this is equivalent to returning leading monomials.

EXAMPLES:

```
sage: P.<x,y,z> = BooleanPolynomialRing(3)
sage: (x+y+y*z).lt()
x
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= P._first_ngens(3)
>>> (x+y+y*z).lt()
x
```

```
sage: P.<x,y,z> = BooleanPolynomialRing(3, order='deglex')
sage: (x+y+y*z).lt()
y*z
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(3), order='deglex', names=('x', 'y', 'z
↪',)); (x, y, z,) = P._first_ngens(3)
>>> (x+y+y*z).lt()
y*z
```

map_every_x_to_x_plus_one()

Map every variable `x_i` in this polynomial to `x_i + 1`.

EXAMPLES:

```
sage: B.<a,b,z> = BooleanPolynomialRing(3)
sage: f = a*b + z + 1; f
a*b + z + 1
sage: f.map_every_x_to_x_plus_one()
a*b + a + b + z + 1
sage: f(a+1,b+1,z+1)
a*b + a + b + z + 1
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(3), names=('a', 'b', 'z',)); (a, b, z,)␣
↪= B._first_ngens(3)
>>> f = a*b + z + Integer(1); f
a*b + z + 1
>>> f.map_every_x_to_x_plus_one()
a*b + a + b + z + 1
>>> f(a+Integer(1),b+Integer(1),z+Integer(1))
a*b + a + b + z + 1
```

monomial_coefficient(*mon*)

Return the coefficient of the monomial `mon` in `self`, where `mon` must have the same parent as `self`.

INPUT:

- `mon` – a monomial

EXAMPLES:

```
sage: P.<x,y> = BooleanPolynomialRing(2)
sage: x.monomial_coefficient(x)
1
sage: x.monomial_coefficient(y)
0
sage: R.<x,y,z,a,b,c>=BooleanPolynomialRing(6)
sage: f=(1-x)*(1+y); f
x*y + x + y + 1
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(2), names=('x', 'y',)); (x, y,) = P._
↪first_ngens(2)
>>> x.monomial_coefficient(x)
1
>>> x.monomial_coefficient(y)
0
>>> R = BooleanPolynomialRing(Integer(6), names=('x', 'y', 'z', 'a', 'b', 'c',
↪)); (x, y, z, a, b, c,) = R._first_ngens(6)
>>> f=(Integer(1)-x)*(Integer(1)+y); f
x*y + x + y + 1
```

```
sage: f.monomial_coefficient(1)
1
```

```
>>> from sage.all import *
>>> f.monomial_coefficient(Integer(1))
1
```

```
sage: f.monomial_coefficient(0)
0
```

```
>>> from sage.all import *
>>> f.monomial_coefficient(Integer(0))
0
```

monomials()

Return a list of monomials appearing in `self` ordered largest to smallest.

EXAMPLES:

```
sage: P.<a,b,c> = BooleanPolynomialRing(3,order='lex')
sage: f = a + c*b
sage: f.monomials()
[a, b*c]
```

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```
sage: P.<a,b,c> = BooleanPolynomialRing(3,order='deglex')
sage: f = a + c*b
sage: f.monomials()
[b*c, a]
sage: P.zero().monomials()
[]
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(3),order='lex', names=('a', 'b', 'c',));
↪ (a, b, c,) = P._first_ngens(3)
>>> f = a + c*b
>>> f.monomials()
[a, b*c]

>>> P = BooleanPolynomialRing(Integer(3),order='deglex', names=('a', 'b', 'c',
↪)); (a, b, c,) = P._first_ngens(3)
>>> f = a + c*b
>>> f.monomials()
[b*c, a]
>>> P.zero().monomials()
[]
```

n_nodes()

Return the number of nodes in the ZDD implementing this polynomial.

EXAMPLES:

```
sage: B = BooleanPolynomialRing(5,'x')
sage: x0,x1,x2,x3,x4 = B.gens()
sage: f = x1*x2 + x2*x3 + 1
sage: f.n_nodes()
4
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(5),'x')
>>> x0,x1,x2,x3,x4 = B.gens()
>>> f = x1*x2 + x2*x3 + Integer(1)
>>> f.n_nodes()
4
```

Note

This function is part of the upstream PolyBoRi interface.

n_vars()

Return the number of variables used to form this boolean polynomial.

EXAMPLES:

```
sage: B.<a,b,c,d> = BooleanPolynomialRing(4)
sage: f = a*b*c + 1
```

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```
sage: f.n_vars()
3
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(4), names=('a', 'b', 'c', 'd',)); (a, b,
↪ c, d,) = B._first_ngens(4)
>>> f = a*b*c + Integer(1)
>>> f.n_vars()
3
```

Note

This function is part of the upstream PolyBoRi interface.

navigation()

Navigators provide an interface to diagram nodes, accessing their index as well as the corresponding then- and else-branches.

You should be very careful and always keep a reference to the original object, when dealing with navigators, as navigators contain only a raw pointer as data. For the same reason, it is necessary to supply the ring as argument, when constructing a set out of a navigator.

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import BooleSet
sage: B = BooleanPolynomialRing(5,'x')
sage: x0,x1,x2,x3,x4 = B.gens()
sage: f = x1*x2+x2*x3*x4+x2*x4+x3+x4+1

sage: nav = f.navigation()
sage: BooleSet(nav, B)
{{x1,x2}, {x2,x3,x4}, {x2,x4}, {x3}, {x4}, {}}

sage: nav.value()
1

sage: nav_else = nav.else_branch()

sage: BooleSet(nav_else, B)
{{x2,x3,x4}, {x2,x4}, {x3}, {x4}, {}}

sage: nav_else.value()
2
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import BooleSet
>>> B = BooleanPolynomialRing(Integer(5),'x')
>>> x0,x1,x2,x3,x4 = B.gens()
>>> f = x1*x2+x2*x3*x4+x2*x4+x3+x4+Integer(1)

>>> nav = f.navigation()
```

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```
>>> BooleSet(nav, B)
{{x1,x2}, {x2,x3,x4}, {x2,x4}, {x3}, {x4}, {}}

>>> nav.value()
1

>>> nav_else = nav.else_branch()

>>> BooleSet(nav_else, B)
{{x2,x3,x4}, {x2,x4}, {x3}, {x4}, {}}

>>> nav_else.value()
2
```

Note

This function is part of the upstream PolyBoRi interface.

nvariables()

Return the number of variables used to form this boolean polynomial.

EXAMPLES:

```
sage: B.<a,b,c,d> = BooleanPolynomialRing(4)
sage: f = a*b*c + 1
sage: f.nvariables()
3
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(4), names=('a', 'b', 'c', 'd',)); (a, b,
↪ c, d,) = B._first_ngens(4)
>>> f = a*b*c + Integer(1)
>>> f.nvariables()
3
```

reduce(*I*)

Return the normal form of `self` w.r.t. `I`, i.e. return the remainder of `self` with respect to the polynomials in `I`. If the polynomial set/list `I` is not a Groebner basis the result is not canonical.

INPUT:

- `I` – list/set of polynomials in `self.parent()`; if I is an ideal, the generators are used

EXAMPLES:

```
sage: B.<x0,x1,x2,x3> = BooleanPolynomialRing(4)
sage: I = B.ideal((x0 + x1 + x2 + x3,
....:              x0*x1 + x1*x2 + x0*x3 + x2*x3,
....:              x0*x1*x2 + x0*x1*x3 + x0*x2*x3 + x1*x2*x3,
....:              x0*x1*x2*x3 + 1))
sage: gb = I.groebner_basis()
sage: f,g,h,i = I.gens()
```

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```
sage: f.reduce(gb)
0
sage: p = f*g + x0*h + x2*i
sage: p.reduce(gb)
0
sage: p.reduce(I)
x1*x2*x3 + x2
sage: p.reduce([])
x0*x1*x2 + x0*x1*x3 + x0*x2*x3 + x2
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(4), names=('x0', 'x1', 'x2', 'x3',));␣
↪(x0, x1, x2, x3,) = B._first_ngens(4)
>>> I = B.ideal((x0 + x1 + x2 + x3,
...              x0*x1 + x1*x2 + x0*x3 + x2*x3,
...              x0*x1*x2 + x0*x1*x3 + x0*x2*x3 + x1*x2*x3,
...              x0*x1*x2*x3 + Integer(1)))
>>> gb = I.groebner_basis()
>>> f,g,h,i = I.gens()
>>> f.reduce(gb)
0
>>> p = f*g + x0*h + x2*i
>>> p.reduce(gb)
0
>>> p.reduce(I)
x1*x2*x3 + x2
>>> p.reduce([])
x0*x1*x2 + x0*x1*x3 + x0*x2*x3 + x2
```

Note

If this function is called repeatedly with the same I then it is advised to use PolyBoRi's *GroebnerStrategy* object directly, since that will be faster. See the source code of this function for details.

reducible_by(*rhs*)

Return `True` if this boolean polynomial is reducible by the polynomial `rhs`.

INPUT:

- `rhs` – boolean polynomial

EXAMPLES:

```
sage: B.<a,b,c,d> = BooleanPolynomialRing(4,order='deglex')
sage: f = (a*b + 1)*(c + 1)
sage: f.reducible_by(d)
False
sage: f.reducible_by(c)
True
sage: f.reducible_by(c + 1)
True
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(4),order='deglex', names=('a', 'b', 'c',
↪ 'd',)); (a, b, c, d,) = B._first_ngens(4)
>>> f = (a*b + Integer(1))*(c + Integer(1))
>>> f.reducible_by(d)
False
>>> f.reducible_by(c)
True
>>> f.reducible_by(c + Integer(1))
True
```

Note

This function is part of the upstream PolyBoRi interface.

ring()

Return the parent of this boolean polynomial.

EXAMPLES:

```
sage: B.<a,b,c,d> = BooleanPolynomialRing(4)
sage: a.ring() is B
True
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(4), names=('a', 'b', 'c', 'd',)); (a, b,
↪ c, d,) = B._first_ngens(4)
>>> a.ring() is B
True
```

set()

Return a `BooleSet` with all monomials appearing in this polynomial.

EXAMPLES:

```
sage: B.<a,b,z> = BooleanPolynomialRing(3)
sage: (a*b+z+1).set()
{{a,b}, {z}, {}}
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(3), names=('a', 'b', 'z',)); (a, b, z,)␣
↪= B._first_ngens(3)
>>> (a*b+z+Integer(1)).set()
{{a,b}, {z}, {}}
```

spoly(*rhs*)

Return the S-Polynomial of this boolean polynomial and the other boolean polynomial `rhs`.

EXAMPLES:

```
sage: B.<a,b,c,d> = BooleanPolynomialRing(4)
sage: f = a*b*c + c*d + a*b + 1
```

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```
sage: g = c*d + b
sage: f.spoly(g)
a*b + a*c*d + c*d + 1
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(4), names=('a', 'b', 'c', 'd',)); (a, b,
↪ c, d,) = B._first_ngens(4)
>>> f = a*b*c + c*d + a*b + Integer(1)
>>> g = c*d + b
>>> f.spoly(g)
a*b + a*c*d + c*d + 1
```

Note

This function is part of the upstream PolyBoRi interface.

stable_hash()

A hash value which is stable across processes.

EXAMPLES:

```
sage: B.<x,y> = BooleanPolynomialRing()
sage: x is B.gen(0)
False
sage: x.stable_hash() == B.gen(0).stable_hash()
True
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(names=('x', 'y',)); (x, y,) = B._first_ngens(2)
>>> x is B.gen(Integer(0))
False
>>> x.stable_hash() == B.gen(Integer(0)).stable_hash()
True
```

Note

This function is part of the upstream PolyBoRi interface. In Sage all hashes are stable.

subs(*in_dict=None*, ***kwds*)

Fixes some given variables in a given boolean polynomial and returns the changed boolean polynomials. The polynomial itself is not affected. The variable, value pairs for fixing are to be provided as dictionary of the form {variable:value} or named parameters (see examples below).

INPUT:

- `in_dict` – (optional) dict with variable:value pairs
- `**kwds` – names parameters

EXAMPLES:

```
sage: P.<x,y,z> = BooleanPolynomialRing(3)
sage: f = x*y + z + y*z + 1
sage: f.subs(x=1)
y*z + y + z + 1
sage: f.subs(x=0)
y*z + z + 1
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= P._first_ngens(3)
>>> f = x*y + z + y*z + Integer(1)
>>> f.subs(x=Integer(1))
y*z + y + z + 1
>>> f.subs(x=Integer(0))
y*z + z + 1
```

```
sage: f.subs(x=y)
y*z + y + z + 1
```

```
>>> from sage.all import *
>>> f.subs(x=y)
y*z + y + z + 1
```

```
sage: f.subs({x:1},y=1)
0
sage: f.subs(y=1)
x + 1
sage: f.subs(y=1,z=1)
x + 1
sage: f.subs(z=1)
x*y + y
sage: f.subs({'x':1},y=1)
0
```

```
>>> from sage.all import *
>>> f.subs({x:Integer(1)},y=Integer(1))
0
>>> f.subs(y=Integer(1))
x + 1
>>> f.subs(y=Integer(1),z=Integer(1))
x + 1
>>> f.subs(z=Integer(1))
x*y + y
>>> f.subs({'x':Integer(1)},y=Integer(1))
0
```

This method can work fully symbolic:

```
sage: f.subs(x=var('a'), y=var('b'), z=var('c'))                          #␣
↪needs sage.symbolic
a*b + b*c + c + 1
sage: f.subs({'x': var('a'), 'y': var('b'), 'z': var('c')})               #␣
```

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```
↪needs sage.symbolic
a*b + b*c + c + 1
```

```
>>> from sage.all import *
>>> f.subs(x=var('a'), y=var('b'), z=var('c'))                               #␣
↪needs sage.symbolic
a*b + b*c + c + 1
>>> f.subs({'x': var('a'), 'y': var('b'), 'z': var('c')})                    #␣
↪needs sage.symbolic
a*b + b*c + c + 1
```

terms()

Return a list of monomials appearing in `self` ordered largest to smallest.

EXAMPLES:

```
sage: P.<a,b,c> = BooleanPolynomialRing(3,order='lex')
sage: f = a + c*b
sage: f.terms()
[a, b*c]

sage: P.<a,b,c> = BooleanPolynomialRing(3,order='deglex')
sage: f = a + c*b
sage: f.terms()
[b*c, a]
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(3),order='lex', names=('a', 'b', 'c',));
↪ (a, b, c,) = P._first_ngens(3)
>>> f = a + c*b
>>> f.terms()
[a, b*c]

>>> P = BooleanPolynomialRing(Integer(3),order='deglex', names=('a', 'b', 'c',
↪)); (a, b, c,) = P._first_ngens(3)
>>> f = a + c*b
>>> f.terms()
[b*c, a]
```

total_degree()

Return the total degree of `self`.

EXAMPLES:

```
sage: P.<x,y> = BooleanPolynomialRing(2)
sage: (x+y).total_degree()
1
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(2), names=('x', 'y',)); (x, y,) = P._
↪first_ngens(2)
>>> (x+y).total_degree()
1
```

```
sage: P(1).total_degree()
0
```

```
>>> from sage.all import *
>>> P(Integer(1)).total_degree()
0
```

```
sage: (x*y + x + y + 1).total_degree()
2
```

```
>>> from sage.all import *
>>> (x*y + x + y + Integer(1)).total_degree()
2
```

univariate_polynomial(*R=None*)

Return a univariate polynomial associated to this multivariate polynomial.

If this polynomial is not in at most one variable, then a `ValueError` exception is raised. This is checked using the *is_univariate()* method. The new Polynomial is over GF(2) and in the variable `x` if no ring `R` is provided.

> sage: R.<x, y> = BooleanPolynomialRing() sage: f = x - y + x*y + 1 sage: f.univariate_polynomial() Traceback (most recent call last): … ValueError: polynomial must involve at most one variable sage: g = f.subs({x:0}); g y + 1 sage: g.univariate_polynomial () y + 1 sage: g.univariate_polynomial(GF(2)['foo']) foo + 1

Here's an example with a constant multivariate polynomial:

```
sage: g = R(1)
sage: h = g.univariate_polynomial(); h
1
sage: h.parent()                                                            #␣
↪needs sage.libs.ntl
Univariate Polynomial Ring in x over Finite Field of size 2 (using GF2X)
```

```
>>> from sage.all import *
>>> g = R(Integer(1))
>>> h = g.univariate_polynomial(); h
1
>>> h.parent()                                                            #␣
↪needs sage.libs.ntl
Univariate Polynomial Ring in x over Finite Field of size 2 (using GF2X)
```

variable(*i=0*)

Return the i-th variable occurring in `self`. The index i is the index in `self.variables()`

EXAMPLES:

```
sage: P.<x,y,z> = BooleanPolynomialRing(3)
sage: f = x*z + z + 1
sage: f.variables()
(x, z)
sage: f.variable(1)
z
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= P._first_ngens(3)
>>> f = x*z + z + Integer(1)
>>> f.variables()
(x, z)
>>> f.variable(Integer(1))
z
```

variables()

Return a tuple of all variables appearing in `self`.

EXAMPLES:

```
sage: P.<x,y,z> = BooleanPolynomialRing(3)
sage: (x + y).variables()
(x, y)

sage: (x*y + z).variables()
(x, y, z)

sage: P.zero().variables()
()

sage: P.one().variables()
()
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= P._first_ngens(3)
>>> (x + y).variables()
(x, y)

>>> (x*y + z).variables()
(x, y, z)

>>> P.zero().variables()
()

>>> P.one().variables()
()
```

vars_as_monomial()

Return a boolean monomial with all variables appearing in `self`.

EXAMPLES:

```
sage: P.<x,y,z> = BooleanPolynomialRing(3)
sage: (x + y).vars_as_monomial()
x*y

sage: (x*y + z).vars_as_monomial()
x*y*z
```

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```
sage: P.zero().vars_as_monomial()
1

sage: P.one().vars_as_monomial()
1
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= P._first_ngens(3)
>>> (x + y).vars_as_monomial()
x*y

>>> (x*y + z).vars_as_monomial()
x*y*z

>>> P.zero().vars_as_monomial()
1

>>> P.one().vars_as_monomial()
1
```

Note

This function is part of the upstream PolyBoRi interface.

zeros_in(*s*)

Return a set containing all elements of `s` where this boolean polynomial evaluates to zero.

If `s` is given as a `BooleSet`, then the return type is also a `BooleSet`. If `s` is a set/list/tuple of tuple this function returns a tuple of tuples.

INPUT:

- `s` – candidate points for evaluation to zero

EXAMPLES:

```
sage: B.<a,b,c,d> = BooleanPolynomialRing(4)
sage: f = a*b + c + d + 1
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(4), names=('a', 'b', 'c', 'd',)); (a, b,
↪ c, d,) = B._first_ngens(4)
>>> f = a*b + c + d + Integer(1)
```

Now we create a set of points:

```
sage: s = a*b + a*b*c + c*d + 1
sage: s = s.set(); s
{{a,b,c}, {a,b}, {c,d}, {}}
```

```
>>> from sage.all import *
>>> s = a*b + a*b*c + c*d + Integer(1)
>>> s = s.set(); s
{{a,b,c}, {a,b}, {c,d}, {}}
```

This encodes the points (1,1,1,0), (1,1,0,0), (0,0,1,1) and (0,0,0,0). But of these only (1,1,0,0) evaluates to zero.

```
sage: f.zeros_in(s)
{{a,b}}
```

```
>>> from sage.all import *
>>> f.zeros_in(s)
{{a,b}}
```

```
sage: f.zeros_in([(1,1,1,0), (1,1,0,0), (0,0,1,1), (0,0,0,0)])
((1, 1, 0, 0),)
```

```
>>> from sage.all import *
>>> f.zeros_in([(Integer(1),Integer(1),Integer(1),Integer(0)), (Integer(1),
↪Integer(1),Integer(0),Integer(0)), (Integer(0),Integer(0),Integer(1),
↪Integer(1)), (Integer(0),Integer(0),Integer(0),Integer(0))])
((1, 1, 0, 0),)
```

class sage.rings.polynomial.pbori.pbori.**BooleanPolynomialEntry**

Bases: object

p

class sage.rings.polynomial.pbori.pbori.**BooleanPolynomialIdeal**(*ring*, *gens=[]*, *coerce=True*)

Bases: *MPolynomialIdeal*

Construct an ideal in the boolean polynomial ring.

INPUT:

- ring – the ring this ideal is defined in
- gens – list of generators
- coerce – coerce all elements to the ring ring (default: True)

EXAMPLES:

```
sage: P.<x0, x1, x2, x3> = BooleanPolynomialRing(4)
sage: I = P.ideal(x0*x1*x2*x3 + x0*x1*x3 + x0*x1 + x0*x2 + x0)
sage: I
Ideal (x0*x1*x2*x3 + x0*x1*x3 + x0*x1 + x0*x2 + x0) of Boolean PolynomialRing in␣
↪x0, x1, x2, x3
sage: loads(dumps(I)) == I
True
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(4), names=('x0', 'x1', 'x2', 'x3',)); (x0,␣
↪x1, x2, x3,) = P._first_ngens(4)
```

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```
>>> I = P.ideal(x0*x1*x2*x3 + x0*x1*x3 + x0*x1 + x0*x2 + x0)
>>> I
Ideal (x0*x1*x2*x3 + x0*x1*x3 + x0*x1 + x0*x2 + x0) of Boolean PolynomialRing in
↪x0, x1, x2, x3
>>> loads(dumps(I)) == I
True
```

dimension()

Return the dimension of `self`, which is always zero.

groebner_basis(*algorithm='polybori'*, ***kwds*)

Return a Groebner basis of this ideal.

INPUT:

- `algorithm` – either `'polybori'` (built-in default) or `'magma'` (requires Magma)
- `red_tail` – tail reductions in intermediate polynomials, this options affects mainly heuristics. The reducedness of the output polynomials can only be guaranteed by the option redsb (default: `True`).
- `minsb` – return a minimal Groebner basis (default: `True`)
- `redsb` – return a minimal Groebner basis and all tails are reduced (default: `True`)
- `deg_bound` – only compute Groebner basis up to a given degree bound (default: `False`)
- `faugere` – turn off or on the linear algebra (default: `False`)
- `linear_algebra_in_last_block` – this affects the last block of block orderings and degree orderings. If it is set to `True` linear algebra takes affect in this block. (default: `True`)
- `gauss_on_linear` – perform Gaussian elimination on linear polynomials (default: `True`)
- `selection_size` – maximum number of polynomials for parallel reductions (default: `1000`)
- `heuristic` – turn off heuristic by setting `heuristic=False` (default: `True`)
- `lazy` – (default: `True`)
- `invert` – setting `invert=True` input and output get a transformation `x+1` for each variable `x`, which should not effect the calculated GB, but the algorithm.
- `other_ordering_first` – possible values are `False` or an ordering code. In practice, many Boolean examples have very few solutions and a very easy Groebner basis. So, a complex walk algorithm (which cannot be implemented using the data structures) seems unnecessary, as such Groebner bases can be converted quite fast by the normal Buchberger algorithm from one ordering into another ordering. (default: `False`)
- `prot` – show protocol (default: `False`)
- `full_prot` – show full protocol (default: `False`)

EXAMPLES:

```
sage: P.<x0, x1, x2, x3> = BooleanPolynomialRing(4)
sage: I = P.ideal(x0*x1*x2*x3 + x0*x1*x3 + x0*x1 + x0*x2 + x0)
sage: I.groebner_basis()
[x0*x1 + x0*x2 + x0, x0*x2*x3 + x0*x3]
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(4), names=('x0', 'x1', 'x2', 'x3',));
↪(x0, x1, x2, x3,) = P._first_ngens(4)
>>> I = P.ideal(x0*x1*x2*x3 + x0*x1*x3 + x0*x1 + x0*x2 + x0)
>>> I.groebner_basis()
[x0*x1 + x0*x2 + x0, x0*x2*x3 + x0*x3]
```

Another somewhat bigger example:

```
sage: sr = mq.SR(2,1,1,4,gf2=True, polybori=True)
sage: while True:  # workaround (see :issue:`31891`)
....:     try:
....:         F, s = sr.polynomial_system()
....:         break
....:     except ZeroDivisionError:
....:         pass
sage: I = F.ideal()
sage: I.groebner_basis()  # not tested, known bug, unstable (see
↪:issue:`32083`)
Polynomial Sequence with 36 Polynomials in 36 Variables
```

```
>>> from sage.all import *
>>> sr = mq.SR(Integer(2),Integer(1),Integer(1),Integer(4),gf2=True,
↪polybori=True)
>>> while True:  # workaround (see :issue:`31891`)
...     try:
...         F, s = sr.polynomial_system()
...         break
...     except ZeroDivisionError:
...         pass
>>> I = F.ideal()
>>> I.groebner_basis()  # not tested, known bug, unstable (see :issue:`32083`)
Polynomial Sequence with 36 Polynomials in 36 Variables
```

We compute the same example with Magma:

```
sage: sr = mq.SR(2,1,1,4,gf2=True, polybori=True)
sage: while True:  # workaround (see :issue:`31891`)
....:     try:
....:         F, s = sr.polynomial_system()
....:         break
....:     except ZeroDivisionError:
....:         pass
sage: I = F.ideal()
sage: I.groebner_basis(algorithm='magma', prot='sage') # optional - magma
Leading term degree:  1. Critical pairs: 148.
...
Highest degree reached during computation:  3.
Polynomial Sequence with ... Polynomials in 36 Variables
```

```
>>> from sage.all import *
>>> sr = mq.SR(Integer(2),Integer(1),Integer(1),Integer(4),gf2=True,
↪polybori=True)
```

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```
>>> while True:  # workaround (see :issue:`31891`)
...     try:
...         F, s = sr.polynomial_system()
...         break
...     except ZeroDivisionError:
...         pass
>>> I = F.ideal()
>>> I.groebner_basis(algorithm='magma', prot='sage') # optional - magma
Leading term degree:  1. Critical pairs: 148.
...
Highest degree reached during computation:  3.
Polynomial Sequence with ... Polynomials in 36 Variables
```

interreduced_basis()

If this ideal is spanned by `(f_1, ..., f_n)` this method returns `(g_1, ..., g_s)` such that:

- `<f_1,...,f_n> = <g_1,...,g_s>`
- `LT(g_i) != LT(g_j)` for all `i != j`
- `LT(g_i)` does not divide `m` for all monomials `m` of `{g_1,...,g_{i-1},g_{i+1},...,g_s}`

EXAMPLES:

```
sage: sr = mq.SR(1, 1, 1, 4, gf2=True, polybori=True)
sage: while True:  # workaround (see :issue:`31891`)
....:     try:
....:         F, s = sr.polynomial_system()
....:         break
....:     except ZeroDivisionError:
....:         pass
sage: I = F.ideal()
sage: g = I.interreduced_basis()
sage: len(g) == len(set(gi.lt() for gi in g))
True
sage: for i in range(len(g)):
....:     lt = g[i].lt()
....:     for j in range(len(g)):
....:         if i == j:
....:             continue
....:         for t in iter(g[j]):
....:             assert lt not in t.divisors()
```

```
>>> from sage.all import *
>>> sr = mq.SR(Integer(1), Integer(1), Integer(1), Integer(4), gf2=True, polybori=True)
>>> while True:  # workaround (see :issue:`31891`)
...     try:
...         F, s = sr.polynomial_system()
...         break
...     except ZeroDivisionError:
...         pass
>>> I = F.ideal()
>>> g = I.interreduced_basis()
```

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```
>>> len(g) == len(set(gi.lt() for gi in g))
True
>>> for i in range(len(g)):
...     lt = g[i].lt()
...     for j in range(len(g)):
...         if i == j:
...             continue
...         for t in iter(g[j]):
...             assert lt not in t.divisors()
```

reduce(*f*)

Reduce an element modulo the reduced Groebner basis for this ideal. This returns 0 if and only if the element is in this ideal. In any case, this reduction is unique up to monomial orders.

EXAMPLES:

```
sage: P = PolynomialRing(GF(2),10, 'x')
sage: B = BooleanPolynomialRing(10,'x')
sage: I = sage.rings.ideal.Cyclic(P)
sage: I = B.ideal([B(f) for f in I.gens()])
sage: gb = I.groebner_basis()
sage: I.reduce(gb[0])
0
sage: I.reduce(gb[0] + 1)
1
sage: I.reduce(gb[0]*gb[1])
0
sage: I.reduce(gb[0]*B.gen(1))
0
```

```
>>> from sage.all import *
>>> P = PolynomialRing(GF(Integer(2)),Integer(10), 'x')
>>> B = BooleanPolynomialRing(Integer(10),'x')
>>> I = sage.rings.ideal.Cyclic(P)
>>> I = B.ideal([B(f) for f in I.gens()])
>>> gb = I.groebner_basis()
>>> I.reduce(gb[Integer(0)])
0
>>> I.reduce(gb[Integer(0)] + Integer(1))
1
>>> I.reduce(gb[Integer(0)]*gb[Integer(1)])
0
>>> I.reduce(gb[Integer(0)]*B.gen(Integer(1)))
0
```

variety(***kwds*)

Return the variety associated to this boolean ideal.

EXAMPLES:

A simple example:

```
sage: R.<x,y,z> = BooleanPolynomialRing()
sage: I = ideal( [ x*y*z + x*z + y + 1, x+y+z+1 ] )
sage: I.variety()
[{z: 0, y: 1, x: 0}, {z: 1, y: 1, x: 1}]
```

```
>>> from sage.all import *
>>> R = BooleanPolynomialRing(names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> I = ideal( [ x*y*z + x*z + y + Integer(1), x+y+z+Integer(1) ] )
>>> I.variety()
[{z: 0, y: 1, x: 0}, {z: 1, y: 1, x: 1}]
```

class sage.rings.polynomial.pbori.pbori.**BooleanPolynomialIterator**

Bases: `object`

Iterator over the monomials of a boolean polynomial.

class sage.rings.polynomial.pbori.pbori.**BooleanPolynomialRing**

Bases: *BooleanPolynomialRing_base*

Construct a boolean polynomial ring with the following parameters:

INPUT:

- `n` – integer > 1; number of variables
- `names` – names of ring variables; may be a string or list/tuple
- `order` – term order (default: lex)

EXAMPLES:

```
sage: R.<x, y, z> = BooleanPolynomialRing()
sage: R
Boolean PolynomialRing in x, y, z
```

```
>>> from sage.all import *
>>> R = BooleanPolynomialRing(names=('x', 'y', 'z',)); (x, y, z,) = R._first_
↪ngens(3)
>>> R
Boolean PolynomialRing in x, y, z
```

```
sage: p = x*y + x*z + y*z
sage: x*p
x*y*z + x*y + x*z
```

```
>>> from sage.all import *
>>> p = x*y + x*z + y*z
>>> x*p
x*y*z + x*y + x*z
```

```
sage: R.term_order()
Lexicographic term order
```

```
>>> from sage.all import *
>>> R.term_order()
Lexicographic term order
```

```
sage: R = BooleanPolynomialRing(5,'x',order='deglex(3),deglex(2)')
sage: R.term_order()
Block term order with blocks:
(Degree lexicographic term order of length 3,
 Degree lexicographic term order of length 2)
```

```
>>> from sage.all import *
>>> R = BooleanPolynomialRing(Integer(5),'x',order='deglex(3),deglex(2)')
>>> R.term_order()
Block term order with blocks:
(Degree lexicographic term order of length 3,
 Degree lexicographic term order of length 2)
```

```
sage: R = BooleanPolynomialRing(3,'x',order='deglex')
sage: R.term_order()
Degree lexicographic term order
```

```
>>> from sage.all import *
>>> R = BooleanPolynomialRing(Integer(3),'x',order='deglex')
>>> R.term_order()
Degree lexicographic term order
```

change_ring(*base_ring=None*, *names=None*, *order=None*)

Return a new multivariate polynomial ring with base ring `base_ring`, variable names set to `names`, and term ordering given by `order`.

When `base_ring` is not specified, this function returns a `BooleanPolynomialRing` isomorphic to `self`. Otherwise, this returns a `MPolynomialRing`. Each argument above is optional.

INPUT:

- `base_ring` – a base ring
- `names` – variable names
- `order` – a term order

EXAMPLES:

```
sage: P.<x, y, z> = BooleanPolynomialRing()
sage: P.term_order()
Lexicographic term order
sage: R = P.change_ring(names=('a', 'b', 'c'), order='deglex')
sage: R
Boolean PolynomialRing in a, b, c
sage: R.term_order()
Degree lexicographic term order
sage: T = P.change_ring(base_ring=GF(3))
sage: T
Multivariate Polynomial Ring in x, y, z over Finite Field of size 3
```

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```
sage: T.term_order()
Lexicographic term order
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(names=('x', 'y', 'z',)); (x, y, z,) = P._first_
↪ngens(3)
>>> P.term_order()
Lexicographic term order
>>> R = P.change_ring(names=('a', 'b', 'c'), order='deglex')
>>> R
Boolean PolynomialRing in a, b, c
>>> R.term_order()
Degree lexicographic term order
>>> T = P.change_ring(base_ring=GF(Integer(3)))
>>> T
Multivariate Polynomial Ring in x, y, z over Finite Field of size 3
>>> T.term_order()
Lexicographic term order
```

clone(*ordering=None*, *names=[]*, *blocks=[]*)

Shallow copy this boolean polynomial ring, but with different ordering, names or blocks if given.

ring.clone(ordering=…, names=…, block=…) generates a shallow copy of ring, but with different ordering, names or blocks if given.

EXAMPLES:

```
sage: B.<a,b,c> = BooleanPolynomialRing()
sage: B.clone()
Boolean PolynomialRing in a, b, c
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c',)); (a, b, c,) = B._first_
↪ngens(3)
>>> B.clone()
Boolean PolynomialRing in a, b, c
```

```
sage: B.<x,y,z> = BooleanPolynomialRing(3,order='deglex')
sage: y*z > x
True
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(3),order='deglex', names=('x', 'y', 'z',
↪)); (x, y, z,) = B._first_ngens(3)
>>> y*z > x
True
```

Now we call the clone method and generate a compatible, but 'lex' ordered, ring:

```
sage: C = B.clone(ordering=0)
sage: C(y*z) > C(x)
False
```

```
>>> from sage.all import *
>>> C = B.clone(ordering=Integer(0))
>>> C(y*z) > C(x)
False
```

Now we change variable names:

```
sage: P.<x0,x1> = BooleanPolynomialRing(2)
sage: P
Boolean PolynomialRing in x0, x1
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(2), names=('x0', 'x1',)); (x0, x1,) = P.
↪_first_ngens(2)
>>> P
Boolean PolynomialRing in x0, x1
```

```
sage: Q = P.clone(names=['t'])
sage: Q
Boolean PolynomialRing in t, x1
```

```
>>> from sage.all import *
>>> Q = P.clone(names=['t'])
>>> Q
Boolean PolynomialRing in t, x1
```

We can also append blocks to block orderings this way:

```
sage: R.<x1,x2,x3,x4> = BooleanPolynomialRing(order='deglex(1),deglex(3)')
sage: x2 > x3*x4
False
```

```
>>> from sage.all import *
>>> R = BooleanPolynomialRing(order='deglex(1),deglex(3)', names=('x1', 'x2',
↪'x3', 'x4',)); (x1, x2, x3, x4,) = R._first_ngens(4)
>>> x2 > x3*x4
False
```

Now we call the internal method and change the blocks:

```
sage: S = R.clone(blocks=[3])
sage: S(x2) > S(x3*x4)
True
```

```
>>> from sage.all import *
>>> S = R.clone(blocks=[Integer(3)])
>>> S(x2) > S(x3*x4)
True
```

Note

This is part of PolyBoRi's native interface.

construction()

A boolean polynomial ring is the quotient of a polynomial ring, in a special implementation.

Before Issue #15223, the boolean polynomial rings returned the construction of a polynomial ring, which was of course wrong.

Now, a `QuotientFunctor` is returned that knows about the "*pbori*" implementation.

EXAMPLES:

```
sage: P.<x0, x1, x2, x3> = BooleanPolynomialRing(4,order='degneglex(2),
↪degneglex(2)')
sage: F,O = P.construction()
sage: O
Multivariate Polynomial Ring in x0, x1, x2, x3 over Finite Field of size 2
sage: F
QuotientFunctor
sage: F(O) is P
True
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(4),order='degneglex(2),degneglex(2)',␣
↪names=('x0', 'x1', 'x2', 'x3',)); (x0, x1, x2, x3,) = P._first_ngens(4)
>>> F,O = P.construction()
>>> O
Multivariate Polynomial Ring in x0, x1, x2, x3 over Finite Field of size 2
>>> F
QuotientFunctor
>>> F(O) is P
True
```

cover_ring()

Return $R = \mathbf{F}_2[x_1, x_2, ..., x_n]$ if `x_1,x_2,...,x_n` is the ordered list of variable names of this ring. `R` also has the same term ordering as this ring.

EXAMPLES:

```
sage: B.<x,y> = BooleanPolynomialRing(2)
sage: R = B.cover_ring(); R
Multivariate Polynomial Ring in x, y over Finite Field of size 2
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(2), names=('x', 'y',)); (x, y,) = B._
↪first_ngens(2)
>>> R = B.cover_ring(); R
Multivariate Polynomial Ring in x, y over Finite Field of size 2
```

```
sage: B.term_order() == R.term_order()
True
```

```
>>> from sage.all import *
>>> B.term_order() == R.term_order()
True
```

The cover ring is cached:

```
sage: B.cover_ring() is B.cover_ring()
True
```

```
>>> from sage.all import *
>>> B.cover_ring() is B.cover_ring()
True
```

defining_ideal()

Return $I =< x_i^2 + x_i >\subset R$ where `R = self.cover_ring()`, and x_i any element in the set of variables of this ring.

EXAMPLES:

```
sage: B.<x,y> = BooleanPolynomialRing(2)
sage: I = B.defining_ideal(); I
Ideal (x^2 + x, y^2 + y) of Multivariate Polynomial Ring
in x, y over Finite Field of size 2
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(2), names=('x', 'y',)); (x, y,) = B._
↪first_ngens(2)
>>> I = B.defining_ideal(); I
Ideal (x^2 + x, y^2 + y) of Multivariate Polynomial Ring
in x, y over Finite Field of size 2
```

gen(*i=0*)

Return the i-th generator of this boolean polynomial ring.

INPUT:

- `i` – integer or a boolean monomial in one variable

EXAMPLES:

```
sage: P.<x,y,z> = BooleanPolynomialRing(3)
sage: P.gen()
x
sage: P.gen(2)
z
sage: m = x.monomials()[0]
sage: P.gen(m)
x
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= P._first_ngens(3)
>>> P.gen()
x
```

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```
>>> P.gen(Integer(2))
z
>>> m = x.monomials()[Integer(0)]
>>> P.gen(m)
x
```

gens()

Return the tuple of variables in this ring.

EXAMPLES:

```
sage: P.<x,y,z> = BooleanPolynomialRing(3)
sage: P.gens()
(x, y, z)
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= P._first_ngens(3)
>>> P.gens()
(x, y, z)
```

```
sage: P = BooleanPolynomialRing(10,'x')
sage: P.gens()
(x0, x1, x2, x3, x4, x5, x6, x7, x8, x9)
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(10),'x')
>>> P.gens()
(x0, x1, x2, x3, x4, x5, x6, x7, x8, x9)
```

get_base_order_code()

EXAMPLES:

```
sage: B.<a,b,c,d,e,f> = BooleanPolynomialRing()
sage: B.get_base_order_code()
0

sage: B.<a,b,c,d,e,f> = BooleanPolynomialRing(order='deglex')
sage: B.get_base_order_code()
1
sage: T = TermOrder('deglex',2) + TermOrder('deglex',2)
sage: B.<a,b,c,d> = BooleanPolynomialRing(4, order=T)
sage: B.get_base_order_code()
1
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd', 'e', 'f',)); (a, b,␣
↪c, d, e, f,) = B._first_ngens(6)
>>> B.get_base_order_code()
0

```

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```
>>> B = BooleanPolynomialRing(order='deglex', names=('a', 'b', 'c', 'd', 'e',
↪'f',)); (a, b, c, d, e, f,) = B._first_ngens(6)
>>> B.get_base_order_code()
1
>>> T = TermOrder('deglex',Integer(2)) + TermOrder('deglex',Integer(2))
>>> B = BooleanPolynomialRing(Integer(4), order=T, names=('a', 'b', 'c', 'd',
↪)); (a, b, c, d,) = B._first_ngens(4)
>>> B.get_base_order_code()
1
```

Note

This function which is part of the PolyBoRi upstream API works with a current global ring. This notion is avoided in Sage.

get_order_code()

EXAMPLES:

```
sage: B.<a,b,c,d,e,f> = BooleanPolynomialRing()
sage: B.get_order_code()
0

sage: B.<a,b,c,d,e,f> = BooleanPolynomialRing(order='deglex')
sage: B.get_order_code()
1
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd', 'e', 'f',)); (a, b,␣
↪c, d, e, f,) = B._first_ngens(6)
>>> B.get_order_code()
0

>>> B = BooleanPolynomialRing(order='deglex', names=('a', 'b', 'c', 'd', 'e',
↪'f',)); (a, b, c, d, e, f,) = B._first_ngens(6)
>>> B.get_order_code()
1
```

Note

This function which is part of the PolyBoRi upstream API works with a current global ring. This notion is avoided in Sage.

has_degree_order()

Return checks whether the order code corresponds to a degree ordering.

EXAMPLES:

```
sage: P.<x,y> = BooleanPolynomialRing(2)
sage: P.has_degree_order()
False
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(2), names=('x', 'y',)); (x, y,) = P._
↪first_ngens(2)
>>> P.has_degree_order()
False
```

id()

Return a unique identifier for this boolean polynomial ring.

EXAMPLES:

```
sage: P.<x,y> = BooleanPolynomialRing(2)
sage: print("id: {}".format(P.id()))
id: ...

sage: P = BooleanPolynomialRing(10, 'x')
sage: Q = BooleanPolynomialRing(20, 'x')

sage: P.id() != Q.id()
True
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(2), names=('x', 'y',)); (x, y,) = P._
↪first_ngens(2)
>>> print("id: {}".format(P.id()))
id: ...

>>> P = BooleanPolynomialRing(Integer(10), 'x')
>>> Q = BooleanPolynomialRing(Integer(20), 'x')

>>> P.id() != Q.id()
True
```

ideal(**gens*, ***kwds*)

Create an ideal in this ring.

INPUT:

- `gens` – list or tuple of generators
- `coerce` – boolean (default: `True`); automatically coerce the given polynomials to this ring to form the ideal

EXAMPLES:

```
sage: P.<x,y,z> = BooleanPolynomialRing(3)
sage: P.ideal(x+y)
Ideal (x + y) of Boolean PolynomialRing in x, y, z
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= P._first_ngens(3)
>>> P.ideal(x+y)
Ideal (x + y) of Boolean PolynomialRing in x, y, z
```

```
sage: P.ideal(x*y, y*z)
Ideal (x*y, y*z) of Boolean PolynomialRing in x, y, z
```

```
>>> from sage.all import *
>>> P.ideal(x*y, y*z)
Ideal (x*y, y*z) of Boolean PolynomialRing in x, y, z
```

```
sage: P.ideal([x+y, z])
Ideal (x + y, z) of Boolean PolynomialRing in x, y, z
```

```
>>> from sage.all import *
>>> P.ideal([x+y, z])
Ideal (x + y, z) of Boolean PolynomialRing in x, y, z
```

interpolation_polynomial(*zeros*, *ones*)

Return the lexicographically minimal boolean polynomial for the given sets of points.

Given two sets of points `zeros` - evaluating to zero - and `ones` - evaluating to one -, compute the lexicographically minimal boolean polynomial satisfying these points.

INPUT:

- `zeros` – the set of interpolation points mapped to zero
- `ones` – the set of interpolation points mapped to one

EXAMPLES:

First we create a random-ish boolean polynomial.

```
sage: B.<a,b,c,d,e,f> = BooleanPolynomialRing(6)
sage: f = a*b*c*e + a*d*e + a*f + b + c + e + f + 1
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(6), names=('a', 'b', 'c', 'd', 'e', 'f',
↪)); (a, b, c, d, e, f,) = B._first_ngens(6)
>>> f = a*b*c*e + a*d*e + a*f + b + c + e + f + Integer(1)
```

Now we find interpolation points mapping to zero and to one.

```
sage: zeros = set([(1, 0, 1, 0, 0, 0), (1, 0, 0, 0, 1, 0),
....:              (0, 0, 1, 1, 1, 1), (1, 0, 1, 1, 1, 1),
....:              (0, 0, 0, 0, 1, 0), (0, 1, 1, 1, 1, 0),
....:              (1, 1, 0, 0, 0, 1), (1, 1, 0, 1, 0, 1)])
sage: ones = set([(0, 0, 0, 0, 0, 0), (1, 0, 1, 0, 1, 0),
....:             (0, 0, 0, 1, 1, 1), (1, 0, 0, 1, 0, 1),
....:             (0, 0, 0, 0, 1, 1), (0, 1, 1, 0, 1, 1),
....:             (0, 1, 1, 1, 1, 1), (1, 1, 1, 0, 1, 0)])
sage: [f(*p) for p in zeros]
[0, 0, 0, 0, 0, 0, 0, 0]
sage: [f(*p) for p in ones]
[1, 1, 1, 1, 1, 1, 1, 1]
```

```
>>> from sage.all import *
>>> zeros = set([(Integer(1), Integer(0), Integer(1), Integer(0), Integer(0),␣
↪Integer(0)), (Integer(1), Integer(0), Integer(0), Integer(0), Integer(1),␣
↪Integer(0)),
...              (Integer(0), Integer(0), Integer(1), Integer(1), Integer(1),␣
↪Integer(1)), (Integer(1), Integer(0), Integer(1), Integer(1), Integer(1),␣
↪Integer(1)),
...              (Integer(0), Integer(0), Integer(0), Integer(0), Integer(1),␣
↪Integer(0)), (Integer(0), Integer(1), Integer(1), Integer(1), Integer(1),␣
↪Integer(0)),
...              (Integer(1), Integer(1), Integer(0), Integer(0), Integer(0),␣
↪Integer(1)), (Integer(1), Integer(1), Integer(0), Integer(1), Integer(0),␣
↪Integer(1))])
>>> ones = set([(Integer(0), Integer(0), Integer(0), Integer(0), Integer(0),␣
↪Integer(0)), (Integer(1), Integer(0), Integer(1), Integer(0), Integer(1),␣
↪Integer(0)),
...             (Integer(0), Integer(0), Integer(0), Integer(1), Integer(1),␣
↪Integer(1)), (Integer(1), Integer(0), Integer(0), Integer(1), Integer(0),␣
↪Integer(1)),
...             (Integer(0), Integer(0), Integer(0), Integer(0), Integer(1),␣
↪Integer(1)), (Integer(0), Integer(1), Integer(1), Integer(0), Integer(1),␣
↪Integer(1)),
...             (Integer(0), Integer(1), Integer(1), Integer(1), Integer(1),␣
↪Integer(1)), (Integer(1), Integer(1), Integer(1), Integer(0), Integer(1),␣
↪Integer(0))])
>>> [f(*p) for p in zeros]
[0, 0, 0, 0, 0, 0, 0, 0]
>>> [f(*p) for p in ones]
[1, 1, 1, 1, 1, 1, 1, 1]
```

Finally, we find the lexicographically smallest interpolation polynomial using PolyBoRi .

```
sage: g = B.interpolation_polynomial(zeros, ones); g
b*f + c + d*f + d + e*f + e + 1
```

```
>>> from sage.all import *
>>> g = B.interpolation_polynomial(zeros, ones); g
b*f + c + d*f + d + e*f + e + 1
```

```
sage: [g(*p) for p in zeros]
[0, 0, 0, 0, 0, 0, 0, 0]
sage: [g(*p) for p in ones]
[1, 1, 1, 1, 1, 1, 1, 1]
```

```
>>> from sage.all import *
>>> [g(*p) for p in zeros]
[0, 0, 0, 0, 0, 0, 0, 0]
>>> [g(*p) for p in ones]
[1, 1, 1, 1, 1, 1, 1, 1]
```

Alternatively, we can work with PolyBoRi's native `BooleSet`'s. This example is from the PolyBoRi tutorial:

```
sage: B = BooleanPolynomialRing(4,"x0,x1,x2,x3")
sage: x = B.gen
sage: V=(x(0)+x(1)+x(2)+x(3)+1).set(); V
{{x0}, {x1}, {x2}, {x3}, {}}
sage: f=x(0)*x(1)+x(1)+x(2)+1
sage: z = f.zeros_in(V); z
{{x1}, {x2}}
sage: o = V.diff(z); o
{{x0}, {x3}, {}}
sage: B.interpolation_polynomial(z,o)
x1 + x2 + 1
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(4),"x0,x1,x2,x3")
>>> x = B.gen
>>> V=(x(Integer(0))+x(Integer(1))+x(Integer(2))+x(Integer(3))+Integer(1)).
↪set(); V
{{x0}, {x1}, {x2}, {x3}, {}}
>>> f=x(Integer(0))*x(Integer(1))+x(Integer(1))+x(Integer(2))+Integer(1)
>>> z = f.zeros_in(V); z
{{x1}, {x2}}
>>> o = V.diff(z); o
{{x0}, {x3}, {}}
>>> B.interpolation_polynomial(z,o)
x1 + x2 + 1
```

ALGORITHM: Calls `interpolate_smallest_lex` as described in the PolyBoRi tutorial.

n_variables()

Return the number of variables in this boolean polynomial ring.

EXAMPLES:

```
sage: P.<x,y> = BooleanPolynomialRing(2)
sage: P.n_variables()
2
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(2), names=('x', 'y',)); (x, y,) = P._
↪first_ngens(2)
>>> P.n_variables()
2
```

```
sage: P = BooleanPolynomialRing(1000, 'x')
sage: P.n_variables()
1000
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(1000), 'x')
>>> P.n_variables()
1000
```

> **Note**
>
> This is part of PolyBoRi's native interface.

ngens()

Return the number of variables in this boolean polynomial ring.

EXAMPLES:

```
sage: P.<x,y> = BooleanPolynomialRing(2)
sage: P.ngens()
2
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(2), names=('x', 'y',)); (x, y,) = P._
↪first_ngens(2)
>>> P.ngens()
2
```

```
sage: P = BooleanPolynomialRing(1000, 'x')
sage: P.ngens()
1000
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(1000), 'x')
>>> P.ngens()
1000
```

one()

EXAMPLES:

```
sage: P.<x0,x1> = BooleanPolynomialRing(2)
sage: P.one()
1
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(2), names=('x0', 'x1',)); (x0, x1,) = P.
↪_first_ngens(2)
>>> P.one()
1
```

random_element(*degree=None*, *terms=None*, *choose_degree=False*, *vars_set=None*)

Return a random boolean polynomial. Generated polynomial has the given number of terms, and at most given degree.

INPUT:

- `degree` – maximum degree (default: 2 for len(var_set) > 1, 1 otherwise)
- `terms` – number of terms requested (default: 5). If more terms are requested than exist, then this parameter is silently reduced to the maximum number of available terms.
- `choose_degree` – choose degree of monomials randomly first, rather than monomials uniformly random

- `vars_set` – list of integer indices of generators of `self` to use in the generated polynomial

EXAMPLES:

```
sage: P.<x,y,z> = BooleanPolynomialRing(3)
sage: f = P.random_element(degree=3, terms=4)
sage: f.degree() <= 3
True
sage: len(f.terms())
4
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= P._first_ngens(3)
>>> f = P.random_element(degree=Integer(3), terms=Integer(4))
>>> f.degree() <= Integer(3)
True
>>> len(f.terms())
4
```

```
sage: f = P.random_element(degree=1, terms=2)
sage: f.degree() <= 1
True
sage: len(f.terms())
2
```

```
>>> from sage.all import *
>>> f = P.random_element(degree=Integer(1), terms=Integer(2))
>>> f.degree() <= Integer(1)
True
>>> len(f.terms())
2
```

In corner cases this function will return fewer terms by default:

```
sage: P = BooleanPolynomialRing(2,'y')
sage: f = P.random_element()
sage: len(f.terms())
2

sage: P = BooleanPolynomialRing(1,'y')
sage: f = P.random_element()
sage: len(f.terms())
1
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(2),'y')
>>> f = P.random_element()
>>> len(f.terms())
2

>>> P = BooleanPolynomialRing(Integer(1),'y')
>>> f = P.random_element()
```

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```
>>> len(f.terms())
1
```

We return uniformly random polynomials up to degree 2:

```
sage: from collections import defaultdict
sage: B.<a,b,c,d> = BooleanPolynomialRing()
sage: counter = 0.0
sage: dic = defaultdict(Integer)
sage: def more_terms():
....:     global counter, dic
....:     for t in B.random_element(terms=Infinity).terms():
....:         counter += 1.0
....:         dic[t] += 1

sage: more_terms()
sage: while any(abs(dic[t]/counter - 1.0/11) > 0.01 for t in dic):
....:     more_terms()
```

```
>>> from sage.all import *
>>> from collections import defaultdict
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd',)); (a, b, c, d,) = B.
↪_first_ngens(4)
>>> counter = RealNumber('0.0')
>>> dic = defaultdict(Integer)
>>> def more_terms():
...     global counter, dic
...     for t in B.random_element(terms=Infinity).terms():
...         counter += RealNumber('1.0')
...         dic[t] += Integer(1)

>>> more_terms()
>>> while any(abs(dic[t]/counter - RealNumber('1.0')/Integer(11)) >␣
↪RealNumber('0.01') for t in dic):
...     more_terms()
```

remove_var(*order=None*, **var*)

Remove a variable or sequence of variables from this ring.

If `order` is not specified, then the subring inherits the term order of the original ring, if possible.

EXAMPLES:

```
sage: R.<x,y,z,w> = BooleanPolynomialRing()
sage: R.remove_var(z)
Boolean PolynomialRing in x, y, w
sage: R.remove_var(z,x)
Boolean PolynomialRing in y, w
sage: R.remove_var(y,z,x)
Boolean PolynomialRing in w
```

```
>>> from sage.all import *
>>> R = BooleanPolynomialRing(names=('x', 'y', 'z', 'w',)); (x, y, z, w,) = R.
```

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```
↪_first_ngens(4)
>>> R.remove_var(z)
Boolean PolynomialRing in x, y, w
>>> R.remove_var(z,x)
Boolean PolynomialRing in y, w
>>> R.remove_var(y,z,x)
Boolean PolynomialRing in w
```

Removing all variables results in the base ring:

```
sage: R.remove_var(y,z,x,w)
Finite Field of size 2
```

```
>>> from sage.all import *
>>> R.remove_var(y,z,x,w)
Finite Field of size 2
```

If possible, the term order is kept:

sage: R.<x,y,z,w> = BooleanPolynomialRing(order='deglex') sage: R.remove_var(y).term_order() Degree lexicographic term order

sage: R.<x,y,z,w> = BooleanPolynomialRing(order='lex') sage: R.remove_var(y).term_order() Lexicographic term order

Be careful with block orders when removing variables:

```
sage: R.<x,y,z,u,v> = BooleanPolynomialRing(order='deglex(2),deglex(3)')
sage: R.remove_var(x,y,z)
Traceback (most recent call last):
...
ValueError: impossible to use the original term order (most likely because it␣
↪was a block order); please specify the term order for the subring
sage: R.remove_var(x,y,z, order='deglex')
Boolean PolynomialRing in u, v
```

```
>>> from sage.all import *
>>> R = BooleanPolynomialRing(order='deglex(2),deglex(3)', names=('x', 'y', 'z
↪', 'u', 'v',)); (x, y, z, u, v,) = R._first_ngens(5)
>>> R.remove_var(x,y,z)
Traceback (most recent call last):
...
ValueError: impossible to use the original term order (most likely because it␣
↪was a block order); please specify the term order for the subring
>>> R.remove_var(x,y,z, order='deglex')
Boolean PolynomialRing in u, v
```

variable(*i=0*)

Return the i-th generator of this boolean polynomial ring.

INPUT:

- `i` – integer or a boolean monomial in one variable

EXAMPLES:

```
sage: P.<x,y,z> = BooleanPolynomialRing(3)
sage: P.variable()
x
sage: P.variable(2)
z
sage: m = x.monomials()[0]
sage: P.variable(m)
x
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,)␣
↪= P._first_ngens(3)
>>> P.variable()
x
>>> P.variable(Integer(2))
z
>>> m = x.monomials()[Integer(0)]
>>> P.variable(m)
x
```

zero()

EXAMPLES:

```
sage: P.<x0,x1> = BooleanPolynomialRing(2)
sage: P.zero()
0
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(2), names=('x0', 'x1',)); (x0, x1,) = P.
↪_first_ngens(2)
>>> P.zero()
0
```

class sage.rings.polynomial.pbori.pbori.**BooleanPolynomialVector**

Bases: object

A vector of boolean polynomials.

EXAMPLES:

```
sage: B.<a,b,c,d,e,f> = BooleanPolynomialRing()
sage: from sage.rings.polynomial.pbori.pbori import BooleanPolynomialVector
sage: l = [B.random_element() for _ in range(3)]
sage: v = BooleanPolynomialVector(l)
sage: len(v)
3
sage: all(vi.parent() is B for vi in v)
True
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd', 'e', 'f',)); (a, b, c, d,
↪ e, f,) = B._first_ngens(6)
>>> from sage.rings.polynomial.pbori.pbori import BooleanPolynomialVector
```

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```
>>> l = [B.random_element() for _ in range(Integer(3))]
>>> v = BooleanPolynomialVector(l)
>>> len(v)
3
>>> all(vi.parent() is B for vi in v)
True
```

append(*el*)

Append the element `el` to this vector.

EXAMPLES:

```
sage: B.<a,b,c,d,e,f> = BooleanPolynomialRing()
sage: from sage.rings.polynomial.pbori.pbori import BooleanPolynomialVector
sage: v = BooleanPolynomialVector()
sage: entries = []
sage: for i in range(5):
....:   entries.append(B.random_element())
....:   v.append(entries[-1])

sage: list(v) == entries
True
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd', 'e', 'f',)); (a, b,
↪c, d, e, f,) = B._first_ngens(6)
>>> from sage.rings.polynomial.pbori.pbori import BooleanPolynomialVector
>>> v = BooleanPolynomialVector()
>>> entries = []
>>> for i in range(Integer(5)):
...   entries.append(B.random_element())
...   v.append(entries[-Integer(1)])

>>> list(v) == entries
True
```

class sage.rings.polynomial.pbori.pbori.**BooleanPolynomialVectorIterator**

Bases: `object`

class sage.rings.polynomial.pbori.pbori.**CCuddNavigator**

Bases: `object`

constant()

else_branch()

terminal_one()

then_branch()

value()

class sage.rings.polynomial.pbori.pbori.**FGLMStrategy**

Bases: `object`

Strategy object for the FGLM algorithm to translate from one Groebner basis with respect to a term ordering A to another Groebner basis with respect to a term ordering B.

main()

Execute the FGLM algorithm.

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import *
sage: B.<x,y,z> = BooleanPolynomialRing()
sage: ideal = BooleanPolynomialVector([x+z, y+z])
sage: list(ideal)
[x + z, y + z]
sage: old_ring = B
sage: new_ring = B.clone(ordering=dp_asc)
sage: list(FGLMStrategy(old_ring, new_ring, ideal).main())
[y + x, z + x]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import *
>>> B = BooleanPolynomialRing(names=('x', 'y', 'z',)); (x, y, z,) = B._first_
↪ngens(3)
>>> ideal = BooleanPolynomialVector([x+z, y+z])
>>> list(ideal)
[x + z, y + z]
>>> old_ring = B
>>> new_ring = B.clone(ordering=dp_asc)
>>> list(FGLMStrategy(old_ring, new_ring, ideal).main())
[y + x, z + x]
```

class sage.rings.polynomial.pbori.pbori.**GroebnerStrategy**

Bases: object

A Groebner strategy is the main object to control the strategy for computing Groebner bases.

> **Note**
>
> This class is mainly used internally.

add_as_you_wish(*p*)

Add a new generator but let the strategy object decide whether to perform immediate interreduction.

INPUT:

- p – a polynomial

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import *
sage: B.<a,b,c,d,e,f> = BooleanPolynomialRing()
sage: gbs = GroebnerStrategy(B)
sage: gbs.add_as_you_wish(a + b)
sage: list(gbs)
[a + b]
sage: gbs.add_as_you_wish(a + c)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd', 'e', 'f',)); (a, b,␣
↪c, d, e, f,) = B._first_ngens(6)
>>> gbs = GroebnerStrategy(B)
>>> gbs.add_as_you_wish(a + b)
>>> list(gbs)
[a + b]
>>> gbs.add_as_you_wish(a + c)
```

Note that nothing happened immediately but that the generator was indeed added:

```
sage: list(gbs)
[a + b]

sage: gbs.symmGB_F2()
sage: list(gbs)
[a + c, b + c]
```

```
>>> from sage.all import *
>>> list(gbs)
[a + b]

>>> gbs.symmGB_F2()
>>> list(gbs)
[a + c, b + c]
```

add_generator(*p*)

Add a new generator.

INPUT:

- `p` – a polynomial

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import *
sage: B.<a,b,c,d,e,f> = BooleanPolynomialRing()
sage: gbs = GroebnerStrategy(B)
sage: gbs.add_generator(a + b)
sage: list(gbs)
[a + b]
sage: gbs.add_generator(a + c)
Traceback (most recent call last):
...
ValueError: strategy already contains a polynomial with same lead
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd', 'e', 'f',)); (a, b,␣
↪c, d, e, f,) = B._first_ngens(6)
>>> gbs = GroebnerStrategy(B)
>>> gbs.add_generator(a + b)
>>> list(gbs)
```

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```
[a + b]
>>> gbs.add_generator(a + c)
Traceback (most recent call last):
...
ValueError: strategy already contains a polynomial with same lead
```

add_generator_delayed(*p*)

Add a new generator but do not perform interreduction immediately.

INPUT:

- p – a polynomial

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import *
sage: B.<a,b,c,d,e,f> = BooleanPolynomialRing()
sage: gbs = GroebnerStrategy(B)
sage: gbs.add_generator(a + b)
sage: list(gbs)
[a + b]
sage: gbs.add_generator_delayed(a + c)
sage: list(gbs)
[a + b]

sage: list(gbs.all_generators())
[a + b, a + c]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd', 'e', 'f',)); (a, b,␣
↪c, d, e, f,) = B._first_ngens(6)
>>> gbs = GroebnerStrategy(B)
>>> gbs.add_generator(a + b)
>>> list(gbs)
[a + b]
>>> gbs.add_generator_delayed(a + c)
>>> list(gbs)
[a + b]

>>> list(gbs.all_generators())
[a + b, a + c]
```

all_generators()

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import *
sage: B.<a,b,c,d,e,f> = BooleanPolynomialRing()
sage: gbs = GroebnerStrategy(B)
sage: gbs.add_as_you_wish(a + b)
sage: list(gbs)
[a + b]
sage: gbs.add_as_you_wish(a + c)
```

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```
sage: list(gbs)
[a + b]

sage: list(gbs.all_generators())
[a + b, a + c]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd', 'e', 'f',)); (a, b,␣
↪c, d, e, f,) = B._first_ngens(6)
>>> gbs = GroebnerStrategy(B)
>>> gbs.add_as_you_wish(a + b)
>>> list(gbs)
[a + b]
>>> gbs.add_as_you_wish(a + c)

>>> list(gbs)
[a + b]

>>> list(gbs.all_generators())
[a + b, a + c]
```

all_spolys_in_next_degree()

clean_top_by_chain_criterion()

contains_one()

Return `True` if 1 is in the generating system.

EXAMPLES:

We construct an example which contains `1` in the ideal spanned by the generators but not in the set of generators:

```
sage: B.<a,b,c,d,e,f> = BooleanPolynomialRing()
sage: from sage.rings.polynomial.pbori.pbori import GroebnerStrategy
sage: gb = GroebnerStrategy(B)
sage: gb.add_generator(a*c + a*f + d*f + d + f)
sage: gb.add_generator(b*c + b*e + c + d + 1)
sage: gb.add_generator(a*f + a + c + d + 1)
sage: gb.add_generator(a*d + a*e + b*e + c + f)
sage: gb.add_generator(b*d + c + d*f + e + f)
sage: gb.add_generator(a*b + b + c*e + e + 1)
sage: gb.add_generator(a + b + c*d + c*e + 1)
sage: gb.contains_one()
False
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd', 'e', 'f',)); (a, b,␣
↪c, d, e, f,) = B._first_ngens(6)
>>> from sage.rings.polynomial.pbori.pbori import GroebnerStrategy
>>> gb = GroebnerStrategy(B)
```

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```
>>> gb.add_generator(a*c + a*f + d*f + d + f)
>>> gb.add_generator(b*c + b*e + c + d + Integer(1))
>>> gb.add_generator(a*f + a + c + d + Integer(1))
>>> gb.add_generator(a*d + a*e + b*e + c + f)
>>> gb.add_generator(b*d + c + d*f + e + f)
>>> gb.add_generator(a*b + b + c*e + e + Integer(1))
>>> gb.add_generator(a + b + c*d + c*e + Integer(1))
>>> gb.contains_one()
False
```

Still, we have that:

```
sage: from sage.rings.polynomial.pbori import groebner_basis
sage: groebner_basis(gb)
[1]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori import groebner_basis
>>> groebner_basis(gb)
[1]
```

faugere_step_dense(*v*)

Reduces a vector of polynomials using linear algebra.

INPUT:

- v – boolean polynomial vector

EXAMPLES:

```
sage: B.<a,b,c,d,e,f> = BooleanPolynomialRing()
sage: from sage.rings.polynomial.pbori.pbori import GroebnerStrategy
sage: gb = GroebnerStrategy(B)
sage: gb.add_generator(a*c + a*f + d*f + d + f)
sage: gb.add_generator(b*c + b*e + c + d + 1)
sage: gb.add_generator(a*f + a + c + d + 1)
sage: gb.add_generator(a*d + a*e + b*e + c + f)
sage: gb.add_generator(b*d + c + d*f + e + f)
sage: gb.add_generator(a*b + b + c*e + e + 1)
sage: gb.add_generator(a + b + c*d + c*e + 1)

sage: from sage.rings.polynomial.pbori.pbori import BooleanPolynomialVector
sage: V= BooleanPolynomialVector([b*d, a*b])
sage: list(gb.faugere_step_dense(V))
[b + c*e + e + 1, c + d*f + e + f]
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd', 'e', 'f',)); (a, b,␣
↪c, d, e, f,) = B._first_ngens(6)
>>> from sage.rings.polynomial.pbori.pbori import GroebnerStrategy
>>> gb = GroebnerStrategy(B)
>>> gb.add_generator(a*c + a*f + d*f + d + f)
>>> gb.add_generator(b*c + b*e + c + d + Integer(1))
```

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```
>>> gb.add_generator(a*f + a + c + d + Integer(1))
>>> gb.add_generator(a*d + a*e + b*e + c + f)
>>> gb.add_generator(b*d + c + d*f + e + f)
>>> gb.add_generator(a*b + b + c*e + e + Integer(1))
>>> gb.add_generator(a + b + c*d + c*e + Integer(1))

>>> from sage.rings.polynomial.pbori.pbori import BooleanPolynomialVector
>>> V= BooleanPolynomialVector([b*d, a*b])
>>> list(gb.faugere_step_dense(V))
[b + c*e + e + 1, c + d*f + e + f]
```

implications(*i*)

Compute "useful" implied polynomials of `i`-th generator, and add them to the strategy, if it finds any.

INPUT:

- `i` – an index

ll_reduce_all()

Use the built-in ll-encoded *BooleSet* of polynomials with linear lexicographical leading term, which coincides with leading term in current ordering, to reduce the tails of all polynomials in the strategy.

minimalize()

Return a vector of all polynomials with minimal leading terms.

> **Note**
>
> Use this function if strat contains a GB.

minimalize_and_tail_reduce()

Return a vector of all polynomials with minimal leading terms and do tail reductions.

> **Note**
>
> Use that if strat contains a GB and you want a reduced GB.

next_spoly()

nf(*p*)

Compute the normal form of `p` with respect to the generating set.

INPUT:

- `p` – boolean polynomial

EXAMPLES:

```
sage: P = PolynomialRing(GF(2),10, 'x')
sage: B = BooleanPolynomialRing(10,'x')
sage: I = sage.rings.ideal.Cyclic(P)
sage: I = B.ideal([B(f) for f in I.gens()])
sage: gb = I.groebner_basis()
```

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```
sage: from sage.rings.polynomial.pbori.pbori import GroebnerStrategy

sage: G = GroebnerStrategy(B)
sage: _ = [G.add_generator(f) for f in gb]
sage: G.nf(gb[0])
0
sage: G.nf(gb[0] + 1)
1
sage: G.nf(gb[0]*gb[1])
0
sage: G.nf(gb[0]*B.gen(1))
0
```

```
>>> from sage.all import *
>>> P = PolynomialRing(GF(Integer(2)),Integer(10), 'x')
>>> B = BooleanPolynomialRing(Integer(10),'x')
>>> I = sage.rings.ideal.Cyclic(P)
>>> I = B.ideal([B(f) for f in I.gens()])
>>> gb = I.groebner_basis()

>>> from sage.rings.polynomial.pbori.pbori import GroebnerStrategy

>>> G = GroebnerStrategy(B)
>>> _ = [G.add_generator(f) for f in gb]
>>> G.nf(gb[Integer(0)])
0
>>> G.nf(gb[Integer(0)] + Integer(1))
1
>>> G.nf(gb[Integer(0)]*gb[Integer(1)])
0
>>> G.nf(gb[Integer(0)]*B.gen(Integer(1)))
0
```

Note

The result is only canonical if the generating set is a Groebner basis.

npairs()

reduction_strategy

select(*m*)

Return the index of the generator which can reduce the monomial `m`.

INPUT:

- `m` – a *BooleanMonomial*

EXAMPLES:

```
sage: B.<a,b,c,d,e> = BooleanPolynomialRing()
sage: f = B.random_element()
```

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```
sage: g = B.random_element()
sage: while g.lt() == f.lt():
....:     g = B.random_element()
sage: from sage.rings.polynomial.pbori.pbori import GroebnerStrategy
sage: strat = GroebnerStrategy(B)
sage: strat.add_generator(f)
sage: strat.add_generator(g)
sage: strat.select(f.lm())
0
sage: strat.select(g.lm())
1
sage: strat.select(e.lm())
-1
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd', 'e',)); (a, b, c, d,␣
→e,) = B._first_ngens(5)
>>> f = B.random_element()
>>> g = B.random_element()
>>> while g.lt() == f.lt():
...     g = B.random_element()
>>> from sage.rings.polynomial.pbori.pbori import GroebnerStrategy
>>> strat = GroebnerStrategy(B)
>>> strat.add_generator(f)
>>> strat.add_generator(g)
>>> strat.select(f.lm())
0
>>> strat.select(g.lm())
1
>>> strat.select(e.lm())
-1
```

small_spolys_in_next_degree(f, n)

some_spolys_in_next_degree(n)

suggest_plugin_variable()

symmGB_F2()

Compute a Groebner basis for the generating system.

> **Note**
>
> This implementation is out of date, but it will revived at some point in time. Use the `groebner_basis()` function instead.

top_sugar()

variable_has_value(v)

Compute whether there exists some polynomial of the form $v + c$ in the Strategy – where `c` is a constant – in the list of generators.

INPUT:

- v – the index of a variable

EXAMPLES:

```
sage: B.<a,b,c,d,e,f> = BooleanPolynomialRing()
sage: from sage.rings.polynomial.pbori.pbori import GroebnerStrategy
sage: gb = GroebnerStrategy(B)
sage: gb.add_generator(a*c + a*f + d*f + d + f)
sage: gb.add_generator(b*c + b*e + c + d + 1)
sage: gb.add_generator(a*f + a + c + d + 1)
sage: gb.add_generator(a*d + a*e + b*e + c + f)
sage: gb.add_generator(b*d + c + d*f + e + f)
sage: gb.add_generator(a*b + b + c*e + e + 1)
sage: gb.variable_has_value(0)
False

sage: from sage.rings.polynomial.pbori import groebner_basis
sage: g = groebner_basis(gb)
sage: list(g)
[a, b + 1, c + 1, d, e + 1, f]

sage: gb = GroebnerStrategy(B)
sage: _ = [gb.add_generator(f) for f in g]
sage: gb.variable_has_value(0)
True
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd', 'e', 'f',)); (a, b,
↪c, d, e, f,) = B._first_ngens(6)
>>> from sage.rings.polynomial.pbori.pbori import GroebnerStrategy
>>> gb = GroebnerStrategy(B)
>>> gb.add_generator(a*c + a*f + d*f + d + f)
>>> gb.add_generator(b*c + b*e + c + d + Integer(1))
>>> gb.add_generator(a*f + a + c + d + Integer(1))
>>> gb.add_generator(a*d + a*e + b*e + c + f)
>>> gb.add_generator(b*d + c + d*f + e + f)
>>> gb.add_generator(a*b + b + c*e + e + Integer(1))
>>> gb.variable_has_value(Integer(0))
False

>>> from sage.rings.polynomial.pbori import groebner_basis
>>> g = groebner_basis(gb)
>>> list(g)
[a, b + 1, c + 1, d, e + 1, f]

>>> gb = GroebnerStrategy(B)
>>> _ = [gb.add_generator(f) for f in g]
>>> gb.variable_has_value(Integer(0))
True
```

class sage.rings.polynomial.pbori.pbori.**MonomialConstruct**

Bases: object

Implement PolyBoRi's Monomial() constructor.

class sage.rings.polynomial.pbori.pbori.**MonomialFactory**

Bases: object

Implement PolyBoRi's Monomial() constructor. If a ring is given is can be used as a Monomial factory for the given ring.

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import *
sage: B.<a,b,c> = BooleanPolynomialRing()
sage: fac = MonomialFactory()
sage: fac = MonomialFactory(B)
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c',)); (a, b, c,) = B._
↪first_ngens(3)
>>> fac = MonomialFactory()
>>> fac = MonomialFactory(B)
```

class sage.rings.polynomial.pbori.pbori.**PolynomialConstruct**

Bases: object

Implement PolyBoRi's Polynomial() constructor.

lead(*x*)

Return the leading monomial of boolean polynomial x, with respect to the order of parent ring.

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import *
sage: B.<a,b,c> = BooleanPolynomialRing()
sage: PolynomialConstruct().lead(a)
a
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c',)); (a, b, c,) = B._first_
↪ngens(3)
>>> PolynomialConstruct().lead(a)
a
```

class sage.rings.polynomial.pbori.pbori.**PolynomialFactory**

Bases: object

Implement PolyBoRi's Polynomial() constructor and a polynomial factory for given rings.

lead(*x*)

Return the leading monomial of boolean polynomial x, with respect to the order of parent ring.

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import *
sage: B.<a,b,c> = BooleanPolynomialRing()
sage: PolynomialFactory().lead(a)
a
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c',)); (a, b, c,) = B._first_
↪ngens(3)
>>> PolynomialFactory().lead(a)
a
```

class sage.rings.polynomial.pbori.pbori.**ReductionStrategy**

Bases: `object`

Functions and options for boolean polynomial reduction.

add_generator(*p*)

Add the new generator `p` to this strategy.

INPUT:

- `p` – boolean polynomial

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import *
sage: B.<x,y,z> = BooleanPolynomialRing()
sage: red = ReductionStrategy(B)
sage: red.add_generator(x)
sage: [f.p for f in red]
[x]
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import *
>>> B = BooleanPolynomialRing(names=('x', 'y', 'z',)); (x, y, z,) = B._first_
↪ngens(3)
>>> red = ReductionStrategy(B)
>>> red.add_generator(x)
>>> [f.p for f in red]
[x]
```

can_rewrite(*p*)

Return `True` if `p` can be reduced by the generators of this strategy.

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import *
sage: B.<a,b,c,d> = BooleanPolynomialRing()
sage: red = ReductionStrategy(B)
sage: red.add_generator(a*b + c + 1)
sage: red.add_generator(b*c + d + 1)
sage: red.can_rewrite(a*b + a)
True
sage: red.can_rewrite(b + c)
False
sage: red.can_rewrite(a*d + b*c + d + 1)
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd',)); (a, b, c, d,) = B.
↪_first_ngens(4)
>>> red = ReductionStrategy(B)
>>> red.add_generator(a*b + c + Integer(1))
>>> red.add_generator(b*c + d + Integer(1))
>>> red.can_rewrite(a*b + a)
True
>>> red.can_rewrite(b + c)
False
>>> red.can_rewrite(a*d + b*c + d + Integer(1))
True
```

cheap_reductions(*p*)

Perform 'cheap' reductions on `p`.

INPUT:

- `p` – boolean polynomial

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import *
sage: B.<a,b,c,d> = BooleanPolynomialRing()
sage: red = ReductionStrategy(B)
sage: red.add_generator(a*b + c + 1)
sage: red.add_generator(b*c + d + 1)
sage: red.add_generator(a)
sage: red.cheap_reductions(a*b + a)
0
sage: red.cheap_reductions(b + c)
b + c
sage: red.cheap_reductions(a*d + b*c + d + 1)
b*c + d + 1
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd',)); (a, b, c, d,) = B.
↪_first_ngens(4)
>>> red = ReductionStrategy(B)
>>> red.add_generator(a*b + c + Integer(1))
>>> red.add_generator(b*c + d + Integer(1))
>>> red.add_generator(a)
>>> red.cheap_reductions(a*b + a)
0
>>> red.cheap_reductions(b + c)
b + c
>>> red.cheap_reductions(a*d + b*c + d + Integer(1))
b*c + d + 1
```

head_normal_form(*p*)

Compute the normal form of `p` with respect to the generators of this strategy but do not perform tail any reductions.

INPUT:

- `p` – a polynomial

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import *
sage: B.<x,y,z> = BooleanPolynomialRing()
sage: red = ReductionStrategy(B)
sage: red.opt_red_tail = True
sage: red.add_generator(x + y + 1)
sage: red.add_generator(y*z + z)

sage: red.head_normal_form(x + y*z)
y + z + 1

sage: red.nf(x + y*z)
y + z + 1
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import *
>>> B = BooleanPolynomialRing(names=('x', 'y', 'z',)); (x, y, z,) = B._first_
↪ngens(3)
>>> red = ReductionStrategy(B)
>>> red.opt_red_tail = True
>>> red.add_generator(x + y + Integer(1))
>>> red.add_generator(y*z + z)

>>> red.head_normal_form(x + y*z)
y + z + 1

>>> red.nf(x + y*z)
y + z + 1
```

nf (*p*)

Compute the normal form of `p` w.r.t. to the generators of this reduction strategy object.

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import *
sage: B.<x,y,z> = BooleanPolynomialRing()
sage: red = ReductionStrategy(B)
sage: red.add_generator(x + y + 1)
sage: red.add_generator(y*z + z)
sage: red.nf(x)
y + 1

sage: red.nf(y*z + x)
y + z + 1
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import *
>>> B = BooleanPolynomialRing(names=('x', 'y', 'z',)); (x, y, z,) = B._first_
↪ngens(3)
```

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```
>>> red = ReductionStrategy(B)
>>> red.add_generator(x + y + Integer(1))
>>> red.add_generator(y*z + z)
>>> red.nf(x)
y + 1

>>> red.nf(y*z + x)
y + z + 1
```

reduced_normal_form(*p*)

Compute the normal form of `p` with respect to the generators of this strategy and perform tail reductions.

INPUT:

- `p` – a polynomial

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import *
sage: B.<x,y,z> = BooleanPolynomialRing()
sage: red = ReductionStrategy(B)
sage: red.add_generator(x + y + 1)
sage: red.add_generator(y*z + z)
sage: red.reduced_normal_form(x)
y + 1

sage: red.reduced_normal_form(y*z + x)
y + z + 1
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import *
>>> B = BooleanPolynomialRing(names=('x', 'y', 'z',)); (x, y, z,) = B._first_
↪ngens(3)
>>> red = ReductionStrategy(B)
>>> red.add_generator(x + y + Integer(1))
>>> red.add_generator(y*z + z)
>>> red.reduced_normal_form(x)
y + 1

>>> red.reduced_normal_form(y*z + x)
y + z + 1
```

sage.rings.polynomial.pbori.pbori.**TermOrder_from_pb_order**(*n*, *order*, *blocks*)

class sage.rings.polynomial.pbori.pbori.**VariableBlock**

Bases: `object`

class sage.rings.polynomial.pbori.pbori.**VariableConstruct**

Bases: `object`

Implement PolyBoRi's `Variable()` constructor.

class sage.rings.polynomial.pbori.pbori.**VariableFactory**

Bases: `object`

Implements PolyBoRi's `Variable()` constructor and a variable factory for given ring sage.rings.polynomial.pbori.pbori.**add_up_polynomials**(*v*, *init*)

Add up all entries in the vector v.

INPUT:

- v – a vector of boolean polynomials

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import *
sage: B.<a,b,c,d> = BooleanPolynomialRing()
sage: v = BooleanPolynomialVector()
sage: l = [B.random_element() for _ in range(5)]
sage: _ = [v.append(e) for e in l]
sage: add_up_polynomials(v, B.zero()) == sum(l)
True
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd',)); (a, b, c, d,) = B._
↪first_ngens(4)
>>> v = BooleanPolynomialVector()
>>> l = [B.random_element() for _ in range(Integer(5))]
>>> _ = [v.append(e) for e in l]
>>> add_up_polynomials(v, B.zero()) == sum(l)
True
```

sage.rings.polynomial.pbori.pbori.**contained_vars**(*m*)

sage.rings.polynomial.pbori.pbori.**easy_linear_factors**(*p*)

sage.rings.polynomial.pbori.pbori.**gauss_on_polys**(*inp*)

Perform Gaussian elimination on the input list of polynomials.

INPUT:

- inp – an iterable

EXAMPLES:

```
sage: B.<a,b,c,d,e,f> = BooleanPolynomialRing()
sage: from sage.rings.polynomial.pbori.pbori import *
sage: l = [B.random_element() for _ in range(B.ngens())]
sage: A, _ = Sequence(l, B).coefficients_monomials()
sage: while A.rank() < 6:
....:     l = [B.random_element() for _ in range(B.ngens())]
....:     A, _ = Sequence(l, B).coefficients_monomials()

sage: e = gauss_on_polys(l)
sage: E, _ = Sequence(e, B).coefficients_monomials()
sage: E == A.echelon_form()
True
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd', 'e', 'f',)); (a, b, c, d,
↪ e, f,) = B._first_ngens(6)
```

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```
>>> from sage.rings.polynomial.pbori.pbori import *
>>> l = [B.random_element() for _ in range(B.ngens())]
>>> A, _ = Sequence(l, B).coefficients_monomials()
>>> while A.rank() < Integer(6):
...     l = [B.random_element() for _ in range(B.ngens())]
...     A, _ = Sequence(l, B).coefficients_monomials()

>>> e = gauss_on_polys(l)
>>> E, _ = Sequence(e, B).coefficients_monomials()
>>> E == A.echelon_form()
True
```

sage.rings.polynomial.pbori.pbori.**get_var_mapping**(*ring*, *other*)

Return a variable mapping between variables of `other` and `ring`. When other is a parent object, the mapping defines images for all variables of other. If it is an element, only variables occurring in other are mapped.

Raises `NameError` if no such mapping is possible.

EXAMPLES:

```
sage: P.<x,y,z> = BooleanPolynomialRing(3)
sage: R.<z,y> = QQ[]
sage: sage.rings.polynomial.pbori.pbori.get_var_mapping(P,R)
[z, y]
sage: sage.rings.polynomial.pbori.pbori.get_var_mapping(P, z^2)
[z, None]
```

```
>>> from sage.all import *
>>> P = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,) = P.
↪_first_ngens(3)
>>> R = QQ['z, y']; (z, y,) = R._first_ngens(2)
>>> sage.rings.polynomial.pbori.pbori.get_var_mapping(P,R)
[z, y]
>>> sage.rings.polynomial.pbori.pbori.get_var_mapping(P, z**Integer(2))
[z, None]
```

```
sage: R.<z,x> = BooleanPolynomialRing(2)
sage: sage.rings.polynomial.pbori.pbori.get_var_mapping(P,R)
[z, x]
sage: sage.rings.polynomial.pbori.pbori.get_var_mapping(P, x^2)
[None, x]
```

```
>>> from sage.all import *
>>> R = BooleanPolynomialRing(Integer(2), names=('z', 'x',)); (z, x,) = R._first_
↪ngens(2)
>>> sage.rings.polynomial.pbori.pbori.get_var_mapping(P,R)
[z, x]
>>> sage.rings.polynomial.pbori.pbori.get_var_mapping(P, x**Integer(2))
[None, x]
```

sage.rings.polynomial.pbori.pbori.**if_then_else**(*root*, *a*, *b*)

The opposite of navigating down a ZDD using navigators is to construct new ZDDs in the same way, namely giving their else- and then-branch as well as the index value of the new node.

INPUT:

- `root` – a variable
- `a` – the if branch, a `BooleSet` or a `BoolePolynomial`
- `b` – the else branch, a `BooleSet` or a `BoolePolynomial`

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import if_then_else
sage: B = BooleanPolynomialRing(6,'x')
sage: x0,x1,x2,x3,x4,x5 = B.gens()
sage: f0 = x2*x3+x3
sage: f1 = x4
sage: if_then_else(x1, f0, f1)
{{x1,x2,x3}, {x1,x3}, {x4}}
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import if_then_else
>>> B = BooleanPolynomialRing(Integer(6),'x')
>>> x0,x1,x2,x3,x4,x5 = B.gens()
>>> f0 = x2*x3+x3
>>> f1 = x4
>>> if_then_else(x1, f0, f1)
{{x1,x2,x3}, {x1,x3}, {x4}}
```

```
sage: if_then_else(x1.lm().index(),f0,f1)
{{x1,x2,x3}, {x1,x3}, {x4}}
```

```
>>> from sage.all import *
>>> if_then_else(x1.lm().index(),f0,f1)
{{x1,x2,x3}, {x1,x3}, {x4}}
```

```
sage: if_then_else(x5, f0, f1)
Traceback (most recent call last):
...
IndexError: index of root must be less than the values of roots of the branches
```

```
>>> from sage.all import *
>>> if_then_else(x5, f0, f1)
Traceback (most recent call last):
...
IndexError: index of root must be less than the values of roots of the branches
```

`sage.rings.polynomial.pbori.pbori.`**`interpolate`**`(`*zero*, *one*`)`

Interpolate a polynomial evaluating to zero on `zero` and to one on `ones`.

INPUT:

- `zero` – the set of zero
- `one` – the set of ones

EXAMPLES:

```
sage: B = BooleanPolynomialRing(4,"x0,x1,x2,x3")
sage: x = B.gen
sage: from sage.rings.polynomial.pbori.interpolate import *
sage: V=(x(0)+x(1)+x(2)+x(3)+1).set()

sage: V
{{x0}, {x1}, {x2}, {x3}, {}}

sage: f=x(0)*x(1)+x(1)+x(2)+1
sage: nf_lex_points(f, V)
x1 + x2 + 1

sage: z=f.zeros_in(V)
sage: z
{{x1}, {x2}}

sage: o=V.diff(z)
sage: o
{{x0}, {x3}, {}}

sage: interpolate(z,o)
x0*x1*x2 + x0*x1 + x0*x2 + x1*x2 + x1 + x2 + 1
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(4),"x0,x1,x2,x3")
>>> x = B.gen
>>> from sage.rings.polynomial.pbori.interpolate import *
>>> V=(x(Integer(0))+x(Integer(1))+x(Integer(2))+x(Integer(3))+Integer(1)).set()

>>> V
{{x0}, {x1}, {x2}, {x3}, {}}

>>> f=x(Integer(0))*x(Integer(1))+x(Integer(1))+x(Integer(2))+Integer(1)
>>> nf_lex_points(f, V)
x1 + x2 + 1

>>> z=f.zeros_in(V)
>>> z
{{x1}, {x2}}

>>> o=V.diff(z)
>>> o
{{x0}, {x3}, {}}

>>> interpolate(z,o)
x0*x1*x2 + x0*x1 + x0*x2 + x1*x2 + x1 + x2 + 1
```

sage.rings.polynomial.pbori.pbori.**interpolate_smallest_lex**(*zero*, *one*)

Interpolate the lexicographical smallest polynomial evaluating to zero on `zero` and to one on `ones`.

INPUT:

- `zero` – the set of zeros

- `one` – the set of ones

EXAMPLES:

Let V be a set of points in $\mathbf{F}_2^n$ and f a Boolean polynomial. V can be encoded as a `BooleSet`. Then we are interested in the normal form of f against the vanishing ideal of V : I(V).

It turns out, that the computation of the normal form can be done by the computation of a minimal interpolation polynomial, which takes the same values as f on V:

```
sage: B = BooleanPolynomialRing(4,"x0,x1,x2,x3")
sage: x = B.gen
sage: from sage.rings.polynomial.pbori.interpolate import *
sage: V=(x(0)+x(1)+x(2)+x(3)+1).set()
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(4),"x0,x1,x2,x3")
>>> x = B.gen
>>> from sage.rings.polynomial.pbori.interpolate import *
>>> V=(x(Integer(0))+x(Integer(1))+x(Integer(2))+x(Integer(3))+Integer(1)).set()
```

We take V = {e0,e1,e2,e3,0}, where ei describes the i-th unit vector. For our considerations it does not play any role, if we suppose V to be embedded in $\mathbf{F}_2^4$ or a vector space of higher dimension:

```
sage: V
{{x0}, {x1}, {x2}, {x3}, {}}

sage: f=x(0)*x(1)+x(1)+x(2)+1
sage: nf_lex_points(f, V)
x1 + x2 + 1
```

```
>>> from sage.all import *
>>> V
{{x0}, {x1}, {x2}, {x3}, {}}

>>> f=x(Integer(0))*x(Integer(1))+x(Integer(1))+x(Integer(2))+Integer(1)
>>> nf_lex_points(f, V)
x1 + x2 + 1
```

In this case, the normal form of f w.r.t. the vanishing ideal of V consists of all terms of f with degree smaller or equal to 1.

It can be easily seen, that this polynomial forms the same function on V as f. In fact, our computation is equivalent to the direct call of the interpolation function `interpolate_smallest_lex`, which has two arguments: the set of interpolation points mapped to zero and the set of interpolation points mapped to one:

```
sage: z=f.zeros_in(V)
sage: z
{{x1}, {x2}}

sage: o=V.diff(z)
sage: o
{{x0}, {x3}, {}}

sage: interpolate_smallest_lex(z,o)
x1 + x2 + 1
```

```
>>> from sage.all import *
>>> z=f.zeros_in(V)
>>> z
{{x1}, {x2}}

>>> o=V.diff(z)
>>> o
{{x0}, {x3}, {}}

>>> interpolate_smallest_lex(z,o)
x1 + x2 + 1
```

sage.rings.polynomial.pbori.pbori.**ll_red_nf_noredsb**(*p*, *reductors*)

Redude the polynomial `p` by the set of `reductors` with linear leading terms.

INPUT:

- `p` – boolean polynomial
- `reductors` – boolean set encoding a Groebner basis with linear leading terms

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import ll_red_nf_noredsb
sage: B.<a,b,c,d> = BooleanPolynomialRing()
sage: p = a*b + c + d + 1
sage: f,g  = a + c + 1, b + d + 1
sage: reductors = f.set().union( g.set() )
sage: ll_red_nf_noredsb(p, reductors)
b*c + b*d + c + d + 1
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import ll_red_nf_noredsb
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd',)); (a, b, c, d,) = B._
↪first_ngens(4)
>>> p = a*b + c + d + Integer(1)
>>> f,g  = a + c + Integer(1), b + d + Integer(1)
>>> reductors = f.set().union( g.set() )
>>> ll_red_nf_noredsb(p, reductors)
b*c + b*d + c + d + 1
```

sage.rings.polynomial.pbori.pbori.**ll_red_nf_noredsb_single_recursive_call**(*p*, *reductors*)

Redude the polynomial `p` by the set of `reductors` with linear leading terms.

`ll_red_nf_noredsb_single_recursive()` call has the same specification as `ll_red_nf_noredsb()`, but a different implementation: It is very sensitive to the ordering of variables, however it has the property, that it needs just one recursive call.

INPUT:

- `p` – boolean polynomial
- `reductors` – boolean set encoding a Groebner basis with linear leading terms

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import ll_red_nf_noredsb_single_
↪recursive_call
sage: B.<a,b,c,d> = BooleanPolynomialRing()
sage: p = a*b + c + d + 1
sage: f,g  = a + c + 1, b + d + 1
sage: reductors = f.set().union( g.set() )
sage: ll_red_nf_noredsb_single_recursive_call(p, reductors)
b*c + b*d + c + d + 1
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import ll_red_nf_noredsb_single_
↪recursive_call
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd',)); (a, b, c, d,) = B._
↪first_ngens(4)
>>> p = a*b + c + d + Integer(1)
>>> f,g  = a + c + Integer(1), b + d + Integer(1)
>>> reductors = f.set().union( g.set() )
>>> ll_red_nf_noredsb_single_recursive_call(p, reductors)
b*c + b*d + c + d + 1
```

`sage.rings.polynomial.pbori.pbori.`**`ll_red_nf_redsb`**(*p*, *reductors*)

Redude the polynomial `p` by the set of `reductors` with linear leading terms. It is assumed that the set `reductors` is a reduced Groebner basis.

INPUT:

- `p` – boolean polynomial
- `reductors` – boolean set encoding a reduced Groebner basis with linear leading terms

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import ll_red_nf_redsb
sage: B.<a,b,c,d> = BooleanPolynomialRing()
sage: p = a*b + c + d + 1
sage: f,g  = a + c + 1, b + d + 1
sage: reductors = f.set().union( g.set() )
sage: ll_red_nf_redsb(p, reductors)
b*c + b*d + c + d + 1
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import ll_red_nf_redsb
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd',)); (a, b, c, d,) = B._
↪first_ngens(4)
>>> p = a*b + c + d + Integer(1)
>>> f,g  = a + c + Integer(1), b + d + Integer(1)
>>> reductors = f.set().union( g.set() )
>>> ll_red_nf_redsb(p, reductors)
b*c + b*d + c + d + 1
```

`sage.rings.polynomial.pbori.pbori.`**`map_every_x_to_x_plus_one`**(*p*)

Map every variable `x_i` in this polynomial to `x_i + 1`.

EXAMPLES:

```
sage: B.<a,b,z> = BooleanPolynomialRing(3)
sage: f = a*b + z + 1; f
a*b + z + 1
sage: from sage.rings.polynomial.pbori.pbori import map_every_x_to_x_plus_one
sage: map_every_x_to_x_plus_one(f)
a*b + a + b + z + 1
sage: f(a+1,b+1,z+1)
a*b + a + b + z + 1
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(3), names=('a', 'b', 'z',)); (a, b, z,) = B.
↪_first_ngens(3)
>>> f = a*b + z + Integer(1); f
a*b + z + 1
>>> from sage.rings.polynomial.pbori.pbori import map_every_x_to_x_plus_one
>>> map_every_x_to_x_plus_one(f)
a*b + a + b + z + 1
>>> f(a+Integer(1),b+Integer(1),z+Integer(1))
a*b + a + b + z + 1
```

sage.rings.polynomial.pbori.pbori.**mod_mon_set**(*a_s*, *v_s*)

sage.rings.polynomial.pbori.pbori.**mod_var_set**(*a*, *v*)

sage.rings.polynomial.pbori.pbori.**mult_fact_sim_C**(*v*, *ring*)

sage.rings.polynomial.pbori.pbori.**nf3**(*s*, *p*, *m*)

sage.rings.polynomial.pbori.pbori.**parallel_reduce**(*inp*, *strat*, *average_steps*, *delay_f*)

sage.rings.polynomial.pbori.pbori.**random_set**(*variables*, *length*)

Return a random set of monomials with `length` elements with each element in the variables `variables`.

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import random_set, set_random_seed
sage: B.<a,b,c,d,e> = BooleanPolynomialRing()
sage: (a*b*c*d).lm()
a*b*c*d
sage: set_random_seed(1337)
sage: random_set((a*b*c*d).lm(),10)
{{a,b,c,d}, {a,b}, {a,c,d}, {a,c}, {b,c,d}, {b,d}, {b}, {c,d}, {c}, {d}}
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import random_set, set_random_seed
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd', 'e',)); (a, b, c, d, e,)␣
↪= B._first_ngens(5)
>>> (a*b*c*d).lm()
a*b*c*d
>>> set_random_seed(Integer(1337))
>>> random_set((a*b*c*d).lm(),Integer(10))
{{a,b,c,d}, {a,b}, {a,c,d}, {a,c}, {b,c,d}, {b,d}, {b}, {c,d}, {c}, {d}}
```

sage.rings.polynomial.pbori.pbori.**recursively_insert**(*n*, *ind*, *m*) sage.rings.polynomial.pbori.pbori.**red_tail**(*s*, *p*)

Perform tail reduction on p using the generators of s.

INPUT:

- s – a reduction strategy
- p – a polynomial

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import *
sage: B.<x,y,z> = BooleanPolynomialRing()
sage: red = ReductionStrategy(B)
sage: red.add_generator(x + y + 1)
sage: red.add_generator(y*z + z)
sage: red_tail(red,x)
x
sage: red_tail(red,x*y + x)
x*y + y + 1
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import *
>>> B = BooleanPolynomialRing(names=('x', 'y', 'z',)); (x, y, z,) = B._first_
↪ngens(3)
>>> red = ReductionStrategy(B)
>>> red.add_generator(x + y + Integer(1))
>>> red.add_generator(y*z + z)
>>> red_tail(red,x)
x
>>> red_tail(red,x*y + x)
x*y + y + 1
```

sage.rings.polynomial.pbori.pbori.**set_random_seed**(*seed*)

Set the PolyBoRi random seed to seed.

EXAMPLES:

```
sage: from sage.rings.polynomial.pbori.pbori import random_set, set_random_seed
sage: B.<a,b,c,d,e> = BooleanPolynomialRing()
sage: (a*b*c*d).lm()
a*b*c*d
sage: set_random_seed(1337)
sage: random_set((a*b*c*d).lm(),2)
{{b}, {c}}
sage: random_set((a*b*c*d).lm(),2)
{{a,c,d}, {c}}

sage: set_random_seed(1337)
sage: random_set((a*b*c*d).lm(),2)
{{b}, {c}}
sage: random_set((a*b*c*d).lm(),2)
{{a,c,d}, {c}}
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import random_set, set_random_seed
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c', 'd', 'e',)); (a, b, c, d, e,)␣
↪= B._first_ngens(5)
>>> (a*b*c*d).lm()
a*b*c*d
>>> set_random_seed(Integer(1337))
>>> random_set((a*b*c*d).lm(),Integer(2))
{{b}, {c}}
>>> random_set((a*b*c*d).lm(),Integer(2))
{{a,c,d}, {c}}

>>> set_random_seed(Integer(1337))
>>> random_set((a*b*c*d).lm(),Integer(2))
{{b}, {c}}
>>> random_set((a*b*c*d).lm(),Integer(2))
{{a,c,d}, {c}}
```

sage.rings.polynomial.pbori.pbori.**substitute_variables**(*parent*, *vec*, *poly*)

`var(i)` is replaced by `vec[i]` in `poly`.

EXAMPLES:

```
sage: B.<a,b,c> = BooleanPolynomialRing()
sage: f = a*b + c + 1
sage: from sage.rings.polynomial.pbori.pbori import substitute_variables
sage: substitute_variables(B, [a,b,c],f)
a*b + c + 1
sage: substitute_variables(B, [a+1,b,c],f)
a*b + b + c + 1
sage: substitute_variables(B, [a+1,b+1,c],f)
a*b + a + b + c
sage: substitute_variables(B, [a+1,b+1,B(0)],f)
a*b + a + b
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c',)); (a, b, c,) = B._first_
↪ngens(3)
>>> f = a*b + c + Integer(1)
>>> from sage.rings.polynomial.pbori.pbori import substitute_variables
>>> substitute_variables(B, [a,b,c],f)
a*b + c + 1
>>> substitute_variables(B, [a+Integer(1),b,c],f)
a*b + b + c + 1
>>> substitute_variables(B, [a+Integer(1),b+Integer(1),c],f)
a*b + a + b + c
>>> substitute_variables(B, [a+Integer(1),b+Integer(1),B(Integer(0))],f)
a*b + a + b
```

Substitution is also allowed with different rings:

```
sage: B.<a,b,c> = BooleanPolynomialRing()
sage: f = a*b + c + 1
```

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```
sage: B.<w,x,y,z> = BooleanPolynomialRing(order='deglex')

sage: from sage.rings.polynomial.pbori.pbori import substitute_variables
sage: substitute_variables(B, [x,y,z], f) * w
w*x*y + w*z + w
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(names=('a', 'b', 'c',)); (a, b, c,) = B._first_
↪ngens(3)
>>> f = a*b + c + Integer(1)
>>> B = BooleanPolynomialRing(order='deglex', names=('w', 'x', 'y', 'z',)); (w, x,
↪ y, z,) = B._first_ngens(4)

>>> from sage.rings.polynomial.pbori.pbori import substitute_variables
>>> substitute_variables(B, [x,y,z], f) * w
w*x*y + w*z + w
```

`sage.rings.polynomial.pbori.pbori.`**`top_index`**(*s*)

Return the highest index in the parameter `s`.

INPUT:

- `s` – `BooleSet`, `BooleMonomial`, `BoolePolynomial`

EXAMPLES:

```
sage: B.<x,y,z> = BooleanPolynomialRing(3)
sage: from sage.rings.polynomial.pbori.pbori import top_index
sage: top_index(x.lm())
0
sage: top_index(y*z)
1
sage: top_index(x + 1)
0
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(3), names=('x', 'y', 'z',)); (x, y, z,) = B.
↪_first_ngens(3)
>>> from sage.rings.polynomial.pbori.pbori import top_index
>>> top_index(x.lm())
0
>>> top_index(y*z)
1
>>> top_index(x + Integer(1))
0
```

`sage.rings.polynomial.pbori.pbori.`**`unpickle_BooleanPolynomial`**(*ring*, *string*)

Unpickle boolean polynomials.

EXAMPLES:

```
sage: T = TermOrder('deglex',2)+TermOrder('deglex',2)
sage: P.<a,b,c,d> = BooleanPolynomialRing(4,order=T)
```

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```
sage: loads(dumps(a+b)) == a+b # indirect doctest
True
```

```
>>> from sage.all import *
>>> T = TermOrder('deglex',Integer(2))+TermOrder('deglex',Integer(2))
>>> P = BooleanPolynomialRing(Integer(4),order=T, names=('a', 'b', 'c', 'd',));␣
↪(a, b, c, d,) = P._first_ngens(4)
>>> loads(dumps(a+b)) == a+b # indirect doctest
True
```

sage.rings.polynomial.pbori.pbori.**unpickle_BooleanPolynomial0**(*ring*, *l*)

Unpickle boolean polynomials.

EXAMPLES:

```
sage: T = TermOrder('deglex',2)+TermOrder('deglex',2)
sage: P.<a,b,c,d> = BooleanPolynomialRing(4,order=T)
sage: loads(dumps(a+b)) == a+b # indirect doctest
True
```

```
>>> from sage.all import *
>>> T = TermOrder('deglex',Integer(2))+TermOrder('deglex',Integer(2))
>>> P = BooleanPolynomialRing(Integer(4),order=T, names=('a', 'b', 'c', 'd',));␣
↪(a, b, c, d,) = P._first_ngens(4)
>>> loads(dumps(a+b)) == a+b # indirect doctest
True
```

sage.rings.polynomial.pbori.pbori.**unpickle_BooleanPolynomialRing**(*n*, *names*, *order*)

Unpickle boolean polynomial rings.

EXAMPLES:

```
sage: T = TermOrder('deglex',2)+TermOrder('deglex',2)
sage: P.<a,b,c,d> = BooleanPolynomialRing(4,order=T)
sage: loads(dumps(P)) == P  # indirect doctest
True
```

```
>>> from sage.all import *
>>> T = TermOrder('deglex',Integer(2))+TermOrder('deglex',Integer(2))
>>> P = BooleanPolynomialRing(Integer(4),order=T, names=('a', 'b', 'c', 'd',));␣
↪(a, b, c, d,) = P._first_ngens(4)
>>> loads(dumps(P)) == P  # indirect doctest
True
```

sage.rings.polynomial.pbori.pbori.**zeros**(*pol*, *s*)

Return a `BooleSet` encoding on which points from `s` the polynomial `pol` evaluates to zero.

INPUT:

- `pol` – boolean polynomial
- `s` – set of points encoded as a `BooleSet`

EXAMPLES:

```
sage: B.<a,b,c,d> = BooleanPolynomialRing(4)
sage: f = a*b + a*c + d + b
```

```
>>> from sage.all import *
>>> B = BooleanPolynomialRing(Integer(4), names=('a', 'b', 'c', 'd',)); (a, b, c,␣
↪d,) = B._first_ngens(4)
>>> f = a*b + a*c + d + b
```

Now we create a set of points:

```
sage: s = a*b + a*b*c + c*d + b*c
sage: s = s.set(); s
{{a,b,c}, {a,b}, {b,c}, {c,d}}
```

```
>>> from sage.all import *
>>> s = a*b + a*b*c + c*d + b*c
>>> s = s.set(); s
{{a,b,c}, {a,b}, {b,c}, {c,d}}
```

This encodes the points (1,1,1,0), (1,1,0,0), (0,0,1,1) and (0,1,1,0). But of these only (1,1,0,0) evaluates to zero.:

```
sage: from sage.rings.polynomial.pbori.pbori import zeros
sage: zeros(f, s)
{{a,b}}
```

```
>>> from sage.all import *
>>> from sage.rings.polynomial.pbori.pbori import zeros
>>> zeros(f, s)
{{a,b}}
```

For comparison we work with tuples:

```
sage: f.zeros_in([(1,1,1,0), (1,1,0,0), (0,0,1,1), (0,1,1,0)])
((1, 1, 0, 0),)
```

```
>>> from sage.all import *
>>> f.zeros_in([(Integer(1),Integer(1),Integer(1),Integer(0)), (Integer(1),
↪Integer(1),Integer(0),Integer(0)), (Integer(0),Integer(0),Integer(1),
↪Integer(1)), (Integer(0),Integer(1),Integer(1),Integer(0))])
((1, 1, 0, 0),)
```

CHAPTER

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