Polynomials

Release 10.3

The Sage Development Team

Mar 20, 2024
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Polynomial Rings</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>Univariate Polynomials</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>Multivariate Polynomials</td>
<td>275</td>
</tr>
<tr>
<td>4</td>
<td>Rational Functions</td>
<td>541</td>
</tr>
<tr>
<td>5</td>
<td>Laurent Polynomials</td>
<td>561</td>
</tr>
<tr>
<td>6</td>
<td>Infinite Polynomial Rings</td>
<td>587</td>
</tr>
<tr>
<td>7</td>
<td>Boolean Polynomials</td>
<td>623</td>
</tr>
<tr>
<td>8</td>
<td>Indices and Tables</td>
<td>625</td>
</tr>
<tr>
<td></td>
<td>Python Module Index</td>
<td>627</td>
</tr>
<tr>
<td></td>
<td>Index</td>
<td>629</td>
</tr>
</tbody>
</table>


1.1 Constructors for polynomial rings

This module provides the function `PolynomialRing()`, which constructs rings of univariate and multivariate polynomials, and implements caching to prevent the same ring being created in memory multiple times (which is wasteful and breaks the general assumption in Sage that parents are unique).

There is also a function `BooleanPolynomialRing_constructor()`, used for constructing Boolean polynomial rings, which are not technically polynomial rings but rather quotients of them (see module `sage.rings.polynomial.pbori` for more details).

Construct a boolean polynomial ring with the following parameters:

**INPUT:**

- `n` – number of variables (an integer > 1)
- `names` – names of ring variables, may be a string or list/tuple of strings
- `order` – term order (default: 'lex')

**EXAMPLES:**

```
sage: # needs sage.rings.polynomial.pbori
sage: R.<x, y, z> = BooleanPolynomialRing(); R  # indirect doctest
Boolean PolynomialRing in x, y, z
sage: p = x*y + x*z + y*z
sage: x*p
x*y*z + x*y + x*z
sage: R.term_order()  # indirect doctest
Lexicographic term order
```

```
sage: R = BooleanPolynomialRing(5, 'x', order='deglex(3),deglex(2)')  # indirect doctest
sage: R.term_order()  # indirect doctest
Block term order with blocks:
(Degree lexicographic term order of length 3,
 Degree lexicographic term order of length 2)
```

(continues on next page)
sage.rings.polynomial.polynomial_ring_constructor.PolynomialRing(base_ring, *args, **kwds)

Return the globally unique univariate or multivariate polynomial ring with given properties and variable name or names.

There are many ways to specify the variables for the polynomial ring:

1. PolynomialRing(base_ring, name, ...)
2. PolynomialRing(base_ring, names, ...)
3. PolynomialRing(base_ring, n, names, ...)
4. PolynomialRing(base_ring, n, ..., var_array=var_array, ...)

The ... at the end of these commands stands for additional keywords, like sparse or order.

INPUT:

- base_ring – a ring
- n – an integer
- name – a string
- names – a list or tuple of names (strings), or a comma separated string
- var_array – a list or tuple of names, or a comma separated string
- sparse – bool: whether or not elements are sparse. The default is a dense representation (sparse=False) for univariate rings and a sparse representation (sparse=True) for multivariate rings.
- order – string or TermOrder object, e.g.,
  - 'degrevlex' (default) – degree reverse lexicographic
  - 'lex' – lexicographic
  - 'deglex' – degree lexicographic
  - TermOrder('deglex',3) + TermOrder('deglex',3) – block ordering
- implementation – string or None; selects an implementation in cases where Sage includes multiple choices (currently \( \mathbb{Z}[x] \) can be implemented with 'NTL' or 'FLINT'; default is 'FLINT'). For many base rings, the "singular" implementation is available. One can always specify implementation="generic" for a generic Sage implementation which does not use any specialized library.
Note: If the given implementation does not exist for rings with the given number of generators and the given sparsity, then an error results.

OUTPUT:

PolynomialRing(base_ring, name, sparse=False) returns a univariate polynomial ring; also, PolynomialRing(base_ring, names, sparse=False) yields a univariate polynomial ring, if names is a list or tuple providing exactly one name. All other input formats return a multivariate polynomial ring.

UNIQUENESS and IMMUTABILITY: In Sage there is exactly one single-variate polynomial ring over each base ring in each choice of variable, sparseness, and implementation. There is also exactly one multivariate polynomial ring over each base ring for each choice of names of variables and term order. The names of the generators can only be temporarily changed after the ring has been created. Do this using the localvars() context.

EXAMPLES:

1. PolynomialRing(base_ring, name, …)

```sage
P = PolynomialRing(QQ, 'w')
Univariate Polynomial Ring in w over Rational Field
P = PolynomialRing(QQ, name='w')
Univariate Polynomial Ring in w over Rational Field
```

Use the diamond brackets notation to make the variable ready for use after you define the ring:

```sage
R.<w> = PolynomialRing(QQ)
R
Univariate Polynomial Ring in w over Rational Field
```

```sage
(w^3 + 3*w^2 + 3*w + 1)^3
w^3 + 3*w^2 + 3*w + 1
```

You must specify a name:

```sage
P = PolynomialRing(QQ)
Traceback (most recent call last):
  ...TypeError: you must specify the names of the variables
```

```sage
R.<abc> = PolynomialRing(QQ, sparse=True); R
Sparse Univariate Polynomial Ring in abc over Rational Field
```

```sage
R.<w> = PolynomialRing(PolynomialRing(GF(7),k)); R
Univariate Polynomial Ring in w over
  Univariate Polynomial Ring in k over Finite Field of size 7
```

The square bracket notation:

```sage
R.<y> = QQ['y']; R
Univariate Polynomial Ring in y over Rational Field
```

```sage
y^2 + y
y^2 + y
```

In fact, since the diamond brackets on the left determine the variable name, you can omit the variable from the square brackets:

```sage
R.<zz> = QQ[]; R
Univariate Polynomial Ring in zz over Rational Field
```

```sage
(zz + 1)^2
zz^2 + 2*zz + 1
```
This is exactly the same ring as what PolynomialRing returns:

```
sage: R is PolynomialRing(QQ, 'zz')
True
```

However, rings with different variables are different:

```
sage: QQ['x'] == QQ['y']
False
```

Sage has two implementations of univariate polynomials over the integers, one based on NTL and one based on FLINT. The default is FLINT. Note that FLINT uses a "more dense" representation for its polynomials than NTL, so in particular, creating a polynomial like $2^{1000000} \times x^{1000000}$ in FLINT may be unwise.

```
sage: ZxNTL = PolynomialRing(ZZ, x, implementation=NTL); ZxNTL
Univariate Polynomial Ring in x over Integer Ring (using NTL)
sage: ZxFLINT = PolynomialRing(ZZ, x, implementation=FLINT); ZxFLINT
Univariate Polynomial Ring in x over Integer Ring
sage: ZxFLINT is ZZ['x']
True
sage: ZxFLINT is PolynomialRing(ZZ, x)
True
sage: xNTL = ZxNTL.gen()

```

There is a coercion from the non-default to the default implementation, so the values can be mixed in a single expression:

```
sage: (xNTL + xFLINT^2)
```

The result of such an expression will use the default, i.e., the FLINT implementation:

```
sage: (xNTL + xFLINT^2).parent()
```

The generic implementation uses neither NTL nor FLINT:

```
sage: Zx = PolynomialRing(ZZ, 'x', implementation='generic'); Zx
Univariate Polynomial Ring in x over Integer Ring
```

2. PolynomialRing(base_ring, names, ...)
All three rings are identical:

```python
sage: R is S
True
sage: S is T
True
```

There is a unique polynomial ring with each term order:

```python
sage: R = PolynomialRing(QQ, 'x,y,z', order='degrevlex'); R
Multivariate Polynomial Ring in x, y, z over Rational Field
sage: S = PolynomialRing(QQ, 'x,y,z', order='invlex'); S
Multivariate Polynomial Ring in x, y, z over Rational Field
sage: S is PolynomialRing(QQ, 'x,y,z', order='invlex')
True
sage: R == S
False
```

Note that a univariate polynomial ring is returned, if the list of names is of length one. If it is of length zero, a multivariate polynomial ring with no variables is returned.

```python
sage: PolynomialRing(QQ, "x")
Univariate Polynomial Ring in x over Rational Field
sage: PolynomialRing(QQ, [])
Multivariate Polynomial Ring in no variables over Rational Field
```

The Singular implementation always returns a multivariate ring, even for 1 variable:

```python
sage: PolynomialRing(QQ, "x", implementation="singular")
# needs sage.libs.singular
Multivariate Polynomial Ring in x over Rational Field
sage: P.<x> = PolynomialRing(QQ, implementation="singular"); P
# needs sage.libs.singular
Multivariate Polynomial Ring in x over Rational Field
```

3. PolynomialRing(base_ring, n, names, ...) (where the arguments n and names may be reversed)

If you specify a single name as a string and a number of variables, then variables labeled with numbers are created.

```python
sage: PolynomialRing(QQ, 'x', 10)
Multivariate Polynomial Ring in x0, x1, x2, x3, x4, x5, x6, x7, x8, x9 over Rational Field
sage: PolynomialRing(QQ, 2, 'alpha0')
Multivariate Polynomial Ring in alpha0, alpha01 over Rational Field
sage: PolynomialRing(GF(7), 'y', 5)
Multivariate Polynomial Ring in y0, y1, y2, y3, y4 over Finite Field of size 7
```

(continues on next page)
Note that a multivariate polynomial ring is returned when an explicit number is given.

```
sage: PolynomialRing(QQ, "x", 1)
Multivariate Polynomial Ring in x over Rational Field
```

It is easy in Python to create fairly arbitrary variable names. For example, here is a ring with generators labeled by the primes less than 100:

```
sage: R = PolynomialRing(ZZ, ["x\%s\%p" for p in primes(100)]); R
Multivariate Polynomial Ring in x2, x3, x5, x7, x11, x13, x17, x19, x23, x29, x31, x37, x41, x43, x47, x53, x59, x61, x67, x71, x73, x79, x83, x89, x97 over Integer Ring
```

By calling the `inject_variables()` method, all those variable names are available for interactive use:

```
sage: R.inject_variables()
# needs sage.libs.pari
Defining x2, x3, x5, x7, x11, x13, x17, x19, x23, x29, x31, x37, x41, x43, x47, x53, x59, x61, x67, x71, x73, x79, x83, x89, x97
```

4. `PolynomialRing(base_ring, n, ...., var_array=var_array, ...)`

This creates an array of variables where each variables begins with an entry in `var_array` and is indexed from 0 to \( n - 1 \).

```
sage: PolynomialRing(ZZ, 3, var_array=[‘x’, ‘y’])
Multivariate Polynomial Ring in x0, y0, x1, y1, x2, y2 over Integer Ring
```

```
sage: PolynomialRing(ZZ, 3, var_array=[‘a’, ‘b’])
Multivariate Polynomial Ring in a0, b0, a1, b1, a2, b2 over Integer Ring
```

It is possible to create higher-dimensional arrays:

```
sage: PolynomialRing(ZZ, 2, 3, var_array=('p', 'q'))
Multivariate Polynomial Ring in p00, q00, p01, q01, p02, q02, p10, q10, p11, q11, p12, q12 over Integer Ring
```

```
sage: PolynomialRing(ZZ, 2, 3, 4, var_array='m')
Multivariate Polynomial Ring in m000, m001, m002, m003, m010, m011, m012, m013, m020, m021, m022, m023, m100, m101, m102, m103, m110, m111, m112, m113, m120, m121, m122, m123 over Integer Ring
```

The array is always at least 2-dimensional. So, if `var_array` is a single string and only a single number \( n \) is given, this creates an \( n \times n \) array of variables:

```
sage: PolynomialRing(ZZ, 2, var_array='m')
Multivariate Polynomial Ring in m00, m01, m10, m11 over Integer Ring
```
Square brackets notation

You can alternatively create a polynomial ring over a ring \( R \) with square brackets:

```
sage: # needs sage.rings.real_mpfr
sage: RR["x"]
Univariate Polynomial Ring in x over Real Field with 53 bits of precision
sage: RR["x,y"]
Multivariate Polynomial Ring in x, y over Real Field with 53 bits of precision
sage: P.<x,y> = RR[]; P
Multivariate Polynomial Ring in x, y over Real Field with 53 bits of precision
```

This notation does not allow to set any of the optional arguments.

Changing variable names

Consider

```
sage: R.<x,y> = PolynomialRing(QQ, 2); R
Multivariate Polynomial Ring in x, y over Rational Field
sage: f = x^2 - 2*y^2
```

You can’t just globally change the names of those variables. This is because objects all over Sage could have pointers to that polynomial ring.

```
sage: R._assign_names(['z','w'])
Traceback (most recent call last):
  ...
ValueError: variable names cannot be changed after object creation.
```

However, you can very easily change the names within a `with` block:

```
sage: with localvars(R, ['z','w']):
    ....:   print(f)
z^2 - 2*w^2
```

After the `with` block the names revert to what they were before:

```
sage: print(f)
x^2 - 2*y^2
```

Choose an appropriate category for a polynomial ring.

It is assumed that the corresponding base ring is nonzero.

INPUT:

- `base_ring_category` – The category of ring over which the polynomial ring shall be defined
- `n_variables` – number of variables

EXAMPLES:

```
sage: from sage.rings.polynomial.polynomial_ring_constructor import polynomial_default_category
sage: polynomial_default_category(Rings(),1) is Algebras(Rings()).Infinite()
True
sage: polynomial_default_category(Rings().Commutative(),1) is Algebras(Rings()).
```

(continues on next page)
sage: QQ['t'].category()  # EuclideanDomains() & CommutativeAlgebras(QQ).category()).Infinite()
True
sage: QQ['s','t'].category()  # UniqueFactorizationDomains() & CommutativeAlgebras(QQ.category()).Infinite()
True
sage: QQ['s']['t'].category()  # UniqueFactorizationDomains() & CommutativeAlgebras(QQ['s'].category()).Infinite()
True

Custom unpickling function for polynomial rings.

This has the same positional arguments as the old `PolynomialRing` constructor before `github issue #23338`. 
2.1 Univariate Polynomials and Polynomial Rings

Sage’s architecture for polynomials ‘under the hood’ is complex, interfacing to a variety of C/C++ libraries for polynomials over specific rings. In practice, the user rarely has to worry about which backend is being used.

The hierarchy of class inheritance is somewhat confusing, since most of the polynomial element classes are implemented as Cython extension types rather than pure Python classes and thus can only inherit from a single base class, whereas others have multiple bases.

2.1.1 Univariate Polynomial Rings

Sage implements sparse and dense polynomials over commutative and non-commutative rings. In the non-commutative case, the polynomial variable commutes with the elements of the base ring.

AUTHOR:
• William Stein
• Kiran Kedlaya (2006-02-13): added macaulay2 option
• Martin Albrecht (2006-08-25): removed it again as it isn’t needed anymore
• Simon King (2011-05): Dense and sparse polynomial rings must not be equal.
• Simon King (2011-10): Choice of categories for polynomial rings.

EXAMPLES:

```python
sage: z = QQ['z'].0
sage: (z^3 + z - 1)^3
z^9 + 3*z^7 - 3*z^6 + 3*z^5 - 6*z^4 + 4*z^3 - 3*z^2 + 3*z - 1
```

Saving and loading of polynomial rings works:

```python
sage: loads(dumps(QQ['x'])) == QQ['x']
True
sage: k = PolynomialRing(QQ,'x','y'); loads(dumps(k))==k
True
sage: k = PolynomialRing(ZZ,'y'); loads(dumps(k)) == k
True
sage: k = PolynomialRing(ZZ,'y', sparse=True); loads(dumps(k))
Sparse Univariate Polynomial Ring in y over Integer Ring
```
Rings with different variable names are not equal; in fact, by github issue #9944, polynomial rings are equal if and only if they are identical (which should be the case for all parent structures in Sage):

```
sage: QQ['y'] != QQ['x']
True
sage: QQ['y'] != QQ['z']
True
```

We create a polynomial ring over a quaternion algebra:

```
sage: # needs sage.combinat sage.modules
sage: A.<i,j,k> = QuaternionAlgebra(QQ, -1,-1)
sage: R.<w> = PolynomialRing(A, sparse=True)
sage: f = w^3 + (i+j)*w + 1
sage: f
w^3 + (i + j)*w + 1
sage: f^2
w^6 + (2*i + 2*j)*w^4 + 2*w^3 - 2*w^2 + (2*i + 2*j)*w + 1
sage: f = w + i ; g = w + j
sage: f * g
w^2 + (i + j)*w + k
sage: g * f
w^2 + (i + j)*w - k
```

github issue #9944 introduced some changes related with coercion. Previously, a dense and a sparse polynomial ring with the same variable name over the same base ring evaluated equal, but of course they were not identical. Coercion maps are cached - but if a coercion to a dense ring is requested and a coercion to a sparse ring is returned instead (since the cache keys are equal!), all hell breaks loose.

Therefore, the coercion between rings of sparse and dense polynomials works as follows:

```
sage: R.<x> = PolynomialRing(QQ, sparse=True)
sage: S.<x> = QQ[]
sage: S == R
False
sage: S.has_coerce_map_from(R)
True
sage: R.has_coerce_map_from(S)
False
sage: (R.0 + S.0).parent()
Univariate Polynomial Ring in x over Rational Field
sage: (S.0 + R.0).parent()
Univariate Polynomial Ring in x over Rational Field
```

It may be that one has rings of dense or sparse polynomials over different base rings. In that situation, coercion works by means of the pushout() formalism:

```
sage: R.<x> = PolynomialRing(GF(5), sparse=True)
sage: S.<x> = PolynomialRing(ZZ)
sage: S.has_coerce_map_from(R)
False
sage: S.has_coerce_map_from(S)
False
sage: (R.0 + S.0).parent()
Univariate Polynomial Ring in x over Rational Field
sage: (S.0 + R.0).parent()
Univariate Polynomial Ring in x over Rational Field of size 5
```

(continues on next page)
Similarly, there is a coercion from the (non-default) NTL implementation for univariate polynomials over the integers to the default FLINT implementation, but not vice versa:

```
sage: R.<x> = PolynomialRing(ZZ, implementation='NTL')
# needs sage.libs.ntl
sage: S.<x> = PolynomialRing(ZZ, implementation='FLINT')
sage: (S.0 + R.0).parent() is S      # needs sage.libs.flint sage.libs.ntl
True
sage: (R.0 + S.0).parent() is S      # needs sage.libs.flint sage.libs.ntl
True
```

```
class sage.rings.polynomial.polynomial_ring.PolynomialRing_cdvf
    (base_ring, name=None, sparse=False, implementation=None, element_class=None, category=None)
Bases: PolynomialRing_cdvr, PolynomialRing_field

A class for polynomial ring over complete discrete valuation fields
```

```
class sage.rings.polynomial.polynomial_ring.PolynomialRing_cdvr
    (base_ring, name=None, sparse=False, implementation=None, element_class=None, category=None)
Bases: PolynomialRing_integral_domain

A class for polynomial ring over complete discrete valuation rings
```

```
class sage.rings.polynomial.polynomial_ring.PolynomialRing_commutative
    (base_ring, name=None, sparse=False, implementation=None, element_class=None, category=None)
Bases: PolynomialRing_general

Univariate polynomial ring over a commutative ring.
```

```
quotient_by_principal_ideal (f, names=None, **kwds)
    Return the quotient of this polynomial ring by the principal ideal (generated by) f.

    INPUT:
    • f - either a polynomial in self, or a principal ideal of self.
```
• further named arguments that are passed to the quotient constructor.

**EXAMPLES:**

```python
sage: R.<x> = QQ[]
sage: I = (x^2 - 1) * R
sage: R.quotient_by_principal_ideal(I)  # needs sage.libs.pari
Univariate Quotient Polynomial Ring in xbar
over Rational Field with modulus x^2 - 1
```

The same example, using the polynomial instead of the ideal, and customizing the variable name:

```python
sage: R.<x> = QQ[]
sage: R.quotient_by_principal_ideal(x^2 - 1, names=('foo',))  # needs sage.libs.pari
Univariate Quotient Polynomial Ring in foo
over Rational Field with modulus x^2 - 1
```

**weyl_algebra()**

Return the Weyl algebra generated from self.

**EXAMPLES:**

```python
sage: R = QQ['x']
sage: W = R.weyl_algebra(); W  # needs sage.modules
Differential Weyl algebra of polynomials in x over Rational Field
sage: W.polynomial_ring() == R  # needs sage.modules
True
```

**class** `sage.rings.polynomial.polynomial_ring.PolynomialRing_dense_finite_field`(*base_ring*, *name='x'*, *element_class=None*, *implementation=None*)

**Bases:** `PolynomialRing_field`

Univariate polynomial ring over a finite field.

**EXAMPLES:**

```python
sage: R = PolynomialRing(GF(27, 'a'), 'x')  # needs sage.rings.finite_rings
sage: type(R)  # needs sage.rings.finite_rings
<class 'sage.rings.polynomial.polynomial_ring.PolynomialRing_dense_finite_field_with_category'>
```

**irreducible_element** (*n*, *algorithm=None*)

Construct a monic irreducible polynomial of degree *n*.

**INPUT:**
• \( n \) – integer: degree of the polynomial to construct

• \texttt{algorithm} – string: algorithm to use, or \texttt{None}
  – 'random' or \texttt{None}: try random polynomials until an irreducible one is found.
  – 'first_lexicographic': try polynomials in lexicographic order until an irreducible one is found.

OUTPUT:

A monic irreducible polynomial of degree \( n \) in \texttt{self}.

EXAMPLES:

```
sage: # needs sage.modules sage.rings.finite_rings
dsage: f = GF(5^3, 'a')['x'].irreducible_element(2)
dsage: f.degree()
2
dsage: f.is_irreducible()
True
dsage: R = GF(19)['x']
dsage: R.irreducible_element(21, algorithm="first_lexicographic")
\texttt{x}^{21} + \texttt{x} + 5
```

AUTHORS:

• Peter Bruin (June 2013)

• Jean-Pierre Flori (May 2014)
Residue field in a
of Principal ideal \((t^3 + t + 1)\) of Univariate Polynomial Ring in \(t\)
over Finite Field of size 2 (using GF2X)

```python
sage: k.list()
[0, a, a^2, a + 1, a^2 + a, a^2 + a + 1, a^2 + 1, 1]
```

```python
sage: R.residue_field(t)
Residue field of Principal ideal \((t)\) of Univariate Polynomial Ring in \(t\)
over Finite Field of size 2 (using GF2X)

sage: P = R.irreducible_element(8) * R
sage: P
Principal ideal \((t^8 + t^4 + t^3 + t^2 + 1)\) of Univariate Polynomial Ring in \(t\)
over Finite Field of size 2 (using GF2X)

sage: k.<a> = R.residue_field(P); k
Residue field in a
of Principal ideal \((t^8 + t^4 + t^3 + t^2 + 1)\) of Univariate Polynomial Ring
over Finite Field of size 2 (using GF2X)

sage: k.cardinality()
256
```

Non-maximal ideals are not accepted:

```python
sage: # needs sage.libsntl
sage: R.residue_field(t^2 + 1)
Traceback (most recent call last):
... ArithmeticError: ideal is not maximal
```

```python
sage: R.residue_field(0)
Traceback (most recent call last):
... ArithmeticError: ideal is not maximal
sage: R.residue_field(1)
Traceback (most recent call last):
... ArithmeticError: ideal is not maximal
```

```
class sage.rings.polynomial.polynomial_ring.PolynomialRing_dense_mod_p
```

**irreducible_element** \((n, \text{algorithm}=\text{None})\)

Construct a monic irreducible polynomial of degree \(n\).

**INPUT:**

- \(n\) – integer: the degree of the polynomial to construct
- \(\text{algorithm}\) – string: algorithm to use, or None. Currently available options are:
  - 'adleman-lenstra': a variant of the Adleman–Lenstra algorithm as implemented in PARI.

14 Chapter 2. Univariate Polynomials
- `'conway'`: look up the Conway polynomial of degree \( n \) over the field of \( p \) elements in the database; raise a `RuntimeError` if it is not found.
- `'ffprimroot'`: use the `pari:ffprimroot` function from PARI.
- `'first_lexicographic'`: return the lexicographically smallest irreducible polynomial of degree \( n \).
- `'minimal_weight'`: return an irreducible polynomial of degree \( n \) with minimal number of non-zero coefficients. Only implemented for \( p = 2 \).
- `'primitive'`: return a polynomial \( f \) such that a root of \( f \) generates the multiplicative group of the finite field extension defined by \( f \). This uses the Conway polynomial if possible, otherwise it uses `'ffprimroot'`.
- `'random'`: try random polynomials until an irreducible one is found.

If `algorithm` is `None`, use \( x - 1 \) in degree 1. In degree > 1, the Conway polynomial is used if it is found in the database. Otherwise, the algorithm `minimal_weight` is used if \( p = 2 \), and the algorithm `adleman-lenstra` if \( p > 2 \).

**OUTPUT:**
A monic irreducible polynomial of degree \( n \) in `self`.

**EXAMPLES:**

```sage
sage: # needs sage.rings.finite_rings
sage: GF(5)['x'].irreducible_element(2)
x^2 + 4*x + 2
sage: GF(5)['x'].irreducible_element(2, algorithm="adleman-lenstra")
x^2 + x + 1
sage: GF(5)['x'].irreducible_element(2, algorithm="primitive")
x^2 + 4*x + 2
sage: GF(5)['x'].irreducible_element(32, algorithm="first_lexicographic")
x^32 + 2
sage: GF(5)['x'].irreducible_element(32, algorithm="conway")
Traceback (most recent call last):
  ...RuntimeError: requested Conway polynomial not in database.
sage: GF(5)['x'].irreducible_element(32, algorithm="primitive")
x^32 + ...
```

In characteristic 2:

```sage
sage: GF(2)['x'].irreducible_element(33)  #...
→ needs sage.rings.finite_rings
x^33 + x^13 + x^12 + x^11 + x^10 + x^8 + x^6 + x^3 + 1
sage: GF(2)['x'].irreducible_element(33, algorithm="minimal_weight")  #...
→ needs sage.rings.finite_rings
x^33 + x^10 + 1
```

In degree 1:

```sage
sage: GF(97)['x'].irreducible_element(1)  #...
→ needs sage.rings.finite_rings
x + 96
sage: GF(97)['x'].irreducible_element(1, algorithm="conway")  #...
→ needs sage.rings.finite_rings
x + 92
sage: GF(97)['x'].irreducible_element(1, algorithm="adleman-lenstra")  #...
```
(continues on next page)
AUTHORS:

- Peter Bruin (June 2013)
- Jeroen Demeyer (September 2014): add “ffprimroot” algorithm, see github issue #8373.

\[\text{class sage.rings.polynomial.polynomial_ring.PolynomialRing_dense_padic_field_capped_relative}\]

\text{Bases: PolynomialRing_dense_padic_field_generic}\]

\[\text{class sage.rings.polynomial.polynomial_ring.PolynomialRing_dense_padic_field_generic}\]

\text{Bases: PolynomialRing_cdvf}\]

A class for dense polynomial ring over p-adic fields

\[\text{class sage.rings.polynomial.polynomial_ring.PolynomialRing_dense_padic_ring_capped_absolute}\]

\text{Bases: PolynomialRing_dense_padic_ring_generic}\]

Chapter 2. Univariate Polynomials
A class for dense polynomial ring over p-adic rings

class sage.rings.polynomial.polynomial_ring.PolynomialRing_field
(base_ring, name='x', sparse=False, implementation=None, element_class=None, category=None)

Bases: PolynomialRing_integral_domain, PrincipalIdealDomain
divided_difference (points, full_table=False)

Return the Newton divided-difference coefficients of the Lagrange interpolation polynomial through points.

INPUT:

• points – a list of pairs \((x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\) of elements of the base ring of self, where \(x_i - x_j\) is invertible for \(i \neq j\). This method converts the \(x_i\) and \(y_i\) into the base ring of self.

• full_table – boolean (default: False): If True, return the full divided-difference table. If False, only return entries along the main diagonal; these are the Newton divided-difference coefficients \(F_{i,i}\).

OUTPUT:

The Newton divided-difference coefficients of the \(n\)-th Lagrange interpolation polynomial \(P_n(x)\) that passes through the points in points (see lagrange_polynomial()). These are the coefficients \(F_{0,0}, F_{1,1}, \ldots, F_{n,n}\) in the base ring of self such that

\[
P_n(x) = \sum_{i=0}^{n} F_{i,i} \prod_{j=0}^{i-1} (x - x_j)
\]

EXAMPLES:

Only return the divided-difference coefficients \(F_{i,i}\). This example is taken from Example 1, page 121 of [BF2005]:

```
sage: # needs sage.rings.real_mpfr
sage: points = [(1.0, 0.7651977), (1.3, 0.6200860), (1.6, 0.4554022),
.....: (1.9, 0.2818186), (2.2, 0.1103623)]
sage: R = PolynomialRing(RR, "x")
sage: R.divided_difference(points)
[0.765197700000000,
-0.483705666666666,
-0.108733888888889,
0.0658783950617283,
0.00182510288066044]
```

Now return the full divided-difference table:

```
sage: # needs sage.rings.real_mpfr
sage: points = [(1.0, 0.7651977), (1.3, 0.6200860), (1.6, 0.4554022),
.....: (1.9, 0.2818186), (2.2, 0.1103623)]
sage: R = PolynomialRing(RR, "x")
sage: R.divided_difference(points, full_table=True)
[[0.765197700000000],
[0.620086000000000, -0.483705666666666],
[0.455402200000000, -0.548946000000000, -0.108733888888889],
[0.281818600000000, -0.578612000000000, -0.0494433333333339, 0.0658783950617283],
[0.110362300000000, -0.571520999999999, 0.01181333333349, 0.0680685185185209, 0.00182510288066044]]
```

The following example is taken from Example 4.12, page 225 of [MF1999]:

```
sage: points = [(1, -3), (2, 0), (3, 15), (4, 48), (5, 105), (6, 192)]
sage: R = PolynomialRing(QQ, "x")
sage: R.divided_difference(points)
[-3, 3, 6, 1, 0, 0]
```

(continues on next page)
fraction_field()

Returns the fraction field of self.

EXAMPLES:

sage: R.<t> = GF(5)[]
sage: R.fraction_field()
Fraction Field of Univariate Polynomial Ring in t
   over Finite Field of size 5

lagrange_polynomial (points, algorithm='divided_difference', previous_row=None)

Return the Lagrange interpolation polynomial through the given points.

INPUT:

• points – a list of pairs \((x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\) of elements of the base ring of self, where \(x_i \neq x_j\) is invertible for \(i \neq j\). This method converts the \(x_i\) and \(y_i\) into the base ring of self.

• algorithm – (default: 'divided_difference'): one of the following:
  – 'divided_difference': use the method of divided differences.
  – 'neville': adapt Neville’s method as described on page 144 of [BF2005] to recursively generate the Lagrange interpolation polynomial. Neville’s method generates a table of approximating polynomials, where the last row of that table contains the \(n\)-th Lagrange interpolation polynomial. The adaptation implemented by this method is to only generate the last row of this table, instead of the full table itself. Generating the full table can be memory inefficient.

• previous_row – (default: None): This option is only relevant if used with algorithm='neville'. If provided, this should be the last row of the table resulting from a previous use of Neville’s method. If such a row is passed, then points should consist of both previous and new interpolating points. Neville’s method will then use that last row and the interpolating points to generate a new row containing an interpolation polynomial for the new points.

OUTPUT:

The Lagrange interpolation polynomial through the points \((x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\). This is the unique polynomial \(P_n\) of degree at most \(n\) in self satisfying \(P_n(x_i) = y_i\) for \(0 \leq i \leq n\).

EXAMPLES:

By default, we use the method of divided differences:

```python
sage: R = PolynomialRing(QQ, 'x')
sage: f = R.lagrange_polynomial([(0,1), (2,2), (3,-2), (-4,9)]); f
-23/84*x^3 - 11/84*x^2 + 13/7*x + 1
```

(continues on next page)
Now use a memory efficient version of Neville’s method:

```
R = PolynomialRing(QQ, 'x')
P = R.lagrange_polynomial([(0,1), (2,2), (3,-2), (-4,9)],
    algorithm="neville")
\[\frac{9}{2}, \frac{-11/7*x + 19/7}{x}, \frac{-17/42*x^2 - 83/42*x + 53/7}{x^2}, \frac{-23/84*x^3 - 11/84*x^2 + 13/7*x + 1}{x^3}\]
```

Repeated use of Neville’s method to get better Lagrange interpolation polynomials:

```
P = R.lagrange_polynomial([(0,1), (2,2)], algorithm="neville")
P = R.lagrange_polynomial([(0,1), (2,2), (3,-2), (-4,9)],
    algorithm="neville", previous_row=P)[-1]
\[-23/84*x^3 - 11/84*x^2 + 13/7*x + 1\]
```

```
P = R.lagrange_polynomial([(a^2+a, a), (a, 1)], algorithm="neville")
P = R.lagrange_polynomial([(a^2+a, a), (a, 1), (a^2, a^2+a+1)],
    algorithm="neville", previous_row=P)[-1]
a^2*x^2 + a^2*x + a^2
```

```python
class sage.rings.polynomial.polynomial_ring.PolynomialRing_general(base_ring,
name=None, sparse=False, implementation=None, element_class=None, category=None)
```
Bases: Ring
Univariate polynomial ring over a ring.

base_extend ($R$)
Return the base extension of this polynomial ring to $R$.

EXAMPLES:

```sage
sage: # needs sage.rings.real_mpfr
sage: R.<x> = RR[]; R
Univariate Polynomial Ring in x over Real Field with 53 bits of precision
sage: R.base_extend(CC)
Univariate Polynomial Ring in x over Complex Field with 53 bits of precision
sage: R.base_extend(QQ)
Traceback (most recent call last):
  ...TypeError: no such base extension
sage: R.change_ring(QQ)
Univariate Polynomial Ring in x over Rational Field
```

change_ring ($R$)
Return the polynomial ring in the same variable as self over $R$.

EXAMPLES:

```sage
sage: # needs sage.rings.finite_rings sage.rings.real_interval_field
sage: R.<ZZZ> = RealIntervalField()[]; R
Univariate Polynomial Ring in ZZZ over Real Interval Field with 53 bits of precision
sage: R.change_ring(GF(19^2, 'b'))
Univariate Polynomial Ring in ZZZ over Finite Field in b of size 19^2
```

close_factor
Return the polynomial ring in variable var over the same base ring.

EXAMPLES:

```sage
sage: R.<x> = ZZ[]; R
Univariate Polynomial Ring in x over Integer Ring
sage: R.change_var('y')
Univariate Polynomial Ring in y over Integer Ring
```

characteristic()
Return the characteristic of this polynomial ring, which is the same as that of its base ring.

EXAMPLES:

```sage
sage: # needs sage.rings.real_mpfr
sage: R.<x> = RealIntervalField()[]; R
Univariate Polynomial Ring in x over Real Interval Field with 53 bits of precision
sage: R.characteristic()
0
sage: S = R.change_ring(GF(19^2, 'b')); S
Univariate Polynomial Ring in x over Finite Field in b of size 19^2
sage: S.characteristic()  # needs sage.rings.finite_rings
(continues on next page)```
completion \((p=\text{None}, \text{prec}=20, \text{extras}=\text{None})\)

Return the completion of \(self\) with respect to the irreducible polynomial \(p\).

Currently only implemented for \(p=\text{self}.\text{gen}()\) (the default), i.e. you can only complete \(R[x]\) with respect to \(x\), the result being a ring of power series in \(x\). The \(\text{prec}\) variable controls the precision used in the power series ring. If \(\text{prec}\) is \(\infty\), then this returns a \texttt{LazyPowerSeriesRing}.

**EXAMPLES:**

```python
sage: P.<x> = PolynomialRing(QQ)
sage: P
Univariate Polynomial Ring in x over Rational Field
sage: PP = P.completion(x)
sage: PP
Power Series Ring in x over Rational Field
sage: f = 1 - x
sage: PP(f)
1 - x
sage: 1 / f
\(-1/(x - 1)\)
sage: g = 1 / PP(f); g
1 + x + x^2 + O(x^3)
sage: 1 / g
\(1 - x + O(x^20)\)
sage: # needs sage.combinat
sage: PP = P.completion(x, prec=oo); PP
Lazy Taylor Series Ring in x over Rational Field
sage: g = 1 / PP(f); g
1 + x + x^2 + O(x^3)
sage: 1 / g == f
\(\text{True}\)
```

cyclotomic_polynomial \((n)\)

Return the \(n\)th cyclotomic polynomial as a polynomial in this polynomial ring. For details of the implementation, see the documentation for \texttt{sage.rings.polynomial.cyclotomic.cyclotomic_coeffs()}.  

**EXAMPLES:**

```python
sage: R = ZZ['x']
sage: R.cyclotomic_polynomial(8)
x^4 + 1
sage: R.cyclotomic_polynomial(12)
x^4 - x^2 + 1
sage: S = PolynomialRing(FiniteField(7), 'x')
sage: S.cyclotomic_polynomial(12)
x^4 + 6*x^2 + 1
```
**extend_variables** *(added_names, order='degrevlex')*

Return a multivariate polynomial ring with the same base ring but with `added_names` as additional variables.

**EXAMPLES:**

```python
sage: R.<x> = ZZ[]; R
Univariate Polynomial Ring in x over Integer Ring
sage: R.extend_variables('y, z')
Multivariate Polynomial Ring in x, y, z over Integer Ring
sage: R.extend_variables(('y', 'z'))
Multivariate Polynomial Ring in x, y, z over Integer Ring
```

**flattening_morphism()**

Return the flattening morphism of this polynomial ring

**EXAMPLES:**

```python
sage: QQ['a','b'][x].flattening_morphism()
Flattening morphism:
  From: Univariate Polynomial Ring in x over Multivariate Polynomial Ring in a, b over Rational Field
  To:  Multivariate Polynomial Ring in a, b, x over Rational Field
sage: QQ['x'].flattening_morphism()
Identity endomorphism of Univariate Polynomial Ring in x over Rational Field
```

**gen**(n=0)

Return the indeterminate generator of this polynomial ring.

**EXAMPLES:**

```python
sage: R.<abc> = Integers(8)[]; R
Univariate Polynomial Ring in abc over Ring of integers modulo 8
sage: t = R.gen(); t
abc
sage: t.is_gen()
True
```

An identical generator is always returned.

```python
sage: t is R.gen()
True
```

**gens_dict()**

Return a dictionary whose entries are \{name:variable,...\}, where `name` stands for the variable names of this object (as strings) and `variable` stands for the corresponding generators (as elements of this object).

**EXAMPLES:**

```python
sage: R.<y,x,a42> = RR[]
sage: R.gens_dict()
{'a42': a42, 'x': x, 'y': y}
```
is_exact()

EXAMPLES:

```
sage: class Foo:
    ....:     def __init__(self, x):
    ....:         self._x = x
    ....: @cached_method
    ....:     def f(self):
    ....:         return self._x^2
sage: a = Foo(2)
sage: print(a.f.cache)
None
sage: a.f()
4
sage: a.f.cache
4
```

is_field (proof=True)

Return False, since polynomial rings are never fields.

EXAMPLES:

```
sage: # needs sage.libsntl
sage: R.<z> = Integers(2)[]; R
Univariate Polynomial Ring in z over Ring of integers modulo 2 (using GF2X)
sage: R.is_field()
False
```

is_integral_domain (proof=True)

EXAMPLES:

```
sage: ZZ['x'].is_integral_domain()
True
sage: Integers(8)['x'].is_integral_domain()
False
```

is_noetherian()

is_sparse()

Return True if elements of this polynomial ring have a sparse representation.

EXAMPLES:

```
sage: R.<z> = Integers(8)[]; R
Univariate Polynomial Ring in z over Ring of integers modulo 8
sage: R.is_sparse()
False
sage: R.<W> = PolynomialRing(QQ, sparse=True); R
Sparse Univariate Polynomial Ring in W over Rational Field
sage: R.is_sparse()
True
```

is_unique_factorization_domain (proof=True)

EXAMPLES:

```
sage: ZZ['x'].is_unique_factorization_domain()
True
```

(continues on next page)
karatsuba_threshold()

Return the Karatsuba threshold used for this ring by the method _mul_karatsuba() to fall back to the schoolbook algorithm.

EXAMPLES:

```sage
sage: K = QQ['x']
sage: K.karatsuba_threshold()
8
sage: K = QQ['x']['y']
sage: K.karatsuba_threshold()
0
```

krull_dimension()

Return the Krull dimension of this polynomial ring, which is one more than the Krull dimension of the base ring.

EXAMPLES:

```sage
sage: R.<x> = QQ[]
sage: R.krull_dimension()
1
sage: # needs sage.rings.finite_rings
sage: R.<z> = GF(9, 'a')[]; R
Univariate Polynomial Ring in z over Finite Field in a of size 3^2
sage: R.krull_dimension()
1
sage: S.<t> = R[]
sage: S.krull_dimension()
2
sage: for n in range(10):
   ....:     S = PolynomialRing(S, 'w')
...:     S.krull_dimension()
     12
```

monics (of_degree=None, max_degree=None)

Return an iterator over the monic polynomials of specified degree.

INPUT: Pass exactly one of:

- max_degree - an int; the iterator will generate all monic polynomials which have degree less than or equal to max_degree
- of_degree - an int; the iterator will generate all monic polynomials which have degree of_degree

OUTPUT: an iterator

EXAMPLES:

```sage
# needs sage.rings.finite_rings
sage: P = PolynomialRing(GF(4, 'a'), 'y')
sage: for p in P.monic(of_degree=2): print(p)
y^2
y^2 + a
```
\begin{verbatim}
Polynomials, Release 10.3

(continued from previous page)

\begin{verbatim}
\begin{verbatim}
y^2 + a + 1
y^2 + 1
y^2 + a*y
y^2 + a*y + a
y^2 + a*y + a + 1
y^2 + a*y + 1
y^2 + (a + 1)*y
y^2 + (a + 1)*y + a
y^2 + (a + 1)*y + a + 1
y^2 + y
y^2 + y + a
y^2 + y + a + 1
y^2 + y + 1
\end{verbatim}
sage: for p in P.monics(max_degree=1): print(p)
1
y
y + a
y + a + 1
y + 1
\end{verbatim}
sage: for p in P.monics(max_degree=1, of_degree=3): print(p)
\begin{verbatim}
Traceback (most recent call last):
...
ValueError: you should pass exactly one of of_degree and max_degree
\end{verbatim}

AUTHORS:

- Joel B. Mohler

\texttt{monomial} (exponent)

Return the monomial with the exponent.

INPUT:

- exponent – nonnegative integer

EXAMPLES:

\begin{verbatim}
sage: R.<x> = PolynomialRing(ZZ)
sage: R.monomial(5)
x^5
sage: e=(10,)
sage: R.monomial(*e)
x^10
sage: m = R.monomial(100)
sage: R.monomial(m.degree()) == m
True
\end{verbatim}

\texttt{ngens} ()

Return the number of generators of this polynomial ring, which is 1 since it is a univariate polynomial ring.

EXAMPLES:

\begin{verbatim}
sage: R.<z> = Integers(8)[[]]; R
Univariate Polynomial Ring in z over Ring of integers modulo 8
sage: R.ngens()
1
\end{verbatim}
\end{verbatim}

26 Chapter 2. Univariate Polynomials
parameter()

Return the generator of this polynomial ring.
This is the same as self.gen().

polynomials (of_degree=None, max_degree=None)

Return an iterator over the polynomials of specified degree.

INPUT: Pass exactly one of:

• max_degree - an int; the iterator will generate all polynomials which have degree less than or equal to max_degree

• of_degree - an int; the iterator will generate all polynomials which have degree of_degree

OUTPUT: an iterator

EXAMPLES:

```
sage: P = PolynomialRing(GF(3), 'y')
sage: for p in P.polynomials(of_degree=2): print(p)
y^2
y^2 + 1
y^2 + 2
y^2 + y
y^2 + y + 1
y^2 + y + 2
y^2 + 2*y
y^2 + 2*y + 1
y^2 + 2*y + 2
2*y^2
2*y^2 + 1
2*y^2 + 2
2*y^2 + y
2*y^2 + y + 1
2*y^2 + y + 2
2*y^2 + 2*y
2*y^2 + 2*y + 1
2*y^2 + 2*y + 2
```

```
sage: for p in P.polynomials(max_degree=1): print(p)
0
1
y
y + 1
y + 2
2*y
2*y + 1
2*y + 2
```

```
sage: for p in P.polynomials(max_degree=1, of_degree=3):
    print(p)
Traceback (most recent call last):
...
ValueError: you should pass exactly one of of_degree and max_degree
```

AUTHORS:

• Joel B. Mohler

random_element (degree=(-1, 2), *args, **kwds)

Return a random polynomial of given degree or with given degree bounds.
INPUT:

- **degree** - optional integer for fixing the degree or a tuple of minimum and maximum degrees. By default set to \((-1, 2)\).
- **args, kwargs** - Passed on to the random_element method for the base ring

EXAMPLES:

```python
sage: R.<x> = ZZ[]
sage: f = R.random_element(10, 5, 10)
sage: f.degree()
10
sage: f.parent() is R
True
sage: all(a in range(5, 10) for a in f.coefficients())
True
sage: R.random_element(6).degree()
6
```

If a tuple of two integers is given for the `degree` argument, a degree is first uniformly chosen, then a polynomial of that degree is given:

```python
sage: R.random_element(degree=(0, 8)).degree() in range(0, 9)
True
sage: found = [False]*9
sage: while not all(found):
....:     found[R.random_element(degree=(0, 8)).degree()] = True
```

Note that the zero polynomial has degree \(-1\), so if you want to consider it set the minimum degree to \(-1\):

```python
sage: while R.random_element(degree=(-1, 2), x=-1, y=1) != R.zero():
....:     pass
```

**set_karatsuba_threshold** *(Karatsuba_threshold)*

Changes the default threshold for this ring in the method `_mul_karatsuba()` to fall back to the schoolbook algorithm.

**Warning:** This method may have a negative performance impact in polynomial arithmetic. So use it at your own risk.

EXAMPLES:

```python
sage: K = QQ['x']
sage: K.karatsuba_threshold()
8
sage: K.set_karatsuba_threshold(0)
sage: K.karatsuba_threshold()
0
```

**some_elements**()

Return a list of polynomials.

This is typically used for running generic tests.

EXAMPLES:
variable_names_recursive \((depth=\text{+Infinity})\)

Return the list of variable names of this ring and its base rings, as if it were a single multi-variate polynomial.

**INPUT:**

- **depth** – an integer or \text{Infinity}.

**OUTPUT:**

A tuple of strings.

**EXAMPLES:**

```python
sage: R = QQ['x']['y']['z']
sage: R.variable_names_recursive()
('x', 'y', 'z')
sage: R.variable_names_recursive(2)
('y', 'z')
```

### class sage.rings.polynomial.polynomial_ring.PolynomialRing_integral_domain

**Bases:** PolynomialRing_commutative, PolynomialRing_singular_repr, IntegralDomain

**construction()**

Return the construction functor.

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.polynomial_ring import PolynomialRing_integral_domain
sage: R = PolynomialRing_integral_domain(ZZ, 'x', sparse=False, implementation=None, element_class=None, category=None)
```

(continues on next page)
weil_polynomials \((d, q, \text{sign}=1, \text{lead}=1)\)

Return all integer polynomials whose complex roots all have a specified absolute value.

Such polynomials \(f\) satisfy a functional equation

\[
T^d f(q/T) = s q^{d/2} f(T)
\]

where \(d\) is the degree of \(f\), \(s\) is a sign and \(q^{1/2}\) is the absolute value of the roots of \(f\).

**INPUT:**

- \(d\) – integer, the degree of the polynomials
- \(q\) – integer, the square of the complex absolute value of the roots
- \(\text{sign}\) – integer (default 1), the sign \(s\) of the functional equation
- \(\text{lead}\) – integer, list of integers or list of pairs of integers (default 1), constraints on the leading few coefficients of the generated polynomials. If pairs \((a, b)\) of integers are given, they are treated as a constraint of the form \(\equiv a \pmod b\); the moduli must be in decreasing order by divisibility, and the modulus of the leading coefficient must be 0.

**See also:**

More documentation and additional options are available using the iterator `sage.rings.polynomial.weil.weil_polynomials.WeilPolynomials` directly. In addition, polynomials have a method `is_weil_polynomial()` to test whether or not the given polynomial is a Weil polynomial.

**EXAMPLES:**

```
sage: # needs sage.libs.flint
sage: R.<T> = ZZ[]
sage: L = R.weil_polynomials(4, 2)
sage: len(L)
35
sage: L[9]
T^4 + T^3 + 2*T^2 + 2*T + 4
sage: all(p.is_weil_polynomial() for p in L)
True
```

Setting multiple leading coefficients:

```
sage: R.<T> = QQ[]
sage: l = R.weil_polynomials(4, 2, lead=((1,0), (2,4), (1,2))); l
[\ldots]
[T^4 + 2*T^3 + 5*T^2 + 4*T + 4,
T^4 - 2*T^3 + 5*T^2 - 4*T + 4,
T^4 - 2*T^3 + 3*T^2 - 4*T + 4]
```

We do not require Weil polynomials to be monic. This example generates Weil polynomials associated to K3 surfaces over \(\mathbb{F}_2\) of Picard number at least 12:

```
sage: R.<T> = QQ[]
sage: l = R.weil_polynomials(10, 1, lead=2); l
[\ldots]
```

(continues on next page)
sage: len(l)
4865
\# needs sage.libs.flint

sage: l[len(l)//2]
2*T^10 + T^8 + T^6 + T^4 + T^2 + 2
\# needs sage.libs.flint

sage.rings.polynomial.polynomial_ring.is_PolynomialRing(x)

Return True if x is a \textit{univariate} polynomial ring (and not a sparse multivariate polynomial ring in one variable).

EXAMPLES:

sage: from sage.rings.polynomial.polynomial_ring import is_PolynomialRing
sage: from sage.rings.polynomial.multi_polynomial_ring import is_MPolynomialRing
sage: is_PolynomialRing(2)
False
This polynomial ring is not univariate.

sage: is_PolynomialRing(ZZ['x,y,z'])
False
sage: is_MPolynomialRing(ZZ['x,y,z'])
True
sage: is_PolynomialRing(ZZ['w'])
True
Univariate means not only in one variable, but is a specific data type. There is a multivariate (sparse) polynomial ring data type, which supports a single variable as a special case.

sage: # needs sage.libs.singular
sage: R.<w> = PolynomialRing(ZZ, implementation="singular"); R
Multivariate Polynomial Ring in w over Integer Ring
sage: is_PolynomialRing(R)
False
sage: type(R)
<class 'sage.rings.polynomial.multi_polynomial_libsingular.MPolynomialRing_libsingular'>

sage.rings.polynomial.polynomial_ring.polygen(ring_or_element, name='x')

Return a polynomial indeterminate.

INPUT:

- \texttt{polygen(base\_ring, name="x")}
- \texttt{polygen(ring\_element, name="x")}

If the first input is a ring, return a polynomial generator over that ring. If it is a ring element, return a polynomial generator over the parent of the element.

EXAMPLES:

sage: z = polygen(QQ, 'z')
sage: z^3 + z +1
z^3 + z + 1
sage: parent(z)
Univariate Polynomial Ring in z over Rational Field
Note: If you give a list or comma-separated string to `polygen()`, you’ll get a tuple of indeterminates, exactly as if you called `polygens()`.

```python
sage.rings.polynomial.polynomial_ring.polysgens(base_ring, names='x', *args)
```
Return indeterminates over the given base ring with the given names.

**EXAMPLES:**

```python
sage: x, y, z = polysgens(QQ, 'x,y,z')
sage: (x+y+z)^2
x^2 + 2*x*y + y^2 + 2*x*z + 2*y*z + z^2
sage: parent(x)
Multivariate Polynomial Ring in x, y, z over Rational Field
sage: t = polysgens(QQ, ['x', 'yz', 'abc'])
sage: t
(x, yz, abc)
```

The number of generators can be passed as a third argument:

```python
sage: polysgens(QQ, 'x', 4)
(x0, x1, x2, x3)
```

### 2.1.2 Ring homomorphisms from a polynomial ring to another ring

This module currently implements the canonical ring homomorphism from $A[x]$ to $B[x]$ induced by a ring homomorphism from $A$ to $B$.

**Todo:** Implement homomorphisms from $A[x]$ to an arbitrary ring $R$, given by a ring homomorphism from $A$ to $R$ and the image of $x$ in $R$.

**AUTHORS:**

- Peter Bruin (March 2014): initial version

```python
class sage.rings.polynomial.polynomial_ring_homomorphism.
PolynomialRingHomomorphism_from_base
```
Bases: RingHomomorphism_from_base

The canonical ring homomorphism from $R[x]$ to $S[x]$ induced by a ring homomorphism from $R$ to $S$.

**EXAMPLES:**

```python
sage: ZZ['x'].coerce_map_from(ZZ['x'])
Ring morphism:
  From: Univariate Polynomial Ring in x over Integer Ring
  To:   Univariate Polynomial Ring in x over Rational Field
  Defn: Induced from base ring by
        Natural morphism:
        From: Integer Ring
        To:   Rational Field
```

```python
sage: is_injective()
Return whether this morphism is injective.
```

**EXAMPLES:**

---

Chapter 2. Univariate Polynomials
```
sage: R.<x> = ZZ[

sage: S.<x> = QQ[

sage: R.hom(S).is_injective()
True
```

**is_surjective()**

Return whether this morphism is surjective.

**EXAMPLES:**
```
sage: R.<x> = ZZ[

sage: S.<x> = Zmod(2)[

sage: R.hom(S).is_surjective()
True
```

### 2.1.3 Univariate polynomial base class

**AUTHORS:**
- William Stein: first version
- Martin Albrecht: added singular coercion
- Robert Bradshaw: moved Polynomial_generic_dense to Cython
- Miguel Marco: implemented resultant in the case where PARI fails
- Simon King: used a faster way of conversion from the base ring
- Kwankyu Lee (2013-06-02): enhanced quo_rem()
- Julian Rueth (2012-05-25,2014-05-09): fixed is_squarefree() for imperfect fields, fixed division without remainder over QQbar; added _cache_key for polynomials with unhashable coefficients
- Simon King (2013-10): implemented copying of PolynomialBaseringInjection
- Bruno Grenet (2014-07-13): enhanced quo_rem()
- Kiran Kedlaya (2016-03): added root counting
- Edgar Costa (2017-07): added rational reconstruction
- Kiran Kedlaya (2017-09): added reciprocal transform, trace polynomial
- David Zureick-Brown (2017-09): added is_weil_polynomial
- Sebastian Oehms (2018-10): made roots() and factor() work over more cases of proper integral domains (see github issue #26421)

**class** `sage.rings.polynomial.polynomial_element.ConstantPolynomialSection`

**Bases:** `Map`

This class is used for conversion from a polynomial ring to its base ring.

Since [github issue #9944](https://github.com/sagemath/sage/issues/9944), it calls the `constant_coefficient()` method, which can be optimized for a particular polynomial type.

**EXAMPLES:**
sage: P0.<y_1> = GF(3)[]  
        sage: P1.<y_2,y_1,y_0> = GF(3)[]  
        sage: P0(-y_1)  
        2*y_1  

sage: phi = GF(3).convert_map_from(P0); phi  
Generic map:  
  From: Univariate Polynomial Ring in y_1 over Finite Field of size 3  
  To:   Finite Field of size 3  
        sage: type(phi)  
<class sage.rings.polynomial.polynomial_element.ConstantPolynomialSection>  
        sage: phi(P0.one())  
1  
        sage: phi(y_1)  
Traceback (most recent call last):  
  ...  
TypeError: not a constant polynomial

```
class sage.rings.polynomial.polynomial_element.Polynomial

Bases: CommutativePolynomial

A polynomial.

EXAMPLES:

sage: R.<y> = QQ['y']  
        sage: S.<x> = R['x']  
        sage: S  
Univariate Polynomial Ring in x over Univariate Polynomial Ring in y over Rational Field  
        sage: f = x*y; f  
y*x  
        sage: type(f)  
<class 'sage.rings.polynomial.polynomial_element.Polynomial_generic_dense'>  
        sage: p = (y+1)^10; p(1)  
1024

__add__(right)

Add two polynomials.

EXAMPLES:

sage: R = ZZ['x']  
        sage: p = R([1,2,3,4])  
        sage: q = R([4,-3,2,-1])  
        sage: p + q  
# indirect doctest  
3*x^3 + 5*x^2 - x + 5

__sub__(other)

Default implementation of subtraction using addition and negation.

__lmul__(left)

Multiply self on the left by a scalar.

EXAMPLES:

sage: R.<x> = ZZ[]  
        sage: f = (x^3 + x + 5)  
(continues on next page)
Polynomials, Release 10.3

```
sage: f._lmul_(7)
7*x^3 + 7*x + 35
sage: 7*f
7*x^3 + 7*x + 35
```

_{rmul_} (right)

Multiply self on the right by a scalar.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: f = (x^3 + x + 5)
sage: f._rmul_(7)
7*x^3 + 7*x + 35
sage: f*7
7*x^3 + 7*x + 35
```

_{mul_} (right)

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: (x - 4) * (x^2 - 8*x + 16)
x^3 - 12*x^2 + 48*x - 64
sage: C.<t> = PowerSeriesRing(ZZ)
sage: D.<s> = PolynomialRing(C)
sage: z = (1 + O(t)) + t*s^2
sage: z*z
t^2*s^4 + (2*t + O(t^2))*s^2 + 1 + O(t)
```

## More examples from trac 2943, added by Kiran S. Kedlaya 2 Dec 09

```
sage: C.<t> = PowerSeriesRing(Integers())
sage: D.<s> = PolynomialRing(C)
sage: z = 1 + (t + O(t^2))*s + (t^2 + O(t^3))*s^2
sage: z*z
(t^4 + O(t^5))*s^4 + (2*t^3 + O(t^4))*s^3 + (3*t^2 + O(t^3))*s^2 + (2*t + O(t^\rightarrow 2))*s + 1
```

_{mul_trunc_} (right, n)

Return the truncated multiplication of two polynomials up to \( n \).

This is the default implementation that does the multiplication and then truncate! There are custom implementations in several subclasses:

- on dense polynomial over integers (via FLINT)
- on dense polynomial over \( \mathbb{Z}/n\mathbb{Z} \) (via FLINT)
- on dense rational polynomial (via FLINT)
- on dense polynomial on \( \mathbb{Z}/n\mathbb{Z} \) (via NTL)

EXAMPLES:

```
sage: R = QQ['x']
sage: y = R.gen()
sage: x = R.base_ring().gen()
sage: p1 = 1 - x*y + 2*y**3
sage: p2 = -1/3 + y**5
```

(continues on next page)
Todo: implement a generic truncated Karatsuba and use it here.

\textbf{adams\_operator}(*\texttt{args}, **\texttt{kwds})

Deprecated: Use \texttt{adams\_operator\_on\_roots()} instead. See github issue \#36396 for details.

\textbf{adams\_operator\_on\_roots}(n, \texttt{monic}=False)

Return the polynomial whose roots are the \(n\)-th powers of the roots of \texttt{self}.

\textbf{INPUT:}

- \(n\) – an integer
- \texttt{monic} – boolean (default False) if set to True, force the output to be monic

\textbf{EXAMPLES:}

\begin{verbatim}
sage: # needs sage.libs.pari sage.libs.singular
sage: f = cyclotomic_polynomial(30)
sage: f.adams_operator_on_roots(7) == f
True
sage: f.adams_operator_on_roots(6) == cyclotomic_polynomial(5)**2
True
sage: f.adams_operator_on_roots(10) == cyclotomic_polynomial(3)**4
True
sage: f.adams_operator_on_roots(15) == cyclotomic_polynomial(2)**8
True
sage: f.adams_operator_on_roots(30) == cyclotomic_polynomial(1)**8
True
sage: x = polygen(QQ)
sage: f = x^2 - 2*x + 2
sage: f.adams_operator_on_roots(10) \rightarrow
# needs sage.libs.singular
x^2 + 1024
\end{verbatim}

When \texttt{self} is monic, the output will have leading coefficient ±1 depending on the degree, but we can force it to be monic:

\begin{verbatim}
sage: R.<a,b,c> = ZZ[]
sage: x = polygen(R)
sage: f = (x - a) * (x - b) * (x - c)
sage: f.adams_operator_on_roots(3).factor() \rightarrow
# needs sage.libs.singular
(-1) * (x - c^3) * (x - b^3) * (x - a^3)
sage: f.adams_operator_on_roots(3, monic=True).factor() \rightarrow
# needs sage.libs.singular
(x - c^3) * (x - b^3) * (x - a^3)
\end{verbatim}

\textbf{add\_bigoh}(\texttt{prec})

Return the power series of precision at most \texttt{prec} got by adding \(O(q^{\texttt{prec}})\) to self, where \(q\) is its variable.

\textbf{EXAMPLES:}
Polynomials, Release 10.3

```sage
sage: R.<x> = ZZ[]
sage: f = 1 + 4*x + x^3
sage: f.add_bigoh(7)
1 + 4*x + x^3 + O(x^7)
sage: f.add_bigoh(2)
1 + 4*x + O(x^2)
sage: f.add_bigoh(2).parent()
Power Series Ring in x over Integer Ring
```

**all_roots_in_interval** $(a=None, b=None)$

Return True if the roots of this polynomial are all real and contained in the given interval.

**EXAMPLES:**

```sage
sage: # needs sage.libs.pari
sage: R.<x> = PolynomialRing(ZZ)
sage: pol = (x - 1)^2 * (x - 2)^2 * (x - 3)
sage: pol.all_roots_in_interval(1, 3)
True
sage: pol.all_roots_in_interval(1.01, 3)
False
sage: pol = chebyshev_T(5, x)
sage: pol.all_roots_in_interval(-1, 1)
True
sage: pol = chebyshev_T(5, x/2)
sage: pol.all_roots_in_interval(-1, 1)
False
sage: pol.all_roots_in_interval()
True
```

**any_irreducible_factor** $(degree=None, assume_squarefree=False, assume_equal_deg=False, ext_degree=None)$

Return an irreducible factor of this polynomial.

**INPUT:**

- **degree** (None or positive integer) – (default: None). Used for polynomials over finite fields. If None, returns the first factor found (usually the smallest). Otherwise, attempts to return an irreducible factor of self of chosen degree degree.
- **assume_squarefree** (boolean) – (default: False). Used for polynomials over finite fields. If True, this polynomial is assumed to be squarefree.
- **assume_equal_deg** (boolean) – (default: False). Used for polynomials over finite fields. If True, this polynomial is assumed to be the product of irreducible polynomials of degree equal to degree.
- **ext_degree** – positive integer or None (default); used for polynomials over finite fields. If not None only returns irreducible factors of self whose degree divides ext_degree.

**EXAMPLES:**

```sage
sage: # needs sage.rings.finite_rings
sage: F = GF(163)
sage: R.<x> = F[]
sage: f = (x + 40)^3 * (x^5 + 76*x^4 + 93*x^3 + 112*x^2 + 22*x + 27)^2 * (x^6 -> 50*x^5 + 143*x^4 + 162*x^2 + 109*x + 140)
sage: f.any_irreducible_factor()
```

(continues on next page)
When the polynomial is known to be squarefree we can optimise the call by setting `assume_squarefree` to be `True`:

```
sage: f.any_irreducible_factor(assume_squarefree=True)
x^5 + 76*x^4 + 93*x^3 + 112*x^2 + 22*x + 27
```

If we ask for an irreducible factor which does not exist, the function will throw a `ValueError`:

```
sage: g.any_irreducible_factor(degree=2, assume_squarefree=True)
Traceback (most recent call last):
  ... ValueError: no irreducible factor of degree 2 could be computed from x^4 + 162*x^3 + 7*x^2 + 154*x + 2
```

If we assume that the polynomial is product of irreducible polynomials of the same degree, we must also supply the degree:

```
sage: h.any_irreducible_factor(degree=1, assume_equal_deg=True)
# random
x + 98
```

Also works for extension fields and even characteristic:

```
sage: f.any_irreducible_factor(degree=2)  # random
x^2 + (z4^3 + z4^2 + z4)*x + z4^2 + z4 + 1
```

We can also use this function for polynomials which are not defined over finite fields, but this simply falls back to a slow method of factorisation:
sage: R.<x> = ZZ[]
sage: f = 3*x^4 + 2*x^3
sage: f.any_irreducible_factor()
3*x + 2

any_root (ring=None, degree=None, assume_squarefree=False, assume_equal_deg=False)

Return a root of this polynomial in the given ring.

INPUT:

• ring – The ring in which a root is sought. By default this is the coefficient ring.
• degree (None or nonzero integer) – Used for polynomials over finite fields. Return a root of degree \( \text{abs}(\text{degree}) \) over the ground field. If negative, also assumes that all factors of this polynomial are of degree \( \text{abs}(\text{degree}) \). If None, returns a root of minimal degree contained within the given ring.
• assume_squarefree (bool) – Used for polynomials over finite fields. If True, this polynomial is assumed to be squarefree.
• assume_equal_deg (bool) – Used for polynomials over finite fields. If True, all factors of this polynomial are assumed to have degree \text{degree}. Note that degree must be set.

Warning: Negative degree input will be deprecated. Instead use assume_equal_deg.

Note: For finite fields, any_root() is non-deterministic when finding linear roots of a polynomial over the base ring. However, if degree is greater than one, or ring is an extension of the base ring, then the root computed is found by attempting to return a root after factorisation. Roots found in this way are deterministic. This may change in the future. For all other rings or fields, roots are found by first fully-factoring self and the output is deterministic.

EXAMPLES:

sage: # needs sage.rings.finite_rings
sage: R.<x> = GF(11)[]
sage: f = 7*x^7 + 8*x^6 + 4*x^5 + x^4 + 6*x^3 + 10*x^2 + 8*x + 5
sage: f.any_root()
2
sage: f.factor()
(7) * (x + 9) * (x^6 + 10*x^4 + 6*x^3 + 5*x^2 + 2*x + 2)
sage: f = x^6 + 10*x^4 + 6*x^3 + 5*x^2 + 2*x + 2
sage: root = f.any_root(GF(11^6, a))
sage: roots = sorted(f.roots(GF(11^6, a), multiplicities=False))
sage: roots
[10*a^5 + 2*a^4 + 8*a^3 + 9*a^2 + a, a^5 + a^4 + 7*a^3 + 2*a^2 + 10*a, 2*a^5 + 8*a^4 + 3*a^3 + 6*a + 2, a^5 + 3*a^4 + 8*a^3 + 2*a^2 + 3*a + 4, 10*a^5 + 3*a^4 + 8*a^3 + a^2 + 10*a + 4]
sage: root in roots
True

sage: # needs sage.rings.finite_rings
sage: g = (x-1) * (x^2 + 3*x + 9) * (x^5 + 5*x^4 + 8*x^3 + 5*x^2 + 3*x + 5)

(continues on next page)
sage: g.any_root(ring=GF(11^10, 'b'), degree=1)
1
sage: root = g.any_root(ring=GF(11^10, 'b'), degree=2)
sage: roots = (x^2 + 3*x + 9).roots(ring=GF(11^10, 'b'), multiplicities=False)
sage: root in roots
True
sage: root = g.any_root(ring=GF(11^10, 'b'), degree=5)
sage: roots = (x^5 + 5*x^4 + 8*x^3 + 5*x^2 + 3*x + 5).roots(ring=GF(11^10, 'b'), multiplicities=False)
sage: root in roots
True

**args()**

Return the generator of this polynomial ring, which is the (only) argument used when calling self.

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: x.args()
(x,)
```

A constant polynomial has no variables, but still takes a single argument.

```python
sage: R(2).args()
(x,)
```

**base_extend(R)**

Return a copy of this polynomial but with coefficients in \( R \), if there is a natural map from the coefficient ring of \( \text{self} \) to \( R \).

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: f = x^3 - 17*x + 3
sage: f.base_extend(GF(7))
Traceback (most recent call last):
  ... TypeError: no such base extension
sage: f.change_ring(GF(7))
x^3 + 4*x + 3
```

**base_ring()**

Return the base ring of the parent of \( \text{self} \).

EXAMPLES:

```python
sage: R.<x> = ZZ[]
sage: x.base_ring()
Integer Ring
sage: (2*x + 3).base_ring()
Integer Ring
```

**change_ring(R)**

Return a copy of this polynomial but with coefficients in \( R \), if at all possible.

INPUT:

- \( R \) - a ring or morphism.
polynomials, release 10.3

examples:

```python
sage: K.<z> = CyclotomicField(3)  # needs sage.rings.number_field
sage: f = K.defining_polynomial()  # needs sage.rings.number_field
sage: f.change_ring(GF(7))  # needs sage.rings.finite_rings sage.rings.number_field
x^2 + x + 1
```

```python
sage: # needs sage.rings.number_field
sage: K.<z> = CyclotomicField(3)
sage: R.<x> = K[]
sage: f = x^2 + z
sage: f.change_ring(K.embeddings(CC)[1])  # needs sage.rings.real_mpfr
x^2 - 0.500000000000000 - 0.866025403784438*I
```

```python
sage: R.<x> = QQ[]
sage: f = x^2 + 1
sage: f.change_ring(QQ.embeddings(CC)[0])  # needs sage.rings.real_mpfr
x^2 + 1.00000000000000
```

**change_variable_name** *(var)*

Return a new polynomial over the same base ring but in a different variable.

**EXAMPLES:**

```python
sage: x = polygen(QQ, 'x')
sage: f = -2/7*x^3 + (2/3)*x - 19/993; f
-2/7*x^3 + 2/3*x - 19/993
sage: f.change_variable_name('theta')
-2/7*theta^3 + 2/3*theta - 19/993
```

**coefficients** *(sparse=True)*

Return the coefficients of the monomials appearing in `self`.

If `sparse=True` (the default), it returns only the non-zero coefficients. Otherwise, it returns the same value as `self.list()` (In this case, it may be slightly faster to invoke `self.list()` directly.) In either case, the coefficients are ordered by increasing degree.

**EXAMPLES:**

```python
sage: _.<x> = PolynomialRing(ZZ)
sage: f = 3*x^4 + 2*x^2 + 1
sage: f.coefficients()
[1, 2, 3]
sage: f.coefficients(sparse=False)
[1, 0, 2, 0, 3]
```

**complex_roots** *

Return the complex roots of this polynomial, without multiplicities.

Calls `self.roots(ring=CC)`, unless this is a polynomial with floating-point coefficients, in which case it is uses the appropriate precision from the input coefficients.

**EXAMPLES:**

```python
```
sage: # needs sage.libs.pari sage.rings.real_mpfr
sage: x = polygen(ZZ)

sage: (x^3 - 1).complex_roots()  # note: low order bits slightly different...
→ on ppc.
[1.00000000000000,
 -0.500000000000000 - 0.86602540378443...*I,
 -0.500000000000000 + 0.86602540378443...*I]

\textbf{ compose_power } (k, algorithm=None, monic=False)

Return the $k$-th iterate of the composed product of this polynomial with itself.

**INPUT:**

- $k$ – a non-negative integer
- algorithm=\text{None (default)}, "resultant" or "BFSS". See \text{composed_op()}
- monic=\text{False (default)} or True. See \text{composed_op()}

**OUTPUT:**

The polynomial of degree $d^k$ where $d$ is the degree, whose roots are all $k$-fold products of roots of this polynomial. That is, $f \ast f \ast \cdots \ast f$ where this is $f$ and $f \ast f = f.\text{composed}_\text{op}(f, \text{operator}_\text{mul})$.

**EXAMPLES:**

\begin{verbatim}
sage: R.<a,b,c> = ZZ[]
sage: x = polygen(R)
sage: f = (x - a) * (x - b) * (x - c)
sage: f.compose_power(2).factor()  # needs sage.libs.singular sage.modules
(x - c^2) * (x - b^2) * (x - a^2) * (x - b*c)^2 * (x - a*c)^2 * (x - a*b)^2

sage: # needs sage.libs.singular sage.modules
sage: x = polygen(QQ)
sage: f = x^2 - 2*x + 2
sage: f2 = f.compose_power(2); f2
x^4 - 4*x^3 + 8*x^2 - 16*x + 16
sage: f2 == f.composed_op(f, operator.mul)
True
sage: f3 = f.compose_power(3); f3
x^8 - 8*x^7 + 32*x^6 - 64*x^5 + 128*x^4 - 512*x^3 + 2048*x^2 - 4096*x + 4096
sage: f3 == f2.composed_op(f, operator.mul)
True
sage: f4 = f.compose_power(4)
sage: f4 == f3.composed_op(f, operator.mul)
True
\end{verbatim}

\textbf{ compose_trunc } (other, n)

Return the composition of self and other, truncated to $O(x^n)$.

This method currently works for some specific coefficient rings only.

**EXAMPLES:**

\begin{verbatim}
sage: Pol.<x> = CBF[]  # needs sage.libs.flint
sage: (1 + x + x^2/2 + x^3/6 + x^4/24 + x^5/120).compose_trunc(1 + x, 2)  # needs sage.libs.flint
([2.70833333333333 +/- ...e-16])*x + [2.7166666666667 +/- ...e-15]
\end{verbatim}
sage: Pol.<x> = QQ['y'][]
sage: (1 + x + x^2/2 + x^3/6 + x^4/24 + x^5/120).compose_trunc(1 + x, 2)
Traceback (most recent call last):
...  
NotImplementedError: truncated composition is not implemented
for this subclass of polynomials

composed_op(p1, p2, op=operator.OP, algorithm=None, monic=False)

Return the composed sum, difference, product or quotient of this polynomial with another one.

In the case of two monic polynomials \( p_1 \) and \( p_2 \) over an integral domain, the composed sum, difference, etc.
are given by

\[
\prod_{p_1(a)=p_2(b)=0} (x - (a \cdot b)), \quad * \in \{+, -, \times, /\}
\]

where the roots \( a \) and \( b \) are to be considered in the algebraic closure of the fraction field of the coefficients and
counted with multiplicities. If the polynomials are not monic this quantity is multiplied by \( \alpha_1^{\deg(p_2)} \alpha_2^{\deg(p_1)} \)
where \( \alpha_1 \) and \( \alpha_2 \) are the leading coefficients of \( p_1 \) and \( p_2 \) respectively.

INPUT:

- \( p2 \) – univariate polynomial belonging to the same polynomial ring as this polynomial
- \( op \) – operator.OP where OP=add or sub or mul or truediv.
- \( algorithm \) – can be "resultant" or "BFSS"; by default the former is used when the polynomials
  have few nonzero coefficients and small degrees or if the base ring is not Z or Q. Otherwise the latter is
  used.
- \( monic \) – whether to return a monic polynomial. If True the coefficients of the result belong to the
  fraction field of the coefficients.

ALGORITHM:

The computation is straightforward using resultants. Indeed for the composed sum it would be \( Res_y(p_1(x - y), p_2(y)) \). However, the method from [BFSS2006] using series expansions is asymptotically much faster.

Note that the algorithm BFSS with polynomials with coefficients in Z needs to perform operations over Q.

Todo:

- The [BFSS2006] algorithm has been implemented here only in the case of polynomials over rationals.
  For other rings of zero characteristic (or if the characteristic is larger than the product of the degrees), one
  needs to implement a generic method _exp_series. In the general case of non-zero characteristic
  there is an alternative algorithm in the same paper.
- The Newton series computation can be done much more efficiently! See [BFSS2006].

EXAMPLES:

sage: x = polygen(ZZ)
sage: p1 = x^2 - 1
sage: p2 = x^4 - 1
sage: p1.composed_op(p2, operator.add)  #...
˓→ needs sage.libs.singular

\( x^8 - 4x^6 + 4x^4 - 16x^2 \)
This function works over any field. However for base rings other than \( \mathbb{Z} \) and \( \mathbb{Q} \) only the resultant algorithm is available:

```python
 sage: x = polygen(QQbar)
sage: p1 = x**2 - AA(2).sqrt()
sage: p2 = x**3 - AA(3).sqrt()
sage: r1 = p1.roots(multiplicities=False)
sage: r2 = p2.roots(multiplicities=False)
sage: p = p1.composed_op(p2, operator.add); p
x^6 - 4.242640687119285?*x^4 - 3.464101615137755?*x^3 + 6*x^2
- 14.69693845669907?*x + 0.1715728752538099?
sage: all(p(x+y).is_zero() for x in r1 for y in r2)
True
```

```python
 sage: x = polygen(GF(2))
sage: p1 = x**2 + x - 1
sage: p2 = x**3 + x - 1
sage: p_add = p1.composed_op(p2, operator.add); p_add
x^6 + x^5 + x^3 + x^2 + 1
```

constant_coefficient()

Return the constant coefficient of this polynomial.

OUTPUT: element of base ring

EXAMPLES:
sage: R.<x> = QQ[]
sage: f = -2*x^3 + 2*x - 1/3
sage: f.constant_coefficient()
-1/3

content_ideal()

Return the content ideal of this polynomial, defined as the ideal generated by its coefficients.

EXAMPLES:

sage: R.<x> = IntegerModRing(4)[]
sage: f = x^4 + 3*x^2 + 2
sage: f.content_ideal()
Ideal (2, 3, 1) of Ring of integers modulo 4

When the base ring is a gcd ring, the content as a ring element is the generator of the content ideal:

sage: R.<x> = ZZ[]
sage: f = 2*x^3 - 4*x^2 + 6*x - 10
sage: f.content_ideal().gen()
2

cyclotomic_part()

Return the product of the irreducible factors of this polynomial which are cyclotomic polynomials.

The algorithm assumes that the polynomial has rational coefficients.

See also:

is_cyclotomic() is_cyclotomic_product() has_cyclotomic_factor()

EXAMPLES:

sage: P.<x> = PolynomialRing(Integers())
sage: pol = 2*(x^4 + 1)
sage: pol.cyclotomic_part()
x^4 + 1
sage: pol = x^4 + 2
sage: pol.cyclotomic_part()
1
sage: pol = (x^4 + 1)^2 * (x^4 + 2)
sage: pol.cyclotomic_part()
x^8 + 2*x^4 + 1

sage: P.<x> = PolynomialRing(QQ)
sage: pol = (x^4 + 1)^2 * (x^4 + 2)
sage: pol.cyclotomic_part()
x^8 + 2*x^4 + 1
sage: pol = (x - 1) * x * (x + 2)
sage: pol.cyclotomic_part()
x - 1

degree (gen=None)

Return the degree of this polynomial. The zero polynomial has degree −1.

EXAMPLES:
AUTHORS:

- Naqi Jaffery (2006-01-24): examples

\texttt{denominator()} 

Return a denominator of \texttt{self}.

First, the lcm of the denominators of the entries of \texttt{self} is computed and returned. If this computation fails, the unit of the parent of \texttt{self} is returned.

Note that some subclasses may implement their own \texttt{denominator()} method. For example, see \texttt{sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint}

\textbf{Warning:} This is not the denominator of the rational function defined by \texttt{self}, which would always be 1 since \texttt{self} is a polynomial.

EXAMPIES:

First we compute the denominator of a polynomial with integer coefficients, which is of course 1.

\begin{verbatim}
sage: R.<x> = ZZ[]
sage: f = x^3 + 17*x + 1
sage: f.denominator()
1
\end{verbatim}

Next we compute the denominator of a polynomial with rational coefficients.

\begin{verbatim}
sage: R.<x> = PolynomialRing(QQ)
sage: f = (1/17)*x^19 - (2/3)*x + 1/3; f
1/17*x^19 - 2/3*x + 1/3
sage: f.denominator()
51
\end{verbatim}

Finally, we try to compute the denominator of a polynomial with coefficients in the real numbers, which is a ring whose elements do not have a \texttt{denominator()} method.
```python
sage: # needs sage.rings.real_mpfr
sage: R.<x> = RR[]
sage: f = x + RR('0.3'); f
x + 0.300000000000000
sage: f.denominator()
1.00000000000000
```

Check that the denominator is an element over the base whenever the base has no `denominator()` method. This closes github issue #9063.

```python
sage: R.<a> = GF(5)[]
sage: x = R(0)
sage: x.denominator()
1
sage: type(x.denominator())
<class 'sage.rings.finite_rings.integer_mod.IntegerMod_int'>
sage: isinstance(x.numerator() / x.denominator(), Polynomial)
True
sage: isinstance(x.numerator() / R(1), Polynomial)
False
```

**derivative(*args)**

The formal derivative of this polynomial, with respect to variables supplied in `args`.

Multiple variables and iteration counts may be supplied; see documentation for the global `derivative()` function for more details.

See also:

`_derivative()`

**EXAMPLES:**

```python
sage: R.<x> = PolynomialRing(QQ)
sage: g = -x^4 + x^2/2 - x
sage: g.derivative()
-4*x^3 + x - 1
sage: g.derivative(x)
-4*x^3 + x - 1
sage: g.derivative(x, x)
-12*x^2 + 1
sage: g.derivative(x, 2)
-12*x^2 + 1
sage: R.<t> = PolynomialRing(ZZ)
sage: S.<x> = PolynomialRing(R)
sage: f = t^3*x^2 + t^4*x^3
sage: f.derivative()
3*t^4*x^2 + 2*t^3*x
sage: f.derivative(x)
3*t^4*x^2 + 2*t^3*x
sage: f.derivative(t)
4*t^3*x^3 + 3*t^2*x^2
```

**dict()**

Return a sparse dictionary representation of this univariate polynomial.

**EXAMPLES:**
**diff** (*args*)

The formal derivative of this polynomial, with respect to variables supplied in `args`.

Multiple variables and iteration counts may be supplied; see documentation for the global `derivative()` function for more details.

**See also:**

 `_derivative()`

**EXAMPLES:**

```python
sage: R.<x> = QQ[]
sage: f = x^3 + -1/7*x + 13
sage: f.dict()
{0: 13, 1: -1/7, 3: 1}
```

```python
diff(*args)
```

The formal derivative of this polynomial, with respect to variables supplied in `args`.

Multiple variables and iteration counts may be supplied; see documentation for the global `derivative()` function for more details.

**See also:**

 `_derivative()`

**EXAMPLES:**

```python
sage: R.<x> = PolynomialRing(QQ)
sage: g = -x^4 + x^2/2 - x
sage: g.derivative()
-4*x^3 + x - 1
```

```python
differentiate(*args)
```

The formal derivative of this polynomial, with respect to variables supplied in `args`.

Multiple variables and iteration counts may be supplied; see documentation for the global `derivative()` function for more details.

**See also:**

 `_derivative()`

**EXAMPLES:**

```python
sage: R.<x> = PolynomialRing(QQ)
sage: g = -x^4 + x^2/2 - x
sage: g.derivative()
-4*x^3 + x - 1
```

```python
differentiate(*args)
```
The discriminant is

\[ R_n := a_n^{2n-2} \prod_{1 \leq i < j \leq n} (r_i - r_j)^2, \]

where \( n \) is the degree of \( \text{self} \), \( a_n \) is the leading coefficient of \( \text{self} \), and the roots of \( \text{self} \) are \( r_1, \ldots, r_n \).

**OUTPUT:** An element of the base ring of the polynomial ring.

**ALGORITHM:**

Uses the identity \( R_n(f) := (-1)^{n(n-1)/2} R(f, f') a_n^{n-k-2} \), where \( n \) is the degree of \( \text{self} \), \( a_n \) is the leading coefficient of \( \text{self} \), \( f' \) is the derivative of \( f \), and \( k \) is the degree of \( f' \). Calls `resultant()`.

**EXAMPLES:**

In the case of elliptic curves in special form, the discriminant is easy to calculate:

\[
sage: R.<x> = QQ[]
sage: f = x^3 + x + 1
\]

\[
sage: d = f.discriminant(); d
\]

\[
\# \text{needs sage.libs.pari}
\]

\[
-31
\]

\[
\text{d.parent()} \text{ is QQ}
\]

\[
\# \text{needs sage.libs.pari}
\]

\[
True
\]

\[
\text{EllipticCurve([1, 1]).discriminant()}/16
\]

\[
\# \text{needs sage.libs.pari sage.schemes}
\]

\[
-31
\]

We can compute discriminants over univariate and multivariate polynomial rings:

\[
sage: R.<a> = QQ[]
sage: S.<x> = R[]
sage: f = a*x + x + a + 1
\]

\[
sage: d = f.discriminant(); d
\]

\[
\# \text{needs sage.libs.pari}
\]

\[
-116
\]

(continues on next page)
Polynomials, Release 10.3

(continued from previous page)

\[ \text{\texttt{sage}: d.parent() \texttt{is R}} \]  
\[ \text{\texttt{# needs sage.libs.pari}} \]

\[ \text{\texttt{True}} \]

\[ \text{\texttt{sage}: R.<a, b> = QQ[]} \]
\[ \text{\texttt{sage}: S.<x> = R[]} \]
\[ \text{\texttt{sage}: f = x^2 + a + b} \]
\[ \text{\texttt{sage}: d = f.discriminant(); d} \]
\[ \text{\texttt{# needs sage.libs.pari}} \]
\[ \text{\texttt{-4*a - 4*b}} \]
\[ \text{\texttt{# needs sage.libs.pari}} \]
\[ \text{\texttt{True}} \]

\[ \text{\texttt{ dispersion \ (other=\texttt{None})}} \]

Compute the dispersion of a pair of polynomials.

The dispersion of \(f\) and \(g\) is the largest nonnegative integer \(n\) such that \(f(x + n)\) and \(g(x)\) have a nonconstant common factor.

When \texttt{other} is \texttt{None}, compute the auto-dispersion of \texttt{self}, i.e., its dispersion with itself.

See also:

\[ \text{\texttt{dispersion\_set()}} \]

EXAMPLES:

\[ \text{\texttt{sage}: Pol.<x> = QQ[]} \]
\[ \text{\texttt{sage}: x.dispersion(x + 1)} \]
\[ \text{\texttt{# needs sage.libs.pari}} \]
\[ \text{\texttt{1}} \]
\[ \text{\texttt{sage}: (x + 1).dispersion(x)} \]
\[ \text{\texttt{# needs sage.libs.pari}} \]
\[ \text{\texttt{-Infinity}} \]

\[ \text{\texttt{sage}: \texttt{\# needs sage.libs.pari sage.rings.number_field sage.symbolic}} \]
\[ \text{\texttt{sage}: Pol.<x> = QQbar[]} \]
\[ \text{\texttt{sage}: pol = Pol([sqrt(5), 1, 3/2])} \]
\[ \text{\texttt{sage}: pol.dispersion()} \]
\[ \text{\texttt{0}} \]
\[ \text{\texttt{sage}: (pol\pol(x+3)).dispersion()} \]
\[ \text{\texttt{3}} \]

\[ \text{\texttt{ dispersion\_set \ (other=\texttt{None})}} \]

Compute the dispersion set of two polynomials.

The dispersion set of \(f\) and \(g\) is the set of nonnegative integers \(n\) such that \(f(x + n)\) and \(g(x)\) have a nonconstant common factor.

When \texttt{other} is \texttt{None}, compute the auto-dispersion set of \texttt{self}, i.e., its dispersion set with itself.

ALGORITHM:

See Section 4 of Man & Wright [MW1994].
See also:

\texttt{dispersion()}

EXAMPLES:

\begin{verbatim}
sage: Pol.<x> = QQ[]
sage: x.dispersion_set(x + 1)  # needs sage.libs.pari
[1]
sage: (x + 1).dispersion_set(x)  # needs sage.libs.pari
[]
sage: pol = x^3 + x - 7
sage: (pol*pol(x+3)^2).dispersion_set()  # needs sage.libs.pari
[0, 3]
\end{verbatim}

\texttt{divides}(\textit{p})

Return True if this polynomial divides \textit{p}.

This method is only implemented for polynomials over an integral domain.

EXAMPLES:

\begin{verbatim}
sage: R.<x> = ZZ[]
sage: (2*x + 1).divides(4*x**2 - 1)
True
sage: (2*x + 1).divides(4*x**2 + 1)
False
sage: (2*x + 1).divides(R(0))
True
sage: R(0).divides(2*x + 1)
False
sage: R(0).divides(R(0))
True
sage: S.<y> = R[]
sage: p = x * y**2 + (2*x + 1) * y + x + 1
sage: q = (x + 1) * y + (3*x + 2)
sage: q.divides(p)
False
sage: q.divides(p * q)
True
sage: R.<x> = Zmod(6)[]
sage: p = 4*x + 3
sage: q = 5*x**2 + x + 2
sage: q.divides(p)
False
sage: p.divides(q)
False
\end{verbatim}

\texttt{euclidean_degree()}

Return the degree of this element as an element of an Euclidean domain.

If this polynomial is defined over a field, this is simply its \texttt{degree()}.

EXAMPLES:
Polynomials, Release 10.3

```python
sage: R.<x> = QQ[]
sage: x.euclidean_degree()
1
sage: R.<x> = ZZ[]
sage: x.euclidean_degree()
Traceback (most recent call last):
  ... Not Implemented Error
```

**exponents()**

Return the exponents of the monomials appearing in `self`.

**EXAMPLES:**

```python
sage: _.<x> = PolynomialRing(ZZ)
sage: f = x^4 + 2*x^2 + 1
sage: f.exponents()
[0, 2, 4]
```

**factor(**kwargs**)**

Return the factorization of `self` over its base ring.

**INPUT:**

- `kwargs` — any keyword arguments are passed to the method `_factor_univariate_polynomial()` of the base ring if it defines such a method.

**OUTPUT:**

A factorization of `self` over its parent into a unit and irreducible factors. If the parent is a polynomial ring over a field, these factors are monic.

**EXAMPLES:**

Factorization is implemented over various rings. Over `Q`:

```python
sage: x = QQ['x'].0
sage: f = (x^3 - 1)^2
sage: f.factor()  # needs sage.libs.pari
(x - 1)^2 * (x^2 + x + 1)^2
```

Since `Q` is a field, the irreducible factors are monic:

```python
sage: f = 10*x^5 - 1
sage: f.factor()  # needs sage.libs.pari
(10) * (x - 1) * (x^4 + x^3 + x^2 + x + 1)
```

Over `Z` the irreducible factors need not be monic:

```python
sage: x = ZZ['x'].0
sage: f = 10*x^5 - 1
sage: f.factor()  # needs sage.libs.pari
(10) * (x - 1) * (x^4 + x^3 + x^2 + x + 1)
```

(continues on next page)
We factor a non-monic polynomial over a finite field of 25 elements:

```
 sage: k.<a> = GF(25)
 sage: R.<x> = k[]
 sage: f = 2*x^10 + 2*x + 2*a
 sage: F = f.factor(); F
 (2) * (x + a + 2) * (x^2 + 3*x + 4*a + 4) * (x^2 + (a + 1)*x + a + 2) *
  (x^5 + (3*a + 4)*x^4 + (3*a + 3)*x^3 + 2*a*x^2 + (3*a + 1)*x + 3*a + 1)
```

Notice that the unit factor is included when we multiply $F$ back out:

```
 sage: expand(F)
 2*x^10 + 2*x + 2*a
```

A new ring. In the example below, we set the special method `_factor_univariate_polynomial()` in the base ring which is called to factor univariate polynomials. This facility can be used to easily extend polynomial factorization to work over new rings you introduce:

```
 sage: R.<x> = PolynomialRing(IntegerModRing(4), implementation="NTL")
 sage: (x^2).factor()
 Traceback (most recent call last):
   ...
 NotImplementedError: factorization of polynomials over rings with
 composite characteristic is not implemented
 sage: def my_factor(f):
     ....:     return f.change_ring(IntegerModRing(2)).factor()
 sage: R.base_ring()._factor_univariate_polynomial = my_factor
 sage: (x^2).factor()
 x^2
 sage: del R.base_ring()._factor_univariate_polynomial
```

Arbitrary precision real and complex factorization:

```
 sage: R.<x> = RealField(100)[]
 sage: F = factor(x^2 - 3); F
 (x - 1.7320508075688772935274463415) * (x + 1.7320508075688772935274463415)
 sage: expand(F)
 x^2 - 3.0000000000000000000000000000
 sage: factor(x^2 + 1)
 x^2 + 1.0000000000000000000000000000
 sage: R.<x> = ComplexField(100)[]
 sage: F = factor(x^2 + 3); F
 (x - 1.7320508075688772935274463415*I) * (x + 1.7320508075688772935274463415*I)
 sage: expand(F)
 x^2 + 3.0000000000000000000000000000
```
sage: factor(x^2 + 1)
(x - I) * (x + I)
sage: f = R(I) * (x^2 + 1); f
I*x^2 + I
sage: F = factor(f); F
(1.00000000000000000000000000000*I) * (x - I) * (x + I)
sage: expand(F)
I*x^2 + I

Over a number field:

sage: # needs sage.rings.number_field
sage: K.<z> = CyclotomicField(15)
sage: x = polygen(K)
sage: ((x^3 + z*x + 1)^3 * (x - z)).factor()
(x - z) * (x^3 + z*x + 1)^3
sage: cyclotomic_polynomial(12).change_ring(K).factor()
(x^2 - z^5 - 1) * (x^2 + z^5)

Over a relative number field:

sage: # needs sage.rings.number_field
sage: x = polygen(QQ)
sage: K.<z> = CyclotomicField(3)
sage: L.<a> = K.extension(x^3 - 2)
sage: t = polygen(L, 't')
sage: f = (t^3 + t + a) * (t^5 + t + z); f
t^8 + t^6 + a*t^5 + t^4 + z*t^3 + t^2 + (a + z)*t + z*a
sage: f.factor()
(t^3 + t + a) * (t^5 + t + z)

Over the real double field:

sage: # needs numpy
sage: R.<x> = RDF[]
sage: (-2*x^2 - 1).factor()
(-2.0) * (x^2 + 0.500000000000000000000000000001)
sage: (-2*x^2 - 1).factor().expand()
-2.0*x^2 - 1.000000000000000000000000000002
sage: f = (x - 1)^3
sage: f.factor()  # abs tol 2e-5
(x - 1.00000065719436413) * (x^2 - 1.9999934280563585*x + 0.9999934280995487)

The above output is incorrect because it relies on the roots() method, which does not detect that all the roots are real:

sage: f.roots()  # abs tol 2e-5
→ needs numpy
[(1.00000065719436413, 1)]

Over the complex double field the factors are approximate and therefore occur with multiplicity 1:
sage: # needs numpy sage.rings.complex_double
sage: R.<x> = CDF[]

sage: f = (x^2 + 2*R(I))^3

sage: F = f.factor()

sage: F
(x - 1.00000138879287663 + 1.00000013435286879*I) * (x - 0.99999421966864997 + 0.9999873009803959*I) * (x - 0.9999918923847313 + 1.00000695624138*I) * (x + 1.00000105947233 + 1.0000006965644053*I)

sage: [f(t[0][0]).abs() for t in F] # abs tol 1e-13
[1.979365054e-14, 1.97936298566e-14, 1.97936990747e-14, 3.6812407475e-14, 3.65211563729e-14, 3.65220890052e-14]

Factoring polynomials over \( \mathbb{Z}/n\mathbb{Z} \) for composite \( n \) is not implemented:

sage: R.<x> = PolynomialRing(Integers(35))

sage: f = (x^2 + 2*x + 2) * (x^2 + 3*x + 9)

sage: f.factor()
Traceback (most recent call last):
...
NotImplementedError: factorization of polynomials over rings with composite characteristic is not implemented

Factoring polynomials over the algebraic numbers (see github issue #8544):

sage: R.<x> = QQbar[]

sage: (x^8 - 1).factor() # needs sage.rings.number_field
(x - 1) * (x - 0.7071067811865475? - 0.7071067811865475?*I) * (x - 0.7071067811865475? + 0.7071067811865475?*I) * (x - I) * (x + I) * (x + 0.7071067811865475? - 0.7071067811865475?*I) * (x + 0.7071067811865475? + 0.7071067811865475?*I) * (x + 1)

Factoring polynomials over the algebraic reals (see github issue #8544):

sage: R.<x> = AA[]

sage: (x^8 + 1).factor() # needs sage.rings.number_field
(x^2 - 1.8477590650225747?*x + 1.000000000000000?) * (x^2 - 0.7653668647301795?*x + 1.000000000000000?) * (x^2 + 0.7653668647301795?*x + 1.000000000000000?) * (x^2 + 1.8477590650225747?*x + 1.000000000000000?)

**gcd**

Return a greatest common divisor of this polynomial and other.

**OUTPUT:**

A greatest common divisor as a polynomial in the same ring as this polynomial. If the base ring is a field, the return value is a monic polynomial.

### 2.1. Univariate Polynomials and Polynomial Rings

55
Note: The actual algorithm for computing greatest common divisors depends on the base ring underlying the polynomial ring. If the base ring defines a method \_gcd_univariate_polynomial(), then this method will be called (see examples below).

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: (2*x^2).gcd(2*x)
x
sage: R.zero().gcd(0)
0
sage: (2*x).gcd(0)
x
```

One can easily add gcd functionality to new rings by providing a method \_gcd_univariate_polynomial:

```python
sage: # needs sage.rings.number_field sage.symbolic
sage: O = ZZ[-sqrt(5)]
sage: R.<x> = O[]
sage: a = O.1
sage: p = x + a
sage: q = x^2 - 5
sage: p.gcd(q)
Traceback (most recent call last):
...
NotImplementedError: Order of conductor 2 generated by a in Number Field in a with defining polynomial x^2 - 5 with a = -2.236067977499790 does not provide a gcd implementation for univariate polynomials
sage: S.<x> = O.number_field()[]
sage: O._gcd_univariate_polynomial = lambda f, g: R(S(f).gcd(S(g)))
sage: p.gcd(q)
x + a
sage: del O._gcd_univariate_polynomial
```

Use multivariate implementation for polynomials over polynomials rings:

```python
sage: R.<x> = ZZ[]
sage: S.<y> = R[]
sage: T.<z> = S[]
sage: r = 2*x*y + z
sage: p = r * (3*x*y*z - 1)
sage: q = r * (x + y + z - 2)
sage: p.gcd(q)  # needs sage.libs.singular
z + 2*x*y
```

```python
sage: R.<x> = QQ[]
sage: S.<y> = R[]
sage: r = 2*x*y + 1
sage: p = r * (x - 1/2 * y)
sage: q = r * (x*y^2 - x + 1/3)
sage: p.gcd(q)  # needs sage.libs.singular
2*x*y + 1
```

\texttt{global}\_\texttt{height}(\texttt{prec}=\texttt{None})
Return the (projective) global height of the polynomial.
This returns the absolute logarithmic height of the coefficients thought of as a projective point.

INPUT:

• \texttt{prec} – desired floating point precision (default: default \texttt{RealField} precision).

OUTPUT: a real number.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: f = 3*x^3 + 2*x^2 + x
sage: exp(f.global_height())
# needs sage.symbolic
3.00000000000000
```

Scaling should not change the result:

```
sage: R.<x> = PolynomialRing(QQ)
sage: f = 1/25*x^2 + 25/3*x + 1
sage: f.global_height()
# needs sage.symbolic
6.43775164973640
sage: g = 100 * f
sage: g.global_height()
# needs sage.symbolic
6.43775164973640
```

```
sage: R.<x> = PolynomialRing(QQbar)
# needs sage.rings.number_field
sage: f = QQbar(i)*x^2 + 3*x
# needs sage.rings.number_field
sage: f.global_height() 
1.09861228866811
```

```
sage: # needs sage.rings.number_field
sage: R.<x> = PolynomialRing(QQ)
# needs sage.rings.number_field
sage: K.<k> = NumberField(x^2 + 5)
# needs sage.rings.number_field
sage: T.<t> = PolynomialRing(K)
# needs sage.rings.number_field
sage: f = 1/1331 * t^2 + 5 * t + 7
sage: f.global_height() 
9.13959596745043
```

```
sage: R.<x> = QQ[]
# needs sage.rings.number_field
sage: f = 1/123*x^2 + 12
sage: f.global_height(prec=2)
# needs sage.symbolic
8.0
```

```
sage: R.<x> = QQ[]
# needs sage.rings.number_field
sage: f = 0*x
# needs sage.rings.real_mpfr
sage: f.global_height() 
0.000000000000000
```
gradient()

Return a list of the partial derivative of self with respect to the variable of this univariate polynomial.

There is only one partial derivative.

EXAMPLES:

```sage
sage: P.<x> = QQ[
sage: f = x^2 + (2/3)*x + 1
sage: f.gradient()
[2*x + 2/3]
sage: f = P(1)
sage: f.gradient()
[0]
```

hamming_weight()

Return the number of non-zero coefficients of self.

Also called weight, Hamming weight or sparsity.

EXAMPLES:

```sage
sage: R.<x> = ZZ[
\[119x695]sage: f = x^3 - x
sage: f.number_of_terms()
2
sage: R(0).number_of_terms()
0
sage: f = (x + 1)^100
sage: f.number_of_terms()
101
sage: S = GF(5)["y"]
\[119x695]sage: S(f).number_of_terms()
5
sage: cyclotomic_polynomial(105).number_of_terms()
33
```

The method hamming_weight() is an alias:

```sage
sage: f.hamming_weight()
101
```

has_cyclotomic_factor()

Return True if the given polynomial has a nontrivial cyclotomic factor.

The algorithm assumes that the polynomial has rational coefficients.

If the polynomial is known to be irreducible, it may be slightly more efficient to call is_cyclotomic() instead.

See also:

is_cyclotomic() is_cyclotomic_product() cyclotomic_part()

EXAMPLES:

```sage
sage: pol.<x> = PolynomialRing(Rationals())
sage: u = x^5 - 1; u.has_cyclotomic_factor()
True
sage: u = x^5 - 2; u.has_cyclotomic_factor()
```

(continues on next page)
homogenize\( (\text{var}='h') \)

Return the homogenization of this polynomial.

The polynomial itself is returned if it is homogeneous already. Otherwise, its monomials are multiplied with the smallest powers of \text{var} such that they all have the same total degree.

INPUT:

\begin{itemize}
  \item \text{var} – a variable in the polynomial ring (as a string, an element of the ring, or 0) or a name for a new variable (default: 'h')
\end{itemize}

OUTPUT:

If \text{var} specifies the variable in the polynomial ring, then a homogeneous element in that ring is returned. Otherwise, a homogeneous element is returned in a polynomial ring with an extra last variable \text{var}.

EXAMPLES:

\begin{verbatim}
sage: R.<x> = QQ[]
sage: f = x^2 + 1
sage: f.homogenize()
x^2 + h^2
\end{verbatim}

The parameter \text{var} can be used to specify the name of the variable:

\begin{verbatim}
sage: g = f.homogenize('z'); g
x^2 + z^2
sage: g.parent()
Multivariate Polynomial Ring in x, z over Rational Field
\end{verbatim}

However, if the polynomial is homogeneous already, then that parameter is ignored and no extra variable is added to the polynomial ring:

\begin{verbatim}
sage: f = x^2
sage: g = f.homogenize('z'); g
x^2
sage: g.parent()
Univariate Polynomial Ring in x over Rational Field
\end{verbatim}

For compatibility with the multivariate case, if \text{var} specifies the variable of the polynomial ring, then the monomials are multiplied with the smallest powers of \text{var} such that the result is homogeneous; in other words, we end up with a monomial whose leading coefficient is the sum of the coefficients of the polynomial:

\begin{verbatim}
sage: f = x^2 + x + 1
sage: f.homogenize('x')
3*x^2
\end{verbatim}

In positive characteristic, the degree can drop in this case:

\begin{verbatim}
sage: R.<x> = GF(2)[]
sage: f = x + 1
sage: f.homogenize(x)
0
\end{verbatim}
For compatibility with the multivariate case, the parameter \emph{var} can also be 0 to specify the variable in the polynomial ring:

\begin{verbatim}
sage: R.<x> = QQ[]
sage: f = x^2 + x + 1
sage: f.homogenize(0)
3*x^2
\end{verbatim}

\textbf{\texttt{integral}} (\emph{var}=None)

Return the integral of this polynomial.

By default, the integration variable is the variable of the polynomial.

Otherwise, the integration variable is the optional parameter \emph{var}

\textbf{Note:} The integral is always chosen so that the constant term is 0.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: R.<x> = ZZ[]
sage: R(0).integral()
0
sage: f = R(2).integral(); f
2*x
\end{verbatim}

Note that the integral lives over the fraction field of the scalar coefficients:

\begin{verbatim}
sage: f.parent()
Univariate Polynomial Ring in x over Rational Field
sage: R(0).integral().parent()
Univariate Polynomial Ring in x over Rational Field
sage: f = x^3 + x - 2
sage: g = f.integral(); g
1/4*x^4 + 1/2*x^2 - 2*x
sage: g.parent()
Univariate Polynomial Ring in x over Rational Field
\end{verbatim}

This shows that the issue at \url{github issue #7711} is resolved:

\begin{verbatim}
sage: # needs \texttt{sage.rings.finite_rings}
sage: P.<x,z> = PolynomialRing(GF(2147483647))
sage: Q.<y> = PolynomialRing(P)
sage: p = x + y + z
sage: p.integral()
-1073741823*y^2 + (x + z)*y
\end{verbatim}

\begin{verbatim}
sage: # needs \texttt{sage.rings.finite_rings}
sage: P.<x,z> = PolynomialRing(GF(next_prime(2147483647)))
sage: Q.<y> = PolynomialRing(P)
sage: p = x + y + z
sage: p.integral()
1073741830*y^2 + (x + z)*y
\end{verbatim}

A truly convoluted example:
```
sage: A.<a1, a2> = PolynomialRing(ZZ)
sage: B.<b> = PolynomialRing(A)
sage: C.<c> = PowerSeriesRing(B)
sage: R.<x> = PolynomialRing(C)
sage: f = a2*x^2 + c*x - a1*b
sage: f.parent()
Univariate Polynomial Ring in x over Power Series Ring in c
over Univariate Polynomial Ring in b over Multivariate Polynomial
Ring in a1, a2 over Integer Ring
sage: f.integral()
1/3*a2*x^3 + 1/2*c*x^2 - a1*b*x
sage: f.integral().parent()
Univariate Polynomial Ring in x over Power Series Ring in c
over Univariate Polynomial Ring in b over Multivariate Polynomial
Ring in a1, a2 over Rational Field
sage: g = 3*a2*x^2 + 2*c*x - a1*b
sage: g.integral()
a2*x^3 + c*x^2 - a1*b*x
sage: g.integral().parent()
Univariate Polynomial Ring in x over Power Series Ring in c
over Univariate Polynomial Ring in b over Multivariate Polynomial
Ring in a1, a2 over Rational Field
```

Integration with respect to a variable in the base ring:

```
sage: R.<x> = QQ[]
sage: t = PolynomialRing(R,'t').gen()
sage: f = x*t + 5*t^2
sage: f.integral(x)
5*x*t^2 + 1/2*x^2*t
```

`inverse_mod(a, m)`

Invert the polynomial `a` with respect to `m`, or raise a `ValueError` if no such inverse exists.

The parameter `m` may be either a single polynomial or an ideal (for consistency with `inverse_mod()` in other rings).

**See also:**

If you are only interested in the inverse modulo a monomial `x^k` then you might use the specialized method `inverse_series_trunc()` which is much faster.

**EXAMPLES:**

```
sage: S.<t> = QQ[]
sage: f = inverse_mod(t^2 + 1, t^3 + 1); f
-1/2*t^2 - 1/2*t + 1/2
sage: f * (t^2 + 1) % (t^3 + 1)
1
sage: f = t.inverse_mod((t + 1)^7); f
-t^6 - 7*t^5 - 21*t^4 - 35*t^3 - 35*t^2 - 21*t - 7
sage: (f * t) % (t + 1)^7
1
sage: t.inverse_mod(S.ideal((t + 1)^7)) == f
True
```

This also works over inexact rings, but note that due to rounding error the product may not always exactly equal the constant polynomial 1 and have extra terms with coefficients close to zero.
ALGORITHM: Solve the system \( as + mt = 1 \), returning \( s \) as the inverse of \( a \mod m \).

Uses the Euclidean algorithm for exact rings, and solves a linear system for the coefficients of \( s \) and \( t \) for inexact rings (as the Euclidean algorithm may not converge in that case).

AUTHORS:


**inverse_of_unit()**

**EXAMPLES:**

```python
sage: R.<x> = QQ[]
sage: f = x - 90283
sage: f.inverse_of_unit()
Traceback (most recent call last):
...
ArithmeticError: x - 90283 is not a unit
in Univariate Polynomial Ring in x over Rational Field
```

**inverse_series_trunc(**prec**)**

Return a polynomial approximation of precision **prec** of the inverse series of this polynomial.

See also:

The method **inverse_mod()** allows more generally to invert this polynomial with respect to any ideal.

**EXAMPLES:**

```python
sage: x = polygen(ZZ)
sage: s = (1 + x).inverse_series_trunc(5)
sage: s
```
Note that the constant coefficient needs to be a unit:

```
sage: ZZx.<x> = ZZ[]
sage: ZZxy.<y> = ZZx[]
sage: (1+x + y**2).inverse_series_trunc(4)
Traceback (most recent call last):
  ...  
ValueError: constant term x + 1 is not a unit
sage: (1+x + y**2).change_ring(ZZx.fraction_field()).inverse_series_trunc(4)
(-1/(x^2 + 2*x + 1))*y^2 + 1/(x + 1)
```

The method works over any polynomial ring:

```
sage: R = Zmod(4)
sage: Rx.<x> = R[]
sage: Rxy.<y> = Rx[]
sage: p = 1 + (1+2*x)*y + x**2*y**4
sage: q = p.inverse_series_trunc(10)
sage: (p*q).truncate(11)
(2*x^4 + 3*x^2 + 3)*y^10 + 1
```

Even noncommutative ones:

```
sage: # needs sage.modules
sage: M = MatrixSpace(ZZ, 2)
sage: x = polygen(M)
sage: p = M([1,2,3,4])*x^3 + M([-1,0,0,1])*x^2 + M([1,3,-1,0])*x + M.one()
sage: q = p.inverse_series_trunc(5)
sage: (p*q).truncation(5) == M.one()  
True
sage: q = p.inverse_series_trunc(13)
sage: (p*q).truncation(13) == M.one()  
True
```

AUTHORS:

- David Harvey (2006-09-09): Newton's method implementation for power series
- Vincent Delecroix (2014-2015): move the implementation directly in polynomial

`is_constant()`

Return True if this is a constant polynomial.

OUTPUT:

- bool - True if and only if this polynomial is constant

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: x.is_constant()  
False
sage: R(2).is_constant()  
```

(continues on next page)
is_cyclotomic\(\text{(certificate=False, algorithm=}\text{`pari`)}\)

Test if this polynomial is a cyclotomic polynomial.

A cyclotomic polynomial is a monic, irreducible polynomial such that all roots are roots of unity.

By default the answer is a boolean. But if certificate\text{ is} True, the result is a non-negative integer: it is 0 if self is not cyclotomic, and a positive integer \(n\) if self is the \(n\)-th cyclotomic polynomial.

See also:

\(\text{is_cyclotomic_product()}\) \(\text{cyclotomic_part()}\) \(\text{has_cyclotomic_factor()\r}

INPUT:

\bullet certificate – boolean, default to False. Only works with algorithm set to "pari".

\bullet algorithm – either "pari" or "sage" (default is "pari")

ALGORITHM:

The native algorithm implemented in Sage uses the first algorithm of [BD1989]. The algorithm in PARI (using pari:poliscyclo) is more subtle since it does compute the inverse of the Euler \(\phi\) function to determine the \(n\) such that the polynomial is the \(n\)-th cyclotomic polynomial.

EXAMPLES:

Quick tests:

\[
\begin{align*}
\text{sage:} & \quad \# \text{ needs sage.libs.pari} \\
\text{sage:} & \quad P.<x> = ZZ['x'] \\
\text{sage:} & \quad (x - 1).is_cyclotomic() \\
\text{sage:} & \quad (x + 1).is_cyclotomic() \\
\text{sage:} & \quad (x^2 - 1).is_cyclotomic() \\
\text{sage:} & \quad (x^2 + x + 1).is_cyclotomic(certificate=\text{True}) \\
\text{sage:} & \quad (x^2 + 2^*x + 1).is_cyclotomic(certificate=\text{True}) \\
\end{align*}
\]

Test first 100 cyclotomic polynomials:

\[
\begin{align*}
\text{sage:} & \quad \text{all(cyclotomic_polynomial(i).is_cyclotomic() \text{ for i in range(1, 101)) \# needs sage.libs.pari} \\
\text{sage:} & \quad True \\
\end{align*}
\]

Some more tests:

\[
\begin{align*}
\text{sage:} & \quad \# \text{ needs sage.libs.pari} \\
\text{sage:} & \quad f = x^{16} + x^{14} - x^{10} + x^8 - x^6 + x^2 + 1 \\
\text{sage:} & \quad f.is_cyclotomic(algorithm=\text{`pari'}) \\
\text{sage:} & \quad \text{False} \\
\text{sage:} & \quad f.is_cyclotomic(algorithm=\text{`sage'}) \\
\text{sage:} & \quad \text{False} \\
\text{sage:} & \quad g = x^{16} + x^{14} - x^{10} - x^8 - x^6 + x^2 + 1 \\
\end{align*}
\]
Invalid arguments:

```
sage: (x - 3).is_cyclotomic(algorithm="sage", certificate=True)  
  → needs sage.libs.pari
Traceback (most recent call last):
  ...
ValueError: no implementation of the certificate within Sage
```

Test using other rings:

```
sage: z = polygen(GF(5))
sage: (z - 1).is_cyclotomic() 
Traceback (most recent call last):
  ...
NotImplementedError: not implemented in non-zero characteristic
```

**is_cyclotomic_product()**

Test whether this polynomial is a product of cyclotomic polynomials.

This method simply calls the function `pari:poliscycloprod` from the Pari library.

**See also:**

`is_cyclotomic()` `cyclotomic_part()` `has_cyclotomic_factor()`

**EXAMPLES:**

```
sage: x = polygen(ZZ)
sage: (x^5 - 1).is_cyclotomic_product()  
  → needs sage.libs.pari
True
sage: (x^5 + x^4 - x^2 + 1).is_cyclotomic_product()  
  → needs sage.libs.pari
False
sage: p = prod(cyclotomic_polynomial(i) for i in [2, 5, 7, 12])
sage: p.is_cyclotomic_product()  
  → needs sage.libs.pari
True
sage: (x^5 - 1/3).is_cyclotomic_product() 
False
sage: x = polygen(Zmod(5))
sage: (x - 1).is_cyclotomic_product() 
Traceback (most recent call last):
```

(continues on next page)
is_gen()

Return True if this polynomial is the distinguished generator of the parent polynomial ring.

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: R(1).is_gen()
False
sage: R(x).is_gen()
True
```

Important - this function doesn’t return True if self equals the generator; it returns True if self is the generator.

```python
sage: f = R([0,1]); f
x
sage: f.is_gen()
False
sage: f is x
False
sage: f == x
True
```

is_homogeneous()

Return True if this polynomial is homogeneous.

EXAMPLES:

```python
sage: P.<x> = PolynomialRing(QQ)
sage: x.is_homogeneous()
True
sage: P(0).is_homogeneous()
True
sage: (x + 1).is_homogeneous()
False
```

is_irreducible()

Return whether this polynomial is irreducible.

EXAMPLES:

```python
sage: R.<x> = ZZ[]
sage: (x^3 + 1).is_irreducible()  # needs sage.libs.pari
False
sage: (x^2 - 1).is_irreducible()  # needs sage.libs.pari
False
sage: (x^3 + 2).is_irreducible()  # needs sage.libs.pari
True
sage: R(0).is_irreducible()
False
```
The base ring does matter: for example, $2x$ is irreducible as a polynomial in $\mathbb{Q}[x]$, but not in $\mathbb{Z}[x]$:

\begin{verbatim}
sage: R.<x> = ZZ[]
sage: R(2*x).is_irreducible() # needs sage.libs.pari
False
sage: R.<x> = QQ[]
sage: R(2*x).is_irreducible() # needs sage.libs.pari
True
\end{verbatim}

\textbf{is_lorentzian} (explain=False)

Return True if this is a Lorentzian polynomial.

A univariate real polynomial is Lorentzian if and only if it is a monomial with positive coefficient, or zero. The definition is more involved for multivariate real polynomials.

INPUT:

- \texttt{explain} – boolean (default: False); if True return a tuple whose first element is the boolean result of the test, and the second element is a string describing the reason the test failed, or None if the test succeeded

EXAMPLES:

\begin{verbatim}
sage: P.<x> = QQ[]
sage: p1 = x^2
sage: p1.is_lorentzian()
True
sage: p2 = 1 + x^2
sage: p2.is_lorentzian()
False
sage: p3 = P.zero()

sage: p3.is_lorentzian()
True
sage: p4 = -2*x^3
sage: p4.is_lorentzian()
False
\end{verbatim}

It is an error to check if a polynomial is Lorentzian if its base ring is not a subring of the real numbers, as the notion is not defined in this case:

\begin{verbatim}
sage: # needs sage.rings.real_mpfr
sage: Q.<y> = CC[]
sage: q = y^2
sage: q.is_lorentzian()
Traceback (most recent call last):
...
NotImplementedError: is_lorentzian only implemented for real polynomials
\end{verbatim}

The method can give a reason for a polynomial failing to be Lorentzian:

\begin{verbatim}
sage: p = x^2 + 2*x
sage: p.is_lorentzian(explain=True)
(False, 'inhomogeneous')
\end{verbatim}

REFERENCES:

For full definitions and related discussion, see [BrHu2019] and [HMMS2019].
is_monic()

Returns True if this polynomial is monic. The zero polynomial is by definition not monic.

EXAMPLES:

```
sage: x = QQ['x'].0
sage: f = x + 33
sage: f.is_monic()
True
sage: f = 0*x
sage: f.is_monic()
False
sage: f = 3*x^3 + x^4 + x^2
sage: f.is_monic()
True
sage: f = 2*x^2 + x^3 + 56*x^5
sage: f.is_monic()
False
```

AUTHORS:

- Naqi Jaffery (2006-01-24): examples

is_monomial()

Return True if self is a monomial, i.e., a power of the generator.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: x.is_monomial()
True
sage: (x + 1).is_monomial()
False
sage: (x^2).is_monomial()
True
sage: R(1).is_monomial()
True
```

The coefficient must be 1:

```
sage: (2*x^5).is_monomial()
False
```

To allow a non-1 leading coefficient, use is_term():

```
sage: (2*x^5).is_term()
True
```

**Warning:** The definition of is_monomial() in Sage up to 4.7.1 was the same as is_term(), i.e., it allowed a coefficient not equal to 1.

is_nilpotent()

Return True if this polynomial is nilpotent.

EXAMPLES:
\begin{verbatim}
sage: R = Integers(12)
sage: S.<x> = R[]
sage: f = 5 + 6*x
sage: f.is_nilpotent()
False
sage: f = 6 + 6*x^2
sage: f.is_nilpotent()
True
sage: f^2
0
\end{verbatim}

EXERCISE (Atiyah-McDonald, Ch 1): Let \( A[x] \) be a polynomial ring in one variable. Then \( f = \sum a_i x^i \in A[x] \) is nilpotent if and only if every \( a_i \) is nilpotent.

\textbf{is\_one()}

Test whether this polynomial is 1.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: R.<x> = QQ[]
sage: (x - 3).is_one()
False
sage: R(1).is_one()
True
sage: R2.<y> = R[]
sage: R2(x).is_one()
False
sage: R2(1).is_one()
True
sage: R2(-1).is_one()
False
\end{verbatim}

\textbf{is\_primitive\( (n=None, n\_prime\_divs=None)\)}

Return True if the polynomial is primitive.

The semantics of “primitive” depend on the polynomial coefficients.

- (field theory) A polynomial of degree \( m \) over a finite field \( \mathbb{F}_q \) is primitive if it is irreducible and its root in \( \mathbb{F}_{q^n} \) generates the multiplicative group \( \mathbb{F}_{q^n}^* \).

- (ring theory) A polynomial over a ring is primitive if its coefficients generate the unit ideal.

Calling \texttt{is\_primitive()} on a polynomial over an infinite field will raise an error.

The additional inputs to this function are to speed up computation for field semantics (see note).

\textbf{INPUT:}

- \( n \) (default: None) - if provided, should equal \( q - 1 \) where \texttt{self.parent()} is the field with \( q \) elements; otherwise it will be computed.

- \texttt{n\_prime\_divs} (default: None) - if provided, should be a list of the prime divisors of \( n \); otherwise it will be computed.

\textbf{Note:} Computation of the prime divisors of \( n \) can dominate the running time of this method, so performing this computation externally (e.g., \texttt{pdivs = n.prime\_divisors()}) is a good idea for repeated calls to \texttt{is\_primitive()} for polynomials of the same degree.
Results may be incorrect if the wrong \( n \) and/or factorization are provided.

**EXAMPLES:**

Field semantics examples.

```
sage: # needs sage.rings.finite_rings
sage: R.<x> = GF(2)[x]
sage: f = x^4 + x^3 + x^2 + x + 1
sage: f.is_irreducible(), f.is_primitive()
(True, False)
sage: f = x^3 + x + 1
sage: f.is_irreducible(), f.is_primitive()
(True, True)
sage: R.<x> = GF(3)[x]
sage: f = x^3 - x + 1
sage: f.is_irreducible(), f.is_primitive()
(True, True)
sage: f = x^2 + 1
sage: f.is_irreducible(), f.is_primitive()
(True, False)
sage: R.<x> = GF(5)[x]
sage: f = x^2 + x + 1
sage: f.is_primitive()  # needs sage.rings.number_field
False
sage: f = x^2 - x + 2
sage: f.is_primitive()  # needs sage.rings.number_field
True
sage: x = polygen(QQ); f = x^2 + 1
sage: f.is_primitive()
Traceback (most recent call last):
  ... NotImplimentedError: isPrimitive() not defined for polynomials over infinite fields.
```

Ring semantics examples.

```
sage: x = polygen(ZZ)
sage: f = 5*x^2 + 2
sage: f.is_primitive()  # needs sage.rings.finite_rings
True
sage: f = 5*x^2 + 5
sage: f.is_primitive()  # needs sage.rings.finite_rings
False
sage: # needs sage.rings.number_field
sage: K = NumberField(x^2 + 5, 'a')
sage: R = K.ring_of_integers()
sage: a = R.gen(1)
sage: a^2  # needs sage.rings.finite_rings
-5
sage: f = a*x + 2
sage: f.is_primitive()  # needs sage.rings.finite_rings
True
sage: f = (1+a)*x + 2
sage: f.is_primitive()  # needs sage.rings.finite_rings
False
```

(continues on next page)
sage: x = polygen(Integers(10))
sage: f = 5*x^2 + 2
sage: #f.is_primitive() #BUG:: elsewhere in Sage, should return True
sage: f = 4*x^2 + 2
sage: #f.is_primitive() #BUG:: elsewhere in Sage, should return False

is_real_rooted()
Return True if the roots of this polynomial are all real.

EXAMPLES:

sage: # needs sage.libs.pari
sage: R.<x> = PolynomialRing(ZZ)
sage: pol = chebyshev_T(5, x)
sage: pol.is_real_rooted() # True
sage: pol = x^2 + 1
sage: pol.is_real_rooted() # False

is_square(root=False)
Return whether or not polynomial is square.

If the optional argument root is set to True, then also returns the square root (or None, if the polynomial is not square).

INPUT:

• root - whether or not to also return a square root (default: False)

OUTPUT:

• bool - whether or not a square

• root - (optional) an actual square root if found, and None otherwise.

EXAMPLES:

sage: R.<x> = PolynomialRing(QQ)
sage: (x^2 + 2*x + 1).is_square() # True
sage: (x^4 + 2*x^3 - x^2 - 2*x + 1).is_square(root=True) # (True, x^2 + x - 1)
sage: f = 12 * (x + 1)^2 * (x + 3)^2
sage: f.is_square() # False
sage: f.is_square(root=True) # (False, None)

sage: h = f/3; h
4*x^4 + 32*x^3 + 88*x^2 + 96*x + 36
sage: h.is_square(root=True) # (True, 2*x^2 + 8*x + 6)

sage: S.<y> = PolynomialRing(RR)
sage: g = 12 * (y + 1)^2 * (y + 3)^2

(continues on next page)
\textbf{is\_squarefree()}

Return \texttt{False} if this polynomial is not square-free, i.e., if there is a non-unit \( g \) in the polynomial ring such that \( g^2 \) divides \textit{self}.

\begin{verbatim}
EXAMPLES:

sage: R.<x> = QQ[]
sage: f = (x-1) * (x-2) * (x^2-5) * (x^17-3); f
x^21 - 3*x^20 - 3*x^19 + 15*x^18 - 10*x^17 - 3*x^4 + 9*x^3 + 9*x^2 - 45*x + 30
sage: f.is_squarefree()
True
sage: (f * (x^2-5)).is_squarefree()
False

A generic implementation is available, which relies on gcd computations:

sage: # needs sage.libs.pari
sage: R.<x> = ZZ[]
sage: (2*x).is_squarefree()
True
sage: (4*x).is_squarefree()
False
sage: (2*x^2).is_squarefree()
False
sage: R(0).is_squarefree()
False

sage: S.<y> = QQ[]
sage: R.<x> = S[]
sage: (2*x*y).is_squarefree()
True
sage: (2*x*y^2).is_squarefree()
False

In positive characteristic, we compute the square-free decomposition or a full factorization, depending on which is available:

sage: K.<t> = FunctionField(GF(3))
sage: R.<x> = K[]

sage: (x^3 - x).is_squarefree()
True
sage: (x^3 - 1).is_squarefree()  # needs sage.libs.pari
False

sage: (x^3 + t).is_squarefree()  # needs sage.libs.pari
True
sage: (x^3 + t^3).is_squarefree()  # needs sage.libs.pari

(continues on next page)

(continued from previous page)

\texttt{sage: g.is\_square\_free()}

\begin{verbatim}
True

\texttt{(is\_squarefren())}

Return \texttt{False} if this polynomial is not square-free, i.e., if there is a non-unit \( g \) in the polynomial ring such that \( g^2 \) divides \textit{self}.

\begin{verbatim}
EXAMPLES:

sage: R.<x> = QQ[]
sage: f = (x-1) * (x-2) * (x^2-5) * (x^17-3); f
x^21 - 3*x^20 - 3*x^19 + 15*x^18 - 10*x^17 - 3*x^4 + 9*x^3 + 9*x^2 - 45*x + 30
sage: f.is_squarefree()
True
sage: (f * (x^2-5)).is_squarefree()
False

A generic implementation is available, which relies on gcd computations:

sage: # needs sage.libs.pari
sage: R.<x> = ZZ[]
sage: (2*x).is_squarefree()
True
sage: (4*x).is_squarefree()
False
sage: (2*x^2).is_squarefree()
False
sage: R(0).is_squarefree()
False

sage: S.<y> = QQ[]
sage: R.<x> = S[]
sage: (2*x*y).is_squarefree()
True
sage: (2*x*y^2).is_squarefree()
False

In positive characteristic, we compute the square-free decomposition or a full factorization, depending on which is available:

sage: K.<t> = FunctionField(GF(3))
sage: R.<x> = K[]

sage: (x^3 - x).is_squarefree()
True
sage: (x^3 - 1).is_squarefree()  # needs sage.libs.pari
False

sage: (x^3 + t).is_squarefree()  # needs sage.libs.pari
True
sage: (x^3 + t^3).is_squarefree()  # needs sage.libs.pari

(continues on next page)

(continued from previous page)
In the following example, $t^2$ is a unit in the base field:

```python
sage: R(t^2).is_squarefree()
True
```

This method is not consistent with `squarefree_decomposition()`:

```python
sage: R.<x> = ZZ[]
sage: f = 4 * x
sage: f.is_squarefree()  # needs sage.libs.pari
False
sage: f.squarefree_decomposition()  # needs sage.libs.pari
(4) * x
```

If you want this method equally not to consider the content, you can remove it as in the following example:

```python
sage: c = f.content()
sage: (f/c).is_squarefree()  # needs sage.libs.pari
True
```

If the base ring is not an integral domain, the question is not mathematically well-defined:

```python
sage: R.<x> = IntegerModRing(9)[]
sage: pol = (x + 3) * (x + 6); pol
x^2
sage: pol.is_squarefree()
Traceback (most recent call last):
...
TypeError: is_squarefree() is not defined for polynomials over Ring of integers modulo 9
```

**is_term()**

Return True if this polynomial is a nonzero element of the base ring times a power of the variable.

**EXAMPLES:**

```python
sage: R.<x> = QQ[]
sage: x.is_term()
True
sage: R(0).is_term()
False
sage: R(1).is_term()
True
sage: (3*x^5).is_term()
True
sage: (1 + 3*x^5).is_term()
False
```

To require that the coefficient is 1, use `is_monomial()` instead:
is_monomial()

Return True if this polynomial is a monomial.

EXAMPLES:

```python
sage: (3*x^5).is_monomial()
False
```

is_unit()

Return True if this polynomial is a unit.

EXAMPLES:

```python
sage: a = Integers(90384098234^3)
sage: b = a(2*191*236607587)
sage: b.is_unit()
True

sage: # needs sage.libs.pari
sage: R.<x> = a[]
sage: f = 3 + b*x + b^2*x^2
sage: f.is_unit()
True
sage: f = 3 + b*x + b^2*x^2 + 17*x^3
sage: f.is_unit()
False
```

EXERCISE (Atiyah-McDonald, Ch 1): Let \( A[x] \) be a polynomial ring in one variable. Then \( f = \sum a_i x^i \in A[x] \) is a unit if and only if \( a_0 \) is a unit and \( a_1, \ldots, a_n \) are nilpotent.

is_weil_polynomial(return_q=False)

Return True if this is a Weil polynomial.

This polynomial must have rational or integer coefficients.

INPUT:

- `self` - polynomial with rational or integer coefficients
- `return_q` - (default False) if True, return a second value \( q \) which is the prime power with respect to which this is \( q \)-Weil, or 0 if there is no such value.

EXAMPLES:

```python
sage: polRing.<x> = PolynomialRing(Rationals())
sage: P0 = x^4 + 5*x^3 + 15*x^2 + 25*x + 25
sage: P1 = x^4 + 25*x^3 + 15*x^2 + 5*x + 25
sage: P2 = x^4 + 5*x^3 + 25*x^2 + 25*x + 25
sage: P0.is_weil_polynomial(return_q=True) # needs sage.libs.pari
(True, 5)
sage: P0.is_weil_polynomial(return_q=False) # needs sage.libs.pari
True
sage: P1.is_weil_polynomial(return_q=True) # needs sage.libs.pari
(True, 0)
sage: P1.is_weil_polynomial(return_q=False)
False
sage: P2.is_weil_polynomial() # needs sage.libs.pari
False
```

See also:

Polynomial rings have a method `weil_polynomials()` to compute sets of Weil polynomials.
als. This computation uses the iterator `sage.rings.polynomial.weil.weil_polynomials`. WeilPolynomials.

AUTHORS:
David Zureick-Brown (2017-10-01)

**is_zero()**
Test whether this polynomial is zero.

EXAMPLES:

```python
sage: R = GF(2)[x][y]
sage: R([0,1]).is_zero()
False
sage: R([0]).is_zero()
True
sage: R([-1]).is_zero()
False
```

**lc()**
Return the leading coefficient of this polynomial.

OUTPUT: element of the base ring
This method is the same as `leading_coefficient()`.

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: f = (-2/5)*x^3 + 2*x - 1/3
sage: f.lc()
-2/5
```

**lcm(other)**
Let \( f \) and \( g \) be two polynomials. Then this function returns the monic least common multiple of \( f \) and \( g \).

**leading_coefficient()**
Return the leading coefficient of this polynomial.

OUTPUT: element of the base ring

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: f = (-2/5)*x^3 + 2*x - 1/3
sage: f.leading_coefficient()
-2/5
```

**list**(copy=True)
Return a new copy of the list of the underlying elements of `self`.

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: f = (-2/5)*x^3 + 2*x - 1/3
sage: v = f.list(); v
[-1/3, 2, 0, -2/5]
```

Note that \( v \) is a list, it is mutable, and each call to the `list()` method returns a new list:

```python
sage: v = f.list(); v
[-1/3, 2, 0, -2/5]
```
Here is an example with a generic polynomial ring:

```python
sage: R.<x> = QQ[]
sage: S.<y> = R[]
sage: f = y^3 + x*y - 3*x; f
y^3 + x*y - 3*x
sage: type(f)
<class 'sage.rings.polynomial.polynomial_element.Polynomial_generic_dense'>
sage: v = f.list(); v
[-3*x, x, 0, 1]
sage: v[0] = 10
sage: f.list()
[-3*x, x, 0, 1]
```

**lm()**

Return the leading monomial of this polynomial.

**EXAMPLES:**

```python
sage: R.<x> = QQ[]
sage: f = (-2/5)*x^3 + 2*x - 1/3
sage: f.lm()
x^3
sage: R(5).lm()
1
sage: R(0).lm()
0
sage: R(0).lm().parent() is R
True
```

**local_height(v, prec=None)**

Return the maximum of the local height of the coefficients of this polynomial.

**INPUT:**

- `v` – a prime or prime ideal of the base ring.
- `prec` – desired floating point precision (default: default RealField precision).

**OUTPUT:** a real number.

**EXAMPLES:**

```python
sage: R.<x> = PolynomialRing(QQ)
sage: f = 1/1331*x^2 + 1/4000*x
sage: f.local_height(1331) # needs sage.rings.ring.integer
7.19368581839511
```

```python
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: K.<k> = NumberField(x^2 - 5)
(continues on next page)
```
sage: T.<t> = K[]
sage: I = K.ideal(3)
sage: f = 1/3*t^2 + 3
sage: f.local_height(I)
1.09861228866811

sage: R.<x> = QQ[]
sage: f = 1/2*x^2 + 2
sage: f.local_height(2, prec=2)
# needs sage.rings.real_mpfr
0.75

local_height_arch(i, prec=None)

Return the maximum of the local height at the i-th infinite place of the coefficients of this polynomial.

INPUT:

• i – an integer.
• prec – desired floating point precision (default: default RealField precision).

OUTPUT: a real number.

EXAMPLES:

sage: R.<x> = PolynomialRing(QQ)
sage: f = 210*x^2
sage: f.local_height_arch(0)  # needs sage.rings.real_mpfr
5.34710753071747

sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: K.<k> = NumberField(x^2 - 5)
sage: T.<t> = K[]
sage: f = 1/2*t^2 + 3
sage: f.local_height_arch(1, prec=52)
1.09861228866811

sage: R.<x> = QQ[]
sage: f = 1/2*x^2 + 3
sage: f.local_height_arch(0, prec=2)  # needs sage.rings.real_mpfr
1.0

lt()

Return the leading term of this polynomial.

EXAMPLES:

sage: R.<x> = QQ[]
sage: f = (-2/5)*x^3 + 2*x - 1/3
sage: f.lt()
-2/5*x^3
sage: R(5).lt()  # R(5) is a constant
5
sage: R(0).lt()  # R(0) is a constant
0
Polynomials, Release 10.3

map_coefficients \((f, new\_base\_ring=\text{None})\)

Return the polynomial obtained by applying \(f\) to the non-zero coefficients of \(self\).

If \(f\) is a \texttt{sage.categories.map.Map}, then the resulting polynomial will be defined over the codomain of \(f\). Otherwise, the resulting polynomial will be over the same ring as \(self\). Set \texttt{new_base_ring} to override this behaviour.

INPUT:

- \(f\) – a callable that will be applied to the coefficients of \(self\).
- \texttt{new_base_ring} (optional) – if given, the resulting polynomial will be defined over this ring.

EXAMPLES:

```python
sage: R.<x> = ZZ[]
sage: f = x^2 + 2
sage: f.map_coefficients(lambda a: a + 42)
43*x^2 + 44
sage: R.<x> = PolynomialRing(ZZ, sparse=True)
sage: f = x*(2^32) + 2
sage: f.map_coefficients(lambda a: a + 42)
43*x^4294967296 + 44
```

```python
sage: # needs sage.symbolic
sage: R.<x> = SR[]
sage: f = (1+I)*x^2 + 3*x - I
sage: f.map_coefficients(lambda z: z.conjugate())
(-I + 1)*x^2 + 3*x + I
```

Examples with different base ring:

```python
sage: R.<x> = ZZ[]
sage: k = GF(2)
sage: residue = lambda x: k(x)
sage: f = 4*x^2 + x + 3
sage: g = f.map_coefficients(residue); g
x + 1
sage: g.parent() # needs sage.libs.ntl
Univariate Polynomial Ring in x over Finite Field of size 2 (using GF2X)
```

(continues on next page)
Univariate Polynomial Ring in x over Finite Field of size 2 (using GF2X)

\textbf{mod} \text{(other)}

Remainder of division of \textit{self} by \textit{other}.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: R.<x> = ZZ[]
sage: x % (x+1)
-1
sage: (x^3 + x - 1) % (x^2 - 1)
2*x - 1
\end{verbatim}

\textbf{monic()} 

Return this polynomial divided by its leading coefficient. Does not change this polynomial.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: x = QQ['x'].0
sage: f = 2*x^2 + x^3 + 56*x^5
sage: f.monic()
x^5 + 1/56*x^3 + 1/28*x^2
sage: f = (1/4)*x^2 + 3*x + 1
sage: f.monic()
x^2 + 12*x + 4
\end{verbatim}

The following happens because \( f = 0 \) cannot be made into a monic polynomial

\begin{verbatim}
sage: f = 0*x
sage: f.monic()
Traceback (most recent call last):
...
ZeroDivisionError: rational division by zero
\end{verbatim}

Notice that the monic version of a polynomial over the integers is defined over the rationals.

\begin{verbatim}
sage: x = ZZ['x'].0
sage: f = 3*x^19 + x^2 - 37
sage: g = f.monic(); g
x^19 + 1/3*x^2 - 37/3
sage: g.parent()
Univariate Polynomial Ring in x over Rational Field
\end{verbatim}

\textbf{AUTHORS:}

- Naqi Jaffery (2006-01-24): examples

\textbf{monomial\_coefficient} \text{(m)}

Return the coefficient in the base ring of the monomial \textit{m} in \textit{self}, where \textit{m} must have the same parent as \textit{self}.

\textbf{INPUT:}

- \textit{m} - a monomial

\textbf{OUTPUT:} Coefficient in base ring.

\textbf{EXAMPLES:}
Polynomials, Release 10.3

```python
sage: P.<x> = QQ[]
sage: f = 2 * x
sage: c = f.monomial_coefficient(x); c
2
sage: c.parent()
Rational Field
sage: f = x^9 - 1/2*x^2 + 7*x + 5/11
sage: f.monomial_coefficient(x^9)
1
sage: f.monomial_coefficient(x^2)
-1/2
sage: f.monomial_coefficient(x)
7
sage: f.monomial_coefficient(x^0)
5/11
sage: f.monomial_coefficient(x^3)
0
```

### monomials()

Return the list of the monomials in `self` in a decreasing order of their degrees.

**EXAMPLES:**

```python
sage: P.<x> = QQ[]
sage: f = x^2 + (2/3)*x + 1
sage: f.monomials()
[x^2, x, 1]
sage: f = P(3/2)
```

### multiplication_trunc(other, n)

Truncated multiplication

**EXAMPLES:**

```python
sage: R.<x> = ZZ[]
sage: (x^10 + 5*x^5 + x^2 - 3).multiplication_trunc(x^7 - 3*x^3 + 1, 11)
x^10 + x^9 - 15*x^8 - 3*x^7 + 2*x^5 + 9*x^3 + x^2 - 3
```
Check that coercion is working:

```
sage: R2 = QQ['x']
sage: x2 = R2.gen()
sage: p1 = (x^3 + 1).multiplication_trunc(x2^3 - 2, 5); p1
-x^3 - 2
sage: p2 = (x2^3 + 1).multiplication_trunc(x^3 - 2, 5); p2
-x^3 - 2
sage: parent(p1) == parent(p2) == R2
True
```

```
newton_raphson \( n, x0 \)

Return a list of \( n \) iterative approximations to a root of this polynomial, computed using the Newton-Raphson method. The Newton-Raphson method is an iterative root-finding algorithm. For \( f(x) \) a polynomial, as is the case here, this is essentially the same as Horner’s method.

**INPUT:**
- \( n \) - an integer (the number of iterations),
- \( x0 \) - an initial guess \( x_0 \).

**OUTPUT:** A list of numbers hopefully approximating a root of \( f(x) = 0 \).

If one of the iterates is a critical point of \( f \), a `ZeroDivisionError` exception is raised.

**EXAMPLES:**

```
sage: x = PolynomialRing(RealField(), 'x').gen()  # needs sage.rings.real_mpfr
sage: f = x^2 - 2  # needs sage.rings.real_mpfr
sage: f.newton_raphson(4, 1)  # needs sage.rings.real_mpfr
[1.50000000000000, 1.41666666666667, 1.41421568627451, 1.41421356237469]
```

**AUTHORS:**
- David Joyner and William Stein (2005-11-28)

```
newton_slopes \( p, \) lengths=False \)

Return the \( p \)-adic slopes of the Newton polygon of \( self \), when this makes sense.

**OUTPUT:**

If `lengths` is `False`, a list of rational numbers. If `lengths` is `True`, a list of couples \((s, l)\) where \( s \) is the slope and \( l \) the length of the corresponding segment in the Newton polygon.

**EXAMPLES:**

```
sage: x = QQ['x'].0
sage: f = x^3 + 2
sage: f.newton_slopes(2)  # needs sage.libs.pari
[1/3, 1/3, 1/3]
```

```
R.<x> = PolynomialRing(ZZ, sparse=True)
sage: p = x^5 + 6*x^2 + 4
sage: p.newton_slopes(2)  # needs sage.libs.pari
```

(continues on next page)
[1/2, 1/2, 1/3, 1/3, 1/3]
sage: p.newton_slopes(2, lengths=True)
[(1/2, 2), (1/3, 3)]
sage: (x^2^100 + 27).newton_slopes(3, lengths=True)
[[(3/126750600228229401496703205376, 126750600228229401496703205376)]

ALGORITHM: Uses PARI if lengths is False.

\textbf{norm} \text{(p)}

Return the \( p \)-norm of this polynomial.

\textbf{DEFINITION:} For integer \( p \), the \( p \)-norm of a polynomial is the \( p \)th root of the sum of the \( p \)th powers of the absolute values of the coefficients of the polynomial.

\textbf{INPUT:}

- \( p \) -(positive integer or +infinity) the degree of the norm

\textbf{EXAMPLES:}

\begin{verbatim}
sage: # needs sage.rings.real_mpfr
sage: R.<x> = RR[]
sage: f = x^6 + x^2 + -x^4 - 2*x^3
sage: f.norm(2)
2.64575131106459
sage: (sqrt(1^2 + 1^2 + (-1)^2 + (-2)^2)).n()  # needs sage.symbolic
2.64575131106459
sage: f.norm(1)  # needs sage.rings.real_mpfr
5.00000000000000
sage: f.norm(infinity)  # needs sage.rings.real_mpfr
2.00000000000000
sage: f.norm(-1)  # needs sage.rings.real_mpfr
Traceback (most recent call last):
  ...
ValueError: The degree of the norm must be positive
\end{verbatim}

\textbf{AUTHORS:}

- Didier Deshommes
- William Stein: fix bugs, add definition, etc.

\textbf{nth_root} \text{(n)}

Return a \( n \)-th root of this polynomial.

This is computed using Newton method in the ring of power series. This method works only when the base ring is an integral domain. Moreover, for polynomial whose coefficient of lower degree is different from 1, the elements of the base ring should have a method \text{nth_root()} implemented.

\textbf{EXAMPLES:}
sage: R.<x> = ZZ[]
sage: a = 27 * (x+3)**6 * (x+5)**3
sage: a.nth_root(3)
3*x^3 + 33*x^2 + 117*x + 135

sage: b = 25 * (x^2 + x + 1)
sage: b.nth_root(2)
Traceback (most recent call last):
... ValueError: not a 2nd power

sage: R.<x> = QQ[]
sage: a = 1/4 * (x/7 + 3/2)^2 * (x/2 + 5/3)^4
sage: a.nth_root(2)
1/56*x^3 + 103/336*x^2 + 365/252*x + 25/12

# needs sage.rings.number_field
sage: K.<sqrt2> = QuadraticField(2)
sage: R.<x> = K[]
sage: a = (x + sqrt2)^3 * ((1+sqrt2)*x - 1/sqrt2)^6
sage: b = a.nth_root(3); b
(2*sqrt2 + 3)*x^3 + (2*sqrt2 + 2)*x^2 + (-2*sqrt2 - 3/2)*x + 1/2*sqrt2
sage: b^3 == a
True

# needs sage.rings.number_field
sage: R.<x> = QQbar[]
sage: p = x**3 + QQbar(2).sqrt() * x - QQbar(3).sqrt()
sage: r = (p**5).nth_root(5)
sage: r * p[0] == p * r[0]
True
sage: p = (x+1)^20 + x^20
sage: p.nth_root(20)
Traceback (most recent call last):
... ValueError: not a 20th power

# needs sage.rings.finite_rings
sage: z = GF(4).gen()
sage: R.<x> = GF(4)[]
sage: p = z*x**4 + 2*x - 1
sage: r = (p**15).nth_root(15)
sage: r * p[0] == p * r[0]
True
sage: ((x+1)**2).nth_root(2)
x + 1
sage: ((x+1)**4).nth_root(4)
x + 1
sage: ((x+1)**12).nth_root(12)
x + 1
sage: (x^4 + x^3 + 1).nth_root(2)
Traceback (most recent call last):
... ValueError: not a 2nd power
sage: p = (x+1)^17 + x^17
sage: r = p.nth_root(17)
(continues on next page)
Here we consider a base ring without \texttt{nth\_root} method. The third example with a non-trivial coefficient of lowest degree raises an error:

\begin{verbatim}
sage: # needs sage.libs.pari
sage: R.<x> = QQ[]
sage: R2 = R.quotient(x**2 + 1)
sage: x = R2.gen()
sage: R3.<y> = R2[]
sage: (y**2 - 2*y + 1).nth_root(2)
-y + 1
sage: (y**3).nth_root(3)
y
sage: (y**2 + x).nth_root(2)
Traceback (most recent call last):
  ... AttributeError: ... has no attribute 'nth_root'...
\end{verbatim}

\textbf{number\_of\_real\_roots}()

Return the number of real roots of this polynomial, counted without multiplicity.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: # needs sage.libs.pari
sage: R.<x> = PolynomialRing(ZZ)
sage: pol = (x - 1)^2 * (x - 2)^2 * (x - 3)
sage: pol.number_of_real_roots()
3
sage: pol = (x - 1) * (x - 2) * (x - 3)
sage: # needs sage.libs.pari sage.rings.real_mpfr
sage: pol2 = pol.change_ring(CC)
sage: pol2.number_of_real_roots()
3
sage: R.<x> = PolynomialRing(CC)
sage: pol = (x - 1) * (x - CC(I))
sage: pol.number_of_real_roots()
1
\end{verbatim}

\textbf{number\_of\_roots\_in\_interval} (\texttt{a=None, b=None})

Return the number of roots of this polynomial in the interval \([a, b]\), counted without multiplicity. The endpoints \(a, b\) default to \(-\text{Infinity}, \text{Infinity}\) (which are also valid input values).

Calls the PARI routine \texttt{pari:polsturm}.

Note that as of version 2.8, PARI includes the left endpoint of the interval (and no longer uses Sturm’s
algorithm on exact inputs). \texttt{pari:polsturm} requires a polynomial with real coefficients; in case PARI returns an error, we try again after taking the GCD of \texttt{self} with its complex conjugate.

**EXAMPLES:**

```python
sage: # needs sage.libs.pari
sage: R.<x> = PolynomialRing(ZZ)
sage: pol = (x - 1)^2 * (x - 2)^2 * (x - 3)
sage: pol.number_of_roots_in_interval(1, 2)
2
sage: pol.number_of_roots_in_interval(1.01, 2)
1
sage: pol.number_of_roots_in_interval(None, 2)
2
sage: pol.number_of_roots_in_interval(1, Infinity)
3
sage: pol.number_of_roots_in_interval()
3
sage: pol = (x - 1) * (x - 2) * (x - 3)
```

**number_of_terms()**

Return the number of non-zero coefficients of \texttt{self}.

Also called weight, Hamming weight or sparsity.

**EXAMPLES:**

```python
sage: R.<x> = ZZ[]
sage: f = x^3 - x
sage: f.number_of_terms()
2
sage: R(0).number_of_terms()
0
sage: f = (x + 1)^100
sage: f.number_of_terms()
101
sage: S = GF(5)['y']
sage: S(f).number_of_terms()
5
sage: cyclotomic_polynomial(105).number_of_terms()
33
```

The method \texttt{hamming_weight()} is an alias:

```python
sage: f.hamming_weight()
101
```

**numerator()**

Return a numerator of \texttt{self}, computed as \texttt{self} * \texttt{self.denominator()}.
Note that some subclasses may implement its own numerator function. For example, see `sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint`.

**Warning:** This is not the numerator of the rational function defined by `self`, which would always be `self` since `self` is a polynomial.

**EXAMPLES:**

First we compute the numerator of a polynomial with integer coefficients, which is of course `self`.

```sage
sage: R.<x> = ZZ[]
sage: f = x^3 + 17*x + 1
sage: f.numerator()
x^3 + 17*x + 1
sage: f == f.numerator()
True
```

Next we compute the numerator of a polynomial with rational coefficients.

```sage
sage: R.<x> = PolynomialRing(QQ)
sage: f = (1/17)*x^19 - (2/3)*x + 1/3; f
1/17*x^19 - 2/3*x + 1/3
sage: f.numerator()
3*x^19 - 34*x + 17
sage: f == f.numerator()
False
```

We try to compute the denominator of a polynomial with coefficients in the real numbers, which is a ring whose elements do not have a denominator method.

```sage
sage: # needs sage.rings.real_mpfr
sage: R.<x> = RR[]
sage: f = x + RR('0.3'); f
x + 0.300000000000000
sage: f.numerator()
x + 0.300000000000000
```

We check that the computation of the numerator and denominator are valid.

```sage
sage: # needs sage.rings.number_field sage.symbolic
sage: K = NumberField(symbolic_expression('x^3+2'), 'a')['s,t']['x']
sage: f = K.random_element()
sage: f.numerator() / f.denominator() == f
True
```

`ord(p=None)`

This is the same as the valuation of `self` at `p`. See the documentation for `valuation()`.

**EXAMPLES:**
sage: R.<x> = ZZ[]
sage: (x^2 + x).ord(x + 1)
1

padded_list (n=None)
Return list of coefficients of self up to (but not including) \( q^n \).
Includes 0's in the list on the right so that the list has length n.

INPUT:
• n - (default: None); if given, an integer that is at least 0

EXAMPLES:

sage: x = polygen(QQ)
sage: f = 1 + x^3 + 23*x^5
sage: f.padded_list()
[1, 0, 0, 1, 0, 23]
sage: f.padded_list(10)
[1, 0, 0, 1, 0, 23, 0, 0, 0, 0]
sage: len(f.padded_list(10))
10
sage: f.padded_list(3)
[1, 0, 0]
sage: f.padded_list(0)
[]
sage: f.padded_list(-1)
Traceback (most recent call last):
... ValueError: n must be at least 0

plot (xmin=None, xmax=None, *args, **kwds)
Return a plot of this polynomial.

INPUT:
• xmin - float
• xmax - float
• *args, **kwds - passed to either plot or point

OUTPUT: returns a graphic object.

EXAMPLES:

sage: x = polygen(GF(389))
sage: plot(x^2 + 1, rgbcolor=(0,0,1))
# needs sage.plot
Graphics object consisting of 1 graphics primitive
sage: x = polygen(QQ)
sage: plot(x^2 + 1, rgbcolor=(1,0,0))
# needs sage.plot
Graphics object consisting of 1 graphics primitive

polynomial (var)
Let var be one of the variables of the parent of self. This returns self viewed as a univariate polynomial
in var over the polynomial ring generated by all the other variables of the parent.
For univariate polynomials, if \( \var \) is the generator of the parent ring, we return this polynomial, otherwise raise an error.

**EXAMPLES:**

```
sage: R.<x> = QQ[]
sage: (x + 1).polynomial(x)
x + 1
```

\[ \text{power} \_\text{trunc}(n, \text{prec}) \]

Truncated \( n \)-th power of this polynomial up to precision \( \text{prec} \)

**INPUT:**

- \( n \) – (non-negative integer) power to be taken
- \( \text{prec} \) – (integer) the precision

**EXAMPLES:**

```
sage: R.<x> = ZZ[]
sage: (3*x^2 - 2*x + 1).power_trunc(5, 8)
-1800*x^7 + 1590*x^6 - 1052*x^5 + 530*x^4 - 200*x^3 + 55*x^2 - 10*x + 1
sage: ((3*x^2 - 2*x + 1)^5).truncate(8)
-1800*x^7 + 1590*x^6 - 1052*x^5 + 530*x^4 - 200*x^3 + 55*x^2 - 10*x + 1
sage: S.<y> = R[]
sage: (x + y).power_trunc(5,5)
5*x*y^4 + 10*x^2*y^3 + 10*x^3*y^2 + 5*x^4*y + x^5
sage: ((x + y)^5).truncate(5)
5*x*y^4 + 10*x^2*y^3 + 10*x^3*y^2 + 5*x^4*y + x^5
```

\[ \text{prec}() \]

Return the precision of this polynomial. This is always infinity, since polynomials are of infinite precision by definition (there is no big-oh).

**EXAMPLES:**

```
sage: x = polygen(ZZ)
sage: (x^5 + x + 1).prec()
+Infinity
sage: x.prec()
+Infinity
```
**pseudo_quo_rem**(other)

Compute the pseudo-division of two polynomials.

**INPUT:**

- other – a nonzero polynomial

**OUTPUT:**

Q and R such that \( \ell \cdot m + 1 \cdot \text{self} = Q \cdot \text{other} + R \) where m is the degree of this polynomial, n is the degree of other, \( \ell \) is the leading coefficient of other. The result is such that \( \deg(R) < \deg(\text{other}) \).

**ALGORITHM:**

Algorithm 3.1.2 in [Coh1993].

**EXAMPLES:**

```python
sage: R.<x> = PolynomialRing(ZZ, sparse=True)
sage: p = x^4 + 6*x^3 + x^2 - x + 2
sage: q = 2*x^2 - 3*x - 1
sage: quo, rem = p.pseudo_quo_rem(q); quo, rem
(4*x^2 + 30*x + 51, 175*x + 67)
sage: 2^(4-2+1)*p == quo*q + rem
True
```

**radical()**

Return the radical of self.

Over a field, this is the product of the distinct irreducible factors of self. (This is also sometimes called the “square-free part” of self, but that term is ambiguous; it is sometimes used to mean the quotient of self by its maximal square factor.)

**EXAMPLES:**

```python
sage: P.<x> = ZZ[]
sage: t = (x^2-x+1)^3 * (3*x-1)^2
sage: t.radical()
3*x^3 - 4*x^2 + 4*x - 1
```

If self has a factor of multiplicity divisible by the characteristic (see github issue #8736):

```python
sage: P.<x> = GF(2)[]
sage: (x^3 + x^2).radical() #...
← needs sage.rings.finite_rings
x^2 + x
```

**rational_reconstruct**(*args, **kwds)

Deprecated: Use **rational_reconstruction**() instead. See github issue #12696 for details.
**rational_reconstruction** (*m*, *n_deg=None, d_deg=None)

Return a tuple of two polynomials (*n*, *d*) where `self * d` is congruent to *n* modulo *m* and *n_.degree() <= *n_deg and d_.degree() <= *d_deg*.

**INPUT:**

- *m* – a univariate polynomial
- *n_deg* – (optional) an integer; the default is ⌊(deg(*m*)) − 1)/2⌋
- *d_deg* – (optional) an integer; the default is ⌊(deg(*m*)) − 1)/2⌋

**ALGORITHM:**

The algorithm is based on the extended Euclidean algorithm for the polynomial greatest common divisor.

**EXAMPLES:**

**Over Q[z]:**

```sage
sage: z = PolynomialRing(QQ, 'z').gen()
sage: p = -z**16 - z**15 - z**14 + z**13 + z**12 + z**11 - z**5 - z**4 - z**3 + z**2 + z + 1
sage: m = z**21
sage: n, d = p.rational_reconstruction(m); n, d
(z^4 + 2*z^3 + 3*z^2 + 2*z + 1, z^10 + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)
sage: ((p*d - n) % m).is_zero()
True
```

**Over Z[z]:**

```sage
sage: z = PolynomialRing(ZZ, 'z').gen()
sage: p = -z**16 - z**15 - z**14 + z**13 + z**12 + z**11 - z**5 - z**4 - z**3 + z**2 + z + 1
sage: m = z**21
sage: n, d = p.rational_reconstruction(m); n, d
(z^4 + 2*z^3 + 3*z^2 + 2*z + 1, z^10 + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)
sage: ((p*d - n) % m).is_zero()
True
```

**Over an integral domain, d might not be monic:**

```sage
sage: P = PolynomialRing(ZZ, 'x')
sage: x = P.gen()
sage: p = 7*x^5 - 10*x^4 + 16*x^3 - 32*x^2 + 128*x + 256
sage: m = x^5
sage: n, d = p.rational_reconstruction(m, 3, 2); n, d
(-32*x^3 + 384*x^2 + 2304*x + 2048, 5*x + 8)
sage: ((p*d - n) % m).is_zero()
True
sage: n, d = p.rational_reconstruction(m, 4, 0); n, d
(-10*x^4 + 16*x^3 - 32*x^2 + 128*x + 256, 1)
sage: ((p*d - n) % m).is_zero()
True
```

**Over Q(t)[z]:**
sage: P = PolynomialRing(QQ, 't')
sage: t = P.gen()
sage: Pz = PolynomialRing(P.fraction_field(), 'z')
sage: z = Pz.gen()
sage: # p = (1 + t^2*z + z^4) / (1 - t*z)
sage: p = (1 + t^2*z + z^4)*(1 - t*z).inverse_mod(z^9)
sage: m = z^9
sage: n, d = p.rational_reconstruction(m); n, d
(-1/t/z^4 - t*z - 1/t, z - 1/t)
sage: ((p*d - n) % m).is_zero()  
True
sage: w = PowerSeriesRing(P.fraction_field(), 'w').gen()
sage: n = -10*t^2*z^4 + (-t^2 + t - 1)*z^3 + (-t - 8)*z^2 + z + 2*t^2 - t
sage: d = z^4 + (2*t + 4)*z^3 + (-t + 5)*z^2 + (t^2 + 2)*z + t^2 + 2*t + 1
sage: prec = 9
sage: x = n.subs(z=w)/d.subs(z=w) + O(w^prec)
sage: (nc, dc) == Pz(x.list()).rational_reconstruction(z^prec)
True

Over \( \mathbb{Q}[t][z] \):

sage: P = PolynomialRing(QQ, 't')
sage: t = P.gen()
sage: z = PolynomialRing(P, 'z').gen()
sage: # p = (1 + t^2*z + z^4) mod z^9
sage: p = (1 + t^2*z + z^4) * sum((t*z)**i for i in range(9))
sage: m = z^9
sage: n, d = p.rational_reconstruction(m); n, d
(-z^4 - t^2*z - 1, t*z - 1)
sage: ((p*d - n) % m).is_zero()  
True

Over \( \mathbb{Q}_5 \):

sage: # needs sage.rings.padics
sage: x = PolynomialRing(Qp(5), 'x').gen()
sage: p = 4*x^5 + 3*x^4 + 2*x^3 + 2*x^2 + 4*x + 2
sage: m = x^6
sage: n, d = p.rational_reconstruction(m, 3, 2)
sage: ((p*d - n) % m).is_zero()  
True

Can also be used to obtain known Padé approximations:

sage: z = PowerSeriesRing(QQ, 'z').gen()
sage: P = PolynomialRing(QQ, 'x')
sage: x = P.gen()
sage: p = P(z.exp().list())
sage: m = x^5
sage: n, d = p.rational_reconstruction(m, 4, 0); n, d
(1/24*x^4 + 1/6*x^3 + 1/2*x^2 + x + 1, 1)
sage: ((p*d - n) % m).is_zero()  
True
sage: m = x^3
sage: n, d = p.rational_reconstruction(m, 1, 1); n, d
(-x - 2, x - 2)

(continues on next page)
Polynomials, Release 10.3

(continued from previous page)

```
sage: ((p*d - n) % m).is_zero()
True
sage: p = P(log(1-z).list())
sage: m = x^9
sage: n, d = p.rational_reconstruction(m, 4, 4); n, d
(25/6*x^4 - 130/3*x^3 + 105*x^2 - 70*x, x^4 - 20*x^3 + 90*x^2 - 140*x + 70)
sage: ((p*d - n) % m).is_zero()
True
sage: p = P(sqrt(1+z).list())
sage: m = x^6
sage: n, d = p.rational_reconstruction(m, 3, 2); n, d
(1/6*x^3 + 3*x^2 + 8*x + 16/3, x^2 + 16/3*x + 16/3)
sage: ((p*d - n) % m).is_zero()
True
sage: p = P((2*z).exp().list())
sage: m = x^7
sage: n, d = p.rational_reconstruction(m, 3, 3); n, d
(-x^3 - 6.0*x^2 - 15.0*x - 15.0, x^3 - 6.0*x^2 + 15.0*x - 15.0)
sage: ((p*d - n) % m).is_zero()
True
```

Over $\mathbb{R}[z]$:

```
sage: # needs sage.rings.real_mpfr
sage: z = PowerSeriesRing(RR, 'z').gen()
sage: P = PolynomialRing(RR, 'x')
sage: x = P.gen()
sage: p = P((2*z).exp().list())
sage: m = x^7
sage: n, d = p.rational_reconstruction(m, 3, 3); n, d
# absolute tolerance

(-x^3 - 6.0*x^2 - 15.0*x - 15.0, x^3 - 6.0*x^2 + 15.0*x - 15.0)
```

See also:

- `sage.matrix.berlekamp_massey`
- `sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint.rational_reconstruction`

**real_roots()**

Return the real roots of this polynomial, without multiplicities.

Calls `self.roots(ring=RR)`, unless this is a polynomial with floating-point real coefficients, in which case it calls `self.roots()`.

**EXAMPLES:**

```
sage: x = polygen(ZZ)
sage: (x^2 - x - 1).real_roots()  
# needs sage.libs.pari sage.rings.real_mpfr
[-0.618033988749895, 1.61803398874989]
```

**reciprocal_transform** $(R=1,q=1)$

Transform a general polynomial into a self-reciprocal polynomial.
The input $Q$ and output $P$ satisfy the relation
\[ P(x) = Q(x + q/x)x^\text{deg}(Q) R(x). \]

In this relation, $Q$ has all roots in the real interval $[-2\sqrt{q}, 2\sqrt{q}]$ if and only if $P$ has all roots on the circle $|x| = \sqrt{q}$ and $R$ divides $x^2 - q$.

See also:
The inverse operation is \texttt{trace_polynomial()}.

INPUT:
- $R$ – polynomial
- $q$ – scalar (default: 1)

EXAMPLES:

```
sage: pol.<x> = PolynomialRing(Rationals())
sage: u = x^2 + x - 1
sage: u.reciprocal_transform()
x^4 + x^3 + x^2 + x + 1
sage: u.reciprocal_transform(R=x-1)
x^5 - 1
sage: u.reciprocal_transform(q=3)
x^4 + x^3 + 5*x^2 + 3*x + 9
```

\texttt{resultant} (other)

Return the resultant of \texttt{self} and \texttt{other}.

INPUT:
- \texttt{other} – a polynomial

OUTPUT: an element of the base ring of the polynomial ring

ALGORITHM:
Uses PARI’s function \texttt{polresultant}. For base rings that are not supported by PARI, the resultant is computed as the determinant of the Sylvester matrix.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: f = x^3 + x + 1; g = x^3 - x - 1
sage: r = f.resultant(g); r
# needs sage.libs.pari
-8
sage: r.parent() is QQ
# needs sage.libs.pari
True
```

We can compute resultants over univariate and multivariate polynomial rings:

```
sage: R.<a> = QQ[]
sage: S.<x> = R[]
sage: f = x^2 + a; g = x^3 + a
sage: r = f.resultant(g); r
# needs sage.libs.pari
a^3 + a^2
sage: r.parent() is R
# needs sage.libs.pari
True
```

(continues on next page)
reverse (degree=None)

Return polynomial but with the coefficients reversed.

If an optional degree argument is given, the coefficient list will be truncated or zero padded as necessary before reversing it. Assuming that the constant coefficient of self is nonzero, the reverse polynomial will have the specified degree.

EXAMPLES:

```python
sage: R.<x> = ZZ[]; S.<y> = R[]
sage: f = y^3 + x*y - 3*x; f
y^3 + x*y - 3*x
sage: f.reverse()
-3*x*y^3 + x*y^2 + 1
sage: f.reverse(degree=2)
-3*x*y^2 + x*y
sage: f.reverse(degree=5)
-3*x*y^5 + x*y^4 + y^2
```

revert_series (n)

Return a polynomial f such that f(self(x)) = self(f(x)) = x mod x^n.

Currently, this is only implemented over some coefficient rings.

EXAMPLES:

```python
sage: Pol.<x> = QQ[]
sage: (x + x^3/6 + x^5/120).revert_series(6)
3/40*x^5 - 1/6*x^3 + x
sage: Pol.<x> = CBF[]
(\[0.075000000000000 +/- ...e-17\])*x^5 + ([-0.1666666666666667 +/- ...e-16])*x^4 + ... + 1
sage: Pol.<x> = SR[]
sage: x.revert_series(6)
Traceback (most recent call last):
... NotImplementedError: only implemented for certain base rings
```

root_field (names, check_irreducible=True)
Return the field generated by the roots of the irreducible polynomial `self`. The output is either a number field, relative number field, a quotient of a polynomial ring over a field, or the fraction field of the base ring.

**EXAMPLES:**

```sage
sage: R.<x> = QQ['x']
sage: f = x^3 + x + 17
sage: f.root_field('a')  # needs sage.rings.number_field
Number Field in a with defining polynomial x^3 + x + 17
```

```sage
sage: R.<x> = QQ['x']
sage: f = x - 3
sage: f.root_field('b')  # needs sage.rings.number_field
Rational Field
```

```sage
sage: R.<x> = ZZ['x']
sage: f = x^3 + x + 17
sage: f.root_field('b')  # needs sage.rings.number_field
Number Field in b with defining polynomial x^3 + x + 17
```

```sage
sage: y = QQ['x'].0
sage: L.<a> = NumberField(y^3 - 2)
sage: R.<x> = L['x']
sage: f = x^3 + x + 17
sage: f.root_field(c)  # needs sage.rings.finite_rings
Number Field in c with defining polynomial x^3 + x + 17 over its base field
```

```sage
sage: R.<x> = PolynomialRing(GF(9, 'a'))
sage: f = x^3 + x^2 + 8
sage: K.<alpha> = f.root_field(); K
Univariate Quotient Polynomial Ring in alpha
over Finite Field in a of size 3^2 with modulus x^3 + x^2 + 2
sage: alpha^2 + 1
alpha^2 + 1
sage: alpha^3 + alpha^2
1
```

```sage
sage: R.<x> = QQ[]
sage: f = x^2
sage: K.<alpha> = f.root_field()  # needs sage.libs.pari
Traceback (most recent call last):
...
ValueError: polynomial must be irreducible
```

**roots** (`ring=None, multiplicities=True, algorithm=None, **kwds`)

Return the roots of this polynomial (by default, in the base ring of this polynomial).

**INPUT:**

- `ring` - the ring to find roots in
• multiplicities - bool (default: True) if True return list of pairs \((r, n)\), where \(r\) is the root and \(n\) is the multiplicity. If False, just return the unique roots, with no information about multiplicities.

• algorithm - the root-finding algorithm to use. We attempt to select a reasonable algorithm by default, but this lets the caller override our choice.

By default, this finds all the roots that lie in the base ring of the polynomial. However, the ring parameter can be used to specify a ring to look for roots in.

If the polynomial and the output ring are both exact (integers, rationals, finite fields, etc.), then the output should always be correct (or raise an exception, if that case is not yet handled).

If the output ring is approximate (floating-point real or complex numbers), then the answer will be estimated numerically, using floating-point arithmetic of at least the precision of the output ring. If the polynomial is ill-conditioned, meaning that a small change in the coefficients of the polynomial will lead to a relatively large change in the location of the roots, this may give poor results. Distinct roots may be returned as multiple roots, multiple roots may be returned as distinct roots, real roots may be lost entirely (because the numerical estimate thinks they are complex roots). Note that polynomials with multiple roots are always ill-conditioned; there’s a footnote at the end of the docstring about this.

If the output ring is a RealIntervalField or ComplexIntervalField of a given precision, then the answer will always be correct (or an exception will be raised, if a case is not implemented). Each root will be contained in one of the returned intervals, and the intervals will be disjoint. (The returned intervals may be of higher precision than the specified output ring.)

At the end of this docstring (after the examples) is a description of all the cases implemented in this function, and the algorithms used. That section also describes the possibilities for the algorithm keyword, for the cases where multiple algorithms exist.

EXAMPLES:

```
sage: # needs sage.libs.pari
sage: x = QQ['x'].0
sage: f = x^3 - 1
sage: f.roots()
\[(1, 1)\]
sage: f.roots(ring=CC)  # ... - low order bits slightly different on ppc  #...
\[(1.00000000000000, 1),
 (-0.500000000000000 - 0.86602540378443...*I, 1),
 (-0.500000000000000 + 0.86602540378443...*I, 1)\]
sage: f = (x^3 - 1)^2
sage: f.roots()
\[(1, 2)\]
sage: f = -19*x + 884736
sage: f.roots()
\[(884736/19, 1)\]
sage: (f^20).roots()
\[(884736/19, 20)\]
sage: # needs sage.rings.number_field
sage: K.<z> = CyclotomicField(3)
sage: f = K.defining_polynomial()
sage: f.roots(ring=GF(7))
\[(4, 1), (2, 1)\]
sage: g = f.change_ring(GF(7))
sage: g.roots()
\[(4, 1), (2, 1)\]
```

(continues on next page)
A new ring. In the example below, we add the special method `_roots_univariate_polynomial()` to the base ring, and observe that this method is called instead to find roots of polynomials over this ring. This facility can be used to easily extend root finding to work over new rings you introduce:

```python
sage: R.<x> = QQ[]
sage: (x^2 + 1).roots()  # needs sage.libs.pari
[]
sage: def my_roots(f, *args, **kwds):
    ....: return f.change_ring(CDF).roots()

sage: QQ._roots_univariate_polynomial = my_roots
sage: (x^2 + 1).roots()  # abs tol 1e-14
# needs numpy
[(2.7755575615628914e-17 - 1.0*I, 1), (0.999999999999997*I, 1)]
sage: del QQ._roots_univariate_polynomial
```

An example over RR, which illustrates that only the roots in RR are returned:

```python
sage: # needs numpy sage.rings.real_mpfr
sage: x = RR['x'].0
sage: f = x^3 - 2
sage: f.roots()
[(1.25992104989487, 1)]
sage: f.factor()
(x - 1.25992104989487) * (x^2 + 1.25992104989487*x + 1.58740105196820)
sage: x = RealField(100)['x'].0
sage: f = x^3 - 2
sage: f.roots()
[(1.2599210498948731647672106073, 1)]
```

Another example showing that only roots in the base ring are returned:

```python
sage: x = polygen(ZZ)
sage: f = (2*x - 3) * (x - 1) * (x + 1)
sage: f.roots()  # needs sage.libs.pari
[(1, 1), (-1, 1)]
sage: f.roots(ring=QQ)  # needs sage.libs.pari
```

(continues on next page)
An example where we compute the roots lying in a subring of the base ring:

```
sage: P = QQ['n']
sage: pol = (n - 1/2)^2 * (n - 1)^2 * (n - 2)
sage: pol.roots(ZZ)
[(2, 1), (1, 2)]
```

An example involving large numbers:

```
sage: # needs numpy sage.rings.real_mpfr
sage: x = RR['x'].0
sage: f = x^2 - 1e100
sage: f.roots()
[(-1.00000000000000e50, 1), (1.00000000000000e50, 1)]
sage: f = x^10 - 2 * (5*x - 1)^2
sage: f.roots(multiplicities=False)
[-1.6772670339941..., 0.19995479628..., 0.20004530611..., 1.5763035161844...]
```

Describing roots using radical expressions:

```
sage: x = QQ['x'].0
sage: f = x^2 + 2
sage: f.roots(SR)
[(-I*sqrt(2), 1), (I*sqrt(2), 1)]
sage: f.roots(SR, multiplicities=False)
[-I*sqrt(2), I*sqrt(2)]
```

The roots of some polynomials cannot be described using radical expressions:

```
sage: (x^5 - x + 1).roots(SR)
[]
```

For some other polynomials, no roots can be found at the moment due to the way roots are computed. github issue #17516 addresses these defects. Until that gets implemented, one such example is the following:

```
sage: f = x^6 - 300*x^5 + 30361*x^4 - 1061610*x^3 + 1141893*x^2 - 915320*x + 101724
sage: f.roots()
[]
```

Chapter 2. Univariate Polynomials
A purely symbolic roots example:

```
sage: # needs sage.symbolic
sage: X = var('X')
sage: f = expand((X - 1) * (X - I)^3 * (X^2 - sqrt(2))); f
X^6 - (3*I + 1)*X^5 - sqrt(2)*X^4 + (3*I - 3)*X^4 + (3*I + 1)*sqrt(2)*X^3
+ (I + 3)*X^3 - (3*I - 3)*sqrt(2)*X^2 - I*X^2 - (I + 3)*sqrt(2)*X + I*sqrt(2)
sage: f.roots()
[(I, 3), (-2^(1/4), 1), (2^(1/4), 1), (1, 1)]
```

The same operation, performed over a polynomial ring with symbolic coefficients:

```
sage: # needs sage.symbolic
sage: X = SR['X'].0
sage: f = (X - 1) * (X - I)^3 * (X^2 - sqrt(2)); f
X^6 + (-3*I - 1)*X^5 + (-sqrt(2) + 3*I - 3)*X^4 + ((3*I + 1)*sqrt(2) + I +
→3)*X^3
+ (-3*I - 3)*sqrt(2) - I)*X^2 + (-I + 3)*sqrt(2))*X + I*sqrt(2)
sage: f.roots()
[(I, 3), (-2^(1/4), 1), (2^(1/4), 1), (1, 1)]
sage: f.roots(multiplicities=False)
[I, -2^(1/4), 2^(1/4), 1]
```

A couple of examples where the base ring does not have a factorization algorithm (yet). Note that this is currently done via a rather naive enumeration, so could be very slow:

```
sage: R = Integers(6)
sage: S.<x> = R['x']
sage: p = x^2 - 1
sage: p.roots()
Traceback (most recent call last):
...
NotImplementedError: root finding with multiplicities for this polynomial
not implemented (try the multiplicities=False option)
sage: p.roots(multiplicities=False) # needs sage.libs.pari
[5, 1]
sage: R = Integers(9)
sage: A = PolynomialRing(R, 'y')
sage: y = A.gen()
sage: f = 10*y^2 - y^3 - 9
sage: f.roots(multiplicities=False) # needs sage.libs.pari
[1, 0, 3, 6]
```

An example over the complex double field (where root finding is fast, thanks to NumPy):

```
sage: # needs numpy sage.rings.complex_double
sage: R.<x> = CDF[]
sage: f = R.cyclotomic_polynomial(5); f
x^4 + x^3 + x^2 + x + 1.0
sage: f.roots(multiplicities=False) # abs tol 1e-9
[-0.8090169943749469 - 0.5877852522924724*I, -0.8090169943749473 + 0.
→3)*X^3
+ (-3*I - 3)*sqrt(2) - I)*X^2 + (-I + 3)*sqrt(2))*X + I*sqrt(2)
sage: f.roots(multiplicities=False)
[(I, 3), (-2^(1/4), 1), (2^(1/4), 1), (1, 1)]
sage: f.roots(multiplicities=False)
[I, -2^(1/4), 2^(1/4), 1]
```

(continues on next page)
Another example over RDF:

```
sage: x = RDF['x'].0
sage: ((x^3 - 1)).roots()  # abs tol 4e-16
\[ \{1.0000000000000002\} \]
```

More examples involving the complex double field:

```
sage: # needs numpy sage.rings.complex_double sage.rings.real_mpfr
sage: x = CDF['x'].0
sage: i = CDF.0
sage: f = x^3 + 2*i; f
x^3 + 2.0*I
sage: f.roots()
\[ (-1.09112363597172\ldots - 0.62996052494743\ldots*I, 1),
   (1.25992104989487\ldots*I, 1),
   (1.09112363597172\ldots - 0.62996052494743\ldots*I, 1) \]
sage: f.roots(multiplicities=False)
\[ -1.09112363597172\ldots - 0.62996052494743\ldots*I,
   1.25992104989487\ldots*I,
   1.09112363597172\ldots - 0.62996052494743\ldots*I \]
sage: [abs(f(z)) for z in f.roots(multiplicities=False)]  # abs tol 1e-14
\[ 8.95090418262362e-16, 8.728374398092689e-16, 1.0235750533041806e-15 \]
sage: f = i*x^3 + 2; f
I*x^3 + 2.0
sage: f.roots()
\[ (-1.09112363597172\ldots + 0.62996052494743\ldots*I, 1),
   (1.25992104989487\ldots*I, 1),
   (1.09112363597172\ldots + 0.62996052494743\ldots*I, 1) \]
sage: abs(f(f.roots()[0][0]))  # abs tol 1e-13
1.1102230246251565e-16
```

Examples using real root isolation:

```
sage: x = polygen(ZZ)
sage: f = x^2 - x - 1
sage: f.roots()  # needs sage.libs.pari
\[
\{\}
\]
sage: f.roots(ring=AA)  # needs sage.rings.number_field
\[ (-0.618033988749895?, 1), (1.618033988749895?, 1) \]
sage: # needs sage.rings.real_interval_field
sage: f.roots(ring=RIF)
```

(continues on next page)
Examples using complex root isolation:

```python
sage: x = polygen(ZZ)
sage: p = x^5 - x - 1
sage: p.roots()
# does not work
[]
sage: p.roots(ring=CIF)
# does not work
[(-0.1666666666666667? - 0.552770798392567?*I, 1), (-0.1666666666666667? + 0.552770798392567?*I, 1),
 (0.083954101371771? - 1.083954101371771?*I, 1),
 (0.083954101371771? + 1.083954101371771?*I, 1),
 (1.167309782641876? - 0.3524715460317263?*I, 1),
 (1.167309782641876? + 0.3524715460317263?*I, 1)]
sage: rts = p.roots(ring=QQbar); rts
[(-0.1666666666666667? - 0.552770798392567?*I, 1),
 (-0.1666666666666667? + 0.552770798392567?*I, 1),
 (0.083954101371771? - 1.083954101371771?*I, 1),
 (0.083954101371771? + 1.083954101371771?*I, 1),
 (1.167309782641876? - 0.3524715460317263?*I, 1),
 (1.167309782641876? + 0.3524715460317263?*I, 1)]
sage: p.roots(ring=AA)
[(-0.1666666666666667? - 0.552770798392567?*I, 1),
 (-0.1666666666666667? + 0.552770798392567?*I, 1),
 (0.083954101371771? - 1.083954101371771?*I, 1),
 (0.083954101371771? + 1.083954101371771?*I, 1),
 (1.167309782641876? - 0.3524715460317263?*I, 1),
 (1.167309782641876? + 0.3524715460317263?*I, 1)]
sage: p = (x - rts[4][0])^2 * (3*x^2 + x + 1)
# does not work
sage: p.roots(ring=QQbar)
[(-0.1666666666666667? - 0.552770798392567?*I, 1),
 (-0.1666666666666667? + 0.552770798392567?*I, 1),
 (0.083954101371771? - 1.083954101371771?*I, 1),
 (0.083954101371771? + 1.083954101371771?*I, 1),
 (1.167309782641876? - 0.3524715460317263?*I, 1),
 (1.167309782641876? + 0.3524715460317263?*I, 1)]
```
In some cases, it is possible to isolate the roots of polynomials over complex ball fields:

```
sage: # needs sage.libs.flint
sage: Pol.<x> = CBF[]
sage: (x^2 + 2).roots(multiplicities=False)
[[+/- ...e-19] + [+/- ...e-17]*I,
[+/- ...e-19] + [+/- ...e-17]*I]
sage: (x^3 - 1/2).roots(RBF, multiplicities=False)
[[0.7397053990773725 +/- ...e-17]]
sage: ((x - 1)^2).roots(multiplicities=False, proof=False)
doctest:...
UserWarning: roots may have been lost...
[[1.00000000000 +/- ...e-12] + [+/- ...e-11]*I,
[1.00000000000 +/- ...e-12] + [+/- ...e-12]*I]
```

Note that coefficients in a number field with defining polynomial \( x^2 + 1 \) are considered to be Gaussian rationals (with the generator mapping to \(+i\)), if you ask for complex roots:

```
sage: # needs sage.rings.number_field
sage: K.<im> = QuadraticField(-1)
sage: y = polygen(K)
sage: p = y^4 - 2 - im
sage: p.roots(ring=CC)
[(-1.214638932244183? - 0.141425052582394?*I, 1),
 (-0.141425052582394? + 1.214638932244183?*I, 1),
 (0.141425052582394? - 1.214638932244183?*I, 1),
 (1.214638932244183? + 0.141425052582394?*I, 1)]
```

Note that one should not use NumPy when wanting high precision output as it does not support any of the high precision types:

```
sage: # needs numpy sage.rings.real_mpfr sage.symbolic
sage: R.<x> = RealField(200)[]
sage: f = x^2 - R(pi)
sage: f.roots()(doctest... UserWarning: NumPy does not support arbitrary precision arithmetic.
The roots found will likely have less precision than you expect.
[(-1.7724538509055160272981674833411451827975494561223871281238, 1),
 (1.7724538509055160272981674833411451827975494561223871281238, 1)]
```
We can also find roots over number fields:

```python
sage: K.<z> = CyclotomicField(15)  # needs sage.rings.number_field
sage: R.<x> = PolynomialRing(K)   # needs sage.rings.number_field
sage: (x^2 + x + 1).roots()     # needs sage.rings.number_field
[(z^5, 1), (-z^5 - 1, 1)]
```

There are many combinations of floating-point input and output types that work. (Note that some of them are quite pointless like using `algorithm='numpy'` with high-precision types.)

```python
sage: # needs sage.rings.complex_double sage.rings.real_mpfr
sage: rflds = (RR, RDF, RealField(100))
sage: cflds = (CC, CDF, ComplexField(100))
sage: def cross(a, b):
    return list(cartesian_product_iterator([a, b]))
sage: flds = cross(rflds, rflds) + cross(rflds, cflds) + cross(cflds, cflds)
sage: for (fld_in, fld_out) in flds:
    x = polygen(fld_in)
    f = x^3 - fld_in(2)
    x2 = polygen(fld_out)
    f2 = x2^3 - fld_out(2)
    for algo in (None, pari, numpy):
        rts = f.roots(ring=fld_out, multiplicities=False)
        rts = sorted(rts, key=lambda x: x.imag())
        if fld_in == fld_out and algo is None:
            print('{} {}'.format(fld_in, rts))
        for rt in rts:
            assert(abs(f2(rt)) <= 1e-10)
            assert(rt.parent() == fld_out)
```

Real Field with 53 bits of precision [1.25992104989487]
Real Double Field [1.25999210498942...]
Real Field with 100 bits of precision [1.2599210498948731647672106073] Complex Field with 53 bits of precision [-0.629960524947433... - 1.0911236359717214035600726142*I, 1.2599210498948731647672106073, -0.6299605249474331647672106073 + 1.0911236359717214035600726142*I]

Notethat wecan findtherootsofapolynomial with algebraic coefficients:

```python
sage: # needs sage.rings.number_field
sage: rt2 = sqrt(AA(2))
sage: rt3 = sqrt(AA(3))
sage: x = polygen(AA)
sage: f = (x - rt2) * (x - rt3); f
x^2 - 3.146264369941973?*x + 2.449489742783178?
sage: rts = f.roots(); rts
[(1.414213562373095?, 1), (1.732050807568878?, 1)]
sage: rts[0][0] == rt2
True
sage: f.roots(ring=RealIntervalField(150))
```

(continues on next page)
We can handle polynomials with huge coefficients.

This number doesn’t even fit in an IEEE double-precision float, but RR and CC allow a much larger range of floating-point numbers:

\[
\begin{align*}
\text{sage: } & \text{bigc} = 2^{1500} \\
\text{sage: } & \text{CDF(bigc)} \quad \text{# } \text{needs sage.rings.complex_double} \\
\text{+infinity} \\
\text{sage: } & \text{CC(bigc)} \quad \text{# } \text{needs sage.rings.real_mpfr} \\
& 3.50746621104340e451
\end{align*}
\]

Polynomials using such large coefficients can’t be handled by numpy, but pari can deal with them:

\[
\begin{align*}
\text{sage: } & \text{x = polygen(QQ)} \\
\text{sage: } & \text{p = x + bigc} \\
\text{sage: } & \text{p.roots(ring=RR, algorithm='numpy')} \quad \text{# needs numpy sage.rings.real_mpfr} \\
\text{Traceback (most recent call last):} \\
\text{...} \\
\text{LinAlgError: Array must not contain infs or NaNs} \\
\text{sage: } & \text{p.roots(ring=RR, algorithm='pari')} \quad \text{# needs sage.libs.pari sage.rings.real_mpfr} \\
& [(-3.50746621104340e451, 1)] \\
\text{sage: } & \text{p.roots(ring=AA)} \quad \text{# needs sage.rings.number_field} \\
& [(-3.5074662110434039e451, 1)] \\
\text{sage: } & \text{p.roots(ring=QQbar)} \quad \text{# needs sage.rings.number_field} \\
& [(-3.5074662110434039e451, 1)] \\
\text{sage: } & \text{p = bigc*x + 1} \\
\text{sage: } & \text{p.roots(ring=RR)} \quad \text{# needs numpy} \\
& [(-2.85106096489671e-452, 1)] \\
\text{sage: } & \text{p.roots(ring=AA)} \quad \text{# needs sage.rings.number_field} \\
& [(-2.8510609648967059e-452, 1)] \\
\text{sage: } & \text{p.roots(ring=QQbar)} \quad \text{# needs sage.rings.number_field} \\
& [(-2.8510609648967059e-452, 1)] \\
\text{sage: } & \text{p = x^2 - bigc} \\
\text{sage: } & \text{p.roots(ring=RR)} \quad \text{# needs numpy} \\
& [(-5.92238652153286e225, 1), (5.92238652153286e225, 1)] \\
\text{sage: } & \text{p.roots(ring=QQbar)} \quad \text{# needs sage.rings.number_field} \\
& [(-5.9223865215328558e225, 1), (5.9223865215328558e225, 1)]
\end{align*}
\]

Check that github issue #30522 is fixed:

\[
\begin{align*}
\text{sage: } & \text{PolynomialRing(SR, names="x")("x^2").roots()} \quad \text{# needs sage.symbolic} \\
& [(0, 2)]
\end{align*}
\]

Check that github issue #30523 is fixed:
ALGORITHM:
For brevity, we will use RR to mean any RealField of any precision; similarly for RIF, CC, and CIF. Since Sage has no specific implementation of Gaussian rationals (or of number fields with embedding, at all), when we refer to Gaussian rationals below we will accept any number field with defining polynomial \(x^2 + 1\), mapping the field generator to \(+i\).

We call the base ring of the polynomial \(K\), and the ring given by the ring argument \(L\). (If ring is not specified, then \(L\) is the same as \(K\).)

If \(K\) and \(L\) are floating-point (RDF, CDF, RR, or CC), then a floating-point root-finder is used. If \(L\) is RDF or CDF, then we default to using NumPy’s \texttt{roots()}; otherwise, we use PARI’s function \texttt{pari:polroots}. This choice can be overridden with algorithm='pari' or algorithm='numpy'. If the algorithm is unspecified and NumPy’s \texttt{roots()} algorithm fails, then we fall back to PARI (NumPy will fail if some coefficient is infinite, for instance).

If \(L\) is SR (or one of its subrings), then the roots will be radical expressions, computed as the solutions of a symbolic polynomial expression. At the moment this delegates to \texttt{sage.symbolic.expression.Expression.solve()} which in turn uses Maxima to find radical solutions. Some solutions may be lost in this approach. Once github issue \#17516 gets implemented, all possible radical solutions should become available.

If \(L\) is AA or RIF, and \(K\) is ZZ, QQ, or AA, then the root isolation algorithm \texttt{sage.rings.polynomial.real_roots.real_roots()} is used. (You can call \texttt{real_roots()} directly to get more control than this method gives.)

If \(L\) is QQbar or CIF, and \(K\) is ZZ, QQ, AA, QQbar, or the Gaussian rationals, then the root isolation algorithm \texttt{sage.rings.polynomial.complex_roots.complex_roots()} is used. (You can call \texttt{complex_roots()} directly to get more control than this method gives.)

If \(L\) is AA and \(K\) is QQbar or the Gaussian rationals, then \texttt{complex_roots()} is used (as above) to find roots in QQbar, then these roots are filtered to select only the real roots.

If \(L\) is floating-point and \(K\) is not, then we attempt to change the polynomial ring to \(L\) (using \texttt{change_ring()}) (or, if \(L\) is complex and \(K\) is not, to the corresponding real field). Then we use either PARI or NumPy as specified above.

For all other cases where \(K\) is different from \(L\), we attempt to use \texttt{.change_ring(L)}. When that fails but \(L\) is a subring of \(K\), we also attempt to compute the roots over \(K\) and filter the ones belonging to \(L\).

The next method, which is used if \(K\) is an integral domain, is to attempt to factor the polynomial. If this succeeds, then for every degree-one factor \(ax + b\), we add \(-b/a\) as a root (as long as this quotient is actually in the desired ring).

If factoring over \(K\) is not implemented (or \(K\) is not an integral domain), and \(K\) is finite, then we find the roots by enumerating all elements of \(K\) and checking whether the polynomial evaluates to zero at that value.

Note: We mentioned above that polynomials with multiple roots are always ill-conditioned; if your input is given to \(n\) bits of precision, you should not expect more than \(n/k\) good bits for a \(k\)-fold root. (You can get solutions that make the polynomial evaluate to a number very close to zero; basically the problem is that with a multiple root, there are many such numbers, and it’s difficult to choose between them.)

To see why this is true, consider the naive floating-point error analysis model where you just pretend that all floating-point numbers are somewhat imprecise - a little ‘fuzzy’, if you will. Then the graph of a floating-point polynomial will be a fuzzy line. Consider the graph of \((x - 1)^3\); this will be a fuzzy line with a horizontal
tangent at $x = 1$, $y = 0$. If the fuzziness extends up and down by about $j$, then it will extend left and right by about $\text{cube_root}(j)$.

**shift** $(n)$

Return this polynomial multiplied by the power $x^n$. If $n$ is negative, terms below $x^n$ will be discarded. Does not change this polynomial (since polynomials are immutable).

**EXAMPLES:**

```
sage: R.<x> = QQ[]
sage: p = x^2 + 2*x + 4
sage: p.shift(0)
x^2 + 2*x + 4
sage: p.shift(-1)
x + 2
sage: p.shift(-5)
0
sage: p.shift(2)
x^4 + 2*x^3 + 4*x^2
```

One can also use the infix shift operator:

```
sage: f = x^3 + x
sage: f >> 2
x
sage: f << 2
x^5 + x^3
```

**AUTHORS:**

- David Harvey (2006-08-06)

**specialization** $(D=None, phi=None)$

Specialization of this polynomial.

Given a family of polynomials defined over a polynomial ring. A specialization is a particular member of that family. The specialization can be specified either by a dictionary or a `SpecializationMorphism`.

**INPUT:**

- $D$ – dictionary (optional)
- $\phi$ – `SpecializationMorphism` (optional)

**OUTPUT:** a new polynomial

**EXAMPLES:**

```
sage: R.<c> = PolynomialRing(ZZ)
sage: S.<z> = PolynomialRing(R)
sage: F = c*z^2 + c^2
sage: F.specialization({c:2})
2*z^2 + 4
```

(continues on next page)
sage: X
1/20*x + 1
sage: X.parent()
Univariate Polynomial Ring in x over Rational Field

\textbf{splitting\_field} (\textit{names}=None, \textit{map}=False, **\textit{kwds})

Compute the absolute splitting field of a given polynomial.

\textbf{INPUT}:

\begin{itemize}
  \item names – (default: None) a variable name for the splitting field.
  \item map – (default: False) also return an embedding of \textit{self} into the resulting field.
  \item kwds – additional keywords depending on the type. Currently, only number fields are implemented.
    See \textit{sage.rings.number_field.splitting\_field.splitting\_field()} for the documentation of these keywords.
\end{itemize}

\textbf{OUTPUT}:

If \textit{map} is False, the splitting field as an absolute field. If \textit{map} is True, a tuple (\textit{K}, \textit{phi}) where \textit{phi} is an embedding of the base field of \textit{self} in \textit{K}.

\textbf{EXAMPLES}:

\begin{verbatim}
sage: R.<x> = PolynomialRing(ZZ)
sage: K.<a> = (x^3 + 2).splitting_field(); K
Number Field in a with defining polynomial x^6 + 3*x^5 + 6*x^4 + 11*x^3 + 12*x^2 - 3*x + 1
sage: K.<a> = (x^3 - 3*x + 1).splitting_field(); K
Number Field in a with defining polynomial x^3 - 3*x + 1
\end{verbatim}

Relative situation:

\begin{verbatim}
sage: # needs sage.rings.number_field
sage: R.<x> = PolynomialRing(QQ)
sage: K.<a> = NumberField(x^3 + 2)
sage: S.<t> = PolynomialRing(K)
sage: L.<b> = (t^2 - a).splitting_field()
sage: L
Number Field in b with defining polynomial t^6 + 2
\end{verbatim}

With \textit{map=\text{True}}, we also get the embedding of the base field into the splitting field:

\begin{verbatim}
sage: L.<b>, phi = (t^2 - a).splitting_field(map=True)
sage: phi
Ring morphism:
  From: Number Field in a with defining polynomial x^3 + 2
  To:   Number Field in b with defining polynomial t^6 + 2
  Defn: a |--> b^2
\end{verbatim}

An example over a finite field:
If the extension is trivial and the generators have the same name, the map will be the identity:

```
sage: t = 24*x^13 + 2*x^12 + 14
sage: t.splitting_field('a', map=True)  #...
(Finite Field in a of size 7^3,
 Identity endomorphism of Finite Field in a of size 7^3)
```

See also:

`sage.rings.number_field.splitting_field.splitting_field()` for more examples over number fields

`square()`

Return the square of this polynomial.

**Todo:**

- This is just a placeholder; for now it just uses ordinary multiplication. But generally speaking, squaring is faster than ordinary multiplication, and it's frequently used, so subclasses may choose to provide a specialised squaring routine.
- Perhaps this even belongs at a lower level? RingElement or something?

**AUTHORS:**

- David Harvey (2006-09-09)

**EXAMPLES:**
```
sage: R.<x> = QQ[]
sage: f = x^3 + 1
sage: f.square()
x^6 + 2*x^3 + 1
sage: f*f
x^6 + 2*x^3 + 1
```

**squarefree_decomposition()**

Return the square-free decomposition of this polynomial. This is a partial factorization into square-free, coprime polynomials.

**EXAMPLES:**
```
sage: x = polygen(QQ)
sage: p = 37 * (x - 1)^3 * (x - 2)^3 * (x - 1/3)^7 * (x - 3/7)
sage: p.squarefree_decomposition()
(37*x - 111/7) * (x^2 - 3*x + 2)^3 * (x - 1/3)^7
sage: p = 37 * (x - 2/3)^2
sage: p.squarefree_decomposition()
(37) * (x - 2/3)^2
sage: x = polygen(GF(3))
sage: x.squarefree_decomposition()
x
sage: f = QQbar['x'](1) # needs sage.rings.number_field
sage: f.squarefree_decomposition() # needs sage.rings.number_field
1
```

**subresultants (other)**

Return the nonzero subresultant polynomials of self and other.

**INPUT:**
- `other` – a polynomial

**OUTPUT:** a list of polynomials in the same ring as self

**EXAMPLES:**
```
sage: R.<x> = ZZ[]
sage: f = x^8 + x^6 - 3*x^4 - 3*x^3 + 8*x^2 + 2*x - 5
sage: g = 3*x^6 + 5*x^4 - 4*x^2 - 9*x + 21
sage: f.subresultants(g)
[260708,
  9326*x - 12300,
  169*x^2 + 325*x - 637,
  65*x^2 + 125*x - 245,
  25*x^4 - 5*x^2 + 15,
  15*x^4 - 3*x^2 + 9]
```

**ALGORITHM:**

We use the schoolbook algorithm with Lazard’s optimization described in [Duc1998]

**REFERENCES:**

Wikipedia article Polynomial_greatest_common_divisor#Subresultants
**subs** *(in_dict=None, *args, **kwds)*

Substitute the variable in self.

**EXAMPLES:**

```
sage: R.<x> = QQ[]
sage: f = x^3 + x - 3
sage: f.subs(x=5)
127
sage: f.subs(5)
127
sage: f.subs({x:2})
7
sage: f.subs({})
x^3 + x - 3
sage: f.subs({'x':2})
Traceback (most recent call last):
...  
TypeError: keys do not match self's parent
```

**sylvester_matrix** *(right, variable=None)*

Return the Sylvester matrix of self and right.

Note that the Sylvester matrix is not defined if one of the polynomials is zero.

**INPUT:**

- **right** – a polynomial in the same ring as self.
- **variable** – optional, included for compatibility with the multivariate case only. The variable of the polynomials.

**EXAMPLES:**

```
sage: R.<x> = PolynomialRing(ZZ)
sage: f = (6*x + 47) * (7*x^2 - 2*x + 38)
sage: g = (6*x + 47) * (3*x^3 + 2*x + 1)
sage: M = f.sylvester_matrix(g); M
[ 42 317 134 1786 0 0 0]
[ 0 42 317 134 1786 0 0]
[ 0 0 42 317 134 1786 0]
[ 0 0 0 42 317 134 1786]
[ 18 141 12 100 47 0 0]
[ 0 18 141 12 100 47 0]
[ 0 0 18 141 12 100 47]
```

If the polynomials share a non-constant common factor then the determinant of the Sylvester matrix will be zero:

```
sage: M.det()  # needs sage.modules
0
```

If self and right are polynomials of positive degree, the determinant of the Sylvester matrix is the resultant of the polynomials:
sage: M1 = h1.sylvester_matrix(h2)  # needs sage.modules
sage: M1.determinant() == h1.resultant(h2)  # needs sage.libs.pari sage.modules
True

The rank of the Sylvester matrix is related to the degree of the gcd of self and right:

sage: f.gcd(g).degree() == f.degree() + g.degree() - M.rank()  # needs sage.modules
True
sage: h1.gcd(h2).degree() == h1.degree() + h2.degree() - M1.rank()  # needs sage.modules
True

**symmetric_power** *(k, monic=False)*

Return the polynomial whose roots are products of *k*-th distinct roots of this.

**EXAMPLES:**

```python
sage: x = polygen(QQ)
sage: f = x^4 - x + 2
sage: [f.symmetric_power(k) for k in range(5)]
[x - 1, x^4 - x + 2, x^6 - 2*x^4 - x^3 - 4*x^2 + 8, x^4 - x^3 + 8, x - 2]
sage: f = x^5 - 2*x + 2
sage: [f.symmetric_power(k) for k in range(6)]
[x - 1, x^5 - 2*x + 2, x^10 + 2*x^8 - 4*x^6 - 8*x^5 - 8*x^4 - 8*x^3 + 16,
 x^10 + 4*x^7 - 8*x^6 + 16*x^5 - 16*x^4 + 32*x^2 + 64,
 x^5 + 2*x^4 - 16,
 x + 2]
sage: R.<a,b,c,d> = ZZ[]
sage: x = polygen(R)
sage: f = (x - a) * (x - b) * (x - c) * (x - d)
sage: [f.symmetric_power(k).factor() for k in range(5)]
[x - 1,
 (-x + d) * (-x + c) * (-x + b) * (-x + a),
 (x - c*d) * (x - b*d) * (x - a*d) * (x - b*c) * (x - a*c) * (x - a*b),
 (x - b*c*d) * (x - a*c*d) * (x - a*b*d) * (x - a*b*c),
 x - a*b*c*d]
```

**trace_polynomial()**

Compute the trace polynomial and cofactor.

The input *P* and output *Q* satisfy the relation

\[ P(x) = Q(x + q/x)x^{\deg(Q)}R(x). \]

In this relation, *Q* has all roots in the real interval \([-2\sqrt{q}, 2\sqrt{q}]\) if and only if *P* has all roots on the circle \(|x| = \sqrt{q}\) and *R* divides \(x^2 - q\). We thus require that the base ring of this polynomial have a coercion to the real numbers.
See also:

The inverse operation is `reciprocal_transform()`.

**OUTPUT:**

- $Q$ – trace polynomial
- $R$ – cofactor
- $q$ – scaling factor

**EXAMPLES:**

```python
sage: pol.<x> = PolynomialRing(Rationals())
sage: u = x^5 - 1; u.trace_polynomial()
(x^2 + x - 1, x - 1, 1)
sage: u = x^4 + x^3 + 5*x^2 + 3*x + 9
sage: u.trace_polynomial()
(x^2 + x - 1, 1, 3)
```

We check that this function works for rings that have a coercion to the reals:

```python
sage: # needs sage.rings.number_field
sage: K.<a> = NumberField(x^2 - 2, embedding=1.4)
sage: u = x^4 + a*x^3 + 3*x^2 + 2*a*x + 4
sage: u.trace_polynomial()
(x^2 + a*x - 1, 1, 2)
sage: (u*(x^2-2)).trace_polynomial()
(x^2 + a*x - 1, x^2 - 2, 2)
sage: (u*(x^2-2)^2).trace_polynomial()
(x^4 + a*x^3 - 9*x^2 - 8*a*x + 8, x^2 - 2, 2)
sage: u = x^4 + a*x^3 + 3*x^2 + 4*a*x + 16
sage: u.trace_polynomial()
(x^2 + a*x - 5, 1, 4)
sage: (u*(x-2)).trace_polynomial()
(x^2 + a*x - 5, x - 2, 4)
sage: (u*(x+2)).trace_polynomial()
(x^2 + a*x - 5, x + 2, 4)
```

**trunc**`ate**\((n)\)**

Return the polynomial of degree $< n$ which is equivalent to self modulo $x^n$.

**EXAMPLES:**

```python
sage: R.<x> = ZZ[]; S.<y> = PolynomialRing(R, sparse=True)
sage: f = y^3 + x*y - 3*x; f
y^3 + x*y - 3*x
sage: f.truncate(2)
x*y - 3*x
sage: f.truncate(1)
-3*x
sage: f.truncate(0)
0
```

**valuation**\((p=None)\)**

If $f = a_rx^r + a_{r+1}x^{r+1} + \cdots$, with $a_r$ nonzero, then the valuation of $f$ is $r$. The valuation of the zero polynomial is $\infty$. 

112 Chapter 2. Univariate Polynomials
If a prime (or non-prime) $p$ is given, then the valuation is the largest power of $p$ which divides $\text{self}$.

The valuation at $\infty$ is $-\text{self}.\text{degree()}$.

**EXAMPLES:**

```sage
sage: P.<x> = ZZ[]
sage: (x^2 + x).valuation()
1
sage: (x^2 + x).valuation(x + 1)
1
sage: (x^2 + 1).valuation()
0
sage: (x^3 + 1).valuation(infinity)
-3
sage: P(0).valuation()
+Infinity
```

**variable_name()**

Return name of variable used in this polynomial as a string.

OUTPUT: string

**EXAMPLES:**

```sage
sage: R.<t> = QQ[]
sage: f = t^3 + 3/2*t + 5
sage: f.variable_name()
"t"
```

**variables()**

Return the tuple of variables occurring in this polynomial.

**EXAMPLES:**

```sage
sage: R.<x> = QQ[]
sage: x.variables()
(x,)
```

A constant polynomial has no variables.

```sage
sage: R(2).variables()
()  
```

**xgcd**(other)

Return an extended gcd of this polynomial and other.

**INPUT:**

- other – a polynomial in the same ring as this polynomial

**OUTPUT:**

A tuple $(r, s, t)$ where $r$ is a greatest common divisor of this polynomial and other, and $s$ and $t$ are such that $r = s*\text{self} + t*\text{other}$ holds.

**Note:** The actual algorithm for computing the extended gcd depends on the base ring underlying the polynomial ring. If the base ring defines a method `_xgcd_univariate_polynomial()`, then this method will be called (see examples below).
EXAMPLES:

```python
sage: # needs sage.rings.number_field
sage: R.<x> = QQbar[]
sage: (2*x^2).gcd(2*x)
x
sage: R.zero().gcd(0)
0
sage: (2*x).gcd(0)
x
```

One can easily add \texttt{gcd} functionality to new rings by providing a method \texttt{gcd} function:

```python
sage: R.<x> = QQ[]
sage: S.<y> = R[]
sage: h1 = y*x
sage: h2 = y^2*x^2
sage: h1.xgcd(h2)
Traceback (most recent call last):
  ...  
NotImplementedError: Univariate Polynomial Ring in x over Rational Field
does not provide an \texttt{gcd} implementation for univariate polynomials
sage: T.<x,y> = QQ[]
sage: def poor_xgcd(f, g):
    ...:     ret = S(T(f).gcd(g))
    ...:     if ret == f: return S.one(), S.zero()
    ...:     if ret == g: return S.zero(), S.one()
    ...:     raise
sage: R._xgcd_univariate_polynomial = poor_xgcd
sage: h1.xgcd(h2)
(x*y, 1, 0)
sage: del R._xgcd_univariate_polynomial
```

class \texttt{sage.rings.polynomial.polynomial_element.PolynomialBaseringInjection}

Bases: \texttt{Morphism}

This class is used for conversion from a ring to a polynomial over that ring.

It calls the \texttt{new_constant_poly()} method on the generator, which should be optimized for a particular polynomial type.

Technically, it should be a method of the polynomial ring, but few polynomial rings are Cython classes, and so, as a method of a Cython polynomial class, it is faster.

EXAMPLES:

We demonstrate that most polynomial ring classes use polynomial base injection maps for coercion. They are supposed to be the fastest maps for that purpose. See github issue \#9944.

```python
sage: # needs sage.rings.padics
sage: R.<x> = Qp(3)[]
Polynomial base injection morphism:
  From: 3-adic Field with capped relative precision 20
  To:   Univariate Polynomial Ring in x over 3-adic Field with capped relative precision 20
sage: R.<x,y> = Qp(3)[]
sage: R.coerce_map_from(R.base_ring())
```

(continues on next page)
Polynomials, Release 10.3

Polynomial base injection morphism:
From: 3-adic Field with capped relative precision 20
To: Multivariate Polynomial Ring in x, y over 3-adic Field with capped relative precision 20

sage: R.<x,y> = QQ[]
sage: R.coerce_map_from(R.base_ring())
Polynomial base injection morphism:
From: Rational Field
To: Multivariate Polynomial Ring in x over Rational Field
sage: R.<x> = QQ[]
sage: R.coerce_map_from(R.base_ring())
Polynomial base injection morphism:
From: Rational Field
To: Univariate Polynomial Ring in x over Rational Field

By github issue #9944, there are now only very few exceptions:

sage: PolynomialRing(QQ,names=[]).coerce_map_from(QQ)
Call morphism:
From: Rational Field
To: Multivariate Polynomial Ring in no variables over Rational Field

is_injective()  
Return whether this morphism is injective.

EXAMPLES:

sage: R.<x> = ZZ[]
sage: S.<y> = R[]
sage: S.coerce_map_from(R).is_injective()
True

Check that github issue #23203 has been resolved:

sage: R.is_subring(S)  # indirect doctest
True

is_surjective()  
Return whether this morphism is surjective.

EXAMPLES:

sage: R.<x> = ZZ[]
sage: R.coerce_map_from(ZZ).is_surjective()
False

section()  

class sage.rings.polynomial.polynomial_element.Polynomial_generic_dense
Bases: Polynomial

A generic dense polynomial.

EXAMPLES:
Polynomials, Release 10.3

```python
sage: f = QQ['x']['y'].random_element()
sage: loads(f.dumps()) == f
True
```

### constant_coefficient()
Return the constant coefficient of this polynomial.

**OUTPUT:** element of base ring

**EXAMPLES:**

```python
sage: R.<t> = QQ[]
sage: S.<x> = R[]
sage: f = x*t + x + t
sage: f.constant_coefficient()
t
```

### degree(gen=None)

**EXAMPLES:**

```python
sage: R.<x> = RDF[]
sage: f = (1+2*x^7)^5
sage: f.degree()
35
```

### is_term()
Return True if this polynomial is a nonzero element of the base ring times a power of the variable.

**EXAMPLES:**

```python
sage: # needs sage.symbolic
sage: R.<x> = SR[]
sage: R(0).is_term()  # False
sage: R(1).is_term()  # True
sage: (3*x^5).is_term()  # True
sage: (1 + 3*x^5).is_term()  # False
```

### list(copy=True)
Return a new copy of the list of the underlying elements of self.

**EXAMPLES:**

```python
sage: R.<x> = GF(17)[]
sage: f = (1+2*x)^3 + 3*x; f
8*x^3 + 12*x^2 + 9*x + 1
sage: f.list()
[1, 9, 12, 8]
```

### quo_rem(other)
Return the quotient and remainder of the Euclidean division of self and other.

Raises a **ZeroDivisionError** if other is zero. Raises an **ArithmeticError** if the division is not exact.

**EXAMPLES:**
Polynomials over noncommutative rings are also allowed (after github issue #34733):

```python
sage: HH = QuaternionAlgebra(QQ, -1, -1)
sage: P.<x> = HH[

```

**shift** ($n$)

Return this polynomial multiplied by the power $x^n$.

If $n$ is negative, terms below $x^n$ will be discarded. Does not change this polynomial.

**EXAMPLES:**

```python
sage: p = x^2 + 2*x + 4
sage: type(p)
<class 'sage.rings.polynomial.polynomial_element.Polynomial_generic_dense'>
sage: p.shift(0)
x^2 + 2*x + 4
sage: p.shift(-1)
x + 2
sage: p.shift(2)
x^4 + 2*x^3 + 4*x^2
```
```
sage: S.<q> = QQ['t']['q']
sage: f = (1 + q^10 + q^11 + q^12).truncate(11); f
q^10 + 1
sage: f = (1 + q^10 + q^100).truncate(50); f
q^10 + 1
sage: f = (1 + q^10 + q^100).truncate(500); f
q^100 + q^10 + 1
```

```python
class sage.rings.polynomial.polynomial_element.Polynomial_generic_dense_inexact
    Bases: Polynomial_generic_dense

    A dense polynomial over an inexact ring.

    AUTHOR:
    • Xavier Caruso (2013-03)

    degree (secure=False)

    INPUT:
    • secure -- a boolean (default: False)

    OUTPUT: The degree of self.

    If secure is True and the degree of this polynomial is not determined (because the leading coefficient is indistinguishable from 0), an error is raised.

    If secure is False, the returned value is the largest \( n \) so that the coefficient of \( x^n \) does not compare equal to 0.

    EXAMPLES:
```
sage: # needs sage.rings.padics
sage: K = Qp(3, 10)
sage: R.<T> = K[]
sage: f = T + 2; f
(1 + O(3^10))*T + 2 + O(3^10)
sage: f.degree()
1
sage: (f - T).degree()
0
sage: (f - T).degree(secure=True)
Traceback (most recent call last):
  ...
PrecisionError: the leading coefficient is indistinguishable from 0
```
sage: # needs sage.rings.padics
sage: x = O(3^5)
sage: li = [3^i * x for i in range(0,5)]; li
[0(3^5), O(3^6), O(3^7), O(3^8), O(3^9)]
sage: f = R(li); f
O(3^9)*T^4 + O(3^8)*T^3 + O(3^7)*T^2 + O(3^6)*T + O(3^5)
sage: f.degree()
-1
sage: f.degree(secure=True)
Traceback (most recent call last):
```
PrecisionError: the leading coefficient is indistinguishable from 0

AUTHOR:
- Xavier Caruso (2013-03)

prec_degree()
Return the largest \( n \) so that precision information is stored about the coefficient of \( x^n \).
Always greater than or equal to degree.

EXAMPLES:

```sage
sage: # needs sage.rings.padics
sage: K = Qp(3, 10)
sage: R.<T> = K[]
sage: f = T + 2; f
(1 + O(3^10))*T + 2 + O(3^10)
sage: f.degree()
1
sage: f.prec_degree()
1

sage: g = f - T; g
# needs sage.rings.padics
O(3^10)*T + 2 + O(3^10)
sage: g.degree()
# needs sage.rings.padics
0
sage: g.prec_degree()
# needs sage.rings.padics
1
```

AUTHOR:
- Xavier Caruso (2013-03)

sage.rings.polynomial.polynomial_element.generic_power_trunc(p, n, prec)
Generic truncated power algorithm

INPUT:
- \( p \) - a polynomial
- \( n \) - an integer (of type sage.rings.integer.Integer)
- \( \text{prec} \) - a precision (should fit into a C long)

sage.rings.polynomial.polynomial_element.is_Polynomial(f)
Return True if \( f \) is of type univariate polynomial.
This function is deprecated.

INPUT:
- \( f \) - an object

EXAMPLES:
sage: from sage.rings.polynomial.polynomial_element import is_Polynomial
sage: R.<x> = ZZ[]
sage: is_Polynomial(x^3 + x + 1)
doctest:...: DeprecationWarning: the function is_Polynomial is deprecated; use isinstance(x, sage.rings.polynomial.polynomial_element.Polynomial) instead
See https://github.com/sagemath/sage/issues/32709 for details.
True
sage: S.<y> = R[]
sage: f = y^3 + x*y - 3*x; f
y^3 + x*y - 3*x
sage: is_Polynomial(f)
True

However this function does not return True for genuine multivariate polynomial type objects or symbolic polynomials, since those are not of the same data type as univariate polynomials:

sage: R.<x,y> = QQ[]
sage: f = y^3 + x*y - 3*x; f
y^3 + x*y - 3*x
sage: is_Polynomial(f)
False

sage: # needs sage.symbolic
sage: var('x,y')
(x, y)
sage: f = y^3 + x*y - 3*x; f
y^3 + x*y - 3*x
sage: is_Polynomial(f)
False

sage.rings.polynomial.polynomial_element.make_generic_polynomial(parent, coeffs)

sage.rings.polynomial.polynomial_element.polynomial_is_variable(x)

Test whether the given polynomial is a variable of its parent ring.

Implemented for instances of Polynomial and MPolynomial.

See also:

• sage.rings.polynomial.polynomial_element.Polynomial.is_gen()
• sage.rings.polynomial.multi_polynomial.MPolynomial.is_generator()

EXAMPLES:

sage: from sage.rings.polynomial.polynomial_element import polynomial_is_variable
sage: R.<x> = QQ[]
sage: polynomial_is_variable(x)
True
sage: polynomial_is_variable(R([0,1]))
True
sage: polynomial_is_variable(x^2)
False
sage: polynomial_is_variable(R(42))
False

Chapter 2. Univariate Polynomials
sage: R.<y,z> = QQ[]
sage: polynomial_is_variable(y)
True
sage: polynomial_is_variable(z)
True
sage: polynomial_is_variable(y^2)
False
sage: polynomial_is_variable(y+z)
False
sage: polynomial_is_variable(R(42))
False
sage: polynomial_is_variable(42)
False

sage.rings.polynomial.polynomial_element.universal_discriminant()

Return the discriminant of the 'universal' univariate polynomial \(a_n x^n + \cdots + a_1 x + a_0\) in \(\mathbb{Z}[a_0, \ldots, a_n][x]\).

INPUT:

- \(n\) - degree of the polynomial

OUTPUT:

The discriminant as a polynomial in \(n + 1\) variables over \(\mathbb{Z}\). The result will be cached, so subsequent computations of discriminants of the same degree will be faster.

EXAMPLES:

sage: # needs sage.libs.pari
sage: from sage.rings.polynomial.polynomial_element import universal_discriminant
sage: universal_discriminant(1)
1
sage: universal_discriminant(2)
a1^2 - 4*a0*a2
sage: universal_discriminant(3)
a1^2*a2^2 - 4*a0*a2^3 - 4*a1^3*a3 + 18*a0*a1*a2*a3 - 27*a0^2*a3^2
sage: universal_discriminant(4).degrees()
(3, 4, 4, 4, 3)

See also:

Polynomial.discriminant()
class sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_cdv (parent, is_gen=False, construct=False)

Bases: Polynomial_generic_domain

A generic class for polynomials over complete discrete valuation domains and fields.

AUTHOR:

• Xavier Caruso (2013-03)

factor_of_slope (slope=None)

INPUT:

• slope – a rational number (default: the first slope in the Newton polygon of self)

OUTPUT:

The factor of self corresponding to the slope slope (i.e. the unique monic divisor of self whose slope is slope and degree is the length of slope in the Newton polygon).

EXAMPLES:

```python
sage: # needs sage.geometry.polyhedron sage.rings.padics
sage: K = Qp(5)
sage: R.<x> = K[]
sage: f = 5 + 3*t + t^4 + 25*t^10
sage: f.newton_slopes()
[1, 0, 0, 0, -1/3, -1/3, -1/3, -1/3, -1/3, -1/3]
sage: g = f.factor_of_slope(0)
sage: g.newton_slopes()
[0, 0, 0]

sage: (f % g).is_zero()
True

sage: h = f.factor_of_slope()
sage: h.newton_slopes()
[1]

sage: (f % h).is_zero()
True
```

If slope is not a slope of self, the corresponding factor is 1:

```python
sage: f.factor_of_slope(-1)  # needs sage.geometry.polyhedron sage.rings.padics
1 + O(5^20)
```

AUTHOR:

• Xavier Caruso (2013-03-20)

hensel_lift (a)

Lift a to a root of this polynomial (using Newton iteration).

If a is not close enough to a root (so that Newton iteration does not converge), an error is raised.

EXAMPLES:
sage: # needs sage.rings.padics
sage: K = Qp(5, 10)
sage: P.<x> = PolynomialRing(K)
sage: f = x^2 + 1
sage: root = f.hensel_lift(2); root
2 + 5 + 2*5^2 + 5^3 + 3*5^4 + 4*5^5 + 2*5^6 + 3*5^7 + 3*5^9 + O(5^10)
sage: f(root)
O(5^10)

sage: g = (x^2 + 1) * (x - 7)  # needs sage.rings.padics
sage: g.hensel_lift(2)  # here, 2 is a multiple root modulo p
Traceback (most recent call last):
  ... ValueError: a is not close enough to a root of this polynomial

AUTHOR:
  • Xavier Caruso (2013-03-23)

**newton_polygon()**

Return a list of vertices of the Newton polygon of this polynomial.

Note: If some coefficients have not enough precision an error is raised.

**EXAMPLES:**

sage: # needs sage.geometry.polyhedron sage.rings.padics
sage: K = Qp(5)
sage: R.<t> = K[]
sage: f = 5 + 3*t + t^4 + 25*t^10
sage: f.newton_polygon()
Finite Newton polygon with 4 vertices: (0, 1), (1, 0), (4, 0), (10, 2)

sage: g = f + K(0,0)*t^4; g
(5^2 + O(5^22))*t^10 + O(5^0)*t^4 + (3 + O(5^20))*t + 5 + O(5^21)

sage: g.newton_polygon()
Traceback (most recent call last):
  ... PrecisionError: The coefficient of t^4 has not enough precision

AUTHOR:
  • Xavier Caruso (2013-03-20)

**newton_slopes**(repetition=True)

Return a list of the Newton slopes of this polynomial.

These are the valuations of the roots of this polynomial.

If repetition is True, each slope is repeated a number of times equal to its multiplicity. Otherwise it appears only one time.

**EXAMPLES:**

(continues on next page)
AUTHOR:

- Xavier Caruso (2013-03-20)

`sage_factorization()`

Return a factorization of self into a product of factors corresponding to each slope in the Newton polygon.

EXAMPLES:

```sage
# needs sage.geometry.polyhedron sage.rings.padics
sage: K = Qp(5)
sage: R.<x> = K[]
sage: K = Qp(5)
sage: R.<t> = K[]
sage: f = 5 + 3*t + t^4 + 25*t^10
sage: f.newton_slopes()
[1, 0, 0, -1/3, -1/3, -1/3, -1/3, -1/3, -1/3]
sage: F = f.slope_factorization()
sage: F.prod() == f
True
sage: for (f,_) in F:
    ....:     print(f.newton_slopes())
[-1/3, -1/3, -1/3, -1/3, -1/3, -1/3]
[0, 0, 0]
[1]
```

AUTHOR:

- Xavier Caruso (2013-03-20)
Bases: `Polynomial_generic_dense_cdv, Polynomial_generic_cdvf`

class sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_dense_cdvr
Bases: `Polynomial_generic_dense_cdv, Polynomial_generic_cdvr`

class sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_dense_field (parent, x=None, check=True, is_gen=False, construct=False)
Bases: `Polynomial_generic_dense, Polynomial_generic_field`

class sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_domain (parent, is_gen=False, construct=False)
Bases: `Polynomial, IntegralDomainElement`

`is_unit()`

Return `True` if this polynomial is a unit.

**EXERCISE** (Atiyah-McDonald, Ch 1): Let $A[x]$ be a polynomial ring in one variable. Then $f = \sum a_i x^i \in A[x]$ is a unit if and only if $a_0$ is a unit and $a_1, \ldots, a_n$ are nilpotent.

**EXAMPLES:**

```python
sage: R.<z> = PolynomialRing(ZZ, sparse=True)
sage: (2 + z^3).is_unit()
False
sage: f = -1 + 3*z^3; f
3*z^3 - 1
sage: f.is_unit()
False
sage: R(-3).is_unit()
False
sage: R(-1).is_unit()
True
sage: R(0).is_unit()
False
```

class sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_field (parent, is_gen=False, construct=False)
Bases: `Polynomial_singular_repr, Polynomial_generic_domain, EuclideanDomainElement`

`quo_rem(other)`

Return a tuple `(quotient, remainder)` where `self = quotient * other + remainder`.

**EXAMPLES:**

2.1. Univariate Polynomials and Polynomial Rings 125
sage: # needs sage.rings.number_field
sage: R.<y> = PolynomialRing(QQ)

sage: K.<t> = NumberField(y^2 - 2)

sage: P.<x> = PolynomialRing(K)

sage: x.quo_rem(K(1))
(x, 0)

sage: x.xgcd(K(1))
(1, 0, 1)

```
class sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_sparse (parent,
 x=None, check=True, is_gen=False, construct=False)

Bases: Polynomial

A generic sparse polynomial.

The Polynomial_generic_sparse class defines functionality for sparse polynomials over any base ring. A sparse polynomial is represented using a dictionary which maps each exponent to the corresponding coefficient. The coefficients must never be zero.

EXAMPLES:

```sage
R.<x> = PolynomialRing(PolynomialRing(QQ, y), sparse=True)
f = x^3 - x + 17
type(f)
<type 'sage.rings.polynomial.polynomial_ring.PolynomialRing_integral_domain_with_category.element_class'>
loads(f.dumps()) == f
True
```

A more extensive example:

```sage
# needs sage.libs.pari
A.<T> = PolynomialRing(Integers(5), sparse=True)
f = T^2 + 1; B = A.quo(f)
C.<s> = PolynomialRing(B)
C
Univariate Polynomial Ring in s over Univariate Quotient Polynomial Ring in Tbar over Ring of integers modulo 5 with modulus T^2 + 1
s + T
s + Tbar
(s + T)**2
s^2 + 2*Tbar*s + 4
```

```coefficients(sparse=True)```

Return the coefficients of the monomials appearing in self.

EXAMPLES:

```sage
R.<w> = PolynomialRing(Integers(8), sparse=True)
f = 5 + w^1997 - w^10000; f
7*w^10000 + w^1997 + 5
f.coefficients()
[5, 1, 7]
```
**degree** *(gen=None)*

Return the degree of this sparse polynomial.

**EXAMPLES:**

```
sage: R.<z> = PolynomialRing(ZZ, sparse=True)
sage: f = 13*z^50000 + 15*z^2 + 17*z
sage: f.degree()
50000
```

**dict()**

Return a new copy of the dict of the underlying elements of *self*.

**EXAMPLES:**

```
sage: R.<w> = PolynomialRing(Integers(8), sparse=True)
sage: f = 5 + w^1997 - w^10000; f
7*w^10000 + w^1997 + 5
sage: d = f.dict(); d
{0: 5, 1997: 1, 10000: 7}
sage: d[0] = 10
sage: f.dict()
{0: 5, 1997: 1, 10000: 7}
```

**exponents()**

Return the exponents of the monomials appearing in *self*.

**EXAMPLES:**

```
sage: R.<w> = PolynomialRing(Integers(8), sparse=True)
sage: f = 5 + w^1997 - w^10000; f
7*w^10000 + w^1997 + 5
sage: f.exponents()
[0, 1997, 10000]
```

**gcd**(other, algorithm=None)

Return the gcd of this polynomial and *other*

**INPUT:**

- *other* — a polynomial defined over the same ring as this polynomial.

**ALGORITHM:**

Two algorithms are provided:

- *generic* — Uses the generic implementation, which depends on the base ring being a UFD or a field.
- *dense* — The polynomials are converted to the dense representation, their gcd is computed and is converted back to the sparse representation.

Default is dense for polynomials over *Z* and generic in the other cases.

**EXAMPLES:**

```
sage: R.<x> = PolynomialRing(ZZ, sparse=True)
sage: p = x^6 + 7*x^5 + 8*x^4 + 6*x^3 + 2*x^2 + x + 2
sage: q = 2*x^4 - x^3 - 2*x^2 - 4*x - 1
sage: gcd(p, q)
x^2 + x + 1
```

(continues on next page)
integral (var=None)

Return the integral of this polynomial.

By default, the integration variable is the variable of the polynomial.
Otherwise, the integration variable is the optional parameter var.

Note: The integral is always chosen so that the constant term is 0.

EXAMPLES:

```python
sage: R.<x> = PolynomialRing(ZZ, sparse=True)
sage: (1 + 3*x^10 - 2*x^100).integral()
-2/101*x^101 + 3/11*x^11 + x
```

list (copy=True)

Return a new copy of the list of the underlying elements of self.

EXAMPLES:

```python
sage: R.<z> = PolynomialRing(Integers(100), sparse=True)
sage: f = 13*z^5 + 15*z^2 + 17*z
sage: f.list()
[0, 17, 15, 0, 0, 13]
```

number_of_terms()

Return the number of nonzero terms.

EXAMPLES:

```python
sage: R.<x> = PolynomialRing(ZZ, sparse=True)
sage: p = x^100 - 3*x^10 + 12
sage: p.number_of_terms()
3
```

quo_rem (other)

Return the quotient and remainder of the Euclidean division of self and other.

Raises ZeroDivisionError if other is zero.

Raises ArithmeticError if other has a nonunit leading coefficient and this causes the Euclidean division to fail.

EXAMPLES:

```python
sage: P.<x> = PolynomialRing(ZZ, sparse=True)
sage: R.<y> = PolynomialRing(P, sparse=True)
```

(continues on next page)
sage: R = PolynomialRing(QQ, 'x', sparse=True)
sage: f = R.random_element(10)
sage: while x.divides(f.leading_coefficient()):
    f = R.random_element(10)
sage: g = y^5 + R.random_element(4)
sage: q, r = f.quo_rem(g)
sage: f == q*g + r and r.degree() < g.degree()
True
sage: g = x*y^5
sage: f.quo_rem(g)
Traceback (most recent call last):
  ... ArithmeticError: Division non exact
  (consider coercing to polynomials over the fraction field)
sage: g = 0
sage: f.quo_rem(g)
Traceback (most recent call last):
  ... ZeroDivisionError: Division by zero polynomial

If the leading coefficient of other is not a unit, Euclidean division may still work:

sage: f = -x*y^10 + 2*x*y^7 + y^3 - 2*x^2*y^2 - y
sage: g = x*y^5
sage: f.quo_rem(g)
(-y^5 + 2*y^2, y^3 - 2*x^2*y^2 - y)

Polynomialsovernoncommutativeringsarealsoallowed:

sage: HH = QuaternionAlgebra(QQ, -1, -1)
sage: P.<x> = PolynomialRing(HH, sparse=True)
sage: f = P.random_element(5)
sage: g = P.random_element((0, 5))
sage: q, r = f.quo_rem(g)
sage: f == q*g + r
True

AUTHORS:

- Bruno Grenet (2014-07-09)

reverse (degree=None)

Return this polynomial but with the coefficients reversed.

If an optional degree argument is given, the coefficient list will be truncated or zero padded as necessary and the reverse polynomial will have the specified degree.

EXAMPLES:

sage: R.<x> = PolynomialRing(ZZ, sparse=True)
sage: p = x^4 + 2*x^2 + 100
sage: p.reverse()
x^1267650600228229401496703205372 + 2
sage: p.reverse(10)
x^6

shift (n)

Return this polynomial multiplied by the power x^n.
If $n$ is negative, terms below $x^n$ will be discarded. Does not change this polynomial.

EXAMPLES:

```python
sage: R.<x> = PolynomialRing(ZZ, sparse=True)
```

```python
sage: p = x^100000 + 2*x + 4
sage: type(p)
<class 'sage.rings.polynomial.polynomial_ring.PolynomialRing_integral_domain_˓
  with_category.element_class'>
```

```python
sage: p.shift(0)
x^100000 + 2*x + 4
sage: p.shift(-1)
x^99999 + 2
sage: p.shift(-100002)
0
sage: p.shift(2)
x^100002 + 2*x^3 + 4*x^2
```

AUTHOR: - David Harvey (2006-08-06)

**truncate** ($n$)

Return the polynomial of degree < $n$ equal to self modulo $x^n$.

EXAMPLES:

```python
sage: R.<x> = PolynomialRing(ZZ, sparse=True)
```

```python
sage: (x^11 + x^10 + 1).truncate(11)
x^10 + 1
sage: (x^2^500 + x^2^100 + 1).truncate(2^101)
x^1267650600228229401496703205376 + 1
```

**valuation** ($p=None$)

Return the valuation of self.

EXAMPLES:

```python
sage: # needs sage.rings.finite_rings
sage: R.<w> = PolynomialRing(GF(9, 'a'), sparse=True)
```

```python
sage: f = w^1997 - w^10000
sage: f.valuation()
1997
sage: R(19).valuation()
0
sage: R(0).valuation()
+Infinity
```

```python
class sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_sparse_cdv(par-
  ent, x=None, check=True, is_gen=False, construct=False)
```

Bases: Polynomial_generic_sparse, Polynomial_generic_cdv
class sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_sparse_cdvf (parent, x=None, check=True, is_gen=False, construct=False)

Bases: Polynomial_generic_sparse_cdv, Polynomial_generic_cdvf

class sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_sparse_cdvr (parent, x=None, check=True, is_gen=False, construct=False)

Bases: Polynomial_generic_sparse_cdv, Polynomial_generic_cdvr

class sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_sparse_field (parent, x=None, check=True, is_gen=False, construct=False)

Bases: Polynomial_generic_sparse, Polynomial_generic_field

EXAMPLES:

sage: R.<x> = PolynomialRing(Frac(RR['t']), sparse=True)
sage: f = x^3 - x + 17
sage: type(f)
<class 'sage.rings.polynomial.polynomial_ring.PolynomialRing_field_with_category.element_class'>
sage: loads(f.dumps()) == f
True

2.1.5 Univariate Polynomials over GF(2) via NTL’s GF2X

AUTHOR: - Martin Albrecht (2008-10) initial implementation

sage.rings.polynomial.polynomial_gf2x.GF2X_BuildIrred_list (n)

Return the list of coefficients of the lexicographically smallest irreducible polynomial of degree \( n \) over the field of 2 elements.

EXAMPLES:

sage: from sage.rings.polynomial.polynomial_gf2x import GF2X_BuildIrred_list
sage: GF2X_BuildIrred_list(2)
[1, 1, 1]
sage: GF2X_BuildIrred_list(3)
[1, 1, 0, 1]
sage: GF2X_BuildIrred_list(4)
[1, 1, 0, 0, 1]

(continues on next page)
sage: GF(2) '['x'] (GF2X_BuildIrred_list(33))
x^33 + x^6 + x^3 + x + 1

sage.rings.polynomial.polynomial_gf2x.GF2X_BuildRandomIrred_list(n)
Return the list of coefficients of an irreducible polynomial of degree \( n \) of minimal weight over the field of 2 elements.

EXAMPLES:

```python
sage: from sage.rings.polynomial.polynomial_gf2x import GF2X_BuildRandomIrred_list
sage: GF2X_BuildRandomIrred_list(2)
[1, 1, 1]
sage: GF2X_BuildRandomIrred_list(3) in [[1, 1, 0, 1], [1, 0, 1, 1]]
True
```

sage.rings.polynomial.polynomial_gf2x.GF2X_BuildSparseIrred_list(n)
Return the list of coefficients of an irreducible polynomial of degree \( n \) of minimal weight over the field of 2 elements.

EXAMPLES:

```python
sage: from sage.rings.polynomial.polynomial_gf2x import GF2X_BuildIrred_list,
   GF2X_BuildSparseIrred_list
sage: all([GF2X_BuildSparseIrred_list(n) == GF2X_BuildIrred_list(n)
   ....:     for n in range(1,33)])
True
sage: GF(2) '['x'] (GF2X_BuildSparseIrred_list(33))
x^33 + x^10 + 1
```

class sage.rings.polynomial.polynomial_gf2x.Polynomial_GF2X
Bases: Polynomial_template
Univariate Polynomials over \( \mathbb{F}_2 \) via NTL’s GF2X.

EXAMPLES:

```python
sage: P.<x> = GF(2)[]
sage: x^3 + x^2 + 1
x^3 + x^2 + 1

is_irreducible()  
Return whether this polynomial is irreducible over \( \mathbb{F}_2 \).

EXAMPLES:

```python
sage: R.<x> = GF(2)[]
sage: (x^2 + 1).is_irreducible()
False
sage: (x^3 + x + 1).is_irreducible()
True
```

Test that caching works:

```python
sage: R.<x> = GF(2)[]
sage: f = x^2 + 1
sage: f.is_irreducible()
False
```
modular_composition (g, h, algorithm=None)

Compute $f(g) \pmod{h}$.


INPUT:

- $g$ – a polynomial
- $h$ – a polynomial
- algorithm – either 'native' or 'ntl' (default: 'native')

EXAMPLES:

```python
sage: P.<x> = GF(2)[]
sage: r = 279
sage: f = x^r + x + 1
sage: g = x^r
sage: g.modular_composition(g, f) == g(g) % f
True
sage: P.<x> = GF(2)[]
sage: f = x^29 + x^24 + x^22 + x^21 + x^20 + x^16 + x^15 + x^14 + x^10 + x^9 + x^8 + x^7 + x^6 + x^5 + x^2
sage: g = x^31 + x^30 + x^28 + x^24 + x^21 + x^19 + x^18 + x^11 + x^10
sage: h = x^30 + x^28 + x^26 + x^24 + x^22 + x^21 + x^18 + x^17 + x^15
sage: f.modular_composition(g, h) == f(g) % h
True
```

AUTHORS:

- Paul Zimmermann (2008-10) initial implementation
- Martin Albrecht (2008-10) performance improvements

```
class sage.rings.polynomial.polynomial_gf2x.Polynomial_template

Template for interfacing to external C / C++ libraries for implementations of polynomials.

AUTHORS:

- Robert Bradshaw (2008-10): original idea for templating
- Martin Albrecht (2008-10): initial implementation
```

This file implements a simple templating engine for linking univariate polynomials to their C/C++ library implementations. It requires a ‘linkage’ file which implements the `element_` functions (see `sage.libsntl.ntl_GF2X_linkage` for an example). Both parts are then plugged together by inclusion of the linkage file when inheriting from this class. See `sage.rings.polynomial.polynomial_gf2x` for an example.

We illustrate the generic gluing using univariate polynomials over GF(2).
Polynomials, Release 10.3

Note: Implementations using this template MUST implement coercion from base ring elements and get_unsafe(). See Polynomial_GF2X for an example.

**degree()**

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: x.degree()
1
sage: P(1).degree()
0
sage: P(0).degree()
-1
```

**gcd(other)**

Return the greatest common divisor of self and other.

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: f = x*(x+1)
sage: f.gcd(x+1)
x + 1
sage: f.gcd(x^2)
x
```

**get_cparent()**

**is_gen()**

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: x.is_gen()
True
sage: (x+1).is_gen()
False
```

**is_one()**

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: P(1).is_one()
True
```

**is_zero()**

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: x.is_zero()
False
```

**list(copy=True)**

EXAMPLES:
\begin{verbatim}
sage: P.<x> = GF(2)[]
sage: x.list()
[0, 1]
sage: list(x)
[0, 1]
\end{verbatim}

\textbf{quo_rem}(right)

\textbf{EXAMPLES:}

\begin{verbatim}
sage: P.<x> = GF(2)[]
sage: f = x^2 + x + 1
sage: f.quo_rem(x + 1)
(x, 1)
\end{verbatim}

\textbf{shift}(n)

\textbf{EXAMPLES:}

\begin{verbatim}
sage: P.<x> = GF(2)[]
sage: f = x^3 + x^2 + 1
sage: f.shift(1)
x^4 + x^3 + x
sage: f.shift(-1)
x^2 + x
\end{verbatim}

\textbf{truncate}(n)

Returns this polynomial mod \(x^n\).

\textbf{EXAMPLES:}

\begin{verbatim}
sage: R.<x> =GF(2)[]
sage: f = sum(x^n for n in range(10)); f
x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1
sage: f.truncate(6)
x^5 + x^4 + x^3 + x^2 + x + 1
\end{verbatim}

If the precision is higher than the degree of the polynomial then the polynomial itself is returned:

\begin{verbatim}
sage: f.truncate(10) is f
True
\end{verbatim}

If the precision is negative, the zero polynomial is returned:

\begin{verbatim}
sage: f.truncate(-1)
0
\end{verbatim}

\textbf{xgcd}(other)

Computes extended \texttt{gcd} of self and other.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: P.<x> = GF(7)[]
sage: f = x*(x+1)
sage: f.xgcd(x+1)
(x + 1, 0, 1)
sage: f.xgcd(x^2)
(x, 1, 6)
\end{verbatim}
Polynomials, Release 10.3

2.1.6 Univariate polynomials over number fields

AUTHOR:


EXAMPLES:

Define a polynomial over an absolute number field and perform basic operations with them:

```python
sage: x = polygen(ZZ, 'x')
sage: N.<a> = NumberField(x^2 - 2)
sage: K.<x> = N[]
sage: f = x - a
sage: g = x^3 - 2*a + 1
sage: f * (x + a)
x^2 - 2
sage: f + g
x^3 + x - 3*a + 1
sage: g // f
x^2 + a*x + 2
sage: g % f
1
sage: factor(x^3 - 2*a*x^2 - 2*x + 4*a)
(x - 2*a) * (x - a) * (x + a)
sage: gcd(f, x - a)
x - a
```

Polynomials are aware of embeddings of the underlying field:

```python
sage: # needs sage.rings.padics
sage: x = polygen(ZZ, 'x')
sage: Q7 = Qp(7)
sage: r1 = Q7(3 + 7 + 2*7^2 + 6*7^3 + 7^4 + 2*7^5 + 7^6 + 2*7^7 + 4*7^8 + 4*7^9 + 6*7^10 + 2*7^11 + 7^12 + 7^13 + 2*7^15 + 7^16 + 7^17 + 4*7^18 + 6*7^19)

sage: N.<b> = NumberField(x^2 - 2, embedding=r1)

sage: K.<t> = N[

sage: f = t^3 - 2*t + 1

sage: f(r1)
1 + O(7^20)
```

We can also construct polynomials over relative number fields:

```python
sage: # needs sage.symbolic
sage: N.<i, s2> = QQ[I, sqrt(2)]

sage: N.<x> = N[

sage: f = x - s2

sage: f * (x + s2)
x^2 - 2

sage: f + g
x^3 - 2*I*x^2 + (sqrt2 + 1)*x - sqrt2

sage: g // f
x^2 + (-2*I + sqrt2)*x - 2*sqrt2*I + sqrt2 + 2
```

(continues on next page)
class sage.rings.polynomial.polynomial_number_field.Polynomial_absolute_number_field_dense

Bases: Polynomial_generic_dense_field

Class of dense univariate polynomials over an absolute number field.

gcd(other)

Compute the monic gcd of two univariate polynomials using PARI.

INPUT:

• other – a polynomial with the same parent as self.

OUTPUT: The monic gcd of self and other.

EXAMPLES:

sage: x = polygen(ZZ, 'x')
sage: N.<a> = NumberField(x^3 - 1/2, 'a')
sage: R.<r> = N[[x]]
sage: f = (5/4*a^2 - 2*a + 4)*r^2 + (5*a^2 - 81/5*a - 17/2)*r + 4/5*a^2 + 24*a + 6
sage: g = (5/4*a^2 - 2*a + 4)*r^2 + (-11*a^2 + 79/5*a - 7/2)*r - 4/5*a^2 - 24*a - 6
sage: gcd(f, g)**2
r - 60808/96625*a^2 - 69936/96625*a - 149212/96625
sage: R = QQ[[I]][x]
sage: f = R.random_element(2)
sage: g = f + 1
sage: h = R.random_element(2).monic()
sage: f *= h
sage: g *= h
sage: gcd(f, g) - h
0
sage: f.gcd(g) - h
0
Class of dense univariate polynomials over a relative number field.

**gcd**(other)

Compute the monic gcd of two polynomials.

Currently, the method checks corner cases in which one of the polynomials is zero or a constant. Then, computes an absolute extension and performs the computations there.

**INPUT:**

- other – a polynomial with the same parent as self.

**OUTPUT:**

The monic gcd of self and other.

See `Polynomial_absolute_number_field_dense.gcd()` for more details.

**EXAMPLES:**

```
sage: # needs sage.symbolic
da = QQ[sqrt(2), sqrt(3)]
sage: x = polygen(da)
sage: f = x^4 - 5*x^2 + 6
sage: g = x^3 + (-2*s2 + s3)*x^2 + (-2*s3*s2 + 2)*x + 2*s3
sage: gcd(f, g)
x^2 + (-sqrt2 + sqrt3)*x - sqrt3*sqrt2
```

### 2.1.7 Dense univariate polynomials over Z, implemented using FLINT

**AUTHORS:**

- David Harvey: rewrote to talk to NTL directly, instead of via ntl.pynx (2007-09); a lot of this was based on Joel Mohler's recent rewrite of the NTL wrapper
- David Harvey: split off from polynomial_element_generic.py (2007-09)
- Burcin Erocal: rewrote to use FLINT (2008-06-16)

**class** `sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint`

**Bases:** `Polynomial`

A dense polynomial over the integers, implemented via FLINT.

**_add_**(right)

Return self plus right.

**EXAMPLES:**

```
sage: R.<x> = PolynomialRing(ZZ)
sage: f = 2*x + 1
sage: g = -3*x^2 + 6
sage: f + g
-3*x^2 + 2*x + 7
```
_sub_(right)
Return self minus right.

EXAMPLES:
```
sage: R.<x> = PolynomialRing(ZZ)
sage: f = 2*x + 1
sage: g = -3*x^2 + 6
sage: f - g
3*x^2 + 2*x - 5
```

_lmul_(right)
Return self multiplied by right, where right is a scalar (integer).

EXAMPLES:
```
sage: R.<x> = PolynomialRing(ZZ)
sage: x*3
3*x
sage: (2*x^2 + 4)*3
6*x^2 + 12
```

_rmul_(right)
Return self multiplied by right, where right is a scalar (integer).

EXAMPLES:
```
sage: R.<x> = PolynomialRing(ZZ)
sage: 3*x
3*x
sage: 3*(2*x^2 + 4)
6*x^2 + 12
```

_mul_(right)
Return self multiplied by right.

EXAMPLES:
```
sage: R.<x> = PolynomialRing(ZZ)
sage: (x - 2)*(x^2 - 8*x + 16)
x^3 - 10*x^2 + 32*x - 32
```

_mul_trunc_(right, n)
Truncated multiplication

See also:
_mul_() for standard multiplication

EXAMPLES:
```
sage: x = polygen(ZZ)
sage: p1 = 1 + x + x^2 + x^4
sage: p2 = -2 + 3*x^2 + 5*x^4
sage: p1._mul_trunc_(p2, 4)
3*x^3 + x^2 - 2*x - 2
sage: (p1*p2).truncate(4)
3*x^3 + x^2 - 2*x - 2
```

(continues on next page)
content()  
Return the greatest common divisor of the coefficients of this polynomial. The sign is the sign of the leading coefficient. The content of the zero polynomial is zero.

EXAMPLES:

```python  
sage: R.<x> = PolynomialRing(ZZ)  
sage: (2*x^2 - 4*x^4 + 14*x^7).content()  
2  
sage: x.content()  
1  
sage: R(1).content()  
1  
sage: R(0).content()  
0  
```

degree(gen=None)  
Return the degree of this polynomial.

The zero polynomial has degree \(-1\).

EXAMPLES:

```python  
sage: R.<x> = PolynomialRing(ZZ)  
sage: x.degree()  
1  
sage: (x^2).degree()  
2  
sage: R(1).degree()  
0  
sage: R(0).degree()  
-1  
```

disc(proof=True)  
Return the discriminant of `self`, which is by definition

\((-1)^{m(m-1)/2} \text{resultant}(a, a')/\text{lc}(a),\)

where \(m = \text{deg}(a)\), and \(\text{lc}(a)\) is the leading coefficient of \(a\). If `proof` is False (the default is True), then this function may use a randomized strategy that errors with probability no more than \(2^{-80}\).

EXAMPLES:

```python  
sage: R.<x> = ZZ[]  
sage: f = 3*x^3 + 2*x + 1  
sage: f.discriminant()  
-339  
sage: f.discriminant(proof=False)  
-339  
```

discriminant(proof=True)  
Return the discriminant of `self`, which is by definition

\((-1)^{m(m-1)/2} \text{resultant}(a, a')/\text{lc}(a),\)
where \( m = \deg(a) \), and \( \text{lc}(a) \) is the leading coefficient of \( a \). If \( \text{proof} \) is False (the default is True), then this function may use a randomized strategy that errors with probability no more than \( 2^{-80} \).

**EXAMPLES:**

```
sage: R.<x> = ZZ[]
sage: f = 3*x^3 + 2*x + 1
sage: f.discriminant()
-339
sage: f.discriminant(proof=False)
-339
```

**factor()**

This function overrides the generic polynomial factorization to make a somewhat intelligent decision to use PARI or NTL based on some benchmarking.

Note: This function factors the content of the polynomial, which can take very long if it’s a really big integer. If you do not need the content factored, divide it out of your polynomial before calling this function.

**EXAMPLES:**

```
sage: R.<x> = ZZ[]
sage: f = x^4 - 1
sage: f.factor()
(x - 1) * (x + 1) * (x^2 + 1)
sage: f = 1 - x
sage: f.factor()
(-1) * (x - 1)
sage: f.factor().unit()
-1
sage: f = -30*x; f.factor()
(-1) * 2 * 3 * 5 * x
```

**factor_mod(p)**

Return the factorization of \( \text{self} \) modulo the prime \( p \).

**INPUT:**

- \( p \) – prime

**OUTPUT:** factorization of \( \text{self} \) reduced modulo \( p \).

**EXAMPLES:**

```
sage: R.<x> = ZZ['x']
sage: f = -3*x*(x-2)*(x-9) + x
sage: f.factor_mod(3)
x
sage: f = -3 * x * (x - 2) * (x - 9)
sage: f.factor_mod(3)
Traceback (most recent call last):
...
ArithmeticError: factorization of 0 is not defined
sage: f = 2 * x * (x - 2) * (x - 9)
sage: f.factor_mod(7)
(2) * x * (x + 5)^2
```

**factor_padic(p, prec=10)**

Return \( p \)-adic factorization of \( \text{self} \) to given precision.
INPUT:
• p – prime
• prec – integer; the precision

OUTPUT: factorization of self over the completion at p.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ)
sage: f = x^2 + 1
sage: f.factor_padic(5, 4)
((1 + O(5^4))*x + 2 + 5 + 2*5^2 + 5^3 + O(5^4))
* ((1 + O(5^4))*x + 3 + 3*5 + 2*5^2 + 3*5^3 + O(5^4))
```

A more difficult example:

```
sage: f = 100 * (5*x + 1)^2 * (x + 5)^2
sage: f.factor_padic(5, 10)
(4 + O(5^10)) * (5 + O(5^11))^2 * ((1 + O(5^10))*x + 5 + O(5^10))^2
* ((5 + O(5^10))*x + 1 + O(5^10))^2
```

\texttt{gcd}(\texttt{right})

Return the GCD of self and right. The leading coefficient need not be 1.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ)
sage: f = (6*x + 47) * (7*x^2 - 2*x + 38)
sage: g = (6*x + 47) * (3*x^3 + 2*x + 1)
sage: f.gcd(g)
6*x + 47
```

\texttt{inverse_series_trunc}(\texttt{prec})

Return a polynomial approximation of precision prec of the inverse series of this polynomial.

EXAMPLES:

```
sage: x = polygen(ZZ)
sage: p = 1 + x + 2*x^2
sage: q5 = p.inverse_series_trunc(5)
sage: q5
-x^4 + 3*x^3 - x^2 - x + 1
sage: p*q5
-2*x^6 + 5*x^5 + 1
sage: (x-1).inverse_series_trunc(5)
-x^4 - x^3 - x^2 - x - 1
sage: q100 = p.inverse_series_trunc(100)
sage: (q100 * p).truncate(100)
1
```

\texttt{is_one}()

Return True if self is equal to one.

EXAMPLES:
sage: R.<x> = ZZ[]
sage: R(0).is_one()
False
sage: R(1).is_one()
True
sage: x.is_one()
False

is_zero()
Return True if self is equal to zero.

EXAMPLES:

sage: R.<x> = ZZ[]
sage: R(0).is_zero()
True
sage: R(1).is_zero()
False
sage: x.is_zero()
False

lcm(right)
Return the LCM of self and right.

EXAMPLES:

sage: R.<x> = PolynomialRing(ZZ)
sage: f = (6*x + 47) * (7*x^2 - 2*x + 38)
sage: g = (6*x + 47) * (3*x^3 + 2*x + 1)
sage: h = f.lcm(g); h
126*x^6 + 951*x^5 + 486*x^4 + 6034*x^3 + 585*x^2 + 3706*x + 1786
sage: h == (6*x + 47) * (7*x^2 - 2*x + 38) * (3*x^3 + 2*x + 1)
True

list(copy=True)
Return a new copy of the list of the underlying elements of self.

EXAMPLES:

sage: x = PolynomialRing(ZZ,'x').0
sage: f = x^3 + 3*x - 17
sage: f.list()
[-17, 3, 0, 1]

pseudo_divrem(B)
Write \( A = \text{self} \). This function computes polynomials \( Q \) and \( R \) and an integer \( d \) such that

\[
\text{lead}(B)^d A = BQ + R
\]

where \( R \) has degree less than that of \( B \).

INPUT:

- \( B \) – a polynomial over \( \mathbb{Z} \)

OUTPUT:
• \(Q, R\) – polynomials
• \(d\) – nonnegative integer

EXAMPLES:

```python
sage: R.<x> = ZZ['x']
sage: A = R(range(10))
sage: B = 3*R([-1, 0, 1])
sage: Q, R, d = A.pseudo_divrem(B)
sage: Q, R, d
(9*x^7 + 8*x^6 + 16*x^5 + 14*x^4 + 21*x^3 + 18*x^2 + 24*x + 20, 75*x + 60, 1)
sage: B.leading_coefficient()^d * A == B*Q + R
True
```

`quo_rem(right)`

Attempts to divide `self` by `right`, and return a quotient and remainder.

EXAMPLES:

```python
sage: R.<x> = PolynomialRing(ZZ)
sage: f = R(range(10)); g = R([-1, 0, 1])
sage: q, r = f.quo_rem(g)
sage: q, r
(9*x^7 + 8*x^6 + 16*x^5 + 14*x^4 + 21*x^3 + 18*x^2 + 24*x + 20, 25*x + 20)
sage: q*g + r == f
True
sage: f = x^2
sage: f.quo_rem(0)
Traceback (most recent call last):
  ...  DivisionError: division by zero polynomial
sage: f = (x^2 + 3) * (2*x - 1)
sage: f.quo_rem(2*x - 1)
(x^2 + 3, 0)
sage: f = x^2
sage: f.quo_rem(2*x - 1)
(0, x^2)
```

`real_root_intervals()`

Return isolating intervals for the real roots of this polynomial.

EXAMPLES: We compute the roots of the characteristic polynomial of some Salem numbers:

```python
sage: R.<x> = PolynomialRing(ZZ)
sage: f = 1 - x^2 - x^3 - x^4 + x^6
sage: f.real_root_intervals()
(1/2, 3/4), 1), ((1, 3/2), 1)
```

`resultant(other, proof=True)`

Return the resultant of `self` and `other`, which must lie in the same polynomial ring.

If `proof=False` (the default is `proof=True`), then this function may use a randomized strategy that errors with probability no more than 2^{-80}.

INPUT:
• other – a polynomial

OUTPUT: an element of the base ring of the polynomial ring

EXAMPLES:

```python
sage: x = PolynomialRing(ZZ,'x').0
sage: f = x^3 + x + 1; g = x^3 - x - 1
sage: r = f.resultant(g); r
-8
```

```python
sage: r.parent() is ZZ
True
```

### reverse (degree=None)

Return a polynomial with the coefficients of this polynomial reversed.

If an optional degree argument is given the coefficient list will be truncated or zero padded as necessary before computing the reverse.

EXAMPLES:

```python
sage: R.<x> = ZZ[]
sage: p = R([1,2,3,4]); p
4*x^3 + 3*x^2 + 2*x + 1
sage: p.reverse()
x^3 + 2*x^2 + 3*x + 4
sage: p.reverse(degree=6)
x^6 + 2*x^5 + 3*x^4 + 4*x^3
sage: p.reverse(degree=2)
x^2 + 2*x + 3
```

### revert_series (n)

Return a polynomial \( f \) such that \( f(\text{self}(x)) = \text{self}(f(x)) = x \mod x^n \).

EXAMPLES:

```python
sage: R.<t> = ZZ[]
sage: f = t - t^3 + t^5
sage: f.revert_series(6)
2*t^5 + t^3 + t
sage: f.revert_series(-1)
Traceback (most recent call last):
  ...
ValueError: argument n must be a non-negative integer, got -1
```

```python
sage: g = - t^3 + t^5
sage: g.revert_series(6)
Traceback (most recent call last):
  ...
ValueError: self must have constant coefficient 0 and a unit for coefficient...
```

### squarefree_decomposition ()

Return the square-free decomposition of \( \text{self} \). This is a partial factorization of \( \text{self} \) into square-free, relatively prime polynomials.

This is a wrapper for the NTL function `SquareFreeDecomp`.

EXAMPLES:
sage: R.<x> = PolynomialRing(ZZ)
sage: p = (x-1)^2 * (x-2)^2 * (x-3)^3 * (x-4)
sage: p.squarefree_decomposition()
(x - 4) * (x^2 - 3*x + 2)^2 * (x - 3)^3
sage: p = 37 * (x-1)^2 * (x-2)^2 * (x-3)^3 * (x-4)
sage: p.squarefree_decomposition()
(37) * (x - 4) * (x^2 - 3*x + 2)^2 * (x - 3)^3

\textbf{xgcd (right)}

Return a triple \((g, s, t)\) such that \(g = s \cdot \text{self} + t \cdot \text{right}\) and such that \(g\) is the \gcd of \text{self} and \text{right} up to a divisor of the resultant of \text{self} and \text{other}.

As integer polynomials do not form a principal ideal domain, it is not always possible given \(a\) and \(b\) to find a pair \(s, t\) such that \(gcd(a, b) = sa + tb\). Take \(a = x + 2\) and \(b = x + 4\) as an example for which the \gcd is 1 but the best you can achieve in the Bezout identity is 2.

If \text{self} and \text{right} are coprime as polynomials over the rationals, then \(g\) is guaranteed to be the resultant of \text{self} and \text{right}, as a constant polynomial.

**EXAMPLES:**

sage: P.<x> = PolynomialRing(ZZ)
sage: (x + 2).xgcd(x + 4)
\((2, -1, 1)\)
sage: (x + 2).resultant(x + 4)
2
sage: (x + 2).gcd(x + 4)
1

sage: F = (x^2 + 2)*x^3; G = (x^2 + 2) * (x - 3)
sage: g, u, v = F.xgcd(G)
sage: g, u, v
\((27*x^2 + 54, -432*x + 8208, 432*x^2 + 864*x + 14256)\)
sage: u*F + v*G
2985984
sage: zero = P(0)
sage: x.xgcd(zero)
\((x, 1, 0)\)
sage: zero.xgcd(x)
\((x, 0, 1)\)

sage: F = (x - 3)^3; G = (x - 15)^2
sage: g, u, v = F.xgcd(G)
sage: g, u, v
\((2985984, -432*x + 8208, 432*x^2 + 864*x + 14256)\)
sage: u*F + v*G
2985984
2.1.8 Dense univariate polynomials over \( \mathbb{Z} \), implemented using NTL.

AUTHORS:

- David Harvey: split off from polynomial_element_generic.py (2007-09)
- David Harvey: rewrote to talk to NTL directly, instead of via ntl.pyx (2007-09); a lot of this was based on Joel Mohler's recent rewrite of the NTL wrapper

Sage includes two implementations of dense univariate polynomials over \( \mathbb{Z} \); this file contains the implementation based on NTL, but there is also an implementation based on FLINT in \texttt{sage.rings.polynomial.polynomial_integer_dense_flint}.

The FLINT implementation is preferred (FLINT’s arithmetic operations are generally faster), so it is the default; to use the NTL implementation, you can do:

```python
sage: K.<x> = PolynomialRing(ZZ, implementation='NTL')
sage: K
Univariate Polynomial Ring in x over Integer Ring (using NTL)
```

class \texttt{sage.rings.polynomial.polynomial_integer_dense_ntl.Polynomial_integer_dense_ntl}

Bases: \texttt{Polynomial}

A dense polynomial over the integers, implemented via NTL.

\texttt{content()}  
Return the greatest common divisor of the coefficients of this polynomial. The sign is the sign of the leading coefficient. The content of the zero polynomial is zero.

EXAMPLES:

```python
sage: R.<x> = PolynomialRing(ZZ, implementation='NTL')
sage: (2*x^2 - 4*x^4 + 14*x^7).content()  
2
sage: (2*x^2 - 4*x^4 - 14*x^7).content()  
-2
sage: x.content()  
1
sage: R(1).content()  
1
sage: R(0).content()  
0
```

\texttt{degree(gen=None)}  
Return the degree of this polynomial. The zero polynomial has degree \(-1\).

EXAMPLES:

```python
sage: R.<x> = PolynomialRing(ZZ, implementation='NTL')
sage: x.degree()  
1
sage: (x^2).degree()  
2
sage: R(1).degree()  
0
sage: R(0).degree()  
-1
```
**discriminant** *(proof=True)*

Return the discriminant of `self`, which is by definition

\[ (-1)^{m(m-1)/2} \text{resultant}(a, a')/lc(a), \]

where \( m = \text{deg}(a) \), and \( lc(a) \) is the leading coefficient of \( a \). If \( proof \) is False (the default is True), then this function may use a randomized strategy that errors with probability no more than \( 2^{-80} \).

**EXAMPLES:**

```python
sage: f = ntl.ZZX([1,2,0,3])
sage: f.discriminant()
-339
sage: f.discriminant(proof=False)
-339
```

**factor()**

This function overrides the generic polynomial factorization to make a somewhat intelligent decision to use PARI or NTL based on some benchmarking.

Note: This function factors the content of the polynomial, which can take very long if it’s a really big integer. If you do not need the content factored, divide it out of your polynomial before calling this function.

**EXAMPLES:**

```python
sage: R.<x> = ZZ[]
sage: f = x^4 - 1
sage: f.factor()  # (x - 1) * (x + 1) * (x^2 + 1)
sage: f = 1 - x
sage: f.factor()  # (-1) * (x - 1)
sage: f.factor().unit()  # -1
sage: f = -30*x; f.factor()  # (-1) * 2 * 3 * 5 * x
```

**factor_mod(p)**

Return the factorization of `self modulo the prime p`.

**INPUT:**

• \( p \) – prime

**OUTPUT:** factorization of `self` reduced modulo \( p \).

**EXAMPLES:**

```python
sage: R.<x> = PolynomialRing(ZZ, 'x', implementation='NTL')
sage: f = -3*x*(x-2)*(x-9) + x
sage: f.factor_mod(3)  # x
sage: f = -3*x*(x-2)*(x-9)
sage: f.factor_mod(3)
Traceback (most recent call last):
  ...
ArithmeticError: factorization of 0 is not defined
sage: f = 2*x*(x-2)*(x-9)
```

(continues on next page)
factor_padic \(p, \text{prec}=10\)

Return \(p\)-adic factorization of \(self\) to given precision.

INPUT:

- \(p\) – prime
- \(\text{prec}\) – integer; the precision

OUTPUT:

- factorization of \(self\) over the completion at \(p\).

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ, implementation='NTL')
sage: f = x^2 + 1
sage: h = f.factor_padic(5)
((1 + O(5^4)) \cdot x + 2 + 5 + 2 \cdot 5^2 + 5 \cdot 3 + O(5^4))
\ast ((1 + O(5^4)) \cdot x + 3 + 3 \cdot 5 + 2 \cdot 5^2 + 3 \cdot 5^3 + O(5^4))
```

A more difficult example:

```
sage: f = 100 \cdot (5 \cdot x + 1)^2 \cdot (x + 5)^2
sage: g = (6 \cdot x + 47) \cdot (3 \cdot x^3 + 2 \cdot x + 1)
```

```
sage: h = f.gcd(g)
6 \cdot x + 47
```

```
sage: h == (6 \cdot x + 47) \cdot (7 \cdot x^2 - 2 \cdot x + 38) \cdot (3 \cdot x^3 + 2 \cdot x + 1)
True
```

gcd \((right)\)

Return the GCD of \(self\) and \(right\). The leading coefficient need not be 1.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ, implementation='NTL')
sage: f = (6 \cdot x + 47) \cdot (7 \cdot x^2 - 2 \cdot x + 38)
sage: g = (6 \cdot x + 47) \cdot (3 \cdot x^3 + 2 \cdot x + 1)
```

```
sage: h = f.lcm(g)
126 \cdot x^6 + 951 \cdot x^5 + 486 \cdot x^4 + 6034 \cdot x^3 + 585 \cdot x^2 + 3706 \cdot x + 1786
```

```
sage: h == (6 \cdot x + 47) \cdot (7 \cdot x^2 - 2 \cdot x + 38) \cdot (3 \cdot x^3 + 2 \cdot x + 1)
True
```

lcm \((right)\)

Return the LCM of \(self\) and \(right\).

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ, implementation='NTL')
sage: f = (6 \cdot x + 47) \cdot (7 \cdot x^2 - 2 \cdot x + 38)
sage: g = (6 \cdot x + 47) \cdot (3 \cdot x^3 + 2 \cdot x + 1)
```

```
sage: h = f.lcm(g)
126 \cdot x^6 + 951 \cdot x^5 + 486 \cdot x^4 + 6034 \cdot x^3 + 585 \cdot x^2 + 3706 \cdot x + 1786
```

```
sage: h == (6 \cdot x + 47) \cdot (7 \cdot x^2 - 2 \cdot x + 38) \cdot (3 \cdot x^3 + 2 \cdot x + 1)
True
```

list \((copy=True)\)

Return a new copy of the list of the underlying elements of \(self\).

EXAMPLES:
sage: x = PolynomialRing(ZZ, 'x', implementation='NTL').0
sage: f = x^3 + 3*x - 17
sage: f.list()
[-17, 3, 0, 1]
sage: f = PolynomialRing(ZZ, 'x', implementation='NTL')(0)

quo_rem(right)

Attempt to divide self by right, and return a quotient and remainder.

If right is monic, then it returns (q, r) where self = q * right + r and deg(r) < deg(right).

If right is not monic, then it returns (q, 0) where q = self/right if right exactly divides self, otherwise it raises an exception.

EXAMPLES:

sage: R.<x> = PolynomialRing(ZZ, implementation='NTL')
sage: f = R(range(10)); g = R([-1, 0, 1])
sage: q, r = f.quo_rem(g)
sage: q, r
(9*x^7 + 8*x^6 + 16*x^5 + 14*x^4 + 21*x^3 + 18*x^2 + 24*x + 20, 25*x + 20)
sage: q*g + r == f
True

sage: 0//(2*x)
0

sage: f = x^2
sage: f.quo_rem(0)
Traceback (most recent call last):
... ArithmeticError: division by zero polynomial

sage: f = (x^2 + 3) * (2*x - 1)
sage: f.quo_rem(2*x - 1)
(x^2 + 3, 0)
sage: f = x^2
sage: f.quo_rem(2*x - 1)
Traceback (most recent call last):
... ArithmeticError: division not exact in Z[x] (consider coercing to Q[x] first)

real_root_intervals()

Returns isolating intervals for the real roots of this polynomial.

EXAMPLES: We compute the roots of the characteristic polynomial of some Salem numbers:

sage: R.<x> = PolynomialRing(ZZ, implementation='NTL')
sage: f = 1 - x^2 - x^3 - x^4 + x^6
sage: f.real_root_intervals()
[['(1/2, 3/4)', 1], ['(1, 3/2)', 1]]

resultant(other, proof=True)

Returns the resultant of self and other, which must lie in the same polynomial ring.
If \( \text{proof}=\text{False} \) (the default is \( \text{proof}=\text{True} \)), then this function may use a randomized strategy that errors with probability no more than \( 2^{-80} \).

INPUT:

- \( \text{other} \) – a polynomial

OUTPUT: an element of the base ring of the polynomial ring

EXAMPLES:

```
sage: x = PolynomialRing(ZZ, 'x', implementation='NTL').0
sage: f = x^3 + x + 1; g = x^3 - x - 1
sage: r = f.resultant(g); r
-8
sage: r.parent() is ZZ
True
```

\textbf{squarefree\_decomposition}()  

Return the square-free decomposition of \( \text{self} \). This is a partial factorization of \( \text{self} \) into square-free, relatively prime polynomials.

This is a wrapper for the NTL function \text{SquareFreeDecomp}.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(ZZ, implementation='NTL')
sage: p = 37 * (x-1)^2 * (x-2)^2 * (x-3)^3 * (x-4)
sage: p.squarefree_decomposition()
(37) * (x - 4) * (x^2 - 3*x + 2)^2 * (x - 3)^3
```

\textbf{xgcd} (\textit{right})

This function can’t in general return \((g,s,t)\) as above, since they need not exist. Instead, over the integers, we first multiply \(g\) by a divisor of the resultant of \(a/g\) and \(b/g\), up to sign, and return \(g, u, v\) such that \(g = s*\text{self} + s*\text{right}\). But note that this \(g\) may be a multiple of the gcd.

If \( \text{self} \) and \( \text{right} \) are coprime as polynomials over the rationals, then \( g \) is guaranteed to be the resultant of \( \text{self} \) and \( \text{right} \), as a constant polynomial.

EXAMPLES:

```
sage: P.<x> = PolynomialRing(ZZ, implementation='NTL')
sage: F = (x^2 + 2)*x^3; G = (x^2+2)*(x-3)
sage: g, u, v = F.xgcd(G)
sage: g, u, v
(27*x^2 + 54, 1, -x^2 - 3*x - 9)
sage: u*F + v*G
27*x^2 + 54
sage: x.xgcd(P(0))
(x, 1, 0)
sage: f = P(0)
sage: f.xgcd(x)
(x, 0, 1)
sage: F = (x-3)^3; G = (x-15)^2
sage: g, u, v = F.xgcd(G)
sage: g, u, v
(2985984, -432*x + 8208, 432*x^2 + 864*x + 14256)
sage: u*F + v*G
2985984
```
2.1.9 Univariate polynomials over \( \mathbb{Q} \) implemented via FLINT

**AUTHOR:**
- Sebastian Pancratz

```python
class sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint
    Bases: Polynomial

    Univariate polynomials over the rationals, implemented via FLINT.
    Internally, we represent rational polynomial as the quotient of an integer polynomial and a positive denominator which is coprime to the content of the numerator.

    _add_ (right)
    Return the sum of two rational polynomials.
    EXAMPLES:
    ```sage```
    sage: R.<t> = QQ[]
    sage: f = 2/3 + t + 2*t^3
    sage: g = -1 + t/3 - 10/11*t^4
    sage: f + g
    -10/11*t^4 + 2*t^3 + 4/3*t - 1/3
    ```

    _sub_ (right)
    Return the difference of two rational polynomials.
    EXAMPLES:
    ```sage```
    sage: R.<t> = QQ[]
    sage: f = -10/11*t^4 + 2*t^3 + 4/3*t - 1/3
    sage: g = 2*t^3
    sage: f - g
    # indirect doctest
    -10/11*t^4 + 4/3*t - 1/3
    ```

    _lmul_ (right)
    Return self * right, where right is a rational number.
    EXAMPLES:
    ```sage```
    sage: R.<t> = QQ[]
    sage: f = 3/2*t^3 - t + 1/3
    sage: f * 6
    # indirect doctest
    9*t^3 - 6*t + 2
    ```

    _rmul_ (left)
    Return left * self, where left is a rational number.
    EXAMPLES:
    ```sage```
    sage: R.<t> = QQ[]
    sage: f = 3/2*t^3 - t + 1/3
    sage: 6 * f
    # indirect doctest
    9*t^3 - 6*t + 2
    ```
```
_polysage: R.<t> = QQ[]
sage: f = -1 + 3*t/2 - t^3
sage: g = 2/3 + 7/3*t + 3*t^2
sage: f * g
-3*t^5 - 7/3*t^4 + 23/6*t^3 + 1/2*t^2 - 4/3*t - 2/3
```

```
_polysage: x = polygen(QQ)
sage: p1 = 1/2 - 3*x + 2/7*x**3
sage: p2 = x + 2/5*x**5 + x**7
sage: p1._mul_trunc_(p2, 5)
2/7*x^4 - 3*x^2 + 1/2*x
sage: p1._mul_trunc_(p2, 1)
0
sage: p1._mul_trunc_(p2, 0)
Traceback (most recent call last):
  ... 
ValueError: n must be > 0
```

```
_sage: R.<t> = QQ[]
sage: f = 1 + t + t^2/2 + t^3/3 + t^4/4
sage: f.degree()
4
sage: g = R(0)
sage: g.degree()
-1
```

```
_sage: R.<t> = QQ[]
sage: f = (3 * t^3 + 1) / -3
sage: f.denominator()
3
```
Return the discriminant of this polynomial.

The discriminant $R_n$ is defined as

$$R_n = a_n^{2n-2} \prod_{1 \leq i < j \leq n} (r_i - r_j)^2,$$

where $n$ is the degree of this polynomial, $a_n$ is the leading coefficient and the roots over $\bar{\mathbb{Q}}$ are $r_1, \ldots, r_n$.

The discriminant of constant polynomials is defined to be $0$.

OUTPUT: Discriminant, an element of the base ring of the polynomial ring

**Note:** Note the identity $R_n(f) := (-1)^{(n(n-1)/2)} R(f, f') a_n^{(n-k-2)}$, where $n$ is the degree of this polynomial, $a_n$ is the leading coefficient, $f'$ is the derivative of $f$, and $k$ is the degree of $f'$. Calls resultant().

**ALGORITHM:**

Use PARI.

**EXAMPLES:**

In the case of elliptic curves in special form, the discriminant is easy to calculate:

```
sage: R.<t> = QQ[]
sage: f = t^3 + t + 1
sage: d = f.discriminant(); d
-31
sage: d.parent() is QQ
True
sage: EllipticCurve([1, 1]).discriminant() / 16  # needs sage.schemes
-31
```

```
sage: R.<t> = QQ[]
sage: f = 2*t^3 + t + 1
sage: d = f.discriminant(); d
-116
```

```
sage: R.<t> = QQ[]
sage: f = t^3 + 3*t - 17
sage: f.discriminant()
-7911
```

Return the discriminant of this polynomial.

The discriminant $R_n$ is defined as

$$R_n = a_n^{2n-2} \prod_{1 \leq i < j \leq n} (r_i - r_j)^2,$$

where $n$ is the degree of this polynomial, $a_n$ is the leading coefficient and the roots over $\bar{\mathbb{Q}}$ are $r_1, \ldots, r_n$.

The discriminant of constant polynomials is defined to be $0$.

OUTPUT: Discriminant, an element of the base ring of the polynomial ring

154 Chapter 2. Univariate Polynomials
Note: Note the identity $R_n(f) := (-1)^{(n(n-1)/2)} R(f, f') a_n^{n-k-2}$, where $n$ is the degree of this polynomial, $a_n$ is the leading coefficient, $f'$ is the derivative of $f$, and $k$ is the degree of $f'$. Calls `resultant()`.

ALGORITHM:
Use PARI.

EXAMPLES:
In the case of elliptic curves in special form, the discriminant is easy to calculate:

```python
sage: R.<t> = QQ[]
sage: f = t^3 + t + 1
sage: d = f.discriminant(); d
-31
sage: d.parent() is QQ
True
sage: EllipticCurve([1, 1]).discriminant() / 16
# needs sage.schemes
-31
```

```python
sage: R.<t> = QQ[]
sage: f = 2*t^3 + t + 1
sage: d = f.discriminant(); d
-116
```

```python
sage: R.<t> = QQ[]
sage: f = t^3 + 3*t - 17
sage: f.discriminant()
-7911
```

`factor_mod(p)`
Return the factorization of `self` modulo the prime $p$.

Assumes that the degree of this polynomial is at least one, and raises a `ValueError` otherwise.

INPUT:

- `p` - Prime number

OUTPUT: Factorization of this polynomial modulo $p$

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: (x^5 + 17*x^3 + x + 3).factor_mod(3)
x * (x^2 + 1)^2
sage: (x^5 + 2).factor_mod(5)
(x + 2)^5
```

Variable names that are reserved in PARI, such as `zeta`, are supported (see github issue #20631):

```python
sage: R.<zeta> = QQ[]
sage: (zeta^2 + zeta + 1).factor_mod(7)
(zeta + 3) * (zeta + 5)
```

`factor_padic(p, prec=10)`
Return the $p$-adic factorization of this polynomial to the given precision.

2.1. Univariate Polynomials and Polynomial Rings 155
INPUT:

- \textit{p} - Prime number
- \textit{prec} - Integer; the precision

OUTPUT: factorization of \texttt{self} viewed as a \(p\)-adic polynomial

EXAMPLES:

```
sage: # needs sage.rings.padic
sage: R.<x> = QQ[]
sage: f = x^3 - 2
sage: f.factor_padic(2)
(1 + O(2^10))*x^3 + O(2^10)*x^2 + O(2^10)*x + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + O(2^10)
sage: f.factor_padic(3)
(1 + O(3^10))*x^3 + O(3^10)*x^2 + O(3^10)*x + 1 + 2*3 + 2*3^2 + 2*3^3 + 2*3^4 + 2*3^5 + 2*3^6 + 2*3^7 + 2*3^8 + O(3^10)
```

The input polynomial is considered to have “infinite” precision, therefore the \(p\)-adic factorization of the polynomial is not the same as first coercing to \(Q_p\) and then factoring (see also github issue #15422):

```
sage: # needs sage.rings.padic
sage: f = x^2 - 3^6
sage: f.factor_padic(3, 5)
((1 + O(3^5))*x + 3^3 + O(3^5)) * ((1 + O(3^5))*x + 2*3^3 + 2*3^4 + O(3^5))
sage: f.change_ring(Qp(3,5)).factor()  
Traceback (most recent call last):
...
PrecisionError: p-adic factorization not well-defined since the discriminant is zero up to the requestion p-adic precision
```

A more difficult example:

```
sage: R.<x> = QQ[]
sage: f = 100 * (5*x + 1)^2 * (x + 5)^2
sage: f.factor_padic(5, 10)
#____
(4*5^4 + O(5^14)) * ((1 + O(5^9))*x + 5^-1 + O(5^9))^2  
* ((1 + O(5^10))*x + (1 + O(5^9))*x + 5^-1 + O(5^9))^2
```

Try some bogus inputs:

```
sage: # needs sage.rings.padic
sage: f.factor_padic(3, -1)
Traceback (most recent call last):
...
ValueError: prec_cap must be non-negative
sage: f.factor_padic(6, 10)
Traceback (most recent call last):
...
```

(continues on next page)
ValueError: p must be prime

sage: f.factor_padic('hello', 'world')
Traceback (most recent call last):
...
TypeError: unable to convert 'hello' to an integer

**galois_group** *(pari_group=False, algorithm='pari')*

Return the Galois group of this polynomial as a permutation group.

**INPUT:**

- **self** - Irreducible polynomial
- **pari_group** - bool (default: False); if True instead return the Galois group as a PARI group. This has a useful label in it, and may be slightly faster since it doesn't require looking up a group in GAP. To get a permutation group from a PARI group $P$, type `PermutationGroup(P)`.
- **algorithm** - 'pari', 'gap', 'kash', 'magma' (default: 'pari', for degrees is at most 11; 'gap', for degrees from 12 to 15; 'kash', for degrees from 16 or more).

**OUTPUT:** Galois group

**ALGORITHM:**

The Galois group is computed using PARI in C library mode, or possibly GAP, KASH, or MAGMA.

**Note:** The PARI documentation contains the following warning: The method used is that of resolvent polynomials and is sensitive to the current precision. The precision is updated internally but, in very rare cases, a wrong result may be returned if the initial precision was not sufficient.

GAP uses the “Transitive Groups Libraries” from the “TransGrp” GAP package which comes installed with the “gap” Sage package.

MAGMA does not return a provably correct result. Please see the MAGMA documentation for how to obtain a provably correct result.

**EXAMPLES:**

```
sage: # needs sage.groups sage.libs.pari
sage: R.<x> = QQ[]
sage: f = x^4 - 17*x^3 - 2*x + 1
sage: G = f.galois_group(); G
Transitive group number 5 of degree 4
sage: G.gens()
((1,2), (1,2,3,4))
sage: G.order()
24
```

It is potentially useful to instead obtain the corresponding PARI group, which is little more than a 4-tuple. See the PARI manual for the exact details. (Note that the third entry in the tuple is in the new standard ordering.)

```
sage: # needs sage.groups sage.libs.pari
sage: f = x^4 - 17*x^3 - 2*x + 1
sage: G = f.galois_group(pari_group=True); G
PARI group [24, -1, 5, "S4"] of degree 4
sage: PermutationGroup(G)
Transitive group number 5 of degree 4
```
You can use KASH or GAP to compute Galois groups as well. The advantage is that KASH (resp. GAP) can compute Galois groups of fields up to degree 23 (resp. 15), whereas PARI only goes to degree 11. (In my not-so-thorough experiments PARI is faster than KASH.)

```
sage: R.<x> = QQ[]
sage: f = x^4 - 17*x^3 - 2*x + 1
sage: f.galois_group(algorithm='kash')  # optional - kash
Transitive group number 5 of degree 4
```

```
sage: # needs sage.libs.gap
sage: f = x^4 - 17*x^3 - 2*x + 1
sage: f.galois_group(algorithm='gap')
Transitive group number 5 of degree 4
```

```
sage: f = x^13 - 17*x^3 - 2*x + 1
sage: f.galois_group(algorithm='gap')
Transitive group number 9 of degree 13
```

```
sage: f = x^12 - 2*x^8 - x^7 + 2*x^6 + 4*x^4 - 2*x^3 - x^2 - x + 1
sage: f.galois_group(algorithm='gap')
Transitive group number 183 of degree 12
```

```
sage: f.galois_group(algorithm='magma')  # optional - magma
Transitive group number 5 of degree 4
```

galois_group_davenport_smith_test (num_trials=50, assume_irreducible=False)
Use the Davenport-Smith test to attempt to certify that \( f \) has Galois group \( A_n \) or \( S_n \).

Return 1 if the Galois group is certified as \( S_n \), 2 if \( A_n \), or 0 if no conclusion is reached.

By default, we first check that \( f \) is irreducible. For extra efficiency, one can override this by specifying assume_irreducible=True; this yields undefined results if \( f \) is not irreducible.

A corresponding function in Magma is IsEasySnAn.

EXAMPLES:

```
sage: P.<x> = QQ[]
sage: u = x^7 + x + 1
sage: u.galois_group_davenport_smith_test()  # optional - magma
1
```

```
sage: u = x^7 - x^4 - x^3 + 3*x^2 - 1
sage: u.galois_group_davenport_smith_test()  # optional - magma
2
```

```
sage: u = x^7 - 2
sage: u.galois_group_davenport_smith_test()  # optional - magma
0
```

gcd (right)
Return the (monic) greatest common divisor of self and right.

Corner cases: if self and right are both zero, returns zero. If only one of them is zero, returns the other polynomial, up to normalisation.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: f = -2 + 3*t/2 + 4*t^2/7 - t^3
sage: g = 1/2 + 4*t + 2*t^4/3
sage: f.gcd(g)
1
```
(continues on next page)
hensel_lift \((p, e)\)
Assuming that this polynomial factors modulo \(p\) into distinct monic factors, computes the Hensel lifts of these factors modulo \(p^e\). We assume that self has integer coefficients.

Return an empty list if this polynomial has degree less than one.

INPUT:
- \(p\) - Prime number; coercable to Integer
- \(e\) - Exponent; coercable to Integer

OUTPUT: Hensel lifts; list of polynomials over \(\mathbb{Z}/p^e\mathbb{Z}\)

EXAMPLES:

```sage
sage: R.<x> = QQ[]
sage: R((x-1)*(x+1)).hensel_lift(7, 2)
[x + 1, x + 48]
```

If the input polynomial \(f\) is not monic, we get a factorization of \(f/lc(f)\):

```sage
sage: R(2*x^2 - 2).hensel_lift(7, 2)
[x + 1, x + 48]
```

inverse_series_trunc \((prec)\)
Return a polynomial approximation of precision \(prec\) of the inverse series of this polynomial.

EXAMPLES:

```sage
sage: x = polygen(QQ)
sage: p = 2 + x - 3/5*x**2
sage: q5 = p.inverse_series_trunc(5)
sage: q5
151/800*x^4 - 17/80*x^3 + 11/40*x^2 - 1/4*x + 1/2
sage: q5 * p
-453/4000*x^6 + 253/800*x^5 + 1
sage: q100 = p.inverse_series_trunc(100)
sage: (q100 * p).truncate(100)
1
```

is_irreducible()
Return whether this polynomial is irreducible.

This method computes the primitive part as an element of \(\mathbb{Z}[t]\) and calls the method is_irreducible for elements of that polynomial ring.

By definition, over any integral domain, an element \(r\) is irreducible if and only if it is non-zero, not a unit and whenever \(r = ab\) then \(a\) or \(b\) is a unit.

EXAMPLES:
is_one()
Return whether or not this polynomial is one.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: R([0,1]).is_one()
False
sage: R([1]).is_one()
True
sage: R([0]).is_one()
False
sage: R([-1]).is_one()
False
sage: R([1,1]).is_one()
False
```

is_zero()
Return whether or not self is the zero polynomial.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: f = 1 - t + 1/2*t^2 - 1/3*t^3
sage: f.is_zero()
False
sage: R(0).is_zero()
True
```

lcm(right)
Return the monic (or zero) least common multiple of self and right.

Corner cases: if either of self and right are zero, returns zero. This behaviour is ensures that the relation lcm(a, b) · gcd(a, b) = a · b holds up to multiplication by rationals.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: f = -2 + 3*t/2 + 4*t^2/7 - t^3
sage: g = 1/2 + 4*t + 2*t^4/3
sage: f.lcm(g)
t^7 - 4/7*t^6 - 3/2*t^5 + 8*t^4 - 75/28*t^3 - 66/7*t^2 + 87/8*t + 3/2
sage: f.lcm(g) * f.gcd(g) // (f * g)
-3/2
```

list (copy=True)
Return a list with the coefficients of self.

EXAMPLES:
sage: R.<t> = QQ[]
sage: f = 1 + t + t^2/2 + t^3/3 + t^4/4
sage: f.list()
[1, 1, 1/2, 1/3, 1/4]
sage: g = R(0)
sage: g.list()
[]

**numerator()**

Return the numerator of self.

Representing self as the quotient of an integer polynomial and a positive integer denominator (coprime to the content of the polynomial), returns the integer polynomial.

**EXAMPLES:**

sage: R.<t> = QQ[]
sage: f = (3 * t^3 + 1) / -3
sage: f.numerator()
-3*t^3 - 1

**quo_rem(right)**

Return the quotient and remainder of the Euclidean division of self and right.

Raises a `ZeroDivisionError` if right is zero.

**EXAMPLES:**

sage: R.<t> = QQ[]
sage: g = R.random_element(1000)
sage: q, r = f.quo_rem(g)
sage: f == q*g + r
True

**real_root_intervals()**

Return isolating intervals for the real roots of self.

**EXAMPLES:**

We compute the roots of the characteristic polynomial of some Salem numbers:

sage: R.<t> = QQ[]
sage: f = 1 - t^2 - t^3 - t^4 + t^6
sage: f.real_root_intervals()
[((1/2, 3/4), 1), ((1, 3/2), 1)]

**resultant(right)**

Return the resultant of self and right.

Enumerating the roots over $\mathbb{Q}$ as $r_1, \ldots, r_m$ and $s_1, \ldots, s_n$ and letting $x$ and $y$ denote the leading coefficients of $f$ and $g$, the resultant of the two polynomials is defined by

$$x^\deg g y^\deg f \prod_{i,j} (r_i - s_j).$$

Corner cases: if one of the polynomials is zero, the resultant is zero. Note that otherwise if one of the polynomials is constant, the last term in the above is the empty product.

**EXAMPLES:**
sage: R.<t> = QQ[]
sage: f = (t - 2/3) * (t + 4/5) * (t - 1)
sage: g = (t - 1/3) * (t + 1/2) * (t + 1)
sage: f.resultant(g)
119/1350
sage: h = (t - 1/3) * (t + 1/2) * (t - 1)
sage: f.resultant(h)
0

reverse (degree=None)
Reverse the coefficients of this polynomial (thought of as a polynomial of degree degree).

INPUT:

- degree (None or integral value that fits in an unsigned long, default: degree of self) - if specified, truncate or zero pad the list of coefficients to this degree before reversing it.

EXAMPLES:

We first consider the simplest case, where we reverse all coefficients of a polynomial and obtain a polynomial of the same degree:

sage: R.<t> = QQ[]
sage: f = 1 + t + t^2 / 2 + t^3 / 3 + t^4 / 4
sage: f.reverse()
t^4 + t^3 + 1/2*t^2 + 1/3*t + 1/4

Next, an example where the returned polynomial has lower degree because the original polynomial has low coefficients equal to zero:

sage: R.<t> = QQ[]
sage: f = 3/4*t^2 + 6*t^7
sage: f.reverse()
3/4*t^5 + 6

The next example illustrates the passing of a value for degree less than the length of self, notationally resulting in truncation prior to reversing:

sage: R.<t> = QQ[]
sage: f = 1 + t + t^2 / 2 + t^3 / 3 + t^4 / 4
sage: f.reverse(2)
t^2 + t + 1/2

Now we illustrate the passing of a value for degree greater than the length of self, notationally resulting in zero padding at the top end prior to reversing:

sage: R.<t> = QQ[]
sage: f = 1 + t + t^2 / 2 + t^3 / 3
sage: f.reverse(4)
t^4 + t^3 + 1/2*t^2 + 1/3*t

collect_coefficient (degree)

Return a polynomial f such that f(self(x)) = self(f(x)) = x mod x^n.

EXAMPLES:

sage: R.<t> = QQ[]
sage: f = t - t^3/6 + t^5/120

(continues on next page)
sage: f.revert_series(6)
3/40*t^5 + 1/6*t^3 + t
sage: f.revert_series(-1)
Traceback (most recent call last):
  ... ValueError: argument n must be a non-negative integer, got -1

sage: g = - t^3/3 + t^5/5
sage: g.revert_series(6)
Traceback (most recent call last):
  ... ValueError: self must have constant coefficient 0 and a unit for coefficient → t^1

**truncate** \(n\)

Return self truncated modulo \(t^n\).

**INPUT:**

- \(n\) - The power of \(t\) modulo which self is truncated

**EXAMPLES:**

```python
sage: R.<t> = QQ[]
sage: f = 1 - t + 1/2*t^2 - 1/3*t^3
sage: f.truncate(0)
0
sage: f.truncate(2)
-t + 1
```

**xgcd** \(\text{right}\)

Return polynomials \(d, s,\) and \(t\) such that \(d == s * \text{self} + t * \text{right}\), where \(d\) is the (monic) greatest common divisor of \(\text{self}\) and \(\text{right}\). The choice of \(s\) and \(t\) is not specified any further.

Corner cases: if \(\text{self}\) and \(\text{right}\) are zero, returns zero polynomials. Otherwise, if only \(\text{self}\) is zero, returns \((d, s, t) = (\text{right}, 0, 1)\) up to normalisation, and similarly if only \(\text{right}\) is zero.

**EXAMPLES:**

```python
sage: R.<t> = QQ[]
sage: f = 2/3 + 3/4 * t - t^2
sage: g = -3 + 1/7 * t
sage: f.xgcd(g)
(1, -12/5095, -84/5095*t - 1701/5095)
```

## 2.1.10 Dense univariate polynomials over \(\mathbb{Z}/n\mathbb{Z}\), implemented using FLINT

This module gives a fast implementation of \((\mathbb{Z}/n\mathbb{Z})[x]\) whenever \(n\) is at most \(\text{sys.maxsize}\). We use it by default in preference to NTL when the modulus is small, falling back to NTL if the modulus is too large, as in the example below.

**EXAMPLES:**

```python
sage: R.<a> = PolynomialRing(Integers(100))
sage: type(a)
<class 'sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint'>
sage: R.<a> = PolynomialRing(Integers(5*2^64))
```

(continues on next page)
AUTHORS:

- Burcin Erocal (2008-11) initial implementation
- Martin Albrecht (2009-01) another initial implementation

**class** `sage.rings.polynomial.polynomial_zmod_flint.Polynomial_template`

Bases: `Polynomial`

Template for interfacing to external C/C++ libraries for implementations of polynomials.

AUTHORS:

- Robert Bradshaw (2008-10): original idea for templating
- Martin Albrecht (2008-10): initial implementation

This file implements a simple templating engine for linking univariate polynomials to their C/C++ library implementa-
tions. It requires a “linkage” file which implements the `element_` functions (see `sage.libsntl.ntl_GF2X_linkage` for an example). Both parts are then plugged together by inclusion of the linkage file when inheriting from this class. See `sage.rings.polynomial.polynomial_gf2x` for an example.

We illustrate the generic glueing using univariate polynomials over GF(2).

**Note:** Implementations using this template MUST implement coercion from base ring elements and `get_unsafe()`. See `Polynomial_GF2X` for an example.

degree()

EXAMPLES:

```sage
sage: P.<x> = GF(2)[]
sage: x.degree()
1
sage: P(1).degree()
0
sage: P(0).degree()
-1
```

gcd(other)

Return the greatest common divisor of self and other.

EXAMPLES:

```sage
sage: P.<x> = GF(2)[]
sage: f = x*(x+1)
sage: f.gcd(x+1)
x + 1
sage: f.gcd(x^2)
x
```
**get_cparent()**

**is_gen()**

EXAMPLES:
```
sage: P.<x> = GF(2)[]
sage: x.is_gen()
True
sage: (x+1).is_gen()
False
```

**is_one()**

EXAMPLES:
```
sage: P.<x> = GF(2)[]
sage: P(1).is_one()
True
```

**is_zero()**

EXAMPLES:
```
sage: P.<x> = GF(2)[]
sage: x.is_zero()
False
```

**list** *(copy=True)*

EXAMPLES:
```
sage: P.<x> = GF(2)[]
sage: x.list()
[0, 1]
sage: list(x)
[0, 1]
```

**quo_rem** *(right)*

EXAMPLES:
```
sage: P.<x> = GF(2)[]
sage: f = x^2 + x + 1
sage: f.quo_rem(x + 1)
(x, 1)
```

**shift** *(n)*

EXAMPLES:
```
sage: P.<x> = GF(2)[]
sage: f = x^3 + x^2 + 1
sage: f.shift(1)
x^4 + x^3 + x
sage: f.shift(-1)
x^2 + x
```

**truncate** *(n)*

Returns this polynomial mod $x^n$.

EXAMPLES:
Polynomials, Release 10.3

```python
sage: R.<x> = GF(2)[]
sage: f = sum(x^n for n in range(10)); f
x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1
sage: f.truncate(6)
x^5 + x^4 + x^3 + x^2 + x + 1
```

If the precision is higher than the degree of the polynomial then the polynomial itself is returned:

```python
sage: f.truncate(10) is f
True
```

If the precision is negative, the zero polynomial is returned:

```python
sage: f.truncate(-1)
0
```

**xgcd**(other)

Computes extended gcd of self and other.

EXAMPLES:

```python
sage: P.<x> = GF(7)[]
sage: f = x*(x+1)
sage: f.xgcd(x+1)
(x + 1, 0, 1)
sage: f.xgcd(x^2)
(x, 1, 6)
```

```
class sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint

Bases: Polynomial_template

Polynomial on \(\mathbb{Z}/n\mathbb{Z}\) implemented via FLINT.

__add__(right)

EXAMPLES:

```python
definition
```

__sub__(right)

EXAMPLES:

```python
definition
```

__lmul__(left)

EXAMPLES:

```python
definition
```

(continues on next page)
Polynomials, Release 10.3

(continued from previous page)

```python
sage: R.<y> = GF(5)[]
sage: u = y^2 + y + 1
sage: 3*u
3*y^2 + 3*y + 3
sage: 5*u
0
sage: (2^81)*u
2*y^2 + 2*y + 2
sage: (-2^81)*u
3*y^2 + 3*y + 3
```

```python
sage: P.<x> = GF(2)[]
sage: t = x^2 + x + 1
sage: t*0
0
sage: t*1
x^2 + x + 1
sage: R.<y> = GF(5)[]
sage: u = y^2 + y + 1
sage: u*3
3*y^2 + 3*y + 3
sage: u*5
0
```

_**rmul** _(right)_
Multiply self on the right by a scalar.

**EXAMPLES:**

```python
sage: R.<x> = ZZ[]
sage: f = (x^3 + x + 5)
sage: f._rmul_(7)
7*x^3 + 7*x + 35
sage: f*7
7*x^3 + 7*x + 35
```

_**mul** _(right)_

**EXAMPLES:**

```python
sage: P.<x> = GF(2)[]
sage: x*(x+1)
x^2 + x
```

_**mul_trunc** _(right, n)_
Return the product of this polynomial and other truncated to the given length n.

This function is usually more efficient than simply doing the multiplication and then truncating. The function is tuned for length n about half the length of a full product.

**EXAMPLES:**

```python
sage: P.<a> = GF(7)[]
sage: a = P(range(10)); b = P(range(5, 15))
sage: a._mul_trunc_(b, 5)
4*a^4 + 6*a^3 + 2*a^2 + 5*a
```

2.1. Univariate Polynomials and Polynomial Rings
factor()  
Return the factorization of the polynomial.

EXAMPLES:

```
sage: R.<x> = GF(5)[]
sage: (x^2 + 1).factor()
(x + 2) * (x + 3)
```

It also works for prime-power moduli:

```
sage: R.<x> = Zmod(23^5)[]
sage: (x^3 + 1).factor()
(x + 1) * (x^2 + 6436340*x + 1)
```

is_irreducible()  
Return whether this polynomial is irreducible.

EXAMPLES:

```
sage: R.<x> = GF(5)[]
sage: (x^2 + 1).is_irreducible()
False
sage: (x^3 + x + 1).is_irreducible()
True
```

Not implemented when the base ring is not a field:

```
sage: S.<s> = Zmod(10)[]
sage: (s^2).is_irreducible()
Traceback (most recent call last):
...
NotImplementedError: checking irreducibility of polynomials
over rings with composite characteristic is not implemented
```

minpoly_mod(other)  
Thin wrapper for sage.rings.polynomial.polynomial_modn_dense_ntl.
Polynomial_dense_mod_n.minpoly_mod().

EXAMPLES:

```
sage: R.<x> = GF(127)[]
sage: type(x)
<class 'sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint'>
sage: (x^5 - 3).minpoly_mod(x^3 + 5*x - 1)
x^3 + 34*x^2 + 125*x + 95
```

monic()  
Return this polynomial divided by its leading coefficient.

Raises ValueError if the leading coefficient is not invertible in the base ring.

EXAMPLES:

```
sage: R.<x> = GF(5)[]
sage: (2*x^2 + 1).monic()
x^2 + 3
```
rational_reconstruct(*args, **kwds)

Deprecated: Use rational_reconstruction() instead. See github issue #12696 for details.

rational_reconstruction(m, n_deg=0, d_deg=0)

Construct a rational function \( n/d \) such that \( p + d \) is equivalent to \( n \) modulo \( m \) where \( p \) is this polynomial.

EXAMPLES:

```
sage: P.<x> = GF(5)[]
sage: p = 4*x^5 + 3*x^4 + 2*x^3 + 2*x^2 + 4*x + 2
sage: n, d = p.rational_reconstruction(x^9, 4, 4); n, d
(3*x^4 + 2*x^3 + x^2 + 2*x, x^4 + 3*x^3 + x^2 + x)
sage: (p*d % x^9) == n
True
```

resultant(other)

Return the resultant of self and other, which must lie in the same polynomial ring.

INPUT:

- other -- a polynomial

OUTPUT: an element of the base ring of the polynomial ring

EXAMPLES:

```
sage: R.<x> = GF(19)[

sage: f = x^3 + x + 1; g = x^3 - x - 1
sage: r = f.resultant(g); r
11
sage: r.parent() is GF(19)
True
```

The following example shows that github issue #11782 has been fixed:

```
sage: R.<x> = ZZ.quo(9)[

sage: f = 2*x^3 + x^2 + x; g = 6*x^2 + 2*x + 1
sage: f.resultant(g)
5
```

reverse(degree=None)

Return a polynomial with the coefficients of this polynomial reversed.

If the optional argument degree is given, the coefficient list will be truncated or zero padded as necessary before computing the reverse.

EXAMPLES:

```
sage: R.<x> = GF(5)[

sage: p = R([1,2,3,4]); p
4*x^3 + 3*x^2 + 2*x + 1
sage: p.reverse()
-x^3 + 2*x^2 + 3*x + 4
sage: p.reverse(degree=6)
-x^6 + 2*x^5 + 3*x^4 + 4*x^3
sage: p.reverse(degree=2)
x^2 + 2*x + 3
sage: R.<x> = GF(101)[
```

(continues on next page)
Note that if $f$ has zero constant coefficient, its reverse will have lower degree.

```sage
sage: f = x^3 + 2*x
sage: f.reverse()
x^2 + 2
```

In this case, reverse is not an involution unless we explicitly specify a degree.

```sage
sage: f = x^3 + 2*x
sage: f.reverse().reverse()
x^3 + 2*x
```

**revert_series** ($n$)

Return a polynomial $f$ such that $f(self(x)) = self(f(x)) = x \pmod{x^n}$.

**EXAMPLES:**

```sage
sage: R.<t> = GF(5)[]
sage: f = t + 2*t^2 - t^3 - 3*t^4
sage: f.revert_series(5)
3*t^4 + 4*t^3 + 3*t^2 + t
```

```sage
sage: g = - t^3 + t^5
sage: g.revert_series(-1)
Traceback (most recent call last):
  ... ValueError: argument n must be a non-negative integer, got -1
```

```sage
sage: g = t + 2*t^2 - t^3 -3*t^4 + t^5
sage: g.revert_series(6)
Traceback (most recent call last):
  ... ValueError: the integers 1 up to n=5 are required to be invertible over the...
```

**small_roots** (*args, **kwds)

See `sage.rings.polynomial.polynomial_modn_dense_ntl.small_roots()` for the documentation of this function.

**EXAMPLES:**
sage: N = 10001
sage: K = Zmod(10001)
sage: P.<x> = PolynomialRing(K)
sage: f = x^3 + 10*x^2 + 5000*x - 222
sage: f.small_roots()
[4]

squarefree_decomposition()
Return the squarefree decomposition of this polynomial.

EXAMPLES:

sage: R.<x> = GF(5)[]
sage: ((x+1)*(x^2+1)^2*x^3).squarefree_decomposition()
(x + 1) * (x^2 + 1)^2 * x^3

sage.rings.polynomial.polynomial_zmod_flint.make_element (parent, args)

2.1.11 Dense univariate polynomials over \( \mathbb{Z}/n\mathbb{Z} \), implemented using NTL

This implementation is generally slower than the FLINT implementation in polynomial_zmod_flint, so we use FLINT by default when the modulus is small enough; but NTL does not require that \( n \) be int-sized, so we use it as default when \( n \) is too large for FLINT.

Note that the classes Polynomial_dense_modn_ntl_zz and Polynomial_dense_modn_ntl_ZZ are different; the former is limited to moduli less than a certain bound, while the latter supports arbitrarily large moduli.

AUTHORS:

• Robert Bradshaw: Split off from polynomial_element_generic.py (2007-09)
• Robert Bradshaw: Major rewrite to use NTL directly (2007-09)

class sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_dense_mod_n
Bases: Polynomial

A dense polynomial over the integers modulo \( n \), where \( n \) is composite, with the underlying arithmetic done using NTL.

EXAMPLES:

sage: R.<x> = PolynomialRing(Integers(16), implementation='NTL')
sage: f = x^3 - x + 17
sage: f^2
x^6 + 14*x^4 + 2*x^3 + x^2 + 14*x + 1
sage: loads(f.dumps()) == f
True

sage: R.<x> = PolynomialRing(Integers(100), implementation='NTL')
sage: p = 3*x
sage: q = 7*x
sage: p + q
10*x
sage: R.<x> = PolynomialRing(Integers(8), implementation='NTL')
sage: parent(p)
Univariate Polynomial Ring in x over Ring of integers modulo 100 (using NTL)
(continues on next page)
sage: p + q
10*x
sage: R((10:-1))
7*x^10

\textbf{\texttt{degree}} (gen=None)

Return the degree of this polynomial.

The zero polynomial has degree -1.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: R.<x> = PolynomialRing(Integers(100), implementation='NTL')
sage: (x^3 + 3*x - 17).degree()
3
sage: R.zero().degree()
-1
\end{verbatim}

\textbf{\texttt{int_list}}()

\textbf{\texttt{list}} (copy=True)

Return a new copy of the list of the underlying elements of \texttt{self}.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: _.<x> = PolynomialRing(Integers(100), implementation='NTL')
sage: f = x^3 + 3*x - 17
sage: f.list()
[83, 3, 0, 1]
\end{verbatim}

\textbf{\texttt{minpoly_mod}} (other)

Compute the minimal polynomial of this polynomial modulo another polynomial in the same ring.

\textbf{ALGORITHM:}

\texttt{NTL's MinPolyMod()}, which uses Shoup's algorithm [Sho1999].

\textbf{EXAMPLES:}

\begin{verbatim}
sage: R.<x> = PolynomialRing(GF(101), implementation='NTL')
sage: f = x^17 + x^2 - 1
sage: (x^2).minpoly_mod(f)
x^17 + 100*x^2 + 2*x + 100
\end{verbatim}

\textbf{\texttt{ntl\_ZZ\_pX}}()

Return underlying \texttt{NTL} representation of this polynomial. Additional “bonus” functionality is available through this function.

\begin{verbatim}
Warning: You must call \texttt{ntl.set_modulus(ntl.ZZ(n))} before doing arithmetic with this object!
\end{verbatim}

\textbf{\texttt{ntl\_set\_directly}} (v)

Set the value of this polynomial directly from a vector or string.

Polynomials over the integers modulo \(n\) are stored internally using \texttt{NTL's ZZ\_pX} class. Use this function to set the value of this polynomial using the \texttt{NTL} constructor, which is potentially \texttt{very} fast. The input \(v\) is either
a vector of ints or a string of the form \([ n_1 \ n_2 \ n_3 \ldots ]\) where the ni are integers and there are no commas between them. The optimal input format is the string format, since that’s what NTL uses by default.

**EXAMPLES:**

```python
sage: R.<x> = PolynomialRing(Integers(100), implementation='NTL')

sage: from sage.rings.polynomial.polynomial_modn_dense_ntl import Polynomial_→dense_mod_n as poly_modn_dense

sage: poly_modn_dense(R, ([1,-2,3]))
3*x^2 + 98*x + 1

sage: f = poly_modn_dense(R, 0)

sage: f.ntl_set_directly([1,-2,3])

sage: f
3*x^2 + 98*x + 1

sage: f.ntl_set_directly('1 -2 3 4')

sage: f
4*x^3 + 3*x^2 + 98*x + 1
```

**quo_rem** *(right)*

Return a tuple \((\text{quotient}, \text{remainder})\) where \(\text{self} = \text{quotient} \times \text{other} + \text{remainder}\).

**shift** *(n)*

Return this polynomial multiplied by the power \(x^n\). If \(n\) is negative, terms below \(x^n\) will be discarded. Does not change this polynomial.

**EXAMPLES:**

```python
sage: R.<x> = PolynomialRing(Integers(12345678901234567890), implementation='NTL')

sage: p = x^2 + 2*x + 4

sage: p.shift(0)
x^2 + 2*x + 4

sage: p.shift(-1)
x + 2

sage: p.shift(-5)
0

sage: p.shift(2)
x^4 + 2*x^3 + 4*x^2
```

**AUTHOR:**

- David Harvey (2006-08-06)

**small_roots** *(\*args, \*\*kwds)*

See `sage.rings.polynomial.polynomial_modn_dense_ntl.small_roots()` for the documentation of this function.

**EXAMPLES:**

```python
sage: N = 10001

sage: K = Zmod(10001)

sage: P.<x> = PolynomialRing(K, implementation='NTL')

sage: f = x^3 + 10*x^2 + 5000*x - 222

sage: f.small_roots()
[4]
```

class `sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_dense_mod_p`

Bases: `Polynomial_dense_mod_n`

A dense polynomial over the integers modulo \(p\), where \(p\) is prime.
**discriminant()**

**EXAMPLES:**

```
sage: _.<x> = PolynomialRing(GF(19), implementation='NTL')
sage: f = x^3 + 3*x - 17
sage: f.discriminant()
sage: 12
```

**gcd(right)**

Return the greatest common divisor of this polynomial and other, as a monic polynomial.

**INPUT:**

- other – a polynomial defined over the same ring as self

**EXAMPLES:**

```
sage: R.<x> = PolynomialRing(GF(3), implementation="NTL")
sage: f, g = x + 2, x^2 - 1
sage: f.gcd(g)
sage: x + 2
```

**resultant(other)**

Return the resultant of self and other, which must lie in the same polynomial ring.

**INPUT:**

- other – a polynomial

**OUTPUT:** an element of the base ring of the polynomial ring

**EXAMPLES:**

```
sage: R.<x> = PolynomialRing(GF(19), implementation='NTL')
sage: f = x^3 + x + 1; g = x^3 - x - 1
sage: r = f.resultant(g); r
sage: 11
sage: r.parent() is GF(19)
sage: True
```

**xgcd(other)**

Compute the extended gcd of this element and other.

**INPUT:**

- other – an element in the same polynomial ring

**OUTPUT:**

A tuple \((r, s, t)\) of elements in the polynomial ring such that \(r = s*self + t*other\).

**EXAMPLES:**

```
sage: R.<x> = PolynomialRing(GF(3), implementation='NTL')
sage: x.xgcd(x)
(x, 0, 1)
sage: (x^2 - 1).xgcd(x - 1)
(x + 2, 0, 1)
sage: R.zero().xgcd(R.one())
(1, 0, 1)
sage: (x^3 - 1).xgcd((x - 1)^2)
```

(continues on next page)
(continued from previous page)

\[(x^2 + x + 1, 0, 1)\]
\[\text{sage: } ((x - 1)*(x + 1)).\gcd(x^*(x - 1))\]
\[(x + 2, 1, 2)\]

```
class sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_dense_modn_ntl_ZZ
    Bases: Polynomial_dense_mod_n

degree()  
EXAMPLES:
    
    sage: R.<x> = PolynomialRing(Integers(14^34), implementation='NTL')
    sage: f = x^4 - x - 1
    sage: f.degree()  
    4
    sage: f = 14^43*x + 1
    sage: f.degree()  
    0

is_gen()  
list (copy=True)

quo_rem(right)
    Return \(q\) and \(r\), with the degree of \(r\) less than the degree of \(right\), such that \(q \cdot right + r = self\).
    
EXAMPLES:
    
    sage: R.<x> = PolynomialRing(Integers(10^30), implementation='NTL')
    sage: f = x^5+1; g = (x+1)^2
    sage: q, r = f.quo_rem(g)
    sage: q
    x^3 + 999999999999999999999999999998*x^2 + 3*x + ...
    (...999999999999999999999999999996
    sage: r
    5*x + 5
    sage: q*g + r
    x^5 + 1

reverse (degree=None)
    Return the reverse of the input polynomial thought as a polynomial of degree \(degree\).
    
If \(f\) is a degree-\(d\) polynomial, its reverse is \(x^df(1/x)\).
    
INPUT:
    
    • degree (None or an integer) - if specified, truncate or zero pad the list of coefficients to this degree before reversing it.
    
EXAMPLES:
    
    sage: R.<x> = PolynomialRing(Integers(12^29), implementation='NTL')
    sage: f = x^4 + 2*x + 5
    sage: f.reverse()
    5*x^4 + 2*x^3 + 1
    sage: f = x^3 + x
    sage: f.reverse()
```

(continues on next page)
\[ x^2 + 1 \]
\[ \text{sage}: \ f.\text{reverse}(1) \]
\[ 1 \]
\[ \text{sage}: \ f.\text{reverse}(5) \]
\[ x^4 + x^2 \]

**shift** \((n)\)

Shift self to left by \(n\), which is multiplication by \(x^n\), truncating if \(n\) is negative.

**EXAMPLES:**

\[ \text{sage}: \ R.<x> = \text{PolynomialRing}(\text{Integers}(12^{30}), \text{implementation}=\text{'}NTL'\text{'}) \]
\[ \text{sage}: \ f = x^7 + x + 1 \]
\[ \text{sage}: \ f.\text{shift}(1) \]
\[ x^8 + x^2 + x \]
\[ \text{sage}: \ f.\text{shift}(-1) \]
\[ x^6 + 1 \]
\[ \text{sage}: \ f.\text{shift}(10).\text{shift}(-10) == f \]
\[ \text{True} \]

**truncate** \((n)\)

Return this polynomial mod \(x^n\).

**EXAMPLES:**

\[ \text{sage}: \ R.<x> = \text{PolynomialRing}(\text{Integers}(15^{30}), \text{implementation}=\text{'}NTL'\text{'}) \]
\[ \text{sage}: \ f = \sum(x^n \text{ for } n \text{ in range(10))); \ f \]
\[ x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \]
\[ \text{sage}: \ f.\text{truncate}(6) \]
\[ x^5 + x^4 + x^3 + x^2 + x + 1 \]

**valuation**()

Return the valuation of self, that is, the power of the lowest non-zero monomial of self.

**EXAMPLES:**

\[ \text{sage}: \ R.<x> = \text{PolynomialRing}(\text{Integers}(10^{50}), \text{implementation}=\text{'}NTL'\text{'}) \]
\[ \text{sage}: \ x.\text{valuation()} \]
\[ 1 \]
\[ \text{sage}: \ f = x - 3; \ f.\text{valuation()} \]
\[ 0 \]
\[ \text{sage}: \ f = x^99; \ f.\text{valuation()} \]
\[ 99 \]
\[ \text{sage}: \ f = x - x; \ f.\text{valuation()} \]
\[ +\text{Infinity} \]

class

```
sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_dense_modn_ntl_zz
```

Bases: \(\text{Polynomial}\_dense\_modn\_ntl}\)

Polynomial on \(\mathbb{Z}/n\mathbb{Z}\) implemented via NTL.

\[ \_\text{add}_\text{(}_\text{right}_) \]
\[ \_\text{sub}_\text{(}_\text{right}_) \]
\[ \_\text{lmul}_\text{(}_c_\text{)} \]
_rmul_(c)
_mul_(right)
_mul_trunc_(right, n)

Return the product of self and right truncated to the given length n

EXAMPLES:

```sage
sage: R.<x> = PolynomialRing(Integers(100), implementation="NTL")
sage: f = x - 2
sage: g = x^2 - 8*x + 16
sage: f*g
x^3 + 90*x^2 + 32*x + 68
sage: f._mul_trunc_(g, 42)
x^3 + 90*x^2 + 32*x + 68
sage: f._mul_trunc_(g, 3)
90*x^2 + 32*x + 68
sage: f._mul_trunc_(g, 2)
32*x + 68
sage: f._mul_trunc_(g, 1)
68
sage: f._mul_trunc_(g, 0)
0
sage: f = x^2 - 8*x + 16
sage: f._mul_trunc_(f, 2)
44*x + 56
```

degree()

EXAMPLES:

```sage
sage: R.<x> = PolynomialRing(Integers(77), implementation='NTL')
sage: f = x^4 - x - 1
go: f.degree()
4
sage: f = 77*x + 1
sage: f.degree()
0
```

int_list()

Return the coefficients of self as efficiently as possible as a list of python ints.

EXAMPLES:

```sage
sage: R.<x> = PolynomialRing(Integers(100), implementation='NTL')
sage: from sage.rings.polynomial.polynomial_modn_dense_ntl import Polynomial_modn_dense
sage: f = poly_modn_dense(R, [5, 0, 0, 1])
sage: f.int_list()
[5, 0, 0, 1]
sage: [type(a) for a in f.int_list()]
[<... 'int'>, <... 'int'>, <... 'int'>, <... 'int'>]
```

is_gen()

ntl_set_directly(v)
**quo_rem** *(right)*

Return \( q \) and \( r \), with the degree of \( r \) less than the degree of \( \text{right} \), such that \( q \cdot \text{right} + r = \text{self} \).  

**EXAMPLES:**

```python
sage: R.<x> = PolynomialRing(Integers(125), implementation='NTL')
sage: f = x^5+1; g = (x+1)^2
sage: q, r = f.quo_rem(g)
sage: q
x^3 + 123*x^2 + 3*x + 121
sage: r
5*x + 5
sage: q*g + r
x^5 + 1
```

**reverse** *(degree=None)*

Return the reverse of the input polynomial thought as a polynomial of degree \( \text{degree} \).

If \( f \) is a degree-\( d \) polynomial, its reverse is \( x^d f(1/x) \).

**INPUT:**

- \text{degree} (None or an integer) - if specified, truncate or zero pad the list of coefficients to this degree before reversing it.

**EXAMPLES:**

```python
sage: R.<x> = PolynomialRing(Integers(77), implementation='NTL')
sage: f = x^4 - x - 1
sage: f.reverse()
76*x^4 + 76*x^3 + 1
sage: f.reverse(2)
76*x^2 + 76*x
sage: f.reverse(5)
76*x^5 + 76*x^4 + x
sage: g = x^3 - x
sage: g.reverse()
76*x^2 + 1
```

**shift** *(n)*

Shift self to left by \( n \), which is multiplication by \( x^n \), truncating if \( n \) is negative.

**EXAMPLES:**

```python
sage: R.<x> = PolynomialRing(Integers(77), implementation='NTL')
sage: f = x^7 + x + 1
sage: f.shift(1)
x^8 + x^2 + x
sage: f.shift(-1)
x^6 + 1
sage: f.shift(10).shift(-10) == f
True
```

**truncate** *(n)*

Return this polynomial mod \( x^n \).

**EXAMPLES:**
sage: R.<x> = PolynomialRing(Integers(77), implementation='NTL')
sage: f = sum(x^n for n in range(10)); f
x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1
sage: f.truncate(6)
x^5 + x^4 + x^3 + x^2 + x + 1

valuation()
Return the valuation of self, that is, the power of the lowest non-zero monomial of self.

EXAMPLES:

sage: R.<x> = PolynomialRing(Integers(10), implementation='NTL')
sage: x.valuation()
1
sage: f = x^-3; f.valuation()
0
sage: f = x^99; f.valuation()
99
sage: f = x-x; f.valuation()
+Infinity

sage.rings.polynomial.polynomial_modn_dense_ntl.make_element(parent, args)
sage.rings.polynomial.polynomial_modn_dense_ntl.small_roots(self, X=None, beta=1.0, epsilon=None, **kwds)

Let $N$ be the characteristic of the base ring this polynomial is defined over: $N = \text{self.base_ring().characteristic()}$. This method returns small roots of this polynomial modulo some factor $b$ of $N$ with the constraint that $b \geq N^\beta$. Small in this context means that if $x$ is a root of $f$ modulo $b$ then $|x| < X$. This $X$ is either provided by the user or the maximum $X$ is chosen such that this algorithm terminates in polynomial time. If $X$ is chosen automatically it is $X = \lceil 1/2N^{\beta^2/d-\epsilon} \rceil$. The algorithm may also return some roots which are larger than $X$. ‘This algorithm’ in this context means Coppersmith's algorithm for finding small roots using the LLL algorithm. The implementation of this algorithm follows Alexander May’s PhD thesis referenced below.

INPUT:

- $X$ – an absolute bound for the root (default: see above)
- $\text{beta}$ – compute a root mod $b$ where $b$ is a factor of $N$ and $b \geq N^\beta$. (Default: 1.0, so $b = N$.)
- $\text{epsilon}$ – the parameter $\epsilon$ described above. (Default: $\beta/8$)
- $\text{**kwds}$ – passed through to method Matrix_integer_dense.LLL().

EXAMPLES:

First consider a small example:

sage: N = 10001
sage: K = Zmod(10001)
sage: P.<x> = PolynomialRing(K, implementation='NTL')
sage: f = x^3 + 10*x^2 + 5000*x - 222

This polynomial has no roots without modular reduction (i.e. over $\mathbb{Z}$):

sage: f.change_ring(ZZ).roots()
[]

To compute its roots we need to factor the modulus $N$ and use the Chinese remainder theorem:
Polynomials, Release 10.3

```
sage: p, q = N.prime_divisors()
sage: f.change_ring(GF(p)).roots()
[(4, 1)]
sage: f.change_ring(GF(q)).roots()
[(4, 1)]
sage: crt(4, 4, p, q)
4
```

This root is quite small compared to $N$, so we can attempt to recover it without factoring $N$ using Coppersmith’s small root method:

```
sage: f.small_roots()
[4]
```

An application of this method is to consider RSA. We are using 512-bit RSA with public exponent $e = 3$ to encrypt a 56-bit DES key. Because it would be easy to attack this setting if no padding was used we pad the key $K$ with 1s to get a large number:

```
sage: Nbits, Kbits = 512, 56
sage: e = 3
```

We choose two primes of size 256-bit each:

```
sage: p = 2^256 + 2^8 + 2^5 + 2^3 + 1
sage: q = 2^256 + 2^8 + 2^5 + 2^3 + 2^2 + 1
sage: N = p*q
sage: ZmodN = Zmod( N )
```

We choose a random key:

```
sage: K = ZZ.random_element(0, 2^Kbits)
```

and pad it with $512 - 56 = 456$ 1s:

```
sage: Kdigits = K.digits(2)
sage: M = [0]*Kbits + [1]*(Nbits-Kbits)
sage: for i in range(len(Kdigits)): M[i] = Kdigits[i]
sage: M = ZZ(M, 2)
```

Now we encrypt the resulting message:

```
sage: C = ZmodN(M)^e
```

To recover $K$ we consider the following polynomial modulo $N$:

```
sage: P.<x> = PolynomialRing(ZmodN, implementation='NTL')
sage: f = (2^Nbits - 2^Kbits + x)^e - C
```

and recover its small roots:

```
sage: Kbar = f.small_roots()[0]
sage: K == Kbar
True
```

The same algorithm can be used to factor $N = pq$ if partial knowledge about $q$ is available. This example is from the Magma handbook:
First, we set up $p$, $q$ and $N$:

```python
sage: length = 512
sage: hidden = 110
sage: p = next_prime(2^int(round(length/2)))
# needs sage.symbolic
sage: q = next_prime(round(pi.n()^p))
# needs sage.symbolic
sage: N = p*q
# needs sage.symbolic
```

Now we disturb the low 110 bits of $q$:

```python
sage: qbar = q + ZZ.random_element(0, 2^hidden - 1)
# needs sage.symbolic
```

And try to recover $q$ from it:

```python
sage: F.<x> = PolynomialRing(Zmod(N), implementation='NTL')
# needs sage.symbolic
sage: f = x - qbar
# needs sage.symbolic
```

We know that the error is $\leq 2^{\text{hidden}} - 1$ and that the modulus we are looking for is $\geq \sqrt{N}$:

```python
sage: from sage.misc.verbose import set_verbose
sage: set_verbose(2)

sage: d = f.small_roots(X=2^hidden-1, beta=0.5)[0]  # time random
# needs sage.symbolic
```

REFERENCES:


2.1.12 Dense univariate polynomials over $\mathbb{R}$, implemented using MPFR

```python
class sage.rings.polynomial.polynomial_real_mpfr_dense.PolynomialRealDense
Bases: Polynomial
change_ring(R)
EXAMPLES:
```

(continues on next page)
degree()

Return the degree of the polynomial.

EXAMPLES:

```sage
sage: from sage.rings.polynomial.polynomial_real_mpfr_dense import PolynomialRealDense
sage: f = PolynomialRealDense(RR['x'], [1, 2, 3]); f
3.00000000000000*x^2 + 2.00000000000000*x + 1.00000000000000
sage: f.degree()
2
```

integral()

EXAMPLES:

```sage
sage: from sage.rings.polynomial.polynomial_real_mpfr_dense import PolynomialRealDense
sage: f = PolynomialRealDense(RR['x'], [3, pi, 1])
# needs sage.symbolic
sage: f.integral()
# needs sage.symbolic
0.333333333333333*x^3 + 1.57079632679490*x^2 + 3.00000000000000*x
```

list (copy=True)

EXAMPLES:

```sage
sage: from sage.rings.polynomial.polynomial_real_mpfr_dense import PolynomialRealDense
sage: f = PolynomialRealDense(RR['x'], [1, 0, -2]); f
-2.00000000000000*x^2 + 1.00000000000000
sage: f.list()
[1.00000000000000, 0.00000000000000, -2.00000000000000]
```

quo_rem(other)

Return the quotient with remainder of self by other.

EXAMPLES:

```sage
sage: from sage.rings.polynomial.polynomial_real_mpfr_dense import PolynomialRealDense
sage: f = PolynomialRealDense(RR['x'], [-2, 0, 1])
sage: g = PolynomialRealDense(RR['x'], [5, 1])
sage: q, r = f.quo_rem(g)
sage: q
x - 5.00000000000000
sage: r
23.0000000000000
sage: q*g + r == f
```

(continues on next page)
True
sage: fg = f*g
sage: fg.quo_rem(f)
(x + 5.00000000000000, 0)
sage: fg.quo_rem(g)
(x^2 - 2.00000000000000, 0)

sage: # needs sage.symbolic
sage: f = PolynomialRealDense(RR['x'], range(5))
sage: g = PolynomialRealDense(RR['x'], [pi,3000,4])
sage: q, r = f.quo_rem(g)
sage: g*q + r == f
True

reverse (degree=None)
Return reverse of the input polynomial thought as a polynomial of
degree degree.
If f is a degree-d polynomial, its reverse is \( x^d f(1/x) \).

INPUT:

- degree (None or an integer) - if specified, truncate or zero pad the list of coefficients to this degree
  before reversing it.

EXAMPLES:

sage: # needs sage.symbolic
sage: f = RR['x']([-3, pi, 0, 1])
sage: f.reverse()
-3.00000000000000*x^3 + 3.14159265358979*x^2 + 1.00000000000000
sage: f.reverse(2)
-3.00000000000000*x^2 + 3.14159265358979*x
sage: f.reverse(5)
-3.00000000000000*x^5 + 3.14159265358979*x^4 + x^2

shift (n)
Returns this polynomial multiplied by the power \( x^n \). If n is negative, terms below \( x^n \) will be discarded. Does
not change this polynomial.

EXAMPLES:

sage: from sage.rings.polynomial.polynomial_real_mpfr_dense import...
PolynomialRealDense
sage: f = PolynomialRealDense(RR['x'], [1, 2, 3]); f
3.00000000000000*x^2 + 2.00000000000000*x + 1.00000000000000
sage: f.shift(10)
3.00000000000000*x^12 + 2.00000000000000*x^11 + x^10
sage: f.shift(-1)
3.00000000000000*x + 2.00000000000000
sage: f.shift(-10)
0

truncate (n)
Returns the polynomial of degree < n which is equivalent to self modulo \( x^n \).

EXAMPLES:
.. code-block:: python

    sage: from sage.rings.polynomial.polynomial_real_mpfr_dense import PolynomialRealDense
    sage: f = PolynomialRealDense(RealField(10)['x'], [1, 2, 4, 8])
    sage: f.truncate(3)
    4.0*x^2 + 2.0*x + 1.0
    sage: f.truncate(100)
    8.0*x^3 + 4.0*x^2 + 2.0*x + 1.0
    sage: f.truncate(1)
    1.0
    sage: f.truncate(0)
    0

.. function:: truncate_abs (bound)

   Truncate all high order coefficients below $bound$.

   EXAMPLES:

   .. code-block:: python

    sage: from sage.rings.polynomial.polynomial_real_mpfr_dense import PolynomialRealDense
    sage: f = PolynomialRealDense(RealField(10)['x'], [10^-k for k in range(10)])
    sage: f
    1.0e-9*x^9 + 1.0e-8*x^8 + 1.0e-7*x^7 + 1.0e-6*x^6 + 0.000010*x^5 + 0.00010*x^4 + 0.0010*x^3 + 0.010*x^2 + 0.10*x + 1.0
    sage: f.truncate_abs(0.5e-6)
    1.0e-6*x^6 + 0.000010*x^5 + 0.00010*x^4 + 0.0010*x^3 + 0.010*x^2 + 0.10*x + 1.0
    sage: f.truncate_abs(10.0)
    0
    sage: f.truncate_abs(1e-100) == f
    True

sage.rings.polynomial.polynomial_real_mpfr_dense.make_PolynomialRealDense (parent, data)

   EXAMPLES:

   .. code-block:: python

    sage: from sage.rings.polynomial.polynomial_real_mpfr_dense import make_PolynomialRealDense
    sage: make_PolynomialRealDense(RR['x'], [1,2,3])
    3.00000000000000*x^2 + 2.00000000000000*x + 1.00000000000000

2.1.13 Polynomial Interfaces to Singular

AUTHORS:

- Martin Albrecht <malb@informatik.uni-bremen.de> (2006-04-21)
- Robert Bradshaw: Re-factor to avoid multiple inheritance vs. Cython (2007-09)
- Syed Ahmad Lavasani: Added function field to _singular_init_ (2011-12-16); Added non-prime finite fields to _singular_init_ (2012-1-22)

class sage.rings.polynomial.polynomial_singular_interface.PolynomialRing_singular_repr

   Bases: object

   Implements methods to convert polynomial rings to Singular.
This class is a base class for all univariate and multivariate polynomial rings which support conversion from and to Singular rings.

```python
class sage.rings.polynomial.polynomial_singular_interface.Polynomial_singular_repr
    Bases: object

    Implements coercion of polynomials to Singular polynomials.

    This class is a base class for all (univariate and multivariate) polynomial classes which support conversion from and to Singular polynomials.

    Due to the incompatibility of Python extension classes and multiple inheritance, this just defers to module-level functions.

sage.rings.polynomial.polynomial_singular_interface.can_convert_to_singular(R)

    Return True if this ring's base field or ring can be represented in Singular, and the polynomial ring has at least one generator.

    The following base rings are supported: finite fields, rationals, number fields, and real and complex fields.

    EXAMPLES:
```
Polynomials, Release 10.3

```python
sage: # needs sage.libs.ntl
sage: K = Zp(13,7)
sage: R.<t> = K[]
sage: f = 13^7*t^3 + K(169,4)*t - 13^4
sage: f.content()
13^2 + O(13^9)
sage: R(0).content()
0
sage: f = R(K(0,3)); f
O(13^3)
sage: f.content()
O(13^3)

sage: # needs sage.libs.ntl
sage: P.<x> = ZZ[]
sage: f = x + 2
sage: f.content()
1
sage: fp = f.change_ring(pAdicRing(2, 10))
sage: fp
(1 + O(2^10))*x + 2 + O(2^11)
sage: fp.content()
1 + O(2^10)

sage: # needs sage.libs.ntl
sage: K = Qp(13,7)
sage: R.<t> = K[]
sage: f = 13^7*t^3 + K(169,4)*t - 13^-4
sage: f.content()
13^-4 + O(13^3)
sage: f = R.zero(); f
0
sage: f.content()
0
sage: f = R(K(0,3))

Over a field it would be sufficient to return only zero or one, as the content is only defined up to multiplication with a unit. However, we return $\pi^k$ where $k$ is the minimal valuation of any coefficient:

```python
sage: # needs sage.libs.ntl
sage: K = Qp(13,7)
sage: R.<t> = K[]
sage: f = 13^7*t^3 + K(169,4)*t - 13^-4
sage: f.content()
13^-4 + O(13^3)
sage: f = R.zero(); f
0
sage: f.content()
0
sage: f = R(K(0,3))

factor()

Return the factorization of this polynomial.

EXAMPLES:

```python
sage: # needs sage.libs.ntl
sage: R.<t> = PolynomialRing(Qp(3, 3, print_mode='terse', print_pos=False))
sage: pol = t^8 - 1
sage: for p,e in pol.factor():
..... print("{} {}").format(e, p))
1 (1 + O(3^3))*t + 1 + O(3^3)
1 (1 + O(3^3))*t - 1 + O(3^3)
```

(continues on next page)
1 (1 + O(3^3))*t^2 + (5 + O(3^3))*t - 1 + O(3^3)
1 (1 + O(3^3))*t^2 + (-5 + O(3^3))*t - 1 + O(3^3)
1 (1 + O(3^3))*t^2 + 0(3^3)*t + 1 + O(3^3)

\texttt{sage: R.<t> = PolynomialRing(Qp(5, 6, print_mode='terse', print_pos=True))}
\texttt{sage: pol = 100 * (5*t - 1) * (t - 5); pol}
(500 + O(5^9))*t^2 + (-2600 + O(5^8))*t + 500 + O(5^9)
\texttt{sage: pol.factor()}
(500 + O(5^9)) * ((1 + O(5^5))*t - 1/5 + O(5^5)) * ((1 + O(5^6))*t - 5 + O(5^6))
\texttt{sage: pol.factor().value()}
(500 + O(5^8))*t^2 + (-2600 + O(5^8))*t + 500 + O(5^8)

The same factorization over \(\mathbb{Z}_p\). In this case, the “unit” part is a \(p\)-adic unit and the power of \(p\) is considered to be a factor:

\texttt{sage: # needs sage.libs.ntl}
\texttt{sage: R.<t> = PolynomialRing(Zp(5, 6, print_mode='terse', print_pos=True))}
\texttt{sage: pol = 100 * (5*t - 1) * (t - 5); pol}
(500 + O(5^9))*t^2 + (-2600 + O(5^8))*t + 500 + O(5^9)
\texttt{sage: pol.factor()}
(4 + O(5^6)) * (5 + O(5^7))^2 * ((1 + O(5^6))*t - 5 + O(5^6)) * ((5 + O(5^6))*t - 1 + O(5^6))
\texttt{sage: pol.factor().value()}
(500 + O(5^8))*t^2 + (-2600 + O(5^8))*t + 500 + O(5^8)

In the following example, the discriminant is zero, so the \(p\)-adic factorization is not well defined:

\texttt{sage: factor(t^2)}
\texttt{Traceback (most recent call last):}
\texttt{...}
PrecisionError: \(p\)-adic factorization not well-defined since the discriminant is zero up to the \(p\)-adic precision

An example of factoring a constant polynomial (see \texttt{github issue #26669}):
sage: # needs sage.libs.ntl sage.schemes
sage: E = EllipticCurve('37a1')
sage: K = Qp(7,10)
sage: EK = E.base_extend(K)
sage: g = EK.division_polynomial_0(3)
sage: g.factor() *(3 + O(7^10))
  * ((1 + O(7^10))*x
  + 1 + 2*7 + 4*7^2 + 2*7^3 + 5*7^4 + 7^5 + 5*7^6 + 3*7^7 + 5*7^8 + 3*7^9 →
    O(7^10))
  * ((1 + O(7^10))*x^3
    + (6 + 4*7 + 2*7^2 + 4*7^3 + 7^4 + 5*7^5
      + 7^6 + 3*7^7 + 7^8 + 3*7^9 + O(7^10))*x^2
    + (6 + 3*7 + 5*7^2 + 2*7^3 + 7^5 + 7^6 + 2*7^8 + 3*7^9 + O(7^10))*x
    + 2 + 5*7 + 4*7^2 + 2*7^3 + 6*7^4 + 3*7^5 + 7^6 + 4*7^7 + O(7^10))

root_field(names, check_irreducible=True, **kwds)

Return the \( p \)-adic extension field generated by the roots of the irreducible polynomial \texttt{self}.

INPUT:

- \texttt{names} – name of the generator of the extension
- \texttt{check_irreducible} – check whether the polynomial is irreducible
- \texttt{kwds} – see \texttt{sage.ring.padics.padic_generic.pAdicGeneric.extension()}

EXAMPLES:

sage: R.<x> = Qp(3,5,print_mode=digits)[]  # needs sage.libs.ntl
sage: f = x^2 - 3  # needs sage.libs.ntl
sage: f.root_field(x)  # needs sage.libs.ntl
3-adic Eisenstein Extension Field in x defined by x^2 - 3

sage: R.<x> = Qp(5,5,print_mode=digits)[]  # needs sage.libs.ntl
sage: f = x^2 - 3  # needs sage.libs.ntl
sage: f.root_field(x, print_mode=bars)  # needs sage.libs.ntl
5-adic Unramified Extension Field in x defined by x^2 - 3

sage: R.<x> = Qp(11,5,print_mode=digits)[]  # needs sage.libs.ntl
sage: f = x^2 - 3  # needs sage.libs.ntl
sage: f.root_field(x, print_mode=bars)  # needs sage.libs.ntl
Traceback (most recent call last):
...
ValueError: polynomial must be irreducible
2.1.15 p-adic Capped Relative Dense Polynomials

class sage.rings.polynomial.padics.polynomial_padic_capped_relative_dense.Polynomial_padic_capped_relative_dense(par-
ent, x=None, check=True, is_gen=False, construc-
t=False, ab-

sprec=+In-
finity, rel-
prec=+In-
finity)

Bases: Polynomial_generic_cdv, Polynomial_padic
degree (secure=False)

Return the degree of self.

INPUT:

  • secure – a boolean (default: False)

If secure is True and the degree of this polynomial is not determined (because the leading coefficient is
indistinguishable from 0), an error is raised.

If secure is False, the returned value is the largest \( n \) so that the coefficient of \( x^n \) does not compare equal
to 0.

EXAMPLES:

sage: K = Qp(3,10)
sage: R.<T> = K[

sage: f = T + 2; f
(1 + O(3^10))*T + 2 + O(3^10)
sage: f.degree()
1
sage: (f-T).degree()
0
sage: (f-T).degree(secure=True)
Traceback (most recent call last):
  ...
PrecisionError: the leading coefficient is indistinguishable from 0

sage: x = O(3^5)
sage: li = [3^i * x for i in range(0,5)]; li
[O(3^5), O(3^6), O(3^7), O(3^8), O(3^9)]
sage: f = R(li); f
O(3^9)*T^4 + O(3^8)*T^3 + O(3^7)*T^2 + O(3^6)*T + O(3^5)
sage: f.degree()
-1
sage: f.degree(secure=True)
Traceback (most recent call last):
  ...
PrecisionError: the leading coefficient is indistinguishable from 0
**disc()**

Return the factorization of self modulo $p$.

**factor_mod()**

Return the factorization of self modulo $p$.

**is_eisenstein (secure=False)**

Return True if this polynomial is an Eisenstein polynomial.

**EXAMPLES:**

```sage
K = Qp(5)
sage: R.<t> = K[]
sage: f = 5 + 5*t + t^4
sage: f.is_eisenstein()
True
```

**AUTHOR:**
• Xavier Caruso (2013-03)

**lift()**

Return an integer polynomial congruent to this one modulo the precision of each coefficient.

**Note:** The lift that is returned will not necessarily be the same for polynomials with the same coefficients (i.e. same values and precisions): it will depend on how the polynomials are created.

**EXAMPLES:**

```sage
K = Qp(13,7)
sage: R.<t> = K[]
sage: a = 13^7*t^3 + K(169,4)*t - 13^4
sage: a.lift()
62748517*t^3 + 169*t - 28561
```

**list (copy=True)**

Return a list of coefficients of self.

**Note:** The length of the list returned may be greater than expected since it includes any leading zeros that have finite absolute precision.

**EXAMPLES:**

```sage
K = Qp(13,7)
sage: R.<t> = K[]
sage: a = 2*t^3 + 169*t - 1
sage: a
(2 + O(13^7))*t^3 + (13^2 + O(13^9))*t + 12 + 12*13 + 12*13^2 + 12*13^3 +...
˓→12*13^4 + 12*13^5 + 12*13^6 + O(13^7)
sage: a.list()
[12 + 12*13 + 12*13^2 + 12*13^3 + 12*13^4 + 12*13^5 + 12*13^6 + O(13^7),
13^2 + O(13^9),
0,
2 + O(13^7)]
```
**lshift_coeffs** *(shift, no_list=False)*

Return a new polynomials whose coefficients are multiplied by $p^{\text{shift}}$.

**EXAMPLES:**

```python
sage: K = Qp(13, 4)
sage: R.<t> = K[]
sage: a = t + 52
sage: a.lshift_coeffs(3)
(13^3 + O(13^7))*t + 4*13^4 + O(13^8)
```

**newton_polygon** *( )*

Return the Newton polygon of this polynomial.

**Note:** If some coefficients have not enough precision an error is raised.

**OUTPUT:**

- a `NewtonPolygon`

**EXAMPLES:**

```python
sage: K = Qp(2, prec=5)
sage: R.<x> = K[]
sage: f = x^4 + 2^3*x^3 + 2^13*x^2 + 2^21*x + 2^37
sage: f.newton_polygon()  # needs sage.geometry.polyhedron
Finite Newton polygon with 4 vertices: (0, 37), (1, 21), (3, 3), (4, 0)
sage: K = Qp(5)
sage: R.<t> = K[]
sage: f = 5 + 3*t + t^4 + 25*t^10
sage: f.newton_polygon()  # needs sage.geometry.polyhedron
Finite Newton polygon with 4 vertices: (0, 1), (1, 0), (4, 0), (10, 2)
```

Here is an example where the computation fails because precision is not sufficient:

```python
sage: g = f + K(0,0)*t^4; g
(5^2 + O(5^22))*t^10 + O(5^0)*t^4 + (3 + O(5^20))*t + 5 + O(5^21)
sage: g.newton_polygon()  # needs sage.geometry.polyhedron
Traceback (most recent call last):
...
PrecisionError: The coefficient of t^4 has not enough precision
```

**AUTHOR:**

- Xavier Caruso (2013-03-20)

**newton_slopes** *(repetition=True)*

Return a list of the Newton slopes of this polynomial.

These are the valuations of the roots of this polynomial.

If `repetition` is `True`, each slope is repeated a number of times equal to its multiplicity. Otherwise it appears only one time.

**INPUT:**
• repetition — boolean (default True)

OUTPUT:

• a list of rationals

EXAMPLES:

```python
sage: K = Qp(5)
sage: R.<t> = K[]
sage: f = 5 + 3*t + t^4 + 25*t^10
sage: f.newton_polygon()  # needs sage.geometry.polyhedron
Finite Newton polygon with 4 vertices: (0, 1), (1, 0), (4, 0), (10, 2)
sage: f.newton_slopes()  # needs sage.geometry.polyhedron
[1, 0, -1/3]
sage: f.newton_slopes(repetition=False)  # needs sage.geometry.polyhedron
[1, 0, -1/3]
```

AUTHOR:

• Xavier Caruso (2013-03-20)

`prec_degree()`

Return the largest $n$ so that precision information is stored about the coefficient of $x^n$.

Always greater than or equal to degree.

EXAMPLES:

```python
sage: K = Qp(3,10)
sage: R.<T> = K[]
sage: f = T + 2; f
(1 + O(3^10))*T + 2 + O(3^10)
sage: f.prec_degree()
1
```

`precision_absolute(n=None)`

Return absolute precision information about `self`.

INPUT:

• `self` — a p-adic polynomial

• `n` — None or an integer (default None).

OUTPUT:

If `n` is None, returns a list of absolute precisions of coefficients. Otherwise, returns the absolute precision of the coefficient of $x^n$.

EXAMPLES:

```python
sage: K = Qp(3,10)
sage: R.<T> = K[]
sage: f = T + 2; f
(1 + O(3^10))*T + 2 + O(3^10)
sage: f.precision_absolute()  # (continues on next page)
```
**precision_relative** *(n=None)*

Return relative precision information about self.

**INPUT:**

- `self` - a p-adic polynomial
- `n` - None or an integer (default None).

**OUTPUT:**

If `n` is None, returns a list of relative precisions of coefficients. Otherwise, returns the relative precision of the coefficient of $x^n$.

**EXAMPLES:**

```python
sage: K = Qp(3,10)
sage: R.<T> = K[]
sage: f = T + 2; f
(1 + O(3^10))*T + 2 + O(3^10)
sage: f.precision_relative()
[10, 10]
```

**quo_rem**(right, secure=False)

Return the quotient and remainder in division of self by right.

**EXAMPLES:**

```python
sage: K = Qp(3,10)
sage: R.<T> = K[]
sage: f = T + 2
sage: g = T**4 + 3*T+22
sage: g.quo_rem(f)
((1 + O(3^10))*T^3 + (1 + 2*3 + 2*3^2 + 2*3^3 + 2*3^4 + 2*3^5 + 2*3^6 + 2*3^7 +
    2*3^8 + 2*3^9 + O(3^10))*T^2 + (1 + 3 + O(3^10))*T + 1 + 3 + 2*3^2 + 2*3^3 +
    2*3^4 + 2*3^5 + 2*3^6 + 2*3^7 + 2*3^8 + 2*3^9 + O(3^10),
    2 + 3 + 3^3 + O(3^10))
```

**rescale**(a)

Return $f(a \cdot x)$.

**Todo:** Need to write this function for integer polynomials before this works.

**EXAMPLES:**

```python
sage: K = Zp(13, 5)
sage: R.<t> = K[]
sage: f = t^3 + K(13, 3) * t
sage: f.rescale(2)  # not implemented
```

**reverse**(degree=None)

Return the reverse of the input polynomial, thought as a polynomial of degree degree.

If $f$ is a degree-$d$ polynomial, its reverse is $x^d f(1/x)$.
INPUT:

- degree (None or an integer) - if specified, truncate or zero pad the list of coefficients to this degree before reversing it.

EXAMPLES:

```python
sage: K = Qp(13, 7)
sage: R.<t> = K[]
sage: f = t^3 + 4*t; f
(1 + O(13^7))*t^3 + (4 + O(13^7))*t
sage: f.reverse()
0*t^3 + (4 + O(13^7))*t^2 + 1 + O(13^7)
sage: f.reverse(3)
0*t^3 + (4 + O(13^7))*t^2 + 1 + O(13^7)
sage: f.reverse(2)
0*t^2 + (4 + O(13^7))*t
sage: f.reverse(4)
0*t^4 + (4 + O(13^7))*t^3 + (1 + O(13^7))*t
sage: f.reverse(6)
0*t^6 + (4 + O(13^7))*t^5 + (1 + O(13^7))*t^3
```

**rshift_coeffs** (*shift*, *no_list=False*)

Return a new polynomial whose coefficients are p-adically shifted to the right by *shift*.

**Note:** Type `Qp(5)(0).__rshift__?` for more information.

EXAMPLES:

```python
sage: K = Zp(13, 4)
sage: R.<t> = K[]
sage: a = t^2 + K(13, 3)*t + 169; a
(1 + O(13^4))*t^2 + (13 + O(13^3))*t + 13^2 + O(13^6)
sage: b = a.rshift_coeffs(1); b
O(13^3)*t^2 + (1 + O(13^2))*t + 13 + O(13^5)
sage: b.list()
[13 + O(13^5), 1 + O(13^2), O(13^3)]
sage: b = a.rshift_coeffs(2); b
O(13^2)*t^2 + O(13)*t + 1 + O(13^4)
sage: b.list()
[1 + O(13^4), O(13), O(13^2)]
```

**valuation** (*val_of_var=None*)

Return the valuation of *self*.

**INPUT:**

- *self* - a p-adic polynomial
- *val_of_var* - None or a rational (default None).

**OUTPUT:**

If *val_of_var* is None, returns the largest power of the variable dividing *self*. Otherwise, returns the valuation of *self* where the variable is assigned valuation *val_of_var*.

EXAMPLES:
\begin{verbatim}
sage: K = Qp(3,10)
sage: R.<T> = K[]
sage: f = T + 2; f
(1 + O(3^10))*T + 2 + O(3^10)
sage: f.valuation()
0
\end{verbatim}

**valuation_of_coefficient**\(^{(n=\text{None})}\)

Return valuation information about self's coefficients.

**INPUT:**

- self – a p-adic polynomial
- n – None or an integer (default None).

**OUTPUT:**

If n is None, returns a list of valuations of coefficients. Otherwise, returns the valuation of the coefficient of \(x^n\).

**EXAMPLES:**

\begin{verbatim}
sage: K = Qp(3,10)
sage: R.<T> = K[]
sage: f = T + 2; f
(1 + O(3^10))*T + 2 + O(3^10)
sage: f.valuation_of_coefficient(1)
0
\end{verbatim}

2.1.16 \textit{p}-adic Flat Polynomials

**class** sage.rings.polynomial.padics.polynomial_padic_flat.Polynomial_padic_flat\(^{(\text{parent}, x=\text{None}, check=\text{True}, is_gen=\text{False}, construct=\text{False}, absprec=\text{None})}}\)

**Bases:** Polynomial_generic_dense, Polynomial_padic
2.1.17 Univariate Polynomials over \( GF(p^e) \) via NTL’s ZZ_pEX

AUTHOR:

• Yann Laigle-Chapuy (2010-01): initial implementation
• Lorenz Panny (2023-01): minpoly_mod()
• Giacomo Pope (2023-08): reverse(), inverse_series_trunc()

class sage.rings.polynomial.polynomial_zz_pex.Polynomial_ZZ_pEX
    Bases: Polynomial_template

Univariate Polynomials over \( F_{p^n} \) via NTL’s ZZ_pEX.

EXAMPLES:

```python
sage: K.<a> = GF(next_prime(2**60)**3)
sage: R.<x> = PolynomialRing(K, implementation='NTL')
sage: (x^3 + a*x^2 + 1) * (x + a)
x^4 + 2*a*x^3 + a^2*x^2 + x + a
```

```python
inverse_series_trunc(prec)

Compute and return the inverse of self modulo \( x^{prec} \).

The constant term of self must be invertible.

EXAMPLES:

```python
sage: R.<x> = GF(101^2)

sage: z2 = R.base_ring().gen()
sage: f = (3*z2 + 57)*x^3 + (13*z2 + 94)*x^2 + (7*z2 + 2)*x + 66*z2 + 15
sage: f.inverse_series_trunc(1)
51*z2 + 92
sage: f.inverse_series_trunc(2)
(30*z2 + 30)*x + 51*z2 + 92
sage: f.inverse_series_trunc(3)
(42*z2 + 94)*x^2 + (30*z2 + 30)*x + 51*z2 + 92
sage: f.inverse_series_trunc(4)
(99*z2 + 96)*x^3 + (42*z2 + 94)*x^2 + (30*z2 + 30)*x + 51*z2 + 92
```

```python
is_irreducible(algorithm='fast_when_false', iter=1)

Return True precisely when self is irreducible over its base ring.

INPUT:

• algorithm – a string (default "fast_when_false"), there are 3 available algorithms: "fast_when_true", "fast_when_false", and "probabilistic".

• iter – (default: 1) if the algorithm is "probabilistic", defines the number of iterations. The error probability is bounded by \( q^{-iter} \) for polynomials in \( F_q[x] \).

EXAMPLES:

```python
sage: K.<a> = GF(next_prime(2**60)**3)
sage: R.<x> = PolynomialRing(K, implementation='NTL')
sage: P = x^3 + (2-a)*x + 1
sage: P.is_irreducible(algorithm="fast_when_false")
True
sage: P.is_irreducible(algorithm="fast_when_true")
True
```

(continues on next page)
Polynomials, Release 10.3

(continued from previous page)

```python
sage: P.is_irreducible(algorithm="probabilistic")
True
sage: Q = (x^2+a)*(x+a^3)
sage: Q.is_irreducible(algorithm="fast_when_false")
False
sage: Q.is_irreducible(algorithm="fast_when_true")
False
sage: Q.is_irreducible(algorithm="probabilistic")
False
```

`list (copy=True)`

Return the list of coefficients.

**EXAMPLES:**

```python
sage: K.<a> = GF(5^3)
sage: P = PolynomialRing(K, 'x')
sage: f = P.random_element(100)
sage: f.list() == [f[i] for i in range(f.degree()+1)]
True
sage: P.0.list()
[0, 1]
```

`minpoly_mod (other)`

Compute the minimal polynomial of this polynomial modulo another polynomial in the same ring.

**ALGORITHM:**

NTL's MinPolyMod(), which uses Shoup's algorithm [Sho1999].

**EXAMPLES:**

```python
sage: R.<x> = GF(101^2)[]
sage: f = x^17 + x^2 - 1
sage: (x^2).minpoly_mod(f)
x^17 + 100*x^2 + 2*x + 100
```

`resultant (other)`

Return the resultant of `self` and `other`, which must lie in the same polynomial ring.

**INPUT:**

- `other` — a polynomial

**OUTPUT:** an element of the base ring of the polynomial ring

**EXAMPLES:**

```python
sage: K.<a> = GF(next_prime(2**60)**3)
sage: R.<x> = PolynomialRing(K, implementation='NTL')
sage: f = (x-a)*(x-a^2)*(x+1)
sage: g = (x-a^3)*(x-a^4)*(x+a)
sage: r = f.resultant(g)
sage: r == prod(u - v for (u,eu) in f.roots() for (v,ev) in g.roots())
True
```

`reverse (degree=None)`

Return the polynomial obtained by reversing the coefficients of this polynomial. If degree is set then this function behaves as if this polynomial has degree `degree`.

2.1. Univariate Polynomials and Polynomial Rings
Polynomials, Release 10.3

EXAMPLES:

```
sage: R.<x> = GF(101^2)[]
sage: f = x^13 + 11*x^10 + 32*x^6 + 4
sage: f.reverse()
4*x^13 + 32*x^7 + 11*x^3 + 1
sage: f.reverse(degree=15)
4*x^15 + 32*x^9 + 11*x^5 + x^2
sage: f.reverse(degree=2)
4*x^2
```

```
shift (n)
```

EXAMPLES:

```
sage: K.<a> = GF(next_prime(2**60)**3)
sage: R.<x> = PolynomialRing(K, implementation='NTL')
sage: f = x^3 + x^2 + 1
sage: f.shift(1)
x^4 + x^3 + x
sage: f.shift(-1)
x^2 + x
```

class `sage.rings.polynomial.polynomial_zz_pex.Polynomial_template`

Bases: `Polynomial`

Template for interfacing to external C / C++ libraries for implementations of polynomials.

AUTHORS:

- Robert Bradshaw (2008-10): original idea for templating
- Martin Albrecht (2008-10): initial implementation

This file implements a simple templating engine for linking univariate polynomials to their C/C++ library implementations. It requires a "linkage" file which implements the `celement_` functions (see `sage.libs.ntl.ntl_GF2X_linkage` for an example). Both parts are then plugged together by inclusion of the linkage file when inheriting from this class. See `sage.rings.polynomial.polynomial_gf2x` for an example.

We illustrate the generic gluing using univariate polynomials over GF(2).

**Note:** Implementations using this template MUST implement coercion from base ring elements and `get_unsafe()`. See `Polynomial_GF2X` for an example.

```
deregree ()
```

EXAMPLES:

```
sage: P.<x> = GF(2)[]
sage: x.degree()
1
sage: P(1).degree()
0
sage: P(0).degree()
-1
```

gcd (other)

Return the greatest common divisor of self and other.

EXAMPLES:
sage: P.<x> = GF(2)[]
sage: f = x*(x+1)
sage: f.gcd(x+1)
x + 1
sage: f.gcd(x^2)
x

get_cparent()
is_gen()
EXAMPLES:

sage: P.<x> = GF(2)[]
sage: x.is_gen()
True
sage: (x+1).is_gen()
False

is_one()
EXAMPLES:

sage: P.<x> = GF(2)[]
sage: P(1).is_one()
True

is_zero()
EXAMPLES:

sage: P.<x> = GF(2)[]
sage: x.is_zero()
False

list (copy=True)
EXAMPLES:

sage: P.<x> = GF(2)[]
sage: x.list()
[0, 1]
sage: list(x)
[0, 1]

quo_rem (right)
EXAMPLES:

sage: P.<x> = GF(2)[]
sage: f = x^2 + x + 1
sage: f.quo_rem(x + 1)
(x, 1)

shift (n)
EXAMPLES:

sage: P.<x> = GF(2)[]
sage: f = x^3 + x^2 + 1
sage: f.shift(1)
\texttt{trunc} (n)

Returns this polynomial mod $x^n$.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: R.<x> = GF(2)[]
sage: f = sum(x^n for n in range(10)); f
x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1
sage: f.truncate(6)
x^5 + x^4 + x^3 + x^2 + x + 1
\end{verbatim}

If the precision is higher than the degree of the polynomial then the polynomial itself is returned:

\begin{verbatim}
sage: f.truncate(10) is f
True
\end{verbatim}

If the precision is negative, the zero polynomial is returned:

\begin{verbatim}
sage: f.truncate(-1)
0
\end{verbatim}

\texttt{xgcd} (other)

Computes extended gcd of self and other.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: P.<x> = GF(7)[]
sage: f = x*(x+1)
sage: f.xgcd(x+1)
(x + 1, 0, 1)
sage: f.xgcd(x^2)
(x, 1, 6)
\end{verbatim}

\section{Isolate Real Roots of Real Polynomials}

\textbf{AUTHOR:}

- Carl Witty (2007-09-19): initial version

This is an implementation of real root isolation. That is, given a polynomial with exact real coefficients, we compute isolating intervals for the real roots of the polynomial. (Polynomials with integer, rational, or algebraic real coefficients are supported.)

We convert the polynomials into the Bernstein basis, and then use de Casteljau’s algorithm and Descartes’ rule of signs on the Bernstein basis polynomial (using interval arithmetic) to locate the roots. The algorithm is similar to that in “A Descartes Algorithm for Polynomials with Bit-Stream Coefficients”, by Eigenwillig, Kettner, Krandick, Mehlhorn, Schmitt, and Wolpert, but has three crucial optimizations over the algorithm in that paper:

- Precision reduction: at certain points in the computation, we discard the low-order bits of the coefficients, widening the intervals.
• Degree reduction: at certain points in the computation, we find lower-degree polynomials that are approximately equal to our high-degree polynomial over the region of interest.

• When the intervals are too wide to continue (either because of a too-low initial precision, or because of precision or degree reduction), and we need to restart with higher precision, we recall which regions have already been proven not to have any roots and do not examine them again.

The best description of the algorithms used (other than this source code itself) is in the slides for my Sage Days 4 talk, currently available from https://wiki.sagemath.org/days4schedule.

exception sage.rings.polynomial.real_roots.PrecisionError

Bases: ValueError

sage.rings.polynomial.real_roots.bernstein_down(d1, d2, s)

Given polynomial degrees d1 and d2 (where d1 < d2), and a number of samples s, computes a matrix bd.

If you have a Bernstein polynomial of formal degree d2, and select s of its coefficients (according to subsample_vec), and multiply the resulting vector by bd, then you get the coefficients of a Bernstein polynomial of formal degree d1, where this second polynomial is a good approximation to the first polynomial over the region of the Bernstein basis.

EXAMPLES:

```python
sage: from sage.rings.polynomial.real_roots import *
sage: bernstein_down(3, 8, 5)
[ 612/245 -348/245 -37/49 338/245 -172/245]
[-124/441 132/49 395/441 -290/147 452/441]
[ 452/441 -290/147 395/441 132/49 -724/441]
[-172/245 338/245 -37/49 -348/245 612/245]
```

sage.rings.polynomial.real_roots.bernstein_expand(c, d2)

Given an integer vector representing a Bernstein polynomial p, and a degree d2, compute the representation of p as a Bernstein polynomial of formal degree d2.

This is similar to multiplying by the result of bernstein_up, but should be faster for large d2 (this has about the same number of multiplies, but in this version all the multiplies are by single machine words).

This returns a pair consisting of the expanded polynomial, and the maximum error E. (So if an element of the returned polynomial is a, and the true value of that coefficient is b, then a <= b < a + E.)

EXAMPLES:

```python
sage: from sage.rings.polynomial.real_roots import *
sage: c = vector(ZZ, [1000, 2000, -3000])
sage: bernstein_expand(c, 3)
((1000, 1666, 333, -3000), 1)
sage: bernstein_expand(c, 4)
((1000, 1500, 1000, -500, -3000), 1)
sage: bernstein_expand(c, 20)
((1000, 1100, 1168, 1205, 1210, 1184, 1126, 1036, 915, 763, 578, 363, 115, -164, -474, -816, -1190, -1595, -2032, -2500, -3000), 1)
```

class sage.rings.polynomial.real_roots.bernstein_polynomial_factory

Bases: object

An abstract base class for bernstein_polynomial factories. That is, elements of subclasses represent Bernstein polynomials (exactly), and are responsible for creating interval_bernstein_polynomial_integer approximations at arbitrary precision.
Supports four methods, coeffs_bitsize(), bernstein_polynomial(), lsign(), and usign(). The coeffs_bitsize() method gives an integer approximation to the log2 of the max of the absolute values of the Bernstein coefficients. The bernstein_polynomial(scale_log2) method gives an approximation where the maximum coefficient has approximately coeffs_bitsize() - scale_log2 bits. The lsign() and usign() methods give the (exact) sign of the first and last coefficient, respectively.

lsign()
Return the sign of the first coefficient of this Bernstein polynomial.

usign()
Return the sign of the last coefficient of this Bernstein polynomial.

class sage.rings.polynomial.real_roots.bernstein_polynomial_factory_ar(poly, neg)
Bases: bernstein_polynomial_factory
This class holds an exact Bernstein polynomial (represented as a list of algebraic real coefficients), and returns arbitrarily-precise interval approximations of this polynomial on demand.

bernstein_polynomial(scale_log2)
Compute an interval_bernstein_polynomial_integer that approximates this polynomial, using the given scale_log2. (Smaller scale_log2 values give more accurate approximations.)

EXAMPLES:

```python
sage: from sage.rings.polynomial.real_roots import *

sage: x = polygen(AA)
sage: p = (x - 1) * (x - sqrt(AA(2))) * (x - 2)
sage: bpf = bernstein_polynomial_factory_ar(p, False)
sage: print(bpf.bernstein_polynomial(0))
degree 3 IBP with 2-bit coefficients

sage: bpf.bernstein_polynomial(-20)
<IBP: ((-2965821, 2181961, -1542880, 1048576) + [0 .. 1)) * 2^-20>

sage: bpf = bernstein_polynomial_factory_ar(p, True)

sage: bpf.bernstein_polynomial(-20)
<IBP: ((-2965821, -2181962, -1542880, -1048576) + [0 .. 1)) * 2^-20>

sage: print(bpf.bernstein_polynomial(-10))
<IBP: ((-1024, 0, 1024) + [0 .. 1)) * 2^-10>
```

coeffs_bitsize()
Computes the approximate log2 of the maximum of the absolute values of the coefficients.

EXAMPLES:

```python
sage: from sage.rings.polynomial.real_roots import *

sage: x = polygen(AA)
sage: p = (x - 1) * (x - sqrt(AA(2))) * (x - 2)
sage: bernstein_polynomial_factory_ar(p, False).coeffs_bitsize()
1
```

class sage.rings.polynomial.real_roots.bernstein_polynomial_factory_intlist(coeffs)
Bases: bernstein_polynomial_factory
This class holds an exact Bernstein polynomial (represented as a list of integer coefficients), and returns arbitrarily-precise interval approximations of this polynomial on demand.
bernstein_polynomial (scale_log2)
Compute an interval_bernstein_polynomial_integer that approximates this polynomial, using the given
scale_log2. (Smaller scale_log2 values give more accurate approximations.)

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: bpf = bernstein_polynomial_factory_intlist([10, -20, 30, -40])
sage: print(bpf.bernstein_polynomial(0))
degree 3 IBP with 6-bit coefficients
sage: bpf.bernstein_polynomial(20)
<IBP: ((0, -1, 0, -1) + [0 .. 1]) * 2^20; lsign 1>
sage: bpf.bernstein_polynomial(0)
<IBP: (10, -20, 30, -40) + [0 .. 1)>
sage: bpf.bernstein_polynomial(-20)
<IBP: ((10485760, -20971520, 31457280, -41943040) + [0 .. 1)) * 2^-20>
```

coeffs_bitsize()
Computes the approximate log2 of the maximum of the absolute values of the coefficients.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: bernstein_polynomial_factory_intlist([1, 2, 3, -60000]).coeffs_bitsize()
16
```

class sage.rings.polynomial.real_roots.bernstein_polynomial_factory_ratlist(coeffs)
Bases: bernstein_polynomial_factory

This class holds an exact Bernstein polynomial (represented as a list of rational coefficients), and returns
arbitrarily-precise interval approximations of this polynomial on demand.

bernstein_polynomial (scale_log2)
Compute an interval_bernstein_polynomial_integer that approximates this polynomial, using the given
scale_log2. (Smaller scale_log2 values give more accurate approximations.)

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: bpf = bernstein_polynomial_factory_ratlist([1/3, -22/7, 193/71, -140/99])
sage: print(bpf.bernstein_polynomial(0))
degree 3 IBP with 3-bit coefficients
sage: bpf.bernstein_polynomial(20)
<IBP: ((0, -1, 0, -1) + [0 .. 1)) * 2^20; lsign 1>
sage: bpf.bernstein_polynomial(0)
<IBP: (0, -4, 2, -2) + [0 .. 1)]; lsign 1>
sage: bpf.bernstein_polynomial(-20)
<IBP: ((349525, -3295525, 2850354, -1482835) + [0 .. 1)) * 2^-20>
```

coeffs_bitsize()
Computes the approximate log2 of the maximum of the absolute values of the coefficients.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: bernstein_polynomial_factory_ratlist([1, 2, 3, -60000]).coeffs_bitsize()
```

(continues on next page)
Given polynomial degrees $d_1$ and $d_2$, where $d_1 < d_2$, compute a matrix $b_u$.

If you have a Bernstein polynomial of formal degree $d_1$, and multiply its coefficient vector by $b_u$, then the result is the coefficient vector of the same polynomial represented as a Bernstein polynomial of formal degree $d_2$.

If $s$ is not None, then it represents a number of samples; then the product only gives $s$ of the coefficients of the new Bernstein polynomial, selected according to subsample_vec.

EXAMPLES:

```python
sage: from sage.rings.polynomial.real_roots import *
sage: bernstein_down(3, 7, 4)
[ 12/5 -4 3 -2/5]
[ -13/15 16/3 -4 8/15]
[ 8/15 -4 16/3 -13/15]
[ -2/5 3 -4 12/5]
```

Given a polynomial represented by a list of its coefficients (as RealIntervalFieldElements), compute an upper bound on its largest real root.

Uses two algorithms of Akritas, Strzeboński, and Vigklas, and picks the better result.

EXAMPLES:

```python
sage: from sage.rings.polynomial.real_roots import *
sage: cl_maximum_root([RIF(-1), RIF(0), RIF(1)])
1.00000000000000
```

Given a polynomial represented by a list of its coefficients (as RealIntervalFieldElements), compute an upper bound on its largest real root.


EXAMPLES:

```python
sage: from sage.rings.polynomial.real_roots import *
sage: cl_maximum_root_first_lambda([RIF(-1), RIF(0), RIF(1)])
1.00000000000000
```

Given a polynomial represented by a list of its coefficients (as RealIntervalFieldElements), compute an upper bound on its largest real root.


EXAMPLES:
```python
class sage.rings.polynomial.real_roots.context:
    Bases: object
    
    A simple context class, which is passed through parts of the real root isolation algorithm to avoid global variables. Holds logging information, a random number generator, and the target machine wordsize.
    
    get_be_log()
    get_dc_log()
```

```python
def de_casteljau_doublevec(c, x):
    
    Given a polynomial in Bernstein form with floating-point coefficients over the region \([0 .. 1]\), and a split point \(x\), use de Casteljau's algorithm to give polynomials in Bernstein form over \([0 .. x]\) and \([x .. 1]\).
    
    This function will work for an arbitrary rational split point \(x\), as long as \(0 < x < 1\); but it has a specialized code path for \(x = 1/2\).
    
    **INPUT:**
    
    - \(c\) – vector of coefficients of polynomial in Bernstein form
    - \(x\) – rational splitting point; \(0 < x < 1\)
    
    **OUTPUT:**
    
    - \(c1\) – coefficients of polynomial over range \([0 .. x]\)
    - \(c2\) – coefficients of polynomial over range \([x .. 1]\)
    - \(err_inc\) – number of half-ulps by which error intervals widened
    
    **EXAMPLES:**
    ```
sage: from sage.rings.polynomial.real_roots import *

sage: c = vector(RDF, [0.7, 0, 0, 0, 0, 0])

sage: de_casteljau_doublevec(c, 1/2)
((0.7, 0.35, 0.175, 0.0875, 0.04375, 0.021875), (0.021875, 0.0, 0.0, 0.0, 0.0, 0.0), 5)

sage: de_casteljau_doublevec(c, 1/3)  # rel tol
((0.7, 0.4666666666666667, 0.3111111111111117, 0.20740740740740746, 0.13827160493827165, 0.09218106995884777), (0.09218106995884777, 0.0, 0.0, 0.0, 0.0, 0.0), 15)

sage: de_casteljau_doublevec(c, 7/22)  # rel tol
((0.7, 0.4772727272727273, 0.3254132323404959, 0.2218726521424724, 0.15127680827812312, 0.1031432783714759), (0.1031432783714759, 0.0, 0.0, 0.0, 0.0, 0.0), 0, 0), 15)
```
Given split points \( x = a/(a+b) \) and \( y = c/(c+d) \), where min(a, b) and min(c, d) fit in the same number of
machine words and a+b and c+d are both powers of two, then x and y should be equally fast split points.

If use_ints is nonzero, then instead of checking whether numerators and denominators fit in machine words, we
check whether they fit in ints (32 bits, even on 64-bit machines). This slows things down, but allows for identical
results across machines.

**INPUT:**
- \( c \) – vector of coefficients of polynomial in Bernstein form
- \( c\_bitsize \) – approximate size of coefficients in \( c \) (in bits)
- \( x \) – rational splitting point; \( 0 < x < 1 \)

**OUTPUT:**
- \( c1 \) – coefficients of polynomial over range \([0.. \ x]\)
- \( c2 \) – coefficients of polynomial over range \([\ x.. \ 1]\)
- \( \text{err\_inc} \) – amount by which error intervals widened

**EXAMPLES:**

```sage
def de_casteljau_intvec(c, n, x, mode):
    # Implementation details...
    return (c1, c2, err_inc)
```

**sage.rings.polynomial.real_roots.\texttt{degree_reduction_next_size}(n)**

Given \( n \) (a polynomial degree), returns either a smaller integer or None. This defines the sequence of degrees
followed by our degree reduction implementation.

**EXAMPLES:**

```sage
def degree_reduction_next_size(n):
    # Implementation details...
    return 30
```

**sage.rings.polynomial.real_roots.\texttt{dprod_imatrow_vec}(m, v, k)**

Computes the dot product of row \( k \) of the matrix \( m \) with the vector \( v \) (that is, compute one element of the product
\( m*v \)).

If \( v \) has more elements than \( m \) has columns, then elements of \( v \) are selected using \texttt{subsample_vec}.

**EXAMPLES:**

```sage
def dprod_imatrow_vec(m, v, k):
    # Implementation details...
    return v[k]
```
sage.rings.polynomial.real_roots.get_realfield_rndu(n)

A simple cache for RealField fields (with rounding set to round-to-positive-infinity).

EXAMPLES:

```python
sage: from sage.rings.polynomial.real_roots import *

sage: get_realfield_rndu(20)
Real Field with 20 bits of precision and rounding RNDU

sage: get_realfield_rndu(53)
Real Field with 53 bits of precision and rounding RNDU

sage: get_realfield_rndu(20)
Real Field with 20 bits of precision and rounding RNDU
```

2.1. Univariate Polynomials and Polynomial Rings

An interval Bernstein polynomial is an approximation to an exact polynomial. This approximation is in the form of a Bernstein polynomial (a polynomial given as coefficients over a Bernstein basis) with interval coefficients.

The Bernstein basis of degree \( n \) over the region \([a \ldots b]\) is the set of polynomials

\[
\binom{n}{k} (x-a)^k (b-x)^{n-k} / (b-a)^n
\]

for \( 0 \leq k \leq n \).

A degree-\( n \) interval Bernstein polynomial \( P \) with its region \([a \ldots b]\) can represent an exact polynomial \( p \) in two different ways: it can “contain” the polynomial or it can “bound” the polynomial.

We say that \( P \) contains \( p \) if, when \( p \) is represented as a degree-\( n \) Bernstein polynomial over \([a \ldots b]\), its coefficients are contained in the corresponding interval coefficients of \( P \). For instance, \([0.9 \ldots 1.1]\)*x^2 (which is a degree-2 interval Bernstein polynomial over \([0 .. 1]\)) contains \( x^2 \).

We say that \( P \) bounds \( p \) if, for all \( a \leq x \leq b \), there exists a polynomial \( p' \) contained in \( P \) such that \( p(x) = p'(x) \). For instance, \([0 .. 1]\)*x is a degree-1 interval Bernstein polynomial which bounds \( x^2 \) over \([0 .. 1]\).

If \( P \) contains \( p \), then \( P \) bounds \( p \); but the converse is not necessarily true. In particular, if \( n < m \), it is possible for a degree-\( n \) interval Bernstein polynomial to bound a degree-\( m \) polynomial; but it cannot contain the polynomial.

In the case where \( P \) bounds \( p \), we maintain extra information, the “slope error”. We say that \( P \) (over \([a \ldots b]\)) bounds \( p \) with a slope error of \( E \) (where \( E \) is an interval) if there is a polynomial \( p' \) contained in \( P \) such that the derivative of \( (p - p') \) is bounded by \( E \) in the range \([a \ldots b]\). If \( P \) bounds \( p \) with a slope error of 0 then \( P \) contains \( p \).

(Note that “contains” and “bounds” are not standard terminology; I just made them up.)
Interval Bernstein polynomials are useful in finding real roots because of the following properties:

- Given an exact real polynomial $p$, we can compute an interval Bernstein polynomial over an arbitrary region containing $p$.
- Given an interval Bernstein polynomial $P$ over $[a .. c]$, where $a < b < c$, we can compute interval Bernstein polynomials $P_1$ over $[a .. b]$ and $P_2$ over $[b .. c]$, where $P_1$ and $P_2$ contain (or bound) all polynomials that $P$ contains (or bounds).
- Given a degree-$n$ interval Bernstein polynomial $P$ over $[a .. b]$, and $m < n$, we can compute a degree-$m$ interval Bernstein polynomial $P'$ over $[a .. b]$ that bounds all polynomials that $P$ bounds.
- It is sometimes possible to prove that no polynomial bounded by $P$ over $[a .. b]$ has any roots in $[a .. b]$. (Roughly, this is possible when no polynomial contained by $P$ has any complex roots near the line segment $[a .. b]$, where “near” is defined relative to the length $b-a$.)
- It is sometimes possible to prove that every polynomial bounded by $P$ over $[a .. b]$ with slope error $E$ has exactly one root in $[a .. b]$. (Roughly, this is possible when every polynomial contained by $P$ over $[a .. b]$ has exactly one root in $[a .. b]$, there are no other complex roots near the line segment $[a .. b]$, and every polynomial contained in $P$ has a derivative which is bounded away from zero over $[a .. b]$ by an amount which is large relative to $E$.)
- Starting from a sufficiently precise interval Bernstein polynomial, it is always possible to split it into polynomials which provably have 0 or 1 roots (as long as your original polynomial has no multiple real roots).

So a rough outline of a family of algorithms would be:

- Given a polynomial $p$, compute a region $[a .. b]$ in which any real roots must lie.
- Compute an interval Bernstein polynomial $P$ containing $p$ over $[a .. b]$.
- Keep splitting $P$ until you have isolated all the roots. Optionally, reduce the degree or the precision of the interval Bernstein polynomials at intermediate stages (to reduce computation time). If this seems not to be working, go back and try again with higher precision.

Obviously, there are many details to be worked out to turn this into a full algorithm, like:

- What initial precision is selected for computing $P$?
- How do you decide when to reduce the degree of intermediate polynomials?
- How do you decide when to reduce the precision of intermediate polynomials?
- How do you decide where to split the interval Bernstein polynomial regions?
- How do you decide when to give up and start over with higher precision?

Each set of answers to these questions gives a different algorithm (potentially with very different performance characteristics), but all of them can use this `interval_bernstein_polynomial` class as their basic building block.

To save computation time, all coefficients in an `interval_bernstein_polynomial` share the same interval width. (There is one exception: when creating an `interval_bernstein_polynomial`, the first and last coefficients can be marked as “known positive” or “known negative”. This has some of the same effect as having a (potentially) smaller interval width for these two coefficients, although it does not affect de Casteljau splitting.) To allow for widely varying coefficient magnitudes, all coefficients in an `interval_bernstein_polynomial` are scaled by $2^n$ (where $n$ may be positive, negative, or zero).

There are two representations for `interval_bernstein_polynomials`, integer and floating-point. These are the two subclasses of this class; `interval_bernstein_polynomial` itself is an abstract class.

`interval_bernstein_polynomial` and its subclasses are not expected to be used outside this file.
region()

region_width()

**try_rand_split (ctx, logging_note)**

Compute a random split point r (using the random number generator embedded in ctx). We require $1/4 \leq r < 3/4$ (to ensure that recursive algorithms make progress).

Then, try doing a de Casteljau split of this polynomial at r, resulting in polynomials p1 and p2. If we see that the sign of this polynomial is determined at r, then return (p1, p2, r); otherwise, return None.

**EXAMPLES:**

```sage
sage: from sage.rings.polynomial.real_roots import *
sage: bp = mk_ibpi([50, 20, -90, -70, 200], error=5)
sage: bp1, bp2, _ = bp.try_rand_split(mk_context(), None)
sage: bp1
<IBP: (50, 29, -27, -56, -11) + [0 .. 6) over [0 .. 43/64]>
sage: bp2
<IBP: (-11, 10, 49, 111, 200) + [0 .. 6) over [43/64 .. 1]>
sage: bp1, bp2, _ = bp.try_rand_split(mk_context(seed=42), None)
sage: bp1
<IBP: (50, 32, -11, -41, -29) + [0 .. 6) over [0 .. 583/1024]>
sage: bp2
<IBP: (-29, -20, 13, 83, 200) + [0 .. 6) over [583/1024 .. 1]>
sage: bp = mk_ibpf([0.5, 0.2, -0.9, -0.7, 0.99], neg_err=-0.1, pos_err=0.01)
sage: bp1, bp2, _ = bp.try_rand_split(mk_context(), None)
sage: bp1
# rel tol
<IBP: (0.5, 0.35, 0.0, -0.2875, -0.369375) + [-0.10000000000000023 .. 0.010000000000000226] over [0 .. 1/2]>
sage: bp2
# rel tol
<IBP: (-0.369375, -0.45125, -0.3275, 0.14500000000000002, 0.99) + [-0.10000000000000023 .. 0.010000000000000226] over [1/2 .. 1]>
```

**try_split (ctx, logging_note)**

Try doing a de Casteljau split of this polynomial at 1/2, resulting in polynomials p1 and p2. If we see that the sign of this polynomial is determined at 1/2, then return (p1, p2, 1/2); otherwise, return None.

**EXAMPLES:**

```sage
sage: from sage.rings.polynomial.real_roots import *
sage: bp = mk_ibpi([50, 20, -90, -70, 200], error=5)
sage: bp1, bp2, _ = bp.try_split(mk_context(), None)
sage: bp1
<IBP: (50, 35, 0, -29, -31) + [0 .. 6) over [0 .. 1/2]>
sage: bp2
<IBP: (-31, -33, -8, 65, 200) + [0 .. 6) over [1/2 .. 1]>
sage: bp = mk_ibpf([0.5, 0.2, -0.9, -0.7, 0.99], neg_err=-0.1, pos_err=0.01)
sage: bp1, bp2, _ = bp.try_split(mk_context(), None)
sage: bp1
<IBP: (0.5, 0.35, 0.0, -0.2875, -0.369375) + [-0.10000000000000023 .. 0.010000000000000226] over [0 .. 1/2]>
sage: bp2
<IBP: (-0.369375, -0.45125, -0.3275, 0.14500000000000002, 0.99) + [-0.10000000000000023 .. 0.010000000000000226] over [1/2 .. 1]>
```
Consider a polynomial (written in either the normal power basis or the Bernstein basis). Take its list of coefficients, omitting zeroes. Count the number of positions in the list where the sign of one coefficient is opposite the sign of the next coefficient.

This count is the number of sign variations of the polynomial. According to Descartes' rule of signs, the number of real roots of the polynomial (counted with multiplicity) in a certain interval is always less than or equal to the number of sign variations, and the difference is always even. (If the polynomial is written in the power basis, the region is the positive reals; if the polynomial is written in the Bernstein basis over a particular region, then we count roots in that region.)

In particular, a polynomial with no sign variations has no real roots in the region, and a polynomial with one sign variation has one real root in the region.

In an interval Bernstein polynomial, we do not necessarily know the signs of the coefficients (if some of the coefficient intervals contain zero), so the polynomials contained by this interval polynomial may not all have the same number of sign variations. However, we can compute a range of possible numbers of sign variations.

This function returns the range, as a 2-tuple of integers.

class sage.rings.polynomial.real_roots.interval_bernstein_polynomial_float
Bases: interval_bernstein_polynomial

This is the subclass of interval_bernstein_polynomial where polynomial coefficients are represented using floating-point numbers.

In the floating-point representation, each coefficient is represented as an IEEE double-precision float A, and the (shared) lower and upper interval widths E1 and E2. These represent the coefficients \((A+E1)*2^n <= c <= (A+E2)*2^n\).

Note that we always have E1 <= 0 <= E2. Also, each floating-point coefficient has absolute value less than one.

(Not that mk_ibpf() is a simple helper function for creating elements of interval_bernstein_polynomial_float in doctests.)

EXAMPLES:

```python
sage: from sage.rings.polynomial.real_roots import *
sage: bp = mk_ibpf([0.1, 0.2, 0.3], pos_err=0.5); print(bp)
degree 2 IBP with floating-point coefficients
sage: bp
<IBP: (0.1, 0.2, 0.3) + [0.0 .. 0.5]>
sage: bp.variations()
(0, 0)
sage: bp = mk_ibpf([-0.3, -0.1, 0.1, -0.1, -0.3, -0.1], # needs sage.symbolic
                  lower=1, upper=5/4, usign=1, pos_err=0.2,
                  scale_log2=-3, level=2, slope_err=RIF(pi)); print(bp)
degree 5 IBP with floating-point coefficients
sage: bp
<IBP: ((-0.3, -0.1, 0.1, -0.1, -0.3, -0.1) + [0.0 .. 0.2]) * 2^-3
     over [1 .. 5/4]; usign 1; level 2; slope_err 3.141592653589794?>
sage: bp.variations()
(3, 3)
sage: as_float()
```
de_casteljau$(ctx, mid, msign=0)$

Uses de Casteljau’s algorithm to compute the representation of this polynomial in a Bernstein basis over new regions.

**INPUT:**
- mid – where to split the Bernstein basis region; $0 < mid < 1$
- msign – default 0 (unknown); the sign of this polynomial at mid

**OUTPUT:**
- bp1, bp2 – the new interval Bernstein polynomials
- ok – a boolean; True if the sign of the original polynomial at mid is known

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.real_roots import *
sage: ctx = mk_context()
sage: bp = mk_ibpf([0.5, 0.2, -0.9, -0.7, 0.99], neg_err=-0.1, pos_err=0.01)
sage: bp.de_casteljau(ctx, 1/2)
sage: bp1
<IBP: (0.5, 0.35, 0.0, -0.2875, -0.369375) + [-0.10000000000000023 .. 0.010000000000000226] over [0 .. 1/2]>
sage: bp2
<IBP: (-0.369375, -0.45125, -0.3275, 0.14500000000000002, 0.99) + [-0.10000000000000023 .. 0.010000000000000226] over [1/2 .. 1]>
sage: bp.de_casteljau(ctx, 2/3)
sage: bp1
# rel tol 2e-16
<IBP: (0.5, 0.30000000000000004, -0.2555555555555555, -0.5444444444444444, -0.1765692706232836) + [-0.10000000000000069 .. 0.010000000000000677] over [0 .. 2/3]>
sage: bp2
# rel tol 2e-15
<IBP: (-0.32172839506172846, -0.21037037037037046, 0.028888888888888797, 0.2656803047927313, -0.1765692706232836) + [-0.10000000000000069 .. 0.010000000000000677] over [2/3 .. 1]>
sage: bp.de_casteljau(ctx, 7/39)
sage: bp1
# rel tol
<IBP: (0.5, 0.4461538461538461, 0.36653517422748183, 0.27328680523946786, -0.19999999999999998) + [-0.10000000000000069 .. 0.010000000000000677] over [0 .. 7/39]>
sage: bp2
# rel tol
<IBP: (0.1765692706232836, -0.2656803047927313, -0.7802038132870364, -0.3966666666666666, 0.99) + [-0.10000000000000069 .. 0.010000000000000677] over [7/39 .. 1]>
```

get_msb_bit()

Return an approximation of the log2 of the maximum of the absolute values of the coefficients, as an integer.

slope_range()

Compute a bound on the derivative of this polynomial, over its region.

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.real_roots import *
sage: bp = mk_ibpf([0.5, 0.2, -0.9, -0.7, 0.99], neg_err=-0.1, pos_err=0.01)
sage: bp.slope_range().str(style='brackets')
'[-4.8400000000000017 .. 7.2000000000000011]'```
class sage.rings.polynomial.real_roots.interval_bernstein_polynomial_integer

Bases: interval_bernstein_polynomial

This is the subclass of interval_bernstein_polynomial where polynomial coefficients are represented using integers.

In this integer representation, each coefficient is represented by a GMP arbitrary-precision integer A, and a (shared) interval width E (which is a machine integer). These represent the coefficients A*2^n <= c < (A+E)*2^n.

(Note that mk_ibpi is a simple helper() function for creating elements of interval_bernstein_polynomial_integer in doctests.)

EXAMPLES:

sage: from sage.rings.polynomial.real_roots import *

sage: bp = mk_ibpi([1, 2, 3], error=5); print(bp)
degree 2 IBP with 2-bit coefficients
sage: bp
<IBP: (1, 2, 3) + [0 .. 5)>
sage: bp.variations()
(0, 0)

sage: bp = mk_ibpi([-3, -1, 1, -1, -3, -1], lower=1, upper=5/4, usign=1, #
˓→needs sage.symbolic
....: error=2, scale_log2=-3, level=2, slope_err=RIF(pi)); print(bp)
degree 5 IBP with 2-bit coefficients
sage: bp  # ...
˓→needs sage.symbolic
<IBP: ((-3, -1, 1, -1, -3, -1) + [0 .. 2)) * 2^-3 over [1 .. 5/4]; usign 1;
˓→level 2; slope_err 3.141592653589794?>
sage: bp.variations()  # ...
˓→needs sage.symbolic
(3, 3)

as_float()

Compute an interval_bernstein_polynomial_float which contains (or bounds) all the polynomials this interval polynomial contains (or bounds).

EXAMPLES:

sage: from sage.rings.polynomial.real_roots import *

sage: bp = mk_ibpi([50, 20, -90, -70, 200], error=5)

sage: print(bp.as_float())
degree 4 IBP with floating-point coefficients
sage: bp.as_float()
<IBP: ((0.1953125, 0.078125, -0.3515625, -0.2734375, 0.78125) + [-1.
˓→1275702593849246e-16 .. 0.0195312500000017]) * 2^8>

de_casteljau(ctx, mid, msign=0)

Uses de Casteljau’s algorithm to compute the representation of this polynomial in a Bernstein basis over new regions.

INPUT:

• mid – where to split the Bernstein basis region; 0 < mid < 1
• msign – default 0 (unknown); the sign of this polynomial at mid

OUTPUT:

• bp1, bp2 – the new interval Bernstein polynomials
• ok – a boolean; True if the sign of the original polynomial at mid is known

EXAMPLES:

```python
sage: from sage.rings.polynomial.real_roots import *
sage: bp = mk_ibpi([50, 20, -90, -70, 200], error=5)
sage: ctx = mk_context()
sage: bp1, bp2, ok = bp.de_casteljau(ctx, 1/2)
sage: bp1
<IBP: (50, 35, 0, -29, -31) + [0 .. 6) over [0 .. 1/2]>
sage: bp2
<IBP: (-31, -33, -8, 65, 200) + [0 .. 6) over [1/2 .. 1]>
sage: bpl, bp2, ok = bp.de_casteljau(ctx, 2/3)
sage: bp1
<IBP: (50, 30, -26, -55, -13) + [0 .. 6) over [0 .. 2/3]>
sage: bp2
<IBP: (-13, 8, 47, 110, 200) + [0 .. 6) over [2/3 .. 1]>
sage: bp1, bp2, ok = bp.de_casteljau(ctx, 7/39)
sage: bp1
<IBP: (50, 44, 36, 27, 17) + [0 .. 6) over [0 .. 7/39]>
sage: bp2
<IBP: (17, -26, -75, -22, 200) + [0 .. 6) over [7/39 .. 1]>
```

down_degree (ctx, max_err, exp_err_shift)

Compute an interval_bernstein_polynomial_integer which bounds all the polynomials this interval polynomial bounds, but is of lesser degree.

During the computation, we find an “expected error” expected_err, which is the error inherent in our approach (this depends on the degrees involved, and is proportional to the error of the current polynomial).

We require that the error of the new interval polynomial be bounded both by max_err, and by expected_err << exp_err_shift. If we find such a polynomial p, then we return a pair of p and some debugging/logging information. Otherwise, we return the pair (None, None).

If the resulting polynomial would have error more than 2^17, then it is downscaled before returning.

EXAMPLES:

```python
sage: from sage.rings.polynomial.real_roots import *
sage: bp = mk_ibpi([0, 100, 400, 903], error=2)
sage: ctx = mk_context()
sage: bp
<IBP: (0, 100, 400, 903) + [0 .. 2)>
sage: dbp, _ = bp.down_degree(ctx, 10, 32)
sage: dbp
<IBP: (-1, 148, 901) + [0 .. 4); level 1; slope_err 0.?e2>
```

down_degree_iter (ctx, max_scale)

Compute a degree-reduced version of this interval polynomial, by iterating down_degree.

We stop when degree reduction would give a polynomial which is too inaccurate, meaning that either we think the current polynomial may have more roots in its region than the degree of the reduced polynomial, or that the least significant accurate bit in the result (on the absolute scale) would be larger than 1 << max_scale.

EXAMPLES:

```python
sage: from sage.rings.polynomial.real_roots import *
sage: bp = mk_ibpi([0, 100, 400, 903, 1600, 2500], error=2)
sage: ctx = mk_context()
```

(continues on next page)
downscale (bits)

Compute an interval_bernstein_polynomial_integer which contains (or bounds) all the polynomials this interval polynomial contains (or bounds), but uses “bits” fewer bits.

EXAMPLES:

\begin{verbatim}
sage: from sage.rings.polynomial.real_roots import *
sage: bp = mk_ibpi([0, 100, 400, 903], error=2)
sage: bp.downscale(5)
<IBP: ((0, 3, 12, 28) + [0 .. 1)) * 2^5>
\end{verbatim}

get_msb_bit()

Return an approximation of the log2 of the maximum of the absolute values of the coefficients, as an integer.

slope_range()

Compute a bound on the derivative of this polynomial, over its region.

EXAMPLES:

\begin{verbatim}
sage: from sage.rings.polynomial.real_roots import *
sage: bp = mk_ibpi([0, 100, 400, 903], error=2)
sage: bp.slope_range().str(style='brackets')
'[294.00000000000000 .. 1515.0000000000000]
\end{verbatim}

sage.rings.polynomial.real_roots.intvec_to_doublevec(b, err)

Given a vector of integers \( A = [a_1, \ldots, a_n] \), and an integer error bound \( E \), returns a vector of floating-point numbers \( B = [b_1, \ldots, b_n] \), lower and upper error bounds \( F_1 \) and \( F_2 \), and a scaling factor \( d \), such that

\[(b_k + F_1) \times 2^d \leq a_k\]

and

\[a_k + E \leq (b_k + F_2) \times 2^d\]

If \( b_j \) is the element of \( B \) with largest absolute value, then \( 0.5 \leq \text{abs}(b_j) < 1.0 \).

EXAMPLES:

\begin{verbatim}
sage: from sage.rings.polynomial.real_roots import *
sage: intvec_to_doublevec(vector(ZZ, [1, 2, 3, 4, 5]), 3)
((0.125, 0.25, 0.375, 0.5, 0.625), -1.1275702593849246e-16, 0.37500000000000017, →3)
\end{verbatim}

class sage.rings.polynomial.real_roots.island

Bases: object

This implements the island portion of my ocean-island root isolation algorithm. See the documentation for class ocean, for more information on the overall algorithm.

Island root refinement starts with a Bernstein polynomial whose region is the whole island (or perhaps slightly more than the island in certain cases). There are two subalgorithms; one when looking at a Bernstein polynomial covering
a whole island (so we know that there are gaps on the left and right), and one when looking at a Bernstein polynomial covering the left segment of an island (so we know that there is a gap on the left, but the right is in the middle of an island). An important invariant of the left-segment subalgorithm over the region \([l.. r]\) is that it always finds a gap \([r0 .. r]\) ending at its right endpoint.

Ignoring degree reduction, downscaling (precision reduction), and failures to split, the algorithm is roughly:

**Whole island:**
1. If the island definitely has exactly one root, then return.
2. Split the island in (approximately) half.
3. If both halves definitely have no roots, then remove this island from its doubly-linked list (merging its left and right gaps) and return.
4. If either half definitely has no roots, then discard that half and call the whole-island algorithm with the other half, then return.
5. If both halves may have roots, then call the left-segment algorithm on the left half.
6. We now know that there is a gap immediately to the left of the right half, so call the whole-island algorithm on the right half, then return.

**Left segment:**
1. Split the left segment in (approximately) half.
2. If both halves definitely have no roots, then extend the left gap over the segment and return.
3. If the left half definitely has no roots, then extend the left gap over this half and call the left-segment algorithm on the right half, then return.
4. If the right half definitely has no roots, then split the island in two, creating a new gap. Call the whole-island algorithm on the left half, then return.
5. Both halves may have roots. Call the left-segment algorithm on the left half.
6. We now know that there is a gap immediately to the left of the right half, so call the left-segment algorithm on the right half, then return.

Degree reduction complicates this picture only slightly. Basically, we use heuristics to decide when degree reduction might be likely to succeed and be helpful; whenever this is the case, we attempt degree reduction.

Precision reduction and split failure add more complications. The algorithm maintains a stack of different-precision representations of the interval Bernstein polynomial. The base of the stack is at the highest (currently known) precision; each stack entry has approximately half the precision of the entry below it. When we do a split, we pop off the top of the stack, split it, then push whichever half we’re interested in back on the stack (so the different Bernstein polynomials may be over different regions). When we push a polynomial onto the stack, we may heuristically decide to push further lower-precision versions of the same polynomial onto the stack.

In the algorithm above, whenever we say “split in (approximately) half”, we attempt to split the top-of-stack polynomial using try_split() and try_rand_split(). However, these will fail if the sign of the polynomial at the chosen split point is unknown (if the polynomial is not known to high enough precision, or if the chosen split point actually happens to be a root of the polynomial). If this fails, then we discard the top-of-stack polynomial, and try again with the next polynomial down (which has approximately twice the precision). This next polynomial may not be over the same region; if not, we split it using de Casteljau’s algorithm to get a polynomial over (approximately) the same region first.

If we run out of higher-precision polynomials (if we empty out the entire stack), then we give up on root refinement for this island. The ocean class will notice this, provide the island with a higher-precision polynomial, and restart root refinement. Basically the only information kept in that case is the lower and upper bounds on the island. Since these are updated whenever we discover a “half” (of an island or a segment) that definitely contains no roots, we
never need to re-examine these gaps. (We could keep more information. For example, we could keep a record of split points that succeeded and failed. However, a split point that failed at lower precision is likely to succeed at higher precision, so it’s not worth avoiding. It could be useful to select split points that are known to succeed, but starting from a new Bernstein polynomial over a slightly different region, hitting such split points would require de Casteljau splits with non-power-of-two denominators, which are much much slower.)

**bp_done** *(bp)*

Examine the given Bernstein polynomial to see if it is known to have exactly one root in its region. (In addition, we require that the polynomial region not include 0 or 1. This makes things work if the user gives explicit bounds to real_roots(), where the lower or upper bound is a root of the polynomial. real_roots() deals with this by explicitly detecting it, dividing out the appropriate linear polynomial, and adding the root to the returned list of roots; but then if the island considers itself “done” with a region including 0 or 1, the returned root regions can overlap with each other.)

**done** *(ctx)*

Check to see if the island is known to contain zero roots or is known to contain one root.

**has_root** ()

Assuming that the island is done (has either 0 or 1 roots), reports whether the island has a root.

**less_bits** *(ancestors, bp)*

Heuristically pushes lower-precision polynomials on the polynomial stack. See the class documentation for class island for more information.

**more_bits** *(ctx, ancestors, bp, rightmost)*

Find a Bernstein polynomial on the “ancestors” stack with more precision than bp; if it is over a different region, then shrink its region to (approximately) match that of bp. (If this is rightmost – if bp covers the whole island – then we only require that the new region cover the whole island fairly tightly; if this is not rightmost, then the new region will have exactly the same right boundary as bp, although the left boundary may vary slightly.)

**refine** *(ctx)*

Attempts to shrink and/or split this island into sub-island that each definitely contain exactly one root.

**refine_recurse** *(ctx, bp, ancestors, history, rightmost)*

This implements the root isolation algorithm described in the class documentation for class island. This is the implementation of both the whole-island and the left-segment algorithms; if the flag rightmost is True, then it is the whole-island algorithm, otherwise the left-segment algorithm.

The precision-reduction stack is (ancestors + [bp]); that is, the top-of-stack is maintained separately.

**reset_root_width** *(target_width)*

Modify the criteria for this island to require that it is not “done” until its width is less than or equal to target_width.

**shrink_bp** *(ctx)*

If the island’s Bernstein polynomial covers a region much larger than the island itself (in particular, if either the island’s left gap or right gap are totally contained in the polynomial’s region) then shrink the polynomial down to cover the island more tightly.

**class** `sage.rings.polynomial.real_roots.linear_map` *(lower, upper)*

Bases: object

A simple class to map linearly between original coordinates (ranging from [lower .. upper]) and ocean coordinates (ranging from [0 .. 1]).

**from_ocean** *(region)*
to_ocean (region)

sage.rings.polynomial.real_roots.max_abs_doublevec(c)
Given a floating-point vector, return the maximum of the absolute values of its elements.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: max_abs_doublevec(vector(RDF, [0.1, -0.767, 0.3, 0.693]))
0.767
```

sage.rings.polynomial.real_roots.max_bitsize_intvec_doctest(b)

sage.rings.polynomial.real_roots.maximum_root_first_lambda(p)
Given a polynomial with real coefficients, computes an upper bound on its largest real root.

This is using the first-lambda algorithm from “Implementations of a New Theorem for Computing Bounds for Positive Roots of Polynomials”, by Akritas, Strzeboński, and Vigklas.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: x = polygen(ZZ)
sage: maximum_root_first_lambda((x-1)*(x-2)*(x-3))
6.00000000000001
sage: maximum_root_first_lambda((x+1)*(x+2)*(x+3))
0.000000000000000
sage: maximum_root_first_lambda(x^2 - 1)
1.00000000000000
```

sage.rings.polynomial.real_roots.maximum_root_local_max(p)
Given a polynomial with real coefficients, computes an upper bound on its largest real root, using the local-max algorithm from “Implementations of a New Theorem for Computing Bounds for Positive Roots of Polynomials”, by Akritas, Strzeboński, and Vigklas.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: x = polygen(ZZ)
sage: maximum_root_local_max((x-1)*(x-2)*(x-3))
12.0000000000001
sage: maximum_root_local_max((x+1)*(x+2)*(x+3))
0.000000000000000
sage: maximum_root_local_max(x^2 - 1)
1.41421356237310
```

sage.rings.polynomial.real_roots.min_max_delta_intvec(a, b)
Given two integer vectors a and b (of equal, nonzero length), return a pair of the minimum and maximum values taken on by a[i] - b[i].

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: a = vector(ZZ, [10, -30])
sage: b = vector(ZZ, [15, -60])
sage: min_max_delta_intvec(a, b)
(30, -5)
```
Given a floating-point vector \( b = (b_0, ..., b_n) \), compute the minimum and maximum values of \( b_{j+1} - b_j \).

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.real_roots import *
sage: min_max_diff_doublevec(vector(RDF, [1, 7, -2]))
(-9.0, 6.0)
```

Given an integer vector \( b = (b_0, ..., b_n) \), compute the minimum and maximum values of \( b_{j+1} - b_j \).

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.real_roots import *
sage: min_max_diff_intvec(vector(ZZ, [1, 7, -2]))
(-9, 6)
```

A simple wrapper for creating `interval_bernstein_polynomial_float` objects with coercions, defaults, etc.

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.real_roots import *
sage: print(mk_ibpf([0.5, 0.2, -0.9, -0.7, 0.99], pos_err=0.1, neg_err=-0.01))
degree 4 IBP with floating-point coefficients
```

A simple wrapper for creating `interval_bernstein_polynomial_integer` objects with coercions, defaults, etc.

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.real_roots import *
sage: print(mk_ibpi([50, 20, -90, -70, 200], error=5))
degree 4 IBP with 8-bit coefficients
```

Given the tools we’ve defined so far, there are many possible root isolation algorithms that differ on where to select split points, what precision to work at when, and when to attempt degree reduction.
Here we implement one particular algorithm, which I call the ocean-island algorithm. We start with an interval Bernstein polynomial defined over the region $[0 .. 1]$. This region is the “ocean”. Using de Casteljau's algorithm and Descartes' rule of signs, we divide this region into subregions which may contain roots, and subregions which are guaranteed not to contain roots. Subregions which may contain roots are “islands”; subregions known not to contain roots are “gaps”.

All the real root isolation work happens in class island. See the documentation of that class for more information.

An island can be told to refine itself until it contains only a single root. This may not succeed, if the island’s interval Bernstein polynomial does not have enough precision. The ocean basically loops, refining each of its islands, then increasing the precision of islands which did not succeed in isolating a single root; until all islands are done.

Increasing the precision of unsuccessful islands is done in a single pass using split_for_target(); this means it is possible to share work among multiple islands.

**all_done()**

Return True iff all islands are known to contain exactly one root.

**approx_bp** (*scale_log2*)

Return an approximation to our Bernstein polynomial with the given scale_log2.

**find_roots()**

Isolate all roots in this ocean.
increase_precision()

Increase the precision of the interval Bernstein polynomial held by any islands which are not done. (In normal
use, calls to this function are separated by calls to self.refine_all().)

EXAMPLES:

```sage
from sage.rings.polynomial.real_roots import *
sage: oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([1/3, -22/7, 193/71, -140/99]), lmap)
sage: oc
```

```sage
ocean with precision 120 and 1 island(s)
sage: oc.increase_precision()
sage: oc.increase_precision()
sage: oc.increase_precision()
sage: oc
```

```sage
```

```sage
```

```sage
```

```sage
```

```sage
```

```sage
```

```sage
```

```sage
```

```sage
```

```sage
```

refine_all()

Refine all islands which are not done (which are not known to contain exactly one root).

EXAMPLES:

```sage
from sage.rings.polynomial.real_roots import *
sage: oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([1/3, -22/7, 193/71, -140/99]), lmap)
sage: oc
```

```sage
```

```sage
```

```sage
```

```sage
```

```sage
```

reset_root_width(isle_num, target_width)

Require that the isle_num island have a width at most target_width.

If this is followed by a call to find_roots(), then the corresponding root will be refined to the specified width.

EXAMPLES:

```sage
from sage.rings.polynomial.real_roots import *
sage: oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([-1, -1, -1]), lmap)
sage: oc.find_roots()
sage: oc.roots()

[(1/2, 3/4)]
sage: oc.reset_root_width(0, 1/2^200)
sage: oc.find_roots()
sage: RR(RealIntervalField(300)(oc.roots()[0]).absolute_diameter()).log2()

-232.668979560890
```

roots()

Return the locations of all islands in this ocean. (If run after find_roots(), this is the location of all roots in
the ocean.)
EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([[1/3, -22/7, 193/71, -140/99]], lmap)
sage: oc.find_roots()
sage: oc.roots()
[(1/32, 1/16), (1/2, 5/8), (3/4, 7/8)]
sage: oc = ocean(mk_context(), bernstein_polynomial_factory_ratlist([1, 0, -1111/2, 0, 11108889/14, 0, 0, 0, -1]], lmap)
sage: oc.find_roots()
sage: oc.roots()
[(95761241267509487747625/9671406556917033397649408, 191522482605387719863145/19342813113834066795298816), (1496269395904347376805/151115727451828646838272, 374067366568272936175/37778931862957161709568),
(31/32, 63/64)]
```

**sage.rings.polynomial.real_roots.precompute_degree_reduction_cache** (*n*)

Compute and cache the matrices used for degree reduction, starting from degree n.

**EXAMPLES:**

```
sage: from sage.rings.polynomial.real_roots import *
sage: precompute_degree_reduction_cache(5)
sage: dr_cache[5]
{
        [ -3/7 37/42 16/21 1/21 -3/7 1/6]
        [ 1/6 -3/7 1/21 16/21 37/42 -3/7]
    2, [ [121/126 8/63 -1/9 -2/63 11/126 -2/63]
        [ -3/7 37/42 16/21 1/21 -3/7 1/6]
        [ 1/6 -3/7 1/21 16/21 37/42 -3/7]
    1, [ [121 16 -14 -4 11 -4]
        [ -54 111 96 6 -54 21]
        [ 21 -54 6 96 111 -54]
        [ -4 11 -4 -14 16 121], 126]
}
```

**sage.rings.polynomial.real_roots.pseudoinverse** (*m*)

**sage.rings.polynomial.real_roots.rational_root_bounds** (*p*)

Given a polynomial p with real coefficients, computes rationals a and b, such that for every real root r of p, a < r < b. We try to find rationals which bound the roots somewhat tightly, yet are simple (have small numerators and denominators).

**EXAMPLES:**

```
sage: from sage.rings.polynomial.real_roots import *
sage: x = polygen(ZZ)
sage: rational_root_bounds((x-1)*(x-2)*(x-3))
(0, 7)
sage: rational_root_bounds(x^2)
(-1/2, 1/2)
sage: rational_root_bounds(x*(x+1))
(-3/2, 1/2)
sage: rational_root_bounds((x+2)*(x-3))
(-3, 6)
sage: rational_root_bounds(x^995 * (x^2 - 9999) - 1)
(-100, 1000/7)
```
Polynomials, Release 10.3

sage: rational_root_bounds(x^995 * (x^2 - 9999) + 1)
(-142, 213/2)

If we can see that the polynomial has no real roots, return None.
sage: rational_root_bounds(x^2 + 7) is None
True

sage.rings.polynomial.real_roots.real_roots(p, bounds=None, seed=None, skip_squarefree=False, do_logging=False, wordsize=32, retval='rational', strategy=None, max_diameter=None)

Compute the real roots of a given polynomial with exact coefficients (integer, rational, and algebraic real coefficients are supported).

This returns a list of pairs of a root and its multiplicity.

The root itself can be returned in one of three different ways. If retval=='rational', then it is returned as a pair of rationals that define a region that includes exactly one root. If retval=='interval', then it is returned as a RealIntervalFieldElement that includes exactly one root. If retval=='algebraic_real', then it is returned as an AlgebraicReal. In the former two cases, all the intervals are disjoint.

An alternate high-level algorithm can be used by selecting strategy='warp'. This affects the conversion into Bernstein polynomial form, but still uses the same ocean-island algorithm as the default algorithm. The ‘warp’ algorithm performs the conversion into Bernstein polynomial form much more quickly, but performs the rest of the computation slightly slower in some benchmarks. The ‘warp’ algorithm is particularly likely to be helpful for low-degree polynomials.

Part of the algorithm is randomized; the seed parameter gives a seed for the random number generator. (By default, the same seed is used for every call, so that results are repeatable.) The random seed may affect the running time, or the exact intervals returned, but the results are correct regardless of the seed used.

The bounds parameter lets you find roots in some proper subinterval of the reals; it takes a pair of a rational lower and upper bound and only roots within this bound will be found. Currently, specifying bounds does not work if you select strategy='warp', or if you use a polynomial with algebraic real coefficients.

By default, the algorithm will do a squarefree decomposition to get squarefree polynomials. The skip_squarefree parameter lets you skip this step. (If this step is skipped, and the polynomial has a repeated real root, then the algorithm will loop forever! However, repeated non-real roots are not a problem.)

For integer and rational coefficients, the squarefree decomposition is very fast, but it may be slow for algebraic reals. (It may trigger exact computation, so it might be arbitrarily slow. The only other way that this algorithm might trigger exact computation on algebraic real coefficients is that it checks the constant term of the input polynomial for equality with zero.)

Part of the algorithm works (approximately) by splitting numbers into word-size pieces (that is, pieces that fit into a machine word). For portability, this defaults to always selecting pieces suitable for a 32-bit machine; the wordsize parameter lets you make choices suitable for a 64-bit machine instead. (This affects the running time, and the exact intervals returned, but the results are correct on both 32- and 64-bit machines even if the wordsize is chosen “wrong”.)

The precision of the results can be improved (at the expense of time, of course) by specifying the max_diameter parameter. If specified, this sets the maximum diameter() of the intervals returned. (Sage defines diameter() to be the relative diameter for intervals that do not contain 0, and the absolute diameter for intervals containing 0.) This directly affects the results in rational or interval return mode; in algebraic_real mode, it increases the precision of the intervals passed to the algebraic number package, which may speed up some operations on that algebraic real.

Some logging can be enabled with do_logging=True. If logging is enabled, then the normal values are not returned; instead, a pair of the internal context object and a list of all the roots in their internal form is returned.
ALGORITHM: We convert the polynomial into the Bernstein basis, and then use de Casteljau's algorithm and Descartes' rule of signs (using interval arithmetic) to locate the roots.

EXAMPLES:

```
sage: from sage.rings.polynomial.real_roots import *
sage: x = polygen(ZZ)
sage: real_roots(x^3 - x^2 - x - 1)
[[(7/4, 19/8), 1]]
sage: real_roots((x-1)*(x-2)*(x-3)*(x-5)*(x-8)*(x-13)*(x-21)*(x-34))
[[(11/16, 33/32), 1], [(11/8, 33/16), 1], [(11/4, 55/16), 1], [(77/16, 165/32), 1], [(11/2, 33/4), 1], [(11, 55/4), 1], [(165/8, 341/16), 1], [(22, 44), 1]]
sage: real_roots(x^5 * (x^2 - 9999)^2 - 1)
[[-(2274496381311/9007199254740992, 419601125186091/2251799813685248), 1], 
 (22656845014584945395100616544153249597/2126764793255865396646912964485513216, 1), 
 (245331690271330018853696359533061621799/24535295865117307932921825928971026432, 1), 
 (1063329226287402282451317352558954186101/106338293629732968330456482242756608, 351664614358685696701454201630854654353, 1)]
sage: real_roots(x^5 * (x^2 - 9999)^2 - 1, seed=42)
[[-(123196838480289/18014398509481984, 293964743458749/9007199254740992), 1], 
 (83072595739795519078416696381986376143/8307647947365572420568478941267521536, 1), 
 (16615191508103378137940378745325503/166153499473114484112975882535043072, 1), 
 (5192203725629521675180152499797434335/519229685853487268253049632200096, 1), 
 (604432689420816080312183/6044629098073145875303808, 1)]
sage: real_roots(x^5 * (x^2 - 9999)^2 - 1, wordsize=64)
[[-(62866503803202151050003/19324813113834066795298816, 90186554512564177624143/48357032784516698824704, 1), ((5444245632733731521499089722890050101157/54445178707350145139937198291383296, 1), 
 (2722128219296614397110639286860007142141/10889035447003083028798743781658266692, 1), 
 (62165404551230519591003933768548566263/1361129467683753853853498429727072845824, 1)]
```
If the polynomial has no real roots, we get an empty list.

```python
sage: (x^2 + 1).real_root_intervals()
[]
```
We can compute Conway's constant (see http://mathworld.wolfram.com/ConwaysConstant.html) to arbitrary precision.

```sage
p = x^71 - x^69 - 2*x^68 - x^67 + 2*x^66 + 2*x^65 + x^64 - x^63 - x^62 - x^61 - x^60 - x^59 + 2*x^58 + 5*x^57 + 3*x^56 - 2*x^55 - 10*x^54 - 3*x^53 - 2*x^52 + 6*x^51 + 6*x^50 + x^49 + 9*x^48 - 3*x^47 - 7*x^46 - 8*x^45 - 10*x^44 + 10*x^43 - 3*x^42 + 10*x^41 - 2*x^40 - 10*x^39 - 3*x^38 - 7*x^37 + 7*x^36 + x^35 - 3*x^34 + 10*x^33 + x^32 - 6*x^31 - 2*x^30 - 10*x^29 - 3*x^28 + 2*x^27 + 9*x^26 - 8*x^25 - 4*x^24 + 12*x^23 - 7*x^22 + 9*x^21 + 3*x^20 - 12*x^19 - 4*x^18 - 2*x^17 - 12*x^16 + 7*x^15 - 4*x^14 + 2*x^13 - 12*x^12 - 4*x^11 - 2*x^10 - 10*x^9 + x^7 - 7*x^6 + 7*x^5 - 4*x^4 + 12*x^3 - 6*x^2 + 3*x - 6
```

```sage
cc = real_roots(p, retval=algebraic_real)[2][0] # long time
RealField(180)(cc) # long time
1.3035772690342963912570991121525518907307025046594049
```

Now we play with algebraic real coefficients.

```sage
x = polygen(AA)
p = (x - 1) * (x - sqrt(AA(2))) * (x - 2)
real_roots(p)
[(499/525, 2171/1925), (1173/875, 2521/1575), (337/175, 849/175), (1.414213562373095?, 1), (2.000000000000000?, 1)]
ar_rts = real_roots(p, retval=algebraic_real); ar_rts
[(1.000000000000000?, 1), (1.414213562373095?, 1), (2.000000000000000?, 1)]
ar_rts[1][0]^2 == 2
True
ar_rts = real_roots(x*(x-1), retval=algebraic_real)
ar_rts[0][0] == 0
True
p2 = p * (p - 1/100); p2
x^6 - 8.82842712474619?*x^5 + 31.97056274847714?*x^4 - 60.77955262170047?*x^3 + ...
real_roots(p2, retval='interval')
[(1.00?, 1), (1.1?, 1), (1.38?, 1), (1.5?, 1), (2.00?, 1), (2.1?, 1)]
real_roots(p, retval=real_field)
[(499/525, 2171/1925), (1173/875, 2521/1575), (337/175, 849/175), (1.414213562373095?, 1), (2.000000000000000?, 1)]
```

```
sage: from sage.rings.polynomial.real_roots import *
sage: relative_bounds((1/7, 1/4), (1/6, 1/5))
(2/9, 8/15)
sage: reverse_intvec(c)
Given a vector of integers, reverse the vector (like the reverse() method on lists).
```

sage.rings.polynomial.real_roots.relative_bounds(a, b)
```

**INPUT:**

- (al, ah) – pair of rationals
- (bl, bh) – pair of rationals

**OUTPUT:**

- (cl, ch) – pair of rationals

Computes the linear transformation that maps (al, ah) to (0, 1); then applies this transformation to (bl, bh) and returns the result.

**EXAMPLES:**

```
sage: from sage.rings.polynomial.real_roots import *
sage: relative_bounds(((1/7, 1/4), (1/6, 1/5))
(2/9, 8/15)
sage.rings.polynomial.real_roots.reverse_intvec(c)
```

**2.1. Univariate Polynomials and Polynomial Rings 225**
Polynomials, Release 10.3

Modifies the input vector; has no return value.

EXAMPLES:

```python
sage: from sage.rings.polynomial.real_roots import *
sage: v = vector(ZZ, [1, 2, 3, 4]); v
(1, 2, 3, 4)
sage: reverse_intvec(v)
sage: v
(4, 3, 2, 1)
```

`sage.rings.polynomial.real_roots.root_bounds(p)`

Given a polynomial with real coefficients, computes a lower and upper bound on its real roots. Uses algorithms of Akritas, Strzeboński, and Vigklas.

EXAMPLES:

```python
sage: from sage.rings.polynomial.real_roots import *
sage: x = polygen(ZZ)
sage: root_bounds((x-1)*(x-2)*(x-3))
(0.545454545454545, 6.00000000000001)
sage: root_bounds(x^2)
(0.000000000000000, 0.000000000000000)
sage: root_bounds(x*(x+1))
(-1.000000000000000, 0.000000000000000)
sage: root_bounds((x+2)*(x-3))
(-2.44948974278317, 3.46410161513776)
sage: root_bounds(x^995 * (x^2 - 9999) - 1)
(-99.9949998749937, 141.414284992713)
sage: root_bounds(x^995 * (x^2 - 9999) + 1)
(-141.414284992712, 99.9949998749938)
```

If we can see that the polynomial has no real roots, return None.

```python
sage: root_bounds(x^2 + 1) is None
True
```

class sage.rings.polynomial.real_roots.rr_gap

Bases: object

A simple class representing the gaps between islands, in my ocean-island root isolation algorithm. Named “rr_gap” for “real roots gap”, because “gap” seemed too short and generic.

```python
sage: region()
```

`sage.rings.polynomial.real_roots.scale_intvec_var(c,k)`

Given a vector of integers c of length n+1, and a rational k == kn / kd, multiplies each element c[i] by (kd^i)*(kn^(n-i)).

Modifies the input vector; has no return value.

EXAMPLES:

```python
sage: from sage.rings.polynomial.real_roots import *
sage: v = vector(ZZ, [1, 1, 1, 1])
sage: scale_intvec_var(v, 3/4)
sage: v
(64, 48, 36, 27)
```
Given an interval Bernstein polynomial over a particular region (assumed to be a (not necessarily proper) subregion of \([0.. 1]\)), and a list of targets, uses de Casteljau’s method to compute representations of the Bernstein polynomial over each target. Uses degree reduction as often as possible while maintaining the requested precision.

Each target is of the form \((\text{lgap}, \text{ugap}, b)\). Suppose \(\text{lgap.region()}\) is \((l_1, l_2)\), and \(\text{ugap.region()}\) is \((u_1, u_2)\). Then we will compute an interval Bernstein polynomial over a region \([l .. u]\), where \(l_1 \leq l \leq l_2\) and \(u_1 \leq u \leq u_2\). (\text{split_for_targets()}\) is free to select arbitrary region endpoints within these bounds; it picks endpoints which make the computation easier.) The third component of the target, \(b\), is the maximum allowed scale \_log2\ of the result; this is used to decide when degree reduction is allowed.

The pair \((l_1, l_2)\) can be replaced by \(\text{None}\), meaning \([-\infty .. 0]\); or, \((u_1, u_2)\) can be replaced by \(\text{None}\), meaning \([1.. \infty]\).

There is another constraint on the region endpoints selected by \text{split_for_targets()}\) for a target \((l_1, l_2), \(u_1, u_2\), \(b)\). We set a size goal \(g\), such that \((u - l) \leq g \cdot (u_1 - l_2)\). Normally \(g\) is \(256/255\), but if \text{precise} is \(\text{True}\), then \(g\) is \(65536/65535\).

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.real_roots import *
sage: bp = mk_ibpi([1000000, -2000000, 3000000, -4000000, -5000000, -6000000])
sage: ctx = mk_context()
sage: bps = split_for_targets(ctx, bp, [(rr_gap(1/1234567893, 1/1234567892, 1), → rr_gap(1/1234567891, 1/1234567890, 1), 12), (rr_gap(1/3, 1/2, -1), rr_gap(2/3, → 3/4, -1), 6)])
sage: bps[0]
<IBP: (999992, 999992, 999992) + [0 .. 15) over [1063823966279326983230456482242756608 .. → 59190818625959349813386527495938294787/ → 73075081865645145910184216358141509872966271488]; level 2; slope_err 0.?e12>
sage: bps[1]
<IBP: (-1562500, -1875001, -2222223, -2592593, -2969137, -3337450) + [0 .. 4) → over [1/2 .. 2863311531/4294967296]>
```

**sage.rings.polynomial.real_roots.subsample_vec_doctest**(a, slen, llen)

**sage.rings.polynomial.real_roots.taylor_shift1_intvec**(c)

Given a vector of integers \(c\) of length \(d+1\), representing the coefficients of a degree-\(d\) polynomial \(p\), modify the vector to perform a Taylor shift by 1 (that is, \(p(x+1)\)).

This is the straightforward algorithm, which is not asymptotically optimal.

Modifies the input vector; has no return value.

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.real_roots import *
sage: x = polygen(ZZ)
sage: p = (x-1)*(x-2)*(x-3)
sage: v = vector(ZZ, p.list())
sage: p, v
(x^3 - 6*x^2 + 11*x - 6, (-6, 11, -6, 1))
sage: taylor_shift1_intvec(v)
sage: p(x+1), v
(x^3 - 3*x^2 + 2*x, (0, 2, -3, 1))
```

**sage.rings.polynomial.real_roots.to_bernstein**(p, low=0, high=1, degree=None)

Given a polynomial \(p\) with integer coefficients, and rational bounds low and high, compute the exact rational Bern-
stein coefficients of \( p \) over the region \([\text{low} .. \text{high}]\). The optional parameter degree can be used to give a formal degree higher than the actual degree.

The return value is a pair \((c, \text{scale})\); \( c \) represents the same polynomial as \( p \times \text{scale} \). (If you only care about the roots of the polynomial, then of course scale can be ignored.)

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.real_roots import *
sage: x = polygen(ZZ)
sage: to_bernstein(x)
([0, 1], 1)
sage: to_bernstein(x, degree=5)
([0, 1/5, 2/5, 3/5, 4/5, 1], 1)
sage: to_bernstein(x^3 + x^2 - x - 1, low=-3, high=3)
([-16, 24, -32, 32], 1)
sage: to_bernstein(x^3 + x^2 - x - 1, low=3, high=22/7)
([296352, 310464, 325206, 340605], 9261)
```

`sage.rings.polynomial.real_roots.to_bernstein_warp(p)`

Given a polynomial \( p \) with rational coefficients, compute the exact rational Bernstein coefficients of \( p(x/(x+1)) \).

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.real_roots import *
sage: x = polygen(ZZ)
sage: to_bernstein_warp(1 + x + x^2 + x^3 + x^4 + x^5)
[1, 1/5, 1/10, 1/10, 1/5, 1]
```

**class** `sage.rings.polynomial.real_roots.warp_map(neg)`

Bases: object

A class to map between original coordinates and ocean coordinates. If neg is False, then the original->ocean transform is \( x \rightarrow x/(x+1) \), and the ocean->original transform is \( x/(1-x) \); this maps between \([0 .. \infty]\) and \([0 .. 1]\). If neg is True, then the original->ocean transform is \( x \rightarrow -x/(1-x) \), and the ocean->original transform is the same thing: \(-x/(1-x)\). This maps between \([0 .. -\infty]\) and \([0 .. 1]\).

**from_ocean** *(region)*

**to_ocean** *(region)*

`sage.rings.polynomial.real_roots.wordsize_rational(a, b, wordsize)`

Given rationals \( a \) and \( b \), select a de Casteljau split point \( r \) between \( a \) and \( b \).

An attempt is made to select an efficient split point (according to the criteria mentioned in the documentation for `de_casteljau_intvec`), with a bias towards split points near \( a \).

In full detail:

This takes as input two rationals, \( a \) and \( b \), such that \( 0 \leq a \leq 1 \), \( 0 \leq b \leq 1 \), and \( a! = b \). This returns rational \( r \), such that \( a \leq r \leq b \) or \( b \leq r \leq a \). The denominator of \( r \) is a power of 2. Let \( m \) be \( \min(r, 1-r) \), \( nm \) be numerator(\( m \)), and \( dml \) be \( \log_2(\text{denominator}(m)) \). The return value \( r \) is taken from the first of the following classes to have any members between \( a \) and \( b \) (except that if \( a \leq 1/8 \), or \( 7/8 \leq a \), then class 2 is preferred to class 1).

1. \( dml < \text{wordsize} \)
2. \( \text{bitsize}(nm) \leq \text{wordsize} \)
3. \( \text{bitsize}(nm) \leq 2^*\text{wordsize} \)
4. \( \text{bitsize}(nm) \leq 3^*\text{wordsize} \)
k. \text{bitsize}(nm) \leq (k-1)\text{wordsize}

From the first class to have members between a and b, r is chosen as the element of the class which is closest to a.

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.real_roots import *
sage: wordsize_rational(1/5, 1/7, 32)
429496729/2147483648
sage: wordsize_rational(1/7, 1/5, 32)
306783379/2147483648
sage: wordsize_rational(1/5, 1/7, 64)
1844674407370955161/9223372036854775808
sage: wordsize_rational(1/7, 1/5, 64)
658812288346769701/4611686018427387904
sage: wordsize_rational(1/17, 1/19, 32)
252645135/4294967296
sage: wordsize_rational(1/17, 1/19, 64)
108510259271150095/18446744073709551616
sage: wordsize_rational(1/1234567890, 1/1234567891, 32)
933866427/1152921504606846976
sage: wordsize_rational(1/1234567890, 1/1234567891, 64)
4010925763784056541/4951760157141521099596496896
```

### 2.1.19 Isolate Complex Roots of Polynomials

**AUTHOR:**

- Carl Witty (2007-11-18): initial version

This is an implementation of complex root isolation. That is, given a polynomial with exact complex coefficients, we compute isolating intervals for the complex roots of the polynomial. (Polynomials with integer, rational, Gaussian rational, or algebraic coefficients are supported.)

We use a simple algorithm. First, we compute a squarefree decomposition of the input polynomial; the resulting polynomials have no multiple roots. Then, we find the roots numerically, using NumPy (at low precision) or Pari (at high precision). Then, we verify the roots using interval arithmetic.

**EXAMPLES:**

```python
sage: x = polygen(ZZ)
sage: (x^5 - x - 1).roots(ring=CIF)
[(1.16730397826142?, 1),
 (-0.764884433600585? - 0.352471546031727?*I, 1),
 (-0.764884433600585? + 0.352471546031727?*I, 1),
 (0.181232444469876? - 1.083954101317711?*I, 1),
 (0.181232444469876? + 1.083954101317711?*I, 1)]
```

```
sage.rings.polynomial.complex_roots.complex_roots(p, skip_squarefree=False, retval='interval', min_prec=0)
```

Compute the complex roots of a given polynomial with exact coefficients (integer, rational, Gaussian rational, and algebraic coefficients are supported). Returns a list of pairs of a root and its multiplicity.

Roots are returned as a ComplexIntervalFieldElement; each interval includes exactly one root, and the intervals are disjoint.
By default, the algorithm will do a squarefree decomposition to get squarefree polynomials. The skip_squarefree parameter lets you skip this step. (If this step is skipped, and the polynomial has a repeated root, then the algorithm will loop forever!)

You can specify retval='interval' (the default) to get roots as complex intervals. The other options are retval='algebraic' to get elements of QQbar, or retval='algebraic_real' to get only the real roots, and to get them as elements of AA.

**EXAMPLES:**

```sage
def from sage.rings.polynomial.complex_roots import complex_roots
def x = polygen(ZZ)
def complex_roots(x^5 - x - 1)
[(1.167303978261419?, 1),
 (-0.764884433600585? - 0.352471546031727?*I, 1),
 (-0.764884433600585? + 0.352471546031727?*I, 1),
 (0.181232444469876? - 1.083954101317711?*I, 1),
 (0.181232444469876? + 1.083954101317711?*I, 1)]
def v = complex_roots(x^2 + 27*x + 181)
def unfortunately due to numerical noise there can be a small imaginary part to each root depending on CPU, compiler, etc. and that affects the printing order. So we verify the real part of each root and check that the imaginary part is small in both cases:
```
sage: def tiny(x):
....:     return x.contains_zero() and x.absolute_diameter() < 1e-14
sage: def smash(x):
....:     x = CIF(x[0]) # discard multiplicity
....:     if tiny(x.imag()):
....:         return x.real()
....:     if tiny(x.real()):
....:         return CIF(0, x.imag())
sage: rts = complex_roots(p, retval=algebraic); type(rts[0][0]), sorted(map(smash, rts))
(<class 'sage.rings.qqbar.AlgebraicNumber'>,
[-1.618033988749895?, -0.618033988749895?*I,
1.618033988749895?*I, 0.618033988749895?])
sage: rts = complex_roots(p, retval=algebraic_real); type(rts[0][0]), rts
(<class 'sage.rings.qqbar.AlgebraicReal'>,
[(-1.618033988749895?, 1), (0.618033988749895?, 1)])

sage.rings.polynomial.complex_roots.interval_roots(p, rts, prec)

We are given a squarefree polynomial p, a list of estimated roots, and a precision.
We attempt to verify that the estimated roots are in fact distinct roots of the polynomial, using interval arithmetic
of precision prec. If we succeed, we return a list of intervals bounding the roots; if we fail, we return None.

EXAMPLES:

sage: x = polygen(ZZ)
sage: p = x^3 - 1
sage: rts = [CC.zeta(3)^i for i in range(0, 3)]
sage: from sage.rings.polynomial.complex_roots import interval_roots
sage: interval_roots(p, rts, 53)
[1, -0.500000000000000? + 0.866025403784439?*I,
-0.500000000000000? - 0.866025403784439?*I]
sage: interval_roots(p, rts, 200)
[1, -0.500000000000000000000000000000000000000000000000000000000000?
+ 0.866025403784438646736723170752936183471402626905190314027904?*I,
-0.500000000000000000000000000000000000000000000000000000000000?
- 0.866025403784438646736723170752936183471402626905190314027904?*I]

sage.rings.polynomial.complex_roots.intervals_disjoint(intvs)

Given a list of complex intervals, check whether they are pairwise disjoint.

EXAMPLES:

sage: from sage.rings.polynomial.complex_roots import intervals_disjoint
sage: a = CIF(RIF(0, 3), 0)
sage: b = CIF(0, RIF(1, 3))
sage: c = CIF(RIF(1, 2), RIF(1, 2))
sage: d = CIF(RIF(2, 3), RIF(2, 3))
sage: intervals_disjoint([a, b, c, d])
False
sage: d2 = CIF(RIF(2, 3), RIF(2.001, 3))
sage: intervals_disjoint([a, b, c, d2])
True
2.1.20 Refine polynomial roots using Newton–Raphson

This is an implementation of the Newton–Raphson algorithm to approximate roots of complex polynomials. The implementation is based on interval arithmetic.

AUTHORS:

- Carl Witty (2007-11-18): initial version

```python
sage.rings.polynomial.refine_root.refine_root(ip, ipd, irt, fld)
```

We are given a polynomial and its derivative (with complex interval coefficients), an estimated root, and a complex interval field to use in computations. We use interval arithmetic to refine the root and prove that we have in fact isolated a unique root.

If we succeed, we return the isolated root; if we fail, we return None.

EXAMPLES:

```python
sage: # needs sage.symbolic
sage: from sage.rings.polynomial.refine_root import refine_root
sage: x = polygen(ZZ)
sage: p = x^9 - 1
sage: ip = CIF[x](p); ip
x^9 - 1
sage: ipd = CIF[x](p.derivative()); ipd
9*x^8
sage: irt = CIF(CC(cos(2*pi/9), sin(2*pi/9))); irt
0.76604444311897802? + 0.64278760968653926?*I
sage: ip(irt)
0.?e-14 + 0.?e-14*I
sage: ipd(irt)
6.89439998807080? - 5.78508848717885?*I
sage: refine_root(ip, ipd, irt, CIF)
0.766044443118978? + 0.642787609686540?*I
```

2.1.21 Ideals in Univariate Polynomial Rings

AUTHORS:

- David Roe (2009-12-14) – initial version.

```python
class sage.rings.polynomial.ideal.Ideal_1poly_field
```

An ideal in a univariate polynomial ring over a field.

```python
change_ring(R)
```

Coerce an ideal into a new ring.

EXAMPLES:

```python
sage: R.<q> = QQ[]
sage: I = R.ideal([q^2+q-1])
sage: I.change_ring(RR['q'])
Principal ideal (q^2 + q - 1.00000000000000) of Univariate Polynomial Ring in q over Real Field with 53 bits of precision
```
**groebner_basis** *(algorithm=None)*

Return a Gröbner basis for this ideal.

The Gröbner basis has 1 element, namely the generator of the ideal. This trivial method exists for compatibility with multi-variate polynomial rings.

**INPUT:**

- algorithm – ignored

**EXAMPLES:**

```python
sage: R.<x> = QQ[]
sage: I = R.ideal([x^2 - 1, x^3 - 1])
sage: G = I.groebner_basis(); G
[x - 1]
sage: type(G)
<class 'sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic'>
sage: list(G)
[x - 1]
```

**residue_class_degree**

Return the degree of the generator of this ideal.

This function is included for compatibility with ideals in rings of integers of number fields.

**EXAMPLES:**

```python
sage: R.<t> = GF(5)[]
sage: P = R.ideal(t^4 + t + 1)
sage: P.residue_class_degree()
4
```

**residue_field** *(names=None, check=True)*

If this ideal is \( P \subset \mathbb{F}_p[t] \), return the quotient \( \mathbb{F}_p[t]/P \).

**EXAMPLES:**

```python
sage: R.<t> = GF(17)[]; P = R.ideal(t^3 + 2*t + 9)
sage: k.<a> = P.residue_field(); k
Univariate Polynomial Ring in a of Principal ideal (t^3 + 2*t + 9) of Univariate Polynomial Ring in t over Finite Field of size 17
```

### 2.1.22 Quotients of Univariate Polynomial Rings

**EXAMPLES:**

```python
sage: R.<x> = QQ[]
sage: S = R.quotient(x**3 - 3*x + 1, alpha)
sage: S.gen()**2 in S
True
sage: x in S
True
sage: S.gen() in R
False
```

(continues on next page)
class
sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRingFactory
Bases: UniqueFactory
Create a quotient of a polynomial ring.

INPUT:

- ring - a univariate polynomial ring
- polynomial - an element of ring with a unit leading coefficient
- names - (optional) name for the variable

OUTPUT: Creates the quotient ring \( R/I \), where \( R \) is the ring and \( I \) is the principal ideal generated by polynomial.

EXAMPLES:

We create the quotient ring \( \mathbb{Z}[x]/(x^3 + 7) \), and demonstrate many basic functions with it:

```
sage: Z = IntegerRing()
sage: R = PolynomialRing(Z, 'x'); x = R.gen()
sage: S = R.quotient(x^3 + 7, 'a'); a = S.gen()
sage: S
Univariate Quotient Polynomial Ring in a over Integer Ring with modulus x^3 + 7
sage: a^3
-7
sage: S.is_field()
False
sage: a in S
True
sage: x in S
True
sage: a in R
False
sage: S.polynomial_ring()
Univariate Polynomial Ring in x over Integer Ring
sage: S.modulus()
x^3 + 7
sage: S.degree()
3
```

We create the “iterated” polynomial ring quotient

\[
R = (\mathbb{F}_2[y]/(y^2 + y + 1))[x]/(x^3 - 5).
\]

```
sage: # needs sage.libs.ntl
sage: A.<y> = PolynomialRing(GF(2)); A
Univariate Polynomial Ring in y over Finite Field of size 2 (using GF2X)
sage: B = A.quotient(y^2 + y + 1, 'y2'); B
Univariate Quotient Polynomial Ring in y2 over Finite Field of size 2
with modulus y^2 + y + 1
sage: C = PolynomialRing(B, 'x'); x = C.gen(); C
```

(continues on next page)
Next we create a number field, but viewed as a quotient of a polynomial ring over \( \mathbb{Q} \):

```python
sage: R = PolynomialRing(RationalField(), 'x'); x = R.gen()
sage: S = R.quotient(x^3 + 2*x - 5, 'a'); S
Univariate Quotient Polynomial Ring in a over Rational Field with modulus x^3 + 2*x - 5
sage: S.is_field()
True
sage: S.degree()
3
```

There are conversion functions for easily going back and forth between quotients of polynomial rings over \( \mathbb{Q} \) and number fields:

```python
sage: K = S.number_field(); K
Number Field in a with defining polynomial x^3 + 2*x - 5
sage: K.polynomial_quotient_ring()
Univariate Quotient Polynomial Ring in a over Rational Field with modulus x^3 + 2*x - 5
```

The leading coefficient must be a unit (but need not be 1).

```python
sage: R = PolynomialRing(Integers(9), 'x'); x = R.gen()
sage: S = R.quotient(2*x^4 + 2*x^3 + x + 2, 'a')
sage: S = R.quotient(3*x^4 + 2*x^3 + x + 2, 'a')
Traceback (most recent call last):
  ...
TypeError: polynomial must have unit leading coefficient
```

Another example:

```python
sage: R.<x> = PolynomialRing(IntegerRing())
sage: f = x^2 + 1
sage: R.quotient(f)
Univariate Quotient Polynomial Ring in xbar over Integer Ring with modulus x^2 + 1
```

This shows that the issue at github issue #5482 is solved:

```python
sage: R.<x> = PolynomialRing(QQ)
sage: f = x^2 - 1
sage: R.quotient_by_principal_ideal(f)
Univariate Quotient Polynomial Ring in xbar over Rational Field with modulus x^2 - 1
```

```
create_key (ring, polynomial, names=None)

Return a unique description of the quotient ring specified by the arguments.
```

2.1. Univariate Polynomials and Polynomial Rings 235
create_object (version, key)

Return the quotient ring specified by key.

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: PolynomialQuotientRing.create_object((8, 0, 0),
....: (R, x^2 - 1, ('xbar')))  
Univariate Quotient Polynomial Ring in xbar over Rational Field with modulus x^2 - 1
```

class sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_coercion

Bases: DefaultConvertMap_unique

A coercion map from a PolynomialQuotientRing to a PolynomialQuotientRing that restricts to the coercion map on the underlying ring of constants.

EXAMPLES:

```python
sage: R.<x> = ZZ[]
sage: S.<x> = QQ[]
sage: f = S.quo(x^2 + 1).coerce_map_from(R.quo(x^2 + 1)); f  
Coercion map:
  From: Univariate Quotient Polynomial Ring in xbar over Integer Ring
  with modulus x^2 + 1
  To:   Univariate Quotient Polynomial Ring in xbar over Rational Field
  with modulus x^2 + 1
```

isInjective ()

Return whether this coercion is injective.

EXAMPLES:

If the modulus of the domain and the codomain is the same and the leading coefficient is a unit in the domain, then the map is injective if the underlying map on the constants is:

```python
sage: R.<x> = ZZ[]
sage: S.<x> = QQ[]
sage: f = S.quo(x^2 + 1).coerce_map_from(R.quo(x^2 + 1))
sage: f.is_injective()  
True
```

isSurjective ()

Return whether this coercion is surjective.

EXAMPLES:

If the underlying map on constants is surjective, then this coercion is surjective since the modulus of the codomain divides the modulus of the domain:
If the modulus of the domain and the codomain is the same, then the map is surjective iff the underlying map on the constants is:

```python
sage: # needs sage.rings.padics
sage: A.<a> = ZqCA(9)
sage: R.<x> = A[]
sage: S.<x> = A.fraction_field()[]
sage: f = S.quo(x^2 + 2).coerce_map_from(R.quo(x^2 + 2))
sage: f.is_surjective()
False
```

```python
class sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_domain(ring, polynomial, name=None, category=None):

    Bases: PolynomialQuotientRing_generic, IntegralDomain

    EXAMPLES:

    sage: R.<x> = PolynomialRing(ZZ)
sage: S.<xbar> = R.quotient(x^2 + 1)
sage: S
    Univariate Quotient Polynomial Ring in xbar
    over Integer Ring with modulus x^2 + 1
    sage: loads(S.dumps()) == S
    True
    sage: loads(xbar.dumps()) == xbar
    True

    field_extension(names)
    
    Take a polynomial quotient ring, and return a tuple with three elements: the NumberField defined by
    the same polynomial quotient ring, a homomorphism from its parent to the NumberField sending the
    generators to one another, and the inverse isomorphism.
    
    OUTPUT:
    
    • field
    • homomorphism from self to field
    • homomorphism from field to self

    EXAMPLES:

    sage: # needs sage.rings.number_field
    sage: R.<x> = PolynomialRing(Rationals())
sage: S.<alpha> = R.quotient(x^3 - 2)
sage: F.<b>, f, g = S.field_extension()
sage: F
```
Number Field in $b$ with defining polynomial $x^3 - 2$

```
sage: a = F.gen()
sage: f(alpha)
b
sage: g(a)
alpha
```

Note that the parent ring must be an integral domain:

```
sage: R.<x> = GF(25, 'f25')[x]
# needs sage.rings.finite_rings
sage: S.<a> = R.quo(x^3 - 2)
# needs sage.rings.finite_rings
sage: F, g, h = S.field_extension('b')
# needs sage.rings.finite_rings
```

Traceback (most recent call last):
...
AttributeError: 'PolynomialQuotientRing_generic_with_category' object has no
attribute 'field_extension'...

Over a finite field, the corresponding field extension is not a number field:

```
sage: # needs sage.modules sage.rings.finite_rings
sage: R.<x> = GF(25, 'a')[x]
sage: S.<a> = R.quo(x^3 + 2*x + 1)
sage: F, g, h = S.field_extension('b')
sage: h(F.0^2 + 3)
a^2 + 3
sage: g(x^2 + 2)
b^2 + 2
```

We do an example involving a relative number field:

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQ['x']
sage: K.<a> = NumberField(x^3 - 2)
sage: S.<X> = K['X']
sage: f = (X+a)^3 + 2*(X+a) + 1
sage: f
```

We slightly change the example above so it works.

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQ['x']
sage: K.<a> = NumberField(x^3 - 2)
sage: S.<X> = K['X']
sage: f = (X+a)^3 + 2*(X+a) + 1
sage: f
```
\[ X^3 + 3*a*X^2 + (3*a^2 + 2)*X + 2*a + 3 \]

sage: Q.<z> = S.quo(f)
sage: F.<w>, g, h = Q.field_extension()
sage: c = g(z)
sage: f(c)
0
sage: h(g(z))
z
sage: g(h(w))
w

AUTHORS:

- Craig Citro (2006-08-07)
- William Stein (2006-08-06)

class sage.rings.polynomial.polynomial_quotient_ring.<br>PolynomialQuotientRing_field(ring, polynomial, name=None, category=None)

Bases: PolynomialQuotientRing_domain, Field

EXAMPLES:

sage: # needs sage.rings.number_field
sage: R.<x> = PolynomialRing(QQ)
sage: S.<xbar> = R.quotient(x^2 + 1)
sage: S
Univariate Quotient Polynomial Ring in xbar over Rational Field
    with modulus x^2 + 1
sage: loads(S.dumps()) == S
True
sage: loads(xbar.dumps()) == xbar
True

base_field()

Alias for base_ring(), when we're defined over a field.

complex_embeddings (prec=53)

Return all homomorphisms of this ring into the approximate complex field with precision prec.

EXAMPLES:

sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: f = x^5 + x + 17
sage: k = R.quotient(f)
sage: v = k.complex_embeddings(100)
sage: [phi(k.0^2) for phi in v]
[2.9757207403766761469671194565,
 -2.4088994371613850098316292196 + 1.9025410530350528612407363802*I,
 -2.4088994371613850098316292196 - 1.9025410530350528612407363802*I, ...]
Polynomials, Release 10.3

(continued from previous page)

0.92103906697304693634806949137 - 3.0755331188457794473265418086*I,
0.92103906697304693634806949137 + 3.0755331188457794473265418086*I]

class sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_generic(ring, polynomial, name=None, category=None)

Bases: QuotientRing_generic

Quotient of a univariate polynomial ring by an ideal.

EXAMPLES:

```python
code
sage: R.<x> = PolynomialRing(Integers(8)); R
Univariate Polynomial Ring in x over Ring of integers modulo 8
code
sage: S.<xbar> = R.quotient(x^2 + 1); S
Univariate Quotient Polynomial Ring in xbar over Ring of integers modulo 8
code
with modulus x^2 + 1
```

We demonstrate object persistence.

```python
code
sage: loads(S.dumps()) == S
True
code
sage: loads(xbar.dumps()) == xbar
True
```

We create some sample homomorphisms:

```python
code
sage: R.<x> = PolynomialRing(ZZ)
code
sage: S = R.quo(x^2 - 4)
code
sage: f = S.hom([2])
code
sage: f
Ring morphism:
  From: Univariate Quotient Polynomial Ring in xbar over Integer Ring
code
  with modulus x^2 - 4
code
  To:   Integer Ring
code
  Defn: xbar |--> 2
code
  sage: f(x)
2
code
  sage: f(x^2 - 4)
0
code
  sage: f(x^2)
4
```

Element

alias of PolynomialQuotientRingElement

S_class_group (S, proof=True)

If `self` is an étale algebra `D` over a number field `K` (i.e. a quotient of `K[x]` by a squarefree polynomial) and `S` is a finite set of places of `K`, return a list of generators of the `S`-class group of `D`.

NOTE:
Since the `ideal` function behaves differently over number fields than over polynomial quotient rings (the quotient does not even know its ring of integers), we return a set of pairs `(gen, order)`, where `gen` is a tuple of generators of an ideal $I$ and `order` is the order of $I$ in the $S$-class group.

**INPUT:**

- $S$ - a set of primes of the coefficient ring
- `proof` - if False, assume the GRH in computing the class group

**OUTPUT:**

A list of generators of the $S$-class group, in the form `(gen, order)`, where `gen` is a tuple of elements generating a fractional ideal $I$ and `order` is the order of $I$ in the $S$-class group.

**EXAMPLES:**

A trivial algebra over $\mathbb{Q}(\sqrt{-5})$ has the same class group as its base:

```
 sage: # needs sage.rings.number_field
 sage: K.<a> = QuadraticField(-5)
 sage: R.<x> = K[]
 sage: S.<xbar> = R.quotient(x)
 sage: S.S_class_group([])
```

When we include the prime $(2, -a + 1)$, the $S$-class group becomes trivial:

```
 sage: S.S_class_group([K.ideal(2, -a+1)])
```

Here is an example where the base and the extension both contribute to the class group:

```
 sage: # needs sage.rings.number_field
 sage: K.<a> = QuadraticField(-5)
 sage: R.<x> = K[]
 sage: S.<xbar> = R.quotient((x^2 + 23) * (x^2 + 31))
 sage: S.S_class_group([])
```

Now we take an example over a nontrivial base with two factors, each contributing to the class group:

```
 sage: # needs sage.rings.number_field
 sage: K.<a> = QuadraticField(-5)
 sage: R.<x> = K[]
 sage: S.<xbar> = R.quotient((x^2 + 23) * (x^2 + 31))
 sage: S.S_class_group([]) # not tested
```

(continues on next page)
By using the ideal \((a)\), we cut the part of the class group coming from \(x^2 + 31\) from 12 to 2, i.e. we lose a generator of order 6 (this was fixed in github issue #14489):

\[
\begin{align*}
&(-1/16*a*xbar^3 + (1/16*a + 1/8)*xbar^2 - 31/16*a*xbar + 31/16*a + 31/8), \\
&(1/8*a - 1/8)*xbar^2 + 23/8*a - 23/8, \\
&-1/16*xbar^3 - 1/16*xbar^2 - 23/16*xbar - 23/16, \\
&1/16*a*xbar^3 + (-1/16*a - 1/8)*xbar^2 + 23/16*a*xbar - 23/16*a - 23/8), \\
&(6), \\
&((-1/4*xbar^2 - 23/4, (1/8*a - 1/8)*xbar^2 + 23/8*a - 23/8, \\
&-1/16*xbar^3 - 1/16*xbar^2 - 23/16*xbar - 23/16, \\
&1/16*a*xbar^3 + (-1/16*a - 1/8)*xbar^2 + 23/16*a*xbar - 23/16*a - 23/8), \\
&(6), \\
&((-5/4*xbar^2 - 115/4, 1/4*a*xbar^2 + 23/4*a, \\
&-1/16*xbar^3 - 7/16*xbar^2 - 23/16*xbar - 161/16, \\
&1/16*a*xbar^3 - 1/16*a*xbar^2 + 23/16*a*xbar - 23/16*a), \\
&2)
\end{align*}
\]

Note that all the returned values live where we expect them to:

```python
sage: # needs sage.rings.number_field
sage: CG = S.S_class_group([])
sage: type(CG[0][0][1])
<class 'sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_generic_with_category.element_class'>
sage: type(CG[0][1])
<class 'sage.rings.integer.Integer'>
```

**S_units** \((S, proof=True)\)

If self is an étale algebra \(D\) over a number field \(K\) (i.e. a quotient of \(K[x]\) by a squarefree polynomial) and \(S\) is a finite set of places of \(K\), return a list of generators of the group of \(S\)-units of \(D\).

**INPUT:**

- \(S\) - a set of primes of the base field
- \(proof\) - if False, assume the GRH in computing the class group

**OUTPUT:**

A list of generators of the \(S\)-unit group, in the form \((\text{gen}, \text{order})\), where \text{gen} is a unit of order \text{order}.

**EXAMPLES:**

```python
sage: K.<a> = QuadraticField(-3)  # needs sage.rings.number_field
sage: K.unit_group()  # needs sage.rings.number_field
Unit group with structure C6 of Number Field in a with defining polynomial x^2 + 3
```

(continues on next page)
with defining polynomial \(x^2 + 3\) with \(a = 1.732050807568878\times i\)

```sage
# needs sage.rings.number_field
sage: x = polygen(ZZ, 'x')
sage: K.<a> = QQ['x'].quotient(x^2 + 3)
sage: u, o = K.S_units([[]])[0]; o
6
sage: 2*u - 1 in {a, -a}
True
sage: u^6
1
sage: u^3
-1
sage: 2*u^2 + 1 in {a, -a}
True

# needs sage.rings.number_field
sage: K.<a> = QuadraticField(-3)
sage: y = polygen(K)
sage: L.<b> = K[y].quotient(y^3 + 5); L
Univariate Quotient Polynomial Ring in b over Number Field in a
with defining polynomial \(x^2 + 3\) with \(a = 1.732050807568878\times i\)
with modulus \(y^3 + 5\)
sage: [u for u, o in L.S_units([]) if o is Infinity]
[(-1/3*a - 1)*b^2 - 4/3*a*b - 5/6*a + 7/2,
  2/3*a*b^2 + (2/3*a - 2)*b - 5/6*a - 7/2]
sage: [u for u, o in L.S_units([K.ideal(1/2*a - 3/2)])]
....: if o is Infinity]
[(-1/6*a - 1/2)*b^2 + (1/3*a - 1)*b + 4/3*a,
  -1/3*a - 1)*b^2 - 4/3*a*b - 5/6*a + 7/2,
  2/3*a*b^2 + (2/3*a - 2)*b - 5/6*a - 7/2]
sage: [u for u, o in L.S_units([K.ideal(2)]) if o is Infinity]
[(1/2*a - 1/2)*b^2 + (a + 1)*b + 3,
  (1/6*a + 1/2)*b^2 + (-1/3*a + 1)*b - 5/6*a + 1/2,
  (1/6*a + 1/2)*b^2 + (-1/3*a + 1)*b - 5/6*a - 1/2,
  -1/3*a - 1)*b^2 - 4/3*a*b - 5/6*a + 7/2,
  2/3*a*b^2 + (2/3*a - 2)*b - 5/6*a - 7/2]
```

Note that all the returned values live where we expect them to:

```sage
# needs sage.rings.number_field
sage: U = L.S_units([])
sage: type(U[0][0])
<class 'sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_field_with_category.element_class'>
sage: type(U[0][1])
<class 'sage.rings.integer.Integer'>
sage: type(U[1][0])
<class 'sage.rings.infinity.PlusInfinity'>
```

**ambient()**

**base_ring()**

Return the base ring of the polynomial ring, of which this ring is a quotient.

**EXAMPLES:**

The base ring of \(\mathbb{Z}[z]/(z^3 + z^2 + z + 1)\) is \(\mathbb{Z}\).
Next we make a polynomial quotient ring over $S$ and ask for its base ring.

```
sage: T.<t> = PolynomialRing(S)
sage: W = T.quotient(t^99 + 99)
sage: W.base_ring()
Univariate Quotient Polynomial Ring in beta
   over Integer Ring with modulus z^3 + z^2 + z + 1
```

### cardinality()

Return the number of elements of this quotient ring.

order is an alias of cardinality.

**EXAMPLES:**

```
sage: R.<x> = ZZ[]
sage: R.quo(1).cardinality()
1
sage: R.quo(x^3 - 2).cardinality()
+Infinity
sage: R.quo(1).order()
1
sage: R.quo(x^3 - 2).order()
+Infinity
```

```
sage: # needs sage.rings.finite_rings
sage: R.<x> = GF(9, 'a')[]
sage: R.quo(2*x^3 + x + 1).cardinality()
729
sage: GF(9, 'a').extension(2*x^3 + x + 1).cardinality()
729
sage: R.quo(2).cardinality()
1
```

### characteristic()

Return the characteristic of this quotient ring.

This is always the same as the characteristic of the base ring.

**EXAMPLES:**

```
sage: R.<z> = PolynomialRing(ZZ)
sage: S.<a> = R.quo(z - 19)
sage: S.characteristic()
0
sage: R.<x> = PolynomialRing(GF(9, 'a'))   # needs sage.rings.finite_rings
sage: S = R.quotient(x^3 + 1)              # needs sage.rings.finite_rings
sage: S.characteristic()                  # needs sage.rings.finite_rings
3
```
class_group (proof=True)

If self is a quotient ring of a polynomial ring over a number field $K$, by a polynomial of nonzero discriminant, return a list of generators of the class group.

NOTE:

Since the ideal function behaves differently over number fields than over polynomial quotient rings (the quotient does not even know its ring of integers), we return a set of pairs (gen, order), where gen is a tuple of generators of an ideal $I$ and order is the order of $I$ in the class group.

INPUT:

- proof - if False, assume the GRH in computing the class group

OUTPUT:

A list of pairs (gen, order), where gen is a tuple of elements generating a fractional ideal and order is the order of $I$ in the class group.

EXAMPLES:

```python
sage: # needs sage.rings.number_field
sage: K.<a> = QuadraticField(-3)
sage: K.class_group()
Class group of order 1 of Number Field in a

sage: x = polygen(QQ, 'x')
sage: K.<a> = QQ['x'].quotient(x^2 + 3)
sage: K.class_group()
[]
```

A trivial algebra over $\mathbb{Q}(\sqrt{-5})$ has the same class group as its base:

```python
sage: # needs sage.rings.number_field
sage: K.<a> = QuadraticField(-5)
sage: R.<x> = K[]
sage: S.<xbar> = R.quotient(x)
sage: S.class_group()
[((2, -a + 1), 2)]
```

The same algebra constructed in a different way:

```python
sage: x = polygen(ZZ, 'x')
sage: K.<a> = QQ['x'].quotient(x^2 + 5)
sage: K.class_group() # needs sage.rings.number_field
[((2, a + 1), 2)]
```

Here is an example where the base and the extension both contribute to the class group:

```python
sage: # needs sage.rings.number_field
sage: K.<a> = QuadraticField(-5)
sage: R.<x> = K[]
sage: S.<xbar> = R.quotient(x^2 + 23)
sage: S.class_group()[(2, -a + 1, 1/2*xbar + 1/2, -1/2*a*xbar + 1/2*a + 1), 6]
```

Here is an example of a product of number fields, both of which contribute to the class group:
Polynomials, Release 10.3

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: S.<xbar> = R.quotient((x^2 + 23) * (x^2 + 47))
sage: S.class_group()
([(1/12*xbar^2 + 47/12,
  1/48*xbar^3 - 1/48*xbar^2 + 47/48*xbar - 47/48),
  3),
  ((-1/12*xbar^2 - 23/12,
   -1/48*xbar^3 - 1/48*xbar^2 - 23/48*xbar - 23/48),
   5))
```

Now we take an example over a nontrivial base with two factors, each contributing to the class group:

```
sage: # needs sage.rings.number_field
sage: K.<a> = QuadraticField(-5)
sage: R.<x> = K[]
sage: S.<xbar> = R.quotient((x^2 + 23) * (x^2 + 31))
sage: S.class_group() # not tested
([(1/4*xbar^2 + 31/4,
  (-1/8*a + 1/8)*xbar^2 - 31/8*a + 31/8,
  1/16*xbar^3 + 1/16*xbar^2 + 31/16*xbar + 31/16,
  -1/16*a*xbar^3 + (1/16*a + 1/8)*xbar^2 - 31/16*a*xbar + 31/16*a + 31/8),
  6),
  ((-1/4*xbar^2 - 23/4,
   (1/8*a - 1/8)*xbar^2 + 23/8*a - 23/8,
   -1/16*xbar^3 - 1/16*xbar^2 - 23/16*xbar - 23/16,
   1/16*a*xbar^3 + (-1/16*a - 1/8)*xbar^2 + 23/16*a*xbar - 23/16*a - 23/8),
   6),
  ((-5/4*xbar^2 - 115/4,
   1/4*a*xbar^2 + 23/4*a,
   -1/16*xbar^3 - 7/16*xbar^2 - 23/16*xbar - 161/16,
   1/16*a*xbar^3 - 1/16*a*xbar^2 + 23/16*a*xbar - 23/16*a),
   2))
```

Note that all the returned values live where we expect them to:

```
sage: # needs sage.rings.number_field
sage: CG = S.class_group()
sage: type(CG[0][0][1])
<class 'sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_generic_with_category.element_class'>
sage: type(CG[0][1])
<class 'sage.rings.integer.Integer'>
```

**construction**

Functorial construction of self

**EXAMPLES:**

```
sage: P.<t> = ZZ[]
sage: Q = P.quo(5 + t^2)
sage: F, R = Q.construction()
sage: F(R) == Q
True
sage: P.<t> = GF(3)[]
sage: Q = P.quo([2 + t^2])
sage: F, R = Q.construction()
```

(continues on next page)
AUTHOR:
– Simon King (2010-05)

**cover_ring()**
Return the polynomial ring of which this ring is the quotient.

**degree()**
Return the degree of this quotient ring. The degree is the degree of the polynomial that we quotiented out by.

**discriminant**(v=None)
Return the discriminant of this ring over the base ring. This is by definition the discriminant of the polynomial that we quotiented out by.

The discriminant of the quotient polynomial ring need not equal the discriminant of the corresponding number field, since the discriminant of a number field is by definition the discriminant of the ring of integers of the number field:

**gen**(n=0)
Return the generator of this quotient ring. This is the equivalence class of the image of the generator of the polynomial ring.
**is_field**(proof=True)
Return whether or not this quotient ring is a field.

**EXAMPLES:**

```
sage: R.<z> = PolynomialRing(ZZ)
sage: S = R.quo(z^2 - 2)
sage: S.is_field()  
False
sage: R.<x> = PolynomialRing(QQ)
sage: S = R.quotient(x^2 - 2)
sage: S.is_field()  
True
```

If proof is True, requires the is_irreducible method of the modulus to be implemented:

```
sage: # needs sage.rings.padics
sage: R1.<x> = Qp(2)[]
sage: F1 = R1.quotient_ring(x^2 + x + 1)
sage: R2.<x> = F1[]
sage: F2 = R2.quotient_ring(x^2 + x + 1)
sage: F2.is_field()  
Traceback (most recent call last):
  ...  
NotImplementedError: cannot rewrite Univariate Quotient Polynomial Ring in xbar over 2-adic Field with capped relative precision 20 with modulus (1 + O(2^20))*x^2 + (1 + O(2^20))*x + 1 + O(2^20) as an isomorphic ring
sage: F2.is_field(proof = False)  
False
```

**is_finite()**
Return whether or not this quotient ring is finite.

**EXAMPLES:**

```
sage: R.<x> = ZZ[]
sage: R.quo(1).is_finite()  
True
sage: R.quo(x^3 - 2).is_finite()  
False
sage: R.<x> = GF(9, 'a')[]  
# needs sage.rings.finite_rings
sage: R.quo(2*x^3 + x + 1).is_finite()  
True
sage: R.quo(2).is_finite()  
True
```

---

Chapter 2. Univariate Polynomials
is_integral_domain(proof=True)

Return whether or not this quotient ring is an integral domain.

EXAMPLES:

```sage
sage: R.<z> = PolynomialRing(ZZ)
sage: S = R.quotient(z^2 - z)
sage: S.is_integral_domain()  # False
sage: T = R.quotient(z^2 + 1)
sage: T.is_integral_domain()  # True
sage: U = R.quotient(-1)
sage: U.is_integral_domain()  # False

sage: # needs sage.libs.singular
sage: R2.<y> = PolynomialRing(R)
sage: S2 = R2.quotient(z^2 - y^3)
sage: S2.is_integral_domain()  # True
sage: S3 = R2.quotient(z^2 - 2*y*z + y^2)
sage: S3.is_integral_domain()  # False
sage: R.<z> = PolynomialRing(ZZ.quotient(4))
sage: S = R.quotient(z - 1)
sage: S.is_integral_domain()  # False
```

krull_dimension()

Return the Krull dimension.

This is the Krull dimension of the base ring, unless the quotient is zero.

EXAMPLES:

```sage
sage: x = polygen(ZZ, 'x')
sage: R = PolynomialRing(ZZ, 'x').quotient(x^6 - 1)
sage: R.krull_dimension()  # 1
sage: R = PolynomialRing(ZZ, 'x').quotient(1)
sage: R.krull_dimension()  # -1
```

lift(x)

Return an element of the ambient ring mapping to the given argument.

EXAMPLES:

```sage
sage: P.<x> = QQ[]
sage: Q = P.quotient(x^2 + 2)
sage: Q.lift(Q.0^3)
```

(continues on next page)
Polynomials, Release 10.3

-2*x
\texttt{sage: Q(-2*x)}
-2*xbar
\texttt{sage: Q.0^3}
-2*xbar

\textbf{modulus()}

Return the polynomial modulus of this quotient ring.

\textbf{EXAMPLES:}

\texttt{sage: R.<x> = PolynomialRing(GF(3))}
\texttt{sage: S = R.quotient(x^2 - 2)}
\texttt{sage: S.modulus()}
\texttt{x^2 + 1}

\textbf{ngens()}

Return the number of generators of this quotient ring over the base ring. This function always returns 1.

\textbf{EXAMPLES:}

\texttt{sage: R.<x> = PolynomialRing(QQ)}
\texttt{sage: S.<y> = PolynomialRing(R)}
\texttt{sage: T.<z> = S.quotient(y + x)}
\texttt{sage: T}
\texttt{Univariate Quotient Polynomial Ring in z over}
\texttt{Univariate Polynomial Ring in x over Rational Field with modulus y + x}
\texttt{sage: T.ngens()}
\texttt{1}

\textbf{number_field()}

Return the number field isomorphic to this quotient polynomial ring, if possible.

\textbf{EXAMPLES:}

\texttt{sage: # needs sage.rings.number_field}
\texttt{sage: R.<x> = PolynomialRing(QQ)}
\texttt{sage: S.<alpha> = R.quotient(x^29 - 17*x - 1)}
\texttt{sage: K = S.number_field(); K}
\texttt{Number Field in alpha with defining polynomial x^29 - 17*x - 1}
\texttt{sage: alpha = K.gen()}
\texttt{sage: alpha^29}
\texttt{17*alpha + 1}

\textbf{order()}

Return the number of elements of this quotient ring.

\textbf{order} is an alias of \textbf{cardinality}.

\textbf{EXAMPLES:}

\texttt{sage: R.<x> = ZZ[]} 
\texttt{sage: R.quo(1).cardinality()}
\texttt{1}
\texttt{sage: R.quo(x^3 - 2).cardinality()}
\texttt{+Infinity}

(continues on next page)
polynomial_ring()

Return the polynomial ring of which this ring is the quotient.

EXAMPLES:

```python
sage: R.<x> = PolynomialRing(QQ)
sage: S = R.quotient(x^2 - 2)
sage: S.polynomial_ring()
Univariate Polynomial Ring in x over Rational Field
```

random_element\(\text{(degree=}\text{None,} \ *args, \ **kwds)\)

Return a random element of this quotient ring.

INPUT:

• degree - Optional argument: either an integer for fixing the degree, or a tuple of the minimum and maximum degree. By default the degree is \(n - 1\) with \(n\) the degree of the polynomial ring. Note that the degree of the polynomial is fixed before the modulo calculation. So when \(\text{degree}\) is bigger than the degree of the polynomial ring, the degree of the returned polynomial would be lower than \(\text{degree}\).

• *args, **kwds - Arguments for randomization that are passed on to the random_element method of the polynomial ring, and from there to the base ring

OUTPUT:

Element of this quotient ring

EXAMPLES:

```python
sage: # needs sage.modules sage.rings.finite_rings
sage: F1.<a> = GF(2^7)
sage: P1.<x> = F1[]
sage: F2 = F1.extension(x^2 + x + 1, 'u')
sage: F2.random_element().parent() is F2
True
```

retract\(\text{(x)}\)

Return the coercion of \(x\) into this polynomial quotient ring.

The rings that coerce into the quotient ring canonically are:

• this ring

• any canonically isomorphic ring

• anything that coerces into the ring of which this is the quotient
selmer_generators \((S, m, \text{proof=True})\)

If self is an étale algebra \(D\) over a number field \(K\) (i.e. a quotient of \(K[x]\) by a squarefree polynomial) and \(S\) is a finite set of places of \(K\), compute the Selmer group \(D(S, m)\). This is the subgroup of \(D^*/(D^*)^m\) consisting of elements \(a\) such that \(D(\sqrt{a})/D\) is unramified at all primes of \(D\) lying above a place outside of \(S\).

INPUT:

- \(S\) - A set of primes of the coefficient ring (which is a number field).
- \(m\) - a positive integer
- \(\text{proof}\) - if False, assume the GRH in computing the class group

OUTPUT:
A list of generators of \(D(S, m)\).

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: K.<a> = QuadraticField(-5)
sage: R.<x> = K[]
sage: D.<T> = R.quotient(x)
sage: D.selmer_generators((), 2)
[-1, 2]
sage: D.selmer_generators((K.ideal(2, -a + 1)), 2)
[2, -1]
sage: D.selmer_generators((K.ideal(2, -a + 1), K.ideal(3, a + 1)), 2)
[2, a + 1, -1]
sage: D.selmer_generators((K.ideal(2, -a + 1), K.ideal(3, a + 1)), 4)
[2, a + 1, -1]
sage: D.selmer_generators((K.ideal(2, -a + 1)), 3)
[2]
sage: D.selmer_generators((K.ideal(2, -a + 1), K.ideal(3, a + 1)), 3)
[2, a + 1]
sage: D.selmer_generators((K.ideal(2, -a + 1),
    K.ideal(3, a + 1),
    K.ideal(a)), 3)
[2, a + 1, -a]
```

selmer_group \((S, m, \text{proof=True})\)

If self is an étale algebra \(D\) over a number field \(K\) (i.e. a quotient of \(K[x]\) by a squarefree polynomial) and \(S\) is a finite set of places of \(K\), compute the Selmer group \(D(S, m)\). This is the subgroup of \(D^*/(D^*)^m\) consisting of elements \(a\) such that \(D(\sqrt{a})/D\) is unramified at all primes of \(D\) lying above a place outside of \(S\).

INPUT:

- \(S\) - A set of primes of the coefficient ring (which is a number field).
- \(m\) - a positive integer
- \(\text{proof}\) - if False, assume the GRH in computing the class group

OUTPUT:
A list of generators of \(D(S, m)\).

EXAMPLES:
units (proof=True)

If this quotient ring is over a number field K, by a polynomial of nonzero discriminant, returns a list of generators of the units.

INPUT:

• proof - if False, assume the GRH in computing the class group

OUTPUT:

A list of generators of the unit group, in the form (gen, order), where gen is a unit of order order.

EXAMPLES:

```python
sage: K.<a> = QuadraticField(-3)  # needs sage.rings.number_field
sage: K.unit_group()  # needs sage.rings.number_field
Unit group with structure C6 of
Number Field in a with defining polynomial x^2 + 3 with a = 1.
7320508075688787

sage: # needs sage.rings.number_field
sage: x = polygen(ZZ, 'x')
sage: K.<a> = QQ['x'].quotient(x^2 + 3)
sage: u = K.units()[0][0]
True
sage: u^6
1
sage: u^3
-a
sage: 2*u^2 + 1 in {a, -a}
True
```

2.1. Univariate Polynomials and Polynomial Rings
Polynomials, Release 10.3

```python
sage: # needs sage.rings.number_field
sage: K.<a> = QuadraticField(-3)
sage: y = polygen(K)
sage: L.<b> = K[y].quotient(y^3 + 5); L
Univariate Quotient Polynomial Ring in b over Number Field in a with defining polynomial x^2 + 3 with a = 1.732050807568878?*I with modulus y^3 + 5
sage: [u for u, o in L.units() if o is Infinity]
[(-1/3*a - 1)*b^2 - 4/3*a*b - 5/6*a + 7/2, 2/3*a*b^2 + (2/3*a - 2)*b - 5/6*a - 7/2]
sage: L.<b> = K.extension(y^3 + 5)
sage: L.unit_group()
Unit group with structure C6 x 2 x 2 of Number Field in b with defining polynomial x^3 + 5 over its base field
sage: L.unit_group().gens()  # abstract generators
(u0, u1, u2)
sage: L.unit_group().gens_values()[1:]
[(-1/3*a - 1)*b^2 - 4/3*a*b - 5/6*a + 7/2, 2/3*a*b^2 + (2/3*a - 2)*b - 5/6*a - 7/2]
```

Note that all the returned values live where we expect them to:

```python
sage: # needs sage.rings.number_field
sage: L.<b> = K[y].quotient(y^3 + 5)
sage: U = L.units()
sage: type(U[0][0])
<class 'sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_field_with_category.element_class'>
sage: type(U[0][1])
<class 'sage.rings.integer.Integer'>
sage: type(U[1][1])
<class 'sage.rings.infinity.PlusInfinity'>
```

```python
sage: sage.rings.polynomial.polynomial_quotient_ring.is_PolynomialQuotientRing(x)
```

### 2.1.23 Elements of Quotients of Univariate Polynomial Rings

**EXAMPLES:** We create a quotient of a univariate polynomial ring over \( \mathbb{Z} \).

```python
sage: R.<x> = ZZ[]
sage: S.<a> = R.quotient(x^3 + 3*x - 1)
sage: 2 * a^3
-6*a + 2
```

Next we make a univariate polynomial ring over \( \mathbb{Z}[x]/(x^3 + 3x - 1) \).

```python
sage: S1.<y> = S[]
```

And, we quotient out that by \( y^2 + a \).

```python
sage: T.<z> = S1.quotient(y^2 + a)
```

In the quotient \( z^2 \) is \(-a\).

```python
sage: z^2
-a
```

254 Chapter 2. Univariate Polynomials
And since $a^3 = -3x + 1$, we have:

```plaintext
sage: z^6
3*a - 1
```

```plaintext
sage: R.<x> = PolynomialRing(Integers(9))
sage: S.<a> = R.quotient(x^4 + 2*x^3 + x + 2)
sage: a^100
7*a^3 + 8*a + 7
```

```plaintext
sage: R.<x> = PolynomialRing(QQ)
sage: S.<a> = R.quotient(x^3 - 2)
sage: a

```

```plaintext
AUTHORS:
• William Stein
```

```plaintext
class sage.rings.polynomial.polynomial_quotient_ring_element.PolynomialQuotientRingElement

Bases: Polynomial_singular_repr, CommutativeRingElement

Element of a quotient of a polynomial ring.

EXAMPLES:
```
```

```plaintext
charpoly(var)

The characteristic polynomial of this element, which is by definition the characteristic polynomial of right
multiplication by this element.

INPUT:

- var - string - the variable name

EXAMPLES:

```python
sage: R.<x> = PolynomialRing(QQ)
sage: S.<a> = R.quo(x^3 - 389*x^2 + 2*x - 5)
sage: a.charpoly('X') # needs sage.modules
X^3 - 389*X^2 + 2*X - 5
```

**fcp** *(var='x')*

Return the factorization of the characteristic polynomial of this element.

EXAMPLES:

```python
sage: R.<x> = PolynomialRing(QQ)
sage: S.<a> = R.quotient(x^3 - 389*x^2 + 2*x - 5)
sage: a.fcp('x') # needs sage.modules
x^3 - 389*x^2 + 2*x - 5
sage: S(1).fcp(y) # needs sage.modules
(y - 1)^3
```

**field_extension** *(names)*

Given a polynomial with base ring a quotient ring, return a 3-tuple: a number field defined by the same polynomial, a homomorphism from its parent to the number field sending the generators to one another, and the inverse isomorphism.

INPUT:

- names - name of generator of output field

OUTPUT:

- field
- homomorphism from self to field
- homomorphism from field to self

EXAMPLES:

```python
sage: # needs sage.rings.number_field
sage: R.<x> = PolynomialRing(QQ)
sage: S.<alpha> = R.quotient(x^3 - 2)
sage: F.<a>, f, g = alpha.field_extension()
sage: F
Number Field in a with defining polynomial x^3 - 2
sage: a = F.gen()
sage: f(alpha)
a
sage: g(a)
alpha
```

Over a finite field, the corresponding field extension is not a number field:
We do an example involving a relative number field:

```sage
# needs sage.rings.number_field
sage: R.<x> = QQ['x']
sage: K.<a> = NumberField(x^3 - 2)
sage: S.<X> = K['X']
sage: f = (X+a)^3 + 2*(X+a) + 1
sage: f
X^3 + 3*a*X^2 + (3*a^2 + 2)*X + 2*a + 3
sage: Q.<z> = S.quotient(f)
sage: F, g, h = z.field_extension()
sage: c = g(z)
sage: f(c)
0
sage: h(g(z))
z
sage: g(h(w))
w
```

AUTHORS:
- Craig Citro (2006-08-06)
- William Stein (2006-08-06)

```python
is_unit()
```
Return True if self is invertible.

**Warning:** Only implemented when the base ring is a field.

**EXAMPLES:**

```sage
sage: R.<x> = QQ[]
sage: S.<y> = R.quotient(x^2 + 2*x + 1)
sage: (2*y).is_unit()
True
sage: (y + 1).is_unit()
False
```
lift()

Return lift of this polynomial quotient ring element to the unique equivalent polynomial of degree less than
the modulus.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: S.<a> = R.quotient(x^3 - 2)
sage: b = a^2 - 3
sage: b
a^2 - 3
sage: b.lift()
x^2 - 3
```

list(
copy=True)

Return list of the elements of self, of length the same as the degree of the quotient polynomial ring.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: S.<a> = R.quotient(x^3 + 2*x - 5)
sage: a^10
-134*a^2 - 35*a + 300
sage: (a^10).list()
[300, -35, -134]
```

matrix()

The matrix of right multiplication by this element on the power basis for the quotient ring.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: S.<a> = R.quotient(x^3 + 2*x - 5)
sage: a.matrix()
[ 0 1 0]
[ 0 0 1]
[ 5 -2 0]
```

minpoly()

The minimal polynomial of this element, which is by definition the minimal polynomial of the matrix() of this element.

ALGORITHM: Use minpoly_mod() if possible, otherwise compute the minimal polynomial of the matrix().

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: S.<a> = R.quotient(x^3 + 2*x - 5)
sage: (a + 123).minpoly()
#_x^3 - 369*x^2 + 45389*x - 1861118
```
```python
sage: # needs sage.rings.finite_rings
sage: F2.<i> = GF((431,2), modulus=[1,0,1])
# needs sage.modules
sage: F6.<u> = F2.extension(3)
# needs sage.modules
sage: (u + 1).minpoly()  # indirect doctest
x^6 + 425*x^5 + 19*x^4 + 125*x^3 + 189*x^2 + 239*x + 302
```

```python
sage: ext = F6.over(F2)  # needs sage.modules
# random
sage: ext(u + 1).minpoly()  # indirect doctest
x^3 + (396*i + 428)*x^2 + (80*i + 39)*x + 9*i + 178
```

### norm()

The norm of this element, which is the determinant of the matrix of right multiplication by this element.

**EXAMPLES:**

```python
sage: R.<x> = PolynomialRing(QQ)
sage: S.<a> = R.quotient(x^3 - 389*x^2 + 2*x - 5)
sage: a.norm()  # needs sage.modules
5
```

### rational_reconstruction(*args, **kwargs)

Compute a rational reconstruction of this polynomial quotient ring element to its cover ring.

This method is a thin convenience wrapper around `Polynomial.rational_reconstruction()`.

**EXAMPLES:**

```python
sage: # needs sage.rings.finite_rings
sage: R.<x> = GF(65537)[[]]
```

```python
sage: m = (x^11 + 25345*x^10 + 10956*x^9 + 13873*x^8 + 23962*x^7
.....: + 17496*x^6 + 30348*x^5 + 7440*x^4 + 65438*x^3 + 7676*x^2
.....: + 54266*x + 47805)
```

```python
sage: f = (20437*x^10 + 62630*x^9 + 63241*x^8 + 12820*x^7 + 42171*x^6
.....: + 63091*x^5 + 15288*x^4 + 32516*x^3 + 2181*x^2 + 45236*x + 2447)
```

```python
sage: f_mod_m = R.quotient(m)(f)
```

```python
sage: f_mod_m.rational_reconstruction()  # random
(51388*x^5 + 29141*x^4 + 59341*x^3 + 7034*x^2 + 14152*x + 23746,
x^5 + 15208*x^4 + 19504*x^3 + 20457*x^2 + 11180*x + 28352)
```

### trace()

The trace of this element, which is the trace of the matrix of right multiplication by this element.

**EXAMPLES:**

```python
sage: R.<x> = PolynomialRing(QQ)
sage: S.<a> = R.quotient(x^3 - 389*x^2 + 2*x - 5)
sage: a.trace()  # needs sage.modules
389
```

---

**2.1. Univariate Polynomials and Polynomial Rings**

259
2.1.24 Polynomial Compilers

AUTHORS:

• Tom Boothby, initial design & implementation
• Robert Bradshaw, bug fixes / suggested & assisted with significant design improvements

```python
class sage.rings.polynomial.polynomial_compiled.CompiledPolynomialFunction
    Bases: object

Builds a reasonably optimized directed acyclic graph representation for a given polynomial. A CompiledPolynomialFunction is callable from python, though it is a little faster to call the eval function from pyrex.

This class is not intended to be called by a user, rather, it is intended to improve the performance of immutable polynomial objects.
```

Todo:

• Recursive calling
• Faster casting of coefficients / argument
• Multivariate polynomials
• Cython implementation of Pippenger’s Algorithm that doesn’t depend heavily upon dicts.
• Computation of parameter sequence suggested by Pippenger
• Univariate exponentiation can use Brauer’s method to improve extremely sparse polynomials of very high degree

```python
class sage.rings.polynomial.polynomial_compiled.abc_pd
    Bases: binary_pd

class sage.rings.polynomial.polynomial_compiled.add_pd
    Bases: binary_pd

class sage.rings.polynomial.polynomial_compiled.binary_pd
    Bases: generic_pd

class sage.rings.polynomial.polynomial_compiled.coeff_pd
    Bases: generic_pd

class sage.rings.polynomial.polynomial_compiled.dummy_pd
    Bases: generic_pd

class sage.rings.polynomial.polynomial_compiled.generic_pd
    Bases: object

class sage.rings.polynomial.polynomial_compiled.mul_pd
    Bases: binary_pd

class sage.rings.polynomial.polynomial_compiled.pow_pd
    Bases: unary_pd

class sage.rings.polynomial.polynomial_compiled.sqr_pd
    Bases: unary_pd
```
2.1.25 Polynomial multiplication by Kronecker substitution

2.1.26 Integer-valued polynomial rings

AUTHORS:
• Frédéric Chapoton (2023-03): Initial version

class sage.rings.polynomial.integer_valued_polynomials.IntegerValuedPolynomialRing(R)
    Bases: UniqueRepresentation, Parent

The integer-valued polynomial ring over a base ring $R$.

Integer-valued polynomial rings are commutative and associative algebras, with a basis indexed by non-negative integers.

There are two natural bases, made of the sequence $\binom{x^n}{n}$ for $n \geq 0$ (the binomial basis) and of the other sequence $\binom{x^n + n}{n}$ for $n \geq 0$ (the shifted basis).

These two bases are available as follows:

```
sage: B = IntegerValuedPolynomialRing(QQ).Binomial()
sage: S = IntegerValuedPolynomialRing(QQ).Shifted()
```

or by using the shortcuts:

```
sage: B = IntegerValuedPolynomialRing(QQ).B()
sage: S = IntegerValuedPolynomialRing(QQ).S()
```

There is a conversion formula between the two bases:

\[
\binom{x^i}{i} = \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \binom{x+k}{k}
\]

with inverse:

\[
\binom{x+i}{i} = \sum_{k=0}^{i} \binom{i}{k} \binom{x}{k}.
\]

REFERENCES:
• Wikipedia article Integer-valued polynomial

B
alias of Binomial

class Bases (parent_with_realization)
    Bases: Category_realization_of_parent
class ElementMethods

    Bases: object

    content()

        Return the content of self.

        This is the gcd of the coefficients.

        EXAMPLES:

        sage: F = IntegerValuedPolynomialRing(ZZ).S()
        sage: B = F.basis()
        sage: (3*B[4]+6*B[7]).content()
        3

    polynomial()

        Convert to a polynomial in x.

        EXAMPLES:

        sage: F = IntegerValuedPolynomialRing(ZZ).S()
        sage: B = F.gen()
        sage: (B+1).polynomial()
        x + 2
        sage: F = IntegerValuedPolynomialRing(ZZ).B()
        sage: B = F.gen()
        sage: (B+1).polynomial()
        x + 1

    shift(j=1)

        Shift all indices by j.

        INPUT:

        • j – integer (default: 1)

        In the binomial basis, the shift by 1 corresponds to a summation operator from 0 to x.

        EXAMPLES:

        sage: F = IntegerValuedPolynomialRing(ZZ).B()
        sage: B = F.gen()
        sage: (B+1).shift()
        sage: (B+1).shift(3)

    sum_of_coefficients()

        Return the sum of coefficients.

        In the shifted basis, this is the evaluation at x = 0.

        EXAMPLES:

        sage: F = IntegerValuedPolynomialRing(ZZ).S()
        sage: B = F.basis()
        sage: (B[2]*B[4]).sum_of_coefficients()
        1
class ParentMethods

Bases: object

algebra_generators()

Return the generators of this algebra.

EXAMPLES:

```sage
A = IntegerValuedPolynomialRing(ZZ).S(); A
Integer-Valued Polynomial Ring over Integer Ring
in the shifted basis
sage: A.algebra_generators()
Family (S[1],)
```

degree_on_basis(m)

Return the degree of the basis element indexed by m.

EXAMPLES:

```sage
A = IntegerValuedPolynomialRing(QQ).S()
sage: A.degree_on_basis(4)
4
```

from_polynomial(p)

Convert a polynomial into the ring of integer-valued polynomials.

This raises a `ValueError` if this is not possible.

INPUT:

- p – a polynomial in one variable

EXAMPLES:

```sage
A = IntegerValuedPolynomialRing(ZZ).S()
sage: S = A.basis()
sage: S[5].polynomial()
1/120*x^5 + 1/8*x^4 + 17/24*x^3 + 15/8*x^2 + 137/60*x + 1
sage: A.from_polynomial(_)
S[5]
sage: x = polygen(QQ, 'x')
sage: A.from_polynomial(x)
-S[0] + S[1]
```

gen(i=0)

Return the generator of this algebra.

The optional argument is ignored.

EXAMPLES:
.. code::

   sage: F = IntegerValuedPolynomialRing(ZZ).B()
   sage: F.gen()
   B[1]

**gens()**

Return the generators of this algebra.

**EXAMPLES:**

.. code::

   sage: A = IntegerValuedPolynomialRing(ZZ).S(); A
   Integer-Valued Polynomial Ring over Integer Ring
   in the shifted basis
   sage: A.algebra_generators()
   Family (S[1],)

**one_basis()**

Return the number 0, which index the unit of this algebra.

**EXAMPLES:**

.. code::

   sage: A = IntegerValuedPolynomialRing(QQ).S()
   sage: A.one_basis()
   0
   sage: A.one()
   S[0]

**super_categories()**

Return the super-categories of self.

**EXAMPLES:**

.. code::

   sage: A = IntegerValuedPolynomialRing(QQ); A
   Integer-Valued Polynomial Ring over Rational Field
   sage: C = A.Bases(); C
   Category of bases of Integer-Valued Polynomial Ring over Rational Field
   sage: C.super_categories()
   [Category of realizations of Integer-Valued Polynomial Ring over Rational Field,
    Join of Category of algebras with basis over Rational Field and
    Category of filtered algebras over Rational Field and
    Category of commutative algebras over Rational Field and
    Category of realizations of unital magmas]

**class** Binomial(A)

**Bases:** CombinatorialFreeModule, BindableClass

The integer-valued polynomial ring in the binomial basis.

The basis used here is given by \( B[i] = {i \choose j} \) for \( i \in \mathbb{N} \).

Assuming \( n_1 \leq n_2 \), the product of two monomials \( B[n_1] \cdot B[n_2] \) is given by the sum

\[
\sum_{k=0}^{n_1} \binom{n_1}{k} \binom{n_1 + n_2 - k}{n_1} B[n_1 + n_2 - k].
\]

The product of two monomials is therefore a positive linear combination of monomials.

**EXAMPLES:**
Integer-valued polynomial rings commute with their base ring:

```
sage: K = IntegerValuedPolynomialRing(QQ).B()
sage: a = K.gen()
sage: K.is_commutative()
True
sage: L = IntegerValuedPolynomialRing(K).B()
sage: c = L.gen()
sage: L.is_commutative()
True
sage: s = a * c^3; s
sage: parent(s)
Integer-Valued Polynomial Ring over Integer-Valued Polynomial Ring over Rational Field in the binomial basis in the binomial basis
```

Integer-valued polynomial rings are commutative:

```
sage: c^3 * a == c * a * c * c
True
```

We can also manipulate elements in the basis:

```
sage: F = IntegerValuedPolynomialRing(QQ).B()
sage: B = F.basis()
```

and coerce elements from our base field:

```
sage: F(4/3)
4/3*B[0]
```

```python
class Element
    Bases: IndexedFreeModuleElement
```
variable_shift \((k=1)\)

Return the image by the shift of variables.

On polynomials, the action is the shift on variables \(x \mapsto x + k\).

INPUT:

• \(k\) – integer (default: 1)

EXAMPLES:

\[
\begin{align*}
\text{sage: } & \text{A = IntegerValuedPolynomialRing(ZZ).B() } \\
\text{sage: } & \text{B = A.basis() } \\
\text{sage: } & \text{B[5].variable_shift()} \\
\text{sage: } & \text{B[5].variable_shift(-1)} \\
\end{align*}
\]

product_on_basis \((n₁, n₂)\)

Return the product of basis elements \(n₁\) and \(n₂\).

INPUT:

• \(n₁, n₂\) – integers

EXAMPLES:

\[
\begin{align*}
\text{sage: } & \text{A = IntegerValuedPolynomialRing(QQ).B() } \\
\text{sage: } & \text{A.product_on_basis(0, 1)} \\
& \text{B[1]} \\
\text{sage: } & \text{A.product_on_basis(1, 2)} \\
\end{align*}
\]

\(S\)

alias of \(Shifted\)

class Shifted(A)

Bases: CombinatorialFreeModule, BindableClass

The integer-valued polynomial ring in the shifted basis.

The basis used here is given by \(S[i] = \binom{i+x}{i}\) for \(i \in \mathbb{N}\).

Assuming \(n₁ \leq n₂\), the product of two monomials \(S[n₁] \cdot S[n₂]\) is given by the sum

\[
\sum_{k=0}^{n₁} (-1)^k \binom{n₁}{k} \binom{n₁ + n₂ - k}{n₁} S[n₁ + n₂ - k].
\]

EXAMPLES:

\[
\begin{align*}
\text{sage: } & \text{F = IntegerValuedPolynomialRing(QQ).S(); F } \\
& \text{Integer-Valued Polynomial Ring over Rational Field } \\
& \text{in the shifted basis } \\
\text{sage: } & \text{F.gen()} \\
& \text{S[1]} \\
\text{sage: } & \text{S = IntegerValuedPolynomialRing(ZZ).S(); S } \\
& \text{Integer-Valued Polynomial Ring over Integer Ring } \\
& \text{in the shifted basis } \\
\text{sage: } & \text{S.base_ring()} \\
& \text{Integer Ring}
\end{align*}
\]
Integer-valued polynomial rings commute with their base ring:

```python
sage: K = IntegerValuedPolynomialRing(QQ).S()
sage: a = K.gen()
sage: K.is_commutative()
True
sage: L = IntegerValuedPolynomialRing(K).S()
sage: c = L.gen()
sage: L.is_commutative()
True
sage: s = a * c^3; s
sage: parent(s)
Integer-Valued Polynomial Ring over Integer-Valued Polynomial Ring over Rational Field in the shifted basis in the shifted basis
```

Integer-valued polynomial rings are commutative:

```python
sage: c^3 * a == c * a * c * c
True
```

We can also manipulate elements in the basis and coerce elements from our base field:

```python
sage: F = IntegerValuedPolynomialRing(QQ).S()
sage: S = F.basis()
```

class Element

**Bases:** `IndexedFreeModuleElement`

**delta()**

Return the image by the difference operator Δ.

The operator Δ is defined on polynomials by

\[ f \mapsto f(x + 1) - f(x). \]

**EXAMPLES:**

```python
sage: F = IntegerValuedPolynomialRing(ZZ).S()
sage: S = F.basis()
sage: S[5].delta()
```

**derivative_at_minus_one()**

Return the derivative at −1.
This is sometimes useful when $-1$ is a root.

**See also:**

`umbra()`

**EXAMPLES:**

```sage
sage: F = IntegerValuedPolynomialRing(ZZ).S()
sage: B = F.gen()
sage: (B+1).derivative_at_minus_one()
1
```

**fraction()**

Return the generating series of values as a fraction.

In the case of Ehrhart polynomials, this is known as the Ehrhart series.

**See also:**

`h_vector()`, `h_polynomial()`

**EXAMPLES:**

```sage
sage: A = IntegerValuedPolynomialRing(ZZ).S()
sage: ex = A.monomial(4)
sage: f = ex.fraction(); f
-1/(t^5 - 5*t^4 + 10*t^3 - 10*t^2 + 5*t - 1)
sage: F = LazyPowerSeriesRing(QQ, 't')
sage: F(f)
1 + 5*t + 15*t^2 + 35*t^3 + 70*t^4 + 126*t^5 + 210*t^6 + O(t^7)
sage: poly = ex.polynomial()
sage: [poly(i) for i in range(6)]
[1, 5, 15, 35, 70, 126]
sage: y = polygen(QQ, 'y')
sage: penta = A.from_polynomial(7/2*y^2 + 7/2*y + 1)
sage: penta.fraction()
(-t^2 - 5*t - 1)/(t^3 - 3*t^2 + 3*t - 1)
```

**h_polynomial()**

Return the $h$-vector as a polynomial.

**See also:**

`h_vector()`, `fraction()`

**EXAMPLES:**

```sage
sage: x = polygen(QQ,'x')
sage: A = IntegerValuedPolynomialRing(ZZ).S()
sage: ex = A.from_polynomial((1+x)**3)
sage: ex.h_polynomial()
z^2 + 4*z + 1
```

**h_vector()**

Return the numerator of the generating series of values.

If `self` is an Ehrhart polynomial, this is the $h$-vector.
See also:

\texttt{h\_polynomial()}, \texttt{\_frac{}ction()}

EXAMPLES:

\begin{verbatim}
sage: x = polygen(QQ,'x')
sage: A = IntegerValuedPolynomialRing(ZZ).S()
sage: ex = A.from_polynomial((1+x)**3)
sage: ex.h_vector()
(0, 1, 4, 1)
\end{verbatim}

\texttt{umbra()}

Return the Bernoulli umbra.

This is the derivative at $-1$ of the shift by one.

See also:

\texttt{\_derivative\_at\_minus\_one()}

EXAMPLES:

\begin{verbatim}
sage: F = IntegerValuedPolynomialRing(ZZ).S()
sage: B = F.gen()
sage: (B+1).umbra()
3/2
\end{verbatim}

\texttt{variable\_shift}(k=1)

Return the image by the shift of variables.

On polynomials, the action is the shift on variables $x \mapsto x + k$.

INPUT:

• $k$ – integer (default: 1)

EXAMPLES:

\begin{verbatim}
sage: A = IntegerValuedPolynomialRing(ZZ).S()
sage: S = A.basis()
sage: S[5].variable_shift()
sage: S[5].variable_shift(-1)
\end{verbatim}

\texttt{from\_h\_vector}(h)

Convert from some $h$-vector.

INPUT:

• $h$ – a tuple or vector

See also:

\texttt{Element.h\_vector()}

EXAMPLES:

\begin{verbatim}
sage: A = IntegerValuedPolynomialRing(ZZ).S()
sage: S = A.basis()
sage: A.from_h_vector(ex.h_vector())
\end{verbatim}
**product_on_basis** \((n1, n2)\)

Return the product of basis elements \(n1\) and \(n2\).

**INPUT:**
- \(n1, n2\) – integers

**EXAMPLES:**

```
sage: A = IntegerValuedPolynomialRing(QQ).S()
sage: A.product_on_basis(0, 1)
S[1]
sage: A.product_on_basis(1, 2)
```

**a_realization()**

Return a default realization.

The Binomial realization is chosen.

**EXAMPLES:**

```
sage: IntegerValuedPolynomialRing(QQ).a_realization()
Integer-Valued Polynomial Ring over Rational Field
in the binomial basis
```

### 2.2 Generic Convolution

Asymptotically fast convolution of lists over any commutative ring in which the multiply-by-two map is injective. (More precisely, if \(x \in R\), and \(x = 2^k \cdot y\) for some \(k \geq 0\), we require that \(R(x/2^k)\) returns \(y\).)

The main function to be exported is **convolution()**.

**EXAMPLES:**

```
sage: convolution([1, 2, 3, 4, 5], [6, 7])
[6, 19, 32, 45, 58, 35]
```

The convolution function is reasonably fast, even though it is written in pure Python. For example, the following takes less than a second:

```
sage: v = convolution(list(range(1000)), list(range(1000)))
```

**ALGORITHM:**

Converts the problem to multiplication in the ring \(S[x]/(x^M - 1)\), where \(S = R[y]/(y^K + 1)\) (where \(R\) is the original base ring). Performs FFT with respect to the roots of unity \(1, y, y^2, \ldots, y^{2K-1}\) in \(S\). The FFT/IFFT are accomplished with just additions and subtractions and rotating python lists. (I think this algorithm is essentially due to Schonhage, not completely sure.) The pointwise multiplications are handled recursively, switching to a classical algorithm at some point.

Complexity is \(O(n \log(n) \log(\log(n)))\) additions/subtractions in \(R\) and \(O(n \log(n))\) multiplications in \(R\).

**AUTHORS:**
- David Harvey (2007-07): first implementation
- William Stein: editing the docstrings for inclusion in Sage.
sage.rings.polynomial.convolution.convolution($L_1, L_2$)
Return convolution of non-empty lists $L_1$ and $L_2$.
$L_1$ and $L_2$ may have arbitrary lengths.

**EXAMPLES:**
sage: convolution([1, 2, 3], [4, 5, 6, 7])
[4, 13, 28, 34, 32, 21]

### 2.3 Fast calculation of cyclotomic polynomials

This module provides a function `cyclotomic_coeffs()`, which calculates the coefficients of cyclotomic polynomials. This is not intended to be invoked directly by the user, but it is called by the method `cyclotomic_polynomial()` method of univariate polynomial ring objects and the top-level `cyclotomic_polynomial()` function.

sage.rings.polynomial.cyclotomic.bateman_bound($nn$)
Reference:
Bateman, P. T.; Pomerance, C.; Vaughan, R. C. *On the size of the coefficients of the cyclotomic polynomial.*

**EXAMPLES:**
sage: from sage.rings.polynomial.cyclotomic import bateman_bound
sage: bateman_bound(2**8 * 1234567893377)  # needs sage.libs.pari
66944986927

sage.rings.polynomial.cyclotomic.cyclotomic_coeffs($nn$, sparse=None)
Return the coefficients of the $n$-th cyclotomic polynomial by using the formula

$$
\Phi_n(x) = \prod_{d|n} (1 - x^n/d)^\mu(d)
$$

where $\mu(d)$ is the Möbius function that is 1 if $d$ has an even number of distinct prime divisors, $-1$ if it has an odd number of distinct prime divisors, and 0 if $d$ is not squarefree.

Multiplications and divisions by polynomials of the form $1 - x^n$ can be done very quickly in a single pass.

If `sparse` is True, the result is returned as a dictionary of the non-zero entries, otherwise the result is returned as a list of python ints.

**EXAMPLES:**
sage: from sage.rings.polynomial.cyclotomic import cyclotomic_coeffs
sage: cyclotomic_coeffs(30)
[1, 1, 0, -1, -1, -1, 0, 1, 1]
sage: cyclotomic_coeffs(10^5)
{0: 1, 10000: -1, 20000: 1, 30000: -1, 40000: 1}
sage: R = QQ['x']
sage: R(cyclotomic_coeffs(30))
x^8 + x^7 - x^5 - x^4 - x^3 + x + 1

Check that it has the right degree:
The coefficients are not always +/-1:

\[
x^6 - x^5 + x^4 - x^3 + x^2 - x + 1
\]

In fact the height is not bounded by any polynomial in \( n \) (Erdos), although takes a while just to exceed linear:

\[
\text{sage: } v = \text{cyclotomic_coeffs}(1181895)
\]
\[
\text{sage: } \text{max}(v)
\]
\[
14102773
\]

The polynomial is a palindrome for any \( n \):

\[
\text{sage: } n = \text{ZZ.random_element}(50000)
\]
\[
\text{sage: } v = \text{cyclotomic_coeffs}(n, \text{sparse=False})
\]
\[
\text{sage: } v == \text{list(reversed}(v))
\]
\[
True
\]

AUTHORS:

• Robert Bradshaw (2007-10-27): initial version (inspired by work of Andrew Arnold and Michael Monagan)

REFERENCE:

• http://www.cecm.sfu.ca/~ada26/cyclotomic/

\text{sage.rings.polynomial.cyclotomic.cyclotomic_value} (n, x)

Return the value of the \( n \)-th cyclotomic polynomial evaluated at \( x \).

INPUT:

• \( n \) – an Integer, specifying which cyclotomic polynomial is to be evaluated
• \( x \) – an element of a ring

OUTPUT:

• the value of the cyclotomic polynomial \( \Phi_n \) at \( x \)

ALGORITHM:

• Reduce to the case that \( n \) is squarefree: use the identity

\[
\Phi_n(x) = \Phi_q(x^{n/q})
\]

where \( q \) is the radical of \( n \).
• Use the identity

\[
\Phi_n(x) = \prod_{d|n}(x^d - 1)^\mu(n/d),
\]

where \( \mu \) is the Möbius function.
• Handles the case that $x^d = 1$ for some $d$, but not the case that $x^d - 1$ is non-invertible: in this case polynomial evaluation is used instead.

**EXAMPLES:**

```sage
cyclotomic_value(51, 3)
1282860140677441
sage: cyclotomic_polynomial(51)(3)
1282860140677441
```

It works for non-integral values as well:

```sage
cyclotomic_value(144, 4/3)
79148745433504023621920372161/79766443076872509863361
sage: cyclotomic_polynomial(144)(4/3)
79148745433504023621920372161/79766443076872509863361
```
3.1 Multivariate Polynomials and Polynomial Rings

Sage implements multivariate polynomial rings through several backends. The most generic implementation uses the classes `sage.rings.polynomial.polydict.PolyDict` and `sage.rings.polynomial.polydict.ETuple` to construct a dictionary with exponent tuples as keys and coefficients as values. Additionally, specialized and optimized implementations over many specific coefficient rings are implemented via a shared library interface to SINGULAR; and polynomials in the boolean polynomial ring

\[ \mathbb{F}_2[x_1, \ldots, x_n]/(x_1^2 + x_1, \ldots, x_n^2 + x_n). \]

are implemented using the PolyBoRi library (cf. `sage.rings.polynomial.pbori`).

3.1.1 Term orders

Sage supports the following term orders:

Lexicographic (lex)

\[ x^a < x^b \text{ if and only if there exists } 1 \leq i \leq n \text{ such that } a_1 = b_1, \ldots, a_{i-1} = b_{i-1}, a_i < b_i. \]

This term order is called ‘lp’ in Singular.

**EXAMPLES:**

```python
sage: P.<x,y,z> = PolynomialRing(QQ, 3, order='lex')
sage: x > y
True
sage: x > y^2
True
sage: x > 1
True
sage: x^1*y^2 > y^3*z^4
True
sage: x^3*y^2*z^4 < x^3*y^2*z^1
False
```

Degree reverse lexicographic (degrevlex)

Let \( \deg(x^a) = a_1 + a_2 + \cdots + a_n \), then \( x^a < x^b \) if and only if \( \deg(x^a) < \deg(x^b) \) or \( \deg(x^a) = \deg(x^b) \) and there exists \( 1 \leq i \leq n \) such that \( a_n = b_n, \ldots, a_{i+1} = b_{i+1}, a_i > b_i \). This term order is called ‘dp’ in Singular.

**EXAMPLES:**
Degree lexicographic (deglex)

Let \( \deg(x^a) = a_1 + a_2 + \cdots + a_n \), then \( x^a < x^b \) if and only if \( \deg(x^a) < \deg(x^b) \) or \( \deg(x^a) = \deg(x^b) \) and there exists \( 1 \leq i \leq n \) such that \( a_1 = b_1, \ldots, a_{i-1} = b_{i-1}, a_i < b_i \). This term order is called ‘Dp’ in Singular.

EXAMPLES:

```python
sage: P.<x,y,z> = PolynomialRing(QQ, 3, order='deglex')
sage: x > y
True
sage: x > y^2*z
False
sage: x > 1
True
sage: x^1*y^2*z^3 > x^3*y^2*z^0
True
sage: x^2*y*z^2 > x*y^3*z
False
```

Inverse lexicographic (invlex)

\( x^a < x^b \) if and only if there exists \( 1 \leq i \leq n \) such that \( a_n = b_n, \ldots, a_{i+1} = b_{i+1}, a_i < b_i \). This order is called ‘rp’ in Singular.

EXAMPLES:

```python
sage: P.<x,y,z> = PolynomialRing(QQ, 3, order='invlex')
sage: x > y
False
sage: y > x^2
True
sage: x > 1
True
sage: x*y > z
False
```

This term order only makes sense in a non-commutative setting because if \( P \) is the ring \( k[x_1, \ldots, x_n] \) and term order ‘invlex’ then it is equivalent to the ring \( k[x_n, \ldots, x_1] \) with term order ‘lex’.

Negative lexicographic (neglex)

\( x^a < x^b \) if and only if there exists \( 1 \leq i \leq n \) such that \( a_1 = b_1, \ldots, a_{i-1} = b_{i-1}, a_i > b_i \). This term order is called ‘ls’ in Singular.

EXAMPLES:

```python
sage: P.<x,y,z> = PolynomialRing(QQ, 3, order='neglex')
sage: x > y
False
```
Negative degree reverse lexicographic (negdegrevlex)
Let \( \text{deg}(x^a) = a_1 + a_2 + \cdots + a_n \), then \( x^a < x^b \) if and only if \( \text{deg}(x^a) > \text{deg}(x^b) \) or \( \text{deg}(x^a) = \text{deg}(x^b) \) and there exists \( 1 \leq i \leq n \) such that \( a_n = b_n, \ldots, a_{i+1} = b_{i+1}, a_i > b_i \). This term order is called ‘ds’ in Singular.

EXAMPLES:

```sage
sage: P.<x,y,z> = PolynomialRing(QQ, 3, order='negdegrevlex')
sage: x > y
True
sage: x > x^2
True
sage: x > 1
False
sage: x^1*y^2 > y^3*z^4
True
sage: x^2*y*z^2 > x*y^3*z
False
```

Negative degree lexicographic (negdeglex)
Let \( \text{deg}(x^a) = a_1 + a_2 + \cdots + a_n \), then \( x^a < x^b \) if and only if \( \text{deg}(x^a) > \text{deg}(x^b) \) or \( \text{deg}(x^a) = \text{deg}(x^b) \) and there exists \( 1 \leq i \leq n \) such that \( a_1 = b_1, \ldots, a_{i-1} = b_{i-1}, a_i < b_i \). This term order is called ‘Ds’ in Singular.

EXAMPLES:

```sage
sage: P.<x,y,z> = PolynomialRing(QQ, 3, order='negdeglex')
sage: x > y
True
sage: x > x^2
True
sage: x > 1
False
sage: x^1*y^2 > y^3*z^4
True
sage: x^2*y*z^2 > x*y^3*z
True
```

Weighted degree reverse lexicographic (wdegrevlex), positive integral weights
Let \( \text{deg}_w(x^a) = a_1 w_1 + a_2 w_2 + \cdots + a_n w_n \) with weights \( w \), then \( x^a < x^b \) if and only if \( \text{deg}_w(x^a) < \text{deg}_w(x^b) \) or \( \text{deg}_w(x^a) = \text{deg}_w(x^b) \) and there exists \( 1 \leq i \leq n \) such that \( a_n = b_n, \ldots, a_{i+1} = b_{i+1}, a_i > b_i \). This term order is called ‘wp’ in Singular.

EXAMPLES:

```sage
sage: P.<x,y,z> = PolynomialRing(QQ, 3, order=TermOrder('wdegrevlex',(1,2,3)))
sage: x > y
False
sage: x > x^2
False
```
Weighted degree lexicographic (wdeglex), positive integral weights

Let $\deg_w(x^a) = a_1 w_1 + a_2 w_2 + \cdots + a_n w_n$ with weights $w$, then $x^a < x^b$ if and only if $\deg_w(x^a) < \deg_w(x^b)$ or $\deg_w(x^a) = \deg_w(x^b)$ and there exists $1 \leq i \leq n$ such that $a_1 = b_1, \ldots, a_{i-1} = b_{i-1}, a_i < b_i$. This term order is called ‘Wp’ in Singular.

EXAMPLES:

```sage
P.<x,y,z> = PolynomialRing(QQ, 3, order=TermOrder('wdeglex', (1,2,3)))
sage: x > y
False
sage: x > x^2
False
sage: x > 1
True
sage: x^1*y^2 > x^2*z
False
sage: y*z > x^3*y
False
```

Negative weighted degree reverse lexicographic (negwdegrevlex), positive integral weights

Let $\deg_w(x^a) = a_1 w_1 + a_2 w_2 + \cdots + a_n w_n$ with weights $w$, then $x^a < x^b$ if and only if $\deg_w(x^a) > \deg_w(x^b)$ or $\deg_w(x^a) = \deg_w(x^b)$ and there exists $1 \leq i \leq n$ such that $a_n = b_n, \ldots, a_{i+1} = b_{i+1}, a_i > b_i$. This term order is called ‘ws’ in Singular.

EXAMPLES:

```sage
P.<x,y,z> = PolynomialRing(QQ, 3, order=TermOrder('negwdegrevlex', (1,2,3)))
sage: x > y
True
sage: x > x^2
True
sage: x > 1
False
sage: x^1*y^2 > x^2*z
True
sage: y*z > x^3*y
False
```

Degree negative lexicographic (degneglex)

Let $\deg(x^a) = a_1 + a_2 + \cdots + a_n$, then $x^a < x^b$ if and only if $\deg(x^a) < \deg(x^b)$ or $\deg(x^a) = \deg(x^b)$ and there exists $1 \leq i \leq n$ such that $a_1 = b_1, \ldots, a_{i-1} = b_{i-1}, a_i > b_i$. This term order is called ‘dp_asc’ in PolyBoRi. Singular has the extra weight vector ordering $(a(1:n), ls)$ for this purpose.

EXAMPLES:

```sage
t = TermOrder('degneglex')
sage: P.<x,y,z> = PolynomialRing(QQ, order=t)
sage: x*y > y*z # indirect doctest
False
```

(continues on next page)
Negative weighted degree lexicographic (negwdeglex), positive integral weights

Let \( \deg_w(x^a) = a_1 w_1 + a_2 w_2 + \cdots + a_n w_n \) with weights \( w \), then \( x^a < x^b \) if and only if \( \deg_w(x^a) > \deg_w(x^b) \) or \( \deg_w(x^n) = \deg_w(x^b) \) and there exists \( 1 \leq i \leq n \) such that \( a_i = b_1, \ldots, a_{i-1} = b_{i-1}, a_i < b_i \). This term order is called ‘Ws’ in Singular.

**EXAMPLES:**

```
sage: P.<x,y,z> = PolynomialRing(QQ, 3, order=TermOrder('negwdeglex', (1,2,3)))
sage: x > y
True
sage: x > x^2
True
sage: x > 1
False
sage: x^1*y^2 > x^2*z
False
sage: y*z > x^3*y
False
```

Of these, only ‘degrevlex’, ‘deglex’, ‘degneglex’, ‘wdegrevlex’, ‘wdeglex’, ‘invlex’ and ‘lex’ are global orders.

Sage also supports matrix term order. Given a square matrix \( A \),

\[
x^a <_A x^b \text{ if and only if } Ax < Ab
\]

where \( < \) is the lexicographic term order.

**EXAMPLES:**

```
sage: # needs sage.modules
sage: m = matrix(ZZ, 2, [2,3,0,1]); m
[2 3]
[0 1]
sage: T = TermOrder(m); T
Matrix term order with matrix
[2 3]
[0 1]
sage: P.<a,b> = PolynomialRing(QQ, 2, order=T)
sage: P
Multivariate Polynomial Ring in a, b over Rational Field
sage: a > b
False
sage: a^3 < b^2
True
sage: S = TermOrder('M(2,3,0,1)')
sage: T == S
True
```

Additionally all these monomial orders may be combined to product or block orders, defined as:

Let \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_m) \) be two ordered sets of variables, \( <_1 \) a monomial order on \( k[x] \) and \( <_2 \) a monomial order on \( k[y] \).

The product order (or block order) \( < := (<_1, <_2) \) on \( k[x,y] \) is defined as: \( x^a y^b < x^A y^B \) if and only if \( x^a <_1 x^A \) or \( (x^a = x^A \text{ and } y^b <_2 y^B) \).
These block orders are constructed in Sage by giving a comma separated list of monomial orders with the length of each block attached to them.

EXAMPLES:

As an example, consider constructing a block order where the first four variables are compared using the degree reverse lexicographical order while the last two variables in the second block are compared using negative lexicographical order.

```
sage: P.<a,b,c,d,e,f> = PolynomialRing(QQ, 6, order='degrevlex(4),neglex(2))
sage: a > c^4
False
sage: a > e^4
True
sage: e > f^2
False
```

The same result can be achieved by:

```
sage: T1 = TermOrder('degrevlex',4)
sage: T2 = TermOrder('neglex',2)
sage: T = T1 + T2
sage: P.<a,b,c,d,e,f> = PolynomialRing(QQ, 6, order=T)
sage: a > c^4
False
sage: a > e^4
True
```

If any other unsupported term order is given the provided string can be forced to be passed through as is to Singular, Macaulay2, and Magma. This ensures that it is for example possible to calculate a Groebner basis with respect to some term order Singular supports but Sage doesn’t:

```
sage: T = TermOrder("royalorder")
Traceback (most recent call last):
  ... ValueError: unknown term order 'royalorder'
sage: T = TermOrder("royalorder", force=True)
sage: T
royalorder term order
sage: T.singular_str()
'royalorder'
```

AUTHORS:

- David Joyner and William Stein: initial version of multi_polynomial_ring
- Kiran S. Kedlaya: added macaulay2 interface
- Martin Albrecht: implemented native term orders, refactoring
- Kwankyu Lee: implemented matrix and weighted degree term orders
- Simon King (2011-06-06): added termorder_from_singular

```
class sage.rings.polynomial.term_order.TermOrder(name='lex', n=0, force=False)
    Bases: sage.rings.polynomial.term_order.SageObject
    A term order.

    See sage.rings.polynomial.term_order for details on supported term orders.
```
blocks()

Return the term order blocks of self.

NOTE:

This method has been added in github issue #11316. There used to be an attribute of the same name and the same content. So, it is a backward incompatible syntax change.

EXAMPLES:

```
sage: t = TermOrder('deglex',2) + TermOrder('lex',2)
sage: t.blocks()
(Degree lexicographic term order, Lexicographic term order)
```

property greater_tuple

The default greater_tuple method for this term order.

EXAMPLES:

```
sage: O = TermOrder()
sage: O.greater_tuple.__func__ is O.greater_tuple_lex.__func__
True
sage: O = TermOrder('deglex')
sage: O.greater_tuple.__func__ is O.greater_tuple_deglex.__func__
True
```

greater_tuple_block(f, g)

Return the greater exponent tuple with respect to the block order as specified when constructing this element.

This method is called by the lm/lc/lt methods of MPolynomial_polydict.

INPUT:

- f - exponent tuple
- g - exponent tuple

EXAMPLES:

```
sage: P.<a,b,c,d,e,f> = PolynomialRing(QQbar, 6, 
                       order='degrevlex(3),degrevlex(3)')
sage: f = a + c^4; f.lm()  # indirect doctest
\# does not need sage.rings.number_field
\# order='degrevlex(3),degrevlex(3)'
sage: g = a + e^4; g.lm()  # does not need sage.rings.number_field
\# order='degrevlex(3),degrevlex(3)'
sage: c^4
```

greater_tuple_deglex(f, g)

Return the greater exponent tuple with respect to the total degree lexicographical term order.

INPUT:

- f - exponent tuple
- g - exponent tuple

EXAMPLES:
This method is called by the \texttt{lm/lc/lt} methods of \texttt{MPolynomial\_polydict}.

\texttt{greater\_tuple\_degneglex}(f, g)

Return the greater exponent tuple with respect to the degree negative lexicographical term order.

INPUT:

- \texttt{f} - exponent tuple
- \texttt{g} - exponent tuple

EXAMPLES:

\begin{verbatim}
sage: P.<x,y,z> = PolynomialRing(QQbar, 3, order='degneglex') # needs sage.rings.number_field
sage: f = x + y; f.lm() # indirect doctest # needs sage.rings.number_field
x
sage: f = x + y^2*z; f.lm() # needs sage.rings.number_field
y^2*z
\end{verbatim}

This method is called by the \texttt{lm/lc/lt} methods of \texttt{MPolynomial\_polydict}.

\texttt{greater\_tuple\_degrevlex}(f, g)

Return the greater exponent tuple with respect to the total degree reversed lexicographical term order.

INPUT:

- \texttt{f} - exponent tuple
- \texttt{g} - exponent tuple

EXAMPLES:

\begin{verbatim}
sage: P.<x,y,z> = PolynomialRing(QQbar, 3, order='degrevlex') # needs sage.rings.number_field
sage: f = x + y; f.lm() # indirect doctest # needs sage.rings.number_field
y
sage: f = x + y^2*z; f.lm() # needs sage.rings.number_field
y^2*z
\end{verbatim}

This method is called by the \texttt{lm/lc/lt} methods of \texttt{MPolynomial\_polydict}.

\texttt{greater\_tuple\_invlex}(f, g)

Return the greater exponent tuple with respect to the inversed lexicographical term order.

INPUT:

- \texttt{f} - exponent tuple
• $g$ - exponent tuple

EXAMPLES:

```python
sage: P.<x,y,z> = PolynomialRing(QQbar, 3, order='invlex')  # needs sage.rings.number_field
sage: f = x + y; f.lm()  # indirect doctest  # needs sage.rings.number_field
y
sage: f = y + x^2; f.lm()  # needs sage.rings.number_field
y
```

This method is called by the \texttt{lm/lc/lt} methods of \texttt{MPolynomial_polydict}.

\textbf{\texttt{greater_tuple_lex}}($f, g$)

Return the greater exponent tuple with respect to the lexicographical term order.

INPUT:

• $f$ - exponent tuple
• $g$ - exponent tuple

EXAMPLES:

```python
sage: P.<x,y,z> = PolynomialRing(QQbar, 3, order='lex')  # needs sage.rings.number_field
sage: f = x + y^2; f.lm()  # indirect doctest  # needs sage.rings.number_field
x
```

This method is called by the \texttt{lm/lc/lt} methods of \texttt{MPolynomial_polydict}.

\textbf{\texttt{greater_tuple_matrix}}($f, g$)

Return the greater exponent tuple with respect to the matrix term order.

INPUT:

• $f$ - exponent tuple
• $g$ - exponent tuple

EXAMPLES:

```python
sage: P.<x,y> = PolynomialRing(QQbar, 2, order='m(1,3,1,0)')  # needs sage.rings.number_field
sage: y > x^2  # indirect doctest  # needs sage.rings.number_field
True
sage: y > x^3  # needs sage.rings.number_field
False
```

\textbf{\texttt{greater_tuple_negdeglex}}($f, g$)

Return the greater exponent tuple with respect to the negative degree lexicographical term order.

INPUT:

• $f$ - exponent tuple
• $g$ - exponent tuple
EXAMPLES:

```python
sage: # needs sage.rings.number_field
sage: P.<x,y,z> = PolynomialRing(QQbar, 3, order='negdeglex')
sage: f = x + y; f.lm() # indirect doctest
x
sage: f = x + x^2; f.lm()
x
sage: f = x^2*y*z^2 + x*y^3*z; f.lm()
x^2*y*z^2
```

This method is called by the lm/lc/lt methods of MPolynomial_polydict.

**greater_tuple_negdegrevlex** *(f, g)*

Return the greater exponent tuple with respect to the negative degree reverse lexicographical term order.

**INPUT:**

- f - exponent tuple
- g - exponent tuple

**EXAMPLES:**

```python
sage: # needs sage.rings.number_field
sage: P.<x,y,z> = PolynomialRing(QQbar, 3, order='negdegrevlex')
sage: f = x + y; f.lm() # indirect doctest
x
sage: f = x + x^2; f.lm()
x
sage: f = x^2*y*z^2 + x*y^3*z; f.lm()
x*y^3*z
```

This method is called by the lm/lc/lt methods of MPolynomial_polydict.

**greater_tuple_neglex** *(f, g)*

Return the greater exponent tuple with respect to the negative lexicographical term order.

This method is called by the lm/lc/lt methods of MPolynomial_polydict.

**INPUT:**

- f - exponent tuple
- g - exponent tuple

**EXAMPLES:**

```python
sage: # needs sage.rings.number_field
sage: P.<a,b,c,d,e,f> = PolynomialRing(QQbar, 6, order='degrevlex(3),degrevlex(3)')
....: # indirect doctest
sage: f = a + c^4; f.lm() # indirect doctest
a
sage: g = a + e^4; g.lm() # indirect doctest
```

**greater_tuple_negwdeglex** *(f, g)*

Return the greater exponent tuple with respect to the negative weighted degree lexicographical term order.

**INPUT:**

This method is called by the lm/lc/lt methods of MPolynomial_polydict.

```python
sage: P.<a,b,c,d,e,f> = PolynomialRing(QQbar, 6, order='degrevlex(3),degrevlex(3)')
```
• \( f \) - exponent tuple
• \( g \) - exponent tuple

EXAMPLES:

```
sage: # needs sage.rings.number_field
ts = TermOrder('negwdeglex', (1,2,3))
P.<x,y,z> = PolynomialRing(QQbar, 3, order=t)
sage: f = x + y; f.lm() # indirect doctest
    x
sage: f = x + x^2; f.lm()
x
sage: f = x^3 + z; f.lm()
x^3
```

This method is called by the \texttt{lm/lc.lt} methods of \texttt{MPolynomial_polydict}.

\textbf{greater\_tuple\_negwdegrevlex} \((f,g)\)

Return the greater exponent tuple with respect to the negative weighted degree reverse lexicographical term order.

INPUT:

• \( f \) - exponent tuple
• \( g \) - exponent tuple

EXAMPLES:

```
sage: # needs sage.rings.number_field
ts = TermOrder('negwdegrevlex', (1,2,3))
P.<x,y,z> = PolynomialRing(QQbar, 3, order=t)
sage: f = x + y; f.lm() # indirect doctest
    x
sage: f = x + x^2; f.lm()
x
sage: f = x^3 + z; f.lm()
x^3
```

This method is called by the \texttt{lm/lc.lt} methods of \texttt{MPolynomial_polydict}.

\textbf{greater\_tuple\_wdeglex} \((f,g)\)

Return the greater exponent tuple with respect to the weighted degree lexicographical term order.

INPUT:

• \( f \) - exponent tuple
• \( g \) - exponent tuple

EXAMPLES:

```
sage: t = TermOrder('wdeglex', (1,2,3))
P.<x,y,z> = PolynomialRing(QQbar, 3, order=t)
sage: f = x + y; f.lm() # indirect doctest
    y
sage: f = x*y + z; f.lm() # indirect doctest
    x*y
```

```
This method is called by the \texttt{lm/lc/lt} methods of \texttt{MPolynomial\_polydict}.

\textbf{\texttt{greater\_tuple\_wdegrevlex}(f,g)}

Return the greater exponent tuple with respect to the weighted degree reverse lexicographical term order.

\textbf{INPUT:}

\begin{itemize}
  \item $f$ - exponent tuple
  \item $g$ - exponent tuple
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: t = TermOrder('wdegrevlex',(1,2,3))
sage: P.<x,y,z> = PolynomialRing(QQbar, 3, order=t) # needs sage.rings.number_field
sage: f = x + y; f.lm() # indirect doctest # needs sage.rings.number_field
y
sage: f = x + y^2*z; f.lm() # needs sage.rings.number_field
y^2*z
\end{verbatim}

This method is called by the \texttt{lm/lc/lt} methods of \texttt{MPolynomial\_polydict}.

\textbf{\texttt{is\_block\_order}()} 

Return \texttt{True} if \texttt{self} is a block term order.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: t = TermOrder('deglex',2) + TermOrder('lex',2)
sage: t.is_block_order()
True
\end{verbatim}

\textbf{\texttt{is\_global}()} 

Return \texttt{True} if this term order is definitely global. Return false otherwise, which includes unknown term orders.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: T = TermOrder('lex')
sage: T.is_global()
True
sage: T = TermOrder('degrevlex', 3) + TermOrder('degrevlex', 3)
sage: T.is_global()
True
sage: T = TermOrder('degrevlex', 3) + TermOrder('negdegrevlex', 3)
sage: T.is_global()
False
sage: T = TermOrder('degneglex', 3)
sage: T.is_global()
True
sage: T = TermOrder('invlex', 3)
sage: T.is_global()
True
\end{verbatim}

\textbf{\texttt{is\_local}()} 

Return \texttt{True} if this term order is definitely local. Return false otherwise, which includes unknown term orders.

\textbf{EXAMPLES:}
sage: T = TermOrder('lex')
sage: T.is_local()
False
sage: T = TermOrder('negdeglex', 3) + TermOrder('negdegrevlex', 3)
sage: T.is_local()
True
sage: T = TermOrder('degrevlex', 3) + TermOrder('negdegrevlex', 3)
sage: T.is_local()
False

is_weighted_degree_order()

Return True if self is a weighted degree term order.

EXAMPLES:

sage: t = TermOrder('wdeglex', (2,3))
sage: t.is_weighted_degree_order()
True

macaulay2_str()

Return a Macaulay2 representation of self.

Used to convert polynomial rings to their Macaulay2 representation.

EXAMPLES:

sage: P = PolynomialRing(GF(127), 8, names='x', order='degrevlex(3),lex(5)')
sage: T = P.term_order()
sage: T.macaulay2_str()
{"GRevLex => 3,Lex => 5}"

magma_str()

Return a MAGMA representation of self.

Used to convert polynomial rings to their MAGMA representation.

EXAMPLES:

sage: P = PolynomialRing(GF(127), 10, names='x', order='degrevlex')
sage: magma(P) # optional - magma
Polynomial ring of rank 10 over GF(127)
Order: Graded Reverse Lexicographical
Variables: x0, x1, x2, x3, x4, x5, x6, x7, x8, x9
sage: T = P.term_order()
sage: T.magma_str() "grevlex"

matrix()

Return the matrix defining matrix term order.

EXAMPLES:
sage: t = TermOrder("M(1,2,0,1)") # needs sage.modules
sage: t.matrix() # needs sage.modules
[1 2]
[0 1]

name()

EXAMPLES:

sage: TermOrder('lex').name() 'lex'

singular_moreblocks()

Return a the number of additional blocks SINGULAR needs to allocate for handling non-native orderings like degneglex.

EXAMPLES:

sage: P = PolynomialRing(GF(127), 10, names='x',
.....:  order='lex(3),deglex(5),lex(2)')
sage: T = P.term_order()
sage: T.singular_moreblocks() 0
dsage: P = PolynomialRing(GF(127), 10, names='x',
.....:  order='lex(3),degneglex(5),lex(2)')
sage: T = P.term_order()
sage: T.singular_moreblocks() 1
dsage: P = PolynomialRing(GF(127), 10, names='x',
.....:  order='degneglex(5),degneglex(5)')
sage: T = P.term_order()
sage: T.singular_moreblocks() 2

singular_str()

Return a SINGULAR representation of self.

Used to convert polynomial rings to their SINGULAR representation.

EXAMPLES:

sage: P = PolynomialRing(GF(127), 10, names='x',
.....:  order='lex(3),deglex(5),lex(2)')
sage: T = P.term_order()
sage: T.singular_str() '(lp(3),Dp(5),lp(2))'
sage: P._singular_() # needs sage.libs.singular
polynomial ring, over a field, global ordering
// coefficients: ZZ/127
// number of vars : 10
// block 1 : ordering lp
//    : names   x0 x1 x2
// block 2 : ordering Dp
//    : names   x3 x4 x5 x6 x7
// block 3 : ordering lp

(continues on next page)
The `degneglex` ordering is somehow special, it looks like a block ordering in SINGULAR:

```python
sage: T = TermOrder("degneglex", 2)
sage: P = PolynomialRing(QQ, 2, names='x', order=T)
sage: T = P.term_order()
sage: T.singular_str()
 '(a(1:2),ls(2))'
```

```python
sage: T = TermOrder("degneglex", 2) + TermOrder("degneglex", 2)
sage: P = PolynomialRing(QQ, 4, names='x', order=T)
sage: T = P.term_order()
sage: T.singular_str()
 '(a(1:2),ls(2),a(1:2),ls(2))'
```

```python
sage: T = TermOrder("degneglex", 2) + TermOrder("degneglex", 2)
```

The position of the ordering C block can be controlled by setting `_singular_ringorder_column` attribute to an integer:

```python
sage: T = TermOrder("degneglex", 2) + TermOrder("degneglex", 2)
sage: T._singular_ringorder_column = 0
sage: P = PolynomialRing(QQ, 4, names='x', order=T)
sage: P._singular()
```

```python
sage: T._singular_ringorder_column = 1
```

(continues on next page)
Polynomials, Release 10.3

(continued from previous page)

```
sage: P = PolynomialRing(QQ, 4, names=y, order=T)
sage: P._singular_()
needs sage.libs.singular polynomial ring, over a field, global ordering
// coefficients: QQ
// number of vars : 4
//   block 1 : ordering c
//   block 2 : ordering a
//     : names  y0 y1
//     : weights 1 1
//   block 3 : ordering ls
//   block 4 : ordering a
//     : names  y0 y1
//     : weights 1 1
//   block 5 : ordering ls
//     : names  y2 y3
```

```
sage: T._singular_ringorder_column = 2
sage: P = PolynomialRing(QQ, 4, names=z, order=T)
sage: P._singular_()
needs sage.libs.singular polynomial ring, over a field, global ordering
// coefficients: QQ
// number of vars : 4
//   block 1 : ordering a
//     : names  z0 z1
//     : weights 1 1
//   block 2 : ordering C
//   block 3 : ordering ls
//   block 4 : ordering a
//     : names  z0 z1
//     : names  z2 z3
//     : weights 1 1
//   block 5 : ordering ls
//     : names  z2 z3
```

**property sortkey**

The default sortkey method for this term order.

**EXAMPLES:**

```
sage: O = TermOrder()
sage: O.sortkey.__func__ is O.sortkey_lex.__func__
True
sage: O = TermOrder('deglex')
sage: O.sortkey.__func__ is O.sortkey_deglex.__func__
True
```

**sortkey_block**

Return the sortkey of an exponent tuple with respect to the block order as specified when constructing this element.

**INPUT:**

- *f* – exponent tuple

**EXAMPLES:**
sage: P.<a,b,c,d,e,f> = PolynomialRing(QQbar, 6, #
.....: order='degrevlex(3),degrevlex(3)')

sage: a > c^4  # indirect doctest
False
sage: a > e^4
True

\textbf{sortkey\_deglex}(f)

Return the sortkey of an exponent tuple with respect to the degree lexicographical term order.

INPUT:

\begin{itemize}
  \item $f$ – exponent tuple
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: P.<x,y> = PolynomialRing(QQbar, 2, order='deglex')

sage: x > y^2  # indirect doctest
False
sage: x > 1
True
\end{verbatim}

\textbf{sortkey\_degneglex}(f)

Return the sortkey of an exponent tuple with respect to the degree negative lexicographical term order.

INPUT:

\begin{itemize}
  \item $f$ – exponent tuple
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: P.<x,y,z> = PolynomialRing(QQbar, 3, order='degneglex')

sage: x*y > y*z  # indirect doctest
False
sage: x*y > x
True
\end{verbatim}

\textbf{sortkey\_degrevlex}(f)

Return the sortkey of an exponent tuple with respect to the degree reversed lexicographical term order.

INPUT:

\begin{itemize}
  \item $f$ – exponent tuple
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: P.<x,y> = PolynomialRing(QQbar, 2, order='degrevlex')

sage: x > y^2  # indirect doctest
False
sage: x > 1
True
\end{verbatim}
False

sage: x > 1
→ needs sage.rings.number_field
True

\textbf{sortkey\_invlex}(f)

Return the sortkey of an exponent tuple with respect to the inverted lexicographical term order.

\textbf{INPUT:}

\begin{itemize}
\item $f$ – exponent tuple
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: P.<x,y> = PolynomialRing(QQbar, 2, order='invlex') #...
→ needs sage.rings.number_field
sage: x > y^2  # indirect doctest
→ needs sage.rings.number_field
False
sage: x > 1
→ needs sage.rings.number_field
True
\end{verbatim}

\textbf{sortkey\_lex}(f)

Return the sortkey of an exponent tuple with respect to the lexicographical term order.

\textbf{INPUT:}

\begin{itemize}
\item $f$ – exponent tuple
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: P.<x,y> = PolynomialRing(QQbar, 2, order='lex')  #...
→ needs sage.rings.number_field
sage: x > y^2  # indirect doctest
→ needs sage.rings.number_field
True
sage: x > 1
→ needs sage.rings.number_field
True
\end{verbatim}

\textbf{sortkey\_matrix}(f)

Return the sortkey of an exponent tuple with respect to the matrix term order.

\textbf{INPUT:}

\begin{itemize}
\item $f$ - exponent tuple
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: P.<x,y> = PolynomialRing(QQbar, 2, order='m(1,3,1,0)') #...
→ needs sage.rings.number_field
sage: y > x^2  # indirect doctest
→ needs sage.rings.number_field
True
sage: y > x^3
→ needs sage.rings.number_field
False
\end{verbatim}
sortkey_negdeglex \((f)\)
Return the sortkey of an exponent tuple with respect to the negative degree lexicographical term order.

INPUT:
- \(f\) – exponent tuple

EXAMPLES:
```
sage: P.<x,y> = PolynomialRing(QQbar, 2, order='negdeglex')
# needs sage.rings.number_field
sage: x > y^2  # indirect doctest
# needs sage.rings.number_field
True
sage: x > 1
# needs sage.rings.number_field
False
```

sortkey_negdegrevlex \((f)\)
Return the sortkey of an exponent tuple with respect to the negative degree reverse lexicographical term order.

INPUT:
- \(f\) – exponent tuple

EXAMPLES:
```
sage: P.<x,y> = PolynomialRing(QQbar, 2, order='negdegrevlex')
# needs sage.rings.number_field
sage: x > y^2  # indirect doctest
# needs sage.rings.number_field
True
sage: x > 1
# needs sage.rings.number_field
False
```

sortkey_neglex \((f)\)
Return the sortkey of an exponent tuple with respect to the negative lexicographical term order.

INPUT:
- \(f\) – exponent tuple

EXAMPLES:
```
sage: P.<x,y> = PolynomialRing(QQbar, 2, order='neglex')
# needs sage.rings.number_field
sage: x > y^2  # indirect doctest
# needs sage.rings.number_field
False
sage: x > 1
# needs sage.rings.number_field
False
```

sortkey_negwdeglex \((f)\)
Return the sortkey of an exponent tuple with respect to the negative weighted degree lexicographical term order.

INPUT:
- \(f\) – exponent tuple
EXAMPLES:

```python
sage: t = TermOrder('negwdeglex', (3, 2))
sage: P.<x,y> = PolynomialRing(QQbar, 2, order=t)  # Indirect doctest
sage: x > y^2  # indirect doctest  # This line is marked as needing sage.rings.number_field
True
sage: x^2 > y^3  # This line is marked as needing sage.rings.number_field
True
```

**sortkey_negwdegrevlex**(*f*)

Return the sortkey of an exponent tuple with respect to the negative weighted degree reverse lexicographical term order.

**INPUT:**

- *f* – exponent tuple

**EXAMPLES:**

```python
sage: t = TermOrder('negwdegrevlex', (3, 2))
sage: P.<x,y> = PolynomialRing(QQbar, 2, order=t)  # Indirect doctest
sage: x > y^2  # indirect doctest  # This line is marked as needing sage.rings.number_field
True
sage: x^2 > y^3  # This line is marked as needing sage.rings.number_field
True
```

**sortkey_wdeglex**(*f*)

Return the sortkey of an exponent tuple with respect to the weighted degree lexicographical term order.

**INPUT:**

- *f* – exponent tuple

**EXAMPLES:**

```python
sage: t = TermOrder('wdeglex', (3, 2))
sage: P.<x,y> = PolynomialRing(QQbar, 2, order=t)  # Indirect doctest
sage: x > y^2  # indirect doctest  # This line is marked as needing sage.rings.number_field
False
sage: x > y  # This line is marked as needing sage.rings.number_field
True
```

**sortkey_wdegrevlex**(*f*)

Return the sortkey of an exponent tuple with respect to the weighted degree reverse lexicographical term order.

**INPUT:**

- *f* – exponent tuple

**EXAMPLES:**

```python
sage: t = TermOrder('wdegrevlex', (3, 2))
sage: P.<x,y> = PolynomialRing(QQbar, 2, order=t)  # Indirect doctest
sage: x > y  # This line is marked as needing sage.rings.number_field
True
```

---

294 Chapter 3. Multivariate Polynomials
\begin{verbatim}
sage: t = TermOrder('wdegrevlex', (3,2))
sage: P.<x,y> = PolynomialRing(QQbar, 2, order=t)  # needs sage.rings.number_field
sage: x > y^2  # indirect doctest  # needs sage.rings.number_field
False
sage: x^2 > y^3  # needs sage.rings.number_field
True

tuple_weight (f)
Return the weight of tuple f.

INPUT:
- f - exponent tuple

EXAMPLES:

\begin{verbatim}
sage: t = TermOrder('wdeglex', (1,2,3))
sage: P.<a,b,c> = PolynomialRing(QQbar, order=t)  # needs sage.rings.number_field
sage: P.term_order().tuple_weight([3,2,1])  # needs sage.rings.number_field
10
\end{verbatim}

weights()
Return the weights for weighted term orders.

EXAMPLES:

\begin{verbatim}
sage: t = TermOrder('wdeglex', (2,3))
sage: t.weights()
(2, 3)
\end{verbatim}

\end{verbatim}

```
sage.rings.polynomial.term_order.termorder_from_singular(S)

Return the Sage term order of the basering in the given Singular interface

INPUT:
- An instance of the Singular interface.

EXAMPLES:

\begin{verbatim}
sage: from sage.rings.polynomial.term_order import termorder_from_singular
sage: singular.eval('ring r1 = (9,x),(a,b,c,d,e,f),(M((1,2,3,0)),wp(2,3),lp)')  # needs sage.libs.singular
's'

sage: termorder_from_singular(singular)  # needs sage.libs.singular
Block term order with blocks:
(Matrix term order with matrix
[1 2]
[3 0],
Weighted degree reverse lexicographic term order with weights (2, 3),
Lexicographic term order of length 2)
\end{verbatim}

```

A term order in Singular also involves information on orders for modules. This information is reflected in `_singu-
lar_ringorder_column` attribute of the term order.
sage: # needs sage.libs.singular
sage: singular.ring(0, '(x,y,z,w)', '(C,dp(2),lp(2))')
polynomial ring, over a field, global ordering
// coefficients: QQ
// number of vars : 4
//   block 1 : ordering C
//   block 2 : ordering dp
//       : names x y
//   block 3 : ordering lp
//       : names z w
sage: T = termorder_from_singular(singular)
sage: T
Block term order with blocks:
(Degree reverse lexicographic term order of length 2,
 Lexicographic term order of length 2)
sage: T._singular_ringorder_column
0

sage: # needs sage.libs.singular
sage: singular.ring(0, '(x,y,z,w)', '(c,dp(2),lp(2))')
polynomial ring, over a field, global ordering
// coefficients: QQ
// number of vars : 4
//   block 1 : ordering c
//   block 2 : ordering dp
//       : names x y
//   block 3 : ordering lp
//       : names z w
sage: T = termorder_from_singular(singular)
sage: T
Block term order with blocks:
(Degree reverse lexicographic term order of length 2,
 Lexicographic term order of length 2)
sage: T._singular_ringorder_column
1

3.1.2 Base class for multivariate polynomial rings

class
sage.rings.polynomial.multi_polynomial_ring_base.BooleanPolynomialRing_base

Bases: MPolynomialRing_base

Abstract base class for BooleanPolynomialRing.

This class is defined for the purpose of isinstance tests. It should not be instantiated.

EXAMPLES:

sage: from sage.rings.polynomial.multi_polynomial_ring_base import...
 s sage.rings.polynomial.multi_polynomial_ring_base BooleanPolynomialRing_base
 s R.<x, y, z> = BooleanPolynomialRing()
 # s needs sage.rings.polynomial.pbori
 s sage: isinstance(R, BooleanPolynomialRing_base)
 s # needs sage.rings.polynomial.pbori
 s True

By design, there is only one direct implementation subclass:
sage: len(BooleanPolynomialRing_base.__subclasses__()) <= 1
True

class sage.rings.polynomial.multi_polynomial_ring_base.MPolynomialRing_base

Create a polynomial ring in several variables over a commutative ring.

EXAMPLES:

```
sage: R.<x,y> = ZZ[x,y]; R
Multivariate Polynomial Ring in x, y over Integer Ring
sage: cat = Rings().Commutative()
sage: class CR(Parent):
....:     def __init__(self):
....:         Parent.__init__(self, self, category=cat)
....:     def __call__(self, x):
....:         return None
sage: cr = CR()
sage: cr.is_commutative()
True
sage: cr['x,y']
Multivariate Polynomial Ring in x, y over <__main__.CR_with_category object at ...>
```  

**change_ring** (base_ring=None, names=None, order=None)

Return a new multivariate polynomial ring which is isomorphic to self, but has a different ordering given by the parameter order or names given by the parameter names.

INPUT:

- base_ring – a base ring
- names – variable names
- order – a term order

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(GF(127), 3, order='lex')
sage: x > y^2
True
sage: Q.<x,y,z> = P.change_ring(order='degrevlex')
sage: x > y^2
False
```  

**characteristic()**

Return the characteristic of this polynomial ring.

EXAMPLES:

```
sage: R = PolynomialRing(QQ, 'x', 3)
sage: R.characteristic()
0
sage: R = PolynomialRing(GF(7), 'x', 20)
sage: R.characteristic()
7
```
**completion** *(names=None, prec=20, extras={}, **kwds)*

Return the completion of *self* with respect to the ideal generated by the variable(s) *names*.

**INPUT:**

- *names* – (optional) variable or list/tuple of variables (given either as elements of the polynomial ring or as strings); the default is all variables of *self*
- *prec* – default precision of resulting power series ring, possibly infinite
- *extras* – passed as keywords to *PowerSeriesRing* or *LazyPowerSeriesRing*; can also be keyword arguments

**EXAMPLES:**

```
sage: P.<x,y,z,w> = PolynomialRing(ZZ)
sage: P.completion('w')
Power Series Ring in w over Multivariate Polynomial Ring in x, y, z over Integer Ring
sage: P.completion((w,x,y))
Multivariate Power Series Ring in w, x, y over Univariate Polynomial Ring in z over Integer Ring
sage: Q.<w,x,y,z> = P.completion(); Q
Multivariate Power Series Ring in w, x, y, z over Integer Ring
sage: H = PolynomialRing(PolynomialRing(ZZ,3,'z'),4,'f'); H
Multivariate Polynomial Ring in f0, f1, f2, f3 over Multivariate Polynomial Ring in z0, z1, z2 over Integer Ring
sage: H.completion(H.gens())
Multivariate Power Series Ring in f0, f1, f2, f3 over Multivariate Polynomial Ring in z0, z1, z2 over Integer Ring
sage: H.completion(H.gens()[2])
Power Series Ring in f2 over Multivariate Polynomial Ring in f0, f1, f3 over Multivariate Polynomial Ring in z0, z1, z2 over Integer Ring
sage: P.<x,y,z,w> = PolynomialRing(ZZ)
sage: P.completion(prec=oo)
 Multivariate Lazy Taylor Series Ring in x, y, z, w over Integer Ring
```

**construction()**

Returns a functor *F* and base ring *R* such that *F(R) == self*.

**EXAMPLES:**

```
sage: S = ZZ['x','y']
sage: F, R = S.construction(); R
Integer Ring
sage: F
MPoly[x,y]
sage: F(R) == S
True
```

(continues on next page)
flattening_morphism()

Return the flattening morphism of this polynomial ring

EXAMPLES:

```
sage: QQ['a','b'][x,y].flattening_morphism()
Flattening morphism:
  From: Multivariate Polynomial Ring in x, y
    over Multivariate Polynomial Ring in a, b over Rational Field
  To:   Multivariate Polynomial Ring in a, b, x, y over Rational Field

sage: QQ['x,y'].flattening_morphism()
Identity endomorphism of
  Multivariate Polynomial Ring in x, y over Rational Field
```

gen(n=0)

interpolation(bound, *args)

Create a polynomial with specified evaluations.

CALL FORMATS:

This function can be called in two ways:

1. interpolation(bound, points, values)
2. interpolation(bound, function)

INPUT:

- bound – either an integer bounding the total degree or a list/tuple of integers bounding the degree of the variables
- points – list/tuple containing the evaluation points
- values – list/tuple containing the desired values at points
- function – evaluable function in \( n \) variables, where \( n \) is the number of variables of the polynomial ring

OUTPUT:

1. A polynomial respecting the bounds and having values as values when evaluated at points.
2. A polynomial respecting the bounds and having the same values as function at exactly so many points so that the polynomial is unique.

EXAMPLES:

```
sage: def F(a,b,c):
....:     return a^3*b + b + c^2 + 25
....:
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: R.interpolation(4, F) #...
# needs sage.modules
x^3*y + z^2 + y + 25
```

sage: def F(a,b,c):
....:
```
....: return a^3*b + b + c^2 + 25
....:
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: R.interpolation([3,1,2], F)  # needs sage.modules
x^3*y + z^2 + y + 25

sage: def F(a,b,c):
....: return a^3*b + b + c^2 + 25
....:
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: points = [(5,1,1),(7,2,2),(8,5,-1),(2,5,3),(1,4,0),(5,9,0),
....: (2,7,0),(1,10,13),(0,0,1),(-1,0),(2,5,3),(1,1,1),(7,4,11),
....: (12,1,9),(1,1,3),(4,1,2),(0,1,5),(5,1,3),(3,1,-2),(2,11,3),
....: (4,12,19),(3,1,1),(5,2,-3),(12,1,1),(2,3,6)]
sage: R.interpolation([3,1,2], points, [F(*x) for x in points])  # needs sage.modules
x^3*y + z^2 + y + 25

ALGORITHM:
Solves a linear system of equations with the linear algebra module. If the points are not specified, it samples exactly as many points as needed for a unique solution.

Note: It will only run if the base ring is a field, even though it might work otherwise as well. If your base ring is an integral domain, let it run over the fraction field.

Also, if the solution is not unique, it spits out one solution, without any notice that there are more.

Lastly, the interpolation function for univariate polynomial rings is called `lagrange_polynomial`.

Warning: If you don’t provide point/value pairs but just a function, it will only use as many points as needed for a unique solution with the given bounds. In particular it will not notice or check whether the result yields the correct evaluation for other points as well. So if you give wrong bounds, you will get a wrong answer without any warning.

sage: def F(a,b,c):
....: return a^3*b + b + c^2 + 25
....:
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: R.interpolation(3, F)  # needs sage.modules
1/2*x^3 + x*y + z^2 - 1/2*x + y + 25

See also:

`lagrange_polynomial`

`irrelevant_ideal()`

Return the irrelevant ideal of this multivariate polynomial ring.

This is the ideal generated by all of the indeterminate generators of this ring.

EXAMPLES:
is_exact()
Test whether this multivariate polynomial ring is defined over an exact base ring.

EXAMPLES:

```sage```
PolynomialRing(QQ, 2, 'x').is_exact()
True
```

```sage```
PolynomialRing(RDF, 2, 'x').is_exact()
False
```

is_field(proof=True)
Test whether this multivariate polynomial ring is a field.

A polynomial ring is a field when there are no variable and the base ring is a field.

EXAMPLES:

```sage```
PolynomialRing(QQ, 'x', 2).is_field()
False
```

```sage```
PolynomialRing(QQ, 'x', 0).is_field()
True
```

```sage```
PolynomialRing(ZZ, 'x', 0).is_field()
False
```

```sage```
PolynomialRing(Zmod(1), names=['x','y']).is_finite()
True
```

is_integral_domain(proof=True)

EXAMPLES:

```sage```
ZZ['x,y'].is_integral_domain()
True
```

```sage```
Integers(8)['x,y'].is_integral_domain()
False
```

is_noetherian()

EXAMPLES:

```sage```
ZZ['x,y'].is_noetherian()
True
```

```sage```
Integers(8)['x,y'].is_noetherian()
True
```

krull_dimension()

macaulay_resultant(*args, **kwds)
Return the Macaulay resultant.

This computes the resultant of universal polynomials as well as polynomials with constant coefficients. This is a project done in sage days 55. It is based on the implementation in Maple by Manfred Minimair, which in turn is based on the references listed below. It calculates the Macaulay resultant for a list of polynomials, up to sign!

REFERENCES:
AUTHORS:

• Hao Chen, Solomon Vishkautsan (7-2014)

INPUT:

• args – a list of $n$ homogeneous polynomials in $n$ variables. works when args[0] is the list of polynomials, or args is itself the list of polynomials

kwds:

• sparse – boolean (optional - default: False) if True, the function creates sparse matrices.

OUTPUT:

• the Macaulay resultant, an element of the base ring of self

**Todo:** Working with sparse matrices should usually give faster results, but with the current implementation it actually works slower. There should be a way to improve performance with regards to this.

EXAMPLES:

The number of polynomials has to match the number of variables:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: R.macaulay_resultant([y, x + z])
# needs sage.modules
Traceback (most recent call last):
...
TypeError: number of polynomials(= 2) must equal number of variables (= 3)
```

The polynomials need to be all homogeneous:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: R.macaulay_resultant([y, x + z, z + x^3])
# needs sage.modules
Traceback (most recent call last):
...
TypeError: resultant for non-homogeneous polynomials is not supported
```

All polynomials must be in the same ring:

```
sage: S.<x,y> = PolynomialRing(QQ, 2)
sage: R.<x,y,z> = PolynomialRing(QQ,3)
sage: S.macaulay_resultant([y, z+x])
# needs sage.modules
Traceback (most recent call last):
...
TypeError: not all inputs are polynomials in the calling ring
```

The following example recreates Proposition 2.10 in Ch.3 in [CLO2005]:

```
sage: K.<x,y> = PolynomialRing(ZZ, 2)
sage: flist, R = K._macaulay_resultant_universal_polynomials([1,1,2])
```
The following example degenerates into the determinant of a $3 \times 3$ matrix:

```sage
sage: K.<x,y> = PolynomialRing(ZZ, 2)
sage: flist,R = K._macaulay_resultant_universal_polynomials([1,1,1])
sage: R.macaulay_resultant(flist)  #...
-\(u_2 u_4 u_6 + u_1 u_5 u_6 + u_2 u_3 u_7 - u_0 u_5 u_7 - u_1 u_3 u_8 + u_0 u_4 u_8\)
```

The following example is by Patrick Ingram (arXiv 1310.4114):

```sage
sage: U = PolynomialRing(ZZ,y,2); y0,y1 = U.gens()
sage: R = PolynomialRing(U,x,3); x0,x1,x2 = R.gens()
sage: f0 = y0*x2^2 - x0^2 + 2*x1*x2
sage: f1 = y1*x2^2 - x1^2 + 2*x0*x2
sage: f2 = x0*x1 - x2^2
sage: flist = [f0,f1,f2]
sage: R.macaulay_resultant(flist)  #...
y0^2*y1^2 - 4*y0^3 - 4*y1^3 + 18*y0*y1 - 27
```

A simple example with constant rational coefficients:

```sage
sage: R.<x,y,z,w> = PolynomialRing(QQ, 4)
sage: R.macaulay_resultant([w, z, y, x])  #...
1
```

An example where the resultant vanishes:

```sage
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: R.macaulay_resultant([x + y, y^2, x])  #...
0
```

An example of bad reduction at a prime $p = 5$:

```sage
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: R.macaulay_resultant([y, x^3 + 25*y^2*x, 5*z])  #...
sage.libs.pari sage.modules
125
```

The input can given as an unpacked list of polynomials:

```sage
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: R.macaulay_resultant(y, x^3 + 25*y^2*x, 5*z)  #...
sage.libs.pari sage.modules
125
```
An example when the coefficients live in a finite field:

```python
sage: F = FiniteField(11)
sage: R.<x,y,z,w> = PolynomialRing(F, 4)
sage: R.macaulay_resultant([z, x^3, 5*y, w])
```

Example when the denominator in the algorithm vanishes (in this case the resultant is the constant term of the quotient of char polynomials of numerator/denominator):

```python
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: R.macaulay_resultant([y, x + z, z^2])
```

When there are only 2 polynomials, the Macaulay resultant degenerates to the traditional resultant:

```python
sage: R.<x> = PolynomialRing(QQ, 1)
sage: f = x^2 + 1; g = x^5 + 1
sage: fh = f.homogenize()
sage: gh = g.homogenize()
sage: RH = fh.parent()
sage: f.resultant(g) == RH.macaulay_resultant([fh, gh])
```

**monomial(**exponents**)**

Return the monomial with given exponents.

**EXAMPLES:**

```python
sage: R.<x,y,z> = PolynomialRing(ZZ, 3)
sage: R.monomial(1,1,1)
x*y*z
sage: e=(1,2,3)
sage: R.monomial(*e)
x*y^2*z^3
```

```python
sage: m = R.monomial(1,2,3)
sage: R.monomial(*m.degrees()) == m
True
```

**monomials_of_degree**(degree)

Return a list of all monomials of the given total degree in this multivariate polynomial ring.

**EXAMPLES:**

```python
sage: # needs sage.combinat
sage: R.<x,y,z> = ZZ[]
sage: mons = R.monomials_of_degree(2)
sage: mons
[z^2, y*z, x*z, y^2, x*y, x^2]
sage: P = PolynomialRing(QQ, 3, 'x, y, z', order=TermOrder('wdeglex', [1, 2, 3]))
sage: P.monomials_of_degree(2)
[z^2, y, x*z, x^2]
sage: P = PolynomialRing(QQ, 3, 'x, y, z', order='lex')
sage: P.monomials_of_degree(2)
[z^2, y, x*z, x^2]
sage: P = PolynomialRing(QQ, 3, 'x, y, z', order='lex')
sage: P.monomials_of_degree(3)
```

(continues on next page)
[z^3, y*z^2, y^2*z, y^3, x*z^2, x*y*z, x*y^2, x^2*z, x^2*y, x^3]

`sage: P = PolynomialRing(QQ, 3, 'x, y, z', order='invlex')`
`sage: P.monomials_of_degree(3)`

[x^3, x^2*y, x*y^2, y^3, x^2*z, x*y*z, y^2*z, x*z^2, y*z^2, z^3]

The number of such monomials equals \(\binom{n+k-1}{k}\) where \(n\) is the number of variables and \(k\) the degree:

```py
sage: len(mons) == binomial(3 + 2 - 1, 2)  # ...needs sage.combinat
True
```

`ngens()`

`random_element (degree=2, terms=None, choose_degree=False, *args, **kwargs)`

Return a random polynomial of at most degree \(d\) and at most \(t\) terms.

First monomials are chosen uniformly random from the set of all possible monomials of degree up to \(d\) (inclusive). This means that it is more likely that a monomial of degree \(d\) appears than a monomial of degree \(d-1\) because the former class is bigger.

Exactly \(t\) distinct monomials are chosen this way and each one gets a random coefficient (possibly zero) from the base ring assigned.

The returned polynomial is the sum of this list of terms.

**INPUT:**

- `degree` – maximal degree (likely to be reached) (default: 2)
- `terms` – number of terms requested (default: 5). If more terms are requested than exist, then this parameter is silently reduced to the maximum number of available terms.
- `choose_degree` – choose degrees of monomials randomly first rather than monomials uniformly random.
- `**kwargs` – passed to the random element generator of the base ring

**EXAMPLES:**

```py
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: f = P.random_element(2, 5)
sage: f.degree() <= 2
True
sage: f.parent() is P
True
sage: len(list(f)) <= 5
True

sage: f = P.random_element(2, 5, choose_degree=True)
sage: f.degree() <= 2
True
sage: f.parent() is P
True
sage: len(list(f)) <= 5
True
```

Stacked rings:
```python
sage: R = QQ['x,y']
sage: S = R['t,u']
sage: f = S._random_nonzero_element(degree=2, terms=1)
sage: len(list(f))
1
sage: f.degree() <= 2
True
sage: f.parent() is S
True
```

Default values apply if no degree and/or number of terms is provided:

```python
sage: # needs sage.modules
sage: M = random_matrix(QQ['x,y,z'], 2, 2)
sage: all(a.degree() <= 2 for a in M.list())
True
sage: all(len(list(a)) <= 5 for a in M.list())
True
sage: M = random_matrix(QQ['x,y,z'], 2, 2, terms=1, degree=2)
sage: all(a.degree() <= 2 for a in M.list())
True
sage: all(len(list(a)) <= 1 for a in M.list())
True
sage: P.random_element(0, 1) in QQ
True
sage: P.random_element(2, 0)
0
sage: R.<x> = PolynomialRing(Integers(3), 1)
sage: f = R.random_element()
sage: f.degree() <= 2
True
sage: len(list(f)) <= 3
True
```

To produce a dense polynomial, pick terms=Infinity:

```python
sage: P.<x,y,z> = GF(127)[]
sage: f = P.random_element(degree=2, terms=Infinity)
sage: while len(list(f)) != 10:
    f = P.random_element(degree=2, terms=Infinity)

sage: f = P.random_element(degree=3, terms=Infinity)
sage: while len(list(f)) != 20:
    f = P.random_element(degree=3, terms=Infinity, choose_degree=True)

sage: f = P.random_element(degree=3, terms=Infinity)
sage: while len(list(f)) != 20:
    f = P.random_element(degree=3, terms=Infinity)
```

The number of terms is silently reduced to the maximum available if more terms are requested:

```python
sage: P.<x,y,z> = GF(127)[]
sage: f = P.random_element(degree=2, terms=1000)
sage: len(list(f)) <= 10
True
```

remove_var(order=None, *var)
Remove a variable or sequence of variables from self.

If order is not specified, then the subring inherits the term order of the original ring, if possible.

**EXAMPLES:**

```python
sage: P.<x,y,z,w> = PolynomialRing(ZZ)
sage: P.remove_var(z)
Multivariate Polynomial Ring in x, y, w over Integer Ring
sage: P.remove_var(z, x)
Multivariate Polynomial Ring in y, w over Integer Ring
sage: P.remove_var(y, z, x)
Univariate Polynomial Ring in w over Integer Ring
```

Removing all variables results in the base ring:

```python
sage: P.remove_var(y, z, x, w)
Integer Ring
```

If possible, the term order is kept:

```python
sage: R.<x,y,z,w> = PolynomialRing(ZZ, order='deglex')
sage: R.remove_var(y).term_order()
Degree lexicographic term order
sage: R.<x,y,z,w> = PolynomialRing(ZZ, order='lex')
sage: R.remove_var(y).term_order()
Lexicographic term order
```

Be careful with block orders when removing variables:

```python
sage: R.<x,y,z,u,v> = PolynomialRing(ZZ, order='deglex(2),lex(3)')
sage: R.remove_var(x, y, z)
Traceback (most recent call last):
  ...:
ValueError: impossible to use the original term order (most likely because it was a block order). Please specify the term order for the subring
sage: R.remove_var(x,y,z, order='degrevlex')
Multivariate Polynomial Ring in u, v over Integer Ring
```

**repr_long()**

Return structured string representation of self.

**EXAMPLES:**

```python
sage: P.<x,y,z> = PolynomialRing(QQ, order=TermOrder('degrevlex',1)
....:     + TermOrder('lex',2))
sage: print(P.repr_long())
Polynomial Ring
  Base Ring : Rational Field
  Size : 3 Variables
  Block  0 : Ordering : degrevlex
      Names : x
  Block  1 : Ordering : lex
      Names : y, z
```
**some_elements()**

Return a list of polynomials.

This is typically used for running generic tests.

**EXAMPLES:**

```python
sage: R.<x,y> = QQ[]
sage: R.some_elements()
[x, y, x + y, x^2 + x*y, 0, 1]
```

**term_order()**

**univariate_ring(x)**

Return a univariate polynomial ring whose base ring comprises all but one variables of `self`.

**INPUT:**

- `x` – a variable of `self`.

**EXAMPLES:**

```python
sage: P.<x,y,z> = QQ[]
sage: P.univariate_ring(y)
Univariate Polynomial Ring in y
over Multivariate Polynomial Ring in x, z over Rational Field
```

**variable_names_recursive** *(depth=None)*

Return the list of variable names of this and its base rings, as if it were a single multi-variate polynomial.

**EXAMPLES:**

```python
sage: R = QQ[\{'x,y\}][\{'z,w\}]
sage: R.variable_names_recursive()
('x', 'y', 'z', 'w')
sage: R.variable_names_recursive(3)
('y', 'z', 'w')
```

**weyl_algebra()**

Return the Weyl algebra generated from `self`.

**EXAMPLES:**

```python
sage: R = QQ[\{'x,y,z\}]
sage: W = R.weyl_algebra(); W
# needs sage.modules
Differential Weyl algebra of polynomials in x, y, z over Rational Field
sage: W.polynomial_ring() == R
# needs sage.modules
True
```

```python
sage.rings.polynomial.multi_polynomial_ring_base.is_MPolynomialRing(x)
sage.rings.polynomial.multi_polynomial_ring_base.unpickle_MPolynomialRing_generic(base_ring, n, names, order)
```
3.1.3 Base class for elements of multivariate polynomial rings

class sage.rings.polynomial.multi_polynomial.MPolynomial

    Bases: CommutativePolynomial

    args()
    Returns the names of the arguments of self, in the order they are accepted from call.

    EXAMPLES:

    sage: R.<x,y> = ZZ[]
sage: x.args()
    (x, y)

    change_ring(R)
    Return this polynomial with coefficients converted to R.

    INPUT:
    • R – a ring or morphism; if a morphism, the coefficients are mapped to the codomain of R

    OUTPUT: a new polynomial with the base ring changed to R.

    EXAMPLES:

    sage: R.<x,y> = QQ[]
sage: f = x^3 + 3/5*y + 1
    sage: f.change_ring(GF(7))
x^3 + 2*y + 1
    sage: g = x^2 + 5*y
    sage: g.change_ring(GF(5))
x^2
    sage: # needs sage.rings.finite_rings
    sage: R.<x,y> = GF(9,'a')[]
sage: (x+2*y).change_ring(GF(3))
x - y
    sage: # needs sage.rings.finite_rings
    sage: F.<a> = GF(7^2)
sage: R.<x,y> = F[]
sage: f = x^2 + a^2*y^2 + a*x + a^3*y
    sage: g = f.change_ring(F.frobenius_endomorphism()); g
    x^2 + (-a - 2)*y^2 + (-a + 1)*x + (2*a + 2)*y
    sage: g.change_ring(F.frobenius_endomorphism()) == f
    True

    sage: # needs sage.rings.number_field
    sage: K.<z> = CyclotomicField(3)
sage: R.<x,y> = K[]

(continues on next page)
sage: f = x^2 + z*y
sage: f.change_ring(K.embeddings(CC)[1])
x^2 + (-0.500000000000000 - 0.866025403784438*I)*y

sage: # needs sage.rings.number_field
sage: K.<w> = CyclotomicField(5)
sage: R.<x,y> = K[

sage: f = x^2 + w*y
sage: f.change_ring(K.embeddings(QQbar)[1])
x^2 + (-0.8090169943749474? + 0.5877852522924731?*I)*y

coefficients()

Return the nonzero coefficients of this polynomial in a list.

The returned list is decreasingly ordered by the term ordering of self.parent(), i.e. the list of coefficients matches the list of monomials returned by sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular.monomials().

EXAMPLES:

sage: R.<x,y,z> = PolynomialRing(QQ, 3, order='degrevlex')
sage: f = 23*x^6*y^7 + x^3*y+6*x^7*z
sage: f.coefficients()
[23, 6, 1]

sage: R.<x,y,z> = PolynomialRing(QQ, 3, order='lex')

sage: f = 23*x^6*y^7 + x^3*y+6*x^7*z
sage: f.coefficients()
[6, 23, 1]

Test the same stuff with base ring Z – different implementation:

sage: R.<x,y,z> = PolynomialRing(ZZ, 3, order='degrevlex')
sage: f = 23*x^6*y^7 + x^3*y+6*x^7*z
sage: f.coefficients()
[23, 6, 1]

sage: R.<x,y,z> = PolynomialRing(ZZ, 3, order='lex')

sage: f = 23*x^6*y^7 + x^3*y+6*x^7*z
sage: f.coefficients()
[6, 23, 1]

AUTHOR:

• Didier Deshommes

ccontent()

Return the content of this polynomial. Here, we define content as the gcd of the coefficients in the base ring.

See also:

ccontent_ideal()

EXAMPLES:

sage: R.<x,y> = ZZ[

sage: f = 4*x+6*y
sage: f.content()
2

(continues on next page)
**content_ideal()**

Return the content ideal of this polynomial, defined as the ideal generated by its coefficients.

**See also:**

- `content()`

**EXAMPLES:**

```sage
sage: R.<x,y> = ZZ[
```
```sage
sage: f = 2*x*y + 6*x - 4*y + 2
```
```sage
sage: f.content_ideal()
```
```sage
Principal ideal (2) of Integer Ring
```
```sage
sage: S.<z,t> = R[
```
```sage
sage: g = x*z + y*t
```
```sage
sage: g.content_ideal()
```
```sage
Ideal (x, y) of Multivariate Polynomial Ring in x, y over Integer Ring
```

**denominator()**

Return a denominator of `self`.

First, the lcm of the denominators of the entries of `self` is computed and returned. If this computation fails, the unit of the parent of `self` is returned.

Note that some subclasses may implement its own denominator function.

**Warning:** This is not the denominator of the rational function defined by `self`, which would always be 1 since `self` is a polynomial.

**EXAMPLES:**

First we compute the denominator of a polynomial with integer coefficients, which is of course 1.

```sage
sage: R.<x,y> = ZZ[
```
```sage
sage: f = x^3 + 17*y + x + y
```
```sage
sage: f.denominator()
```
```sage
1
```

Next we compute the denominator of a polynomial over a number field.

```sage
sage: # needs sage.rings.number_field sage.symbolic
```
```sage
sage: R.<x,y> = NumberField(symbolic_expression(x^2+3),'a')['x,y']
```
```sage
sage: f = (1/17)*x^19 + (1/6)*y - (2/3)*x + 1/3; f
```
```sage
1/17*x^19 - 2/3*x + 1/6*y + 1/3
```
```sage
sage: f.denominator()
```
```sage
102
```

Finally, we try to compute the denominator of a polynomial with coefficients in the real numbers, which is a ring whose elements do not have a denominator method.

```sage
sage: # needs sage.rings.real_mpfr
```
```sage
sage: R.<a,b,c> = RR[]
```
```sage
sage: f = a + b + RR('0.3'); f
```

(continues on next page)
Check that the denominator is an element over the base whenever the base has no denominator function. This closes github issue #9063:

```
sage: R.<a,b,c> = GF(5)[]
sage: x = R(0)
sage: x.denominator()
1
sage: type(x.denominator())
<class 'sage.rings.finite_rings.integer_mod.IntegerMod_int'>
sage: type(a.denominator())
<class 'sage.rings.finite_rings.integer_mod.IntegerMod_int'>
sage: from sage.rings.polynomial.multi_polynomial_element import MPolynomial
sage: isinstance(a / b, MPolynomial)
False
sage: isinstance(a.numerator() / a.denominator(), MPolynomial)
True
```

**derivative (*args)**

The formal derivative of this polynomial, with respect to variables supplied in `args`.

Multiple variables and iteration counts may be supplied; see documentation for the global function `derivative()` for more details.

See also:

_derivative()

**EXAMPLES:**

Polynomials implemented via Singular:

```
sage: # needs sage.libs.singular
sage: R.<x, y> = PolynomialRing(FiniteField(5))
sage: f = x^3*y^5 + x^7*y
sage: type(f)
<class 'sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular'>
sage: f.derivative(x)
2*x^6*y - 2*x^2*y^5
sage: f.derivative(y)
x^7
```

Generic multivariate polynomials:

```
sage: R.<t> = PowerSeriesRing(QQ)
sage: S.<x, y> = PolynomialRing(R)
sage: f = (t^2 + O(t^3))*x^2*y^3 + (37*t^4 + O(t^5))*x^3
sage: type(f)
<class 'sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict'>
sage: f.derivative(x)  # with respect to x
(2*t^2 + O(t^3))*x*y^3 + (111*t^4 + O(t^5))*x^2
sage: f.derivative(y)  # with respect to y
(3*t^2 + O(t^3))*x^2*y^2
sage: f.derivative(t)  # with respect to t (recurses into base ring)
```
Polynomials, Release 10.3

(continued from previous page)

\[(2t + O(t^2))x^2y^3 + (148t^3 + O(t^4))x^3\]
\[\text{sage: } f \text{.derivative}(x, y) \quad \# \text{ with respect to } x \text{ and then } y\]
\[6t^2 + O(t^3)x^2y^2\]
\[\text{sage: } f \text{.derivative}(y, 3) \quad \# \text{ with respect to } y \text{ three times}\]
\[6t^2 + O(t^3)x^2\]
\[\text{sage: } f \text{.derivative()} \quad \# \text{ can't figure out the variable}\]
Traceback (most recent call last):
...
ValueError: must specify which variable to differentiate with respect to

Polynomials over the symbolic ring (just for fun….):

\[\text{sage: } \# \text{ needs sage.symbolic}\]
\[\text{sage: } x = \text{var}("x")\]
\[\text{sage: } S.<u, v> = \text{PolynomialRing}(\text{SR})\]
\[\text{sage: } f = u*v*x\]
\[\text{sage: } f \text{.derivative}(x) == u*v\]
True
\[\text{sage: } f \text{.derivative}(u) == v*x\]
True

\textbf{discriminant (variable)}

Returns the discriminant of \textit{self} with respect to the given variable.

INPUT:

\begin{itemize}
\item \textbf{variable} \text{- The variable with respect to which we compute the discriminant}
\end{itemize}

OUTPUT: An element of the base ring of the polynomial ring.

EXAMPLES:

\[\text{sage: } R.<x,y,z> = \text{QQ}[]\]
\[\text{sage: } f = 4*x*y^2 + 1/4*x*y*z + 3/2*x*z^2 - 1/2*z^2\]
\[\text{sage: } f \text{.discriminant}(x) \quad \#\]
\[\rightarrow \text{needs sage.libs.singular}\]
\[1\]
\[\text{sage: } f \text{.discriminant}(y) \quad \#\]
\[\rightarrow \text{needs sage.libs.singular}\]
\[-383/16*x^2*z^2 + 8*x*z^2\]
\[\text{sage: } f \text{.discriminant}(z) \quad \#\]
\[\rightarrow \text{needs sage.libs.singular}\]
\[-383/16*x^2*y^2 + 8*x*y^2\]

Note that, unlike the univariate case, the result lives in the same ring as the polynomial:

\[\text{sage: } R.<x,y> = \text{QQ}[]\]
\[\text{sage: } f = x^5*y + 3*x^2*y^2 - 2*x + y - 1\]
\[\text{sage: } f \text{.discriminant}(y) \quad \#\]
\[\rightarrow \text{needs sage.libs.singular}\]
\[x^{10} + 2*x^5 + 24*x^3 + 12*x^2 + 1\]
\[\text{sage: } f \text{.polynomial}(y) \text{.discriminant}() \quad \#\]
\[\rightarrow \text{needs sage.libs.pari sage.modules}\]
\[x^{10} + 2*x^5 + 24*x^3 + 12*x^2 + 1\]
\[\text{sage: } f \text{.discriminant}(y) \text{.parent}() == f \text{.polynomial}(y) \text{.discriminant}() \text{.parent}() \quad \rightarrow \]
False
\[\rightarrow \# \text{ needs sage.libs.singular sage.modules}\]

3.1. Multivariate Polynomials and Polynomial Rings 313
gcd \text{(other)}

Return a greatest common divisor of this polynomial and other.

INPUT:

- other – a polynomial with the same parent as this polynomial

EXAMPLES:

```sage
Q.<z> = Frac(QQ['z'])
R.<x,y> = Q[]
r = x*y - (2*z-1)/(z^2+z+1) * x + y/z
p = r * (x + z*y - 1/z^2)
q = r * (x*y*z + 1)
gcd(p, q)
```

\[(z^3 + z^2 + z)*x*y + (-2*z^2 + z)*x + (z^2 + z + 1)*y\]

Polynomials over polynomial rings are converted to a simpler polynomial ring with all variables to compute the gcd:

```sage
A.<z,t> = ZZ[]
B.<x,y> = A[]
r = x*y*z*t + 1
p = r * (x - y + z - t + 1)
q = r * (x*z - y*t)
gcd(p, q)
```

\[z*t*x*y + 1\]

```sage
_.parent()
```

Multivariate Polynomial Ring in x, y over
Multivariate Polynomial Ring in z, t over Integer Ring

Some multivariate polynomial rings have no gcd implementation:

```sage
R.<x,y> = GaussianIntegers()[]

x.gcd(x)
```

Traceback (most recent call last):
...
NotImplementedError: GCD is not implemented for multivariate polynomials over Gaussian Integers generated by I in Number Field in I with defining_polynomial x^2 + 1 with I = 1*I

gradient()

Return a list of partial derivatives of this polynomial, ordered by the variables of self.parent().

EXAMPLES:

```sage
P.<x,y,z> = PolynomialRing(ZZ, 3)
f = x*y + 1
f.gradient()
```

\[[y, x, 0]\]

homogeneous_components()

Return the homogeneous components of this polynomial.

OUTPUT:
A dictionary mapping degrees to homogeneous polynomials.

**EXAMPLES:**

```
sage: R.<x,y> = QQ[]
sage: (x^3 + 2*x*y^3 + 4*y^3 + y).homogeneous_components()
{1: y, 3: x^3 + 4*y^3, 4: 2*x*y^3}
sage: R.zero().homogeneous_components()
{}
```

In case of weighted term orders, the polynomials are homogeneous with respect to the weights:

```
sage: S.<a,b,c> = PolynomialRing(ZZ, order=TermOrder('wdegrevlex', (1,2,3)))
sage: (a^6 + b^3 + b*c + a^2*c + c + a + 1).homogeneous_components()
{0: 1, 1: a, 3: c, 5: a^2*c + b*c, 6: a^6 + b^3}
```

**homogenize** *(var='h')*

Return the homogenization of this polynomial.

The polynomial itself is returned if it is homogeneous already. Otherwise, the monomials are multiplied with the smallest powers of \(\text{var}\) such that they all have the same total degree.

**INPUT:**

- \(\text{var}\) – a variable in the polynomial ring (as a string, an element of the ring, or a zero-based index in the list of variables) or a name for a new variable (default: 'h')

**OUTPUT:**

If \(\text{var}\) specifies a variable in the polynomial ring, then a homogeneous element in that ring is returned. Otherwise, a homogeneous element is returned in a polynomial ring with an extra last variable \(\text{var}\).

**EXAMPLES:**

```
sage: R.<x,y> = QQ[]
sage: f = x^2 + y + 1 + 5*x*y^10
sage: f.homogenize()
5*x*y^10 + x^2*h^9 + y*h^10 + h^11
```

The parameter \(\text{var}\) can be used to specify the name of the variable:

```
sage: g = f.homogenize('z'); g
5*x*y^10 + x^2*z^9 + y*z^10 + z^11
```

However, if the polynomial is homogeneous already, then that parameter is ignored and no extra variable is added to the polynomial ring:

```
sage: f = x^2 + y^2
sage: g = f.homogenize('z'); g
x^2 + y^2
```

If you want the ring of the result to be independent of whether the polynomial is homogenized, you can use \(\text{var}\) to use an existing variable to homogenize:
sage: R.<x,y,z> = QQ[]
sage: f = x^2 + y^2
sage: g = f.homogenize(z); g
x^2 + y^2
sage: g.parent()
Multivariate Polynomial Ring in x, y, z over Rational Field
sage: f = x^2 - y
sage: g = f.homogenize(z); g
x^2 - y*z
sage: g.parent()
Multivariate Polynomial Ring in x, y, z over Rational Field

The parameter `var` can also be given as a zero-based index in the list of variables:

sage: g = f.homogenize(2); g
x^2 - y*z

If the variable specified by `var` is not present in the polynomial, then setting it to 1 yields the original polynomial:

sage: g(x,y,1)
x^2 - y

If it is present already, this might not be the case:

sage: g = f.homogenize(x); g
x^2 - x*y
sage: g(1,y,z)
-y + 1

In particular, this can be surprising in positive characteristic:

sage: R.<x,y> = GF(2)[]
sage: f = x + 1
sage: f.homogenize(x)
0

`inverse_mod(I)`

Returns an inverse of `self` modulo the polynomial ideal `I`, namely a multivariate polynomial `f` such that `self * f - 1` belongs to `I`.

INPUT:

- `I` – an ideal of the polynomial ring in which `self` lives

OUTPUT:

a multivariate polynomial representing the inverse of `f` modulo `I`

EXAMPLES:

sage: R.<x1,x2> = QQ[]
sage: I = R.ideal(x2**2 + x1 - 2, x1**2 - 1)
sage: f = x1 + 3*x2**2; g = f.inverse_mod(I); g
→# needs sage.libs.singular
1/16*x1 + 3/16
sage: (f*g).reduce(I)
→# needs sage.libs.singular
1
Test a non-invertible element:

```python
sage: R.<x1,x2> = QQ[]
sage: I = R.ideal(x2**2 + x1 - 2, x1**2 - 1)
sage: f = x1 + x2
sage: f.inverse_mod(I)
˓→# needs sage.libs.singular
Traceback (most recent call last):
... ArithmeticError: element is non-invertible
```

**is_generator()**

Returns **True** if this polynomial is a generator of its parent.

**EXAMPLES:**

```python
sage: R.<x,y> = ZZ[]
sage: x.is_generator()
True
sage: (x + y - y).is_generator()
True
sage: (x*y).is_generator()
False
sage: R.<x,y> = QQ[]
sage: x.is_generator()
True
sage: (x + y - y).is_generator()
True
sage: (x*y).is_generator()
False
```

**is_homogeneous()**

Return **True** if `self` is a homogeneous polynomial.

**Note:** This is a generic implementation which is likely overridden by subclasses.

**is_lorentzian**(explain=False)

Return whether this is a Lorentzian polynomial.

**INPUT:**

- `explain` – boolean (default: False); if **True** return a tuple whose first element is the boolean result of the test, and the second element is a string describing the reason the test failed, or **None** if the test succeeded.

Lorentzian polynomials are a class of polynomials connected with the area of discrete convex analysis. A polynomial `f` with positive real coefficients is Lorentzian if:

- `f` is homogeneous;
- the support of `f` is `M`-convex
- `f` has degree less than 2, or if its degree is at least two, the collection of sequential partial derivatives of `f` which are quadratic forms have Gram matrices with at most one positive eigenvalue.

Note in particular that the zero polynomial is Lorentzian. Examples of Lorentzian polynomials include homogeneous stable polynomials, volume polynomials of convex bodies and projective varieties, and Schur polynomials after renormalizing the coefficient of each monomial `x^α` by `1/α!`.

---

3.1. Multivariate Polynomials and Polynomial Rings  317
EXAMPLES:

Renormalized Schur polynomials are Lorentzian, but not in general if the renormalization is skipped:

```sage
P.<x,y> = QQ[]
p = (x^2 / 2) + x*y + (y^2 / 2)
p.is_lorentzian()
True
p = x^2 + x*y + y^2
p.is_lorentzian()
False
```

Homogeneous linear forms and constant polynomials with positive coefficients are Lorentzian, as well as the zero polynomial:

```sage
p = x + 2*y
p.is_lorentzian()
True
p = P(5)
p.is_lorentzian()
True
P.zero().is_lorentzian()
True
```

Inhomogeneous polynomials and polynomials with negative coefficients are not Lorentzian:

```sage
p = x^2 + 2*x + y^2
p.is_lorentzian()
False
p = 2*x^2 - y^2
p.is_lorentzian()
False
```

It is an error to check if a polynomial is Lorentzian if its base ring is not a subring of the real numbers, as the notion is not defined in this case:

```sage
# needs sage.rings.real_mpfr
Q.<z,w> = CC[
q = z^2 + w^2
q.is_lorentzian()
Traceback (most recent call last):
... Not Implemented Error: is_lorentzian only implemented for real polynomials
```

The method can give a reason for a polynomial failing to be Lorentzian:

```sage
p = x^2 + 2*x + y^2
p.is_lorentzian(explain=True)
(False, 'inhomogeneous')
```

REFERENCES:

For full definitions and related discussion, see [BrHu2019] and [HMMS2019]. The second reference gives the characterization of Lorentzian polynomials applied in this implementation explicitly.
```python
sage: R.<x,y> = QQbar[]
# needs sage.rings.number_field
sage: (x + y).is_nilpotent()  # needs sage.rings.number_field
False
sage: R(0).is_nilpotent()  # needs sage.rings.number_field
True
sage: _.<x,y> = Zmod(4)[]
sage: (2*x).is_nilpotent()
True
sage: (2 + y*x).is_nilpotent()
False
sage: _.<x,y> = Zmod(36)[]
sage: (4 + 6*x).is_nilpotent()
False
sage: (6*x + 12*y + 18*x*y + 24*(x^2+y^2)).is_nilpotent()
True
```

**is_square** (root=False)

Test whether this polynomial is a square.

**INPUT:**

- root - if set to True, return a pair (True, root) where root is a square root or (False, None) if it is not a square.

**EXAMPLES:**

```python
sage: R.<a,b> = QQ[]
sage: a.is_square()
False
sage: ((1+a*b^2)^2).is_square()
True
sage: ((1+a*b^2)^2).is_square(root=True)
(True, a*b^2 + 1)
```

**is_symmetric** (group=None)

Return whether this polynomial is symmetric.

**INPUT:**

- group (default: symmetric group) – if set, test whether the polynomial is invariant with respect to the given permutation group

**EXAMPLES:**

```python
sage: R.<x,y,z> = QQ[]
sage: p = (x+y+z)**2 - 3 * (x+y)*(x+z)*(y+z)
sage: p.is_symmetric()
True
sage: (x + y - z).is_symmetric()
False
sage: R.one().is_symmetric()
True
sage: p = (x-y)*(y-z)*(z-x)
sage: p.is_symmetric()
False
```

(continues on next page)
An example with a GAP permutation group (here the quaternions):

```python
sage: R = PolynomialRing(QQ, 'x', 8)
sage: x = R(gens())
sage: p = sum(prod(x[i] for i in e) for e in [(0,1,2), (0,1,7), (0,2,7), (1,2,7),
... (3,4,5), (3,4,6), (3,5,6), (4,5,6)])
sage: p.is_symmetric(libgap.TransitiveGroup(8, 5))
# needs sage.groups
True
```

```
```

\textbf{is\_unit}()

Return \texttt{True} if \texttt{self} is a unit, that is, has a multiplicative inverse.

\textbf{EXAMPLES}:

```python
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: (x + y).is_unit()
False
sage: R(0).is_unit()
False
sage: R(-1).is_unit()
True
sage: R(-1 + x).is_unit()
False
sage: R(2).is_unit()
True
```

Check that \texttt{github issue #22454} is fixed:

```python
sage: _.<x,y> = Zmod(4)[]
sage: (1 + 2*x).is_unit()
True
sage: (x*y).is_unit()
False
sage: _.<x,y> = Zmod(36)[]
```

(continues on next page)
Polynomials, Release 10.3

(continued from previous page)

```
sage: (7 + 6*x + 12*y - 18*x*y).is_unit()
True
```

**iterator_exp_coeff**(as_ETuples=True)

Iterate over self as pairs of ((E)Tuple, coefficient).

**INPUT:**

- as_ETuples – (default: True) if True, iterate over pairs whose first element is an ETuple, otherwise as tuples

**EXAMPLES:**

```
sage: R.<a,b,c> = QQ[]
sage: f = a*c^3 + a^2*b + 2*b^4
sage: list(f.iterator_exp_coeff())
[((0, 4, 0), 2), ((1, 0, 3), 1), ((2, 1, 0), 1)]
sage: list(f.iterator_exp_coeff(as_ETuples=False))
[((0, 4, 0), 2), ((1, 0, 3), 1), ((2, 1, 0), 1)]
sage: R.<a,b,c> = PolynomialRing(QQ, 3, order='lex')
sage: f = a*c^3 + a^2*b + 2*b^4
sage: list(f.iterator_exp_coeff())
[((2, 1, 0), 1), ((1, 0, 3), 1), ((0, 4, 0), 2)]
```

**jacobian_ideal()**

Return the Jacobian ideal of the polynomial self.

**EXAMPLES:**

```
sage: R.<x,y,z> = QQ[]
sage: f = x^3 + y^3 + z^3
sage: f.jacobian_ideal()
Ideal (3*x^2, 3*y^2, 3*z^2) of
Multivariate Polynomial Ring in x, y, z over Rational Field
```

**lift**(I)

Given an ideal \( I = (f_1, \ldots, f_r) \) that contains self, find \( s_1, \ldots, s_r \) such that \( \text{self} = s_1 f_1 + \ldots + s_r f_r \).

**EXAMPLES:**

```
sage: # needs sage.rings.real_mpfr
sage: A.<x,y> = PolynomialRing(CC, 2, order='degrevlex')
sage: I = A.ideal([x^10 + x^9*y^2, y^8 - x^2*y^7 ])
sage: f = x*y^13 + y^12
sage: M = f.lift(I); M  # needs sage.libs.singular
[y^7, x^7*y^2 + x^8 + x^5*y^3 + x^6*y + x^3*y^4 + x^4*y^2 + x*y^5 + x^2*y^3 + x*y^4]
sage: sum(map(mul, zip(M, I.gens()))) == f  # needs sage/libs/singular
True
```

**macaulay_resultant**(\*args)

This is an implementation of the Macaulay resultant. It computes the resultant of universal polynomials as well as polynomials with constant coefficients. This is a project done in sage days 55. It’s based on the implementation in Maple by Manfred Minimair, which in turn is based on the references [CLO], [Can], [Mac]. It calculates the Macaulay resultant for a list of Polynomials, up to sign!
AUTHORS:
• Hao Chen, Solomon Vishkautsan (7-2014)

INPUT:
• args – a list of \( n - 1 \) homogeneous polynomials in \( n \) variables. works when \( \text{args[0]} \) is the list of polynomials, or \( \text{args} \) is itself the list of polynomials

OUTPUT:
• the Macaulay resultant

EXAMPLES:
The number of polynomials has to match the number of variables:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: y.macaulay_resultant(x + z)  # needs sage.modules
Traceback (most recent call last):
... TypeError: number of polynomials(= 2) must equal number of variables (= 3)
```

The polynomials need to be all homogeneous:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: y.macaulay_resultant([x + z, z + x^3])  # needs sage.modules
Traceback (most recent call last):
... TypeError: resultant for non-homogeneous polynomials is not supported
```

All polynomials must be in the same ring:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: S.<x,y> = PolynomialRing(QQ, 2)
sage: y.macaulay_resultant(z + x, z)  # needs sage.modules
Traceback (most recent call last):
... TypeError: not all inputs are polynomials in the calling ring
```

The following example recreates Proposition 2.10 in Ch.3 of Using Algebraic Geometry:

```
sage: K.<x,y> = PolynomialRing(ZZ, 2)
sage: flist, R = K._macaulay_resultant_universal_polynomials([1,1,2])
sage: flist[0].macaulay_resultant(flist[1:])  # needs sage.modules
u2^2*u4^2*u6 - 2*u1*u2*u4*u5*u6 + u1^2*u5^2*u6 - u2^2*u3*u4*u7 +
  u1*u2*u3*u5*u7 + u0*u2*u4*u5*u7 - u0*u1*u5^2*u7 + u1*u2*u3*u4*u8 - u0*u2*u4^2*u8 - u1^2*u3*u5*u8
+ u0*u1*u4*u5*u8 + u2^2*u3^2*u9 - 2*u0*u2*u3*u5*u9 + u0^2*u5^2*u9 - u1*u2*u3^2*u10 + u0*u2*u3*u4*u10 + u0*u1*u3*u5*u10 - u0^2*u4*u5*u10 + u1^2*u3^2*u11 - 2*u0*u1*u3*u4*u11 + u0^2*u4^2*u11
```

The following example degenerates into the determinant of a \( 3 \times 3 \) matrix:
Polynomials, Release 10.3

```
sage: K.<x,y> = PolynomialRing(ZZ, 2)
sage: flist, R = K._macaulay_resultant_universal_polynomials([1,1,1])
sage: flist[0].macaulay_resultant(flist[1:])  # needs sage.modules
-u2*u4*u6 + u1*u5*u6 + u2*u3*u7 - u0*u5*u7 - u1*u3*u8 + u0*u4*u8
```

The following example is by Patrick Ingram (arXiv 1310.4114):

```
sage: U = PolynomialRing(ZZ,y,2); y0,y1 = U.gens()
sage: R = PolynomialRing(U,x,3); x0,x1,x2 = R.gens()
sage: f0 = y0*x2^2 - x0^2 + 2*x1*x2
sage: f1 = y1*x2^2 - x1^2 + 2*x0*x2
sage: f2 = x0*x1 - x2^2
sage: f0.macaulay_resultant(f1, f2)  # needs sage.modules
y0^2*y1^2 - 4*y0^3 - 4*y1^3 + 18*y0*y1 - 27
```

a simple example with constant rational coefficients:

```
sage: R.<x,y,z,w> = PolynomialRing(QQ, 4)
sage: w.macaulay_resultant([z, y, x])  # needs sage.modules
1
```

an example where the resultant vanishes:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: (x + y).macaulay_resultant([y^2, x])  # needs sage.modules
0
```

an example of bad reduction at a prime \( p = 5 \):

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: y.macaulay_resultant([x^3 + 25*y^2*x, 5*z])  # needs sage.libs.pari sage.modules
125
```

The input can given as an unpacked list of polynomials:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: y.macaulay_resultant([x^3 + 25*y^2*x, 5*z])  # needs sage.libs.pari sage.modules
125
```

an example when the coefficients live in a finite field:

```
sage: F = FiniteField(11)
sage: R.<x,y,z,w> = PolynomialRing(F, 4)
sage: z.macaulay_resultant([x^3, 5*y, w])  # needs sage.modules sage.rings.finite_rings
4
```

example when the denominator in the algorithm vanishes(in this case the resultant is the constant term of the quotient of char polynomials of numerator/denominator):

3.1. Multivariate Polynomials and Polynomial Rings 323
When there are only 2 polynomials, the Macaulay resultant degenerates to the traditional resultant:

```python
sage: R.<x> = PolynomialRing(QQ, 1)
sage: f = x^2 + 1; g = x^5 + 1
sage: fh = f.homogenize()
sage: gh = g.homogenize()
sage: RH = fh.parent()
sage: f.resultant(g) == fh.macaulay_resultant(gh) # needs sage.modules
True
```

\textbf{map\_coefficients}(f, new\_base\_ring=None)

Returns the polynomial obtained by applying \( f \) to the non-zero coefficients of \( \text{self} \).

If \( f \) is a \texttt{sage.categories.map.Map}, then the resulting polynomial will be defined over the codomain of \( f \). Otherwise, the resulting polynomial will be over the same ring as \( \text{self} \). Set \texttt{new\_base\_ring} to override this behaviour.

\textbf{INPUT:}

- \( f \) – a callable that will be applied to the coefficients of \( \text{self} \).

- \texttt{new\_base\_ring} (optional) – if given, the resulting polynomial will be defined over this ring.

\textbf{EXAMPLES:}

```python
sage: k.<a> = GF(9); R.<x,y> = k[]; f = x*a + 2*x^3*y*a + a
# needs sage.rings.finite_rings
sage: f.map_coefficients(lambda a: a + 1) # needs sage.rings.finite_rings
(-a + 1)*x^3*y + (a + 1)*x + (a + 1)
```

Examples with different base ring:

```python
sage: # needs sage.rings.finite_rings
sage: R.<r> = GF(9); S.<s> = GF(81)
sage: h = Hom(R,S)[0]; h
Ring morphism:
    From: Finite Field in r of size 3^2
to:    Finite Field in s of size 3^4
defn: r |--> 2*s^3 + 2*s^2 + 1
sage: T.<X,Y> = R[]
sage: f = r*X + Y
sage: g = f.map_coefficients(h); g
(-s^3 - s^2 + 1)*X + Y
```

(continues on next page)
newton_polytope()  
Return the Newton polytope of this polynomial.

EXAMPLES:

```sage
sage: R.<x,y> = QQ[]
sage: f = 1 + x*y + x^3 + y^3
sage: P = f.newton_polytope(); P  
# needs sage.geometry.polyhedron
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
sage: P.is_simple()  
# needs sage.geometry.polyhedron
True
```

nth_root(n)  
Return a \( n \)-th root of this element.

If there is no such root, a ValueError is raised.

EXAMPLES:

```sage
sage: R.<x,y,z> = QQ[]
sage: a = 32 * (x*y + 1)^5 * (x+y+z)^5
sage: a.nth_root(5)
2*x^2*y + 2*x*y^2 + 2*x*y*z + 2*x + 2*y + 2*z
sage: b = x + 2*y + 3*z
sage: b.nth_root(42)
Traceback (most recent call last):
...  
ValueError: not a 42nd power
```

numerator()  
Return a numerator of \( self \), computed as \( self \cdot self.denominator() \).

Note that some subclasses may implement its own numerator function.

**Warning:** This is not the numerator of the rational function defined by \( self \), which would always be \( self \) since \( self \) is a polynomial.

EXAMPLES:
First we compute the numerator of a polynomial with integer coefficients, which is of course \texttt{self}.

```
sage: R.<x, y> = ZZ[]
sage: f = x^3 + 17*x + y + 1
sage: f.numerator()
x^3 + 17*x + y + 1
sage: f == f.numerator()
True
```

Next we compute the numerator of a polynomial over a number field.

```
sage: # needs sage.rings.number_field sage.symbolic
sage: R.<x,y> = NumberField(symbolic_expression(x^2+3), 'a')[x,y]
sage: f = (1/17)*y^19 - (2/3)*x + 1/3; f
1/17*y^19 - 2/3*x + 1/3
sage: f.numerator()
3*y^19 - 34*x + 17
sage: f == f.numerator()
False
```

We try to compute the numerator of a polynomial with coefficients in the finite field of 3 elements.

```
sage: K.<x,y,z> = GF(3)[']x', y', z']
sage: f = 2*x*z + 2*z^2 + 2*y + 1; f
-x*z - z^2 - y + 1
sage: f.numerator()
-x*z - z^2 - y + 1
```

We check that the computation the numerator and denominator are valid.

```
sage: # needs sage.rings.number_field sage.symbolic
sage: K = NumberField(symbolic_expression('x^3+2'), 'a')['x']['s,t']
sage: f = K.random_element()
sage: f.numerator() / f.denominator() == f
True
sage: R = RR['x,y,z']
sage: f = R.random_element()
sage: f.numerator() / f.denominator() == f
True
```

\texttt{polynomial(\textit{var})}

Let \textit{var} be one of the variables of the parent of \texttt{self}. This returns \texttt{self} viewed as a univariate polynomial in \textit{var} over the polynomial ring generated by all the other variables of the parent.

EXAMPLES:

```
sage: R.<x,w,z> = QQ[]
sage: f = x^3 + 3*w*x + w^5 + (17*w^3)*x + z^5
sage: f.polynomial(x)
x^3 + (17*w^3 + 3*w)*x + w^5 + z^5
sage: parent(f.polynomial(x))
Univariate Polynomial Ring in x over Multivariate Polynomial Ring in w, z over Rational Field
sage: f.polynomial(w)
w^5 + 17*x*w^3 + 3*x*w + z^5 + x^3
sage: f.polynomial(z)
z^5 + w^5 + 17*x*w^3 + x^3 + 3*x*w
```

(continues on next page)
sage: R.<x,w,z,k> = ZZ[]
sage: f = x^3 + 3*w*x + w^5 + (17*w^3)*x + z^5 + x*w*z*k + 5
sage: f.polynomial(x)
x^3 + (17*w^3 + w*z*k + 3*w)*x + w^5 + z^5 + 5
sage: f.polynomial(w)
w^5 + 17*x*w^3 + (x*z*k + 3*x)*w + z^5 + x^3 + 5
sage: f.polynomial(z)
z^5 + x*w*k*z + w^5 + 17*x*w^3 + x^3 + 3*x*w + 5
sage: f.polynomial(k)
x*w*z*k + w^5 + z^5 + 17*x*w^3 + x^3 + 3*x*w + 5
sage: R.<x,y> = GF(5)[]
sage: f = x^2 + x + y
sage: f.polynomial(x)
x^2 + x + y
sage: f.polynomial(y)
y + x^2 + x

reduced_form(**kwds)

Return a reduced form of this polynomial.

The algorithm is from Stoll and Cremona’s “On the Reduction Theory of Binary Forms” [CS2003]. This takes a two variable homogeneous polynomial and finds a reduced form. This is a $SL(2,\mathbb{Z})$-equivalent binary form whose covariant in the upper half plane is in the fundamental domain. If the polynomial has multiple roots, they are removed and the algorithm is applied to the portion without multiple roots.

This reduction should also minimize the sum of the squares of the coefficients, but this is not always the case. By default the coefficient minimizing algorithm in [HS2018] is applied. The coefficients can be minimized either with respect to the sum of their squares or the maximum of their global heights.

A portion of the algorithm uses Newton’s method to find a solution to a system of equations. If Newton’s method fails to converge to a point in the upper half plane, the function will use the less precise $z_0$ covariant from the $Q_0$ form as defined on page 7 of [CS2003]. Additionally, if this polynomial has a root with multiplicity at least half the total degree of the polynomial, then we must also use the $z_0$ covariant. See [CS2003] for details.

Note that, if the covariant is within error_limit of the boundary but outside the fundamental domain, our function will erroneously move it to within the fundamental domain, hence our conjugation will be off by 1. If you don’t want this to happen, decrease your error_limit and increase your precision.

Implemented by Rebecca Lauren Miller as part of GSOC 2016. Smallest coefficients added by Ben Hutz July 2018.

INPUT:

keywords:

• prec – integer, sets the precision (default: 300)
• return_conjugation – boolean. Returns element of $SL(2,\mathbb{Z})$ (default: True)
• error_limit – sets the error tolerance (default: 0.000001)
• smallest_coeffs – (default: True), boolean, whether to find the model with smallest coefficients
• norm_type – either 'norm' or 'height'. What type of norm to use for smallest coefficients
• emb – (optional) embedding of based field into $\mathbb{C}$

OUTPUT:

• a polynomial (reduced binary form)
• a matrix (element of $SL(2, \mathbb{Z})$)

**Todo:** When Newton’s Method doesn’t converge to a root in the upper half plane. Now we just return $z_0$. It would be better to modify and find the unique root in the upper half plane.

**EXAMPLES:**

```python
sage: R.<x,h> = PolynomialRing(QQ)
sage: f = 19*x^8 - 262*x^7*h + 1507*x^6*h^2 - 4784*x^5*h^3 + 9202*x^4*h^4
    - 10962*x^3*h^5 + 7844*x^2*h^6 - 3040*x*h^7 + 475*h^8
sage: f.reduced_form(prec=200, smallest_coeffs=False)  # needs sage.modules sage.rings.complex_interval_field

(−x^8 − 2*x^7*h + 7*x^6*h^2 + 16*x^5*h^3 + 2*x^4*h^4 − 2*x^3*h^5 + 4*x^2*h^6 − 5*h^8,
  [ 1 -2]
  [ 1 -1])
```

An example where the multiplicity is too high:

```python
sage: R.<x,y> = PolynomialRing(QQ)
sage: f = x^3 + 378666*x^2*y - 12444444*x*y^2 + 1234567890*y^3
sage: j = f * (x-545*y)^9
sage: j.reduced_form(prec=200, smallest_coeffs=False)  # needs sage.modules sage.rings.complex_interval_field

Traceback (most recent call last):
  ... ValueError: cannot have a root with multiplicity >= 12/2
```

An example where Newton’s Method does not find the right root:

```python
sage: R.<x,y> = PolynomialRing(QQ)
sage: F = x^6 + 3*x^5*y - 8*x^4*y^2 - 2*x^3*y^3 - 44*x^2*y^4 - 8*x*y^5
sage: F.reduced_form(smallest_coeffs=False, prec=400)  # needs sage.modules sage.rings.complex_interval_field

Traceback (most recent call last):
  ... ArithmeticError: Newton's method converged to z not in the upper half plane
```

An example with covariant on the boundary, therefore a non-unique form:

```python
sage: R.<x,y> = PolynomialRing(QQ)
sage: F = 5*x^2*y - 5*x*y^2 - 30*y^3
sage: F.reduced_form(smallest_coeffs=False)  # needs sage.modules sage.rings.complex_interval_field

(5*x^2*y + 5*x*y^2 - 30*y^3, [0 1])
```

An example where precision needs to be increased:

```python
sage: R.<x,y> = PolynomialRing(QQ)
sage: F = (-16*x^7 - 114*x^6*y - 345*x^5*y^2 - 599*x^4*y^3

(continues on next page)
Polynomials, Release 10.3

(continued from previous page)

....:  - 666*x^3*y^4 - 481*x^2*y^5 - 207*x*y^6 - 40*y^7)
sage: F.reduced_form(prec=50, smallest_coeffs=False)  # ...
→ needs sage.modules sage.rings.complex_interval_field
Traceback (most recent call last):
...
ValueError: accuracy of Newton's root not within tolerance(0.000012... > 1e-06),
increase precision
sage: F.reduced_form(prec=100, smallest_coeffs=False)  # ...
→ needs sage.modules sage.rings.complex_interval_field
([-1 -1]
-666*x^3*y^4 - 481*x^2*y^5 - 207*x*y^6 - 40*y^7, [ 1 0])

sage: R.<x,y> = PolynomialRing(QQ)
sage: F = -8*x^4 - 3933*x^3*y - 725085*x^2*y^2 - 59411592*x*y^3 -
→ 1825511633*y^4
sage: F.reduced_form(return_conjugation=False)  # ...
→ needs sage.modules sage.rings.complex_interval_field
x^4 + 9*x^3*y - 3*x*y^3 - 8*y^4

sage: R.<x,y> = QQ[]
sage: F = -2*x^3 + 2*x^2*y + 3*x*y^2 + 127*y^3
sage: F.reduced_form()  # ...
→ needs sage.modules sage.rings.complex_interval_field
(1 4)
-2*x^3 - 22*x^2*y - 77*x*y^2 + 43*y^3, [0 1])

sage: R.<x,y> = QQ[]
sage: F = -2*x^3 + 2*x^2*y + 3*x*y^2 + 127*y^3
sage: F.reduced_form(norm_type='height')  # ...
(5 4)
-58*x^3 - 47*x^2*y + 52*x*y^2 + 43*y^3, [1 1])

sage: R.<x,y,z> = PolynomialRing(QQ)
sage: F = x^4 + x^3*y*z + y^2*z
sage: F.reduced_form()  # ...
→ needs sage.modules sage.rings.complex_interval_field
Traceback (most recent call last):
...
ValueError: (=x^3*y*z + x^4 + y^2*z) must have two variables

sage: R.<x,y> = PolynomialRing(ZZ)
sage: F = -8*x^6 - 3933*x^3*y - 725085*x^2*y^2 - 59411592*x*y^3 - 99*y^6
sage: F.reduced_form(return_conjugation=False)  # ...
→ needs sage.modules sage.rings.complex_interval_field
Traceback (most recent call last):
...
ValueError: (=-8*x^6 - 99*y^6 - 3933*x^3*y - 725085*x^2*y^2 -
...
59411592*x*y^3) must be homogeneous

\begin{verbatim}
sage: R.<x,y> = PolynomialRing(RR)
sage: F = (217.992172373276*x^3 + 96023.1505442490*x^2*y +
    ...:     1.40987971253579e7*x*y^2 + 6.90016027113216e8*y^3)
sage: F.reduced_form(smallest_coeffs=False)  # tol 1e-8
˓→ needs sage.modules sage.rings.complex_interval_field
(39.5673942565918*x^3 + 111.874026298523*x^2*y +
  231.052762985229*x*y^2 - 138.380829811096*y^3,
[-147 -148]
[ 1  1])
\end{verbatim}

\begin{verbatim}
sage: R.<x,y> = PolynomialRing(CC)
˓→ needs sage.rings.real_mpfr
sage: F = ((0.759099196558145 + 0.845425869641446*CC.0)*x^3 +
    ...:     (84.8317207268542 + 93.8840848648033*CC.0)*x^2*y +
    ...:     (3159.07040755858 + 3475.330377779*CC.0)*x*y^2 +
    ...:     (39202.5965389079 + 42882.5139724962*CC.0)*y^3)
sage: F.reduced_form(smallest_coeffs=False)  # tol 1e-11
˓→ needs sage.modules sage.rings.complex_interval_field sage.rings.real_mpfr
(-0.759099196558145 - 0.845425869641446*I)*x^3 +
  (-0.571709908900118 - 0.0418133346027929*I)*x^2*y +
  (0.856525964330103 - 0.0721403997649759*I)*x*y^2 +
  (-0.9655310444130330 + 0.754252314465703*I)*y^3,
[-1  37]
[ 0 -1]
\end{verbatim}

\textbf{specialization (D=None, phi=None)}

Specialization of this polynomial.

Given a family of polynomials defined over a polynomial ring. A specialization is a particular member of that family. The specialization can be specified either by a dictionary or a \texttt{SpecializationMorphism}.

**INPUT:**

- \texttt{D} – dictionary (optional)
- \texttt{phi} – \texttt{SpecializationMorphism} (optional)

**OUTPUT:** a new polynomial

**EXAMPLES:**

\begin{verbatim}
sage: R.<c> = PolynomialRing(QQ)
sage: S.<x,y> = PolynomialRing(R)
sage: F = x^2 + c*y^2
sage: F.specialization({c:2})
x^2 + 2*y^2
\end{verbatim}
Polynomials, Release 10.3

```
sage: S.<a,b> = PolynomialRing(QQ)
sage: P.<x,y,z> = PolynomialRing(S)
sage: RR.<c,d> = PolynomialRing(P)
sage: f = a*x^2 + b*y^3 + c*y^2 - b*a*d + d^2 - a*c*b*z^2
sage: f.specialization({a:2, z:4, d:2})
(y^2 - 32*b)*c + b*y^3 + 2*x^2 - 4*b + 4
```

Check that we preserve multi-versus uni-variate:

```
sage: R.<l> = PolynomialRing(QQ, 1)
sage: S.<k> = PolynomialRing(R)
sage: K.<a, b, c> = PolynomialRing(S)
sage: F = a*k^2 + b*l + c^2
sage: F.specialization({b:56, c:5}).parent()
Univariate Polynomial Ring in a over Univariate Polynomial Ring in k
over Multivariate Polynomial Ring in l over Rational Field
```

**subresultants** *(other, variable=None)*

Return the nonzero subresultant polynomials of self and other.

**INPUT:**
- other – a polynomial

**OUTPUT:** a list of polynomials in the same ring as self

**EXAMPLES:**

```
sage: R.<x,y> = QQ[]
sage: p = (y^2 + 6)*(x - 1) - y*(x^2 + 1)
sage: q = (x^2 + 6)*(y - 1) - x*(y^2 + 1)
sage: p.subresultants(q, y)
[2*x^6 - 22*x^5 + 102*x^4 - 274*x^3 + 488*x^2 - 552*x + 288,
 -x^3 - x^2*y + 6*x^2 + 5*x*y - 11*x - 6*y + 6]
sage: p.subresultants(q, x)
[2*y^6 - 22*y^5 + 102*y^4 - 274*y^3 + 488*y^2 - 552*y + 288,
 x*y^2 + y^3 - 5*x*y - 6*y^2 + 6*x + 11*y - 6]
```

**sylvester_matrix** *(right, variable=None)*

Given two nonzero polynomials self and right, return the Sylvester matrix of the polynomials with respect to a given variable.

Note that the Sylvester matrix is not defined if one of the polynomials is zero.

**INPUT:**
- self, right – multivariate polynomials
- variable – optional, compute the Sylvester matrix with respect to this variable. If variable is not provided, the first variable of the polynomial ring is used.

**OUTPUT:**
- The Sylvester matrix of self and right.

**EXAMPLES:**

```
sage: R.<x, y> = PolynomialRing(ZZ)
sage: f = (y + 1)*x + 3*x^2
sage: g = (y + 2)*x + 4*x^2
```

(continues on next page)
If the polynomials share a non-constant common factor then the determinant of the Sylvester matrix will be zero:

```python
sage: M = f.sylvester_matrix(g, x)  # needs sage.modules
sage: M
[3 y + 1 0 0]
[0 3 y + 1 0]
[4 y + 2 0 0]
[0 4 y + 2 0]
```

If both polynomials are of positive degree with respect to variable, the determinant of the Sylvester matrix is the resultant:

```python
sage: M.determinant()  # needs sage.modules
0
sage: f.sylvester_matrix(1 + g, x).determinant()  # needs sage.modules
y^2 - y + 7
```

`truncate`(var, n)

Returns a new multivariate polynomial obtained from `self` by deleting all terms that involve the given variable to a power at least n.

`weighted_degree`(*weights)

Return the weighted degree of `self`, which is the maximum weighted degree of all monomials in `self`; the weighted degree of a monomial is the sum of all powers of the variables in the monomial, each power multiplied with its respective weight in `weights`.

This method is given for convenience. It is faster to use polynomial rings with weighted term orders and the standard `degree` function.

**INPUT:**

- `weights`: Either individual numbers, an iterable or a dictionary, specifying the weights of each variable. If it is a dictionary, it maps each variable of `self` to its weight. If it is a sequence of individual numbers or a tuple, the weights are specified in the order of the generators as given by `self.parent().gens()`.

**EXAMPLES:**

```python
sage: R.<x,y,z> = GF(7)[]
sage: p = x^3 + y + x*z^2
sage: p.weighted_degree((z:0, x:1, y:2))
3
sage: p.weighted_degree((1, 2, 0))
3
sage: p.weighted_degree((1, 4, 2))
```

(continues on next page)
You may also work with negative weights

```python
sage: p.weighted_degree(-1, -2, -1)
-2
```

Note that only integer weights are allowed

```python
sage: p.weighted_degree(x, 1, 1)
Traceback (most recent call last):
...TypeError: unable to convert non-constant polynomial x to Integer Ring
```

The `weighted_degree()` coincides with the `degree()` of a weighted polynomial ring, but the latter is faster.

```python
sage: K = PolynomialRing(QQ, 'x,y', order=TermOrder('wdegrevlex', (2,3)))
sage: p = K.random_element(10)
sage: p.degree() == p.weighted_degree(2,3)
True
```

```
class sage.rings.polynomial.multi_polynomial.MPolynomial_libsingular
    Bases: MPolynomial

    Abstract base class for MPolynomial_libsingular

    This class is defined for the purpose of isinstance() tests. It should not be instantiated.

    EXAMPLES:

    sage: from sage.rings.polynomial.multi_polynomial import MPolynomial_libsingular
    sage: R1.<x> = QQ[]
    sage: isinstance(x, MPolynomial_libsingular)
    False
    sage: R2.<y,z> = QQ[]
    sage: isinstance(y, MPolynomial_libsingular)  # needs sage.libs.singular
    True

    By design, there is a unique direct subclass:

    sage: len(sage.rings.polynomial.multi_polynomial.MPolynomial_libsingular.__subclasses__()) <= 1
    True
```

```python
sage.rings.polynomial.multi_polynomial.is_MPolynomial(x)
```
3.1.4 Multivariate Polynomial Rings over Generic Rings

Sage implements multivariate polynomial rings through several backends. This generic implementation uses the classes \texttt{PolyDict} and \texttt{ETuple} to construct a dictionary with exponent tuples as keys and coefficients as values.

**AUTHORS:**

- David Joyner and William Stein
- Kiran S. Kedlaya (2006-02-12): added Macaulay2 analogues of Singular features
- Martin Albrecht (2006-04-21): reorganize class hierarchy for singular rep
- Martin Albrecht (2007-04-20): reorganized class hierarchy to support Pyrex implementations

**EXAMPLES:**

We construct the Frobenius morphism on \(F_5[x, y, z]\) over \(F_5\):

```python
sage: R.<x,y,z> = GF(5)[]
sage: frob = R.hom([x^5, y^5, z^5])
sage: frob(x^2 + 2*y - z^4)
-z^20 + x^10 + 2*y^5
sage: frob((x + 2*y)^3)  # needs sage.rings.finite_rings
x^15 + x^10*y^5 + 2*x^5*y^10 - 2*y^15
```

We make a polynomial ring in one variable over a polynomial ring in two variables:

```python
sage: R.<x, y> = PolynomialRing(QQ, 2)
sage: S.<t> = PowerSeriesRing(R)
sage: t*(x+y)
(x + y)*t
```

**class**

\texttt{sage.rings.polynomial.multi_polynomial_ring.MPolynomialRing_macaulay2_repr}

\texttt{Bases: object}

A mixin class for polynomial rings that support conversion to Macaulay2.

**class** \texttt{sage.rings.polynomial.multi_polynomial_ring.MPolynomialRing_polydict}(\textit{base_ring, n, names, order})

\texttt{Bases: MPolynomialRing_macaulay2_repr, PolynomialRing_singular_repr, MPolynomialRing_base}

Multivariable polynomial ring.

**EXAMPLES:**

```python
sage: R = PolynomialRing(Integers(12), 'x', 5); R
Multivariate Polynomial Ring in x0, x1, x2, x3, x4 over Ring of integers modulo 12
sage: loads(R.dumps()) == R
True
```
Element_hidden

alias of MPolynomial_polydict

monomial_all_divisors(t)
Return a list of all monomials that divide t, coefficients are ignored.

INPUT:
• t - a monomial.

OUTPUT: a list of monomials.

EXAMPLES:
sage: from sage.rings.polynomial.multi_polynomial_ring import MPolynomialRing_polydict_domain
sage: P.<x,y,z> = MPolynomialRing_polydict_domain(QQ,3, order='degrevlex')
sage: P.monomial_all_divisors(x^2*z^3)
[x, x^2, z, x*z, x^2*z, z^2, x*z^2, x^2*z^2, z^3, x*z^3, x^2*z^3]

ALGORITHM: addwithcarry idea by Toon Segers

monomial_divides(a, b)
Return False if a does not divide b and True otherwise.

INPUT:
• a - monomial
• b - monomial

OUTPUT: Boolean

EXAMPLES:
sage: P.<x,y,z> = PolynomialRing(ZZ,3, order='degrevlex')
sage: P.monomial_divides(x*y*z, x^3*y^2*z^4)
True
sage: P.monomial_divides(x^3*y^2*z^4, x*y*z)
False

monomial_lcm(f, g)
LCM for monomials. Coefficients are ignored.

INPUT:
• f - monomial.
• g - monomial.

OUTPUT: monomial.

EXAMPLES:
sage: from sage.rings.polynomial.multi_polynomial_ring import MPolynomialRing_polydict_domain
sage: P.<x,y,z> = MPolynomialRing_polydict_domain(QQ,3, order='degrevlex')
sage: P.monomial_lcm(3/2*x*y, x)
x*y
\textbf{monomial\_pairwise\_prime}(h, g)

Return \texttt{True} if \(h\) and \(g\) are pairwise prime.

Both are treated as monomials.

\textbf{INPUT:}

- \(h\) - monomial.
- \(g\) - monomial.

\textbf{OUTPUT:} Boolean.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: from sage.rings.polynomial.multi_polynomial_ring import MPolynomialRing_
sage: →polydict_domain
sage: P.<x,y,z> = MPolynomialRing_polydict_domain(QQ, 3, order='degrevlex')
sage: P.monomial_pairwise_prime(x^2*z^3, y^4)
True
sage: P.monomial_pairwise_prime(1/2*x^3*y^2, 3/4*y^3)
False
\end{verbatim}

\textbf{monomial\_quotient}(f, g, \texttt{coeff=False})

Return \(f/g\), where both \(f\) and \(g\) are treated as monomials.

Coefficients are ignored by default.

\textbf{INPUT:}

- \(f\) - monomial.
- \(g\) - monomial.
- \texttt{coeff} - divide coefficients as well (default: \texttt{False}).

\textbf{OUTPUT:} monomial.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: from sage.rings.polynomial.multi_polynomial_ring import MPolynomialRing_
sage: →polydict_domain
sage: P.<x,y,z> = MPolynomialRing_polydict_domain(QQ, 3, order='degrevlex')
sage: P.monomial_quotient(3/2*x*y, x)
y
sage: P.monomial_quotient(3/2*x*y, 2*x, \texttt{coeff=True})
3/4*y
\end{verbatim}

\textbf{Note:} Assumes that the head term of \(f\) is a multiple of the head term of \(g\) and return the multiplicant \(m\). If this rule is violated, funny things may happen.

\textbf{monomial\_reduce}(f, G)

Try to find a \(g\) in \(G\) where \(g.\text{lm}()\) divides \(f\).

If found, \((\texttt{flt}, g)\) is returned, \((0, 0)\) otherwise, where \(\texttt{flt}\) is \(f/g.\text{lm}()\). It is assumed that \(G\) is iterable and contains ONLY elements in this ring.

\textbf{INPUT:}
• \( f \) - monomial
• \( G \) - list/set of mpolynomials

EXAMPLES:

```python
sage: from sage.rings.polynomial.multi_polynomial_ring import MPolynomialRing_
˓→polydict_domain
sage: P.<x,y,z>=MPolynomialRing_polydict_domain(QQ,3, order='degrevlex')
sage: f = x*y^2
sage: G = [3/2*x^3 + y^2 + 1/2, 1/4*x*y + 2/7, P(1/2)]
sage: P.monomial_reduce(f,G)
(y, 1/4*x*y + 2/7)
```

```python
sage: from sage.rings.polynomial.multi_polynomial_ring import MPolynomialRing_
˓→polydict_domain
sage: P.<x,y,z> = MPolynomialRing_polydict_domain(Zmod(23432),3, order=
˓→'degrevlex')
sage: f = x*y^2
sage: G = [3*x^3 + y^2 + 2, 4*x*y + 7, P(2)]
sage: P.monomial_reduce(f,G)
(y, 4*x*y + 7)
```

**sum** *(terms)*

Return a sum of elements of this multipolynomial ring.

This is method is much faster than the Python builtin \texttt{sum()}.

EXAMPLES:

```python
sage: R = QQ['x']
sage: S = R['y, z']
sage: x = R.gen()
sage: y, z = S.gens()
sage: S.sum([x*y, 2*x^2*z - 2*x*y, 1 + y + z])
(-x + 1)*y + (2*x^2 + 1)*z + 1
```

Comparison with builtin \texttt{sum}:

```python
sage: sum([x*y, 2*x^2*z - 2*x*y, 1 + y + z])
(-x + 1)*y + (2*x^2 + 1)*z + 1
```

**class** \texttt{sage.rings.polynomial.multi_polynomial_ring.MPolynomialRing_polydict_domain(\texttt{base_ring}, \texttt{n}, \texttt{names}, \texttt{order})}

**ideal** *(\texttt{*gens}, \texttt{**kwds})*

Create an ideal in this polynomial ring.

**is_field** *(\texttt{proof=True})*

**is_integral_domain** *(\texttt{proof=True})*
3.1.5 Generic Multivariate Polynomials

AUTHORS:

- David Joyner: first version
- William Stein: use dict’s instead of lists
- Martin Albrecht malb@informatik.uni-bremen.de: some functions added
- Kiran S. Kedlaya (2006-02-12): added Macaulay2 analogues of some Singular features
- William Stein (2006-04-19): added e.g., \( f[1, 3] \) to get coeff of \( xy^3 \); added examples of the new \( R.x, y = \text{PolynomialRing}(\mathbb{Q}, 2) \) notation.
- Martin Albrecht: improved singular coercions (restructured class hierarchy) and added ETuples
- Robert Bradshaw (2007-08-14): added support for coercion of polynomials in a subset of variables (including multi-level univariate rings)

EXAMPLES:

We verify Lagrange’s four squares identity:

```python
sage: R.<a0,a1,a2,a3,b0,b1,b2,b3> = QQbar[]
# needs sage.rings.number_field
sage: ((a0^2 + a1^2 + a2^2 + a3^2) * (b0^2 + b1^2 + b2^2 + b3^2) == # needs sage.rings.number_field
    (a0*b0 - a1*b1 - a2*b2 - a3*b3)^2 + (a0*b1 + a1*b0 + a2*b3 - a3*b2)^2 +
    (a0*b2 - a1*b3 + a2*b0 + a3*b1)^2 + (a0*b3 + a1*b2 - a2*b1 + a3*b0)^2)
True
```

```
class sage.rings.polynomial.multi_polynomial_element.MPolynomial_element (parent, x)
Bases: MPolynomial

Generic multivariate polynomial.

This implementation is based on the PolyDict.

Todo: As mentioned in their docstring, PolyDict objects never clear zeros. In all arithmetic operations on MPolynomial_element there is an additional call to the method remove_zeros to clear them. This is not ideal because of the presence of inexact zeros, see github issue #35174.

```
2
sage: R(0).number_of_terms()
0
sage: f = (x+y)^100
sage: f.number_of_terms()
101

The method `hamming_weight()` is an alias:

```
sage: f.hamming_weight()  # Needs sage.rings.real_mpfr
101
```

### number_of_terms()

Return the number of non-zero coefficients of this polynomial.

This is also called weight, `hamming_weight()` or sparsity.

**EXAMPLES:**

```
sage: # Needs sage.rings.real_mpfr
sage: R.<x, y> = CC[

sage: f = x^3 - y
sage: f.number_of_terms()
2
sage: R(0).number_of_terms()
0
sage: f = (x+y)^100
sage: f.number_of_terms()
101
```

The method `hamming_weight()` is an alias:

```
sage: f.hamming_weight()  # Needs sage.rings.real_mpfr
101
```

class `sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict` (parent, x)

**Bases:** `Polynomial_singular_repr, MPolynomial_element`

Multivariate polynomials implemented in pure python using polyncts.

**coefficient (degrees)**

Return the coefficient of the variables with the degrees specified in the python dictionary `degrees`. Mathematically, this is the coefficient in the base ring adjoined by the variables of this ring not listed in `degrees`. However, the result has the same parent as this polynomial.

This function contrasts with the function `monomial_coefficient` which returns the coefficient in the base ring of a monomial.

**INPUT:**

- **degrees** - Can be any of:
  - a dictionary of degree restrictions
  - a list of degree restrictions (with None in the unrestricted variables)
Polynomials, Release 10.3

– a monomial (very fast, but not as flexible)

OUTPUT: element of the parent of self

See also:

For coefficients of specific monomials, look at \texttt{monomial\_coefficient()}.

EXAMPLES:

\begin{verbatim}
  sage: # needs sage.rings.number_field
  sage: R.<x, y> = QQbar[]
  sage: f = 2 * x * y
  sage: c = f.coefficient({x: 1, y: 1}); c
  2
  sage: c.parent()
  Multivariate Polynomial Ring in x, y over Algebraic Field
  sage: c in PolynomialRing(QQbar, 2, names=['x', 'y'])
  True
  sage: f = y^2 - x^9 - 7*x + 5*x*y
  sage: f.coefficient({y: 1})
  5*x
  sage: f.coefficient({y: 0})
  -x^9 + (-7)*x
  sage: f.coefficient({x: 0, y: 0})
  0
  sage: f = (1+y+y^2) * (1+x+x^2)
  sage: f.coefficient({x: 0})
  y^2 + y + 1
  sage: f.coefficient([0, None])
  y^2 + y + 1
  sage: f.coefficient(x)
  y^2 + y + 1
  sage: # Be aware that this may not be what you think!
  sage: # The physical appearance of the variable x is deceiving --
  # particularly if the exponent would be a variable.
  sage: f.coefficient(x^0) # outputs the full polynomial
  x^2*y^2 + x^2*y + x*y^2 + x^2 + x*y + y^2 + x + y + 1

  sage: # needs sage.rings.real_mpfr
  sage: R.<x,y> = RR[]
  sage: f = x*y + 5
  sage: c = f.coefficient({x: 0, y: 0}); c
  5.00000000000000
  sage: parent(c)
  Multivariate Polynomial Ring in x, y over Real Field with 53 bits of precision
\end{verbatim}

AUTHORS:

• Joel B. Mohler (2007-10-31)

\texttt{constant\_coefficient()}.

Return the constant coefficient of this multivariate polynomial.

EXAMPLES:

\begin{verbatim}
  sage: # needs sage.rings.number_field
  sage: R.<x,y> = QQbar[]
  sage: f = 3*x^2 - 2*y + 7*x^2*y^2 + 5
  sage: f.constant_coefficient()
\end{verbatim}

(continues on next page)
degree (\texttt{x=None, std\_grading=False})

Return the degree of \texttt{self} in \texttt{x}, where \texttt{x} must be one of the generators for the parent of \texttt{self}.

INPUT:

\begin{itemize}
\item \texttt{x} - multivariate polynomial (a generator of the parent of \texttt{self}). If \texttt{x} is not specified (or is \texttt{None}), return the total degree, which is the maximum degree of any monomial. Note that a weighted term ordering alters the grading of the generators of the ring; see the tests below. To avoid this behavior, set the optional argument \texttt{std\_grading=True}.
\end{itemize}

OUTPUT: integer

EXAMPLES:

\begin{verbatim}
sage: R.<x,y> = RR[]
sage: f = y^2 - x^9 - x
sage: f.degree(x)
9
sage: f.degree(y)
2
sage: (y^10*x - 7*x^2*y^5 + 5*x^3).degree(x)
3
sage: (y^10*x - 7*x^2*y^5 + 5*x^3).degree(y)
10
\end{verbatim}

Note that total degree takes into account if we are working in a polynomial ring with a weighted term order.

\begin{verbatim}
sage: R = PolynomialRing(QQ, 'x,y', order=TermOrder('wdeglex', (2,3)))
sage: x,y = R.gens()
sage: x.degree()
2
sage: y.degree()
3
sage: x.degree(y), x.degree(x), y.degree(x), y.degree(y)
(0, 1, 0, 1)
sage: f = x^2*y + x*y^2
sage: f.degree(x)
2
sage: f.degree(y)
2
sage: f.degree()
8
sage: f.degree(std_grading=True)
3
\end{verbatim}

Note that if \texttt{x} is not a generator of the parent of \texttt{self}, for example if it is a generator of a polynomial algebra which maps naturally to this one, then it is converted to an element of this algebra. (This fixes the problem reported in github issue #17366.)

\begin{verbatim}
sage: x, y = ZZ['x','y'].gens()
sage: GF(3037000453)['x','y'].gen(0).degree(x) #...
\end{verbatim}

\texttt{needs sage.rings.finite\_rings}
degrees()  
Returns a tuple (precisely - an ETuple) with the degree of each variable in this polynomial. The list of degrees is, of course, ordered by the order of the generators.

EXAMPLES:

sage: # needs sage.rings.number_field
sage: R.<x,y,z> = PolynomialRing(QQbar)
sage: f = 3*x^2 - 2*y + 7*x^2*y^2 + 5
sage: f.degrees()
(2, 2, 0)
sage: f = x^2 + z^2
sage: f.degrees()
(2, 0, 2)
sage: f.total_degree()  # this simply illustrates that total degree is not...
                   # the sum of the degrees
2
sage: R.<x,y,z,u> = PolynomialRing(QQbar)
sage: f = (1-x) * (1+y+z+x^3)^5
sage: f.degrees()
(16, 5, 5, 0)
sage: R(0).degrees()
(0, 0, 0, 0)

dict()  
Return underlying dictionary with keys the exponents and values the coefficients of this polynomial.

exponents (as_ETuples=True)  
Return the exponents of the monomials appearing in self.

INPUT:

• as_ETuples – (default: True): return the list of exponents as a list of ETuples

OUTPUT:

The list of exponents as a list of ETuples or tuples.
Polynomials, Release 10.3

(continued from previous page)

\[
\text{sage: } \text{f.exponents()}
\]
\[
[(3, 0, 0), (0, 2, 0), (0, 1, 0)]
\]

By default the list of exponents is a list of ETuples:

\[
\text{sage: type(f.exponents()[0])}
\]
\[
<\text{class 'sage.rings.polynomial.polydict.ETuple'>}
\]
\[
\text{sage: type(f.exponents(as_ETuples=False)[0])}
\]
\[
<\text{... 'tuple'}>
\]

\textbf{factor (proof=None)}

Compute the irreducible factorization of this polynomial.

**INPUT:**

- \texttt{proof} - insist on provably correct results (default: True unless explicitly disabled for the "polynomial" subsystem with \texttt{sage.structure.proof.proof.WithProof}.)

\textbf{global_height (prec=None)}

Return the (projective) global height of the polynomial.

This returns the absolute logarithmic height of the coefficients thought of as a projective point.

**INPUT:**

- \texttt{prec} – desired floating point precision (default: default \texttt{RealField} precision).

**OUTPUT:** a real number.

**EXAMPLES:**

\[
\text{sage: R.<x,y> = PolynomialRing(QQbar, 2)}
\]
\[
\text{sage: f = QQbar(i)*x^2 + 3*x*y}
\]
\[
\text{sage: f.global_height()}
\]
\[
1.09861228866811
\]

Scaling should not change the result:

\[
\text{sage: R.<x, y> = PolynomialRing(QQbar, 2)}
\]
\[
\text{sage: f = 1/25*x^2 + 25/3*x + 1 + QQbar(sqrt(2))*y^2}
\]
\[
\text{sage: f.global_height()}
\]
\[
6.43775164973640
\]
\[
\text{sage: g = 100 * f}
\]
\[
\text{sage: g.global_height()}
\]
\[
6.43775164973640
\]

(continues on next page)
Polynomials, Release 10.3

```python
sage: f = 12 * q
sage: f.global_height()
0.000000000000000

sage: R.<x,y> = PolynomialRing(QQ, implementation='generic')
sage: f = 1/123*x*y + 12
sage: f.global_height(prec=2)
#␣˓→needs sage.symbolic
8.0

sage: R.<x,y> = PolynomialRing(QQ, implementation='generic')
sage: f = 0*x*y
sage: f.global_height()
˓→needs sage.rings.real_mpfr
0.000000000000000
```

integral(var=None)

Integrate self with respect to variable var.

**Note:** The integral is always chosen so the constant term is 0.

If var is not one of the generators of this ring, integral(var) is called recursively on each coefficient of this polynomial.

**EXAMPLES:**

On polynomials with rational coefficients:

```python
sage: x, y = PolynomialRing(QQ, 'x, y').gens()
sage: ex = x*y + x - y
sage: it = ex.integral(x); it
1/2*x^2*y + 1/2*x^2 - x*y
sage: it.parent() == x.parent()
True
sage: R = ZZ['x']['y, z']
sage: y, z = R.gens()
sage: R.an_element().integral(y).parent()
Multivariate Polynomial Ring in y, z
over Univariate Polynomial Ring in x over Rational Field
```

On polynomials with coefficients in power series:

```python
sage: # needs sage.rings.number_field
sage: R.<t> = PowerSeriesRing(QQbar)
sage: S.<x, y> = PolynomialRing(R)
sage: f = (t^2 + O(t^3))*x^2*y^3 + (37*t^4 + O(t^5))*x^3
sage: f.parent()
Multivariate Polynomial Ring in x, y
over Power Series Ring in t over Algebraic Field
sage: f.integral(x)  # with respect to x
(1/3*t^2 + O(t^3))*x^3*y^3 + (37/4*t^4 + O(t^5))*x^4
sage: f.integral(x).parent()
Multivariate Polynomial Ring in x, y
over Power Series Ring in t over Algebraic Field
```
inverse_of_unit()

Return the inverse of a unit in a ring.

is_constant()

Return True if self is a constant and False otherwise.

EXAMPLES:

```sage
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: f = 3*x^2 - 2*y + 7*x^2*y^2 + 5
sage: f.is_constant()
False
sage: g = 10*x^0
sage: g.is_constant()
True
```

is_generator()

Return True if self is a generator of its parent.

EXAMPLES:

```sage
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: x.is_generator()
True
sage: (x + y - y).is_generator()
True
sage: (x*y).is_generator()
False
```

is_homogeneous()

Return True if self is a homogeneous polynomial.

EXAMPLES:

```sage
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: (x + y).is_homogeneous()
True
sage: (x.parent()(0)).is_homogeneous()
True
sage: (x + y^2).is_homogeneous()
False
sage: (x^2 + y^2).is_homogeneous()
False
sage: (x^2 + y^2*x).is_homogeneous()
False
sage: (x^2*y + y^2*x).is_homogeneous()
True
```
is_monomial()
Return True if self is a monomial, which we define to be a product of generators with coefficient 1.
Use is_term() to allow the coefficient to not be 1.
EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: x.is_monomial()
True
sage: (x + 2*y).is_monomial()
False
sage: (2*x).is_monomial()
False
sage: (x*y).is_monomial()
True
```

To allow a non-1 leading coefficient, use is_term():

```
sage: (2*x*y).is_term()  # needs sage.rings.number_field
True
sage: (2*x*y).is_monomial()  # needs sage.rings.number_field
False
```

is_term()
Return True if self is a term, which we define to be a product of generators times some coefficient, which need not be 1.
Use is_monomial() to require that the coefficient be 1.
EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: x.is_term()
True
sage: (x + 2*y).is_term()
False
sage: (2*x).is_term()
True
sage: (7*x^5*y).is_term()
True
```

To require leading coefficient 1, use is_monomial():

```
sage: (2*x^y).is_monomial()  # needs sage.rings.number_field
False
sage: (2*x^y).is_term()  # needs sage.rings.number_field
True
```

is_univariate()
Return True if this multivariate polynomial is univariate and False otherwise.
EXAMPLES:
```
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: f = 3*x^2 - 2*y + 7*x^2*y^2 + 5
sage: f.is_univariate()
False
sage: g = f.subs({x: 10}); g
700*y^2 + (-2)*y + 305
sage: g.is_univariate()
True
sage: f = x^0
sage: f.is_univariate()
True
```

**iterator_exp_coeff** *(as_ETuples=True)*

Iterate over `self` as pairs of ((E)Tuple, coefficient).

**INPUT:**

- `as_ETuples` *(default: True)* if True iterate over pairs whose first element is an ETuple, otherwise as a tuples

**EXAMPLES:**

```
sage: R.<x,y,z> = PolynomialRing(QQbar, order='lex')
# needs sage.rings.number_field
sage: f = (x^1*y^5*z^2 + x^2*z + x^4*y^1*z^3)
# needs sage.rings.number_field
sage: list(f.iterator_exp_coeff())
# needs sage.rings.number_field
[((4, 1, 3), 1), ((2, 0, 1), 1), ((1, 5, 2), 1)]
sage: R.<x,y,z> = PolynomialRing(QQbar, order='deglex')
# needs sage.rings.number_field
sage: f = (x^1*y^5*z^2 + x^2*z + x^4*y^1*z^3)
# needs sage.rings.number_field
sage: list(f.iterator_exp_coeff(as_ETuples=False))
# needs sage.rings.number_field
[((4, 1, 3), 1), ((1, 5, 2), 1), ((2, 0, 1), 1)]
```

**lc()**

Returns the leading coefficient of `self`, i.e., `self.coefficient(self.lm())`

**EXAMPLES:**

```
sage: R.<x,y,z> = QQbar[]
# needs sage.rings.number_field
sage: f = 3*x^2 - y^2 - x*y
# needs sage.rings.number_field
sage: f.lc()
# needs sage.rings.number_field
3
```

**lift(I)**

Given an ideal $I = (f_1, ..., f_r)$ and some $g (= self)$ in $I$, find $s_1, ..., s_r$ such that $g = s_1f_1 + ... + s_rf_r$.

**ALGORITHM:** Use Singular.

**EXAMPLES:**
```python
sage: # needs sage.rings.real_mpfr
sage: A.<x,y> = PolynomialRing(CC, 2, order='degrevlex')
sage: I = A.ideal([x^10 + x^9*y^2, y^8 - x^2*y^7])
sage: f = x*y^13 + y^12
sage: M = f.lift(I); M
˓→ needs sage.libs.singular
[y^7, x^7*y^2 + x^8 + x^5*y^3 + x^6*y + x^3*y^4 + x^4*y^2 + x*y^5 + x^2*y^3 +
˓→ y^4]
sage: sum(map(mul, zip(M, I.gens()))) == f
˓→ needs sage.libs.singular
True
```

lm()

Return the lead monomial of self with respect to the term order of self.parent().

EXAMPLES:

```python
sage: R.<x,y,z> = PolynomialRing(QQ, implementation=generic)
sage: f = 1/1331*x^2 + 1/4000*y
text(sage: f.local_height(1331)
˓→ needs sage.rings.real_mpfr
7.19368581839511
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
```

local_height(v, prec=None)

Return the maximum of the local height of the coefficients of this polynomial.

INPUT:

- v – a prime or prime ideal of the base ring.
- prec – desired floating point precision (default: default RealField precision).

OUTPUT: a real number.

EXAMPLES:

```python
sage: R.<x,y,z> = PolynomialRing(QQ, implementation='generic')
sage: f = 1/1331*x^2 + 1/4000*y
sage: f.local_height(1331)
˓→ needs sage.rings.real_mpfr
7.19368581839511
```

(continues on next page)
```python
sage: K.<k> = NumberField(x^2 - 5)
sage: T.<t,w> = PolynomialRing(K, implementation='generic')
sage: I = K.ideal(3)
sage: f = 1/3*t*w + 3
sage: f.local_height(I)
# needs sage.symbolic
1.09861228866811
```

```python
sage: R.<x,y> = PolynomialRing(QQ, implementation='generic')
sage: f = 1/2*x*y + 3
sage: f.local_height(I)
# needs sage.rings.real_mpfr
0.75
```

**local_height_arch**(i, prec=None)

Return the maximum of the local height at the i-th infinite place of the coefficients of this polynomial.

**INPUT:**

- i – an integer.
- prec – desired floating point precision (default: default RealField precision).

**OUTPUT:** a real number.

**EXAMPLES:**

```python
sage: R.<x,y> = PolynomialRing(QQ, implementation='generic')
sage: f = 210*x*y
sage: f.local_height_arch(0)
# needs sage.rings.real_mpfr
5.34710753071747
```

```python
sage: # needs sage.rings.number_field
sage: R.<x,y,z> = PolynomialRing(QQbar)
sage: f = 3*x^2 - y^2 - x*y
sage: f.lt()
```

**lt()**

Return the leading term of self i.e., self.lc()*self.lm(). The notion of “leading term” depends on the ordering defined in the parent ring.

**EXAMPLES:**

```python
sage: # needs sage.rings.number_field
sage: R.<x,y,z> = PolynomialRing(QQbar)
sage: f = 3*x^2 - y^2 - x*y
sage: f.lt()
```
Polynomials, Release 10.3

(continued from previous page)

```python
3*x^2
sage: R.<x,y,z> = PolynomialRing(QQbar, order="invlex")
sage: f = 3*x^2 - y^2 - x*y
sage: f.lt()
-y^2
```

**monomial_coefficient (mon)**

Return the coefficient in the base ring of the monomial mon in self, where mon must have the same parent as self.

This function contrasts with the function `coefficient` which returns the coefficient of a monomial viewing this polynomial in a polynomial ring over a base ring having fewer variables.

**INPUT:**

- mon - a monomial

**OUTPUT:** coefficient in base ring

**See also:**

For coefficients in a base ring of fewer variables, look at `coefficient()`.

**EXAMPLES:**

```python
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: f = 2 * x * y
sage: c = f.monomial_coefficient(x*y); c
2
sage: c.parent()
Algebraic Field

sage: # needs sage.rings.number_field
sage: f = y^2 + y^2*x - x^9 - 7*x + 5*x*y
sage: f.monomial_coefficient(y^2)
1
sage: f.monomial_coefficient(x*y)
5
sage: f.monomial_coefficient(x^9)
-1
sage: f.monomial_coefficient(x^10)
0

sage: # needs sage.rings.number_field
sage: a = polygen(ZZ, 'a')
sage: K.<a> = NumberField(a^2 + a + 1)
sage: P.<x,y> = K[]
sage: f = (a*x - 1) * ((a+1)*y - 1); f
-x*y + (-a)*x + (-a - 1)*y + 1
sage: f.monomial_coefficient(x)
-a
```

**monomials ()**

Return the list of monomials in self. The returned list is decreasingly ordered by the term ordering of self.parent().

**OUTPUT:** list of `MPolynomial` instances, representing monomials

350 Chapter 3. Multivariate Polynomials
EXAMPLES:

```python
sage: R.<x,y> = QQbar[]  # needs sage.rings.number_field
sage: f = 3*x^2 - 2*y + 7*x^2*y^2 + 5  # needs sage.rings.number_field
sage: f.monomials()  # needs sage.rings.number_field
[x^2*y^2, x^2, y, 1]
```

```python
sage: # needs sage.rings.number_field
sage: R.<fx,fy,gx,gy> = QQbar[]
sage: F = (fx*gy - fy*gx)^3; F
-fy^3*gx^3 + 3*fx*fy^2*gx^2*gy + (-3)*fx^2*fy*gx*gy^2 + fx^3*gy^3
sage: F.monomials()
[fy^3*gx^3, fx*fy^2*gx^2*gy, fx^2*fy*gx*gy^2, fx^3*gy^3]
sage: F.coefficients()
[-1, 3, -3, 1]
sage: sum(map(mul, zip(F.coefficients(), F.monomials()))) == F
True
```

`nvariables()`

Return the number of variables in this polynomial.

EXAMPLES:

```python
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: f = 3*x^2 - 2*y + 7*x^2*y^2 + 5
sage: f.nvariables()
2
```

```python
sage: g = f.subs({x: 10}); g
700*y^2 + (-2)*y + 305
sage: g.nvariables()
1
```

`quo_rem(right)`

Returns quotient and remainder of `self` and `right`.

EXAMPLES:

```python
sage: R.<x,y> = CC[]  # needs sage.rings.real_mpfr
sage: f = y*x^2 + x + 1  # needs sage.rings.real_mpfr
sage: f.quo_rem(x)  # needs sage.libs.singular sage.rings.real_mpfr
(x*y + 1.00000000000000, 1.00000000000000)
sage: R = QQ['a','b']['x','y','z']
sage: p1 = R('a + (1+2*b)*x*y + (3-a^2)*z')
sage: p2 = R('x-1')
sage: p1.quo_rem(p2)  # needs sage.libs.singular
((2*b + 1)*y, (2*b + 1)*y + (-a^2 + 3)*z + a)
sage: R.<x,y> = Qp(5)[]  # needs sage.rings.padics
```

(continues on next page)
sage: x.quo_rem(y)  # needs sage.libs.singular sage.rings.padics
Traceback (most recent call last):
...
TypeError: no conversion of this ring to a Singular ring defined

ALGORITHM: Use Singular.

reduce(I)
Reduce this polynomial by the polynomials in I.

INPUT:
- I - a list of polynomials or an ideal

EXAMPLES:

sage: # needs sage.rings.number_field
sage: P.<x,y,z> = QQbar[]

sage: f1 = -2 * x^2 + x^3
sage: f2 = -2 * y + x * y
sage: f3 = -x^2 + y^2
sage: F = Ideal([f1, f2, f3])

sage: g = x*y - 3*x*y^2
sage: g.reduce(F)  # needs sage.libs.singular
(-6)*y^2 + 2*y
sage: g.reduce(F.gens())  # needs sage.libs.singular
(-6)*y^2 + 2*y

sage: f = 3*x
# needs sage.rings.number_field

sage: f.reduce([2*x, y])  # needs sage.rings.number_field
0

sage: # needs sage.rings.number_field
sage: k.<w> = CyclotomicField(3)

sage: A.<y9,y12,y13,y15> = PolynomialRing(k)

sage: J = [y9 + y12]

sage: f = y9 - y12; f.reduce(J)
-2*y12
sage: f = y13*y15; f.reduce(J)
y13*y15
sage: f = y13*y15 + y9 - y12; f.reduce(J)
y13*y15 - 2*y12

Make sure the remainder returns the correct type, fixing github issue #13903:

sage: R.<y1,y2> = PolynomialRing(Qp(5), 2, order='lex')  # needs sage.rings.padics

sage: G = [y1^2 + y2^2, y1*y2 + y2^2, y2^3]  # needs sage.rings.padics

sage: type((y2^3).reduce(G))  # needs sage.rings.padics
<class 'sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict'>
resultant (other, variable=None)

Compute the resultant of self and other with respect to variable.
If a second argument is not provided, the first variable of self.parent() is chosen.
For inexact rings or rings not available in Singular, this computes the determinant of the Sylvester matrix.

INPUT:
- other – polynomial in self.parent()
- variable – (optional) variable (of type polynomial) in self.parent()

EXAMPLES:

```sage
g = P.<x,y> = PolynomialRing(QQ, 2)
g = a + b
g = b - a
```

subresultants (other, variable=None)

Return the nonzero subresultant polynomials of self and other.

INPUT:
- other – a polynomial

OUTPUT: a list of polynomials in the same ring as self

EXAMPLES:

```sage`
q = (y^2 + 6)*(x - 1) - y*(x^2 + 1)
p = (x^2 + 6)*(y - 1) - x*(y^2 + 1)
```

subs (fixed=None, **kwds)

Fix some given variables in a given multivariate polynomial and return the changed multivariate polynomials.
The polynomial itself is not affected. The variable, value pairs for fixing are to be provided as a dictionary of
the form {variable: value}.

This is a special case of evaluating the polynomial with some of the variables constants and the others the
original variables.

INPUT:
- fixed – (optional) dictionary of inputs
- **kwds – named parameters

OUTPUT: new MPolynomial
EXAMPLES:

```python
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: f = x^2 + y + x^2*y^2 + 5
sage: f((5, y))
25*y^2 + y + 30
sage: f.subs({x: 5})
25*y^2 + y + 30
```

**total_degree()**

Return the total degree of `self`, which is the maximum degree of any monomial in `self`.

EXAMPLES:

```python
sage: # needs sage.rings.number_field
sage: R.<x,y,z> = QQbar[]
sage: f = 2*x*y^3*z^2
sage: f.total_degree()
6
sage: f = 4*x^2*y^2*z^3
sage: f.total_degree()
7
sage: f = 99*x^6*y^3*z^9
sage: f.total_degree()
18
sage: f = x*y^3*z^6 + 3*x^2
sage: f.total_degree()
10
sage: f = z^3 + 8*x^4*y^5*z
sage: f.total_degree()
10
sage: f = z^9 + 10*x^4 + y^8*x^2
sage: f.total_degree()
10
```

**univariate_polynomial(R=None)**

Returns a univariate polynomial associated to this multivariate polynomial.

INPUT:

- `[R]` (default: None) *PolynomialRing*

If this polynomial is not in at most one variable, then a `ValueError` exception is raised. This is checked using the method `is_univariate()`. The new *Polynomial* is over the same base ring as the given *MPolynomial*.

EXAMPLES:

```python
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: f = 3*x^2 - 2*y + 7*x^2*y^2 + 5
sage: f.univariate_polynomial()
Traceback (most recent call last):
...
TypeError: polynomial must involve at most one variable
sage: g = f.subs({x: 10}); g
700*y^2 + (-2)*y + 305
sage: g.univariate_polynomial()
(continues on next page)```
variable \((i)\)

Return the \(i\)-th variable occurring in this polynomial.

EXAMPLES:

```python
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: f = 3*x^2 - 2*y + 7*x^2*y^2 + 5
sage: f.variable(0)  # x
sage: f.variable(1)  # y
```

variables

Returns the tuple of variables occurring in this polynomial.

EXAMPLES:

```python
sage: # needs sage.rings.number_field
sage: R.<x,y> = QQbar[]
sage: f = 3*x^2 - 2*y + 7*x^2*y^2 + 5
sage: f.variables()
(x, y)
sage: g = f.subs({x: 10}); g
700*y^2 + (-2)*y + 305
sage: g.variables()
(y,)
```

The function should be made a method of the FractionFieldElement class.

EXAMPLES:

```python
sage: R1 = PolynomialRing(FiniteField(5), 3, names=["a", "b", "c"])
sage: F = FractionField(R1)
sage: a,b,c = R1.gens()
sage: f = 3*a*b^2*c^3 + 4*a*b*c
sage: g = a^2*b*c^2 + 2*a^2*b^4*c^7
sage: f/g
```

Consider the quotient \(f/g = \frac{4+3bc^2}{ac+2ab^2c^4}\) (note the cancellation).

3.1. Multivariate Polynomials and Polynomial Rings
3.1.6 Ideals in multivariate polynomial rings

Sage has a powerful system to compute with multivariate polynomial rings. Most algorithms dealing with these ideals are centered on the computation of Groebner bases. Sage mainly uses Singular to implement this functionality. Singular is widely regarded as the best open-source system for Groebner basis calculation in multivariate polynomial rings over fields.

EXAMPLES:

We compute a Groebner basis for some given ideal. The type returned by the groebner_basis method is PolynomialSequence, i.e., it is not a MPolynomialIdeal:

```
sage: x,y,z = QQ[['x,y,z']].gens()
sage: I = ideal(x^5 + y^4 + z^3 - 1, x^3 + y^3 + z^2 - 1)
sage: B = I.groebner_basis()
sage: type(B)
<class 'sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic'>
```

Groebner bases can be used to solve the ideal membership problem:

```
sage: f,g,h = B
sage: (2*x*f + g).reduce(B)
0
sage: (2*x*f + g) in I
True
sage: (2*x*f + 2*z*h + y^3).reduce(B)
y^3
sage: (2*x*f + 2*z*h + y^3) in I
False
```

We compute a Groebner basis for Cyclic 6, which is a standard benchmark and test ideal.

```
sage: R.<x,y,z,t,u,v> = QQ['x,y,z,t,u,v']
sage: I = sage.rings.ideal.Cyclic(R,6)
sage: B = I.groebner_basis()
sage: len(B)
45
```

We compute in a quotient of a polynomial ring over Z/17Z:

```
sage: R.<x,y> = ZZ[]
sage: S.<a,b> = R.quotient((x^2 + y^2, 17))
sage: S
Quotient of Multivariate Polynomial Ring in x, y over Integer Ring
```

(continues on next page)
by the ideal \((x^2 + y^2, 17)\)

\[
\begin{align*}
sage: & \quad a^2 + b^2 == 0 \\
& \quad \text{True} \\
sage: & \quad a^3 - b^2 \\
& \quad -a*b^2 - b^2 \\
\end{align*}
\]

Note that the result of a computation is not necessarily reduced:

\[
\begin{align*}
sage: & \quad (a+b)^{17} \\
& \quad a*b^{16} + b^{17} \\
sage: & \quad S(17) == 0 \\
& \quad \text{True} \\
\end{align*}
\]

Or we can work with \(\mathbb{Z}/17\mathbb{Z}\) directly:

\[
\begin{align*}
sage: & \quad R.<x,y> = \mathbb{Z}/17[] \\
sage: & \quad S.<a,b> = R.quotient((x^2 + y^2,)) \\
sage: & \quad S \\
& \quad \text{Quotient of Multivariate Polynomial Ring in x, y over Ring of integers modulo 17 by the ideal (x^2 + y^2)} \\
sage: & \quad a^2 + b^2 == 0 \\
& \quad \text{True} \\
sage: & \quad a^3 - b^2 == -a*b^2 - b^2 == 16*a*b^2 + 16*b^2 \\
& \quad \text{True} \\
sage: & \quad (a+b)^{17} \\
& \quad a*b^{16} + b^{17} \\
sage: & \quad S(17) == 0 \\
& \quad \text{True} \\
\end{align*}
\]

Working with a polynomial ring over \(\mathbb{Z}\):

\[
\begin{align*}
sage: & \quad R.<x,y,z,w> = \mathbb{Z}[] \\
sage: & \quad I = \text{ideal}(x^2 + y^2 - z^2 - w^2, x-y) \\
sage: & \quad J = I^2 \\
sage: & \quad J.groebner_basis() \\
& \quad [4*y^4 - 4*y^2*z^2 + z^4 - 4*y^2*w^2 + 2*z^2*w^2 + w^4, \\
& \quad 2*x*y^2 - 2*y^3 - x*z^2 + y*z^2 - x*w^2 + y*w^2, \\
& \quad x^2 - 2*x*y + y^2] \\
sage: & \quad y^2 - 2*x*y + x^2 \text{ in } J \\
& \quad \text{True} \\
sage: & \quad 0 \text{ in } J \\
& \quad \text{True} \\
\end{align*}
\]

We do a Groebner basis computation over a number field:

\[
\begin{align*}
sage: & \quad K.<\zeta> = \text{CyclotomicField}(3) \\
sage: & \quad R.<x,y,z> = K[]; R \\
& \quad \text{Multivariate Polynomial Ring in x, y, z over Cyclotomic Field of order 3 and degree 2} \\
\end{align*}
\]

\[
\begin{align*}
sage: & \quad i = \text{ideal}(x - \zeta*y + 1, x^3 - \zeta*y^3); i \\
& \quad \text{Ideal (x + (-zeta)*y + 1, x^3 + (-zeta)*y^3) of Multivariate Polynomial Ring in x, y, z over Cyclotomic Field of order 3 and degree 2} \\
sage: & \quad i.groebner_basis() \\
\end{align*}
\]
Polynomials, Release 10.3

(continued from previous page)

\[ y^3 + (2zeta + 1)y^2 + (zeta - 1)y + (-1/3zeta - 2/3), x + (-zeta)y + 1 \]

\texttt{sage: } S = R.quotient(i); S
Quotient of Multivariate Polynomial Ring in x, y, z over
Cyclotomic Field of order 3 and degree 2 by the ideal (x +
(-zeta)y + 1, x^3 + (-zeta)y^3)

\texttt{sage: } S.0 - zeta*S.1
-1
\texttt{sage: } S.0^3 - zeta*S.1^3
0

Two examples from the Mathematica documentation (done in Sage):

We compute a Groebner basis:

\texttt{sage: } R.<x,y> = PolynomialRing(QQ, order='lex')
\texttt{sage: } ideal(x^2 - 2*y^2, x*y - 3).groebner_basis()
\[ [x - 2/3*y^3, y^4 - 9/2] \]

We show that three polynomials have no common root:

\texttt{sage: } R.<x,y> = QQ[]
\texttt{sage: } ideal(x+y, x^2 - 1, y^2 - 2*x).groebner_basis()
\[ [1] \]

The next example shows how we can use Groebner bases over \( \mathbb{Z} \) to find the primes modulo which a system of equations has a solution, when the system has no solutions over the rationals.

We first form a certain ideal \( I \) in \( \mathbb{Z}[x,y,z] \), and note that the Groebner basis of \( I \) over \( \mathbb{Q} \) contains 1, so there are no solutions over \( \mathbb{Q} \) or an algebraic closure of it (this is not surprising as there are 4 equations in 3 unknowns).

\texttt{sage: } P.<x,y,z> = PolynomialRing(ZZ,order='lex')
\texttt{sage: } I = ideal(-y^2 - 3*y + z^2 + 3, -2*y*z + z^2 + 2*z + 1,
.....: x*z + y*z + z^2, -3*x*y + 2*y*z + 6*z^2)
\texttt{sage: } I.change_ring(P.change_ring(QQ)).groebner_basis()
\[ [x + y + 57119*z + 4, y^2 + 3*y + 17220, y*z + ...,
2*y + 158864, z^2 + 17223, 2*z + 41856, 164878] \]

However, when we compute the Groebner basis of \( I \) (defined over \( \mathbb{Z} \)), we note that there is a certain integer in the ideal which is not 1.

\texttt{sage: } I.groebner_basis()
\[ [x + y + 57119*z + 4, y^2 + 3*y + 17220, y*z + ...,
2*y + 158864, z^2 + 17223, 2*z + 41856, 164878] \]

Now for each prime \( p \) dividing this integer 164878, the Groebner basis of \( I \) modulo \( p \) will be non-trivial and will thus give a solution of the original system modulo \( p \).

\texttt{sage: } factor(164878)
2 * 7 * 11777
\texttt{sage: } I.change_ring(P.change_ring(GF(2))).groebner_basis()
\[ [x + y + z, y^2 + y, y*z + y, z^2 + 1] \]
\texttt{sage: } I.change_ring(P.change_ring(GF(7))).groebner_basis()
\[ [x + y + 3*z + 4, y^2 + 2*y + 4, y*z + 3y + 3z + 1] \]

(continues on next page)
The Groebner basis modulo any product of the prime factors is also non-trivial:

```sage
sage: I.change_ring(P.change_ring(IntegerModRing(2 * 7))).groebner_basis()
[x + 9*y + 13*z, y^2 + 3*y, y*z + 7*y + 6, 2*y + 6, z^2 + 3, 2*z + 10]
```

Modulo any other prime the Groebner basis is trivial so there are no other solutions. For example:

```sage
sage: I.change_ring(P.change_ring(GF(3))).groebner_basis()
[1]
```

**Note:** Sage distinguishes between lists or sequences of polynomials and ideals. Thus an ideal is not identified with a particular set of generators. For sequences of multivariate polynomials see `sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic`.

**AUTHORS:**

- William Stein: initial version
- Kiran S. Kedlaya (2006-02-12): added Macaulay2 analogues of some Singular features
- Martin Albrecht (2009): added Groebner basis over rings functionality from Singular 3.1
- John Perry (2012): bug fixing equality & containment of ideals

```python
class sage.rings.polynomial.multi_polynomial_ideal.MPolynomialIdeal
```
**property basis**
Shortcut to `gens()`.

**EXAMPLES:**
```
sage: P.<x,y> = PolynomialRing(QQ,2)
sage: I = Ideal([x, y + 1])
sage: I.basis
[x, y + 1]
```

**change_ring**\((P)\)
Return the ideal \(I\) in \(P\) spanned by the generators \(g_1, ..., g_n\) of self as returned by `self.gens()`.

**INPUT:**
- \(P\) - a multivariate polynomial ring

**EXAMPLES:**
```
sage: P.<x,y,z> = PolynomialRing(QQ,3,order='lex')
sage: I = sage.rings.ideal.Cyclic(P)
sage: I.groebner_basis()
[x + y + z, y^2 + y*z + z^2, z^3 - 1]
sage: Q.<x,y,z> = P.change_ring(order='degrevlex'); Q
Multivariate Polynomial Ring in x, y, z over Rational Field
sage: Q.term_order()
Degree reverse lexicographic term order
sage: J = I.change_ring(Q); J
Ideal (x + y + z, x*y + x*z + y*z, x*y*z - 1) of
Multivariate Polynomial Ring in x, y, z over Rational Field
sage: J.groebner_basis()
[z^3 - 1, y^2 + y*z + z^2, x + y + z]
```

**degree_of_semi_regularity**\()\)
Return the degree of semi-regularity of this ideal under the assumption that it is semi-regular.

Let \(\{f_1, ..., f_m\} \subset K[x_1, ..., x_n]\) be homogeneous polynomials of degrees \(d_1, ..., d_m\) respectively. This sequence is semi-regular if:
- \(\{f_1, ..., f_m\} \neq K[x_1, ..., x_n]\)
- for all \(1 \leq i \leq m\) and \(g \in K[x_1, ..., x_n]\): \(\deg(g \cdot pi) < D\) and \(g \cdot f_i < f_1, ..., f_{i-1}\) implies that \(g \in < f_1, ..., f_{i-1} >\) where \(D\) is the degree of regularity.

This notion can be extended to affine polynomials by considering their homogeneous components of highest degree.

The degree of regularity of a semi-regular sequence \(f_1, ..., f_m\) of respective degrees \(d_1, ..., d_m\) is given by the index of the first non-positive coefficient of:

\[
\sum c_k z^k = \prod (1-z^{d_i})
\]
EXAMPLES:

We consider a homogeneous example:

```python
sage: n = 8
sage: K = GF(127)
sage: P = PolynomialRing(K, n, 'x')
sage: s = [K.random_element() for _ in range(n)]
sage: L = []
sage: for i in range(2 * n):
    f = P.random_element(degree=2, terms=binomial(n, 2))
    f -= f(*s)
    L.append(f.homogenize())
sage: I = Ideal(L)
sage: I.degree_of_semi_regularity()
4
```

From this, we expect a Groebner basis computation to reach at most degree 4. For homogeneous systems this is equivalent to the largest degree in the Groebner basis:

```python
sage: max(f.degree() for f in I.groebner_basis())
4
```

We increase the number of polynomials and observe a decrease the degree of regularity:

```python
sage: for i in range(2 * n):
    ...:     f = P.random_element(degree=2, terms=binomial(n, 2))
    ...:     f -= f(*s)
    ...:     L.append(f.homogenize())
    sage: I = Ideal(L)
    sage: I.degree_of_semi_regularity()
3
    sage: max(f.degree() for f in I.groebner_basis())
3
```

The degree of regularity approaches 2 for quadratic systems as the number of polynomials approaches $n^2$:

```python
sage: for i in range((n-4) * n):
    ...:     f = P.random_element(degree=2, terms=binomial(n, 2))
    ...:     f -= f(*s)
    ...:     L.append(f.homogenize())
    sage: I = Ideal(L)
    sage: I.degree_of_semi_regularity()
2
    sage: max(f.degree() for f in I.groebner_basis())
2
```

Note: It is unknown whether semi-regular sequences exist. However, it is expected that random systems are semi-regular sequences. For more details about semi-regular sequences see [BFS2004].


gens ()

Return a set of generators / a basis of this ideal. This is usually the set of generators provided during object creation.

EXAMPLES:
sage: P.<x,y> = PolynomialRing(QQ,2)
sage: I = Ideal([x, y + 1]); I
Ideal (x, y + 1) of Multivariate Polynomial Ring in x, y over Rational Field
sage: I.gens()
[x, y + 1]

\textbf{groebner\_basis}(\text{algorithm}=\text{\textbackslash ''}, \text{deg\_bound}=None, \text{mult\_bound}=None, \text{prot}=False, *\text{args}, **\text{kwds})

Return the reduced Groebner basis of this ideal.

A Groebner basis \(g_1, \ldots, g_n\) for an ideal \(I\) is a generating set such that \(<LM(g_i)> = LM(I)\), i.e., the leading monomial ideal of \(I\) is spanned by the leading terms of \(g_1, \ldots, g_n\). Groebner bases are the key concept in computational ideal theory in multivariate polynomial rings which allows a variety of problems to be solved.

Additionally, a \textit{reduced} Groebner basis \(G\) is a unique representation for the ideal \(<G>\) with respect to the chosen monomial ordering.

**INPUT:**

\begin{itemize}
\item \textbf{algorithm} - determines the algorithm to use, see below for available algorithms.
\item \textbf{deg\_bound} - only compute to degree \text{deg\_bound}, that is, ignore all S-polynomials of higher degree. (default: None)
\item \textbf{mult\_bound} - the computation is stopped if the ideal is zero-dimensional in a ring with local ordering and its multiplicity is lower than \text{mult\_bound}. Singular only. (default: None)
\item \textbf{prot} - if set to True the computation protocol of the underlying implementation is printed. If an algorithm from the \texttt{singular}: or \texttt{magma}: family is used, \texttt{prot} may also be \texttt{sage} in which case the output is parsed and printed in a common format where the amount of information printed can be controlled via calls to \texttt{set\_verbose()}.
\item *\text{args} - additional parameters passed to the respective implementations
\item **\text{kwds} - additional keyword parameters passed to the respective implementations
\end{itemize}

**ALGORITHMS:**

\begin{itemize}
\item \texttt{\textbackslash ''} - autoselect (default)
\item \texttt{'\texttt{singular}:	exttt{groebner}'}
  \quad Singular's \texttt{groebner} command
\item \texttt{'\texttt{singular}:	exttt{std}'}
  \quad Singular's \texttt{std} command
\item \texttt{'\texttt{singular}:	exttt{stdhilb}'}
  \quad Singular's \texttt{stdhilb} command
\item \texttt{'\texttt{singular}:	exttt{stdfglm}'}
  \quad Singular's \texttt{stdfglm} command
\item \texttt{'\texttt{singular}:	exttt{slimgb}'}
  \quad Singular's \texttt{slimgb} command
\item \texttt{'\texttt{libsingular}:	exttt{groebner}'}
  \quad \texttt{libSingular}'s \texttt{groebner} command
\item \texttt{'\texttt{libsingular}:	exttt{std}'}
  \quad \texttt{libSingular}'s \texttt{std} command
\item \texttt{'\texttt{libsingular}:	exttt{slimgb}'}
  \quad \texttt{libSingular}'s \texttt{slimgb} command
\end{itemize}
Polynomials, Release 10.3

'libsingular:stdhilb'
libSingular's stdhilb command

'libsingular:stdfglm'
libSingular's stdfglm command

'toy:buchberger'
Sage's toy/educational buchberger without Buchberger criteria

'toy:buchberger2'
Sage's toy/educational buchberger with Buchberger criteria

'toy:d_basis'
Sage's toy/educational algorithm for computation over PIDs

'macaulay2:gb'
Macaulay2's gb command (if available)

'macaulay2:f4'
Macaulay2's GroebnerBasis command with the strategy “F4” (if available)

'macaulay2:mgb'
Macaulay2's GroebnerBasis command with the strategy “MGB” (if available)

'msolve'
optional package msolve (degrevlex order)

'magma:GroebnerBasis'
Magma's Groebnerbasis command (if available)

'ginv:TQ', 'ginv:TQBlockHigh', 'ginv:TQBlockLow' and 'ginv:TQDegree'
One of GINV's implementations (if available)

'giac:gbasis'
Giac's gbasis command (if available)

If only a system is given - e.g. 'magma' - the default algorithm is chosen for that system.

Note: The Singular and libSingular versions of the respective algorithms are identical, but the former calls an external Singular process while the latter calls a C function, and thus the calling overhead is smaller. However, the libSingular interface does not support pretty printing of computation protocols.

EXAMPLES:

Consider Katsura-3 over $\mathbb{Q}$ with lexicographical term ordering. We compute the reduced Groebner basis using every available implementation and check their equality.

\begin{verbatim}
sage: P.<a,b,c> = PolynomialRing(QQ,3, order='lex')
sage: I = sage.rings.ideal.Katsura(P,3)  # regenerate to prevent caching sage: I.groebner_basis() [a - 60*c^3 + 158/7*c^2 + 8/7*c - 1, b + 30*c^3 - 79/7*c^2 + 3/7*c, c^4 - 10/21*c^3 + 1/84*c^2 + 1/84*c]
\end{verbatim}

\begin{verbatim}
sage: I = sage.rings.ideal.Katsura(P,3)  # regenerate to prevent caching sage: I.groebner_basis('libsingular:groebner') [a - 60*c^3 + 158/7*c^2 + 8/7*c - 1, b + 30*c^3 - 79/7*c^2 + 3/7*c, c^4 - 10/21*c^3 + 1/84*c^2 + 1/84*c]
\end{verbatim}
Although Giac does support lexicographical ordering, we use degree reverse lexicographical ordering here, in order to test against github issue #21884:

```
sage: I = sage.rings.ideal.Katsura(P,3)  # regenerate to prevent caching
sage: gb = I.groebner_basis('toy:buchberger')
sage: gb.is_groebner()
True
sage: gb == gb.reduced()
False
```

but that `toy:buchberger2` does.

Giac’s `gbasis` over $\mathbb{Q}$ can benefit from a probabilistic lifting and multi threaded operations:

```
sage: A9 = PolynomialRing(QQ, 9, 'x')
sage: I9 = sage.rings.ideal.Katsura(A9)
sage: print("possible output from giac", flush=True); I9.groebner_basis("giac", proba_epsilon=1e-7)  # long time (3s)
possible output...
Polynomial Sequence with 143 Polynomials in 9 Variables
```

The list of available Giac options is provided at `sage.libs.giac.groebner_basis()`.

Note that `toy:buchberger` options is provided at `sage.libs.giac.groebner_basis()`.
We use Macaulay2 with three different strategies over a finite field.

```python
sage: R.<a,b,c> = PolynomialRing(GF(101), 3)
sage: I = sage.rings.ideal.Katsura(R,3)  # regenerate to prevent caching
sage: I.groebner_basis('macaulay2:gb')
[c^3 + 28*c^2 - 37*b + 13*c, b^2 - 41*c^2 + 20*b - 20*c,
b*c - 19*c^2 + 10*b + 40*c, a + 2*b + 2*c - 1]
sage: I.groebner_basis('macaulay2:f4')
[c^3 + 28*c^2 - 37*b + 13*c, b*c - 19*c^2 + 10*b + 40*c, a + 2*b + 2*c - 1]
sage: I.groebner_basis('macaulay2:mgb')
[c^3 + 28*c^2 - 37*b + 13*c, b*c - 19*c^2 + 10*b + 40*c, a + 2*b + 2*c - 1]
```

Over prime fields of small characteristic, we can also use the optional package msolve:

```python
sage: R.<a,b,c> = PolynomialRing(GF(101), 3)
sage: I = sage.rings.ideal.Katsura(R,3)  # regenerate to prevent caching
sage: I.groebner_basis('msolve')  # optional - msolve
[a + 2*b + 2*c - 1, b*c - 19*c^2 + 10*b + 40*c, b^2 - 41*c^2 + 20*b - 20*c, c^3 + 28*c^2 - 37*b + 13*c]
```

Singular and libSingular can compute Groebner basis with degree restrictions.

```python
sage: R.<x,y> = QQ[]
sage: I = R*[x^3 + y^2, x^2*y + 1]
sage: I.groebner_basis()  # not tested
[x^3 + y^2, x^2*y + 1, y^3 - x]
```

A protocol is printed, if the verbosity level is at least 2, or if the argument prot is provided. Historically, the protocol did not appear during doctests, so, we skip the examples with protocol output.

```python
sage: from sage.misc.verbose import setVerbose
sage: setVerbose(2)
sage: I = R*[x^3+y^2,x^2*y+1]
sage: I.groebner_basis()  # not tested
```

(continues on next page)
Polynomials, Release 10.3

```
std in (QQ),(x,y),(dp(2),C)
[...2]3ss4s6
(S:2)--
product criterion:1 chain criterion:0
[x^3 + y^2, x^2*y + 1, y^3 - x]
sage: I.groebner_basis(prot=False)
std in (QQ),(x,y),(dp(2),C)
[...2]3ss4s6
(S:2)--
product criterion:1 chain criterion:0
[x^3 + y^2, x^2*y + 1, y^3 - x]
sage: set_verbose(0)
sage: I.groebner_basis(prot=True)  # not tested
std in (QQ),(x,y),(dp(2),C)
[...2]3ss4s6
(S:2)--
product criterion:1 chain criterion:0
[x^3 + y^2, x^2*y + 1, y^3 - x]
```

The list of available options is provided at `LibSingularOptions`.

Note that Groebner bases over \( \mathbb{Z} \) can also be computed.

```
sage: P.<a,b,c> = PolynomialRing(ZZ,3)
sage: I = P * (a + 2*b + 2*c - 1, a^2 - a + 2*b^2 + 2*c^2, 2*a*b + 2*b*c - b)
sage: I.groebner_basis()
\[ b^3 + b*c^2 + 12*c^3 + b^2 + b*c - 4*c^2, \\
2*b*c^2 - 6*c^3 + b^2 - b*c + 2*c^2, \\
42*c^3 + b^2 + 2*b*c - 14*c^2 + b, \\
2*b^2 + 6*b*c + 6*c^2 - b - 2*c, \\
10*b*c + 12*c^2 - b - 4*c, \\
a + 2*b + 2*c - 1 \]
```

```
sage: I.groebner_basis('macaulay2')  # optional - macaulay2
\[ b^3 + b*c^2 + 12*c^3 + b^2 + b*c - 4*c^2, \\
2*b*c^2 - 6*c^3 + b^2 + 5*b*c + 8*c^2 - b - 2*c, \\
42*c^3 + b^2 + 2*b*c - 14*c^2 + b, \\
2*b^2 - 4*b*c - 6*c^2 + 2*c, 10*b*c + 12*c^2 - b - 4*c, \\
a + 2*b + 2*c - 1 \]
```

Groebner bases over \( \mathbb{Z}/n\mathbb{Z} \) are also supported:

```
sage: P.<a,b,c> = PolynomialRing(Zmod(1000), 3)
sage: I = P * (a + 2*b + 2*c - 1, a^2 - a + 2*b^2 + 2*c^2, 2*a*b + 2*b*c - b)
sage: I.groebner_basis()
\[ b^3 + c^2 + 732*b*c + 808*b, \\
2*c^3 + 884*b*c + 666*c^2 + 320*b, \\
b^2 + 438*b*c + 281*b, \\
5*b*c + 156*c^2 + 112*b + 948*c, \\
50*c^2 + 600*b + 650*c, \\
a + 2*b + 2*c + 999, \\
125*b \]
```

```
sage: R.<x,y,z> = PolynomialRing(Zmod(233497349584))
sage: I = R.ideal([z*(x-3*y), 3^2*x^2-y*z, z^2+y^2])
sage: I.groebner_basis()
```

(continues on next page)
Sage also supports local orderings:

```python
sage: P.<x,y,z> = PolynomialRing(QQ, 3, order='negdegrevlex')
```

```python
t = P *( x*y*z + z^5, 2*x^2 + y^3 + z^7, 3*z^5 + y^5 )
```

```python
sage: I = t *groebner_basis()
```

```python
[x^2 + 1/2*y^3, x*y*z + z^5, y^5 + 3*z^5, y^4*z - 2*x*z^5, z^6]
```

We can represent every element in the ideal as a combination of the generators using the `lift()` method:

```python
sage: I = P * ( x*y*z + z^5, 2*x^2 + y^3 + z^7, 3*z^5 + y^5 )
```

```python
sage: J = Ideal(I.groebner_basis())
```

```python
sage: f = sum(P.random_element(terms=2)*f for f in I.gens())
```

```python
sage: f.lift(I.gens())
```

```python
[sage: l = f.lift(J.gens()); l]
```

Groebner bases over fraction fields of polynomial rings are also supported:

```python
sage: R.<a,b> = QQ[]; I = R.ideal(a^2+b^2-1)
```

```python
sage: Q = QuotientRing(R,I); K = Frac(Q)
```

```python
sage: R2.<x,y> = K[]; J = R2.ideal([(a^2+b^2)*x + y, x+y])
```

```python
sage: J.groebner_basis()
```

```python
verbose 0 (...: multi_polynomial_ideal.py, groebner_basis) Warning: falling back to very slow toy implementation.
```

```python
[x + y]
```

**ALGORITHM:**

Uses Singular, one of the other systems listed above (if available), or a toy implementation.

`groebner_cover()`

Compute the Gröbner cover of the ideal, over a field with parameters.

The Groebner cover is a partition of the space of parameters, such that the Groebner basis in each part is given by the same expression.
EXAMPLES:

```python
sage: F = PolynomialRing(QQ, 'a').fraction_field()
sage: F.inject_variables()
Defining a
sage: R.<x,y,z> = F[]
sage: I = R.ideal([-x+3*y+z-5, 2*x+a*z+4, 4*x-3*z-1/a])
sage: I.groebner_cover()
{Quasi-affine subscheme X - Y of Affine Space of dimension 1 over Rational Field,
 where X is defined by:
  0
 and Y is defined by:
  2*a^2 + 3*a: [(2*a^2 + 3*a)*z + (8*a + 1),
  (12*a^2 + 18*a)*y + (-20*a^2 - 35*a - 2), (4*a + 6)*x + 1],
Quasi-affine subscheme X - Y of Affine Space of dimension 1 over Rational Field,
 where X is defined by:
   ...
 and Y is defined by:
  1: [1],
Quasi-affine subscheme X - Y of Affine Space of dimension 1 over Rational Field,
 where X is defined by:
   ...
 and Y is defined by:
  1: [1]}
```

`groebner_fan` (**is_groebner_basis**=False, **symmetry**=None, **verbose**=False)

Return the Groebner fan of this ideal.

The base ring must be `Q` or a finite field `F_p` of with `p ≤ 32749`.

EXAMPLES:

```python
sage: P.<x,y> = PolynomialRing(QQ)
sage: i = ideal(x^2 - y^2 + 1)
sage: g = i.groebner_fan()
sage: g.reduced_groebner_bases()
[[x^2 - y^2 + 1], [-x^2 + y^2 - 1]]
```

**INPUT:**

- **is_groebner_basis** - bool (default False). if True, then `I.gens()` must be a Groebner basis with respect to the standard degree lexicographic term order.
- **symmetry** - default: None; if not None, describes symmetries of the ideal
- **verbose** - default: False; if True, printout useful info during computations

`homogenize` (**var**='h')

Return homogeneous ideal spanned by the homogeneous polynomials generated by homogenizing the generators of this ideal.

**INPUT:**

- **h** - variable name or variable in cover ring (default: ‘h’)

**EXAMPLES:**
sage: P.<x,y,z> = PolynomialRing(GF(2))
sage: I = Ideal([x^2*y + z + 1, x + y^2 + 1]); I
Ideal (x^2*y + z + 1, y^2 + x + 1) of Multivariate
Polynomial Ring in x, y, z over Finite Field of size 2

sage: I.homogenize()
Ideal (x^2*y + z*h^2 + h^3, y^2 + x*h + h^2) of
Multivariate Polynomial Ring in x, y, z, h over Finite
Field of size 2

sage: I.homogenize(y)
Ideal (x^2*y + y^3 + y^2*z, x*y) of Multivariate
Polynomial Ring in x, y, z over Finite Field of size 2

sage: I = Ideal([x^2*y + z^3 + y^2*x, x + y^2 + 1])
sage: I.homogenize()
Ideal (x^2*y + x*y^2 + z^3, y^2 + x*h + h^2) of
Multivariate Polynomial Ring in x, y, z, h over Finite
Field of size 2

is_homogeneous()

Return True if this ideal is spanned by homogeneous polynomials, i.e., if it is a homogeneous ideal.

EXAMPLES:

sage: P.<x,y,z> = PolynomialRing(QQ,3)
sage: I = sage.rings.ideal.Katsura(P)
sage: I
Ideal (x + 2*y + 2*z - 1, x^2 + 2*y^2 + 2*z^2 - x, 2*x*y +
2*y*z - y) of Multivariate Polynomial Ring in x, y, z over
Rational Field

sage: I.is_homogeneous()
False

sage: J = I.homogenize()
sage: J
Ideal (x + 2*y + 2*z - h, x^2 + 2*y^2 + 2*z^2 - x*h, 2*x*y +
2*y*z - y*h) of Multivariate Polynomial Ring in x, y, z,
h over Rational Field

sage: J.is_homogeneous()
True

plot (*args, **kwds)

Plot the real zero locus of this principal ideal.

INPUT:

• self - a principal ideal in 2 variables
• algorithm - set this to ‘surf’ if you want ‘surf’ to plot the ideal (default: None)
• *args - optional tuples (variable, minimum, maximum) for plotting dimensions
• **kwds - optional keyword arguments passed on to implicit_plot
EXAMPLES:

Implicit plotting in 2-d:

```python
sage: R.<x,y> = PolynomialRing(QQ, 2)
sage: I = R.ideal([y^3 - x^2])
sage: I.plot()  # cusp
needs sage.plot
Graphics object consisting of 1 graphics primitive
```

```python
sage: I = R.ideal([y^2 - x^2 - 1])
sage: I.plot((x,-3, 3), (y, -2, 2))  # hyperbola
needs sage.plot
Graphics object consisting of 1 graphics primitive
```

```python
sage: I = R.ideal([y^2 + x^2*(1/4) - 1])
sage: I.plot()  # ellipse
needs sage.plot
Graphics object consisting of 1 graphics primitive
```

```python
sage: I = R.ideal([y^2-(x^2-1)*(x-2)])
sage: I.plot()  # elliptic curve
needs sage.plot
Graphics object consisting of 1 graphics primitive
```

```python
sage: f = ((x+3)^3 + 2*(x+3)^2 - y^2)*(x^3 - y^2)*((x-3)^3-2*(x-3)^2-y^2)
sage: I = R.ideal(f)
sage: I.plot()  # the Singular logo
needs sage.plot
Graphics object consisting of 1 graphics primitive
```

```python
sage: R.<x,y> = PolynomialRing(QQ, 2)
sage: I = R.ideal([x - 1])
sage: I.plot((y, -2, 2))  # vertical line
needs sage.plot
Graphics object consisting of 1 graphics primitive
```

```python
sage: I = R.ideal([-x^2*y + 1])
sage: I.plot()  # blow up
needs sage.plot
Graphics object consisting of 1 graphics primitive
```

random_element (degree, compute_gb=False, *args, **kwds)

Return a random element in this ideal as \( r = \sum h_i f_i \).

INPUT:

- `compute_gb` - if True then a Gröbner basis is computed first and \( f_i \) are the elements in the Gröbner basis. Otherwise whatever basis is returned by `self.gens()` is used.

- `*args` and `**kwds` are passed to R.random_element() with `R = self.ring()`.

EXAMPLES:

We compute a uniformly random element up to the provided degree.
Note that sampling uniformly at random from the ideal at some large enough degree is equivalent to computing a Gröbner basis. We give an example showing how to compute a Gröbner basis if we can sample uniformly at random from an ideal:

1. We sample $n^d$ uniformly random elements in the ideal:

2. We linearize and compute the echelon form:

3. The result is the desired Gröbner basis:

We return some element in the ideal with no guarantee on the distribution:

We show that the default method does not sample uniformly at random from the ideal:

If `degree` equals the degree of the generators, a random linear combination of the generators is returned:
Polynomials, Release 10.3

```python
sage: P.<x,y> = QQ[]
sage: I = P.ideal([x^2,y^2])
sage: set_random_seed(5)
sage: I.random_element(degree=2)
-2*x^2 + 2*y^2
```

**reduce** (*f*)

Reduce an element modulo the reduced Groebner basis for this ideal. This returns 0 if and only if the element is in this ideal. In any case, this reduction is unique up to monomial orders.

**EXAMPLES:**

```python
sage: R.<x,y> = PolynomialRing(QQ, 2)
sage: I = (x^3 + y, y) * R
sage: I.reduce(y)
0
sage: I.reduce(x^3)
0
sage: I.reduce(x - y)
x
sage: I = (y^2 - (x^3 + x)) * R
sage: I.reduce(x^3)
y^2 - x
sage: I.reduce(x^6)
y^4 - 2*x*y^2 + x^2
sage: (y^2 - x)^2
y^4 - 2*x*y^2 + x^2
```

**Note:** Requires computation of a Groebner basis, which can be a very expensive operation.

**subs** (*in_dict=None, **kwds*)

Substitute variables.

This method substitutes some variables in the polynomials that generate the ideal with given values. Variables that are not specified in the input remain unchanged.

**INPUT:**

- *in_dict* – (optional) dictionary of inputs
- **kwds** – named parameters

**OUTPUT:**

A new ideal with modified generators. If possible, in the same polynomial ring. Raises a `TypeError` if no common polynomial ring of the substituted generators can be found.

**EXAMPLES:**

```python
sage: R.<x,y> = PolynomialRing(ZZ, 2, 'xy')
sage: I = R.ideal(x^5 + y^5, x^2 + y + x^2*y^2 + 5); I
Ideal (x^5 + y^5, x^2 + y + x^2*y^2 + 5) of Multivariate Polynomial Ring in x, y over Integer Ring
sage: I.subs(x=y)
Ideal (2*y^5, y^4 + y^2 + y + 5) of Multivariate Polynomial Ring in x, y over Integer Ring
```

(continues on next page)
The new ideal can be in a different ring:

```sage
sage: R.<a,b> = PolynomialRing(QQ, 2)
sage: S.<x,y> = PolynomialRing(QQ, 2)
sage: I = R.ideal(a^2 + b^2 + a - b + 2); I
Ideal (a^2 + b^2 + a - b + 2)
of Multivariate Polynomial Ring in a, b over Rational Field
sage: I.subs(a=x, b=y)
Ideal (x^2 + y^2 + x - y + 2)
of Multivariate Polynomial Ring in x, y over Rational Field
```

The resulting ring need not be a multivariate polynomial ring:

```sage
sage: T.<t> = PolynomialRing(QQ)
sage: I.subs(a=t, b=t)
Principal ideal (t^2 + 1) of Univariate Polynomial Ring in t over Rational Field
sage: var("z")
# needs sage.symbolic
z
sage: I.subs(a=z, b=z)
# needs sage.symbolic
Principal ideal (2*z^2 + 2) of Symbolic Ring
```

Variables that are not substituted remain unchanged:

```sage
sage: R.<x,y> = PolynomialRing(QQ, 2)
sage: I = R.ideal(x^2 + y^2 + x - y + 2); I
Ideal (x^2 + y^2 + x - y + 2)
of Multivariate Polynomial Ring in x, y over Rational Field
sage: I.subs(x=1)
Ideal (y^2 - y + 4) of Multivariate Polynomial Ring in x, y over Rational Field
```

**weil_restriction()**

Compute the Weil restriction of this ideal over some extension field. If the field is a finite field, then this computes the Weil restriction to the prime subfield.

A Weil restriction of scalars - denoted $\text{Res}_{L/k}$ - is a functor which, for any finite extension of fields $L/k$ and any algebraic variety $X$ over $L$, produces another corresponding variety $\text{Res}_{L/k}(X)$, defined over $k$. It is useful for reducing questions about varieties over large fields to questions about more complicated varieties over smaller fields.

This function does not compute this Weil restriction directly but computes on generating sets of polynomial ideals:

Let $d$ be the degree of the field extension $L/k$, let $a$ a generator of $L/k$ and $p$ the minimal polynomial of $L/k$. Denote this ideal by $I$.

Specifically, this function first maps each variable $x$ to its representation over $k$: $\sum_{i=0}^{d-1} a^i x_i$. Then each generator of $I$ is evaluated over these representations and reduced modulo the minimal polynomial $p$. The result is interpreted as a univariate polynomial in $a$ and its coefficients are the new generators of the returned ideal.

### 3.1. Multivariate Polynomials and Polynomial Rings
If the input and the output ideals are radical, this is equivalent to the statement about algebraic varieties above.

**OUTPUT:** \texttt{MPolynomialIdeal}

**EXAMPLES:**

```
sage: # needs sage.rings.finite_rings
cpy.<a> = GF(2^2)
sp.<x,y> = PolynomialRing(k, 2)
sage: I = Ideal([x*y + 1, a*x + 1])
sage: I.variety()
[(y: a, x: a + 1)]
sage: J = I.weil_restriction()
sage: J
Ideal (x0*y0 + x1*y1 + 1, x1*y0 + x0*y1 + x1*y1, x1 + 1, x0 + x1) of Multivariate Polynomial Ring in x0, x1, y0, y1 over Finite Field of size 2
sage: J += sage.rings.ideal.FieldIdeal(J.ring())  # ensure radical ideal
sage: J.variety()
[(y1: 1, y0: 0, x1: 1, x0: 1)]
sage: J.weil_restriction()  # returns J
Ideal (x0*y0 + x1*y1 + 1, x1*y0 + x0*y1 + x1*y1, x1 + 1, x0 + x1,
x0^2 + x0, x1^2 + x1, y0^2 + y0, y1^2 + y1) of Multivariate Polynomial Ring in x0, x1, y0, y1 over Finite Field of size 2

sage: # needs sage.rings.finite_rings
榜首inka.<a> = GF(3^5)
sage: P.<x,y,z> = PolynomialRing(k)
sage: I = sage.rings.ideal.Katsura(P)
sage: I.dimension()
0
sage: I.variety()
[(z: 0, y: 0, x: 1)]
sage: J = I.weil_restriction(); J
Ideal (x0 - y0 - z0 - 1,
- x1 - y1 - z1, x2 - y2 - z2, x3 - y3 - z3, x4 - y4 - z4,
x0^2 + x2*x3 + x1*x4 - y0^2 - y2*y3 - y1*y4 - z0^2 - z2*z3 - z1*z4 -
- x0, -x0*x1 - x2*x3 - x3^2 - x1*x4 + x2*x4 + y0*y1 + y2*y3 +
y3^2 + y1*y4 - y2*y4 + z0*z1 + z2*z3 + z3^2 + z1*z4 - z2*z4 -
- x1, x1^2 - x0*x2 + x3^2 - x2*x4 + x3*x4 - y1^2 + y0*y2 -
y3^2 + y2*y4 - y3*y4 - z1^2 + z0*z2 - z3^2 + z2*z4 - z3*z4 -
- x2, -x1*x2 - x0*x3 - x3*x4 - x4^2 +
y1*y2 + y0*y3 + y3*y4 + y4^2 - z1*z2 + z0*z3 + z3^2 + z4^2 -
- x3, x2^2 - x1*x3 - x0*x4 + x4^2 - y2^2 +
y1*y3 + y0*y4 - y4^2 - z2^2 + z1*z3 + z0*z4 - z4^2 -
- x4, -x0*y0 + x4*y1 + x3*y2 + x2*y3 +
x1*y4 - y0*z0 + y4*z1 + y3*z2 + y2*z3 + y1*z4 -
- y0, -x1*y0 - x0*y1 - x4*y2 + x2*y4 - x2*y3 + x3*y3 -
x1*y4 + x2*y4 - y1*z0 - y0*z1 - y4*z1 - y3*z2 +
y4*z2 - y2*z3 + y3*z3 - y1*z4 + y2*z4 -
- y1, -x2*y0 - x1*y1 - x0*y2 - x4*y2 + x3*y3 -
y2*z0 - y1*z1 - y0*z2 - y4*z2 - y3*z3 + y4*z3 - y2*z4 + y3*z4 -
- y2,
```

(continues on next page)
Polynomials, Release 10.3

(continued from previous page)

\[-x^3y0 - x^2y1 - x^1y2 - x^0y3 - x^4y3 - x^3y4 + x^4y4
  - y^3z0 - y^2z1 - y^1z2 - y^0z3 - y^4z3 - y^3z4 + y^4z4 -\]
\[-y^3,
  -x^4y0 - x^3y1 - x^2y2 - x^1y3 - x^0y4 - x^4y4
  - y^4z0 - y^3z1 - y^2z2 - y^1z3 - y^0z4 - y^4z4 -\]
\[-y^4)\]

of Multivariate Polynomial Ring in \(x_0, x_1, x_2, x_3, x_4, y_0, y_1, y_2, y_3, y_4,\)
\(z_0, z_1, z_2, z_3, z_4\) over Finite Field of size 3

\[\text{sage}: J += \text{sage.rings.ideal.FieldIdeal}(J.ring()); \# \text{ ensure radical ideal}\]
\[\text{sage}: \text{from sage.doctest.fixtures import reproducible_repr}\]
\[\text{sage}: \text{print}(\text{reproducible_repr}(J.variety()))\]
\[
\{\{x_0: 1, x_1: 0, x_2: 0, x_3: 0, x_4: 0, \\
y_0: 0, y_1: 0, y_2: 0, y_3: 0, y_4: 0, \\
z_0: 0, z_1: 0, z_2: 0, z_3: 0, z_4: 0\}\}
\]

Weil restrictions are often used to study elliptic curves over extension fields so we give a simple example involving those:

\[\text{sage}: K.<a> = \text{QuadraticField}(1/3) \quad \# \text{needs sage.rings.number_field}\]
\[\text{sage}: E = \text{EllipticCurve}(K, [1,2,3,4,5]) \quad \# \text{needs sage.rings.number_field}\]

We pick a point on \(E\):

\[\text{sage}: p = E.lift_x(1); p \quad \# \text{needs sage.rings.number_field}\]
\[
(1 : -6 : 1)\]
\[\text{sage}: I = E.defining_ideal(); I \quad \# \text{needs sage.rings.number_field}\]

Ideal \((-x^3 - 2*x^2*z + x*y*z + y^2*z - 4*x*z^2 + 3*y*z^2 - 5*z^3)\)

of Multivariate Polynomial Ring in \(x, y, z\)

over Number Field in \(a\) with defining polynomial \(x^2 - 1/3\)

with \(a = 0.5773502691896258?\)

Of course, the point \(p\) is a root of all generators of \(I\):

\[\text{sage}: I.subs(x=1, y=2, z=1) \quad \# \text{needs sage.rings.number_field}\]

Ideal \((0)\) of Multivariate Polynomial Ring in \(x, y, z\)

over Number Field in \(a\) with defining polynomial \(x^2 - 1/3\)

with \(a = 0.5773502691896258?\)

I is also radical:

\[\text{sage}: I.radical() == I \quad \# \text{needs sage.rings.number_field}\]

True

So we compute its Weil restriction:

\[\text{sage}: J = I.weil_restriction(); J \quad \# \text{needs sage.rings.number_field}\]

Ideal \((-x^0^3 - x^0*x^1^2 - 2*x^0*z^0 - 2/3*x^1^2*z^0 + x^0*y^0*z^0 + y^0^2*z^0 + 1/3*x^1*y^1*z^0 + 1/3*y^1^2*z^0 - 4*x^0*z^0^2 + 3*y^0*z^0^2 - 5*z^0^3\)

(continues on next page)
- 4/3*x0*x1*z1 + 1/3*x1*y0*z1 + 1/3*x0*y1*z1 + 2/3*y0*y1*z1
- 8/3*x1*z0*z1 + 2*y1*z0*z1 - 4/3*x0*z1^2 + y0*z1^2 - 5*z0*z1^2,
- 3*x0^2*x1 - 1/3*x1^3 - 4*x0*x1*z0 + x1*y0*z0 + x0*y1*z0
+ 2*y0*y1*z0 - 4*x1*z0^2 + 3*y1*z0^2 - 2*x0^2*z1 - 2/3*x1^2*z1
+ x0*y0*z1 + y0^2*z1 + 1/3*x1*y1*z1 + 1/3*y1^2*z1 - 8*x0*z0*z1
+ 6*y0*y1*z0 - 15*z0^2*z1 - 4/3*x1*z1^2 + y1*z1^2 - 5/3*z1^3)
of Multivariate Polynomial Ring in x0, x1, y0, y1, z0, z1 over Rational Field

We can check that the point $p$ is still a root of all generators of $J$:

```sage
sage: J.subs(x0=1, y0=2, z0=1, x1=0, y1=0, z1=0)  # needs sage.rings.number_field
Ideal (0, 0) of Multivariate Polynomial Ring in x0, x1, y0, y1, z0, z1
over Rational Field
```

Example for relative number fields:

```sage
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: K.<w> = NumberField(x^5 - 2)
sage: R.<x> = K[]
sage: L.<v> = K.extension(x^2 + 1)
sage: S.<x,y> = L[]
sage: I = S.ideal([y^2 - x^3 - 1])
sage: I.weil_restriction()
Ideal (-x0^3 + 3*x0*x1^2 + y0^2 - y1^2 - 1, -3*x0^2*x1 + x1^3 + 2*y0*y1) of
Multivariate Polynomial Ring in x0, x1, y0, y1, y over Number Field in w with defining polynomial x^5 - 2
```

**Note:** Based on a Singular implementation by Michael Brickenstein

```python
class sage.rings.polynomial.multi_polynomial_ideal.MPolynomialIdeal_macaulay2_repr
Bases: object

An ideal in a multivariate polynomial ring, which has an underlying Macaulay2 ring associated to it.

EXAMPLES:

```sage
sage: R.<x,y,z,w> = PolynomialRing(ZZ, 4)
sage: I = ideal(x*y-z^2, y^2-w^2)
sage: I
Ideal (x*y - z^2, y^2 - w^2) of Multivariate Polynomial Ring in x, y, z, w over...

```

```python
class sage.rings.polynomial.multi_polynomial_ideal.MPolynomialIdeal_magma_repr
Bases: object

class sage.rings.polynomial.multi_polynomial_ideal.MPolynomialIdeal_quotient

Bases: MPolynomialIdeal

An ideal in a quotient of a multivariate polynomial ring.

```
EXAMPLES:

```python
sage: Q.<x,y,z,w> = QQ[x,y,z,w].quotient([x*y-z^2, y^2-w^2])
sage: I = ideal(x + y^2 + z - 1)
sage: I
Ideal (w^2 + x + z - 1) of Quotient
of Multivariate Polynomial Ring in x, y, z, w over Rational Field
by the ideal (x*y - z^2, y^2 - w^2)
```

`reduce(f)`

Reduce an element modulo a Gröbner basis for this ideal. This returns 0 if and only if the element is in this ideal. In any case, this reduction is unique up to monomial orders.

EXAMPLES:

```python
sage: R.<T,U,V,W,X,Y,Z> = PolynomialRing(QQ, order='lex')
sage: I = R.ideal([T^2 + U^2 - 1, V^2 + W^2 - 1, X^2 + Y^2 + Z^2 - 1])
sage: Q.<t,u,v,w,x,y,z> = R.quotient(I)
sage: J = Q.ideal([u*v - x, u*w - y, t - z])
sage: J.reduce(t^2 - z^2)
0
sage: J.reduce(u^2)
-z^2 + 1
sage: t^2 - z^2 in J
True
sage: u^2 in J
False
```

```python
class sage.rings.polynomial.multi_polynomial_ideal.MPolynomialIdeal_singular_base_repr
Bases: object

syzygy_module()

Computes the first syzygy (i.e., the module of relations of the given generators) of the ideal.

EXAMPLES:

```python
sage: R.<x,y> = PolynomialRing(QQ)
sage: f = 2*x^2 + y
sage: g = y
sage: h = 2*f + g
sage: I = Ideal([f,g,h])
sage: M = I.syzygy_module(); M
[ -2 -1 1]
[ -y 2*x^2 + y 0]
```

ALGORITHM: Uses Singular's syz command

```python
class sage.rings.polynomial.multi_polynomial_ideal.MPolynomialIdeal_singular_repr
Bases: MPolynomialIdeal_singular_base_repr

An ideal in a multivariate polynomial ring, which has an underlying Singular ring associated to it.

3.1. Multivariate Polynomials and Polynomial Rings 377
**associated_primes** *(algorithm='sy')*

Return a list of the associated primes of primary ideals of which the intersection is \( I = \text{self} \).

An ideal \( Q \) is called primary if it is a proper ideal of the ring \( R \) and if whenever \( ab \in Q \) and \( a \notin Q \) then \( b^n \in Q \) for some \( n \in \mathbb{Z} \).

If \( Q \) is a primary ideal of the ring \( R \), then the radical ideal \( P \) of \( Q \), i.e. \( P = \{ a \in R, a^n \in Q \} \) for some \( n \in \mathbb{Z} \), is called the associated prime of \( Q \).

If \( I \) is a proper ideal of the ring \( R \) then there exists a decomposition in primary ideals \( Q_i \), such that

- their intersection is \( I \)
- none of the \( Q_i \) contain the intersection of the rest, and
- the associated prime ideals of \( Q_i \) are pairwise different.

This method returns the associated primes of the \( Q_i \).

**INPUT:**

- `algorithm` - string:
  - 'sy' – (default) use the Shimoyama-Yokoyama algorithm
  - 'gtz' – use the Gianni-Trager-Zacharias algorithm

**OUTPUT:** a list of associated primes

**EXAMPLES:**

```sage
sage: R.<x,y,z> = PolynomialRing(QQ, 3, order='lex')
sage: p = z^2 + 1; q = z^3 + 2
sage: I = (p*q^2, y - z^2) * R
sage: pd = I.associated_primes(); sorted(pd, key=str)
[ Ideal (z^2 + 1, y + 1) of Multivariate Polynomial Ring in x, y, z over Rational Field,
  Ideal (z^3 + 2, y - z^2) of Multivariate Polynomial Ring in x, y, z over Rational Field]
```

**ALGORITHM:**

Uses Singular.

**REFERENCES:**


**basis_is_groebner** *(singular=Singular)*

Return True if the generators of this ideal (`self.gens()`) form a Groebner basis.

Let \( I \) be the set of generators of this ideal. The check is performed by trying to lift \( \text{Syz}(LM(I)) \) to \( \text{Syz}(I) \) as \( I \) forms a Groebner basis if and only if for every element \( S \) in \( \text{Syz}(LM(I)) \):

\[
S \ast G = \sum_{i=0}^{m} h_i g_i \succ 0.
\]

**ALGORITHM:**

Uses Singular.

**EXAMPLES:**
A more complicated example:

```
sage: R.<U6,U5,U4,U3,U2, u6,u5,u4,u3,u2, h> = PolynomialRing(GF(7583))  # needs sage.rings.fin...
  U6 + U5 + U4 + U3 + U2 - 3791*h,
  U2*u2 - U3 + h^2, U3*u3 - h^2, U4*u4 - h^2,  
  U5*u5 + U4 + U3 + U2 + U3*r + U2 - U3 + U3*r + U2
<...>
  U5^2*U4*U3*U2*h + U5*U4^2*U3*U2*h + U5*U4*U3^2*U2*h + U5*U4*U3*U2^2*h
<...>
  U4^2*U3^2*h^2 + 1515*u5*u3^2*u2*h + 1515*u5*u4*u2^2*h + 1521*u5*u4*u3*h^3 - 3028*u4^2*u3*h^3 - 3028*u4*u3^2*h^3 + 1521*u5*u4*u2*h^3 - 3028*u4^2*u2*h^3 + 1521*u5*u3*u2*h^3
  U4^2*U3^2*h^2 - U5^2*U3^2*h^2 - U4*U3^2*h^2 - U5^2*U3^2*h^2 - U4*U3^2*h^2 - 1422*U5^3*h^3  
```

(continues on next page)
\[\begin{align*}
&\quad + 2U5^2U4U3h^2 + 2U5U4^2U3h^2 + 2U5U4U3^2h^2 \\
&\quad + 2U5U4^2U2h^2 + 2U5^2U3U2h^2 \\
&\quad - 2U4^2U3U2h^2 - 2U5U3^2U2h^2 - 2U4U3^2U2h^2 \\
&\quad - U5U4U3h^3 - U5U4U2h^3 - U5U3U2h^3 - U4U3U2h^3
\end{align*}\]

```
sage: Ideal(l).basis_is_groebner()  # needs sage.rings.finite_rings
False
sage: gb = Ideal(l).groebner_basis()  # needs sage.rings.finite_rings
sage: Ideal(gb).basis_is_groebner()  # needs sage.rings.finite_rings
True
```

**Note:** From the Singular Manual for the reduce function we use in this method: ‘The result may have no meaning if the second argument (self) is not a standard basis’. I (malb) believe this refers to the mathematical fact that the results may have no meaning if self is not a standard basis, i.e., Singular doesn’t ‘add’ any additional ‘nonsense’ to the result. So we may actually use reduce to determine if self is a Groebner basis.

**complete_primary_decomposition()**

A decorator that creates a cached version of an instance method of a class.

**Note:** For proper behavior, the method must be a pure function (no side effects). Arguments to the method must be hashable or transformed into something hashable using key or they must define `sage.structure.sage_object.SageObject._cache_key()`.

**EXAMPLES:**

```
sage: class Foo():
    ....:     @cached_method
    ....:     def f(self, t, x=2):
    ....:         print('computing')
    ....:         return t**x
sage: a = Foo()
```

The example shows that the actual computation takes place only once, and that the result is identical for equivalent input:

```
sage: res = a.f(3, 2); res
computing
9
sage: a.f(t = 3, x = 2) is res
True
sage: a.f(3) is res
True
```

**Note,** however, that the `CachedMethod` is replaced by a `CachedMethodCaller` or `CachedMethodCallerNoArgs` as soon as it is bound to an instance or class:

```
sage: P.<a,b,c,d> = QQ[]
sage: I = P*[a,b]
```
So, you would hardly ever see an instance of this class alive.

The parameter \texttt{key} can be used to pass a function which creates a custom cache key for inputs. In the following example, this parameter is used to ignore the \texttt{algorithm} keyword for caching:

\begin{verbatim}
sage: class A():
    ....:    def _f_normalize(self, x, algorithm):
    ....:        return x
    ....:    @cached_method(key=_f_normalize)
    ....:    def f(self, x, algorithm='default'):
    ....:        return x
sage: a = A()
sage: a.f(1, algorithm='default')
is a.f(1) is a.f(1, algorithm='algorithm')
True
\end{verbatim}

The parameter \texttt{do_pickle} can be used to enable pickling of the cache. Usually the cache is not stored when pickling:

\begin{verbatim}
sage: class A():
    ....:    @cached_method
    ....:    def f(self, x):
    ....:        return None
sage: import __main__
sage: __main__.A = A
sage: a = A()
sage: a.f(1)

sage: len(a.f.cache)
1
sage: b = loads(dumps(a))

sage: len(b.f.cache)
0
\end{verbatim}

When \texttt{do_pickle} is set, the pickle contains the contents of the cache:

\begin{verbatim}
sage: class A():
    ....:    @cached_method(do_pickle=True)
    ....:    def f(self, x):
    ....:        return None
sage: __main__.A = A
sage: a = A()
sage: a.f(1)

sage: len(a.f.cache)
1
sage: b = loads(dumps(a))

sage: len(b.f.cache)
1
\end{verbatim}

Cached methods cannot be copied like usual methods, see \url{github issue #12603}. Copying them can lead to very surprising results:

\begin{verbatim}
sage: class A:
    ....:    @cached_method
    ....:    def f(self):
    ....:        return 1
sage: class B:
    ....:    g=A.f
    ....:    def f(self):
    ....:        return 2
\end{verbatim}

(continues on next page)
dimension (singular='singular_default')

The dimension of the ring modulo this ideal.

EXAMPLES:

```python
sage: P.<x,y,z> = PolynomialRing(GF(32003), order='degrevlex') # needs sage.rings.finite_rings
sage: I = ideal(x^2 - y, x^3) # needs sage.rings.finite_rings
sage: I.dimension() # needs sage.rings.finite_rings
1
```

If the ideal is the total ring, the dimension is $-1$ by convention.

For polynomials over a finite field of order too large for Singular, this falls back on a toy implementation of Buchberger to compute the Groebner basis, then uses the algorithm described in Chapter 9, Section 1 of Cox, Little, and O'Shea's "Ideals, Varieties, and Algorithms".

EXAMPLES:

```python
sage: R.<x,y> = PolynomialRing(GF(2147483659^2), order='lex')
sage: I = R.ideal([x*y, x*y + 1])
sage: I.dimension()  # needs sage.rings.finite_rings
-1
```

ALGORITHM:

Uses Singular, unless the characteristic is too large.
**Note:** Requires computation of a Groebner basis, which can be a very expensive operation.

### elimination_ideal\( \text{variables, algorithm=None, *args, **kwds} \)

Return the elimination ideal of this ideal with respect to the variables given in `variables`.

**INPUT:**

- `variables` – a list or tuple of variables in `self.ring()`
- `algorithm` - determines the algorithm to use, see below for available algorithms.

**ALGORITHMS:**

- `'libsingular:eliminate'` – libSingular's `eliminate` command (default)
- `'giac:eliminate'` – Giac's `eliminate` command (if available)

If only a system is given - e.g. 'giac' - the default algorithm is chosen for that system.

**EXAMPLES:**

```
sage: R.<x,y,t,s,z> = PolynomialRing(QQ,5)
sage: I = R * [x - t, y - t^2, z - t^3, s - x + y^3]
sage: J = I.elimination_ideal([t, s]); J
Ideal (y^2 - x*z, x*y - z, x^2 - y)
of Multivariate Polynomial Ring in x, y, t, s, z over Rational Field
```

You can use Giac to compute the elimination ideal:

```
sage: print("possible output from giac", flush=True); I.elimination_ideal([t,˓→s], algorithm="giac") == J
possible output...
True
```

The list of available Giac options is provided at `sage.libs.giac.groebner_basis()`.

**ALGORITHM:**

Uses Singular, or Giac (if available).

**Note:** Requires computation of a Groebner basis, which can be a very expensive operation.

### free_resolution\( *args, **kwds \)

Return a free resolution of `self`.

For input options, see `FreeResolution`.

**EXAMPLES:**

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: f = 2*x^2 + y
sage: g = y
sage: h = 2*f + g
sage: I = R.ideal([f,g,h])
sage: res = I.free_resolution()
sage: res
S^1 <-- S^2 <-- S^1 <-- 0
sage: ascii_art(res.chain_complex())
```

(continues on next page)
genus()

A decorator that creates a cached version of an instance method of a class.

**Note:** For proper behavior, the method must be a pure function (no side effects). Arguments to the method must be hashable or transformed into something hashable using `key` or they must define `sage.structure.sage_object.SageObject._cache_key()`.

**EXAMPLES:**

```python
sage: class Foo():
    ....:     @cached_method
    ....:     def f(self, t, x=2):
    ....:         print('computing')
    ....:         return t**x
sage: a = Foo()
```

The example shows that the actual computation takes place only once, and that the result is identical for equivalent input:

```python
sage: res = a.f(3, 2); res
computing
9
sage: a.f(t = 3, x = 2) is res
True
sage: a.f(3) is res
True
```

Note, however, that the `CachedMethod` is replaced by a `CachedMethodCaller` or `CachedMethodCallerNoArgs` as soon as it is bound to an instance or class:

```python
sage: P.<a,b,c,d> = QQ[]
sage: I = P*[a,b]
sage: type(I._cached__gens)
<class 'sage.misc.cachefunc.CachedMethodCallerNoArgs'>
```

So, you would hardly ever see an instance of this class alive.

The parameter `key` can be used to pass a function which creates a custom cache key for inputs. In the following example, this parameter is used to ignore the `algorithm` keyword for caching:

```python
sage: class A():
    ....:     def _f_normalize(self, x, algorithm):
    ....:         return x
    ....:     @cached_method(key=_f_normalize)
```

(continues on next page)
The parameter `do_pickle` can be used to enable pickling of the cache. Usually the cache is not stored when pickling:

```python
sage: class A():
....:     @cached_method
....:     def f(self, x): return None
sage: __main__.A = A
sage: a = A()
sage: a.f(1)
1
sage: len(a.f.cache)
1
sage: b = loads(dumps(a))
sage: len(b.f.cache)
0
```

When `do_pickle` is set, the pickle contains the contents of the cache:

```python
sage: class A():
....:     @cached_method(do_pickle=True)
....:     def f(self, x): return None
sage: __main__.A = A
sage: a = A()
sage: a.f(1)
1
sage: len(a.f.cache)
1
sage: b = loads(dumps(a))
sage: len(b.f.cache)
1
```

Cached methods cannot be copied like usual methods, see [github issue #12603](https://github.com). Copying them can lead to very surprising results:

```python
sage: class A:
....:     @cached_method
....:     def f(self):
....:         return 1
sage: class B:
....:     g=A.f
....:     def f(self):
....:         return 2
sage: b=B()
sage: b.f()
2
sage: b.g()
1
sage: b.f()
1
```

**graded_free_resolution**(*args, **kwds*)

Return a graded free resolution of `self`.  

3.1. Multivariate Polynomials and Polynomial Rings
For input options, see GradedFreeResolution.

EXAMPLES:

```python
sage: R.<x,y> = PolynomialRing(QQ)
sage: f = 2*x^2 + y^2
sage: g = y^2
sage: h = 2*f + g
sage: I = R.ideal([f,g,h])
sage: I.graded_free_resolution(shifts=[1])
S(-1) <-- S(-3)⊕S(-3) <-- S(-5) <-- 0
sage: f = 2*x^2 + y
sage: g = y
sage: h = 2*f + g
sage: I = R.ideal([f,g,h])
sage: I.graded_free_resolution(degrees=[1,2])
S(0) <-- S(-2)⊕S(-2) <-- S(-4) <-- 0
sage: q = ZZ['q'].fraction_field().gen()
sage: R.<x,y,z> = q.parent()
```

hilbert_numerator (grading=None, algorithm='sage')

Return the Hilbert numerator of this ideal.

INPUT:

- grading - (optional) a list or tuple of integers
- algorithm - (default: 'sage') must be either 'sage' or 'singular'

Let $I$ (which is self) be a homogeneous ideal and $R = \bigoplus_d R_d$ (which is self.ring()) be a graded commutative algebra over a field $K$. Then the Hilbert function is defined as $H(d) = \dim_K R_d$ and the Hilbert series of $I$ is defined as the formal power series $HS(t) = \sum_{d=0}^{\infty} H(d)t^d$.

This power series can be expressed as $HS(t) = Q(t)/(1 - t)^n$ where $Q(t)$ is a polynomial over $\mathbb{Z}$ and $n$ the number of variables in $R$. This method returns $Q(t)$, the numerator; hence the name, hilbert_numerator. An optional grading can be given, in which case the graded (or weighted) Hilbert numerator is given.

EXAMPLES:

```python
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: I = Ideal([x^3*y^2 + 3*x^2*y^2*z + y^3*z^2 + z^5])
sage: I.hilbert_numerator()
# needs sage.libs.flint
-t^5 + 1
sage: R.<a,b> = PolynomialRing(QQ)
sage: J = R.ideal([a^2*b, a*b^2])
sage: J.hilbert_numerator()
# needs sage.libs.flint
t^4 - 2*t^3 + 1
```

Chapter 3. Multivariate Polynomials
hilbert_polynomial (algorithm='sage')

Return the Hilbert polynomial of this ideal.

INPUT:

- algorithm= (default: 'sage') must be either 'sage' or 'singular'

Let $I$ (which is self) be a homogeneous ideal and $R = \bigoplus_d R_d$ (which is self.ring()) be a graded commutative algebra over a field $K$. The Hilbert polynomial is the unique polynomial $HP(t)$ with rational coefficients such that $HP(d) = \dim_K R_d$ for all but finitely many positive integers $d$.

EXAMPLES:

```python
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: I = Ideal([x^3*y^2 + 3*x^2*y^2*z + y^3*z^2 + z^5])
sage: I.hilbert_polynomial()
5*t - 5
```

Of course, the Hilbert polynomial of a zero-dimensional ideal is zero:

```python
sage: J0 = Ideal([x^3*y^2 + 3*x^2*y^2*z + y^3*z^2 + z^5, ....: y^3 - 2*x*z^2 + x*y, x^4 + x*y - y*z^2])
sage: J = P*[m.lm() for m in J0.groebner_basis()]
sage: J.dimension()
0
sage: J.hilbert_polynomial()
0
```

It is possible to request a computation using the Singular library:

```python
sage: I.hilbert_polynomial(algorithm='singular') == I.hilbert_polynomial()  # needs sage.libs.flint
True
sage: J.hilbert_polynomial(algorithm='singular') == J.hilbert_polynomial()  # needs sage.libs.flint
True
```

Here is a bigger examples:

```python
sage: n = 4; m = 11; P = PolynomialRing(QQ, n * m, "x"); x = P.gens()
sage: M = Matrix(n, x)
sage: Minors = P.ideal(M.minors(2))
sage: hp = Minors.hilbert_polynomial(); hp  # needs sage.libs.flint
1/21772800*t^13 + 61/21772800*t^12 + 1661/21772800*t^11 + 26681/21772800*t^10 + 93841/7257600*t^9 + 685421/7257600*t^8 + 1524809/3110400*t^7 + 39780323/21772800*t^6 + 6638071/1360800*t^5 + 12509761/1360800*t^4 + 2689031/226800*t^3 + 1494509/151200*t^2 + 12001/2520*t + 1
```

Because Singular uses 32-bit integers, the above example would fail with Singular. We don’t test it here, as it has a side-effect on other tests that is not understood yet (see github issue #26300):

```python
sage: Minors.hilbert_polynomial(algorithm='singular')  # not tested
Traceback (most recent call last):
  ...
RuntimeError: error in Singular function call 'hilbPoly':
```

(continues on next page)
int overflow in hilb 1
error occurred in or before poly.lib::hilbPoly line 58: `intvec v=hilb(I,
-2);`
expected intvec-expression. type 'help intvec;'

Note that in this example, the Hilbert polynomial gives the coefficients of the Hilbert-Poincaré series in all
degrees:

```python
sage: P = PowerSeriesRing(QQ, 't', default_prec=50)
sage: hs = Minors.hilbert_series()  # needs sage.libs.flint
sage: list(P(hs.numerator()) / P(hs.denominator())) == [hp(t=k)  # needs sage.libs.flint
....: for k in range(50)]
True
```

`hilbert_series(grading=None, algorithm='sage')`

Return the Hilbert series of this ideal.

INPUT:

- grading – (optional) a list or tuple of integers
- algorithm – (default: 'sage') must be either 'sage' or 'singular'

Let \( I \) (which is self) be a homogeneous ideal and \( R = \bigoplus_d R_d \) (which is self.ring()) be a graded
commutative algebra over a field \( K \). Then the Hilbert function is defined as \( H(d) = \dim_K R_d \) and the Hilbert
series of \( I \) is defined as the formal power series \( HS(t) = \sum_{d=0}^\infty H(d) t^d \).

This power series can be expressed as \( HS(t) = Q(t)/(1 - t)^n \) where \( Q(t) \) is a polynomial over \( \mathbb{Z} \) and \( n \) the
number of variables in \( R \). This method returns \( Q(t)/(1 - t)^n \), normalised so that the leading monomial of
the numerator is positive.

An optional grading can be given, in which case the graded (or weighted) Hilbert series is given.

EXAMPLES:

```python
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: I = Ideal([x^3*y^2 + 3*x^2*y^2*z + y^3*z^2 + z^5])
sage: I.hilbert_series()  # needs sage.libs.flint
(t^4 + t^3 + t^2 + t + 1)/(t^2 - 2*t + 1)
sage: R.<a,b> = PolynomialRing(QQ)
sage: J = R.ideal([a^2*b, a*b^2])
sage: J.hilbert_series()  # needs sage.libs.flint
(t^3 - t^2 - t - 1)/(t^12 + t^11 + t^10 - t^2 - t - 1)
sage: K = R.ideal([a^2*b^3, a*b^4 + a^3*b^2])
sage: K.hilbert_series(grading=[1,2])  # needs sage.libs.flint
(t^11 + t^8 - t^6 - t^5 - t^4 - t^3 - t^2 - t - 1)/(t^2 - 1)
```

(continues on next page)
integral_closure \( p=0, r=True, \text{ singular='singular_default'} \)

Let \( I = self \).

Return the integral closure of \( I, \ldots, IP \), where \( sI \) is an ideal in the polynomial ring \( R = k[x(1), \ldots x(n)] \). If \( p \) is not given, or \( p = 0 \), compute the closure of all powers up to the maximum degree in \( t \) occurring in the closure of \( R[It] \) (so this is the last power whose closure is not just the sum/product of the smaller). If \( r \) is given and \( r \) is True, \( I.\text{integral_closure}() \) starts with a check whether \( I \) is already a radical ideal.

**INPUT:**

- \( p \) - powers of \( I \) (default: 0)
- \( r \) - check whether \( self \) is a radical ideal first (default: True)

**EXAMPLES:**

```python
sage: R.<x,y> = QQ[]
sage: I = ideal([x^2, x*y^4, y^5])
sage: I.integral_closure()
[x^2, x*y^4, y^5, x*y^3]
```

**ALGORITHM:**

Uses libSINGULAR.

**interreduced_basis()**

If this ideal is spanned by \( (f_1, \ldots, f_n) \), return \( (g_1, \ldots, g_s) \) such that:

- \( (f_1, \ldots, f_n) = (g_1, \ldots, g_s) \)
- \( LT(g_i) \neq LT(g_j) \) for all \( i \neq j \)
- \( LT(g_i) \) does not divide \( m \) for all monomials \( m \) of \( \{g_1, \ldots, g_i-1, g_i+1, \ldots, g_s\} \)
- \( LC(g_i) = 1 \) for all \( i \) if the coefficient ring is a field.

**EXAMPLES:**

```python
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: I = Ideal([z*x + y^3, z + y^3, z + x*y])
sage: I.interreduced_basis()
[y^3 + z, x*y + z, x*z - z]
```

**Note that tail reduction for local orderings is not well-defined:**

```python
sage: R.<x,y,z> = PolynomialRing(QQ, order='negdegrevlex')
sage: I = Ideal([z*x + y^3, z + y^3, z + x*y])
sage: I.interreduced_basis()
[z + x*y, x*y - y^3, x^2*y - y^3]
```

A fixed error with nonstandard base fields:

```python
sage: R.<t> = QQ['t']
sage: K.<x,y> = R.fraction_field() ['x,y']
sage: I = t*x + K
(continues on next page)"
The interreduced basis of 0 is 0:

```python
sage: P.<x,y,z> = GF(2)[]
sage: Ideal(P(0)).interreduced_basis()
[0]
```

**ALGORITHM:**

Uses Singular’s `interred` command or `sage.rings.polynomial.toy_buchberger.inter_reduction()` if conversion to Singular fails.

**intersection** (*others*)

Return the intersection of the arguments with this ideal.

**EXAMPLES:**

```python
sage: R.<x,y> = PolynomialRing(QQ, 2, order='lex')
sage: I = x*R
sage: J = y*R
sage: I.intersection(J)
Ideal (x*y) of Multivariate Polynomial Ring in x, y over Rational Field
```

The following simple example illustrates that the product need not equal the intersection.

```python
sage: I = (x^2, y) * R
sage: J = (y^2, x) * R
sage: K = I.intersection(J); K
Ideal (y^2, x*y, x^2) of Multivariate Polynomial Ring in x, y over Rational Field
```

```python
sage: IJ = I*J; IJ
Ideal (x^2*y^2, x^3, y^3, x*y) of Multivariate Polynomial Ring in x, y over Rational Field
```

```python
sage: IJ == K
False
```

**Intersection of several ideals:**

```python
sage: R.<x,y,z> = PolynomialRing(QQ, 3, order='lex')
sage: I1 = x * R
sage: I2 = y * R
sage: I3 = (x, y) * R
sage: I4 = (x^2 + x*y*z, y^2 - z^3*y, z^3 + y^5*x*z) * R
sage: I1.intersection(I3, I4).groebner_basis()
[x^2*y + x*y*z^4, x*y^2 - x*y*z^3, x*y*z^20 - x*y*z^3]
```

The ideals must share the same ring:

```python
sage: R2.<x,y> = PolynomialRing(QQ, 2, order='lex')
sage: R3.<x,y,z> = PolynomialRing(QQ, 3, order='lex')
sage: I2 = x*R2
sage: I3 = x*R3
sage: I2.intersection(I3)
Traceback (most recent call last):
  ...
TypeError: Intersection is only available for ideals of the same ring.
```
**is_prime(**

Return True if this ideal is prime.

**INPUT:**

- keyword arguments are passed on to `complete_primary_decomposition()`; in this way you can specify the algorithm to use.

**EXAMPLES:**

```python
sage: R.<x, y> = PolynomialRing(QQ, 2)
sage: I = (x^2 - y^2 - 1) * R
sage: I.is_prime()
True
sage: (I^2).is_prime()
False
sage: J = (x^2 - y^2) * R
sage: J.is_prime()
False
sage: (J^3).is_prime()
False
sage: (I * J).is_prime()
False
```

The following is `github issue #5982`. Note that the quotient ring is not recognized as being a field at this time, so the fraction field is not the quotient ring itself:

```python
sage: Q = R.quotient(I); Q
Quotient of Multivariate Polynomial Ring in x, y over Rational Field
   by the ideal (x^2 - y^2 - 1)
sage: Q.fraction_field()
Fraction Field of
   Quotient of Multivariate Polynomial Ring in x, y over Rational Field
   by the ideal (x^2 - y^2 - 1)
```

**minimal_associated_primes()**

**OUTPUT:**

- list - a list of prime ideals

**EXAMPLES:**

```python
sage: R.<x,y,z> = PolynomialRing(QQ, 3, 'xyz')
sage: p = z^2 + 1; q = z^3 + 2
sage: I = (p*q^2, y - z^2) * R
sage: sorted(I.minimal_associated_primes(), key=str)
[Ideal (z^2 + 1, -z^2 + y)
   of Multivariate Polynomial Ring in x, y, z over Rational Field,
   Ideal (z^2 - 1, -z^2 + y)
   of Multivariate Polynomial Ring in x, y, z over Rational Field]
```

**ALGORITHM:**

Uses Singular.

**normal_basis**(degree=\texttt{None}, algorithm='\texttt{libsingular}', singular='\texttt{libsingular}\_default')

Return a vector space basis of the quotient ring of this ideal.

**INPUT:**
• **degree** – integer (default: None)

• **algorithm** – string (default: "libsingular"); if not the default, this will use the kbase() or weightKB() command from Singular

• **singular** – the singular interpreter to use when algorithm is not "libsingular" (default: the default instance)

**OUTPUT:**

Monomials in the basis. If **degree** is given, only the monomials of the given degree are returned.

**EXAMPLES:**

```python
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: I = R.ideal(x^2 + y^2 + z^2 - 4, x^2 + 2*y^2 - 5, x*z - 1)
sage: I.normal_basis()
y*z^2, z^2, y*z, z, x*y, y, x, 1
sage: I.normal_basis(algorithm=singular)
y*z^2, z^2, y*z, z, x*y, y, x, 1
```

The result can be restricted to monomials of a chosen degree, which is particularly useful when the quotient ring is not finite-dimensional as a vector space.

```python
sage: J = R.ideal(x^2 + y^2 + z^2 - 4, x^2 + 2*y^2 - 5)
sage: J.dimension()
1
sage: [J.normal_basis(d) for d in (0..3)]
[[1], [z, y, x], [z^2, y*z, x*z, x*y], [z^3, y*z^2, x*z^2, x*y*z]]
sage: [J.normal_basis(d, algorithm=singular) for d in (0..3)]
[[1], [z, y, x], [z^2, y*z, x*z, x*y], [z^3, y*z^2, x*z^2, x*y*z]]
```

In case of a polynomial ring with a weighted term order, the degree of the monomials is taken with respect to the weights.

```python
sage: T = TermOrder('wdegrevlex', (1, 2, 3))
sage: R.<x,y,z> = PolynomialRing(QQ, order=T)
sage: B = R.ideal(x*y^2 + x^5, z*y + x^3*y).normal_basis(9); B
x^2*y^2*z, x^3*z^2, x*y*z^2, z^3
sage: all(f.degree() == 9 for f in B)
True
```

**plot** *(singular=Singular)*

If you somehow manage to install surf, perhaps you can use this function to implicitly plot the real zero locus of this ideal (if principal).

**INPUT:**

• **self** - must be a principal ideal in 2 or 3 vars over Q.

**EXAMPLES:**

Implicit plotting in 2-d:

```python
sage: R.<x,y> = PolynomialRing(QQ,2)
sage: I = R.ideal([y^3 - x^2])
sage: I.plot() # cusp
Graphics object consisting of 1 graphics primitive
sage: I = R.ideal([y^2 - x^2 - 1])
sage: I.plot() # hyperbola
```

(continues on next page)
Implicit plotting in 3-d:

```
sage: R.<x,y,z> = PolynomialRing(QQ,3)
sage: I = R.ideal([y^2 + x^2*(1/4) - z])
sage: I.plot()  # a cone; optional - surf
sage: I = R.ideal([y^2 + z^2*(1/4) - x])
sage: I.plot()  # same code, from a different angle; optional - surf
sage: I = R.ideal([x^2*y^2 + x^2*z^2 + y^2*z^2 - 16*x*y*z])
sage: I.plot()  # Steiner surface; optional - surf
```

AUTHORS:
• David Joyner (2006-02-12)

**primary_decomposition(algorithm='sy')**

Return a list of primary ideals such that their intersection is self.

An ideal \( Q \) is called primary if it is a proper ideal of the ring \( R \), and if whenever \( ab \in Q \) and \( a \notin Q \), then \( b^n \in Q \) for some \( n \in \mathbb{Z} \).

If \( Q \) is a primary ideal of the ring \( R \), then the radical ideal \( P \) of \( Q \) (i.e., the ideal consisting of all \( a \in R \) with \( a^n \in Q \) for some \( n \in \mathbb{Z} \)), is called the associated prime of \( Q \).

If \( I \) is a proper ideal of a Noetherian ring \( R \), then there exists a finite collection of primary ideals \( Q_i \) such that the following hold:

• the intersection of the \( Q_i \) is \( I \);

• none of the \( Q_i \) contains the intersection of the others;

• the associated prime ideals of the \( Q_i \) are pairwise distinct.

INPUT:
• algorithm -- string:
  – 'sy' – (default) use the Shimoyama-Yokoyama algorithm
  – 'gtz' – use the Gianni-Trager-Zacharias algorithm

OUTPUT:
• a list of primary ideals \( Q_i \) forming a primary decomposition of self.

**EXAMPLES:**

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3, order='lex')
sage: p = z^2 + 1; q = z^3 + 2
sage: I = (p^5*q^2, y - z^2) * R
sage: pd = I.primary_decomposition(); sorted(pd, key=str)
[Ideal (z^2 + 1, y + 1) of Multivariate Polynomial Ring in x, y, z over Rational Field, Ideal (z^6 + 4*z^3 + 4, y - z^2) of Multivariate Polynomial Ring in x, y, z over Rational Field]
```
ALGORITHM:
Uses Singular.

REFERENCES:

primary_decomposition_complete()
A decorator that creates a cached version of an instance method of a class.

Note: For proper behavior, the method must be a pure function (no side effects). Arguments to the method must be hashable or transformed into something hashable using key or they must define sage.structure.sage_object.SageObject._cache_key().

EXAMPLES:
sage: class Foo():
......:   @cached_method
......:   def f(self, t, x=2):
......:       print('computing')
......:       return t**x
sage: a = Foo()

The example shows that the actual computation takes place only once, and that the result is identical for equivalent input:
sage: res = a.f(3, 2); res
computing
9
sage: a.f(t = 3, x = 2) is res
True
sage: a.f(3) is res
True

Note, however, that the CachedMethod is replaced by a CachedMethodCaller or CachedMethodCallerNoArgs as soon as it is bound to an instance or class:

sage: P.<a,b,c,d> = QQ[]
sage: I = P*[a,b]
sage: type(I._class__gens)
<class 'sage.misc.cachefunc.CachedMethodCallerNoArgs'>

So, you would hardly ever see an instance of this class alive.

The parameter key can be used to pass a function which creates a custom cache key for inputs. In the following example, this parameter is used to ignore the algorithm keyword for caching:

sage: class A():
......:    def _f_normalize(self, x, algorithm): return x
......:    @cached_method(key=_f_normalize)
......:    def f(self, x, algorithm='default'): return x
The parameter `do_pickle` can be used to enable pickling of the cache. Usually the cache is not stored when pickling:

```python
sage: class A():
    ....:     @cached_method
    ....:     def f(self, x): return None
sage: import __main__
sage: __main__.A = A
sage: a = A()
sage: a.f(1)
len(a.f.cache) 1
sage: b = loads(dumps(a))
sage: len(b.f.cache) 0
```

When `do_pickle` is set, the pickle contains the contents of the cache:

```python
sage: class A():
    ....:     @cached_method(do_pickle=True)
    ....:     def f(self, x): return None
sage: __main__.A = A
sage: a = A()
sage: a.f(1)
len(a.f.cache) 1
sage: b = loads(dumps(a))
sage: len(b.f.cache) 1
```

Cached methods cannot be copied like usual methods, see github issue #12603. Copying them can lead to very surprising results:

```python
sage: class A:
    ....:     @cached_method
    ....:     def f(self):
    ....:         return 1
sage: class B:
    ....:     g=A.f
    ....:     def f(self):
    ....:         return 2
sage: b=B()
sage: b.f() 2
sage: b.g() 1
sage: b.f() 1
```

**quotient** (*J*)

Given ideals *I* = `self` and *J* in the same polynomial ring *P*, return the ideal quotient of *I* by *J* consisting of the polynomials *a* of *P* such that \{a*J* ⊂ *I*\}. 

3.1. Multivariate Polynomials and Polynomial Rings 395
This is also referred to as the colon ideal \( (I:J) \).

**INPUT:**

- \( J \) - multivariate polynomial ideal

**EXAMPLES:**

```python
sage: # needs sage.rings.finite_rings
sage: R.<x,y,z> = PolynomialRing(GF(181), 3)
sage: I = Ideal([x^2 + x*y*z, y^2 - z^3*y, z^3 + y^5*x*z])
sage: J = Ideal([x])
sage: Q = I.quotient(J)
sage: y*z + x in I
False
sage: x in J
True
sage: x * (y*z + x) in I
True
```

**radical()**

The radical of this ideal.

**EXAMPLES:**

This is an obviously not radical ideal:

```python
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: I = (x^2, y^3, (x*z)^4 + y^3 + 10*x^2) * R
sage: I.radical()
Ideal (y, x) of Multivariate Polynomial Ring in x, y, z over Rational Field
```

That the radical is correct is clear from the Groebner basis.

```python
sage: I.groebner_basis()
[y^3, x^2]
```

This is the example from the Singular manual:

```python
sage: p = z^2 + 1; q = z^3 + 2
sage: I = (p*q^2, y - z^2) * R
sage: I.radical()
Ideal (z^2 - y, y^2*z + y*z + 2*y + 2)
of Multivariate Polynomial Ring in x, y, z over Rational Field
```

**Note:** From the Singular manual: A combination of the algorithms of Krick/Logar and Kemper is used. Works also in positive characteristic (Kemper's algorithm).

```python
sage: # needs sage.rings.finite_rings
sage: R.<x,y,z> = PolynomialRing(GF(37), 3)
sage: p = z^2 + 1; q = z^3 + 2
sage: I = (p*q^2, y - z^2) * R
sage: I.radical()
Ideal (z^2 - y, y^2*z + y*z + 2*y + 2)
of Multivariate Polynomial Ring in x, y, z over Finite Field of size 37
```
**saturation**(other)

Return the saturation (and saturation exponent) of the ideal self with respect to the ideal other

**INPUT:**

- other – another ideal in the same ring

**OUTPUT:** a pair (ideal, integer)

**EXAMPLES:**

```python
sage: R.<x, y, z> = QQ[]
sage: I = R.ideal(x^5*z^3, x*y*z, y*z^4)
sage: J = R.ideal(z)
sage: I.saturation(J)
(Ideal (y, x^5) of Multivariate Polynomial Ring in x, y, z over Rational Field, 4)
```

**syzygy_module()**

Computes the first syzygy (i.e., the module of relations of the given generators) of the ideal.

**EXAMPLES:**

```python
sage: R.<x, y> = PolynomialRing(QQ)
sage: f = 2*x^2 + y
sage: g = y
sage: h = 2*f + g
sage: I = Ideal([f, g, h])
sage: M = I.syzygy_module(); M
[-2 -1  1]
[-y 2*x^2 + y 0]
sage: G = vector(I.gens())
sage: M*G
(0, 0)
```

**ALGORITHM:**

Uses Singular's syz command.

**transformed_basis**(algorithm='gwalk', other_ring=None, singular='singular_default')

Return a lex or other_ring Groebner Basis for this ideal.

**INPUT:**

- algorithm - see below for options.
- other_ring - only valid for algorithm='fglm'; if provided, conversion will be performed to this ring. Otherwise a lex Groebner basis will be returned.

**ALGORITHMS:**

- "fglm" - FGLM algorithm. The input ideal must be given with a reduced Groebner Basis of a zero-dimensional ideal
- "gwalk" - Groebner Walk algorithm (default)
- "awalk1" - ‘first alternative’ algorithm
- "awalk2" - 'second alternative' algorithm
- "twalk" - Tran algorithm
- "fwalk" - Fractal Walk algorithm
EXAMPLES:

```python
sage: R.<x,y,z> = PolynomialRing(QQ,3)
sage: I = Ideal([y^3*x^2,x^2*y^2, x^3-x^2, z^4-x^2-y])
sage: J = Ideal(I.groebner_basis())
sage: J = Ideal(J.transformed_basis('fglm',S))
sage: J
Ideal (z^4 + y^3 - y, x^2 + y^3, x*y^3 - y^3, y^4 + y^3)
of Multivariate Polynomial Ring in z, x, y over Rational Field
```

```python
sage: R.<z,y,x> = PolynomialRing(GF(32003), 3, order='lex')

```

needs sage.rings.finite_rings

```python
sage: I = Ideal([y^3 + x*y*z + y^2*z + x*z^3, 3 + x*y + x^2*y + y^2*z])

```

needs sage.rings.finite_rings

```python
sage: I.transformed_basis(gwalk)

```

needs sage.rings.finite_rings

```
[y^9 - y^7*x^2 - y^7*x - y^6*x^3 - y^6*x^2 - 3*y^6 - 3*y^5*x - y^3*x^7 - 3*y^3*x^6 - 3*y^3*x^5 - y^3*x^4 - 9*y^2*x^5 - 18*y^2*x^4 - 9*y^2*x^3 - 27*y*x^3 - 27*y*x^2 - 27*x]
```

ALGORITHM:

Uses Singular.

`triangular_decomposition(algorithm=None, singular='singular_default')`

Decompose zero-dimensional ideal `self` into triangular sets.

This requires that the given basis is reduced w.r.t. to the lexicographical monomial ordering. If the basis of `self` does not have this property, the required Groebner basis is computed implicitly.

INPUT:

- `algorithm` - string or None (default: None)

ALGORITHMS:

- "singular:triangL" - decomposition of `self` into triangular systems (Lazard).
- "singular:triangLfak" - decomposition of `self` into triangular systems plus factorization.
- "singular:triangM" - decomposition of `self` into triangular systems (Moeller).

OUTPUT: a list `T` of lists `t` such that the variety of `self` is the union of the varieties of `t` in `L` and each `t` is in triangular form.

EXAMPLES:

```python
sage: P.<e,d,c,b,a> = PolynomialRing(QQ, 5, order='lex!')
sage: I = sage.rings.ideal.Cyclic(P)
sage: GB = Ideal(I.groebner_basis('libsingular:stdfglm!'))
sage: GB.triangular_decomposition('singular:triangLfak')

[Ideal (a - 1, b - 1, c - 1, d^2 + 3*d + 1, e + d + 3) of Multivariate...]
```

(continues on next page)
Polynomials, Release 10.3

(continued from previous page)

sage: R.<x1,x2> = PolynomialRing(QQ, 2, order='lex')
sage: f1 = 1/2*((x1^2 + 2*x1 - 4)*x2^2 + 2*(x1^2 + x1)*x2 + x1^2)
sage: f2 = 1/2*((x1^2 + 2*x1 + 1)*x2^2 + 2*(x1^2 + x1)*x2 - 4*x1^2)
sage: I = Ideal(f1,f2)
sage: I.triangular_decomposition()
Ideal (x2, x1^2) of Multivariate Polynomial Ring in x1, x2 over Rational Field,
Ideal (x2, x1^2) of Multivariate Polynomial Ring in x1, x2 over Rational Field,
Ideal (x2^4 + 4*x2^3 - 6*x2^2 - 20*x2 + 5, 8*x1 - x2^3 + x2^2 + 13*x2 - 5) of Multivariate Polynomial Ring in x1, x2 over Rational Field]

variety (ring=None)
Return the variety of this ideal.

Given a zero-dimensional ideal $I$ (= self) of a polynomial ring $P$ whose order is lexicographic, return the variety of $I$ as a list of dictionaries with (variable, value) pairs. By default, the variety of the ideal over its coefficient field $K$ is returned; ring can be specified to find the variety over a different ring.

These dictionaries have cardinality equal to the number of variables in $P$ and represent assignments of values to these variables such that all polynomials in $I$ vanish.

If ring is specified, then a triangular decomposition of self is found over the original coefficient field $K$; then the triangular systems are solved using root-finding over ring. This is particularly useful when $K$ is $\mathbb{Q}$ (to allow fast symbolic computation of the triangular decomposition) and ring is $\mathbb{R}$, $\mathbb{A}$, $\mathbb{C}$, or $\mathbb{Q}$bar (to compute the whole real or complex variety of the ideal).

Note that with ring=$\mathbb{R}$ or $\mathbb{C}$, computation is done numerically and potentially inaccurately; in particular, the number of points in the real variety may be miscomputed. With ring=$\mathbb{A}$ or $\mathbb{Q}$bar, computation is done exactly (which may be much slower, of course).

INPUT:

- ring - return roots in the ring instead of the base ring of this ideal (default: None)
- algorithm - algorithm or implementation to use; see below for supported values
- proof - return a provably correct result (default: True)

EXAMPLES:

```
sage: # needs sage.rings.finite_rings
sage: K.<w> = GF(27)  # this example is from the MAGMA handbook
sage: P.<x, y> = PolynomialRing(K, 2, order='lex')
sage: I = Ideal([x^8 + y + 2, y^6 + x*y^5 + x^2])
sage: I = Ideal(I.groebner_basis()); I
Ideal (x - y^47 - y^45 + y^44 - y^43 + y^41 - y^39 - y^38 - y^37 - y^36 + y^35 - y^34 - y^33 + y^32 - y^31 + y^30 + y^29 + y^28 + y^27 + y^26 + y^25 - y^23 + y^22 + y^21 - y^19 - y^18 - y^16 + y^15 + y^13 + y^12 - y^10 + y^9 + y^8 + y^7 - y^6 + y^4 + y^3 + y^2 + y - 1,
y^48 + y^41 - y^40 + y^37 - y^36 - y^33 + y^32 - y^29 + y^28 - y^25 + y^24 + y^2 + y + 1)
of Multivariate Polynomial Ring in x, y over Finite Field in w of size 3^3
```

```
sage: V = I.variety();
sage: sorted(V, key=str)
[{y: w^2 + 2*w, x: 2*w + 2}, {y: w^2 + 2, x: 2*w}, {y: w^2 + w, x: 2*w + 1}]
sage: [f.subs(v) # check that all polynomials vanish
...:   for f in I.gens() for v in V]
[0, 0, 0, 0, 0, 0]
sage: [I.subs(v).is_zero() for v in V]  # same test, but nicer syntax
[True, True, True]
```

However, we only account for solutions in the ground field and not in the algebraic closure:

```
sage: I.vector_space_dimension()  # needs sage.rings.finite_rings
48
```

Here we compute the points of intersection of a hyperbola and a circle, in several fields:

```
sage: K.<x, y> = PolynomialRing(QQ, 2, order='lex')
sage: I = Ideal([x*y - 1, (x-2)^2 + (y-1)^2 - 1])
sage: I = Ideal(I.groebner_basis()); I
```

(continues on next page)
These two curves have one rational intersection:

```
sage: I.variety()
[({y: 1, x: 1})]
```

There are two real intersections:

```
sage: sorted(I.variety(ring=RR), key=str)
[({y: 0.361103080528647, x: 2.769292354238636},
 {y: 1.000000000000000, x: 1.000000000000000})]
sage: I.variety(ring=AA)  
# needs sage.rings.number_field
[({y: 1, x: 1},
 {y: 0.3611030805286474?, x: 2.769292354238632?})]
```

and a total of four intersections:

```
sage: sorted(I.variety(ring=CC), key=str)
[({y: 0.3194484597356763? + 1.633170240915238?*I,
  x: 0.11535382288068... - 0.589742805022055?*I},
 {y: 0.3194484597356763? - 1.633170240915238?*I,
  x: 0.11535382288068... + 0.589742805022055?*I},
 {y: 0.3611030805286474?, x: 2.769292354238632?},
 {y: 1.000000000000000, x: 1.000000000000000})]
sage: sorted(I.variety(ring=QQbar), key=str)  
# needs sage.rings.number_field
[({y: 0.3194484597356763? + 1.633170240915238?*I,
  x: 0.11535382288068... - 0.589742805022055?*I},
 {y: 0.3194484597356763? - 1.633170240915238?*I,
  x: 0.11535382288068... + 0.589742805022055?*I},
 {y: 0.3611030805286474?, x: 2.769292354238632?},
 {y: 1, x: 1})]
```

We can also use the optional package msolve to compute the variety. See msolve for more information.

```
sage: I.variety(RBF, algorithm='msolve', proof=False)  
# optional - msolve
[({x: 2.769292354238636 +/- 2.08e-15}, {y: 0.361103080528647 +/- 4.53e-16}),
 {x: 1.000000000000000, y: 1.000000000000000})]
```

Computation over floating point numbers may compute only a partial solution, or even none at all. Notice that x values are missing from the following variety:

```
sage: R.<x,y> = CC[]  
sage: I = ideal([x^2+y^2-1,x*y-1])  
sage: I.variety(), key=str)
verbose 0 (...: multi_polynomial_ideal.py, variety) Warning: computations in
  the complex field are inexact; variety may be computed partially or
  incorrectly.
verbose 0 (...: multi_polynomial_ideal.py, variety) Warning: falling back to
  very slow toy implementation.
[({y: -0.86602540378443... + 0.500000000000000*I},
 {y: -0.86602540378443... - 0.500000000000000*I},
 {y: 0.86602540378443... + 0.500000000000000*I},
 {y: 0.86602540378443... - 0.500000000000000*I})
```

3.1. Multivariate Polynomials and Polynomial Rings
This is due to precision error, which causes the computation of an intermediate Groebner basis to fail. If the ground field's characteristic is too large for Singular, we resort to a toy implementation:

```sage
# needs sage.rings.finite_rings
R.<x,y> = PolynomialRing(GF(2147483659^3), order='lex')
I = ideal([x^3 - 2*y^2, 3*x + y^4])
I.variety()
```

The dictionary expressing the variety will be indexed by generators of the polynomial ring after changing to the target field. But the mapping will also accept generators of the original ring, or even generator names as strings, when provided as keys:

```sage
# needs sage.rings.number_field
K.<x,y> = QQ[]
I = ideal([x^2 + 2*y - 5, x + y + 3])
v = I.variety(AA)[0]; v[x], v[y]
```

msolve also works over finite fields:

```sage
R.<x, y> = PolynomialRing(GF(536870909), 2, order='lex')
I = Ideal([x^2 - 1, y^2 - 1])
sorted(I.variety(algorithm='msolve', proof=False), key=str)
```

but may fail in small characteristic, especially with ideals of high degree with respect to the characteristic:

```sage
R.<x, y> = PolynomialRing(GF(3), 2, order='lex')
I = Ideal([x^2 - 1, y^2 - 1])
I.variety(algorithm='msolve', proof=False) # optional - msolve
```

ALGORITHM:

- With *algorithm* = "triangular_decomposition" (default), uses triangular decomposition,
via Singular if possible, falling back on a toy implementation otherwise.

- With algorithm = "msolve", uses the optional package msolve. Note that msolve uses heuristics and therefore requires setting the proof flag to False. See msolve for more information.

\textbf{vector_space_dimension}()

Return the vector space dimension of the ring modulo this ideal. If the ideal is not zero-dimensional, a \texttt{TypeError} is raised.

\textbf{ALGORITHM:}

Uses Singular.

\textbf{EXAMPLES:}

```
sage: R.<u,v> = PolynomialRing(QQ)
sage: g = u^4 + v^4 + u^3 + v^3
sage: I = ideal(g) + ideal(g.gradient())
sage: I.dimension()
0
sage: I.vector_space_dimension()
4
```

When the ideal is not zero-dimensional, we return infinity:

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: I = R.ideal(x)
sage: I.dimension()
1
sage: I.vector_space_dimension()
+Infinity
```

Due to integer overflow, the result is correct only modulo $2^{32}$, see \texttt{github} issue \#8586:

```
sage: P.<x,y,z> = PolynomialRing(GF(32003), 3)
# needs sage.rings.finite_rings
sage: sage.rings.ideal.FieldIdeal(P).vector_space_dimension()  # known
32777216864027
```

\textbf{class} \texttt{sage.rings.polynomial.multi_polynomial_ideal.NCPolynomialIdeal} \texttt{(ring, gens, coerce=True, side='left')}

\textbf{Bases:} \texttt{MPolynomialIdeal_singular_repr, Ideal_nc}

Creates a non-commutative polynomial ideal.

\textbf{INPUT:}

- \texttt{ring} - the \texttt{g}-algebra to which this ideal belongs
- \texttt{gens} - the generators of this ideal
- \texttt{coerce} (optional - default True) - generators are coerced into the ring before creating the ideal
- \texttt{side} - optional string, either "left" (default) or "twosided": defines whether this ideal is left of two-sided.

\textbf{EXAMPLES:}
sage: # needs sage.combinat sage.modules
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H = A.g_algebra({y*x: x*y-z, z*x: x*z+2*x, z*y: y*z-2*y})
sage: H.inject_variables()
Defining x, y, z
sage: I = H.ideal([y^2, x^2, z^2 - H.one()], # indirect doctest
.....:          coerc=False)
sage: I  # random
Left Ideal (y^2, x^2, z^2 - 1) of
Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - z}
sage: sorted(I.gens(), key=str)
x^2, y^2, z^2 - 1
sage: H.ideal([y^2, x^2, z^2 - H.one()], side="twosided")  # random
Twosided Ideal (y^2, x^2, z^2 - 1) of
Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - z}
sage: sorted(H.ideal([y^2, x^2, z^2 - H.one()], side="twosided").gens(),
.....:          key=str)
x^2, y^2, z^2 - 1
sage: H.ideal([y^2, x^2, z^2 - H.one()], side="right")
Traceback (most recent call last):
...
ValueError: Only left and two-sided ideals are allowed.

elimination_ideal (variables)

Return the elimination ideal of this ideal with respect to the variables given in variables.

EXAMPLES:

sage: # needs sage.combinat sage.modules
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H = A.g_algebra({y*x: x*y-z, z*x: x*z+2*x, z*y: y*z-2*y})
sage: H.inject_variables()
Defining x, y, z
sage: I = H.ideal([y^2, x^2, z^2 - H.one()], coerc=False)
sage: I.elimination_ideal([x, z])
Left Ideal (y^2) of
Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
nc-relations: {...}
sage: J = I.twostd(); J
Twosided Ideal (z^2 - 1, y*z - y, x*z + x, y^2, 2*x*y - z - 1, x^2) of
Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
nc-relations: {...}
sage: J.elimination_ideal([x, z])
Twosided Ideal (y^2) of
Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
nc-relations: {...}

ALGORITHM: Uses Singular's eliminate command

reduce (p)

Reduce an element modulo a Groebner basis for this ideal.

It returns 0 if and only if the element is in this ideal. In any case, this reduction is unique up to monomial orders.

EXAMPLES:
Here, we see that the relation that we just found in the quotient is actually a consequence of the given relations:

```
sage: H.2^2 - H.one() in I.std().gens()  # indirect doctest
```

```
True
```

Here is the corresponding direct test:

```
sage: I.reduce(z^2)  # indirect doctest
```

```
1
```

**res** *(length)*

Compute the resolution up to a given length of the ideal.

**NOTE:**

Only left syzygies can be computed. So, even if the ideal is two-sided, then the resolution is only one-sided. In that case, a warning is printed.

**EXAMPLES:**

```
sage: # needs sage.combinat sage.modules
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H = A.g_algebra({y*x: x*y-z, z*x: x*z+2*x, z*y: y*z-2*y})
sage: H.inject_variables()
Defining x, y, z
sage: I = H.ideal([y^2, x^2, z^2 - H.one()])
sage: I.res(3)  # indirect doctest
<Resolution>
```

**std()**

Compute a GB of the ideal. It is two-sided if and only if the ideal is two-sided.

**EXAMPLES:**

```
sage: # needs sage.combinat sage.modules
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H = A.g_algebra({y*x: x*y-z, z*x: x*z+2*x, z*y: y*z-2*y})
sage: H.inject_variables()
Defining x, y, z
sage: I = H.ideal([y^2, x^2, z^2 - H.one()])
sage: I.std()  # random
Left Ideal (z^2 - 1, y*z - y, x*z + x, y^2, 2*x*y - z - 1, x^2) of Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
```

(continues on next page)
nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - z}
sage: sorted(I.std().gens(), key=str)
[2*x*y - z - 1, x*z + x, x^2, y*z - y, y^2, z^2 - 1]

If the ideal is a left ideal, then std() returns a left Groebner basis. But if it is a two-sided ideal, then the output of std() and twostd() coincide:

sage: # needs sage.combinat sage.modules
sage: JL = H.ideal([x^3, y^3, z^3 - 4*z])
sage: JL
Left Ideal (x^3, y^3, z^3 - 4*z) of
Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - z}
sage: sorted(JL.gens(), key=str)
[x^3, y^3, z^3 - 4*z]
sage: JL.std() # random
Left Ideal (z^3 - 4*z, y*z^2 - 2*y*z, x*z^2 + 2*x*z, 2*x*y*z - z^2 - 2*z, y^3, x^3) of
Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - z}
sage: sorted(JL.std().gens(), key=str)
[2*x*y*z - z^2 - 2*z, x*z^2 + 2*x*z, 2*x^2, y^3, x*y^2 - y*z, x^2*y - x*z - 2*x, x^3, y^2*z - 2*y*z, y^3, z^3 - 4*z]
sage: JT = H.ideal([x^3, y^3, z^3 - 4*z], side='twosided')
sage: JT
Twosided Ideal (x^3, y^3, z^3 - 4*z) of
Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field,
nc-relations: {z*x: x*z + 2*x, z*y: y*z - 2*y, y*x: x*y - z}
sage: sorted(JT.gens(), key=str)
[x^3, y^3, z^3 - 4*z]
sage: JT.std() # random
Twosided Ideal (z^3 - 4*z, y*z^2 - 2*y*z, x*z^2 + 2*x*z, y^2*z - 2*y^2, x^2*y - x*z - 2*x, y^3, x*y^2 - y*z, x^2*y - x*z - 2*x, x^3, y^2*z - 2*y*z, y^3, z^3 - 4*z)
sage: sorted(JT.std().gens(), key=str)
[2*x*y*z - z^2 - 2*z, x*y^2 - y*z, x*z^2 + 2*x*z, x^2*y - x*z - 2*x, x^2*z + 2*x^2, x^3, y*z^2 - 2*y*z, y^2*z - 2*y^2, y^3, z^3 - 4*z]
sage: JT.std() == JL.twostd()
True

ALGORITHM: Uses Singular's std command

syzygy_module()
Compute the first syzygy (i.e., the module of relations of the given generators) of the ideal.

NOTE:
Only left syzygies can be computed. So, even if the ideal is two-sided, then the syzygies are only one-sided. In that case, a warning is printed.

EXAMPLES:

sage: # needs sage.combinat sage.modules
sage: A, x, y, z = FreeAlgebra(QQ, 3)
sage: H = A.g_algebra({y*x: x*y-z, z*x: x*z+2*x, z*y: y*z-2*y})
sage: H.inject_variables()
Defining x, y, z
sage: I = H.ideal([y^2, x^2, z^2-H.one()], coerce=False)
sage: G = vector(I.gens()); G
d...: UserWarning: You are constructing a free module over a noncommutative ring. Sage does not have a concept of left/right and both sided modules, so be careful. It's also not guaranteed that all multiplications are done from the right side.
d...: UserWarning: You are constructing a free module over a noncommutative ring. Sage does not have a concept of left/right and both sided modules, so be careful. It's also not guaranteed that all multiplications are done from the right side.
(y^2, x^2, z^2 - 1)
sage: M = I.syzygy_module(); M
[ -z^2 + 8*z - 15  x^2]
[ 0 x^2*z^2 + 8*x^2]
[-2*z + 15*x  y^2]
[ 0 y^2*z^2 + 8*y^2]
[ 282*x*z - 360*x -4*x*y*z + 2*z^2 + 2*z]
[282*x*z - 360*x -4*x*y*z + 2*z^2 + 2*z]
[-y^3*z^2 + 7*y^3*z - 12*y^3 6*y*z^2]
[ -3*z + 12*x^3 -x*y^2*z^2 + 9*x*y^2*z - 6*x*y^3 + 20*x^2*y - 72*x*z^2 -]
[ 52*y*z^2 - 224*y*z + 320*y -6*x*z^2]
[ x^2*y^2*z + 4*x^2*y^2 - 8*x*y*z + 12*x^2 - 64*x*y + 108*z^2 +]
[-312*x + 288 -y^4*z + 4*y^4]
[ 0 2*x^3*y^3 + 8*x^3*y + 9*x^3]
[ 2*x^2 + 27*x^2 +2*x*y^3*z + 8*x*y^3 - 12*y^3]
[ 2*z^2 + 99*y^2*z + 195*y^2 0]
[ 0 -36*x*y*z + 24*z^2 + 18*z]
[4*z + 4*x^4 -x^2*y^2*z + 4*x^2*y^2 - 4*x*y*z + 2 + 32*x*y^2 - 6*z^3 -]
[ 64*x*y + 66*z^2 - 240*z + 288 0]
[ 0 1656*x*z - 2052*x -x*y^4*z + 4*x*y^4]
[8*y^3*z^2 + 62*y^3*z - 114*y^3 48*y*z^2 - 36*y*z]
[ 0 0, 0, 0, 0, 0, 0, 0]

ALGORITHM: Uses Singular's syz command
twostd()
Compute a two-sided GB of the ideal (even if it is a left ideal).

**EXAMPLES:**

```plaintext
sage: # needs sage.combinat sage.modules
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
```

```plaintext
sage: H = A.g_algebra({y*x: x*y-z, z*x: x*z+2*x, z*y: y*z-2*y})
```

```plaintext
sage: H.inject_variables()
```

```plaintext
Defining x, y, z
```

```plaintext
sage: I = H.ideal([y^2, x^2, z^2 - H.one()], coerce=False)
```

```plaintext
sage: I.twostd()
```

```plaintext
#random
```

Twosided Ideal (z^2 - 1, y*z - y, x*z + x, y^2, 2*x*y - z - 1, x^2) of Noncommutative Multivariate Polynomial Ring in x, y, z over Rational Field...
```

```plaintext
sage: sorted(I.twostd().gens(), key=str)
```

```plaintext
[2*x*y - z - 1, x*z + x, x^2, y*z - y, y^2, z^2 - 1]
```

**ALGORITHM:** Uses Singular’s `twostd` command

```plaintext
class sage.rings.polynomial.multi_polynomial_ideal.RequireField(f)
```

**Bases:** MethodDecorator

Decorator which throws an exception if a computation over a coefficient ring which is not a field is attempted.

**Note:** This decorator is used automatically internally so the user does not need to use it manually.

```plaintext
sage.rings.polynomial.multi_polynomial_ideal.is_MPolynomialIdeal(x)
```

Return `True` if the provided argument `x` is an ideal in a multivariate polynomial ring.

**INPUT:**

- `x`: an arbitrary object

**EXAMPLES:**

```plaintext
sage: from sage.rings.polynomial.multi_polynomial_ideal import is_MPolynomialIdeal
sage: P.<x,y,z> = PolynomialRing(QQ)
```

```plaintext
sage: I = [x^2 + 2*y + 2*z - 1, x^2 + 2*y^2 + 2*z^2 - x, 2*x*y + 2*y*z - y]
```

Sage distinguishes between a list of generators for an ideal and the ideal itself. This distinction is inconsistent with Singular but matches Magma’s behavior.

```plaintext
sage: is_MPolynomialIdeal(I)
```

```plaintext
False
```

```plaintext
sage: I = Ideal(I)
```

```plaintext
sage: is_MPolynomialIdeal(I)
```

```plaintext
True
```

```plaintext
sage.rings.polynomial.multi_polynomial_ideal.require_field
```

alias of `RequireField`
3.1.7 Polynomial Sequences

We call a finite list of polynomials a Polynomial Sequence.

Polynomial sequences in Sage can optionally be viewed as consisting of various parts or sub-sequences. These kind of polynomial sequences which naturally split into parts arise naturally for example in algebraic cryptanalysis of symmetric cryptographic primitives. The most prominent examples of these systems are: the small scale variants of the AES [CMR2005] (cf. sage.crypto.mq.sr.SR()) and Flurry/Curry [BPW2006]. By default, a polynomial sequence has exactly one part.

AUTHORS:

- Martin Albrecht (2007ff): initial version
- Martin Albrecht (2009): refactoring, clean-up, new functions
- Martin Albrecht (2011): refactoring, moved to sage.rings.polynomial
- Alex Raichev (2011-06): added algebraic_dependence()
- Charles Bouillaguet (2013-1): added solve()

EXAMPLES:

As an example consider a small scale variant of the AES:

```python
sage: sr = mq.SR(2, 1, 2, 4, gf2=True, polybori=True)  # needs sage.rings.polynomial.pbori
sage: sr
SR(2,1,2,4)
```

We can construct a polynomial sequence for a random plaintext-ciphertext pair and study it:

```python
sage: set_random_seed(1)
sage: while True:  # workaround (see :issue:`A31891A`)
....: try:
....:     F, s = sr.polynomial_system()
....:     break
....: except ZeroDivisionError:
....:     pass
sage: F  # needs sage.rings.polynomial.pbori
Polynomial Sequence with 112 Polynomials in 64 Variables
```

(continues on next page)
We separate the system in independent subsystems:

```
sage: C = Sequence(r2).connected_components(); C
[ ]  # needs sage.rings.polynomial.pbori
[w200 + k100 + x100 + x102 + x103, 
  w201 + k101 + x100 + x101 + x103 + 1, 
  w202 + k102 + x100 + x101 + x102 + 1, 
  w203 + k103 + x101 + x102 + x103, 
  x100*w100 + x100*w103 + x101*w101 + x101*w102 + x102*w100 + x102*w103 + x103*w101 +... 
  x100, 
  x100*w101 + x100*w103 + x101*w101 + x101*w102 + x102*w100 + x102*w103 + x103*w101 +...
```

(continues on next page)
and compute the coefficient matrix:

```python
sage: A, v = Sequence(r2).coefficients_monomials()
# needs sage.rings.polynomial.pbori
sage: A.rank()
# needs sage.rings.polynomial.pbori
32
```

Using these building blocks we can implement a simple XL algorithm easily:

```python
sage: sr = mq.SR(1,1,1,4, gf2=True, polybori=True, order='lex')
# needs sage.rings.polynomial.pbori
sage: while True:
    try:
        F, s = sr.polynomial_system()
        break
    except ZeroDivisionError:
        pass

sage: monomials = [a*b for a in F.variables() for b in F.variables() if a < b]

sage: len(monomials)
190

sage: F2 = Sequence(map(mul, cartesian_product_iterator((monomials, F))))

sage: A, v = F2.coefficients_monomials(sparse=False)

sage: A.echelonize()

sage: A
6840 x 4474 dense matrix over Finite Field of size 2...
```

(continues on next page)
4056

sage: A[4055] * v
k001*k003

Note: In many other computer algebra systems (cf. Singular) this class would be called Ideal but an ideal is a very distinct object from its generators and thus this is not an ideal in Sage.

Classes

sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence(arg1, 
arg2=None, immutable=False, 

Construct a new polynomial sequence object.

INPUT:

- arg1 - a multivariate polynomial ring, an ideal or a matrix
- arg2 - an iterable object of parts or polynomials (default: None)
  - immutable - if True the sequence is immutable (default: False)
  - cr - print a line break after each element (default: False)
  - cr_str - print a line break after each element if `str` is called (default: None)

EXAMPLES:

```python
sage: P.<a,b,c,d> = PolynomialRing(GF(127), 4)
sage: I = sage.rings.ideal.Katsura(P)  # indirect doctest
[2] → needs sage.libs.singular
sage: Sequence(I.gens(), I.gens()); F
[a + 2*b + 2*c + 2*d - 1, 
a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a, 
2*a*b + 2*b*c + 2*c*d - b, 
b^2 + 2*a*c + 2*b*d - c, 
a + 2*b + 2*c + 2*d - 1, 
a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a, 
2*a*b + 2*b*c + 2*c*d - b, 
b^2 + 2*a*c + 2*b*d - c]
sage: F.nparts()
2
```

If a list of tuples is provided, those form the parts:

```python
sage: F = Sequence([(I.gens(), I.gens()), I.ring()]); F  # indirect doctest
[2] → needs sage.libs.singular
[a + 2*b + 2*c + 2*d - 1, 
a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a, 
2*a*b + 2*b*c + 2*c*d - b, 
b^2 + 2*a*c + 2*b*d - c, 
a + 2*b + 2*c + 2*d - 1, 
a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a, 
2*a*b + 2*b*c + 2*c*d - b, 
b^2 + 2*a*c + 2*b*d - c]
sage: F.nparts()
2
```

If an ideal is provided, the generators are used:

```python
sage: Sequence(I)  # indirect doctest
[a + 2*b + 2*c + 2*d - 1, 
```

(continues on next page)
If a list of polynomials is provided, the system has only one part:

```plaintext
sage: F = Sequence(I.gens(), I.ring()); F
[a + 2*b + 2*c + 2*d - 1,
 a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
 2*a*b + 2*b*c + 2*c*d - b,
 b^2 + 2*a*c + 2*b*d - c]
```

We test that the ring is inferred correctly:

```plaintext
class sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic(parts, ring, immutable=False, cr=False, cr_str=None)

Bases: Sequence_generic

Construct a new system of multivariate polynomials.

INPUT:

- `part` - a list of lists with polynomials
- `ring` - a multivariate polynomial ring
- `immutable` - if True the sequence is immutable (default: False)
- `cr` - print a line break after each element (default: False)
- `cr_str` - print a line break after each element if ‘str’ is called (default: None)

EXAMPLES:

```plaintext
sage: P.<a,b,c,d> = PolynomialRing(GF(127), 4)
sage: I = sage.rings.ideal.Katsura(P)  # indirect doctest
```

3.1. Multivariate Polynomials and Polynomial Rings 413
If an ideal is provided, the generators are used:

```
sage: Sequence(I)  # needs sage.libs.singular
[a + 2*b + 2*c + 2*d - 1, a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
  2*a*b + 2*b*c + 2*c*d - b, b^2 + 2*a*c + 2*b*d - c]
```

If a list of polynomials is provided, the system has only one part:

```
sage: Sequence(I.gens(), I.ring())  # needs sage.libs.singular
[a + 2*b + 2*c + 2*d - 1, a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
  2*a*b + 2*b*c + 2*c*d - b, b^2 + 2*a*c + 2*b*d - c]
```

```
algebraic_dependence()
```

Returns the ideal of annihilating polynomials for the polynomials in `self`, if those polynomials are algebraically dependent. Otherwise, returns the zero ideal.

**OUTPUT:**

If the polynomials \( f_1, \ldots, f_r \) in `self` are algebraically dependent, then the output is the ideal \( \{ F \in K[T_1, \ldots, T_r] : F(f_1, \ldots, f_r) = 0 \} \) of annihilating polynomials of \( f_1, \ldots, f_r \). Here \( K \) is the coefficient ring of polynomial ring of \( f_1, \ldots, f_r \) and \( T_1, \ldots, T_r \) are new indeterminates. If \( f_1, \ldots, f_r \) are algebraically independent, then the output is the zero ideal in \( K[T_1, \ldots, T_r] \).

**EXAMPLES:**

```
sage: # needs sage.libs.singular
sage: R.<x,y> = PolynomialRing(QQ)
sage: S = Sequence([x, x*y])
sage: I = S.algebraic_dependence(); I
Ideal (0) of Multivariate Polynomial Ring in T0, T1 over Rational Field
```

```
sage: # needs sage.libs.singular
sage: R.<x,y> = PolynomialRing(QQ)
sage: S = Sequence([x, (x^2 + y^2 - 1)^2, x*y - 2])
sage: I = S.algebraic_dependence(); I
Ideal (16 + 32*T2 - 8*T0^2 + 24*T2^2 - 8*T0^2*T2 + 8*T2^3 + 9*T0^4 - 2*T0^2*T2^2 + T2^4 - T0^4*T1 + 8*T0^4*T2 - 2*T0^6 + 2*T0^4*T2^2 + T0^8)
  of Multivariate Polynomial Ring in T0, T1, T2 over Rational Field
sage: [F(S) for F in I.gens()]
[0]
```

```
sage: # needs sage.libs.singular
sage: R.<x,y> = PolynomialRing(GF(7))
sage: S = Sequence([x, (x^2 + y^2 - 1)^2, x*y - 2])
sage: I = S.algebraic_dependence(); I
Ideal (2 - 3*T2 - T0^2 + 3*T2^2 - T0^2*T2 + T2^3 + 2*T0^4 - 2*T0^2*T2^2 + T2^4 - T0^4*T1 + 12*T0^4*T2 - 2*T0^6 + 2*T0^4*T2^2 + T0^8)
  of Multivariate Polynomial Ring in T0, T1, T2 over Finite Field of size 7
sage: [F(S) for F in I.gens()]
[0]
```

**Note:** This function’s code also works for sequences of polynomials from a univariate polynomial ring, but i don’t know where in the Sage codebase to put it to use it to that effect.
AUTHORS:

- Alex Raichev (2011-06-22)

**coefficient_matrix**(sparse=True)

Return tuple \((A, v)\) where \(A\) is the coefficient matrix of this system and \(v\) the matching monomial vector.

Thus value of \(A[i,j]\) corresponds the coefficient of the monomial \(v[j]\) in the \(i\)-th polynomial in this system.

Monomials are order w.r.t. the term ordering of `self.ring()` in reverse order, i.e. such that the smallest entry comes last.

**INPUT:**

- **sparse** - construct a sparse matrix (default: True)

**EXAMPLES:**

```
sage: # needs sage.libs.singular
sage: P.<a,b,c,d> = PolynomialRing(GF(127), 4)
sage: I = sage.rings.ideal.Katsura(P)
sage: I.gens()
[ a + 2*b + 2*c + 2*d - 1,
  a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a,
  2*a*b + 2*b*c + 2*c*d - b,
  b^2 + 2*a*c + 2*b*d - c]
sage: F = Sequence(I)
sage: A, v = F.coefficient_matrix()
doctest:warning...
DeprecationWarning: the function coefficient_matrix is deprecated; use...
˓→coefficients_monomials instead
See https://github.com/sagemath/sage/issues/37035 for details.
sage: A
[ 0 0 0 0 0 0 0 0 0 1 2 2 2 126]
[ 1 0 2 0 0 2 0 0 2 126 0 0 0 0]
[ 0 2 0 0 2 0 0 2 0 0 126 0 0 0]
[ 0 0 1 2 0 0 2 0 0 0 0 126 0 0]
sage: v
[a^2]
[a*b]
[b^2]
[a*c]
[b*c]
[c^2]
[b*d]
[c*d]
[d^2]
[a]
[b]
[c]
[d]
[1]
sage: A*v
[a + 2*b + 2*c + 2*d - 1]
[a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a]
[2*a*b + 2*b*c + 2*c*d - b]
[ b^2 + 2*a*c + 2*b*d - c]
```

**coefficients_monomials**(order=None, sparse=True)
Return the matrix of coefficients \( A \) and the matching vector of monomials \( v \), such that \( A \cdot v = \text{vector}(\text{self}) \).

Thus value of \( A[i, j] \) corresponds the coefficient of the monomial \( v[j] \) in the \( i \)-th polynomial in this system.

Monomials are ordered w.r.t. the term ordering of \text{order} if given; otherwise, they are ordered w.r.t. \text{self}. \text{ring()} in reverse order, i.e., such that the smallest entry comes last.

INPUT:

- \text{sparse} - construct a sparse matrix (default: True)
- \text{order} - a list or tuple specifying the order of monomials (default: None)

EXAMPLES:

```python
sage: # needs sage.libs.singular
sage: P.<a,b,c,d> = PolynomialRing(GF(127), 4)
sage: I = sage.rings.ideal.Katsura(P)
sage: I.gens()
[a + 2*b + 2*c + 2*d - 1, a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a, 2*a*b + 2*b*c + 2*c*d - b, b^2 + 2*a*c + 2*b*d - c]
sage: F = Sequence(I)
sage: A, v = F.coefficients_monomials()
sage: A
[[0 0 0 0 0 0 0 0 0 0 0 0 0 1 2 2 2 2 126],
 [1 0 2 0 0 2 0 0 2 126 0 0 0 0 0 0 0 0],
 [0 2 0 0 2 0 0 2 0 0 126 0 0 0 0 0 0 0],
 [0 0 1 2 0 0 2 0 0 0 0 0 0 0 0 126 0 0 0]]
sage: v
(a^2, a*b, b^2, a*c, b*c, c^2, b*d, c*d, d^2, a, b, c, d, 1)
sage: A*v
(a + 2*b + 2*c + 2*d - 1, a^2 + 2*b^2 + 2*c^2 + 2*d^2 - a, 2*a*b + 2*b*c + 2*c*d - b, b^2 + 2*a*c + 2*b*d - c)
```

\text{connected\_components()} ()

Split the polynomial system in systems which do not share any variables.

EXAMPLES:

As an example consider one part of AES, which naturally splits into four subsystems which are independent:

```python
sage: # needs sage.rings.polynomial.pbori
sage: sr = mq.SR(2, 4, 4, 8, gf2=True, polybori=True)
sage: while True:
    ....:     try:
    ....:         F, s = sr.polynomial_system()
    ....:     except ZeroDivisionError:
    ....:         break
    ....: else:
    ....:     Fz = Sequence(F.part(2))
    ....:     Fz.connected_components()
[Polynomial Sequence with 128 Polynomials in 128 Variables,
 Polynomial Sequence with 128 Polynomials in 128 Variables,
 Polynomial Sequence with 128 Polynomials in 128 Variables,
 Polynomial Sequence with 128 Polynomials in 128 Variables]
```
connection_graph()
Return the graph which has the variables of this system as vertices and edges between two variables if they appear in the same polynomial.

EXAMPLES:

```python
sage: B.<x,y,z> = BooleanPolynomialRing()
sage: F = Sequence([x*y + y + 1, z + 1])
sage: G = F.connection_graph(); G
Graph on 3 vertices
sage: G.is_connected()
False
sage: F = Sequence([x])
sage: F.connection_graph()
Graph on 1 vertex
```

groebner_basis(*args,**kwargs)
Compute and return a Groebner basis for the ideal spanned by the polynomials in this system.

INPUT:

- args - list of arguments passed to MPolynomialIdeal.groebner_basis call
- kwargs - dictionary of arguments passed to MPolynomialIdeal.groebner_basis call

EXAMPLES:

```python
sage: sr = mq.SR(allow_zero_inversions=True)
sage: F, s = sr.polynomial_system()
sage: gb = F.groebner_basis()
Ideal(gb).basis_is_groebner()
True
```

ideal()
Return ideal spanned by the elements of this system.

EXAMPLES:

```python
sage: sr = mq.SR(allow_zero_inversions=True)
sage: F, s = sr.polynomial_system()
sage: P = F.ring()
sage: I = F.ideal()
sage: J = I.elimination_ideal(P.gens()[4:-4])
J <= I
True
```

is_groebner(singular=Singular)
Returns True if the generators of this ideal (self.gens()) form a Groebner basis.

Let I be the set of generators of this ideal. The check is performed by trying to lift Syz(LM(I)) to Syz(I) as I forms a Groebner basis if and only if for every element S in Syz(LM(I)):

\[ S \cdot G = \sum_{i=0}^{m} h_i g_i \cdot G \geq 0. \]

EXAMPLES:
maximal_degree()  
Return the maximal degree of any polynomial in this sequence.

EXAMPLES:

```python
sage: P.<x,y,z> = PolynomialRing(GF(7))
sage: F = Sequence([x*y + x, x])
sage: F.maximal_degree()
2
sage: P.<x,y,z> = PolynomialRing(GF(7))
sage: F = Sequence([], universe=P)
sage: F.maximal_degree()
-1
```

monomials()  
Return an unordered tuple of monomials in this polynomial system.

EXAMPLES:

```python
sage: sr = mq.SR(allow_zero_inversions=True)  # needs sage.rings.polynomial.pbori
sage: F, s = sr.polynomial_system()  # needs sage.rings.polynomial.pbori
sage: len(F.monomials())  # needs sage.rings.polynomial.pbori
49
```

nmonomials()  
Return the number of monomials present in this system.

EXAMPLES:

```python
sage: sr = mq.SR(allow_zero_inversions=True)  # needs sage.rings.polynomial.pbori
sage: F, s = sr.polynomial_system()  # needs sage.rings.polynomial.pbori
sage: F.nmonomials()  # needs sage.rings.polynomial.pbori
49
```

nparts()  
Return number of parts of this system.

EXAMPLES:

```python
sage: sr = mq.SR(allow_zero_inversions=True)  # needs sage.rings.polynomial.pbori
sage: F, s = sr.polynomial_system()  # needs sage.rings.polynomial.pbori
```
nvariables()  
Return number of variables present in this system.  

EXAMPLES:  

```python  
sage: sr = mq.SR(allow_zero_inversions=True)  
sage: F, s = sr.polynomial_system()  
sage: F.nvariables()  
20  
```

part(i)  
Return i-th part of this system.  

EXAMPLES:  

```python  
sage: sr = mq.SR(allow_zero_inversions=True)  
sage: F, s = sr.polynomial_system()  
sage: R0 = F.part(1)  
sage: R0  
k000^2 + k001, k001^2 + k002, k002^2 + k003, k003^2 + k000  
```

parts()  
Return a tuple of parts of this system.  

EXAMPLES:  

```python  
sage: sr = mq.SR(allow_zero_inversions=True)  
sage: F, s = sr.polynomial_system()  
sage: l = F.parts()  
sage: len(l)  
4  
```

reduced()  
If this sequence is \((f_1, ..., f_n)\) then this method returns \((g_1, ..., g_s)\) such that:  

- \((f_1, ..., f_n) = (g_1, ..., g_s)\)  
- \(LT(g_i) \neq LT(g_j)\) for all \(i \neq j\)  
- \(LT(g_i)\) does not divide \(m\) for all monomials \(m\) of \(\{g_1, ..., g_{i-1}, g_{i+1}, ..., g_s\}\)  
- \(LC(g_i) = 1\) for all \(i\) if the coefficient ring is a field.  

EXAMPLES:  

```python  
sage: R.<x,y,z> = PolynomialRing(QQ)  
sage: F = Sequence([z*x+y^3, z+y^3, z+x*y])  
sage: F.reduced()  
[y^3 + z, x*y + z, x*z - z]  
```
Note that tail reduction for local orderings is not well-defined:

```
sage: R.<x,y,z> = PolynomialRing(QQ, order='negdegrevlex')
sage: F = Sequence([z*x+y^3,z+y^3,z+x*y])
sage: F.reduced()
[z + x*y, x*y - y^3, x^2*y - y^3]
```

A fixed error with nonstandard base fields:

```
sage: R.<t>=QQ['t']
sage: K.<x,y>=R.fraction_field()['x,y']
sage: I=t*x*K
sage: I.basis.reduced()  
[x]
```

The interreduced basis of 0 is 0:

```
sage: P.<x,y,z> = GF(2)[]
sage: Sequence([P(0)]).reduced()
[0]
```

Leading coefficients are reduced to 1:

```
sage: P.<x,y> = QQ[]
sage: Sequence([2*x,y]).reduced()
[x, y]
sage: P.<x,y> = CC[]
#→ needs sage.rings.real_mpfr
sage: Sequence([2*x,y]).reduced()
[x, y]
```

**ALGORITHM:**

Uses Singular’s interred command or `sage.rings.polynomial.toy_buchberger.inter_reduction()` if conversion to Singular fails.

**ring()**

Return the polynomial ring all elements live in.

**EXAMPLES:**

```
sage: sr = mq.SR(allow_zero_inversions=True, gf2=True, order='block')  #→ needs sage.rings.polynomial.pbori
sage: F, s = sr.polynomial_system()
#→ needs sage.rings.polynomial.pbori
sage: print(F.ring().repr_long())
#→ needs sage.rings.polynomial.pbori
Polynomial Ring
  Base Ring : Finite Field of size 2
  Size : 20 Variables
  Block 0 : Ordering : deglex
     Names : k100, k101, k102, k103, x100, x101, x102, x103, w100,
    w101, w102, w103, s000, s001, s002, s003
  Block 1 : Ordering : deglex
     Names : k000, k001, k002, k003
```
Substitute variables for every polynomial in this system and return a new system. See `MPolynomial.subs()` for calling convention.

**INPUT:**

- `args` - arguments to be passed to `MPolynomial.subs()`
- `kwargs` - keyword arguments to be passed to `MPolynomial.subs()`

**EXAMPLES:**

```python
sage: sr = mq.SR(allow_zero_inversions=True)  # needs sage.rings.polynomial.pbori
sage: F, s = sr.polynomial_system(); F
Polynomial Sequence with 40 Polynomials in 20 Variables

sage: F = F.subs(s); F
Polynomial Sequence with 40 Polynomials in 16 Variables
```

**universe()**

Return the polynomial ring all elements live in.

**EXAMPLES:**

```python
sage: sr = mq.SR(allow_zero_inversions=True, gf2=True, order='block')  # needs sage.rings.polynomial.pbori
sage: F, s = sr.polynomial_system()

sage: print(F.ring().repr_long())  # needs sage.rings.polynomial.pbori
Polynomial Ring
Base Ring : Finite Field of size 2
Size : 20 Variables
Block 0 : Ordering : deglex
   Names : k003, k002, k001, k000, s003, s002, s001, s000, w103, w102

sage: F.ring().repr_long()  # needs sage.rings.polynomial.pbori
Polynomial Ring
Base Ring : Finite Field of size 2
Size : 20 Variables
Block 0 : Ordering : deglex
   Names : k003, k002, k001, k000, s003, s002, s001, s000, w103, w102

sage: print(F.ring().repr_long())  # needs sage.rings.polynomial.pbori
Polynomial Ring
Base Ring : Finite Field of size 2
Size : 20 Variables
Block 0 : Ordering : deglex
   Names : k003, k002, k001, k000, s003, s002, s001, s000, w103, w102

sage: print(F.ring().repr_long())  # needs sage.rings.polynomial.pbori
Polynomial Ring
Base Ring : Finite Field of size 2
Size : 20 Variables
Block 0 : Ordering : deglex
   Names : k003, k002, k001, k000, s003, s002, s001, s000, w103, w102

sage: print(F.ring().repr_long())  # needs sage.rings.polynomial.pbori
Polynomial Ring
Base Ring : Finite Field of size 2
Size : 20 Variables
Block 0 : Ordering : deglex
   Names : k003, k002, k001, k000, s003, s002, s001, s000, w103, w102
```

**variables()**

Return all variables present in this system. This tuple may or may not be equal to the generators of the ring of this system.

**EXAMPLES:**

```python
sage: sr = mq.SR(allow_zero_inversions=True)  # needs sage.rings.polynomial.pbori
sage: F, s = sr.polynomial_system()

sage: F.variables()[:10]  # needs sage.rings.polynomial.pbori
(k003, k002, k001, k000, s003, s002, s001, s000, w103, w102)
```
class sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_gf2 (parts, 
    ring, 
    immutable=False, 
    cr=False, 
    cr_str=None)

Bases: PolynomialSequence_generic

Polynomial Sequences over \( \mathbb{F}_2 \).

coefficients_monomials (order=None, sparse=True)

Return the matrix of coefficients \( A \) and the matching vector of monomials \( v \), such that \( A \cdot v = \text{vector}(self) \).

Thus value of \( A[i,j] \) corresponds the coefficient of the monomial \( v[j] \) in the \( i \)-th polynomial in this system.

Monomials are ordered w.r.t. the term ordering of \( order \) if given; otherwise, they are ordered w.r.t. \( self.ring() \) in reverse order, i.e., such that the smallest entry comes last.

INPUT:

- \( sparse \) - construct a sparse matrix (default: True)
- \( order \) - a list or tuple specifying the order of monomials (default: None)

EXAMPLES:

```python
sage: # needs sage.rings.polynomial.pbori
sage: B.<x,y,z> = BooleanPolynomialRing()
sage: F = Sequence([x*y + y + 1, z + 1])

sage: A, v = F.coefficients_monomials()
sage: A
[1 1 0 1]
[0 0 1 1]
sage: v
(x*y, y, z, 1)
sage: A*v
(x*y + y + 1, z + 1)
```

eliminate_linear_variables (maxlength=+Infinity, skip=None, return_reductors=False, 
                        use_polybori=False)

Return a new system where linear leading variables are eliminated if the tail of the polynomial has length at most \( \text{maxlength} \).

INPUT:

- \( \text{maxlength} \) - an optional upper bound on the number of monomials by which a variable is replaced. If \( \text{maxlength}==+\text{Infinity} \) then no condition is checked. (default: +\text{Infinity}).
- \( \text{skip} \) - an optional callable to skip eliminations. It must accept two parameters and return either True or False. The two parameters are the leading term and the tail of a polynomial (default: None).
- \( \text{return_reductors} \) - if True the list of polynomials with linear leading terms which were used for reduction is also returned (default: False).
- \( \text{use_polybori} \) - if True then polybori.ll.eliminate is called. While this is typically faster than what is implemented here, it is less flexible (\( \text{skip} \) is not supported) and may increase the degree (default: False)
OUTPUT:

With \texttt{return\_reductors=True}, a pair of sequences of boolean polynomials are returned, along with the promises that:

1. The union of the two sequences spans the same boolean ideal as the argument of the method.

2. The second sequence only contains linear polynomials, and it forms a reduced Groebner basis (they all have pairwise distinct leading variables, and the leading variable of a polynomial does not occur anywhere in other polynomials).

3. The leading variables of the second sequence do not occur anywhere in the first sequence (these variables have been eliminated).

With \texttt{return\_reductors=False}, only the first sequence is returned.

EXAMPLES:

```python
sage: # needs sage.rings.polynomial.pbori
sage: B.<a,b,c,d> = BooleanPolynomialRing()

sage: F = Sequence([c + d + b + 1, a + c + d, a*b + c, b*c*d + c])

sage: F.eliminate_linear_variables()  # everything vanishes
[]

sage: F.eliminate_linear_variables(maxlength=2)
[b + c + d + 1, b*c + b*d + c, b*c*d + c]

sage: F.eliminate_linear_variables(skip=lambda lm, tail: str(lm) == 'a')
[a + c + d, a*c + a*d + a + c, c*d + c]
```

The list of reductors can be requested by setting \texttt{return\_reductors} to \texttt{True}:

```python
sage: # needs sage.rings.polynomial.pbori
sage: B.<a,b,c,d> = BooleanPolynomialRing()

sage: F, R = F.eliminate_linear_variables(return_reductors=True)

sage: F
[]

sage: R
[a + b + d, c + d]
```

If the input system is detected to be inconsistent then \([1]\) is returned, and the list of reductors is empty:

```python
sage: # needs sage.rings.polynomial.pbori
sage: R.<x,y,z> = BooleanPolynomialRing()

sage: S = Sequence([x*y*z + x*y + z*y + x*z, x + y + z + 1, x + y + z])

sage: S.eliminate_linear_variables()
[1]

sage: R.<x,y,z> = BooleanPolynomialRing()

sage: S = Sequence([x*y*z + x*y + z*y + x*z, x + y + z + 1, x + y + z])

sage: S.eliminate_linear_variables(return_reductors=True)
([(1)], [])
```

\textbf{Note:} This is called “massaging” in [BCJ2007].

\textit{reduced}()  

If this sequence is \(f_1, ..., f_n\), return \(g_1, ..., g_s\) such that:

- \((f_1, ..., f_n) = (g_1, ..., g_s)\)
- \(LT(g_i) \neq LT(g_j)\) for all \(i \neq j\)
\( LT(g_i) \) does not divide \( m \) for all monomials \( m \) of \( g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_s \)

EXAMPLES:

```
sage: # needs sage.rings.polynomial.pbori
sage: sr = mq.SR(1, 1, 1, 4, gf2=True, polybori=True)
sage: while True:
....:     try:
....:         F, s = sr.polynomial_system()
....:         break
....:     except ZeroDivisionError:
....:         pass
sage: g = F.reduced()
sage: len(g) == len(set(gi.lt() for gi in g))
True
sage: for i in range(len(g)):
....:     for j in range(len(g)):
....:         if i == j:
....:             continue
....:         for t in list(g[j]):
....:             assert g[i].lt() not in t.divisors()
```

\texttt{solve(algorithm='pbori', n=1, eliminate_linear_variables=True, verbose=False, **kwds)}

Find solutions of this boolean polynomial system.

This function provide a unified interface to several algorithms dedicated to solving systems of boolean equations. Depending on the particular nature of the system, some might be much faster than some others.

INPUT:

- \texttt{self} - a sequence of boolean polynomials
- \texttt{algorithm} - the method to use. Possible values are \texttt{pbori, sat} and \texttt{exhaustive_search}. (default: \texttt{pbori}, since it is always available)
- \texttt{n} - number of solutions to return. If \( n = +\infty \) then all solutions are returned. If \( n < \infty \) then \( n \) solutions are returned if the equations have at least \( n \) solutions. Otherwise, all the solutions are returned. (default: \( 1 \))
- \texttt{eliminate_linear_variables} - whether to eliminate variables that appear linearly. This reduces the number of variables (makes solving faster a priori), but is likely to make the equations denser (may make solving slower depending on the method).
- \texttt{verbose} - whether to display progress and (potentially) useful information while the computation runs. (default: \texttt{False})

EXAMPLES:

Without argument, a single arbitrary solution is returned:

```
sage: # needs sage.rings.polynomial.pbori
sage: from sage.doctest.fixtures import reproducible_repr
sage: R.<x,y,z> = BooleanPolynomialRing()
sage: S = Sequence([x*y + z, y*z + x, x + y + z + 1])
sage: sol = S.solve()
sage: print(reproducible_repr(sol))
[(x: 0, y: 1, z: 0)]
```

We check that it is actually a solution:
We obtain all solutions:

```
sage: sols = S.solve(n=Infinity)  # needs sage.rings.polynomial.pbori
sage: print(reproducible_repr(sols))  # needs sage.rings.polynomial.pbori
[{x: 0, y: 1, z: 0}, {x: 1, y: 1, z: 1}]
sage: [S.subs(x) for x in sols]  # needs sage.rings.polynomial.pbori
[[0, 0, 0], [0, 0, 0]]
```

We can force the use of exhaustive search if the optional package `FES` is present:

```
sage: sol = S.solve(algorithm='exhaustive_search')  # optional - fes
sage: print(reproducible_repr(sol))  # optional - fes
[{x: 1, y: 1, z: 1}]
sage: S.subs(sol[0])  # optional - fes
[0, 0, 0]
```

And we may use SAT-solvers if they are available:

```
sage: sol = S.solve(algorithm='sat')  # optional - pycryptosat
sage: print(reproducible_repr(sol))  # optional - pycryptosat
[{x: 0, y: 1, z: 0}]
sage: S.subs(sol[0])  # optional - pycryptosat
[0, 0, 0]
```

```python
class sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_gf2e(parts, ring, immutable=False, cr=False, cr_str=None)

Bases: PolynomialSequence_generic
PolynomialSequence over \( F_{2^e} \), i.e extensions over \( F_2 \).

weil_restriction()

Project this polynomial system to \( F_2 \).

That is, compute the Weil restriction of scalars for the variety corresponding to this polynomial system and express it as a polynomial system over \( F_2 \).

EXAMPLES:

```
sage: # needs sage.rings.finite_rings
sage: k.<a> = GF(2^2)
```
```
sage: P.<x,y> = PolynomialRing(k, 2)
sage: a = P.base_ring().gen()
sage: F = Sequence([x*y + 1, a*x + 1], P)
sage: F2 = F.weil_restriction()
sage: F2
[x0*y0 + x1*y1 + 1, x1*y0 + x0*y1 + x1*y1, x1 + 1, x0 + x1, x0^2 + x0,
 x1^2 + x1, y0^2 + y0, y1^2 + y1]

Another bigger example for a small scale AES:

sage: # needs sage.rings.polynomial.pbori
sage: sr = mq.SR(1, 1, 1, 4, gf2=False)
sage: while True:
    try:
        F, s = sr.polynomial_system()
        break
    except ZeroDivisionError:
        pass
sage: F
Polynomial Sequence with 40 Polynomials in 20 Variables
sage: F2 = F.weil_restriction(); F2
Polynomial Sequence with 240 Polynomials in 80 Variables

sage.rings.polynomial.multi_polynomial_sequence.is_PolynomialSequence(F)

Return True if F is a PolynomialSequence.

INPUT:

• F - anything

EXAMPLES:

sage: P.<x,y> = PolynomialRing(QQ)
sage: I = [[x^2 + y^2], [x^2 - y^2]]
sage: F = Sequence(I, P); F
[x^2 + y^2, x^2 - y^2]

sage: from sage.rings.polynomial.multi_polynomial_sequence import is_PolynomialSequence
sage: is_PolynomialSequence(F)
True

3.1.8 Multivariate Polynomials via libSINGULAR

This module implements specialized and optimized implementations for multivariate polynomials over many coefficient rings, via a shared library interface to SINGULAR. In particular, the following coefficient rings are supported by this implementation:

• the rational numbers \( \mathbb{Q} \),
• the ring of integers \( \mathbb{Z} \),
• \( \mathbb{Z}/n\mathbb{Z} \) for any integer \( n \),
• finite fields \( \mathbb{F}_p^m \) for \( p \) prime and \( n > 0 \),
• and absolute number fields \( \mathbb{Q}(\alpha) \).
EXAMPLES:

We show how to construct various multivariate polynomial rings:

```python
sage: P.<x,y,z> = QQ[]
sage: P
Multivariate Polynomial Ring in x, y, z over Rational Field

sage: f = 27/113 * x^2 + y*z + 1/2; f
27/113*x^2 + y*z + 1/2

sage: P.term_order()
Degree reverse lexicographic term order

sage: P = PolynomialRing(GF(127), 3, names='abc', order='lex'); P
Multivariate Polynomial Ring in a, b, c over Finite Field of size 127

sage: a,b,c = P.gens()
sage: f = 57 * a^2*b + 43 * c + 1; f
57*a^2*b + 43*c + 1

sage: P.term_order()
Lexicographic term order

sage: z = QQ['z'].0
sage: K.<s> = NumberField(z^2 - 2) # needs sage.rings.number_field
sage: P.<x,y> = PolynomialRing(K, 2); P
Multivariate Polynomial Ring in x, y over Number Field in s with defining polynomial z^2 - 2

sage: 1/2*s*x^2 + 3/4*s
(1/2*s)*x^2 + (3/4*s)

sage: P.<x,y,z> = ZZ[]; P
Multivariate Polynomial Ring in x, y, z over Integer Ring

sage: P.<x,y,z> = Zmod(2^10)[]; P
Multivariate Polynomial Ring in x, y, z over Ring of integers modulo 1024

sage: P.<x,y,z> = Zmod(3^10)[]; P
Multivariate Polynomial Ring in x, y, z over Ring of integers modulo 59049

sage: P.<x,y,z> = Zmod(2^100)[]; P
Multivariate Polynomial Ring in x, y, z over Ring of integers modulo 1267650600228229401496703205376

sage: P.<x,y,z> = Zmod(2521352)[]; P
Multivariate Polynomial Ring in x, y, z over Ring of integers modulo 2521352

sage: type(P)
<class 'sage.rings.polynomial.multi_polynomial_libsingular.MPolynomialRing_libsingular'>

sage: P.<x,y,z> = Zmod(2521352135151515232)[]; P
Multivariate Polynomial Ring in x, y, z over Ring of integers modulo 2521352135151515232

sage: type(P)
<class 'sage.rings.polynomial.multi_polynomial_ring.MPolynomialRing_polydict_with_category'>
```

We construct the Frobenius morphism on \( \mathbb{F}_5[x, y, z] \) over \( \mathbb{F}_5 \):

3.1. Multivariate Polynomials and Polynomial Rings 427
We make a polynomial ring in one variable over a polynomial ring in two variables:

```
sage: R.<x, y> = PolynomialRing(QQ, 2)
sage: S.<t> = PowerSeriesRing(R)
sage: t*(x+y)
(x + y)*t
```

Todo: Implement Real, Complex coefficient rings via libSINGULAR

AUTHORS:

- Martin Albrecht (2007-01): initial implementation
- Joel Mohler (2008-01): misc improvements, polishing
- Martin Albrecht (2008-08): added \(\mathbb{Q}(a)\) and \(\mathbb{Z}\) support
- Simon King (2009-04): improved coercion
- Martin Albrecht (2009-05): added \(\mathbb{Z}/n\mathbb{Z}\) support, refactoring
- Martin Albrecht (2009-06): refactored the code to allow better re-use
- Simon King (2011-03): use a faster way of conversion from the base ring.
- Volker Braun (2011-06): major cleanup, refcount singular rings, bugfixes.
Element

alias of \texttt{MPolynomial_libsingular}

gen \((n=0)\)

Returns the \(n\)-th generator of this multivariate polynomial ring.

INPUT:
• \( n \) – an integer \( \geq 0 \)

EXAMPLES:

```python
sage: P.<x,y,z> = QQ[]
sage: P.gen(), P.gen(1)
(x, y)
sage: P = PolynomialRing(GF(127), 1000, 'x')
sage: P.gen(500)
x500
sage: P.<SAGE,SINGULAR> = QQ[]  # weird names
sage: P.gen(1)
SINGULAR
```

ideal (**\*gens, **\*kwds**)

Create an ideal in this polynomial ring.

INPUT:

• \*\*gens\* - list or tuple of generators (or several input arguments)

• \*\*coerce\* - bool (default: True); this must be a keyword argument. Only set it to \texttt{False} if you are certain that each generator is already in the ring.

EXAMPLES:

```python
sage: P.<x,y,z> = QQ[]
sage: sage.rings.ideal.Katsura(P)  # needs sage.rings.function_field
Ideal (x + 2*y + 2*z - 1, x^2 + 2*y^2 + 2*z^2 - x, 2*x*y + 2*y*z - y)
of Multivariate Polynomial Ring in x, y, z over Rational Field
sage: P.ideal([x + 2*y + 2*z-1, 2*x*y + 2*y*z-y, x^2 + 2*y^2 + 2*z^2-x])
Ideal (x + 2*y + 2*z - 1, 2*x*y + 2*y*z - y, x^2 + 2*y^2 + 2*z^2 - x)
of Multivariate Polynomial Ring in x, y, z over Rational Field
```

monomial_all_divisors (**\*t**)

Return a list of all monomials that divide \( t \).

Coefficients are ignored.

INPUT:

• \*\*t\* - a monomial

OUTPUT:

• a list of monomials

EXAMPLES:

```python
sage: P.<x,y,z> = QQ[]
sage: P.monomial_all_divisors(x^2*z^3)
[x, x^2, z, x*z, x^2*z, z^2, x*z^2, x^2*z^2, z^3, x*z^3, x^2*z^3]
```

ALGORITHM: addwithcarry idea by Toon Segers
\textbf{monomial\_divides}(a, b)

Return \texttt{False} if \textit{a} does not divide \textit{b} and \texttt{True} otherwise.

Coefficients are ignored.

\textbf{INPUT:}

\begin{itemize}
  \item \textit{a} – monomial
  \item \textit{b} – monomial
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: P.<x,y,z> = QQ[]
sage: P.monomial_divides(x*y*z, x^3*y^2*z^4)
True
sage: P.monomial_divides(x^3*y^2*z^4, x*y*z)
False
\end{verbatim}

\textbf{monomial\_lcm}(f, g)

\texttt{LCM} for monomials. Coefficients are ignored.

\textbf{INPUT:}

\begin{itemize}
  \item \textit{f} - monomial
  \item \textit{g} - monomial
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: P.<x,y,z> = QQ[]
sage: P.monomial_lcm(3/2*x*y,x)
x*y
\end{verbatim}

\textbf{monomial\_pairwise\_prime}(g, h)

Return \texttt{True} if \textit{h} and \textit{g} are pairwise prime. Both are treated as monomials.

Coefficients are ignored.

\textbf{INPUT:}

\begin{itemize}
  \item \textit{h} - monomial
  \item \textit{g} - monomial
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: P.<x,y,z> = QQ[]
sage: P.monomial_pairwise_prime(x^2*z^3, y^4)
True
sage: P.monomial_pairwise_prime(1/2*x^3*y^2, 3/4*y^3)
False
\end{verbatim}

\textbf{monomial\_quotient}(f, g, coeff=False)

Return \textit{f} / \textit{g}, where both \textit{f} and \textit{g} are treated as monomials.

Coefficients are ignored by default.

\textbf{INPUT:}

\begin{itemize}
  \item \textit{f} - monomial
  \item \textit{g} - monomial
\end{itemize}
• coeff - divide coefficients as well (default: False)

EXAMPLES:

```python
sage: P.<x,y,z> = QQ[]
sage: P.monomial_quotient(3/2*x*y,x)  
   y
sage: P.monomial_quotient(3/2*x*y,x,coeff=True)  
   3/2*y
```

Note, that \( \mathbb{Z} \) behaves different if coeff=True:

```python
sage: P.monomial_quotient(2*x,3*x)  
   1
sage: P.<x,y> = PolynomialRing(ZZ)
sage: P.monomial_quotient(2*x,3*x,coeff=True)
  Traceback (most recent call last):
  ...
  ArithmeticError: Cannot divide these coefficients.
```

**Warning:** Assumes that the head term of \( f \) is a multiple of the head term of \( g \) and return the multiplicant \( m \). If this rule is violated, funny things may happen.

**monomial_reduce** \((f,G)\)

Try to find a \( g \) in \( G \) where \( g.lm() \) divides \( f \). If found \((flt,g)\) is returned, \((0,0)\) otherwise, where \( flt = f/g.lm() \).

It is assumed that \( G \) is iterable and contains only elements in this polynomial ring.

Coefficients are ignored.

**INPUT:**

• \( f \) - monomial
• \( G \) - list/set of \text{mpolynomials}

**EXAMPLES:**

```python
sage: P.<x,y,z> = QQ[]
sage: f = x*y^2
sage: G = [ 3/2*x^3 + y^2 + 1/2, 1/4*x*y + 2/7, 1/2 ]
sage: P.monomial_reduce(f,G)  
   (y, 1/4*x*y + 2/7)
```

**ngens**

Returns the number of variables in this multivariate polynomial ring.

**EXAMPLES:**

```python
sage: P.<x,y> = QQ[]
sage: P.ngens()  
   2
```

```python
sage: k.<a> = GF(2^16)  
   #...  
   \#needs sage.rings.finite_rings
```
class sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular

Bases: MPolynomial_libsingular

A multivariate polynomial implemented using libSINGULAR.

add_m_mul_q (m, q)

Return self + m*q, where m must be a monomial and q a polynomial.

INPUT:

• m – a monomial
• q – a polynomial

EXAMPLES:

sage: P.<x,y,z> = PolynomialRing(QQ,3)
sage: x.add_m_mul_q(y,z)
y*z + x

coefficient (degrees)

Return the coefficient of the variables with the degrees specified in the python dictionary degrees. Mathematically, this is the coefficient in the base ring adjoined by the variables of this ring not listed in degrees. However, the result has the same parent as this polynomial.

This function contrasts with the function monomial_coefficient which returns the coefficient in the base ring of a monomial.

INPUT:

• degrees - Can be any of: - a dictionary of degree restrictions - a list of degree restrictions (with None in the unrestricted variables) - a monomial (very fast, but not as flexible)

OUTPUT: element of the parent of this element.

Note: For coefficients of specific monomials, look at monomial_coefficient().

EXAMPLES:

sage: R.<x,y> = QQ[]
sage: f=x*y+y+5
sage: f.coefficient({x:0,y:1})
1
sage: f.coefficient({x:0})
y + 5
sage: f=(1+y+y^2)*(1+x+x^2)
sage: f.coefficient({x:0})
y^2 + y + 1
sage: f.coefficient([0,None])
y^2 + y + 1
Note that exponents have all variables specified:

```sage
tax.coefficient(x.exponents()[0])
1
tax = f.coefficient([1,0])
1
tax = f.coefficient({x:1,y:0})
1
```

Be aware that this may not be what you think! The physical appearance of the variable `x` is deceiving – particularly if the exponent would be a variable.

```sage
tax = f.coefficient(x^0)  # outputs the full polynomial
x^2*y^2 + x^2*y + x*y^2 + x^2 + x*y + y^2 + x + y + 1
tax = R.<x,y> = GF(389)[]
tax = f = x*y + 5
tax = c = f.coefficient({x:0, y:0}); c
5
tax = parent(c)
Multivariate Polynomial Ring in x, y over Finite Field of size 389
```

AUTHOR:

- Joel B. Mohler (2007.10.31)

`coefficients()`

Return the nonzero coefficients of this polynomial in a list. The returned list is decreasingly ordered by the term ordering of the parent.

EXAMPLES:

```sage
tax = R.<x,y,z> = PolynomialRing(QQ, order='degrevlex')
tax = f = 23*x^6*y^7 + x^3*y + 6*x^7*z
tax = f.coefficients()  
[23, 6, 1]
tax = R.<x,y,z> = PolynomialRing(QQ, order='lex')
tax = f = 23*x^6*y^7 + x^3*y + 6*x^7*z
tax = f.coefficients()  
[6, 23, 1]
```

AUTHOR:

- Didier Deshommes

`constant_coefficient()`

Return the constant coefficient of this multivariate polynomial.

EXAMPLES:

```sage
tax = P.<x, y> = QQ[]
tax = f = 3*x^2 - 2*y + 7*x^2*y^2 + 5
tax = f.constant_coefficient()  
5
tax = f = 3*x^2
```

(continues on next page)
degree (x=None, std_grading=False)

Return the degree of this polynomial.

INPUT:

- x – (default: None) a generator of the parent ring

OUTPUT:

If x is not given, return the maximum degree of the monomials of the polynomial. Note that the degree of a monomial is affected by the gradings given to the generators of the parent ring. If x is given, it is (or coercible to) a generator of the parent ring and the output is the maximum degree in x. This is not affected by the gradings of the generators.

EXAMPLES:

```sage
R.<x, y> = QQ[]
sage: f = y^2 - x^9 - x
default: None)) for x in R.gens()]
[1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4]
```

A matrix term ordering determines the grading of the generators by the first row of the matrix.

```sage
m = matrix(3, [3, 2, 1, 1, 1, 0, 1, 0, 0])
sage: x, y, z = PolynomialRing(QQ, order=TermOrder(m))
sage: x.degree(), y.degree(), z.degree()
(3, 2, 1)
sage: f = x^3*y + x*z^4
sage: f.degree()
11
```

If the first row contains zero, the grading becomes the standard one.

```sage
m = matrix(3, [3, 0, 1, 1, 1, 0, 1, 0, 0])
sage: m
```
To get the degree with the standard grading regardless of the term ordering of the parent ring, use `std_grading=True`.

```
sage: f.degree(std_grading=True)
5
```

### degrees()

Returns a tuple with the maximal degree of each variable in this polynomial. The list of degrees is ordered by the order of the generators.

**EXAMPLES:**

```
sage: R.<y0,y1,y2> = PolynomialRing(QQ,3)
sage: q = 3*y0*y1*y2; q
3*y0*y1*y2
sage: q.degrees()
(1, 2, 1)
sage: (q + y0^5).degrees()
(5, 2, 1)
```

### dict()

Return a dictionary representing self. This dictionary is in the same format as the generic MPolynomial: The dictionary consists of ETuple:coefficient pairs.

**EXAMPLES:**

```
sage: R.<x,y,z> = QQ[]
sage: f=2*x*y^3*z^2 + 1/7*x^2 + 2/3
sage: f.dict()
{(0, 0, 0): 2/3, (1, 3, 2): 2, (2, 0, 0): 1/7}
```

### divides(other)

Return `True` if this polynomial divides `other`.

**EXAMPLES:**

```
sage: R.<x,y,z> = QQ[]
sage: p = 3*x*y + 2*y*z + x*z
sage: q = x + y + z + 1
sage: r = p * q
sage: p.divides(r)
True
sage: q.divides(p)
False
sage: r.divides(0)
True
sage: R.zero().divides(r)
False
sage: R.zero().divides(0)
True
```
**exponents** *(as_ETuples=True)*

Return the exponents of the monomials appearing in this polynomial.

**INPUT:**

- as_ETuples – (default: True) if True returns the result as a list of ETuples, otherwise returns a list of tuples

**EXAMPLES:**

```python
sage: R.<a,b,c> = QQ[]
sage: f = a^3 + b + 2*b^2
sage: f.exponents() [(3, 0, 0), (0, 2, 0), (0, 1, 0)]
sage: f.exponents(as_ETuples=False) [(3, 0, 0), (0, 2, 0), (0, 1, 0)]
```

**factor** *(proof=None)*

Return the factorization of this polynomial.

**INPUT:**

- proof – ignored.

**EXAMPLES:**

```python
sage: R.<x, y> = QQ[]
sage: f = (x^3 + 2*y^2*x) * (x^2 + x + 1); f
x^5 + 2*x^3*y^2 + x^4 + 2*x^2*y^2 + x^3 + 2*x*y^2
sage: F = f.factor(); F
x * (x^2 + x + 1) * (x^2 + 2*y^2)
```

Next we factor the same polynomial, but over the finite field of order 3.:

```python
sage: R.<x, y> = GF(3)[]
sage: f = (x^3 + 2*y^2*x) * (x^2 + x + 1); f
x^5 - x^3*y^2 + x^4 - x^2*y^2 + x^3 - x*y^2
sage: F = f.factor(); F
(-1) * x * (-x + y) * (x + y) * (x - 1)^2
```

Next we factor a polynomial, but over a finite field of order 9.:

```python
sage: # needs sage.rings.finite_rings
sage: K.<a> = GF(3^2)
sage: R.<x, y> = K[]
sage: f = (x^3 + 2*a*y^2*x) * (x^2 + x + 1); f
x^5 + (-a)*x^3*y^2 + x^4 + (-a)*x^2*y^2 + x^3 + (-a)*x*y^2
sage: F = f.factor(); F
((-a)) * x * (x - 1)^2 * ((-a + 1)*x^2 + y^2)
```

Next we factor a polynomial over a number field.:

```python
sage: # needs sage.rings.number_field
sage: p = polygen(ZZ, 'p')
sage: K.<s> = NumberField(p^3 - 2)
sage: KXY.<x,y> = K[]
```

(continues on next page)
sage: factor(x^3 - 2*y^3)
(x + (-s)*y) * (x^2 + s*x*y + (s^2)*y^2)
sage: k = (x^3-2*y^3)^5*(x+s*y)^2*(2/3 + s^2)
sage: k.factor()
((s^2 + 2/3)) * (x + s*y)^2 * (x + (-s)*y)^5 * (x^2 + s*x*y + (s^2)*y^2)^5

This shows that issue github issue #2780 is fixed, i.e. that the unit part of the factorization is set correctly:

sage: # needs sage.rings.number_field
sage: x = polygen(ZZ, 'x')
sage: K.<a> = NumberField(x^2 + 1)
sage: R.<y, z> = PolynomialRing(K)
sage: f = 2*y^2 + 2*z^2
sage: F = f.factor(); F.unit()
2

Another example:

sage: R.<x,y,z> = GF(32003)[]
# needs sage.rings.finite_rings
sage: p = (4*v^4*u^2 - 16*v^2*u^4 + 16*u^6 - 4*v^4*u + 8*v^2*u^3 + v^4)
# needs sage.rings.finite_rings
sage: p.factor()
(-2*v^2*u + 4*u^3 + v^2)^2

Constant elements are factorized in the base rings.

sage: P.<x,y> = ZZ[]
sage: P(2^3*7).factor()
2^3 * 7

Factorization for finite prime fields with characteristic $> 2^{29}$ is not supported

sage: q = 1073741789
sage: T.<aa, bb> = PolynomialRing(GF(q))
# needs sage.rings.finite_rings
sage: f = aa^2 + 12124343*bb*aa + 32434598*bb^2
# needs sage.rings.finite_rings
sage: f.factor()
Traceback (most recent call last):
...
NotImplementedError: Factorization of multivariate polynomials over prime fields with characteristic $> 2^{29}$ is not implemented.

Factorization over the integers is now supported, see github issue #17840:

```
sage: P.<x,y> = PolynomialRing(ZZ)
sage: f = 12 * (3*x*y + 4) * (5*x - 2) * (2*y + 7)^2
sage: f.factor()
2^2 * 3 * (2*y + 7)^2 * (5*x - 2) * (3*x*y + 4)
sage: g = -12 * (x^2 - y^2)
sage: g.factor()
(-1) * 2^2 * 3 * (x - y) * (x + y)
sage: factor(-4*x*y - 2*x + 2*y + 1)
(-1) * (2*y + 1) * (2*x - 1)
```

Factorization over non-integral domains is not supported

```
sage: R.<x,y> = PolynomialRing(Zmod(4))
sage: f = (2*x + 1) * (x^2 + x + 1)
sage: f.factor()
Traceback (most recent call last):
...
NotImplementedError: Factorization of multivariate polynomials over Ring of integers modulo 4 is not implemented.
```

**gcd** (*right*, *algorithm=None*, **kwds**)

Return the greatest common divisor of self and right.

**INPUT:**

- *right* – polynomial
- *algorithm* - 'ezgcd' – EZGCD algorithm - 'modular' – multi-modular algorithm (default)
- **kwds** – ignored

**EXAMPLES:**

```
sage: P.<x,y,z> = QQ[]
sage: f = (x*y*z)^6 - 1
go: g = (x*y*z)^4 - 1
go: f.gcd(g)
x^2*y^2*z^2 - 1
sage: GCD([x^3 - 3*x + 2, x^4 - 1, x^6 -1])
x - 1
sage: R.<x,y> = QQ[]
sage: f = (x^3 + 2*y^2*x)^2
go: g = x^2*y^2
sage: f.gcd(g)
x^2
```

We compute a gcd over a finite field:

```
sage: # needs sage.rings.finite_rings
sage: F.<u> = GF(31^2)
```
sage: R.<x,y,z> = F[]
sage: p = x^3 + (1+u)*y^3 + z^3
sage: q = p^3 * (x - y + z*u)
sage: gcd(p,q)
x^3 + (u + 1)*y^3 + z^3
sage: gcd(p,q)  # yes, twice -- tests that singular ring is properly set.
x^3 + (u + 1)*y^3 + z^3

We compute a gcd over a number field:

sage: # needs sage.rings.number_field
sage: x = polygen(QQ)
sage: F.<u> = NumberField(x^3 - 2)
sage: R.<x,y,z> = F[]
sage: p = x^3 + (1+u)*y^3 + z^3
sage: q = p^3 * (x - y + z*u)
sage: gcd(p,q)
x^3 + (u + 1)*y^3 + z^3

global_height (prec=None)

Return the (projective) global height of the polynomial.

This returns the absolute logarithmic height of the coefficients thought of as a projective point.

INPUT:

  • prec – desired floating point precision (default: default RealField precision).

OUTPUT: a real number.

EXAMPLES:

sage: R.<x,y> = PolynomialRing(QQ)
sage: f = 3*x^3 + 2*x*y^2
sage: exp(f.global_height())  # needs sage.symbolic
3.00000000000000

sage: # needs sage.rings.number_field
sage: K.<k> = CyclotomicField(3)
sage: R.<x,y> = PolynomialRing(K, sparse=True)
sage: f = k*x*y + 1
sage: exp(f.global_height())
1.00000000000000

Scaling should not change the result:

sage: R.<x,y> = PolynomialRing(QQ)
sage: f = 1/25*x^2 + 25/3*x*y + y^2
sage: f.global_height()  # needs sage.symbolic
6.43775164973640
sage: g = 100 * f
sage: g.global_height()  # needs sage.symbolic
6.43775164973640
```python
sage: R.<x> = PolynomialRing(QQ)
sage: K.<k> = NumberField(x^2 + 5)
sage: T.<t,w> = PolynomialRing(K)
sage: f = 1/1331 * t^2 + 5 * w + 7
sage: f.global_height()
9.13959596745043
```

```python
sage: R.<x,y> = QQ[]
sage: f = 1/123*x*y + 12
sage: f.global_height(prec=2)
8.0
```

```python
sage: R.<x,y> = QQ[]
sage: f = 0*x*y
sage: f.global_height()
0.000000000000000
```

**gradient()**

Return a list of partial derivatives of this polynomial, ordered by the variables of the parent.

**EXAMPLES:**

```python
sage: P.<x,y,z> = PolynomialRing(QQ,3)
sage: f= x*y + 1
sage: f.gradient()
[y, x, 0]
```

**hamming_weight()**

Return the number of non-zero coefficients of this polynomial.

This is also called weight, *hamming_weight()* or sparsity.

**EXAMPLES:**

```python
sage: R.<x, y> = ZZ[]
sage: f = x^3 - y
sage: f.number_of_terms()
2
sage: R(0).number_of_terms()
0
sage: f = (x+y)^100
sage: f.number_of_terms()
101
```

The method *hamming_weight()* is an alias:

```python
sage: f.hamming_weight()
101
```

**in_subalgebra** (*J*, *algorithm='algebra_containment'*)

Return whether this polynomial is contained in the subalgebra generated by *J*

**INPUT:**

- *J* – list of elements of the parent polynomial ring
• algorithm – can be "algebra_containment" (the default), "inSubring", or "groebner".

- "algebra_containment": use Singular's algebra_containment function, https://www.singular.uni-kl.de/Manual/4-2-1/sing_1247.htm#SEC1328. The Singular documentation suggests that this is frequently faster than the next option.


- "groebner": use the algorithm described in Singular's documentation, but within Sage: if the subalgebra generators are \( y_1, \ldots, y_m \), then create a new polynomial algebra with the old generators along with new ones: \( z_1, \ldots, z_m \). Create the ideal \( (z_1 - y_1, \ldots, z_m - y_m) \), and reduce the polynomial modulo this ideal. The polynomial is contained in the subalgebra if and only if the remainder involves only the new variables \( z_i \).

EXAMPLES:

```
sage: P.<x,y,z> = QQ[]
sage: J = [x^2 + y^2, x^2 + z^2]
sage: (y^2).in_subalgebra(J)
False
sage: a = (x^2 + y^2) * (x^2 + z^2)
sage: a.in_subalgebra(J, algorithm='inSubring')
True
sage: a^2.in_subalgebra(J, algorithm='groebner')
True
```

**integral**(var)**

Integrate this polynomial with respect to the provided variable.

One requires that \( \mathbb{Q} \) is contained in the ring.

INPUT:

• variable – the integral is taken with respect to variable

EXAMPLES:

```
sage: R.<x, y> = PolynomialRing(QQ, 2)
sage: f = 3*x^3*y^2 + 5*y^2 + 3*x + 2
sage: f.integral(x)
3/4*x^4*y^2 + 5/2*y^2 + 3/2*x^2 + 2*x
sage: f.integral(y)
x^3*y^3 + 5/3*y^3 + 3*x*y^2 + 2*y
```

Check that github issue #15896 is solved:

```
sage: s = x+y
sage: s.integral(x)+x
1/2*x^2 + x*y + x
sage: s.integral(x)*s
1/2*x^3 + 3/2*x^2*y + x*y^2
```

**inverse_of_unit**()

Return the inverse of this polynomial if it is a unit.

EXAMPLES:
sage: R.<x,y> = QQ[]
sage: x.inverse_of_unit()
Traceback (most recent call last):
  ...  
ArithmeticError: Element is not a unit.
sage: R(1/2).inverse_of_unit()
2

is_constant()
Return True if this polynomial is constant.

EXAMPLES:

sage: P.<x,y,z> = PolynomialRing(GF(127))
sage: x.is_constant()
False
sage: P(1).is_constant()
True

is_homogeneous()
Return True if this polynomial is homogeneous.

EXAMPLES:

sage: P.<x,y> = PolynomialRing(RationalField(), 2)
sage: (x+y).is_homogeneous()
True
sage: (x.parent()(0)).is_homogeneous()
True
sage: (x+y^2).is_homogeneous()
False
sage: (x^2 + y^2).is_homogeneous()
True
sage: (x^2 + y^2*x).is_homogeneous()
False
sage: (x^2*y + y^2*x).is_homogeneous()
True

is_monomial()
Return True if this polynomial is a monomial. A monomial is defined to be a product of generators with coefficient 1.

EXAMPLES:

sage: P.<x,y,z> = PolynomialRing(QQ)
sage: x.is_monomial()
True
sage: (2*x).is_monomial()
False
sage: (x*y).is_monomial()
True
sage: (x*y + x).is_monomial()
False
sage: P(2).is_monomial()
False
sage: P.zero().is_monomial()
False
**is_squarefree()**

Return **True** if this polynomial is square free.

**EXAMPLES:**

```python
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: f= x^2 + 2*x*y + 1/2*z
sage: f.is_squarefree()
True
sage: h = f^2
sage: h.is_squarefree()
False
```

**is_term()**

Return **True** if **self** is a term, which we define to be a product of generators times some coefficient, which need not be 1.

Use **is_monomial()** to require that the coefficient be 1.

**EXAMPLES:**

```python
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: x.is_term()
True
sage: (2*x).is_term()
True
sage: (x*y).is_term()
True
sage: (x*y + x).is_term()
False
sage: P(2).is_term()
True
sage: P.zero().is_term()
False
```

**is_univariate()**

Return **True** if **self** is a univariate polynomial, that is if **self** contains only one variable.

**EXAMPLES:**

```python
sage: P.<x,y,z> = GF(2)[]
sage: f = x^2 + 1
sage: f.is_univariate()
True
sage: f = y*x^2 + 1
sage: f.is_univariate()
False
sage: f = P(0)
sage: f.is_univariate()
True
```

**is_zero()**

Return **True** if this polynomial is zero.

**EXAMPLES:**

```python
sage: P.<x,y> = PolynomialRing(QQ)
sage: x.is_zero()
```

(continues on next page)
False
sage: (x - x).is_zero()
True

**iterator_exp_coeff** (*as_ETuples=True*)

Iterate over self as pairs of ((E)Tuple, coefficient).

**INPUT:**

- *as_ETuples* – (default: True) if True iterate over pairs whose first element is an ETuple, otherwise as a tuples

**EXAMPLES:**

```
sage: R.<a,b,c> = QQ[]
sage: f = a*c^3 + a^2*b + 2*b^4
sage: list(f.iterator_exp_coeff())
[((0, 4, 0), 2), ((1, 0, 3), 1), ((2, 1, 0), 1)]
sage: list(f.iterator_exp_coeff(as_ETuples=False))
[((0, 4, 0), 2), ((1, 0, 3), 1), ((2, 1, 0), 1)]
```

```
sage: R.<a,b,c> = PolynomialRing(QQ, 3, order='lex')
sage: f = a*c^3 + a^2*b + 2*b^4
sage: list(f.iterator_exp_coeff())
[((2, 1, 0), 1), ((1, 0, 3), 1), ((0, 4, 0), 2)]
```

**lc**

Leading coefficient of this polynomial with respect to the term order of `self.parent()`.

**EXAMPLES:**

```
sage: R.<x,y,z> = PolynomialRing(GF(7), 3, order='lex')
sage: f = 3*x^1*y^2 + 2*y^3*z^4
sage: f.lc()
3
```

```
sage: f = 5*x^3*y^2*z^4 + 4*x^3*y^2*z^1
sage: f.lc()
5
```

**lcm** (*g*)

Return the least common multiple of `self` and `g`.

**EXAMPLES:**

```
sage: P.<x,y,z> = QQ[]
sage: p = (x+y)*(y+z)
sage: q = (z^4+2)*(y+z)
sage: lcm(p,q)
x*y*z^4 + y^2*z^4 + x*z^5 + y*z^5 + 2*x*y + 2*y^2 + 2*x*z + 2*y*z
```

```
sage: P.<x,y,z> = ZZ[]
sage: p = 2*(x+y)*(y+z)
sage: q = 3*(z^4+2)*(y+z)
sage: lcm(p,q)
6*x*y*z^4 + 6*y^2*z^4 + 6*x*z^5 + 6*y*z^5 + 12*x*y + 12*y^2 + 12*x*z + 12*y*z
```

(continues on next page)
lift(I)

given an ideal \( I = (f_1, \ldots, f_r) \) and some \( g \ (== \text{self}) \) in \( I \), find \( s_1, \ldots, s_r \) such that \( g = s_1 f_1 + \ldots + s_r f_r \).

A ValueError exception is raised if \( g \ (== \text{self}) \) does not belong to \( I \).

EXAMPLES:

\[
\begin{align*}
\text{sage: } & \ x, y > = \text{PolynomialRing} (\text{QQ}, 2, \text{order}='\text{degrevlex}') \\
\text{sage: } & \ I = \text{A.ideal}([x^{10} + x^9 y^2, y^8 - x^2 y^7]) \\
\text{sage: } & \ f = x^13 + y^{12} \\
\text{sage: } & \ M = f.lift(I) \\
\text{sage: } & \ M \\
\text{sage: } & \ [y^7, x^9 y^2 + x^8 + x^5 y^3 + x^6 y + x^3 y^4 + x^4 y^2 + x^5 y + x^2 y^3 + \ldots - y^4] \\
\text{sage: } & \ \text{sum( map( mul, zip( M, I.gens() ) ) ) == f} \\
\text{sage: } & \ True
\end{align*}
\]

Check that github issue #13671 is fixed:

\[
\begin{align*}
\text{sage: } & \ R.<x1,x2> = \text{QQ[]} \\
\text{sage: } & \ I = R.ideal(x2**2 + x1 - 2, x1**2 - 1) \\
\text{sage: } & \ f = I\.\text{gen}(0) + x2*I\.\text{gen}(1) \\
\text{sage: } & \ f.lift(I) \\
\text{sage: } & \ [1, x2] \\
\text{sage: } & \ (f+1).lift(I) \\
\text{Traceback (most recent call last):} \\
\text{...} \\
\text{ValueError: polynomial is not in the ideal}
\end{align*}
\]

Check that we can work over the integers:

\[
\begin{align*}
\text{sage: } & \ R.<x1,x2> = \text{ZZ[]} \\
\text{sage: } & \ I = R.ideal(x2**2 + x1 - 2, x1**2 - 1) \\
\text{sage: } & \ f = I\.\text{gen}(0) + x2*I\.\text{gen}(1) \\
\text{sage: } & \ f.lift(I) \\
\text{sage: } & \ [1, x2] \\
\text{sage: } & \ (f+1).lift(I) \\
\text{Traceback (most recent call last):} \\
\text{...} \\
\text{ValueError: polynomial is not in the ideal}
\end{align*}
\]
 sage: I = A.ideal([x^10 + x^9*y^2, y^8 - x^2*y^7 ])
 sage: f = x*y^13 + y^12
 sage: M = f.lift(I)
 saga: M
\[y^7, x^7*y^2 + x^8 + x^5*y^3 + x^6*y + x^3*y^4 + x^4*y^2 + x*y^5 + x^2*y^3 +\cdots\]
\[\longrightarrow y^4\]

\textbf{lm()}\

Returns the lead monomial of \texttt{self} with respect to the term order of \texttt{self.parent()}. In Sage a monomial is a product of variables in some power without a coefficient.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: R.<x,y,z> = PolynomialRing(GF(7), 3, order='lex')
sage: f = x^1*y^2 + y^3*z^4
sage: f.lm()
x*y^2
sage: f = x^3*y^2*z^4 + x^3*y^2*z^1
sage: f.lm()
x^3*y^2*z^4
sage: R.<x,y,z>=PolynomialRing(QQ, 3, order=deglex)
sage: f = x^1*y^2*z^3 + x^3*y^2*z^0
sage: f.lm()
x*y^2*z^3
sage: f = x^1*y^2*z^4 + x^1*y^1*z^5
sage: f.lm()
x*y^2*z^4
sage: R.<x,y,z>=PolynomialRing(GF(127), 3, order=degrevlex)
sage: f = x^1*y^5*z^2 + x^4*y^1*z^3
sage: f.lm()
x*y^5*z^2
sage: f = x^4*y^7*z^1 + x^4*y^2*z^3
sage: f.lm()
x^4*y^7*z
\end{verbatim}

\textbf{local\_height}(v, prec=None)\

Return the maximum of the local height of the coefficients of this polynomial.

\textbf{INPUT:}

\begin{itemize}
\item v – a prime or prime ideal of the base ring.
\item prec – desired floating point precision (default: default \texttt{RealField} precision).
\end{itemize}

\textbf{OUTPUT:} a real number.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: R.<x,y> = PolynomialRing(QQ)
sage: f = 1/1331*x^2 + 1/4000*y^2
sage: f.local_height(1331)
7.19368581839511
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: K.<k> = NumberField(x^2 - 5)
\end{verbatim}
sage: T.<t,w> = K[]
sage: I = K.ideal(3)
sage: f = 1/3*t*w + 3
sage: f.local_height(I)
1.09861228866811

sage: R.<x,y> = QQ[]
sage: f = 1/2*x*y + 2
sage: f.local_height(2, prec=2)
0.75

local_height_arch \(i, \text{prec=}\text{None}\)

Return the maximum of the local height at the \(i\)-th infinite place of the coefficients of this polynomial.

INPUT:

• \(i\) – an integer.

• \text{prec} – desired floating point precision (default: default \texttt{RealField} precision).

OUTPUT: a real number.

EXAMPLES:

sage: R.<x,y> = PolynomialRing(QQ)
sage: f = 210*x*y
sage: f.local_height_arch(0)
5.34710753071747

sage: # needs \texttt{sage.rings.number_field}
sage: R.<x> = QQ[]
sage: K.<k> = NumberField(x^2 - 5)
sage: T.<t,w> = K[]
sage: f = 1/2*t*w + 3
sage: f.local_height_arch(1, prec=52)
1.09861228866811

sage: R.<x,y> = QQ[]
sage: f = 1/2*x*y + 3
sage: f.local_height_arch(0, prec=2)
1.0

lt()

Leading term of this polynomial. In Sage a term is a product of variables in some power and a coefficient.

EXAMPLES:

sage: R.<x,y,z> = PolynomialRing(GF(7), 3, order='lex')
sage: f = 3*x^1*y^2 + 2*y^3*z^4
sage: f.lt()  
3*x*y^2

sage: f = 5*x^3*y^2*z^4 + 4*x^3*y^2*z^1
sage: f.lt()  
-2*x^3*y^2*z^4
monomial_coefficient (mon)

Return the coefficient in the base ring of the monomial mon in self, where mon must have the same parent as self.

This function contrasts with the function coefficient which returns the coefficient of a monomial viewing this polynomial in a polynomial ring over a base ring having fewer variables.

INPUT:
- mon - a monomial

OUTPUT:
coefficient in base ring

See also:
For coefficients in a base ring of fewer variables, look at coefficient.

EXAMPLES:

```
sage: P.<x,y> = QQ[]
The parent of the return is a member of the base ring.
sage: f = 2 * x * y
sage: c = f.monomial_coefficient(x*y); c
2
sage: c.parent()
Rational Field
sage: f = y^2 + y^2*x - x^9 - 7*x + 5*x*y
sage: f.monomial_coefficient(y^2)
1
sage: f.monomial_coefficient(x*y)
5
sage: f.monomial_coefficient(x^9)
-1
sage: f.monomial_coefficient(x^10)
0
```

monomials()

Return the list of monomials in self. The returned list is decreasingly ordered by the term ordering of self. parent().

EXAMPLES:

```
sage: P.<x,y,z> = QQ[]
sage: f = x + 3/2*y*z^2 + 2/3
sage: f.monomials()
[y*z^2, x, 1]
sage: f = P(3/2)
sage: f.monomials()
[1]
```

number_of_terms()

Return the number of non-zero coefficients of this polynomial.

This is also called weight, hamming_weight() or sparsity.

EXAMPLES:
The method `hamming_weight()` is an alias:

```python
sage: f.hamming_weight()
101
```

`numerator()`

Return a numerator of self computed as self * self.denominator()

If the base_field of self is the Rational Field then the numerator is a polynomial whose base_ring is the Integer Ring, this is done for compatibility to the univariate case.

**Warning:** This is not the numerator of the rational function defined by self, which would always be self since self is a polynomial.

**EXAMPLES:**

First we compute the numerator of a polynomial with integer coefficients, which is of course self.

```python
sage: R.<x, y> = ZZ[]
sage: f = x^3 + 17*y + 1
sage: f.numerator()
x^3 + 17*y + 1
sage: f == f.numerator()
True
```

Next we compute the numerator of a polynomial with rational coefficients.

```python
sage: R.<x,y> = PolynomialRing(QQ)
sage: f = (1/17)*x^19 - (2/3)*y + 1/3; f
1/17*x^19 - 2/3*y + 1/3
sage: f.numerator()
3*x^19 - 34*y + 17
sage: f == f.numerator()
False
sage: f.numerator().base_ring()
Integer Ring
```

We check that the computation of numerator and denominator is valid.

```python
sage: K=QQ['x,y']
sage: f=K.random_element()
sage: f.numerator() / f.denominator() == f
True
```

The following tests against a bug fixed in github issue #11780:
sage: P.<foo,bar> = ZZ[]
sage: Q.<foo,bar> = QQ[]
sage: f = Q.random_element()
sage: f.numerator().parent() is P
True

nvariables()

Return the number variables in this polynomial.

EXAMPLES:

sage: P.<x,y,z> = PolynomialRing(GF(127))
sage: f = x*y + z
sage: f.nvariables()
3
sage: f = x + y
sage: f.nvariables()
2

quo_rem(right)

Return quotient and remainder of self and right.

EXAMPLES:

sage: R.<x,y> = QQ[]
sage: f = y*x^2 + x + 1
sage: f.quo_rem(x)
(x*y + 1, 1)
sage: f.quo_rem(y)
(x^2, x + 1)

sage: R.<x,y> = ZZ[]
sage: f = 2*y*x^2 + x + 1
sage: f.quo_rem(x)
(2*x*y + 1, 1)
sage: f.quo_rem(y)
(2*x^2, x + 1)
sage: f.quo_rem(3*x)
(0, 2*x^2*y + x + 1)

reduce(I)

Return a remainder of this polynomial modulo the polynomials in I.

INPUT:

- I - an ideal or a list/set/iterable of polynomials.

OUTPUT:

A polynomial r such that:

- self - r is in the ideal generated by I.
- No term in r is divisible by any of the leading monomials of I.

The result r is canonical if:

- I is an ideal, and Sage can compute a Groebner basis of it.
• $I$ is a list/set/iterable that is a (strong) Groebner basis for the term order of $\text{self}$. (A strong Groebner basis is such that for every leading term $t$ of the ideal generated by $I$, there exists an element $g$ of $I$ such that the leading term of $g$ divides $t$.)

The result $r$ is implementation-dependent (and possibly order-dependent) otherwise. If $I$ is an ideal and no Groebner basis can be computed, its list of generators $I.gens()$ is used for the reduction.

**EXAMPLES:**

```python
sage: P.<x,y,z> = QQ[

sage: f1 = -2 * x^2 + x^3

sage: f2 = -2 * y + x* y

sage: f3 = -x^2 + y^2

sage: F = Ideal([f1,f2,f3])

sage: g = x*y - 3*x*y^2

sage: g.reduce(F)

-6*y^2 + 2*y

sage: g.reduce(F.gens())

-6*y^2 + 2*y
```

$\mathbb{Z}$ is also supported.

```python
sage: P.<x,y,z> = ZZ[

sage: f1 = -2 * x^2 + x^3

sage: f2 = -2 * y + x* y

sage: f3 = -x^2 + y^2

sage: F = Ideal([f1,f2,f3])

sage: g = x*y - 3*x*y^2

sage: g.reduce(F)

-6*y^2 + 2*y

sage: g.reduce(F.gens())

-6*y^2 + 2*y

sage: f = 3*x

sage: f.reduce([2*x,y])

x
```

The reduction is not canonical when $I$ is not a Groebner basis:

```python
sage: A.<x,y> = QQ[

sage: (x+y).reduce([x+y, x-y])

2*y

sage: (x+y).reduce([x-y, x+y])

0
```

**resultant**(other, variable=None)

Compute the resultant of this polynomial and the first argument with respect to the variable given as the second argument.

If a second argument is not provide the first variable of the parent is chosen.

**INPUT:**

• other - polynomial

• variable - optional variable (default: None)

**EXAMPLES:**
As demonstrated in the provided code examples, polynomials can be evaluated using methods such as `subs` and `resultant` in SageMath. These methods allow for the manipulation of polynomials over various rings, including the integers and finite fields.

For instance, to evaluate a polynomial with some variables fixed, you can use the `subs` method. Here's how you might do it:

```python
sage: P.<x,y> = PolynomialRing(QQ,2)
sage: a = x+y
sage: b = x^3-y^3
sage: x.subs(y=2)
```

This will substitute `y` with `2` in the polynomial `x + y`. Similarly, the `resultant` method can be used to compute the resultant of two polynomials, which is a measure of their relationship.

```python
sage: c = a.resultant(b); c
```

This calculates the resultant of `a` and `b`.

The `resultant` method can also be used with specific variables. For example:

```python
sage: d = a.resultant(b,y); d
```

This calculates the resultant of `a` and `b` with respect to the variable `y`.

The SINGULAR example also showcases the ability to work with polynomials over finite fields:

```python
sage: R.<x,y,z> = PolynomialRing(GF(32003),3)
sage: f = 3 * (x+2)^3 + y
sage: g = x + y + z
sage: f.resultant(g, x)
```

This computes the resultant of `f` and `g` with respect to `x` over the finite field `GF(32003)`. The `resultant` method is particularly useful in algebraic computations, where it can help determine if two polynomials have a common root.

Subtraction of a monomial from a polynomial, as shown in the `sub_m_mul_q` method, is another fundamental operation:

```python
sage: P.<x,y,z> = PolynomialRing(QQ,3)
sage: x.sub_m_mul_q(y, z)
```

This subtracts `m*q` from `self`, where `m` is a monomial and `q` is a polynomial.

The `subs` method is a powerful tool for fixing variables in a polynomial, as it allows for the evaluation of the polynomial under certain conditions:

```python
sage: x.subs fixed=None, **kw)
```

This method fixes some variables and returns the changed multivariate polynomial, which can be particularly useful in algebraic manipulations.
OUTPUT: a new multivariate polynomial

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: f = x^2 + y + x^2*y^2 + 5
sage: f(5,y)
25*y^2 + y + 30
sage: f.subs((x: 5))
25*y^2 + y + 30
sage: f.subs(x=5)
25*y^2 + y + 30
sage: P.<x,y,z> = PolynomialRing(GF(2), 3)
sage: f = x + y + 1
sage: f.subs({x:y+1})
0
sage: f.subs(x=y)
1
sage: f.subs(x=x)
x + y + 1
sage: f.subs((x: z))
y + z + 1
sage: f.subs(x=z + 1)
y + z
sage: f.subs(x=1/y)
(y^2 + y + 1)/y
sage: f.subs({x: 1/y})
(y^2 + y + 1)/y
```

The parameters are substituted in order and without side effects:

```
sage: R.<x,y>=QQ[]
sage: g=x+y
sage: g.subs({x:x+1,y:x*y})
x*y + x + 1
sage: g.subs({x:x+1}).subs({y:x*y})
x*y + x + 1
sage: g.subs({y:x*y}).subs({x:x+1})
x*y + x + y + 1
```

**total_degree**(std_grading=False)

Return the total degree of self, which is the maximum degree of all monomials in self.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: f = 2*x*y^3*z^2
sage: f.total_degree()
6
sage: f = 4*x^2*y^2*z^3
sage: f.total_degree()
6
```

(continues on next page)
Polynomials, Release 10.3

A matrix term ordering changes the grading. To get the total degree using the standard grading, use **std_grading=True**:

sage: tord = TermOrder(matrix(3, [3, 2, 1, 1, 1, 0, 1, 0, 0]))
sage: tord
Matrix term order with matrix
<table>
<thead>
<tr>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
sage: R.<x,y,z> = PolynomialRing(QQ, order=tord)
sage: f = x^2*y
sage: f.total_degree()
8
sage: f.total_degree(std_grading=True)
3

### univariate_polynomial ($R=None$)

Returns a univariate polynomial associated to this multivariate polynomial.

**INPUT:**

- $R$ - (default: None) PolynomialRing

If this polynomial is not in at most one variable, then a **ValueError** exception is raised. This is checked using the **is_univariate()** method. The new Polynomial is over the same base ring as the given MPolynomial and in the variable $x$ if no ring $R$ is provided.

**EXAMPLES:**

sage: R.<x, y> = QQ[]
sage: f = 3*x^2 - 2*y + 7*x^2*y^2 + 5
sage: f.univariate_polynomial()
Traceback (most recent call last):
...  
TypeError: polynomial must involve at most one variable
sage: g = f.subs({x:10}); g
700*y^2 - 2*y + 305
sage: g.univariate_polynomial()
700*y^2 - 2*y + 305
sage: g.univariate_polynomial(PolynomialRing(QQ,'z'))
700*z^2 - 2*z + 305

Here's an example with a constant multivariate polynomial:
variable \( (i=0) \)
Return the \( i \)-th variable occurring in \( \text{self} \). The index \( i \) is the index in \( \text{self}.\text{variables}() \).

EXAMPLES:

```
sage: P.<x,y,z> = GF(2)[]
sage: f = x*z^2 + z + 1
sage: f.variables()
(x, z)
sage: f.variable(1)
z
```

variables()
Return a tuple of all variables occurring in \( \text{self} \).

EXAMPLES:

```
sage: P.<x,y,z> = GF(2)[]
sage: f = x*z^2 + z + 1
sage: f.variables()
(x, z)
```

sage.rings.polynomial.multi_polynomial_libsingular.unpickle_MPolynomialRing_libsingular(base_ring, names, term_order)
inverse function for \( \text{MPolynomialRing_libsingular}.\text{__reduce__} \)

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQ)
sage: loads(dumps(P)) \is\ P \ #\ indirect\ doctest
True
```

sage.rings.polynomial.multi_polynomial_libsingular.unpickle_MPolynomial_libsingular(R, d)
Deserialize an \( \text{MPolynomial_libsingular} \) object

INPUT:

- \( R \) - the base ring
- \( d \) - a Python dictionary as returned by \( \text{MPolynomial_libsingular}.\text{dict}() \)

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQ)
sage: loads(dumps(x)) == x \ #\ indirect\ doctest
True
```
3.1.9 Direct low-level access to SINGULAR’s Groebner basis engine via libSINGULAR

AUTHOR:
- Martin Albrecht (2007-08-08): initial version

EXAMPLES:

```sage
x, y, z = QQ['x, y, z'].gens()
x^5 + y^4 + z^3 - 1, x^3 + y^3 + z^2 - 1
I = ideal(x^5 + y^4 + z^3 - 1, x^3 + y^3 + z^2 - 1)
I.groebner_basis('libsingular:std')
[y^6 + x*y^4 + 2*x^2*z^2 + x*z^3 + z^4 - 2*y^3 - 2*z^2 - x + 1,
x^2*y^3 - y^4 + x^2*z^2 - z^3 - x^2 + 1, x^3 + y^3 + z^2 - 1]
```

We compute a Groebner basis for cyclic 6, which is a standard benchmark and test ideal:

```sage
R.<x,y,z,t,u,v> = QQ['x,y,z,t,u,v']
I = sage.rings.ideal.Cyclic(R, 6)
B = I.groebner_basis('libsingular:std')
len(B)
```

Two examples from the Mathematica documentation (done in Sage):

- We compute a Groebner basis:
  ```sage
  R.<x,y> = PolynomialRing(QQ, order='lex')
  ideal(x^2 - 2*y^2, x*y - 3).groebner_basis('libsingular:slimgb')
  [x - 2/3*y^3, y^4 - 9/2]
  ```

- We show that three polynomials have no common root:
  ```sage
  R.<x, y> = QQ[]
  ideal(x+y, x^2 - 1, y^2 - 2*x).groebner_basis('libsingular:slimgb')
  [1]
  ```

SINGULAR’s `interred()` command.

INPUT:
- I – a Sage ideal

EXAMPLES:

```sage
P.<x,y,z> = PolynomialRing(ZZ)
I = ideal( x^2 - 3*y, y^3 - x*y, z^3 - x, x^4 - y*z + 1 )
I.interreduced_basis()
[y^3 - x, x^4 - y*z + 1]
```

```sage
P.<x,y,z> = PolynomialRing(QQ)
I = ideal( x^2 - 3*y, y^3 - x*y, z^3 - x, x^4 - y*z + 1 )
I.interreduced_basis()
[y^3 - x, x^4 - 3*y, y^2 - 1/9*y*z + 1/9]
```

SINGULAR’s `kbase()` command.

INPUT:
- I – a Sage ideal

EXAMPLES:

```sage
P.<x,y,z> = PolynomialRing(ZZ)
P.<x,y,z> = PolynomialRing(QQ)
I = ideal( x^2 - 3*y, y^3 - x*y, z^3 - x, x^4 - y*z + 1 )
P.<x,y,z> = PolynomialRing(QQ)
I = ideal( x^2 - 3*y, y^3 - x*y, z^3 - x, x^4 - y*z + 1 )
P.<x,y,z> = PolynomialRing(QQ)
I = ideal( x^2 - 3*y, y^3 - x*y, z^3 - x, x^4 - y*z + 1 )
```

```sage
sage.rings.polynomial.multi_polynomial_ideal_libsingular.interreduced_libsingular(I)
```

```sage
sage.rings.polynomial.multi_polynomial_ideal_libsingular.kbase_libsingular(I,
  degree=None)
```
SINGULAR's \texttt{kbase()} algorithm.

**INPUT:**
- \( I \) – a groebner basis of an ideal
- \( \text{degree} \) – integer (default: None); if not None, return only the monomials of the given degree

**OUTPUT:**
Computes a vector space basis (consisting of monomials) of the quotient ring by the ideal, resp. of a free module by the module, in case it is finite dimensional and if the input is a standard basis with respect to the ring ordering. If the input is not a standard basis, the leading terms of the input are used and the result may have no meaning.

With two arguments: computes the part of a vector space basis of the respective quotient with degree of the monomials equal to the second argument. Here, the quotient does not need to be finite dimensional.

**EXAMPLES:**

```python
sage: R.<x,y> = PolynomialRing(QQ, order='lex')
sage: I = R.ideal(x^2-2*y^2, x*y-3)
sage: I.normal_basis()  # indirect doctest
[y^3, y^2, y, 1]
sage: J = R.ideal(x^2-2*y^2)
sage: [J.normal_basis(d) for d in (0..4)]  # indirect doctest
[[1], [y, x], [y^2, x*y], [y^3, x*y^2], [y^4, x*y^3]]
```

SINGULAR's \texttt{slimgb()}/\texttt{slimgb()} algorithm.

**INPUT:**
- \( I \) – a Sage ideal

SINGULAR's \texttt{std()} algorithm.

**INPUT:**
- \( I \) – a Sage ideal

### 3.1.10 Solution of polynomial systems using msolve

msolve is a multivariate polynomial system solver based on Gröbner bases.

This module provide implementations of some operations on polynomial ideals based on msolve.

Note that the optional package msolve must be installed.

See also:
- \texttt{sage.features.msolve}
- \texttt{sage.rings.polynomial.multi_polynomial_ideal}\texttt{\_libsingular.groebner\_basis\_degrevlex(ideal, proof=True)}

Compute a degrevlex Gröbner basis using msolve

**EXAMPLES:**
sage: from sage.rings.polynomial.msolve import groebner_basis_degrevlex

sage: R.<a,b,c> = PolynomialRing(GF(101), 3, order='lex')
sage: I = sage.rings.ideal.Katsura(R,3)
sage: gb = groebner_basis_degrevlex(I); gb  # optional - msolve
[a + 2*b + 2*c - 1, b*c - 19*c^2 + 10*b + 40*c,
b^2 - 41*c^2 + 20*b - 20*c, c^3 + 28*c^2 - 37*b + 13*c]
sage: gb.universe() is R  # optional - msolve
False
sage: gb.universe().term_order()  # optional - msolve
Degree reverse lexicographic term order
sage: ideal(gb).transformed_basis(other_ring=R)  # optional - msolve
[c^4 + 38*c^3 - 6*c^2 - 6*c, 30*c^3 + 32*c^2 + b - 14*c,
a + 2*b + 2*c - 1]

Gröbner bases over the rationals require \texttt{proof = False}:

sage: R.<x, y> = PolynomialRing(QQ, 2)
sage: I = Ideal([x*y - 1, (x-2)^2 + (y-1)^2 - 1])
sage: I.groebner_basis(algorithm='msolve')  # optional - msolve
Traceback (most recent call last):
  ...
ValueError: msolve relies on heuristics; please use proof=False

sage: I.groebner_basis(algorithm='msolve', proof=False)  # optional - msolve
[x*y - 1, x^2 + y^2 - 4*x - 2*y + 4, y^3 - 2*y^2 + x + 4*y - 4]

\texttt{sage.rings.polynomial.msolve.variety(ideal, ring, proof)}

Compute the variety of a zero-dimensional ideal using msolve.

Part of the initial implementation was loosely based on the example interfaces available as part of msolve, with the authors’ permission.

EXAMPLES:

sage: from sage.rings.polynomial.msolve import variety
sage: p = 536870909
sage: R.<x, y> = PolynomialRing(GF(p), 2, order='lex')
sage: I = Ideal([x*y - 1, (x-2)^2 + (y-1)^2 - 1])
sage: sorted(variety(I, GF(p^2), proof=False), key=str)  # optional - msolve
[[x: 1, y: 1],
 {x: 254228855*z2 + 114981228, y: 23249571*z2 + 402714189},
 {x: 267525699, y: 473946006},
 {x: 282642054*z2 + 154363985, y: 304421338*z2 + 197081624}]

3.1.11 Generic data structures for multivariate polynomials

This module provides an implementation of a generic data structure \texttt{PolyDict} and the underlying arithmetic for multi-variate polynomial rings. It uses an sparse representation of polynomials encoded as a Python dictionary where keys are exponents and values coefficients.

\{(e1,...,er):c1,...\} \leftrightarrow c1*x1^{e1}...*xr^{er}+....

The exponent \((e1,...,er)\) in this representation is an instance of the class \texttt{ETuple}.

AUTHORS:

- William Stein
• David Joyner
• Martin Albrecht (ETuple)
• Joel B. Mohler (2008-03-17) – ETuple rewrite as sparse C array

class sage.rings.polynomial.polydict.ETuple
    Bases: object

Representation of the exponents of a polydict monomial. If (0,0,3,0,5) is the exponent tuple of x_2^3*x_4^5 then this class only stores {2:3, 4:5} instead of the full tuple. This sparse information may be obtained by provided methods.

The index/value data is all stored in the _data C int array member variable. For the example above, the C array would contain 2,3,4,5. The indices are interlaced with the values.

This data structure is very nice to work with for some functions implemented in this class, but tricky for others. One reason that I really like the format is that it requires a single memory allocation for all of the values. A hash table would require more allocations and presumably be slower. I didn’t benchmark this question (although, there is no question that this is much faster than the prior use of python dicts).

combine_to_positives (other)
    Given a pair of ETuples (self, other), returns a triple of ETuples (a, b, c) so that self = a + b, other = a + c and b and c have all positive entries.

    EXAMPLES:

    sage: from sage.rings.polynomial.polydict import ETuple
    sage: e = ETuple([-2, 1, -5, 3, 1, 0])
    sage: f = ETuple([1, -3, -3, 4, 0, 2])
    sage: e.combine_to_positives(f)
    ((-2, -3, -5, 3, 0, 0), (0, 4, 0, 0, 1, 0), (3, 0, 2, 1, 0, 2))

common_nonzero_positions (other, sort=False)
    Returns an optionally sorted list of non zero positions either in self or other, i.e. the only positions that need to be considered for any vector operation.

    EXAMPLES:

    sage: from sage.rings.polynomial.polydict import ETuple
    sage: e = ETuple([1, 0, 2])
    sage: f = ETuple([0, 0, 1])
    sage: e.common_nonzero_positions(f)
    {0, 2}
    sage: e.common_nonzero_positions(f, sort=True)
    [0, 2]

divide_by_gcd (other)
    Return self / gcd(self, other).

    The entries of the result are the maximum of 0 and the difference of the corresponding entries of self and other.

divide_by_var (pos)
    Return division of self by the variable with index pos.

    If self[pos] == 0 then a ArithmeticError is raised. Otherwise, an ETuple is returned that is zero in position pos and coincides with self in the other positions.

    EXAMPLES:
from sage.rings.polynomial.polydict import ETuple

e = ETuple([1, 2, 0, 1])

e.divide_by_var(0)
(0, 2, 0, 1)

e.divide_by_var(1)
(1, 1, 0, 1)

e.divide_by_var(3)
(1, 2, 0, 0)

e.divide_by_var(2)
Traceback (most recent call last):
...
ArithmeticError: not divisible by this variable

divides (other)

Return whether self divides other, i.e., no entry of self exceeds that of other.

EXAMPLES:

sage: ETuple([1, 1, 0, 1, 0]).divides(ETuple([2, 2, 2, 2, 2]))
True

sage: ETuple([0, 3, 0, 1, 0]).divides(ETuple([2, 2, 2, 2, 2]))
False

sage: ETuple([0, 3, 0, 1, 0]).divides(ETuple([0, 3, 2, 2, 2]))
True

sage: ETuple([0, 0, 0, 0, 0]).divides(ETuple([2, 2, 2, 2, 2]))
True

False

True

dotprod (other)

Return the dot product of this tuple by other.

EXAMPLES:

sage: e = ETuple([1, 0, 2])
sage: f = ETuple([0, -2, 1])
sage: e.dotprod(f)
2

sage: e = ETuple([1, 1, -1])
sage: f = ETuple([0, -2, 1])
sage: e.dotprod(f)
-3

eadd (other)

Return the vector addition of self with other.

EXAMPLES:

sage: e = ETuple([1, 0, 2])

(continues on next page)
sage: f = ETuple([0, 1, 1])
sage: e.eadd(f)
(1, 1, 3)

Verify that github issue #6428 has been addressed:

sage: # needs sage.libs.singular
sage: R.<y, z> = Frac(QQ['x'])[]
sage: type(y)
<class 'sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular'>
sage: y^(2^32)
Traceback (most recent call last):
... OverflowError: exponent overflow (...) # 64-bit
OverflowError: Python int too large to convert to C unsigned long # 32-bit

eadd_p(other, pos)

Add other to self at position pos.

EXAMPLES:

sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 0, 2])
sage: e.eadd_p(5, 1)
(1, 5, 2)
sage: e = ETuple([0]*7)
sage: e.eadd_p(5, 4)
(0, 0, 0, 0, 5, 0, 0)
sage: ETuple([0,1]).eadd_p(1, 0) == ETuple([1,1])
True
sage: e = ETuple([0, 1, 0])
sage: e.eadd_p(0, 0).nonzero_positions()
[1]
sage: e.eadd_p(0, 1).nonzero_positions()
[1]
sage: e.eadd_p(0, 2).nonzero_positions()
[1]

eadd_scaled(other, scalar)

Vector addition of self with scalar * other.

EXAMPLES:

sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 0, 2])
sage: f = ETuple([0, 1, 1])
sage: e.eadd_scaled(f, 3)
(1, 3, 5)

emax(other)

Vector of maximum of components of self and other.

EXAMPLES:
emin \( \text{(other)} \)

Vector of minimum of components of \text{self} and \text{other}.

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 0, 2])
sage: f = ETuple([0, 1, 1])
... e.emin(f)
(0, 0, 1)
sage: e = ETuple([1, 0, -1])
... f = ETuple([0, -2, 1])
... e.emin(f)
(0, -2, -1)
```

emul \( \text{(factor)} \)

Scalar Vector multiplication of \text{self}.

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 0, 2])
... e.emul(2)
(2, 0, 4)
```

escalar_div \( n \)

Divide each exponent by \( n \).

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.polydict import ETuple
sage: ETuple([1, 0, 2]).escalar_div(2)
(0, 0, 1)
sage: ETuple([0, 3, 12]).escalar_div(3)
(0, 1, 4)
sage: ETuple([1, 5, 2]).escalar_div(0)
Traceback (most recent call last):
  ...
ZeroDivisionError
```
**esub** (other)

Vector subtraction of `self` with `other`.

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 0, 2])
sage: f = ETuple([0, 1, 1])
sage: e.esub(f)
(1, -1, 1)
```

**is_constant** ()

Return if all exponents are zero in the tuple.

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 0, 2])
sage: e.is_constant()  # False
sage: e = ETuple([0, 0])
sage: e.is_constant()  # True
```

**is_multiple_of** (n)

Test whether each entry is a multiple of `n`.

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.polydict import ETuple
sage: ETuple([0, 0]).is_multiple_of(3)  # True
sage: ETuple([0, 3, 12, 0, 6]).is_multiple_of(3)  # True
sage: ETuple([0, 0, 2]).is_multiple_of(3)  # False
```

**nonzero_positions** (sort=False)

Return the positions of non-zero exponents in the tuple.

**INPUT:**

- `sort` — (default: False) if True a sorted list is returned; if False an unsorted list is returned

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 0, 2])
sage: e.nonzero_positions()  # [0, 2]
```

**nonzero_values** (sort=True)

Return the non-zero values of the tuple.

**INPUT:**

- `sort` — (default: True) if True the values are sorted by their indices; otherwise the values are returned unsorted
EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([2, 0, 1])
sage: e.nonzero_values()
[2, 1]
sage: f = ETuple([0, -1, 1])
sage: f.nonzero_values(sort=True)
[-1, 1]
```

reversed()  
Return the reversed ETuple of self.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 2, 3])
sage: e.reversed()
(3, 2, 1)
```

sparse_iter()  
Iterator over the elements of self where the elements are returned as (i, e) where i is the position of e in the tuple.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 0, 2, 0, 3])
sage: list(e.sparse_iter())
[(0, 1), (2, 2), (4, 3)]
```

unweighted_degree()  
Return the sum of entries.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: ETuple([1, 1, 0, 2, 0]).unweighted_degree()
4
sage: ETuple([-1, 1]).unweighted_degree()
0
```

unweighted_quotient_degree(other)  
Return the degree of self divided by its gcd with other.
It amounts to counting the non-negative entries of self.esub(other).

weighted_degree(w)  
Return the weighted sum of entries.

INPUT:

• w – tuple of non-negative integers

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple
sage: e = ETuple([1, 1, 0, 2, 0])
sage: e.weighted_degree((1, 2, 3, 4, 5))
(continues on next page)
```
weighted_quotient_degree \( (\text{other}, w) \)

Return the weighted degree of \( \text{self} \) divided by its gcd with \( \text{other} \).

INPUT:

- \( \text{other} \) – an \( \text{ETuple} \)
- \( w \) – tuple of non-negative integers.

class \( \text{sage.rings.polynomial.polydict.PolyDict} \)

Bases: \text{object}

Data structure for multivariate polynomials.

A \( \text{PolyDict} \) holds a dictionary all of whose keys are \( \text{ETuple} \) and whose values are coefficients on which it is implicitly assumed that arithmetic operations can be performed.

No arithmetic operation on \( \text{PolyDict} \) clear zero coefficients as of now there is no reliable way of testing it in the most general setting, see \text{github} issue \#35319. For removing zero coefficients from a \( \text{PolyDict} \) you can use the method \( \text{remove_zeros()} \) which can be parametrized by a zero test.

apply_map \( (f) \)

Apply the map \( f \) on the coefficients (inplace).

EXAMPLES:

```python
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(1, 0): 1, (1, 1): -2})
sage: f.apply_map(lambda x: x^2)
sage: f
PolyDict with representation {(1, 0): 1, (1, 1): 4}
```

coefficient \( (\text{mon}) \)

Return a polydict that defines a polynomial in 1 less number of variables that gives the coefficient of \( \text{mon} \) in this polynomial.

The coefficient is defined as follows. If \( f \) is this polynomial, then the coefficient is the sum \( T/\text{mon} \) where the sum is over terms \( T \) in \( f \) that are exactly divisible by \( \text{mon} \).

coefficients()\n
Return the coefficients of \( \text{self} \).

EXAMPLES:

```python
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: sorted(f.coefficients())
[(2, 3, 4)]
```
coerce_coefficients(A)

Coerce the coefficients in the parent A

EXAMPLES:

```python
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 0})
sage: f
PolyDict with representation {(2, 3): 0}
sage: f.coerce_coefficients(QQ)
doctest:warning
...  
DeprecationWarning: coerce_coefficients is deprecated; use apply_map instead
See https://github.com/sagemath/sage/issues/34000 for details.
sage: f
PolyDict with representation {(2, 3): 0}
```

degree(x=None)

Return the total degree or the maximum degree in the variable x.

EXAMPLES:

```python
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.degree()
5
sage: f.degree(PolyDict({(1, 0): 1}))
2
sage: f.degree(PolyDict({(0, 1): 1}))
3
```

derivative(x)

Return the derivative of self with respect to x

EXAMPLES:

```python
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.derivative(PolyDict({(1, 0): 1}))
PolyDict with representation {(0, 2): 3, (1, 1): 8, (2, 0): 4, (2, 2): 6}
sage: f.derivative(PolyDict({(0, 1): 1}))
PolyDict with representation {(1, 1): 6, (2, 0): 4, (2, 2): 6}
sage: PolyDict({(-1,): 1}).derivative(PolyDict({(1,): 1}))
PolyDict with representation {(-2,): -1}
sage: PolyDict({(-2,): 1}).derivative(PolyDict({(1,): 1}))
PolyDict with representation {(-3,): -2}
sage: PolyDict({}).derivative(PolyDict({(1, 1): 1}))
Traceback (most recent call last):
...  
ValueError: x must be a generator
```

derivative_i(i)

Return the derivative of self with respect to the i-th variable.

EXAMPLES:
Polynomials, Release 10.3

```python
sage: from sage.rings.polynomial.polydict import PolyDict
sage: PolyDict({(1, 1): 1}).derivative_i(0)
PolyDict with representation {(0, 1): 1}
```

**dict()**

Return a copy of the dict that defines self.

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.dict()
{(1, 2): 3, (2, 1): 4, (2, 3): 2}
```

**exponents()**

Return the exponents of self.

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: sorted(f.exponents())
[(1, 2), (2, 1), (2, 3)]
```

**get(e, default=None)**

Return the coefficient of the ETuple e if present and default otherwise.

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.polydict import PolyDict, ETuple
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.get(ETuple([1,2]))
3
sage: f.get(ETuple([1,1]), 'hello')
'hello'
```

**homogenize(var)**

Return the homogeneization of self by increasing the degree of the variable var.

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(0, 0): 1, (2, 1): 3, (1, 1): 5})
sage: f.homogenize(0)
PolyDict with representation {(2, 1): 8, (3, 0): 1}
sage: f.homogenize(1)
PolyDict with representation {(0, 3): 1, (1, 2): 5, (2, 1): 3}
sage: PolyDict({(0, 1): 1, (1, 1): -1}).homogenize(0)
PolyDict with representation {(1, 1): 0}
```

**integral(x)**

Return the integral of self with respect to x

**EXAMPLES:**
```python
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.integral(PolyDict({(1, 0): 1}))
PolyDict with representation {(1, 3): 1, (2, 2): 2, (2, 4): 1/2}
sage: PolyDict({(-1,): 1}).integral(PolyDict({(1,): 1}))
Traceback (most recent call last):
... ArithmeticError: integral of monomial with exponent -1
sage: PolyDict({(-2,): 1}).integral(PolyDict({(1,): 1}))
PolyDict with representation {(-1,): -1}
sage: PolyDict({}).integral(PolyDict({(1, 1): 1}))
Traceback (most recent call last):
... ValueError: x must be a generator
```

**integral** 

Return the derivative of self with respect to the i-th variable.

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.polydict import PolyDict
sage: PolyDict({(1, 1): 1}).integral_i(0)
PolyDict with representation {(2, 1): 1/2}
```

**is_constant**

Return whether this polynomial is constant.

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.polydict import PolyDict
sage: PolyDict({}).is_constant()
True
sage: PolyDict({(1, 2): 1, (0, 3): -2}).is_constant()
True
sage: PolyDict({(1, 0): 1, (1, 2): 3}).is_constant()
False
```

**is_homogeneous**

Return whether this polynomial is homogeneous.

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.polydict import PolyDict
sage: PolyDict().is_homogeneous()  # True
sage: PolyDict({(1, 2): 1, (0, 3): -2}).is_homogeneous()  # True
sage: PolyDict({(1, 0): 1, (1, 2): 3}).is_homogeneous()  # False
```

**latex** *(vars, atomic_exponents=True, atomic_coefficients=True, sortkey=None)*

Return a nice polynomial latex representation of this PolyDict, where the vars are substituted in.
INPUT:

- `vars` – list
- `atomic_exponents` – bool (default: True)
- `atomic_coefficients` – bool (default: True)

EXAMPLES:

```python
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.latex([a, WW])
'2 a^{2} WW^{3} + 4 a^{2} WW + 3 a WW^{2}'
```

`lcmt(greater_etuple)`

Provides functionality of lc, lm, and lt by calling the tuple compare function on the provided term order T.

INPUT:

- `greater_etuple` – a term order

EXAMPLES:

```python
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: sorted(f.list())
[[2, [2, 3]], [3, [1, 2]], [4, [2, 1]]]
```

`max_exp()`

Returns an ETuple containing the maximum exponents appearing. If there are no terms at all in the PolyDict, it returns None.

The nvars parameter is necessary because a PolyDict doesn’t know it from the data it has (and an empty PolyDict offers no clues).

EXAMPLES:

```python
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.max_exp()
(2, 3)
sage: PolyDict({}).max_exp() # returns None
```

`min_exp()`

Returns an ETuple containing the minimum exponents appearing. If there are no terms at all in the PolyDict, it returns None.

The nvars parameter is necessary because a PolyDict doesn’t know it from the data it has (and an empty PolyDict offers no clues).

EXAMPLES:

```python
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.min_exp()
(1, 1)
sage: PolyDict({}).min_exp() # returns None
```
**monomial_coefficient** *(mon)*

Return the coefficient of the monomial *mon*.

**INPUT:**

- *mon* – a PolyDict with a single key

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.monomial_coefficient(PolyDict({(2, 1): 1}).dict())
doctest:warning...
DeprecationWarning: PolyDict.monomial_coefficient is deprecated; use PolyDict.get instead
See https://github.com/sagemath/sage/issues/34000 for details.
```

**poly_repr** *(vars, atomic_exponents=True, atomic_coefficients=True, sortkey=None)*

Return a nice polynomial string representation of this PolyDict, where the vars are substituted in.

**INPUT:**

- *vars* – list
- *atomic_exponents* – bool (default: True)
- *atomic_coefficients* – bool (default: True)

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.poly_repr(['a', 'WW'])
2*a^2*WW^3 + 4*a^2*WW + 3*a*WW^2
```

We check to make sure that when we are in characteristic two, we don’t put negative signs on the generators.

```python
sage: Integers(2)['x, y'].gens()
(x, y)
```

We make sure that intervals are correctly represented.

```python
sage: f = PolyDict({(2, 3): RIF(1/2, 3/2), (1, 2): RIF(-1, 1)})  # needs sage.rings.real_interval_field
sage: f.poly_repr(['x', 'y'])  # needs sage.rings.real_interval_field
'1.?*x^2*y^3 + 0.?*x*y^2'
```

**polynomial_coefficient** *(degrees)*

Return a polydict that defines the coefficient in the current polynomial viewed as a tower of polynomial extensions.

**INPUT:**

- *degrees* – a list of degree restrictions; list elements are None if the variable in that position should be unrestricted

**EXAMPLES:**

```python
sage: f = PolyDict({(2, 3): RIF(1/2, 3/2), (1, 2): RIF(-1, 1)})  # needs sage.rings.real_interval_field
sage: f.poly_coefficient([None, 2])  # needs sage.rings.real_interval_field
'1.?*x^2*y^3 + 0.?*x*y^2'
```
from sage.rings.polynomial.polydict import PolyDict

f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
f.polynomial_coefficient([2, None])  # PolyDict with representation {(0, 1): 4, (0, 3): 2}
f = PolyDict({(0, 3): 2, (0, 2): 3, (2, 1): 4})
f.polynomial_coefficient([0, None])  # PolyDict with representation {(0, 2): 3, (0, 3): 2}

remove_zeros (zero_test=None)
Remove the entries with zero coefficients.

INPUT:
• zero_test – optional function that performs test to zero of a coefficient

EXAMPLES:
from sage.rings.polynomial.polydict import PolyDict
f = PolyDict({(0, 3): 0})
f.remove_zeros()  # PolyDict with representation {}

The following example shows how to remove only exact zeros from a PolyDict containing univariate power series:
R.<t> = PowerSeriesRing(QQ)
f = PolyDict({(1, 1): O(t), (1, 0): R.zero()})
f.remove_zeros(lambda s: s.is_zero() and s.prec() is Infinity)  # PolyDict with representation {(1, 1): O(t^1)}

rich_compare (other, op, sortkey=None)
Compare two PolyDicts using specified term ordering "sortkey".

EXAMPLES:
from sage.rings.polynomial.polydict import PolyDict
from sage.structure.richcmp import op_EQ, op_NE, op_LT
p1 = PolyDict({(0,): 1})  # PolyDict with representation {(0,): 1})
p2 = PolyDict({(0,): 2})  # PolyDict with representation {(0,): 2})
O = TermOrder()
p1.rich_compare(PolyDict({(0,): 1}), op_EQ, O.sortkey)  # True
p1.rich_compare(p2, op_EQ, O.sortkey)  # False
p1.rich_compare(p2, op_NE, O.sortkey)  # True
p1.rich_compare(p2, op_LT, O.sortkey)  # True
p3 = PolyDict({(3, 2, 4): 1, (3, 2, 5): 2})
p4 = PolyDict({(3, 2, 4): 1, (3, 2, 3): 2})
p3.rich_compare(p4, op_LT, O.sortkey)  # False
scalar_lmult\((s)\)

Return the left scalar multiplication of self by s.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: x, y = FreeMonoid(2, 'x, y').gens()  # a strange object to live in a
˓→polydict, but non-commutative!  # needs sage.combinat
sage: f = PolyDict({(2,3):x})  # needs sage.combinat
˓→needs sage.combinat
sage: f.scalar_lmult(y)  # needs sage.combinat
PolyDict with representation {(2, 3): y*x}
```

```
sage: f = PolyDict({(2,3):2, (1,2):3, (2,1):4})
sage: f.scalar_lmult(-2)
PolyDict with representation {(1, 2): -6, (2, 1): -8, (2, 3): -4}
sage: f.scalar_lmult(RIF(-1,1))  # needs sage.rings.real_interval_field
PolyDict with representation {(1, 2): 0.?e1, (2, 1): 0.?e1, (2, 3): 0.?e1}
```

scalar_rmult\((s)\)

Return the right scalar multiplication of self by s.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict
sage: x, y = FreeMonoid(2, 'x, y').gens()  # a strange object to live in a
˓→polydict, but non-commutative!  # needs sage.combinat
sage: f = PolyDict({(2,3):x})  # needs sage.combinat
˓→needs sage.combinat
sage: f.scalar_rmult(y)  # needs sage.combinat
PolyDict with representation {(2, 3): x*y}
```

```
sage: f = PolyDict({(2,3):2, (1,2):3, (2,1):4})
sage: f.scalar_rmult(-2)
PolyDict with representation {(1, 2): -6, (2, 1): -8, (2, 3): -4}
sage: f.scalar_rmult(RIF(-1,1))  # needs sage.rings.real_interval_field
PolyDict with representation {(1, 2): 0.?e1, (2, 1): 0.?e1, (2, 3): 0.?e1}
```

term_lmult\((\text{exponent}, s)\)

Return this element multiplied by s on the left and with exponents shifted by exponent.

INPUT:

- exponent – a ETuple
- s – a scalar

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import ETuple, PolyDict
sage: x, y = FreeMonoid(2, 'x, y').gens()  # a strange object to live in a
˓→polydict, but non-commutative!  # needs sage.combinat
```

(continues on next page)
term_rmult (exponent, s)

Return this element multiplied by \( s \) on the right and with exponents shifted by \( \text{exponent} \).

INPUT:

- \( \text{exponent} \) – a ETuple
- \( s \) – a scalar

EXAMPLES:

```python
sage: from sage.rings.polynomial.polydict import ETuple, PolyDict
sage: x, y = FreeMonoid(2, 'x, y').gens()  # a strange object to live in a...
               # polydict, but non-commutative!  # needs sage.combinat
sage: f = PolyDict({(2, 3): x})          #...
               # needs sage.combinat
sage: f.term_rmult(ETuple((1, 2)), y)    #...
               # needs sage.combinat
PolyDict with representation {(3, 5): x*y}
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.term_rmult(ETuple((1, 2)), -2)
PolyDict with representation {(2, 4): -6, (3, 3): -8, (3, 5): -4}
```

total_degree \( (w=\text{None}) \)

Return the total degree.

INPUT:

- \( w \) – (optional) a tuple of weights

EXAMPLES:

```python
sage: from sage.rings.polynomial.polydict import PolyDict
sage: f = PolyDict({(2, 3): 2, (1, 2): 3, (2, 1): 4})
sage: f.total_degree()
5
sage: f.total_degree((3, 1))
9
sage: PolyDict({}).degree()
-1
```

sage.rings.polynomial.polydict.gen_index(x)

Return the index of the variable represented by \( x \) or \(-1\) if \( x \) is not a monomial of degree one.

EXAMPLES:
sage: from sage.rings.polynomial.polydict import PolyDict, gen_index
sage: gen_index(PolyDict({(1, 0): 1}))
0
sage: gen_index(PolyDict({(0, 1): 1}))
1
sage: gen_index(PolyDict({}))
-1

sage.rings.polynomial.polydict.make_ETuple(data, length)
Ensure support for pickled data from older sage versions.

sage.rings.polynomial.polydict.make_PolyDict(data)
Ensure support for pickled data from older sage versions.

sage.rings.polynomial.polydict.monomial_exponent(p)
Return the unique exponent of \( p \) if it is a monomial or raise a ValueError.

EXAMPLES:

```
sage: from sage.rings.polynomial.polydict import PolyDict, monomial_exponent
sage: monomial_exponent(PolyDict({(2, 3): 1}))
(2, 3)
sage: monomial_exponent(PolyDict({(2, 3): 3}))
Traceback (most recent call last):
  ... ValueError: not a monomial
sage: monomial_exponent(PolyDict({(1, 0): 1, (0, 1): 1}))
Traceback (most recent call last):
  ... ValueError: not a monomial
```

3.1.12 Compute Hilbert series of monomial ideals

This implementation was provided at github issue #26243 and is supposed to be a way out when Singular fails with an int overflow, which will regularly be the case in any example with more than 34 variables.

```
class sage.rings.polynomial.hilbert.Node
    Bases: object
    A node of a binary tree

    It has slots for data that allow to recursively compute the first
    Hilbert series of a monomial ideal.
	sage.rings.polynomial.hilbert.first_hilbert_series(I, grading=None, return_grading=False)
    Return the first Hilbert series of the given monomial ideal.

    INPUT:
    
    • I – a monomial ideal (possibly defined in singular)
    • grading – (optional) a list or tuple of integers used as degree weights
    • return_grading – (default: False) whether to return the grading

    OUTPUT:

    A univariate polynomial, namely the first Hilbert function of \( I \), and if return_grading==True also the
    grading used to compute the series.

    EXAMPLES:
```
Polynomials, Release 10.3

```python
sage: from sage.rings.polynomial.hilbert import first_hilbert_series

sage: # needs sage.libs.singular
sage: R = singular.ring(0, '(x,y,z)', 'dp')

sage: I = singular.ideal(['x^2','y^2','z^2'])

sage: first_hilbert_series(I)
-t^6 + 3*t^4 - 3*t^2 + 1

sage: first_hilbert_series(I, return_grading=True)
(-t^6 + 3*t^4 - 3*t^2 + 1, (1, 1, 1))

sage: first_hilbert_series(I, grading=(1,2,3))
-t^12 + t^10 + t^8 - t^4 - t^2 + 1

sage.rings.polynomial.hilbert.hilbert_poincare_series(I, grading=None)

Return the Hilbert Poincaré series of the given monomial ideal.

INPUT:

- I – a monomial ideal (possibly defined in Singular)
- grading – (optional) a tuple of degree weights

EXAMPLES:

```python
sage: from sage.rings.polynomial.hilbert import hilbert_poincare_series

sage: R = PolynomialRing(QQ, x, 9)

sage: I = [m.lm() for m in (matrix(R, 3, R.gens())^2).list() * R].groebner_basis() * R

sage: hilbert_poincare_series(I)

(t^7 - 3*t^6 + 2*t^5 + 2*t^4 - 2*t^3 + 6*t^2 + 5*t + 1)/(t^4 - 4*t^3 + 6*t^2 - 4*t + 1)

sage: hilbert_poincare_series((R * R.gens())^2, grading=range(1,10))
t^9 + t^8 + t^7 + t^6 + t^5 + t^4 + t^3 + t^2 + t + 1

The following example is taken from github issue #20145:

```python
sage: from sage.rings.polynomial.hilbert import first_hilbert_series

sage: I = P.ideal(M.minors(2))

sage: J = P * [m.lm() for m in I.groebner_basis()]

sage: hilbert_poincare_series(J).numerator()
120*t^3 + 135*t^2 + 30*t + 1

sage: hilbert_poincare_series(J).denominator().factor()
(t - 1)^14
```

3.1.13 Class to flatten polynomial rings over polynomial ring

For example `QQ[['a', 'b'], ['x', 'y']]` flattens to `QQ[['a', 'b', 'x', 'y']]`.

EXAMPLES:

```python
sage: R = QQ[['x']][['y']][['s','t']]['X']

sage: from sage.rings.polynomial.flatten import FlatteningMorphism

sage: phi = FlatteningMorphism(R); phi
Flattening morphism:
  From: Univariate Polynomial Ring in X
  To:   Multivariate Polynomial Ring in x, y, s, t over Univariate Polynomial Ring in X over Rational Field

(continues on next page)
```
over Multivariate Polynomial Ring in s, t
over Univariate Polynomial Ring in x over Rational Field
To: Multivariate Polynomial Ring in x, y, s, t, X over Rational Field
sage: phi('x*y*s + t*X').parent()
Multivariate Polynomial Ring in x, y, s, t, X over Rational Field

Authors:
Vincent Delecroix, Ben Hutz (July 2016): initial implementation

```python
class sage.rings.polynomial.flatten.FlatteningMorphism(domain):
    Bases: Morphism

    EXAMPLES:

    sage: R = QQ['a','b'][['x','y'],['z','t1','t2']]
    sage: from sage.rings.polynomial.flatten import FlatteningMorphism
    sage: f = FlatteningMorphism(R)
    sage: f.codomain()
    Multivariate Polynomial Ring in a, b, x, y, z, t1, t2 over Rational Field
    sage: p = R('(a+b)*x + (a^2-b)*t2*(z+y)')
    sage: f(p)
    a^2*y*t2 + a^2*z*t2 - b*y*t2 - b*z*t2 + a*x + b*x
    sage: f(p).parent()
    Multivariate Polynomial Ring in a, b, x, y, z, t1, t2 over Rational Field

    Also works when univariate polynomial ring are involved:

    sage: R = QQ['x']['y']['s','t']['X']
    sage: from sage.rings.polynomial.flatten import FlatteningMorphism
    sage: f = FlatteningMorphism(R)
    sage: f.codomain()
    Multivariate Polynomial Ring in x, y, s, t, X over Rational Field
    sage: p = R('((x^2 + 1) + (x+2)*y + x*y^3)*(s+t) + x*y*X')
    sage: f(p)
    x*y^3*s + x*y^3*t + x^2*s + x*y*s + x^2*t + x*y*t + x*y*X + 2*y*s + 2*y*t + s + t
    sage: f(p).parent()
    Multivariate Polynomial Ring in x, y, s, t, X over Rational Field

    inverse()
    Return the inverse of this flattening morphism.
    This is the same as calling section().

    EXAMPLES:

    sage: f = QQ['x','y']['u','v'].flattening_morphism()
    sage: f.inverse()
    Unflattening morphism:
    From: Multivariate Polynomial Ring in x, y, u, v over Rational Field
    To: Multivariate Polynomial Ring in u, v over Multivariate Polynomial Ring in x, y over Rational Field
```

3.1. Multivariate Polynomials and Polynomial Rings
section()

Inverse of this flattening morphism.

EXAMPLES:

```python
sage: R = QQ['a','b','c'][['x','y','z']]
sage: from sage.rings.polynomial.flatten import FlatteningMorphism
sage: h = FlatteningMorphism(R)
sage: h.section()
Unflattening morphism:
From: Multivariate Polynomial Ring in a, b, c, x, y, z over Rational Field
To: Multivariate Polynomial Ring in x, y, z over Multivariate Polynomial Ring in a, b, c over Rational Field
```

```python
sage: R = ZZ['a']['b']['c']
sage: from sage.rings.polynomial.flatten import FlatteningMorphism
sage: FlatteningMorphism(R).section()
Unflattening morphism:
From: Multivariate Polynomial Ring in a, b, c over Integer Ring
To: Univariate Polynomial Ring in c over Univariate Polynomial Ring in b over Univariate Polynomial Ring in a over Integer Ring
```

class sage.rings.polynomial.flatten.FractionSpecializationMorphism(domain, D)

Bases: Morphism

A specialization morphism for fraction fields over (stacked) polynomial rings

class sage.rings.polynomial.flatten.SpecializationMorphism(domain, D)

Bases: Morphism

Morphisms to specialize parameters in (stacked) polynomial rings

EXAMPLES:

```python
sage: R.<c> = PolynomialRing(QQ)
sage: S.<x,y,z> = PolynomialRing(R)
sage: D = dict({c:1})
sage: from sage.rings.polynomial.flatten import SpecializationMorphism
sage: f = SpecializationMorphism(S, D)
sage: g = f(x^2 + c*y^2 - z^2); g
x^2 + y^2 - z^2
sage: g.parent()
Multivariate Polynomial Ring in x, y, z over Rational Field
```

```python
sage: R.<c> = PolynomialRing(QQ)
sage: S.<z> = PolynomialRing(R)
sage: from sage.rings.polynomial.flatten import SpecializationMorphism
sage: xi = SpecializationMorphism(S, {c:0}); xi
Specialization morphism:
From: Univariate Polynomial Ring in z over Univariate Polynomial Ring in c over Rational Field
To: Univariate Polynomial Ring in z over Rational Field
sage: xi(z^2+c)
z^2
```

```python
sage: R1.<u,v> = PolynomialRing(QQ)
sage: R2.<a,b,c> = PolynomialRing(R1)
sage: S.<x,y,z> = PolynomialRing(R2)
```

(continues on next page)
### 3.1.14 Monomials

`sage.rings.monomials.monomials(v, n)`

Given two lists `v` and `n`, of exactly the same length, return all monomials in the elements of `v`, where variable `i` (i.e., `v[i]`) in the monomial appears to degree strictly less than `n[i].`

**INPUT:**
- `v` – list of ring elements
- `n` – list of integers

**EXAMPLES:**

```python
sage: monomials([x], [3])  # needs sage.symbolic
[1, x, x^2]
sage: R.<x,y,z> = QQ[]
```

(continues on next page)
3.2 Classical Invariant Theory

3.2.1 Classical Invariant Theory

This module lists classical invariants and covariants of homogeneous polynomials (also called algebraic forms) under the action of the special linear group. That is, we are dealing with polynomials of degree \(d\) in \(n\) variables. The special linear group \(SL(n, \mathbb{C})\) acts on the variables \((x_1, \ldots, x_n)\) linearly,

\[
(x_1, \ldots, x_n)^t \rightarrow A(x_1, \ldots, x_n)^t, \quad A \in SL(n, \mathbb{C})
\]

The linear action on the variables transforms a polynomial \(p\) generally into a different polynomial \(g p\). We can think of it as an action on the space of coefficients in \(p\). An invariant is a polynomial in the coefficients that is invariant under this action. A covariant is a polynomial in the coefficients and the variables \((x_1, \ldots, x_n)\) that is invariant under the combined action.

For example, the binary quadratic \(p(x, y) = ax^2 + bxy + cy^2\) has as its invariant the discriminant \(disc(p) = b^2 - 4ac\). This means that for any \(SL(2, \mathbb{C})\) coordinate change

\[
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \alpha \delta - \beta \gamma = 1
\]

the discriminant is invariant, \(disc(p(x', y')) = disc(p(x, y))\).

To use this module, you should use the factory object \textit{invariant\_theory}. For example, take the quartic:

\begin{verbatim}
sage: R.<x,y> = QQ[]
sage: q = x^4 + y^4
sage: quartic = invariant_theory.binary_quartic(q); quartic
Binary quartic with coefficients (1, 0, 0, 0, 1)
\end{verbatim}

One invariant of a quartic is known as the Eisenstein D-invariant. Since it is an invariant, it is a polynomial in the coefficients (which are integers in this example):

\begin{verbatim}
sage: quartic.EisensteinD()
1
\end{verbatim}

One example of a covariant of a quartic is the so-called g-covariant (actually, the Hessian). As with all covariants, it is a polynomial in \(x, y\) and the coefficients:

\begin{verbatim}
sage: quartic.g_covariant()
-x^2*y^2
\end{verbatim}

As usual, use tab completion and the online help to discover the implemented invariants and covariants.

In general, the variables of the defining polynomial cannot be guessed. For example, the zero polynomial can be thought of as a homogeneous polynomial of any degree. Also, since we also want to allow polynomial coefficients we cannot just take all variables of the polynomial ring as the variables of the form. This is why you will have to specify the variables explicitly if there is any potential ambiguity. For example:
Finally, it is often convenient to use inhomogeneous polynomials where it is understood that one wants to homogenize them. This is also supported, just define the form with an inhomogeneous polynomial and specify one less variable:

```
sage: R.<x,t> = QQ[]
sage: invariant_theory.binary_quartic(x^4 + 1 + t*x^2, [x])
```

```
Binary quartic with coefficients (1, 0, t, 0, 1)
```

REFERENCES:

- Wikipedia article Glossary_of.Invariant_theory

AUTHORS:

- Volker Braun (2013-01-24): initial version
- Jesper Noordsij (2018-05-18): support for binary quintics added

```python
class sage.rings.invariants.invariant_theory.AlgebraicForm(n, d, polynomial, *args, **kwds)
```

Bases: `FormsBase`

The base class of algebraic forms (i.e. homogeneous polynomials).

You should only instantiate the derived classes of this base class.

Derived classes must implement `coeffs()` and `scaled_coeffs()`

INPUT:

- `n` – The number of variables.
- `d` – The degree of the polynomial.
- `polynomial` – The polynomial.
- `*args` – The variables, as a single list/tuple, multiple arguments, or `None` to use all variables of the polynomial.

Derived classes must implement the same arguments for the constructor.

EXAMPLES:

```
sage: from sage.rings.invariants.invariant_theory import AlgebraicForm
sage: R.<x,y> = QQ[]
sage: p = x^2 + y^2
sage: AlgebraicForm(2, 2, p).variables()
(x, y)
sage: AlgebraicForm(2, 2, p, None).variables()
(x, y)
sage: AlgebraicForm(3, 2, p).variables()
(x, y, None)
sage: AlgebraicForm(3, 2, p, None).variables()
```

(continues on next page)
Polynomials, Release 10.3

(continued from previous page)

```python
sage: from sage.rings.invariants.invariant_theory import AlgebraicForm
sage: R.<x,y,s,t> = QQ[]
sage: p = s*x^2 + t*y^2
sage: AlgebraicForm(2, 2, p, [x,y]).variables()
(x, y)
sage: AlgebraicForm(2, 2, p, x,y).variables()
(x, y)
sage: AlgebraicForm(3, 2, p, [x,y,None]).variables()
(x, y, None)
sage: AlgebraicForm(3, 2, p, x,y,None).variables()
(x, y, None)
sage: AlgebraicForm(2, 1, p, [x,y]).variables()
Traceback (most recent call last):
  ... ValueError: polynomial is of the wrong degree
sage: AlgebraicForm(2, 2, x^2 + y, [x,y]).variables()
Traceback (most recent call last):
  ... ValueError: polynomial is not homogeneous
```

**coefficients()**

Alias for coeffxs().

See the documentation for coeffxs() for details.

**EXAMPLES:**

```python
sage: R.<a,b,c,d,e,f,g, x,y,z> = QQ[]
sage: p = a*x^2 + b*y^2 + c*z^2 + d*x*y + e*x*z + f*y*z
sage: q = invariant_theory.quadratic_form(p, x,y,z)
sage: q.coefficients()
(a, b, c, d, e, f)
sage: q.coeffs()
(a, b, c, d, e, f)
```

**form()**

Return the defining polynomial.

**OUTPUT:**

The polynomial used to define the algebraic form.

**EXAMPLES:**

```python
sage: R.<x,y> = QQ[]
sage: quartic = invariant_theory.binary_quartic(x^4 + y^4)
sage: quartic.form()
x^4 + y^4
sage: quartic.polynomial()
x^4 + y^4
```

**homogenized**(var='h')

Return form as defined by a homogeneous polynomial.
INPUT:
- \texttt{var} – either a variable name, variable index or a variable (default: \texttt{'h'}).

OUTPUT:
The same algebraic form, but defined by a homogeneous polynomial.

EXAMPLES:

\begin{verbatim}
sage: T.<t> = QQ[]
sage: quadratic = invariant_theory.binary_quadratic(t^2 + 2*t + 3)
sage: quadratic
Binary quadratic with coefficients (1, 3, 2)
sage: quadratic.homogenized()
Binary quadratic with coefficients (1, 3, 2)
sage: quadratic == quadratic.homogenized()
True
sage: quadratic.form()
t^2 + 2*t + 3
sage: quadratic.homogenized().form()
t^2 + 2*t*h + 3*h^2
sage: R.<x,y,z> = QQ[]
sage: quadratic = invariant_theory.ternary_quadratic(x^2 + 1, [x,y])
sage: quadratic.homogenized().form()
x^2 + h^2
sage: R.<x> = QQ[]
sage: quintic = invariant_theory.binary_quintic(x^4 + 1, x)
sage: quintic.homogenized().form()
x^4*h + h^5
\end{verbatim}

\texttt{polynomial()}  
Return the defining polynomial.

OUTPUT:
The polynomial used to define the algebraic form.

EXAMPLES:

\begin{verbatim}
sage: R.<x,y> = QQ[]
sage: quartic = invariant_theory.binary_quartic(x^4 + y^4)
sage: quartic.form()
x^4 + y^4
sage: quartic.polynomial()
x^4 + y^4
\end{verbatim}

\texttt{transformed(g)}  
Return the image under a linear transformation of the variables.

INPUT:
- \texttt{g} – a \texttt{GL(n,C)} matrix or a dictionary with the
  variables as keys. A matrix is used to define the linear transformation of homogeneous variables, a
dictionary acts by substitution of the variables.

OUTPUT:
A new instance of a subclass of \texttt{AlgebraicForm} obtained by replacing the variables of the homogeneous
polynomial by their image under \texttt{g}.
Polynomials, Release 10.3

EXAMPLES:

```python
sage: R.<x,y,z> = QQ[]
sage: cubic = invariant_theory.ternary_cubic(x^3 + 2*y^3 + 3*z^3 + 4*x*y*z)
sage: cubic.transformed({x: y, y: z, z: x}).form()
3*x^3 + y^3 + 4*x*y*z + 2*z^3
sage: cyc = matrix([[0,1,0], [0,0,1], [1,0,0]])
sage: cubic.transformed(cyc) == cubic.transformed({x:y, y:z, z:x})
True
sage: g = matrix(QQ, [[1, 0, 0], [-1, 1, -3], [-5, -5, 16]])
sage: cubic.transformed(g)
Ternary cubic with coefficients (-356, -373, 12234, -1119, 3578, -1151, 3582, -11766, -11466, 7360)
sage: cubic.transformed(g).transformed(g.inverse()) == cubic
True
```

```python
class sage.rings.invariants.invariant_theory.BinaryQuartic(n, d, polynomial, *args)
    Bases: AlgebraicForm

Invariant theory of a binary quartic.

You should use the invariant_theory factory object to construct instances of this class. See binary_quartic() for details.

**EisensteinD()**

One of the Eisenstein invariants of a binary quartic.

OUTPUT:

The Eisenstein D-invariant of the quartic.

\[
f(x) = a_0 x_1^4 + 4a_1 x_0 x_1^3 + 6a_2 x_0^2 x_1^2 + 4a_3 x_0^3 x_1 + a_4 x_0^4
\]

\[\Rightarrow D(f) = a_0 a_4 + 3a_2^2 - 4a_1 a_3\]

EXAMPLES:

```python
sage: R.<a0, a1, a2, a3, a4, x0, x1> = QQ[]
sage: f = a0*x1^4 + 4*a1*x0*x1^3 + 6*a2*x0^2*x1^2 + 4*a3*x0^3*x1 + a4*x0^4
sage: inv = invariant_theory.binary_quartic(f, x0, x1)
sage: inv.EisensteinD()
a2^3 - 2*a1*a3 + a0*a4
```

**EisensteinE()**

One of the Eisenstein invariants of a binary quartic.

OUTPUT:

The Eisenstein E-invariant of the quartic.

\[
f(x) = a_0 x_1^4 + 4a_1 x_0 x_1^3 + 6a_2 x_0^2 x_1^2 + 4a_3 x_0^3 x_1 + a_4 x_0^4
\]

\[\Rightarrow E(f) = a_0 a_3^2 + a_2^2 a_4 - a_0 a_2 a_4 - 2a_1 a_2 a_3 + a_2^3\]

EXAMPLES:

```python
sage: R.<a0, a1, a2, a3, a4, x0, x1> = QQ[]
sage: f = a0*x1^4 + 4*a1*x0*x1^3 + 6*a2*x0^2*x1^2 + 4*a3*x0^3*x1 + a4*x0^4
sage: inv = invariant_theory.binary_quartic(f, x0, x1)
sage: inv.EisensteinE()
a2^3 - 2*a1*a2*a3 + a0*a3^2 + a1^2*a4 - a0*a2*a4
```
coeffs ()
The coefficients of a binary quartic.

Given
\[ f(x) = a_0 x_1^4 + a_1 x_0 x_1^3 + a_2 x_0^2 x_1^2 + a_3 x_0^3 x_1 + a_4 x_0^4 \]
this function returns \( a = (a_0, a_1, a_2, a_3, a_4) \)

**EXAMPLES:**
```python
sage: R.<a0, a1, a2, a3, a4, x0, x1> = QQ[]
sage: p = a0*x1^4 + a1*x1^3*x0 + a2*x0^2*x1^2 + a3*x0^3*x1 + a4*x0^4
sage: quartic = invariant_theory.binary_quartic(p, x0, x1)
sage: quartic.coeffs()
(a0, a1, a2, a3, a4)
```

**g_covariant ()**
The g-covariant of a binary quartic.

**OUTPUT:**
The g-covariant of the quartic.
\[ f(x) = a_0 x_1^4 + 4a_1 x_0 x_1^3 + 6a_2 x_0^2 x_1^2 + 4a_3 x_0^3 x_1 + a_4 x_0^4 \]
\[ \Rightarrow D(f) = \frac{1}{144} \left( \frac{\partial^2 f}{\partial x \partial y} \right) \]

**EXAMPLES:**
```python
sage: R.<a0, a1, a2, a3, a4, x, y> = QQ[]
sage: p = a0*x^4 + 4*a1*x^3*y + 6*a2*x^2*y^2 + 4*a3*x*y^3 + a4*y^4
sage: inv = invariant_theory.binary_quartic(p, x, y)
sage: g = inv.g_covariant(); g
a1^2*x^4 - a0*a2*x^4 + 2*a1*a2*x^3*y - 2*a0*a3*x^3*y + 3*a2^2*x^2*y^2
- 2*a1*a3*x^2*y^2 - a0*a4*x^2*y^2 + 2*a2*a3*x*y^3
- 2*a1*a4*x*y^3 + a3^2*y^4 - a2*a4*y^4
sage: inv_inhomogeneous = invariant_theory.binary_quartic(p.subs(y=1), x)
sage: inv_inhomogeneous.g_covariant()
a1^2*x^4 - a0*a2*x^4 + 2*a1*a2*x^3 - 2*a0*a3*x^3 + 3*a2^2*x^2
- 2*a1*a3*x^2 - a0*a4*x^2 + 2*a2*a3*x + a3^2 - a2*a4
sage: g == 1/144 * (p.derivative(x,y)^2 - p.derivative(x,x)*p.derivative(y,y))
True
```

**h_covariant ()**
The h-covariant of a binary quartic.

**OUTPUT:**
The h-covariant of the quartic.
\[ f(x) = a_0 x_1^4 + 4a_1 x_0 x_1^3 + 6a_2 x_0^2 x_1^2 + 4a_3 x_0^3 x_1 + a_4 x_0^4 \]
\[ \Rightarrow D(f) = \frac{1}{144} \left( \frac{\partial^2 f}{\partial x \partial y} \right) \]
EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, x, y> = QQ[]  
sage: p = a0*x^4 + 4*a1*x^3*y + 6*a2*x^2*y^2 + 4*a3*x*y^3 + a4*y^4  
sage: inv = invariant_theory.binary_quartic(p, x, y)  
sage: h = inv.h_covariant(); h  
-2*a1^3*x^6 + 3*a0*a1*a2*x^6 - a0^2*a3*x^6 - 6*a1^2*a2*x^5*y + 9*a0*a2^2*x^5*y  
- 2*a0*a1*a3*x^5*y - a0^2*a4*x^5*y - 10*a1^2*a3*x^4*y^2 + 15*a0*a2*a3*x^4*y^2  
- 5*a0*a1*a4*x^4*y^2 + 10*a0*a3^2*x^3*y^3 - 10*a1^2*a4*x^3*y^3  
+ 10*a1*a3^2*x^2*y^4 - 15*a1*a2*a4*x^2*y^4 + 5*a0*a3*a4*x^2*y^4  
+ 6*a2*a3^2*x*y^5 - 9*a2^2*a4*x*y^5 + 2*a1*a3*a4*x*y^5 + a0*a4^2*x*y^5  
+ 2*a3^3*y^6 - 3*a2*a3*a4*y^6 + a1*a4^2*y^6  
sage: inv_inhomogeneous = invariant_theory.binary_quartic(p.subs(y=1), x)  
sage: inv_inhomogeneous.h_covariant()  
-2*a1^3*x^6 + 3*a0*a1*a2*x^6 - a0^2*a3*x^6 - 6*a1^2*a2*x^5 + 9*a0*a2^2*x^5  
- 2*a0*a1*a3*x^5 - a0^2*a4*x^5 - 10*a1^2*a3*x^4 + 10*a1*a3^2*x^2  
- 15*a1*a2*a4*x^2 + 5*a0*a3*a4*x^2 - 9*a2^2*a4*x  
+ 2*a1*a3*a4*x + a0*a4^2*x + 2*a3^3 - 3*a2*a3*a4 + a1*a4^2  
sage: g = inv.g_covariant()  
sage: h == 1/8 * (p.derivative(x)*g.derivative(y) - p.derivative(y)*g.derivative(x))  
True
```

monomials()

List the basis monomials in the form.

OUTPUT:

A tuple of monomials. They are in the same order as `coeffs()`.

EXAMPLES:

```
sage: R.<x,y> = QQ[]  
sage: quartic = invariant_theory.binary_quartic(x^4 + y^4)  
sage: quartic.monomials()  
(y^4, x*y^3, x^2*y^2, x^3*y, x^4)
```

scaled_coeffs()

The coefficients of a binary quartic.

Given

\[ f(x) = a_0 x_1^4 + 4a_1 x_0 x_1^3 + 6a_2 x_0^2 x_1^2 + 4a_3 x_0^3 x_1 + a_4 x_0^4 \]

this function returns \( a = (a_0, a_1, a_2, a_3, a_4) \).

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, x0, x1> = QQ[]  
sage: quartic = a0*x1^4 + 4*a1*x1^3*x0 + 6*a2*x1^2*x0^2 + 4*a3*x1^3*x0^3 + a4*x0^4  
sage: inv = invariant_theory.binary_quartic(quartic, x0, x1)  
sage: inv.scaled_coeffs()  
(a0, a1, a2, a3, a4)
```
Polynomials, Release 10.3

(continued from previous page)

```python
sage: quartic = a0 + 4*a1*x + 6*a2*x^2 + 4*a3*x^3 + a4*x^4
sage: inv = invariant_theory.binary_quartic(quartic, x)
sage: inv.scaled_coeffs()
(a0, a1, a2, a3, a4)
```

class sage.rings.invariants.invariant_theory.BinaryQuintic(n, d, polynomial, *args)

Bases: AlgebraicForm

Invariant theory of a binary quintic form.

You should use the invariant_theory factory object to construct instances of this class. See binary_quintic() for details.

REFERENCES:

For a description of all invariants and covariants of a binary quintic, see section 73 of [Cle1872].

A_invariant()

Return the invariant $A$ of a binary quintic.

OUTPUT:

The $A$-invariant of the binary quintic.

EXAMPLES:

```python
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 + a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.A_invariant()
4/625*a2^2*a3^2 - 12/625*a1*a3^3 - 12/625*a2^3*a4 + 38/625*a1*a2*a3*a4 + 6/125*a0*a3^2*a4 - 18/625*a1^2*a4^2 - 16/125*a0*a2*a4^2 + 6/125*a1*a2^2*a5 - 16/125*a1^2*a3*a5 - 2/25*a0*a2*a3*a5 + 4/5*a0*a1*a4*a5 - 2*a0^2*a5^2
```

B_invariant()

Return the invariant $B$ of a binary quintic.

OUTPUT:

The $B$-invariant of the binary quintic.

EXAMPLES:

```python
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 + a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.B_invariant()
1/1562500*a2^4*a3^4 - 3/781250*a1*a2^2*a3^5 + 9/1562500*a1^2*a3^6 - 3/781250*a2^5*a3^2*a4 + 37/1562500*a1*a2^3*a3^3*a4 - 57/1562500*a1^2*a2*a3^4*a4 + 3/312500*a0*a2^2*a3^4*a4 - 8/625*a0^2*a2*a3^5 + 4/5*a0*a1^2*a3^5 + 8/625*a0^2*a1^2*a3^4*a4 - 4/125*a0^3*a2*a4^2*a5^2 - 16/3125*a1^5*a5^3 + 4/125*a0*a1^3*a2*a5^3 - 6/125*a0^2*a1^2*a5^3 - 2/25*a0^3*a2*a3*a5^3
```

3.2. Classical Invariant Theory 487
\textbf{C\_invariant()}

Return the invariant $C$ of a binary quintic.

\textbf{OUTPUT:}

The $C$-invariant of the binary quintic.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 + a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.C_invariant()
sage: quintic.C_invariant()  
-3/1953125000*a2^6*a3^6 + 27/1953125000*a1*a2^4*a3^7
- 249/7812500000*a1^2*a2^2*a3^8 - 3/78125000*a0*a2^3*a3^8
+ 3/976562500*a1^3*a3^9 + 27/156250000*a0*a1*a2*a3^9
+ 192/15625*a0^2*a1^3*a2^2*a3*a5^4
- 36/3125*a0^3*a1*a2^3*a3^4 - 4/25*a0^2*a1^4*a3^2*a5^4
+ 24/25*a0*a1^2*a2^2*a3^2*a5^4
+ 6/25*a0^4*a2^2*a3^2*a5^4
\end{verbatim}

\textbf{H\_covariant (as\_form=False)}

Return the covariant $H$ of a binary quintic.

\textbf{INPUT:}

- \texttt{as\_form} – if \texttt{as\_form} is False, the result will be returned as polynomial (default). If it is True the result is returned as an object of the class \texttt{AlgebraicForm}.

\textbf{OUTPUT:}

The $H$-covariant of the binary quintic as polynomial or as binary form.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 + a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.H_covariant()  
-2/25*a4^2*x0^6 + 1/5*a3*a5*x0^6 - 3/25*a3*a4*x0^5*x1
+ 3/5*a2*a5*x0^5*x1 - 3/25*a3^2*x0^4*x1^2 + 3/25*a2*a4*x0^4*x1^2
+ 6/5*a1*a5*x0^4*x1^2 - 4/25*a2*a3*x0^3*x1^3 + 14/25*a1*a4*x0^3*x1^3
+ 2*a0*a5*x0^3*x1^3 - 3/25*a2^2*x0^2*x1^4 + 3/25*a1*a3*x0^2*x1^4
+ 6/5*a0*a4*x0^2*x1^4 - 3/25*a1*a2*x0*x1^5 + 3/5*a0*a3*x0*x1^5
- 2/25*a1^2*x1^6 + 1/5*a0*a2*x1^6
sage: quintic.H_covariant(as_form=True)
Binary sextic given by ...
\end{verbatim}

\textbf{R\_invariant()}

Return the invariant $R$ of a binary quintic.

\textbf{OUTPUT:}

The $R$-invariant of the binary quintic.

\textbf{EXAMPLES:}
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 + ...
    +a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.R_invariant()
3/39062500000000*a1^2*a2^5*a3^11 - 3/9765625000000*a0*a2^6*a3^11 -
    51/7812500000000*a1^3*a2^3*a3^12 + 27/9765625000000*a0*a1*a2^4*a3^12 +
    27/19531250000000*a1^4*a2*a3^13 - 81/1562500000000*a0*a1^2*a2^2*a3^13 +
    384/9765625*a0*a1^10*a5^7 - 192/390625*a0^2*a1^8*a2*a5^7 + 192/78125*a0^3*a1^6*a2^2*a5^7 - 96/15625*a0^4*a1^4*a2^3*a5^7 +
    24/3125*a0^5*a1^2*a2^4*a5^7 - 12/3125*a0^6*a2^5*a5^7

\textbf{T\_covariant} \textit{(as\_form=False)}

Return the covariant $T$ of a binary quintic.

\textbf{INPUT:}

- \textit{as\_form} – if \textit{as\_form} is False, the result will be returned as polynomial (default). If it is True the result is returned as an object of the class \texttt{AlgebraicForm}.

\textbf{OUTPUT:}

The $T$-covariant of the binary quintic as polynomial or as binary form.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 +...
    +a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.T_covariant()
2/125*a4^3*x0^9 - 3/50*a3*a4*a5*x0^9 + 1/10*a2*a5^2*x0^9 + 9/250*a3*a4^2*x0^8*x1 - 3/25*a3^2*a5*x0^8*x1 + 1/50*a2*a4*a5*x0^8*x1 + 2/5*a1*a5^2*x0^8*x1 + 3/250*a3^2*a4*x0^7*x1^2 + 8/125*a2*a4^2*x0^7*x1^2 +...
11/25*a0*a1*a4*x0^2*x1^7 - a0^2*a5*x0^2*x1^7 - 9/250*a1^2*a2*x0*x1^8 + 3/25*a0^2*a2*x0*x1^8 - 1/50*a0*a1*a3*x0*x1^8 - 2/5*a0^2*a4*x0*x1^8 - 2/125*a1^3*x1^9 + 3/50*a0*a1*a2*x1^9 - 1/10*a0^2*a3*x1^9
sage: quintic.T_covariant(as\_form=True)
Binary nonic given by ...
\end{verbatim}

\textbf{alpha\_covariant} \textit{(as\_form=False)}

Return the covariant $\alpha$ of a binary quintic.

\textbf{INPUT:}

- \textit{as\_form} – if \textit{as\_form} is False, the result will be returned as polynomial (default). If it is True the result is returned as an object of the class \texttt{AlgebraicForm}.

\textbf{OUTPUT:}

The $\alpha$-covariant of the binary quintic as polynomial or as binary form.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 +...
    +a5*x0^5
\end{verbatim}

(continues on next page)
arithmetic_invariants()

Return a set of generating arithmetic invariants of a binary quintic.

An arithmetic invariants is an invariant whose coefficients are integers for a general binary quintic. They are linear combinations of the Clebsch invariants, such that they still generate the ring of invariants.

OUTPUT:

The arithmetic invariants of the binary quintic. They are given by

\[ I_4 = 2^{-1} \cdot 5^4 \cdot A \]
\[ I_8 = 5^5 \cdot (2^{-1} \cdot 47 \cdot A^2 - 2^2 \cdot B) \]
\[ I_{12} = 5^{10} \cdot (2^{-1} \cdot 3 \cdot A^3 - 5 \cdot 3^{-1} \cdot C) \]
\[ I_{18} = 2^8 \cdot 3^{-1} \cdot 5^{15} \cdot R \]

where \( A, B, C \) and \( R \) are the \texttt{BinaryQuintic.clebsch_invariants()}.

EXAMPLES:

```python
sage: R.<x0, x1> = QQ[]
sage: p = 2*x1^5 + 4*x1^4*x0 + 5*x1^3*x0^2 + 7*x1^2*x0^3 - 11*x1*x0^4 + x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.arithmetic_invariants()
{'I12': -1156502613073152, 'I18': -12712872348048797642752, 'I4': -138016, 'I8': 14164936192}
```

We can check that the coefficients of the invariants have no common divisor for a general quintic form:

```python
sage: R.<a0,a1,a2,a3,a4,a5,x0,x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 + a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: invs = quintic.arithmetic_invariants()
sage: [invs[x].content() for x in invs]
[1, 1, 1, 1]
```

beta_covariant (as_form=False)

Return the covariant \( \beta \) of a binary quintic.

INPUT:

- as_form – if as_form is False, the result will be returned as polynomial (default). If it is True the result is returned as an object of the class \texttt{AlgebraicForm}.
The $\beta$-covariant of the binary quintic as polynomial or as binary form.

**EXAMPLES:**

```python
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 +...
\quad ... + a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.beta_covariant()
-1/62500*a2^3*a3^4*x0 + 9/125000*a1*a2*a3^5*x0 - 27/125000*a0*a3^6*x0
+ 13/125000*a2^4*a3^2*a4*x0 - 31/625000*a1*a2^2*a3^3*a4*x0
- 3/625000*a1^2*a3^4*a4*x0 + 27/15625*a0*a2*a3^4*a4*x0
... - 16/125*a0^2*a1*a3^2*a5^2*x1 - 28/625*a0*a1^3*a4*a5^2*x1
+ 6/125*a0^2*a1*a2*a4*a5^2*x1 + 8/25*a0^3*a3*a4*a5^2*x1
+ 4/25*a0^2*a1^2*a5^3*x1 - 2/5*a0^3*a2*a5^3*x1
sage: quintic.beta_covariant(as_form=True)
Binary monic given by ...
```

**canonical_form**(reduce_gcd=False)

Return a canonical representative of the quintic.

Given a binary quintic $f$ with coefficients in a field $K$, returns a canonical representative of the $GL(2, \bar{K})$-orbit of the quintic, where $\bar{K}$ is an algebraic closure of $K$. This means that two binary quintics $f$ and $g$ are $GL(2, \bar{K})$-equivalent if and only if their canonical forms are the same.

**INPUT:**

- **reduce_gcd** – If set to True, then a variant of this canonical form is computed where the coefficients are coprime integers. The obtained form is then unique up to multiplication by a unit. See also `binary_quintic_from_invariants()`.

**OUTPUT:**

A canonical $GL(2, \bar{K})$-equivalent binary quintic.

**EXAMPLES:**

```python
sage: R.<x0, x1> = QQ[]
sage: p = 2*x1^5 + 4*x1^4*x0 + 5*x1^3*x0^2 + 7*x1^2*x0^3 - 11*x1*x0^4 + x0^5
sage: f = invariant_theory.binary_quintic(p, x0, x1)
sage: g = matrix(QQ, [[[11, 5], [7, 2]])
sage: gf = f.transformed(g)
sage: f.canonical_form() == gf.canonical_form()
True
sage: h = f.canonical_form(reduce_gcd=True)
sage: gcd(h.coeffs())
1
```

**clebsch_invariants**(as_tuple=False)

Return the invariants of a binary quintic as described by Clebsch.

The following invariants are returned: $A$, $B$, $C$ and $R$.

**OUTPUT:**

The Clebsch invariants of the binary quintic.

**EXAMPLES:**

```python
3.2. Classical Invariant Theory
```

491
sage: R.<x0, x1> = QQ[]
sage: p = 2*x1^5 + 4*x1^4*x0 + 5*x1^3*x0^2 + 7*x1^2*x0^3 - 11*x1*x0^4 + x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.clebsch_invariants()
{'A': -276032/625, 'B': 4983526016/390625, 'C': -247056495846408/244140625, 'R': -148978972828696847376/30517578125}
sage: quintic.clebsch_invariants(as_tuple=True)
(-276032/625, 4983526016/390625, -247056495846408/244140625, -148978972828696847376/30517578125)
sage: coeffs()
The coefficients of a binary quintic.
Given
\[ f(x) = a_0 x_1^5 + a_1 x_0 x_1^4 + a_2 x_0^2 x_1^3 + a_3 x_0^3 x_1^2 + a_4 x_0^4 x_1 + a_5 x_0^5 \]
this function returns \( a = (a_0, a_1, a_2, a_3, a_4, a_5) \)

EXAMPLES:

sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 + a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.coeffs()
(a0, a1, a2, a3, a4, a5)
sage: R.<a0, a1, a2, a3, a4, a5, x> = QQ[]
sage: p = a0 + a1*x + a2*x^2 + a3*x^3 + a4*x^4 + a5*x^5
sage: quintic = invariant_theory.binary_quintic(p, x)
sage: quintic.coeffs()
(a0, a1, a2, a3, a4, a5)

delta_covariant (as_form=False)
Return the covariant \( \delta \) of a binary quintic.

INPUT:

* as_form -- if as_form is False, the result will be returned as polynomial (default). If it is True the result is returned as an object of the class AlgebraicForm.

OUTPUT:

The \( \delta \)-covariant of the binary quintic as polynomial or as binary form.

EXAMPLES:

sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 + a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.delta_covariant()
1/1562500000*a2^6*a3^7*x0 - 9/1562500000*a1*a2^4*a3^8*x0
(continues on next page)
classmethod `from_invariants` *(invariants, x, z, *args, **kwargs)*

Construct a binary quintic from its invariants.

This function constructs a binary quintic whose invariants equal the ones provided as argument up to scaling.

**INPUT:**

- `invariants` – A list or tuple of invariants that are used to reconstruct the binary quintic.

**OUTPUT:**

A BinaryQuintic.

**EXAMPLES:**

```python
sage: R.<x, y> = QQ[]
sage: from sage.rings.invariants.invariant_theory import BinaryQuintic
go to the class
sage: BinaryQuintic.from_invariants([3, 6, 12], x, y)
Binary quintic with coefficients (0, 1, 0, 0, 1, 0)
```

`gamma_covariant` *(as_form=False)*

Return the covariant $\gamma$ of a binary quintic.

**INPUT:**

- `as_form` – if `as_form` is `False`, the result will be returned as polynomial (default). If it is `True` the result is returned as an object of the class `AlgebraicForm`.

**OUTPUT:**

The $\gamma$-covariant of the binary quintic as polynomial or as binary form.

**EXAMPLES:**

```python
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 +...
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.gamma_covariant() 1/156250000*a2^5*a3^6*x0 - 3/62500000*a1*a2^3*a3^7*x0 +...
sage: quintic.gamma_covariant(as_form=True) Binary monic given by ...
```
i_covariant (as_form=False)

Return the covariant $i$ of a binary quintic.

INPUT:

• as_form – if as_form is False, the result will be returned as polynomial (default). If it is True the result is returned as an object of the class AlgebraicForm.

OUTPUT:

The $i$-covariant of the binary quintic as polynomial or as binary form.

EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 + ...
    - a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.i_covariant()
3/50*a3^2*x0^2 - 4/25*a2*a4*x0^2 + 2/5*a1*a5*x0^2 + 1/25*a2*a3*x0*x1
   - 6/25*a1*a4*x0*x1 + 2*a0*a5*x0*x1 + 3/50*a2^2*x1^2 - 4/25*a1*a3*x1^2
   + 2/5*a0*a4*x1^2
sage: quintic.i_covariant(as_form=True)
Binary quadratic given by ...
```

invariants (type='clebsch')

Return a tuple of invariants of a binary quintic.

INPUT:

• type – The type of invariants to return. The default choice is to return the Clebsch invariants.

OUTPUT:

The invariants of the binary quintic.

EXAMPLES:

```
sage: R.<x0, x1> = QQ[]
sage: p = 2*x1^5 + 4*x1^4*x0 + 5*x1^3*x0^2 + 7*x1^2*x0^3 - 11*x1*x0^4 + x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.invariants()
(-276032/625,
  4983526016/390625,
 -247056495846408/244140625,
 -148978972828696847376/30517578125)
sage: quintic.invariants('unknown')
Traceback (most recent call last):
...
ValueError: unknown type of invariants unknown for a binary quintic
```

j_covariant (as_form=False)

Return the covariant $j$ of a binary quintic.

INPUT:

• as_form – if as_form is False, the result will be returned as polynomial (default). If it is True the result is returned as an object of the class AlgebraicForm.

OUTPUT:

The $j$-covariant of the binary quintic as polynomial or as binary form.
EXAMPLES:

```
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 +
   → a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.j_covariant()
-3/500*a3^3*x0^3 + 3/125*a2*a3*a4*x0^3 - 6/125*a1*a4^2*x0^3
- 3/50*a2^2*a5*x0^3 + 3/25*a1*a3*a5*x0^3 - 3/500*a2*a3^2*x0*x1^2
+ 3/250*a1*a3^2*x0*x1^2 + 3/125*a1*a2*a4*x0*x1^2 - 3/25*a0*a3*a4*x0*x1^2
- 6/25*a1^2*a5*x0*x1^2 + 3/5*a0*a2*a5*x0*x1^2 - 3/500*a2^3*x1^3
+ 3/125*a1*a2*a3*x1^3 - 3/50*a0*a3^2*x1^3 - 6/125*a1^2*a4*x1^3
+ 3/25*a0*a2*a4*x1^3

sage: quintic.j_covariant(as_form=True)
Binary cubic given by ...
```

**monomials()**

List the basis monomials of the form.

This function lists a basis of monomials of the space of binary quintics of which this form is an element.

**OUTPUT:**

A tuple of monomials. They are in the same order as `coeffs()`.

**EXAMPLES:**

```
sage: R.<x,y> = QQ[]
sage: quintic = invariant_theory.binary_quintic(x^5 + y^5)
sage: quintic.monomials()
(y^5, x*y^4, x^2*y^3, x^3*y^2, x^4*y, x^5)
```

**scaled_coeffs()**

The coefficients of a binary quintic.

Given

\[ f(x) = a_0 x_1^5 + 5a_1 x_0 x_1^4 + 10a_2 x_0^2 x_1^3 + 10a_3 x_0^3 x_1^2 + 5a_4 x_0^4 x_1 + a_5 x_0^5 \]

this function returns \( a = (a_0, a_1, a_2, a_3, a_4, a_5) \)

**EXAMPLES:**

```
sage: R.<a0, a1, a2, a3, a4, a5, x> = QQ[]
sage: p = a0*x1^5 + 5*a1*x1^4*x0 + 10*a2*x0^2*x1^3 + 10*a3*x0^3*x1^2 + 5*a4*x0^4*x1 + a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.scaled_coeffs()
(a0, a1, a2, a3, a4, a5)
sage: quintic = invariant_theory.binary_quintic(p, x)
sage: quintic.scaled_coeffs()
(a0, a1, a2, a3, a4, a5)
```
**tau_covariant** *(as_form=False)*

Return the covariant $\tau$ of a binary quintic.

**INPUT:**

- `as_form` – if `as_form` is `False`, the result will be returned as polynomial (default). If it is `True` the result is returned as an object of the class `AlgebraicForm`.

**OUTPUT:**

The $\tau$-covariant of the binary quintic as polynomial or as binary form.

**EXAMPLES:**

```
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 + ...
        a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.tau_covariant()
1/62500*a2^2*a3^4*x0^2 - 3/62500*a1*a3^5*x0^2
- 1/15625*a2^3*a3^2*a4*x0^2 + 1/6250*a1*a2*a3^3*a4*x0^2
+ 3/6250*a0*a3^4*a4*x0^2 - 1/31250*a2^4*a4*x0^2
... - 2/125*a0*a1*a2^2*a4*a5*x1^2 - 4/125*a0*a1^2*a3*a4*a5*x1^2
+ 2/25*a0^2*a3*a4*a5^2*x1^2 - 8/625*a1^4*a5^2*x1^2
+ 8/125*a0*a1^2*a2*a5^2*x1^2 - 2/25*a0^2*a2^2*a5^2*x1^2
sage: quintic.tau_covariant(as_form=True)
Binary quadratic given by ...
```

**theta_covariant** *(as_form=False)*

Return the covariant $\theta$ of a binary quintic.

**INPUT:**

- `as_form` – if `as_form` is `False`, the result will be returned as polynomial (default). If it is `True` the result is returned as an object of the class `AlgebraicForm`.

**OUTPUT:**

The $\theta$-covariant of the binary quintic as polynomial or as binary form.

**EXAMPLES:**

```
sage: R.<a0, a1, a2, a3, a4, a5, x0, x1> = QQ[]
sage: p = a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 + ...
        a5*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: quintic.theta_covariant()
-1/625000*a2^3*a3^5*x0^2 + 9/1250000*a1*a2*a3^6*x0^2
- 27/1250000*a0*a3^7*x0^2 + 3/250000*a2^4*a3^3*a4*x0^2
+ 7/1250000*a1*a2^2*a3^4*a4*x0^2 - 3/312500*a1^2*a3^5*a4*x0^2
... + 6/625*a0^2*a1*a2^2*a4*a5^2*x1^2 - 24/625*a0^2*a1^2*a3*a4*a5^2*x1^2
- 12/125*a0^3*a2*a3*a4*a5^2*x1^2 + 8/625*a0*a1^4*a5^3*x1^2
- 8/125*a0^2*a1^2*a2*a5^3*x1^2 + 2/25*a0^3*a2^2*a5^3*x1^2
sage: quintic.theta_covariant(as_form=True)
Binary quadratic given by ...
```
class sage.rings.invariants.invariant_theory.FormsBase (n, homogeneous, ring, variables)

Bases: SageObject

The common base class of AlgebraicForm and SeveralAlgebraicForms.

This is an abstract base class to provide common methods. It does not make much sense to instantiate it.

is_homogeneous ()

Return whether the forms were defined by homogeneous polynomials.

OUTPUT:

Boolean. Whether the user originally defined the form via homogeneous variables.

EXAMPLES:

```
sage: R.<x,y,t> = QQ[

sage: quartic = invariant_theory.binary_quartic(x^4 + y^4 + t*x^2*y^2, [x,y])

sage: quartic.is_homogeneous()
True

sage: quartic.form()

x^2*y^2*t + x^4 + y^4

sage: R.<x,y,t> = QQ[

sage: quartic = invariant_theory.binary_quartic(x^4 + 1 + t*x^2, [x])

sage: quartic.is_homogeneous()
False

sage: quartic.form()

x^4 + x^2*t + 1
```

ring ()

Return the polynomial ring.

OUTPUT:

A polynomial ring. This is where the defining polynomial(s) live. Note that the polynomials may be homogeneous or inhomogeneous, depending on how the user constructed the object.

EXAMPLES:

```
sage: R.<x,y,t> = QQ[

sage: quartic = invariant_theory.binary_quartic(x^4 + y^4 + t*x^2*y^2, [x,y])

sage: quartic.ring()

Multivariate Polynomial Ring in x, y, t over Rational Field

sage: R.<x,y,t> = QQ[

sage: quartic = invariant_theory.binary_quartic(x^4 + 1 + t*x^2, [x])

sage: quartic.ring()

Multivariate Polynomial Ring in x, y, t over Rational Field
```

variables ()

Return the variables of the form.

OUTPUT:

A tuple of variables. If inhomogeneous notation is used for the defining polynomial then the last entry will be None.

EXAMPLES:
class sage.rings.invariants.invariant_theory.InvariantTheoryFactory

Factory object for invariants of multilinear forms.

Use the invariant_theory object to construct algebraic forms. These can then be queried for invariant and covariants.

EXAMPLES:

```sage
R.<x,y,z> = QQ[]
print(invariant_theory.ternary_cubic(x^3 + y^3 + z^3))
Ternary cubic with coefficients (1, 1, 1, 0, 0, 0, 0, 0, 0, 0)
print(invariant_theory.ternary_cubic(x^3 + y^3 + z^3).J_covariant())
x^6*y^3 - x^3*y^6 - x^6*z^3 + y^6*z^3 + x^3*z^6 - y^3*z^6
```

`binary_form_from_invariants`(degree, invariants, variables=None, as_form=True, *args, **kwargs)

Reconstruct a binary form from the values of its invariants.

INPUT:

- `degree` – The degree of the binary form.
- `invariants` – A list or tuple of values of the invariants of the binary form.
- `variables` – A list or tuple of two variables that are used for the resulting form (only if `as_form` is True). If no variables are provided, two abstract variables x and z will be used.
- `as_form` – boolean. If False, the function will return a tuple of coefficients of a binary form.

OUTPUT:

A binary form or a tuple of its coefficients, whose invariants are equal to the given `invariants` up to a scaling.

EXAMPLES:

In the case of binary quadratics and cubics, the form is reconstructed based on the value of the discriminant. See also `binary_quadratic_coefficients_from_invariants()` and `binary_cubic_coefficients_from_invariants()`. These methods will always return the same result if the discriminant is non-zero:

```sage
discriminant = 1
print(invariant_theory.binary_form_from_invariants(2, [discriminant]))
Binary quadratic with coefficients (1, -1/4, 0)
print(invariant_theory.binary_form_from_invariants(3, [discriminant], as_form=False))
(0, 1, -1, 0)
```

For binary cubics, there is no class implemented yet, so `as_form=True` will yield a `NotImplementedError`:  

```
for binary cubics, there is no class implemented yet, so as_form=True will yield a NotImplemented...
For binary quintics, the three Clebsch invariants of the form should be provided to reconstruct the form. For more details about these invariants, see `clebsch_invariants()`:

```
sage: invariants = [1, 0, 0]
sage: invariant_theory.binary_form_from_invariants(5, invariants)
Binary quintic with coefficients (1, 0, 0, 0, 1)
```

An optional scaling argument may be provided in order to scale the resulting quintic. For more details, see `binary_quintic_coefficients_from_invariants()`:

```
sage: invariants = [3, 4, 7]
sage: invariant_theory.binary_form_from_invariants(5, invariants)
Binary quintic with coefficients (-37725479487783/1048576, 565882192316745/8388608, 0, 103386676532693115/67108864, 1284948694093328715/268435456, -23129076493685391687/2147483648)
sage: invariant_theory.binary_form_from_invariants(5, invariants, scaling='normalized')
Binary quintic with coefficients (24389/892616806656, 4205/11019960576, 0, 1015/209952, -145/1296, -3/16)
sage: invariant_theory.binary_form_from_invariants(5, invariants, scaling='coprime')
Binary quintic with coefficients (-2048, 3840, 0, 876960, 2724840, -613089)
```

The invariants can also be computed using the invariants of a given binary quintic. The resulting form has the same invariants up to scaling, is $GL(2, \mathbb{Q})$-equivalent to the provided form and hence has the same canonical form (see `canonical_form()`):

```
sage: R.<x0, x1> = QQ[]
sage: p = 3*x1^5 + 6*x1^4*x0 + 3*x1^3*x0^2 + 4*x1^2*x0^3 - 5*x1*x0^4 + 4*x0^5
sage: quintic = invariant_theory.binary_quintic(p, x0, x1)
sage: invariants = quintic.clebsch_invariants(as_tuple=True)
sage: newquintic = invariant_theory.binary_form_from_invariants(5, invariants, variables=quintic.variables())
sage: newquintic
Binary quintic with coefficients (9592267437341790539005557/244140625000000, 214929628207625556323004064707/61035156250000000000, 11149651890347700974453304786783/76293945312500000000, 122650775751894638395648891202734239/476837158203125000000, 32399663094570652847286334593218447/11920928955078125000000, 150450650364608395841632538558481466127/149011611938476562500000)
sage: quintic.canonical_form() == newquintic.canonical_form()
True
```

For binary forms of other degrees, no reconstruction has been implemented yet. For forms of degree 6, see github issue #26462:

```
sage: invariant_theory.binary_form_from_invariants(6, invariants)
Traceback (most recent call last):
  ...                      → implemented
```
**binary_quadratic** *(quadratic, *args)*

Invariant theory of a quadratic in two variables.

**INPUT:**
- **quadratic** — a quadratic form.
- **x, y** — the homogeneous variables. If **y** is **None**, the quadratic is assumed to be inhomogeneous.

**REFERENCES:**
- Wikipedia article Invariant_of_a_binary_form

**EXAMPLES:**

```python
sage: R.<x,y> = QQ[]
sage: invariant_theory.binary_quadratic(x^2 + y^2)
Binary quadratic with coefficients (1, 1, 0)
sage: T.<t> = QQ[]
sage: invariant_theory.binary_quadratic(t^2 + 2*t + 1, [t])
Binary quadratic with coefficients (1, 1, 2)
```

**binary_quartic** *(quartic, *args, **kwds)*

Invariant theory of a quartic in two variables.

The algebra of invariants of a quartic form is generated by invariants $i, j$ of degrees 2, 3. This ring is naturally isomorphic to the ring of modular forms of level 1, with the two generators corresponding to the Eisenstein series $E_4$ (see `EisensteinD()`) and $E_6$ (see `EisensteinE()`). The algebra of covariants is generated by these two invariants together with the form $f$ of degree 1 and order 4, the Hessian $g$ (see `g_covariant()`) of degree 2 and order 4, and a covariant $h$ (see `h_covariant()`) of degree 3 and order 6. They are related by a syzygy

$$jf^3 - g f^2 i + 4g^3 + h^2 = 0$$

of degree 6 and order 12.

**INPUT:**
- **quartic** — a quartic.
- **x, y** — the homogeneous variables. If **y** is **None**, the quartic is assumed to be inhomogeneous.

**REFERENCES:**
- Wikipedia article Invariant_of_a_binary_form

**EXAMPLES:**

```python
sage: R.<x,y> = QQ[]
sage: quartic = invariant_theory.binary_quartic(x^4 + y^4)
sage: quartic
Binary quartic with coefficients (1, 0, 0, 0, 1)
sage: type(quartic)
<class 'sage.rings.invariants.invariant_theory.BinaryQuartic'>
```

**binary_quintic** *(quintic, *args, **kwds)*

Create a binary quintic for computing invariants.

A binary quintic is a homogeneous polynomial of degree 5 in two variables. The algebra of invariants of a binary quintic is generated by the invariants $A$, $B$ and $C$ of respective degrees 4, 8 and 12 (see `A_invariant()`, `B_invariant()` and `C_invariant()`).
INPUT:

• `quintic` — a homogeneous polynomial of degree five in two variables or a (possibly inhomogeneous) polynomial of degree at most five in one variable.

• `*args` — the two homogeneous variables. If only one variable is given, the polynomial `quintic` is assumed to be univariate. If no variables are given, they are guessed.

REFERENCES:

• Wikipedia article Invariant_of_a_binary_form

• [Cle1872]

EXAMPLES:

If no variables are provided, they will be guessed:

```sage
R.<x,y> = QQ[]
quintic = invariant_theory.binary_quintic(x^5 + y^5)
quintic
```

Binary quintic with coefficients (1, 0, 0, 0, 0, 1)

If only one variable is given, the quintic is the homogenisation of the provided polynomial:

```sage
quintic = invariant_theory.binary_quintic(x^5 + y^5, x)
quintic
```

Binary quintic with coefficients (y^5, 0, 0, 0, 0, 1)

```sage
quintic.is_homogeneous() False
```

If the polynomial has three or more variables, the variables should be specified:

```sage
R.<x,y,z> = QQ[]
quintic = invariant_theory.binary_quintic(x^5 + z*y^5)
Traceback (most recent call last):
  ...:
ValueError: need 2 or 1 variables, got (x, y, z)
quintic = invariant_theory.binary_quintic(x^5 + z*y^5, x, y)
quintic
```

Binary quintic with coefficients (z, 0, 0, 0, 0, 1)

```sage
type(quintic) <class 'sage.rings.invariants.invariant_theory.BinaryQuintic'>
```

**inhomogeneous_quadratic_form** *(polynomial, *args)*

Invariants of an inhomogeneous quadratic form.

INPUT:

• `polynomial` — an inhomogeneous quadratic form.

• `*args` — the variables as multiple arguments, or as a single list/tuple.

EXAMPLES:

```sage
R.<x,y,z> = QQ[]
quadratic = x^2 + 2*y^2 + 3*x*y + 4*x + 5*y + 6
inv3 = invariant_theory.inhomogeneous_quadratic_form(quadratic)
type(inv3) <class 'sage.rings.invariants.invariant_theory.TernaryQuadratic'>
```

(continues on next page)
quadratic_form (polynomial, *args)

Invariants of a homogeneous quadratic form.

INPUT:

- polynomial – a homogeneous or inhomogeneous quadratic form.
- *args – the variables as multiple arguments, or as a single list/tuple. If the last argument is None, the cubic is assumed to be inhomogeneous.

EXAMPLES:

```python
sage: R.<x,y,z> = QQ[]
sage: quadratic = x^2 + y^2 + z^2
sage: inv = invariant_theory.quadratic_form(quadratic)
sage: type(inv)
<class 'sage.rings.invariants.invariant_theory.TernaryQuadratic'>
```

If some of the ring variables are to be treated as coefficients you need to specify the polynomial variables:

```python
sage: R.<x,y,z, a,b> = QQ[]
sage: quadratic = a*x^2 + b*y^2 + z^2 + 2*y*z
sage: invariant_theory.quadratic_form(quadratic, x,y,z)
Ternary quadratic with coefficients (a, b, 1, 0, 0, 2)
sage: invariant_theory.quadratic_form(quadratic, [x,y,z])  # alternate syntax
Ternary quadratic with coefficients (a, b, 1, 0, 0, 2)
```

Inhomogeneous quadratic forms (see also inhomogeneous_quadratic_form()) can be specified by passing None as the last variable:

```python
sage: inhom = quadratic.subs(z=1)
sage: invariant_theory.quadratic_form(inhom, x,y,None)
Ternary quadratic with coefficients (a, b, 1, 0, 0, 2)
```

quaternary_biquadratic (quadratic1, quadratic2, *args, **kwds)

Invariants of two quadratics in four variables.

INPUT:

- quadratic1, quadratic2 – two polynomials. Either homogeneous quadratic in 4 homogeneous variables, or inhomogeneous quadratic in 3 variables.
- w, x, y, z – the variables. If z is None, the quadratics are assumed to be inhomogeneous.

EXAMPLES:

```python
sage: R.<w,x,y,z> = QQ[]
sage: q1 = w^2 + x^2 + y^2 + z^2
sage: q2 = w*x + y*z
sage: inv = invariant_theory.quaternary_biquadratic(q1, q2)
sage: type(inv)
<class 'sage.rings.invariants.invariant_theory.TwoQuaternaryQuadratics'>
```

Distance between two spheres [Sal1958], [Sal1965]
```
sage: R.<x,y,z, a,b,c, r1,r2> = QQ[]
sage: S1 = -r1^2 + x^2 + y^2 + z^2
sage: S2 = -r2^2 + (x-a)^2 + (y-b)^2 + (z-c)^2
sage: inv = invariant_theory.quaternary_biquadratic(S1, S2, [x, y, z])
sage: inv.Delta_invariant()
r1^2
sage: inv.Delta_prime_invariant()
r2^2
sage: inv.Theta_invariant()
a^2 + b^2 + c^2 - 3*r1^2 - r2^2
sage: inv.Theta_prime_invariant()
a^2 + b^2 + c^2 - r1^2 - 3*r2^2
sage: inv.Phi_invariant()
2*a^2 + 2*b^2 + 2*c^2 - 3*r1^2 - 3*r2^2
sage: inv.J_covariant()
0
```

quaternary_quadratic\((quadric, *args)\)

Invariant theory of a quadratic in four variables.

INPUT:

- quadric – a quadratic form.
  
- w, x, y, z – the homogeneous variables. If z is None, the quadratic is assumed to be inhomogeneous.

REFERENCES:

- Wikipedia article Invariant_of_a_binary_form

EXAMPLES:

```
sage: R.<w,x,y,z> = QQ[]
sage: inv = invariant_theory.quaternary_quadratic(w^2 + x^2 + y^2 + z^2)
sage: inv.quadrics
\([1, 1, 1, 1, 0, 0, 0, 0, 0, 0]\)
```

ternary_biquadratic\((quadric1, quadric2, *args, **kwds)\)

Invariants of two quadratics in three variables.

INPUT:

- quadric1, quadric2 – two polynomials. Either homogeneous quadratic in 3 homogeneous variables, or inhomogeneous quadratic in 2 variables.
  
- x, y, z – the variables. If z is None, the quadratics are assumed to be inhomogeneous.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: q1 = x^2 + y^2 + z^2
sage: q2 = x*y + y*z + x*z
sage: inv = invariant_theory.ternary_biquadratic(q1, q2)
sage: type(inv)
<class 'sage.rings.invariants.invariant_theory.TwoTernaryQuadratics'>
```

Distance between two circles:
**ternary_cubic** *(cubic, *args, **kwds)*

Invariants of a cubic in three variables.

The algebra of invariants of a ternary cubic under $SL_3(\mathbb{C})$ is a polynomial algebra generated by two invariants $S$ (see `S_invariant()`) and $T$ (see `T_invariant()`) of degrees 4 and 6, called Aronhold invariants.

The ring of covariants is given as follows. The identity covariant $U$ of a ternary cubic has degree 1 and order 3. The Hessian $H$ (see `Hessian()`) is a covariant of ternary cubics of degree 3 and order 3. There is a covariant $\Theta$ (see `Theta_covariant()`) of ternary cubics of degree 8 and order 6 that vanishes on points $x$ lying on the Salmon conic of the polar of $x$ with respect to the curve and its Hessian curve. The Brioussch covariant $J$ (see `J_covariant()`) is the Jacobian of $U, \Theta,$ and $H$ of degree 12, order 9. The algebra of covariants of a ternary cubic is generated over the ring of invariants by $U, \Theta, H,$ and $J,$ with a relation

$$J^2 = 4\Theta^3 + TU^2\Theta^2 + \Theta(-4S^3U^4 + 2STU^3H - 72S^2U^2H^2$$

$$- 18TUH^3 + 108SH^4) - 16S^4U^5H - 11S^3TU^4H^2$$

$$- 4T^2U^3H^3 + 54STU^2H^4 - 432S^2UH^5 - 27TH^6$$

**REFERENCES:**

- Wikipedia article Ternary_cubic

**INPUT:**

- `cubic` – a homogeneous cubic in 3 homogeneous variables, or an inhomogeneous cubic in 2 variables.
- `x, y, z` – the variables. If `z` is `None`, the cubic is assumed to be inhomogeneous.

**EXAMPLES:**

```python
sage: R.<x,y,z> = QQ[]
sage: cubic = invariant_theory.ternary_cubic(x^3 + y^3 + z^3)
sage: type(cubic)
<class 'sage.rings.invariants.invariant_theory.TernaryCubic'>
```
**ternary_quadratic** *(quadratic, *args, **kwds)*

Invariants of a quadratic in three variables.

**INPUT:**

- **quadratic** – a homogeneous quadratic in 3 homogeneous variables, or an inhomogeneous quadratic in 2 variables.
- **x, y, z** – the variables. If z is None, the quadratic is assumed to be inhomogeneous.

**REFERENCES:**

- Wikipedia article Invariant_of_a_binary_form

**EXAMPLES:**

```
sage: R.<x,y,z> = QQ[]
sage: invariant_theory.ternary_quadratic(x^2 + y^2 + z^2)
Ternary quadratic with coefficients (1, 1, 1, 0, 0, 0)
sage: T.<u, v> = QQ[]
sage: invariant_theory.ternary_quadratic(1 + u^2 + v^2)
Ternary quadratic with coefficients (1, 1, 1, 0, 0, 0)
sage: quadratic = x^2 + y^2 + z^2
sage: inv = invariant_theory.ternary_quadratic(quadratic)
sage: type(inv)
<class 'sage.rings.invariants.invariant_theory.TernaryQuadratic'>
```

class sage.rings.invariants.invariant_theory.QuadraticForm *(n, d, polynomial, *args)*

Bases: AlgebraicForm

Invariant theory of a multivariate quadratic form.

You should use the **invariant_theory** factory object to construct instances of this class. See **quadratic_form()** for details.

**as_QuadraticForm()**

Convert into a **QuadraticForm**.

**OUTPUT:**

Sage has a special quadratic forms subsystem. This method converts **self** into this **QuadraticForm** representation.

**EXAMPLES:**

```
sage: R.<x,y,z> = QQ[]
sage: p = x^2 + y^2 + z^2 + 2*x*y + 3*x*z
sage: quadratic = invariant_theory.ternary_quadratic(p)
sage: matrix(quadratic)
[ 1 1 3/2]
[ 1 1 0]
[3/2 0 1]
sage: quadratic.as_QuadraticForm()
Quadratic form in 3 variables over Multivariate Polynomial Ring in x, y, z over Rational Field with coefficients:
[ 1 2 3 ]
[ * 1 0 ]
[ * * 1 ]
sage: _.polynomial(['X','Y','Z'])
X^2 + 2*X*Y + Y^2 + 3*X*Z + Z^2
```

3.2. Classical Invariant Theory 505
coefs ()

The coefficients of a quadratic form.

Given

\[ f(x) = \sum_{0 \leq i < n} a_i x_i^2 + \sum_{0 \leq j < k < n} a_{jk} x_j x_k \]

this function returns \( a = (a_0, \ldots, a_n, a_{00}, a_{01}, \ldots, a_{n-1,n}) \)

EXAMPLES:

```python
sage: R.<a,b,c,d,e,f,g, x,y,z> = QQ[]
sage: p = a*x^2 + b*y^2 + c*z^2 + d*x*y + e*x*z + f*y*z
sage: inv = invariant_theory.quadratic_form(p, x,y,z); inv
Ternary quadratic with coefficients (a, b, c, d, e, f)
sage: inv.coeffs()
(a, b, c, d, e, f)
sage: inv.scaled_coeffs()
(a, b, c, 1/2*d, 1/2*e, 1/2*f)
```

discriminant ()

Return the discriminant of the quadratic form.

Up to an overall constant factor, this is just the determinant of the defining matrix, see matrix(). For a quadratic form in \( n \) variables, the overall constant is \( 2^{n-1} \) if \( n \) is odd and \((-1)^{n/2}2^n\) if \( n \) is even.

EXAMPLES:

```python
sage: R.<a,b,c, x,y> = QQ[]
sage: p = a*x^2 + b*x*y + c*y^2
sage: quadratic = invariant_theory.quadratic_form(p, x,y)
sage: quadratic.discriminant()
b^2 - 4*a*c
sage: R.<a,b,c,d,e,f,g, x,y,z> = QQ[]
sage: p = a*x^2 + b*y^2 + c*z^2 + d*x*y + e*x*z + f*y*z
sage: quadratic = invariant_theory.quadratic_form(p, x,y,z)
sage: quadratic.discriminant()
4*a*b*c - c*d^2 - b*e^2 + d*e*f - a*f^2
```
dual ()

Return the dual quadratic form.

OUTPUT:

A new quadratic form (with the same number of variables) defined by the adjoint matrix.

EXAMPLES:

```python
sage: R.<a,b,c,x,y,z> = QQ[]
sage: cubic = x^2+y^2+z^2
sage: quadratic = invariant_theory.ternary_quadratic(a*x^2+b*y^2+c*z^2, [x,y,˓→z])
sage: quadratic.form()
a*x^2 + b*y^2 + c*z^2
sage: quadratic.dual().form()
b*c*x^2 + a*c*y^2 + a*b*z^2
sage: R.<x,y,z, t> = QQ[]
```
class method from_invariants(\texttt{discriminant, x, z, *args, **kwargs})

Construct a binary quadratic from its discriminant.

This function constructs a binary quadratic whose discriminant equal the one provided as argument up to scaling.

INPUT:

\begin{itemize}
  \item discriminant – Value of the discriminant used to reconstruct the binary quadratic.
\end{itemize}

OUTPUT:

A QuadraticForm with 2 variables.

EXAMPLES:

\begin{verbatim}
sage: R.<x,y> = QQ[]
sage: from sage.rings.invariants.invariant_theory import QuadraticForm
data: QuadraticForm(1, x, y)
Binary quadratic with coefficients (1, -1/4, 0)
\end{verbatim}

invariants(\texttt{type='discriminant'})

Return a tuple of invariants of a binary quadratic.

INPUT:

\begin{itemize}
  \item type – The type of invariants to return. The default choice is to return the discriminant.
\end{itemize}

OUTPUT:

The invariants of the binary quadratic.

EXAMPLES:

\begin{verbatim}
sage: R.<x0, x1> = QQ[]
sage: p = 2*x1^2 + 5*x0*x1 + 3*x0^2
sage: quadratic = invariant_theory.binary_quadratic(p, x0, x1)
sage: quadratic.invariants()
(1,)
sage: quadratic.invariants('unknown')
Traceback (most recent call last):
  ... ValueError: unknown type of invariants unknown for a binary quadratic
\end{verbatim}

matrix()

Return the quadratic form as a symmetric matrix.

OUTPUT:
This method returns a symmetric matrix $A$ such that the quadratic $Q$ equals

$$Q(x, y, z, \ldots) = (x, y, \ldots) A (x, y, \ldots)^t$$

**EXAMPLES:**

```python
sage: R.<x,y,z> = QQ[]
sage: quadratic = invariant_theory.ternary_quadratic(x^2+y^2+z^2+x*y)
sage: matrix(quadratic)
[ 1 1/2 0]
[1/2 1 0]
[ 0 0 1]
sage: quadratic._matrix_() == matrix(quadratic)
True
```

**monomials()**

List the basis monomials in the form.

**OUTPUT:**

A tuple of monomials. They are in the same order as `coeffs()`.

**EXAMPLES:**

```python
sage: R.<x,y> = QQ[]
sage: quadratic = invariant_theory.quadratic_form(x^2 + y^2)
sage: quadratic.monomials()
(x^2, y^2, x*y)
sage: quadratic = invariant_theory.inhomogeneous_quadratic_form(x^2 + y^2)
sage: quadratic.monomials()
(x^2, y^2, 1, x*y, x, y)
```

**scaled_coeffs()**

The scaled coefficients of a quadratic form.

Given

$$f(x) = \sum_{0 \leq i \leq n} a_i x_i^2 + \sum_{0 \leq j < k < n} 2a_{j,k} x_j x_k$$

this function returns $a = (a_0, \ldots, a_n, a_{00}, a_{01}, \ldots, a_{n-1,n})$.

**EXAMPLES:**

```python
sage: R.<a,b,c,d,e,f,g, x,y,z> = QQ[]
sage: p = a*x^2 + b*y^2 + c*z^2 + d*x*y + e*x*z + f*y*z
sage: inv = invariant_theory.quadratic_form(p, x,y,z); inv
Ternary quadratic with coefficients (a, b, c, d, e, f)
sage: inv.coeffs()
(a, b, c, d, e, f)
sage: inv.scaled_coeffs()
(a, b, c, 1/2*d, 1/2*e, 1/2*f)
```

**class** `sage.rings.invariants.invariant_theory.SeveralAlgebraicForms` *(forms)*

**Bases:** `FormsBase`

The base class of multiple algebraic forms (i.e. homogeneous polynomials).

You should only instantiate the derived classes of this base class.
See :class:`AlgebraicForm` for the base class of a single algebraic form.

**INPUT:**

- **forms** – a list/tuple/iterable of at least one :class:`AlgebraicForm` object, all with the same number of variables. Interpreted as multiple homogeneous polynomials in a common polynomial ring.

**EXAMPLES:**

```python
define get_form(i)
    Return the i-th form.

define homogenized(var='h')
    Return form as defined by a homogeneous polynomial.
```

**EXAMPLES:**

```python
R.<x,y> = QQ[]
quadratic_form(x^2 + y^2)
quadratic_form(x*y)
SeveralAlgebraicForms([q1, q2])
g12.get_form(0).is(q1)
q12[0].is(q12.get_form(0))  # syntactic sugar
q12[1].is(q12.get_form(1))  # syntactic sugar
```

**INPUT:**

- **var** – either a variable name, variable index or a variable (default: `'h'`).

**OUTPUT:**

The same algebraic form, but defined by a homogeneous polynomial.

**EXAMPLES:**

```python
R.<x,y,z> = QQ[]
quaternary_biquadratic(x^2 + 1, y^2 + 1, [x,y,z])
q = q.homogenized()
type(q).is(type(q.homogenized()))
```
n_forms()

Return the number of forms.

EXAMPLES:

```python
sage: R.<x,y> = QQ[]
sage: q1 = invariant_theory.quadratic_form(x^2 + y^2)
sage: q2 = invariant_theory.quadratic_form(x*y)
sage: from sage.rings.invariants.invariant_theory import SeveralAlgebraicForms
sage: q12 = SeveralAlgebraicForms([q1, q2])
sage: q12.n_forms()
2
sage: len(q12) == q12.n_forms()  # syntactic sugar
True
```

class sage.rings.invariants.invariant_theory.TernaryCubic(n, d, polynomial, *args)

Bases: AlgebraicForm

Invariant theory of a ternary cubic.

You should use the `invariant_theory` factory object to construct instances of this class. See `ternary_cubic()` for details.

Hessian()

Return the Hessian covariant.

OUTPUT:

The Hessian matrix multiplied with the conventional normalization factor 1/216.

EXAMPLES:

```python
sage: R.<x,y,z> = QQ[]
sage: cubic = invariant_theory.ternary_cubic(x^3 + y^3 + z^3)
sage: cubic.Hessian()
x*y*z
sage: R.<x,y> = QQ[]
sage: cubic = invariant_theory.ternary_cubic(x^3 + y^3 + 1)
sage: cubic.Hessian()
x*y
```

J_covariant()

Return the J-covariant of the ternary cubic.

EXAMPLES:

```python
sage: R.<x,y,z> = QQ[]
sage: cubic = invariant_theory.ternary_cubic(x^3 + y^3 + z^3)
sage: cubic.J_covariant()
x^6*y^3 - x^3*y^6 - x^6*z^3 + y^6*z^3 + x^3*z^6 - y^3*z^6
sage: R.<x,y> = QQ[]
sage: cubic = invariant_theory.ternary_cubic(x^3 + y^3 + 1)
sage: cubic.J_covariant()
x^6*y^3 - x^3*y^6 - x^6 + y^6 + x^3 - y^3
```

S_invariant()

Return the S-invariant.
EXAMPLES:

```python
sage: R.<x,y,z> = QQ[]
sage: cubic = invariant_theory.ternary_cubic(x^2*y + y^3 + z^3 + x*y*z)
sage: cubic.S_invariant()
-1/1296
```

**T_invariant()**

Return the T-invariant.

**EXAMPLES:**

```python
sage: R.<x,y,z> = QQ[]
sage: cubic = invariant_theory.ternary_cubic(x^3 + y^3 + z^3)
sage: cubic.T_invariant()
1
sage: R.<x,y,z,t> = GF(7)[]
sage: cubic = invariant_theory.ternary_cubic(x^3 + y^3 + z^3 + t*x*y*z, [x,y,->z])
sage: cubic.T_invariant()
-t^6 - t^3 + 1
```

**Theta_covariant()**

Return the Θ covariant.

**EXAMPLES:**

```python
sage: R.<x,y,z> = QQ[]
sage: cubic = invariant_theory.ternary_cubic(x^3 + y^3 + z^3)
sage: cubic.Theta_covariant()
-x^3*y^3 - x^3*z^3 - y^3*z^3
sage: R.<x,y> = QQ[]
sage: cubic = invariant_theory.ternary_cubic(x^3 + y^3 + 1)
sage: cubic.Theta_covariant()
-x^3*y^3 - x^3 - y^3
sage: R.<x,y,z,a30,a21,a12,a03,a20,a10,a01,a00> = QQ[]
sage: p = ( a30*x^3 + a21*x^2*y + a12*x*y^2 + a03*y^3 + a20*x^2*z +
       a11*x*y*z + a02*y^2*z + a10*x*z^2 + a01*y*z^2 + a00*z^3 )
sage: cubic = invariant_theory.ternary_cubic(p, x,y,z)
sage: len(list(cubic.Theta_covariant()))
6952
```

**coeffs()**

Return the coefficients of a cubic.

Given

\[
p(x, y) = a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{20}x^2 +
\]

\[
a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}
\]

this function returns \( a = (a_{30}, a_{03}, a_{00}, a_{21}, a_{20}, a_{12}, a_{02}, a_{10}, a_{01}, a_{11}) \)

**EXAMPLES:**
**Polynomials, Release 10.3**

```python
sage: R.<x,y,z,a30,a21,a12,a03,a20,a11,a02,a10,a01,a00> = QQ[]
sage: p = ( a30*x^3 + a21*x^2*y + a12*x*y^2 + a03*y^3 + a20*x^2*z +
.....:     a11*x*y*z + a02*y^2*z + a10*x*z^2 + a01*y*z^2 + a00*z^3 )
sage: invariant_theory.ternary_cubic(p, x,y,z).coeffs()
(a30, a03, a00, a21, a20, a12, a02, a10, a01, a11)
sage: invariant_theory.ternary_cubic(p.subs(z=1), x, y).coeffs()
(a30, a03, a00, a21, a20, a12, a02, a10, a01, a11)
```

**monomials()**

List the basis monomials of the form.

**OUTPUT:**

A tuple of monomials. They are in the same order as `coeffs()`.

**EXAMPLES:**

```python
sage: R.<x,y,z> = QQ[]
sage: cubic = invariant_theory.ternary_cubic(x^3+y*z^2)
sage: cubic.monomials()
(x^3, y^3, z^3, x^2*y, x^2*z, x*y^2, y^2*z, x*z^2, y*z^2, x*y*z)
```

**polar_conic()**

Return the polar conic of the cubic.

**OUTPUT:**

Given the ternary cubic \( f(X, Y, Z) \), this method returns the symmetric matrix \( A(x, y, z) \) defined by

\[
x f_X + y f_Y + z f_Z = (X, Y, Z) \cdot A(x, y, z) \cdot (X, Y, Z)^t
\]

**EXAMPLES:**

```python
sage: R.<x,y,z,X,Y,Z,a30,a21,a12,a03,a20,a11,a02,a10,a01,a00> = QQ[]
sage: p = ( a30*x^3 + a21*x^2*y + a12*x*y^2 + a03*y^3 + a20*x^2*z +
.....:     a11*x*y*z + a02*y^2*z + a10*x*z^2 + a01*y*z^2 + a00*z^3 )
sage: cubic = invariant_theory.ternary_cubic(p, x,y,z)
sage: cubic.polar_conic()

\[
\begin{bmatrix}
3*x*a30 + y*a21 + z*a20 & x*a21 + y*a12 + 1/2*z*a11 & x*a20 + 1/2*y*a11 + 1/2*z*a10 \\
\rightarrow z*a10 \\
x*a21 + y*a12 + 1/2*z*a11 & x*a12 + 3*y*a03 + z*a02 & 1/2*x*a11 + y*a02 + 1/2*z*a01 \\
\rightarrow z*a01 \\
x*a20 + 1/2*y*a11 + z*a10 & 1/2*x*a11 + y*a02 + z*a01 & x*a10 + y*a01 + 3*z*a00
\end{bmatrix}
\]
sage: polar_eqn = X*p.derivative(x) + Y*p.derivative(y) + Z*p.derivative(z)
sage: polar = invariant_theory.ternary_quadratic(polar_eqn, [x,y,z])
sage: polar.matrix().subs(X=x,Y=y,Z=z) == cubic.polar_conic()
True
```

**scaled_coeffs()**

Return the coefficients of a cubic.

Compared to `coeffs()`, this method returns rescaled coefficients that are often used in invariant theory.

Given

\[
p(x, y) = a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{20}x^2 +
\]

\[a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}\]
this function returns \( a = (a_{30}, a_{03}, a_{00}, a_{21}/3, a_{20}/3, a_{12}/3, a_{02}/3, a_{10}/3, a_{01}/3, a_{11}/6) \)

**EXAMPLES:**

```
sage: R.<x,y,z,a30,a21,a12,a03,a20,a10,a01,a00> = QQ[]
sage: p = (a30*x^3 + a21*x^2*y + a12*x*y^2 + a03*y^3 + a20*x^2*z +
     ....:     a11*x*y*z + a02*y^2*z + a10*x*z^2 + a01*y*z^2 + a00*z^3)
sage: invariant_theory.ternary_cubic(p, x,y,z).scaled_coeffs()
(a30, a03, a00, 1/3*a21, 1/3*a20, 1/3*a12, 1/3*a02, 1/3*a10, 1/3*a01, 1/6*a11)
```

**syzygy** \((U, S, T, H, \Theta, J)\)

Return the syzygy of the cubic evaluated on the invariants and covariants.

**INPUT:**

- \( U, S, T, H, \Theta, J \) – polynomials from the same polynomial ring.

**OUTPUT:**

0 if evaluated for the form, the S invariant, the T invariant, the Hessian, the \( \Theta \) covariant and the J-covariant of a ternary cubic.

**EXAMPLES:**

```
sage: R.<x,y,z> = QQ[]
sage: monomials = (x^3, y^3, z^3, x^2*y, x^2*z, x*y^2, 
     ....:     y^2*z, x*z^2, y*z^2, x*y*z)
sage: random_poly = sum([randint(0,10000) * m for m in monomials])
sage: cubic = invariant_theory.ternary_cubic(random_poly)
sage: U = cubic.form()
sage: S = cubic.S_invariant()
sage: T = cubic.T_invariant()
sage: H = cubic.Hessian()
sage: Theta = cubic.Theta_covariant()
sage: J = cubic.J_covariant()
sage: cubic.syzygy(U, S, T, H, Theta, J)
0
```

**class** \( \text{Sage} \cdot \text{Rings} \cdot \text{Invariants} \cdot \text{Invariant Theory} \cdot \text{TernaryQuadratic} \(n, d, \text{polynomial}, \ast \text{args})\)

Bases: \( \text{QuadraticForm} \)

Invariant theory of a ternary quadratic.

You should use the \( \text{Invariant Theory} \) factory object to construct instances of this class. See \text{ternary_quadratic}() for details.

**coeffs()**

Return the coefficients of a quadratic.

Given

\[
p(x, y) = a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}
\]

this function returns \( a = (a_{20}, a_{02}, a_{00}, a_{11}, a_{10}, a_{01}) \)

**EXAMPLES:**

```
sage: R.<x,y,z,a20,a11,a02,a10,a01,a00> = QQ[]
sage: p = (a20*x^2 + a11*x*y + a02*y^2 +
     ....:     a10*x*z + a01*y*z + a00*z^2)
sage: invariant_theory.ternary_quadratic(p, x,y,z).coeffs()
```
(continued from previous page)

```
(a20, a02, a00, a11, a10, a01)
sage: invariant_theory.ternary_quadratic(p.subs(z=1), x, y).coeffs()
(a20, a02, a00, a11, a10, a01)
```

covariant_conic(other)
Return the ternary quadratic covariant to self and other.

INPUT:

• other – Another ternary quadratic.

OUTPUT:
The so-called covariant conic, a ternary quadratic. It is symmetric under exchange of self and other.

EXAMPLES:
```
sage: ring.<x,y,z> = QQ[]
sage: Q = invariant_theory.ternary_quadratic(x^2 + y^2 + z^2)
sage: R = invariant_theory.ternary_quadratic(x*y + x*z + y*z)
sage: Q.covariant_conic(R)
-x*y - x*z - y*z
sage: R.covariant_conic(Q)
-x*y - x*z - y*z
```

monomials()
List the basis monomials of the form.

OUTPUT:
A tuple of monomials. They are in the same order as coeffs().

EXAMPLES:
```
sage: R.<x,y,z> = QQ[]
sage: quadratic = invariant_theory.ternary_quadratic(x^2 + y*z)
sage: quadratic.monomials()
(x^2, y^2, z^2, x*y, x*z, y*z)
```

scaled_coeffs()
Return the scaled coefficients of a quadratic.
Given

\[ p(x, y) = a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00} \]

this function returns \( a = (a_{20}, a_{02}, a_{00}, a_{11}/2, a_{10}/2, a_{01}/2, ) \).

EXAMPLES:
```
sage: R.<x,y,z,a20,a11,a02,a10,a01,a00> = QQ[]
sage: p = ( a20*x^2 + a11*x*y + a02*y^2 +
.....: a10*x*z + a01*y*z + a00*z^2 )
sage: invariant_theory.ternary_quadratic(p, x,y,z).scaled_coeffs()
(a20, a02, a00, 1/2*a11, 1/2*a10, 1/2*a01)
sage: invariant_theory.ternary_quadratic(p.subs(z=1), x, y).scaled_coeffs()
(a20, a02, a00, 1/2*a11, 1/2*a10, 1/2*a01)
```
class sage.rings.invariants.invariant_theory.TwoAlgebraicForms(forms)
    Bases: SeveralAlgebraicForms

    first()
    Return the first of the two forms.
    OUTPUT:
    The first algebraic form used in the definition.
    EXAMPLES:

    sage: R.<x,y> = QQ[]
    sage: q0 = invariant_theory.quadratic_form(x^2 + y^2)
    sage: q1 = invariant_theory.quadratic_form(x*y)
    sage: from sage.rings.invariants.invariant_theory import TwoAlgebraicForms
    sage: q = TwoAlgebraicForms([q0, q1])
    sage: q.first() is q0
    True
    sage: q.get_form(0) is q0
    True
    sage: q.first().polynomial()
    x^2 + y^2

    second()
    Return the second of the two forms.
    OUTPUT:
    The second form used in the definition.
    EXAMPLES:

    sage: R.<x,y> = QQ[]
    sage: q0 = invariant_theory.quadratic_form(x^2 + y^2)
    sage: q1 = invariant_theory.quadratic_form(x*y)
    sage: from sage.rings.invariants.invariant_theory import TwoAlgebraicForms
    sage: q = TwoAlgebraicForms([q0, q1])
    sage: q.second() is q1
    True
    sage: q.get_form(1) is q1
    True
    sage: q.second().polynomial()
    x*y

class sage.rings.invariants.invariant_theory.TwoQuaternaryQuadratics(forms)
    Bases: TwoAlgebraicForms

   Invariant theory of two quaternary quadratics.
    You should use the invariant_theory factory object to construct instances of this class. See quater-
    nary_biquadratics() for details.
    REFERENCES:
    • section on “Invariants and Covariants of Systems of Quadrics” in [Sal1958], [Sal1965]

    Delta_invariant()
    Return the $\Delta$ invariant.
    EXAMPLES:
Delta_prime_invariant():
Return the $\Delta'$ invariant.

EXAMPLES:

```
sage: R.<x,y,z,t,a0,a1,a2,a3,b0,b1,b2,b3,b4,b5,A0,A1,A2,A3,B0,B1,B2,B3,B4,B5> = QQ[]
sage: p1 = a0*x^2 + a1*y^2 + a2*z^2 + a3
sage: p1 += b0*x*y + b1*x*z + b2*x + b3*y*z + b4*y + b5*z
sage: p2 = A0*x^2 + A1*y^2 + A2*z^2 + A3
sage: p2 += B0*x*y + B1*x*z + B2*x + B3*y*z + B4*y + B5*z
sage: q = invariant_theory.quaternary_biquadratic(p1, p2, [x, y, z])
sage: coeffs = det(t * q[0].matrix() + q[1].matrix()).polynomial(t).
\rightarrow coefficients(sparse=False)
True
```

J_covariant():
The J-covariant.

This is the Jacobian determinant of the two biquadratics, the $T$-covariant, and the $T'$-covariant with respect to the four homogeneous variables.

EXAMPLES:

```
sage: R.<w,x,y,z,a0,a1,a2,a3,A0,A1,A2,A3> = QQ[]
sage: p1 = a0*x^2 + a1*y^2 + a2*z^2 + a3*w^2
sage: p2 = A0*x^2 + A1*y^2 + A2*z^2 + A3*w^2
sage: q = invariant_theory.quaternary_biquadratic(p1, p2, [w, x, y, z])
sage: q.J_covariant().factor()
z * y * x * w * (a3*A2 - a2*A3) * (a3*A1 - a1*A3) * (-a2*A1 + a1*A2)
* (a3*A0 - a0*A3) * (-a2*A0 + a0*A2) * (-a1*A0 + a0*A1)
```

Phi_invariant():
Return the $\Phi'$ invariant.

EXAMPLES:

```
sage: R.<x,y,z,t,a0,a1,a2,a3,b0,b1,b2,b3,b4,b5,A0,A1,A2,A3,B0,B1,B2,B3,B4,B5> = QQ[]
sage: p1 = a0*x^2 + a1*y^2 + a2*z^2 + a3
sage: p1 += b0*x*y + b1*x*z + b2*x + b3*y*z + b4*y + b5*z
sage: p2 = A0*x^2 + A1*y^2 + A2*z^2 + A3
sage: p2 += B0*x*y + B1*x*z + B2*x + B3*y*z + B4*y + B5*z
sage: q = invariant_theory.quaternary_biquadratic(p1, p2, [x, y, z])
sage: coeffs = det(t * q[0].matrix() + q[1].matrix()).polynomial(t).
\rightarrow\coefficients(sparse=False)
sage: q.Phi_invariant() == coeffs[4]
True
```

(continues on next page)
\[ \text{coefficients(sparse=False)} \]

```python
sage: q.Phi_invariant() == coeffs[2]
True
```

**T_covariant()**

The $T$-covariant.

**EXAMPLES:**

```python
sage: R.<x,y,z,t,a0,a1,a2,a3,b0,b1,b2,b3,b4,b5,A0,A1,A2,A3,B0,B1,B2,B3,B4,B5> = QQ[]
sage: p1 = a0*x^2 + a1*y^2 + a2*z^2 + a3
sage: p1 += b0*x*y + b1*x*z + b2*x + b3*y*z + b4*y + b5*z
sage: p2 = A0*x^2 + A1*y^2 + A2*z^2 + A3
sage: p2 += B0*x*y + B1*x*z + B2*x + B3*y*z + B4*y + B5*z
sage: q = invariant_theory.quaternary_biquadratic(p1, p2, [x, y, z])
sage: T = invariant_theory.quaternary_quadratic(q.T_covariant(), [x,y,z]).matrix()
sage: M = q[0].matrix().adjugate() + t*q[1].matrix().adjugate()
sage: M = M.adjugate().apply_map( # long time (4s on my thinkpad W530)
    lambda m: m.coefficient(t))
sage: M == q.Delta_invariant()*T
# long time
True
```

**T_prime_covariant()**

The $T'$-covariant.

**EXAMPLES:**

```python
sage: R.<x,y,z,t,a0,a1,a2,a3,b0,b1,b2,b3,b4,b5,A0,A1,A2,A3,B0,B1,B2,B3,B4,B5> = QQ[]
sage: p1 = a0*x^2 + a1*y^2 + a2*z^2 + a3
sage: p1 += b0*x*y + b1*x*z + b2*x + b3*y*z + b4*y + b5*z
sage: p2 = A0*x^2 + A1*y^2 + A2*z^2 + A3
sage: p2 += B0*x*y + B1*x*z + B2*x + B3*y*z + B4*y + B5*z
sage: q = invariant_theory.quaternary_biquadratic(p1, p2, [x, y, z])
sage: Tprime = invariant_theory.quaternary_quadratic(q.T_prime_covariant(), [x,y,z]).matrix()
sage: M = q[0].matrix().adjugate() + t^2*q[1].matrix().adjugate()
sage: M = M.adjugate().apply_map( # long time (4s on my thinkpad W530)
    lambda m: m.coefficient(t^2))
sage: M == q.Delta_prime_invariant() * Tprime
# long time
True
```

**Theta_invariant()**

Return the $\Theta$ invariant.

**EXAMPLES:**

```python
sage: R.<x,y,z,t,a0,a1,a2,a3,b0,b1,b2,b3,b4,b5,A0,A1,A2,A3,B0,B1,B2,B3,B4,B5> = QQ[]
sage: p1 = a0*x^2 + a1*y^2 + a2*z^2 + a3
sage: p1 += b0*x*y + b1*x*z + b2*x + b3*y*z + b4*y + b5*z
sage: p2 = A0*x^2 + A1*y^2 + A2*z^2 + A3
sage: p2 += B0*x*y + B1*x*z + B2*x + B3*y*z + B4*y + B5*z
```

(continues on next page)
**Theta_prime_invariant()**

Return the $\Theta'$ invariant.

**EXAMPLES:**

```python
sage: R.<x,y,z,t,a0,a1,a2,a3,b0,b1,b2,b3,b4,b5,A0,A1,A2,A3,B0,B1,B2,B3,B4,B5> = QQ[]
sage: p1 = a0*x^2 + a1*y^2 + a2*z^2 + a3
sage: p1 += b0*x*y + b1*x*z + b2*x + b3*y*z + b4*y + b5*z
sage: p2 = A0*x^2 + A1*y^2 + A2*z^2 + A3
sage: p2 += B0*x*y + B1*x*z + B2*x + B3*y*z + B4*y + B5*z
sage: q = invariant_theory.quaternary_biquadratic(p1, p2, [x, y, z])
sage: coeffs = det(t * q[0].matrix() + q[1].matrix()).polynomial(t).coefficients(sparse=False)
sage: q.Theta_prime_invariant() == coeffs[1]
True
```

**syzygy** *(Delta, Theta, Phi, Theta_prime, Delta_prime, U, V, T, T_prime, J)*

Return the syzygy evaluated on the invariants and covariants.

**INPUT:**

- Delta, Theta, Phi, Theta_prime, Delta_prime, U, V, T, T_prime, J - polynomials from the same polynomial ring.

**OUTPUT:**

Zero if the $U$ is the first polynomial, $V$ the second polynomial, and the remaining input are the invariants and covariants of a quaternary biquadratic.

**EXAMPLES:**

```python
sage: R.<w,x,y,z> = QQ[]
sage: monomials = [x^2, x*y, y^2, x*z, y*z, z^2, x*w, y*w, z*w, w^2]
sage: def q_rnd(): return sum(randint(-1000,1000)*m for m in monomials)
sage: biquadratic = invariant_theory.quaternary_biquadratic(q_rnd(), q_rnd())
sage: Delta = biquadratic.Delta_invariant()
sage: Theta = biquadratic.Theta_invariant()
sage: Phi = biquadratic.Phi_invariant()
sage: Theta_prime = biquadratic.Theta_prime_invariant()
sage: Delta_prime = biquadratic.Delta_prime_invariant()
sage: U = biquadratic.first().polynomial()
sage: V = biquadratic.second().polynomial()
sage: T = biquadratic.T_covariant()
sage: T_prime = biquadratic.T_prime_covariant()
sage: J = biquadratic.J_covariant()
sage: biquadratic.syzygy(Delta, Theta, Phi, Theta_prime, Delta_prime, U, V, T, T_prime, J)
0
```

If the arguments are not the invariants and covariants then the output is some (generically non-zero) polynomial:
class sage.rings.invariants.invariant_theory.TwoTernaryQuadratics(form)

Bases: TwoAlgebraicForms

Invariant theory of two ternary quadratics.

You should use the invariant_theory factory object to construct instances of this class. See ternary_biquadratics() for details.

REFERENCES:

• Section on “Invariants and Covariants of Systems of Conics”, Art. 388 (a) in [Sal1954]

Delta_invariant()

Return the $\Delta$ invariant.

EXAMPLES:

sage: R.<a00, a01, a11, a02, a12, b00, b01, b11, b02, b12, y0, y1, y2, t> = QQ[]

sage: p1 = a00*y0^2 + 2*a01*y0*y1 + a11*y1^2 + 2*a02*y0*y2 + 2*a12*y1*y2 + ... + a22*y2^2
sage: p2 = b00*y0^2 + 2*b01*y0*y1 + b11*y1^2 + 2*b02*y0*y2 + 2*b12*y1*y2 + ... + b22*y2^2
sage: q = invariant_theory.ternary_biquadratic(p1, p2, [y0, y1, y2])

sage: coeffs = det(t * q[0].matrix() + q[1].matrix()).polynomial(t).

sage: q.Delta_invariant() == coeffs[3]
True

Delta_prime_invariant()

Return the $\Delta'$ invariant.

EXAMPLES:

sage: R.<a00, a01, a11, a02, a12, b00, b01, b11, b02, b12, y0, y1, y2, t> = QQ[]

sage: p1 = a00*y0^2 + 2*a01*y0*y1 + a11*y1^2 + 2*a02*y0*y2 + 2*a12*y1*y2 + ... + a22*y2^2
sage: p2 = b00*y0^2 + 2*b01*y0*y1 + b11*y1^2 + 2*b02*y0*y2 + 2*b12*y1*y2 + ... + b22*y2^2
sage: q = invariant_theory.ternary_biquadratic(p1, p2, [y0, y1, y2])

sage: coeffs = det(t * q[0].matrix() + q[1].matrix()).polynomial(t).

sage: q.Delta_prime_invariant() == coeffs[0]
True

F_covariant()

Return the $F$ covariant.

EXAMPLES:

sage: R.<a00, a01, a11, a02, a12, a22, b00, b01, b11, b02, b12, b22, x, y> = QQ[]

sage: p1 = 73*x^2 + 96*x*y - 11*y^2 + 4*x + 63*y + 57
sage: p2 = 61*x^2 - 100*x*y - 72*y^2 - 81*x + 39*y - 7

sage: q = invariant_theory.ternary_biquadratic(p1, p2, [x, y])

(continues on next page)
Polynomials, Release 10.3

sage: q.F_covariant()
-32566577*x^2 + 29060637/2*x*y + 20153633/4*y^2 -
30250497/2*x - 241241273/4*y - 323820473/16

J_covariant()

Return the $J$ covariant.

EXAMPLES:

sage: R.<a00, a01, a11, a02, a12, b00, b01, b11, b02, b12, b22, x, y> =
    QQ[]

sage: p1 = 73*x^2 + 96*x*y - 11*y^2 + 4*x + 63*y + 57
sage: p2 = 61*x^2 - 100*x*y - 72*y^2 - 81*x + 39*y - 7
sage: q = invariant_theory.ternary_biquadratic(p1, p2, [x, y])

sage: q.J_covariant()
1057324024445*x^3 + 1209531088209*x^2*y + 942116599708*x*y^2 +
984553030871*y^3 + 543715345505/2*x^2 - 306509356021/2*x*y +
755263948570*y^2 - 1118430692650*x - 509948695327/4*y + 3369951531745/8

Theta_invariant()

Return the $\Theta$ invariant.

EXAMPLES:

sage: R.<a00, a01, a11, a02, a12, b00, b01, b11, b02, b12, b22, y0, y1, y2, t> = QQ[]

sage: p1 = a00*y0^2 + 2*a01*y0*y1 + a11*y1^2 + 2*a02*y0*y2 + 2*a12*y1*y2 +
    a22*y2^2
sage: p2 = b00*y0^2 + 2*b01*y0*y1 + b11*y1^2 + 2*b02*y0*y2 + 2*b12*y1*y2 +
    b22*y2^2
sage: q = invariant_theory.ternary_biquadratic(p1, p2, [y0, y1, y2])

sage: coeffs = det(t * q[0].matrix() + q[1].matrix()).polynomial(t).
    coefficients(sparse=False)

sage: q.Theta_invariant() == coeffs[2]
True

Theta_prime_invariant()

Return the $\Theta'$ invariant.

EXAMPLES:

sage: R.<a00, a01, a11, a02, a12, b00, b01, b11, b02, b12, b22, y0, y1, y2, t> = QQ[]

sage: p1 = a00*y0^2 + 2*a01*y0*y1 + a11*y1^2 + 2*a02*y0*y2 + 2*a12*y1*y2 +
    a22*y2^2
sage: p2 = b00*y0^2 + 2*b01*y0*y1 + b11*y1^2 + 2*b02*y0*y2 + 2*b12*y1*y2 +
    b22*y2^2
sage: q = invariant_theory.ternary_biquadratic(p1, p2, [y0, y1, y2])

sage: coeffs = det(t * q[0].matrix() + q[1].matrix()).polynomial(t).
    coefficients(sparse=False)

sage: q.Theta_prime_invariant() == coeffs[1]
True

syzygy (Delta, Theta, Theta_prime, Delta_prime, S, S_prime, F, J)

Return the syzygy evaluated on the invariants and covariants.

INPUT:
• Delta, Theta, Theta_prime, Delta_prime, S, S_prime, F, J – polynomials from the same polynomial ring.

OUTPUT:
Zero if S is the first polynomial, S_prime the second polynomial, and the remaining input are the invariants and covariants of a ternary biquadratic.

EXAMPLES:

sage: R.<x,y,z> = QQ[]
sage: monomials = [x^2, x*y, y^2, x*z, y*z, z^2]
sage: def q_rnd():
    return sum(randint(-1000,1000)*m for m in monomials)
sage: biquadratic = invariant_theory.ternary_biquadratic(q_rnd(), q_rnd(), [x, y, z])
sage: Delta = biquadratic.Delta_invariant()
sage: Theta = biquadratic.Theta_invariant()
sage: Theta_prime = biquadratic.Theta_prime_invariant()
sage: Delta_prime = biquadratic.Delta_prime_invariant()
sage: S = biquadratic.first().polynomial()
sage: S_prime = biquadratic.second().polynomial()
sage: F = biquadratic.F_covariant()
sage: J = biquadratic.J_covariant()
sage: biquadratic.syzygy(Delta, Theta, Theta_prime, Delta_prime, S, S_prime, F, J)
0

If the arguments are not the invariants and covariants then the output is some (generically non-zero) polynomial:

sage: biquadratic.syzygy(1, 1, 1, 1, 1, 1, 1, x)
1/64*x^2 + 1

sage.rings.invariants.invariant_theory.transvectant(f, g, h=1, scale='default')
Return the h-th transvectant of f and g.

INPUT:
• f, g – two homogeneous binary forms in the same polynomial ring.
• h – the order of the transvectant. If it is not specified, the first transvectant is returned.
• scale – the scaling factor applied to the result. Possible values are 'default' and 'none'. The 'default' scaling factor is the one that appears in the output statement below, if the scaling factor is 'none' the quotient of factorials is left out.

OUTPUT:
The h-th transvectant of the listed forms f and g:

$$(f, g)_h = \frac{(d_f - h)! \cdot (d_g - h)!}{d_f! \cdot d_g!} \left( \frac{\partial}{\partial x} \frac{\partial}{\partial z} - \frac{\partial}{\partial x'} \frac{\partial}{\partial z'} \right)^h \left( f(x, z) \cdot g(x', z') \right)_{(x', z')=(x, z)}$$

EXAMPLES:

sage: from sage.rings.invariants.invariant_theory import AlgebraicForm, transvectant
sage: R.<x,y> = QQ[]
sage: f = AlgebraicForm(2, 5, x^5 + 5*x^4*y + 5*x*y^4 + y^5)
sage: transvectant(f, f, 4)
(continues on next page)
Binary quadratic given by \(2x^2 - 4xy + 2y^2\)

\[
sage: \text{transvectant}(f, f, 8)
\]

Binary form of degree \(-6\) given by \(0\)

The default scaling will yield an error for fields of positive characteristic below \(d_f\) or \(d_g\) as the denominator of the scaling factor will not be invertible in that case. The scale argument 'none' can be used to compute the transvectant in this case:

\[
sage: \text{# needs sage.rings.finite_rings}
\]

\[
sage: R.<a0,a1,a2,a3,a4,a5,x0,x1> = GF(5)[]
\]

\[
sage: f = \text{AlgebraicForm}(2, 5, a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 + a5*x0^5, x0, x1)
\]

\[
sage: \text{transvectant}(f, f, 4)
\]

Traceback (most recent call last):
... ZeroDivisionError

\[
sage: \text{transvectant}(f, f, 4, scale='none')
\]

Binary quadratic given by \(-a3^2x0^2 + a2*a4*x0^2 + a2*a3*x0*x1 - a1*a4*x0*x1 - a2^2*x1^2 + a1*a3*x1^2\)

The additional factors that appear when scale='none' is used can be seen if we consider the same transvectant over the rationals and compare it to the scaled version:

\[
sage: R.<a0,a1,a2,a3,a4,a5,x0,x1> = QQ[]
\]

\[
sage: f = \text{AlgebraicForm}(2, 5, a0*x1^5 + a1*x1^4*x0 + a2*x1^3*x0^2 + a3*x1^2*x0^3 + a4*x1*x0^4 + a5*x0^5, x0, x1)
\]

\[
sage: \text{transvectant}(f, f, 4)
\]

Binary quadratic given by \(3/50*a3^2*x0^2 - 4/25*a2*a4*x0^2 + 2/5*a0*a5*x0*x1 + 3/50*a2^2*x1^2 - 4/25*a1*a3*x1^2 + 2/5*a0*a4*x1^2\)

\[
sage: \text{transvectant}(f, f, 4, scale='none')
\]

Binary quadratic given by \(864*a3^2*x0^2 - 2304*a2*a4*x0^2 + 5760*a1*a5*x0^2 + 28800*a0*a5*x0*x1 + 864*a2^2*x1^2 - 2304*a1*a3*x1^2 + 5760*a0*a4*x1^2\)

If the forms are given as inhomogeneous polynomials, the homogenisation might fail if the polynomial ring has multiple variables. You can circumvent this by making sure the base ring of the polynomial has only one variable:

\[
sage: R.<x,y> = QQ[]
\]

\[
sage: \text{quintic = invariant_theory.binary_quintic(x^5 + x^3 + 2*x^2 + y^5, x)}
\]

\[
sage: \text{transvectant(quintic, quintic, 2)}
\]

Traceback (most recent call last):
... ValueError: polynomial is not homogeneous

\[
sage: R.<y> = QQ[]
\]

\[
sage: S.<x> = R[]
\]

\[
sage: \text{quintic = invariant_theory.binary_quintic(x^5 + x^3 + 2*x^2 + y^5, x)}
\]

\[
sage: \text{transvectant(quintic, quintic, 2)}
\]

Binary sextic given by \(1/5*x^6 + 6/5*x^5*h - 3/25*x^4*h^2 + (50*y^5 - 8)/25*x^3*h^3 - 12/25*x^2*h^4 + (3*y^5)/5*x*h^5 + (2*y^5)/5*h^6\)
3.2.2 Reconstruction of Algebraic Forms

This module reconstructs algebraic forms from the values of their invariants. Given a set of (classical) invariants, it returns a form that attains this values as invariants (up to scaling).

AUTHORS:
- Jesper Noordsij (2018-06): initial version

sage.rings.invariants.reconstruction.binary_cubic_coefficients_from_invariants

Reconstruct a binary cubic from the value of its discriminant.

INPUT:
- discriminant – The value of the discriminant of the binary cubic.
- invariant_choice – The type of invariants provided. The accepted options are 'discriminant' and 'default', which are the same. No other options are implemented.

OUTPUT:
A set of coefficients of a binary cubic, whose discriminant is equal to the given discriminant up to a scaling.

EXAMPLES:

sage: from sage.rings.invariants.reconstruction import binary_cubic_coefficients_from_invariants
sage: coeffs = binary_cubic_coefficients_from_invariants(1)
sage: coeffs
(0, 1, -1, 0)
sage: R.<x> = QQ[]
sage: R(coeffs).discriminant()  # needs sage.libs.pari
1

The two non-equivalent cubics $x^3$ and $x^2 \cdot z$ with discriminant 0 can’t be distinguished based on their discriminant, hence an error is raised:

sage: binary_cubic_coefficients_from_invariants(0)
Traceback (most recent call last):
  ... ValueError: no unique reconstruction possible for binary cubics with a double root

sage.rings.invariants.reconstruction.binary_quadratic_coefficients_from_invariants

Reconstruct a binary quadratic from the value of its discriminant.
INPUT:

- **discriminant** – The value of the discriminant of the binary quadratic.
- **invariant_choice** – The type of invariants provided. The accepted options are 'discriminant' and 'default', which are the same. No other options are implemented.

OUTPUT:

A set of coefficients of a binary quadratic, whose discriminant is equal to the given discriminant up to a scaling.

EXAMPLES:

```python
sage: from sage.rings.invariants.reconstruction import binary_quadratic_coefficients_from_invariants
sage: quadratic = invariant_theory.binary_form_from_invariants(2, [24])  # indirect doctest
sage: quadratic
Binary quadratic with coefficients (1, -6, 0)
sage: quadratic.discriminant()
24
sage: binary_quadratic_coefficients_from_invariants(0)
(1, 0, 0)
```

Reconstruct a binary quintic from the values of its (Clebsch) invariants.

INPUT:

- **invariants** – A list or tuple of values of the three or four invariants. The default option requires the Clebsch invariants $A$, $B$, $C$ and $R$ of the binary quintic.
- **K** – The field over which the quintic is defined.
- **invariant_choice** – The type of invariants provided. The accepted options are 'clebsch' and 'default', which are the same. No other options are implemented.
- **scaling** – How the coefficients should be scaled. The accepted values are 'none' for no scaling, 'normalized' to scale in such a way that the resulting coefficients are independent of the scaling of the input invariants and 'coprime' which scales the input invariants by dividing them by their gcd.

OUTPUT:

A set of coefficients of a binary quintic, whose invariants are equal to the given invariants up to a scaling.

EXAMPLES:

First we check the general case, where the invariant $M$ is non-zero:

```python
sage: R.<x0, x1> = QQ[]
sage: p = 3*x1^5 + 6*x1^4*x0 + 3*x1^3*x0^2 + 4*x1^2*x0^3 - 5*x1*x0^4 + 4*x0^5
```
We can see that the invariants of the reconstructed form match the ones of the original form by scaling the invariants $B$ and $C$:

```python
sage: scale = invs[0]/reconstructed.A_invariant()
True
True
```

If we compare the form obtained by this reconstruction to the one found by letting the covariants $\alpha$ and $\beta$ be the coordinates of the form, we find the forms are the same up to a power of the determinant of $\alpha$ and $\beta$:

```python
sage: alpha = quintic.alpha_covariant()
sage: beta = quintic.beta_covariant()
sage: g = matrix([[alpha(x0=1,x1=0), alpha(x0=0,x1=1)],
              [beta(x0=1,x1=0), beta(x0=0,x1=1)]])^-1
sage: transformed = tuple([g.determinant()^-5*x
                        for x in quintic.transformed(g).coeffs()])
sage: transformed == reconstructed.coeffs()
True
```

This can also be seen by computing the $\alpha$ covariant of the obtained form:

```python
sage: reconstructed.alpha_covariant().coefficient(x1)
0
sage: reconstructed.alpha_covariant().coefficient(x0) != 0
True
```

If the invariant $M$ vanishes, then the coefficients are computed in a different way:

```python
sage: [A,B,C] = [3,1,2]
sage: M = 2*A*B - 3*C
sage: M
0
sage: from sage.rings.invariants.reconstruction import binary_quintic_...
    coefficients_from_invariants
sage: reconstructed = binary_quintic_coefficients_from_invariants([A,B,C])
sage: reconstructed
(-66741943359375/2097152,
 -12514143798828125/134217728,
 0,
 52793920040130615234375/34359738368,
 19797720015048980712890625/1099511627776,)
```

(continues on next page)

```python
-4454487003386020660400390625/17592186044416
sage: newform = sum([ reconstructed[i]*x0^i*x1^(5-i) for i in range(6) ])  
sage: newquintic = invariant_theory.binary_quintic(newform, x0, x1)  
sage: scale = 3/newquintic.A_invariant()  
sage: [3, newquintic.B_invariant()*scale^2, newquintic.C_invariant()*scale^3]  
[3, 1, 2]
```

Several special cases:

```python
sage: quintic = invariant_theory.binary_quintic(x0^5 - x1^5, x0, x1)  
sage: invs = quintic.clebsch_invariants(as_tuple=True)  
sage: binary_quintic_coefficients_from_invariants(invs)  
(1, 0, 0, 0, 1)
```

```python
sage: quintic = invariant_theory.binary_quintic(x0*x1*(x0^3-x1^3), x0, x1)  
sage: invs = quintic.clebsch_invariants(as_tuple=True)  
sage: binary_quintic_coefficients_from_invariants(invs)  
(0, 1, 0, 0, 1, 0)
```

```python
sage: quintic = invariant_theory.binary_quintic(x0^5 + 10*x0^3*x1^2 - 15*x0*x1^4, x0, x1)  
sage: invs = quintic.clebsch_invariants(as_tuple=True)  
sage: binary_quintic_coefficients_from_invariants(invs)  
(1, 0, 10, 0, -15, 0)
```

```python
sage: quintic = invariant_theory.binary_quintic(x0^2*(x0^3 + x1^3), x0, x1)  
sage: invs = quintic.clebsch_invariants(as_tuple=True)  
sage: binary_quintic_coefficients_from_invariants(invs)  
(1, 0, 0, 1, 0, 0)
```

```python
sage: quintic = invariant_theory.binary_quintic(x0*(x0^4 + x1^4), x0, x1)  
sage: invs = quintic.clebsch_invariants(as_tuple=True)  
sage: binary_quintic_coefficients_from_invariants(invs)  
(1, 0, 0, 0, 1, 0)
```

For fields of characteristic 2, 3 or 5, there is no reconstruction implemented. This is part of [github issue #26786](https://github.com/sagemath/sage/issues/26786):}

```python
sage: binary_quintic_coefficients_from_invariants([3,1,2], K=GF(5))  
Traceback (most recent call last):  
...  
NotImplementedError: no reconstruction of binary quintics implemented for fields of characteristic 2, 3 or 5
```

### 3.3 Educational Versions of Groebner Basis Related Algorithms

#### 3.3.1 Educational versions of Groebner basis algorithms

Following [BW1993], the original Buchberger algorithm (algorithm GROEBNER in [BW1993]) and an improved version of Buchberger's algorithm (algorithm GROEBNERNEW2 in [BW1993]) are implemented.

No attempt was made to optimize either algorithm as the emphasis of these implementations is a clean and easy presentation. To compute a Groebner basis most efficiently in Sage, use the `MPolynomialIdeal.groebner_basis()` method on multivariate polynomial objects instead.

**Note:** The notion of ‘term’ and ‘monomial’ in [BW1993] is swapped from the notion of those words in Sage (or the other way around, however you prefer it). In Sage a term is a monomial multiplied by a coefficient, while in [BW1993] a
monomial is a term multiplied by a coefficient. Also, what is called LM (the leading monomial) in Sage is called HT (the head term) in [BW1993].

EXAMPLES:
Consider Katsura-6 with respect to a degrevlex ordering.

```
sage: # needs sage.libs.singular sage.rings.finite_rings
sage: from sage.rings.polynomial.toy_buchberger import *
sage: P.<a,b,c,e,f,g,h,i,j,k> = PolynomialRing(GF(32003))
sage: I = sage.rings.ideal.Katsura(P, 6)
sage: g1 = buchberger(I)
sage: g2 = buchberger_improved(I)
sage: g3 = I.groebner_basis()
```

All algorithms actually compute a Groebner basis:

```
sage: # needs sage.libs.singular sage.rings.finite_rings
sage: Ideal(g1).basis_is_groebner()
True
sage: Ideal(g2).basis_is_groebner()
True
sage: Ideal(g3).basis_is_groebner()
True
```

The results are correct:

```
sage: # needs sage.libs.singular sage.rings.finite_rings
sage: Ideal(g1) == Ideal(g2) == Ideal(g3)
True
```

If `get_verbose()` is $\geq 1$, a protocol is provided:

```
sage: # needs sage.libs.singular sage.rings.finite_rings
sage: from sage.misc.verbose import set_verbose
sage: set_verbose(1)
sage: P.<a,b,c> = PolynomialRing(GF(127))
// sage... ideal
sage: I
Ideal (a + 2*b + 2*c - 1, a^2 + 2*b^2 + 2*c^2 - a, 2*a*b + 2*b*c - b)
of Multivariate Polynomial Ring in a, b, c over Finite Field of size 127
sage: buchberger(I) # random
(a + 2*b + 2*c - 1, a^2 + 2*b^2 + 2*c^2 - a) => -2*b^2 - 6*b*c - 6*c^2 + b + 2*c
G: set([a + 2*b + 2*c - 1, 2*a*b + 2*b*c - b, a^2 + 2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c])
(a^2 + 2*b^2 + 2*c^2 - a, a + 2*b + 2*c - 1) => 0
G: set([a + 2*b + 2*c - 1, 2*a*b + 2*b*c - b, a^2 + 2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c])
(a + 2*b + 2*c - 1, 2*a*b + 2*b*c - b) => -5*b*c - 6*c^2 - 63*b + 2*c
G: set([a + 2*b + 2*c - 1, 2*a*b + 2*b*c - b, -5*b*c - 6*c^2 - 63*b + 2*c, a^2 + 2*b^2 - 2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c])
(2*a*b + 2*b*c - b, a + 2*b + 2*c - 1) => 0
G: set([a + 2*b + 2*c - 1, 2*a*b + 2*b*c - b, -5*b*c - 6*c^2 - 63*b + 2*c, a^2 + 2*b^2 - 2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c])
(continues on next page)
→ 2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c))

(2*a*b + 2*b*c - b, -5*b*c - 6*c^2 - 63*b + 2*c) => -22*c^3 + 24*c^2 - 60*b - 62*c
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 -
→ 2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

(2*a*b + 2*b*c - b, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
→ 2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

(a + 2*b + 2*c - 1, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
→ 2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

(a + 2*b + 2*c - 1, -5*b*c - 6*c^2 - 63*b + 2*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
→ 2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

(-5*b*c - 6*c^2 - 63*b + 2*c, -22*c^3 + 24*c^2 - 60*b - 62*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
→ 2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

(a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
→ 2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

(a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
→ 2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

(a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
→ 2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

(a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
→ 2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

(a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
→ 2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])

(a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c) => 0
G: set([a + 2*b + 2*c - 1, -22*c^3 + 24*c^2 - 60*b - 62*c, 2*a*b + 2*b*c - b, a^2 +
→ 2*b^2 + 2*c^2 - a, -2*b^2 - 6*b*c - 6*c^2 + b + 2*c, -5*b*c - 6*c^2 - 63*b + 2*c])
The original Buchberger algorithm performs 15 useless reductions to zero for this example:

```
sage: # needs sage.libs.singular sage.rings.finite_rings
sage: gb = buchberger(I)
...  
15 reductions to zero.
```

The ‘improved’ Buchberger algorithm in contrast only performs 1 reduction to zero:

```
sage: # needs sage.libs.singular sage.rings.finite_rings
sage: gb = buchberger_improved(I)
...  
1 reductions to zero.
sage: sorted(gb)
[a + 2*b + 2*c - 1, b*c + 52*c^2 + 38*b + 25*c, 
b^2 - 26*c^2 - 51*b + 51*c, c^3 + 22*c^2 - 55*b + 49*c]
```

AUTHORS:

- Marshall Hampton (2009-07-08): some doctest additions

Note: The verbosity of this function may be controlled with a `set_verbose()` call. Any value >=1 will result in this function printing intermediate bases.

EXAMPLES:

```
sage: from sage.rings.polynomial.toy_buchberger import buchberger
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: I = R.ideal([x^2 - z - 1, z^2 - y - 1, x*y^2 - x - 1])
sage: set_verbose(0)
sage: gb = buchberger(I)  # needs sage.libs.singular
sage: gb.is_groebner()  # needs sage.libs.singular
sage: gb
```

(continues on next page)
sage.rings.polynomial.toy_buchberger.buchberger_improved($F$)

Compute a Groebner basis using an improved version of Buchberger’s algorithm as presented in [BW1993], page 232.

This variant uses the Gebauer-Moeller Installation to apply Buchberger’s first and second criterion to avoid useless pairs.

**INPUT:**

- $F$ – an ideal in a multivariate polynomial ring

**OUTPUT:** a Groebner basis for $F$

**Note:** The verbosity of this function may be controlled with a `setVerbose()` call. Any value $\geq 1$ will result in this function printing intermediate Groebner bases.

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.toy_buchberger import buchberger_improved
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: setVerbose(0)
sage: sorted(buchberger_improved(R.ideal([x^4 - y - z, x*y*z - 1]))) #...
[x*y*z - 1, x^3 - y^2*z - y*z^2, y^3*z^2 + y^2*z^3 - x^2]
```

sage.rings.polynomial.toy_buchberger.inter_reduction($Q$)

Compute inter-reduced polynomials from a set of polynomials.

**INPUT:**

- $Q$ – a set of polynomials

**OUTPUT:**

if $Q$ is the set $f_1, ..., f_n$, this method returns $g_1, ..., g_s$ such that:

- $(f_1, ..., f_n) = (g_1, ..., g_s)$
- $LM(g_i) \neq LM(g_j)$ for all $i \neq j$
- $LM(g_i)$ does not divide $m$ for all monomials $m$ of $\{g_1, ..., g_{i-1}, g_{i+1}, ..., g_s\}$
- $LC(g_i) = 1$ for all $i$.

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.toy_buchberger import inter_reduction
sage: inter_reduction(set())
set()
```

```python
sage: P.<x,y> = QQ[]
sage: reduced = inter_reduction(set([x^2 - 5*y^2, x^3])) #...
Needs sage.libs.singular
```
Polynomials, Release 10.3

sage: reduced == set([x*y^2, x^2 - 5*y^2])  #...
needs sage.libs.singular
True
sage: reduced == inter_reduction(set([2*(x^2 - 5*y^2), x^3]))  #...
needs sage.libs.singular
True

sage.rings.polynomial.toy_buchberger.select(P)

Select a polynomial using the normal selection strategy.

INPUT:

- P – a list of critical pairs

OUTPUT: an element of P

EXAMPLES:

```python
sage: from sage.rings.polynomial.toy_buchberger import select
sage: R.<x,y,z> = PolynomialRing(QQ, order='lex')
sage: ps = [x^3 - z - 1, z^3 - y - 1, x^5 - y - 2]
sage: pairs = [[ps[i], ps[j]] for i in range(3) for j in range(i + 1, 3)]
sage: select(pairs)
[x^3 - z - 1, -y + z^3 - 1]
```

sage.rings.polynomial.toy_buchberger.spol(f, g)

Compute the S-polynomial of f and g.

INPUT:

- f, g – polynomials

OUTPUT: the S-polynomial of f and g

EXAMPLES:

```python
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: from sage.rings.polynomial.toy_buchberger import spol
sage: spol(x^2 - z - 1, z^2 - y - 1)
x^2*y - z^3 + x^2 - z^2
```

sage.rings.polynomial.toy_buchberger.update(G, B, h)

Update G using the set of critical pairs B and the polynomial h as presented in [BW1993], page 230. For this, Buchberger's first and second criterion are tested.

This function implements the Gebauer-Moeller Installation.

INPUT:

- G – an intermediate Groebner basis
- B – a set of critical pairs
- h – a polynomial

OUTPUT: a tuple of

- an intermediate Groebner basis
- a set of critical pairs

EXAMPLES:
sage: from sage.rings.polynomial.toy_buchberger import update
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: set_verbose(0)
sage: update(set(), set(), x*y*z)
({x*y*z}, set())
sage: G, B = update(set(), set(), x*y*z - 1)

3.3.2 Educational versions of Groebner basis algorithms: triangular factorization

In this file is the implementation of two algorithms in [Laz1992].

The main algorithm is Triangular; a secondary algorithm, necessary for the first, is ElimPolMin. As per Lazard's formulation, the implementation works with any term ordering, not only lexicographic.

Lazard does not specify a few of the subalgorithms implemented as the functions

- `is_triangular`
- `is_linearly_dependent`, and
- `linear_representation`.

The implementations are not hard, and the choice of algorithm is described with the relevant function.

No attempt was made to optimize these algorithms as the emphasis of this implementation is a clean and easy presentation.

Examples appear with the appropriate function.

AUTHORS:

- John Perry (2009-02-24): initial version, but some words of documentation were stolen shamelessly from Martin Albrecht’s toy_buchberger.py.

sage.rings.polynomial.toy_variety.coefficient_matrix(polys)

Generate the matrix \( M \) whose entries are the coefficients of \( \text{polys} \).

The entries of row \( i \) of \( M \) consist of the coefficients of \( \text{polys}[i] \).

INPUT:

- `polys` – a list/tuple of polynomials

OUTPUT:

A matrix \( M \) of the coefficients of \( \text{polys} \)

EXAMPLES:

```python
sage: from sage.rings.polynomial.toy_variety import coefficient_matrix
sage: R.<x,y> = PolynomialRing(QQ)
sage: coefficient_matrix([x^2 + 1, y^2 + 1, x*y + 1])
```

needs sage.modules

\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1
\end{bmatrix}
\]
sage.rings.polynomial.toy_variety.elim_pol($B, n=-1$)

Find the unique monic polynomial of lowest degree and lowest variable in the ideal described by $B$.

For the purposes of the triangularization algorithm, it is necessary to preserve the ring, so $n$ specifies which variable to check. By default, we check the last one, which should also be the smallest.

The algorithm may not work if you are trying to cheat: $B$ should describe the Groebner basis of a zero-dimensional ideal. However, it is not necessary for the Groebner basis to be lexicographic.

The algorithm is taken from a 1993 paper by Lazard [Laz1992].

INPUT:

- $B$ – a list/tuple of polynomials or a multivariate polynomial ideal
- $n$ – the variable to check (see above) (default: -1)

EXAMPLES:

```sage
sage: # needs sage.rings.finite_rings
sage: from sage.misc.verbose import set_verbose
sage: set_verbose(0)

sage: from sage.rings.polynomial.toy_variety import elim_pol

sage: R.<x,y,z> = PolynomialRing(GF(32003))

sage: p1 = x^2*(x-1)^3*y^2*(z-3)^3
sage: p2 = z^2 - z
sage: p3 = (x-2)^2*(y-1)^3

sage: I = R.ideal(p1,p2,p3)

sage: elim_pol(I.groebner_basis())
# ... needs sage.libs.singular
z^2 - z
```

sage.rings.polynomial.toy_variety.is_linearly_dependent($polys$)

Decide whether the polynomials of $polys$ are linearly dependent.

Here $polys$ is a collection of polynomials.

The algorithm creates a matrix of coefficients of the monomials of $polys$. It computes the echelon form of the matrix, then checks whether any of the rows is the zero vector.

Essentially this relies on the fact that the monomials are linearly independent, and therefore is building a linear map from the vector space of the monomials to the canonical basis of $\mathbb{R}^n$, where $n$ is the number of distinct monomials in $polys$. There is a zero vector if there is a linear dependence among $polys$.

The case where $polys=[]$ is considered to be not linearly dependent.

INPUT:

- $polys$ – a list/tuple of polynomials

OUTPUT:

True if the elements of $polys$ are linearly dependent; False otherwise.

EXAMPLES:
sage: from sage.rings.polynomial.toy_variety import is_linearly_dependent
sage: R.<x,y> = PolynomialRing(QQ)
sage: B = [x^2 + 1, y^2 + 1, x*y + 1]
sage: is_linearly_dependent(B + [p])  # needs sage.modules
True
sage: p = x*B[0]
sage: is_linearly_dependent(B + [p])  # needs sage.modules
False
sage: R.<x> = PolynomialRing(QQ)
sage: B = [x^147 + x^99,
....: 2*x^123 + x^75,
....: x^147 + 2*x^123 + 2*x^75,
....: 2*x^147 + x^99 + x^75]
sage: is_linearly_dependent(B)
True

sage.rings.polynomial.toy_variety.is_triangular(B)
Check whether the basis \( B \) of an ideal is triangular.
That is: check whether the largest variable in \( B[i] \) with respect to the ordering of the base ring \( R \) is \( R.\ gens()[i] \).
The algorithm is based on the definition of a triangular basis, given by Lazard in 1992 in [Laz1992].

INPUT:
- \( B \) – a list/tuple of polynomials or a multivariate polynomial ideal

OUTPUT:
True if the basis is triangular; False otherwise.

EXAMPLES:

sage: from sage.rings.polynomial.toy_variety import is_triangular
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: p1 = x^2*y + z^2
sage: p2 = y*z + z^3
sage: p3 = y+z
sage: is_triangular(R.ideal(p1,p2,p3))
False
sage: p3 = z^2 - 3
sage: is_triangular(R.ideal(p1,p2,p3))
True

sage.rings.polynomial.toy_variety.linear_representation(p, polys)
Assuming that \( p \) is a linear combination of \( \text{polys} \), determine coefficients that describe the linear combination.

This probably does not work for any inputs except \( p \), a polynomial, and \( \text{polys} \), a sequence of polynomials. If \( p \) is not in fact a linear combination of \( \text{polys} \), the function raises an exception.

The algorithm creates a matrix of coefficients of the monomials of \( \text{polys} \) and \( p \), with the coefficients of \( p \) in the last row. It augments this matrix with the appropriate identity matrix, then computes the echelon form of the augmented matrix. The last row should contain zeroes in the first columns, and the last columns contain a linear dependence relation. Solving for the desired linear relation is straightforward.
Polynomials, Release 10.3

INPUT:
• p – a polynomial
• polys – a list/tuple of polynomials

OUTPUT:
If \( n == \text{len}(\text{polys}) \), returns \([a[0], a[1], \ldots, a[n-1]]\) such that \( p == a[0]*\text{poly}[0] + \ldots + a[n-1]*\text{poly}[n-1] \).

EXAMPLES:

```python
sage: # needs sage.modules sage.rings.finite_rings
sage: from sage.rings.polynomial.toy_variety import linear_representation
sage: R.<x,y> = PolynomialRing(GF(32003))
sage: B = [x^2 + 1, y^2 + 1, x*y + 1]
```

sage.rings.polynomial.toy_variety.triangular_factorization(B, n=-1)

Compute the triangular factorization of the Groebner basis \( B \) of an ideal.

This will not work properly if \( B \) is not a Groebner basis!

The algorithm used is that described in a 1992 paper by Daniel Lazard [Laz1992]. It is not necessary for the term ordering to be lexicographic.

INPUT:
• \( B \) – a list/tuple of polynomials or a multivariate polynomial ideal
• \( n \) – the recursion parameter (default: \(-1\))

OUTPUT:
A list \( T \) of triangular sets \( T_0, T_1, \ldots \).

EXAMPLES:

```python
sage: # needs sage.rings.finite_rings
sage: from sage.misc.verbose import set_verbose
sage: set_verbose(0)
```
3.3.3 Educational version of the \(d\)-Groebner basis algorithm over PIDs

No attempt was made to optimize this algorithm as the emphasis of this implementation is a clean and easy presentation.

**Note:** The notion of ‘term’ and ‘monomial’ in [BW1993] is swapped from the notion of those words in Sage (or the other way around, however you prefer it). In Sage a term is a monomial multiplied by a coefficient, while in [BW1993] a monomial is a term multiplied by a coefficient. Also, what is called LM (the leading monomial) in Sage is called HT (the head term) in [BW1993].

**EXAMPLES:**

```
sage: from sage.rings.polynomial.toy_d_basis import d_basis
```

First, consider an example from arithmetic geometry:

```
sage: A.<x,y> = PolynomialRing(ZZ, 2)
sage: B.<X,Y> = PolynomialRing(Rationals(), 2)
sage: f = -y^2 - y + x^3 + 7*x + 1
sage: fx = f.derivative(x)
sage: fy = f.derivative(y)
sage: I = B.ideal([B(f), B(fx), B(fy)])
sage: I.groebner_basis()
```

Since the output is 1, we know that there are no generic singularities.

To look at the singularities of the arithmetic surface, we need to do the corresponding computation over \(\mathbb{Z}\):

```
sage: I = A.ideal([f, fx, fy])
sage: gb = d_basis(I); gb
```

This Groebner Basis gives a lot of information. First, the only fibers (over \(\mathbb{Z}\)) that are not smooth are at 11 = 0, and 17 = 0. Examining the Groebner Basis, we see that we have a simple node in both the fiber at 11 and at 17. From the factorization, we see that the node at 17 is regular on the surface (an \(I_1\) node), but the node at 11 is not. After blowing up this non-regular point, we find that it is an \(I_3\) node.

Another example. This one is from the Magma Handbook:

```
sage: # needs sage.libs.singular
sage: P.<x, y, z> = PolynomialRing(IntegerRing(), 3, order='lex')
sage: I = ideal(x^2 - 1, y^2 - 1, 2*x*y - z)
sage: I = Ideal(d_basis(I))
sage: x.reduce(I)
```

To compute modulo 4, we can add the generator 4 to our basis.
A third example is also from the Magma Handbook.

This example shows how one can use Groebner bases over the integers to find the primes modulo which a system of equations has a solution, when the system has no solutions over the rationals.

We first form a certain ideal $I$ in $\mathbb{Z}[x, y, z]$, and note that the Groebner basis of $I$ over $\mathbb{Q}$ contains 1, so there are no solutions over $\mathbb{Q}$ or an algebraic closure of it (this is not surprising as there are 4 equations in 3 unknowns).

However, when we compute the Groebner basis of $I$ (defined over $\mathbb{Z}$), we note that there is a certain integer in the ideal which is not 1:

Now for each prime $p$ dividing this integer 282687803443, the Groebner basis of $I$ modulo $p$ will be non-trivial and will thus give a solution of the original system modulo $p$.

Of course, modulo any other prime the Groebner basis is trivial so there are no other solutions. For example:

\begin{verbatim}
AUTHOR:
  • Martin Albrecht (2008-08): initial version

sage.rings.polynomial.toy_d_basis.LC(f)
\end{verbatim}
sage.rings.polynomial.toy_d_basis.LM(f)

sage.rings.polynomial.toy_d_basis.d_basis(F, strat=True)
Return the $d$-basis for the Ideal $F$ as defined in [BW1993].

**INPUT:**
- $F$ – an ideal
- $\text{strat}$ – use update strategy (default: True)

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.toy_d_basis import d_basis
sage: A.<x,y> = PolynomialRing(ZZ, 2)
sage: f = -y^2 - y + x^3 + 7*x + 1
sage: fx = f.derivative(x)
sage: fy = f.derivative(y)
sage: I = A.ideal([f,fx,fy])
sage: gb = d_basis(I); gb
# needs sage.libs.singular
[x - 2020, y - 11313, 22627]
```

sage.rings.polynomial.toy_d_basis.gpol(g1, g2)
Return the G-Polynomial of $g_1$ and $g_2$.

Let $a_i \, t_i$ be $\text{LT}(g_i), \quad a = a_i * c_i + a_j * c_j$ with $a = \text{GCD}(a_i, a_j)$, and $s_i = t/t_i$ with $t = \text{LCM}(t_i, t_j)$. Then the G-Polynomial is defined as: $c_1 s_1 g_1 - c_2 s_2 g_2$.

**INPUT:**
- $g1$ – polynomial
- $g2$ – polynomial

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.toy_d_basis import gpol
sage: P.<x, y, z> = PolynomialRing(IntegerRing(), 3, order='lex')
sage: f = x^2 - 1
sage: g = 2*x*y - z
sage: gpol(f, g)
x^2*y - y
```

sage.rings.polynomial.toy_d_basis.select(P)
The normal selection strategy.

**INPUT:**
- $P$ – a list of critical pairs

**OUTPUT:**
an element of $P$

**EXAMPLES:**

```python
sage: from sage.rings.polynomial.toy_d_basis import select
sage: A.<x,y> = PolynomialRing(ZZ, 2)
sage: f = -y^2 - y + x^3 + 7*x + 1
sage: fx = f.derivative(x)
sage: fy = f.derivative(y)
```
sage: G = [f, fx, fy]
sage: B = set((f1, f2) for f1 in G for f2 in G if f1 != f2)
sage: select(B)
(-2*y - 1, 3*x^2 + 7)

sage.rings.polynomial.toy_d_basis.spol(g1, g2)

Return the S-Polynomial of \(g_1\) and \(g_2\).

Let \(a_it_i\) be \(LT(g_i)\), \(b_i = a_i/a_i\) with \(a = LCM(a_i,a_j)\), and \(s_i = t/t_i\) with \(t = LCM(t_i,t_j)\). Then the S-Polynomial is defined as: \(b_1s_1g_1 - b_2s_2g_2\).

INPUT:

* \(g_1\) – polynomial
* \(g_2\) – polynomial

EXAMPLES:

```sage
sage: from sage.rings.polynomial.toy_d_basis import spol
sage: P.<x, y, z> = PolynomialRing(IntegerRing(), 3, order='lex')
sage: f = x^2 - 1
sage: g = 2*x*y - z
sage: spol(f, g)
x*z - 2*y
```

sage.rings.polynomial.toy_d_basis.update(G, B, h)

Update \(G\) using the list of critical pairs \(B\) and the polynomial \(h\) as presented in [BW1993], page 230. For this, Buchberger’s first and second criterion are tested.

This function uses the Gebauer-Moeller Installation.

INPUT:

* \(G\) – an intermediate Groebner basis
* \(B\) – a list of critical pairs
* \(h\) – a polynomial

OUTPUT:

\(G, B\) where \(G\) and \(B\) are updated

EXAMPLES:

```sage
sage: from sage.rings.polynomial.toy_d_basis import update
sage: A.<x,y> = PolynomialRing(ZZ, 2)
sage: G = set([3*x^2 + 7, 2*y + 1, x^3 - y^2 + 7*x - y + 1])
sage: B = set()
sage: h = x^2*y - x^2 + y - 3
sage: update(G, B, h)
({2*y + 1, 3*x^2 + 7, x^2*y - x^2 + y + 3, x^3 - y^2 + 7*x - y + 1},
 {x^2*y - x^2 + y - 3, 2*y + 1},
 {x^2*y - x^2 + y - 3, 3*x^2 + 7},
 {x^2*y - x^2 + y - 3, x^3 - y^2 + 7*x - y + 1})
```
4.1 Fraction Field of Integral Domains

AUTHORS:

- William Stein (with input from David Joyner, David Kohel, and Joe Wetherell)
- Burcin Erocal
- Julian Rüth (2017-06-27): embedding into the field of fractions and its section

EXAMPLES:

Quotienting is a constructor for an element of the fraction field:

```sage
definition of sage.rings.fraction_field.FractionField(R, names=None)
Create the fraction field of the integral domain R.

INPUT:

- R – an integral domain
- names – ignored

EXAMPLES:

We create some example fraction fields:
```
Polynomials, Release 10.3

(continued from previous page)

Fraction Field of Univariate Polynomial Ring in x over Rational Field
\texttt{sage: FractionField(PolynomialRing(IntegerRing(), 'x'))}
Fraction Field of Univariate Polynomial Ring in x over Integer Ring
\texttt{sage: FractionField(PolynomialRing(RationalField(),2, 'x'))}
Fraction Field of Multivariate Polynomial Ring in x0, x1 over Rational Field

Dividing elements often implicitly creates elements of the fraction field:

\texttt{sage: x = PolynomialRing(RationalField(), 'x').gen()}
\texttt{sage: f = x/(x+1)}
\texttt{sage: g = x**3/(x+1)}
\texttt{sage: f/g}
\texttt{1/x^2}
\texttt{sage: g/f}
\texttt{x^2}

The input must be an integral domain:

\texttt{sage: Frac(Integers(4))}
Traceback (most recent call last):
...:
TypeError: R must be an integral domain.

\texttt{class sage.rings.fraction_field.FractionFieldEmbedding}

Bases: DefaultConvertMap_unique

The embedding of an integral domain into its field of fractions.

\texttt{EXAMPLES:}

\texttt{sage: R.<x> = QQ[]}
\texttt{sage: f = R.fraction_field().coerce_map_from(R); f}
Coercion map:
  From: Univariate Polynomial Ring in x over Rational Field
  To: Fraction Field of Univariate Polynomial Ring in x over Rational Field

\texttt{is_injective()}

Return whether this map is injective.

\texttt{EXAMPLES:}

The map from an integral domain to its fraction field is always injective:

\texttt{sage: R.<x> = QQ[]}
\texttt{sage: R.fraction_field().coerce_map_from(R).is_injective()}
\texttt{True}

\texttt{is_surjective()}

Return whether this map is surjective.

\texttt{EXAMPLES:}

\texttt{sage: R.<x> = QQ[]}
\texttt{sage: R.fraction_field().coerce_map_from(R).is_surjective()}
\texttt{False}
section()

Return a section of this map.

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: R.fraction_field().coerce_map_from(R).section()
Section map:
   From: Fraction Field of Univariate Polynomial Ring in x over Rational Field
   To:   Univariate Polynomial Ring in x over Rational Field
```

class sage.rings.fraction_field.FractionFieldEmbeddingSection

Bases: Section

The section of the embedding of an integral domain into its field of fractions.

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: f = R.fraction_field().coerce_map_from(R).section(); f
Section map:
   From: Fraction Field of Univariate Polynomial Ring in x over Rational Field
   To:   Univariate Polynomial Ring in x over Rational Field
```

class sage.rings.fraction_field.FractionField_1poly_field(R, element_class=<class 'sage.rings.fraction_field_element.FractionFieldElement_1poly_field'>)

Bases: FractionField_generic

The fraction field of a univariate polynomial ring over a field.

Many of the functions here are included for coherence with number fields.

class_number()

Here for compatibility with number fields and function fields.

EXAMPLES:

```python
sage: R.<t> = GF(5)[]; K = R.fraction_field()
sage: K.class_number()
1
```

function_field()

Return the isomorphic function field.

EXAMPLES:

```python
sage: R.<t> = GF(5)[]
```
```
sage: K = FractionField(GF(5)['t'])
sage: K.maximal_order()
Univariate Polynomial Ring in t over Finite Field of size 5
```

**ring_of_integers()**

Return the ring of integers in this fraction field.

**EXAMPLES:**

```
sage: K = FractionField(GF(5)['t'])
sage: K.ring_of_integers()
Univariate Polynomial Ring in t over Finite Field of size 5
```

**class sage.rings.fraction_field.FractionField_generic**

```
class sage.rings.fraction_field.FractionField_generic(R, element_class=<class 'sage.rings.fraction_field_element.FractionFieldElement'>, category=Category of quotient fields)
```

**Bases:** Field

The fraction field of an integral domain.

**base_ring()**

Return the base ring of self.

This is the base ring of the ring which this fraction field is the fraction field of.

**EXAMPLES:**

```
sage: R = Frac(ZZ['t'])
sage: R.base_ring()
Integer Ring
```

**characteristic()**

Return the characteristic of this fraction field.

**EXAMPLES:**

```
sage: R = Frac(ZZ['t'])
sage: R.base_ring()
Integer Ring
sage: R = Frac(ZZ['t']); R.characteristic()
0
sage: R = Frac(GF(5)['w']); R.characteristic()
5
```

**construction()**

**EXAMPLES:**

```
sage: Frac(ZZ['x']).construction()
(FractionField, Univariate Polynomial Ring in x over Integer Ring)
sage: K = Frac(GF(3)['t'])
sage: f, R = K.construction()
sage: f(R)
Fraction Field of Univariate Polynomial Ring in t
  over Finite Field of size 3
sage: f(R) == K
True
```
\textbf{gen} \((i=0)\)

Return the \(i\)-th generator of \texttt{self}.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: R = Frac(PolynomialRing(QQ,'z',10)); R
Fraction Field of Multivariate Polynomial Ring
  in z0, z1, z2, z3, z4, z5, z6, z7, z8, z9 over Rational Field
sage: R.0
z0
sage: R.gen(3)
z3
sage: R.3
z3
\end{verbatim}

\textbf{is\_exact}()

Return if \texttt{self} is exact which is if the underlying ring is exact.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: Frac(ZZ['x']).is_exact()
True
sage: Frac(CDF['x']).is_exact() # needs sage.rings.complex_double
False
\end{verbatim}

\textbf{is\_field} \((proof=True)\)

Return \texttt{True}, since the fraction field is a field.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: Frac(ZZ).is_field()
True
\end{verbatim}

\textbf{is\_finite}()

Tells whether this fraction field is finite.

\textbf{Note:} A fraction field is finite if and only if the associated integral domain is finite.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: Frac(QQ['a','b','c']).is_finite()
False
\end{verbatim}

\textbf{ngens}()

This is the same as for the parent object.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: R = Frac(PolynomialRing(QQ,'z',10)); R
Fraction Field of Multivariate Polynomial Ring
  in z0, z1, z2, z3, z4, z5, z6, z7, z8, z9 over Rational Field
sage: R.ngens()
10
\end{verbatim}
**random_element** (*args, **kwds*)

Return a random element in this fraction field.

The arguments are passed to the random generator of the underlying ring.

**EXAMPLES:**

```python
sage: F = ZZ['x'].fraction_field()
sage: F.random_element()  # random
(2*x - 8)/(-x^2 + x)
```

```python
sage: f = F.random_element(degree=5)
sage: f.numerator().degree() == f.denominator().degree()  # True
sage: f.denominator().degree() <= 5  # True
sage: while f.numerator().degree() != 5:
    f = F.random_element(degree=5)
```

**ring()**

Return the ring that this is the fraction field of.

**EXAMPLES:**

```python
sage: R = Frac(QQ['x,y'])
sage: R
Fraction Field of Multivariate Polynomial Ring in x, y over Rational Field
sage: R.ring()
Multivariate Polynomial Ring in x, y over Rational Field
```

**some_elements()**

Return some elements in this field.

**EXAMPLES:**

```python
sage: R.<x> = QQ[]
sage: R.fraction_field().some_elements()
[0, 1, x, 2*x, x/(x^2 + 2*x + 1), 1/x^2, ...
(2*x^2 + 2)/(x^2 + 2*x + 1),
(2*x^2 + 2)/x^3,
(2*x^2 + 2)/(x^2 - 1), 2]
```

`sage.rings.fraction_field.is_FractionField(x)`

Test whether or not x inherits from `FractionField_generic`.

**EXAMPLES:**

```python
sage: from sage.rings.fraction_field import is_FractionField
sage: is_FractionField(Frac(ZZ['x']))
True
sage: is_FractionField(QQ)
False
```
4.2 Fraction Field Elements

AUTHORS:

- William Stein (input from David Joyner, David Kohel, and Joe Wetherell)
- Sebastian Pancratz (2010-01-06): Rewrite of addition, multiplication and derivative to use Henrici’s algorithms [Hor1972]

class sage.rings.fraction_field_element.FractionFieldElement
Bases: FieldElement

EXAMPLES:

```
sage: K = FractionField(PolynomialRing(QQ, 'x'))
sage: K
Fraction Field of Univariate Polynomial Ring in x over Rational Field
sage: loads(K.dumps()) == K
True
sage: x = K.gen()
sage: f = (x^3 + x)/(17 - x^19); f
(-x^3 - x)/(x^19 - 17)
sage: loads(f.dumps()) == f
True
```

denominator()
Return the denominator of self.

EXAMPLES:

```
sage: R.<x,y> = ZZ[]
sage: f = x/y + 1; f
(x + y)/y
sage: f.denominator()
y
```

is_one()
Return True if this element is equal to one.

EXAMPLES:

```
sage: F = ZZ['x,y'].fraction_field()
sage: x,y = F.gens()
sage: (x/x).is_one()  # True
True
sage: (x/y).is_one()  # False
False
```

is_square(root=False)
Return whether or not self is a perfect square.

If the optional argument root is True, then also returns a square root (or None, if the fraction field element is not square).

INPUT:

- root – whether or not to also return a square root (default: False)

OUTPUT:
• bool - whether or not a square
• object - (optional) an actual square root if found, and None otherwise.

EXAMPLES:

```python
sage: R.<t> = QQ[]
sage: (1/t).is_square()  False
sage: (1/t^6).is_square()  True
sage: ((1+t)^4/t^6).is_square()  True
sage: (4*(1+t)^4/t^6).is_square()  True
sage: (2*(1+t)^4/t^6).is_square()  False
sage: ((1+t)/t^6).is_square()  False
sage: (4*(1+t)^4/t^6).is_square(root=True)  (True, (2*t^2 + 4*t + 2)/t^3)
```

```python
is_zero()
Return True if this element is equal to zero.

EXAMPLES:

```python
sage: F = ZZ['x,y'].fraction_field()
sage: x,y = F.gens()
sage: t = F(0)/x
sage: t.is_zero()  True
sage: u = 1/x - 1/x
sage: u.is_zero()  True
sage: u.parent().is F  True
```

```python
nth_root(n)
Return a n-th root of this element.

EXAMPLES:

```python
sage: R = QQ['t'].fraction_field()
sage: t = R.gen()
sage: p = (t+1)^3 / (t^2+t-1)^3
sage: p.nth_root(3)  (t + 1)/(t^2 + t - 1)
(continues on next page)```
\begin{verbatim}
sage: p = (t+1) / (t-1)
sage: p.nth_root(2)
Traceback (most recent call last):
...
ValueError: not a 2nd power
\end{verbatim}

**numerator()**

Return the numerator of \texttt{self}.

**EXAMPLES:**

\begin{verbatim}
sage: R.<x,y> = ZZ[]
sage: f = x/y + 1; f
(x + y)/y
sage: f.numerator()
x + y
\end{verbatim}

**reduce()**

Reduce this fraction.

Divides out the gcd of the numerator and denominator. If the denominator becomes a unit, it becomes 1. Additionally, depending on the base ring, the leading coefficients of the numerator and the denominator may be normalized to 1.

Automatically called for exact rings, but because it may be numerically unstable for inexact rings it must be called manually in that case.

**EXAMPLES:**

\begin{verbatim}
sage: R.<x> = RealField(10)[]  # needs sage.rings.real_mpfr
sage: f = (x^2+2*x+1)/(x+1); f
(x^2 + 2.0*x + 1.0)/(x + 1.0)
sage: f.reduce(); f
x + 1.0
\end{verbatim}

**specialization** \((D=\text{None}, \phi=\text{None})\)

Returns the specialization of a fraction element of a polynomial ring

**subs** \((\text{in} \_ \text{dict} = \text{None}, *\text{args}, **\text{kwds})\)

Substitute variables in the numerator and denominator of \texttt{self}.

If a dictionary is passed, the keys are mapped to generators of the parent ring. Otherwise, the arguments are transmitted unchanged to the method \texttt{subs} of the numerator and the denominator.

**EXAMPLES:**

\begin{verbatim}
sage: x, y = PolynomialRing(ZZ, 2, 'xy').gens()
sage: f = x^2 + y + x^2*y^2 + 5
sage: (1/f).subs(x=5)
1/(25*y^2 + y + 30)
\end{verbatim}

**valuation** \((v=\text{None})\)

Return the valuation of \texttt{self}, assuming that the numerator and denominator have valuation functions defined on them.

4.2. Fraction Field Elements 549
EXAMPLES:

```python
sage: x = PolynomialRing(RationalField(), 'x').gen()
sage: f = (x^3 + x) / (x^2 - 2*x^3)
sage: f
-1/2*x^2 - 1/2/(x^2 - 1/2*x)
sage: f.valuation()
-1
sage: f.valuation(x^2 + 1)
1
```

```text
class sage.rings.fraction_field_element.FractionFieldElement_1poly_field

Bases: FractionFieldElement

A fraction field element where the parent is the fraction field of a univariate polynomial ring over a field.

Many of the functions here are included for coherence with number fields.

**is_integral()**

Returns whether this element is actually a polynomial.

**reduce()**

Pick a normalized representation of self.

In particular, for any \(a = b\), after normalization they will have the same numerator and denominator.

**support()**

Returns a sorted list of primes dividing either the numerator or denominator of this element.
```
sage.rings.fraction_field_element.is_FractionFieldElement(x)

Return whether or not x is a FractionFieldElement.

EXAMPLES:

sage: from sage.rings.fraction_field_element import is_FractionFieldElement
sage: R.<x> = ZZ[]
sage: is_FractionFieldElement(x/2)
False
sage: is_FractionFieldElement(2/x)
True
sage: is_FractionFieldElement(1/3)
False

sage.rings.fraction_field_element.make_element(parent, numerator, denominator)

Used for unpicking FractionFieldElement objects (and subclasses).

EXAMPLES:

sage: from sage.rings.fraction_field_element import make_element
sage: R = ZZ['x,y']
sage: x,y = R.gens()
sage: F = R.fraction_field()
sage: make_element(F, 1 + x, 1 + y)
(x + 1)/(y + 1)

4.3 Univariate rational functions over prime fields

class sage.rings.fraction_field_FpT.FpT(R, names=None)

Bases: FractionField_1poly_field

This class represents the fraction field \( \mathbb{F}_p(T) \) for \( 2 < p < \sqrt{2^{31} - 1} \).

EXAMPLES:

sage: R.<T> = GF(71)[]
sage: K = FractionField(R); K
Fraction Field of Univariate Polynomial Ring in T over Finite Field of size 71

```
sage: 1-1/T
(T + 70)/T
sage: parent(1-1/T) is K
True
```

**INTEGRAL_LIMIT** = 46341

**iter**(bound=None, start=None)

**EXAMPLES:**

```
sage: from sage.rings.fraction_field_FpT import *
sage: R.<t> = FpT(GF(5)['t'])
sage: list(R.iter(2))[350:355]
[(t^2 + t + 1)/(t + 2),
 (t^2 + t + 2)/(t + 2),
 (t^2 + t + 4)/(t + 2),
 (t^2 + 2*t + 1)/(t + 2),
 (t^2 + 2*t + 2)/(t + 2)]
```

**class** sage.rings.fraction_field_FpT.FpTElement

Bases: FieldElement

An element of an $FpT$ fraction field.

**denom**()

Return the denominator of this element, as an element of the polynomial ring.

**EXAMPLES:**

```
sage: K = GF(11)['t'].fraction_field()
sage: t = K.gen(); a = (t + 1/t)^3 - 1
sage: a.denom()
t^3
```

**denominator**()

Return the denominator of this element, as an element of the polynomial ring.

**EXAMPLES:**

```
sage: K = GF(11)['t'].fraction_field()
sage: t = K.gen(); a = (t + 1/t)^3 - 1
sage: a.denominator()
t^3
```

**factor**()

**EXAMPLES:**

```
sage: K = Frac(GF(5)['t'])
sage: t = K.gen()
sage: f = 2 * (t+1) * (t^2+t+1)^2 / (t-1)
sage: factor(f)
(2) * (t + 4)^-1 * (t + 1) * (t^2 + t + 1)^2
```

**is_square**()

Return **True** if this element is the square of another element of the fraction field.

**EXAMPLES:**
Next ()

Iterate through all polynomials, returning the “next” polynomial after this one.

The strategy is as follows:

- We always leave the denominator monic.
- We progress through the elements with both numerator and denominator monic, and with the denominator less than the numerator. For each such, we output all the scalar multiples of it, then all of the scalar multiples of its inverse.
- So if the leading coefficient of the numerator is less than \( p - 1 \), we scale the numerator to increase it by 1.
- Otherwise, we consider the multiple with numerator and denominator monic.
  - If the numerator is less than the denominator (lexicographically), we return the inverse of that element.
  - If the numerator is greater than the denominator, we invert, and then increase the numerator (remaining monic) until we either get something relatively prime to the new denominator, or we reach the new denominator. In this case, we increase the denominator and set the numerator to 1.

Examples:

```python
sage: from sage.rings.fraction_field_FpT import *
sage: R.<t> = FpT(GF(3)[t])
sage: a = R(0)
sage: for _ in range(30):
    ....:     a = a.next()
    ....:     print(a)
1
2
1/t
2/t
t
2*t
1/(t + 1)
2/(t + 1)
t + 1
2*t + 2
t/(t + 1)
2*t/(t + 1)
(t + 1)/t
(2*t + 2)/t
1/(t + 2)
2/(t + 2)
t + 2
2*t + 1
t/(t + 2)
2*t/(t + 2)
```

(continues on next page)
Polynomials, Release 10.3

(continued from previous page)

\[(t + 2)/t\]
\[(2*t + 1)/t\]
\[(t + 1)/(t + 2)\]
\[(2*t + 2)/(t + 2)\]
\[(t + 2)/(t + 1)\]
\[(2*t + 1)/(t + 1)\]
\[1/t^2\]
\[2/t^2\]
\[t^2\]
\[2*t^2\]

\textbf{numer()}\texttt{
}

Return the numerator of this element, as an element of the polynomial ring.

**EXAMPLES:**

\begin{verbatim}
sage: K = GF(11)['t'].fraction_field()
sage: t = K.gen(0); a = (t + 1/t)^3 - 1
sage: a.numer()
t^6 + 3*t^4 + 10*t^3 + 3*t^2 + 1
\end{verbatim}

\textbf{numerator()}\texttt{
}

Return the numerator of this element, as an element of the polynomial ring.

**EXAMPLES:**

\begin{verbatim}
sage: K = GF(11)['t'].fraction_field()
sage: t = K.gen(0); a = (t + 1/t)^3 - 1
sage: a.numerator()
t^6 + 3*t^4 + 10*t^3 + 3*t^2 + 1
\end{verbatim}

\textbf{sqrt}\texttt{(extend=True, all=False)}\texttt{
}

Return the square root of this element.

**INPUT:**

\begin{itemize}
\item extend\ - bool (default: True); if True, return a square root in an extension ring, if necessary. Otherwise, raise a ValueError if the square is not in the base ring.
\item all\ - bool (default: False); if True, return all square roots of self, instead of just one.
\end{itemize}

**EXAMPLES:**

\begin{verbatim}
sage: from sage.rings.fraction_field_FpT import *
sage: K = GF(7)['t'].fraction_field(); t = K.gen(0)
sage: p = (t + 2)^2/(3*t^3 + 1)^4
sage: p.sqrt()
(3*t + 6)/(t^6 + 3*t^4 + 10*t^3 + 3*t^2 + 1)
sage: p.sqrt()^2 == p
True
\end{verbatim}

\textbf{subs}\texttt{(in_dict=None, *args, **kwds)}\texttt{
}

**EXAMPLES:**

\begin{verbatim}
sage: K = Frac(GF(11)['t'])
sage: t = K.gen()
sage: f = (t+1)/(t-1)
\end{verbatim}
valuation \( (v) \)

Return the valuation of self at \( v \).

EXAMPLES:

```
sage: R.<t> = GF(5)[]
sage: f = (t+1)^2 * (t^2+t+1) / (t-1)^3
sage: f.valuation(t+1)
2
sage: f.valuation(t-1)
-3
sage: f.valuation(t)
0
```
Polynomials, Release 10.3

(continued from previous page)

```python
sage: f = R.convert_map_from(K); f
Section map:
  From: Fraction Field of Univariate Polynomial Ring in t over Finite Field of size 5
  To: Univariate Polynomial Ring in t over Finite Field of size 5
sage: type(f)
<class 'sage.rings.fraction_field_FpT.FpT_Polyring_section'>
```

**Warning:** Comparison of `FpT_Polyring_section` objects is not currently implemented. See [github issue #23469](https://github.com/).

```python
sage: fprime = loads(dumps(f))
sage: fprime == f
False
sage: fprime(1+t) == f(1+t)
True
```

class `sage.rings.fraction_field_FpT.FpT_iter`

Bases: `object`

Return a class that iterates over all elements of an FpT.

**EXAMPLES:**

```python
sage: K = GF(3)['t'].fraction_field()
sage: I = K.iter(1)
sage: list(I)
[0, 1, 2,
t, t + 1, t + 2, 2*t, 2*t + 1, 2*t + 2, 1/t, 2/t, (t + 1)/t,
(t + 2)/t, (2*t + 1)/t, (2*t + 2)/t, 1/(t + 1), 2/(t + 1), t/(t + 1),
(t + 2)/(t + 1), 2*t/(t + 1), (2*t + 1)/(t + 1), 1/(t + 2), 2/(t + 2),
t/(t + 2), (t + 1)/(t + 2), 2*t/(t + 2), (2*t + 2)/(t + 2)]
```

Chapter 4. Rational Functions
class sage.rings.fraction_field_FpT.Fp_FpT_coerce

Bases: RingHomomorphism

This class represents the coercion map from GF(p) to GF(p(t))

EXAMPLES:

```
sage: R.<t> = GF(5)[]
sage: K = R.fraction_field()
sage: f = K.coerce_map_from(GF(5)); f
Ring morphism:
  From: Finite Field of size 5
  To:   Fraction Field of Univariate Polynomial Ring in t over Finite Field of...
        size 5
sage: type(f)
<class 'sage.rings.fraction_field_FpT.Fp_FpT_coerce'>
```

section()

Return the section of this inclusion: the partially defined map from GF(p(t)) back to GF(p), defined on constant elements.

EXAMPLES:

```
sage: R.<t> = GF(5)[]
sage: K = R.fraction_field()
sage: f = K.coerce_map_from(GF(5))
sage: g = f.section(); g
Section map:
  From: Fraction Field of Univariate Polynomial Ring in t over Finite Field...
  of size 5
  To:   Finite Field of size 5
sage: t = K.gen()
sage: g(f(1,3,reduce=False))
2
sage: g(t)
Traceback (most recent call last):
  ... ValueError: not constant
sage: g(1/t)
Traceback (most recent call last):
  ... ValueError: not integral
```

class sage.rings.fraction_field_FpT.Polyring_FpT_coerce

Bases: RingHomomorphism

This class represents the coercion map from GF(p)[t] to GF(p(t))

EXAMPLES:

```
sage: R.<t> = GF(5)[]
sage: K = R.fraction_field()
sage: f = K.coerce_map_from(R); f
Ring morphism:
  From: Univariate Polynomial Ring in t over Finite Field of size 5
  To:   Fraction Field of Univariate Polynomial Ring in t over Finite Field of...
        size 5
sage: type(f)
<class 'sage.rings.fraction_field_FpT.Polyring_FpT_coerce'>
```
section()

Return the section of this inclusion: the partially defined map from \( GF(p)(t) \) back to \( GF(p)[t] \), defined on elements with unit denominator.

EXAMPLES:

```python
sage: R.<t> = GF(5)[]
sage: K = R.fraction_field()
sage: f = K.coerce_map_from(R)
sage: g = f.section(); g
Section map:
  From: Fraction Field of Univariate Polynomial Ring in t over Finite Field
  \rightarrow of size 5
  To:  Univariate Polynomial Ring in t over Finite Field of size 5
sage: t = K.gen()
sage: g(t)
t
sage: g(1/t)
Traceback (most recent call last):
  ... ValueError: not integral
```

```python
class sage.rings.fraction_field_FpT.ZZ_FpT_coerce

 Bases: RingHomomorphism

This class represents the coercion map from ZZ to GF(p(t)

EXAMPLES:

```python
sage: R.<t> = GF(17)[]
sage: K = R.fraction_field()
sage: f = K.coerce_map_from(ZZ); f
Ring morphism:
  From: Integer Ring
  To:  Fraction Field of Univariate Polynomial Ring in t over Finite Field of...
  \rightarrow size 17
sage: type(f)
<class 'sage.rings.fraction_field_FpT.ZZ_FpT_coerce'>
```

section()

Return the section of this inclusion: the partially defined map from \( GF(p)(t) \) back to ZZ, defined on constant elements.

EXAMPLES:

```python
sage: R.<t> = GF(5)[]
sage: K = R.fraction_field()
sage: f = K.coerce_map_from(ZZ)
sage: g = f.section(); g
Composite map:
  From: Fraction Field of Univariate Polynomial Ring in t over Finite Field...
  \rightarrow of size 5
  To:  Integer Ring
  Defn: Section map:
        From: Fraction Field of Univariate Polynomial Ring in t over Finite...
        \rightarrow Field of size 5
        To:  Finite Field of size 5
        then
```
Lifting map:
  From: Finite Field of size 5
  To:  Integer Ring

```
sage: t = K.gen()
sage: g(f(1,3,reduce=False))
2
sage: g(t)
Traceback (most recent call last):
  ...  
ValueError: not constant  
sage: g(1/t)
Traceback (most recent call last):
  ...  
ValueError: not integral
```

`sage.rings.fraction_field_FpT.unpickle_FpT_element(K, numer, denom)`

Used for pickling.

4.3. Univariate rational functions over prime fields
LAURENT POLYNOMIALS

5.1 Ring of Laurent Polynomials (base class)

If $R$ is a commutative ring, then the ring of Laurent polynomials in $n$ variables over $R$ is $R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$.

AUTHORS:

- David Roe (2008-2-23): created
- David Loeffler (2009-07-10): cleaned up docstrings

class sage.rings.polynomial.laurent_polynomial_ring_base.LaurentPolynomialRing_generic(R)

Bases: CommutativeRing, Parent

Laurent polynomial ring (base class).

EXAMPLES:

This base class inherits from CommutativeRing. Since github issue #11900, it is also initialised as such:

```
sage: R.<x1,x2> = LaurentPolynomialRing(QQ)
sage: R.category()
Join of Category of unique factorization domains
and Category of commutative algebras
   over (number fields and quotient fields and metric spaces)
and Category of infinite sets
sage: TestSuite(R).run()
```

change_ring (base_ring=None, names=None, sparse=False, order=None)

EXAMPLES:

```
sage: R = LaurentPolynomialRing(QQ, 2, 'x')
sage: R.change_ring(ZZ)
Multivariate Laurent Polynomial Ring in x0, x1 over Integer Ring
```

Check that the distinction between a univariate ring and a multivariate ring with one generator is preserved:

```
sage: P.<x> = LaurentPolynomialRing(QQ, 1)
sage: P
Multivariate Laurent Polynomial Ring in x over Rational Field
sage: K.<i> = CyclotomicField(4)  # needs sage.rings.number_field
sage: P.change_ring(K)  # needs sage.rings.number_field
Multivariate Laurent Polynomial Ring in x over
Cyclotomic Field of order 4 and degree 2
```
characteristic()
Returns the characteristic of the base ring.
EXAMPLES:

```
sage: LaurentPolynomialRing(QQ, 2, 'x').characteristic()
0
sage: LaurentPolynomialRing(GF(3), 2, 'x').characteristic()
3
```

completion (p=None, prec=20, extras=None)
Return the completion of self.
Currently only implemented for the ring of formal Laurent series. The prec variable controls the precision
used in the Laurent series ring. If prec is ∞, then this returns a LazyLaurentSeriesRing.
EXAMPLES:

```
sage: P.<x> = LaurentPolynomialRing(QQ); P
Univariate Laurent Polynomial Ring in x over Rational Field
sage: PP = P.completion(x); PP
Laurent Series Ring in x over Rational Field
sage: f = 1 - 1/x
sage: PP(f)
-x^-1 + 1
sage: g = 1 / PP(f); g
-x^-2 - x^-3 - x^-4 - x^-5 - x^-6 - x^-7 - x^-8 - x^-9 - x^-10 - x^-11
   - x^-12 - x^-13 - x^-14 - x^-15 - x^-16 - x^-17 - x^-18 - x^-19 - x^-20 + O(x^-21)
sage: 1 / g
-x^-1 + 1 + O(x^19)
sage: # needs sage.combinat
sage: PP = P.completion(x, prec=oo); PP
Lazy Laurent Series Ring in x over Rational Field
sage: g = 1 / PP(f); g
-x^-2 - x^-3 + O(x^-4)
sage: 1 / g == f
True
```

collection()
Return the construction of self.
EXAMPLES:

```
sage: LaurentPolynomialRing(QQ, 2, 'x,y').construction()
(LaurentPolynomialFunctor,
   Univariate Laurent Polynomial Ring in x over Rational Field)
```

fraction_field()
The fraction field is the same as the fraction field of the polynomial ring.
EXAMPLES:

```
sage: L.<x> = LaurentPolynomialRing(QQ)
sage: L.fraction_field()
Fraction Field of Univariate Polynomial Ring in x over Rational Field
sage: (x^-1 + 2) / (x - 1)
(2*x + 1)/(x^2 - x)
```
**gen** (*i=0*)

Returns the \(i\)th generator of self. If \(i\) is not specified, then the first generator will be returned.

**EXAMPLES:**

```
sage: LaurentPolynomialRing(QQ, 2, 'x').gen()
x0
sage: LaurentPolynomialRing(QQ, 2, 'x').gen(0)
x0
sage: LaurentPolynomialRing(QQ, 2, 'x').gen(1)
x1
```

**ideal** (*args, **kwds*)

**EXAMPLES:**

```
sage: LaurentPolynomialRing(QQ, 2, 'x').ideal([1])
Ideal (1) of Multivariate Laurent Polynomial Ring in x0, x1 over Rational Field
```

**is_exact** ()

Return True if the base ring is exact.

**EXAMPLES:**

```
sage: LaurentPolynomialRing(QQ, 2, 'x').is_exact()
True
sage: LaurentPolynomialRing(RDF, 2, 'x').is_exact()
False
```

**is_field** (*proof=True*)

**EXAMPLES:**

```
sage: LaurentPolynomialRing(QQ, 2, 'x').is_field()
False
```

**is_finite** ()

**EXAMPLES:**

```
sage: LaurentPolynomialRing(QQ, 2, 'x').is_finite()
False
```

**is_integral_domain** (*proof=True*)

Return True if self is an integral domain.

**EXAMPLES:**

```
sage: LaurentPolynomialRing(QQ, 2, 'x').is_integral_domain()
True
```

The following used to fail; see github issue #7530:

```
sage: L = LaurentPolynomialRing(ZZ, 'X')
sage: L['Y']
Univariate Polynomial Ring in Y over Univariate Laurent Polynomial Ring in X over Integer Ring
```
**is_noetherian()**

Return True if self is Noetherian.

EXAMPLES:

```
sage: LaurentPolynomialRing(QQ, 2, 'x').is_noetherian()
True
```

**krull_dimension()**

EXAMPLES:

```
sage: LaurentPolynomialRing(QQ, 2, 'x').krull_dimension()
Traceback (most recent call last):
...  
NotImplementedError
```

**ngens()**

Return the number of generators of self.

EXAMPLES:

```
sage: LaurentPolynomialRing(QQ, 2, 'x').ngens()
2
sage: LaurentPolynomialRing(QQ, 1, 'x').ngens()
1
```

**polynomial_ring()**

Returns the polynomial ring associated with self.

EXAMPLES:

```
sage: LaurentPolynomialRing(QQ, 2, 'x').polynomial_ring()
Multivariate Polynomial Ring in x0, x1 over Rational Field
sage: LaurentPolynomialRing(QQ, 1, 'x').polynomial_ring()
Multivariate Polynomial Ring in x over Rational Field
```

**random_element** (*low_degree*=-2, *high_degree*=2, *terms*=5, *choose_degree=False, *args, **kwds*)

EXAMPLES:

```
sage: LaurentPolynomialRing(QQ, 2, 'x').random_element()
Traceback (most recent call last):
...  
NotImplementedError
```

**remove_var** (*var*)

EXAMPLES:

```
sage: R = LaurentPolynomialRing(QQ,'x,y,z')
sage: R.remove_var('x')
Multivariate Laurent Polynomial Ring in y, z over Rational Field
sage: R.remove_var('x').remove_var('y')
Univariate Laurent Polynomial Ring in z over Rational Field
```

**term_order()**

Returns the term order of self.

EXAMPLES:
variable_names_recursive (depth=+Infinity)

Return the list of variable names of this ring and its base rings, as if it were a single multi-variate Laurent polynomial.

INPUT:

• depth – an integer or Infinity.

OUTPUT:

A tuple of strings.

EXAMPLES:

```sage
t = LaurentPolynomialRing(QQ, 'x')
s = LaurentPolynomialRing(t, 'y')
r = LaurentPolynomialRing(s, 'z')
sage: r.variable_names_recursive()
('x', 'y', 'z')
sage: r.variable_names_recursive(2)
('y', 'z')
```

5.2 Ring of Laurent Polynomials

If $R$ is a commutative ring, then the ring of Laurent polynomials in $n$ variables over $R$ is $R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$. We implement it as a quotient ring

$$R[x_1, y_1, x_2, y_2, \ldots, x_n, y_n]/(x_1y_1 - 1, x_2y_2 - 1, \ldots, x_ny_n - 1).$$

AUTHORS:

- David Roe (2008-2-23): created
- David Loeffler (2009-07-10): cleaned up docstrings

Return the globally unique univariate or multivariate Laurent polynomial ring with given properties and variable name or names.

There are four ways to call the Laurent polynomial ring constructor:

1. LaurentPolynomialRing(base_ring, name, sparse=False)
2. LaurentPolynomialRing(base_ring, names, order='degrevlex')
3. LaurentPolynomialRing(base_ring, name, n, order='degrevlex')
4. LaurentPolynomialRing(base_ring, n, name, order='degrevlex')

The optional arguments sparse and order must be explicitly named, and the other arguments must be given positionally.

INPUT:

• base_ring – a commutative ring
• name – a string
• names – a list or tuple of names, or a comma separated string
• n – a positive integer
• sparse – bool (default: False), whether or not elements are sparse
• order – string or TermOrder, e.g.,
  – 'degrevlex' (default) – degree reverse lexicographic
  – 'lex' – lexicographic
  – 'deglex' – degree lexicographic
  – TermOrder('deglex',3) + TermOrder('deglex',3) – block ordering

OUTPUT:
LaurentPolynomialRing(base_ring, name, sparse=False) returns a univariate Laurent polynomial ring; all other input formats return a multivariate Laurent polynomial ring.

UNIQUENESS and IMMUTABILITY: In Sage there is exactly one single-variate Laurent polynomial ring over each base ring in each choice of variable and sparseness. There is also exactly one multivariate Laurent polynomial ring over each base ring for each choice of names of variables and term order.

Examples:
1. LaurentPolynomialRing(base_ring, name, sparse=False)

You can’t just globally change the names of those variables. This is because objects all over Sage could have pointers to that polynomial ring.

Examples:
1. LaurentPolynomialRing(base_ring, name, sparse=False)
Polynomials, Release 10.3

(continued from previous page)

```python
sage: \texttt{R.<abc> = LaurentPolynomialRing(QQ, sparse=True);} R
\text{Univariate Laurent Polynomial Ring in abc over Rational Field}

sage: \texttt{R.<w> = LaurentPolynomialRing(PolynomialRing(GF(7),'k')); R}
\text{Univariate Laurent Polynomial Ring in w over}
\text{Univariate Polynomial Ring in k over Finite Field of size 7}
```

Rings with different variables are different:

```python
sage: \texttt{LaurentPolynomialRing(QQ, 'x') == LaurentPolynomialRing(QQ, 'y')}
\text{False}
```

2. `LaurentPolynomialRing\texttt{(base\_ring, names, order='degrevlex')}`

```python
sage: \texttt{R = LaurentPolynomialRing(QQ, 'a,b,c'); R}
\text{Multivariate Laurent Polynomial Ring in a, b, c over Rational Field}

sage: \texttt{S = LaurentPolynomialRing(QQ, ['a','b','c']); S}
\text{Multivariate Laurent Polynomial Ring in a, b, c over Rational Field}

sage: \texttt{T = LaurentPolynomialRing(QQ, ('a','b','c')); T}
\text{Multivariate Laurent Polynomial Ring in a, b, c over Rational Field}
```

All three rings are identical.

```python
sage: \texttt{\texttt{(R is S) and (S is T)}}
\text{True}
```

There is a unique Laurent polynomial ring with each term order:

```python
sage: \texttt{# needs sage.modules}
\texttt{R = LaurentPolynomialRing(QQ, 'x,y,z', order='degrevlex'); R}
\text{Multivariate Laurent Polynomial Ring in x, y, z over Rational Field}

sage: \texttt{S = LaurentPolynomialRing(QQ, 'x,y,z', order='invlex'); S}
\text{Multivariate Laurent Polynomial Ring in x, y, z over Rational Field}

sage: \texttt{S} \texttt{is LaurentPolynomialRing(QQ, 'x,y,z', order='invlex')}
\text{True}

sage: \texttt{R \texttt{== S}}
\text{False}
```

3. `LaurentPolynomialRing\texttt{(base\_ring, name, n, order='degrevlex')}`

If you specify a single name as a string and a number of variables, then variables labeled with numbers are created.

```python
sage: \texttt{LaurentPolynomialRing(QQ, 'x', 10)}
\text{Multivariate Laurent Polynomial Ring in x0, x1, x2, x3, x4, x5, x6, x7, x8, x9 over Rational Field}

sage: \texttt{LaurentPolynomialRing(GF(7), 'y', 5)}
\text{Multivariate Laurent Polynomial Ring in y0, y1, y2, y3, y4}
```

(continues on next page)
over Finite Field of size 7

\texttt{sage}: \texttt{LaurentPolynomialRing(QQ, 'y', 3, sparse=True)}
\texttarrow{}# needs sage.modules
Multivariate Laurent Polynomial Ring in y0, y1, y2 over Rational Field

By calling the \texttt{inject_variables()} method, all those variable names are available for interactive use:

\texttt{sage}: R = LaurentPolynomialRing(GF(7), 15, 'w'); R
\texttarrow{}# needs sage.modules
Multivariate Laurent Polynomial Ring in w0, w1, w2, w3, w4, w5, w6, w7,
w8, w9, w10, w11, w12, w13, w14 over Finite Field of size 7
\texttt{sage}: R.inject_variables()
\texttarrow{}# needs sage.modules
Defining w0, w1, w2, w3, w4, w5, w6, w7, w8, w9, w10, w11, w12, w13, w14
\texttt{sage}: (w0 + 2*w8 + w13)^2
\texttarrow{}# needs sage.modules
w0^2 + 4*w0*w8 + 4*w8^2 + 2*w0*w13 + 4*w8*w13 + w13^2

\texttt{class sage.rings.polynomial.laurent_polynomial_ring.LaurentPolynomialRing_mpair}(R)
\texttt{Bases: LaurentPolynomialRing_generic}

\texttt{EXAMPLES:}

\texttt{sage}: L = LaurentPolynomialRing(QQ, 2, 'x')
\texttarrow{}# needs sage.modules
\texttt{sage}: type(L)
\texttarrow{}# needs sage.modules
\texttt{<class}
't.sage.rings.polynomial.laurent_polynomial_ring.LaurentPolynomialRing_mpairy_with_}
\texttarrow{}\texttt{category'>}
\texttt{sage}: L == loads(dumps(L))
\texttarrow{}# needs sage.modules
\texttt{True}

\texttt{Element}

\texttt{alias of LaurentPolynomial_mpairy}

\texttt{monomial}(\texttt{**args})

Return the monomial whose exponents are given in argument.

\texttt{EXAMPLES:}

\texttt{sage}: # needs sage.modules
\texttt{sage}: L = LaurentPolynomialRing(QQ, 'x', 2)
\texttt{sage}: L.monomial(-3, 5)
x0^-3*x1^5
\texttt{sage}: L.monomial(1, 1)
x0*x1
\texttt{sage}: L.monomial(0, 0)
1
\texttt{sage}: L.monomial(-2, -3)
x0^-2*x1^-3
\texttt{sage}: x0, x1 = L.gens()
\texttarrow{}# needs sage.modules
\texttt{sage}: L.monomial(-1, 2) == x0^-1 * x1^2
\texttarrow{}#
needs sage.modules
True

sage: L.monomial(1, 2, 3)

Traceback (most recent call last):
  ...
TypeError: tuple key must have same length as ngens

class sage.rings.polynomial.laurent_polynomial_ring.LaurentPolynomialRing_univariate(R)

Bases: LaurentPolynomialRing_generic

EXAMPLES:

sage: L = LaurentPolynomialRing(QQ,'x')
sage: type(L)
<class 'sage.rings.polynomial.laurent_polynomial_ring.LaurentPolynomialRing_'
˓→univariate_with_category'>
sage: L == loads(dumps(L))
True

Element

alias of LaurentPolynomial_univariate

sage.rings.polynomial.laurent_polynomial_ring.from_fraction_field(L,x)

Helper function to construct a Laurent polynomial from an element of its parent's fraction field.

INPUT:

- L – an instance of LaurentPolynomialRing_generic
- x – an element of the fraction field of L

OUTPUT:

An instance of the element class of L. If the denominator fails to be a unit in L an error is raised.

EXAMPLES:

sage: # needs sage.modules
sage: from sage.rings.polynomial.laurent_polynomial_ring import from_fraction_field
sage: L.<x, y> = LaurentPolynomialRing(ZZ)
sage: F = L.fraction_field()
sage: xi = F(~x)
sage: from_fraction_field(L, xi) == ~x
True

sage.rings.polynomial.laurent_polynomial_ring.is_LaurentPolynomialRing(R)

Return True if and only if R is a Laurent polynomial ring.

EXAMPLES:

sage: from sage.rings.polynomial.laurent_polynomial_ring import is_
˓→LaurentPolynomialRing
sage: P = PolynomialRing(QQ, 2, 'x')
sage: is_LaurentPolynomialRing(P)
doctest:warning...
5.3 Elements of Laurent polynomial rings

class sage.rings.polynomial.laurent_polynomial.LaurentPolynomial
Bases: CommutativeAlgebraElement

Base class for Laurent polynomials.

change_ring (R)

Return a copy of this Laurent polynomial, with coefficients in R.

EXAMPLES:

sage: R.<x> = LaurentPolynomialRing(QQ)  # needs sage.modules
sage: a = x^2 + 3*x^3 + 5*x^-1
sage: a.change_ring(GF(3))
2*x^-1 + x^2

Check that github issue #22277 is fixed:

sage: # needs sage.modules
sage: R.<x, y> = LaurentPolynomialRing(QQ)
Sage: a = 2*x^2 + 3*x^3 + 4*x^-1
sage: a.change_ring(GF(3))
-x^2 + x^-1

dict ()

Abstract dict method.

EXAMPLES:

sage: R.<x> = LaurentPolynomialRing(ZZ)
sage: from sage.rings.polynomial.laurent_polynomial import LaurentPolynomial
sage: LaurentPolynomial.dict(x)
Traceback (most recent call last):
... Not Implemented Error

hamming_weight ()

Return the hamming weight of self.

The hamming weight is number of non-zero coefficients and also known as the weight or sparsity.

EXAMPLES:
```python
sage: R.<x> = LaurentPolynomialRing(ZZ)
sage: f = x^3 - 1
sage: f.hamming_weight()
2
```

**map_coefficients** *(f, new_base_ring=None)*

Apply *f* to the coefficients of *self*.

If *f* is a `sage.categories.map.Map`, then the resulting polynomial will be defined over the codomain of *f*. Otherwise, the resulting polynomial will be over the same ring as *self*. Set `new_base_ring` to override this behavior.

**INPUT:**

- *f* – a callable that will be applied to the coefficients of *self*.
- `new_base_ring` (optional) – if given, the resulting polynomial will be defined over this ring.

**EXAMPLES:**

```python
sage: # needs sage.rings.finite_rings
sage: k.<a> = GF(9)
sage: R.<x> = LaurentPolynomialRing(k)
sage: f = x*a + a
sage: f.map_coefficients(lambda a: a + 1)
(a + 1)*x + (a + 1)*a

sage: R.<x,y> = LaurentPolynomialRing(k, 2)

sage: f = x*a + 2*x^3*y*a + a
sage: f.map_coefficients(lambda a: a + 1)
(2*a + 1)*x^3*y + (a + 1)*x + a + 1
```

Examples with different base ring:

```python
sage: # needs sage.modules sage.rings.finite_rings
sage: R.<r> = GF(9); S.<s> = GF(81)
sage: h = Hom(R, S)[0]; h
Ring morphism:
  From: Finite Field in r of size 3^2
  To:   Finite Field in s of size 3^4
  Defn: r |--> 2*s^3 + 2*s^2 + 1
sage: T.<X,Y> = LaurentPolynomialRing(R, 2)
sage: f = r*X + Y
sage: g = f.map_coefficients(h); g
(2*s^3 + 2*s^2 + 1)*X + Y
sage: g.parent()
Multivariate Laurent Polynomial Ring in X, Y
  over Finite Field in s of size 3^4
sage: h = lambda x: x.trace()
sage: g = f.map_coefficients(h); g
X - Y
sage: g.parent()
Multivariate Laurent Polynomial Ring in X, Y
  over Finite Field in r of size 3^2
sage: g = f.map_coefficients(h, new_base_ring=GF(3)); g
X - Y
```

(continues on next page)
sage: g.parent()
Multivariate Laurent Polynomial Ring in X, Y over Finite Field of size 3

number_of_terms()
Abstract method for number of terms

EXAMPLES:

sage: R.<x> = LaurentPolynomialRing(ZZ)
sage: from sage.rings.polynomial.laurent_polynomial import LaurentPolynomial
sage: LaurentPolynomial.number_of_terms(x)
Traceback (most recent call last):
... 
NotImplementedError

class sage.rings.polynomial.laurent_polynomial.LaurentPolynomial_univariate
Bases: LaurentPolynomial

A univariate Laurent polynomial in the form of $t^n \cdot f$ where $f$ is a polynomial in $t$.

INPUT:

• parent – a Laurent polynomial ring
• f – a polynomial (or something can be coerced to one)
• n – (default: 0) an integer

AUTHORS:

• Tom Boothby (2011) copied this class almost verbatim from laurent_series_ring_element.pyx, so most of the credit goes to William Stein, David Joyner, and Robert Bradshaw
• Travis Scrimshaw (09-2013): Cleaned-up and added a few extra methods

coefficients()
Return the nonzero coefficients of self.

EXAMPLES:

sage: R.<t> = LaurentPolynomialRing(QQ)
sage: f = -5/t^(2) + t + t^2 - 10/3*t^3
sage: f.coefficients()
[-5, 1, 1, -10/3]

constant_coefficient()
Return the coefficient of the constant term of self.

EXAMPLES:

sage: R.<t> = LaurentPolynomialRing(QQ)
sage: f = 3*t^-2 - t^-1 + 3 + t^2
sage: f.constant_coefficient()
3
sage: g = -2*t^-2 + t^-1 + 3*t
sage: g.constant_coefficient()
0
**degree()**

Return the degree of `self`.

**EXAMPLES:**

```
sage: R.<x> = LaurentPolynomialRing(ZZ)
sage: g = x^2 - x^4
g.degree()  
4
sage: g = -10/x^5 + x^2 - x^7
g.degree()  
7
```

**derivative(*args)**

The formal derivative of this Laurent polynomial, with respect to variables supplied in `args`.

Multiple variables and iteration counts may be supplied. See documentation for the global `derivative()` function for more details.

**See also:**

 `_derivative()`

**EXAMPLES:**

```
sage: R.<x> = LaurentPolynomialRing(QQ)
sage: g = 1/x^10 - x + x^2 - x^4
g.derivative()  
-10*x^-11 - 1 + 2*x - 4*x^3
g.derivative(x)  
-10*x^-11 - 1 + 2*x - 4*x^3
```

```
sage: R.<t> = PolynomialRing(ZZ)
sage: S.<x> = LaurentPolynomialRing(R)
sage: f = 2*t/x + (3*t^2 + 6*t)*x
f.derivative()  
-2*t*x^-2 + (3*t^2 + 6*t)
f.derivative(x)  
-2*t*x^-2 + (3*t^2 + 6*t)
f.derivative(t)  
2*x^-1 + (6*t + 6)*x
```

**dict()**

Return a dictionary representing `self`.

**EXAMPLES:**

```
sage: R.<x,y> = ZZ[]
sage: Q.<t> = LaurentPolynomialRing(R)
sage: f = (x^3 + y/t^3)^3 + t^2; f
y^3*3^t^-9 + 3*x^3*y^2*t^-6 + 3*x^6*y*t^-3 + x^9 + t^2
f.dict()  
{-9: y^3, -6: 3*x^3*y^2, -3: 3*x^6*y, 0: x^9, 2: 1}
```

**divides(other)**

Return `True` if `self` divides `other`.

**EXAMPLES:**
```python
sage: R.<x> = LaurentPolynomialRing(ZZ)
sage: (2*x**-1 + 1).divides(4*x**-2 - 1)
True
sage: (2*x + 1).divides(4*x**2 + 1)
False
sage: (2*x + x**-1).divides(R(0))
True
sage: R(0).divides(2*x ** -1 + 1)
False
sage: R(0).divides(R(0))
True
sage: R.<x> = LaurentPolynomialRing(Zmod(6))
sage: p = 4*x + 3*x^-1
sage: q = 5*x^2 + x + 2*x^-2
sage: p.divides(q)
False
sage: R.<x,y> = GF(2)[]
sage: S.<z> = LaurentPolynomialRing(R)
sage: p = (x+y+1) * z^-1 + x*y
sage: q = (y^2-x^2) * z^-2 + z + x-y
sage: p.divides(q), p.divides(p*q)
(False, True)
```

### `exponents()`

Return the exponents appearing in `self` with nonzero coefficients.

**EXAMPLES:**

```python
sage: R.<t> = LaurentPolynomialRing(QQ)
sage: f = -5/t^(2) + t + t^2 - 10/3*t^3
sage: f.exponents()
[-2, 1, 2, 3]
```

### `factor()`

Return a Laurent monomial (the unit part of the factorization) and a factored polynomial.

**EXAMPLES:**

```python
sage: R.<t> = LaurentPolynomialRing(QQ)
sage: f = 4*t^-7 + 3*t^3 + 2*t^4 + t^-6
sage: f.factor()  # needs sage.libs.pari
(t^-7) * (4 + t + 3*t^10 + 2*t^11)
```

### `gcd(right)`

Return the gcd of `self` with `right` where the common divisor `d` makes both `self` and `right` into polynomials with the lowest possible degree.

**EXAMPLES:**

```python
sage: R.<t> = LaurentPolynomialRing(QQ)
sage: t.gcd(2)
1
sage: gcd(t^-2 + 1, t^-4 + 3*t^-1)
t^-4
```

(continues on next page)
\begin{verbatim}
sage: gcd((t^-2 + t)*(t + t^-1), (t^5 + t^8)*(1 + t^-2))
t^-3 + t^-1 + 1 + t^2
\end{verbatim}

```
integral()
```

The formal integral of this Laurent series with 0 constant term.

EXAMPLES:

The integral may or may not be defined if the base ring is not a field.

```
sage: t = LaurentPolynomialRing(ZZ, 't').0
sage: f = 2*t^-3 + 3*t^2
sage: f.integral()
-t^-2 + t^3
```

```
sage: f = t^3
sage: f.integral()
Traceback (most recent call last):
... ArithmeticError: coefficients of integral cannot be coerced into the base ring
```

The integral of $1/t$ is $\log(t)$, which is not given by a Laurent polynomial:

```
sage: t = LaurentPolynomialRing(ZZ,'t').0
sage: f = -1/t^3 - 31/t
sage: f.integral()
Traceback (most recent call last):
... ArithmeticError: the integral of is not a Laurent polynomial, since t^-1 has
˓→ nonzero coefficient
```

Another example with just one negative coefficient:

```
sage: A.<t> = LaurentPolynomialRing(QQ)
sage: f = -2*t^(-4)
sage: f.integral()
2/3*t^-3
sage: f.integral().derivative() == f
True
```

```
inverse_mod(a, m)
```

Invert the polynomial $a$ with respect to $m$, or raise a \texttt{ValueError} if no such inverse exists.

The parameter $m$ may be either a single polynomial or an ideal (for consistency with \texttt{inverse_mod()} in other rings).

ALGORITHM: Solve the system $as + mt = 1$, returning $s$ as the inverse of $a$ mod $m$.

EXAMPLES:

```
sage: S.<t> = LaurentPolynomialRing(QQ)
sage: f = inverse_mod(t^-2 + 1, t^-3 + 1); f
1/2*t^2 - 1/2*t^3 - 1/2*t^4
sage: f * (t^-2 + 1) + (1/2*t^4 + 1/2*t^3) * (t^-3 + 1)
1
```
inverse_of_unit()

Return the inverse of self if a unit.

EXAMPLES:

```
sage: R.<t> = LaurentPolynomialRing(QQ)
sage: (t^-2).inverse_of_unit()
t^2
```

is_constant()

Return whether this Laurent polynomial is constant.

EXAMPLES:

```
sage: R.<x> = LaurentPolynomialRing(QQ)
sage: x.is_constant()
False
```

is_monomial()

Return True if self is a monomial; that is, if self is $x^n$ for some integer $n$.

EXAMPLES:

```
sage: k.<z> = LaurentPolynomialRing(QQ)
sage: z.is_monomial()
True
```

is_square (root=False)

Return whether this Laurent polynomial is a square.

If root is set to True then return a pair made of the boolean answer together with None or a square root.
EXAMPLES:

```
sage: R.<t> = LaurentPolynomialRing(QQ)
sage: R.one().is_square()  
True
sage: R(2).is_square()  
False
sage: t.is_square()  
False
sage: (t**-2).is_square()  
True
```

Usage of the root option:

```
sage: p = (1 + t^-1 - 2*t^3)
sage: p.is_square(root=True)  
(False, None)
sage: (p**2).is_square(root=True)  
(True, -t^-1 - 1 + 2*t^3)
```

The answer is dependent of the base ring:

```
sage: # needs sage.rings.number_field
sage: S.<u> = LaurentPolynomialRing(QQbar)
sage: (2 + 4*t + 2*t^2).is_square()  
False
sage: (2 + 4*u + 2*u^2).is_square()  
True
```

is_unit()

Return True if this Laurent polynomial is a unit in this ring.

EXAMPLES:

```
sage: R.<t> = LaurentPolynomialRing(QQ)
sage: (2 + t).is_unit()  
False
sage: f = 2*t
sage: f.is_unit()  
True
sage: 1/f  
1/2*t^-1
sage: R(0).is_unit()  
False
sage: R.<s> = LaurentPolynomialRing(ZZ)
sage: g = 2*s
sage: g.is_unit()  
False
sage: 1/g  
1/2*s^-1
```

ALGORITHM: A Laurent polynomial is a unit if and only if its “unit part” is a unit.

is_zero()

Return 1 if self is 0, else return 0.

EXAMPLES:
monomial_reduction()

Return the decomposition as a polynomial and a power of the variable. Constructed for compatibility with the multivariate case.

OUTPUT:

A tuple $\langle u, t^n \rangle$ where $u$ is the underlying polynomial and $n$ is the power of the exponent shift.

EXAMPLES:

```sage
sage: R.<x> = LaurentPolynomialRing(QQ)
sage: f = 1/x + x^2 + 3*x^4
sage: f.monomial_reduction()
(3*x^5 + x^3 + 1, x^-1)
```

number_of_terms()

Return the number of non-zero coefficients of self.

Also called weight, hamming weight or sparsity.

EXAMPLES:

```sage
sage: R.<x> = LaurentPolynomialRing(ZZ)
sage: f = x^3 - 1
sage: f.number_of_terms()
2
sage: R(0).number_of_terms()
0
sage: f = (x+1)^100
sage: f.number_of_terms()
101
```

The method hamming_weight() is an alias:

```sage
sage: f.hamming_weight()
101
```

polynomial_construction()

Return the polynomial and the shift in power used to construct the Laurent polynomial $t^n u$.

OUTPUT:

A tuple $\langle u, n \rangle$ where $u$ is the underlying polynomial and $n$ is the power of the exponent shift.

EXAMPLES:

```sage
sage: R.<x> = LaurentPolynomialRing(QQ)
sage: f = 1/x + x^2 + 3*x^4
sage: f.polynomial_construction()
(3*x^5 + x^3 + 1, -1)
```
\textbf{quo_rem}(\textit{other})

Divide \textit{self} by \textit{other} and return a quotient \(q\) and a remainder \(r\) such that \(\textit{self} = q \times \textit{other} + r\).

**EXAMPLES:**

```python
sage: R.<t> = LaurentPolynomialRing(QQ)
sage: (t^-3 - t^3).quo_rem(t^-1 - t)
(t^-2 + 1 + t^2, 0)
sage: (t^-2 + 3 + t).quo_rem(t^-4)
(t^2 + 3t^4 + t^5, 0)
sage: num = t^-2 + t
sage: den = t^-2 + 1
sage: q, r = num.quo_rem(den)
sage: num == q * den + r
True
```

\textbf{residue}(\textit{})

Return the residue of \textit{self}.

The residue is the coefficient of \(t^{-1}\).

**EXAMPLES:**

```python
sage: R.<t> = LaurentPolynomialRing(QQ)
sage: f = 3*t^-2 - t^-1 + 3 + t^2
sage: f.residue()
-1
sage: g = -2*t^-2 + 4 + 3*t
sage: g.residue()
0
sage: f.residue().parent()
Rational Field
```

\textbf{shift}(\textit{k})

Return this Laurent polynomial multiplied by the power \(t^n\). Does not change this polynomial.

**EXAMPLES:**

```python
sage: R.<t> = LaurentPolynomialRing(QQ['y'])
sage: f = (t+t^-1)^4; f
\text{t}^{-4} + 4\text{t}^{-2} + 6 + 4\text{t} + \text{t}^{4}
sage: f.shift(10)
\text{t}^{6} + 4\text{t}^{8} + 6\text{t}^{10} + 4\text{t}^{12} + \text{t}^{14}
sage: f >> 10
\text{t}^{-14} + 4\text{t}^{-12} + 6\text{t}^{-10} + 4\text{t}^{-8} + \text{t}^{-6}
sage: f << 4
1 + 4\text{t}^{2} + 6\text{t}^{4} + 4\text{t}^{6} + \text{t}^{8}
```

\textbf{truncate}(\textit{n})

Return a polynomial with degree at most \(n - 1\) whose \(j\)-th coefficients agree with \textit{self} for all \(j < n\).

**EXAMPLES:**

```python
sage: R.<x> = LaurentPolynomialRing(QQ)
sage: f = 1/x^12 + x^3 + x^5 + x^9
sage: f.truncate(10)
```

(continues on next page)
valuation \((p=None)\)

Return the valuation of \(self\).

The valuation of a Laurent polynomial \(t^n u\) is \(n\) plus the valuation of \(u\).

EXAMPLES:

```
sage: R.<x> = LaurentPolynomialRing(ZZ)
sage: f = 1/x + x^2 + 3*x^4
sage: g = 1 - x + x^2 - x^4
sage: f.valuation()
-1
sage: g.valuation()
0
```

variable_name()

Return the name of variable of \(self\) as a string.

EXAMPLES:

```
sage: R.<x> = LaurentPolynomialRing(QQ)
sage: f = 1/x + x^2 + 3*x^4
sage: f.variable_name()
'x'
```

variables()

Return the tuple of variables occurring in this Laurent polynomial.

EXAMPLES:

```
sage: R.<x> = LaurentPolynomialRing(QQ)
sage: f = 1/x + x^2 + 3*x^4
sage: f.variables()
(x,)
sage: R.one().variables()
()  
```

xgcd \((other)\)

Extended \(gcd\) for univariate Laurent polynomial rings over a field.

EXAMPLES:

```
sage: S.<t> = LaurentPolynomialRing(QQ)
sage: (t^-2 + 1).xgcd(t^-3 + 1)
(1, 1/2*t^2 - 1/2*t^3 - 1/2*t^4, 1/2*t^3 + 1/2*t^4)
```
5.4 MacMahon's Partition Analysis Omega Operator

This module implements MacMahon's Omega Operator [Mac1915], which takes a quotient of Laurent polynomials and removes all negative exponents in the corresponding power series.

5.4.1 Examples

In the following example, all negative exponents of \( \mu \) are removed. The formula
\[
\Omega \geq \frac{1}{(1 - x\mu)(1 - y/\mu)} = \frac{1}{(1 - x)(1 - xy)}
\]
can be calculated and verified by

\[
\text{sage: } L.<\mu, x, y> = \text{LaurentPolynomialRing}(\mathbb{Z}) \\
\text{sage: } \text{MacMahonOmega}(\mu, 1, [1 - x*\mu, 1 - y/\mu]) \\
1 * (-x + 1)^-1 * (-x*y + 1)^-1
\]

5.4.2 Various

AUTHORS:
• Daniel Krenn (2016)

ACKNOWLEDGEMENT:
• Daniel Krenn is supported by the Austrian Science Fund (FWF): P 24644-N26.

5.4.3 Functions

\text{sage.rings.polynomial.omega.} \text{MacMahonOmega} (\var, \text{expression}, \text{denominator} = \text{None}, \text{op} = <\text{built-in function ge}>, \text{Factorization_sort} = \text{False}, \text{Factorization_simplify} = \text{True})

Return \( \Omega_{op} \) of expression with respect to \var.

To be more precise, calculate
\[
\Omega_{op} \frac{n}{d_1 \ldots d_n}
\]
for the numerator \( n \) and the factors \( d_1, \ldots, d_n \) of the denominator, all of which are Laurent polynomials in \var and return a (partial) factorization of the result.

INPUT:

• \var – a variable or a representation string of a variable
• \text{expression} – a \text{Factorization} of Laurent polynomials or, if \text{denominator} is specified, a Laurent polynomial interpreted as the numerator of the expression
• \text{denominator} – a Laurent polynomial or a \text{Factorization} (consisting of Laurent polynomial factors) or a tuple/list of factors (Laurent polynomials)
• \text{op} – (default: \text{operator.ge}) an operator
  At the moment only \text{operator.ge} is implemented.
Polynomials, Release 10.3

- Factorization_sort (default: False) and Factorization_simplify (default: True) – are passed on to sage.structure.factorization.Factorization when creating the result

OUTPUT:

A (partial) Factorization of the result whose factors are Laurent polynomials

Note: The numerator of the result may not be factored.

REFERENCES:
- [Mac1915]
- [APR2001]

EXAMPLES:

```
sage: L.<mu, x, y, z, w> = LaurentPolynomialRing(ZZ)
sage: MacMahonOmega(mu, 1, [1 - x*mu, 1 - y/mu])
1 * (-x + 1)^-1 * (-x*y + 1)^-1
sage: MacMahonOmega(mu, 1, [1 - x*mu, 1 - y/mu, 1 - z/mu])
1 * (-x + 1)^-1 * (-x*y + 1)^-1 * (-x*z + 1)^-1
sage: MacMahonOmega(mu, 1, [1 - x*mu, 1 - y*mu, 1 - z/mu])
(-x*y*z + 1) * (-x + 1)^-1 * (-y + 1)^-1 * (-x*z + 1)^-1 * (-y*z + 1)^-1
sage: MacMahonOmega(mu, 1, [1 - x*mu, 1 - y*mu^2])
1 * (-x + 1)^-1 * (-x^2*y + 1)^-1
sage: MacMahonOmega(mu, 1, [1 - x*mu^2, 1 - y/mu])
(x*y + 1) * (-x + 1)^-1 * (-x*y^2 + 1)^-1
sage: MacMahonOmega(mu, 1, [1 - x*mu^2, 1 - y*mu, 1 - z/mu])
(-x*y*z^2 - x*y*z + x*z + 1) * (-x + 1)^-1 * (-y + 1)^-1 * (-x*y^2 + 1)^-1 * (-y^2*z + 1)^-1
sage: MacMahonOmega(mu, 1, [1 - x*mu^2, 1 - y*mu^3])
(x*y^2 + x*y + 1) * (-x + 1)^-1 * (-x*y^3 + 1)^-1
sage: MacMahonOmega(mu, 1, [1 - x*mu^4, 1 - y/mu])
(x*y^3 + x*y^2 + x*y + 1) * (-x + 1)^-1 * (-x*y^4 + 1)^-1
sage: MacMahonOmega(mu, 1, [1 - x*mu, 1 - y*mu, 1 - z*mu, 1 - w/mu])
(x^2*y*z*w + x*y^2*z*w - x*y*z*w - x*y*z*w + 1) * (-x + 1)^-1 * (-y + 1)^-1 * (-z*w + 1)^-1 * (-y*z + 1)^-1
```

(continues on next page)
We demonstrate the different allowed input variants:

\[
(-x^2 - x^2y + y^2 + y + 1) * (-x + 1)^{-1} * (-x*y + 1)^{-1}
\]

sage: MacMahonOmega(mu, mu, [1 - x*mu, 1 - y/mu])
(-x^2 - x^2y + y^2 + y + 1) * (-x + 1)^{-1} * (-x*y + 1)^{-1}

sage: MacMahonOmega(mu, mu^2, [1 - x*mu, 1 - y/mu])
(-x^2y + y + 1) * (-x + 1)^{-1} * (-x*y + 1)^{-1}

sage: MacMahonOmega(mu, mu^2 / ((1 - x*mu)*(1 - y/mu)))
# not tested because not fully implemented
(-x^2y + y + 1) * (-x + 1)^{-1} * (-x*y + 1)^{-1}

sage.rings.polynomial.omega.Omega_ge(exponents)
Return \(\Omega \geq\) of the expression specified by the input.

To be more precise, calculate

\[
\Omega \geq \frac{\mu^a}{(1 - z_0\mu^{e_0}) \ldots (1 - z_{n-1}\mu^{e_{n-1}})}
\]

and return its numerator and a factorization of its denominator. Note that \(z_0, \ldots, z_{n-1}\) only appear in the output, but not in the input.

INPUT:
- \(a\) – an integer
- \(\text{exponents}\) – a tuple of integers

OUTPUT:
A pair representing a quotient as follows: Its first component is the numerator as a Laurent polynomial, its second component a factorization of the denominator as a tuple of Laurent polynomials, where each Laurent polynomial \(z\) represents a factor \(1 - z\).

The parents of these Laurent polynomials is always a Laurent polynomial ring in \(z_0, \ldots, z_{n-1}\) over \(\mathbb{Z}\), where \(n\) is the length of \(\text{exponents}\).

EXAMPLES:
sage: from sage.rings.polynomial.omega import Omega_ge
sage: Omega_ge(0, (1, -2))
(1, (z0, z0^2*z1))
sage: Omega_ge(0, (1, -3))
(1, (z0, z0^3*z1))
sage: Omega_ge(0, (1, -4))
(1, (z0, z0^4*z1))
sage: Omega_ge(0, (2, -1))
(z0*z1 + 1, (z0, z0*z1^2))
sage: Omega_ge(0, (3, -1))
(z0*z1^2 + z0*z1 + 1, (z0, z0*z1^3))
sage: Omega_ge(0, (4, -1))
(z0*z1^3 + z0*z1^2 + z0*z1 + 1, (z0, z0*z1^4))
sage: Omega_ge(0, (1, 1, -2))
(-z0^2*z1*z2 - z0*z1^2*z2 + z0*z1*z2 + 1, (z0, z1, z0^2*z2, z1^2*z2))
sage: Omega_ge(0, (2, -1, -1))
(z0*z1*z2 + z0*z1 + z0*z2 + 1, (z0, z0*z1^2, z0*z2^2))
sage: Omega_ge(0, (2, 1, -1))
(-z0*z1*z2^2 - z0*z1*z2 + z0*z2 + 1, (z0, z1, z0*z2^2, z1*z2))
sage: Omega_ge(0, (2, -2))
(z0^2*z1 + 1, (z0, z0*z1, z0*z1))
sage: Omega_ge(0, (2, -3))
(z0^3*z1 + 1, (z0, z0^3*z1^2))
sage: Omega_ge(0, (3, 1, -3))
(-z0^3*z1^3*z2^3 + 2*z0^2*z1^3*z2^2 - z0*z1^3*z2 + z0^2*z2^2 - 2*z0*z2 + 1, (z0, z1, z0*z2, z0*z2, z1^3*z2))
sage: Omega_ge(0, (3, 6, -1))
(-z0*z1*z2^8 - z0*z1*z2^7 - z0*z1*z2^6 - z0*z1*z2^5 - z0*z1*z2^4 + z1*z2^5 - z0*z1*z2^3 + z1*z2^4 - z0*z1*z2^2 + z1*z2^3 - z0*z1*z2 + z0*z2^2 + z1*z2^2 + z0*z2 + z1*z2 + 1, (z0, z1, z0*z2^3, z1*z2^6))

sage: from sage.rings.polynomial.omega import homogeneous_symmetric_function
sage: homogeneous_symmetric_function(0, x)
1
sage: homogeneous_symmetric_function(1, x)
(continues on next page)
\[ X_0 + X_1 + X_2 \]

\texttt{sage: homogeneous_symmetric_function(2, P.gens())}
\[ X_0^2 + X_0*X_1 + X_1^2 + X_0*X_2 + X_1*X_2 + X_2^2 \]

\texttt{sage: homogeneous_symmetric_function(3, P.gens())}
\[ X_0^3 + X_0^2*X_1 + X_0*X_1^2 + X_1^3 + X_0^2*X_2 + X_0*X_1*X_2 + X_1^2*X_2 + X_0*X_2^2 + X_1*X_2^2 + X_2^3 \]

\texttt{sage.rings.polynomial.omega.partition(items, predicate=<class 'bool'>)}

Split \texttt{items} into two parts by the given \texttt{predicate}.

\begin{itemize}
  \item \texttt{item} – an iterator
  \item \texttt{predicate} – a function
\end{itemize}

\textbf{OUTPUT:}

A pair of iterators; the first contains the elements not satisfying the \texttt{predicate}, the second the elements satisfying the \texttt{predicate}.

\textbf{ALGORITHM:}

Source of the code: \url{http://nedbatchelder.com/blog/201306/filter_a_list_into_two_parts.html}

\textbf{EXAMPLES:}

\texttt{sage: from sage.rings.polynomial.omega import partition}
\texttt{sage: E, O = partition(srange(10), is_odd)}
\texttt{sage: tuple(E), tuple(O)}
\[ ((0, 2, 4, 6, 8), (1, 3, 5, 7, 9)) \]
6.1 Infinite Polynomial Rings

By Infinite Polynomial Rings, we mean polynomial rings in a countably infinite number of variables. The implementation consists of a wrapper around the current finite polynomial rings in Sage.

AUTHORS:

• Simon King <simon.king@nuigalway.ie>
• Mike Hansen <mhansen@gmail.com>

An Infinite Polynomial Ring has finitely many generators \( x_*, y_* \) and infinitely many variables of the form \( x_0, x_1, x_2, \ldots, y_0, y_1, y_2, \ldots, \ldots \). We refer to the natural number \( n \) as the index of the variable \( x_n \).

INPUT:

• \( R \), the base ring. It has to be a commutative ring, and in some applications it must even be a field
• \( \text{names} \), a finite list of generator names. Generator names must be alpha-numeric.
• \( \text{order} \) (optional string). The default order is 'lex' (lexicographic). 'deglex' is degree lexicographic, and 'degrevlex' (degree reverse lexicographic) is possible but discouraged.

Each generator \( x \) produces an infinite sequence of variables \( x[1], x[2], \ldots \) which are printed on screen as \( x_1, x_2, \ldots \) and are latex typeset as \( x_1, x_2 \). Then, the Infinite Polynomial Ring is formed by polynomials in these variables.

By default, the monomials are ordered lexicographically. Alternatively, degree (reverse) lexicographic ordering is possible as well. However, we do not guarantee that the computation of Groebner bases will terminate in this case.

In either case, the variables of a Infinite Polynomial Ring \( X \) are ordered according to the following rule:

\[
X\text{.gen}(i)[m] > X\text{.gen}(j)[n] \text{ if and only if } i<j \text{ or } (i==j \text{ and } m>n)
\]

We provide a 'dense' and a 'sparse' implementation. In the dense implementation, the Infinite Polynomial Ring carries a finite polynomial ring that comprises all variables up to the maximal index that has been used so far. This is potentially a very big ring and may also comprise many variables that are not used.

In the sparse implementation, we try to keep the underlying finite polynomial rings small, using only those variables that are really needed. By default, we use the dense implementation, since it usually is much faster.

EXAMPLES:

\[
sage: X.<x,y> = InfinitePolynomialRing(ZZ, implementation='sparse')
sage: A.<alpha,beta> = InfinitePolynomialRing(QQ, order='deglex')
\]

\[
sage: f = x[5] + 2; f
x_5 + 2
\]
It has some advantages to have an underlying ring that is not univariate. Hence, we always have at least two variables:

```python
sage: g._p.parent()
Multivariate Polynomial Ring in y_1, y_0 over Integer Ring
sage: f2 = alpha[5] + 2; f2
alpha_5 + 2
sage: g2 = 3*beta[1]; g2
3*beta_1
```

Of course, we provide the usual polynomial arithmetic:

```python
sage: f + g
x_5 + 3*y_1 + 2
sage: p = x[10]^2*(f+g); p
x_10^2*x_5 + 3*x_10^2*y_1 + 2*x_10^2
sage: p2 = alpha[10]^2*(f2+g2); p2
alpha_10^2*alpha_5 + 3*alpha_10^2*beta_1 + 2*alpha_10^2
```

There is a permutation action on the variables, by permuting positive variable indices:

```python
sage: P = Permutation(((10,1)))
sage: p^P
alpha_5*alpha_1^2 + 3*alpha_1^2*beta_10 + 2*alpha_1^2
```

Note that $x_P^0 = x_0$, since the permutations only change positive variable indices.

We also implemented ideals of Infinite Polynomial Rings. Here, it is thoroughly assumed that the ideals are set-wise invariant under the permutation action. We therefore refer to these ideals as Symmetric Ideals. Symmetric Ideals are finitely generated modulo addition, multiplication by ring elements and permutation of variables. If the base ring is a field, one can compute Symmetric Groebner Bases:

```python
sage: J = A * (alpha[1]*beta[2])
sage: J.groebner_basis()
[alpha_1*beta_2, alpha_2*beta_1]
```

For more details, see `SymmetricIdeal`.

Infinite Polynomial Rings can have any commutative base ring. If the base ring of an Infinite Polynomial Ring is a (classical or infinite) Polynomial Ring, then our implementation tries to merge everything into one ring. The basic requirement is that the monomial orders match. In the case of two Infinite Polynomial Rings, the implementations must match. Moreover, name conflicts should be avoided. An overlap is only accepted if the order of variables can be uniquely inferred, as in the following example:

```python
sage: A.<a,b,c> = InfinitePolynomialRing(ZZ)
sage: B.<b,c,d> = InfinitePolynomialRing(A)
sage: B
Infinite polynomial ring in a, b, c, d over Integer Ring
```
This is also allowed if finite polynomial rings are involved:

```python
sage: A.<a_3,a_1,b_1,c_2,c_0> = ZZ[]
sage: B.<b,c,d> = InfinitePolynomialRing(A, order='degrevlex')
sage: B
Infinite polynomial ring in b, c, d over
Multivariate Polynomial Ring in a_3, a_1 over Integer Ring
```

It is no problem if one generator of the Infinite Polynomial Ring is called x and one variable of the base ring is also called x. This is since no variable of the Infinite Polynomial Ring will be called x. However, a problem arises if the underlying classical Polynomial Ring has a variable x_1, since this can be confused with a variable of the Infinite Polynomial Ring. In this case, an error will be raised:

```python
sage: X.<x,y_1> = ZZ[]
sage: Y.<x,z> = InfinitePolynomialRing(X)
```

Note that X is not merged into Y; this is since the monomial order of X is ‘degrevlex’, but of Y is ‘lex’.

```python
sage: Y
Infinite polynomial ring in x, z over
Multivariate Polynomial Ring in x, y_1 over Integer Ring
```

The variable x of X can still be interpreted in Y, although the first generator of Y is called x as well:

```python
sage: x
x
sage: x
x
sage: X(x)
x
sage: Y(X(x))
x
sage: Y(x)
x
```

But there is only merging if the resulting monomial order is uniquely determined. This is not the case in the following examples, and thus an error is raised:

```python
sage: X.<y_1,x> = ZZ[]
sage: Y.<y,z> = InfinitePolynomialRing(X)
Traceback (most recent call last):
... CoercionException: Overlapping variables (('y', 'z'),['y_1']) are incompatible
sage: Y.<z,y> = InfinitePolynomialRing(X)
Traceback (most recent call last):
... CoercionException: Overlapping variables (('z', 'y'),['y_1']) are incompatible
sage: X.<x_3,y_1,y_2> = PolynomialRing(ZZ, order='lex')
sage: # y_1 and y_2 would be in opposite order in an Infinite Polynomial Ring
sage: Y.<y> = InfinitePolynomialRing(X)
Traceback (most recent call last):
... CoercionException: Overlapping variables (('y',),['y_1', 'y_2']) are incompatible
```

If the type of monomial orderings (e.g., ‘degrevlex’ versus ‘lex’) or if the implementations do not match, there is no simplified construction available:

```python
sage: X.<x,y> = InfinitePolynomialRing(ZZ)
sage: Y.<z> = InfinitePolynomialRing(X, order='degrevlex')
```

(continues on next page)
sage: Y
Infinite polynomial ring in z over Infinite polynomial ring in x, y over Integer Ring

sage: Y.<z> = InfinitePolynomialRing(X, implementation='sparse')

sage: Y
Infinite polynomial ring in z over Infinite polynomial ring in x, y over Integer Ring

class sage.rings.polynomial.infinite_polynomial_ring.GenDictWithBasering(parent, start)

Bases: object

A dictionary-like class that is suitable for usage in sage_eval.

This pseudo-dictionary accepts strings as index, and then walks down a chain of base rings of (infinite) polynomial rings until it finds one ring that has the given string as variable name, which is then returned.

EXAMPLES:

```python
sage: R.<a,b> = InfinitePolynomialRing(ZZ)
sage: D = R.gens_dict()  # indirect doctest
sage: D
GenDict of Infinite polynomial ring in a, b over Integer Ring

sage: D['a_15']
a_15
sage: type(_)
<class sage.rings.polynomial.infinite_polynomial_element.InfinitePolynomial_dense
˓→>

sage: sage_eval(3*a_3*b_5-1/2*a_7, D)
-1/2*a_7 + 3*a_3*b_5
```

next()

Return a dictionary that can be used to interprete strings in the base ring of self.

EXAMPLES:

```python
sage: R.<a,b> = InfinitePolynomialRing(QQ['t'])
sage: D = R.gens_dict()
sage: D
GenDict of Infinite polynomial ring in a, b over Univariate Polynomial Ring
˓→ in t over Rational Field

sage: next(D)
GenDict of Univariate Polynomial Ring in t over Rational Field

sage: sage_eval('t^2', next(D))
t^2
```

class sage.rings.polynomial.infinite_polynomial_ring.InfiniteGenDict(Gens)

Bases: object

A dictionary-like class that is suitable for usage in sage_eval.

The generators of an Infinite Polynomial Ring are not variables. Variables of an Infinite Polynomial Ring are returned by indexing a generator. The purpose of this class is to return a variable of an Infinite Polynomial Ring, given its string representation.

EXAMPLES:

```python
sage: R.<a,b> = InfinitePolynomialRing(ZZ)
sage: D = R.gens_dict()  # indirect doctest
sage: D._D
```
[InfiniteGenDict defined by ['a', 'b'], {'1': 1}]

sage: D._D[0]['a_15']
a_15

sage: type(_)
<class 'sage.rings.polynomial.infinite_polynomial_element.InfinitePolynomial_dense'>

sage: sage_eval(3*a_3*b_5 - 1/2*a_7, D._D[0])
-1/2*a_7 + 3*a_3*b_5

class sage.rings.polynomial.infinite_polynomial_ring.InfinitePolynomialGen(parent, name)

Bases: SageObject

This class provides the object which is responsible for returning variables in an infinite polynomial ring (implemented in __getitem__).

EXAMPLES:

sage: X.<x1,x2> = InfinitePolynomialRing(RR)
sage: x1
x1_*
sage: x1[5]
x1_5
sage: x1 == loads(dumps(x1))
True

class sage.rings.polynomial.infinite_polynomial_ring.InfinitePolynomialRingFactory

Bases: UniqueFactory

A factory for creating infinite polynomial ring elements. It handles making sure that they are unique as well as handling pickling. For more details, see UniqueFactory and infinite_polynomial_ring.

EXAMPLES:

sage: A.<a> = InfinitePolynomialRing(QQ)
sage: B.<b> = InfinitePolynomialRing(A)
sage: B.construction()
[InfPoly([a,b], "lex", "dense"), Rational Field]
sage: R.<a,b> = InfinitePolynomialRing(QQ)
sage: R is B
True
sage: X.<x> = InfinitePolynomialRing(QQ)
sage: X2.<x> = InfinitePolynomialRing(QQ, implementation='sparse')
sage: X is X2
False
sage: X is loads(dumps(X))
True

create_key (R, names=('x',), order='lex', implementation='dense')

Creates a key which uniquely defines the infinite polynomial ring.

create_object (version, key)

Return the infinite polynomial ring corresponding to the key key.
class sage.rings.polynomial.infinite_polynomial_ring.InfinitePolynomialRing_dense(R, names, order)

Bases: InfinitePolynomialRing_sparse

Dense implementation of Infinite Polynomial Rings

Compared with InfinitePolynomialRing_sparse, from which this class inherits, it keeps a polynomial ring that comprises all elements that have been created so far.

collection()

Return the construction of self.

OUTPUT:

A pair \( F, R \), where \( F \) is a construction functor and \( R \) is a ring, so that \( F(R) \) is self.

EXAMPLES:

```
sage: R.<x,y> = InfinitePolynomialRing(GF(5))
sage: R.construction()
[InfPoly([x,y], "lex", "dense"), Finite Field of size 5]
```

collection()()

Return the underlying finite polynomial ring.

Note: The ring returned can change over time as more variables are used.

Since the rings are cached, we create here a ring with variable names that do not occur in other doc tests, so that we avoid side effects.

EXAMPLES:

```
sage: X.<xx, yy> = InfinitePolynomialRing(ZZ)
sage: X.polynomial_ring()
Multivariate Polynomial Ring in xx_0, yy_0 over Integer Ring
sage: a = yy[3]
sage: X.polynomial_ring()
Multivariate Polynomial Ring in xx_3, xx_2, xx_1, xx_0, yy_3, yy_2, yy_1, yy_0 over Integer Ring
```
	
tensor_with_ring(R)

Return the tensor product of self with another ring.

INPUT:

\( R \) - a ring.

OUTPUT:

An infinite polynomial ring that, mathematically, can be seen as the tensor product of self with \( R \).

NOTE:

It is required that the underlying ring of self coerces into \( R \). Hence, the tensor product is in fact merely an extension of the base ring.

EXAMPLES:
```python
sage: R.<a,b> = InfinitePolynomialRing(ZZ, implementation='sparse')
sage: R.tensor_with_ring(QQ)
Infinite polynomial ring in a, b over Rational Field
sage: R
Infinite polynomial ring in a, b over Integer Ring
```

The following tests against a bug that was fixed at github issue #10468:

```python
sage: R.<x,y> = InfinitePolynomialRing(QQ, implementation='sparse')
sage: R.tensor_with_ring(QQ) == R
True
```

```python
class sage.rings.polynomial.infinite_polynomial_ring.InfinitePolynomialRing_sparse(R, names, order)

Bases: CommtutativeRing

Sparse implementation of Infinite Polynomial Rings.

An Infinite Polynomial Ring with generators \(x_0, y_*, ...\) over a field \(F\) is a free commutative \(F\)-algebra generated by \(x_0, x_1, x_2, ..., y_0, y_1, y_2, ....\) and is equipped with a permutation action on the generators, namely \(x_n^P = x_{P(n)}, y_n^P = y_{P(n)}\) for any permutation \(P\) (note that variables of index zero are invariant under such permutation).

It is known that any permutation invariant ideal in an Infinite Polynomial Ring is finitely generated modulo the permutation action – see SymmetricIdeal for more details.

Usually, an instance of this class is created using InfinitePolynomialRing with the optional parameter implementation='sparse'. This takes care of uniqueness of parent structures. However, a direct construction is possible, in principle:

```python
sage: X.<x,y> = InfinitePolynomialRing(QQ, implementation='sparse')
sage: Y.<x,y> = InfinitePolynomialRing(QQ, implementation='sparse')
sage: X == Y
True
sage: from sage.rings.polynomial.infinite_polynomial_ring import InfinitePolynomialRing_sparse
sage: Z = InfinitePolynomialRing_sparse(QQ, ['x','y'], 'lex')
```

Nevertheless, since infinite polynomial rings are supposed to be unique parent structures, they do not evaluate equal.

```python
sage: Z == X
False
```

The last parameter (‘lex’ in the above example) can also be ‘deglex’ or ‘degrevlex'; this would result in an Infinite Polynomial Ring in degree lexicographic or degree reverse lexicographic order.

See infinite_polynomial_ring for more details.

```python
characteristic()
```

Return the characteristic of the base field.

EXAMPLES:

```python
sage: X.<x,y> = InfinitePolynomialRing(GF(25,'a'))  #...
needs sage.rings.finite_rings
sage: X
```

(continues on next page)
construction()

Return the construction of self.

OUTPUT:

A pair \( F, R \), where \( F \) is a construction functor and \( R \) is a ring, so that \( F(R) \) is self.

EXAMPLES:

```python
sage: R.<x,y> = InfinitePolynomialRing(GF(5))
sage: R.construction()
[InfPoly{[x,y], "lex", "dense"}, Finite Field of size 5]
```


gen\( (i=None) \)

Return the \( i \)th 'generator' (see the description in \( \text{ngens()} \) of this infinite polynomial ring.

EXAMPLES:

```python
sage: X = InfinitePolynomialRing(QQ)
sage: x = X.gen()
sage: x[1]
x_1
sage: X.gen() is X.gen(0)
True
sage: XX = InfinitePolynomialRing(GF(5))
sage: XX.gen(0) is XX.gen()
True
```

gens_dict()

Return a dictionary-like object containing the infinitely many \( \{\text{var\_name:variable}\} \) pairs.

EXAMPLES:

```python
sage: R = InfinitePolynomialRing(ZZ, 'a')
sage: D = R.gens_dict()
sage: D
GenDict of Infinite polynomial ring in a over Integer Ring
sage: D['a_5']
a_5
```

is_field\( (*\text{args, **kwds}) \)

Return False since Infinite Polynomial Rings are never fields.

Since Infinite Polynomial Rings must have at least one generator, they have infinitely many variables and thus never are fields.

EXAMPLES:

```python
sage: R.<x, y> = InfinitePolynomialRing(QQ)
sage: R.is_field()
False
```
**is_integral_domain**(*args, **kwds*)

An infinite polynomial ring is an integral domain if and only if the base ring is. Arguments are passed to is_integral_domain method of base ring.

EXAMPLES:

```
sage: R.<x, y> = InfinitePolynomialRing(QQ)
sage: R.is_integral_domain()
True
```

**is_noetherian**()

Return False, since polynomial rings in infinitely many variables are never Noetherian rings.

Since Infinite Polynomial Rings must have at least one generator, they have infinitely many variables and are thus not noetherian, as a ring.

**Note:** Infinite Polynomial Rings over a field $F$ are noetherian as $F(G)$ modules, where $G$ is the symmetric group of the natural numbers. But this is not what the method is_noetherian() is answering.

**key_basis**()

Return the basis of self given by key polynomials.

EXAMPLES:

```
sage: R.<x> = InfinitePolynomialRing(GF(2))
sage: R.key_basis()
# needs sage.combinat sage.modules
Key polynomial basis over Finite Field of size 2
```

**krull_dimension**(*args, **kwds*)

Return Infinity, since polynomial rings in infinitely many variables have infinite Krull dimension.

EXAMPLES:

```
sage: R.<x, y> = InfinitePolynomialRing(QQ)
sage: R.krull_dimension() +Infinity
```

**ngens**()

Return the number of generators for this ring.

Since there are countably infinitely many variables in this polynomial ring, by 'generators' we mean the number of infinite families of variables. See infinite_polynomial_ring for more details.

EXAMPLES:

```
sage: X.<x> = InfinitePolynomialRing(ZZ)
sage: X.ngens()
1

sage: X.<x1,x2> = InfinitePolynomialRing(QQ)
sage: X.ngens()
2
```

**one**()
order()

Return Infinity, since polynomial rings have infinitely many elements.

EXAMPLES:

```sage
sage: R.<x> = InfinitePolynomialRing(GF(2))
sage: R.order()
+Infinity
```

tensor_with_ring(R)

Return the tensor product of self with another ring.

INPUT:
R - a ring.

OUTPUT:
An infinite polynomial ring that, mathematically, can be seen as the tensor product of self with R.

NOTE:
It is required that the underlying ring of self coerces into R. Hence, the tensor product is in fact merely an extension of the base ring.

EXAMPLES:

```sage
sage: R.<a,b> = InfinitePolynomialRing(ZZ)
sage: R.tensor_with_ring(QQ)
Infinite polynomial ring in a, b over Rational Field
sage: R
Infinite polynomial ring in a, b over Integer Ring
```

The following tests against a bug that was fixed at github issue #10468:

```sage
sage: R.<x,y> = InfinitePolynomialRing(QQ)
sage: R.tensor_with_ring(QQ) is R
True
```

varname_key(x)

Key for comparison of variable names.

INPUT:

x - a string of the form a+'_'+str(n), where a is the name of a generator, and n is an integer

RETURN:
a key used to sort the variables

THEORY:
The order is defined as follows:

\[ x < y \iff \text{the string } x.split('_')[0] \text{ is later in the list of generator names of self than } y.split('_')[0]. \text{ or } (x.split('_')[0] == y.split('_')[0] \text{ and } \text{int}(x.split('_')[1]) < \text{int}(y.split('_')[1])) \]

EXAMPLES:
6.2 Elements of Infinite Polynomial Rings

AUTHORS:
- Simon King <simon.king@nuigalway.ie>
- Mike Hansen <mhansen@gmail.com>

An Infinite Polynomial Ring has generators \(x, y, \ldots\), so that the variables are of the form \(x_0, x_1, x_2, \ldots, y_0, y_1, y_2, \ldots, \ldots\) (see `infinite_polynomial_ring`). Using the generators, we can create elements as follows:

```
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: a = x[3]
sage: b = y[4]
sage: a
x_3
sage: b
y_4
sage: c = a*b + a^3 - 2*b^4
sage: c
x_3^3 + x_3 y_4 - 2 y_4^4
```

Any Infinite Polynomial Ring \(X\) is equipped with a monomial ordering. We only consider monomial orderings in which:

\[ X\text{.gen}(i)[m] > X\text{.gen}(j)[n] \iff i < j, \text{or } i = j \text{ and } m > n \]

Under this restriction, the monomial ordering can be lexicographic (default), degree lexicographic, or degree reverse lexicographic. Here, the ordering is lexicographic, and elements can be compared as usual:

```
sage: X._order
'lex'
sage: a > b
True
```

Note that, when a method is called that is not directly implemented for ‘InfinitePolynomial’, it is tried to call this method for the underlying classical polynomial. This holds, e.g., when applying the `latex` function:

```
sage: latex(c)
x_{3}^{3} + x_{3} y_{4} - 2 y_{4}^{4}
```

There is a permutation action on Infinite Polynomial Rings by permuting the indices of the variables:
Polynomials, Release 10.3

\begin{verbatim}
sage: P = Permutation(((4,5),(2,3)))
sage: c^P
x_2^3 + x_2*y_5 - 2*y_5^4
\end{verbatim}

Note that \( P(0) == 0 \), and thus variables of index zero are invariant under the permutation action. More generally, if \( P \) is any callable object that accepts non-negative integers as input and returns non-negative integers, then \( c^P \) means to apply \( P \) to the variable indices occurring in \( c \).

If you want to substitute variables you can use the standard polynomial methods, such as \texttt{subs()}:

\begin{verbatim}
sage: R.<x,y> = InfinitePolynomialRing(QQ)
sage: f = x[1] + x[1]*x[2]*x[3]
sage: f.subs({x[1]: x[0]})
x_3*x_2*x_0 + x_0
sage: g = x[0] + x[1] + y[0]
sage: g.subs({x[0]: y[0]})
x_1 + 2*y_0
\end{verbatim}

\textbf{class} \texttt{sage.rings.polynomial.infinite_polynomial_element.InfinitePolynomial} \texttt{(A, P)}

\textbf{Bases:} \texttt{CommutativePolynomial}

Create an element of a Polynomial Ring with a Countably Infinite Number of Variables.

Usually, an \texttt{InfinitePolynomial} is obtained by using the generators of an Infinite Polynomial Ring (see \texttt{infinite_polynomial_ring}) or by conversion.

\textbf{INPUT:}

- \( A \) – an Infinite Polynomial Ring.
- \( p \) – a \textit{classical} polynomial that can be interpreted in \( A \).

\textbf{ASSUMPTIONS:}

In the dense implementation, it must be ensured that the argument \( p \) coerces into \( A._P \) by a name preserving conversion map.

In the sparse implementation, in the direct construction of an infinite polynomial, it is \textit{not} tested whether the argument \( p \) makes sense in \( A \).

\textbf{EXAMPLES:}

\begin{verbatim}
sage: from sage.rings.polynomial.infinite_polynomial_element import...
\end{verbatim}

Currently, \( P \) and \( X._P \) (the underlying polynomial ring of \( X \)) both have two variables:

\begin{verbatim}
sage: X._P
Multivariate Polynomial Ring in alpha_1, alpha_0 over Integer Ring
\end{verbatim}

By default, a coercion from \( P \) to \( X._P \) would not be name preserving. However, this is taken care for; a name preserving conversion is impossible, and by consequence an error is raised:

\begin{verbatim}
sage: InfinitePolynomial(X, (alpha_1+alpha_2)^2)
Traceback (most recent call last):
  ... TypeError: Could not find a mapping of the passed element to this ring.
\end{verbatim}
When extending the underlying polynomial ring, the construction of an infinite polynomial works:

```sage
alpha_2
```

```sage
InfinitePolynomial(X, (alpha_1+alpha_2)^2)
alpha_2^2 + 2*alpha_2*alpha_1 + alpha_1^2
```

In the sparse implementation, it is not checked whether the polynomial really belongs to the parent, and when it does not, the results may be unexpected due to coercions:

```sage
Y.<alpha,beta> = InfinitePolynomialRing(GF(2), implementation='sparse')
a = (alpha_1+alpha_2)^2
InfinitePolynomial(Y, a)
alpha_0^2 + beta_0^2
```

However, it is checked when doing a conversion:

```sage
Y(a)
alpha_2^2 + alpha_1^2
```

**coefficient (monomial)**

Returns the coefficient of a monomial in this polynomial.

**INPUT:**

- A monomial (element of the parent of self) or
- a dictionary that describes a monomial (the keys are variables of the parent of self, the values are the corresponding exponents)

**EXAMPLES:**

We can get the coefficient in front of monomials:

```sage
X.<x> = InfinitePolynomialRing(QQ)
a.coefficient(x[0])
2*x_1
a.coefficient(x[1])
2*x_0 + 1
a.coefficient(x[2])
1
a.coefficient(x[0]*x[1])
2
```

We can also pass in a dictionary:

```sage
a.coefficient({x[0]:1, x[1]:1})
2
```

**footprint ()**

Leading exponents sorted by index and generator.

**OUTPUT:**

D – a dictionary whose keys are the occurring variable indices.

D[s] is a list [i_1, ..., i_n], where i_j gives the exponent of self.parent().gen(j)[s] in the leading term of self.

**EXAMPLES:**
Polynomials, Release 10.3

sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: sorted(p.footprint().items())
[(1, [2, 3]), (30, [1, 0])]

gcd(x)
computes the greatest common divisor

EXAMPLES:

sage: R.<x>=InfinitePolynomialRing(QQ)
sage: p1=x[0] + x[1]^2
sage: gcd(p1,p1+3)
1
sage: gcd(p1,p1)==p1
True

is_nilpotent()
Return True if self is nilpotent, i.e., some power of self is 0.

EXAMPLES:

sage: R.<x> = InfinitePolynomialRing(QQbar)  # needs sage.rings.number_field
sage: (x[0] + x[1]).is_nilpotent()  # needs sage.rings.number_field
False
sage: R(0).is_nilpotent()  # needs sage.rings.number_field
True
sage: _.<x> = InfinitePolynomialRing(Zmod(4))
True
sage: (2∗x[0])∗x[4]∗x[7]).is_nilpotent()
False
sage: _.<y> = InfinitePolynomialRing(Zmod(100))
False
True

is_unit()
Answer whether self is a unit.

EXAMPLES:

sage: R1.<x,y> = InfinitePolynomialRing(ZZ)
sage: R2.<a,b> = InfinitePolynomialRing(QQ)
sage: (1 + x[2]).is_unit()
False
sage: R1(1).is_unit()
True
sage: R1(2).is_unit()
False
sage: R2(2).is_unit()
True
sage: (1 + a[2]).is_unit()
False
Check that `github issue #22454` is fixed:

```
sage: _.<x> = InfinitePolynomialRing(Zmod(4))
sage: (1 + 2*x[0]).is_unit()
  True
sage: (x[0]*x[1]).is_unit()
  False
sage: _.<x> = InfinitePolynomialRing(Zmod(900))
sage: (7+150*x[0] + 30*x[1] + 120*x[1]*x[100]).is_unit()
  True
```

**lc()**

The coefficient of the leading term of `self`.

**EXAMPLES:**

```
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: p.lc()
  3
```

**lm()**

The leading monomial of `self`.

**EXAMPLES:**

```
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: p.lm()
  x_10*x_1^2*y_1^3
```

**lt()**

The leading term (= product of coefficient and monomial) of `self`.

**EXAMPLES:**

```
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: p.lt()
  3*x_10*x_1^2*y_1^3
```

**max_index()**

Return the maximal index of a variable occurring in `self`, or -1 if `self` is scalar.

**EXAMPLES:**

```
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: p.max_index()
  4
sage: x[0].max_index()
  0
sage: X(10).max_index()
  -1
```

**polynomial()**

Return the underlying polynomial.

**EXAMPLES:**

```
```
Polynomials, Release 10.3

sage: X.<x,y> = InfinitePolynomialRing(GF(7))
sage: p = x[2]*y[1] + 3*y[0]
sage: p
x_2*y_1 + 3*y_0
sage: p.polynomial()
x_2*y_1 + 3*y_0
sage: p.polynomial().parent()
Multivariate Polynomial Ring in x_2, x_1, x_0, y_2, y_1, y_0
over Finite Field of size 7
sage: p.parent()
Infinite polynomial ring in x, y over Finite Field of size 7

reduce(I, tailreduce=False, report=None)

Symmetrical reduction of self with respect to a symmetric ideal (or list of Infinite Polynomials).

INPUT:

• I – a SymmetricIdeal or a list of Infinite Polynomials.
• tailreduce – (bool, default False) Tail reduction is performed if this parameter is True.
• report – (object, default None) If not None, some information on the progress of computation is printed, since reduction of huge polynomials may take a long time.

OUTPUT:

Symmetrical reduction of self with respect to I, possibly with tail reduction.

THEORY:

Reducing an element $p$ of an Infinite Polynomial Ring $X$ by some other element $q$ means the following:

1. Let $M$ and $N$ be the leading terms of $p$ and $q$.
2. Test whether there is a permutation $P$ that does not diminish the variable indices occurring in $N$ and preserves their order, so that there is some term $T \in X$ with $TN^P = M$. If there is no such permutation, return $p$.
3. Replace $p$ by $p - Tq^P$ and continue with step 1.

EXAMPLES:

sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: p.reduce([y[2]*x[1]^2])
x_3^3*y_2 + y_3*y_1^2

The preceding is correct: If a permutation turns $y[2]*x[1]^2$ into a factor of the leading monomial $y[2]*x[3]^3$ of $p$, then it interchanges the variable indices 1 and 2; this is not allowed in a symmetric reduction. However, reduction by $y[1]*x[2]^2$ works, since one can change variable index 1 into 2 and 2 into 3:

sage: p.reduce([y[1]*x[2]^2])
# needs sage.libs.singular
y_3^3*y_1^2

The next example shows that tail reduction is not done, unless it is explicitly advised. The input can also be a Symmetric Ideal:
**Polynomials, Release 10.3**

```python
sage: I = (y[3])*X
sage: p.reduce(I)
x_3^3*y_2 + y_3*y_1^2
sage: p.reduce(I, tailreduce=True)  # needs sage.libs.singular
x_3^3*y_2
```

Last, we demonstrate the **report** option:

```python
sage: p.reduce(I, tailreduce=True, report=True)  # needs sage.libs.singular
: T[2]: >
x_1^2 + y_2^2
```

The output `:` means that there was one reduction of the leading monomial. `'T[2]'` means that a tail reduction was performed on a polynomial with two terms. At `'>', one round of the reduction process is finished (there could only be several non-trivial rounds if `I` was generated by more than one polynomial).

**ring()**

The ring which `self` belongs to.

This is the same as `self.parent()`.

**EXAMPLES:**

```python
sage: X.<x,y> = InfinitePolynomialRing(ZZ, implementation='sparse')
sage: p.ring()
Infinite polynomial ring in x, y over Integer Ring
```

**squeezed()**

Reduce the variable indices occurring in `self`.

**OUTPUT:**

Apply a permutation to `self` that does not change the order of the variable indices of `self` but squeezes them into the range 1,2,...

**EXAMPLES:**

```python
sage: X.<x,y> = InfinitePolynomialRing(QQ, implementation='sparse')
sage: p = x[1]*y[100] + x[50]*y[1000]
sage: p.squeezed()
x_2*y_4 + x_1*y_3
```

**stretch(k)**

Stretch `self` by a given factor.

**INPUT:**

`k` – an integer.

**OUTPUT:**

Replace $v_n$ with $v_{n,k}$ for all generators $v_n$ occurring in `self`.

**EXAMPLES:**
Polynomials, Release 10.3

sage: X.<x> = InfinitePolynomialRing(QQ)
sage: a.stretch(2)
x_4 + x_2 + x_0

sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: a = x[0] + x[1] + y[0]*y[1]; a
x_1 + x_0 + y_1*y_0
sage: a.stretch(2)
x_2 + x_0 + y_2*y_0

subs (fixed=None, **kwargs)
Substitute variables in self.

INPUT:
• fixed—(optional) dict with {variable: value} pairs
• **kwargs—named parameters

OUTPUT:
the resulting substitution

EXAMPLES:
sage: R.<x,y> = InfinitePolynomialRing(QQ)
sage: f = x[1] + x[1]*x[2]*x[3]
Passing fixed={x[1]: x[0]}. Note that the keys may be given using the generators of the infinite polynomial ring or as a string:

sage: f.subs({x[1]: x[0]})
x_3*x_2*x_0 + x_0
sage: f.subs({x_1: x[0]})
x_3*x_2*x_0 + x_0
Passing the variables as names parameters:

sage: f.subs(x_1=y[1])
x_3*x_2*y_1 + y_1
sage: f.subs(x_1=y[1], x_2=2)
2*x_3*y_1 + y_1
The substitution returns the original polynomial if you try to substitute a variable not present:

sage: g = x[0] + x[1]
sage: g.subs({y[0]: x[0]})
x_1 + x_0

The substitution can also handle matrices:

sage: # needs sage.modules
sage: M = matrix([[1,0], [0,2]])
sage: N = matrix([[0,3], [4,0]])
sage: g = x[0]^2 + 3*x[1]
sage: g.subs({'x_0': M})
[3*x_1 + 1 0]
[ 0 3*x_1 + 4]
If you pass both fixed and kwargs, any conflicts will defer to fixed:

```python
sage: R.<x,y> = InfinitePolynomialRing(QQ)
sage: f = x[0]
sage: f.subs({x[0]: 1})
1
sage: f.subs(x_0=5)
5
sage: f.subs({x[0]: 1}, x_0=5)
1
```

**symmetric_cancellation_order**(other)
Comparison of leading terms by Symmetric Cancellation Order, \(<_{sc}\).

**INPUT:**
- self, other – two Infinite Polynomials

**ASSUMPTION:**
Both Infinite Polynomials are non-zero.

**OUTPUT:**
(c, sigma, w), where
- c = -1,0,1, or None if the leading monomial of self is smaller, equal, greater, or incomparable with respect to other in the monomial ordering of the Infinite Polynomial Ring
- sigma is a permutation witnessing self \(<_{sc}\) other (resp. self \(\geq_{sc}\) other) or is 1 if self.lm()==other.lm()
- w is 1 or is a term so that w*self.lt()^sigma == other.lt() if c \leq 0, and w*other.lt()^sigma == self.lt() if c = 1

**THEORY:**
If the Symmetric Cancellation Order is a well-quasi-ordering then computation of Groebner bases always terminates. This is the case, e.g., if the monomial order is lexicographic. For that reason, lexicographic order is our default order.

**EXAMPLES:**

```python
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: (x[2]*x[1]).symmetric_cancellation_order(x[2]^2)
(None, 1, 1)
sage: (x[2]*x[1]).symmetric_cancellation_order(x[2]^2*y[1])
(-1, [2, 3, 1], y_1)
sage: (x[2]*x[1]*y[1]).symmetric_cancellation_order(x[2]^2*y[1])
(None, 1, 1)
sage: (x[2]*x[1]*y[1]).symmetric_cancellation_order(x[2]^2*x[3]*y[2])
(-1, [2, 3, 1], 1)
```

**tail()**
The tail of self (this is self minus its leading term).

**EXAMPLES:**
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: p.tail()
2*x_10*y_30

variables()

Return the variables occurring in self (tuple of elements of some polynomial ring).

EXAMPLES:

sage: X.<x> = InfinitePolynomialRing(QQ)
sage: p.variables()
(x_3, x_2, x_1)
sage: x[1].variables()
(x_1,)
sage: X(1).variables()
()
6.3 Symmetric Ideals of Infinite Polynomial Rings

This module provides an implementation of ideals of polynomial rings in a countably infinite number of variables that are invariant under variable permutation. Such ideals are called ‘Symmetric Ideals’ in the rest of this document. Our implementation is based on the theory of M. Aschenbrenner and C. Hillar.

AUTHORS:

- Simon King <simon.king@nuigalway.ie>

EXAMPLES:

Here, we demonstrate that working in quotient rings of Infinite Polynomial Rings works, provided that one uses symmetric Groebner bases.

```
sage: R.<x> = InfinitePolynomialRing(QQ)
sage: I = R.ideal([x[1]*x[2] + x[3]])
```

Note that I is not a symmetric Groebner basis:

```
sage: # needs sage.combinat
sage: G = R * I.groebner_basis()
sage: G
Symmetric Ideal (x_1^2 + x_1, x_2 - x_1) of
    Infinite polynomial ring in x over Rational Field
sage: Q = R.quotient(G)
sage: Q(p)
-2*x_1 + 3
```

By the second generator of G, variable $x_n$ is equal to $x_1$ for any positive integer $n$. By the first generator of G, $x_3^2$ is equal to $x_1$ in Q. Indeed, we have

```
True
```

```python
class sage.rings.polynomial.symmetric_ideal.SymmetricIdeal (ring, gens, coerce=True)

Bases: Ideal_generic

Ideal in an Infinite Polynomial Ring, invariant under permutation of variable indices

THEORY:

An Infinite Polynomial Ring with finitely many generators $x_*, y_*, ...$ over a field $F$ is a free commutative $F$-algebra generated by infinitely many ‘variables’ $x_0, x_1, x_2, ..., y_0, y_1, y_2, ...$. We refer to the natural number $n$ as the index of the variable $x_n$. See more detailed description at infinite_polynomial_ring

Infinite Polynomial Rings are equipped with a permutation action by permuting positive variable indices, i.e., $x_P^n = x_{P(n)}$, $y_P^n = y_{P(n)}$, ... for any permutation $P$. Note that the variables $x_0, y_0, ...$ of index zero are invariant under that action.

A Symmetric Ideal is an ideal in an infinite polynomial ring $X$ that is invariant under the permutation action. In other words, if $\mathcal{S}_\infty$ denotes the symmetric group of 1, 2, ..., then a Symmetric Ideal is a right $X[\mathcal{S}_\infty]$-submodule of $X$.

It is known by work of Aschenbrenner and Hillar [AB2007] that an Infinite Polynomial Ring $X$ with a single generator $x_*$ is Noetherian, in the sense that any Symmetric Ideal $I \subseteq X$ is finitely generated modulo addition, multiplication by elements of $X$, and permutation of variable indices (hence, it is a finitely generated right $X[\mathcal{S}_\infty]$-module).
Moreover, if $X$ is equipped with a lexicographic monomial ordering with $x_1 < x_2 < x_3 \ldots$ then there is an algorithm of Buchberger type that computes a Groebner basis $G$ for $I$ that allows for computation of a unique normal form, that is zero precisely for the elements of $I$ – see [AB2008]. See `groebner_basis()` for more details.

Our implementation allows more than one generator and also provides degree lexicographic and degree reverse lexicographic monomial orderings – we do, however, not guarantee termination of the Buchberger algorithm in these cases.

**EXAMPLES:**

```sage
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: I == loads(dumps(I))
True
sage: latex(I)
\left(x_{1} y_{2} y_{1} + 2 x_{1} y_{2}\right)\mathbb{Q}[x_{\ast}, y_{\ast}]\rightarrow
```

The default ordering is lexicographic. We now compute a Groebner basis:

```sage
sage: J = I.groebner_basis(); J # about 3 seconds
\rightarrow
\mathbb{Q}[x_{2}, y_{2}, y_{1}, x_{1} y_{2}, y_{1} + 2 x_{1} y_{2}, x_{2} y_{2} y_{1} + 2 x_{2} y_{1},
x_{2} x_{1} y_{1}^2 + 2 x_{2} x_{1} y_{1}, x_{2} x_{1} y_{2} - x_{2} x_{1} y_{1}]
```

Note that even though the symmetric ideal can be generated by a single polynomial, its reduced symmetric Groebner basis comprises four elements. Ideal membership in $I$ can now be tested by commuting symmetric reduction modulo $J$:

```sage
sage: I.reduce(J) # needs sage.combinat
Symmetric Ideal (0) of Infinite polynomial ring in x, y over Rational Field
```

The Groebner basis is not point-wise invariant under permutation:

```sage
sage: P = Permutation([2, 1])
sage: J[2]^P
\rightarrow
x_{2} x_{1} y_{2}^2 + 2 x_{2} x_{1} y_{2}
```

However, any element of $J$ has symmetric reduction zero even after applying a permutation. This even holds when the permutations involve higher variable indices than the ones occurring in $J$:

```sage
sage: [[(p^P).reduce(J) for p in J] for P in Permutations(3)] # needs sage.combinat
[[0, 0, 0, 0], [0, 0, 0, 0], [0, 0, 0, 0], [0, 0, 0, 0], [0, 0, 0, 0], [0, 0, 0, 0], [0, 0, 0, 0]]
```

Since $I$ is not a Groebner basis, it is no surprise that it cannot detect ideal membership:

```sage
sage: [p.reduce(I) for p in J] # needs sage.combinat
(continues on next page)
```
Note that we give no guarantee that the computation of a symmetric Groebner basis will terminate in any order different from lexicographic.

When multiplying Symmetric Ideals or raising them to some integer power, the permutation action is taken into account, so that the product is indeed the product of ideals in the mathematical sense.

```python
sage: I = X * (x[1])
sage: I * I  #...
Symmetric Ideal (x_1^2, x_2*x_1) of
Infinite polynomial ring in x, y over Rational Field
sage: I^3  #...
Symmetric Ideal (x_1^3, x_2*x_1^2, x_2^2*x_1, x_3*x_2*x_1) of
Infinite polynomial ring in x, y over Rational Field
sage: I * I == X * (x[1]^2)  #...
needs sage.combinat False
```

**groebner_basis** *(tailreduce=False, reduced=True, algorithm=None, report=None, use_full_group=False)*

Return a symmetric Groebner basis (type Sequence) of self.

**INPUT:**

- `tailreduce` – (bool, default False) If True, use tail reduction in intermediate computations
- `reduced` – (bool, default True) If True, return the reduced normalised symmetric Groebner basis.
- `algorithm` – (string, default None) Determine the algorithm (see below for available algorithms).
- `report` – (object, default None) If not None, print information on the progress of computation.
- `use_full_group` – (bool, default False) If True then proceed as originally suggested by [AB2008]. Our default method should be faster; see `symmetrisation()` for more details.

The computation of symmetric Groebner bases also involves the computation of classical Groebner bases, i.e., of Groebner bases for ideals in polynomial rings with finitely many variables. For these computations, Sage provides the following ALGORITHMS:

```
"'
    autoselect (default)

    ‘singular:groebner’
        Singular’s groebner command

    ‘singular:std’
        Singular’s std command

    ‘singular:stdhilb’
        Singular’s stdhilb command

    ‘singular:stdfglm’
        Singular’s stdfglm command

    ‘singular:slimgb’
        Singular’s slimgb command
```
Polynomials, Release 10.3

'libsingular:std'
libSingular's std command

'libsingular:slimgb'
libSingular's slimgb command

'toy:buchberger'
Sage's toy/educational buchberger without strategy

'toy:buchberger2'
Sage's toy/educational buchberger with strategy

'toy:d_basis'
Sage's toy/educational d_basis algorithm

'macaulay2:gb'
Macaulay2's gb command (if available)

'magma:GroebnerBasis'
Magma's Groebnerbasis command (if available)

If only a system is given - e.g. 'magma' - the default algorithm is chosen for that system.

Note: The Singular and libSingular versions of the respective algorithms are identical, but the former calls an external Singular process while the later calls a C function, i.e. the calling overhead is smaller.

EXAMPLES:

```
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: I1 = X * (x[1] + x[2], x[1]*x[2])
sage: I1.groebner_basis()  #...
→ needs sage.combinat
[x_1]
sage: I2.groebner_basis()  #...
→ needs sage.combinat
[x_1*y_2 + y_2^2*y_1, x_2*y_1 + y_2*y_1^2]
```

Note that a symmetric Groebner basis of a principal ideal is not necessarily formed by a single polynomial. When using the algorithm originally suggested by Aschenbrenner and Hillar, the result is the same, but the computation takes much longer:

```
sage: I2.groebner_basis(use_full_group=True)  #...
→ needs sage.combinat
[x_1*y_2 + y_2^2*y_1, x_2*y_1 + y_2*y_1^2]
```

Last, we demonstrate how the report on the progress of computations looks like:

```
sage: I1.groebner_basis(report=True, reduced=True)  #...
→ needs sage.combinat
Symmetric interreduction
[1/2] >
[2/2] :>
[1/2] >
[2/2] >
Symmetrise 2 polynomials at level
Apply permutations
```
(continues on next page)
Symmetric interreduction

Symmetrisation done

Classical Groebner basis

Symmetric interreduction

Symmetrise 2 polynomials at level 3
Apply permutations

Symmetric interreduction

Apply permutations

Symmetric interreduction

Symmetrise 1 polynomials at level 4
Apply permutations

Symmetric interreduction

Symmetric interreduction

Symmetric interreduction
The Aschenbrenner-Hillar algorithm is only guaranteed to work if the base ring is a field. So, we raise a \texttt{TypeError} if this is not the case:

\begin{verbatim}
sage: R.<x,y> = InfinitePolynomialRing(ZZ)
sage: I = R * [x[1] + x[2], y[1]]
sage: I.groebner_basis()  
...
Traceback (most recent call last):
  ...
TypeError: The base ring (= Integer Ring) must be a field
\end{verbatim}

\textbf{interreduced\_basis()}  
A fully symmetrically reduced generating set (type \texttt{Sequence}) of self.

This does essentially the same as \texttt{interreduction()} with the option ‘tailreduce’, but it returns a \texttt{Sequence} rather than a \texttt{SymmetricIdeal}.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: X.<x> = InfinitePolynomialRing(QQ)
sage: I = X * (x[1] + x[2], x[1]*x[2])
sage: I.interreduced\_basis()  
[-x_1^2, x_2 + x_1]
\end{verbatim}

\textbf{interreduction}(\texttt{tailreduce=True}, \texttt{sorted=False}, \texttt{report=None}, \texttt{RStrat=None})  
Return symmetrically interreduced form of self.

\textbf{INPUT:}

- \texttt{tailreduce} – (bool, default True) If True, the interreduction is also performed on the non-leading monomials.
- \texttt{sorted} – (bool, default False) If True, it is assumed that the generators of \texttt{self} are already increasingly sorted.
- \texttt{report} – (object, default None) If not None, some information on the progress of computation is printed
- \texttt{RStrat} – \texttt{(SymmetricReductionStrategy, default None)} A reduction strategy to which the polynomials resulting from the interreduction will be added. If \texttt{RStrat} already contains some polynomials, they will be used in the interreduction. The effect is to compute in a quotient ring.

\textbf{OUTPUT:}

A Symmetric Ideal \(J\) (sorted list of generators) coinciding with \texttt{self} as an ideal, so that any generator is symmetrically reduced w.r.t. the other generators. Note that the leading coefficients of the result are not necessarily 1.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: X.<x> = InfinitePolynomialRing(QQ)
sage: I = X * (x[1] + x[2], x[1]*x[2])
sage: I.interreduction()  
Symmetric Ideal (-x_1^2, x_2 + x_1) of
Infinite polynomial ring in x over Rational Field
\end{verbatim}

Here, we show the \texttt{report} option:
[m/n] indicates that polynomial number m is considered and the total number of polynomials under consideration is n. ‘-> 0’ is printed if a zero reduction occurred. The rest of the report is as described in sage.rings.polynomial.symmetric_reduction.SymmetricReductionStrategy.reduce().

Last, we demonstrate the use of the optional parameter RStrat:

```python
sage: from sage.rings.polynomial.symmetric_reduction import...
    SymmetricReductionStrategy
sage: R = SymmetricReductionStrategy(X); R
Symmetric Reduction Strategy in
Infinite polynomial ring in x over Rational Field
sage: I.interreduction(RStrat=R)  #...
    Symmetric Ideal (-x_1^2, x_2 + x_1) of
    Infinite polynomial ring in x over Rational Field
sage: R  #...
    Symmetric Reduction Strategy in
    Infinite polynomial ring in x over Rational Field, modulo
    x_1^2,
    x_2 + x_1
sage: R = SymmetricReductionStrategy(X, [x[1]^2])
```
Infinite polynomial ring in x, y over Rational Field
\[ \text{# needs sage.combinat} \]
sage: I.is_maximal() \[ \text{# needs sage.combinat} \]
False

The preceding answer is wrong, since it is not the case that \( I \) is given by a symmetric Groebner basis:

\[ \text{sage: I = R * I.groebner_basis(); I} \]
\[ \text{# needs sage.combinat} \]
\[ \text{Symmetric Ideal (y_1, x_1) of Infinite polynomial ring in x, y over Rational Field} \]
\[ \text{sage: I.is_maximal()} \]
\[ \text{# needs sage.combinat} \]
\[ \text{True} \]

**normalisation()**

Return an ideal that coincides with \texttt{self}, so that all generators have leading coefficient 1.

Possibly occurring zeroes are removed from the generator list.

**EXAMPLES:**

\[ \text{sage: X.<x> = InfinitePolynomialRing(QQ)} \]
\[ \text{sage: I = X*(1/2*x[1] + 2/3*x[2], 0, 4/5*x[1]*x[2])} \]
\[ \text{Symmetric Ideal (x_2 + 3/4*x_1, x_2*x_1) of Infinite polynomial ring in x over Rational Field} \]

**reduce**(\( I \), tailreduce=False)

Symmetric reduction of \texttt{self} by another Symmetric Ideal or list of Infinite Polynomials, or symmetric reduction of a given Infinite Polynomial by \texttt{self}.

**INPUT:**

- \( I \)-- an Infinite Polynomial, or a Symmetric Ideal or a list of Infinite Polynomials.
- \texttt{tailreduce} -- (bool, default False) If True, the non-leading terms will be reduced as well.

**OUTPUT:**

Symmetric reduction of \texttt{self} with respect to \( I \).

**THEORY:**

Reduction of an element \( p \) of an Infinite Polynomial Ring \( X \) by some other element \( q \) means the following:

1. Let \( M \) and \( N \) be the leading terms of \( p \) and \( q \).
2. Test whether there is a permutation \( P \) that does not diminish the variable indices occurring in \( N \) and preserves their order, so that there is some term \( T \in X \) with \( TN^P = M \). If there is no such permutation, return \( p \)
3. Replace \( p \) by \( p - Tq^P \) and continue with step 1.

**EXAMPLES:**

\[ \text{sage: X.<x,y> = InfinitePolynomialRing(QQ)} \]
The preceding is correct, since any permutation that turns $x[1]^2*y[2]$ into a factor of $x[3]^2*y[2]$ interchanges the variable indices 1 and 2 – which is not allowed. However, reduction by $x[2]^2*y[1]$ works, since one can change variable index 1 into 2 and 2 into 3:

```sage
sage: I.reduce([x[2]^2*y[1]])
Symmetric Ideal (y_3*y_1^2) of
Infinite polynomial ring in x, y over Rational Field
```

The next example shows that tail reduction is not done, unless it is explicitly advised. The input can also be a symmetric ideal:

```sage
sage: J = (y[2]) * X
sage: I.reduce(J)
Symmetric Ideal (x_3^2*y_1 + y_3*y_1^2) of
Infinite polynomial ring in x, y over Rational Field
sage: I.reduce(J, tailreduce=True)
Symmetric Ideal (x_3^2*y_1) of
Infinite polynomial ring in x, y over Rational Field
```

### squeezed()

Reduce the variable indices occurring in `self`.

**OUTPUT:**

A Symmetric Ideal whose generators are the result of applying `squeezed()` to the generators of `self`.

**NOTE:**

The output describes the same Symmetric Ideal as `self`.

**EXAMPLES:**

```sage
sage: X.<x,y> = InfinitePolynomialRing(QQ, implementation='sparse')
sage: I = X * (x[1000]*y[100], x[50]*y[1000])
sage: I.squeezed()
Symmetric Ideal (x_2*y_1, x_1*y_2) of
Infinite polynomial ring in x, y over Rational Field
```

### symmetric_basis()

A symmetrised generating set (type `Sequence`) of `self`.

This does essentially the same as `symmetrisation()` with the option `tailreduce`, and it returns a `Sequence` rather than a `SymmetricIdeal`.

**EXAMPLES:**

```sage
sage: X.<x> = InfinitePolynomialRing(QQ)
sage: I = X * (x[1] + x[2], x[1]*x[2])
sage: I.symmetric_basis()    #...
[ x_1^2, x_2 + x_1 ]
```
**symmetrisation** \((N=None,\ tailreduce=False,\ report=None, use_full_group=False)\)

Apply permutations to the generators of *self* and interreduce.

**INPUT:**

- \(N\) – (integer, default None) Apply permutations in \(Sym(N)\). If it is not given then it will be replaced by the maximal variable index occurring in the generators of \(self.interreduction().squeezed()\).
- \(\text{tailreduce}\) – (bool, default False) If True, perform tail reductions.
- \(\text{report}\) – (object, default None) If not None, report on the progress of computations.
- \(\text{use_full_group}\) (optional) – If True, apply all elements of \(Sym(N)\) to the generators of \(self\) (this is what [AB2008] originally suggests). The default is to apply all elementary transpositions to the generators of \(self.squeezed()\), interreduce, and repeat until the result stabilises, which is often much faster than applying all of \(Sym(N)\), and we are convinced that both methods yield the same result.

**OUTPUT:**

A symmetrically interreduced symmetric ideal with respect to which any \(Sym(N)\)-translate of a generator of \(self\) is symmetrically reducible, where by default \(N\) is the maximal variable index that occurs in the generators of \(self.interreduction().squeezed()\).

**NOTE:**

If \(I\) is a symmetric ideal whose generators are monomials, then \(I.symmetrisation()\) is its reduced Groebner basis. It should be noted that without symmetrisation, monomial generators, in general, do not form a Groebner basis.

**EXAMPLES:**

```python
sage: X.<x> = InfinitePolynomialRing(QQ)
sage: I = X * (x[1] + x[2], x[1]*x[2])
sage: I.symmetrisation()  #...
→ needs sage.combinat
Symmetric Ideal (-x_1^2, x_2 + x_1) of
Infinite polynomial ring in x over Rational Field
sage: I.symmetrisation(N=3)  #...
→ needs sage.combinat
Symmetric Ideal (-2*x_1) of Infinite polynomial ring in x over Rational Field
sage: I.symmetrisation(N=3, use_full_group=True)  #...
→ needs sage.combinat
Symmetric Ideal (-2*x_1) of Infinite polynomial ring in x over Rational Field
```

### 6.4 Symmetric Reduction of Infinite Polynomials

*SymmetricReductionStrategy* provides a framework for efficient symmetric reduction of Infinite Polynomials, see *infinite_polynomial_element*.

**AUTHORS:**

- Simon King <simon.king@nuigalway.ie>

**THEORY:**

According to M. Aschenbrenner and C. Hillar [AB2007], Symmetric Reduction of an element \(p\) of an Infinite Polynomial Ring \(X\) by some other element \(q\) means the following:
1. Let $M$ and $N$ be the leading terms of $p$ and $q$.

2. Test whether there is a permutation $P$ that does not diminish the variable indices occurring in $N$ and preserves their order, so that there is some term $T \in X$ with $TN^P = M$. If there is no such permutation, return $p$.

3. Replace $p$ by $p - Tq^P$ and continue with step 1.

When reducing one polynomial $p$ with respect to a list $L$ of other polynomials, there usually is a choice of order on which the efficiency crucially depends. Also it helps to modify the polynomials on the list in order to simplify the basic reduction steps.

The preparation of $L$ may be expensive. Hence, if the same list is used many times then it is reasonable to perform the preparation only once. This is the background of SymmetricReductionStrategy.

Our current strategy is to keep the number of terms in the polynomials as small as possible. For this, we sort $L$ by increasing number of terms. If several elements of $L$ allow for a reduction of $p$, we choose the one with the smallest number of terms. Later on, it should be possible to implement further strategies for choice.

When adding a new polynomial $q$ to $L$, we first reduce $q$ with respect to $L$. Then, we test heuristically whether it is possible to reduce the number of terms of the elements of $L$ by reduction modulo $q$. That way, we see best chances to keep the number of terms in intermediate reduction steps relatively small.

**EXAMPLES:**

First, we create an infinite polynomial ring and one of its elements:

```
sage: X, <x, y> = InfinitePolynomialRing(QQ)
```

We want to symmetrically reduce it by another polynomial. So, we put this other polynomial into a list and create a Symmetric Reduction Strategy object:

```
sage: from sage.rings.polynomial.symmetric_reduction import SymmetricReductionStrategy
sage: S = SymmetricReductionStrategy(X, [y[2]^2*x[1]])
sage: S
Symmetric Reduction Strategy in
   Infinite polynomial ring in x, y over Rational Field, modulo
   x_1*y_2^2
sage: S.reduce(p)
x_3*y_1^2 + y_3*y_1
```

The preceding is correct, since any permutation that turns $y[2]^2*x[1]$ into a factor of $y[1]^2*x[3]$ interchanges the variable indices 1 and 2 – which is not allowed in a symmetric reduction. However, reduction by $y[1]^2*x[2]$ works, since one can change variable index 1 into 2 and 2 into 3. So, we add this to $S$:

```
sage: S.add_generator(y[1]^2*x[2])
sage: S
Symmetric Reduction Strategy in
   Infinite polynomial ring in x, y over Rational Field, modulo
   x_2*y_1^2, x_1*y_2^2
sage: S.reduce(p)
y_3*y_1
```

The next example shows that tail reduction is not done, unless it is explicitly advised:

```
needs sage.combinat
x_3 + 2*x_2*y_1^2 + 3*x_1*y_2^2
```

(continues on next page)
However, it is possible to ask for tailreduction already when the Symmetric Reduction Strategy is created:

```python
sage: S2
Symmetric Reduction Strategy in
   Infinite polynomial ring in x, y over Rational Field, modulo
   x_2*y_1^2,
   x_1*y_2^2
with tailreduction
#...
-> needs sage.combinat
x_3
```
EXAMPLES:

```python
sage: from sage.rings.polynomial.symmetric_reduction import SymmetricReductionStrategy
sage: X.<x,y> = InfinitePolynomialRing(QQ)
sage: S = SymmetricReductionStrategy(X)
sage: S
Symmetric Reduction Strategy in
Infinite polynomial ring in x, y over Rational Field
sage: S
Symmetric Reduction Strategy in
Infinite polynomial ring in x, y over Rational Field, modulo
  x_3*y_1 + x_1*y_1 + y_3
```

Note that the first added polynomial will be simplified when adding a suitable second polynomial:

```python
sage: S.add_generator(x[2] + x[1])  # needs sage.combinat
sage: S
Symmetric Reduction Strategy in
Infinite polynomial ring in x, y over Rational Field, modulo
  y_3,
  x_2 + x_1
```

By default, reduction is applied to any newly added polynomial. This can be avoided by specifying the optional parameter 'good_input':

```python
sage: # needs sage.combinat
sage: S.add_generator(y[2] + y[1]*x[2])
```

```python
sage: S.add_generator(x[3] + x[2], good_input=True)
```

In the previous example, `x[3] + x[2]` is added without being reduced to zero.

```python
gens()
```

Return the list of Infinite Polynomials modulo which self reduces.

EXAMPLES:

```python
sage: X.<y> = InfinitePolynomialRing(QQ)
sage: from sage.rings.polynomial.symmetric_reduction import SymmetricReductionStrategy
(continues on next page)```
reduce \((p, \text{notail=False, report=None})\)

Symmetric reduction of an infinite polynomial.

INPUT:

- \(p\) – an element of the underlying infinite polynomial ring.
- \(\text{notail}=(\text{bool, default False})\) If True, tail reduction is avoided (but there is no guarantee that there will be no tail reduction at all).
- \(\text{report}=(\text{object, default None})\) If not None, print information on the progress of the computation.

OUTPUT:

Reduction of \(p\) with respect to \(self\).

Note: If tail reduction shall be forced, use \(tailreduce()\).

EXAMPLES:

\[
\begin{align*}
\text{sage: } & \text{from sage.rings.polynomial.symmetric_reduction import } \_
\rightarrow\text{SymmetricReductionStrategy} \\
\text{sage: } & X.<x,y> = \text{InfinitePolynomialRing}(\text{QQ}) \\
\text{sage: } & S = \text{SymmetricReductionStrategy}(X, [y[3]], \text{tailreduce=True}) \\
\text{sage: } & S.\text{reduce}(y[4]*x[1] + y[1]*x[4]) \\
& x_4*y_1 \\
\text{sage: } & S.\text{reduce}(y[4]*x[1] + y[1]*x[4], \text{notail=True}) \\
& x_4*y_1 + x_1*y_4 \\
\end{align*}
\]

Last, we demonstrate the report option:

\[
\begin{align*}
\text{sage: } & S = \text{SymmetricReductionStrategy}(X, [x[2] + y[1],} \\
& \text{....:} \quad y[3] + y[2]}) \\
\text{sage: } & S \\
\end{align*}
\]

Symmetric Reduction Strategy in
Infinite polynomial ring in x, y over Rational Field, modulo
\[y_3 + y_2, \quad x_2 + y_1, \quad x_1*y_2 + y_4 - y_3*y_1\]
\[
\begin{align*}
\text{sage: } & S.\text{reduce}(x[3] + x[1]*y[3] + x[1]*y[1], \text{report=True}) \\
& ::::>
\end{align*}
\]

\[x_1*y_1 + y_4 - y_3*y_1 - y_1\]

Each ‘:’ indicates that one reduction of the leading monomial was performed. Eventually, the ‘>’ indicates that the computation is finished.
**reset()**

Remove all polynomials from `self`.

**EXAMPLES:**

```
sage: X.<y> = InfinitePolynomialRing(QQ)
sage: from sage.rings.polynomial.symmetric_reduction import SymmetricReductionStrategy
sage: S
Symmetric Reduction Strategy in Infinite polynomial ring in y over Rational Field, modulo y_2*y_1^2, y_2^2*y_1
sage: S.reset()
sage: S
Symmetric Reduction Strategy in Infinite polynomial ring in y over Rational Field
```

**setgens(L)**

Define the list of Infinite Polynomials modulo which `self` reduces.

**INPUT:**

$L$ – a list of elements of the underlying infinite polynomial ring.

**Note:** It is not tested if $L$ is a good input. That method simply assigns a `copy` of $L$ to the generators of `self`.

**EXAMPLES:**

```
sage: from sage.rings.polynomial.symmetric_reduction import SymmetricReductionStrategy
sage: X.<y> = InfinitePolynomialRing(QQ)
sage: R = SymmetricReductionStrategy(X)
sage: R.setgens(S.gens())
sage: R
Symmetric Reduction Strategy in Infinite polynomial ring in y over Rational Field, modulo y_2*y_1^2, y_2^2*y_1
sage: R.gens() == S.gens()
False
sage: R.gens() == S.gens()
True
```

**tailreduce(p, report=None)**

Symmetric reduction of an infinite polynomial, with forced tail reduction.

**INPUT:**

- $p$ – an element of the underlying infinite polynomial ring.
- `report` – (object, default None) If not None, print information on the progress of the computation.

**OUTPUT:**

Reduction (including the non-leading elements) of $p$ with respect to `self`.

**EXAMPLES:**

6.4. Symmetric Reduction of Infinite Polynomials 621
Last, we demonstrate the ‘report’ option:

```
sage: S = SymmetricReductionStrategy(X, [x[2] + y[1],
.....: y[3] + y[2]])
T[3]: x_1*y_1 - y_2 + y_1^2 - y_1
```

The protocol means the following.

- ‘T[3]’ means that we currently do tail reduction for a polynomial with three terms.
- ‘:::>’ means that there were three reductions of leading terms.
- The tail of the result of the preceding reduction still has three terms. One reduction of leading terms was possible, and then the final result was obtained.
CHAPTER
SEVEN

BOOLEAN POLYNOMIALS
CHAPTER EIGHT

INDICES AND TABLES

- Index
- Module Index
- Search Page
sage.rings.fraction_field, 541
sage.rings.fraction_field_element, 547
sage.rings.fraction_field_FpT, 551
sage.rings.invariants.invariant_theory, 480
sage.rings.invariants.reconstruction, 523
sage.rings.monomials, 479
sage.rings.polynomial.complex_roots, 229
sage.rings.polynomial.convolution, 270
sage.rings.polynomial.cyclotomic, 271
sage.rings.polynomial.flatten, 476
sage.rings.polynomial.hilbert, 475
sage.rings.polynomial.ideal, 232
sage.rings.polynomial.infinite_polynomial_element, 597
sage.rings.polynomial.infinite_polynomial_ring, 587
sage.rings.polynomial.integer_values_polynomials, 261
sage.rings.polynomial.laurent_polynomial, 570
sage.rings.polynomial.laurent_polynomial_ring, 565
sage.rings.polynomial.laurent_polynomial_ring_base, 561
sage.rings.polynomial.msolve, 458
sage.rings.polynomial.multi_polynomial, 309
sage.rings.polynomial.multi_polynomial_element, 338
sage.rings.polynomial.multi_polynomial_ideal, 356
sage.rings.polynomial.multi_polynomial_ideal_libsingular, 457
sage.rings.polynomial.multi_polynomial_libsingular, 426
sage.rings.polynomial.multi_polynomial_ring, 334
sage.rings.polynomial.multi_polynomial_ring_base, 296
sage.rings.polynomial.multi_polynomial_sequence, 409
sage.rings.polynomial.omega, 581
sage.rings.polynomial.padics.polynomial_padic, 185
sage.rings.polynomial.padics.polynomial_padic_capped_relative_dense, 189
sage.rings.polynomial.padics.polynomial_padic_flint, 195
sage.rings.polynomial.pbori.pbori, ??
sage.rings.polynomial.polydict, 459
sage.rings.polynomial.polynomial_compiled, 260
sage.rings.polynomial.polynomial_element, 33
sage.rings.polynomial.polynomial_element_generic, 121
sage.rings.polynomial.polynomial_free, 261
sage.rings.polynomial.polynomial_integer_dense_flint, 138
sage.rings.polynomial.polynomial_integer_dense_ntl, 147
sage.rings.polynomial.polynomial_modn_dense_ntl, 171
sage.rings.polynomial.polynomial_number_field, 136
sage.rings.polynomial.polynomial_quotient_ring, 233
sage.rings.polynomial.polynomial_quotient_ring_element, 254
sage.rings.polynomial.polynomial_rational_flint, 152
sage.rings.polynomial.polynomial_real_mpfr_dense, 181
sage.rings.polynomial.polynomial_ring, 9
sage.rings.polynomial.polynomial_ring_constructor, 1
sage.rings.polynomial.polynomial_ring_homomorphism, 32
sage.rings.polynomial.polynomial_singular_interface, 184
sage.rings.polynomial.polynomial_zmod_flint, 163
sage.rings.polynomial.polynomial_zz_pex, 196
sage.rings.polynomial.real_roots, 200
sage.rings.polynomial.refine_root, 232
sage.rings.polynomial.symmetric_ideal, 607
sage.rings.polynomial.symmetric_reduction, 616
sage.rings.polynomial.term_order, 275
sage.rings.polynomial.toy_buchberger, 526
sage.rings.polynomial.toy_d_basis, 536
sage.rings.polynomial.toy_variety, 532
Non-alphabetical

_add_() (sage.rings.polynomial.polynomial_element.Polynomial method), 34
_add_() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 138
_add_() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_modn_dense_ntl method), 176
_add_() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 152
_add_() (sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint method), 166
_mul_() (sage.rings.polynomial.polynomial_element.Polynomial method), 35
_mul_() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 139
_mul_() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_modn_dense_ntl method), 177
_mul_() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 153
_mul_() (sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint method), 167
_mul_trunc_() (sage.rings.polynomial.polynomial_element.Polynomial method), 35
_mul_trunc_() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 139
_mul_trunc_() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_modn_dense_ntl method), 177
_mul_trunc_() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 153
_mul_trunc_() (sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint method), 167
_rmul_() (sage.rings.polynomial.polynomial_element.Polynomial method), 35
_rmul_() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 139
_rmul_() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_modn_dense_ntl method), 176
_rmul_() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 152
_rmul_() (sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint method), 167
_sub_() (sage.rings.polynomial.polynomial_element.Polynomial method), 34
_sub_() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 138
_sub_() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_modn_dense_ntl method), 177
_sub_() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 152
_sub_() (sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint method)
A

A_invariant() (sage.rings.invariants.invariant_theory.BinaryQuintic method), 487

a_realization() (sage.rings.polynomial.integer_valued_polynomials.IntegerValuedPolynomialRing method), 270

abc_pd (class in sage.rings.polynomial.polynomial_compiled), 260

adams_operator() (sage.rings.polynomial.polynomial_element.Polynomial method), 36

adams_operator_on_roots() (sage.rings.polynomial.polynomial_element.Polynomial method), 36

add_bigoh() (sage.rings.polynomial.polynomial_element.Polynomial method), 36

add_generator() (sage.rings.polynomial.symmetric_reduction.SymmetricReductionStrategy method), 618

add_m_mul_q() (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method), 433

add_pd (class in sage.rings.polynomial.polynomial_compiled), 260


algebraic_dependence() (sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic method), 414

AlgebraicForm (class in sage.rings.invariants.invariant_theory), 481

all_done() (sage.rings.polynomial.real_roots.ocean method), 219

all_roots_in_interval() (sage.rings.polynomial.polynomial_element.Polynomial method), 37

alpha_covariant() (sage.rings.invariants.invariant_theory.BinaryQuintic method), 489

ambient() (sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_generic method), 243

any_irreducible_factor() (sage.rings.polynomial.polynomial_element.Polynomial method), 37

any_root() (sage.rings.polynomial.polynomial_element.Polynomial method), 39

apply_map() (sage.rings.polynomial.polydict.PolyDict method), 466

approx_bp() (sage.rings.polynomial.real_roots.ocean method), 219

arithmetic_invariants() (sage.rings.invariants.invariant_theory.BinaryQuintic method), 490

as_float() (sage.rings.polynomial.real_roots.interval_bernstein_polynomial_float method), 210

as_float() (sage.rings.polynomial.real_roots.interval_bernstein_polynomial_integer method), 212

as_QuadraticForm() (sage.rings.invariants.invariant_theory.QuadraticForm method), 505

associated_primes() (sage.rings.polynomial.multi_polynomial_ideal.MPolynomial_ideal_singular_repr method), 377

B

B (sage.rings.polynomial.integer_valued_polynomials.IntegerValuedPolynomialRing attribute), 261

B_invariant() (sage.rings.invariants.invariant_theory.BinaryQuintic method), 487

base_extend() (sage.rings.polynomial.polynomial_element.Polynomial method), 40

base_extend() (sage.rings.polynomial.polynomial_ring.PolynomialRing_general method), 21

base_field() (sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_field method), 239

base_ring() (sage.rings.fraction_field.FractionField_generic method), 544

base_ring() (sage.rings.polynomial.polynomial_element.Polynomial method), 40

base_ring() (sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_generic method), 243

basis (sage.rings.polynomial.multi_polynomial_ideal.MPolynomialIdeal property), 359

basis_is_groebner() (sage.rings.polynomial.multi_polynomial_ideal.MPolynomialIdeal Singular_repr method), 378

bateman_bound() (in module sage.rings.polynomial.cyclotomic), 271

bernstein_down() (in module sage.rings.polynomial.real_roots), 201

bernstein_expand() (in module sage.rings.polynomial.real_roots), 201

bernstein_polynomial() (sage.rings.polynomial.real_roots.bernstein_polynomial_factory_ar method), 202

bernstein_polynomial() (sage.rings.polynomial.real_roots.bernstein_polynomial fact-
characteristic() (sage.rings.polynomial.multi_polynomial_ring_base.MPolynomialRing_base method), 297
characteristic() (sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_generic method), 244
characteristic() (sage.rings.polynomial.polynomial_ring.PolynomialRing_general method), 21
charpoly() (sage.rings.polynomial.polynomial_quotient_ring_element.PolynomialQuotientRingElement method), 255
c1_maximum_root() (in module sage.rings.polynomial.real_roots), 204
c1_maximum_root_first_lambda() (in module sage.rings.polynomial.real_roots), 204
c1_maximum_root_local_max() (in module sage.rings.polynomial.real_roots), 204
class_group() (sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_generic method), 244
class_number() (sage.rings.fraction_field.FractionField_Fp_field method), 543
clebsch_invariants() (sage.rings.invariants.invariant_theory.BinaryQuintic method), 491
coeff_pd (class in sage.rings.polynomial.polynomial_ring_compiled), 260
coefficient() (sage.rings.polynomial.infinite_polynomial_element.InfinitePolynomial method), 599
coefficient() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_libdict method), 339
coefficient() (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method), 433
coefficient() (sage.rings.polynomial.polydict.PolyDict method), 466
coefficient_matrix() (in module sage.rings.polynomial.toy_variety), 532
coefficient_matrix() (sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic method), 415
coefficients() (sage.rings.invariants.invariant_theory.AlgebraicForm method), 482
coefficients() (sage.rings.polynomial.laurent_polynomial.LaurentPolynomial_univariate method), 572
coefficients() (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method), 434
coefficients() (sage.rings.polynomial.multi_polynomial.MPolynomial method), 310
coefficients() (sage.rings.polynomial.polydict.PolyDict method), 466
coefficients() (sage.rings.polynomial.polynomial_element_generic.Polynomial_element_sparse method), 126
coefficients() (sage.rings.polynomial.polynomial_element.Polynomial method), 41
coefficients() (sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic method), 415
coefficients_monomials() (sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic method), 422
coeffs() (sage.rings.invariants.invariant_theory.BinaryQuartic method), 484
coeffs() (sage.rings.invariants.invariant_theory.BinaryQuintic method), 492
coeffs() (sage.rings.invariants.invariant_theory.QuadraticForm method), 505
coeffs() (sage.rings.invariants.invariant_theory.TernaryCubic method), 511
coeffs() (sage.rings.invariants.invariant_theory.TernaryQuadratic method), 513
coeffs_bitsize() (sage.rings.polynomial.real_roots.bernstein_polynomial_factor_ar method), 202
coeffs_bitsize() (sage.rings.polynomial.real_roots.bernstein_polynomial_factor_intlist method), 203
coeffs_bitsize() (sage.rings.polynomial.real_roots.bernstein_polynomial_factor_ratlist method), 203
coeerce_coefficients() (sage.rings.polynomial.polydict.PolyDict method), 466
combine_to_positives() (sage.rings.polynomial.polydict.ETuple method), 460
common_nonzero_positions() (sage.rings.polynomial.polydict.ETuple method), 460
CompiledPolynomialFunction (class in sage.rings.polynomial.polynomial_ring_compiled), 260
complete_primary_decomposition() (sage.rings.polynomial.multi_polynomial_ideal.MPolynomialIdeal_singular_repr method), 380
completion() (sage.rings.polynomial.laurent_polynomial.laurent_polynomial.PolynomialRing_base.LaurentPolynomialRing_generic method), 562
completion() (sage.rings.polynomial.multi_polynomial_ring_base.MPolynomialRing_base method), 297
completion() (sage.rings.polynomial.multi_polynomial_ring.PolynomialRing_general method), 22
complex_embeddings() (sage.rings.polynomial.polynomial_quotient_ring_element.PolynomialQuotientRingElement method), 255

632 Index
tientRing_field method), 239
complex_roots() (in module sage.rings.polynomial.complex_roots), 229
complex_roots() (sage.rings.polynomial.polynomial_element.Polynomial method), 41
compose_power() (sage.rings.polynomial.polynomial_element.Polynomial method), 42
compose_trunc() (sage.rings.polynomial.polynomial_element.Polynomial method), 42
composed_op() (sage.rings.polynomial.polynomial_element.Polynomial method), 43
connected_components() (sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic method), 416
connection_graph() (sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic method), 416
constant_coefficient() (sage.rings.polynomial.laurent_polynomial.LaurentPolynomial_univariate method), 572
constant_coefficient() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_libsingular method), 340
constant_coefficient() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_libsingular method), 434
constant_coefficient() (sage.rings.polynomial.polynomial_element.Polynomial method), 44
constant_coefficient() (sage.rings.polynomial.polynomial_element.Polynomial_generic_dense method), 116
ConstantPolynomialSection (class in sage.rings.polynomial.polynomial_element), 33
construction() (sage.rings.polynomial.fraction_field.FractionField_generic method), 544
construction() (sage.rings.polynomial.infinite_polynomial_ring.InfinitePolynomialRing_dense method), 592
construction() (sage.rings.polynomial.infinite_polynomial_ring.InfinitePolynomialRing_sparse method), 594
construction() (sage.rings.polynomial.laurent_polynomial_ring_base.LaurentPolynomialRing_generic method), 562
construction() (sage.rings.polynomial.multi_polynomial_ring_base.MPolynomialRing_base method), 298
construction() (sage.rings.polynomial.polynomial_quotient_ring.QuotientRing_generic method), 246
d_basis() (in module sage.rings.polynomial.toy_d_basis), 538

d constructor() (sage.rings.polynomial.polynomial_ring.PolynomialRing_integral_domain method), 29
content() (sage.rings.polynomial.integer_valued_polynomials.IntegerValuedPolynomialRing.Bases.ElementMethods method), 262
content() (sage.rings.polynomial.multi_polynomial.MPolynomial method), 310
content() (sage.rings.polynomial.padics.polynomial_padic.Polynomial_padic method), 185
content() (sage.rings.polynomial.integer_dense_flint.Polynomial_integer_dense_flint method), 140
content() (sage.rings.polynomial.integer_dense_ntl.Polynomial_integer_dense_ntl method), 147
content_ideal() (sage.rings.polynomial.multi_polynomial_element.Polynomial method), 45
context (class in sage.rings.polynomial.real_roots), 205
convolution() (in module sage.rings.polynomial.convolution), 270
covariant_conic() (sage.rings.invariants.invariant_theory.TernaryQuadratic method), 514
cover_ring() (sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_generic method), 247
create_key() (sage.rings.polynomial.integer_values.PolynomialRing_integral_domain method), 29
create_key() (sage.rings.polynomial.quotient_ring.PolynomialQuotientRingFactory method), 235
create_object() (sage.rings.polynomial.integer_values.Polynomial Ring method), 591
create_object() (sage.rings.polynomial.quotient_ring.PolynomialQuotientRingFactory method), 591
cyclotomic_coeff() (in module sage.rings.polynomial.quotient_ring), 236
cyclotomic_part() (sage.rings.polynomial.polynomial_element.Polynomial method), 45
cyclotomic_polynomial() (sage.rings.polynomial.integer_ring.PolynomialRing_general method), 22
cyclotomic_value() (in module sage.rings.polynomial.cyclotomic), 272

D

Index 633
de_casteljau() (sage.rings.polynomial.real_roots.interval_bernstein_polynomial_float method), 210
degree() (sage.rings.polynomial.real_roots.interval_bernstein_polynomial_integer method), 212
de_casteljau_doublevec() (in module sage.rings.polynomial.real_roots), 205
de_casteljau_intvec() (in module sage.rings.polynomial.real_roots), 205
degree() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict method), 572
degree() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict method), 341
degree() (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method), 435
degree() (sage.rings.polynomial.padics.polynomial_padic_capped_relative_dense.Polynomial_padic_capped_relative_dense method), 189
degree() (sage.rings.polynomial.polydict.PolyDict method), 467
degree() (sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_sparse method), 126
degree() (sage.rings.polynomial.polynomial_element.Polynomial method), 45
degree() (sage.rings.polynomial.polynomial_element.Polynomial_generic_dense method), 116
degree() (sage.rings.polynomial.polynomial_element.Polynomial_generic_dense_inexact method), 118
degree() (sage.rings.polynomial.polynomial_gf2x.Polynomial_template method), 134
degree() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 140
degree() (sage.rings.polynomial.polynomial_integer_dense_ntl.Polynomial_integer_dense_ntl method), 147
degree() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_modn_dense_ntl method), 172
degree() (sage.rings.polynomial.polynomial_modn_dense_mod_n method), 175
degree() (sage.rings.polynomial.polynomial_modn_dense_modn_ZZ method), 177
degree() (sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_generic method), 247
degree() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 153
degree() (sage.rings.polynomial.polynomial_real_mpfr_dense.PolynomialRealDense method), 182
degree() (sage.rings.polynomial.polynomial_zmod_flint.Polynomial_template method), 164
degree() (sage.rings.polynomial.polynomial_zz_pex.Polynomial_template method), 198
degree() (sage.rings.polynomial.multi_polynomial_element_integer_valued_polynomials.IntegerValuedPolynomialRing_Shifted.Element method), 263
degree_reduction_next_size() (in module sage.rings.polynomial.real_roots), 206
degrees() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict method), 342
degrees() (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method), 436
delta() (sage.rings.polynomial.integer_valued_polynomials.IntegerValuedPolynomialRing_Shifted.Element method), 267
delta_covariant() (sage.rings.invariants.invariant_theory.BinaryQuintic method), 492
Delta_invariant() (sage.rings.invariants.invariant_theory.TwoQuaternaryQuadratics method), 515
Delta_invariant() (sage.rings.invariants.invariant_theory.TwoTernaryQuadratics method), 519
Delta_prime_invariant() (sage.rings.invariants.invariant_theory.TwoQuaternaryQuadratics method), 516
Delta_prime_invariant() (sage.rings.invariants.invariant_theory.TwoTernaryQuadratics method), 519
denom() (sage.rings.fraction_field_FpT.FpTElement method), 552
denominator() (sage.rings.fraction_field_element.FractionFieldElement method), 547
denominator() (sage.rings.fraction_field_FpT.FpTElement method), 552
denominator() (sage.rings.polynomial.multi_polynomial_element_integer_valued_polynomials.IntegerValuedPolynomialRing_Shifted.Element method), 46
denominator() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 153
derivative() (sage.rings.polynomial.laurent_polynomial.LaurentPolynomial_univariate method), 573
derivative() (sage.rings.polynomial.multi_polynomial.MPolynomial method), 312
derivative() (sage.rings.polynomial.polydict.PolyDict method), 467
derivative() (sage.rings.polynomial.polynomial_element.MPolynomial method), 47
derivative_at_minus_one() (sage.rings.polynomial.integer_valued_polynomials.IntegerValuedPolynomialRing.Shifted.Element method), 267
derivative_i() (sage.rings.polynomial.polydict.PolyDict method), 467
dict() (sage.rings.polynomial.laurent_polynomial.LaurentPolynomial method), 570
dict() (sage.rings.polynomial.laurent_polynomial.LaurentPolynomial_univariate method), 573
dict() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_libpoldict method), 342
dict() (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method), 436
dict() (sage.rings.polynomial.polydict.PolyDict method), 468
dict() (sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_sparse method), 127
dict() (sage.rings.polynomial.polynomial_element.Polynomial method), 47
diff() (sage.rings.polynomial.polynomial_element.Polynomial method), 48
differentiate() (sage.rings.polynomial.polynomial_element.Polynomial method), 48
dimension() (sage.rings.polynomial.multi_polynomial_ideal.MPolynomialIdeal_singular_repr method), 382
disc() (sage.rings.polynomial.padics.polynomial_padic_capped_relative_dense.Polynomial_padic_capped_relative_dense method), 189
disc() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 140
disc() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 153
discriminant() (sage.rings.invariants.invariant_theory.QuadraticForm method), 506
discriminant() (sage.rings.polynomial.multi_polynomial.MPolynomial method), 313
discriminant() (sage.rings.polynomial.polynomial_element.Polynomial method), 49
discriminant() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 140
discriminant() (sage.rings.polynomial.polynomial_integer_dense_ntl.Polynomial_integer_dense_ntl method), 147
discriminant() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_dense_mod_p method), 173
discriminant() (sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_generic method), 247
discriminant() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 154
dispersion() (sage.rings.polynomial.polynomial_element.Polynomial method), 50
dispersion_set() (sage.rings.polynomial.polynomial_element.Polynomial method), 50
divide_by_gcd() (sage.rings.polynomial.polydict.ETuple method), 460
divide_by_var() (sage.rings.polynomial.polydict.ETuple method), 460
divided_difference() (sage.rings.polynomial.polynomial_ring.PolynomialRing_field method), 17
divides() (sage.rings.polynomial.laurent_polynomial.LaurentPolynomial_univariate method), 573
divides() (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method), 436
divides() (sage.rings.polynomial.polydict.ETuple method), 461
divides() (sage.rings.polynomial.polynomial_element.Polynomial method), 51
done() (sage.rings.polynomial.real_roots.island method), 216
dotprod() (sage.rings.polynomial.polydict.ETuple method), 461
down_degree() (sage.rings.polynomial.real_roots.interval_bernstein_polynomial_integer method), 213
down_degree_iter() (sage.rings.polynomial.real_roots.interval_bernstein_polynomial_integer method), 213
downsquare() (sage.rings.polynomial.real_roots.interval_bernstein_polynomial_integer method), 214
dprod_imatrow_vec() (in module sage.rings.polynomial.real_roots), 206
dual() (sage.rings.invariants.invariant_theory.QuadraticForm method), 506

dummypd (class in sage.rings.polynomial.polynomial compiled), 260

E
eadd() (sage.rings.polynomial.polydict.ETuple method), 461
eadd_p() (sage.rings.polynomial.polydict.ETuple method), 462
eadd_scaled() (sage.rings.polynomial.polydict.ETuple method), 462

eisensteinD() (sage.rings.invariants.invariant_theory.BinaryQuartic method), 484
eisensteinE() (sage.rings.invariants.invariant_theory.BinaryQuartic method), 484

element (sage.rings.polynomial.laurent_polynomial.LaurentPolynomialRing_univariate attribute), 569
element (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomialRing_libsingular attribute), 429

element (sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_generic attribute), 240
element() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_element method), 338

E_hidden (sage.rings.polynomial.multi_polynomial_element.MPolynomial_element method), 334

elim_pol() (in module sage.rings.polynomial.toy_variety), 533

eliminate_linear_variables() (sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_gf2 method), 422

eliminat_ideal() (sage.rings.polynomial.multi_polynomial_ideal.MPolynomial_ideal_singular_repr method), 383
eimax() (sage.rings.polynomial.polydict.ETuple method), 462
eimin() (sage.rings.polynomial.polydict.ETuple method), 463
eimul() (sage.rings.polynomial.polydict.ETuple method), 463

escalar_div() (sage.rings.polynomial.polydict.ETuple method), 463

esub() (sage.rings.polynomial.polydict.ETuple method), 463

ETuple (class in sage.rings.polynomial.polydict), 460
euclidean_degree() (sage.rings.polynomial.polynomial_element.Polynomial method), 51

exponents() (sage.rings.polynomial.laurent_polynomial.LaurentPolynomial_univariate method), 574

exponents() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_polynomial method), 342

exponents() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_libsingular method), 436

exponents() (sage.rings.polynomial.polydict.PolyDict method), 468

exponents() (sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_sparse method), 127

exponents() (sage.rings.polynomial.polynomial_element.Polynomial method), 52

extend_variables() (sage.rings.polynomial.polynomial_ring.PolynomialRing_general method), 23

F
F_covariant() (sage.rings.invariants.invariant_theory.TwoTernaryQuadratics method), 519

factor() (sage.rings.fraction_field_FpT.FpTElement method), 552

factor() (sage.rings.polynomial.laurent_polynomial.LaurentPolynomial_univariate method), 574

factor() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_element method), 343

factor() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_libsingular method), 437

factor() (sage.rings.polynomial.padics.polynomial_padic.Polynomial_padic method), 186

factor() (sage.rings.polynomial.polynomial_element.Polynomial method), 52

factor() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 141

factor() (sage.rings.polynomial.polynomial_integer_dense_ntl.Polynomial_integer_dense_ntl method), 148

factor() (sage.rings.polynomial.polynomial_integer_dense_prime_field.Polynomial_integer_dense_prime_field method), 167

factor_mod() (sage.rings.polynomial.padics.polynomial_padic_capped_relative_dense.Polynomial_padic_capped_relative_dense method), 190
get_dc_log() (sage.rings.polynomial.real_roots.context method), 205
get_form() (sage.rings.invariants.invariant_theory.SeveralAlgebraicForms method), 509
greater_tuple_matrix() (sage.rings.polynomial.real_roots.interval_bernstein_polynomial_float method), 211
greater_tuple() (sage.rings.polynomial.real_roots.interval_bernstein_polynomial_integer method), 214
greater_tuple_realfield_rndu() (in module sage.rings.polynomial.real_roots), 207
GF2X_BuildIrred_list() (in module sage.rings.polynomial.polynomial_gf2x), 131
GF2X_BuildRandomIrred_list() (in module sage.rings.polynomial.polynomial_gf2x), 132
GF2X_BuildSparseIrred_list() (in module sage.rings.polynomial.polynomial_gf2x), 132
global_height() (sage.rings.polynomial.multi_polynomial.MPolynomialMultivariateDicts method), 343
global_height() (sage.rings.polynomial.multi_polynomial.libsingular.MPolynomial_libsingular method), 440
global_height() (sage.rings.polynomial.multi_polynomial_element.MPolynomialPolydict method), 343
global_height() (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method), 441
global_height() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_libsingular method), 314
gradient() (sage.rings.polynomial.multi_polynomial_element.MPolynomialElement method), 57
greater_tuple() (sage.rings.polynomial.multi_polynomial_element.MPolynomialElement term_order.TermOrder property), 281
greater_tuple_block() (sage.rings.polynomial.multi_polynomial_element.MPolynomialElement term_order.TermOrder method), 281
greater_tuple_deglex() (sage.rings.polynomial.multi_polynomial_element.MPolynomialElement term_order.TermOrder method), 281
greater_tuple_degneglex() (sage.rings.polynomial.multi_polynomial_element.MPolynomialElement term_order.TermOrder method), 282
greater_tuple_degrevlex() (sage.rings.polynomial.multi_polynomial_element.MPolynomialElement term_order.TermOrder method), 282
greater_tuple_invlex() (sage.rings.polynomial.multi_polynomial_element.MPolynomialElement term_order.TermOrder method), 282
greater_tuple_lex() (sage.rings.polynomial.multi_polynomial_element.MPolynomialElement term_order.TermOrder method), 282
greater_tuple_matrix() (sage.rings.polynomial.multi_polynomial_element.MPolynomialElement term_order.TermOrder method), 283

H
h_covariant() (sage.rings.invariants.invariant_theory.BinaryQuartic method), 485
H_covariant() (sage.rings.invariants.invariant_theory.BinaryQuintic method), 488
h_polynomial() (sage.rings.polynomial.integer_valued_polynomials.IntegerValuedPolynomialRing.Shifted.Element method), 268
h_vector() (sage.rings.polynomial.integer_valued_polynomials.IntegerValuedPolynomialRing.Shifted.Element method), 268
hamming_weight() (sage.rings.polynomial.laurent_polynomial.LaurentPolynomial method), 570
hamming_weight() (sage.rings.polynomial.multi_polynomial_element.MPolynomialElement method), 338
hamming_weight() (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomialElement method), 207

Index 639
Polynomials, Release 10.3

hamming_weight() (sage.rings.polynomial.polynomial_element.Polynomial method), 58
has_cyclotomic_factor() (sage.rings.polynomial.polynomial_element.Polynomial method), 58
has_root() (sage.rings.polynomial.real_roots.island method), 216
hensel_lift() (sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_cdv method), 122
hensel_lift() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 159
Hessian() (sage.rings.invariants.invariant_theory.TernaryCubic method), 510
hilbert_numerator() (sage.rings.polynomial.multipolynomial_ideal.MPolynomialIdeal_singular_repr method), 386
hilbert_poincare_series() (in module sage.rings.polynomial.hilbert), 476
hilbert_polynomial() (sage.rings.polynomial.multipolynomial_element.MPolynomialRing_libsingular method), 314
homogeneous_components() (sage.rings.polynomial.multipolynomial.MPolynomial method), 314
homogeneous_symmetric_function() (in module sage.rings.polynomial.homogenization), 584
homogenize() (sage.rings.polynomial.multipolynomial_element.MPolynomialIdeal method), 368
homogenize() (sage.rings.polynomial.multipolynomial_element.MPolynomial method), 315
homogenize() (sage.rings.polynomial.polydict.PolyDict method), 468
homogenize() (sage.rings.polynomial.multipolynomial_element.Polynomial method), 59
homogenized() (sage.rings.invariants.invariant_theory.AlgebraicForm method), 482
homogenized() (sage.rings.invariants.invariant_theory.SeveralAlgebraicForms method), 509
ideal() (sage.rings.polynomial.multipolynomial_ring.MPolynomialRing_libsingular method), 430
ideal() (sage.rings.polynomial.multipolynomial_ring.MPolynomialRing_polydict_domain method), 337
Ideal_1poly_field (class in sage.rings.polynomial.integer_polynomial), 590
InfinitePolynomial (class in sage.rings.polynomial.integer_polynomial_element), 598
InfinitePolynomial_dense (class in sage.rings.polynomial.integer_polynomial_element), 606
InfinitePolynomial_sparse (class in sage.rings.polynomial.integer_polynomial_element), 606
IntegerValuedPolynomial (class in sage.rings.polynomial.integer_polynomial_element), 591
IntegerValuedPolynomialRing_dense (class in sage.rings.polynomial.integer_polynomial_element), 591
IntegerValuedPolynomialRing_sparse (class in sage.rings.polynomial.integer_polynomial_element), 593
IntegerValuedPolynomialRingFactory (class in sage.rings.polynomial.integer_polynomial_element), 591
inhomogeneous_quadratic_form() (sage.rings.polynomial.integer_valued_polynomials.InvariantTheoryFactory method), 501
int_list() (sage.rings.polynomial.multipolynomial_modn_dense_ntl.Polynomial_dense_mod_n method), 172
int_list() (sage.rings.polynomial.multipolynomial_modn_dense_ntl.Polynomial_dense_mod_n method), 177
INTEGER_LIMIT (sage.rings.fraction_field_FpT.FpT attribute), 552
IntegerValuedPolynomialRing (class in sage.rings.polynomial.integer_valued_polynomials), 261
IntegerValuedPolynomialRing.Bases (class in sage.rings.polynomial.integer_valued_polynomials), 261
IntegerValuedPolynomialRing.Bases.ElementMethods (class in sage.rings.polynomial.integer_valued_polynomials), 261
irreducible_element() (sage.rings.polynomial.polynomial_ring.PolynomialRing_dense_finite_field method), 12
irreducible_element() (sage.rings.polynomial.polynomial_ring.PolynomialRing_dense_mod_p method), 14
irrelevant_ideal() (sage.rings.polynomial.multi_polynomial_ring_base.MPolynomialRing_base method), 300
is_block_order() (sage.rings.polynomial.term_order.TermOrder method), 286
is_constant() (sage.rings.polynomial.laurent_polynomial.LaurentPolynomial_univariate method), 576
is_constant() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict method), 345
is_constant() (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method), 443
is_constant() (sage.rings.polynomial.polydict.ETuple method), 464
is_constant() (sage.rings.polynomial.polydict.PolyDict method), 469
is_constant() (sage.rings.polynomial.polydict.PolyDict method), 469
is_field() (sage.rings.polynomial.multi_polynomial_ring_base.MPolynomialRing_base method), 301
is_field() (sage.rings.polynomial.multi_polynomial_ring.MPolynomialRing_polydict_domain method), 337
is_field() (sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_generic method), 248
is_field() (sage.rings.polynomial.polynomial_ring.PolynomialRing_general method), 24
is_finite() (sage.rings.fraction_field.FractionField_generic method), 545
is_finite() (sage.rings.polynomial.laurent_polynomial.LaurentPolynomialRing_base method), 563
is_finite() (sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_generic method), 248
is_FractionField() (in module sage.rings.fraction_field), 546
is_FractionFieldElement() (in module sage.rings.fraction_field_element), 551
is_gen() (sage.rings.polynomial.polynomial_element.Polynomial method), 66
is_gen() (sage.rings.polynomial.polynomial_gf2x.Polynomial_template method), 134
is_gen() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_dense_modn_ntl_ZZ method), 175
is_gen() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_dense_modn_ntl_zz method), 177
is_gen() (sage.rings.polynomial.polynomial_zmod_flint.Polynomial_template method), 165
is_gen() (sage.rings.polynomial.polynomial_zz_pex.Polynomial_template method), 199
is_generator() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict method), 345
is_generator() (sage.rings.polynomial.multi_polynomial.MPolynomial method), 317
is_global() (sage.rings.polynomial.term_order.TermOrder method), 286
is_groebner() (sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic method), 417
is_homogeneous() (sage.rings.invariants.invariant_theory.FormsBase method), 497
is_homogeneous() (sage.rings.polynomial.multi_polynomial_element.MPolynomial...
mial_polydict method), 345

is_homogeneous() (sage.rings.polynomial.multivariate_polynomial ideal.MPolynomialIdeal method), 369

is_homogeneous() (sage.rings.polynomial.multivariate_polynomial_libsingular.MPolynomial_libsingular method), 443

is_homogeneous() (sage.rings.polynomial.multivariate_polynomial.MPolynomial method), 317

is_homogeneous() (sage.rings.polynomial.polydict.PolyDict method), 469

is_homogeneous() (sage.rings.polynomial.polynomial_element.Polynomial method), 66

is_injective() (sage.rings.fraction_field.FractionFieldEmbedding method), 542

is_injective() (sage.rings.polynomial.polynomial_element_polynomial_element.PolynomialBaseringInjection method), 115

is_injective() (sage.rings.polynomial.polynomial_quotient_ring PolynomialQuotientRing_coercion method), 236

is_injective() (sage.rings.polynomial.polynomial_ring.homomorphism.PolynomialRingHomomorphism_from_base method), 32

is_integral() (sage.rings.fraction_field_element.FractionFieldElement1poly_field method), 550

is_integral_domain() (sage.rings.polynomial.infinite_polynomial_ring.InfinitePolynomialRing_sparse method), 594

is_integral_domain() (sage.rings.polynomial.laurent_polynomial_ring_base.LaurentPolynomialRing_generic method), 563

is_integral_domain() (sage.rings.polynomial.multi_polynomial_ring_base.MPolynomialRing_base method), 301

is_integral_domain() (sage.rings.polynomial.multi_polynomial_ring.MPolynomialRing_polydict_domain method), 337

is_integral_domain() (sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_generic method), 249

is_integral_domain() (sage.rings.polynomial_polynomial_ring.PolynomialRing_general method), 24

is_irreducible() (sage.rings.polynomial.polynomial_element.Polynomial method), 66

is_irreducible() (sage.rings.polynomial.polynomial_gf2x.Polynomial_GF2X method), 132

is_irreducible() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 159

is_irreducible() (sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint method), 168

is_irreducible() (sage.rings.polynomial.polynomial_zz_pex.Polynomial_ZZ_pEX method), 196

is_LaurentPolynomialRing() (in module sage.rings.polynomial.laurent_polynomial_ring), 569

is_linearly_dependent() (in module sage.rings.polynomial.toy_variety), 533

is_local() (sage.rings.polynomial.term_order.TermandOrder method), 286

is_lorentzian() (sage.rings.polynomial.multi_polynomial.Polynomial method), 317

is_lorentzian() (sage.rings.polynomial.polynomial_element.Polynomial method), 67

is_maximal() (sage.rings.polynomial.symmetric_ideal.SymmetricIdeal method), 613

is_monic() (sage.rings.polynomial.polynomial_element.Polynomial method), 67

is_monomial() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict method), 345

is_monomial() (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method), 443

is_monomial() (sage.rings.polynomial.polynomial_element.Polynomial method), 68

is_MPolynomial() (in module sage.rings.polynomial.multi_polynomial), 333

is_MPolynomialIdeal() (in module sage.rings.polynomial.multi_polynomial_ideal), 408

is_MPolynomialRing() (in module sage.rings.polynomial.multi_polynomial_ring_base), 308

is_multiple_of() (sage.rings.polynomial.polydict.ETuple method), 464

is_nilpotent() (sage.rings.polynomial.infinite_polynomial_element.InfinitePolynomialMethod), 600

is_nilpotent() (sage.rings.polynomial.multi_polynomial.MPolynomial method), 318

is_nilpotent() (sage.rings.polynomial.polynomial_element.Polynomial method), 68

is_noetherian() (sage.rings.polynomial.infinite_polynomial_ring.InfinitePolynomialRing_sparse method), 595

is_noetherian() (sage.rings.polynomial.laurent_polynomial_ring_base.LaurentPolynomialRing_generic method), 563

is_noetherian() (sage.rings.polynomial.multi_polynomial_ring_base.MPolynomialRing_base method), 434...
Polynomials, Release 10.3

is_noetherian() (sage.rings.polynomial.polynomial_ring.PolynomialRing_general method), 301

is_one() (sage.rings.fraction_field_element.FractionFieldElement method), 24

is_squarefree() (sage.rings.polynomial.polynomial_element.Polynomial method), 443

isSquare() (sage.rings.fraction_field_element.FractionFieldElement method), 547

is_square() (sage.rings.polynomial.polynomial_element.Polynomial method), 69

is_square_free() (sage.rings.polynomial.polynomial_element.Polynomial method), 443

is_surjective() (sage.rings.fraction_field_element.FractionFieldElement method), 542

is_surjective() (sage.rings.polynomial.polynomial_element.PolynomialBaseringInjection method), 115

is_term() (sage.rings.polynomial.polynomial_element.PolynomialQuotientRing_coercion method), 236

is_term() (sage.rings.polynomial.polynomial_element.PolynomialRing_homomorphism.PolynomialRingHomomorphism_from_base method), 33

is_symmetric() (sage.rings.polynomial.multi_polynomial.MPolynomial method), 319

is_term() (sage.rings.polynomial.polynomial_element.PolynomialRing_homomorphism.PolynomialRingHomomorphism_from_base method), 33
is_well_polynomial()  (sage.rings.polynomial.polynomial_element.Polynomial method), 74
is_zero()  (sage.rings.fraction_field_element.FractionFieldElement method), 548
is_zero()  (sage.rings.polynomial.laurent_polynomial.LaurentPolynomial_univariate method), 577
is_zero()  (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method), 444
is_zero()  (sage.rings.polynomial.polynomial_element.Polynomial method), 75
is_zero()  (sage.rings.polynomial.polynomial_gf2x.Polynomial_template method), 134
is_zero()  (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 143
is_zero()  (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 160
is_zero()  (sage.rings.polynomial.polynomial_zz_pex.Polynomial_template method), 165
is_zero()  (sage.rings.polynomial.polynomial_zz_pseudofirst.Polynomial_template method), 199
island  (class in sage.rings.polynomial.real_roots), 214
iter()  (sage.rings.fraction_field_FpT.FpT method), 552
iterator_exp_coeff()  (sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict method), 347
iterator_exp_coeff()  (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method), 445
iterator_exp_coeff()  (sage.rings.polynomial.multi_polynomial.MPolynomial method), 321

J
j_covariant()  (sage.rings.invariants.invariant_theory.BinaryQuintic method), 494
J_covariant()  (sage.rings.invariants.invariant_theory.TernaryCubic method), 510
J_covariant()  (sage.rings.invariants.invariant_theory.TwoQuaternaryQuadratics method), 516
J_covariant()  (sage.rings.invariants.invariant_theory.TwoTernaryQuadratics method), 520
jacobian_ideal()  (sage.rings.polynomial.multi_polynomial.MPolynomial method), 321

K
karatsuba_threshold()  (sage.rings.polynomial.polynomial_ring.PolynomialRing_general method), 25
kbase_libsingular()  (in module sage.rings.polynomial.multi_polynomial_ideal_libsingular), 457
key_basis()  (sage.rings.polynomial.infinite_polynomial_ring.InfinitePolynomialRing_sparse method), 595
krull_dimension()  (sage.rings.polynomial.infinite_polynomial_ring.InfinitePolynomialRing_sparse method), 595
krull_dimension()  (sage.rings.polynomial.laurent_polynomial_ring_base.LaurentPolynomialRing_generic method), 564
krull_dimension()  (sage.rings.polynomial.multi_polynomial_ring_base.MPolynomialRing_base method), 301
krull_dimension()  (sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_generic method), 249
krull_dimension()  (sage.rings.polynomial.polynomial_ring.PolynomialRing_general method), 25

L
lagrange_polynomial()  (sage.rings.polynomial.polynomial_ring.PolynomialRing_field method), 19
latex()  (sage.rings.polynomial.polydict.PolyDict method), 469
LaurentPolynomial  (class in sage.rings.polynomial.laurent_polynomial), 570
LaurentPolynomial_univariate  (class in sage.rings.polynomial.laurent_polynomial), 572
LaurentPolynomialRing()  (in module sage.rings.polynomial.laurent_polynomial_ring), 565
LaurentPolynomialRing_generic  (class in sage.rings.polynomial.laurent_polynomial_ring_base), 561
LaurentPolynomialRing_mpair  (class in sage.rings.polynomial.laurent_polynomial_ring), 568
LaurentPolynomialRing_univariate  (class in sage.rings.polynomial.laurent_polynomial_ring), 569
LC()  (in module sage.rings.polynomial.toy_d_basis), 537
lc()  (sage.rings.polynomial.infinite_polynomial_element.InfinitePolynomial method), 601
lc()  (sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict method), 347
lc()  (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method), 445

Index 645
lc() (sage.rings.polynomial.polynomial_element.Polynomial method), 75
LCM() (in module sage.rings.polynomial.toy_buchberger), 529
lcm() (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method), 445
lcm() (sage.rings.polynomial.polynomial_element.Polynomial method), 75
lcm() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 143
lcm() (sage.rings.polynomial.polynomial_integer_dense_ntl.Polynomial_integer_dense_ntl method), 149
lcm() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 160
lcm() (sage.rings.polynomial.polydict.PolyDict method), 470
leading_coefficient() (sage.rings.polynomial.polynomial_element.Polynomial method), 75
less_bits() (sage.rings.polynomial.real_roots.island method), 216
lift() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict method), 347
lift() (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method), 446
lift() (sage.rings.polynomial.multi_polynomial.MPolynomial method), 321
lift() (sage.rings.polynomial.padics.polynomial_padic_capped_relative_dense.Polynomial_padic_capped_relative_dense method), 190
lift() (sage.rings.polynomial.polynomial_quotient_ring_element.PolynomialQuotientRingElement method), 257
lift() (sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_generic method), 249
linear_map (class in sage.rings.polynomial.real_roots), 216
linear_representation() (in module sage.rings.polynomial.toy_variety), 534
list() (sage.rings.polynomial.padics.polynomial_padic_capped_relative_dense.Polynomial_padic_capped_relative_dense method), 190
list() (sage.rings.polynomial.polydict.PolyDict method), 470
list() (sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_sparse method), 128
list() (sage.rings.polynomial.polynomial_element.Polynomial method), 75
list() (sage.rings.polynomial.polynomial_element.Polynomial_generic_dense method), 116
list() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 143
list() (sage.rings.polynomial.polynomial_integer_dense_ntl.Polynomial_integer_dense_ntl method), 149
list() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_modn_dense_ntl method), 172
list() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_modn_dense_ntl method), 175
list() (sage.rings.polynomial.polynomial_quotient_ring_element.PolynomialQuotientRingElement method), 258
list() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 160
list() (sage.rings.polynomial.polynomial_real_mpfr_dense.PolynomialRealDense method), 182
list() (sage.rings.polynomial.polynomial_zmod_flint.Polynomial_template method), 165
list() (sage.rings.polynomial.polynomial_zz_pex.Polynomial_template method), 199
list() (sage.rings.polynomial.polynomial_zz_pex.Polynomial_ZZ_pEX method), 197
LM() (in module sage.rings.polynomial.toy_buchberger), 529
LM() (in module sage.rings.polynomial.toy_d_basis), 537
lm() (sage.rings.polynomial.infinite_polynomial_element.InfinitePolynomial method), 601
lm() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict method), 348
lm() (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method), 447
lm() (sage.rings.polynomial.polynomial_element.Polynomial method), 76
local_height() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict method), 348
local_height() (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method), 447
local_height() (sage.rings.polynomial.polynomial_element.Polynomial method), 76
local_height_arch() (sage.rings.polyno-
Index
sage.rings.polynomial.polynomial_ring_homomorphism, 32
sage.rings.polynomial.polynomial_singular_interface, 184
sage.rings.polynomial.polynomial_zmod_flint, 163
sage.rings.polynomial.polynomial_zz_pex, 196
sage.rings.polynomial.real_roots, 200
sage.rings.polynomial.refine_root, 232
sage.rings.polynomial.symmetric_ideal, 607
sage.rings.polynomial.symmetric_reduction, 616
sage.rings.polynomial.term_order, 275
sage.rings.polynomial.toy_buchberger, 526
sage.rings.polynomial.toy_d_basis, 536
sage.rings.polynomial.toy_variety, 532

modulus() (sage.rings.polynomial.polynomial_quotient_ring PolynomialQuotientRing_generic method), 250

modulus() (sage.rings.polynomial.polynomial_ring PolynomialRing_dense_mod_n method), 13

monic() (sage.rings.polynomial.polynomial_element Polynomial method), 79

monic() (sage.rings.polynomial.polynomial_zmod_flint Polynomial_zmod_flint method), 168

monics() (sage.rings.polynomial.polynomial_ring PolynomialRing_general method), 25

monomial() (sage.rings.polynomial.laurent_polynomial laurent_polynomial.LaurentPolynomialRing_impair method), 568

monomial() (sage.rings.polynomial.multi_polynomial_ring PolynomialRing_base method), 304

monomial() (sage.rings.polynomial.multi_polynomial_ring PolynomialRing_general method), 26

monomial_all_divisors() (sage.rings.polynomial.multi_polynomial_libsingular MPolynomialRing_libsingular method), 430

monomial_all_divisors() (sage.rings.polynomial.multi_polynomial_ring PolynomialRing_polydict method), 335

monomial_coefficient() (sage.rings.polynomial.multi_polynomial_element MPoly-
mial_polydict method), 350

monomial_coefficient() (sage.rings.polynomial.multi_polynomial_libsingular MPolynomialRing_libsingular method), 448

monomial_coefficient() (sage.rings.polynomial.polydict.PolyDict method), 470

monomial_coefficient() (sage.rings.polynomial.polynomial_element Polynomial method), 79

monomial_divides() (sage.rings.polynomial.multi_polynomial_libsingular MPolynomialRing_libsingular method), 430

monomial_divides() (sage.rings.polynomial.multi_polynomial_ring PolynomialRing_polydict method), 335

monomial_divides() (sage.rings.polynomial.multi_polynomial_ring PolynomialRing_libsingular method), 431

monomial_lcm() (sage.rings.polynomial.multi_polynomial_libsingular MPolynomialRing_libsingular method), 431

monomial_lcm() (sage.rings.polynomial.multi_polynomial_ring PolynomialRing_polydict method), 335

monomial_pairwise_prime() (sage.rings.polynomial.multi_polynomial_libsingular MPolynomialRing_libsingular method), 431

monomial_pairwise_prime() (sage.rings.polynomial.multi_polynomial_ring PolynomialRing_polydict method), 335

monomial_quotient() (sage.rings.polynomial.multi_polynomial_libsingular MPolynomialRing_libsingular method), 431

monomial_quotient() (sage.rings.polynomial.multi_polynomial_ring PolynomialRing_polydict method), 336

monomial_reduce() (sage.rings.polynomial.multi_polynomial_libsingular MPolynomialRing_libsingular method), 432

monomial_reduce() (sage.rings.polynomial.multi_polynomial_ring PolynomialRing_polydict method), 336

monomial_reduction() (sage.rings.polynomial.laurent_polynomial LaurentPolynomial_univariate method), 578

monomials() (in module sage.rings.monomials), 479

monomials() (sage.rings.invariants.invariant_theory.BinaryQuartic method), 486

monomials() (sage.rings.invariants.invariant_theory.BinaryQuintic method), 495

monomials() (sage.rings.invariants.invariant_theory.QuadraticForm method), 508

monomials() (sage.rings.invariants.invariant_theory.TernaryCubic method), 512

monomials() (sage.rings.invariants.invariant_the-
ory.TernaryQuadratic method), 514
monomials() (sage.rings.polynomial.multi_poli-
momial_element.MPolynomial_polydict method), 350
monomials() (sage.rings.polynomial.multi_poli-
momial_libsingular.MPolynomial_libsingular method), 449
monomials() (sage.rings.polynomial.multi_poli-
momial_sequence.PolynomialSequence_generic method), 418
monomials() (sage.rings.polynomial.polynomial_element.Polynomial method), 80
monomials_of_degree() (sage.rings.polyno-
mial.multi_polynomial_ring_base.MPolynomial-
Ring_base method), 304
more_bits() (sage.rings.polynomial.real_roots.island method), 216
MPolynomial (class in sage.rings.polyno-
mial.multi_polynomial), 309
MPolynomial_element (class in sage.rings.polyno-
mial.multi_polynomial), 338
MPolynomial_libsingular (class in sage.rings.polyno-
mial.multi_polynomial), 333
MPolynomial_libsingular (class in sage.rings.polyno-
mial.multi_polynomial_libsingular), 433
MPolynomial_polydict (class in sage.rings.polyno-
mial.multi_polynomial), 343
MPolynomialRing (class in sage.rings.polyno-
mial.multi_polynomial_ring_base.MPolynomial-
Ring_base method), 359
MPolynomialRing_libsingular (class in sage.rings.polynomial.multi_polynomial_lib-
singular), 428
MPolynomialRing_macauly2_repr (class in sage.rings.polynomial.multi_polynomial_ideal), 376
MPolynomialRing_magma_repr (class in sage.rings.polynomial.multi_polynomial_ideal), 376
MPolynomialRing_quotient (class in sage.rings.polynomial.multi_polynomial_ideal), 376
MPolynomialRing_singular_base_repr (class in sage.rings.polynomial.multi_polynomial_ideal), 377
MPolynomialRing_singular_repr (class in sage.rings.polynomial.multi_polynomial_ideal), 377
MPolynomialRing_base (class in sage.rings.polyno-
mial.multi_polynomial_ring_base), 297
MPolynomialRing_libsingular (class in sage.rings.polynomial.multi_polynomial_lib-
singular), 428
MPolynomialRing_macauly2_repr (class in sage.rings.polynomial.multi_polynomial_ring), 334
MPolynomialRing_polydict (class in sage.rings.polynomial.multi_polynomial_ring), 334
MPolynomialRing_polydict_domain (class in sage.rings.polynomial.multi_polynomial_ring), 337
mul_pd (class in sage.rings.polynomial.polynomial_compiled), 260
multiplication_trunc() (sage.rings.polynomial.
multi_polynomial_element.Polynomial method), 80
n_forms() (sage.rings.invariants.invariant_theory.
SeveralAlgebraicForms method), 509
name() (sage.rings.polynomial.term_order.TermOrder method), 288
NCPolynomialIdeal (class in sage.rings.polyno-
mial.multi_polynomial_ideal), 403
newton_polygon() (sage.rings.polynomial.padi-
cs.polynomial_padic_capped_relative_dense.
Polynomial_padic_capped_relative_dense method), 191
newton_polytope() (sage.rings.polynomial.poin-
tial.element_generic.Polynomial_generic_cd
method), 123
newton_polytope() (sage.rings.polynomial.multi_-
polynomial.MPolynomial method), 325
newton_raphson() (sage.rings.polynomial.poin-
tial.element.Polynomial method), 81
newton_slopes() (sage.rings.polynomial.padi-
cs.polynomial_padic_capped_relative_dense.
Polynomial_padic_capped_relative_dense method), 191
newton_slopes() (sage.rings.polynomial.poin-
tial.element_generic.Polynomial_generic_cd
method), 123
newton_slopes() (sage.rings.polynomial.poin-
tial.element.Polynomial method), 81
next() (sage.rings.fraction_field_FpT.FpTElement method), 553
next() (sage.rings.polynomial.infinite_polyno-
mial_ring.GenDictWithBasering method), 590
ngens() (sage.rings.fraction_field.FractionField_generic method), 545
ngens() (sage.rings.polynomial.infinite_polyno-
mial_ring.InfinitePolynomialRing_sparse method), 595
ngens() (sage.rings.polynomial.laurent_polyno-
mial_ring.LaurentPolynomialRing_generic method), 564
ngens() (sage.rings.polynomial.multi_polynomial_lib-
singular.MPolynomialRing_libsingular method),
<table>
<thead>
<tr>
<th>Method</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>order()</td>
<td>(sage.rings.polynomial.infinite_polynomial_ring.InfinitePolynomialRing_sparse method), 595</td>
</tr>
<tr>
<td>order()</td>
<td>(sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_generic method), 250</td>
</tr>
<tr>
<td>padded_list()</td>
<td>(sage.rings.polynomial.polynomial_element.Polynomial method), 87</td>
</tr>
<tr>
<td>parameter()</td>
<td>(sage.rings.polynomial.polynomial_ring.PolynomialRing_general method), 26</td>
</tr>
<tr>
<td>part()</td>
<td>(sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic method), 419</td>
</tr>
<tr>
<td>partition()</td>
<td>(in module sage.rings.polynomial.omega), 585</td>
</tr>
<tr>
<td>parts()</td>
<td>(sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic method), 419</td>
</tr>
<tr>
<td>Phi_invariant()</td>
<td>(sage.rings.invariants.invariant_theory.TwoQuaternaryQuadratics method), 516</td>
</tr>
<tr>
<td>plot()</td>
<td>(sage.rings.polynomial.multi_polynomial_ideal.MPolynomialIdeal method), 369</td>
</tr>
<tr>
<td>plot()</td>
<td>(sage.rings.polynomial.multi_polynomial_ideal.MPolynomialIdeal_singular_repr method), 392</td>
</tr>
<tr>
<td>plot()</td>
<td>(sage.rings.polynomial.polynomial_element.Polynomial method), 87</td>
</tr>
<tr>
<td>polar_conic()</td>
<td>(sage.rings.invariants.invariant_theory.TernaryCubic method), 512</td>
</tr>
<tr>
<td>poly_repr()</td>
<td>(sage.rings.polynomial.polydict.PolyDict method), 471</td>
</tr>
<tr>
<td>PolyDict</td>
<td>(class in sage.rings.polynomial.polydict), 466</td>
</tr>
<tr>
<td>polygen()</td>
<td>(in module sage.rings.polynomial.polynomial_ring), 31</td>
</tr>
<tr>
<td>polygens()</td>
<td>(in module sage.rings.polynomial.polynomial_ring), 32</td>
</tr>
<tr>
<td>Polynomial</td>
<td>(class in sage.rings.polynomial.polynomial_element), 34</td>
</tr>
<tr>
<td>polynomial()</td>
<td>(sage.rings.invariants.invariant_theory.AlgebraicForm method), 483</td>
</tr>
<tr>
<td>polynomial()</td>
<td>(sage.rings.polynomial.infinite_polynomial_element.Infinite Polynomial method), 601</td>
</tr>
<tr>
<td>polynomial()</td>
<td>(sage.rings.polynomial.integer_values.IntegerValuedPolynomialRing.Bases.ElementMethods method), 262</td>
</tr>
<tr>
<td>polynomial()</td>
<td>(sage.rings.polynomial.multi_polynomial.MPolynomial method), 326</td>
</tr>
<tr>
<td>polynomial()</td>
<td>(sage.rings.polynomial.polynomial_element.Polynomial method), 87</td>
</tr>
<tr>
<td>Polynomial_absolute_number_field_dense</td>
<td>(class in sage.rings.polynomial.polynomial_number_field), 137</td>
</tr>
<tr>
<td>polynomial_coefficient()</td>
<td>(sage.rings.polynomial.polydict.PolyDict method), 471</td>
</tr>
<tr>
<td>polynomial_construction()</td>
<td>(sage.rings.polynomial.polynomial_univariate.PolynomialUnivariate method), 578</td>
</tr>
<tr>
<td>polynomial_default_category()</td>
<td>(in module sage.rings.polynomial.polynomial_ring_constructor), 7</td>
</tr>
<tr>
<td>Polynomial_dense_mod_n</td>
<td>(class in sage.rings.polynomial.polynomial_modn_dense_ntl), 171</td>
</tr>
<tr>
<td>Polynomial_dense_mod_p</td>
<td>(class in sage.rings.polynomial.polynomial_modn_dense_ntl), 173</td>
</tr>
<tr>
<td>Polynomial_dense_mod_ntl_ZZ</td>
<td>(class in sage.rings.polynomial.polynomial_modn_dense_ntl), 175</td>
</tr>
<tr>
<td>Polynomial_dense_mod_ntl_ZZ</td>
<td>(class in sage.rings.polynomial.polynomial_modn_dense_ntl), 176</td>
</tr>
<tr>
<td>Polynomial_generic_cdv</td>
<td>(class in sage.rings.polynomial.polynomial_element_generic), 121</td>
</tr>
<tr>
<td>Polynomial_generic_cdvf</td>
<td>(class in sage.rings.polynomial.polynomial_element_generic), 124</td>
</tr>
<tr>
<td>Polynomial_generic_cdvr</td>
<td>(class in sage.rings.polynomial.polynomial_element_generic), 124</td>
</tr>
<tr>
<td>Polynomial_generic_dense</td>
<td>(class in sage.rings.polynomial.polynomial_element), 115</td>
</tr>
<tr>
<td>Polynomial_generic_dense_cdv</td>
<td>(class in sage.rings.polynomial.polynomial_element_generic), 124</td>
</tr>
<tr>
<td>Polynomial_generic_dense_cdvf</td>
<td>(class in sage.rings.polynomial.polynomial_element_generic), 124</td>
</tr>
<tr>
<td>Polynomial_generic_dense_cdvr</td>
<td>(class in sage.rings.polynomial.polynomial_element_generic), 125</td>
</tr>
<tr>
<td>Polynomial_generic_dense_field</td>
<td>(class in sage.rings.polynomial.polynomial_element_generic), 125</td>
</tr>
<tr>
<td>Polynomial_generic_dense_inexact</td>
<td>(class in sage.rings.polynomial.polynomial_element), 118</td>
</tr>
<tr>
<td>Polynomial_generic_domain</td>
<td>(class in sage.rings.polynomial.polynomial_element_generic), 125</td>
</tr>
<tr>
<td>Polynomial_generic_field</td>
<td>(class in sage.rings.polynomial.polynomial_element_generic), 125</td>
</tr>
<tr>
<td>Polynomial_generic_sparse</td>
<td>(class in sage.rings.polynomial.polynomial_element), 87</td>
</tr>
</tbody>
</table>
Polynomials, Release 10.3

PolynomialRing_dense_padic_ring_fixed_mod (class in sage.rings.polynomial.polynomial_ring), 17
PolynomialRing_dense_padic_ring_generic (class in sage.rings.polynomial.polynomial_ring), 17
PolynomialRing_field (class in sage.rings.polynomial.polynomial_ring), 17
PolynomialRing_general (class in sage.rings.polynomial.polynomial_ring), 20
PolynomialRing_integral_domain (class in sage.rings.polynomial.polynomial_ring), 29
PolynomialRing_sage_rings.polynomial.polynomial singular_interface) (class in sage.rings.polynomial.polynomial singular_interface), 184
polynomials() (sage.rings.polynomial.polynomial_ring, 27
PolynomialSequence() (in module sage.rings.polynomial.polynomial_sequence), 412
PolynomialSequence_generic (class in sage.rings.polynomial.multi_polynomial_sequence), 413
PolynomialSequence_gf2 (class in sage.rings.polynomial.multi_polynomial_sequence), 421
PolynomialSequence_gf2e (class in sage.rings.polynomial.multi_polynomial_sequence), 425
Polyring_FpT_coerce (class in sage.rings.fracition_field_FpT), 557
pow_pd (class in sage.rings.polynomial.polynomial_compiled), 260
power_trunc() (sage.rings.polynomial.polynomial_element.Polynomial method), 88
prec() (sage.rings.polynomial.polynomial_element.Polynomial method), 88
prec_degree() (sage.rings.polynomial.padics.polynomial_padic_capped_relative_dense.Polynomial_padic_capped_relative_dense method), 192
prec_degree() (sage.rings.polynomial.polynomial_element.Polynomial_generic_dense_inexact method), 119
precision_absolute() (sage.rings.polynomial.padics.polynomial_padic_capped_relative_dense.Polynomial_padic_capped_relative_dense method), 192
precision_relative() (sage.rings.polynomial.padics.polynomial_padic_capped_relative_dense.Polynomial_padic_capped_relative_dense method), 193
precompute_degree_reduction_cache() (in module sage.rings.polynomial.real_roots), 221
primary_decomposition() (sage.rings.polynomial.multi_polynomial_ideal.MPolynomial_ideal_singular_repr method), 393
primary_decomposition_complete() (sage.rings.polynomial.multi_polynomial_ideal.MPolynomial_ideal_singular_repr method), 394
product_on_basis() (sage.rings.polynomial.integer_valued_polynomials.IntegerValuedPolynomialRing, 266
product_on_basis() (sage.rings.polynomial.integer_valued_polynomials.IntegerValuedPolynomialRing, 269
pseudo_divrem() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 143
pseudo_quo_rem() (sage.rings.polynomial.polynomial_element.Polynomial method), 88
pseudoinverse() (in module sage.rings.polynomial.real_roots), 221

Q

quadratic_form() (sage.rings.invariants.invariant_theory.InvariantTheoryFactory method), 502
QuadraticForm (class in sage.rings.invariants.invariant_theory), 505
quaternary_biquadratic() (sage.rings.invariants.invariant_theory.InvariantTheoryFactory method), 502
quaternary_quadratic() (sage.rings.invariants.invariant_theory.InvariantTheoryFactory method), 503
quo_rem() (sage.rings.polynomial.laurent_polynomial.LaurentPolynomial_univariate method), 578
quo_rem() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict method), 351
quo_rem() (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method), 451
quo_rem() (sage.rings.polynomial.padics.polynomial_padic_capped_relative_dense.Polynomial_padic_capped_relative_dense method), 193
quo_rem() (sage.rings.polynomial.padics.polynomial_padic_capped_relative_dense.Polynomial_padic_capped_relative_dense method), 193
quo_rem() (sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_field method), 125
quo_rem() (sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_sparse method), 128
<table>
<thead>
<tr>
<th>Function</th>
<th>Module</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>quo_rem()</td>
<td>(sage.rings.polynomial.polynomial_element.Polynomial_generic_dense method), 116</td>
<td></td>
</tr>
<tr>
<td>quo_rem()</td>
<td>(sage.rings.polynomial.polynomial_gf2x.Polynomial_template method), 135</td>
<td></td>
</tr>
<tr>
<td>quo_rem()</td>
<td>(sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 144</td>
<td></td>
</tr>
<tr>
<td>quo_rem()</td>
<td>(sage.rings.polynomial.polynomial_integer_dense_ntl.Polynomial_integer_dense_ntl method), 150</td>
<td></td>
</tr>
<tr>
<td>quo_rem()</td>
<td>(sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_modn_dense_ntl method), 173</td>
<td></td>
</tr>
<tr>
<td>quo_rem()</td>
<td>(sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_modn_dense_ntl_ZZ method), 175</td>
<td></td>
</tr>
<tr>
<td>quo_rem()</td>
<td>(sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 161</td>
<td></td>
</tr>
<tr>
<td>quo_rem()</td>
<td>(sage.rings.polynomial.polynomial_real_mpfr_dense.PolynomialRealDense method), 182</td>
<td></td>
</tr>
<tr>
<td>quo_rem()</td>
<td>(sage.rings.polynomial.polynomial_zmod_flint.Polynomial_template method), 165</td>
<td></td>
</tr>
<tr>
<td>quo_rem()</td>
<td>(sage.rings.polynomial.polynomial_zz_pex.Polynomial_template method), 199</td>
<td></td>
</tr>
<tr>
<td>quotient()</td>
<td>(sage.rings.polynomial.multi_polynomial_ideal.MPolynomial_ideal_singular_repr method), 395</td>
<td></td>
</tr>
<tr>
<td>quotient_by_principal_ideal()</td>
<td>(sage.rings.polynomial.polynomial_ring.PolynomialRing_commutative method), 11</td>
<td></td>
</tr>
</tbody>
</table>

**R**

<table>
<thead>
<tr>
<th>Function</th>
<th>Module</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>R_invariant()</td>
<td>(sage.rings.invariants.invariant_theory.BinaryQuintic method), 488</td>
<td></td>
</tr>
<tr>
<td>radical()</td>
<td>(sage.rings.polynomial.multi_polynomial_ideal.MPolynomial_ideal_singular_repr method), 396</td>
<td></td>
</tr>
<tr>
<td>radical()</td>
<td>(sage.rings.polynomial.polynomial_element.Polynomial method), 89</td>
<td></td>
</tr>
<tr>
<td>random_element()</td>
<td>(sage.rings.fraction_field.FractionField_generic method), 545</td>
<td></td>
</tr>
<tr>
<td>random_element()</td>
<td>(sage.rings.polynomial.laurent_polynomial_ring_base.LaurentPolynomialRing_generic method), 564</td>
<td></td>
</tr>
<tr>
<td>random_element()</td>
<td>(sage.rings.polynomial.multi_polynomial_ideal.MPolynomial_ideal_singular_repr method), 370</td>
<td></td>
</tr>
<tr>
<td>random_element()</td>
<td>(sage.rings.polynomial.multi_polynomial_ring_base.MPolynomialRing_base method), 305</td>
<td></td>
</tr>
<tr>
<td>random_element()</td>
<td>(sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_generic method), 251</td>
<td></td>
</tr>
<tr>
<td>random_element()</td>
<td>(sage.rings.polynomial.polynomial_ring.PolynomialRing_general method), 27</td>
<td></td>
</tr>
<tr>
<td>rational_reconstruct()</td>
<td>(sage.rings.polynomial.polynomial_element.Polynomial method), 89</td>
<td></td>
</tr>
<tr>
<td>rational_reconstruct()</td>
<td>(sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint method), 168</td>
<td></td>
</tr>
<tr>
<td>rational_reconstruction()</td>
<td>(sage.rings.polynomial.polynomial_element.Polynomial method), 90</td>
<td></td>
</tr>
<tr>
<td>rational_reconstruction()</td>
<td>(sage.rings.polynomial.polynomial_quotient_ring_element.PolynomialQuotientRingElement method), 259</td>
<td></td>
</tr>
<tr>
<td>rational_reconstruction()</td>
<td>(sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint method), 169</td>
<td></td>
</tr>
<tr>
<td>rational_root_bounds()</td>
<td>(in module sage.rings.polynomial.real_roots), 221</td>
<td></td>
</tr>
<tr>
<td>real_root_intervals()</td>
<td>(sage.rings.polynomial.multi_polynomial_zz_int.Polynomial_zz_int method), 144</td>
<td></td>
</tr>
<tr>
<td>real_root_intervals()</td>
<td>(sage.rings.polynomial.multi_polynomial_zz_int.Polynomial_zz_int method), 150</td>
<td></td>
</tr>
<tr>
<td>real_root_intervals()</td>
<td>(sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 161</td>
<td></td>
</tr>
<tr>
<td>real_roots()</td>
<td>(in module sage.rings.polynomial.real_roots), 222</td>
<td></td>
</tr>
<tr>
<td>real_roots()</td>
<td>(sage.rings.polynomial.polynomial_element.Polynomial method), 92</td>
<td></td>
</tr>
<tr>
<td>reciprocal_transform()</td>
<td>(sage.rings.polynomial.polynomial_element.Polynomial method), 92</td>
<td></td>
</tr>
<tr>
<td>reduce()</td>
<td>(sage.rings.fraction_field_element.FractionFieldElement method), 549</td>
<td></td>
</tr>
<tr>
<td>reduce()</td>
<td>(sage.rings.fraction_field_element.FractionFieldElement_1poly_field method), 550</td>
<td></td>
</tr>
<tr>
<td>reduce()</td>
<td>(sage.rings.polynomial.infinite_polynomial_element.InfinitePolynomial method), 602</td>
<td></td>
</tr>
<tr>
<td>reduce()</td>
<td>(sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict method), 352</td>
<td></td>
</tr>
<tr>
<td>reduce()</td>
<td>(sage.rings.polynomial.multi_polynomial_ideal.MPolynomialIdeal method), 372</td>
<td></td>
</tr>
<tr>
<td>reduce()</td>
<td>(sage.rings.polynomial.multi_polyno-</td>
<td></td>
</tr>
</tbody>
</table>

Index 655
Index

P

Polynomials, Release 10.3

mial_ideal.MPolynomialIdeal_quotient method), 377
reduce() (sage.rings.polynomial.multi_polynomial_ideal.NCPolynomialIdeal method), 404
reduce() (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method), 451
reduce() (sage.rings.polynomial.symmetric_ideal.SymmetricIdeal method), 614
reduce() (sage.rings.polynomial.symmetric_reduction.SymmetricReductionStrategy method), 620
reduced() (sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic method), 419
reduced() (sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_gf2 method), 423
reduced_form() (sage.rings.polynomial.multi_polynomial.MPolynomial method), 327
refine() (sage.rings.polynomial.real_roots.island method), 216
refine_all() (sage.rings.polynomial.real_roots.ocean method), 220
refine_recurse() (sage.rings.polynomial.real_roots.island method), 216
refine_root() in module sage.rings.polynomial.refine_root, 232
region() (sage.rings.polynomial.real_roots.interval_bernstein_polynomial method), 208
region() (sage.rings.polynomial.real_roots.rr_gap method), 226
region_width() (sage.rings.polynomial.real_roots.interval_bernstein_polynomial method), 209
relative_bounds() in module sage.rings.polynomial.real_roots, 225
remove_var() (sage.rings.polynomial.laurent_polynomial_ring_base.LaurentPolynomialRing_generic method), 564
remove_var() (sage.rings.polynomial.multi_polynomial_ring_base.MPolynomialRing_base method), 306
remove_zeros() (sage.rings.polynomial.polydict.PolyDict method), 472
repr_long() (sage.rings.polynomial.multi_polynomial_ring_base.MPolynomialRing_base method), 307
require_field (in module sage.rings.polynomial.multi_polynomial_multi_polynomial_ideal), 408
RequireField (class in sage.rings.polynomial.multi_polynomial_multi_polynomial_ideal), 408
res() (sage.rings.polynomial.padic.polynomial_padic_capped_relative_dense.Polynomial_padic_capped_relative_dense method), 193
rescale() (sage.rings.polynomial.padic.polynomial_padic_capped_relative_dense.Polynomial_padic_capped_relative_dense method), 193
reset() (sage.rings.polynomial.symmetric_reduction.SymmetricReductionStrategy method), 620
reset_root_width() (sage.rings.polynomial.real_roots.island method), 216
reset_root_width() (sage.rings.polynomial.real_roots.ocean method), 220
residue() (sage.rings.polynomial.laurent_polynomial.LaurentPolynomial_univariate method), 579
residue_class_degree() (sage.rings.polynomial.ideal.Ideal_1poly_field method), 233
residue_field() (sage.rings.polynomial.multi_polynomial.ideal.Ideal_1poly_field method), 233
residue_field() (sage.rings.polynomial.multi_polynomial.multi_polynomial_element.MPolynomial_polydict method), 352
resultant() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_libsingular method), 452
resultant() (sage.rings.polynomial.polynomial_element.Polynomial_element method), 93
resultant() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 144
resultant() (sage.rings.polynomial.polynomial_integer_dense_ntl.Polynomial_integer_dense_ntl method), 150
resultant() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_modn_dense_mod_p method), 174
resultant() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 161
resultant() (sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint method), 169
resultant() (sage.rings.polynomial.polynomial_zz_pex.Polynomial_ZZ_pEX method), 197
retract() (sage.rings.polynomial.polynomial_quotient_ring.Polynomial_quotient_ring_generic method), 251
reverse() (sage.rings.polynomial.padic.polynomial_padic_capped_relative_dense.Polynomial_padic_capped_relative_dense method), 193
reverse() (sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_sparse method), 129
reverse() (sage.rings.polynomial.polynomial_element.Polynomial method), 94
reverse() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 144
reverse() (sage.rings.polynomial.polynomial_integer_dense_ntl.Polynomial_integer_dense_ntl method), 150
resultant() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_modn_dense_mod_p method), 174
resultant() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 161
resultant() (sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint method), 169
resultant() (sage.rings.polynomial.polynomial_zz_pex.Polynomial_ZZ_pEX method), 197
retract() (sage.rings.polynomial.polynomial_quotient_ring.Polynomial_quotient_ring_generic method), 251
reverse() (sage.rings.polynomial.padic.polynomial_padic_capped_relative_dense.Polynomial_padic_capped_relative_dense method), 193
reverse() (sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_sparse method), 129
reverse() (sage.rings.polynomial.polynomial_element.Polynomial method), 94
reverse() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 144
resultant() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_modn_dense_mod_p method), 174
resultant() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 161
resultant() (sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint method), 169
resultant() (sage.rings.polynomial.polynomial_zz_pex.Polynomial_ZZ_pEX method), 197
retract() (sage.rings.polynomial.polynomial_quotient_ring.Polynomial_quotient_ring_generic method), 251
reverse() (sage.rings.polynomial.padic.polynomial_padic_capped_relative_dense.Polynomial_padic_capped_relative_dense method), 193
reverse() (sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_sparse method), 129
reverse() (sage.rings.polynomial.polynomial_element.Polynomial method), 94
reverse() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 144
resultant() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_modn_dense_mod_p method), 174
resultant() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 161
resultant() (sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint method), 169
resultant() (sage.rings.polynomial.polynomial_zz_pex.Polynomial_ZZ_pEX method), 197
retract() (sage.rings.polynomial.polynomial_quotient_ring.Polynomial_quotient_ring_generic method), 251
reverse() (sage.rings.polynomial.padic.polynomial_padic_capped_relative_dense.Polynomial_padic_capped_relative_dense method), 193
reverse() (sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_sparse method), 129
reverse() (sage.rings.polynomial.polynomial_element.Polynomial method), 94
reverse() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 144
resultant() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_modn_dense_mod_p method), 174
resultant() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 161
resultant() (sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint method), 169
resultant() (sage.rings.polynomial.polynomial_zz_pex.Polynomial_ZZ_pEX method), 197
retract() (sage.rings.polynomial.polynomial_quotient_ring.Polynomial_quotient_ring_generic method), 251
reverse() (sage.rings.polynomial.padic.polynomial_padic_capped_relative_dense.Polynomial_padic_capped_relative_dense method), 193
reverse() (sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_sparse method), 129
reverse() (sage.rings.polynomial.polynomial_element.Polynomial method), 94
reverse() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 144
resultant() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_modn_dense_mod_p method), 174
resultant() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 161
resultant() (sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint method), 169
resultant() (sage.rings.polynomial.polynomial_zz_pex.Polynomial_ZZ_pEX method), 197
retract() (sage.rings.polynomial.polynomial_quotient_ring.Polynomial_quotient_ring_generic method), 251
reverse() (sage.rings.polynomial.padic.polynomial_padic_capped_relative_dense.Polynomial_padic_capped_relative_dense method), 193
reverse() (sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_sparse method), 129
reverse() (sage.rings.polynomial.polynomial_element.Polynomial method), 94
reverse() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 144
resultant() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_modn_dense_mod_p method), 174
resultant() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 161
resultant() (sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint method), 169
resultant() (sage.rings.polynomial.polynomial_zz_pex.Polynomial_ZZ_pEX method), 197
retract() (sage.rings.polynomial.polynomial_quotient_ring.Polynomial_quotient_ring_generic method), 251
reverse() (sage.rings.polynomial.padic.polynomial_padic_capped_relative_dense.Polynomial_padic_capped_relative_dense method), 193
reverse() (sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_sparse method), 129
reverse() (sage.rings.polynomial.polynomial_element.Polynomial method), 94
reverse() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 144
resultant() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_modn_dense_mod_p method), 174
resultant() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 161
resultant() (sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint method), 169
resultant() (sage.rings.polynomial.polynomial_zz_pex.Polynomial_ZZ_pEX method), 197
retract() (sage.rings.polynomial.polynomial_quotient_ring.Polynomial_quotient_ring_generic method), 251
reverse() (sage.rings.polynomial.padic.polynomial_padic_capped_relative_dense.Polynomial_padic_capped_relative_dense method), 193
reverse() (sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_sparse method), 129
reverse() (sage.rings.polynomial.polynomial_element.Polynomial method), 94
reverse() (sage.rings.polynomial.polynomial_integer_dense_flin
ger_dense_flint.Polynomial_integer_dense_flint

reverse() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_dense_modn_ntl_ZZ method), 175

reverse() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_dense_modn_ntl_zz method), 178

reverse() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 162

reverse() (sage.rings.polynomial.polynomial_real_mpfr_dense.PolynomialRealDense method), 183

reverse() (sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint method), 170

reverse_intvec() (in module sage.rings.polynomial.real_roots), 225

reversed() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 145

revert_series() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 145

revert_series() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 162

revert_series() (sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint method), 169

revert_series() (sage.rings.polynomial.polynomial_zz_pex.Polynomial_zz_pex method), 197

ring() (in module sage.rings.polynomial.real_roots), 225

ring() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 145

ring() (sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint method), 170

rich_compare() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 145

ring() (sage.rings.ring.RealField method), 403

ring() (sage.rings.ring.RealField method), 403

ring() (sage.rings.ring.RealField method), 403

ring() (sage.rings.ring.RealField method), 403

ring() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 145

ring() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 145

ring_of_integers() (sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint method), 145

root_bounds() (in module sage.rings.polynomial.real_roots), 226

root_field() (sage.rings.polynomial.padics.polynomial_padic_capped_relative_dense.Polynomial_padic_capped_relative_dense method), 194

S

S (sage.rings.polynomial.integer_valued_polynomials.IntegerValuedPolynomialRing attribute), 266

S_class_group() (sage.rings.polynomial.polynomial_quotient_ring_quotient_ring.Generic method), 240

S_invariant() (sage.rings.invariants.invariant_theory.TernaryCubic method), 510

S_units() (sage.rings.polynomial.polynomial_quotient_ring_quotient_ring.Generic method), 242

Sage.rings.fraction_field module, 541

Sage.rings.fraction_field_element module, 547

Sage.rings.fraction_field_FpT module, 551

Sage.rings.invariants.invariant_theory module, 480

Sage.rings.invariants.reconstruction module, 523

Sage.rings.monomials module, 479

Sage.rings.polynomial.complex_roots module, 229

Sage.rings.polynomial.convolution module, 270

Sage.rings.polynomial.cyclotomic module, 271

Sage.rings.polynomial.flatten module, 476

Sage.rings.polynomial.hilbert module, 475

Sage.rings.polynomial.ideal module, 232

Sage.rings.polynomial.polynomial_flatten module, 597

Sage.rings.polynomial.polynomial_integer Dense module, 587

Sage.rings.polynomial.polynomial_integer_valued module, 657

Index
Polynomials, Release 10.3

module, 261
sage.rings.polynomial.laurent_polynomial
module, 570
sage.rings.polynomial.laurent_polynomial_ring
module, 565
sage.rings.polynomial.laurent_polynomial_ring_base
module, 561
sage.rings.polynomial.msolve
module, 458
sage.rings.polynomial.multi_polynomial
module, 309
sage.rings.polynomial.multi_polynomial_element
module, 338
sage.rings.polynomial.multi_polynomial_ideal
module, 356
sage.rings.polynomial.multi_polynomial_ideal_libsingular
module, 457
sage.rings.polynomial.multi_polynomial_libsingular
module, 426
sage.rings.polynomial.multi_polynomial_ring
module, 334
sage.rings.polynomial.multi_polynomial_ring_base
module, 296
sage.rings.polynomial.multi_polynomial_sequence
module, 409
sage.rings.polynomial.omega
module, 581
sage.rings.polynomial.padics.polynomial_padic
module, 185
sage.rings.polynomial.padics.polynomial_padic_capped_relative_dense
module, 189
sage.rings.polynomial.padics.polynomial_padic_flat
module, 195
sage.rings.polynomial.polydict
module, 459
sage.rings.polynomial.polynomial_complied
module, 260
sage.rings.polynomial.polynomial_element
module, 33
sage.rings.polynomial.polynomial_element_generic
module, 121
sage.rings.polynomial.polynomial_fate
module, 261
sage.rings.polynomial.polynomial_gf2x
module, 131
sage.rings.polynomial.polynomial_integer_dense_flint
module, 138
sage.rings.polynomial.polynomial_integer_dense_ntl
module, 147
sage.rings.polynomial.polynomial_modn_dense_ntl
module, 171
sage.rings.polynomial.polynomial_number_field
module, 136
sage.rings.polynomial.polynomial_quotient_ring
module, 233
sage.rings.polynomial.polynomial_quotient_ring_element
module, 254
sage.rings.polynomial.polynomial_rational_flint
module, 152
sage.rings.polynomial.polynomial_real_mpfr_dense
module, 181
sage.rings.polynomial.polynomial_ring
module, 9
sage.rings.polynomial.polynomial_ring_constructor
module, 1
sage.rings.polynomial.polynomial_ring_homomorphism
module, 32
sage.rings.polynomial.polynomial_singular_interface
module, 184
sage.rings.polynomial.polynomial_zmod_flint
module, 163
sage.rings.polynomial.polynomial_zz_pex
module, 196
sage.rings.polynomial.real_roots
module, 200
sage.rings.polynomial.refine_root
module, 232
mial.polynomial_element_generic.Polynomial_element_generic_cdv method, 124
slope_range() (sage.rings.polynomial.real_roots.interval_bernstein_polynomial_float method), 211
slope_range() (sage.rings.polynomial.real_roots.interval_bernstein_polynomial_integer method), 214
small_roots() (in module sage.rings.polynomial.polynomial_modn_dense_ntl), 179
small_roots() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_modn_dense_ntl method), 173
small_roots() (sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint method), 170
solve() (sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_gf2 method), 424
some_elements() (sage.rings.fraction_field.FractionField_element method), 554
some_elements() (sage.rings.fraction_field.FractionField_generic method), 546
some_elements() (sage.rings.polynomial.multi_polynomial_ring_base.MPolynomialRing_base method), 307
some_elements() (sage.rings.polynomial.polynomial_ring.PolynomialRing_general method), 28
sortkey (sage.rings.polynomial.term_order.TermOrder method), 290
sortkey_block() (sage.rings.polynomial.term_order.TermOrder method), 290
sortkey_deglex() (sage.rings.polynomial.term_order.TermOrder method), 290
sortkey_degneglex() (sage.rings.polynomial.term_order.TermOrder method), 291
sortkey_degrevlex() (sage.rings.polynomial.term_order.TermOrder method), 291
sortkey_invlex() (sage.rings.polynomial.term_order.TermOrder method), 292
sortkey_lex() (sage.rings.polynomial.term_order.TermOrder method), 292
sortkey_matrix() (sage.rings.polynomial.term_order.TermOrder method), 292
sortkey_negdeglex() (sage.rings.polynomial.term_order.TermOrder method), 292
sortkey_negdegrevlex() (sage.rings.polynomial.term_order.TermOrder method), 292
sortkey_neglex() (sage.rings.polynomial.term_order.TermOrder method), 293
sortkey_negwdeglex() (sage.rings.polynomial.term_order.TermOrder method), 293
sortkey_negwdegrevlex() (sage.rings.polynomial.term_order.TermOrder method), 293
sortkey_wdeglex() (sage.rings.polynomial.term_order.TermOrder method), 294
sortkey_wdegrevlex() (sage.rings.polynomial.term_order.TermOrder method), 294
sparse_iter() (sage.rings.polynomial.polydict.ETuple method), 465
specialization() (sage.rings.rational_field.RationalField method), 28
specialization() (sage.rings.rational_field.RationalField method), 459
specialization() (sage.rings.rational_field.RationalField method), 549
specialization() (sage.rings.rational_field.RationalField method), 330
specialization() (sage.rings.rational_field.RationalField method), 106
SpecializationMorphism (class in sage.rings.polynomial.polynomial_flatten), 478
split_for_targets() (in module sage.rings.polynomial.real_roots), 226
splitting_field() (sage.rings.polynomial.polynomial_integer_dense_ntl PolynomialIntegerDenseNTL method), 107
spol() (in module sage.rings.polynomial.toy_buchberger), 531
spol() (in module sage.rings.polynomial.toy_d_basis), 539
sqr pd (class in sage.rings.polynomial.polynomial_composite), 260
sqrt() (sage.rings.fraction_field.FractionFieldElement method), 554
square() (sage.rings.polynomial.polynomial_element.Polynomial method), 28
square() (sage.rings.polynomial.polynomial_element.Polynomial method), 108
squarefree_decomposition() (sage.rings.polynomial.polynomial_element.Polynomial method), 109
squarefree_decomposition() (sage.rings.polynomial.polynomial_integer_dense_flint PolynomialIntegerDenseFlint method), 145
squarefree_decomposition() (sage.rings.polynomial.polynomial_integer_dense_ntl PolynomialIntegerDenseNTL method), 151
squarefree_decomposition() (sage.rings.polynomial.polynomial_modn_dense_flint PolynomialModnDenseFlint method), 171
squeezed() (sage.rings.polynomial.infinity_polynomial_element.InfinitePolynomial method), 603
squeezed() (sage.rings.polynomial.symmetricIdeal.SymmetricIdeal method), 615
std() (sage.rings.polynomial.multi_polynomial.NCPolynomialIdeal method), 405
std_libsingular() (in module sage.rings.polynomial.multi_polynomial_libsingular PolynomialLibSingular method), 458
stretch() (sage.rings.polynomial.infinity_polynomial_element.InfinitePolynomial method), 603
sub m mul q() (sage.rings.polynomial.multi_polynomial_libsingular PolynomialLibSingular method), 453
subresultants() (sage.rings.polynomial.multi_poly-
polynomial_element.MPolynomial_polydict method), 353
subresultants() (sage.rings.polynomial.multi_polynomial_element.MPolynomial method), 331
subresultants() (sage.rings.polynomial.polynomial_element.Polynomial method), 109
subs() (sage.rings.fraction_field_element.FractionFieldElement method), 549
subs() (sage.rings.fraction_field_element.FractionField_FpT.FpTElement method), 554
subs() (sage.rings.polynomial.infinite_polynomial_element.InfinitePolynomial method), 604
subs() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict method), 353
subs() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_libsingular.MPolynomial_libsingular method), 453
subs() (sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic method), 420
subs() (sage.rings.polynomial.polynomial_element.Polynomial method), 109
subsample_vec_doctest() (in module sage.rings.polynomial.real_roots), 227
sum() (sage.rings.polynomial.multi_polynomial_ring.MPolynomialRing_polydict method), 337
super_categories() (sage.rings.polynomial.integer_valued_polynomials.IntegerValuedPolynomialRing.Bases method), 264
support() (sage.rings.fraction_field_element.FractionFieldElement_Lpoly_field method), 550
sylvester_matrix() (sage.rings.polynomial.multi_polynomial_element.MPolynomial method), 331
sylvester_matrix() (sage.rings.polynomial.polynomial_element.Polynomial method), 110
symmetric_basis() (sage.rings.polynomial.symmetricIdeal.SymmetricIdeal method), 615
symmetric_cancellation_order() (sage.rings.polynomial.infinite_polynomial_element.InfinitePolynomial method), 605
symmetric_power() (sage.rings.polynomial.polynomial_element.Polynomial method), 111
SymmetricIdeal (class in sage.rings.polynomial.symmetricIdeal), 607
SymmetricReductionStrategy (class in sage.rings.polynomial.symmetric_reduction), 618
symmetrisation() (sage.rings.polynomial.symmetricIdeal.SymmetricIdeal method), 615
syzygy() (sage.rings.invariants.invariant_theory.TernaryCubic method), 513
syzygy() (sage.rings.invariants.invariant_theory.TwoQuaternaryQuadratics method), 518
syzygy() (sage.rings.invariants.invariant_theory.TwoTernaryQuadratics method), 520
syzygy_module() (sage.rings.polynomial.multi_polynomial_ideal.MPolynomialIdeal_singular_base_repr method), 377
syzygy_module() (sage.rings.polynomial.multi_polynomial_ideal.MPolynomialIdeal_singular_repr method), 397
syzygy_module() (sage.rings.polynomial.multi_polynomial_ideal.NCPolynomialIdeal method), 406

T

covariant() (sage.rings.invariants.invariant_theory.BinaryQuintic method), 489
covariant() (sage.rings.invariants.invariant_theory.TwoQuaternaryQuadratics method), 517
covariant() (sage.rings.invariants.invariant_theory.TernaryCubic method), 511
covariant() (sage.rings.invariants.invariant_theory.TwoQuaternaryQuadratics method), 517
tail() (sage.rings.polynomial.infinite_polynomial_element.InfinitePolynomial method), 605
tailreduce() (sage.rings.polynomial.symmetric_reduction.SymmetricReductionStrategy method), 621
tau_covariant() (sage.rings.invariants.invariant_theory.BinaryQuintic method), 495
taylor_shift1_intvec() (in module sage.rings.polynomial.real_roots), 227
tensor_with_ring() (sage.rings.polynomial.infinite_polynomial_ring.InfinitePolynomialRing_dense method), 592
tensor_with_ring() (sage.rings.polynomial.infinite_polynomial_ring.InfinitePolynomialRing_sparse method), 596
term_lmult() (sage.rings.polynomial.polynomial.PolyDict method), 473
term_order() (sage.rings.polynomial.laurent_polynomial.laurent_polynomial_laurent_ring_base.LaurentPolynomialRing_generic method), 564
term_lmult() (sage.rings.polynomial.multi_polynomial_ring.laurent_ring_base.MPolynomialRing_base method), 308
term_rmult() (sage.rings.polynomial.polynomial.PolyDict method), 474
TermOrder (class in sage.rings.polynomial.term_order), 280

termorder_from_singular() (in module sage.rings.polynomial.term_order), 295

ternary_biquadratic() (sage.rings.invariants.invariant_theory.InvariantTheoryFactory method), 503
ternary_cubic() (sage.rings.invariants.invariant_theory.InvariantTheoryFactory method), 504

ternary_quadratic() (sage.rings.invariants.invariant_theory.InvariantTheoryFactory method), 504

TernaryCubic (class in sage.rings.invariants.invariant_theory), 510

TernaryQuadratic (class in sage.rings.invariants.invariant_theory), 513

theta_covariant() (sage.rings.invariants.invariant_theory.BinaryQuintic method), 496

Theta_covariant() (sage.rings.invariants.invariant_theory.TernaryCubic method), 511

Theta_invariant() (sage.rings.invariants.invariant_theory.TwoQuaternaryQuadratics method), 517

Theta_invariant() (sage.rings.invariants.invariant_theory.TwoTernaryQuadratics method), 520

Theta_prime_invariant() (sage.rings.invariants.invariant_theory.TwoQuaternaryQuadratics method), 518

Theta_prime_invariant() (sage.rings.invariants.invariant_theory.TwoTernaryQuadratics method), 520

to_bernstein() (in module sage.rings.polynomial.real_roots), 227

to_bernstein_warp() (in module sage.rings.polynomial.real_roots), 228

to_ocean() (sage.rings.polynomial.real_roots.linear_map method), 216

to_ocean() (sage.rings.polynomial.real_roots.warp_map method), 228

total_degree() (sage.rings.polynomial.multi_polynomial_element.MPolynomial_persistence method), 354

total_degree() (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method), 454

total_degree() (sage.rings.polynomial.polydict.PolyDict method), 474

trace() (sage.rings.polynomial.polynomial_quotient_ring_element.PolynomialQuotientRingElement method), 259

trace_polynomial() (sage.rings.polynomial.polynomial_element.Polynomial method), 111

transformed() (sage.rings.invariants.invariant_theory.AlgebraicForm method), 483

transformed_basis() (sage.rings.polynomial.multi_polynomial_ideal.MPolynomial_ideal Singular method), 397

transvectant() (in module sage.rings.invariants.invariant_theory), 521

triangular_decomposition() (sage.rings.polynomial.multi_polynomial_ideal.MPolynomial_ideal Singular method), 398

triangular_factorization() (in module sage.rings.polynomial.toy_variety), 535

truncate() (sage.rings.polynomial.laurent_polynomial.LaurentPolynomial_univariate method), 579

truncate() (sage.rings.polynomial.multi_polynomial.MPolynomial method), 332

truncate() (sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_sparse method), 130

truncate() (sage.rings.polynomial.polynomial_element.Polynomial method), 112

truncate() (sage.rings.polynomial.polynomial_element.Polynomial_generic_dense method), 117

truncate() (sage.rings.polynomial.polynomial_modn_dense_ntl.Polynomial_dense_modn_nTL method), 176

truncate() (sage.rings.polynomial.polynomial_modn_dense_modn_nTL method), 178

truncate() (sage.rings.polynomial.polynomial_rational_flint.Polynomial_rational_flint method), 163

truncate() (sage.rings.polynomial.polynomial_real_mpfr_dense.PolynomialRealDense method), 183

truncate() (sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint method), 165

truncate() (sage.rings.polynomial.polynomial_zz_pex.Polynomial_zz_pex method), 200

truncate_abs() (sage.rings.polynomial.polynomial_real_mpfr_dense.PolynomialRealDense method), 184

try_rand_split() (sage.rings.polynomial.real_roots.interval_bernstein_polynomial method), 209

try_split() (sage.rings.polynomial.real_roots.interval_bernstein_polynomial method), 209
tuple_weight() (sage.rings.polynomial.term_order.TermOrder method), 295
TwoAlgebraicForms (class in sage.rings.invariants.invariant_theory), 514
TwoQuaternaryQuadratics (class in sage.rings.invariants.invariant_theory), 515
twostd() (sage.rings.polynomial.multi_polynomial_ideal.NCPolynomialIdeal method), 407
TwoTernaryQuadratics (class in sage.rings.invariants.invariant_theory), 519

U
umbra() (sage.rings.polynomial.integer_valued_polynomials.IntegerValuedPolynomialRing.Shifted.Element method), 269
unary_pd (class in sage.rings.polynomial.polynomial_compiled), 260
UnflatteningMorphism (class in sage.rings.polynomial.flatten), 479
units() (sage.rings.polynomial.polynomial_quotient_ring.PolynomialQuotientRing_generic method), 253
univar_pd (class in sage.rings.polynomial.polynomial_compiled), 261
univariate_polynomial() (sage.rings.polynomial.multi_polynomial_element.MPolynomialPolydict method), 354
univariate_polynomial() (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method), 455
univariate_ring() (sage.rings.polynomial.multi_polynomial_ring_base.MPolynomialRing_base method), 308
universal_discriminant() (in module sage.rings.polynomial.polynomial_element), 121
universal() (sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic method), 421
unpickle_FpT_element() (in module sage.rings.fraction_field_FpT), 559
unpickle_MPolynomial_libsingular() (in module sage.rings.polynomial.multi_polynomial_libsingular), 456
unpickle_MPolynomialRing_generic() (in module sage.rings.polynomial.multi_polynomial_ring_base), 308
unpickle_MPolynomialRing_generic_v1() (in module sage.rings.polynomial.multi_polynomial_ring_base), 308
unpickle_MPolynomialRing_libsingular() (in module sage.rings.polynomial.multi_polynomial_libsingular), 456
unpickle_PolynomialRing() (in module sage.rings.polynomial.polynomial_ring_constructor), 8
unweighted_degree() (sage.rings.polynomial.polynomial_dict.ETuple method), 465
unweighted_quotient_degree() (sage.rings.polynomial.polynomial_dict.ETuple method), 465
update() (in module sage.rings.polynomial.toy_buchberger), 531
update() (in module sage.rings.polynomial.toy_d_basis), 539
usign() (sage.rings.polynomial.real_roots.bernstein_polynomial_factory method), 202

V
valuation() (sage.rings.fraction_field_element.FractionFieldElement method), 549
valuation() (sage.rings.fraction_field_FpT.FpTElement method), 555
valuation() (sage.rings.polynomial.laurent_polynomial.LaurentPolynomial_univariate method), 580
valuation() (sage.rings.polynomial.padics.polynomial_padic_capped_relative_dense.PolynomialPadicCappedRelativeDense method), 194
valuation() (sage.rings.polynomial.polynomial_element_generic.Polynomial_generic_sparse method), 130
valuation() (sage.rings.polynomial.polynomial_element.Polynomial method), 112
valuation() (sage.rings.polynomial.polynomial_modn_dense_ntl.PolynomialModnDenseModnntlZZ method), 176
valuation() (sage.rings.polynomial.polynomial_modn_dense_ntl.PolynomialModnDenseModnntlZZ method), 179
valuation_of_coefficient() (sage.rings.polynomial.padics.polynomial_padic_capped_relative_dense.PolynomialPadicCappedRelativeDense method), 195
var_pd (class in sage.rings.polynomial.polynomial_compiled), 261
variable() (sage.rings.polynomial.multi_polynomial_element.MPolynomialPolydict method), 355
variable() (sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method), 456
variable_name() (sage.rings.polynomial.laurent_polynomial.LaurentPolynomial_univariate method), 580
variable_name() (sage.rings.polynomial.polynomial_element.Polynomial method), 113

Index 663
<table>
<thead>
<tr>
<th>Method</th>
<th>Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>variable_names_recursive()</td>
<td>sage.rings.polynomial.laurent_polynomial_ring_base.LaurentPolynomialRing_generic method, 565</td>
</tr>
<tr>
<td>variable_names_recursive()</td>
<td>sage.rings.polynomial.multi_polynomial_ring_base.MPolynomialRing_base method, 308</td>
</tr>
<tr>
<td>variable_names_recursive()</td>
<td>sage.rings.polynomial.polynomial_ring.PolynomialRing_general method, 29</td>
</tr>
<tr>
<td>variable_shift()</td>
<td>sage.rings.polynomial.integer_valued_polynomials.IntegerValuedPolynomialRing.Binomial.Element method, 265</td>
</tr>
<tr>
<td>variable_shift()</td>
<td>sage.rings.polynomial.integer_valued_polynomials.IntegerValuedPolynomialRing.Shifted.Element method, 269</td>
</tr>
<tr>
<td>variables()</td>
<td>sage.rings.invariants.invariant_theory.FormsBase method, 497</td>
</tr>
<tr>
<td>variables()</td>
<td>sage.rings.polynomial.infinite_polynomial.InfinitePolynomial method, 606</td>
</tr>
<tr>
<td>variables()</td>
<td>sage.rings.polynomial.laurent_polynomial.LaurentPolynomial_univariate method, 580</td>
</tr>
<tr>
<td>variables()</td>
<td>sage.rings.polynomial.multi_polynomial_element.MPolynomial_polydict method, 355</td>
</tr>
<tr>
<td>variables()</td>
<td>sage.rings.polynomial.multi_polynomial_libsingular.MPolynomial_libsingular method, 456</td>
</tr>
<tr>
<td>variables()</td>
<td>sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic method, 421</td>
</tr>
<tr>
<td>variables()</td>
<td>sage.rings.polynomial.polynomial_element.Polynomial method, 113</td>
</tr>
<tr>
<td>variations()</td>
<td>sage.rings.polynomial.real_roots.interval_bernstein_polynomial method, 209</td>
</tr>
<tr>
<td>variety()</td>
<td>in module sage.rings.polynomial.msolve, 459</td>
</tr>
<tr>
<td>variety()</td>
<td>sage.rings.polynomial.multi_polynomial_ideal.MPolynomialIdeal_singular_repr method, 399</td>
</tr>
<tr>
<td>varname_key()</td>
<td>sage.rings.polynomial.infinite_polynomial_ring.InfinitePolynomialRing_sparse method, 596</td>
</tr>
<tr>
<td>vector_space_dimension()</td>
<td>sage.rings.polynomial.multi_polynomial_ideal.MPolynomialIdeal_singular_repr method, 403</td>
</tr>
<tr>
<td>weighted_degree()</td>
<td>sage.rings.polynomial.polydict.ETuple method, 465</td>
</tr>
<tr>
<td>weighted_quotient_degree()</td>
<td>sage.rings.polynomial.polydict.ETuple method, 466</td>
</tr>
<tr>
<td>weights()</td>
<td>sage.rings.polynomial.term_order.TermOrder method, 295</td>
</tr>
<tr>
<td>weil_polynomials()</td>
<td>sage.rings.polynomial.multi_polynomial_ring_integral_domain method, 30</td>
</tr>
<tr>
<td>weil_restriction()</td>
<td>sage.rings.polynomial.multi_polynomialIdeal.MPolynomialIdeal method, 373</td>
</tr>
<tr>
<td>weil_restriction()</td>
<td>sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_gf2e method, 425</td>
</tr>
<tr>
<td>weyl_algebra()</td>
<td>sage.rings.polynomial.multi_polynomial_ring_base.MPolynomialRing_base method, 308</td>
</tr>
<tr>
<td>weyl_algebra()</td>
<td>sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_gf2e_method, 425</td>
</tr>
<tr>
<td>weyl_algebra()</td>
<td>sage.rings.polynomial.commutative.PolynomialRing_commutative method, 12</td>
</tr>
<tr>
<td>wordsize_rational()</td>
<td>in module sage.rings.polynomial.real_roots, 228</td>
</tr>
<tr>
<td>xgcd()</td>
<td>sage.rings.polynomial.laurent_polynomial.LaurentPolynomial_univariate method, 580</td>
</tr>
<tr>
<td>xgcd()</td>
<td>sage.rings.polynomial.polynomial_element.Polynomial method, 113</td>
</tr>
<tr>
<td>xgcd()</td>
<td>sage.rings.polynomial.gf2x.PolynomialTemplate method, 135</td>
</tr>
<tr>
<td>xgcd()</td>
<td>sage.rings.polynomial.integer_dense_flint.Polynomial_integer_dense_flint method, 146</td>
</tr>
<tr>
<td>xgcd()</td>
<td>sage.rings.polynomial.integer_dense_ntl.Polynomial_integer_dense_ntl method, 151</td>
</tr>
<tr>
<td>xgcd()</td>
<td>sage.rings.polynomial.modn_dense_ntl.Polynomial_dense_mod_p method, 174</td>
</tr>
<tr>
<td>xgcd()</td>
<td>sage.rings.polynomial.rational_flint.Polynomial_rational_flint method, 163</td>
</tr>
<tr>
<td>xgcd()</td>
<td>sage.rings.polynomial.polynomial_template method, 166</td>
</tr>
<tr>
<td>xgcd()</td>
<td>sage.rings.polynomial.polynomial_template method, 200</td>
</tr>
<tr>
<td>ZZ_FpT_coerce</td>
<td>class in sage.rings.fraction_field_FpT, 558</td>
</tr>
</tbody>
</table>