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BASE CLASSES FOR RINGS, ALGEBRAS AND FIELDS

1.1 Rings

This module provides the abstract base class `Ring` from which all rings in Sage (used to) derive, as well as a selection of more specific base classes.

**Warning:** Those classes, except maybe for the lowest ones like `Ring`, `CommutativeRing`, `Algebra` and `CommutativeAlgebra`, are being progressively deprecated in favor of the corresponding categories, which are more flexible, in particular with respect to multiple inheritance.

The class inheritance hierarchy is:

- `Ring`
  - `Algebra`
  - `CommutativeRing`
    - `NoetherianRing`
    - `CommutativeAlgebra`
    - `IntegralDomain`
      - `DedekindDomain`
      - `PrincipalIdealDomain`

Subclasses of `PrincipalIdealDomain` are

- `EuclideanDomain`
- `Field`
  - `FiniteField`

Some aspects of this structure may seem strange, but this is an unfortunate consequence of the fact that Cython classes do not support multiple inheritance. Hence, for instance, `Field` cannot be a subclass of both `NoetherianRing` and `PrincipalIdealDomain`, although all fields are Noetherian PIDs.

(A distinct but equally awkward issue is that sometimes we may not know in advance whether or not a ring belongs in one of these classes; e.g., some orders in number fields are Dedekind domains, but others are not, and we still want to offer a unified interface, so orders are never instances of the `DedekindDomain` class.)

**AUTHORS:**

- David Harvey (2006-10-16): changed `CommutativeAlgebra` to derive from `CommutativeRing` instead of from `Algebra`. 
class sage.rings.ring.Algebra
    Bases: sage.rings.ring.Ring

Generic algebra

characteristic()
    Return the characteristic of this algebra, which is the same as the characteristic of its base ring.

    See objects with the base_ring attribute for additional examples. Here are some examples that explicitly use the Algebra class.

    EXAMPLES:

    sage: A = Algebra(ZZ); A
    <sage.rings.ring.Algebra object at ...>
    sage: A.characteristic()
    0
    sage: A = Algebra(GF(7^3, 'a'))
    sage: A.characteristic()
    7

has_standard_involution()
    Return True if the algebra has a standard involution and False otherwise. This algorithm follows Algorithm 2.10 from John Voight’s Identifying the Matrix Ring. Currently the only type of algebra this will work for is a quaternion algebra. Though this function seems redundant, once algebras have more functionality, in particular have a method to construct a basis, this algorithm will have more general purpose.

    EXAMPLES:

    sage: B = QuaternionAlgebra(2)
    sage: B.has_standard_involution()
    True
    sage: R.<x> = PolynomialRing(QQ)
    sage: K.<u> = NumberField(x**2 - 2)
    sage: A = QuaternionAlgebra(K,-2,5)
    sage: A.has_standard_involution()
    True
    sage: L.<a,b> = FreeAlgebra(QQ,2)
    sage: L.has_standard_involution()
    Traceback (most recent call last):
    ... NotImplementedError: has_standard_involution is not implemented for this algebra

class sage.rings.ring.CcommutativeAlgebra
    Bases: sage.rings.ring.CommutativeRing

Generic commutative algebra

is_commutative()
    Return True since this algebra is commutative.

    EXAMPLES:

    Any commutative ring is a commutative algebra over itself:
Trying to create a commutative algebra over a non-commutative ring will result in a `TypeError`.

**class** `sage.rings.ring.CommutativeRing`

Bases: `sage.rings.ring.Ring`

Generic commutative ring.

**derivation** (`arg=None, twist=None`)

Return the twisted or untwisted derivation over this ring specified by `arg`.

**Note:** A twisted derivation with respect to $\theta$ (or a $\theta$-derivation for short) is an additive map $d$ satisfying the following axiom for all $x, y$ in the domain:

$$d(xy) = \theta(x)d(y) + d(x)y.$$  

**INPUT:**

- `arg` – (optional) a generator or a list of coefficients that defines the derivation
- `twist` – (optional) the twisting homomorphism

**EXAMPLES:**

```python
sage: R.<x,y,z> = QQ[]
sage: R.derivation()
d/dx
```

In that case, `arg` could be a generator:

```python
sage: R.derivation(y)
d/dy
```

or a list of coefficients:

```python
sage: R.derivation([1,2,3])
d/dx + 2*d/dy + 3*d/dz
```

It is not possible to define derivations with respect to a polynomial which is not a variable:

```python
sage: R.derivation(x^2)
Traceback (most recent call last):
...
ValueError: unable to create the derivation
```

Here is an example with twisted derivations:

```python
sage: R.<x,y,z> = QQ[]
sage: theta = R.hom([x^2, y^2, z^2])
sage: f = R.derivation(twist=theta); f
```
Specifying a scalar, the returned twisted derivation is the corresponding multiple of $\theta - \text{id}$:

```
sage: R.derivation(1, twist=theta)
[x |---> x^2, y |---> y^2, z |---> z^2] - id
sage: R.derivation(x, twist=theta)
x*(x |---> x^2, y |---> y^2, z |---> z^2) - id
```

**derivation_module** *(codomain=\text{None}, twist=\text{None})*

Returns the module of derivations over this ring.

**INPUT:**

- **codomain** – an algebra over this ring or a ring homomorphism whose domain is this ring or \text{None} (default: \text{None}); if it is a morphism, the codomain of derivations will be the codomain of the morphism viewed as an algebra over \text{self} through the given morphism; if \text{None}, the codomain will be this ring

- **twist** – a morphism from this ring to \text{codomain} or \text{None} (default: \text{None}); if \text{None}, the coercion map from this ring to \text{codomain} will be used

**Note:** A twisted derivation with respect to $\theta$ (or a $\theta$-derivation for short) is an additive map $d$ satisfying the following axiom for all $x, y$ in the domain:

$$d(xy) = \theta(x)d(y) + d(x)y.$$
Ring morphism:
From: Multivariate Polynomial Ring in x, y, z over Rational Field
To: Rational Field
Defn: x |--> 0
    y |--> 1
    z |--> 2

```
sage: M = R.derivation_module(ev)
sage: M
Module of derivations from Multivariate Polynomial Ring in x, y, z over Rational Field to Rational Field
```

Elements in $M$ acts as derivations at $(0, 1, 2)$:

```
sage: Dx = M.gen(0); Dx
d/dx
sage: Dy = M.gen(1); Dy
d/dy
sage: Dz = M.gen(2); Dz
d/dz
sage: f = x^2 + y^2 + z^2
sage: Dx(f) # = 2*x evaluated at (0,1,2)
0
sage: Dy(f) # = 2*y evaluated at (0,1,2)
2
sage: Dz(f) # = 2*z evaluated at (0,1,2)
4
```

An example with a twisting homomorphism:

```
sage: theta = R.hom([x^2, y^2, z^2])
sage: M = R.derivation_module(twist=theta); M
Module of twisted derivations over Multivariate Polynomial Ring in x, y, z over Rational Field (twisting morphism: x |--> x^2, y |--> y^2, z |--> z^2)
```

See also:
```
derivation()
```

```
extension(poly, name=None, names=None, **kwds)
Algebraically extends self by taking the quotient self[x] / (f(x)).
```

INPUT:
```
• poly – A polynomial whose coefficients are coercible into self
• name – (optional) name for the root of f
```

Note: Using this method on an algebraically complete field does not return this field; the construction self[x] / (f(x)) is done anyway.

EXAMPLES:

```
sage: R = QQ['x']
sage: y = polygen(R)
```

(continues on next page)
sage: R.extension(y^2 - 5, 'a')
Univariate Quotient Polynomial Ring in a over Univariate Polynomial Ring in x
˓→ over Rational Field with modulus a^2 - 5

sage: P.<x> = PolynomialRing(GF(5))
sage: F.<a> = GF(5).extension(x^2 - 2)
sage: P.<t> = F[]
sage: R.<b> = F.extension(t^2 - a); R
Univariate Quotient Polynomial Ring in b over Finite Field in a of size 5^2
˓→ with modulus b^2 + 4*a

fraction_field()
Return the fraction field of self.

EXAMPLES:

sage: R = Integers(389)[x,y]
sage: Frac(R)
Fraction Field of Multivariate Polynomial Ring in x, y over Ring of integers
˓→ modulo 389
sage: R.fraction_field()
Fraction Field of Multivariate Polynomial Ring in x, y over Ring of integers
˓→ modulo 389

frobenius_endomorphism(n=1)
INPUT:
  • n – a nonnegative integer (default: 1)

OUTPUT:
The n-th power of the absolute arithmetic Frobenius endomorphism on this finite field.

EXAMPLES:

sage: K.<u> = PowerSeriesRing(GF(5))
sage: Frob = K.frobenius_endomorphism(); Frob
Frobenius endomorphism x |--> x^5 of Power Series Ring in u over Finite Field
˓→ of size 5
sage: Frob(u)
u^5

We can specify a power:

sage: f = K.frobenius_endomorphism(2); f
Frobenius endomorphism x |--> x^(5^2) of Power Series Ring in u over Finite Field
˓→ of size 5
sage: f(1+u)
1 + u^25

ideal_monoid()
Return the monoid of ideals of this ring.

EXAMPLES:
**sage**: ZZ.ideal_monoid()
Monoid of ideals of Integer Ring

**sage**: R.<x>=QQ[]; R.ideal_monoid()
Monoid of ideals of Univariate Polynomial Ring in x over Rational Field

### is_commutative()
Return True, since this ring is commutative.

**EXAMPLES**:

```python
**sage**: QQ.is_commutative()
True
**sage**: ZpCA(7).is_commutative()
True
**sage**: A = QuaternionAlgebra(QQ, -1, -3, names=('i', 'j', 'k')); A
Quaternion Algebra (-1, -3) with base ring Rational Field
**sage**: A.is_commutative()
False
```

### krull_dimension()
Return the Krull dimension of this commutative ring.

The Krull dimension is the length of the longest ascending chain of prime ideals.

### localization(
additional_units, names=None, normalize=True, category=None)
Return the localization of self at the given additional units.

**EXAMPLES**:

```python
**sage**: R.<x, y> = GF(3)[]
**sage**: R.localization((x*y, x**2+y**2))
Multivariate Polynomial Ring in x, y over Finite Field of size 3 localized at...
˓
→(y, x, x^2 + y^2)
**sage**: ~y in _
True
```

### class sage.rings.ring.DedekindDomain
Bases: sage.rings.ring.IntegralDomain

Generic Dedekind domain class.

A Dedekind domain is a Noetherian integral domain of Krull dimension one that is integrally closed in its field of fractions.

This class is deprecated, and not actually used anywhere in the Sage code base. If you think you need it, please create a category DedekindDomains, move the code of this class there, and use it instead.

### integral_closure()
Return self since Dedekind domains are integrally closed.

**EXAMPLES**:

```python
**sage**: K = NumberField(x**2 + 1, 's')
**sage**: OK = K.ring_of_integers()
**sage**: OK.integral_closure()
Gaussian Integers in Number Field in s with defining polynomial x^2 + 1
**sage**: OK.integral_closure() == OK
True
```
sage: QQ.integral_closure() == QQ
True

is_integrally_closed()
Return True since Dedekind domains are integrally closed.

EXAMPLES:
The following are examples of Dedekind domains (Noetherian integral domains of Krull dimension one that are integrally closed over its field of fractions).

sage: ZZ.is_integrally_closed()
True
sage: K = NumberField(x^2 + 1, 's')
Sage: OK = K.ring_of_integers()
Sage: OK.is_integrally_closed()
True

These, however, are not Dedekind domains:

sage: QQ.is_integrally_closed()
True
sage: S = ZZ[sqrt(5)]; S.is_integrally_closed()
False
sage: T.<x,y> = PolynomialRing(QQ,2); T
Multivariate Polynomial Ring in x, y over Rational Field
sage: T.is_integral_domain()
True

is_noetherian()
Return True since Dedekind domains are Noetherian.

EXAMPLES:
The integers, Z, and rings of integers of number fields are Dedekind domains:

sage: ZZ.is_noetherian()
True
sage: K = NumberField(x^2 + 1, 's')
Sage: OK = K.ring_of_integers()
Sage: OK.is_noetherian()
True
sage: QQ.is_noetherian()
True

krull_dimension()
Return 1 since Dedekind domains have Krull dimension 1.

EXAMPLES:
The following are examples of Dedekind domains (Noetherian integral domains of Krull dimension one that are integrally closed over its field of fractions):

sage: ZZ.krull_dimension()
1
sage: K = NumberField(x^2 + 1, 's')
sage: OK = K.ring_of_integers()
sage: OK.krull_dimension()
1

The following are not Dedekind domains but have a krull_dimension function:

sage: QQ.krull_dimension()
0
sage: T.<x,y> = PolynomialRing(QQ,2); T
Multivariate Polynomial Ring in x, y over Rational Field
sage: T.krull_dimension()
2
sage: U.<x,y,z> = PolynomialRing(ZZ,3); U
Multivariate Polynomial Ring in x, y, z over Integer Ring
sage: U.krull_dimension()
4

sage: K.<i> = QuadraticField(-1)
sage: R = K.order(2*i); R
Order in Number Field in i with defining polynomial x^2 + 1 with i = 1*I
sage: R.is_maximal()
False
sage: R.krull_dimension()
1

class sage.rings.ring.EuclideanDomain

Bases: sage.rings.ring.PrincipalIdealDomain

Generic Euclidean domain class.

This class is deprecated. Please use the EuclideanDomains category instead.

parameter()

Return an element of degree 1.

EXAMPLES:

sage: R.<x>=QQ[]
sage: R.parameter()
x

class sage.rings.ring.Field

Bases: sage.rings.ring.PrincipalIdealDomain

Generic field

algebraic_closure()

Return the algebraic closure of self.

Note: This is only implemented for certain classes of field.

EXAMPLES:
sage: K = PolynomialRing(QQ, 'x').fraction_field(); K
Fraction Field of Univariate Polynomial Ring in x over Rational Field
sage: K.algebraic_closure()
Traceback (most recent call last):
... NotImplementedError: Algebraic closures of general fields not implemented.

\texttt{divides}(x, y, \textit{coerce=True})

Return True if $x$ divides $y$ in this field (usually True in a field!). If \textit{coerce} is True (the default), first coerce $x$ and $y$ into self.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: QQ.divides(2, 3/4)
True
sage: QQ.divides(0, 5)
False
\end{verbatim}

\texttt{fraction\_field()}

Return the fraction field of self.

\textbf{EXAMPLES:}

Since fields are their own field of fractions, we simply get the original field in return:

\begin{verbatim}
sage: QQ.fraction_field()
Rational Field
sage: RR.fraction_field()
Real Field with 53 bits of precision
sage: CC.fraction_field()
Complex Field with 53 bits of precision
sage: F = NumberField(x^2 + 1, 'i')
sage: F.fraction_field()
Number Field in i with defining polynomial x^2 + 1
\end{verbatim}

\texttt{ideal}(\textit{*gens, **kwds})

Return the ideal generated by gens.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: QQ.ideal(2)
Principal ideal (1) of Rational Field
sage: QQ.ideal(0)
Principal ideal (0) of Rational Field
\end{verbatim}

\texttt{integral\_closure()}

Return this field, since fields are integrally closed in their fraction field.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: QQ.integral_closure()
Rational Field
sage: Frac(ZZ['x,y']).integral_closure()
Fraction Field of Multivariate Polynomial Ring in x, y over Integer Ring
\end{verbatim}
is_field (proof=True)
Return True since this is a field.

EXAMPLES:

```
sage: Frac(ZZ['x,y']).is_field()
True
```

is_integrally_closed()
Return True since fields are trivially integrally closed in their fraction field (since they are their own fraction field).

EXAMPLES:

```
sage: Frac(ZZ['x,y']).is_integrally_closed()
True
```

is_noetherian()
Return True since fields are Noetherian rings.

EXAMPLES:

```
sage: QQ.is_noetherian()
True
```

krull_dimension()
Return the Krull dimension of this field, which is 0.

EXAMPLES:

```
sage: QQ.krull_dimension()
0
sage: Frac(QQ['x,y']).krull_dimension()
0
```

prime_subfield()
Return the prime subfield of self.

EXAMPLES:

```
sage: k = GF(9, 'a')
sage: k.prime_subfield()
Finite Field of size 3
```

class sage.rings.ring.IntegralDomain
Bases: sage.rings.ring.CommutativeRing

Generic integral domain class.

This class is deprecated. Please use the sage.categories.integral_domains.IntegralDomains category instead.

is_field (proof=True)
Return True if this ring is a field.

EXAMPLES:

```
sage: GF(7).is_field()
True
```
The following examples have their own `is_field` implementations:

```
sage: ZZ.is_field(); QQ.is_field()
False
True
sage: R.<x> = PolynomialRing(QQ); R.is_field()
False
```

### `is_integral_domain(proof=True)`

Return `True`, since this ring is an integral domain.

(This is a naive implementation for objects with type `IntegralDomain`)

**EXAMPLES:**

```
sage: ZZ.is_integral_domain()
True
sage: QQ.is_integral_domain()
True
sage: ZZ['x'].is_integral_domain()
True
sage: R = ZZ.quotient(ZZ.ideal(10)); R.is_integral_domain()
False
```

### `is_integrally_closed()`

Return `True` if this ring is integrally closed in its field of fractions; otherwise return `False`.

When no algorithm is implemented for this, then this function raises a `NotImplementedError`.

Note that `is_integrally_closed` has a naive implementation in fields. For every field $F$, $F$ is its own field of fractions, hence every element of $F$ is integral over $F$.

**EXAMPLES:**

```
sage: ZZ.is_integrally_closed()
True
sage: QQ.is_integrally_closed()
True
sage: QQbar.is_integrally_closed()
True
sage: GF(5).is_integrally_closed()
True
sage: Z5 = Integers(5); Z5
Ring of integers modulo 5
sage: Z5.is_integrally_closed()
Traceback (most recent call last):
...
AttributeError: 'IntegerModRing_generic_with_category' object has no attribute '__is_integrally_closed'
```

```python
class sage.rings.ring.NoetherianRing
Bases: sage.rings.ring.CommutativeRing

Generic Noetherian ring class.

A Noetherian ring is a commutative ring in which every ideal is finitely generated.

This class is deprecated, and not actually used anywhere in the Sage code base. If you think you need it, please create a category `NoetherianRings`, move the code of this class there, and use it instead.
```
is_noetherian()

Return True since this ring is Noetherian.

EXAMPLES:

```
sage: ZZ.is_noetherian()
True
sage: QQ.is_noetherian()
True
sage: R.<x> = PolynomialRing(QQ)
sage: R.is_noetherian()
True
```

class sage.rings.ring.PrincipalIdealDomain

Bases: sage.rings.ring.IntegralDomain

Generic principal ideal domain.

This class is deprecated. Please use the PrincipalIdealDomains category instead.

class_group()

Return the trivial group, since the class group of a PID is trivial.

EXAMPLES:

```
sage: QQ.class_group()
Trivial Abelian group
```

content(x, y, coerce=True)

Return the content of \( x \) and \( y \), i.e. the unique element \( c \) of self such that \( x/c \) and \( y/c \) are coprime and integral.

EXAMPLES:

```
sage: QQ.content(ZZ(42), ZZ(48)); type(QQ.content(ZZ(42), ZZ(48)))
6
<type 'sage.rings.rational.Rational'>
sage: QQ.content(1/2, 1/3)
1/6
sage: factor(1/2); factor(1/3); factor(1/6)
2^-1
3^-1
2^-1 * 3^-1
sage: a = (2*3)/(7*11); b = (13*17)/(19*23)
sage: factor(a); factor(b); factor(QQ.content(a,b))
2 * 3 * 7^-1 * 11^-1
13 * 17 * 19^-1 * 23^-1
7^-1 * 11^-1 * 19^-1 * 23^-1
```

Note the changes to the second entry:

```
sage: c = (2*3)/(7*11); d = (13*17)/(7*19*23)
sage: factor(c); factor(d); factor(QQ.content(c,d))
2 * 3 * 7^-1 * 11^-1
7^-1 * 13 * 17 * 19^-1 * 23^-1
7^-1 * 11^-1 * 19^-1 * 23^-1
sage: e = (2*3)/(7*11); f = (13*17)/(7^3*19*23)
```

(continues on next page)
sage: factor(e); factor(f); factor(QQ.content(e,f))
2 * 3 * 7^-1 * 11^-1
7^-3 * 13 * 17 * 19^-1 * 23^-1
7^-3 * 11^-1 * 19^-1 * 23^-1

gcd(x, y, coerce=True)
Return the greatest common divisor of x and y, as elements of self.

EXAMPLES:
The integers are a principal ideal domain and hence a GCD domain:

sage: ZZ.gcd(42, 48)
6
sage: ZZ.gcd(2^4*7^2*11, 2^3*11*13)
88

In a field, any nonzero element is a GCD of any nonempty set of nonzero elements. In previous versions, Sage used to return 1 in the case of the rational field. However, since trac ticket #10771, the rational field is considered as the fraction field of the integer ring. For the fraction field of an integral domain that provides both GCD and LCM, it is possible to pick a GCD that is compatible with the GCD of the base ring:

sage: QQ.gcd(1/2, 1/3)
1/6

Polynomial rings over fields are GCD domains as well. Here is a simple example over the ring of polynomials over the rationals as well as over an extension ring. Note that gcd requires x and y to be coercible:

sage: R.<x> = PolynomialRing(QQ)
sage: S.<a> = NumberField(x^2 - 2, 'a')
sage: f = (x - a)*(x + a); g = (x - a)*(x^2 - 2)
sage: print(f); print(g)
x^2 - 2
x^3 - a*x^2 - 2*x + 2*a

sage: f in R
True
sage: g in R
False
sage: R.gcd(f,g)
Traceback (most recent call last):
...
TypeError: Unable to coerce 2*a to a rational
sage: R.base_extend(S).gcd(f,g)
x^2 - 2
sage: R.base_extend(S).gcd(f, (x - a)*(x^2 - 3))
x - a
is_noetherian()  
Every principal ideal domain is noetherian, so we return True.

EXAMPLES:

```
sage: Zp(5).is_noetherian()
True
```

class sage.rings.ring.Ring
Bases: sage.structure.parent_gens.ParentWithGens

Generic ring class.

base_extend(R)

EXAMPLES:

```
sage: QQ.base_extend(GF(7))
Traceback (most recent call last):
...  
TypeError: no base extension defined
sage: ZZ.base_extend(GF(7))
Finite Field of size 7
```

category()

Return the category to which this ring belongs.

Note: This method exists because sometimes a ring is its own base ring. During initialisation of a ring \( R \), it may be checked whether the base ring (hence, the ring itself) is a ring. Hence, it is necessary that \( R.category() \) tells that \( R \) is a ring, even before its category is properly initialised.

EXAMPLES:

```
sage: FreeAlgebra(QQ, 3, 'x').category()  # todo: use a ring which is not an...
Category of algebras with basis over Rational Field
```

Since a quotient of the integers is its own base ring, and during initialisation of a ring it is tested whether the base ring belongs to the category of rings, the following is an indirect test that the category() method of rings returns the category of rings even before the initialisation was successful:

```
sage: I = Integers(15)
sage: I.base_ring()  is I
True
sage: I.category()
Join of Category of finite commutative rings
    and Category of subquotients of monoids
    and Category of quotients of semigroups
    and Category of finite enumerated sets
```

epsilonion()

Return the precision error of elements in this ring.

EXAMPLES:
For exact rings, zero is returned:

```python
sage: ZZ.epsilon()
0
```

This also works over derived rings:

```python
sage: RR['x'].epsilon()
2.22044604925031e-16
sage: QQ['x'].epsilon()
0
```

For the symbolic ring, there is no reasonable answer:

```python
sage: SR.epsilon()
Traceback (most recent call last):
...
NotImplementedError
```

### `ideal(*args, **kwds)`

Return the ideal defined by `x`, i.e., generated by `x`.

**INPUT:**

- `*x` – list or tuple of generators (or several input arguments)
- `coerce` – bool (default: `True`); this must be a keyword argument. Only set it to `False` if you are certain that each generator is already in the ring.
- `ideal_class` – callable (default: `self._ideal_class_`); this must be a keyword argument. A constructor for ideals, taking the ring as the first argument and then the generators. Usually a subclass of `Ideal_generic` or `Ideal_nc`.
- Further named arguments (such as `side` in the case of non-commutative rings) are forwarded to the ideal class.

**EXAMPLES:**

```python
sage: R.<x,y> = QQ[]
sage: R.ideal(x,y)
Ideal (x, y) of Multivariate Polynomial Ring in x, y over Rational Field
sage: R.ideal(x+y^2)
Ideal (y^2 + x) of Multivariate Polynomial Ring in x, y over Rational Field
sage: R.ideal([x^3,y^3+x^3])
Ideal (x^3, x^3 + y^3) of Multivariate Polynomial Ring in x, y over Rational Field
```

Here is an example over a non-commutative ring:
```python
sage: A = SteenrodAlgebra(2)
sage: A.ideal(A.1,A.2^2)
Twosided Ideal (Sq(2), Sq(2,2)) of mod 2 Steenrod algebra, milnor basis
sage: A.ideal(A.1,A.2^2,side='left')
Left Ideal (Sq(2), Sq(2,2)) of mod 2 Steenrod algebra, milnor basis
```

**ideal_monoid()**

Return the monoid of ideals of this ring.

**EXAMPLES:**

```python
sage: F.<x,y,z> = FreeAlgebra(ZZ, 3)
sage: I = F*[x*y+y*z,x^2+x*y-y*x-y^2]*F
sage: Q = sage.rings.ring.Ring.quotient(F,I)
sage: Q.ideal_monoid()
Monoid of ideals of Quotient of Free Algebra on 3 generators (x, y, z) over → Integer Ring by the ideal (x*y + y*z, x^2 + x*y - y*x - y^2)
sage: F.<x,y,z> = FreeAlgebra(ZZ, implementation='letterplace')
sage: I = F*[x*y+y*z,x^2+x*y-y*x-y^2]*F
sage: Q = F.quotient(I)
sage: Q.ideal_monoid()
Monoid of ideals of Quotient of Free Associative Unital Algebra on 3 generators, → (x, y, z) over Integer Ring by the ideal (x*y + y*z, x*x + x*y - y*x - y*y)
```

**is_commutative()**

Return True if this ring is commutative.

**EXAMPLES:**

```python
sage: QQ.is_commutative()
True
sage: QQ['x,y,z'].is_commutative()
True
sage: Q.<i,j,k> = QuaternionAlgebra(QQ, -1,-1)
sage: Q.is_commutative()
False
```

**is_exact()**

Return True if elements of this ring are represented exactly, i.e., there is no precision loss when doing arithmetic.

**Note:** This defaults to True, so even if it does return True you have no guarantee (unless the ring has properly overloaded this).

**EXAMPLES:**

```python
sage: QQ.is_exact()  # indirect doctest
True
sage: ZZ.is_exact()
True
sage: Qp(7).is_exact()
False
sage: Zp(7, type='capped-abs').is_exact()
False
```
**is_field**(proof=True)
Return True if this ring is a field.

**INPUT:**
- proof – (default: True) Determines what to do in unknown cases

**ALGORITHM:**
If the parameter proof is set to True, the returned value is correct but the method might throw an error. Otherwise, if it is set to False, the method returns True if it can establish that self is a field and False otherwise.

**EXAMPLES:**
```
sage: QQ.is_field()
True
sage: GF(9,'a').is_field()
True
sage: ZZ.is_field()
False
sage: QQ['x'].is_field()
False
sage: Frac(QQ['x']).is_field()
True
```
This illustrates the use of the proof parameter:
```
sage: R.<a,b> = QQ[]
sage: S.<x,y> = R.quo((b^3))
sage: S.is_field(proof = True)
Traceback (most recent call last):
... NotImplementedError
sage: S.is_field(proof = False)
False
```

**is_integral_domain**(proof=True)
Return True if this ring is an integral domain.

**INPUT:**
- proof – (default: True) Determines what to do in unknown cases

**ALGORITHM:**
If the parameter proof is set to True, the returned value is correct but the method might throw an error. Otherwise, if it is set to False, the method returns True if it can establish that self is an integral domain and False otherwise.

**EXAMPLES:**
```
sage: QQ.is_integral_domain()
True
sage: ZZ.is_integral_domain()
True
sage: ZZ['x,y,z'].is_integral_domain()
True
sage: Integers(8).is_integral_domain()
```
(continues on next page)
This illustrates the use of the \texttt{proof} parameter:

```
sage: R.<a,b> = ZZ[]
sage: S.<x,y> = R.quo((b^3))
sage: S.is_integral_domain(proof = True)
Traceback (most recent call last):
  ...  
NotImplementedError
sage: S.is_integral_domain(proof = False)
False
```

\textbf{is_noetherian}()  
Return \texttt{True} if this ring is Noetherian.  

\textbf{EXAMPLES:}

```
sage: QQ.is_noetherian()  
True
sage: ZZ.is_noetherian()  
True
```

\textbf{is_prime_field}()  
Return \texttt{True} if this ring is one of the prime fields $\mathbb{Q}$ or $\mathbb{F}_p$.  

\textbf{EXAMPLES:}

```
sage: QQ.is_prime_field()  
True
sage: GF(3).is_prime_field()  
True
sage: GF(9,'a').is_prime_field()  
False
sage: ZZ.is_prime_field()  
False
sage: QQ['x'].is_prime_field()  
False
sage: Qp(19).is_prime_field()  
False
```

\textbf{is_subring}(\texttt{other})  
Return \texttt{True} if the canonical map from \texttt{self} to \texttt{other} is injective.  

Raises a \texttt{NotImplementedError} if not known.  

\textbf{EXAMPLES:}

```python
sage: ZZ.is_subring(QQ)
True
sage: ZZ.is_subring(GF(19))
False
```

**one()**

Return the one element of this ring (cached), if it exists.

EXAMPLES:

```python
sage: ZZ.one()
1
sage: QQ.one()
1
sage: QQ['x'].one()
1
```

The result is cached:

```python
sage: ZZ.one() is ZZ.one()
True
```

**order()**

The number of elements of self.

EXAMPLES:

```python
sage: GF(19).order()
19
sage: QQ.order()
+Infinity
```

**principal_ideal**(gen, coerce=True)

Return the principal ideal generated by gen.

EXAMPLES:

```python
sage: R.<x,y> = ZZ[]
sage: R.principal_ideal(x+2*y)
Ideal (x + 2*y) of Multivariate Polynomial Ring in x, y over Integer Ring
```

**quo**(I, names=None, **kwds)

Create the quotient of \( R \) by the ideal \( I \). This is a synonym for \texttt{quotient()}.

EXAMPLES:

```python
sage: R.<x,y> = PolynomialRing(QQ,2)
sage: S.<a,b> = R.quo((x^2, y))
sage: S.gens()
(a, 0)
sage: a == b
False
```
\textbf{quotient}(I, \texttt{names} = \texttt{None}, **\texttt{kwds})

Create the quotient of this ring by a twosided ideal \( I \).

INPUT:

\begin{itemize}
  \item \( I \) – a twosided ideal of this ring, \( R \).
  \item \texttt{names} – (optional) names of the generators of the quotient (if there are multiple generators, you can specify a single character string and the generators are named in sequence starting with 0).
  \item further named arguments that may be passed to the quotient ring constructor.
\end{itemize}

EXAMPLES:

\begin{verbatim}
sage: R.<x> = PolynomialRing(ZZ)
sage: I = R.ideal([4 + 3*x + x^2, 1 + x^2])
sage: S = R.quotient(I, 'a')
sage: S.gens()
(a,)

sage: R.<x,y> = PolynomialRing(QQ,2)
sage: S.<a,b> = R.quotient((x^2, y))
sage: S
Quotient of Multivariate Polynomial Ring in x, y over Rational Field by the ideal (x^2, y)
sage: S.gens()
(a, 0)
sage: a == b
False
\end{verbatim}

\textbf{quotient\_ring}(I, \texttt{names} = \texttt{None}, **\texttt{kwds})

Return the quotient of self by the ideal \( I \) of \texttt{self}. (Synonym for \texttt{self.quotient}(I).)

INPUT:

\begin{itemize}
  \item \( I \) – an ideal of \( R \)
  \item \texttt{names} – (optional) names of the generators of the quotient. (If there are multiple generators, you can specify a single character string and the generators are named in sequence starting with 0.)
  \item further named arguments that may be passed to the quotient ring constructor.
\end{itemize}

OUTPUT:

\begin{itemize}
  \item \( R/I \) – the quotient ring of \( R \) by the ideal \( I \)
\end{itemize}

EXAMPLES:

\begin{verbatim}
sage: R.<x> = PolynomialRing(ZZ)
sage: I = R.ideal([4 + 3*x + x^2, 1 + x^2])
sage: S = R.quotient\_ring(I, 'a')
sage: S.gens()
(a,)

sage: R.<x,y> = PolynomialRing(QQ,2)
sage: S.<a,b> = R.quotient\_ring((x^2, y))
sage: S
Quotient of Multivariate Polynomial Ring in x, y over Rational Field by the ideal (x^2, y)
sage: S.gens()
\end{verbatim}
(a, 0)
sage: a == b
False

**random_element** *(bound=2)*

Return a random integer coerced into this ring, where the integer is chosen uniformly from the interval \([-bound, bound]\).

**INPUT:**

- **bound** – integer (default: 2)

**ALGORITHM:**

Uses Python’s randint.

**unit_ideal()**

Return the unit ideal of this ring.

**EXAMPLES:**

```python
sage: Zp(7).unit_ideal()
Principal ideal (1 + O(7^20)) of 7-adic Ring with capped relative precision 20
```

**zero()**

Return the zero element of this ring (cached).

**EXAMPLES:**

```python
sage: ZZ.zero()
0
sage: QQ.zero()
0
sage: QQ['x'].zero()
0
```

The result is cached:

```python
sage: ZZ.zero() is ZZ.zero()
True
```

**zero_ideal()**

Return the zero ideal of this ring (cached).

**EXAMPLES:**

```python
sage: ZZ.zero_ideal()
Principal ideal (0) of Integer Ring
sage: QQ.zero_ideal()
Principal ideal (0) of Rational Field
sage: QQ['x'].zero_ideal()
Principal ideal (0) of Univariate Polynomial Ring in x over Rational Field
```

The result is cached:

```python
sage: ZZ.zero_ideal() is ZZ.zero_ideal()
True
```
\textbf{zeta}(n=2, \text{all}=\text{False})

Return a primitive \(n\)-th root of unity in \textit{self} if there is one, or raise a \texttt{ValueError} otherwise.

**INPUT:**

- \(n\) – positive integer
- \texttt{all} – bool (default: \texttt{False}) - whether to return a list of all primitive \(n\)-th roots of unity. If True, raise a \texttt{ValueError} if \textit{self} is not an integral domain.

**OUTPUT:**

Element of \textit{self} of finite order

**EXAMPLES:**

\begin{verbatim}
sage: QQ.zeta()
-1
sage: QQ.zeta(1)
1
sage: CyclotomicField(6).zeta(6)
zeta6
sage: CyclotomicField(3).zeta(3)
zeta3
sage: CyclotomicField(3).zeta(3).multiplicative_order()
3
sage: a = GF(7).zeta(); a
3
sage: a.multiplicative_order()
6
sage: a = GF(49,'z').zeta(); a
z
sage: a.multiplicative_order()
48
sage: a = GF(49,'z').zeta(2); a
6
sage: a.multiplicative_order()
2
sage: QQ.zeta(3)
Traceback (most recent call last):
... ValueError: no n-th root of unity in rational field
sage: Zp(7, prec=8).zeta()
3 + 4*7 + 6*7^2 + 3*7^3 + 2*7^5 + 6*7^6 + 2*7^7 + O(7^8)
\end{verbatim}

\textbf{zeta\_order()} 

Return the order of the distinguished root of unity in \textit{self}.

**EXAMPLES:**

\begin{verbatim}
sage: CyclotomicField(19).zeta_order()
38
sage: GF(19).zeta_order()
18
sage: GF(5^3,'a').zeta_order()
124
\end{verbatim}

(continues on next page)
sage: Zp(7, prec=8).zeta_order()
6

sage.rings.ring.is_Ring(x)
    Return True if x is a ring.

EXAMPLES:

sage: from sage.rings.ring import is_Ring
sage: is_Ring(ZZ)
True
sage: MS = MatrixSpace(QQ,2)
sage: is_Ring(MS)
True
2.1 Ideals of commutative rings

Sage provides functionality for computing with ideals. One can create an ideal in any commutative or non-commutative ring \( R \) by giving a list of generators, using the notation \( R.\text{ideal}([a, b, \ldots]) \). The case of non-commutative rings is implemented in \texttt{noncommutative_ideals}.

A more convenient notation may be \( R*[a, b, \ldots] \) or \([a, b, \ldots]*R\). If \( R \) is non-commutative, the former creates a left and the latter a right ideal, and \( R*[a, b, \ldots]*R \) creates a two-sided ideal.

\texttt{sage.rings.ideal.Cyclic(R, n=None, homog=False, singular=None)}
Ideal of cyclic \( n \)-roots from 1-st \( n \) variables of \( R \) if \( R \) is coercible to \texttt{Singular}.

\begin{itemize}
  \item \( R \) – base ring to construct ideal for
  \item \( n \) – number of cyclic roots (default: \texttt{None}). If \texttt{None}, then \( n \) is set to \( R.\text{ngens}() \).
  \item \( \text{homog} \) – (default: \texttt{False}) if \texttt{True} a homogeneous ideal is returned using the last variable in the ideal
  \item \( \text{singular} \) – singular instance to use
\end{itemize}

\textbf{Note:} \( R \) will be set as the active ring in \texttt{Singular}

\textbf{EXAMPLES:}
An example from a multivariate polynomial ring over the rationals:

\begin{verbatim}
sage: P.<x,y,z> = PolynomialRing(QQ,3,order='lex')
sage: I = sage.rings.ideal.Cyclic(P)
sage: I
Ideal (x + y + z, x*y + x*z + y*z, x*y*z - 1) of Multivariate Polynomial Ring in x, y, z over Rational Field
sage: I.groebner_basis()
[x + y + z, y^2 + y*z + z^2, z^3 - 1]
\end{verbatim}

We compute a Groebner basis for cyclic 6, which is a standard benchmark and test ideal:

\begin{verbatim}
sage: R.<x,y,z,t,u,v> = QQ['x,y,z,t,u,v']
sage: I = sage.rings.ideal.Cyclic(R,6)
sage: B = I.groebner_basis()
sage: len(B)
45
\end{verbatim}
sage.rings.ideal.FieldIdeal(
    \( R \)
)

Let \( q = R.\text{base\_ring()}.\text{order()} \) and \((x_0, \ldots, x_n) = R.\text{gens()}\) then if \( q \) is finite this constructor returns

\[
(x_0^q - x_0, \ldots, x_n^q - x_n).
\]

We call this ideal the field ideal and the generators the field equations.

EXAMPLES:
The field ideal generated from the polynomial ring over two variables in the finite field of size 2:

```
sage: P.<x,y> = PolynomialRing(GF(2),2)
sage: I = sage.rings.ideal.FieldIdeal(P); I
Ideal (x^2 + x, y^2 + y) of Multivariate Polynomial Ring in x, y over
Finite Field of size 2
```

Another, similar example:

```
sage: Q.<x1,x2,x3,x4> = PolynomialRing(GF(2^4,name='alpha'), 4)
sage: J = sage.rings.ideal.FieldIdeal(Q); J
Ideal (x1^16 + x1, x2^16 + x2, x3^16 + x3, x4^16 + x4) of
Multivariate Polynomial Ring in x1, x2, x3, x4 over Finite
Field in alpha of size 2^4
```

sage.rings.ideal.Ideal(*args, **kwds)
Create the ideal in ring with given generators.

There are some shorthand notations for creating an ideal, in addition to using the \texttt{Ideal()} function:

- \texttt{R.ideal(gens, coerce=True)}
- \texttt{gens*R}
- \texttt{R*gens}

INPUT:
- \( R \) - A ring (optional; if not given, will try to infer it from \texttt{gens})
- \texttt{gens} - list of elements generating the ideal
- \texttt{coerce} - bool (optional, default: \texttt{True}); whether \texttt{gens} need to be coerced into the ring.

OUTPUT: The ideal of ring generated by \texttt{gens}.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: I = R.ideal([4 + 3*x + x^2, 1 + x^2])
sage: I
Ideal (x^2 + 3*x + 4, x^2 + 1) of Univariate Polynomial Ring in x over Integer Ring
```

```
sage: ideal(x^2-2*x+1, x^2-1)
Ideal (x^2 - 2*x + 1, x^2 - 1) of Univariate Polynomial Ring in x over Integer Ring
```

(continues on next page)
sage: l = [x^2-2*x+1, x^2-1]
sage: ideal(f^2 for f in l)
Ideal (x^4 - 4*x^3 + 6*x^2 - 4*x + 1, x^4 - 2*x^2 + 1) of Univariate Polynomial Ring in x over Integer Ring

This example illustrates how Sage finds a common ambient ring for the ideal, even though 1 is in the integers (in this case).

sage: R.<t> = ZZ['t']
sage: i = ideal(1,t,t^2)
sage: i
Ideal (1, t, t^2) of Univariate Polynomial Ring in t over Integer Ring
sage: ideal(1/2,t,t^2)
Principal ideal (1) of Univariate Polynomial Ring in t over Rational Field

This shows that the issues at trac ticket #1104 are resolved:

sage: Ideal(3, 5)
Principal ideal (1) of Integer Ring
sage: Ideal(ZZ, 3, 5)
Principal ideal (1) of Integer Ring
sage: Ideal(2, 4, 6)
Principal ideal (2) of Integer Ring

You have to provide enough information that Sage can figure out which ring to put the ideal in.

sage: I = Ideal([])
Traceback (most recent call last):
... ValueError: unable to determine which ring to embed the ideal in
sage: I = Ideal()
Traceback (most recent call last):
... ValueError: need at least one argument

Note that some rings use different ideal implementations than the standard, even if they are PIDs:

sage: R.<x> = GF(5)[]
sage: I = R(x^2+3)
sage: type(I)
<class 'sage.rings.polynomial.ideal.Ideal_1poly_field'>

You can also pass in a specific ideal type:

sage: from sage.rings.ideal import Ideal_pid
sage: I = Ideal(x^2+3,ideal_class=Ideal_pid)
sage: type(I)
<class 'sage.rings.ideal.Ideal_pid'>

class sage.rings.ideal.Ideal_fractional

2.1. Ideals of commutative rings
class sage.rings.ideal.Ideal_generic(ring, gens, coerce=True)

Bases: sage.structure.element.MonoidElement

An ideal.

See :meth:`Ideal()`.

.. automethod:: absolute_norm

.. automethod:: apply_morphism

.. automethod:: associated_primes
```sage
R = ZZ['x']
sage: I = R.ideal(7)
sage: I.associated_primes()
Traceback (most recent call last):
  ...  
NotImplementedError
```

### base_ring()

Returns the base ring of this ideal.

**EXAMPLES:**

```sage
sage: R = ZZ
sage: I = 3*R; I
Principal ideal (3) of Integer Ring
sage: J = 2*I; J
Principal ideal (6) of Integer Ring
sage: I.base_ring(); J.base_ring()
Integer Ring
Integer Ring
```

We construct an example of an ideal of a quotient ring:

```sage
R = PolynomialRing(QQ, 'x'); x = R.gen()
sage: I = R.ideal(x^2 - 2)
```

```sage
sage: I.base_ring()
Rational Field
```

And $p$-adic numbers:

```sage
R = Zp(7, prec=10); R
```

```sage
7-adic Ring with capped relative precision 10
```

```sage
sage: I = 7*R; I
Principal ideal (7 + O(7^11)) of 7-adic Ring with capped relative precision 10
```

```sage
sage: I.base_ring()
7-adic Ring with capped relative precision 10
```

### category()

Return the category of this ideal.

**Note:** category is dependent on the ring of the ideal.

**EXAMPLES:**

```sage
sage: P.<x> = ZZ[]
sage: I = ZZ.ideal(7)
sage: J = P.ideal(7,x)
sage: K = P.ideal(7)
sage: I.category()
Category of ring ideals in Integer Ring
dsage: J.category()
Category of ring ideals in Univariate Polynomial Ring in x over Integer Ring
```

(continues on next page)
embedded_primes()
Return the list of embedded primes of this ideal.

EXAMPLES:

```
sage: R.<x, y> = QQ[]
sage: I = R.ideal(x^2, x^2)
sage: I.embedded_primes()
[Ideal (y, x) of Multivariate Polynomial Ring in x, y over Rational Field]
```

gen()
Return the i-th generator in the current basis of this ideal.

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQ,2)
sage: I = Ideal([x,y+1]); I
Ideal (x, y + 1) of Multivariate Polynomial Ring in x, y over Rational Field
sage: I.gen(1)
y + 1
sage: ZZ.ideal(5,10).gen()
5
```

gens()
Return a set of generators / a basis of self.
This is the set of generators provided during creation of this ideal.

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQ,2)
sage: I = Ideal([x,y+1]); I
Ideal (x, y + 1) of Multivariate Polynomial Ring in x, y over Rational Field
sage: I.gens()
[x, y + 1]
sage: ZZ.ideal(5,10).gens()
(5,)
```

gens_reduced()
Same as gens() for this ideal, since there is currently no special gens_reduced algorithm implemented for this ring.
This method is provided so that ideals in \( \mathbb{Z} \) have the method gens_reduced(), just like ideals of number fields.

EXAMPLES:

```
sage: ZZ.ideal(5).gens_reduced()
(5,)
```
**is_maximal()**

Return True if the ideal is maximal in the ring containing the ideal.

**Todo:** This is not implemented for many rings. Implement it!

**EXAMPLES:**

```sage
R = ZZ
I = R.ideal(7)
I.is_maximal()  # True
R.ideal(16).is_maximal()  # False
S = Integers(8)
S.ideal(0).is_maximal()  # False
S.ideal(2).is_maximal()  # True
S.ideal(4).is_maximal()  # False
```

**is_primary**(\(P=None\))

Returns True if this ideal is primary (or \(P\)-primary, if a prime ideal \(P\) is specified).

Recall that an ideal \(I\) is primary if and only if \(I\) has a unique associated prime (see page 52 in [AM1969]). If this prime is \(P\), then \(I\) is said to be \(P\)-primary.

**INPUT:**

- \(P\) - (default: None) a prime ideal in the same ring

**EXAMPLES:**

```sage
R.<x, y> = QQ[]
I = R.ideal([x^2, x*y])
I.is_primary()  # False
J = I.primary_decomposition()[1]; J
Ideal (y, x^2) of Multivariate Polynomial Ring in x, y over Rational Field
J.is_primary()  # True
J.is_prime()  # False
```

Some examples from the Macaulay2 documentation:

```sage
R.<x, y, z> = GF(101)[]
I = R.ideal([y^6])
I.is_primary()  # True
I.is_primary(R.ideal([y]))  # True
I = R.ideal([x^4, y^7])
I.is_primary()  # True
```

(continues on next page)
sage: I = R.ideal([x*y, y^2])
sage: I.is_primary()
False

Note: This uses the list of associated primes.

is_prime()
Return True if this ideal is prime.

EXAMPLES:

sage: R.<x, y> = QQ[]
sage: I = R.ideal([x, y])
sage: I.is_prime() # a maximal ideal
True
sage: I = R.ideal([x^2-y])
sage: I.is_prime() # a non-maximal prime ideal
True
sage: I = R.ideal([x^2, y])
sage: I.is_prime() # a non-prime primary ideal
False
sage: I = R.ideal([x^2, x*y])
sage: I.is_prime() # a non-prime non-primary ideal
False

sage: S = Integers(8)
sage: S.ideal(0).is_prime()
False
sage: S.ideal(2).is_prime()
True
sage: S.ideal(4).is_prime()
False

Note that this method is not implemented for all rings where it could be:

sage: R.<x> = ZZ[

sage: I = R.ideal(7)
sage: I.is_prime() # when implemented, should be True
Traceback (most recent call last):
  ...
NotImplementedError

Note: For general rings, uses the list of associated primes.

is_principal()
Returns True if the ideal is principal in the ring containing the ideal.

Todo: Code is naive. Only keeps track of ideal generators as set during initialization of the ideal. (Can the base ring change? See example below.)
EXAMPLES:

```python
sage: R = ZZ['x']
sage: I = R.ideal(2,x)
sage: I.is_principal()
Traceback (most recent call last):
... 
NotImplementedError
sage: J = R.base_extend(QQ).ideal(2,x)
sage: J.is_principal()
True
```

`is_trivial()`
Return True if this ideal is (0) or (1).

`minimal_associated_primes()`
Return the list of minimal associated prime ideals of this ideal.

```python
sage: R = ZZ['x']
sage: I = R.ideal(7)
sage: I.minimal_associated_primes()
Traceback (most recent call last):
... 
NotImplementedError
```

`ngens()`
Return the number of generators in the basis.

```python
sage: P.<x,y> = PolynomialRing(QQ,2)
sage: I = Ideal([x,y+1]); I
Ideal (x, y + 1) of Multivariate Polynomial Ring in x, y over Rational Field
sage: I.ngens()
2
```

`norm()`
Returns the norm of this ideal.

In the general case, this is just the ideal itself, since the ring it lies in can't be implicitly assumed to be an extension of anything.

We include this function for compatibility with cases such as ideals in number fields.

```python
sage: R.<t> = GF(8, names='a')[]
sage: I = R.ideal(t^4 + t + 1)
sage: I.norm()
Principal ideal (t^4 + t + 1) of Univariate Polynomial Ring in t over Finite \n\rightarrow Field in a of size 2^3
```

2.1. Ideals of commutative rings
primary_decomposition()  
Return a decomposition of this ideal into primary ideals.

EXAMPLES:

```
sage: R = ZZ['x']
sage: I = R.ideal(7)
sage: I.primary_decomposition()
Traceback (most recent call last):
...  
NotImplementedError
```

random_element(*args, **kwds)  
Return a random element in this ideal.

EXAMPLES:

```
sage: P.<a,b,c> = GF(5)[[]]
sage: I = P.ideal([a^2, a*b + c, c^3])
sage: I.random_element()  
# random  
2*a^5*c + a^2*b*c^4 + ... + O(a, b, c)^13
```

reduce(f)  
Return the reduction of the element of \( f \) modulo self.

This is an element of \( R \) that is equivalent modulo \( I \) to \( f \) where \( I \) is self.

EXAMPLES:

```
sage: ZZ.ideal(5).reduce(17)
2
sage: parent(ZZ.ideal(5).reduce(17))
Integer Ring
```

ring()  
Return the ring containing this ideal.

EXAMPLES:

```
sage: R = ZZ
sage: I = 3*R; I
Principal ideal (3) of Integer Ring
sage: J = 2*I; J
Principal ideal (6) of Integer Ring
sage: I.ring(); J.ring()
Integer Ring
Integer Ring
```

Note that \( \text{self.ring()} \) is different from \( \text{self.base_ring()} \)

```
sage: R = PolynomialRing(QQ, 'x'); x = R.gen()
sage: I = R.ideal(x^2 - 2)
sage: I.base_ring()
Rational Field
sage: I.ring()
Univariate Polynomial Ring in x over Rational Field
```

Another example using polynomial rings:
```python
sage: R = PolynomialRing(QQ, 'x'); x = R.gen()
sage: I = R.ideal(x^2 - 3)
sage: I.ring()  
Univariate Polynomial Ring in x over Rational Field
sage: Rbar = R.quotient(I, names='a')
sage: S = PolynomialRing(Rbar, 'y'); y = Rbar.gen(); S
    Univariate Polynomial Ring in y over Univariate Quotient Polynomial Ring in a___
    → over Rational Field with modulus x^2 - 3
sage: J = S.ideal(y^2 + 1)
sage: J.ring()  
Univariate Polynomial Ring in y over Univariate Quotient Polynomial Ring in a___
    → over Rational Field with modulus x^2 - 3
```

```python
class sage.rings.ideal.Ideal_pid(ring, gen)
Bases: sage.rings.ideal.Ideal_principal

An ideal of a principal ideal domain.

See Ideal().

gcd(other)

Returns the greatest common divisor of the principal ideal with the ideal other; that is, the largest principal ideal contained in both the ideal and other

Todo: This is not implemented in the case when other is neither principal nor when the generator of self is contained in other. Also, it seems that this class is used only in PIDs–is this redundant?

Note: The second example is broken.

EXAMPLES:

An example in the principal ideal domain \( \mathbb{Z} \):

```python
sage: R = ZZ
sage: I = R.ideal(42)
sage: J = R.ideal(70)
sage: I.gcd(J)
Principal ideal (14) of Integer Ring
sage: J.gcd(I)
Principal ideal (14) of Integer Ring
```

```python
is_maximal()

Returns whether this ideal is maximal.

Principal ideal domains have Krull dimension 1 (or 0), so an ideal is maximal if and only if it’s prime (and nonzero if the ring is not a field).

EXAMPLES:

```python
sage: R.<t> = GF(5)[]
sage: p = R.ideal(t^2 + 2)
sage: p.is_maximal()
True
```
```
is_prime()
Return True if the ideal is prime.

This relies on the ring elements having a method is_irreducible() implemented, since an ideal \( (a) \) is prime iff \( a \) is irreducible (or 0).

EXAMPLES:

```
sage: ZZ.ideal(2).is_prime()
True
sage: ZZ.ideal(-2).is_prime()
True
sage: ZZ.ideal(4).is_prime()
False
sage: ZZ.ideal(0).is_prime()
True
sage: R.<x> = QQ[]
sage: P = R.ideal(x^2+1); P
Principal ideal (x^2 + 1) of Univariate Polynomial Ring in x over Rational Field
sage: P.is_prime()
True
```

In fields, only the zero ideal is prime:

```
sage: RR.ideal(0).is_prime()
True
sage: RR.ideal(7).is_prime()
False
```

reduce(f)
Return the reduction of \( f \) modulo self.

EXAMPLES:

```
sage: I = 8*ZZ
sage: I.reduce(10)
2
sage: n = 10; n.mod(I)
2
```

residue_field()
Return the residue class field of this ideal, which must be prime.
Todo: Implement this for more general rings. Currently only defined for $\mathbb{Z}$ and for number field orders.

EXAMPLES:

```sage
sage: P = ZZ.ideal(61); P
Principal ideal (61) of Integer Ring
sage: F = P.residue_field(); F
Residue field of Integers modulo 61
sage: pi = F.reduction_map(); pi
Partially defined reduction map:
   From: Rational Field
   To:   Residue field of Integers modulo 61
sage: pi(123/234)
  6
sage: pi(1/61)
Traceback (most recent call last):
... ZeroDivisionError: Cannot reduce rational 1/61 modulo 61: it has negative valuation
sage: lift = F.lift_map(); lift
Lifting map:
   From: Residue field of Integers modulo 61
   To:   Integer Ring
sage: lift(F(12345/67890))
  33
sage: (12345/67890) % 61
  33
```

class `sage.rings.ideal.Ideal_principal`(ring, gens, coerce=True)
Bases: `sage.rings.ideal.Ideal_generic`

A principal ideal.

See `Ideal()`.

`divides(other)`
Return True if self divides other.

EXAMPLES:

```sage
sage: P.<x> = PolynomialRing(QQ)
sage: I = P.ideal(x)
sage: J = P.ideal(x^2)
sage: I.divides(J)
True
sage: J.divides(I)
False
```

gen()
Returns the generator of the principal ideal. The generators are elements of the ring containing the ideal.

EXAMPLES:

A simple example in the integers:

```sage
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```

```python
sage: R = ZZ
sage: I = R.ideal(7)
sage: J = R.ideal(7, 14)
sage: I.gen(); J.gen()
7
7
```

Note that the generator belongs to the ring from which the ideal was initialized:

```python
sage: R.<x> = ZZ[]
... sage: I = R.ideal(x)
... sage: J = R.base_extend(QQ).ideal(2,x)
... sage: a = I.gen(); a
... x
... sage: b = J.gen(); b
... 1
... sage: a.base_ring()
... Integer Ring
... sage: b.base_ring()
... Rational Field
```

**is_principal()**

Returns True if the ideal is principal in the ring containing the ideal. When the ideal construction is explicitly principal (i.e. when we define an ideal with one element) this is always the case.

**EXAMPLES:**

Note that Sage automatically coerces ideals into principal ideals during initialization:

```python
sage: R.<x> = ZZ[]
... sage: I = R.ideal(x)
... sage: J = R.base_extend(QQ).ideal(2,x)
... sage: K = R.base_extend(QQ).ideal(2,x)
... sage: I
... Principal ideal (x) of Univariate Polynomial Ring in x over Integer Ring
... sage: J
... Ideal (2, x) of Univariate Polynomial Ring in x over Integer Ring
... sage: K
... Principal ideal (1) of Univariate Polynomial Ring in x over Rational Field
... sage: I.is_principal()
... True
... sage: K.is_principal()
... True
```

```python
sage.rings.ideal.Katsura(R, n=None, homog=False, singular=None)
```

n-th katsura ideal of R if R is coercible to Singular.

**INPUT:**

- R – base ring to construct ideal for
- n – (default: None) which katsura ideal of R. If None, then n is set to R.ngens().
- homog – if True a homogeneous ideal is returned using the last variable in the ideal (default: False)
• **singular** – singular instance to use

**EXAMPLES:**

```python
sage: P.<x,y,z> = PolynomialRing(QQ,3)
sage: I = sage.rings.ideal.Katsura(P,3); I
Ideal (x + 2*y + 2*z - 1, x^2 + 2*y^2 + 2*z^2 - x, 2*x*y + 2*y*z - y)
of Multivariate Polynomial Ring in x, y, z over Rational Field

sage: Q.<x> = PolynomialRing(QQ, implementation="singular")
sage: J = sage.rings.ideal.Katsura(Q,1); J
Ideal (x - 1) of Multivariate Polynomial Ring in x over Rational Field
```

`sage.rings.ideal.is_Ideal(x)`

Return True if object is an ideal of a ring.

**EXAMPLES:**

A simple example involving the ring of integers. Note that Sage does not interpret rings objects themselves as ideals. However, one can still explicitly construct these ideals:

```python
sage: from sage.rings.ideal import is_Ideal
sage: R = ZZ
sage: is_Ideal(R)
False
sage: 1*R; is_Ideal(1*R)
Principal ideal (1) of Integer Ring
True
sage: 0*R; is_Ideal(0*R)
Principal ideal (0) of Integer Ring
True
```

Sage recognizes ideals of polynomial rings as well:

```python
sage: R = PolynomialRing(QQ, 'x'); x = R.gen()
sage: I = R.ideal(x^2 + 1); I
Principal ideal (x^2 + 1) of Univariate Polynomial Ring in x over Rational Field
sage: is_Ideal(I)
True
sage: is_Ideal((x^2 + 1)*R)
True
```

### 2.2 Monoid of ideals in a commutative ring

**WARNING:** This is used by some rings that are not commutative!

```python
sage: MS = MatrixSpace(QQ,3,3)
sage: type(MS.ideal(MS.one()).parent())
<class 'sage.rings.ideal_monoid.IdealMonoid_c_with_category'>
```

`sage.rings.ideal_monoid.IdealMonoid(R)`

Return the monoid of ideals in the ring R.

**EXAMPLES:**
```python
sage: R = QQ['x']
sage: sage.rings.ideal_monoid.IdealMonoid(R)
Monoid of ideals of Univariate Polynomial Ring in x over Rational Field
```

```python
class sage.rings.ideal_monoid.IdealMonoid_c(R)
    Bases: sage.structure.parent.Parent

    The monoid of ideals in a commutative ring.

    Element
       alias of sage.rings.ideal.Ideal_generic

    ring()
       Return the ring of which this is the ideal monoid.

    EXAMPLES:
        sage: R = QuadraticField(-23, 'a')
sage: M = sage.rings.ideal_monoid.IdealMonoid(R); M.ring()
        is R
        True
```

## 2.3 Ideals of non-commutative rings

Generic implementation of one- and two-sided ideals of non-commutative rings.

**AUTHOR:**

- Simon King (2011-03-21), <simon.king@uni-jena.de>, trac ticket #7797.

**EXAMPLES:**

```python
sage: MS = MatrixSpace(ZZ,2,2)
sage: MS*MS([0,1,-2,3])
Left Ideal
    ( [ 0 1]
    [ -2 3]
) of Full MatrixSpace of 2 by 2 dense matrices over Integer Ring
sage: MS([0,1,-2,3])*MS
Right Ideal
    ( [ 0 1]
    [ -2 3]
) of Full MatrixSpace of 2 by 2 dense matrices over Integer Ring
sage: MS*MS([0,1,-2,3])*MS
Twosided Ideal
    ( [ 0 1]
    [ -2 3]
) of Full MatrixSpace of 2 by 2 dense matrices over Integer Ring
```

See `letterplace_ideal` for a more elaborate implementation in the special case of ideals in free algebras.
class sage.rings.noncommutative_ideals.IdealMonoid_nc(R)

Bases: sage.rings.ideal_monoid.IdealMonoid_c

Base class for the monoid of ideals over a non-commutative ring.

Note: This class is essentially the same as IdealMonoid_c, but does not complain about non-commutative rings.

EXAMPLES:

```
sage: MS = MatrixSpace(ZZ,2,2)
sage: MS.ideal_monoid()
Monoid of ideals of Full MatrixSpace of 2 by 2 dense matrices over Integer Ring
```

class sage.rings.noncommutative_ideals.Ideal_nc(ring, gens, coerce=True, side='twosided')

Bases: sage.rings.ideal.Ideal_generic

Generic non-commutative ideal.

All fancy stuff such as the computation of Groebner bases must be implemented in sub-classes. See LetterplaceIdeal for an example.

EXAMPLES:

```
sage: MS = MatrixSpace(QQ,2,2)
sage: I = MS*[MS.1,MS.2]; I
Left Ideal
(  
[0 1]
[0 0],
  
[0 0]
[1 0]
)
of Full MatrixSpace of 2 by 2 dense matrices over Rational Field
sage: [MS.1,MS.2]*MS
Right Ideal
(  
[0 1]
[0 0],
  
[0 0]
[1 0]
)
of Full MatrixSpace of 2 by 2 dense matrices over Rational Field
sage: MS*[MS.1,MS.2]*MS
Twosided Ideal
(  
[0 1]
[0 0],
  
[0 0]
[1 0]
)
of Full MatrixSpace of 2 by 2 dense matrices over Rational Field
```
**side()**

Return a string that describes the sidedness of this ideal.

**EXAMPLES:**

```python
sage: A = SteenrodAlgebra(2)
sage: IL = A*[A.1+A.2,A.1^2]
sage: IR = [A.1+A.2,A.1^2]^A
sage: IT = A*[A.1+A.2,A.1^2]^A
sage: IL.side()
'left'
sage: IR.side()
'right'
sage: IT.side()
'twosided'
```
3.1 Homomorphisms of rings

We give a large number of examples of ring homomorphisms.

EXAMPLES:

Natural inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$:

```
sage: H = Hom(ZZ, QQ)
sage: phi = H([1])
sage: phi(10)
10
sage: phi(3/1)
3
sage: phi(2/3)
Traceback (most recent call last):
  ...TypeError: 2/3 fails to convert into the map's domain Integer Ring, but a 'pushforward'_method is not properly implemented
```

There is no homomorphism in the other direction:

```
sage: H = Hom(QQ, ZZ)
sage: H([1])
Traceback (most recent call last):
  ...ValueError: relations do not all (canonically) map to 0 under map determined by images_of generators
```

EXAMPLES:

Reduction to finite field:

```
sage: H = Hom(ZZ, GF(9, 'a'))
sage: phi = H([1])
sage: phi(5)
2
sage: psi = H([4])
sage: psi(5)
2
```

Map from single variable polynomial ring:
Identity map on the real numbers:

```
sage: f = RR.hom([RR(1)]); f
Ring endomorphism of Real Field with 53 bits of precision
    Defn: 1.00000000000000 |--> 1.00000000000000
sage: f(2.5)
2.50000000000000
```

Homomorphism from one precision of field to another.

From smaller to bigger doesn’t make sense:

```
sage: R200 = RealField(200)
sage: f = RR.hom( R200 )
Traceback (most recent call last):
  ...TypeError: natural coercion morphism from Real Field with 53 bits of precision to Real
    _Field with 200 bits of precision not defined
```

From bigger to small does:

```
sage: f = RR.hom( RealField(15) )
sage: f(2.5)
2.500
sage: f(RR.pi())
3.142
```

Inclusion map from the reals to the complexes:

```
sage: i = RR.hom([CC(1)]); i
Ring morphism:
    From: Real Field with 53 bits of precision
    To:    Complex Field with 53 bits of precision
    Defn: 1.00000000000000 |--> 1.00000000000000
sage: i(RR('3.1'))
3.10000000000000
```

A map from a multivariate polynomial ring to itself:
sage: R.<x,y,z> = PolynomialRing(QQ,3)
sage: phi = R.hom([y,z,x^2]); phi
Ring endomorphism of Multivariate Polynomial Ring in x, y, z over Rational Field
  Defn: x |--> y
       y |--> z
       z |--> x^2
sage: phi(x+y+z)
x^2 + y + z

An endomorphism of a quotient of a multi-variate polynomial ring:

sage: R.<x,y> = PolynomialRing(QQ)
sage: S.<a,b> = quo(R, ideal(1 + y^2))
sage: phi = S.hom([a^2, -b])
sage: phi
Ring endomorphism of Quotient of Multivariate Polynomial Ring in x, y over Rational Field by the ideal (y^2 + 1)
  Defn: a |--> a^2
       b |--> -b
sage: phi(b)
-b
sage: phi(a^2 + b^2)
a^4 - 1

The reduction map from the integers to the integers modulo 8, viewed as a quotient ring:

sage: R = ZZ.quo(8*ZZ)
sage: pi = R.cover()
sage: pi
Ring morphism:
  From: Integer Ring
  To: Ring of integers modulo 8
  Defn: Natural quotient map
sage: pi.domain()
Integer Ring
sage: pi.codomain()
Ring of integers modulo 8
sage: pi(10)
2
sage: pi.lift()
Set-theoretic ring morphism:
  From: Ring of integers modulo 8
  To: Integer Ring
  Defn: Choice of lifting map
sage: pi.lift(13)
5

Inclusion of GF(2) into GF(4, 'a'):

sage: k = GF(2)
sage: i = k.hom(GF(4, 'a'))
sage: i
Ring morphism:

3.1. Homomorphisms of rings
We next compose the inclusion with reduction from the integers to \( \text{GF}(2) \):

\[
\text{sage}: \quad \text{i(0)} \\
\text{sage}: \quad i(0) \\
\text{sage}: \quad \text{a = i(1); a.parent()} \\
\text{Finite Field in a of size 2^2}
\]

Inclusion from \( \mathbb{Q} \) to the 3-adic field:

\[
\text{sage}: \quad \text{phi = QQ.hom(Qp(3, print_mode = 'series'))} \\
\text{sage}: \quad \text{phi} \\
\text{Ring morphism:} \\
\text{From: Rational Field} \\
\text{To: 3-adic Field with capped relative precision 20} \\
\text{sage}: \quad \text{phi.codomain()} \\
\text{3-adic Field with capped relative precision 20} \\
\text{sage}: \quad \text{phi(394)} \\
1 + 2*3 + 3^2 + 2*3*3 + 3^4 + 3^5 + O(3^20)
\]

An automorphism of a quotient of a univariate polynomial ring:

\[
\text{sage}: \quad R.<x> = PolynomialRing(QQ) \\
\text{sage}: \quad S.<sqrt2> = R.quo(x^2-2) \\
\text{sage}: \quad \text{sqrt2^2} \\
2 \\
\text{sage}: \quad (3+sqrt2)^10 \\
993054*sqrt2 + 1404491
\]
Note that Sage verifies that the morphism is valid:

```python
sage: c = S.hom([-sqrt2])
sage: c(1+sqrt2)
-sqrt2 + 1
```

Endomorphism of power series ring:

```python
sage: R.<t> = PowerSeriesRing(QQ, default_prec=10); R
Power Series Ring in t over Rational Field
sage: f = R.hom([t^2]); f
Ring endomorphism of Power Series Ring in t over Rational Field
  Defn: t |--> t^2
sage: s = 1/(1 + t); s
1 - t + t^2 - t^3 + t^4 - t^5 + t^6 - t^7 + t^8 - t^9 + O(t^10)
sage: f(s)
1 - t^2 + t^4 - t^6 + t^8 - t^10 + t^12 - t^14 + t^16 - t^18 + O(t^20)
```

Frobenius on a power series ring over a finite field:

```python
sage: R.<t> = PowerSeriesRing(GF(5))
sage: f = R.hom([t^5]); f
Ring endomorphism of Power Series Ring in t over Finite Field of size 5
  Defn: t |--> t^5
sage: a = 2 + t + 3*t^2 + 4*t^3 + O(t^4)
sage: b = 1 + t + 2*t^2 + t^3 + O(t^5)
sage: f(a)
2 + t^5 + 3*t^10 + 4*t^15 + O(t^20)
sage: f(b)
1 + t^5 + 2*t^10 + t^15 + O(t^25)
sage: f(a*b)
2 + 3*t^5 + 3*t^10 + t^15 + O(t^20)
sage: f(a)*f(b)
2 + 3*t^5 + 3*t^10 + t^15 + O(t^20)
```

Homomorphism of Laurent series ring:

```python
sage: R.<t> = LaurentSeriesRing(QQ, 10)
sage: f = R.hom([t^3 + t]); f
Ring endomorphism of Laurent Series Ring in t over Rational Field
  Defn: t |--> t + t^3
sage: s = 2/t^2 + 1/(1 + t); s
2*t^-2 + 1 - t + t^2 - t^3 + t^4 - t^5 + t^6 - t^7 + t^8 - t^9 + O(t^10)
sage: f(s)
2*t^-2 - 3 - t + 7*t^2 - 2*t^3 - 5*t^4 - 4*t^5 + 16*t^6 - 9*t^7 + O(t^8)
```

(continues on next page)
sage: f = R.hom([t^3]); f
Ring endomorphism of Laurent Series Ring in t over Rational Field
  Defn: t |--> t^3
sage: f(s)
2*t^-6 + 1 - t^3 + t^6 - t^9 + t^12 - t^15 + t^18 - t^21 + t^24 - t^27 + O(t^30)

Note that the homomorphism must result in a converging Laurent series, so the valuation of the image of the generator
must be positive:

sage: R.hom([[1/t]])
Traceback (most recent call last):
  ...<ValueError: relations do not all (canonically) map to 0 under map determined by images of generators
sage: R.hom([1])
Traceback (most recent call last):
  ...<ValueError: relations do not all (canonically) map to 0 under map determined by images of generators

Complex conjugation on cyclotomic fields:

sage: K.<zeta7> = CyclotomicField(7)
sage: c = K.hom([1/zeta7]); c
Ring endomorphism of Cyclotomic Field of order 7 and degree 6
  Defn: zeta7 |--> -zeta7^5 - zeta7^4 - zeta7^3 - zeta7^2 - zeta7 - 1
sage: a = (1+zeta7)^5; a
zeta7^5 + 5*zeta7^4 + 10*zeta7^3 + 10*zeta7^2 + 5*zeta7 + 1
sage: c(a)
5*zeta7^5 + 5*zeta7^4 - 4*zeta7^2 - 5*zeta7 - 4
sage: c(zeta7 + 1/zeta7)
# this element is obviously fixed by inversion
-zeta7^5 - zeta7^4 - zeta7^3 - zeta7^2 - 1
sage: zeta7 + 1/zeta7
-zeta7^5 - zeta7^4 - zeta7^3 - zeta7^2 - 1

Embedding a number field into the reals:

sage: R.<x> = PolynomialRing(QQ)
sage: K.<beta> = NumberField(x^3 - 2)
sage: alpha = RR(2)^(1/3); alpha
1.25992104989487
sage: i = K.hom([alpha],check=False); i
Ring morphism:
  From: Number Field in beta with defining polynomial x^3 - 2
  To:   Real Field with 53 bits of precision
  Defn: beta |--> 1.25992104989487
sage: i(beta)
1.25992104989487
sage: i(beta^3)
2.00000000000000
sage: i(beta^2 + 1)
2.58740105196820
sage: i(beta^2 + 1)
2.58740105196820

48 Chapter 3. Ring Morphisms
An example from Jim Carlson:

```
sage: K = QQ # by the way :)  
sage: R.<a,b,c,d> = K[]; R  
Multivariate Polynomial Ring in a, b, c, d over Rational Field  
sage: S.<u> = K[]; S  
Univariate Polynomial Ring in u over Rational Field  
sage: f = R.hom([0,0,0,u], S); f  
Ring morphism:  
    From: Multivariate Polynomial Ring in a, b, c, d over Rational Field  
    To:    Univariate Polynomial Ring in u over Rational Field  
    Defn: a |--> 0  
          b |--> 0  
          c |--> 0  
          d |--> u  
sage: f(a+b+c+d)  
u  
sage: f((a+b+c+d)^2)  
u^2  ```

```
class sage.rings.morphism.FrobeniusEndomorphism_generic  
    Bases: sage.rings.morphism.RingHomomorphism  

A class implementing Frobenius endomorphisms on rings of prime characteristic.

power()  
Return an integer $n$ such that this endomorphism is the $n$-th power of the absolute (arithmetic) Frobenius.

EXAMPLES:

```
sage: K.<u> = PowerSeriesRing(GF(5))  
sage: Frob = K.frobenius_endomorphism()  
sage: Frob.power()  
1  
sage: (Frob^9).power()  
9  ```
```
class sage.rings.morphism.RingHomomorphism  
    Bases: sage.rings.morphism.RingMap  

Homomorphism of rings.

inverse()  
Return the inverse of this ring homomorphism if it exists.

Raises a ZeroDivisionError if the inverse does not exist.

ALGORITHM:

By default, this computes a Gröbner basis of the ideal corresponding to the graph of the ring homomorphism.

EXAMPLES:

```
sage: R.<t> = QQ[]  
sage: f = R.hom([2*t - 1], R)  
sage: f.inverse()  
Ring endomorphism of Univariate Polynomial Ring in t over Rational Field  
    Defn: t |--> 1/2*t + 1/2  ```

3.1. Homomorphisms of rings
The following non-linear homomorphism is not invertible, but it induces an isomorphism on a quotient ring:

```python
sage: R.<x,y,z> = QQ[]
sage: f = R.hom([y*z, x*z, x*y], R)
sage: f.inverse()
Traceback (most recent call last):
...
ZeroDivisionError: ring homomorphism not surjective
sage: f.is_injective()
True
sage: Q.<x,y,z> = R.quotient(x*y*z - 1)
sage: g = Q.hom([y*z, x*z, x*y], Q)
sage: g.inverse()
Ring endomorphism of Quotient of Multivariate Polynomial Ring in x, y, z over Rational Field by the ideal (x*y*z - 1)
  Defn: x |--> y*z
        y |--> x*z
        z |--> x*y
```

Homomorphisms over the integers are supported:

```python
sage: S.<x,y> = ZZ[]
sage: f = S.hom([x + 2*y, x + 3*y], S)
sage: f.inverse()
Ring endomorphism of Multivariate Polynomial Ring in x, y over Integer Ring
  Defn: x |--> 3*x - 2*y
        y |--> -x + y
sage: (f.inverse() * f).is_identity()
True
```

The following homomorphism is invertible over the rationals, but not over the integers:

```python
sage: g = S.hom([x + y, x - y - 2], S)
sage: g.inverse()
Traceback (most recent call last):
...
ZeroDivisionError: ring homomorphism not surjective
sage: R.<x,y> = QQ[x,y]
sage: h = R.hom([x + y, x - y - 2], R)
sage: (h.inverse() * h).is_identity()
True
```

This example by M. Nagata is a wild automorphism:

```python
sage: R.<x,y,z> = QQ[]
sage: sigma = R.hom([x - 2*y*(z*x+y^2) - z*(z*x+y^2)^2, ....: y + z*(z*x+y^2), z], R)
sage: tau = sigma.inverse(); tau
Ring endomorphism of Multivariate Polynomial Ring in x, y, z over Rational Field
  Defn: x |--> -y^4*z - 2*x*y^2*z^2 - x^2*z^3 + 2*y^3 + 2*x*y*z + x
        y |--> -y^2*z - x*z^2 + y
        z |--> z
```
We compute the triangular automorphism that converts moments to cumulants, as well as its inverse, using the moment generating function. The choice of a term ordering can have a great impact on the computation time of a Gröbner basis, so here we choose a weighted ordering such that the images of the generators are homogeneous polynomials.

```python
sage: d = 12
sage: T = TermOrder('wdegrevlex', [1..d])

sage: R = PolynomialRing(QQ, ['x%s' % j for j in (1..d)], order=T)

sage: S.<t> = PowerSeriesRing(R)

sage: egf = S([0] + list(R.gens())).ogf_to_egf().exp(prec=d+1)

sage: phi = R.hom(egf.egf_to_ogf().list()[1:], R)

sage: phi.im_gens()[:5]
[x1, x1^2 + x2, x1^3 + 3*x1*x2 + x3, x1^4 + 6*x1^2*x2 + 3*x2^2 + 4*x1*x3 + x4, x1^5 + 10*x1^3*x2 + 15*x1*x2^2 + 10*x1^2*x3 + 10*x2*x3 + 5*x1*x4 + x5]

sage: all(p.is_homogeneous() for p in phi.im_gens())
True

sage: phi.inverse().im_gens()[:5]
[x1, -x1^2 + x2, 2*x1^3 - 3*x1*x2 + x3, -6*x1^4 + 12*x1^2*x2 - 3*x2^2 - 4*x1*x3 + x4, 24*x1^5 - 60*x1^3*x2 + 30*x1*x2^2 + 20*x1^2*x3 - 10*x2*x3 - 5*x1*x4 + x5]

sage: (phi.inverse() * phi).is_identity()
True
```

Automorphisms of number fields as well as Galois fields are supported:

```python
sage: K.<zeta7> = CyclotomicField(7)

sage: c = K.hom([1/zeta7])

sage: (c.inverse() * c).is_identity()
True

sage: F.<t> = GF(7^3)

sage: f = F.hom(t^7, F)

sage: (f.inverse() * f).is_identity()
True
```

An isomorphism between the algebraic torus and the circle over a number field:

```python
sage: K.<i> = QuadraticField(-1)

sage: A.<z,w> = K['z,w'].quotient('z*w - 1')

sage: B.<x,y> = K['x,y'].quotient('x^2 + y^2 - 1')

sage: f = A.hom([x + i*y, x - i*y], B)

sage: g = f.inverse()

sage: g.morphism_from_cover().im_gens()
[1/2*z + 1/2*w, (-1/2*i)*z + (1/2*i)*w]

sage: all(g(f(z)) == z for z in A.gens())
True
```
**inverse_image(I)**

Return the inverse image of an ideal or an element in the codomain of this ring homomorphism.

**INPUT:**

- I – an ideal or element in the codomain

**OUTPUT:**

For an ideal \( I \) in the codomain, this returns the largest ideal in the domain whose image is contained in \( I \).

Given an element \( b \) in the codomain, this returns an arbitrary element \( a \) in the domain such that \( \text{self}(a) = b \) if one such exists. The element \( a \) is unique if this ring homomorphism is injective.

**EXAMPLES:**

```sage
sage: R.<x,y,z> = QQ[]
sage: S.<u,v> = QQ[]
sage: f = R.hom([u^2, u*v, v^2], S)
sage: I = S.ideal([u^6, u^5*v, u^4*v^2, u^3*v^3])
sage: J = f.inverse_image(I); J
Ideal (y^2 - x*z, x*y*z, x^2*z, x^2*y, x^3)
of Multivariate Polynomial Ring in x, y, z over Rational Field
sage: f(J) == I
True
```

Under the above homomorphism, there exists an inverse image for every element that only involves monomials of even degree:

```sage
sage: [f.inverse_image(p) for p in [u^2, u^4, u^3*v + u^3*v^3]]
x, x^2, x*y*z + y
sage: f.inverse_image(u^v^2)
Traceback (most recent call last):
... 
ValueError: element u*v^2 does not have preimage
```

The image of the inverse image ideal can be strictly smaller than the original ideal:

```sage
sage: S.<u,v> = QQ['u,v'].quotient('v^2 - 2')
sage: f = QuadraticField(2).hom([v], S)
sage: I = S.ideal(u + v)
sage: J = f.inverse_image(I)
sage: J.is_zero()
True
sage: f(J) < I
True
```

Fractional ideals are not yet fully supported:

```sage
sage: K.<a> = NumberField(QQ['x'](x^2+2))
sage: f = K.hom([-a], K)
sage: I = K.ideal([a + 1])
sage: f.inverse_image(I)
Traceback (most recent call last):
... 
NotImplementedError: inverse image not implemented...
sage: f.inverse_image(K.ideal(0)).is_zero()
```

(continues on next page)
True

\texttt{sage: f.inverse()()(I)}
Fractional ideal (-a + 1)

**ALGORITHM:**

By default, this computes a Gröbner basis of an ideal related to the graph of the ring homomorphism.

**REFERENCES:**

- Proposition 2.5.12 [DS2009]

\texttt{is\_invertible()}

Return whether this ring homomorphism is bijective.

**EXAMPLES:**

\texttt{sage: R.<x,y,z> = QQ[]}
\texttt{sage: R.hom([y*z, x*z, x*y], R).is\_invertible()}
False
\texttt{sage: Q.<x,y,z> = R.quotient(x*y*z - 1)}
\texttt{sage: Q.hom([y*z, x*z, x*y], Q).is\_invertible()}
True

**ALGORITHM:**

By default, this requires the computation of a Gröbner basis.

\texttt{is\_surjective()}

Return whether this ring homomorphism is surjective.

**EXAMPLES:**

\texttt{sage: R.<x,y,z> = QQ[]}
\texttt{sage: R.hom([y*z, x*z, x*y], R).is\_surjective()}
False
\texttt{sage: Q.<x,y,z> = R.quotient(x*y*z - 1)}
\texttt{sage: Q.hom([y*z, x*z, x*y], Q).is\_surjective()}
True

**ALGORITHM:**

By default, this requires the computation of a Gröbner basis.

\texttt{kernel()}

Return the kernel ideal of this ring homomorphism.

**EXAMPLES:**

\texttt{sage: A.<x,y> = QQ[]}
\texttt{sage: B.<t> = QQ[]}
\texttt{sage: f = A.hom([t^4, t^3 - t^2], B)}
\texttt{sage: f.kernel()}
Ideal (y^4 - x^3 + 4*x^2*y - 2*x*y^2 + x^2)
of Multivariate Polynomial Ring in x, y over Rational Field

We express a Veronese subring of a polynomial ring as a quotient ring:
sage: A.<a,b,c,d> = QQ[]
sage: B.<u,v> = QQ[]
sage: f = A.hom([u^3, u^2*v, u*v^2, v^3],B)
sage: f.kernel() == A.ideal(matrix.hankel([a, b, c], [d]).minors(2))
True
sage: Q = A.quotient(f.kernel())

sage: Q.hom(f.im_gens(), B).is_injective()
True

The Steiner-Roman surface:

sage: R.<x,y,z> = QQ[]
sage: S = R.quotient(x^2 + y^2 + z^2 - 1)
sage: f = R.hom([[x*y, x*z, y*z], S)
sage: f.kernel()
Ideal (x^2*y^2 + x^2*z^2 + y^2*z^2 - x*y*z)
of Multivariate Polynomial Ring in x, y, z over Rational Field

lift(x=None)

Return a lifting map associated to this homomorphism, if it has been defined.

If x is not None, return the value of the lift morphism on x.

EXAMPLES:

sage: R.<x,y> = QQ[]
sage: f = R.hom([x,x])
sage: f(x+y)
2*x
sage: f.lift()
Traceback (most recent call last):
...
ValueError: no lift map defined
sage: g = R.hom(R)
sage: f._set_lift(g)
sage: f.lift() == g
True
sage: f.lift(x)
x

pushforward(I)

Returns the pushforward of the ideal I under this ring homomorphism.

EXAMPLES:

sage: R.<x,y> = QQ[]; S.<xx,yy> = R.quo([x^2,y^2]); f = S.cover()
sage: f.pushforward(R.ideal([x,3*x+x*y+y^2]))
Ideal (xx, xx*yy + 3*xx) of Quotient of Multivariate Polynomial Ring in x, y over Rational Field by the ideal (x^2, y^2)

class sage.rings.morphism.RingHomomorphism_coercion

Bases: sage.rings.morphism.RingHomomorphism

A ring homomorphism that is a coercion.
Warning: This class is obsolete. Set the category of your morphism to a subcategory of Rings instead.

class sage.rings.morphism.RingHomomorphism_cover

Bases: sage.rings.morphism.RingHomomorphism

A homomorphism induced by quotienting a ring out by an ideal.

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQ, 2)
sage: S.<a,b> = R.quo(x^2 + y^2)
sage: phi = S.cover(); phi
Ring morphism:
  From: Multivariate Polynomial Ring in x, y over Rational Field
  To:  Quotient of Multivariate Polynomial Ring in x, y over Rational Field by the
             → ideal (x^2 + y^2)
    Defn: Natural quotient map
sage: phi(x+y)
a + b
```

kernel()

Return the kernel of this covering morphism, which is the ideal that was quotiented out by.

EXAMPLES:

```
sage: f = Zmod(6).cover()
sage: f.kernel()
Principal ideal (6) of Integer Ring
```

class sage.rings.morphism.RingHomomorphism_from_base

Bases: sage.rings.morphism.RingHomomorphism

A ring homomorphism determined by a ring homomorphism of the base ring.

AUTHOR:
• Simon King (initial version, 2010-04-30)

EXAMPLES:

We define two polynomial rings and a ring homomorphism:

```
sage: R.<x,y> = QQ[]
sage: S.<z> = QQ[]
sage: f = R.hom([2*z,3*z],S)
```

Now we construct polynomial rings based on R and S, and let f act on the coefficients:

```
sage: PR.<t> = R[]
sage: PS = S['t']
sage: Pf = PR.hom(f,PS)
sage: Pf
Ring morphism:
  From: Univariate Polynomial Ring in t over Multivariate Polynomial Ring in x, y
             → over Rational Field
  To:  Univariate Polynomial Ring in t over Univariate Polynomial Ring in z over...
```

(continues on next page)
Defn: Induced from base ring by

Ring morphism:

From: Multivariate Polynomial Ring in x, y over Rational Field
To: Univariate Polynomial Ring in z over Rational Field
Defn: \( x \mapsto 2z \)
\( y \mapsto 3z \)

sage: p = (x - 4*y + 1/13)*t^2 + (1/2*x^2 - 1/3*y^2)*t + 2*y^2 + x
sage: Pf(p)
(-10*z + 1/13)*t^2 - z^2*t + 18*z^2 + 2*z

Similarly, we can construct the induced homomorphism on a matrix ring over our polynomial rings:

sage: MR = MatrixSpace(R,2,2)
sage: MS = MatrixSpace(S,2,2)
sage: M = MR([[x^2 + 1/7*x*y - y^2, - 1/2*y^2 + 2*y + 1/6, 4*x^2 - 14*x, 1/2*y^2 + 13/4*x - 2/11*y]])
sage: Mf = MR.hom(f,MS)
sage: Mf
Ring morphism:

From: Full MatrixSpace of 2 by 2 dense matrices over Multivariate Polynomial Ring in x, y over Rational Field
To: Full MatrixSpace of 2 by 2 dense matrices over Univariate Polynomial Ring in z over Rational Field
Defn: Induced from base ring by

Ring morphism:

From: Multivariate Polynomial Ring in x, y over Rational Field
To: Univariate Polynomial Ring in z over Rational Field
Defn: \( x \mapsto 2z \)
\( y \mapsto 3z \)

sage: Mf(M)
[[-29/7*z^2 - 9/2*z^2 + 6*z + 1/6]
[16*z^2 - 28*z 9/2*z^2 + 131/22*z]

The construction of induced homomorphisms is recursive, and so we have:

sage: MPR = MatrixSpace(PR, 2)
sage: MPS = MatrixSpace(PS, 2)
sage: M = MPR([[(- x + y)*t^2 + 58*t - 3*x^2 + x*y, (- 1/7*x*y - 1/40*x)*t^2 + (5*x^2 + y^2)*t + 2*y^2 + 5/2*y + 1/4, (x + 6*y + 1)*t^2]])
sage: MPf = MPR.hom(f,MPS); MPf
Ring morphism:

From: Full MatrixSpace of 2 by 2 dense matrices over Univariate Polynomial Ring in t over Multivariate Polynomial Ring in x, y over Rational Field
To: Full MatrixSpace of 2 by 2 dense matrices over Univariate Polynomial Ring in t over Univariate Polynomial Ring in z over Rational Field
Defn: Induced from base ring by

Ring morphism:

From: Univariate Polynomial Ring in t over Multivariate Polynomial Ring in x, y over Rational Field
To: Univariate Polynomial Ring in t over Univariate Polynomial Ring in z over Rational Field
Defn: \( x \mapsto 2z \)
\( y \mapsto 3z \)
Defn: Induced from base ring by
Ring morphism:
  From: Multivariate Polynomial Ring in x, y over Rational Field
  To:  Univariate Polynomial Ring in z over Rational Field
  Defn: x |--> 2*z
        y |--> 3*z

sage: MPf(M)
\[
\begin{align*}
  z^2 t^2 & + 58 t - 6 z^2 \\
-6/7 z^2 - 1/20 z & + 29 z^2 t + 6 z
\end{align*}
\]
\[
\begin{align*}
  -z + 1 & + 11 z^2 + 15/2 z + 1/4 \\
20 z + 1 & + t^2
\end{align*}
\]

inverse()
Return the inverse of this ring homomorphism if the underlying homomorphism of the base ring is invertible.

EXAMPLES:

sage: R.<x,y> = QQ[]
sage: S.<a,b> = QQ[]
sage: f = R.hom([a+b, a-b], S)
sage: PR.<t> = R[]
sage: PS = S['t']
sage: Pf = PR.hom(f, PS)
sage: Pf.inverse()
Ring morphism:
  From: Univariate Polynomial Ring in t over Multivariate Polynomial Ring in a, b over Rational Field
  To:  Univariate Polynomial Ring in t over Multivariate Polynomial Ring in x, y over Rational Field
  Defn: Induced from base ring by
  Ring morphism:
    From: Multivariate Polynomial Ring in a, b over Rational Field
    To:  Multivariate Polynomial Ring in x, y over Rational Field
    Defn: a |--> 1/2*x + 1/2*y
          b |--> 1/2*x - 1/2*y
sage: Pf.inverse()(Pf(x*t^2 + y*t))
x*t^2 + y*t

underlying_map()
Return the underlying homomorphism of the base ring.

EXAMPLES:

sage: R.<x,y> = QQ[]
sage: S.<z> = QQ[]
sage: f = R.hom([2*z,3*z], S)
sage: MR = MatrixSpace(R,2)
sage: MS = MatrixSpace(S,2)
sage: g = MR.hom(f,MS)
sage: g.underlying_map() == f
True

class sage.rings.morphism.RingHomomorphism_from_fraction_field
  Bases: sage.rings.morphism.RingHomomorphism

Morphisms between fraction fields.

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inverse()

Return the inverse of this ring homomorphism if it exists.

EXAMPLES:

```
sage: S.<x> = QQ[]
sage: f = S.hom([2*x - 1])
sage: g = f.extend_to_fraction_field()
sage: g.inverse()
Ring endomorphism of Fraction Field of Univariate Polynomial Ring
in x over Rational Field
Defn: x |--> 1/2*x + 1/2
```

class sage.rings.morphism.RingHomomorphism_from_quotient

Bases: `sage.rings.morphism.RingHomomorphism`

A ring homomorphism with domain a generic quotient ring.

INPUT:

- `parent` – a ring homset \( \text{Hom}(R, S) \)
- `phi` – a ring homomorphism \( C \rightarrow S \), where \( C \) is the domain of \( R\text{.cover()} \)

OUTPUT: a ring homomorphism

The domain \( R \) is a quotient object \( C \rightarrow R \), and \( R\text{.cover()} \) is the ring homomorphism \( \varphi : C \rightarrow R \). The condition on the elements \( \text{im}_\text{gens} \) of \( S \) is that they define a homomorphism \( C \rightarrow S \) such that each generator of the kernel of \( \varphi \) maps to 0.

EXAMPLES:

```
sage: R.<x, y, z> = PolynomialRing(QQ, 3)
sage: S.<a, b, c> = R.quo(x^3 + y^3 + z^3)
sage: phi = S.hom([b, c, a]); phi
Ring endomorphism of Quotient of Multivariate Polynomial Ring in x, y, z over Rational Field by the ideal (x^3 + y^3 + z^3)
Defn: a |--> b
b |--> c
c |--> a
sage: phi(a+b+c)
a + b + c
sage: loads(dumps(phi)) == phi
True
```

Validity of the homomorphism is determined, when possible, and a `TypeError` is raised if there is no homomorphism sending the generators to the given images:

```
sage: S.hom([[b^2, c^2, a^2]])
Traceback (most recent call last):
... ValueError: relations do not all (canonically) map to 0 under map determined by images of generators
```

morphism_from_cover()

Underlying morphism used to define this quotient map, i.e., the morphism from the cover of the domain.

EXAMPLES:
```python
sage: R.<x,y> = QQ[]; S.<xx,yy> = R.quo([x^2,y^2])
sage: S.hom([yy,xx]).morphism_from_cover()
Ring morphism:
  From: Multivariate Polynomial Ring in x, y over Rational Field
  To:   Quotient of Multivariate Polynomial Ring in x, y over Rational Field by
        the ideal (x^2, y^2)
  Defn: x |--> yy
        y |--> xx
```

class sage.rings.morphism.RingHomomorphism_im_gens

Bases: `sage.rings.morphism.RingHomomorphism`

A ring homomorphism determined by the images of generators.

```python
def base_map()
    Return the map on the base ring that is part of the defining data for this morphism. May return None if a
    coercion is used.

    EXAMPLES:

    sage: R.<x> = ZZ[]
sage: K.<i> = NumberField(x^2 + 1)
sage: cc = K.hom([-i])
sage: S.<y> = K[]
sage: phi = S.hom([y^2], base_map=cc)
sage: phi
    Ring endomorphism of Univariate Polynomial Ring in y over Number Field in i with defining polynomial x^2 + 1
    Defn: y |--> y^2
    with map of base ring
sage: phi(y)
y^2
sage: phi(i*y)
-i*y^2
sage: phi.base_map()
    Composite map:
        From: Number Field in i with defining polynomial x^2 + 1
        To:   Univariate Polynomial Ring in y over Number Field in i with defining polynomial x^2 + 1
        Defn: Ring endomorphism of Number Field in i with defining polynomial x^2 + 1
        then
        Polynomial base injection morphism:
            From: Number Field in i with defining polynomial x^2 + 1
            To:   Univariate Polynomial Ring in y over Number Field in i with defining polynomial x^2 + 1
```

```python
def im_gens()
    Return the images of the generators of the domain.

    OUTPUT:
    - list – a copy of the list of gens (it is safe to change this)

    EXAMPLES:
```
We verify that the returned list of images of gens is a copy, so changing it doesn’t change $f$:

```
sage: f.im_gens()[0] = 5
sage: f.im_gens()
[x, x + y]
```

```
class sage.rings.morphism.RingMap
    Bases: sage.categories.morphism.Morphism

    Set-theoretic map between rings.

class sage.rings.morphism.RingMap_lift
    Bases: sage.rings.morphism.RingMap

    Given rings $R$ and $S$ such that for any $x \in R$ the function $x.lift()$ is an element that naturally coerces to $S$, this returns the set-theoretic ring map $R \to S$ sending $x$ to $x.lift()$.

    EXAMPLES:

    sage: R.<x,y> = QQ[]
    sage: S.<xbar,ybar> = R.quo( (x^2 + y^2, y) )
    sage: S.lift()
    Set-theoretic ring morphism:
    From: Quotient of Multivariate Polynomial Ring in x, y over Rational Field by the
    \ideal (x^2 + y^2, y)
    To: Multivariate Polynomial Ring in x, y over Rational Field
    Defn: Choice of lifting map
    sage: S.lift() == 0
    False
```

Since trac ticket #11068, it is possible to create quotient rings of non-commutative rings by two-sided ideals. It was needed to modify `RingMap_lift` so that rings can be accepted that are no instances of `sage.rings.ring.Ring`, as in the following example:

```
sage: MS = MatrixSpace(GF(5),2,2)
sage: I = MS*[MS.0*MS.1,MS.2+MS.3]*MS
sage: Q = MS.quo(I)
sage: Q.0*Q.1  # indirect doctest
[0 1]
[0 0]
```

\texttt{sage.rings.morphism.is\_RingHomomorphism(\textit{phi})}

Return True if \textit{phi} is of type \texttt{RingHomomorphism}.

EXAMPLES:

```
sage: f = Zmod(8).cover()
sage: sage.rings.morphism.is\_RingHomomorphism(f)
doctest:warning
...  
DeprecationWarning: is\_RingHomomorphism() should not be used anymore. Check whether the category\_for() your morphism is a subcategory of Rings() instead
```
See http://trac.sagemath.org/23204 for details.

True

sage: sage.rings.morphism.is_RingHomomorphism(2/3)
False

3.2 Space of homomorphisms between two rings

```python
sage.rings.homset.RingHomset(R, S, category=None)
```

Construct a space of homomorphisms between the rings R and S.

For more on homsets, see `Hom()`.

**EXAMPLES:**

```python
sage: Hom(ZZ, QQ)  # indirect doctest
Set of Homomorphisms from Integer Ring to Rational Field
```

```python
class sage.rings.homset.RingHomset_generic(R, S, category=None)
```

Bases: `sage.categories.homset.HomsetWithBase`

A generic space of homomorphisms between two rings.

**EXAMPLES:**

```python
sage: Hom(ZZ, QQ)
Set of Homomorphisms from Integer Ring to Rational Field
sage: QQ.Hom(ZZ)
Set of Homomorphisms from Rational Field to Integer Ring
```

**Element**

alias of `sage.rings.morphism.RingHomomorphism`

**has_coerce_map_from(x)**

The default for coercion maps between ring homomorphism spaces is very restrictive (until more implementation work is done).

Currently this checks if the domains and the codomains are equal.

**EXAMPLES:**

```python
sage: H = Hom(ZZ, QQ)
sage: H2 = Hom(QQ, ZZ)
sage: H.has_coerce_map_from(H2)
False
```

**natural_map()**

Returns the natural map from the domain to the codomain.

The natural map is the coercion map from the domain ring to the codomain ring.

**EXAMPLES:**

```python
sage: H = Hom(ZZ, QQ)
sage: H.natural_map()
Natural morphism:
```

(continues on next page)
From: Integer Ring
To: Rational Field

zero()

Return the zero element of this homset.

EXAMPLES:

Since a ring homomorphism maps 1 to 1, there can only be a zero morphism when mapping to the trivial ring:

```
sage: Hom(ZZ, Zmod(1)).zero()
Ring morphism:
  From: Integer Ring
  To:   Ring of integers modulo 1
  Defn: 1 |--> 0

sage: Hom(ZZ, Zmod(2)).zero()
Traceback (most recent call last):
  ... ValueError: homset has no zero element
```

class sage.rings.homset.RingHomset_quo_ring(R, S, category=None)

Bases: sage.rings.homset.RingHomset_generic

Space of ring homomorphisms where the domain is a (formal) quotient ring.

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQ, 2)
sage: S.<a,b> = R.quotient(x^2 + y^2)
sage: phi = S.hom([b,a]); phi
Ring endomorphism of Quotient of Multivariate Polynomial Ring in x, y over Rational Field by the ideal (x^2 + y^2)
  Defn: a |--> b
  b |--> a
sage: phi(a)
b
sage: phi(b)
a
```

Element

alias of sage.rings.morphism.RingHomomorphism_from_quotient

sage.rings.homset.is_RingHomset(H)

Return True if H is a space of homomorphisms between two rings.

EXAMPLES:

```
sage: from sage.rings.homset import is_RingHomset as is_RH
sage: is_RH(Hom(ZZ, QQ))
True
sage: is_RH(ZZ)
False
sage: is_RH(Hom(RR, CC))
True
```
sage: is_RH(Hom(FreeModule(ZZ,1), FreeModule(QQ,1)))
False
4.1 Quotient Rings

AUTHORS:
• William Stein
• Simon King (2011-04): Put it into the category framework, use the new coercion model.
• Simon King (2011-04): Quotients of non-commutative rings by twosided ideals.

Todo: The following skipped tests should be removed once trac ticket #13999 is fixed:

```
sage: TestSuite(S).run(skip=['_test_nonzero_equal', '_test_elements', '_test_zero'])
```

In trac ticket #11068, non-commutative quotient rings $R/I$ were implemented. The only requirement is that the two-sided ideal $I$ provides a reduce method so that $I$.reduce(x) is the normal form of an element $x$ with respect to $I$ (i.e., we have $I$.reduce(x) == $I$.reduce(y) if $x - y \in I$, and $x - I$.reduce(x) in $I$). Here is a toy example:

```
sage: from sage.rings.noncommutative_ideals import Ideal_nc
sage: from itertools import product
class PowerIdeal(Ideal_nc):
    def __init__(self, R, n):
        self._power = n
        self._power = n
        Ideal_nc.__init__(self, R, [R.prod(m) for m in product(R.gens(), repeat=n)])
    def reduce(self,x):
        R = self.ring()
        return add([c*R(m) for m,c in x if len(m)<self._power],R(0))
sage: F.<x,y,z> = FreeAlgebra(QQ, 3)
sage: I3 = PowerIdeal(F,3); I3
Twosided Ideal (x^3, x^2*y, x^2*z, x*y^2, x*y*z, x^2*x, x^2*y, x*z^2, y*x^2, y*x*y, y*x*z, y^2*x, y^2*z, y*z*x, y*z*y, y*z^2, z*x^2, z*x*y, z*x*z, z*y^2, z*y*z, z^2*x, z^2*y, z^2) of Free Algebra on 3 generators (x, y, z) over Rational Field
```

Free algebras have a custom quotient method that serves at creating finite dimensional quotients defined by multiplication matrices. We are bypassing it, so that we obtain the default quotient:
Even though $Q_3$ is not commutative, there is commutativity for products of degree three:

$$a(b+c) - (b+c)a = F.\text{zero}()$$

True

If we quotient out all terms of degree two then of course the resulting quotient ring is commutative:

$$(a+b+2)^4$$

16 + 32*a + 32*b

Since trac ticket #7797, there is an implementation of free algebras based on Singular’s implementation of the Letterplace Algebra. Our letterplace wrapper allows to provide the above toy example more easily:

$$Q_2 = F.\text{quo}(F.\prod(m \text{ for } m \text{ in } \text{product}(F.\text{gens}(), \text{repeat}=2)), F)$$

True

`sage.rings.quotient_ring.QuotientRing`(R, I, names=None, **kwds)

Creates a quotient ring of the ring $R$ by the twosided ideal $I$.

Variables are labeled by names (if the quotient ring is a quotient of a polynomial ring). If names isn’t given, ‘bar’ will be appended to the variable names in $R$.

INPUT:

- **R** – a ring.
• $I$ – a twosided ideal of $R$.
• names – (optional) a list of strings to be used as names for the variables in the quotient ring $R/I$.
• further named arguments that will be passed to the constructor of the quotient ring instance.

OUTPUT: $R/I$ - the quotient ring $R$ mod the ideal $I$

ASSUMPTION:
I has a method $I.reduce(x)$ returning the normal form of elements $x \in R$. In other words, it is required that $I.reduce(x) == I.reduce(y) \iff x - y \in I$, and $x - I.reduce(x) \in I$, for all $x, y \in R$.

EXAMPLES:
Some simple quotient rings with the integers:

```sage
sage: R = QuotientRing(ZZ, 7*ZZ); R
Quotient of Integer Ring by the ideal (7)
sage: R.gens()
(1,)
sage: 1*R(3); 6*R(3); 7*R(3)
3
4
0
```

```sage
sage: S = QuotientRing(ZZ, ZZ.ideal(8)); S
Quotient of Integer Ring by the ideal (8)
sage: 2*S(4)
0
```

With polynomial rings (note that the variable name of the quotient ring can be specified as shown below):

```sage
sage: P.<x> = QQ[]
sage: R.<xx> = QuotientRing(P, P.ideal(x^2 + 1))
sage: R
Univariate Quotient Polynomial Ring in xx over Rational Field with modulus x^2 + 1
sage: R.gens(); R.gen()
(xx,)
xx
sage: for n in range(4): xx^n
1
xx
-1
-xx
```

```sage
sage: P.<x> = QQ[]
sage: S = QuotientRing(P, P.ideal(x^2 - 2))
sage: S
Univariate Quotient Polynomial Ring in xbar over Rational Field with modulus x^2 - 2
sage: xbar = S.gen(); S.gen()
xbar
sage: for n in range(3): xbar^n
1
xbar
2
```

4.1. Quotient Rings 67
Sage coerces objects into ideals when possible:

```sage
sage: P.<x> = QQ[]
sage: R = QuotientRing(P, x^2 + 1); R
Univariate Quotient Polynomial Ring in xbar over Rational Field with modulus x^2 + 1
```

By Noether’s homomorphism theorems, the quotient of a quotient ring of $R$ is just the quotient of $R$ by the sum of the ideals. In this example, we end up modding out the ideal $(x)$ from the ring $\mathbb{Q}[x, y]$:

```sage
sage: R.<x,y> = PolynomialRing(QQ,2)
sage: S.<a,b> = QuotientRing(R,R.ideal(1 + y^2))
sage: T.<c,d> = QuotientRing(S,S.ideal(a))
sage: T
Quotient of Multivariate Polynomial Ring in x, y over Rational Field by the ideal (x, y^2 + 1)
sage: R.gens(); S.gens(); T.gens()
(x, y)
(a, b)
(0, d)
sage: for n in range(4): d^n
1
d
-d
```

**class** `sage.rings.quotient_ring.QuotientRing_generic(R, I, names=None, category=None)`

Bases: `sage.rings.quotient_ring.QuotientRing_nc, sage.rings.ring.CommutativeRing`

Creates a quotient ring of a commutative ring $R$ by the ideal $I$.

**EXAMPLES:**

```sage
sage: R.<x> = PolynomialRing(ZZ)
sage: I = R.ideal([4 + 3*x + x^2, 1 + x^2])
sage: S = R.quotient_ring(I); S
Quotient of Univariate Polynomial Ring in x over Integer Ring by the ideal (x^2 + 3*x + 4, x^2 + 1)
sage: a*b
-b*c
```

**class** `sage.rings.quotient_ring.QuotientRing_nc(R, I, names=None, category=None)`

Bases: `sage.rings.ring.Ring, sage.structure.parent_gens.ParentWithGens`

The quotient ring of $R$ by a twosided ideal $I$.

This class is for rings that do not inherit from `CommutativeRing`.

**EXAMPLES:**

Here is a quotient of a free algebra by a twosided homogeneous ideal:

```sage
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace')
sage: I = F*[x*y+y*z,x^2+y-y*x-y^2]*F
sage: Q.<a,b,c> = F.quo(I); Q
Quotient of Free Associative Unital Algebra on 3 generators (x, y, z) over Rational Field by the ideal (x*y + y*z, x^2 + y^2 - y*x - y^2)
sage: a*b
-a*b
```

(continues on next page)
A quotient of a quotient is just the quotient of the original top ring by the sum of two ideals:

\[ J = Q[a^3-b^3]Q \]
\[ R.\langle i,j,k \rangle = Q.quo(J); R \]
\[ \text{Quotient of Free Associative Unital Algebra on 3 generators } (x, y, z) \text{ over Rational Field by the ideal } (-y*y*z - y*z*x - 2*y*z*z, x*y + y*z, x*x + x*y - y*x - y*y) \]
\[ i^3 \]
\[ -j*k*i - j*k*j - j*k*k \]
\[ j^3 \]
\[ -j*k*i - j*k*j - j*k*k \]

For rings that do inherit from `CommutativeRing`, we provide a subclass `QuotientRing_generic`, for backwards compatibility.

**EXAMPLES:**

<table>
<thead>
<tr>
<th>Sage Code</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>R.&lt;x&gt; = PolynomialRing(ZZ,'x')</code></td>
<td></td>
</tr>
<tr>
<td><code>I = R.ideal([4 + 3*x + x^2, 1 + x^2])</code></td>
<td></td>
</tr>
<tr>
<td><code>S = R.quotient_ring(I); S</code></td>
<td>Quotient of Univariate Polynomial Ring in x over Integer Ring by the ideal (x^2 + 3*x + 4, x^2 + 1)</td>
</tr>
<tr>
<td><code>R.&lt;x,y&gt; = PolynomialRing(QQ)</code></td>
<td></td>
</tr>
<tr>
<td><code>S.&lt;a,b&gt; = R.quo(x^2 + y^2)</code></td>
<td></td>
</tr>
<tr>
<td><code>a^2 + b^2 == 0</code></td>
<td>True</td>
</tr>
<tr>
<td><code>S(0) == a^2 + b^2</code></td>
<td>True</td>
</tr>
</tbody>
</table>

Again, a quotient of a quotient is just the quotient of the original top ring by the sum of two ideals.

<table>
<thead>
<tr>
<th>Sage Code</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>R.&lt;x,y&gt; = PolynomialRing(QQ,2)</code></td>
<td></td>
</tr>
<tr>
<td><code>S.&lt;a,b&gt; = R.quo(1 + y^2)</code></td>
<td></td>
</tr>
<tr>
<td><code>T.\langle c,d \rangle = S.quo(a)</code></td>
<td></td>
</tr>
<tr>
<td><code>T</code></td>
<td>Quotient of Multivariate Polynomial Ring in x, y over Rational Field by the ideal (x, y^2 + 1)</td>
</tr>
<tr>
<td><code>T gens()</code></td>
<td>(0, d)</td>
</tr>
</tbody>
</table>

**Element**

- alias of `sage.rings.quotient_ring_element.QuotientRingElement`

**ambient()**

Returns the cover ring of the quotient ring: that is, the original ring \( R \) from which we modded out an ideal, \( I \).

**EXAMPLES:**

<table>
<thead>
<tr>
<th>Sage Code</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>Q = QuotientRing(ZZ,7*ZZ)</code></td>
<td></td>
</tr>
<tr>
<td><code>Q. cover_ring()</code></td>
<td></td>
</tr>
</tbody>
</table>
Integer Ring

```
sage: P.<x> = QQ[]
sage: Q = QuotientRing(P, x^2 + 1)
sage: Q.cover_ring()
Univariate Polynomial Ring in x over Rational Field
```

### characteristic()

Return the characteristic of the quotient ring.

**Todo:** Not yet implemented!

**EXAMPLES:**

```
sage: Q = QuotientRing(ZZ,7*ZZ)
sage: Q.characteristic()
Traceback (most recent call last):
  ...  
NotImplementedError
```

### construction()

Returns the functorial construction of `self`.

**EXAMPLES:**

```
sage: R.<x> = PolynomialRing(ZZ,'x')
sage: I = R.ideal([4 + 3*x + x^2, 1 + x^2])
sage: R.quotient_ring(I).construction()
(QuotientFunctor, Univariate Polynomial Ring in x over Integer Ring)
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace')
sage: I = F*[x*y+y*z,x^2+x*y-y*x-y^2]*F
sage: Q = F.quo(I)
sage: Q.construction()
(QuotientFunctor, Free Associative Unital Algebra on 3 generators (x, y, z) \to over Rational Field)
```

### cover()

The covering ring homomorphism \( R \to R/I \), equipped with a section.

**EXAMPLES:**

```
sage: R = ZZ.quo(3*ZZ)
sage: pi = R.cover()
sage: pi
Ring morphism:
  From: Integer Ring
  To:   Ring of integers modulo 3
  Defn: Natural quotient map
sage: pi(5)
2
sage: l = pi.lift()
```

Chapter 4. Quotient Rings
cover_ring()

Returns the cover ring of the quotient ring: that is, the original ring \( R \) from which we modded out an ideal, \( I \).

EXAMPLES:

```python
sage: Q = QuotientRing(ZZ, 7*ZZ)
sage: Q.cover_ring()
Integer Ring
```

```python
sage: P.<x> = QQ[]
sage: Q = QuotientRing(P, x^2 + 1)
sage: Q.cover_ring()
Univariate Polynomial Ring in x over Rational Field
```

defining_ideal()

Returns the ideal generating this quotient ring.

EXAMPLES:

In the integers:

```python
sage: Q = QuotientRing(ZZ, 7*ZZ)
sage: Q.defining_ideal()
Principal ideal (7) of Integer Ring
```

An example involving a quotient of a quotient. By Noether’s homomorphism theorems, this is actually a quotient by a sum of two ideals:

```python
sage: R.<x,y> = PolynomialRing(QQ,2)
sage: S.<a,b> = QuotientRing(R,R.ideal(1 + y^2))
sage: T.<c,d> = QuotientRing(S,S.ideal(a))
sage: T.defining_ideal()
Ideal (y^2 + 1) of Multivariate Polynomial Ring in x, y over Rational Field
sage: T.defining_ideal()
Ideal (x, y^2 + 1) of Multivariate Polynomial Ring in x, y over Rational Field
```
\texttt{gen(i=0)}
\begin{quote}
Returns the $i$-th generator for this quotient ring.
\end{quote}
\begin{verbatim}
sage: R = QuotientRing(ZZ,7*ZZ)
sage: R.gen(0)
1
sage: R.<x,y> = PolynomialRing(QQ,2)
sage: S.<a,b> = QuotientRing(R,R.ideal(1 + y^2))
sage: T.<c,d> = QuotientRing(S,S.ideal(a))
sage: T
Quotient of Multivariate Polynomial Ring in x, y over Rational Field by the
˓→ideal (x, y^2 + 1)
sage: R.gen(0); R.gen(1)
x
y
sage: S.gen(0); S.gen(1)
a
b
sage: T.gen(0); T.gen(1)
0
d
\end{verbatim}
\texttt{ideal(*gens, **kwds)}
\begin{quote}
Return the ideal of self with the given generators.
\end{quote}
\begin{verbatim}
sage: R.<x,y> = PolynomialRing(QQ)
sage: S = R.quotient_ring(x^2+y^2)
sage: S.ideal()
Ideal (0) of Quotient of Multivariate Polynomial Ring in x, y over Rational Field by the ideal (x^2 + y^2)
sage: S.ideal(x+y+1)
Ideal (xbar + ybar + 1) of Quotient of Multivariate Polynomial Ring in x, y over Rational Field by the ideal (x^2 + y^2)
\end{verbatim}
\texttt{is_commutative()}
\begin{quote}
Tell whether this quotient ring is commutative.
\end{quote}
\begin{quote}
\textbf{Note:} This is certainly the case if the cover ring is commutative. Otherwise, if this ring has a finite number of generators, it is tested whether they commute. If the number of generators is infinite, a \texttt{NotImplementedError} is raised.
\end{quote}
\begin{quote}
\textbf{AUTHOR:}
\begin{itemize}
\item Simon King (2011-03-23): See trac ticket #7797.
\end{itemize}
\end{quote}
\begin{quote}
\textbf{EXAMPLES:}
\begin{verbatim}
Any quotient of a commutative ring is commutative:
\end{verbatim}
\end{quote}
The non-commutative case is more interesting:

```python
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace')
sage: I = F*[x*y+y*z,x^2+x*y-y*x-y^2]*F
sage: Q = F.quo(I)
sage: Q.is_commutative()
False
sage: Q.1*Q.2==Q.2*Q.1
False
```

In the next example, the generators apparently commute:

```python
sage: J = F*[x*y-y*x,x*z-z*x,y*z-z*y,x^3-y^3]*F
sage: R = F.quo(J)
sage: R.is_commutative()
True
```

**is_field** *(proof=True)*

Returns True if the quotient ring is a field. Checks to see if the defining ideal is maximal.

**is_integral_domain** *(proof=True)*

With proof equal to True (the default), this function may raise a `NotImplementedError`. When proof is False, if True is returned, then self is definitely an integral domain. If the function returns False, then either self is not an integral domain or it was unable to determine whether or not self is an integral domain.

**EXAMPLES:**

```python
sage: R.<x,y> = QQ[]
sage: R.quo(x^2 - y).is_integral_domain()
True
sage: R.quo(x^2 - y^2).is_integral_domain()
False
sage: R.quo(x^2 - y^2).is_integral_domain(proof=False)
False
sage: R.<a,b,c> = ZZ[]
sage: Q = R.quotient_ring([a, b])
sage: Q.is_integral_domain()
Traceback (most recent call last):
...  
NotImplementedError
sage: Q.is_integral_domain(proof=False)
False
```

**is_noetherian()**

Return True if this ring is Noetherian.

**EXAMPLES:**

```
sage: R = QuotientRing(ZZ, 102*ZZ)
sage: R.is_noetherian()
True

sage: P.<x> = QQ[]
sage: R = QuotientRing(P, x^2+1)
sage: R.is_noetherian()
True

If the cover ring of self is not Noetherian, we currently have no way of testing whether self is Noetherian, so we raise an error:

sage: R.<x> = InfinitePolynomialRing(QQ)
sage: R.is_noetherian()
False

sage: I = R.ideal([x[1]^2, x[2]])
sage: S = R.quotient(I)
sage: S.is_noetherian()
Traceback (most recent call last):
... Not ImplementedError

lift(x=none)

Return the lifting map to the cover, or the image of an element under the lifting map.

Note: The category framework imposes that Q.lift(x) returns the image of an element x under the lifting map. For backwards compatibility, we let Q.lift() return the lifting map.

EXAMPLES:

sage: R.<x,y> = PolynomialRing(QQ, 2)
sage: S = R.quotient(x^2 + y^2)
sage: S.lift()
Set-theoretic ring morphism:
From: Quotient of Multivariate Polynomial Ring in x, y over Rational Field by...
To: Multivariate Polynomial Ring in x, y over Rational Field
Defn: Choice of lifting map
sage: S.lift(S.0) == x
True

lifting_map()

Return the lifting map to the cover.

EXAMPLES:

sage: R.<x,y> = PolynomialRing(QQ, 2)
sage: S = R.quotient(x^2 + y^2)
sage: pi = S.cover(); pi
Ring morphism:
From: Multivariate Polynomial Ring in x, y over Rational Field
To: Quotient of Multivariate Polynomial Ring in x, y over Rational Field by...
Defn: Choice of lifting map
sage: pi.lifting_map()
(continues on next page)
Defn: Natural quotient map

```sage
L = S.lifting_map(); L
```

Set-theoretic ring morphism:

- From: Quotient of Multivariate Polynomial Ring in x, y over Rational Field by
  the ideal \( (x^2 + y^2) \)
- To: Multivariate Polynomial Ring in x, y over Rational Field

Defn: Choice of lifting map

```sage
L(S.0)
x
L(S.1)
y
```

Note that some reduction may be applied so that the lift of a reduction need not equal the original element:

```sage
z = pi(x^3 + 2*y^2); z
-xbar*ybar^2 + 2*ybar^2
sage: L(z)
-x*y^2 + 2*y^2
sage: L(z) == x^3 + 2*y^2
False
```

Test that there also is a lift for rings that are no instances of `Ring` (see trac ticket #11068):

```sage
MS = MatrixSpace(GF(5),2,2)
I = MS*[MS.0*MS.1,MS.2+MS.3]*MS
Q = MS.quo(I)
Q.lift()
```

Set-theoretic ring morphism:

- From: Quotient of Full MatrixSpace of 2 by 2 dense matrices over Finite Field of size 5 by the ideal
  \[
  \begin{pmatrix}
  0 & 1 \\
  0 & 0
  \end{pmatrix},
  \begin{pmatrix}
  0 & 0 \\
  1 & 1
  \end{pmatrix}
  \]
- To: Full MatrixSpace of 2 by 2 dense matrices over Finite Field of size 5

Defn: Choice of lifting map

```sage
ngens()
```

Returns the number of generators for this quotient ring.

**Todo:** Note that `ngens` counts 0 as a generator. Does this make sense? That is, since 0 only generates itself and the fact that this is true for all rings, is there a way to “knock it off” of the generators list if a generator of some original ring is modded out?

**EXAMPLES:**

```sage
R = QuotientRing(ZZ,7*ZZ)
sage: R.ngens(); R.ngens()
```

(continues on next page)
sage: R.<x,y> = PolynomialRing(QQ,2)
sage: S.<a,b> = QuotientRing(R,R.ideal(1 + y^2))
sage: T.<c,d> = QuotientRing(S,S.ideal(a))
sage: T
Quotient of Multivariate Polynomial Ring in x, y over Rational Field by the
→ ideal (x, y^2 + 1)
sage: R.gens(); S.gens(); T.gens()
(x, y)
(a, b)
(0, d)
sage: R.ngens(); S.ngens(); T.ngens()
2
2
2

retract(x)
The image of an element of the cover ring under the quotient map.

INPUT:
• x – An element of the cover ring

OUTPUT:
The image of the given element in self.

EXAMPLES:

sage: R.<x,y> = PolynomialRing(QQ, 2)
sage: S = R.quotient(x^2 + y^2)
sage: S.retract((x+y)^2)
2*xbar*ybar

term_order()
Return the term order of this ring.

EXAMPLES:

sage: P.<a,b,c> = PolynomialRing(QQ)
sage: I = Ideal([a^2 - a, b^2 - b, c^2 - c])
sage: Q = P.quotient(I)
sage: Q.term_order()
Degree reverse lexicographic term order

sage.rings.quotient_ring.is_QuotientRing(x)
Tests whether or not x inherits from QuotientRing_nc.

EXAMPLES:

sage: from sage.rings.quotient_ring import is_QuotientRing
sage: R.<x> = PolynomialRing(ZZ,'x')
sage: I = R.ideal([4 + 3*x + x^2, 1 + x^2])
sage: S = R.quotient_ring(I)
sage: is_QuotientRing(S)
True
sage: is_QuotientRing(R)
False

\[
\begin{align*}
\text{sage: } & F.<x,y,z> = FreeAlgebra(QQ, \text{implementation=}'letterplace') \\
\text{sage: } & I = F^*[x*y+y*z,x^2+x*y-y*x-y^2]*F \\
\text{sage: } & Q = F.quo(I) \\
\text{sage: } & is_\text{QuotientRing}(Q) \\
& \text{True} \\
\text{sage: } & is_\text{QuotientRing}(F) \\
& \text{False}
\end{align*}
\]

4.2 Quotient Ring Elements

AUTHORS:

• William Stein

class sage.rings.quotient_ring_element.QuotientRingElement(parent, rep, reduce=True)

Bases: sage.structure.element.RingElement

An element of a quotient ring \( R/I \).

INPUT:

• parent - the ring \( R/I \)
• rep - a representative of the element in \( R \); this is used as the internal representation of the element
• reduce - bool (optional, default: True) - if True, then the internal representation of the element is rep reduced modulo the ideal \( I \)

EXAMPLES:

\[
\begin{align*}
\text{sage: } & R.<x> = PolynomialRing(ZZ) \\
\text{sage: } & S.<\text{xbar}> = R.quo((4 + 3*x + x^2, 1 + x^2)); S \\
\text{Quotient of Univariate Polynomial Ring in x over Integer Ring by the ideal (x^2 +} \\
\text{→3*x + 4, x^2 + 1)} \\
\text{sage: } & v = S.gens(); v \\
& (xbar,)
\end{align*}
\]

\[
\begin{align*}
\text{sage: } & \text{loads(v[0].dumps()) == v[0]} \\
& \text{True}
\end{align*}
\]

\[
\begin{align*}
\text{sage: } & R.<x,y> = PolynomialRing(QQ, 2) \\
\text{sage: } & S = R.quo(x^2 + y^2); S \\
\text{Quotient of Multivariate Polynomial Ring in x, y over Rational Field by the ideal} \\
\text{→(x^2 + y^2)} \\
\text{sage: } & S.gens() \\
& (xbar, ybar)
\end{align*}
\]

We name each of the generators.
```sage
S.<a,b> = R.quotient(x^2 + y^2)
sage: a
a
data type
sage: b
b
data type
sage: a^2 + b^2 == 0
True
sage: b.lift()
y
sage: (a^3 + b^2).lift()
-x*y^2 + y^2
```

### is_unit()

Return True if self is a unit in the quotient ring.

**EXAMPLES:**

```sage
sage: R.<x,y> = QQ[]; S.<a,b> = R.quo(1 - x*y); type(a)
<class 'sage.rings.quotient_ring.QuotientRing_generic_with_category.element_class'>
sage: a*b
1
sage: S(2).is_unit()
True
```

Check that trac ticket #29469 is fixed:

```sage
sage: a.is_unit()
True
sage: (a+b).is_unit()
False
```

### lcl()

Return the leading coefficient of this quotient ring element.

**EXAMPLES:**

```sage
sage: R.<x,y,z>=PolynomialRing(GF(7),3,order='lex')
sage: I = sage.rings.ideal.FieldIdeal(R)
sage: Q = R.quo( I )
sage: f = Q( z*y + 2*x )
sage: f.lc()
2
```

### lift()

If self is an element of \( R/I \), then return self as an element of \( R \).

**EXAMPLES:**

```sage
sage: R.<x,y> = QQ[]; S.<a,b> = R.quo(x^2 + y^2); type(a)
<class 'sage.rings.quotient_ring.QuotientRing_generic_with_category.element_class'>
sage: a.lift()
x
```

lm()
Return the leading monomial of this quotient ring element.

EXAMPLES:

```
sage: R.<x,y,z>=PolynomialRing(GF(7),3,order='lex')
sage: I = sage.rings.ideal.FieldIdeal(R)
sage: Q = R.quo( I )
sage: f = Q( z*y + 2*x )
sage: f.lm()
xbar
```

lt()
Return the leading term of this quotient ring element.

EXAMPLES:

```
sage: R.<x,y,z>=PolynomialRing(GF(7),3,order='lex')
sage: I = sage.rings.ideal.FieldIdeal(R)
sage: Q = R.quo( I )
sage: f = Q( z*y + 2*x )
sage: f.lt()
2*xbar
```

monomials()
Return the monomials in self.

OUTPUT:
A list of monomials.

EXAMPLES:

```
sage: R.<x,y> = QQ[ ]; S.<a,b> = R.quo(x^2 + y^2); type(a)
<class 'sage.rings.quotient_ring.QuotientRing_generic_with_category.element_˓
class'>
sage: a.monomials()
[a]
sage: (a+a*b).monomials()
[a*b, a]
sage: R.zero().monomials()
[]
```

reduce(G)
Reduce this quotient ring element by a set of quotient ring elements G.

INPUT:
• G - a list of quotient ring elements

EXAMPLES:

```
sage: P.<a,b,c,d,e> = PolynomialRing(GF(2), 5, order='lex')
sage: I1 = ideal([a*b + c*d + 1, a*c*e + d*e, a*b*e + c*e, b*c + c*d*e + 1])
```
variables()

Return all variables occurring in self.

OUTPUT:

A tuple of linear monomials, one for each variable occurring in self.

EXAMPLES:

\begin{verbatim}
sage: R.<x,y> = QQ[]; S.<a,b> = R.quo(x^2 + y^2); type(a) <class 'sage.rings.quotient_ring.QuotientRing_generic_with_category.element_class'>
sage: a.variables()
(a,)
sage: b.variables()
(b,)
sage: s = a^2 + b^2 + 1; s
1
sage: s.variables()
()
sage: (a+b).variables()
(a, b)
\end{verbatim}
5.1 Fraction Field of Integral Domains

AUTHORS:

- William Stein (with input from David Joyner, David Kohel, and Joe Wetherell)
- Burcin Erocal
- Julian Rüth (2017-06-27): embedding into the field of fractions and its section

EXAMPLES:

Quotienting is a constructor for an element of the fraction field:

\[
\text{sage: } \text{R.<x> = QQ[]}
\]
\[
\text{sage: } (x^2-1)/(x+1)
\]
\[
x - 1
\]
\[
\text{sage: } \text{parent)((x^2-1)/(x+1))
\]
\[
\text{Fraction Field of Univariate Polynomial Ring in x over Rational Field}
\]

The GCD is not taken (since it doesn’t converge sometimes) in the inexact case:

\[
\text{sage: } Z.<z> = \text{CC[]}
\]
\[
\text{sage: } I = \text{CC}.\text{gen()}
\]
\[
\text{sage: } (1+I*z)/(z+0.1*I)
\]
\[
(z + 1.00000000000000 + I)/(z + 0.100000000000000*I)
\]
\[
\text{sage: } (1+I^2*z)/(z+1.1)
\]
\[
(I^2z + 1.00000000000000)/(z + 1.10000000000000)
\]

\sage\text {.rings.fraction_field.}\text {FractionField}(R, \text {names=None})

Create the fraction field of the integral domain \( R \).

INPUT:

- \( R \) – an integral domain
- \( \text{ names } \) – ignored

EXAMPLES:

We create some example fraction fields:

\sage: \text{FractionField(IntegerRing())}
Rational Field
\sage: \text{FractionField(PolynomialRing(RationalField(),'x'))}

(continues on next page)
Fraction Field of Univariate Polynomial Ring in x over Rational Field
\texttt{sage: FractionField(PolynomialRing(IntegerRing(),'x'))}
Fraction Field of Univariate Polynomial Ring in x over Integer Ring
\texttt{sage: FractionField(PolynomialRing(RationalField(),2,'x'))}
Fraction Field of Multivariate Polynomial Ring in x0, x1 over Rational Field
\texttt{dividing elements often implicitly creates elements of the fraction field:}
\texttt{sage: x = PolynomialRing(RationalField(), 'x').gen()}
\texttt{sage: f = x/(x+1)}
\texttt{sage: g = x**3/(x+1)}
\texttt{sage: f/g}
\texttt{1/x^2}
\texttt{sage: g/f}
\texttt{x^2}
The input must be an integral domain:
\texttt{sage: Frac(Integers(4))}
\texttt{Traceback (most recent call last):}
\texttt{...}
\texttt{TypeError: R must be an integral domain.}
\texttt{class} \texttt{sage.rings.fraction_field.FractionFieldEmbedding}
\texttt{Bases: sage.structure.coerce_maps.DefaultConvertMap_unique}
The embedding of an integral domain into its field of fractions.
EXAMPLES:
\texttt{sage: R.<x> = QQ[]}
\texttt{sage: f = R.fraction_field().coerce_map_from(R); f}
\texttt{Coercion map:}
\texttt{From: Univariate Polynomial Ring in x over Rational Field}
\texttt{To: Fraction Field of Univariate Polynomial Ring in x over Rational Field}
\texttt{is_injective()}
Return whether this map is injective.
EXAMPLES:
The map from an integral domain to its fraction field is always injective:
\texttt{sage: R.<x> = QQ[]}
\texttt{sage: R.fraction_field().coerce_map_from(R).is_injective()}
\texttt{True}
\texttt{is_surjective()}
Return whether this map is surjective.
EXAMPLES:
\texttt{sage: R.<x> = QQ[]}
\texttt{sage: R.fraction_field().coerce_map_from(R).is_surjective()}
\texttt{False}
section()
Return a section of this map.

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: R.fraction_field().coerce_map_from(R).section()
Section map:
  From: Fraction Field of Univariate Polynomial Ring in x over Rational Field
  To:  Univariate Polynomial Ring in x over Rational Field
```

class sage.rings.fraction_field.FractionFieldEmbeddingSection
Bases: sage.categories.map.Section

The section of the embedding of an integral domain into its field of fractions.

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: f = R.fraction_field().coerce_map_from(R).section(); f
Section map:
  From: Fraction Field of Univariate Polynomial Ring in x over Rational Field
  To:  Univariate Polynomial Ring in x over Rational Field
```

class sage.rings.fraction_field.FractionField_1poly_field(R, element_class=<class 'sage.rings.fraction_field_element.FractionFieldElement_1poly_field'>)
Bases: sage.rings.fraction_field.FractionField_generic

The fraction field of a univariate polynomial ring over a field.

Many of the functions here are included for coherence with number fields.

class_number()
Here for compatibility with number fields and function fields.

EXAMPLES:

```python
sage: R.<t> = GF(5)[]; K = R.fraction_field()
sage: K.class_number()
1
```

function_field()
Return the isomorphic function field.

EXAMPLES:

```python
sage: R.<t> = GF(5)[]
sage: K = R.fraction_field()
sage: K.function_field()
Rational function field in t over Finite Field of size 5
```

See also:
sage.rings.function_field.RationalFunctionField.field()

maximal_order()
Return the maximal order in this fraction field.

EXAMPLES:
sage: K = FractionField(GF(5)['t'])
sage: K.maximal_order()
Univariate Polynomial Ring in t over Finite Field of size 5

ring_of_integers()
Return the ring of integers in this fraction field.

EXAMPLES:

sage: K = FractionField(GF(5)['t'])
sage: K.ring_of_integers()
Univariate Polynomial Ring in t over Finite Field of size 5

class sage.rings.fraction_field.FractionField_generic(R, element_class=<class 'sage.rings.fraction_field_element.FractionFieldElement'>, category=Category of quotient fields)
Bases: sage.rings.ring.Field
The fraction field of an integral domain.

base_ring()
Return the base ring of self.
This is the base ring of the ring which this fraction field is the fraction field of.

EXAMPLES:

sage: R = Frac(ZZ['t'])
sage: R.base_ring()
Integer Ring

class sage.rings.fraction_field.FractionField_generic()

characteristic()
Return the characteristic of this fraction field.

EXAMPLES:

sage: R = Frac(ZZ['t'])
sage: R.base_ring()
Integer Ring
sage: R = Frac(ZZ['t']); R.characteristic()
0
sage: R = Frac(GF(5)['w']); R.characteristic()
5

construction()
EXAMPLES:

sage: Frac(ZZ['x']).construction()
(FractionField, Univariate Polynomial Ring in x over Integer Ring)
sage: K = Frac(GF(3)['t'])
sage: f, R = K.construction()
sage: f(R)
Fraction Field of Univariate Polynomial Ring in t over Finite Field of size 3
sage: f(R) == K
True
gen(i=0)
   Return the i-th generator of self.

   EXAMPLES:
   \begin{verbatim}
   sage: R = Frac(PolynomialRing(QQ,'z',10)); R
   Fraction Field of Multivariate Polynomial Ring in z0, z1, z2, z3, z4, z5, z6,... → z7, z8, z9 over Rational Field
   sage: R.0
   z0
   sage: R.gen(3)
   z3
   sage: R.3
   z3
   \end{verbatim}

is_exact()
   Return if self is exact which is if the underlying ring is exact.

   EXAMPLES:
   \begin{verbatim}
   sage: Frac(ZZ['x']).is_exact()
   True
   sage: Frac(CDF['x']).is_exact()
   False
   \end{verbatim}

is_field(proof=True)
   Return True, since the fraction field is a field.

   EXAMPLES:
   \begin{verbatim}
   sage: Frac(ZZ).is_field()
   True
   \end{verbatim}

is_finite()
   Tells whether this fraction field is finite.

   Note: A fraction field is finite if and only if the associated integral domain is finite.

   EXAMPLES:
   \begin{verbatim}
   sage: Frac(QQ['a','b','c']).is_finite()
   False
   \end{verbatim}

ngens()
   This is the same as for the parent object.

   EXAMPLES:
   \begin{verbatim}
   sage: R = Frac(PolynomialRing(QQ,'z',10)); R
   Fraction Field of Multivariate Polynomial Ring in z0, z1, z2, z3, z4, z5, z6,... → z7, z8, z9 over Rational Field
   sage: R.ngens()
   10
   \end{verbatim}

random_element(*args, **kwds)
   Return a random element in this fraction field.
The arguments are passed to the random generator of the underlying ring.

EXAMPLES:

```
sage: F = ZZ['x'].fraction_field()
sage: F.random_element()  # random
(2*x - 8)/(-x^2 + x)
```

```sage:
f = F.random_element(degree=5)
sage: f.numerator().degree()
5
sage: f.denominator().degree()
5```

```
ring()
Return the ring that this is the fraction field of.

EXAMPLES:

```sage:
R = Frac(QQ['x,y'])
sage: R
Fraction Field of Multivariate Polynomial Ring in x, y over Rational Field
sage: R.ring()
Multivariate Polynomial Ring in x, y over Rational Field```

```
some_elements()
Return some elements in this field.

EXAMPLES:

```sage:
R.<x> = QQ[]
sage: R.fraction_field().some_elements()
[0,
 1,
x,
2*x,
x/(x^2 + 2*x + 1),
1/x^2,
...
(2*x^2 + 2)/(x^2 + 2*x + 1),
(2*x^2 + 2)/x^3,
(2*x^2 + 2)/(x^2 - 1),
2]
```

```
sage.rings.fraction_field.is_FractionField(x)
Test whether or not x inherits from FractionField_generic.

EXAMPLES:

```sage: from sage.rings.fraction_field import is_FractionField
sage: is_FractionField(Frac(ZZ['x']))
True
sage: is_FractionField(QQ)
False```
5.2 Fraction Field Elements

AUTHORS:

- William Stein (input from David Joyner, David Kohel, and Joe Wetherell)
- Sebastian Pancratz (2010-01-06): Rewrite of addition, multiplication and derivative to use Henrici’s algorithms [Hor1972]

class sage.rings.fraction_field_element.FractionFieldElement
Bases: sage.structure.element.FieldElement

EXAMPLES:

```
sage: K = FractionField(PolynomialRing(QQ, 'x'))
sage: K
Fraction Field of Univariate Polynomial Ring in x over Rational Field
sage: loads(K.dumps()) == K
True
sage: x = K.gen()
sage: f = (x^3 + x)/(17 - x^19); f
(-x^3 - x)/(x^19 - 17)
sage: loads(f.dumps()) == f
True
```

denominator()

Return the denominator of self.

EXAMPLES:

```
sage: R.<x,y> = ZZ[]
sage: f = x/y+1; f
(x + y)/y
sage: f.denominator()
y
```

is_one()

Return True if this element is equal to one.

EXAMPLES:

```
sage: F = ZZ['x,y'].fraction_field()
sage: x,y = F.gens()
sage: (x/x).is_one()
True
sage: (x/y).is_one()
False
```

is_square(root=False)

Return whether or not self is a perfect square.

If the optional argument root is True, then also returns a square root (or None, if the fraction field element is not square).

INPUT:

- **root** – whether or not to also return a square root (default: False)

OUTPUT:
• bool - whether or not a square
• object - (optional) an actual square root if found, and None otherwise.

EXAMPLES:

```python
sage: R.<t> = QQ[]
sage: (1/t).is_square()
False
sage: (1/t^6).is_square()
True
sage: ((1+t)^4/t^6).is_square()
True
sage: (4*(1+t)^4/t^6).is_square()
True
sage: (2*(1+t)^4/t^6).is_square()
False
sage: ((1+t)/t^6).is_square()
False
```

```python
sage: (4*(1+t)^4/t^6).is_square(root=True)
(True, (2*t^2 + 4*t + 2)/t^3)
```

```python
sage: (2*(1+t)^4/t^6).is_square(root=True)
(False, None)
```

```python
sage: R.<x> = QQ[]
sage: a = 2*(x+1)^2 / (2*(x-1)^2); a
(x^2 + 2*x + 1)/(x^2 - 2*x + 1)
sage: a.is_square()
True
sage: (0/x).is_square()
True
```

is_zero()

Return True if this element is equal to zero.

EXAMPLES:

```python
sage: F = ZZ['x,y'].fraction_field()
sage: x,y = F.gens()
sage: t = F(0)/x
sage: t.is_zero()
True
sage: u = 1/x - 1/x
sage: u.is_zero()
True
sage: u.parent() is F
True
```

nth_root(n)

Return a n-th root of this element.

EXAMPLES:

```python
sage: R = QQ['t'].fraction_field()
sage: t = R.gen()
(continues on next page)
sage: p = (t+1)^3 / (t^2+t-1)^3
sage: p.nth_root(3)
(t + 1)/(t^2 + t - 1)

sage: p = (t+1) / (t-1)
sage: p.nth_root(2)
Traceback (most recent call last):
  ...  
ValueError: not a 2nd power

**numerator()**

Return the numerator of self.

**EXAMPLES:**

```python
sage: R.<x,y> = ZZ[]
sage: f = x/y+1; f
(x + y)/y
sage: f.numerator()
x + y
```

**reduce()**

Reduce this fraction.

Divides out the gcd of the numerator and denominator. If the denominator becomes a unit, it becomes 1. Additionally, depending on the base ring, the leading coefficients of the numerator and the denominator may be normalized to 1.

Automatically called for exact rings, but because it may be numerically unstable for inexact rings it must be called manually in that case.

**EXAMPLES:**

```python
sage: R.<x> = RealField(10)[]
sage: f = (x^2+2*x+1)/(x+1); f
(x^2 + 2.0*x + 1.0)/(x + 1.0)
sage: f.reduce(); f
x + 1.0
```

**specialization**(D=None, phi=None)

Returns the specialization of a fraction element of a polynomial ring

**valuation**(v=None)

Return the valuation of self, assuming that the numerator and denominator have valuation functions defined on them.

**EXAMPLES:**

```python
sage: x = PolynomialRing(RationalField(),'x').gen()
sage: f = (x^3 + x)/(x^2 - 2*x^3)
sage: f
(-1/2*x^2 - 1/2)/(x^2 - 1/2*x)
sage: f.valuation()
-1
sage: f.valuation(x^2+1)
1
```
class sage.rings.fraction_field_element.FractionFieldElement_ipoly_field
Bases: sage.rings.fraction_field_element.FractionFieldElement

A fraction field element where the parent is the fraction field of a univariate polynomial ring over a field.

Many of the functions here are included for coherence with number fields.

is_integral()

Returns whether this element is actually a polynomial.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: elt = (t^2 + t - 2) / (t + 2); elt
# == (t + 2)*(t - 1)/(t + 2)
t - 1
sage: elt.is_integral()
True
sage: elt = (t^2 - t) / (t + 2); elt
# == t*(t - 1)/(t + 2)
(t^2 - t)/(t + 2)
sage: elt.is_integral()
False
```

reduce()

Pick a normalized representation of self.

In particular, for any a == b, after normalization they will have the same numerator and denominator.

EXAMPLES:

For univariate rational functions over a field, we have:

```
sage: R.<x> = QQ[]
sage: (2 + 2*x) / (4*x)
# indirect doctest
(1/2*x + 1/2)/x
```

Compare with:

```
sage: R.<x> = ZZ[]
sage: (2 + 2*x) / (4*x)
(x + 1)/(2*x)
```

support()

Returns a sorted list of primes dividing either the numerator or denominator of this element.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: h = (t^14 + 2*t^12 - 4*t^11 - 8*t^9 + 6*t^8 + 12*t^6 - 4*t^5 - 8*t^3 + t^2 + 2)/(t^6 + 6*t^5 + 9*t^4 - 2*t^2 - 12*t - 18)
sage: h.support()
[t - 1, t + 3, t^2 + 2, t^2 + t + 1, t^4 - 2]
```

sage.rings.fraction_field_element.is_FractionFieldElement(x)

Return whether or not x is a FractionFieldElement.

EXAMPLES:
sage: from sage.rings.fraction_field_element import is_FractionFieldElement
sage: R.<x> = ZZ[]
sage: is_FractionFieldElement(x/2)
False
sage: is_FractionFieldElement(2/x)
True
sage: is_FractionFieldElement(1/3)
False

sage.rings.fraction_field_element.make_element(parent, numerator, denominator)
Used for unpickling FractionFieldElement objects (and subclasses).

EXAMPLES:

sage: from sage.rings.fraction_field_element import make_element
sage: R = ZZ['x,y']
sage: x,y = R.gens()
sage: F = R.fraction_field()
sage: make_element(F, 1+x, 1+y)
(x + 1)/(y + 1)

sage.rings.fraction_field_element.make_element_old(parent, cdict)
Used for unpickling old FractionFieldElement pickles.

EXAMPLES:

sage: from sage.rings.fraction_field_element import make_element_old
sage: R.<x,y> = ZZ[]
sage: F = R.fraction_field()
sage: make_element_old(F, {'_FractionFieldElement__numerator':x+y,'_
˓FractionFieldElement__denominator':x-y})
(x + y)/(x - y)
6.1 Localization

Localization is an important ring construction tool. Whenever you have to extend a given integral domain such that it contains the inverses of a finite set of elements but should allow non injective homomorphic images this construction will be needed. See the example on Ariki-Koike algebras below for such an application.

**EXAMPLES:**

```sage
sage: LZ = Localization(ZZ, (5,11))
sage: m = matrix(LZ, [[5, 7], [0,11]])
sage: m.parent()
Full MatrixSpace of 2 by 2 dense matrices over Integer Ring localized at (5, 11)
sage: ~m  # parent of inverse is different: see documentation of m.__invert__
[ 1/5 -7/55]
[ 0 1/11]
sage: _.parent()
Full MatrixSpace of 2 by 2 dense matrices over Rational Field
sage: mi = matrix(LZ, ~m)
sage: mi.parent()
Full MatrixSpace of 2 by 2 dense matrices over Integer Ring localized at (5, 11)
sage: mi == ~m
True
```

The next example defines the most general ring containing the coefficients of the irreducible representations of the Ariki-Koike algebra corresponding to the three colored permutations on three elements:

```sage
sage: R.<u0, u1, u2, q> = ZZ[]
sage: u = [u0, u1, u2]
sage: S = Set(u)
sage: I = S.cartesian_product(S)
sage: add_units = u + [q, q+1] + [ui -uj for ui, uj in I if ui != uj]
+ [q*ui -uj for ui, uj in I if ui != uj]
sage: L = R.localization(tuple(add_units)); L
Multivariate Polynomial Ring in u0, u1, u2, q over Integer Ring localized at (q, q + 1, u2, u1, u1 - u2, u0, u0 - u2, u0 - u1, u2*q - u1, u2*q - u0, u1*q - u2, u1*q - u0, u0*q - u2, u0*q - u1)
```

Define the representation matrices (of one of the three dimensional irreducible representations):
m1 = matrix(L, [[u1, 0, 0],[0, u0, 0],[0, 0, u0]])
m2 = matrix(L, [[(u0*q - u0)/(u0 - u1), (u0*q - u1)/(u0 - u1), 0],
                [(-u1*q + u0)/(u0 - u1), (-u1*q + u1)/(u0 - u1), 0],
                [0, 0, -1]])
m3 = matrix(L, [[-1, 0, 0],
                [0, u0*(1 - q)/(u1*q - u0), q*(u1 - u0)/(u1*q - u0)],
                [0, (u1*q^2 - u0)/(u1*q - u0), (u1*q^2 - u1*q)/(u1*q - u0)]]

m1.base_ring() == L
True

Check relations of the Ariki-Koike algebra:

m1*m2*m1*m2 == m2*m1*m2*m1
True
m2*m3*m2 == m3*m2*m3
True
m1*m3 == m3*m1
True
m1**3 - (u0+u1+u2)*m1**2 + (u0*u1+u0*u2+u1*u2)*m1 - u0*u1*u2 == 0
True
m2**2 - (q-1)*m2 - q == 0
True
m3**2 - (q-1)*m3 - q == 0
True
~m1 in m1.parent()
True
~m2 in m2.parent()
True
~m3 in m3.parent()
True

Obtain specializations in positive characteristic:

Fp = GF(17)
f = L.hom((3,5,7,11), codomain=Fp); f
Ring morphism:
    From: Multivariate Polynomial Ring in u0, u1, u2, q over Integer Ring localized at (q, q + 1, u2, u1 - u2, u0 - u2, u0 - u1, u2*q - u1, u2*q - u0, u1*q - u2, u1*q - u0, u0*q - u2, u0*q - u1)
    To: Finite Field of size 17
    Defn: u0 |--> 3
          u1 |--> 5
          u2 |--> 7
          q |--> 11
mFp1 = matrix({k:f(v) for k, v in m1.dict().items()}); mFp1
[5 0 0]
[0 3 0]
[0 0 3]
mFp1.base_ring()
Finite Field of size 17
mFp2 = matrix({k:f(v) for k, v in m2.dict().items()}); mFp2
[2 3 0]
[9 8 0]

(continues on next page)
Obtain specializations in characteristic 0:

```python
sage: fQ = L.hom((3,5,7,11), codomain=QQ); fQ
Ring morphism:
  From: Multivariate Polynomial Ring in u0, u1, u2, q over Integer Ring localized at
   (q, q + 1, u2, u1 - u2, u0, u0 - u2, u0 - u1, u2*q - u1, u2*q - u0, u1*q - u2, u1*q - u0, u0*q - u2, u0*q - u1)
  To:    Rational Field
  Defn: u0 |--> 3
         u1 |--> 5
         u2 |--> 7
         q |--> 11
sage: mQ1 = matrix({k:fQ(v) for k, v in m1.dict().items()}); mQ1
[5 0 0]
[0 3 0]
[0 0 3]
sage: mQ1.base_ring() == F
Rational Field
sage: mQ2 = matrix({k:fQ(v) for k, v in m2.dict().items()}); mQ2
[-15 -14 0]
[ 26 25 0]
[ 0 0 -1]
sage: mQ3 = matrix({k:fQ(v) for k, v in m3.dict().items()}); mQ3
[-1 0 0]
[0 -15/26 11/26]
[0 301/26 275/26]
sage: S.<x, y, z, t> = QQ[]
sage: T = S.quo(x+y+z)
sage: F = T.fraction_field()
sage: fF = L.hom((x, y, z, t), codomain=F); fF
Ring morphism:
  From: Multivariate Polynomial Ring in u0, u1, u2, q over Integer Ring localized at
   (q, q + 1, u2, u1 - u2, u0, u0 - u2, u0 - u1, u2*q - u1, u2*q - u0, u1*q - u2, u1*q - u0, u0*q - u2, u0*q - u1)
  To:    Fraction Field of Quotient of Multivariate Polynomial Ring in x, y, z, t over
          Rational Field by the ideal (x + y + z)
  Defn: u0 |--> -ybar - zbar
         u1 |--> ybar
         u2 |--> zbar
         q |--> tbar
sage: mF1 = matrix({k:fF(v) for k, v in m1.dict().items()}); mF1
[ybar 0 0]
[0 -ybar - zbar 0]
[0 0 -ybar - zbar]
sage: mF1.base_ring() == F
```

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AUTHORS:
• Sebastian Oehms 2019-12-09: initial version.

class sage.rings.localization.Localization(base_ring, additional_units, names=None, normalize=True, category=None, warning=True)

Bases: sage.rings.ring.IntegralDomain, sage.structure.unique_representation.UniqueRepresentation

The localization generalizes the construction of the field of fractions of an integral domain to an arbitrary ring. Given a (not necessarily commutative) ring \( R \) and a subset \( S \) of \( R \), there exists a ring \( R[S^{-1}] \) together with the ring homomorphism \( R \rightarrow R[S^{-1}] \) that “inverts” \( S \); that is, the homomorphism maps elements in \( S \) to unit elements in \( R[S^{-1}] \) and, moreover, any ring homomorphism from \( R \) that “inverts” \( S \) uniquely factors through \( R[S^{-1}] \).

The ring \( R[S^{-1}] \) is called the localization of \( R \) with respect to \( S \). For example, if \( R \) is a commutative ring and \( f \) an element in \( R \), then the localization consists of elements of the form \( r/f, r \in R, n \geq 0 \) (to be precise, \( R[f^{-1}] = R[t]/(ft - 1) \).

The above text is taken from Wikipedia. The construction here used for this class relies on the construction of the field of fraction and is therefore restricted to integral domains.

Accordingly, this class is inherited from IntegralDomain and can only be used in that context. Furthermore, the base ring should support sage.structure.element.CommutativeRingElement.divides() and the exact division operator \((sage.structure.element.Element.__floordiv__())\) in order to guarantee an successful application.

INPUT:
• base_ring – an instance of Ring allowing the construction of fraction_field() (that is an integral domain)
• additional_units – tuple of elements of base_ring which should be turned into units
• names – passed to IntegralDomain
• normalize – (optional, default: True) passed to IntegralDomain
• category – (optional, default: None) passed to IntegralDomain
• warning – (optional, default: True) to suppress a warning which is thrown if self cannot be represented uniquely

REFERENCES:
• Wikipedia article Ring_(mathematics)#Localization

EXAMPLES:

```
sage: L = Localization(ZZ, (3,5))
sage: 1/45 in L
   True
sage: 1/43 in L
   False
```

(continues on next page)
True

```python
sage: L(~7)
Traceback (most recent call last):
... 
ValueError: factor 7 of denominator is not a unit
```

```python
sage: Localization(Zp(7), (3, 5))
Traceback (most recent call last):
... 
ValueError: all given elements are invertible in 7-adic Ring with capped relative →
˓precision 20
```

```python
sage: R.<x> = ZZ[]
sage: L = R.localization(x**2+1)
sage: s = (x+5)/(x**2+1)
sage: s in L
True
sage: t = (x+5)/(x**2+2)
sage: t in L
False
sage: L(t)
Traceback (most recent call last):
... 
TypeError: fraction must have unit denominator
```

```python
sage: L(s) in R
False
sage: y = L(x)
sage: g = L(s)
sage: g.parent()
Univariate Polynomial Ring in x over Integer Ring localized at (x^2 + 1,)
sage: f = (y+5)/(y**2+1); f
(x + 5)/(x^2 + 1)
sage: f == g
True
sage: (y+5)/(y**2+2)
Traceback (most recent call last):
... 
ValueError: factor x^2 + 2 of denominator is not a unit
```

More examples will be shown typing sage.rings.localization?

**Element**

alias of `LocalizationElement`

**characteristic()**

Return the characteristic of self.

**EXAMPLES:**

```python
sage: R.<a> = GF(5)[]
sage: L = R.localization((a**2-3, a))
sage: L.characteristic()
5
```
fraction_field()
Return the fraction field of self.

EXAMPLES:

```sage
sage: R.<a> = GF(5)[]
sage: L = Localization(R, (a**2-3, a))
sage: L.fraction_field()
Fraction Field of Univariate Polynomial Ring in a over Finite Field of size 5
sage: L.is_subring(_)  # _ is some object
True
```

gen(i)
Return the i-th generator of self which is the i-th generator of the base ring.

EXAMPLES:

```sage
sage: R.<x, y> = ZZ[]
sage: R.localization((x**2+1, y-1)).gen(0)
x
sage: ZZ.localization(2).gen(0)
1
```
gens()
Return a tuple whose entries are the generators for this object, in order.

EXAMPLES:

```sage
sage: R.<x, y> = ZZ[]
sage: Localization(R, (x**2+1, y-1)).gens()
(x, y)
sage: Localization(ZZ, 2).gens()
(1,)
```

is_field(proof=True)
Return True if this ring is a field.

INPUT:
- proof – (default: True) Determines what to do in unknown cases

ALGORITHM:
If the parameter proof is set to True, the returned value is correct but the method might throw an error. Otherwise, if it is set to False, the method returns True if it can establish that self is a field and False otherwise.

EXAMPLES:

```sage
sage: R = ZZ.localization((2,3))
sage: R.is_field()  # _ is some object
False
```

krull_dimension()
Return the Krull dimension of this localization.
Since the current implementation just allows integral domains as base ring and localization at a finite set of elements the spectrum of \texttt{self} is open in the irreducible spectrum of its base ring. Therefore, by density we may take the dimension from there.

**EXAMPLES:**

```
sage: R = ZZ.localization((2,3))
sage: R.krull_dimension()
1
```

\texttt{ngens()}

Return the number of generators of \texttt{self} according to the same method for the base ring.

**EXAMPLES:**

```
sage: R.<x, y> = ZZ[]
sage: Localization(R, (x**2+1, y-1)).ngens()
2
sage: Localization(ZZ, 2).ngens()
1
```

class \texttt{sage.rings.localization.LocalizationElement}(\texttt{parent, x})

Bases: \texttt{sage.structure.element.IntegralDomainElement}

Element class for localizations of integral domains

**INPUT:**

- \texttt{parent} – instance of \texttt{Localization}
- \texttt{x} – instance of \texttt{FractionFieldElement} whose parent is the fraction field of the parent’s base ring

**EXAMPLES:**

```
sage: from sage.rings.localization import LocalizationElement
sage: P.<x,y,z> = GF(5)[]
sage: L = P.localization((x, y*z-x))
sage: LocalizationElement(L, 4/(y*z-x)**2)
(-1)/(y^2*z^2 - 2*x*y*z + x^2)
sage: _.parent()
Multivariate Polynomial Ring in x, y, z over Finite Field of size 5 localized at (x, y*z - x)
```

denominator()

Return the denominator of \texttt{self}.

**EXAMPLES:**

```
sage: L = Localization(ZZ, (3,5))
sage: L(7/15).denominator()
15
```

\texttt{inverse_of_unit()}

Return the inverse of \texttt{self}.

**EXAMPLES:**
sage: P.<x,y,z> = ZZ[
 sage: L = Localization(P, x*y*z)
sage: L(x*y*z).inverse_of_unit()
 1/(x*y*z)
sage: L(z).inverse_of_unit()
 1/z

is_unit()
Return True if self is a unit.

EXAMPLES:

sage: P.<x,y,z> = QQ[
 sage: L = P.localization((x, y*z))
sage: L(y*z).is_unit()  
 True
 sage: L(z).is_unit()  
 True
 sage: L(x*y*z).is_unit()  
 True

numerator()
Return the numerator of self.

EXAMPLES:

sage: L = ZZ.localization((3,5))
sage: L(7/15).numerator()
 7

sage.rings.localization.normalize_additional_units(base_ring, add_units, warning=True)
Function to normalize input data.
The given list will be replaced by a list of the involved prime factors (if possible).

INPUT:

• base_ring – an instance of IntegralDomain
• add_units – list of elements from base ring
• warning – (optional, default: True) to suppress a warning which is thrown if no normalization was possible

OUTPUT:
List of all prime factors of the elements of the given list.

EXAMPLES:

sage: from sage.rings.localization import normalize_additional_units
sage: normalize_additional_units(ZZ, [3, -15, 45, 9, 2, 50])
[2, 3, 5]
sage: P.<x,y,z> = ZZ[
 sage: normalize_additional_units(P, [3*x, z*y**2, 2*z, 18*(x*y*z)**2, x*z, 6*x*z, ...
   → 5])
[2, 3, 5, z, y, x]
sage: P.<x,y,z> = QQ[
 sage: normalize_additional_units(P, [3*x, z*y**2, 2*z, 18*(x*y*z)**2, x*z, 6*x*z, ...
   → 5])
(continues on next page)
\[ z, y, x \]

\begin{verbatim}
  sage: R.<x, y> = ZZ[]
  sage: Q.<a, b> = R.quo(x^2-5)
  sage: p = b^2-5
  sage: p == (b-a)*(b+a)
  True
  sage: normalize_additional_units(Q, [p])
  doctest:...: UserWarning: Localization may not be represented uniquely
  [b^2 - 5]
  sage: normalize_additional_units(Q, [p], warning=False)
  [b^2 - 5]
\end{verbatim}
CHAPTER SEVEN

RING EXTENSIONS

7.1 Extension of rings

Sage offers the possibility to work with ring extensions $L/K$ as actual parents and perform meaningful operations on them and their elements.

The simplest way to build an extension is to use the method `sage.categories.commutative_rings.CommutativeRings.ParentMethods.over()` on the top ring, that is $L$. For example, the following line constructs the extension of finite fields $F_{5^4}/F_{5^2}$:

```
sage: GF(5^4).over(GF(5^2))
Field in z4 with defining polynomial x^2 + (4*z2 + 3)*x + z2 over its base
```

By default, Sage reuses the canonical generator of the top ring (here $z_4 \in F_{5^4}$), together with its name. However, the user can customize them by passing in appropriate arguments:

```
sage: F = GF(5^2)
sage: k = GF(5^4)
sage: z4 = k.gen()
sage: K.<a> = k.over(F, gen = 1-z4)
sage: K
Field in a with defining polynomial x^2 + z2*x + 4 over its base
```

The base of the extension is available via the method `base()` (or equivalently `base_ring()`):

```
sage: K.base()
Finite Field in z2 of size 5^2
```

It is also possible to build an extension on top of another extension, obtaining this way a tower of extensions:

```
sage: L.<b> = GF(5^8).over(K)
sage: L
Field in b with defining polynomial x^2 + (4*z2 + 3*a)*x + 1 - a over its base
sage: L.base()
Field in a with defining polynomial x^2 + z2*x + 4 over its base
sage: L.base().base()
Finite Field in z2 of size 5^2
```

The method `bases()` gives access to the complete list of rings in a tower:

```
sage: L.bases()
[Field in b with defining polynomial x^2 + (4*z2 + 3*a)*x + 1 - a over its base,]
(continues on next page)
Field in a with defining polynomial $x^2 + z2*x + 4$ over its base, Finite Field in z2 of size $5^2$]

Once we have constructed an extension (or a tower of extensions), we have interesting methods attached to it. As a basic example, one can compute a basis of the top ring over any base in the tower:

```
sage: L.basis_over(K)
[1, b]
sage: L.basis_over(F)
[1, a, b, a*b]
```

When the base is omitted, the default is the natural base of the extension:

```
sage: L.basis_over()
[1, b]
```

The method `sage.rings.ring_extension_element.RingExtensionWithBasis.vector()` computes the coordinates of an element according to the above basis:

```
sage: u = a + 2*b + 3*a*b
sage: u.vector()  # over K
(a, 2 + 3*a)
sage: u.vector(F)
(0, 1, 2, 3)
```

One can also compute traces and norms with respect to any base of the tower:

```
sage: u.trace()  # over K
(2*z2 + 1) + (2*z2 + 1)*a
eras: u.trace(F)
z2 + 1
sage: u.trace().trace()  # over K, then over F
z2 + 1
```

```
sage: u.norm()  # over K
(z2 + 1) + (4*z2 + 2)*a
eras: u.norm(F)
2*z2 + 2
```

And minimal polynomials:

```
sage: u.minpoly()
x^2 + ((3*z2 + 4) + (3*z2 + 4)*a)*x + (z2 + 1) + (4*z2 + 2)*a
sage: u.minpoly(F)
x^4 + (4*z2 + 4)*x^3 + x^2 + (z2 + 1)*x + 2*z2 + 2
```

AUTHOR:

- Xavier Caruso (2019)

class `sage.rings.ring_extension.RingExtensionFactory`
  Bases: `sage.structure.factory.UniqueFactory`  
  Factory for ring extensions.
**create_key_and_extra_args**(*ring*, *defining_morphism=None*, *gens=None*, *names=None*, *constructors=None*)

Create a key and return it together with a list of constructors of the object.

**INPUT:**

- **ring** – a commutative ring
- **defining_morphism** – a ring homomorphism or a commutative ring or **None** (default: **None**); the defining morphism of this extension or its base (if it coerces to **ring**)
- **gens** – a list of generators of this extension (over its base) or **None** (default: **None**);
- **names** – a list or a tuple of variable names or **None** (default: **None**)
- **constructors** – a list of constructors; each constructor is a pair (**class**, **arguments**) where **class** is the class implementing the extension and **arguments** is the dictionary of arguments to pass in to init function

**create_object**(*version*, *key*, **extra_args*)

Return the object associated to a given key.

**class** **sage.rings.ring_extension.RingExtensionFractionField**

Bases: **sage.rings.ring_extension.RingExtension_generic**

A class for ring extensions of the form \( \text{Frac}(A)/A \).

**Element**

alias of **sage.rings.ring_extension_element.RingExtensionFractionFieldElement**

**ring**()

Return the ring whose fraction field is this extension.

**EXAMPLES:**

```python
sage: A.<a> = ZZ.extension(x^2 - 2)
sage: OK = A.over()
sage: K = OK.fraction_field()
sage: K
Fraction Field of Order in Number Field in a with defining polynomial x^2 - 2 → over its base
sage: K.ring()
Order in Number Field in a with defining polynomial x^2 - 2 over its base
sage: K.ring() is OK
True
```

**class** **sage.rings.ring_extension.RingExtensionWithBasis**

Bases: **sage.rings.ring_extension.RingExtension_generic**

A class for finite free ring extensions equipped with a basis.

**Element**

alias of **sage.rings.ring_extension_element.RingExtensionWithBasisElement**

**basis_over**(*base=None*)

Return a basis of this extension over **base**.

**INPUT:**

- **base** – a commutative ring (which might be itself an extension)

**EXAMPLES:**

---

7.1. Extension of rings
If base is omitted, it is set to its default which is the base of the extension:

\[
\begin{align*}
\text{sage: L.basis_over()} & \quad [1, c, c^2] \\
\text{sage: K.basis_over()} & \quad [1, b]
\end{align*}
\]

Note that base must be an explicit base over which the extension has been defined (as listed by the method bases()):

\[
\begin{align*}
\text{sage: L.degree_over(GF(5^6))} & \quad \text{Traceback (most recent call last):} \\
& \quad \text{...} \\
& \quad \text{ValueError: not (explicitly) defined over Finite Field in z6 of size 5^6}
\end{align*}
\]

\textbf{fraction_field}(extend\_base=False)

Return the fraction field of this extension.

**INPUT:**

- extend\_base – a boolean (default: False);

If extend\_base is False, the fraction field of the extension \(L/K\) is defined as \(\text{Frac}(L)/L/K\), except is \(L\) is already a field in which base the fraction field of \(L/K\) is \(L/K\) itself.

If extend\_base is True, the fraction field of the extension \(L/K\) is defined as \(\text{Frac}(L)/\text{Frac}(K)\) (provided that the defining morphism extends to the fraction fields, i.e. is injective).

**EXAMPLES:**

\[
\begin{align*}
\text{sage: A.<a> = ZZ.extension(x^2 - 5)} \\
\text{sage: OK = A.over()} \quad \text{# over ZZ} \\
\text{sage: OK} & \quad \text{Order in Number Field in a with defining polynomial x^2 - 5 over its base} \\
\text{sage: K1 = OK.fraction_field()} \\
\text{sage: K1} & \quad \text{Fraction Field of Order in Number Field in a with defining polynomial x^2 - 5 \over its base} \\
\text{sage: K1.bases()} \\
& \quad \text{[Fraction Field of Order in Number Field in a with defining polynomial x^2 - 5 \over its base,}
\end{align*}
\]
Order in Number Field in a with defining polynomial \( x^2 - 5 \) over its base, Integer Ring]

\[
\text{sage: K2 = OK.fraction_field(extend_base=True)} \\
\text{sage: K2}
\]

Fraction Field of Order in Number Field in a with defining polynomial \( x^2 - 5 \) over its base

\[
\text{sage: K2.bases()}
\]

[Fraction Field of Order in Number Field in a with defining polynomial \( x^2 - 5 \) over its base, Rational Field]

Note that there is no coercion map between \( K_1 \) and \( K_2 \):

\[
\text{sage: K1.has_coerce_map_from(K2)} \\
\text{False} \\
\text{sage: K2.has_coerce_map_from(K1)} \\
\text{False}
\]

We check that when the extension is a field, its fraction field does not change:

\[
\text{sage: K1.fraction_field() is K1} \\
\text{True} \\
\text{sage: K2.fraction_field() is K2} \\
\text{True}
\]

\texttt{free_module}(\textit{base}=None, \textit{map}=True)

Return a free module \( V \) over \( base \) which is isomorphic to this ring

INPUT:

\begin{itemize}
  \item base – a commutative ring (which might be itself an extension) or None (default: None)
  \item map – boolean (default True); whether to return isomorphisms between this ring and \( V \)
\end{itemize}

OUTPUT:

\begin{itemize}
  \item A finite-rank free module \( V \) over \( base \)
  \item The isomorphism from \( V \) to this ring corresponding to the basis output by the method basis_over() (only included if \( map \) is True)
  \item The reverse isomorphism of the isomorphism above (only included if \( map \) is True)
\end{itemize}

EXAMPLES:

\[
\text{sage: F = GF(11)} \\
\text{sage: K.<a> = GF(11^2).over()} \\
\text{sage: L.<b> = GF(11^6).over(K)}
\]

Forgetting a part of the multiplicative structure, the field \( L \) can be viewed as a vector space of dimension 3 over \( K \), equipped with a distinguished basis, namely \( (1, b, b^2) \):

\[
\text{sage: V, i, j = L.free_module(K)} \\
\text{sage: V}
\]

Vector space of dimension 3 over Field in a with defining polynomial \( x^2 + 7*x + 2 \) over its base
Similarly, one can view $L$ as a $F$-vector space of dimension 6:

\begin{verbatim}
sage: V, i, j, = L.free_module(F)
sage: V
Vector space of dimension 6 over Finite Field of size 11
\end{verbatim}

In this case, the isomorphisms between $V$ and $L$ are given by the basis $(1, a, b, ab, b^2, ab^2)$:

\begin{verbatim}
sage: j(a*b) (0, 0, 0, 1, 0, 0) sage: i((1,2,3,4,5,6)) (1 + 2*a) + (3 + 4*a)*b + (5 + 6*a)*b^2
\end{verbatim}

When base is omitted, the default is the base of this extension:

\begin{verbatim}
sage: L.free_module(map=False)
Vector space of dimension 3 over Field in a with defining polynomial x^2 + 7*x + 2 over its base
\end{verbatim}

Note that base must be an explicit base over which the extension has been defined (as listed by the method bases()):

\begin{verbatim}
sage: L.degree(GF(11^3))
Traceback (most recent call last):
  ... ValueError: not (explicitly) defined over Finite Field in z3 of size 11^3
\end{verbatim}

**class** `sage.rings.ring_extension.RingExtensionWithGen`

Bases: `sage.rings.ring_extension.RingExtensionWithBasis`

A class for finite free ring extensions generated by a single element

**fraction_field**(extend_base=False)

Return the fraction field of this extension.

**INPUT:**

- extend_base – a boolean (default: False);

If extend_base is False, the fraction field of the extension $L/K$ is defined as $\text{Frac}(L)/L/K$, except is $L$ is already a field in which case the fraction field of $L/K$ is $L/K$ itself.
If `extend_base` is `True`, the fraction field of the extension $L/K$ is defined as $\text{Frac}(L)/\text{Frac}(K)$ (provided that the defining morphism extends to the fraction fields, i.e. is injective).

**EXAMPLES:**

```python
sage: A.<a> = ZZ.extension(x^2 - 5)
sage: OK = A.over()  # over ZZ
sage: OK
Order in Number Field in a with defining polynomial x^2 - 5 over its base
sage: K1 = OK.fraction_field()
sage: K1
Fraction Field of Order in Number Field in a with defining polynomial x^2 - 5 over its base
sage: K1.bases()
[Fraction Field of Order in Number Field in a with defining polynomial x^2 - 5 over its base,
 Order in Number Field in a with defining polynomial x^2 - 5 over its base, Integer Ring]
sage: K2 = OK.fraction_field(extend_base=True)
sage: K2
Fraction Field of Order in Number Field in a with defining polynomial x^2 - 5 over its base
sage: K2.bases()
[Fraction Field of Order in Number Field in a with defining polynomial x^2 - 5 over its base,
 Rational Field]
```

Note that there is no coercion map between $K_1$ and $K_2$:

```python
sage: K1.has_coerce_map_from(K2)
False
sage: K2.has_coerce_map_from(K1)
False
```

We check that when the extension is a field, its fraction field does not change:

```python
sage: K1.fraction_field() is K1
True
sage: K2.fraction_field() is K2
True
```

**gens** *(base=None)*

Return the generators of this extension over `base`.

**INPUT:**

- `base` – a commutative ring (which might be itself an extension) or `None` (default: `None`)

**EXAMPLES:**

```python
sage: K.<a> = GF(5^2).over()  # over GF(5)
sage: K.gens()
(a,)
```

(continues on next page)
sage: L.<b> = GF(5^4).over(K)
sage: L.gens()
(b,)
sage: L.gens(GF(5))
(b, a)

**modulus**(var='x')
Return the defining polynomial of this extension, that is the minimal polynomial of the given generator of this extension.

**INPUT:**
- var – a variable name (default: x)

**EXAMPLES:**

sage: K.<u> = GF(7^10).over(GF(7^2))
sage: K
Field in u with defining polynomial x^5 + (6*z2 + 4)*x^4 + (3*z2 + 5)*x^3 + (2*z2 + 2)*x^2 + 4*x + 6*z2 over its base

sage: P = K.modulus(); P
x^5 + (6*z2 + 4)*x^4 + (3*z2 + 5)*x^3 + (2*z2 + 2)*x^2 + 4*x + 6*z2
sage: P(u)
0

We can use a different variable name:

sage: K.modulus('y')
y^5 + (6*z2 + 4)*y^4 + (3*z2 + 5)*y^3 + (2*z2 + 2)*y^2 + 4*y + 6*z2

**class** sage.rings.ring_extension.RingExtension_generic
Bases: sage.rings.ring.CommutativeAlgebra

A generic class for all ring extensions.

**Element**
alias of sage.rings.ring_extension_element.RingExtensionElement

**absolute_base()**
Return the absolute base of this extension.

By definition, the absolute base of an iterated extension $K_n/\cdots/K_2/K_1$ is the ring $K_1$.

**EXAMPLES:**

sage: F = GF(5^2).over()  # over GF(5)
sage: K = GF(5^4).over(F)
sage: L = GF(5^12).over(K)

sage: F.absolute_base()
Finite Field of size 5
sage: K.absolute_base()
Finite Field of size 5
sage: L.absolute_base()
Finite Field of size 5
See also:

`base()`, `bases()`, `is_defined_over()`

**absolute_degree()**

Return the degree of this extension over its absolute base

EXAMPLES:

```python
sage: A = GF(5^4).over(GF(5^2))
sage: B = GF(5^12).over(A)
sage: A.absolute_degree()
2
sage: B.absolute_degree()
6
```

See also:

`degree()`, `relative_degree()`

**base()**

Return the base of this extension.

EXAMPLES:

```python
sage: F = GF(5^2)
sage: K = GF(5^4).over(F)
sage: K.base()
Finite Field in z2 of size 5^2
```

In case of iterated extensions, the base is itself an extension:

```python
sage: L = GF(5^8).over(K)
sage: L.base()
Field in z4 with defining polynomial x^2 + (4*z2 + 3)*x + z2 over its base
sage: L.base() is K
True
```

See also:

`bases()`, `absolute_base()`, `is_defined_over()`

**bases()**

Return the list of successive bases of this extension (including itself).

EXAMPLES:

```python
sage: F = GF(5^2).over() # over GF(5)
sage: K = GF(5^4).over(F)
sage: L = GF(5^12).over(K)
sage: F.bases()
[Field in z2 with defining polynomial x^2 + 4*x + 2 over its base,
 Finite Field of size 5]
sage: K.bases()
[Field in z4 with defining polynomial x^2 + (3 - z2)*x + z2 over its base,
 Field in z4 with defining polynomial x^2 + (4*z2 + 3)*x + z2 over its base,
 Finite Field of size 5]
```

(continues on next page)
Field in \( \mathbb{Z}_2 \) with defining polynomial \( x^2 + 4\cdot x + 2 \) over its base, Finite Field of size 5

\begin{verbatim}
sage: L.bases()
[Field in \( \mathbb{Z}_{12} \) with defining polynomial \( x^3 + (1 + (2 - z_2)z_4)x^2 + (2 + 2z_4)x - z_4 \) over its base, Field in \( \mathbb{Z}_4 \) with defining polynomial \( x^2 + (3 - z_2)x + z_2 \) over its base, Field in \( \mathbb{Z}_2 \) with defining polynomial \( x^2 + 4\cdot x + 2 \) over its base, Finite Field of size 5]
\end{verbatim}

See also:

\begin{verbatim}
base(), absolute_base(), is_defined_over()
\end{verbatim}

construction()

Return the functorial construction of this extension, if defined.

EXAMPLES:

\begin{verbatim}
sage: E = GF(5^3).over()
sage: E.construction()
\end{verbatim}

\textbf{defining_morphism}(\textit{base=None})

Return the defining morphism of this extension over \textit{base}.

\textbf{INPUT:}

- \textbf{base} – a commutative ring (which might be itself an extension) or None (default: None)

\textbf{EXAMPLES:}

\begin{verbatim}
sage: F = GF(5^2)
sage: K = GF(5^4).over(F)
sage: L = GF(5^12).over(K)
sage: K.defining_morphism()
Ring morphism:
  From: Finite Field in \( z_2 \) of size 5^2
  To: Field in \( z_4 \) with defining polynomial \( x^2 + (4\cdot z_2 + 3)x + z_2 \) over its base
  Defn: \( z_2 \rightarrow z_2 \)
sage: L.defining_morphism()
Ring morphism:
  From: Field in \( z_4 \) with defining polynomial \( x^2 + (4\cdot z_2 + 3)x + z_2 \) over its base
  To: Field in \( \mathbb{Z}_{12} \) with defining polynomial \( x^3 + (1 + (4\cdot z_2 + 2)z_4)x^2 + (2 + 2z_4)x - z_4 \) over its base
  Defn: \( z_4 \rightarrow z_4 \)
\end{verbatim}

One can also pass in a base over which the extension is explicitly defined (see also \textit{is_defined_over()}):

\begin{verbatim}
sage: L.defining_morphism(F)
Ring morphism:
  From: Finite Field in \( z_2 \) of size 5^2
\end{verbatim}
To: Field in z12 with defining polynomial x^3 + (1 + (4*z2 + 2)*z4)*x^2 + (2 + 2*z4)*x - z4 over its base
Defn: z2 |--> z2

```
sage: L.defining_morphism(GF(5))
Traceback (most recent call last):
... ValueError: not (explicitly) defined over Finite Field of size 5
```

**degree**(base)
Return the degree of this extension over base.

**INPUT:**
- base – a commutative ring (which might be itself an extension)

**EXAMPLES:**

```
sage: A = GF(5^4).over(GF(5^2))
sage: B = GF(5^12).over(A)
sage: A.degree(GF(5^2))
2
sage: B.degree(A)
3
sage: B.degree(GF(5^2))
6
```

Note that base must be an explicit base over which the extension has been defined (as listed by the method **bases()**):

```
sage: A.degree(GF(5))
Traceback (most recent call last):
... ValueError: not (explicitly) defined over Finite Field of size 5
```

**See also:**
- **relative_degree()**, **absolute_degree()**

**degree_over**(base=None)
Return the degree of this extension over base.

**INPUT:**
- base – a commutative ring (which might be itself an extension) or None (default: None)

**EXAMPLES:**

```
sage: F = GF(5^2)
sage: K = GF(5^4).over(F)
sage: L = GF(5^12).over(K)
sage: K.degree_over(F)
2
sage: L.degree_over(K)
```

(continues on next page)
If base is omitted, the degree is computed over the base of the extension:

```
sage: K.degree_over()
sage: L.degree_over()
```

Note that base must be an explicit base over which the extension has been defined (as listed by the method `bases()`):

```
sage: K.degree_over(GF(5))
```

```
Traceback (most recent call last):
...
ValueError: not (explicitly) defined over Finite Field of size 5
```

### fraction_field(extend_base=False)

Return the fraction field of this extension.

**INPUT:**

- `extend_base` – a boolean (default: False);

If `extend_base` is False, the fraction field of the extension $L/K$ is defined as Frac($L$)/$L/K$, except if $L$ is already a field in which base the fraction field of $L/K$ is $L/K$ itself.

If `extend_base` is True, the fraction field of the extension $L/K$ is defined as Frac($L$)/Frac($K$) (provided that the defining morphism extends to the fraction fields, i.e. is injective).

**EXAMPLES:**

```
sage: A.<a> = ZZ.extension(x^2 - 5)
sage: OK = A.over()  # over ZZ
sage: OK
Order in Number Field in a with defining polynomial x^2 - 5 over its base

sage: K1 = OK.fraction_field()
sage: K1
Fraction Field of Order in Number Field in a with defining polynomial x^2 - 5 over its base
sage: K1.bases()
[Fraction Field of Order in Number Field in a with defining polynomial x^2 - 5 over its base,
 Order in Number Field in a with defining polynomial x^2 - 5 over its base, 
 Integer Ring]

sage: K2 = OK.fraction_field(extend_base=True)
sage: K2
Fraction Field of Order in Number Field in a with defining polynomial x^2 - 5 over its base
sage: K2.bases()
```
[Fraction Field of Order in Number Field in a with defining polynomial \( x^2 - 5 \) over its base, Rational Field]

Note that there is no coercion between \( K_1 \) and \( K_2 \):

```python
sage: K1.has_coerce_map_from(K2)
False
sage: K2.has_coerce_map_from(K1)
False
```

We check that when the extension is a field, its fraction field does not change:

```python
sage: K1.fraction_field() is K1
True
sage: K2.fraction_field() is K2
True
```

**from_base_ring\( (r) \)**

Return the canonical embedding of \( r \) into this extension.

**INPUT:**

- \( r \) – an element of the base of the ring of this extension

**EXAMPLES:**

```python
sage: k = GF(5)
sage: K.<u> = GF(5^2).over(k)
sage: L.<v> = GF(5^4).over(K)
sage: x = L.from_base_ring(k(2)); x
2
sage: x.parent()
Field in v with defining polynomial \( x^2 + (3 - u)x + u \) over its base
sage: x = L.from_base_ring(u); x
u
sage: x.parent()
Field in v with defining polynomial \( x^2 + (3 - u)x + u \) over its base
```

**gen()**

Return the first generator of this extension.

**EXAMPLES:**

```python
sage: K = GF(5^2).over()  # over GF(5)
sage: x = K.gen(); x
z2
```

Observe that the generator lives in the extension:

```python
sage: x.parent()
Field in z2 with defining polynomial \( x^2 + 4x + 2 \) over its base
```
sage: x.parent() is K
True

gens(base=None)

Return the generators of this extension over base.

INPUT:

- base – a commutative ring (which might be itself an extension) or None (default: None); if omitted, use the base of this extension

EXAMPLES:

sage: K.<a> = GF(5^2).over()  # over GF(5)
sage: K.gens()
(a,)
sage: L.<b> = GF(5^4).over(K)
sage: L.gens()
(b,)
sage: L.gens(GF(5))
(b, a)
sage: S.<x> = QQ[]
sage: T.<y> = S[]
sage: T.over(S).gens()
(y,)
sage: T.over(QQ).gens()
(y, x)

hom(im_gens, codomain=None, base_map=None, category=None, check=True)

Return the unique homomorphism from this extension to codomain that sends self.gens() to the entries of im_gens and induces the map base_map on the base ring.

INPUT:

- im_gens – the images of the generators of this extension
- codomain – the codomain of the homomorphism; if omitted, it is set to the smallest parent containing all the entries of im_gens
- base_map – a map from one of the bases of this extension into something that coerces into the codomain; if omitted, coercion maps are used
- category – the category of the resulting morphism
- check – a boolean (default: True); whether to verify that the images of generators extend to define a map (using only canonical coercions)

EXAMPLES:

sage: K.<a> = GF(5^2).over()  # over GF(5)
sage: L.<b> = GF(5^6).over(K)

We define (by hand) the relative Frobenius endomorphism of the extension $L/K$:
Defining the absolute Frobenius of $L$ is a bit more complicated because it is not a homomorphism of $K$-algebras. For this reason, the construction $L.hom([b^5])$ fails:

```
sage: L.hom([b^5])
Traceback (most recent call last):
...
ValueError: images do not define a valid homomorphism
```

What we need is to specify a base map:

```
sage: FrobK = K.hom([a^5])
sage: FrobL = L.hom([b^5], base_map=FrobK)
sage: FrobL
Ring endomorphism of Field in b with defining polynomial x^3 + (2 + 2*a)*x - a␣
˓→ over its base
  Defn: b |--> (-1 + a) + (1 + 2*a)*b + a*b^2
  with map on base ring:
    a |--> 1 - a
```

As a shortcut, we may use the following construction:

```
sage: phi = L.hom([b^5, a^5])
sage: phi
Ring endomorphism of Field in b with defining polynomial x^3 + (2 + 2*a)*x - a␣
˓→ over its base
  Defn: b |--> (-1 + a) + (1 + 2*a)*b + a*b^2
  with map on base ring:
    a |--> 1 - a
sage: phi == FrobL
True
```

**is_defined_over**(base)

Return whether or not base is one of the bases of this extension.

**INPUT:**

- base – a commutative ring, which might be itself an extension

**EXAMPLES:**

```
sage: A = GF(5^4).over(GF(5^2))
sage: B = GF(5^12).over(A)
sage: A.is_defined_over(GF(5^2))
True
sage: A.is_defined_over(GF(5))
False
sage: B.is_defined_over(A)
True
```

(continues on next page)
Note that an extension is defined over itself:

```sage
sage: B.is_defined_over(GF(5^4))
True
sage: B.is_defined_over(GF(5^2))
True
sage: B.is_defined_over(GF(5))
False
```

See also:

- `bases()`, `absolute_base()`

**is_field**(proof=True)

Return whether or not this extension is a field.

INPUT:

- proof – a boolean (default: False)

EXAMPLES:

```sage
sage: K = GF(5^2).over()  # over GF(5)
sage: K.is_field()
True
sage: S.<x> = QQ[]
sage: A = S.over(QQ)
sage: A.is_field()
False
sage: B = A.fraction_field()
sage: B.is_field()
True
```

**is_finite_over**(base=None)

Return whether or not this extension is finite over base (as a module).

INPUT:

- base – a commutative ring (which might be itself an extension) or None (default: None)

EXAMPLES:

```sage
sage: K = GF(5^2).over()  # over GF(5)
sage: L = GF(5^4).over(K)
sage: L.is_finite_over(K)
True
sage: L.is_finite_over(GF(5))
True
```
If `base` is omitted, it is set to its default which is the base of the extension:

```python
sage: L.is_finite_over()
True
```

### is_free_over(base=None)

Return `True` if this extension is free (as a module) over `base`

**INPUT:**
- `base` – a commutative ring (which might be itself an extension) or `None` (default: `None`)

**EXAMPLES:**

```python
sage: K = GF(5^2).over()  # over GF(5)
sage: L = GF(5^4).over(K)
sage: L.is_free_over(K)
True
sage: L.is_free_over(GF(5))
True
```

If `base` is omitted, it is set to its default which is the base of the extension:

```python
sage: L.is_free_over()
True
```

### ngens(base=None)

Return the number of generators of this extension over `base`.

**INPUT:**
- `base` – a commutative ring (which might be itself an extension) or `None` (default: `None`)

**EXAMPLES:**

```python
sage: K = GF(5^2).over()  # over GF(5)
sage: K.gens()
(z2,)
sage: K.ngens()
1
sage: L = GF(5^4).over(K)
sage: L.gens(GF(5))
(z4, z2)
sage: L.ngens(GF(5))
2
```

### print_options(**options)

Update the printing options of this extension.

**INPUT:**
- `over` – an integer or `Infinity` (default: `0`); the maximum number of bases included in the printing of this extension
- `base` – a base over which this extension is finite free; elements in this extension will be printed as a linear combinasion of a basis of this extension over the given base

**EXAMPLES:**
sage: A.<a> = GF(5^2).over()  # over GF(5)
sage: B.<b> = GF(5^4).over(A)
sage: C.<c> = GF(5^12).over(B)
sage: D.<d> = GF(5^24).over(C)

Observe what happens when we modify the option over:

sage: D
Field in d with defining polynomial x^2 + ((1 - a) + ((1 + 2*a) - b)*c + ((2 + a) - a) + (1 - a)*b)*c^2)*x + c over its base

sage: D.print_options(over=2)

sage: D
Field in d with defining polynomial x^2 + ((1 - a) + ((1 + 2*a) - b)*c + ((2 + a) - a) + (1 - a)*b)*c^2)*x + c over Field in c with defining polynomial x^3 + (1 + (2 - a)*b)*x^2 + (2 + 2*b)*x - b over Field in b with defining polynomial x^2 + (3 - a)*x + a over its base

sage: D.print_options(over=Infinity)

sage: D
Field in d with defining polynomial x^2 + ((1 - a) + ((1 + 2*a) - b)*c + ((2 + a) - a) + (1 - a)*b)*c^2)*x + c over Field in c with defining polynomial x^3 + (1 + (2 - a)*b)*x^2 + (2 + 2*b)*x - b over Field in b with defining polynomial x^2 + (3 - a)*x + a over Field in a with defining polynomial x^2 + 4*x + 2 over Finite Field of size 5

Now the option base:

sage: d^2
-c + ((-1 + a) + ((-1 + 3*a) + b)*c + ((3 - a) + (-1 + a)*b)*c^2)*d

sage: D.basis_over(B)
[1, c, c^2, d, c*d, c^2*d]

sage: D.print_options(base=B)

sage: d^2
-c + (-1 + a)*d + ((-1 + 3*a) + b)*c*d + ((3 - a) + (-1 + a)*b)*c^2*d

sage: D.basis_over(A)
[1, b, c, b*c, c^2, b*c^2, d, b*d, c*d, b*c*d, c^2*d, b*c^2*d]

sage: D.print_options(base=A)

sage: d^2
-c + (-1 + a)*d + (-1 + 3*a)*c*d + b*c*d + (3 - a)*c^2*d + (-1 + a)*b*c^2*d

random_element()
Return a random element in this extension.

EXAMPLES:

sage: K = GF(5^2).over()  # over GF(5)
sage: x = K.random_element(); x  # random
3 + z2

(continues on next page)
relative_degree()

Return the degree of this extension over its base

EXAMPLES:

sage: A = GF(5^4).over(GF(5^2))
sage: A.relative_degree()
2

See also:

degree(), absolute_degree()

sage.rings.ring_extension.common_base(K, L, degree)

Return a common base on which K and L are defined.

INPUT:

- K – a commutative ring
- L – a commutative ring
- degree – a boolean; if true, return the degree of K and L over their common base

EXAMPLES:

sage: from sage.rings.ring_extension import common_base

sage: common_base(GF(5^3), GF(5^7), False)
Finite Field of size 5
sage: common_base(GF(5^3), GF(5^7), True)
(Finite Field of size 5, 3, 7)
sage: common_base(GF(5^3), GF(7^5), False)
Traceback (most recent call last):
  ...ImplementerError: unable to find a common base

When degree is set to True, we only look up for bases on which both K and L are finite:

sage: S.<x> = QQ[]
sage: common_base(S, QQ, False)
Rational Field
sage: common_base(S, QQ, True)
Traceback (most recent call last):
  ...ImplementerError: unable to find a common base

sage.rings.ring_extension.generators(ring, base)

Return the generators of ring over base.

INPUT:
• ring – a commutative ring
• base – a commutative ring

EXAMPLES:

```python
sage: from sage.rings.ring_extension import generators
sage: S.<x> = QQ[]
sage: T.<y> = S[]
```
```python
sage: generators(T, S)
(y,)
sage: generators(T, QQ)
(y, x)
```

```
sage.rings.ring_extension.tower_bases(ring, degree)
```
Return the list of bases of ring (including itself); if degree is True, restrict to finite extensions and return in addition the degree of ring over each base.

INPUT:
• ring – a commutative ring
• degree – a boolean

EXAMPLES:

```python
sage: from sage.rings.ring_extension import tower_bases
sage: S.<x> = QQ[]
sage: T.<y> = S[]
```
```python
sage: tower_bases(T, False)
([Univariate Polynomial Ring in y over Univariate Polynomial Ring in x over Rational Field, Univariate Polynomial Ring in x over Rational Field, Rational Field], [])
sage: tower_bases(T, True)
([Univariate Polynomial Ring in y over Univariate Polynomial Ring in x over Rational Field], [1])
```
```python
sage: K.<a> = Qq(5^2)
sage: L.<w> = K.extension(x^3 - 5)
sage: tower_bases(L, True)
([5-adic Eisenstein Extension Field in w defined by x^3 - 5 over its base field, 5-adic Unramified Extension Field in a defined by x^2 + 4*x + 2, 5-adic Field with capped relative precision 20], [1, 3, 6])
```

```
sage.rings.ring_extension.variable_names(ring, base)
```
Return the variable names of the generators of ring over base.

INPUT:
• ring – a commutative ring
• base – a commutative ring

EXAMPLES:

```python
```
7.2 Elements lying in extension of rings

AUTHOR:

• Xavier Caruso (2019)

class sage.rings.ring_extension_element.RingExtensionElement

    Bases: sage.structure.element.CommutativeAlgebraElement

    Generic class for elements lying in ring extensions.

    additive_order()
        Return the additive order of this element.

        EXAMPLES:

        sage: K.<a> = GF(5^4).over(GF(5^2))
        sage: a.additive_order()
        5

    is_nilpotent()
        Return whether if this element is nilpotent in this ring.

        EXAMPLES:

        sage: A.<x> = PolynomialRing(QQ)
        sage: E = A.over(QQ)
        sage: E(0).is_nilpotent()  # doctest: +NORMALIZE_WHITESPACE
        True
        sage: E(x).is_nilpotent()
        False

    is_prime()
        Return whether this element is a prime element in this ring.

        EXAMPLES:

        sage: A.<x> = PolynomialRing(QQ)
        sage: E = A.over(QQ)
        sage: E(x^2+1).is_prime()  # doctest: +NORMALIZE_WHITESPACE
        True
        sage: E(x^2-1).is_prime()
        False

    is_square(root=False)
        Return whether this element is a square in this ring.
INPUT:

- root – a boolean (default: False); if True, return also a square root

EXAMPLES:

```python
sage: K.<a> = GF(5^3).over()
sage: a.is_square()
False
sage: a.is_square(root=True)
(False, None)

sage: b = a + 1
sage: b.is_square()
True
sage: b.is_square(root=True)
(True, 2 + 3*a + a^2)
```

is_unit()

Return whether if this element is a unit in this ring.

EXAMPLES:

```python
sage: A.<x> = PolynomialRing(QQ)
sage: E = A.over(QQ)
sage: E(4).is_unit()
True
sage: E(x).is_unit()
False
```

multiplicative_order()

Return the multiplicative order of this element.

EXAMPLES:

```python
sage: K.<a> = GF(5^4).over(GF(5^2))
sage: a.multiplicative_order()
624
```

sqrt(extend=True, all=False, name=None)

Return a square root or all square roots of this element.

INPUT:

- extend – a boolean (default: True); if “True”, return a square root in an extension ring, if necessary. Otherwise, raise a ValueError if the root is not in the ring
- all – a boolean (default: False); if True, return all square roots of this element, instead of just one.
- name – Required when extend=True and self is not a square. This will be the name of the generator extension.

Note: The option extend = True is often not implemented.

EXAMPLES:
```python
sage: K.<a> = GF(5^3).over()
sage: b = a + 1
sage: b.sqrt()
2 + 3*a + a^2
sage: b.sqrt(all=True)
[2 + 3*a + a^2, 3 + 2*a - a^2]
```

class `sage.rings.ring_extension_element.RingExtensionFractionFieldElement`

Bases: `sage.rings.ring_extension_element.RingExtensionElement`

A class for elements lying in fraction fields of ring extensions.

**denominator()**

Return the denominator of this element.

EXAMPLES:

```python
sage: R.<x> = ZZ[]
sage: A.<a> = ZZ.extension(x^2 - 2)
sage: OK = A.over() # over ZZ
sage: K = OK.fraction_field()
sage: K
Fraction Field of Order in Number Field in a with defining polynomial x^2 - 2 over its base
sage: x = K(1/a); x
a/2
sage: denom = x.denominator(); denom
2
```

The denominator is an element of the ring which was used to construct the fraction field:

```python
sage: denom.parent()
Order in Number Field in a with defining polynomial x^2 - 2 over its base
sage: denom.parent() is OK
True
```

**numerator()**

Return the numerator of this element.

EXAMPLES:

```python
sage: A.<a> = ZZ.extension(x^2 - 2)
sage: OK = A.over() # over ZZ
sage: K = OK.fraction_field()
sage: K
Fraction Field of Order in Number Field in a with defining polynomial x^2 - 2 over its base
sage: x = K(1/a); x
a/2
sage: num = x.numerator(); num
a
```

The numerator is an element of the ring which was used to construct the fraction field:
sage: num.parent()
Order in Number Field in a with defining polynomial x^2 - 2 over its base
sage: num.parent() is OK
True

class sage.rings.ring_extension_element.RingExtensionWithBasisElement
Bases: sage.rings.ring_extension_element.RingExtensionElement

A class for elements lying in finite free extensions.

charpoly\(base=\text{None}, var='x')\)  
Return the characteristic polynomial of this element over \(base\).

INPUT:
- \(base\) – a commutative ring (which might be itself an extension) or \text{None}

EXAMPLES:

```
sage: F = GF(5)
sage: K.<a> = GF(5^3).over(F)
sage: L.<b> = GF(5^6).over(K)
sage: u = a/(1+b)

sage: chi = u.charpoly(K); chi
x^2 + (1 + 2*a + 3*a^2)*x + 3 + 2*a^2
```

We check that the charpoly has coefficients in the base ring:

```
sage: chi.base_ring()
Field in a with defining polynomial x^3 + 3*x + 3 over its base
sage: chi.base_ring() is K
True
```

and that it annihilates \(u\):

```
sage: chi(u)
0
```

Similarly, one can compute the characteristic polynomial over \(F\):

```
sage: u.charpoly(F)
x^6 + x^4 + 2*x^3 + 3*x + 4
```

A different variable name can be specified:

```
sage: u.charpoly(F, var='t')
t^6 + t^4 + 2*t^3 + 3*t + 4
```

If \(base\) is omitted, it is set to its default which is the base of the extension:

```
sage: u.charpoly()
x^2 + (1 + 2*a + 3*a^2)*x + 3 + 2*a^2
```

Note that \(base\) must be an explicit base over which the extension has been defined (as listed by the method \(bases()\)):
```plaintext
sage: u.charpoly(GF(5^2))
Traceback (most recent call last):
...
ValueError: not (explicitly) defined over Finite Field in z2 of size 5^2
```

**matrix** *(base=None)*

Return the matrix of the multiplication by this element (in the basis output by `basis_over()`).

**INPUT:**

- `base` – a commutative ring (which might be itself an extension) or `None`

**EXAMPLES:**

```plaintext
sage: K.<a> = GF(5^3).over()  # over GF(5)
sage: L.<b> = GF(5^6).over(K)
sage: u = a/(1+b)
sage: u
(2 + a + 3*a^2) + (3 + 3*a + a^2)*b
sage: b*u
(3 + 2*a^2) + (2 + 2*a - a^2)*b
sage: u.matrix(K)
[2 + a + 3*a^2 3 + 3*a + a^2]
[ 3 + 2*a^2 2 + 2*a - a^2]
sage: u.matrix(GF(5))
[2 1 3 3 3 1]
[1 3 1 2 0 3]
[2 3 3 1 3 0]
[3 0 2 2 2 4]
[4 2 0 3 0 2]
[0 4 2 4 2 0]
```

If `base` is omitted, it is set to its default which is the base of the extension:

```plaintext
sage: u.matrix()
[2 + a + 3*a^2 3 + 3*a + a^2]
[ 3 + 2*a^2 2 + 2*a - a^2]
```

Note that `base` must be an explicit base over which the extension has been defined (as listed by the method `bases()`):

```plaintext
sage: u.matrix(GF(5^2))
Traceback (most recent call last):
...
ValueError: not (explicitly) defined over Finite Field in z2 of size 5^2
```

**minpoly** *(base=None, var='x')*

Return the minimal polynomial of this element over `base`.

**INPUT:**

- `base` – a commutative ring (which might be itself an extension) or `None`

**EXAMPLES:**
```python
sage: F = GF(5)
sage: K.<a> = GF(5^3).over(F)
sage: L.<b> = GF(5^6).over(K)
sage: u = 1 / (a+b)
sage: chi = u.minpoly(K); chi
x^2 + (2*a + a^2)*x - 1 + a
```

We check that the minimal polynomial has coefficients in the base ring:

```python
sage: chi.base_ring()
Field in a with defining polynomial x^3 + 3*x + 3 over its base
sage: chi.base_ring() is K
True
```

and that it annihilates u:

```python
sage: chi(u)
0
```

Similarly, one can compute the minimal polynomial over F:

```python
sage: u.minpoly(F)
x^6 + 4*x^5 + x^4 + 2*x^2 + 3
```

A different variable name can be specified:

```python
sage: u.minpoly(F, var='t')
t^6 + 4*t^5 + t^4 + 2*t^2 + 3
```

If `base` is omitted, it is set to its default which is the base of the extension:

```python
sage: u.minpoly()
x^2 + (2*a + a^2)*x - 1 + a
```

Note that `base` must be an explicit base over which the extension has been defined (as listed by the method `bases()`):

```python
sage: u.minpoly(GF(5^2))
Traceback (most recent call last):
  ...
ValueError: not (explicitly) defined over Finite Field in z2 of size 5^2
```

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```
```
sage: u = a/(1+b)
sage: nr = u.norm(K); nr
3 + 2*a^2

We check that the norm lives in the base ring:

```
sage: nr.parent()
Field in a with defining polynomial x^3 + 3*x + 3 over its base
sage: nr.parent() is K
True
```

Similarly, one can compute the norm over F:

```
sage: u.norm(F)
4
```

We check the transitivity of the norm:

```
sage: u.norm(F) == nr.norm(F)
True
```

If `base` is omitted, it is set to its default which is the base of the extension:

```
sage: u.norm()
3 + 2*a^2
```

Note that `base` must be an explicit base over which the extension has been defined (as listed by the method `bases()`):

```
sage: u.norm(GF(5^2))
Traceback (most recent call last):
...  
ValueError: not (explicitly) defined over Finite Field in z2 of size 5^2
```

`polynomial(base=None, var='x')`

Return a polynomial (in one or more variables) over `base` whose evaluation at the generators of the parent equals this element.

**INPUT:**
- `base` – a commutative ring (which might be itself an extension) or `None`

**EXAMPLES:**

```
sage: F.<a> = GF(5^2).over()  # over GF(5)
sage: K.<b> = GF(5^4).over(F)
sage: L.<c> = GF(5^12).over(K)
sage: u = 1/(a + b + c); u
(2 + (-1 - a)*b) + ((2 + 3*a) + (1 - a)*b)*c + ((-1 - a) - a*b)*c^2
sage: P = u.polynomial(K); P
((-1 - a) - a*b)*x^2 + ((2 + 3*a) + (1 - a)*b)*x + 2 + (-1 - a)*b
sage: P.base_ring() is K
True
```
When the base is $F$, we obtain a bivariate polynomial:

```
sage: P = u.polynomial(F); P
(-a)*x0^2*x1 + (-1 - a)*x0^2 + (1 - a)*x0*x1 + (2 + 3*a)*x0 + (-1 - a)*x1 + 2
```

We check that its value at the generators is the element we started with:

```
sage: L.gens(F)
(c, b)
sage: P(c, b) == u
True
```

Similarly, when the base is $\text{GF}(5)$, we get a trivariate polynomial:

```
sage: P = u.polynomial(GF(5)); P
-x0^2*x1*x2 - x0^2*x2 - x0*x1*x2 - x0^2*x1 - 2*x0*x2 - x1*x2 + 2*x0 - x1 + 2
```

Different variable names can be specified:

```
sage: u.polynomial(GF(5), var='y')
-y0^2*y1*y2 - y0^2*y2 - y0*y1*y2 - y0^2 + y0*y2 - y1*y2 - 2*y0 - y1 + 2
```

```
sage: u.polynomial(GF(5), var=['x', 'y', 'z'])
-x^2*y*z - x^2*z - x*y*z - x^2 + x*y - 2*x*z - y*z + 2*x - y + 2
```

If base is omitted, it is set to its default which is the base of the extension:

```
sage: u.polynomial()
((-1 - a) - a*b)*x^2 + ((2 + 3*a) + (1 - a)*b)*x + 2 + (-1 - a)*b
```

Note that base must be an explicit base over which the extension has been defined (as listed by the method `bases()`):

```
sage: u.polynomial(GF(5^3))
Traceback (most recent call last):
  ... ValueError: not (explicitly) defined over Finite Field in z3 of size 5^3
```

`trace(base=None)`

Return the trace of this element over base.

**INPUT:**

- `base` – a commutative ring (which might be itself an extension) or `None`

**EXAMPLES:**

```
sage: F = GF(5)
sage: K.<a> = GF(5^3).over(F)
sage: L.<b> = GF(5^6).over(K)
sage: u = a/(1+b)
```

(continues on next page)
We check that the trace lives in the base ring:

```
sage: tr.parent()
Field in a with defining polynomial x^3 + 3*x + 3 over its base
```

```
sage: tr.parent() is K
True
```

Similarly, one can compute the trace over F:

```
sage: u.trace(F)
0
```

We check the transitivity of the trace:

```
sage: u.trace(F) == tr.trace(F)
True
```

If base is omitted, it is set to its default which is the base of the extension:

```
sage: u.trace()
-1 + 3*a + 2*a^2
```

Note that base must be an explicit base over which the extension has been defined (as listed by the method bases()).

```
sage: u.trace(GF(5^2))
Traceback (most recent call last):
...
ValueError: not (explicitly) defined over Finite Field in z2 of size 5^2
```

```
vector(base=None)
```

Return the vector of coordinates of this element over base (in the basis output by the method basis_over()).

**INPUT:**

- `base` – a commutative ring (which might be itself an extension) or None

**EXAMPLES:**

```
sage: F = GF(5)
sage: K.<a> = GF(5^2).over()  # over F
sage: L.<b> = GF(5^6).over(K)
sage: x = (a+b)^4; x
(-1 + a) + (3 + a)*b + (1 - a)*b^2
sage: x.vector(K)  # basis is (1, b, b^2)
(-1 + a, 3 + a, 1 - a)
sage: x.vector(F)  # basis is (1, a, b, a*b, b^2, a*b^2)
(4, 1, 3, 1, 1, 4)
```

If base is omitted, it is set to its default which is the base of the extension:
Note that base must be an explicit base over which the extension has been defined (as listed by the method bases()):

```python
sage: x.vector(GF(5^3))
Traceback (most recent call last):
...:
ValueError: not (explicitly) defined over Finite Field in z3 of size 5^3
```

### 7.3 Morphisms between extension of rings

**AUTHOR:**

- Xavier Caruso (2019)

**class** `sage.rings.ring_extension_morphism.MapFreeModuleToRelativeRing`

```
Bases: sage.categories.map.Map
```

Base class of the module isomorphism between a ring extension and a free module over one of its bases.

**is_injective()**

Return whether this morphism is injective.

**EXAMPLES:**

```python
sage: K = GF(11^6).over(GF(11^3))
sage: V, i, j = K.free_module()
sage: i.is_injective()
True
```

**is_surjective()**

Return whether this morphism is surjective.

**EXAMPLES:**

```python
sage: K = GF(11^6).over(GF(11^3))
sage: V, i, j = K.free_module()
sage: i.is_surjective()
True
```

**class** `sage.rings.ring_extension_morphism.MapRelativeRingToFreeModule`

```
Bases: sage.categories.map.Map
```

Base class of the module isomorphism between a ring extension and a free module over one of its bases.

**is_injective()**

Return whether this morphism is injective.

**EXAMPLES:**

```python
sage: K = GF(11^6).over(GF(11^3))
sage: V, i, j = K.free_module()
sage: j.is_injective()
True
```
**is_surjective()**
Return whether this morphism is injective.

**EXAMPLES:**

```
sage: K = GF(11^6).over(GF(11^3))  
sage: V, i, j = K.free_module()  
sage: j.is_surjective()  
True
```

**class** sage.rings.ring_extension_morphism.RingExtensionBackendIsomorphism
Bases: sage.rings.ring_extension_morphism.RingExtensionHomomorphism
A class for implementing isomorphisms taking an element of the backend to its ring extension.

**class** sage.rings.ring_extension_morphism.RingExtensionBackendReverseIsomorphism
Bases: sage.rings.ring_extension_morphism.RingExtensionHomomorphism
A class for implementing isomorphisms from a ring extension to its backend.

**class** sage.rings.ring_extension_morphism.RingExtensionHomomorphism
Bases: sage.rings.morphism.RingMap
A class for ring homomorphisms between extensions.

**base_map()**
Return the base map of this morphism or just None if the base map is a coercion map.

**EXAMPLES:**

```
sage: F = GF(5)  
sage: K.<a> = GF(5^2).over(F)  
sage: L.<b> = GF(5^6).over(K)  
We define the absolute Frobenius of L:
```

```
sage: FrobL = L.hom([b^5, a^5])  
sage: FrobL  
Ring endomorphism of Field in b with defining polynomial x^3 + (2 + 2*a)*x - a over its base  
  Defn: b |--> (-1 + a) + (1 + 2*a)*b + a*b^2  
  with map on base ring:
  a |--> 1 - a  
sage: FrobL.base_map()  
Ring morphism:
  From: Field in a with defining polynomial x^2 + 4*x + 2 over its base  
  To: Field in b with defining polynomial x^3 + (2 + 2*a)*x - a over its base  
  Defn: a |--> 1 - a
```

The square of FrobL acts trivially on K; in other words, it has a trivial base map:

```
sage: phi = FrobL^2  
sage: phi  
Ring endomorphism of Field in b with defining polynomial x^3 + (2 + 2*a)*x - a over its base  
  Defn: b |--> 2 + 2*a*b + (2 - a)*b^2  
sage: phi.base_map()  
```

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is_identity()  
Return whether this morphism is the identity.

EXAMPLES:

```
sage: K.<a> = GF(5^2).over()  # over GF(5)
sage: FrobK = K.hom([a^5])
sage: FrobK.is_identity()  
False
sage: (FrobK^2).is_identity()  
True
```

Coercion maps are not considered as identity morphisms:

```
sage: L.<b> = GF(5^6).over(K)
sage: iota = L.defining_morphism()
sage: iota  
Ring morphism:
  From: Field in a with defining polynomial x^2 + 4*x + 2 over its base
  To:   Field in b with defining polynomial x^3 + (2 + 2*a)*x - a over its base
  Defn: a |--> a
sage: iota.is_identity()  
False
```

is_injective()  
Return whether this morphism is injective.

EXAMPLES:

```
sage: K = GF(5^10).over(GF(5^5))
sage: iota = K.defining_morphism()
sage: iota  
Ring morphism:
  From: Finite Field in z5 of size 5^5
  To:   Field in z10 with defining polynomial x^2 + (2*z5^3 + 2*z5^2 + 4*z5 +...
        -a)*x + z5 over its base
  Defn: z5 |--> z5
sage: iota.is_injective()  
True
sage: K = GF(7).over(ZZ)
sage: iota = K.defining_morphism()
sage: iota  
Ring morphism:
  From: Integer Ring
  To:   Finite Field of size 7 over its base
  Defn: 1 |--> 1
sage: iota.is_injective()  
False
```

is_surjective()  
Return whether this morphism is surjective.

EXAMPLES:

<table>
<thead>
<tr>
<th>sage:</th>
<th>K = GF(5^10).over(GF(5^5))</th>
</tr>
</thead>
<tbody>
<tr>
<td>sage:</td>
<td>iota = K.defining_morphism()</td>
</tr>
<tr>
<td>sage:</td>
<td>iota</td>
</tr>
<tr>
<td></td>
<td>Ring morphism:</td>
</tr>
<tr>
<td></td>
<td>From: Finite Field in z5 of size 5^5</td>
</tr>
<tr>
<td></td>
<td>To: Field in z10 with defining polynomial x^2 + (2<em>z5^3 + 2</em>z5^2 + 4*z5 + 4)*x + z5 over its base</td>
</tr>
<tr>
<td></td>
<td>Defn: z5</td>
</tr>
<tr>
<td>sage:</td>
<td>iota.is_surjective()</td>
</tr>
<tr>
<td></td>
<td>False</td>
</tr>
</tbody>
</table>

| sage: | K = GF(7).over(ZZ) |
| sage: | iota = K.defining_morphism() |
| sage: | iota |
|      | Ring morphism: |
|      |   From: Integer Ring |
|      |   To: Finite Field of size 7 over its base |
|      |   Defn: 1 |--> 1 |
| sage: | iota.is_surjective() |
|      | True |

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8.1 Big O for various types (power series, p-adics, etc.)

See also:
- asymptotic expansions
- p-adic numbers
- power series
- polynomials

`sage.rings.big_oh.O(*x, **kwds)`

Big O constructor for various types.

**EXAMPLES:**

This is useful for writing power series elements:

```python
sage: R.<t> = ZZ[[t]]
sage: (1+t)^10 + O(t^5)
1 + 10*t + 45*t^2 + 120*t^3 + 210*t^4 + O(t^5)
```

A power series ring is created implicitly if a polynomial element is passed:

```python
sage: R.<x> = QQ[['x']]
sage: O(x^100)
O(x^100)
sage: 1/(1+x+O(x^5))
1 - x + x^2 - x^3 + x^4 + O(x^5)
sage: R.<u,v> = QQ[]
sage: 1 + u + v^2 + O(u, v)^5
1 + u + v^2 + O(u, v)^5
```

This is also useful to create p-adic numbers:

```python
sage: O(7^6)
O(7^6)
sage: 1/3 + O(7^6)
5 + 4*7 + 4*7^2 + 4*7^3 + 4*7^4 + 4*7^5 + O(7^6)
```

It behaves well with respect to adding negative powers of $p$: 
There are problems if you add a rational with very negative valuation to an \( O \)-Term:

\[
sage: 11^{-12} + O(11^{15})
\]
\[
11^{-12} + 0(11^{8})
\]

The reason that this fails is that the constructor doesn’t know the right precision cap to use. If you cast explicitly or use other means of element creation, you can get around this issue:

\[
sage: K = Qp(11, 30)
sage: K(11^{-12}) + O(11^{15})
\]
\[
11^{-12} + 0(11^{15})
\]

We can also work with asymptotic expansions:

\[
sage: A.<n> = AsymptoticRing(growth_group='QQ^n * n^{QQ} * \log(n)^{QQ}', coefficient_ring=QQ); A
\]

Asymptotic Ring \(<QQ^n * n^{QQ} * \log(n)^{QQ} * \text{Signs}^n>\) over Rational Field

\[
sage: 0(n)
\]

Application with Puiseux series:

\[
sage: P.<y> = PuiseuxSeriesRing(ZZ)
sage: y^(1/5) + O(y^(1/3))
\]
\[
y^{(1/5)} + 0(y^{(1/3)})
\]

8.2 Signed and Unsigned Infinities

The unsigned infinity “ring” is the set of two elements

1. infinity
2. A number less than infinity

The rules for arithmetic are that the unsigned infinity ring does not canonically coerce to any other ring, and all other rings canonically coerce to the unsigned infinity ring, sending all elements to the single element “a number less than infinity” of the unsigned infinity ring. Arithmetic and comparisons then take place in the unsigned infinity ring, where all arithmetic operations that are well-defined are defined.

The infinity “ring” is the set of five elements

1. plus infinity
2. a positive finite element
3. zero
4. a negative finite element
5. negative infinity

The infinity ring coerces to the unsigned infinity ring, sending the infinite elements to infinity and the non-infinite elements to “a number less than infinity.” Any ordered ring coerces to the infinity ring in the obvious way.

**Note:** The shorthand oo is predefined in Sage to be the same as +Infinity in the infinity ring. It is considered equal to, but not the same as Infinity in the UnsignedInfinityRing.

**EXAMPLES:**

We fetch the unsigned infinity ring and create some elements:

```python
sage: P = UnsignedInfinityRing; P
The Unsigned Infinity Ring
sage: P(5)
A number less than infinity
sage: P.ngens()
1
sage: unsigned_oo = P.0; unsigned_oo
Infinity
```

We compare finite numbers with infinity:

```python
sage: 5 < unsigned_oo
True
sage: 5 > unsigned_oo
False
sage: unsigned_oo < 5
False
sage: unsigned_oo > 5
True
```

Demonstrating the shorthand oo versus Infinity:

```python
sage: oo
+Infinity
sage: oo is InfinityRing.0
True
sage: oo is UnsignedInfinityRing.0
False
sage: oo == UnsignedInfinityRing.0
True
```

We do arithmetic:

```python
sage: unsigned_oo + 5
Infinity
```

We make 1 / unsigned_oo return the integer 0 so that arithmetic of the following type works:
Note that many operations are not defined, since the result is not well-defined:

```
sage: unsigned_oo/0
Traceback (most recent call last):
  ...  
ValueError: quotient of number < oo by number < oo not defined
```

What happened above is that 0 is canonically coerced to “A number less than infinity” in the unsigned infinity ring. Next, Sage tries to divide by multiplying with its inverse. Finally, this inverse is not well-defined.

```
sage: 0/unsigned_oo
0
sage: unsigned_oo * 0
Traceback (most recent call last):
  ...  
ValueError: unsigned oo times smaller number not defined
sage: unsigned_oo/unsigned_oo
Traceback (most recent call last):
  ...  
ValueError: unsigned oo times smaller number not defined
```

In the infinity ring, we can negate infinity, multiply positive numbers by infinity, etc.

```
sage: P = InfinityRing; P
The Infinity Ring
sage: P(5)
A positive finite number
```

The symbol oo is predefined as a shorthand for +Infinity:

```
sage: oo
+Infinity
```

We compare finite and infinite elements:

```
sage: 5 < oo
True
sage: P(-5) < P(5)
True
sage: P(2) < P(3)
False
sage: -oo < oo
True
```

We can do more arithmetic than in the unsigned infinity ring:

```
sage: 2 * oo
+Infinity
sage: -2 * oo
```
(continues on next page)
We make \( \frac{1}{\infty} \) and \( \frac{1}{-\infty} \) return the integer 0 instead of the infinity ring Zero so that arithmetic of the following type works:

```
sage: (1/oo) + 2
2
sage: 32/5 - (2.439/-oo)
32/5
```

If we try to subtract infinities or multiply infinity by zero we still get an error:

```
sage: oo - oo
Traceback (most recent call last):
  ...
SignError: cannot add infinity to minus infinity
sage: 0 * oo
Traceback (most recent call last):
  ...
SignError: cannot multiply infinity by zero
sage: P(2) + P(-3)
Traceback (most recent call last):
  ...
SignError: cannot add positive finite value to negative finite value
```

Signed infinity can also be represented by \( \mathbb{R} \) / \( \mathbb{R} \)D elements. But unsigned infinity cannot:

```
sage: oo in RR, oo in RDF
(True, True)
```

```class sage.rings.infinity.AnInfinity
Bases: object
lcm(x)
    Return the least common multiple of \( \infty \) and \( x \), which is by definition \( \infty \) unless \( x \) is 0.
```

```
EXAMPLES:
sage: oo.lcm(0)
0
sage: oo.lcm(oo)
+Infinity
sage: oo.lcm(-oo)
+Infinity
sage: oo.lcm(10)
+Infinity
```

sage: (-oo).lcm(10)
+Infinity

class sage.rings.infinity.FiniteNumber(parent, x)
Bases: sage.structure.element.RingElement

Initialize self.

sign()
Return the sign of self.

EXAMPLES:

sage: sign(InfinityRing(2))
1
sage: sign(InfinityRing(0))
0
sage: sign(InfinityRing(-2))
-1

sqrt()
EXAMPLES:

sage: InfinityRing(7).sqrt()
A positive finite number
sage: InfinityRing(0).sqrt()
Zero
sage: InfinityRing(-0.001).sqrt()
Traceback (most recent call last):
... SignError: cannot take square root of a negative number

sage.rings.infinity.InfinityRing = The Infinity Ring

class sage.rings.infinity.InfinityRing_class
Bases: sage.misc.fast_methods.Singleton, sage.rings.ring.Ring

Initialize self.

fraction_field()
This isn’t really a ring, let alone an integral domain.

gen(n=0)
The two generators are plus and minus infinity.

EXAMPLES:

sage: InfinityRing.gen(0)
+Infinity
sage: InfinityRing.gen(1)
-Infinity
sage: InfinityRing.gen(2)
Traceback (most recent call last):
... IndexError: n must be 0 or 1
The two generators are plus and minus infinity.

```
sage: InfinityRing.gens()
[+Infinity, -Infinity]
```

The Infinity Ring is commutative

```
sage: InfinityRing.is_commutative()
True
```

The Infinity Ring is not zero

```
sage: InfinityRing.is_zero()
False
```

The two generators are plus and minus infinity.

```
sage: InfinityRing.ngens()
2
sage: len(InfinityRing.gens())
2
```

**class** sage.rings.infinity.LessThanInfinity(**parent=The Unsigned Infinity Ring**)

Bases: sage.rings.infinity._uniq, sage.structure.element.RingElement

Initialize self.

```
sage: sage.rings.infinity.LessThanInfinity() is UnsignedInfinityRing(5)
True
```

Raise an error because the sign of self is not well defined.

```
sage: sign(UnsignedInfinityRing(2))
Traceback (most recent call last):
  ...:
NotImplementedError: sign of number < oo is not well defined
sage: sign(UnsignedInfinityRing(0))
Traceback (most recent call last):
  ...:
NotImplementedError: sign of number < oo is not well defined
sage: sign(UnsignedInfinityRing(-2))
Traceback (most recent call last):
  ...:
NotImplementedError: sign of number < oo is not well defined
```

(continues on next page)
Traceback (most recent call last):
...
NotImplementedError: sign of number < oo is not well defined

class sage.rings.infinity.MinusInfinity
Bases: sage.rings.infinity._uniq, sage.rings.infinity.AnInfinity, sage.structure.element.InfinityElement

Initialize self.

sqrt()

EXAMPLES:

sage: (-oo).sqrt()
Traceback (most recent call last):
...
SignError: cannot take square root of negative infinity

class sage.rings.infinity.PlusInfinity
Bases: sage.rings.infinity._uniq, sage.rings.infinity.AnInfinity, sage.structure.element.InfinityElement

Initialize self.

sqrt()

The square root of self.

The square root of infinity is infinity.

EXAMPLES:

sage: oo.sqrt()
+Infinity

exception sage.rings.infinity.SignError
Bases: ArithmeticError

Sign error exception.

class sage.rings.infinity.UnsignedInfinity
Bases: sage.rings.infinity._uniq, sage.rings.infinity.AnInfinity, sage.structure.element.InfinityElement

Initialize self.

sage.rings.infinity.UnsignedInfinityRing = The Unsigned Infinity Ring

class sage.rings.infinity.UnsignedInfinityRing_class
Bases: sage.misc.fast_methods.Singleton, sage.rings.ring.Ring

Initialize self.

fraction_field()

The unsigned infinity ring isn’t an integral domain.

EXAMPLES:

sage: UnsignedInfinityRing.fraction_field()
Traceback (most recent call last):
TypeError: infinity 'ring' has no fraction field

gen(n=0)
The “generator” of self is the infinity object.

EXAMPLES:

```python
sage: UnsignedInfinityRing.gen()
Infinity
sage: UnsignedInfinityRing.gen(1)
Traceback (most recent call last):
...
IndexError: UnsignedInfinityRing only has one generator
```

gens()
The “generator” of self is the infinity object.

EXAMPLES:

```python
sage: UnsignedInfinityRing.gens()
[Infinity]
```

less_than_infinity()
This is the element that represents a finite value.

EXAMPLES:

```python
sage: UnsignedInfinityRing.less_than_infinity()
A number less than infinity
sage: UnsignedInfinityRing(5) is UnsignedInfinityRing.less_than_infinity()
True
```

ngens()
The unsigned infinity ring has one “generator.”

EXAMPLES:

```python
sage: UnsignedInfinityRing.ngens()
1
sage: len(UnsignedInfinityRing.gens())
1
```

sage.rings.infinity.is_Infinite(x)
This is a type check for infinity elements.

EXAMPLES:

```python
sage: sage.rings.infinity.is_Infinite(oo)
True
sage: sage.rings.infinity.is_Infinite(-oo)
True
sage: sage.rings.infinity.is_Infinite(unsigned_infinity)
True
sage: sage.rings.infinity.is_Infinite(3)
(continues on next page)
sage.rings.infinity.test_comparison(ring)
Check comparison with infinity

INPUT:

• ring – a sub-ring of the real numbers

OUTPUT:

Various attempts are made to generate elements of ring. An assertion is triggered if one of these elements does not compare correctly with plus/minus infinity.

EXAMPLES:

```
sage: from sage.rings.infinity import test_comparison
sage: rings = [ZZ, QQ, RR, RealField(200), RDF, RLF, AA, RIF]
sage: for R in rings:
    ....:     print('testing {}'.format(R))
    ....:     test_comparison(R)

testing Integer Ring
testing Rational Field
testing Real Field with 53 bits of precision
testing Real Field with 200 bits of precision
testing Real Double Field
testing Real Lazy Field
testing Algebraic Real Field
testing Real Interval Field with 53 bits of precision
```

Comparison with number fields does not work:

```
sage: K.<sqrt3> = NumberField(x^2-3)
sage: (-oo < 1+sqrt3) and (1+sqrt3 < oo)  # known bug
False
```

The symbolic ring handles its own infinities, but answers False (meaning: cannot decide) already for some very elementary comparisons:

```
sage: test_comparison(SR)  # known bug
Traceback (most recent call last):
  ...AssertionError: testing -1000.0 in Symbolic Ring: id = ...
```

sage.rings.infinity.test_signed_infinity(pos_inf)
Test consistency of infinity representations.

There are different possible representations of infinity in Sage. These are all consistent with the infinity ring, that is, compare with infinity in the expected way. See also trac ticket #14045

INPUT:

• pos_inf – a representation of positive infinity.
An assertion error is raised if the representation is not consistent with the infinity ring.

Check that trac ticket #14045 is fixed:

```
sage: InfinityRing(float('+inf'))
+Infinity
sage: InfinityRing(float('-inf'))
-Infinity
sage: oo > float('+inf')
False
sage: oo == float('+inf')
True
```

EXAMPLES:

```
sage: from sage.rings.infinity import test_signed_infinity
sage: for pos_inf in [oo, float('+inf'), RLF(oo), RIF(oo), SR(oo)]:
....:     test_signed_infinity(pos_inf)
```

### 8.3 Support Python’s numbers abstract base class

See also:

PEP 3141 for more information about numbers.

```
sage.rings.numbers_abc.register_sage_classes()
```
Register all relevant Sage classes in the numbers hierarchy.

EXAMPLES:

```
sage: import numbers
sage: isinstance(5, numbers.Integral)
True
sage: isinstance(5, numbers.Number)
True
sage: isinstance(5/1, numbers.Integral)
False
sage: isinstance(22/7, numbers.Rational)
True
sage: isinstance(CC(1.3), numbers.Real)
True
sage: isinstance(CC(1.3 + I), numbers.Complex)
True
sage: isinstance(RDF(1.3), numbers.Real)
True
sage: isinstance(CDF(1.3, 4), numbers.Complex)
True
sage: isinstance(AA(sqrt(2)), numbers.Real)
True
```
(continues on next page)
sage: isinstance(QQbar(I), numbers.Complex)
True

This doesn’t work with symbolic expressions at all:

sage: isinstance(pi, numbers.Real)
False
sage: isinstance(I, numbers.Complex)
False
sage: isinstance(sqrt(2), numbers.Real)
False

Because we do this, NumPy’s isscalar() recognizes Sage types:

sage: from numpy import isscalar
sage: isscalar(3.141)
True
sage: isscalar(4/17)
True
9.1 Derivations

Let $A$ be a ring and $B$ be a bimodule over $A$. A derivation $d : A \to B$ is an additive map that satisfies the Leibniz rule

$$d(xy) = x d(y) + d(x)y.$$

If $B$ is an algebra over $A$ and if we are given in addition a ring homomorphism $\theta : A \to B$, a twisted derivation with respect to $\theta$ (or a $\theta$-derivation) is an additive map $d : A \to B$ such that

$$d(xy) = \theta(x)d(y) + d(x)y.$$

When $\theta$ is the morphism defining the structure of $A$-algebra on $B$, a $\theta$-derivation is nothing but a derivation. In general, if $\iota : A \to B$ denotes the defining morphism above, one easily checks that $\theta - \iota$ is a $\theta$-derivation.

This file provides support for derivations and twisted derivations over commutative rings with values in algebras (i.e. we require that $B$ is a commutative $A$-algebra). In this case, the set of derivations (resp. $\theta$-derivations) is a module over $B$.

Given a ring $A$, the module of derivations over $A$ can be created as follows:

```
sage: A.<x,y,z> = QQ[]
sage: M = A.derivation_module()
sage: M
Module of derivations over Multivariate Polynomial Ring in x, y, z over Rational Field
```

The method `gens()` returns the generators of this module:

```
sage: A.<x,y,z> = QQ[]
sage: M = A.derivation_module()
sage: M.gens()
(d/dx, d/dy, d/dz)
```

We can combine them in order to create all derivations:

```
sage: d = 2*M.gen(0) + z*M.gen(1) + (x^2 + y^2)*M.gen(2)
sage: d
2*d/dx + z*d/dy + (x^2 + y^2)*d/dz
```

and now play with them:
Alternatively we can use the method `derivation()` of the ring $A$ to create derivations:

```python
sage: Dx = A.derivation(x); Dx
d/dx
sage: Dy = A.derivation(y); Dy
d/dy
sage: Dz = A.derivation(z); Dz
d/dz
sage: A.derivation([2, z, x^2+y^2])
2*d/dx + z*d/dy + (x^2 + y^2)*d/dz
```

Sage knows moreover that $M$ is a Lie algebra:

```python
sage: M.category()
Join of Category of lie algebras with basis over Rational Field
   and Category of modules with basis over Multivariate Polynomial Ring in x, y, z over Rational Field
```

Computations of Lie brackets are implemented as well:

```python
sage: Dx.bracket(Dy)
0
sage: d.bracket(Dx)
-2*x*d/dz
```

At the creation of a module of derivations, a codomain can be specified:

```python
sage: B = A.fraction_field()
sage: A.derivation_module(B)
Module of derivations from Multivariate Polynomial Ring in x, y, z over Rational Field
to Fraction Field of Multivariate Polynomial Ring in x, y, z over Rational Field
```

Alternatively, one can specify a morphism $f$ with domain $A$. In this case, the codomain of the derivations is the codomain of $f$ but the latter is viewed as an algebra over $A$ through the homomorphism $f$. This construction is useful, for example, if we want to work with derivations on $A$ at a certain point, e.g. $(0, 1, 2)$. Indeed, in order to achieve this, we first define the evaluation map at this point:

```python
sage: ev = A.hom([QQ(0), QQ(1), QQ(2)])
sage: ev
Ring morphism:
  From: Multivariate Polynomial Ring in x, y, z over Rational Field
  To:   Rational Field
     Defn: x |--> 0
            y |--> 1
            z |--> 2
```

Now we use this ring homomorphism to define a structure of $A$-algebra on $Q$ and then build the following module of derivations:
sage: M = A.derivation_module(ev)
sage: M
Module of derivations from Multivariate Polynomial Ring in x, y, z over Rational Field to Rational Field
sage: M.gens()
(d/dx, d/dy, d/dz)

Elements in $M$ then acts as derivations at $(0,1,2)$:

sage: Dx = M.gen(0)
sage: Dy = M.gen(1)
sage: Dz = M.gen(2)
sage: f = x^2 + y^2 + z^2
sage: Dx(f)  # = 2*x evaluated at (0,1,2)
0
sage: Dy(f)  # = 2*y evaluated at (0,1,2)
2
sage: Dz(f)  # = 2*z evaluated at (0,1,2)
4

Twisted derivations are handled similarly:

sage: theta = B.hom([B(y),B(z),B(x)])
sage: theta
Ring endomorphism of Fraction Field of Multivariate Polynomial Ring in x, y, z over Rational Field
Defn: x |--> y
       y |--> z
       z |--> x
sage: M = B.derivation_module(twist=theta)
sage: M
Module of twisted derivations over Fraction Field of Multivariate Polynomial Ring in x, y, z over Rational Field (twisting morphism: x |--> y, y |--> z, z |--> x)

Over a field, one proves that every $\theta$-derivation is a multiple of $\theta - id$, so that:

sage: d = M.gen(); d
[x |--> y, y |--> z, z |--> x] - id

and then:

sage: d(x)
-x + y
sage: d(y)
-y + z
sage: d(z)
x - z
sage: d(x + y + z)
0

AUTHOR:

- Xavier Caruso (2018-09)
class sage.rings.derivation.RingDerivation
    Bases: sage.structure.element.ModuleElement

    An abstract class for twisted and untwisted derivations over commutative rings.

codomain()
    Return the codomain of this derivation.
    EXAMPLES:
    sage: R.<x> = QQ[
    sage: f = R.derivation(); f
    d/dx
    sage: f.codomain()
    Univariate Polynomial Ring in x over Rational Field
    sage: f.codomain() is R
    True
    sage: S.<y> = R[
    sage: M = R.derivation_module(S)
    sage: M.random_element().codomain().codomain() is S
    True

domain()
    Return the domain of this derivation.
    EXAMPLES:
    sage: R.<x,y> = ZZ[
    sage: f = R.derivation(y); f
d/dy
    sage: f.domain()
    Multivariate Polynomial Ring in x, y over Rational Field
    sage: f.domain() is R
    True

class sage.rings.derivation.RingDerivationModule(domain, codomain, twist=None)
    Bases: sage.modules.module.Module, sage.structure.unique_representation.UniqueRepresentation

    A class for modules of derivations over a commutative ring.

    basis()
    Return a basis of this module of derivations.
    EXAMPLES:
    sage: R.<x,y> = ZZ[
    sage: M = R.derivation_module()
    sage: M.basis()
    Family (d/dx, d/dy)

codomain()
    Return the codomain of the derivations in this module.
EXAMPLES:

```sage
sage: R.<x,y> = ZZ[]
sage: M = R.derivation_module(); M
Module of derivations over Multivariate Polynomial Ring in x, y over Integer Ring
sage: M.codomain()
Multivariate Polynomial Ring in x, y over Integer Ring
```

`defining_morphism()`  
Return the morphism defining the structure of algebra of the codomain over the domain.

EXAMPLES:

```sage
sage: R.<x> = QQ[]
sage: M = R.derivation_module()
sage: M.defining_morphism()
Identity endomorphism of Univariate Polynomial Ring in x over Rational Field
sage: S.<y> = R[]
sage: M = R.derivation_module(S)
sage: M.defining_morphism()
Polynomial base injection morphism:
  From: Univariate Polynomial Ring in x over Rational Field
  To:   Univariate Polynomial Ring in y over Univariate Polynomial Ring in x over Rational Field
sage: ev = R.hom([QQ(0)])
sage: M = R.derivation_module(ev)
sage: M.defining_morphism()
Ring morphism:
  From: Univariate Polynomial Ring in x over Rational Field
  To:   Rational Field
  Defn: x |--> 0
```

`domain()`  
Return the domain of the derivations in this module.

EXAMPLES:

```sage
sage: R.<x,y> = ZZ[]
sage: M = R.derivation_module(); M
Module of derivations over Multivariate Polynomial Ring in x, y over Integer Ring
sage: M.domain()
Multivariate Polynomial Ring in x, y over Integer Ring
```

dual_basis()  
Return the dual basis of the canonical basis of this module of derivations (which is that returned by the method `basis()`).

**Note:** The dual basis of \((d_1, \ldots, d_n)\) is a family \((x_1, \ldots, x_n)\) of elements in the domain such that \(d_i(x_i) = 1\) and \(d_i(x_j) = 0\) if \(i \neq j\).

EXAMPLES:
sage: R.<x,y> = ZZ[]
sage: M = R.derivation_module()
sage: M.basis()
Family (d/dx, d/dy)
sage: M.dual_basis()
Family (x, y)

\textbf{gen}(n=0)

Return the \(n\)-th generator of this module of derivations.

INPUT:

• \(n\) – an integer (default: 0)

EXAMPLES:

sage: R.<x,y> = ZZ[]
sage: M = R.derivation_module(); M
Module of derivations over Multivariate Polynomial Ring in x, y over Integer\rightarrow Ring
sage: M.gen()
d/dx
sage: M.gen(1)
d/dy

gens()

Return the generators of this module of derivations.

EXAMPLES:

sage: R.<x,y> = ZZ[]
sage: M = R.derivation_module(); M
Module of derivations over Multivariate Polynomial Ring in x, y over Integer\rightarrow Ring
sage: M.gens()
(d/dx, d/dy)

We check that, for a nontrivial twist over a field, the module of twisted derivation is a vector space of dimension 1 generated by \texttt{twist - id}:

sage: K = R.fraction_field()
sage: theta = K.hom([K(y),K(x)])
sage: M = K.derivation_module(twist=theta); M
Module of twisted derivations over Fraction Field of Multivariate Polynomial Ring in x, y over Integer Ring (twisting morphism: x |--> y, y |--> x)
sage: M.gens()
([x |--> y, y |--> x] - id,)

\textbf{ngens}()

Return the number of generators of this module of derivations.

EXAMPLES:

sage: R.<x,y> = ZZ[]
sage: M = R.derivation_module(); M
Module of derivations over Multivariate Polynomial Ring in x, y over Integer\rightarrow Ring
(continues on next page)
Indeed, generators are:

```python
sage: M.gens()
(d/dx, d/dy)
```

We check that, for a nontrivial twist over a field, the module of twisted derivation is a vector space of dimension 1 generated by \( \text{twist - id} \):

```python
sage: K = R.fraction_field()
sage: theta = K.hom([K(y),K(x)])
sage: M = K.derivation_module(twist=theta); M
Module of twisted derivations over Fraction Field of Multivariate Polynomial Ring in x, y over Integer Ring (twisting morphism: x |--> y, y |--> x)
sage: M.ngens()
1
sage: M.gen()
[x |--> y, y |--> x] - id
```

**random_element\(*\text{args, **kwargs}\)**

Return a random derivation in this module.

**ring_of_constants**

Return the subring of the domain consisting of elements \( x \) such that \( d(x) = 0 \) for all derivation \( d \) in this module.

**some_elements**

Return a list of elements of this module.
twisting_morphism()  
Return the twisting homomorphism of the derivations in this module.

EXAMPLES:

```python
sage: R.<x,y> = ZZ[]
sage: theta = R.hom([y,x])
sage: M = R.derivation_module(twist=theta); M
Module of twisted derivations over Multivariate Polynomial Ring in x, y over Integer Ring (twisting morphism: x |--> y, y |--> x)
sage: M.twisting_morphism()
Ring endomorphism of Multivariate Polynomial Ring in x, y over Integer Ring
  Defn: x |--> y
  y |--> x
```

When the derivations are untwisted, this method returns nothing:

```python
sage: M = R.derivation_module()
sage: M.twisting_morphism()
```

class sage.rings.derivation.RingDerivationWithTwist_generic(parent, scalar=0)
Bases: sage.rings.derivation.RingDerivation

The class handles $\theta$-derivations of the form $\lambda(\theta - \iota)$ (where $\iota$ is the defining morphism of the codomain over the domain) for a scalar $\lambda$ varying in the codomain.

extend_to_fraction_field()
Return the extension of this derivation to fraction fields of the domain and the codomain.

EXAMPLES:

```python
sage: R.<x,y> = ZZ[]
sage: theta = R.hom([y,x])
sage: d = R.derivation(x, twist=theta)
sage: d
x*(x |--> y, y |--> x) - id
sage: D = d.extend_to_fraction_field()
sage: D
x*(x |--> y, y |--> x) - id
sage: D.domain()
Fraction Field of Multivariate Polynomial Ring in x, y over Integer Ring
sage: D(1/x)
(x - y)/y
```

list()
Return the list of coefficient of this twisted derivation on the canonical basis.

EXAMPLES:

```python
sage: R.<x,y> = QQ[]
sage: K = R.fraction_field()
sage: theta = K.hom([y,x])
sage: M = K.derivation_module(twist=theta)
sage: M.basis()
(continues on next page)
```
postcompose(morphism)
Return the twisted derivation obtained by applying first this twisted derivation and then morphism.

INPUT:
- morphism – a homomorphism of rings whose domain is the codomain of this derivation or a ring into which the codomain of this derivation

EXAMPLES:

```
sage: R.<x,y> = ZZ[]
sage: theta = R.hom([y,x])
sage: D = R.derivation(x, twist=theta); D
x*(x |--> y, y |--> x) - id)
sage: f = R.hom([[x^2, y^3]])
sage: g = D.precompose(f); g
x^2*(x |--> y^3, y |--> x^2) - [x |--> x^2, y |--> y^3])
```

Observe that the \( g \) is no longer a \( \theta \)-derivation but a \( (\theta \circ f) \)-derivation:

```
sage: g.parent().twisting_morphism()
Ring endomorphism of Multivariate Polynomial Ring in x, y over Integer Ring
  Defn: x |--> y^2
       y |--> x^3
```

precompose(morphism)
Return the twisted derivation obtained by applying first morphism and then this twisted derivation.

INPUT:
- morphism – a homomorphism of rings whose codomain is the domain of this derivation or a ring that coerces to the domain of this derivation

EXAMPLES:

```
sage: R.<x,y> = ZZ[]
sage: theta = R.hom([y,x])
sage: D = R.derivation(x, twist=theta); D
x*(x |--> y, y |--> x) - id)
sage: f = R.hom([[x^2, y^3]])
sage: g = D.postcompose(f); g
x^2*(x |--> y^3, y |--> x^2) - [x |--> x^2, y |--> y^3])
```

Observe that the \( g \) is no longer a \( \theta \)-derivation but a \( (f \circ \theta) \)-derivation:

```
sage: g.parent().twisting_morphism()
Ring endomorphism of Multivariate Polynomial Ring in x, y over Integer Ring
  Defn: x |--> y^2
       y |--> x^3
```
class sage.rings.derivation.RingDerivationWithoutTwist

Bases: sage.rings.derivation.RingDerivation

An abstract class for untwisted derivations.

extend_to_fraction_field()

Return the extension of this derivation to fraction fields of the domain and the codomain.

EXAMPLES:

```
sage: S.<x> = QQ[]
sage: d = S.derivation()
sage: d
d/dx
sage: D = d.extend_to_fraction_field()
sage: D
d/dx
sage: D.domain()
Fraction Field of Univariate Polynomial Ring in x over Rational Field
sage: D(1/x)
-1/x^2
```

is_zero()

Return True if this derivation is zero.

EXAMPLES:

```
sage: R.<x,y> = ZZ[]
sage: f = R.derivation(); f
d/dx
sage: f.is_zero()
False
sage: (f-f).is_zero()
True
```

list()

Return the list of coefficient of this derivation on the canonical basis.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: M = R.derivation_module()
sage: M.basis()
Family (d/dx, d/dy)
sage: R.derivation(x).list()
[1, 0]
sage: R.derivation(y).list()
[0, 1]
```
monomial_coefficients()

Return dictionary of nonzero coordinates (on the canonical basis) of this derivation.

More precisely, this returns a dictionary whose keys are indices of basis elements and whose values are the corresponding coefficients.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: M = R.derivation_module()
sage: M.basis()
Family (d/dx, d/dy)
sage: R.derivation(x).monomial_coefficients()
{0: 1}
sage: R.derivation(y).monomial_coefficients()
{1: 1}
sage: f = x*R.derivation(x) + y*R.derivation(y); f
x*d/dx + y*d/dy
sage: f.monomial_coefficients()
{0: x, 1: y}
```

postcompose(morphism)

Return the derivation obtained by applying first this derivation and then morphism.

INPUT:

- morphism – a homomorphism of rings whose domain is the codomain of this derivation or a ring into which the codomain of this derivation coerces

EXAMPLES:

```
sage: A.<x,y>= QQ[]
sage: ev = A.hom([QQ(0), QQ(1)])
sage: Dx = A.derivation(x)
sage: Dy = A.derivation(y)

We can define the derivation at (0, 1) just by postcomposing with ev:

sage: dx = Dx.postcompose(ev)
sage: dy = Dy.postcompose(ev)
sage: f = x^2 + y^2
sage: dx(f)
0
sage: dy(f)
2
```

Note that we cannot avoid the creation of the evaluation morphism: if we pass in QQ instead, an error is raised since there is no coercion morphism from A to QQ:
precompose(morphism)
Return the derivation obtained by applying first morphism and then this derivation.

INPUT:
• morphism – a homomorphism of rings whose codomain is the domain of this derivation or a ring that coerces to the domain of this derivation

EXAMPLES:

sage: A.<x> = QQ[]
sage: B.<x,y> = QQ[]
sage: D = B.derivation(x) - 2*x*B.derivation(y); D
d/dx - 2*x*d/dy

When restricting to A, the term d/dy disappears (since it vanishes on A):

sage: D.precompose(A)
d/dx

If we restrict to another well chosen subring, the derivation vanishes:

sage: C.<t> = QQ[]
sage: f = C.hom([x^2 + y]); f
Ring morphism:
  From: Univariate Polynomial Ring in t over Rational Field
  To:   Multivariate Polynomial Ring in x, y over Rational Field
  Defn: t |--> x^2 + y
sage: D.precompose(f)
0

Note that this method cannot be used to compose derivations:

sage: D.precompose(D)
Traceback (most recent call last):
  ...TypeError: you must give an homomorphism of rings

pth_power()
Return the $p$-th power of this derivation where $p$ is the characteristic of the domain.

Note:  Leibniz rule implies that this is again a derivation.
EXAMPLES:

```python
sage: R.<x,y> = GF(5)[]
sage: Dx = R.derivation(x)
sage: Dx.pth_power(0)
0
sage: (x*Dx).pth_power()
x*d/dx
sage: (x^6*Dx).pth_power()
x^26*d/dx
sage: Dy = R.derivation(y)
sage: (x*Dx + y*Dy).pth_power()
x*d/dx + y*d/dy
```

An error is raised if the domain has characteristic zero:

```python
sage: R.<x,y> = QQ[]
sage: Dx = R.derivation(x)
sage: Dx.pth_power()
Traceback (most recent call last):
  ...TypeError: the domain of the derivation must have positive and prime
  characteristic
```

or if the characteristic is not a prime number:

```python
sage: R.<x,y> = Integers(10)[]
sage: Dx = R.derivation(x)
sage: Dx.pth_power()
Traceback (most recent call last):
  ...TypeError: the domain of the derivation must have positive and prime
  characteristic
```

class sage.rings.derivation.RingDerivationWithoutTwist_fraction_field(parent, arg=None)

This class handles derivations over fraction fields.

class sage.rings.derivation.RingDerivationWithoutTwist_function(parent, arg=None)

A class for untwisted derivations over rings whose elements are either polynomials, rational fractions, power series or Laurent series.

is_zero()

Return True if this derivation is zero.

EXAMPLES:

```python
sage: R.<x,y> = ZZ[]
sage: f = R.derivation(); f
d/dx
sage: f.is_zero()
False
```
list()
Return the list of coefficient of this derivation on the canonical basis.

EXAMPLES:

```sage
er.<x,y> = GF(5)[[]]
er.m = er.derivation_module()
er.m.basis()
Family (d/dx, d/dy)
er.derivation(x).list()
[1, 0]
er.derivation(y).list()
[0, 1]
er = er^*er.derivation(x) + y^*er.derivation(y); er
x*d/dx + y*d/dy
er.f = er.list()
[x, y]
```

class sage.rings.derivation.RingDerivationWithoutTwist_quotient(parent, arg=None)
Bases: sage.rings.derivation.RingDerivationWithoutTwist_wrapper
This class handles derivations over quotient rings.

class sage.rings.derivation.RingDerivationWithoutTwist_wrapper(parent, arg=None)
Bases: sage.rings.derivation.RingDerivationWithoutTwist
This class is a wrapper for derivation.
It is useful for changing the parent without changing the computation rules for derivations. It is used for derivations over fraction fields and quotient rings.
list()
Return the list of coefficient of this derivation on the canonical basis.

EXAMPLES:

```sage
er.<x,y> = GF(5)[[]]
er.s = er.quo([x^5, y^5])
er.m = er.derivation_module()
er.m.basis()
Family (d/dx, d/dy)
er.derivation(x).list()
[1, 0]
er.derivation(y).list()
[0, 1]
er = er^*er.derivation(x) + y^*er.derivation(y); er
x*d/dx + y*d/dy
er.f = er.list()
[x, y]
```
sage: f.list()
[x, y]

class sage.rings.derivation.RingDerivationWithoutTwist_zero(parent, arg=None)
Bases: sage.rings.derivation.RingDerivationWithoutTwist

This class can only represent the zero derivation.

It is used when the parent is the zero derivation module (e.g., when its domain is \(\mathbb{Z}\), \(\mathbb{Q}\), a finite field, etc.)

\textbf{is_zero()}

Return True if this derivation vanishes.

EXAMPLES:

\begin{verbatim}
sage: M = QQ.derivation_module()
sage: M().is_zero()
True
\end{verbatim}

\textbf{list()}

Return the list of coefficient of this derivation on the canonical basis.

EXAMPLES:

\begin{verbatim}
sage: M = QQ.derivation_module()
sage: M().list()
[]
\end{verbatim}
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