Schemes

*Release 10.2*

The Sage Development Team

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Various parts of schemes were implemented by different authors. This document aims to give an overview of the different classes of schemes working together coherently.

1.1 Generic

- **Scheme**: A scheme whose datatype might not be defined in terms of algebraic equations: e.g. the Jacobian of a curve may be represented by means of a Scheme.

- **AlgebraicScheme**: A scheme defined by means of polynomial equations, which may be reducible or defined over a ring other than a field. In particular, the defining ideal need not be a radical ideal, and an algebraic scheme may be defined over \( \text{Spec}(R) \).

- **AmbientSpaces**: Most effective models of algebraic scheme will be defined not by generic gluings, but by embeddings in some fixed ambient space.

1.2 Ambients

- **AffineSpace**: Affine spaces and their affine subschemes form the most important universal objects from which algebraic schemes are built. The affine spaces form universal objects in the sense that a morphism is uniquely determined by the images of its coordinate functions and any such images determine a well-defined morphism.

  By default affine spaces will embed in some ordinary projective space, unless it is created as an affine patch of another object.

- **ProjectiveSpace**: Projective spaces are the most natural ambient spaces for most projective objects. They are locally universal objects.

- **ProjectiveSpace_ordinary (not implemented)**: The ordinary projective spaces have the standard weights \([1, \ldots, 1]\) on their coefficients.

- **ProjectiveSpace_weighted (not implemented)**: A special subtype for non-standard weights.

- **ToricVariety**: Toric varieties are (partial) compactifications of algebraic tori \((\mathbb{C}^*)^n\) compatible with torus action. Affine and projective spaces are examples of toric varieties, but it is not envisioned that these special cases should inherit from ToricVariety.
1.3 Subschemes

- **AlgebraicScheme_subscheme_affine**: An algebraic scheme defined by means of an embedding in a fixed ambient affine space.

- **AlgebraicScheme_subscheme_projective**: An algebraic scheme defined by means of an embedding in a fixed ambient projective space.

- **QuasiAffineScheme (not yet implemented)**: An open subset $U = X \setminus Z$ of a closed subset $X$ of affine space; note that this is mathematically a quasi-projective scheme, but its ambient space is an affine space and its points are represented by affine rather than projective points.

  **Note**: AlgebraicScheme_quasi is implemented, as a base class for this.

- **QuasiProjectiveScheme (not yet implemented)**: An open subset of a closed subset of projective space; this datatype stores the defining polynomial, polynomials, or ideal defining the projective closure $X$ plus the closed subscheme $Z$ of $X$ whose complement $U = X \setminus Z$ is the quasi-projective scheme.

  **Note**: The quasi-affine and quasi-projective datatype lets one create schemes like the multiplicative group scheme $G_m = \mathbb{A}^1 \setminus \{(0)\}$ and the non-affine scheme $\mathbb{A}^2 \setminus \{(0,0)\}$. The latter is not affine and is not of the form $\text{Spec}(R)$.

1.4 Point sets

- **PointSets and points over a ring (to do)**: For algebraic schemes $X/S$ and $T/S$ over $S$, one can form the point set $X(T)$ of morphisms from $T \to X$ over $S$.

  A projective space object in the category of schemes is a locally free object – the images of the generator functions locally determine a point. Over a field, one can choose one of the standard affine patches by the condition that a coordinate function $X_i \neq 0$.

  ```python
  sage: PP.<X,Y,Z> = ProjectiveSpace(2, QQ)
  sage: PP
  Projective Space of dimension 2 over Rational Field
  sage: PP(QQ)
  Set of rational points of Projective Space of dimension 2 over Rational Field
  sage: PP(QQ)([-2, 3, 5])
  (-2/5 : 3/5 : 1)
  ```

  Over a ring, this is not true anymore. For example, even over an integral domain which is not a PID, there may be no single affine patch which covers a point.

  ```python
  sage: R.<x> = ZZ[]
  sage: S.<t> = R.quo(x^2 + 5)
  sage: P.<X,Y,Z> = ProjectiveSpace(2, S)
  sage: P(S)
  Set of rational points of Projective Space of dimension 2 over Univariate Quotient Polynomial Ring in t over Integer Ring with modulus x^2 + 5
  ```
In order to represent the projective point \((2 : 1 + t) = (1 - t : 3)\) we note that the first representative is not well-defined at the prime \(p = (2, 1 + t)\) and the second element is not well-defined at the prime \(q = (1 - t, 3)\), but that \(p + q = (1)\), so globally the pair of coordinate representatives is well-defined.

\[
\text{sage: } P([2, 1 + t])
\]
\[
(2 : t + 1 : 1)
\]

In fact, we need a test \(R.\text{ideal}([2, 1 + t]) == R.\text{ideal}([1])\) in order to make this meaningful.

### 1.5 Berkovich Analytic Spaces

- **Berkovich Analytic Space (not yet implemented)** The construction of analytic spaces from schemes due to Berkovich. Any Berkovich space should inherit from Berkovich.

- **Berkovich Analytic Space over \(C_p\)** A special case of the general Berkovich analytic space construction. Affine Berkovich space over \(C_p\) is the set of seminorms on the polynomial ring \(C_p[x]\), while projective Berkovich space over \(C_p\) is the one-point compactification of affine Berkovich space \(C_p\). Points are represented using the classification (due to Berkovich) of a corresponding decreasing sequence of disks in \(C_p\).

**AUTHORS:**

- David Kohel, William Stein (2006-01-03): initial version
- Andrey Novoseltsev (2010-09-24): updated due to addition of toric varieties
AUTHORS:


class sage.schemes.generic.scheme.AffineScheme(R, S=None, category=None)

    Bases: UniqueRepresentation, Scheme

    Class for general affine schemes.

    See also:

    For affine spaces over a base ring and subschemes thereof, see sage.schemes.generic.algebraic_scheme.AffineSpace.

Element

    alias of SchemeTopologicalPoint_prime_ideal

base_extend(R)

    Extend the base ring/scheme.

    INPUT:

    - R -- an affine scheme or a commutative ring

    EXAMPLES:

    sage: Spec_ZZ = Spec(ZZ); Spec_ZZ
    Spectrum of Integer Ring
    sage: Spec_ZZ.base_extend(QQ)
    Spectrum of Rational Field

    sage: Spec(ZZ['x']).base_extend(Spec(QQ))
    Spectrum of Univariate Polynomial Ring in x over Rational Field

coordinate_ring()

    Return the underlying ring of this scheme.

    OUTPUT:

    A commutative ring.

    EXAMPLES:
sage: Spec(QQ).coordinate_ring()
Rational Field
sage: Spec(PolynomialRing(QQ, 3, 'x')).coordinate_ring()
Multivariate Polynomial Ring in \texttt{x0}, \texttt{x1}, \texttt{x2} over Rational Field

dimension()
Return the absolute dimension of this scheme.
OUTPUT:
Integer.
EXAMPLES:

sage: S = Spec(ZZ)
sage: S.dimension_absolute()
1
sage: S.dimension()
1

dimension_absolute()
Return the absolute dimension of this scheme.
OUTPUT:
Integer.
EXAMPLES:

sage: S = Spec(ZZ)
sage: S.dimension_absolute()
1
sage: S.dimension()
1

dimension_relative()
Return the relative dimension of this scheme over its base.
OUTPUT:
Integer.
EXAMPLES:

sage: S = Spec(ZZ)
sage: S.dimension_relative()
0

hom(x, Y=None)
Return the scheme morphism from self to Y defined by x.

INPUT:

- \texttt{x} – anything that determines a scheme morphism; if \texttt{x} is a scheme, try to determine a natural map to \texttt{x}
- \texttt{Y} – the codomain scheme (optional); if \texttt{Y} is not given, try to determine \texttt{Y} from context
- \texttt{check} – boolean (optional, default: True); whether to check the defining data for consistency

Chapter 2. Schemes
**OUTPUT:**

The scheme morphism from \texttt{self} to \texttt{Y} defined by \texttt{x}.

**EXAMPLES:**

We construct the inclusion from \texttt{Spec(\mathbb{Q})} into \texttt{Spec(\mathbb{Z})} induced by the inclusion from \texttt{\mathbb{Z}} into \texttt{\mathbb{Q}}:

```
sage: X = Spec(QQ)
sage: X.hom(ZZ.hom(QQ))
```

Affine Scheme morphism:
From: Spectrum of Rational Field
To: Spectrum of Integer Ring
Defn: Natural morphism:
From: Integer Ring
To: Rational Field

**is_noetherian()**

Return True if \texttt{self} is Noetherian, False otherwise.

**EXAMPLES:**

```
sage: Spec(ZZ).is_noetherian()
True
```

**class** \texttt{sage.schemes.generic.scheme.Scheme\(X=None,\ category=None\)}

**Bases:** \texttt{Parent}

The base class for all schemes.

**INPUT:**

- \texttt{X} – a scheme, scheme morphism, commutative ring, commutative ring morphism, or \texttt{None} (optional). Determines the base scheme. If a commutative ring is passed, the spectrum of the ring will be used as base.
- \texttt{category} – the category (optional). Will be automatically constructed by default.

**EXAMPLES:**

```
sage: from sage.schemes.generic.scheme import Scheme
sage: Scheme(ZZ)
<sage.schemes.generic.scheme.Scheme_with_category object at ...>
```

A scheme is in the category of all schemes over its base:

```
sage: ProjectiveSpace(4, QQ).category()
Category of schemes over Rational Field
```

There is a special and unique \texttt{Spec(Z)} that is the default base scheme:

```
sage: Spec(ZZ).base_scheme() is Spec(QQ).base_scheme()
True
```

**base_extend\(Y\)**

Extend the base of the scheme.

Derived classes must override this method.

**EXAMPLES:**
```python
sage: from sage.schemes.generic.scheme import Scheme
sage: X = Scheme(ZZ)
sage: X.base_scheme()
Spectrum of Integer Ring
sage: X.base_extend(QQ)
Traceback (most recent call last):
... Not Implemented Error
```

**base_morphism()**

Return the structure morphism from `self` to its base scheme.

OUTPUT:

A scheme morphism.

EXAMPLES:

```python
sage: A = AffineSpace(4, QQ)
sage: A.base_morphism()
Scheme morphism:
  From: Affine Space of dimension 4 over Rational Field
  To:   Spectrum of Rational Field
  Defn: Structure map
sage: X = Spec(QQ)
sage: X.base_morphism()
Scheme morphism:
  From: Spectrum of Rational Field
  To:   Spectrum of Integer Ring
  Defn: Structure map
```

**base_ring()**

Return the base ring of the scheme `self`.

OUTPUT:

A commutative ring.

EXAMPLES:

```python
sage: A = AffineSpace(4, QQ)
sage: A.base_ring()
Rational Field
sage: X = Spec(QQ)
sage: X.base_ring()
Integer Ring
```

**base_scheme()**

Return the base scheme.

OUTPUT:

A scheme.

EXAMPLES:
Sage:

```python
sage: A = AffineSpace(4, QQ)
sage: A.base_scheme()
Spectrum of Rational Field
sage: X = Spec(QQ)
sage: X.base_scheme()
Spectrum of Integer Ring
```

**coordinate_ring()**

Return the coordinate ring.

**OUTPUT:**

The global coordinate ring of this scheme, if defined. Otherwise raise a `ValueError`.

**EXAMPLES:**

```python
sage: R.<x, y> = QQ[]
sage: I = (x^2 - y^2)*R
sage: X = Spec(R.quotient(I))
sage: X.coordinate_ring()
Quotient of Multivariate Polynomial Ring in x, y over Rational Field by the ideal (x^2 - y^2)
```

**count_points(n)**

Count points over finite fields.

**INPUT:**

* n – integer.

**OUTPUT:**

An integer. The number of points over $\mathbb{F}_q, \ldots, \mathbb{F}_{q^n}$ on a scheme over a finite field $\mathbb{F}_q$.

**EXAMPLES:**

```python
sage: # needs sage.schemes
sage: P.<x> = PolynomialRing(GF(3))
sage: C = HyperellipticCurve(x^3 + x^2 + 1)
sage: C.count_points(4)
[6, 12, 18, 96]
sage: C.base_extend(GF(9, 'a')).count_points(2)  # needs sage.rings.finite_rings
[12, 96]
```

**dimension()**

Return the absolute dimension of this scheme.
OUTPUT:

Integer.

EXAMPLES:

```python
sage: R.<x, y> = QQ[]
sage: I = (x^2 - y^2)*R
sage: X = Spec(R.quotient(I))
sage: X.dimension_absolute()
Traceback (most recent call last):
... Not Implemented Error
sage: X.dimension()
Traceback (most recent call last):
... Not Implemented Error
```

dimension_absolute()
Return the absolute dimension of this scheme.

OUTPUT:

Integer.

EXAMPLES:

```python
sage: R.<x, y> = QQ[]
sage: I = (x^2 - y^2)*R
sage: X = Spec(R.quotient(I))
sage: X.dimension_absolute()
Traceback (most recent call last):
... Not Implemented Error
sage: X.dimension()
Traceback (most recent call last):
... Not Implemented Error
```

dimension_relative()
Return the relative dimension of this scheme over its base.

OUTPUT:

Integer.

EXAMPLES:

```python
sage: R.<x, y> = QQ[]
sage: I = (x^2 - y^2)*R
sage: X = Spec(R.quotient(I))
sage: X.dimension_relative()
Traceback (most recent call last):
... Not Implemented Error
```

hom(x, Y=None, check=True)
Return the scheme morphism from self to Y defined by x.
INPUT:
- \( x \) – anything that determines a scheme morphism; if \( x \) is a scheme, try to determine a natural map to \( x \)
- \( Y \) – the codomain scheme (optional); if \( Y \) is not given, try to determine \( Y \) from context
- \( \text{check} \) – boolean (optional, default: \( \text{True} \)); whether to check the defining data for consistency

OUTPUT:
The scheme morphism from \( \text{self} \) to \( Y \) defined by \( x \).

EXAMPLES:

```python
sage: P = ProjectiveSpace(ZZ, 3)
sage: P.hom(Spec(ZZ))
Scheme morphism:
  From: Projective Space of dimension 3 over Integer Ring
  To:    Spectrum of Integer Ring
  Defn: Structure map

identity_morphism()

Return the identity morphism.

OUTPUT:
The identity morphism of the scheme \( \text{self} \).

EXAMPLES:

```python
sage: X = Spec(QQ)
sage: X.identity_morphism()
Scheme endomorphism of Spectrum of Rational Field
  Defn: Identity map

point(\( v \), \( \text{check=\text{True}} \))

Create a point.

INPUT:
- \( v \) – anything that defines a point
- \( \text{check} \) – boolean (optional, default: \( \text{True} \)); whether to check the defining data for consistency

OUTPUT:
A point of the scheme.

EXAMPLES:

```python
sage: A2 = AffineSpace(QQ, 2)
sage: A2.point([4, 5])
(4, 5)

sage: R.<t> = PolynomialRing(QQ)
sage: E = EllipticCurve([t + 1, t, t, 0, 0])
# needs sage.schemes
sage: E.point([0, 0])
# needs sage.schemes
(0 : 0 : 1)"
point_homset($S=\text{None}$)
Return the set of $S$-valued points of this scheme.

INPUT:

- $S$ – a commutative ring.

OUTPUT:

The set of morphisms $Spec(S)\circ X$.

EXAMPLES:

```
sage: P = ProjectiveSpace(ZZ, 3)
sage: P.point_homset(ZZ)
Set of rational points of Projective Space of dimension 3 over Integer Ring
sage: P.point_homset(QQ)
Set of rational points of Projective Space of dimension 3 over Rational Field
sage: P.point_homset(GF(11))
Set of rational points of Projective Space of dimension 3 over
  Finite Field of size 11
```

point_set($S=\text{None}$)
Return the set of $S$-valued points of this scheme.

INPUT:

- $S$ – a commutative ring.

OUTPUT:

The set of morphisms $Spec(S)\circ X$.

EXAMPLES:

```
sage: P = ProjectiveSpace(ZZ, 3)
sage: P.point_homset(ZZ)
Set of rational points of Projective Space of dimension 3 over Integer Ring
sage: P.point_homset(QQ)
Set of rational points of Projective Space of dimension 3 over Rational Field
sage: P.point_homset(GF(11))
Set of rational points of Projective Space of dimension 3 over
  Finite Field of size 11
```

structure_morphism()
Return the structure morphism from self to its base scheme.

OUTPUT:

A scheme morphism.

EXAMPLES:

```
sage: A = AffineSpace(4, QQ)
sage: A.structure_morphism()
Scheme morphism:
  From: Affine Space of dimension 4 over Rational Field
  To:    Spectrum of Rational Field
  Defn: Structure map
```

(continues on next page)
Return the disjoint union of the schemes self and X.

EXAMPLES:

```python
sage: S = Spec(QQ)
sage: X = AffineSpace(1, QQ)
sage: S.union(X)
Traceback (most recent call last):
  ... Not ImplementedError
```

zeta_function()

Compute the zeta function of a generic scheme.

Derived classes should override this method.

OUTPUT: rational function in one variable.

EXAMPLES:

```python
sage: P.<x,y,z> = ProjectiveSpace(GF(4, 't'), 2)  # needs sage.rings.finite_rings
sage: X = P.subscheme([y^2*z - x^3 - z^3])  # needs sage.rings.finite_rings
sage: X.zeta_function()  # needs sage.rings.finite_rings
Traceback (most recent call last):
  ... Not ImplementedError
```

zeta_series(n, t)

Return the zeta series.

Compute a power series approximation to the zeta function of a scheme over a finite field.

INPUT:

- n – the number of terms of the power series to compute
- t – the variable which the series should be returned

OUTPUT:

A power series approximating the zeta function of self

EXAMPLES:
If the scheme has a method `zeta_function`, this is used to provide the required approximation. Otherwise this function depends on `count_points`, which is only defined for prime order fields for general schemes. Nonetheless, since `github issue #15108` and `github issue #15148`, it supports hyperelliptic curves over non-prime fields:

```python
sage: C.base_extend(GF(9, 'a')).zeta_series(4, t)  # needs sage.rings.finite_rings sage.schemes
1 + 12*t + 120*t^2 + 1092*t^3 + 9840*t^4 + O(t^5)
```

```
1 + 5*t + 21*t^2 + 0(t^3)
```

### Examples:

```python
sage: from sage.schemes.generic.scheme import is_AffineScheme
sage: is_AffineScheme(5)
False
sage: E = Spec(QQ)
sage: is_AffineScheme(E)
True
```

### Test whether `x` is a scheme.

**INPUT:**

* `x` – anything.

**OUTPUT:**

Boolean. Whether `x` derives from `Scheme`.

**Examples:**
<table>
<thead>
<tr>
<th>Sage Code</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>from sage.schemes.generic.scheme import is_Scheme</code></td>
<td>Import the <code>is_Scheme</code> function</td>
</tr>
<tr>
<td><code>is_Scheme(5)</code></td>
<td>Evaluate if 5 is a scheme, returns <code>False</code></td>
</tr>
<tr>
<td><code>X = Spec(QQ)</code></td>
<td>Define <code>X</code> as the spectrum of QQ</td>
</tr>
<tr>
<td><code>is_Scheme(X)</code></td>
<td>Evaluate if <code>X</code> is a scheme, returns <code>True</code></td>
</tr>
</tbody>
</table>
ALTER THE SPEC FUNCTOR

AUTHORS:

• William Stein (2006): initial implementation
• Peter Bruin (2014): rewrite Spec as a functor

`sage.schemes.generic.spec.Spec(R, S=None)`

Apply the Spec functor to $R$.

INPUT:

• $R$ – either a commutative ring or a ring homomorphism
• $S$ – a commutative ring (optional), the base ring

OUTPUT:

• `AffineScheme` – the affine scheme `Spec(R)`

EXAMPLES:

```python
sage: Spec(QQ)
Spectrum of Rational Field
sage: Spec(PolynomialRing(QQ, 'x'))
Spectrum of Univariate Polynomial Ring in x over Rational Field
sage: Spec(PolynomialRing(QQ, 'x', 3))
Spectrum of Multivariate Polynomial Ring in x0, x1, x2 over Rational Field
sage: X = Spec(PolynomialRing(GF(49, 'a'), 3, 'x'))
```

Applying `Spec` twice to the same ring gives identical output (see github issue #17008):

```python
sage: A = Spec(ZZ); B = Spec(ZZ)
sage: A is B
True
```

A `TypeError` is raised if the input is not a commutative ring:

```python
sage: Spec(5)
Traceback (most recent call last):
...
```

(continues on next page)
TypeError: x (=5) is not in Category of commutative rings

```
sage: Spec(FreeAlgebra(QQ, 2, 'x'))  
needs sage.combinat sage.modules
...
Traceback (most recent call last):
...
TypeError: x (=Free Algebra on 2 generators (x0, x1) over Rational Field) 
is not in Category of commutative rings
```

```
class sage.schemes.generic.spec.SpecFunctor(base_ring=None)
    Bases: Functor, UniqueRepresentation
    The Spec functor.
```
SCHEME OBTAINED BY GLUING TWO OTHER SCHEMES

class sage.schemes.generic.glue.GluedScheme(f, g, check=True):
    Bases: Scheme

    INPUT:
    • f - open immersion from a scheme U to a scheme X
    • g - open immersion from U to a scheme Y

    OUTPUT: The scheme obtained by gluing X and Y along the open set U.

    Note: Checking that f and g are open immersions is not implemented.

gluing_maps()
Chapter 4. Scheme obtained by gluing two other schemes
class sage.schemes.generic.point.SchemePoint(S, parent=None)

Bases: Element

Base class for points on a scheme, either topological or defined by a morphism.

scheme()

Return the scheme on which self is a point.

EXAMPLES:

```
sage: from sage.schemes.generic.point import SchemePoint
sage: S = Spec(ZZ)
sage: P = SchemePoint(S)
sage: P.scheme()
Spectrum of Integer Ring
```

class sage.schemes.generic.point.SchemeRationalPoint(f)

Bases: SchemePoint

INPUT:

• f - a morphism of schemes

morphism()

class sage.schemes.generic.point.SchemeTopologicalPoint(S)

Bases: SchemePoint

Base class for topological points on schemes.

class sage.schemes.generic.point.SchemeTopologicalPoint_affine_open(u, x)

Bases: SchemeTopologicalPoint

INPUT:

• u – morphism with domain an affine scheme U
• x – topological point on U

affine_open()

Return the affine open subset U.

embedding_of_affine_open()

Return the embedding from the affine open subset U into this scheme.
point_on_affine()

Return the scheme point on the affine open $U$.

class sage.schemes.generic.point.SchemeTopologicalPoint_prime_ideal(S, P, check=False)

Bases: SchemeTopologicalPoint

INPUT:

- S – an affine scheme
- P – a prime ideal of the coordinate ring of $S$, or anything that can be converted into such an ideal

prime_ideal()

Return the prime ideal that defines this scheme point.

EXAMPLES:

```python
sage: from sage.schemes.generic.point import SchemeTopologicalPoint_prime_ideal
sage: P2.<x, y, z> = ProjectiveSpace(2, QQ)
sage: pt = SchemeTopologicalPoint_prime_ideal(P2, y*z - x^2)
sage: pt.prime_ideal()
Ideal (-x^2 + y*z) of Multivariate Polynomial Ring in x, y, z over Rational Field
```

sage.schemes.generic.point.is_SchemeRationalPoint(x)

sage.schemes.generic.point.is_SchemeTopologicalPoint(x)
class sage.schemes.generic.ambient_space.AmbientSpace(n, R=Integer Ring)

Bases: Scheme

Base class for ambient spaces over a ring.

INPUT:

• n - dimension
• R - ring

ambient_space()

Return the ambient space of the scheme self, in this case self itself.

EXAMPLES:

    sage: P = ProjectiveSpace(4, ZZ)
    sage: P.ambient_space() is P
    True

    sage: A = AffineSpace(2, GF(3))
    sage: A.ambient_space()
    Affine Space of dimension 2 over Finite Field of size 3

base_extend(R)

Return the natural extension of self over R.

INPUT:

• R – a commutative ring, such that there is a natural map from the base ring of self to R.

OUTPUT:

• an ambient space over R of the same structure as self.

Note: A ValueError is raised if there is no such natural map. If you need to drop this condition, use self.change_ring(R).

EXAMPLES:

    sage: P.<x, y, z> = ProjectiveSpace(2, ZZ)
    sage: PQ = P.base_extend(QQ); PQ
    Projective Space of dimension 2 over Rational Field
    sage: PQ.base_extend(GF(5))
Traceback (most recent call last):
...
ValueError: no natural map from the base ring (=Rational Field) to R (=Finite Field of size 5)!

**change_ring(R)**
Return an ambient space over ring $R$ and otherwise the same as self.

**INPUT:**
- $R$ – commutative ring

**OUTPUT:**
- ambient space over $R$

**Note:** There is no need to have any relation between $R$ and the base ring of self, if you want to have such a relation, use `self.base_extend(R)` instead.

**defining_polynomials()**
Return the defining polynomials of the scheme self. Since self is an ambient space, this is an empty list.

**EXAMPLES:**

```python
sage: ProjectiveSpace(2, QQ).defining_polynomials()
()  
sage: AffineSpace(0, ZZ).defining_polynomials()
()
```

**dimension()**
Return the absolute dimension of this scheme.

**EXAMPLES:**

```python
sage: A2Q = AffineSpace(2, QQ)
sage: A2Q.dimension_absolute()  
2  
sage: A2Q.dimension()  
2  
sage: A2Z = AffineSpace(2, ZZ)
sage: A2Z.dimension_absolute()  
3  
sage: A2Z.dimension()  
3
```

**dimension_absolute()**
Return the absolute dimension of this scheme.

**EXAMPLES:**

```python
sage: A2Q = AffineSpace(2, QQ)
sage: A2Q.dimension_absolute()  
2  
sage: A2Q.dimension()  
2
```
sage: A2Z = AffineSpace(2, ZZ)
sage: A2Z.dimension_absolute()
3
sage: A2Z.dimension()
3

**dimension_relative()**

Return the relative dimension of this scheme over its base.

**EXAMPLES:**

```python
sage: A2Q = AffineSpace(2, QQ)
sage: A2Q.dimension_relative()
2
sage: A2Z = AffineSpace(2, ZZ)
sage: A2Z.dimension_relative()
2
```

**gen**(\(n=0\))

Return the \(n\)-th generator of the coordinate ring of the scheme self.

**EXAMPLES:**

```python
sage: P.<x, y, z> = ProjectiveSpace(2, ZZ)
sage: P.gen(1)
y
```

**gens()**

Return the generators of the coordinate ring of the scheme self.

**EXAMPLES:**

```python
sage: AffineSpace(0, QQ).gens()
()  
sage: P.<x, y, z> = ProjectiveSpace(2, GF(5))
sage: P.gens()
(x, y, z)
```

**identity_morphism()**

Return the identity morphism.

**OUTPUT:** the identity morphism of the scheme self

**EXAMPLES:**

```python
sage: A = AffineSpace(2, GF(3))
sage: A.identity_morphism()
Scheme endomorphism of Affine Space of dimension 2 over Finite Field of size 3
  Defn: Identity map
sage: P = ProjectiveSpace(3, ZZ)
sage: P.identity_morphism()
```

(continues on next page)
Scheme endomorphism of Projective Space of dimension 3 over Integer Ring
Defn: Identity map

**is_projective()**
Return whether this ambient space is projective n-space.

**EXAMPLES:**
```python
sage: AffineSpace(3, QQ).is_projective()
False
sage: ProjectiveSpace(3, QQ).is_projective()
True
```

**ngens()**
Return the number of generators of the coordinate ring of the scheme self.

**EXAMPLES:**
```python
sage: AffineSpace(0, QQ).ngens()
0
sage: ProjectiveSpace(50, ZZ).ngens()
51
```

sage.schemes.generic.ambient_space.is_AmbientSpace(x)
Return True if x is an ambient space.

**EXAMPLES:**
```python
sage: from sage.schemes.generic.ambient_space import is_AmbientSpace
sage: is_AmbientSpace(ProjectiveSpace(3, ZZ))
True
sage: is_AmbientSpace(AffineSpace(2, QQ))
True
sage: P.<x, y, z> = ProjectiveSpace(2, ZZ)
sage: is_AmbientSpace(P.subscheme([x + y + z]))
False
```
CHAPTER
SEVEN

ALGEBRAIC SCHEMES

An algebraic scheme is defined by a set of polynomials in some suitable affine or projective coordinates. Possible ambient spaces are

- Affine spaces \((\text{AffineSpace})\),
- Projective spaces \((\text{ProjectiveSpace})\), or
- Toric varieties \((\text{ToricVariety})\).

Note that while projective spaces are of course toric varieties themselves, they are implemented differently in Sage due to efficiency considerations. You still can create a projective space as a toric variety if you wish.

In the following, we call the corresponding subschemes affine algebraic schemes, projective algebraic schemes, or toric algebraic schemes. In the future other ambient spaces, perhaps by means of gluing relations, may be introduced.

Generally, polynomials \(p_0, p_1, \ldots, p_n\) define an ideal \(I = \langle p_0, p_1, \ldots, p_n \rangle\). In the projective and toric case, the polynomials (and, therefore, the ideal) must be homogeneous. The associated subscheme \(V(I)\) of the ambient space is, roughly speaking, the subset of the ambient space on which all polynomials vanish simultaneously.

**Warning:** You should not construct algebraic scheme objects directly. Instead, use `.subscheme()` methods of ambient spaces. See below for examples.

**EXAMPLES:**

We first construct the ambient space, here the affine space \(\mathbb{Q}^2\):

```
sage: A2 = AffineSpace(2, QQ, 'x, y')
sage: A2.coordinate_ring().inject_variables()
Defining x, y
```

Now we can write polynomial equations in the variables \(x\) and \(y\). For example, one equation cuts out a curve (a one-dimensional subscheme):

```
sage: V = A2.subscheme([x^2 + y^2 - 1]); V
Closed subscheme of Affine Space of dimension 2 over Rational Field defined by:
  x^2 + y^2 - 1
sage: V.dimension()
# needs sage.libs.singular
1
```

Here is a more complicated example in a projective space:
Note that there are 3 equations in the 3-dimensional ambient space, yet the subscheme is 1-dimensional. One can show that it is not possible to eliminate any of the equations, that is, the twisted cubic is not a complete intersection of two polynomial equations.

Let us look at one affine patch, for example the one where \( x_0 = 1 \)

```
sage: patch = twisted_cubic.affine_patch(0); patch
```
```
Closed subscheme of Affine Space of dimension 3 over Rational Field defined by:
- \( x_1^2 + x_2 \),
- \( -x_1^3 + x_3 \),
- \( -x_2^2 + x_1^3 \)

sage: patch.embedding_morphism()
```
```
Scheme morphism:
From: Closed subscheme of Affine Space of dimension 3 over Rational Field
defined by: \( -x_1^2 + x_2 \), \( -x_1^3 + x_3 \), \( -x_2^2 + x_1^3 \)
To: Closed subscheme of Projective Space of dimension 3 over Rational Field
defined by: \( x_1^2 - x_0^2 x_2 \), \( x_1^3 - x_0^3 x_3 \), \( x_2^2 - x_1^3 \)
Defn: Defined on coordinates by sending \( (x_1, x_2, x_3) \) to \( (1 : x_1 : x_2 : x_3) \)
```

AUTHORS:

- David Kohel, William Stein (2005): initial version
- Andrey Novoseltsev (2010-05-17): subschemes of toric varieties
- Volker Braun (2010-12-24): documentation of schemes and refactoring; added coordinate neighborhoods and is_smooth()
- Ben Hutz (2014): subschemes of Cartesian products of projective space
- Ben Hutz (2017): split subschemes types into respective folders

```
class sage.schemes.generic.algebraic_scheme.AlgebraicScheme(A)
    Bases: Scheme

An algebraic scheme presented as a subscheme in an ambient space.
This is the base class for all algebraic schemes, that is, schemes defined by equations in affine, projective, or toric ambient spaces.
```
ambient_space()  
Return the ambient space of this algebraic scheme.

EXAMPLES:

```
sage: A.<x, y> = AffineSpace(2, GF(5))
sage: S = A.subscheme([])
sage: S.ambient_space()  
Affine Space of dimension 2 over Finite Field of size 5
```

coordinate_ring()  
Return the coordinate ring of this algebraic scheme. The result is cached.

OUTPUT:

The coordinate ring. Usually a polynomial ring, or a quotient thereof.

EXAMPLES:

```
sage: P.<x, y, z> = ProjectiveSpace(2, ZZ)
sage: S = P.subscheme([x - y, x - z])
sage: S.coordinate_ring()  
Quotient of Multivariate Polynomial Ring in x, y, z over Integer Ring  
by the ideal (x - y, x - z)
```

embedding_center()  
Return the distinguished point, if there is any.

If the scheme \( Y \) was constructed as a neighbourhood of a point \( p \in X \), then \( \text{embedding_morphism}() \) returns a local isomorphism \( f : Y \to X \) around the preimage point \( f^{-1}(p) \). The latter is returned by \( \text{embedding_center}() \).

OUTPUT:

A point of \( \text{self} \). Raises \( \text{AttributeError} \) if there is no distinguished point, depending on how \( \text{self} \) was constructed.

EXAMPLES:

```
sage: P3.<w,x,y,z> = ProjectiveSpace(QQ, 3)
sage: X = P3.subscheme( (w^2-x^2)*(y^2-z^2) )
sage: p = [1,-1,3,4]
sage: nbhd = X.neighborhood(p); nbhd
Closed subscheme of Affine Space of dimension 3 over Rational Field defined by:
    w^2*y^2 - x^2*y^2 + 6*w^2*y - 6*x^2*y + 2*w*y + 2*x*y - 7*w^2 + 7*x^2 + 12*w*y + 12*x*y - 14*w - 14*x
sage: nbhd.embedding_center()  
(0, 0, 0)
sage: nbhd.embedding_morphism()(nbhd.embedding_center())  
(1/4 : -1/4 : 3/4 : 1)
sage: nbhd.embedding_morphism()  
Scheme morphism:
```

(continues on next page)
From: Closed subscheme of Affine Space of dimension 3 over Rational Field
defined by:
\[w^2y^2 - x^2y^2 + 6w^2y + 2w^2y + 2x^2y^2 + 2x^2y^2 - 7w^2 + 7x^2 + 12w + 12x + 14w - 14x\]

To: Closed subscheme of Projective Space of dimension 3 over Rational Field
defined by:
\[w^2y^2 - x^2y^2 - w^2z^2 + x^2z^2\]

Defn: Defined on coordinates by sending \((w, x, y)\) to
\[(w + 1 : x - 1 : y + 3 : 4)\]

**embedding_morphism()**

Return the default embedding morphism of `self`.

If the scheme \(Y\) was constructed as a neighbourhood of a point \(p \in X\), then `embedding_morphism()` returns a local isomorphism \(f : Y \to X\) around the preimage point \(f^{-1}(p)\). The latter is returned by `embedding_center()`.

If the algebraic scheme \(Y\) was not constructed as a neighbourhood of a point, then the embedding in its `ambient_space()` is returned.

**OUTPUT:**

A scheme morphism whose `domain()` is `self`.

- By default, it is the tautological embedding into its own ambient space `ambient_space()`.
- If the algebraic scheme (which itself is a subscheme of an auxiliary `ambient_space()`) was constructed as a patch or neighborhood of a point then the embedding is the embedding into the original scheme.
- A `NotImplementedError` is raised if the construction of the embedding morphism is not implemented yet.

**EXAMPLES:**

```sage
sage: A2.<x,y> = AffineSpace(QQ, 2)
sage: C = A2.subscheme(x^2 + y^2 - 1)
sage: C.embedding_morphism()
```

```
Needs sage.libs.singular
```

```
Scheme morphism:
From: Closed subscheme of Affine Space of dimension 2 over Rational Field
defined by: x^2 + y^2 - 1
To: Affine Space of dimension 2 over Rational Field
Defn: Defined on coordinates by sending (x, y) to (x, y)
```

```sage
sage: # needs sage.graphs sage.geometry.polyhedron sage.libs.singular
sage: P1xP1.<x,y,u,v> = toric_varieties.P1xP1()
sage: P1 = P1xP1.subscheme(x - y)
sage: P1.embedding_morphism()
```

```
Scheme morphism:
From: Closed subscheme of 2-d CPR-Fano toric variety covered by 4 affine patches defined by: x - y
To: 2-d CPR-Fano toric variety covered by 4 affine patches
```

So far, the embedding was just in the own ambient space. Now a bit more interesting examples:

---

**Chapter 7. Algebraic schemes**
```python
sage: P2.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: X = P2.subscheme((x^2-y^2)*z)
sage: p = (1,1,0)
sage: nbhd = X.neighborhood(p); nbhd
Closed subscheme of Affine Space of dimension 2 over Rational Field defined by:
   -y^2*z - 2*y*z

Note that \( p = (1,1,0) \) is a singular point of \( X \). So the neighborhood of \( p \) is not just affine space. The \texttt{neighborhood()} method returns a presentation of the neighborhood as a subscheme of an auxiliary 2-dimensional affine space:

```python
sage: nbhd.ambient_space()
Affine Space of dimension 2 over Rational Field
```

But its \texttt{embedding_morphism()} is not into this auxiliary affine space, but the original subscheme \( X \):

```python
sage: nbhd.embedding_morphism()
Scheme morphism:
   From: Closed subscheme of Affine Space of dimension 2 over Rational Field
defined by: -y^2*z - 2*y*z
   To: Closed subscheme of Projective Space of dimension 2 over Rational Field
defined by: x^2*z - y^2*z
   Defn: Defined on coordinates by sending \((y, z)\) to \((1 : y + 1 : z)\)
```

A couple more examples:

```python
sage: # needs sage.geometry.polyhedron sage.graphs sage.libs.singular
sage: patch1 = P1xP1.affine_patch(1); patch1
2-d affine toric variety
sage: patch1.embedding_morphism()
Scheme morphism:
   From: 2-d affine toric variety
   To: 2-d CPR-Fano toric variety covered by 4 affine patches
   Defn: Defined on coordinates by sending \([y : u]\) to \([1 : y : u : 1]\)
sage: subpatch = P1.affine_patch(1); subpatch
Closed subscheme of 2-d affine toric variety defined by:
   -y + 1
sage: subpatch.embedding_morphism()
Scheme morphism:
   From: Closed subscheme of 2-d affine toric variety defined by: -y + 1
   To: Closed subscheme of 2-d CPR-Fano toric variety covered by 4 affine patches defined by: x - y
   Defn: Defined on coordinates by sending \([y : u]\) to \([1 : y : u : 1]\)
```

\texttt{identity_morphism()} 
Return the identity morphism.

OUTPUT: the identity morphism of the scheme \texttt{self}

EXAMPLES:

```python
sage: X = Spec(QQ)
sage: X.identity_morphism()
Scheme endomorphism of Spectrum of Rational Field
   Defn: Identity map
```
**is_projective()**

Return True if self is presented as a subscheme of an ambient projective space.

**OUTPUT:**

Boolean.

**EXAMPLES:**

```python
sage: PP.<x,y,z,w> = ProjectiveSpace(3, QQ)
sage: f = x^3 + y^3 + z^3 + w^3
sage: R = f.parent()
sage: I = [f] + [f.derivative(zz) for zz in PP.gens()]
sage: V = PP.subscheme(I)
sage: V.is_projective()
True
sage: AA.<x,y,z,w> = AffineSpace(4, QQ)
sage: V = AA.subscheme(I)
sage: V.is_projective()
False
```

Note that toric varieties are implemented differently than projective spaces. This is why this method returns False for toric varieties:

```python
sage: # needs sage.geometry.polyhedron sage.graphs
data: PP.<x,y,z,w> = toric_varieties.P(3)
sage: V = PP.subscheme(x^3 + y^3 + z^3 + w^3)
sage: V.is_projective()
False
```

**ngens()**

Return the number of generators of the ambient space of this algebraic scheme.

**EXAMPLES:**

```python
data: A.<x, y> = AffineSpace(2, GF(5))
sage: S = A.subscheme([])
sage: S.ngens()
2
sage: P.<x, y, z> = ProjectiveSpace(2, ZZ)
sage: S = P.subscheme([x^2 - y, x - z])
sage: S.ngens()
3
```

**class** `sage.schemes.generic.algebraic_scheme.AlgebraicScheme_quasi(X, Y)`

**Bases:** `AlgebraicScheme`

The quasi-affine or quasi-projective scheme $X - Y$, where $X$ and $Y$ are both closed subschemes of a common ambient affine or projective space.

**Warning:** You should not create objects of this class directly. The preferred method to construct such subschemes is to use `complement()` method of algebraic schemes.

**OUTPUT:**

An instance of `AlgebraicScheme_quasi`. 
EXAMPLES:

```python
sage: P.<x, y, z> = ProjectiveSpace(2, ZZ)
sage: S = P.subscheme([])
sage: T = P.subscheme([x - y])
sage: T.complement(S)
Quasi-projective subscheme X - Y
of Projective Space of dimension 2 over Integer Ring,
where X is defined by:
  (no polynomials)
and Y is defined by:
  x - y
```

```python
sage: P.<x, y, z> = ProjectiveSpace(2, ZZ)
sage: S = P.subscheme([])
sage: T = P.subscheme([x - y])
sage: U = T.complement(S)
sage: U.X() is S
True
```

```python
sage: P.<x, y, z> = ProjectiveSpace(2, ZZ)
sage: S = P.subscheme([])
sage: T = P.subscheme([x - y])
sage: U = T.complement(S)
sage: U.Y() is T
True
```

```python
sage: A.<x, y> = AffineSpace(2, GF(7))
sage: S = A.subscheme([x^2 - y])
sage: T = A.subscheme([x - y])
sage: U = T.complement(S)
sage: U.rational_points()
[[2, 4], [3, 2], [4, 2], [5, 4], [6, 1]]
```
sage: U.rational_points(F=GF(7^2, 'b'))  
˓→ needs sage.rings.finite_rings
[(2, 4), (3, 2), (4, 2), (5, 4), (6, 1), (b, b + 4), (b + 1, 3*b + 5),
(b + 2, 5*b + 1), (b + 3, 6), (b + 4, 2*b + 6), (b + 5, 4*b + 1),
(b + 6, 6*b + 5), (2*b, 4*b + 2), (2*b + 1, b + 3), (2*b + 2, 5*b + 6),
(2*b + 3, 2*b + 4), (2*b + 4, 6*b + 4), (2*b + 5, 3*b + 6), (2*b + 6, 3),
(3*b, 2*b + 1), (3*b + 1, b + 2), (3*b + 2, 5), (3*b + 3, 6*b + 3),
(3*b + 4, 5*b + 3), (3*b + 5, 4*b + 5), (3*b + 6, 3*b + 2),
(4*b, 2*b + 1), (4*b + 1, 3*b + 2), (4*b + 2, 4*b + 5),
(4*b + 3, 5*b + 3), (4*b + 4, 6*b + 3), (4*b + 5, 5), (4*b + 6, b + 2),
(5*b, 4*b + 2), (5*b + 1, 3), (5*b + 2, 3*b + 6), (5*b + 3, 6*b + 4),
(5*b + 4, 2*b + 4), (5*b + 5, 5*b + 6), (5*b + 6, b + 3), (6*b, b + 4),
(6*b + 1, 6*b + 5), (6*b + 2, 4*b + 1), (6*b + 3, 2*b + 6), (6*b + 4, 6),
(6*b + 5, 5*b + 1), (6*b + 6, 3*b + 5)]

class sage.schemes.generic.algebraic_scheme.AlgebraicScheme_subscheme(A, polynomials)

Bases: AlgebraicScheme

An algebraic scheme presented as a closed subscheme is defined by explicit polynomial equations. This is as opposed to a general scheme, which could, e.g., be the Neron model of some object, and for which we do not want to give explicit equations.

INPUT:

- A - ambient space (e.g. affine or projective n-space)
- polynomials - single polynomial, ideal or iterable of defining polynomials; in any case polynomials must belong to the coordinate ring of the ambient space and define valid polynomial functions (e.g. they should be homogeneous in the case of a projective space)

OUTPUT:

- algebraic scheme

EXAMPLES:

sage: from sage.schemes.generic.algebraic_scheme import AlgebraicScheme_subscheme
sage: P.<x, y, z> = ProjectiveSpace(2, QQ)

sage: P.subscheme([x^2 - y*z])
Closed subscheme of Projective Space of dimension 2 over Rational Field defined by: x^2 - y*z

sage: AlgebraicScheme_subscheme(P, [x^2 - y*z])
Closed subscheme of Projective Space of dimension 2 over Rational Field defined by: x^2 - y*z

Jacobian()

Return the Jacobian ideal.

This is the ideal generated by

- the d x d minors of the Jacobian matrix, where d is the codimension() of the algebraic scheme, and
- the defining polynomials of the algebraic scheme. Note that some authors do not include these in the definition of the Jacobian ideal. An example of a reference that does include the defining equations is [Laz2004], p. 181.

OUTPUT:
An ideal in the coordinate ring of the ambient space.

EXAMPLES:

```python
sage: P3.<w,x,y,z> = ProjectiveSpace(3, QQ)
sage: twisted_cubic = P3.subscheme(matrix([[w, x, y],...
˓→needs sage.libs.singular
....: [x, y, z])).minors(2))
sage: twisted_cubic.Jacobian()  # needs sage.libs.singular
Ideal (-x^2 + w*y, -x*y + w*z, -y^2 + x*z, x*z, -2*w*z, w*y, 3*w*y, x*z, 3*x*z, -2*w*z, w*y)
of Multivariate Polynomial Ring in w, x, y, z over Rational Field
sage: twisted_cubic.defining_ideal()  # needs sage.libs.singular
Ideal (-x^2 + w*y, -x*y + w*z, -y^2 + x*z)
of Multivariate Polynomial Ring in w, x, y, z over Rational Field
```

This example addresses issue github issue #20512:

```python
sage: X = P3.subscheme([])
sage: X.Jacobian() == P3.coordinate_ring().unit_ideal()  # needs sage.libs.singular
True
```

Jacobian_matrix()

Return the matrix $\frac{\partial f_i}{\partial x_j}$ of (formal) partial derivatives.

OUTPUT:

A matrix of polynomials.

EXAMPLES:

```python
sage: P3.<w,x,y,z> = ProjectiveSpace(3, QQ)
sage: twisted_cubic = P3.subscheme(matrix([[w, x, y],...
˓→needs sage.libs.singular
....: [x, y, z])).minors(2))
sage: twisted_cubic.Jacobian_matrix()  # needs sage.libs.singular
[ y -2*x w 0]
[ z -y -x w]
[ 0 z -2*y x]
```

This example addresses issue github issue #20512:

```python
sage: X = P3.subscheme([])
sage: X.Jacobian_matrix().base_ring() == P3.coordinate_ring()  # needs sage.libs.singular
True
```

base_extend($R$)

Return the base change to the ring $R$ of this scheme.

EXAMPLES:
change_ring($R$)

Returns a new algebraic subscheme which is this subscheme coerced to $R$.

INPUT:

- $R$ – ring or morphism.

OUTPUT:

- A new algebraic subscheme which is this subscheme coerced to $R$.

EXAMPLES:

```python
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: X = P.subscheme([3*x^2 - y^2])
sage: H = Hom(X, X)
sage: X.change_ring(GF(3))
Closed subscheme of Projective Space of dimension 1
over Finite Field of size 3 defined by: -y^2

sage: # needs sage.rings.number_field
sage: K.<w> = QuadraticField(2)
sage: R.<z> = K[]
sage: L.<v> = K.extension(z^3 - 5)
sage: P.<x,y> = ProjectiveSpace(K, 1)
sage: X = P.subscheme(x - w*y)
# needs sage.libs.singular
sage: X.change_ring(L)
# needs sage.libs.singular
Closed subscheme of Projective Space of dimension 1 over
Number Field in v with defining polynomial z^3 - 5 over its base field
defined by: x + (-w)*y
```

(continues on next page)
Closed subscheme of Affine Space of dimension 3 over Algebraic Field
defined by:
\[ x + (-1.4142135623730957 + 0.7e-16*I)*y, \\
  z^2 + (0.85498797333834857 + 1.4808826096823657*I)*x \]

Sage:
```python
X.change_ring(emb[0])
```
(continues on next page)
needs sage.libs.singular
sage: X.change_ring(CC) # needs sage.libs.singular
Closed subscheme of Projective Space of dimension 1 
over Complex Field with 53 bits of precision defined by: 
x^2 + (0.623489801858734 + 0.781831482468030*I)*y^2 
sage: X.change_ring(K).change_ring(K.embeddings(QQbar)[3]) 
# needs sage.libs.singular
Closed subscheme of Projective Space of dimension 1 
over Algebraic Field defined by: 
x^2 + (-0.900968867902419 - 0.43383791175581*I)*y^2
sage: # needs sage.rings.number_field 
sage: R.<x> = QQ[] 
sage: f = x^6 - 2 
sage: L.<b> = NumberField(f, embedding=f.roots(CC)[2][0]) 
sage: A.<x,y> = AffineSpace(L, 2) 
sage: H = Hom(A, A) 
sage: X = A.subscheme([b*x^2, y^2]) # needs sage.libs.singular 
sage: X.change_ring(CC) # needs sage.libs.singular
Closed subscheme of Affine Space of dimension 2 
over Complex Field with 53 bits of precision defined by: 
(-0.561231024154687 - 0.972080648619833*I)*x^2, 
y^2
codimension()

Return the codimension of the algebraic subscheme.

OUTPUT:

Integer.

EXAMPLES:
sage: PP.<x,y,z,w,v> = ProjectiveSpace(4, QQ)
sage: V = PP.subscheme(x*y) 
sage: V.codimension() # needs sage.libs.singular 
1 
sage: V.dimension() # needs sage.libs.singular 
3
complement(other=None)

Return the scheme-theoretic complement other - self, where self and other are both closed algebraic sub-
schemes of the same ambient space.

If other is unspecified, it is taken to be the ambient space of self.

EXAMPLES:
sage: A.<x, y, z> = AffineSpace(3, ZZ)
sage: X = A.subscheme([x + y - z])
sage: Y = A.subscheme([x - y + z])
sage: Y.complement(X)
Quasi-affine subscheme X - Y of
Affine Space of dimension 3 over Integer Ring,
where X is defined by:
  x + y - z
and Y is defined by:
  x - y + z
sage: Y.complement()
Quasi-affine subscheme X - Y of
Affine Space of dimension 3 over Integer Ring,
where X is defined by:
  (no polynomials)
and Y is defined by:
  x - y + z
sage: P.<x, y, z> = ProjectiveSpace(2, QQ)
sage: X = P.subscheme([x^2 + y^2 + z^2])
sage: Y = P.subscheme([x*y + x*z + y*z])
sage: Y.complement(X)
Quasi-projective subscheme X - Y of
Projective Space of dimension 2 over Rational Field,
where X is defined by:
  x^2 + y^2 + z^2
and Y is defined by:
  x*y + x*z + y*z
sage: Y.complement(P)
Quasi-projective subscheme X - Y of
Projective Space of dimension 2 over Rational Field,
where X is defined by:
  (no polynomials)
and Y is defined by:
  x*y + x*z + y*z

\textbf{defining\_ideal()} 
Return the ideal that defines this scheme as a subscheme of its ambient space.

\textbf{OUTPUT:}
An ideal in the coordinate ring of the ambient space.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: P.<x, y, z> = ProjectiveSpace(2, ZZ)
sage: S = P.subscheme([x^2 - y*z, x^3 + z^3])
sage: S.defining_ideal()
Ideal (x^2 - y*z, x^3 + z^3) of Multivariate Polynomial Ring in x, y, z
over Integer Ring
\end{verbatim}

\textbf{defining\_polynomials()} 
Return the polynomials that define this scheme as a subscheme of its ambient space.

\textbf{OUTPUT:}
A tuple of polynomials in the coordinate ring of the ambient space.
EXAMPLES:

```sage
sage: P.<x, y, z> = ProjectiveSpace(2, ZZ)
sage: S = P.subscheme([x^2 - y*z, x^3 + z^3])
sage: S.defining_polynomials()
(x^2 - y*z, x^3 + z^3)
```

**intersection**(other)

Return the scheme-theoretic intersection of self and other in their common ambient space.

**EXAMPLES:**

```sage
sage: A.<x, y> = AffineSpace(2, ZZ)
sage: X = A.subscheme([x^2 - y])
sage: Y = A.subscheme([y])
sage: X.intersection(Y)
Closed subscheme of Affine Space of dimension 2 over Integer Ring defined by:
    x^2 - y,
y
```

**irreducible_components()**

Return the irreducible components of this algebraic scheme, as subschemes of the same ambient space.

**OUTPUT:**

an immutable sequence of irreducible subschemes of the ambient space of this scheme

The components are cached.

**EXAMPLES:**

We define what is clearly a union of four hypersurfaces in $\mathbb{P}^4$ then find the irreducible components:

```sage
sage: PP.<x,y,z,w,v> = ProjectiveSpace(4, QQ)
sage: V = PP.subscheme((x^2 - y^2 - z^2) * (w^5 - 2*v^2*z^3) * w * (v^3 - x^2*z))
sage: V.irreducible_components()
# needs sage.libs.singular
[Closed subscheme of Projective Space of dimension 4 over Rational Field defined by:
    w,
Closed subscheme of Projective Space of dimension 4 over Rational Field defined by:
    x^2 - y^2 - z^2,
Closed subscheme of Projective Space of dimension 4 over Rational Field defined by:
    x^2*z - v^3,
Closed subscheme of Projective Space of dimension 4 over Rational Field defined by:
    w^5 - 2*z^3*v^2]
```

We verify that the irrelevant ideal is not accidentally returned (see github issue #6920):

```sage
sage: PP.<x,y,z,w> = ProjectiveSpace(3, QQ)
sage: f = x^3 + y^3 + z^3 + w^3
```

(continues on next page)
The same polynomial as above defines a scheme with a nontrivial irreducible component in affine space (instead of the empty scheme as above):

```python
sage: AA.<x,y,z,w> = AffineSpace(4, QQ)
sage: V = AA.subscheme(I)
sage: V.irreducible_components()  # needs sage.libs.singular
[Closed subscheme of Affine Space of dimension 4 over Rational Field defined by:
  w,
  z,
  y,
  x]
```

### is_irreducible()

Return whether this subscheme is or is not irreducible.

**OUTPUT:** Boolean.

**EXAMPLES:**

```python
sage: # needs sage.rings.number_field
sage: K = QuadraticField(-3)
sage: P.<x,y,z,w,t,u> = ProjectiveSpace(K, 5)
sage: X = P.subscheme([x*y - z^2 - K.0*t^2, t*w*x + y*z^2 - u^3])
sage: X.is_irreducible()  # needs sage.libs.singular
True

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: X = P.subscheme([(y + x - z)^2])
sage: X.is_irreducible()  # needs sage.libs.singular
False

sage: A.<x,y,z,w> = AffineSpace(GF(17), 4)
sage: X = A.subscheme([x*y*z^2 - x*y*z*w - z*w^2 + w^3,
  x^3*y*z*w - x*y^3*z - x^2*y^2*z*w - x^2*w^3 + y^2*w^2 + x*w^3])
sage: X.is_irreducible()  # needs sage.libs.singular
False
```
normalize_defining_polynomials()

Function to normalize the coefficients of defining polynomials of given subscheme.

Normalization as in removing denominator from all the coefficients, and then removing any common factor between the coefficients. It takes LCM of denominators and then removes common factor among coefficients, if any.

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(2, QQ)
sage: S = A.subscheme([2*x^2 + 4*x*y, 1/8*x + 1/3*y])
sage: S.normalize_defining_polynomials()
sage: S.defining_polynomials()
(x^2 + 2*x*y, 3*x + 8*y)
```

rational_points(**kwds)

Return the rational points on the algebraic subscheme.

For a dimension 0 subscheme, if the base ring is a numerical field such as the ComplexField the results returned could be very far from correct. If the polynomials defining the subscheme are defined over a number field, you will get better results calling rational points with $F$ defined as the number field and the base ring as the field of definition. If the base ring is a number field, the embedding into $F$ must be known.

In the case of numerically approximated points, the points are returned over as points of the ambient space.

For a dimension greater than 0 scheme, depending on bound size, either the points in the ambient space are enumerated or a sieving algorithm lifting points modulo primes is used. See the documentation in homset for the details of the sieving algorithm.

INPUT:

kwds:

- **bound** - integer (optional, default=0). The bound for the coordinates for subschemes with dimension at least 1.
- **prec** - integer (optional, default=53). The precision to use to compute the elements of bounded height for number fields.
- **F** - field (optional, default=base ring). The field to compute the rational points over.
- **point_tolerance** - positive real number (optional, default=10^(-10)). For numerically inexact fields, two points are considered the same if their coordinates are within tolerance.
- **zero_tolerance** - positive real number (optional, default=10^(-10)). For numerically inexact fields, points are on the subscheme if they satisfy the equations to within tolerance.
- **tolerance** - a rational number in (0,1] used in doyle-krumm algorithm

OUTPUT: list of points in subscheme or ambient space

**Warning:** For numerically inexact fields such as ComplexField or RealField the list of points returned is very likely to be incomplete at best.

EXAMPLES:

Enumerate over a projective scheme over a number field:
One can enumerate points up to a given bound on a projective scheme over the rationals:

```plaintext
sage: E = EllipticCurve('37a')
needs sage.schemes
sage: E.rational_points(bound=8)  
needs sage.schemes
[(−1 : -1 : 1), (−1 : 0 : 1), (0 : -1 : 1), (0 : 0 : 1), (0 : 1 : 0),
 (1/4 : -5/8 : 1), (1/4 : -3/8 : 1), (1 : -1 : 1), (1 : 0 : 1),
 (2 : -3 : 1), (2 : 2 : 1)]
```

For a small finite field, the complete set of points can be enumerated.

```plaintext
sage: Etilde = E.base_extend(GF(3))  
needs sage.schemes
sage: Etilde.rational_points()  
needs sage.schemes
[0 : 0 : 1], (0 : 1 : 0), (0 : 2 : 1), (1 : 0 : 1),
 (1 : 2 : 1), (2 : 0 : 1), (2 : 2 : 1)]
```

The class of hyperelliptic curves does not (yet) support desingularization of the places at infinity into two points:

```plaintext
sage: FF = FiniteField(7)

sage: P.<x> = PolynomialRing(FiniteField(7))

sage: C = HyperellipticCurve(x^8 + x + 1)  
needs sage.schemes
sage: C.rational_points()  
needs sage.schemes
[(0 : 1 : 0), (0 : 1 : 1), (0 : 6 : 1), (2 : 0 : 1),
 (4 : 0 : 1), (6 : 1 : 1), (6 : 6 : 1)]
```

(continues on next page)
Todo: Implement Stoll’s model in weighted projective space to resolve singularities and find two points $(1:1:0)$ and $(-1:1:0)$ at infinity.

reduce()

Return the corresponding reduced algebraic space associated to this scheme.

EXAMPLES: First we construct the union of a doubled and tripled line in the affine plane over $\mathbb{Q}$

```
sage: A.<x,y> = AffineSpace(2, QQ)
sage: X = A.subscheme([(x-1)^2*(x-y)^3]); X
Closed subscheme of Affine Space of dimension 2 over Rational Field defined by:
  x^5 - 3*x^4*y + 3*x^3*y^2 - x^2*y^3 - 2*x^4 + 6*x^3*y
- 6*x^2*y^2 + 2*x*y^3 + x^3 - 3*x^2*y + 3*x*y^2 - y^3
sage: X.dimension()  # needs sage.libs.singular
1
```

Then we compute the corresponding reduced scheme:

```
sage: Y = X.reduce(); Y  # needs sage.libs.singular
Closed subscheme of Affine Space of dimension 2 over Rational Field defined by:
  x^2 - x*y - x + y
```

Finally, we verify that the reduced scheme $Y$ is the union of those two lines:

```
sage: W = L1.union(L2); W  # taken in ambient space
Closed subscheme of Affine Space of dimension 2 over Rational Field defined by:
  x^2 - x*y - x + y
sage: Y == W
True
```

specialization($D$=$None$, $phi$=$None$)

Specialization of this subscheme.

Given a family of maps defined over a polynomial ring. A specialization is a particular member of that family. The specialization can be specified either by a dictionary or a SpecializationMorphism.

INPUT:

- $D$ – dictionary (optional)
- $phi$ – SpecializationMorphism (optional)
OUTPUT: SchemeMorphism_polynomial

EXAMPLES:

```
sage: R.<c> = PolynomialRing(QQ)
sage: P.<x,y> = ProjectiveSpace(R, 1)
sage: X = P.subscheme([x^2 + c*y^2])
sage: X.specialization(dict({c:2}))
Closed subscheme of Projective Space of dimension 1 over Rational Field defined by:
 x^2 + 2*y^2
```

```
sage: R.<c> = PolynomialRing(QQ)
sage: S.<a,b> = R[]
sage: P.<x,y,z> = AffineSpace(S, 3)
sage: X = P.subscheme([x^2 + a*c*y^2 - b*z^2])
sage: from sage.rings.polynomial.flatten import SpecializationMorphism
sage: phi = SpecializationMorphism(P.coordinate_ring(),
....: dict({c: 2, a: 1}))
sage: X.specialization(phi=phi)
Closed subscheme of Affine Space of dimension 3 over Univariate Polynomial Ring in b over Rational Field defined by:
 x^2 + 2*y^2 + (-b)*z^2
```

union(other)

Return the scheme-theoretic union of self and other in their common ambient space.

EXAMPLES: We construct the union of a line and a tripled-point on the line.

```
sage: A.<x,y> = AffineSpace(2, QQ)
sage: I = ideal([x, y])^3
sage: P = A.subscheme(I)
sage: L = A.subscheme([y - 1])
sage: S = L.union(P); S
```

Closed subscheme of Affine Space of dimension 2 over Rational Field defined by:
 y^4 - y^3,
 x*y^3 - x*y^2,
 x^2*y^2 - x^2*y,
 x^3*y - x^3

```
sage: S.dimension()  # needs sage.libs.singular
1
sage: S.reduce()  # needs sage.libs.singular
```

Closed subscheme of Affine Space of dimension 2 over Rational Field defined by:
 y^2 - y,
 x*y - x

We can also use the notation “+” for the union:

```
sage: A.subscheme([x]) + A.subscheme([y^2 - (x^3+1)])  # needs sage.libs.singular
```

(continues on next page)
Closed subscheme of Affine Space of dimension 2 over Rational Field defined by:
\(x^4 - x^2 y^2 + x\)

Saving and loading:

```python
sage: loads(S.dumps()) == S
# needs sage.libs.singular
True
```

**weil_restriction()**

Compute the Weil restriction of this variety over some extension field. If the field is a finite field, then this computes the Weil restriction to the prime subfield.

A Weil restriction of scalars - denoted \(R_{L/k}\) - is a functor which, for any finite extension of fields \(L/k\) and any algebraic variety \(X\) over \(L\), produces another corresponding variety \(R_{L/k}(X)\), defined over \(k\). It is useful for reducing questions about varieties over large fields to questions about more complicated varieties over smaller fields.

This function does not compute this Weil restriction directly but computes on generating sets of polynomial ideals:

Let \(d\) be the degree of the field extension \(L/k\), let \(a\) a generator of \(L/k\) and \(p\) the minimal polynomial of \(L/k\). Denote this ideal by \(I\).

Specifically, this function first maps each variable \(x\) to its representation over \(k\): \(\sum_{i=0}^{d-1} a^i x_i\). Then each generator of \(I\) is evaluated over these representations and reduced modulo the minimal polynomial \(p\). The result is interpreted as a univariate polynomial in \(a\) and its coefficients are the new generators of the returned ideal.

If the input and the output ideals are radical, this is equivalent to the statement about algebraic varieties above.

**OUTPUT:** Affine subscheme - the Weil restriction of \(self\).

**EXAMPLES:**

```python
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: K.<w> = NumberField(x^5 - 2)
sage: R.<x> = K[]
sage: L.<v> = K.extension(x^2 + 1)
sage: A.<x,y> = AffineSpace(L, 2)
sage: X = A.subscheme([y^2 - L(w)*x^3 - v])
# needs sage.libs.singular
sage: X.weil_restriction() # needs sage.libs.singular
Closed subscheme of Affine Space of dimension 4 over Number Field in w with defining polynomial x^5 - 2 defined by:
(-w)*z0^3 + (3*w)*z0*z1^2 + z2^2 - z3^2,
(-3*w)*z0^2*z1 + w*z1^3 + 2*z2*z3 - 1
sage: X.weil_restriction().ambient_space() is A.weil_restriction() # needs sage.libs.singular
True
```
\begin{verbatim}
sage: A.<x,y,z> = AffineSpace(GF(5^2, 't'), 3)  # needs sage.rings.finite_rings
sage: X = A.subscheme([y^2 - x^2*z, z^2 + 2*y])  # needs sage.libs.singular sage.rings.finite_rings
sage: X.weil_restriction()  # needs sage.libs.singular sage.rings.finite_rings
Closed subscheme of Affine Space of dimension 6 over Finite Field of size 5 defined by:
    z2^2 - 2*z3^2 - z0*z4 + 2*z1*z5,
    2*z2*z3 + z3^2 - z1*z4 - z0*z5 - z1*z5,
    z4^2 - 2*z5^2 + 2*z2,
    2*z4*z5 + z5^2 + 2*z3
\end{verbatim}

sage.schemes.generic.algebraic_scheme.is_AlgebraicScheme(x)

Test whether \( x \) is an algebraic scheme.

INPUT:

\bullet \ x \ – anything.

OUTPUT:

Boolean. Whether \( x \) is an algebraic scheme, that is, a subscheme of an ambient space over a ring defined by
polynomial equations.

EXAMPLES:

\begin{verbatim}
sage: A2 = AffineSpace(2, QQ, 'x, y')
sage: A2.coordinate_ring().inject_variables()
Defining x, y
sage: V = A2.subscheme([x^2 + y^2]); V
Closed subscheme of Affine Space of dimension 2 over Rational Field defined by:
    x^2 + y^2
sage: from sage.schemes.generic.algebraic_scheme import is_AlgebraicScheme
sage: is_AlgebraicScheme(V)
True
\end{verbatim}

Affine space is itself not an algebraic scheme, though the closed subscheme defined by no equations is:

\begin{verbatim}
sage: from sage.schemes.generic.algebraic_scheme import is_AlgebraicScheme
sage: is_AlgebraicScheme(AffineSpace(10, QQ))
False
sage: V = AffineSpace(10, QQ).subscheme([]); V
Closed subscheme of Affine Space of dimension 10 over Rational Field defined by:
    (no polynomials)
sage: is_AlgebraicScheme(V)
True
\end{verbatim}

We create a more complicated closed subscheme:

\begin{verbatim}
sage: A,x = AffineSpace(10, QQ).objgens()
sage: X = A.subscheme([sum(x)]); X
Closed subscheme of Affine Space of dimension 10 over Rational Field defined by:
    x0 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
sage: is_AlgebraicScheme(X)
True
\end{verbatim}
\begin{verbatim}
sage: is_AlgebraicScheme(QQ)
False
 sage: S = Spec(QQ)
sage: is_AlgebraicScheme(S)
False
\end{verbatim}
CHAPTER
EIGHT

HYPERSURFACES IN AFFINE AND PROJECTIVE SPACE

AUTHORS:

- William Stein <wstein@gmail.com> (2005-12-08)
- David Kohel <kohel@maths.usyd.edu.au> (2005-12-08)
- Alex Ghitza <aghitza@alum.mit.edu> (2009-04-17)

class sage.schemes.generic.hypersurface.AffineHypersurface(poly, ambient=None)

Bases: AlgebraicScheme_subscheme_affine

The affine hypersurface defined by the given polynomial.

EXAMPLES:

```python
sage: A.<x, y, z> = AffineSpace(ZZ, 3)
sage: AffineHypersurface(x*y - z^3, A)
Affine hypersurface defined by -z^3 + x*y
in Affine Space of dimension 3 over Integer Ring
```

```python
sage: A.<x, y, z> = QQ[]
sage: AffineHypersurface(x*y - z^3)
Affine hypersurface defined by -z^3 + x*y
in Affine Space of dimension 3 over Rational Field
```

defining_polynomial()

Return the polynomial equation that cuts out this affine hypersurface.

EXAMPLES:

```python
sage: R.<x, y, z> = ZZ[]
sage: H = AffineHypersurface(x*z + y^2)
sage: H.defining_polynomial()
y^2 + x*z
```

class sage.schemes.generic.hypersurface.ProjectiveHypersurface(poly, ambient=None)

Bases: AlgebraicScheme_subscheme_projective

The projective hypersurface defined by the given polynomial.

EXAMPLES:

```python
sage: P.<x, y, z> = ProjectiveSpace(ZZ, 2)
sage: ProjectiveHypersurface(x - y, P)
```

(continues on next page)
Projective hypersurface defined by $x - y$
in Projective Space of dimension 2 over Integer Ring

```sage
R.<x, y, z> = QQ[]
H = ProjectiveHypersurface(x - y)
```

Projective hypersurface defined by $x - y$
in Projective Space of dimension 2 over Rational Field

```python
defining_polynomial()
    Return the polynomial equation that cuts out this projective hypersurface.
    EXAMPLES:
    
    sage: R.<x, y, z> = ZZ[]
    sage: H = ProjectiveHypersurface(x*z + y^2)
    sage: H.defining_polynomial()
    y^2 + x*z
```

sage.schemes.generic.hypersurface.is_Hypersurface(self)
    Return True if self is a hypersurface, i.e. an object of the type ProjectiveHypersurface or AffineHypersurface.
    EXAMPLES:
    
    sage: from sage.schemes.generic.hypersurface import is_Hypersurface
    sage: R.<x, y, z> = ZZ[]
    sage: H = ProjectiveHypersurface(x*z + y^2)
    sage: is_Hypersurface(H)
    True

    sage: H = AffineHypersurface(x*z + y^2)
    sage: is_Hypersurface(H)
    True

    sage: H = ProjectiveSpace(QQ, 5)
    sage: is_Hypersurface(H)
    False
SET OF HOMOMORPHISMS BETWEEN TWO SCHEMES

For schemes $X$ and $Y$, this module implements the set of morphisms $Hom(X, Y)$. This is done by $\texttt{SchemeHomset_generic}$.

As a special case, the Hom-sets can also represent the points of a scheme. Recall that the $K$-rational points of a scheme $X$ over $k$ can be identified with the set of morphisms $\text{Spec}(K) \to X$. In Sage the rational points are implemented by such scheme morphisms. This is done by $\texttt{SchemeHomset_points}$ and its subclasses.

**Note:** You should not create the Hom-sets manually. Instead, use the $\texttt{Hom()}$ method that is inherited by all schemes.

**AUTHORS:**

- Volker Braun (2011-08-11): significant improvement and refactoring.
- Ben Hutz (June 2012): added support for projective ring

**class** $\texttt{sage.schemes.generic.homset.SchemeHomsetFactory}$

Bases: $\texttt{UniqueFactory}$

Factory for Hom-sets of schemes.

**EXAMPLES:**

```
sage: A2 = AffineSpace(QQ, 2)
sage: A3 = AffineSpace(QQ, 3)
sage: Hom = A3.Hom(A2)
```

The Hom-sets are uniquely determined by domain and codomain:

```
sage: Hom is copy(Hom)
True
sage: Hom is A3.Hom(A2)
True
```

The Hom-sets are identical if the domains and codomains are identical:

```
sage: loads(Hom.dumps()) is Hom
True
sage: A3_iso = AffineSpace(QQ, 3)
sage: A3_iso is A3
True
sage: Hom_iso = A3_iso.Hom(A2)
```

(continues on next page)
create_key_and_extra_args($X, Y, category=None, base=None, check=True, as_point_homset=False$)

Create a key that uniquely determines the Hom-set.

**INPUT:**

- $X$ – a scheme. The domain of the morphisms.
- $Y$ – a scheme. The codomain of the morphisms.
- category – a category for the Hom-sets (default: schemes over given base).
- base – a scheme or a ring. The base scheme of domain and codomain schemes. If a ring is specified, the spectrum of that ring will be used as base scheme.
- check – boolean (default: True).

**EXAMPLES:**

```
sage: A2 = AffineSpace(QQ, 2)
sage: A3 = AffineSpace(QQ, 3)
sage: A3.Hom(A2)  # indirect doctest
Set of morphisms
From: Affine Space of dimension 3 over Rational Field
To:  Affine Space of dimension 2 over Rational Field
sage: from sage.schemes.generic.homset import SchemeHomsetFactory
sage: SHOMfactory = SchemeHomsetFactory('test')
sage: key, extra = SHOMfactory.create_key_and_extra_args(A3, A2, check=False)
sage: key
(..., ..., Category of schemes over Rational Field, False)
sage: extra
{'X': Affine Space of dimension 3 over Rational Field,
 'Y': Affine Space of dimension 2 over Rational Field,
 'base_ring': Rational Field,
 'check': False}
```

create_object($version, key, **extra_args$)

Create a $\text{SchemeHomset\_generic}$.  

**INPUT:**

- key – a key created by create_key_and_extra_args().
- extra_args – a dictionary of extra keyword arguments.

**EXAMPLES:**

```
sage: A2 = AffineSpace(QQ, 2)
sage: A3 = AffineSpace(QQ, 3)
sage: A3.Hom(A2) is A3.Hom(A2)  # indirect doctest
True
sage: from sage.schemes.generic.homset import SchemeHomsetFactory
sage: SHOMfactory = SchemeHomsetFactory('test')
sage: SHOMfactory.create_object(0, [id(A3), id(A2), A3.category()], False),
```
class sage.schemes.generic.homset.SchemeHomset_generic(X, Y, category=None, check=True, base=None)

Bases: HomsetWithBase

The base class for Hom-sets of schemes.

INPUT:

• X – a scheme. The domain of the Hom-set.
• Y – a scheme. The codomain of the Hom-set.
• category – a category (optional). The category of the Hom-set.
• check – boolean (optional, default: True). Whether to check the defining data for consistency.

EXAMPLES:

sage: from sage.schemes.generic.homset import SchemeHomset_generic
sage: A2 = AffineSpace(QQ,2)
sage: Hom = SchemeHomset_generic(A2, A2); Hom
Set of morphisms

From: Affine Space of dimension 2 over Rational Field
To:  Affine Space of dimension 2 over Rational Field

Element

alias of SchemeMorphism

natural_map()

Return a natural map in the Hom space.

OUTPUT:

A SchemeMorphism if there is a natural map from domain to codomain. Otherwise, a
NotImplementedError is raised.

EXAMPLES:

sage: A = AffineSpace(4, QQ)
sage: A.structure_morphism()  # indirect doctest
Scheme morphism:

From: Affine Space of dimension 4 over Rational Field
To:  Spectrum of Rational Field
Defn: Structure map

class sage.schemes.generic.homset.SchemeHomset_points(X, Y, category=None, check=True, base= Integer Ring)

Bases: SchemeHomset_generic

Set of rational points of the scheme.
Recall that the $K$-rational points of a scheme $X$ over $k$ can be identified with the set of morphisms $Spec(K) \circ X$. In Sage, the rational points are implemented by such scheme morphisms.

If a scheme has a finite number of points, then the homset is supposed to implement the Python iterator interface. See $SchemeHomset_points_toric_field$ for example.

**INPUT:**

See $SchemeHomset_generic$.

**EXAMPLES:**

```python
sage: from sage.schemes.generic.homset import SchemeHomset_points
sage: SchemeHomset_points(Spec(QQ), AffineSpace(ZZ,2))
Set of rational points of Affine Space of dimension 2 over Rational Field
```

cardinality()

Return the number of points.

**OUTPUT:**

An integer or infinity.

**EXAMPLES:**

```python
sage: toric_varieties.P2().point_set().cardinality()  # needs sage.geometry.polyhedron sage.graphs
+Infinity
sage: P2 = toric_varieties.P2(base_ring=GF(3))  # needs sage.geometry.polyhedron sage.graphs
sage: P2.point_set().cardinality()  # needs sage.geometry.polyhedron sage.graphs
13
```

extended_codomain()

Return the codomain with extended base, if necessary.

**OUTPUT:**

The codomain scheme, with its base ring extended to the codomain. That is, the codomain is of the form $Spec(R)$ and the base ring of the domain is extended to $R$.

**EXAMPLES:**

```python
sage: # needs sage.rings.number_field
sage: P2 = ProjectiveSpace(QQ, 2)
sage: x = polygen(ZZ, 'x')
sage: K.<a> = NumberField(x^2 + x - (3^3-3))
sage: K_points = P2(K); K_points
Set of rational points of Projective Space of dimension 2 over Number Field in a with defining polynomial x^2 + x - 24
sage: K_points.codomain()
Projective Space of dimension 2 over Rational Field
sage: K_points.extended_codomain()
Projective Space of dimension 2 over Number Field in a with defining polynomial x^2 + x - 24
```
list()

Return a tuple containing all points.

OUTPUT:

A tuple containing all points of the toric variety.

EXAMPLES:

```
sage: P1 = toric_varieties.P1(base_ring=GF(3))  # needs sage.geometry.polyhedron sage.graphs
sage: P1.point_set().list()  # needs sage.geometry.polyhedron sage.graphs
([0 : 1], [1 : 0], [1 : 1], [1 : 2])
```

value_ring()

Return $R$ for a point $X(Spec(R))$.

OUTPUT:

A commutative ring.

EXAMPLES:

```
sage: P2 = ProjectiveSpace(ZZ, 2)
sage: P2(QQ).value_ring()
Rational Field
```

`sage.schemes.generic.homset.is_SchemeHomset(H)`

Test whether $H$ is a scheme Hom-set.

EXAMPLES:

```
sage: f = Spec(QQ).identity_morphism(); f
Scheme endomorphism of Spectrum of Rational Field
  Defn: Identity map
sage: from sage.schemes.generic.homset import is_SchemeHomset
sage: is_SchemeHomset(f)
False
sage: is_SchemeHomset(f.parent())
True
sage: is_SchemeHomset('a string')
False
```
Note: You should never create the morphisms directly. Instead, use the \texttt{hom()} and \texttt{Hom()} methods that are inherited by all schemes.

If you want to extend the Sage library with some new kind of scheme, your new class (say, \texttt{MyScheme}) should provide a method

- \texttt{MyScheme._morphism(*args, **kwds)} returning a morphism between two schemes in your category, usually defined via polynomials. Your morphism class should derive from \texttt{SchemeMorphism\_polynomial}. These morphisms will usually be elements of the Hom-set \texttt{SchemeHomset\_generic}.

Optionally, you can also provide a special Hom-set class for your subcategory of schemes. If you want to do this, you should also provide a method

- \texttt{MyScheme._homset(*args, **kwds)} returning a Hom-set, which must be an element of a derived class of \texttt{SchemeHomset\_generic}. If your new Hom-set class does not use \texttt{MyScheme._morphism} then you do not have to provide it.

Note that points on schemes are morphisms \textit{Spec}(K) \to X, too. But we typically use a different notation, so they are implemented in a different derived class. For this, you should implement a method

- \texttt{MyScheme._point(*args, **kwds)} returning a point, that is, a morphism \textit{Spec}(K) \to X. Your point class should derive from \texttt{SchemeMorphism\_point}.

Optionally, you can also provide a special Hom-set for the points, for example the point Hom-set can provide a method to enumerate all points. If you want to do this, you should also provide a method

- \texttt{MyScheme._point\_homset(*args, **kwds)} returning the \texttt{homset} of points. The Hom-sets of points are implemented in classes named \texttt{SchemeHomset\_points}.... If your new Hom-set class does not use \texttt{MyScheme._point} then you do not have to provide it.

AUTHORS:

- David Kohel, William Stein
- William Stein (2006-02-11): fixed bug where P(0,0,0) was allowed as a projective point.
- Volker Braun (2011-08-08): Renamed classes, more documentation, misc cleanups.
- Ben Hutz (June 2012): added support for projective ring
- Simon King (2013-10): copy the changes of \texttt{Morphism} that have been introduced in github issue \#14711.

\texttt{class sage.schemes.generic.morphism.SchemeMorphism(parent, codomain=None)}

\hspace{1cm} Bases: \texttt{Element}

\hspace{1cm} Base class for scheme morphisms
INPUT:

- parent – the parent of the morphism.

Todo: For historical reasons, `SchemeMorphism` copies code from `Map` rather than inheriting from it. Proper inheritance should be used instead. See github issue #14711.

EXAMPLES:

```python
sage: X = Spec(ZZ)
sage: Hom = X.Hom(X)
sage: from sage.schemes.generic.morphism import SchemeMorphism
sage: f = SchemeMorphism(Hom)
sage: type(f)
<class 'sage.schemes.generic.morphism.SchemeMorphism'>
```

`base_ring()`

Return the base ring of self, that is, the ring over which the defining polynomials of self are defined.

OUTPUT:

- ring

EXAMPLES:

```python
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: H = Hom(P, P)
sage: f = H([3/5*x^2, 6*y^2])
sage: f.base_ring()
Rational Field
```

```python
sage: R.<t> = PolynomialRing(ZZ, 1)
sage: P.<x,y> = ProjectiveSpace(R, 1)
sage: H = Hom(P, P)
sage: f = H([3*x^2, y^2])
sage: f.base_ring()
Multivariate Polynomial Ring in t over Integer Ring
```

Points have correct base rings too (github issue #34336):

```python
sage: x = P(t, 5); x
(t : 5)
sage: x.base_ring()
Multivariate Polynomial Ring in t over Integer Ring
```

```python
sage: # needs sage.rings.finite_rings sage.schemes
sage: E = EllipticCurve(GF((17,2)), [1,2,3,4,5])
sage: P = E.random_point()
sage: P.base_ring()
Finite Field in z2 of size 17^2
```

category()

Return the category of the Hom-set.

OUTPUT:
A category.

EXAMPLES:

\begin{verbatim}
sage: A2 = AffineSpace(QQ, 2)
sage: A2.structure_morphism().category()
Category of homsets of schemes
\end{verbatim}

category_for()

Return the category which this morphism belongs to.

EXAMPLES:

\begin{verbatim}
sage: A2 = AffineSpace(QQ, 2)
sage: A2.structure_morphism().category_for()
Category of schemes
\end{verbatim}

codomain()

The constant function from the codomain.

EXAMPLES:

\begin{verbatim}
sage: A.<x,y> = AffineSpace(QQ['x,y'])
sage: H = A.Hom(A)
sage: f = H([y, x^2 + y])
sage: f.codomain() is A
True
\end{verbatim}

domain()

The constant function from the domain.

EXAMPLES:

\begin{verbatim}
sage: A.<x,y> = AffineSpace(QQ['x,y'])
sage: H = A.Hom(A)
sage: f = H([y, x^2 + y])
sage: f.domain() is A
True
\end{verbatim}

glue_along_domains(other)

Glue two morphism

INPUT:

- other – a scheme morphism with the same domain.

OUTPUT:

Assuming that self and other are open immersions with the same domain, return scheme obtained by gluing along the images.

EXAMPLES:

We construct a scheme isomorphic to the projective line over Spec(\mathbb{Q}) by gluing two copies of \mathbb{A}^1 minus a point:

\begin{verbatim}
sage: # needs sage.libs.singular
sage: R.<x,y> = PolynomialRing(QQ, 2)
\end{verbatim}
sage: S.<xbar, ybar> = R.quotient(x*y - 1)
sage: Rx = PolynomialRing(QQ, 'x')
sage: i1 = Rx.hom([xbar])
sage: Ry = PolynomialRing(QQ, 'y')
sage: i2 = Ry.hom([ybar])
sage: Sch = Schemes()
sage: f1 = Sch(i1)
sage: f2 = Sch(i2)

Now f1 and f2 have the same domain, which is a $\mathbb{A}^1$ minus a point. We glue along the domain:

```
sage: # needs sage.libs.singular
sage: P1 = f1.glue_along_domains(f2); P1
Scheme obtained by gluing X and Y along U, where
  X: Spectrum of Univariate Polynomial Ring in x over Rational Field
  Y: Spectrum of Univariate Polynomial Ring in y over Rational Field
  U: Spectrum of Quotient of Multivariate Polynomial Ring in x, y
     over Rational Field by the ideal (x*y - 1)
sage: a, b = P1.gluing_maps()
sage: a
Affine Scheme morphism:
  From: Spectrum of Quotient of Multivariate Polynomial Ring in x, y
        over Rational Field by the ideal (x*y - 1)
  To:   Spectrum of Univariate Polynomial Ring in x over Rational Field
        Defn: Ring morphism:
              From: Univariate Polynomial Ring in x over Rational Field
              To:   Quotient of Multivariate Polynomial Ring in x, y
                     over Rational Field by the ideal (x*y - 1)
              Defn: x |--> xbar
sage: b
Affine Scheme morphism:
  From: Spectrum of Quotient of Multivariate Polynomial Ring in x, y
        over Rational Field by the ideal (x*y - 1)
  To:   Spectrum of Univariate Polynomial Ring in y over Rational Field
        Defn: Ring morphism:
              From: Univariate Polynomial Ring in y over Rational Field
              To:   Quotient of Multivariate Polynomial Ring in x, y
                     over Rational Field by the ideal (x*y - 1)
              Defn: y |--> ybar
```

```python
is_endomorphism()
```

Return whether the morphism is an endomorphism.

OUTPUT:

Boolean. Whether the domain and codomain are identical.

EXAMPLES:

```
sage: X = AffineSpace(QQ, 2)
sage: X.structure_morphism().is_endomorphism()
False
sage: X.identity_morphism().is_endomorphism()
True
```
class sage.schemes.generic.morphism.SchemeMorphism_id(X)

Bases: SchemeMorphism

Return the identity morphism from $X$ to itself.

INPUT:

- $X$ – the scheme.

EXAMPLES:

```
sage: X = Spec(ZZ)
sage: X.identity_morphism() # indirect doctest
Scheme endomorphism of Spectrum of Integer Ring
  Defn: Identity map
```

class sage.schemes.generic.morphism.SchemeMorphism_point(parent, codomain=None)

Bases: SchemeMorphism

Base class for rational points on schemes.

Recall that the $K$-rational points of a scheme $X$ over $k$ can be identified with the set of morphisms $Spec(K)\circ X$.

In Sage, the rational points are implemented by such scheme morphisms.

EXAMPLES:

```
sage: from sage.schemes.generic.morphism import SchemeMorphism
sage: f = SchemeMorphism(Spec(ZZ).Hom(Spec(ZZ)))
sage: type(f)
<class 'sage.schemes.generic.morphism.SchemeMorphism'>
```

change_ring($R$, check=True)

Returns a new SchemeMorphism_point which is this point coerced to $R$.

If check is true, then the initialization checks are performed.

INPUT:

- $R$ – ring or morphism.
- check – Boolean

OUTPUT: SchemeMorphism_point

EXAMPLES:

```
sage: P.<x,y,z> = ProjectiveSpace(ZZ, 2)
sage: X = P.subscheme(x^2 - y^2)
sage: X(23,23,1).change_ring(GF(13))
(10 : 10 : 1)
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: P(-2/3,1).change_ring(CC)
# needs sage.rings.real_mpfr
(-0.666666666666667 : 1.00000000000000)
sage: P.<x,y> = ProjectiveSpace(ZZ, 1)
sage: P(152,113).change_ring(Zp(5))
# needs sage.rings.padics
(2 + 5^2 + 5^3 + 0(5^20) : 3 + 2*5 + 4*5^2 + 0(5^20))
```
```python
sage: # needs sage.rings.number_field
sage: K.<v> = QuadraticField(-7)
sage: O = K.maximal_order()
sage: P.<x,y> = ProjectiveSpace(O, 1)
sage: H = End(P)
sage: F = H([x^2 + O(v)*y^2, y^2])
sage: F.change_ring(K).change_ring(K.embeddings(QQbar)[0])
Scheme endomorphism of Projective Space of dimension 1 over Algebraic Field
Defn: Defined on coordinates by sending (x : y) to
(x^2 + (-2.645751311064591?*I)*y^2 : y^2)
```

```python
sage: # needs sage.rings.number_field
sage: R.<x> = PolynomialRing(QQ)
sage: K.<a> = NumberField(x^2 - x + 1)
sage: P.<x,y> = ProjectiveSpace(K, 1)
sage: Q = P([a + 1, 1])
sage: emb = K.embeddings(QQbar)
sage: Q.change_ring(emb[0])
(1.5000000000000000? - 0.866025403784439?*I : 1)
sage: Q.change_ring(emb[1])
(1.5000000000000000? + 0.866025403784439?*I : 1)
```

```python
sage: # needs sage.rings.number_field
sage: K.<v> = QuadraticField(2)
sage: P.<x,y> = ProjectiveSpace(K, 1)
sage: Q = P([v,1])
sage: Q.change_ring(K.embeddings(QQbar)[0])
(-1.414213562373095? : 1)
```

```python
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: f = x^6 - 2
sage: L.<b> = NumberField(f, embedding=f.roots(QQbar)[1][0])
sage: A.<x,y> = AffineSpace(L, 2)
sage: P = A([b,1])
sage: P.change_ring(QQbar)
(1.122462048309373?, 1)
```

```
sage: scheme()
Return the scheme whose point is represented.

OUTPUT:
A scheme.

EXAMPLES:

sage: A = AffineSpace(2, QQ)
sage: a = A(1,2)
sage: a.scheme()
Affine Space of dimension 2 over Rational Field
```

```python
specialization(D=None, phi=None, ambient=None)
```
Specialization of this point.

Given a family of points defined over a polynomial ring. A specialization is a particular member of that family. The specialization can be specified either by a dictionary or a `SpecializationMorphism`.

**INPUT:**

- `D` – dictionary (optional)
- `phi` – `SpecializationMorphism` (optional)
- `ambient` – ambient space of specialized point (optional)

**OUTPUT:** `SchemeMorphism_polynomial`

**EXAMPLES:**

```python
sage: R.<c> = PolynomialRing(QQ)
sage: P.<x,y> = ProjectiveSpace(R, 1)
sage: Q = P([c,1])
sage: Q.specialization({c: 1})
(1 : 1)

sage: R.<a,b> = PolynomialRing(QQ)
sage: P.<x,y> = ProjectiveSpace(R, 1)
sage: Q = P([a^2 + 2*a*b + 34, 1])
sage: from sage.rings.polynomial.flatten import SpecializationMorphism
sage: phi = SpecializationMorphism(P.coordinate_ring(), {a: 2, b: -1})
sage: T = Q.specialization(phi=phi); T
(34 : 1)
sage: Q2 = P([a,1])
sage: T2 = Q2.specialization(phi=phi)
sage: T2.codomain() is T.codomain()
True
sage: T3 = Q2.specialization(phi=phi, ambient=T.codomain())
sage: T3.codomain() is T.codomain()
True
```

```python
sage: R.<c> = PolynomialRing(QQ)
sage: P.<x,y> = ProjectiveSpace(R, 1)
sage: X = P.subscheme([x - c*y])
sage: Q = X([c, 1])
sage: Q2 = Q.specialization({c:2}); Q2
(2 : 1)
sage: Q2.codomain()
Closed subscheme of Projective Space of dimension 1 over Rational Field
defined by: x - 2*y
```

```python
sage: R.<l> = PolynomialRing(QQ)
sage: S.<k,j> = PolynomialRing(R)
sage: K.<a,b,c,d> = S[]
sage: P.<x,y> = ProjectiveSpace(K, 1)
sage: H = End(P)
sage: Q = P([a^2, b^2])
```

(continues on next page)
sage: Q.specialization({a: 2})
(4 : b^2)

class sage.schemes.generic.morphism.SchemeMorphism_polynomial(parent, polys, check=True)
Bases: SchemeMorphism

A morphism of schemes determined by polynomials that define what the morphism does on points in the ambient space.

INPUT:

• parent – Hom-set whose domain and codomain are affine or projective schemes.

• polys – a list/tuple/iterable of polynomials defining the scheme morphism.

• check – boolean (optional, default:True). Whether to check the input for consistency.

EXAMPLES:

An example involving the affine plane:

```python
sage: R.<x,y> = QQ[]
sage: A2 = AffineSpace(R)
sage: H = A2.Hom(A2)
sage: f = H([x - y, x*y])
sage: f([0, 1])
(-1, 0)
```

An example involving the projective line:

```python
sage: R.<x,y> = QQ[]
sage: P1 = ProjectiveSpace(R)
sage: H = P1.Hom(P1)
sage: f = H([x^2 + y^2, x*y])
sage: f([0, 1])
(1 : 0)
```

Some checks are performed to make sure the given polynomials define a morphism:

```python
sage: f = H([exp(x),exp(y)])
needs sage.symbolic
Traceback (most recent call last):
...
TypeError: polys (=e^x, e^y) must be elements of Multivariate Polynomial Ring in x, y over Rational Field
```

change_ring(R, check=True)

Returns a new SchemeMorphism_polynomial which is this map coerced to R.

If check is True, then the initialization checks are performed.

INPUT:

• R – ring or morphism.

• check – Boolean

OUTPUT:
• A new `SchemeMorphism_polynomial` which is this map coerced to \( \mathbb{R} \).

**EXAMPLES:**

```python
sage: P.<x,y> = ProjectiveSpace(ZZ, 1)
sage: H = Hom(P, P)
sage: f = H([3*x^2, y^2])
sage: f.change_ring(GF(3))
Scheme endomorphism of Projective Space of dimension 1 over Finite Field of size 3
Defn: Defined on coordinates by sending (x : y) to (0 : y^2)
```

```python
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: H = Hom(P, P)
sage: f = H([5/2*x^3 + 3*x*y^2 - y^3, 3*z^3 + y*x^2, x^3 - z^3])
sage: f.change_ring(GF(3))
Scheme endomorphism of Projective Space of dimension 2 over Finite Field of size 3
Defn: Defined on coordinates by sending (x : y : z) to (x^3 - y^3 : x^2*y : x^3 - z^3)
```

```python
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: X = P.subscheme([5*x^2 - y^2])
sage: H = Hom(X, X)
sage: f = H([x, y])
sage: f.change_ring(GF(3))
Scheme endomorphism of Closed subscheme of Projective Space of dimension 1 over Finite Field of size 3 defined by: -x^2 - y^2
Defn: Defined on coordinates by sending (x : y) to (x : y)
```

Check that github issue #16834 is fixed:

```python
sage: # needs sage.rings.real_mpfr
sage: A.<x,y,z> = AffineSpace(RR, 3)
sage: h = Hom(A, A)
sage: f = h([x^2 + 1.5, y^3, z^5 - 2.0])
sage: f.change_ring(CC)
Scheme endomorphism of Affine Space of dimension 3 over Complex Field with 53 bits of precision
Defn: Defined on coordinates by sending (x, y, z) to (x^2 + 1.50000000000000, y^3, z^5 - 2.00000000000000)
```

```python
sage: A.<x,y> = AffineSpace(ZZ, 2)
sage: B.<u,v> = ProjectiveSpace(QQ, 1)
sage: h = Hom(A,B)
sage: f = h([x^2, y^2])
sage: f.change_ring(QQ)
Scheme morphism:
From: Affine Space of dimension 2 over Rational Field
To:   Projective Space of dimension 1 over Rational Field
Defn: Defined on coordinates by sending (x, y) to (x^2 : y^2)
```
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: H = Hom(A, A)
sage: f = H([3*x^2/y, y^2/x])
sage: f.change_ring(RR) # needs sage.rings.real_mpfr
Scheme endomorphism of Affine Space of dimension 2 over Real Field with 53 bits of precision
Defn: Defined on coordinates by sending (x, y) to
(3.00000000000000*x^2/y, y^2/x)

sage: # needs sage.rings.number_field
sage: R.<x> = PolynomialRing(QQ)
sage: K.<a> = NumberField(x^3 - x + 1)
sage: P.<x,y> = ProjectiveSpace(K, 1)
sage: H = End(P)
sage: f = H([x^2 + a*x*y + a^2*y^2, y^2])
sage: emb = K.embeddings(QQbar)
sage: f.change_ring(emb[0])
Scheme endomorphism of Projective Space of dimension 1 over Algebraic Field
Defn: Defined on coordinates by sending (x : y) to
(x^2 + (-1.324717957244746?)*x*y + 1.754877666246693?*y^2 : y^2)
sage: f.change_ring(emb[1])
Scheme endomorphism of Projective Space of dimension 1 over Algebraic Field
Defn: Defined on coordinates by sending (x : y) to
(x^2 + (0.6623589786223013? - 0.5622795120623013?*I)*x*y
 + (0.1225611668766537? - 0.744861766619745?*I)*y^2 : y^2)

sage: # needs sage.rings.number_field sage.symbolic
sage: K.<v> = QuadraticField(2, embedding=QQbar(sqrt(2)))
sage: P.<x,y> = ProjectiveSpace(K, 1)
sage: H = End(P)
sage: f = H([x^2 + v*y^2, y^2])
sage: f.change_ring(QQbar)
Scheme endomorphism of Projective Space of dimension 1 over Algebraic Field
Defn: Defined on coordinates by sending (x : y) to
(x^2 + 1.414213562373095?*y^2 : y^2)

sage: # needs sage.rings.number_field sage.symbolic
sage: from sage.misc.verbose import set_verbose
sage: set_verbose(-1)
sage: K.<w> = QuadraticField(2, embedding=QQbar(-sqrt(2)))
sage: P.<x,y> = ProjectiveSpace(K, 1)
sage: X = P.subscheme(x - y)
sage: H = End(X)
sage: f = H([6*x^2 + 2*x*y + 16*y^2, -w*x^2 - 4*x*y - 4*y^2])
sage: f.change_ring(QQbar)
Scheme endomorphism of Closed subscheme of Projective Space of dimension 1 over Algebraic Field defined by: x - y
Defn: Defined on coordinates by sending (x : y) to
(continued from previous page)

(6*x^2 + 2*x*y + 16*y^2 : 1.414213562373095?*x^2 + (-4)*x*y + (-4)*y^2)

```python
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: f = x^6 - 2
sage: L.<b> = NumberField(f, embedding=f.roots(QQbar)[1][0])
sage: A.<x,y> = AffineSpace(L, 2)
sage: H = Hom(A, A)
sage: F = H([b*x/y, 1 + y])
sage: F.change_ring(QQbar)
Scheme endomorphism of Affine Space of dimension 2 over Algebraic Field
  Defn: Defined on coordinates by sending (x, y) to
        (1.122462048309373?*x/y, y + 1)
```

```python
sage: # needs sage.rings.number_field
sage: K.<a> = QuadraticField(-1)
sage: A.<x,y> = AffineSpace(K, 2)
sage: H = End(A)
sage: phi = H([x/y, y])
sage: emb = K.embeddings(QQbar)[0]
sage: phi.change_ring(emb)
Scheme endomorphism of Affine Space of dimension 2 over Algebraic Field
  Defn: Defined on coordinates by sending (x, y) to (x/y, y)
```

**coordinate_ring()**

Return the coordinate ring of the ambient projective space.

**OUTPUT:** A multivariable polynomial ring over the base ring.

**EXAMPLES:**

```python
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: H = Hom(P, P)
sage: f = H([3/5*x^2, 6*y^2])
sage: f.coordinate_ring()
Multivariate Polynomial Ring in x, y over Rational Field
```

```python
sage: R.<t> = PolynomialRing(ZZ, 1)
sage: P.<x,y> = ProjectiveSpace(R, 1)
sage: H = Hom(P, P)
sage: f = H([3*x^2, y^2])
sage: f.coordinate_ring()
Multivariate Polynomial Ring in x, y over Multivariate Polynomial Ring
in t over Integer Ring
```

**defining_polynomials()**

Return the defining polynomials.

**OUTPUT:**

An immutable sequence of polynomials that defines this scheme morphism.

**EXAMPLES:**
sage: R.<x,y> = QQ[]
sage: A.<x,y> = AffineSpace(R)
sage: H = A.Hom(A)
sage: H([x^3 + y, 1 - x - y]).defining_polynomials()
(x^3 + y, -x - y + 1)

specialization($D=None, \phi=None, \text{homset}=None$)

Specialization of this map.

Given a family of maps defined over a polynomial ring. A specialization is a particular member of that family. The specialization can be specified either by a dictionary or a SpecializationMorphism.

INPUT:

- $D$ – dictionary (optional)
- $\phi$ – SpecializationMorphism (optional)
- $\text{homset}$ – homset of specialized map (optional)

OUTPUT: SchemeMorphism_polynomial

EXAMPLES:

sage: R.<c> = PolynomialRing(QQ)
sage: P.<x,y> = ProjectiveSpace(R, 1)
sage: H = End(P)
sage: f = H([x^2 + c*y^2, y^2])
sage: f.specialization({c: 1})
Scheme endomorphism of Projective Space of dimension 1 over Rational Field
Defn: Defined on coordinates by sending (x : y) to (x^2 + y^2 : y^2)

sage: R.<a,b> = PolynomialRing(QQ)
sage: P.<x,y> = ProjectiveSpace(R, 1)
sage: H = End(P)
sage: f = H([x^3 + a*x*y^2 + b*y^3, y^3])
sage: from sage.rings.polynomial.flatten import SpecializationMorphism
sage: phi = SpecializationMorphism(P.coordinate_ring(), {a: 2, b: -1})
sage: F = f.specialization(phi=phi); F
Scheme endomorphism of Projective Space of dimension 1 over Rational Field
Defn: Defined on coordinates by sending (x : y) to (x^3 + 2*x*y^2 - y^3 : y^3)

sage: g = H([x^2 + a*y^2, y^2])
sage: G = g.specialization(phi=phi)
sage: G.parent() is F.parent()
True
sage: G = g.specialization(phi=phi, homset=F.parent())
sage: G.parent() is F.parent()
True

sage: R.<c> = PolynomialRing(QQ)
sage: P.<x,y> = ProjectiveSpace(R, 1)
sage: X = P.subscheme([x - c*y])
sage: H = End(X)
sage: f = H([x^2, c*y^2])

\texttt{sage}: \texttt{f.specialization({c: 2})}
Scheme endomorphism of Closed subscheme of Projective Space of dimension 1
over Rational Field defined by: \( x - 2*y \)
\hspace{1cm} Defn: Defined on coordinates by sending (x : y) to (x^2 : 2*y^2)

\texttt{sage}: \texttt{R.<c> = QQ[]}
\texttt{sage}: \texttt{P.<x,y> = ProjectiveSpace(R, 1)}
\texttt{sage}: \texttt{f = DynamicalSystem_projective([x^2 + c*y^2, y^2], domain=P)}
\texttt{sage}: \texttt{F = f.dynatomic_polynomial(3)}  
\hspace{1cm} \rightarrow \texttt{needs sage.libs.pari}
\texttt{sage}: \texttt{g = F.specialization({c: 1}); g}
x^6 + x^5*y + 4*x^4*y^2 + 3*x^3*y^3 + 7*x^2*y^4 + 4*x*y^5 + 5*y^6
\texttt{sage}: \texttt{g == f.specialization({c:1}).dynatomic_polynomial(3)}  
\hspace{1cm} \rightarrow \texttt{needs sage.libs.pari}
\hspace{1cm} True

\texttt{sage}: \texttt{R1.<alpha, beta> = QQ[]}
\texttt{sage}: \texttt{A.<x> = AffineSpace(Frac(R1), 1)}
\texttt{sage}: \texttt{f = DynamicalSystem_affine([alpha/(x^2 + 1/alpha)/(x - 1/beta^2)])}
\texttt{sage}: \texttt{f.specialization({alpha: 5, beta: 10})}
\hspace{1cm} \text{Dynamical System of Affine Space of dimension 1 over Rational Field}
\hspace{1cm} Defn: Defined on coordinates by sending (x) to
\hspace{1cm} \( (5/(x^3 - 1/100*x^2 + 1/5*x - 1/500)) \)
\texttt{sage}: \texttt{f_5_10 = f.specialization({alpha: 5}).specialization({beta: 10})}
\texttt{sage}: \texttt{f_5_10 == f.specialization({alpha: 5, beta: 10})}
\hspace{1cm} True

\textbf{class \texttt{sage.schemes.generic.morphism.SchemeMorphism_polynomial_id}(X)}

\quad \textit{Bases: \texttt{SchemeMorphism.id}, \texttt{SchemeMorphism_polynomial}}

\quad Return the identity morphism from \( X \) to itself.

\quad \textbf{INPUT:}

\quad \hspace{1cm} • \( X \) – an affine or projective scheme

\quad \textbf{EXAMPLES:}

\texttt{sage}: \texttt{X = Spec(ZZ)}
\texttt{sage}: \texttt{X.identity_morphism()}  \hspace{1cm} \# \textit{indirect doctest}
\quad Scheme endomorphism of Spectrum of Integer Ring
\hspace{1cm} Defn: Identity map

\textbf{class \texttt{sage.schemes.generic.morphism.SchemeMorphism_spec}(parent, phi, check=True)}

\quad \textit{Bases: \texttt{SchemeMorphism}}

\quad Morphism of spectra of rings

\quad \textbf{INPUT:}

\quad \hspace{1cm} • \( \texttt{parent} \) – Hom-set whose domain and codomain are affine schemes.

\quad \hspace{1cm} • \( \texttt{phi} \) – a ring morphism with matching domain and codomain.

\quad \hspace{1cm} • \( \texttt{check} \) – boolean (optional, default:True). Whether to check the input for consistency.

\quad \textbf{EXAMPLES:}
sage: R.<x> = PolynomialRing(QQ)
sage: phi = R.hom([QQ(7)]); phi
Ring morphism:
    From: Univariate Polynomial Ring in x over Rational Field
    To:   Rational Field
    Defn: x |--> 7

sage: X = Spec(QQ); Y = Spec(R)
sage: f = X.hom(phi); f
Affine Scheme morphism:
    From: Spectrum of Rational Field
    To:   Spectrum of Univariate Polynomial Ring in x over Rational Field
    Defn: Ring morphism:
        From: Univariate Polynomial Ring in x over Rational Field
        To:   Rational Field
        Defn: x |--> 7

sage: f.ring_homomorphism()
Ring morphism:
    From: Univariate Polynomial Ring in x over Rational Field
    To:   Rational Field
    Defn: x |--> 7

class sage.schemes.generic.morphism.SchemeMorphism_structure_map(parent, codomain=None)
Bases: SchemeMorphism
The structure morphism

INPUT:
    • parent – Hom-set with codomain equal to the base scheme of the domain.

EXAMPLES:

sage: Spec(ZZ).structure_morphism()  # indirect doctest
Scheme endomorphism of Spectrum of Integer Ring
    Defn: Structure map
sage.schemes.generic.morphism.is_SchemeMorphism(f)

Test whether $f$ is a scheme morphism.

INPUT:

- $f$ – anything.

OUTPUT:

Boolean. Return True if $f$ is a scheme morphism or a point on an elliptic curve.

EXAMPLES:

```python
sage: A.<x,y> = AffineSpace(QQ, 2); H = A.Hom(A)
sage: f = H([y, x^2 + y]); f
Scheme endomorphism of Affine Space of dimension 2 over Rational Field
  Defn: Defined on coordinates by sending (x, y) to (y, x^2 + y)
sage: from sage.schemes.generic.morphism import is_SchemeMorphism
sage: is_SchemeMorphism(f)
True
```
AUTHORS:
- William Stein
- David Kohel
- David Joyner

EXAMPLES:

```python
sage: x,y,z = ProjectiveSpace(2, GF(5), names='x,y,z').gens()
sage: C = Curve(y^2*z^7 - x^9 - x*z^8)
sage: pts = C.rational_points(); pts
[(0 : 0 : 1), (0 : 1 : 0), (2 : 2 : 1), (2 : 3 : 1), (3 : 1 : 1), (3 : 4 : 1)]
sage: D1 = C.divisor(pts[0])*3
sage: D2 = C.divisor(pts[1])
sage: D3 = 10*C.divisor(pts[5])
sage: D = D1 - D2 + D3; D
3*(x, y) - (x, z) + 10*(x + 2*z, y + z)
sage: D[1][0]
-1
```

```
sage.schemes.generic.divisor.CurvePointToIdeal(C, P)
Return the vanishing ideal of a point on a curve.

EXAMPLES:
```
```
sage: x,y = AffineSpace(2, QQ, names='xy').gens()
sage: C = Curve(y^2 - x^9 - x)
sage: from sage.schemes.generic.divisor import CurvePointToIdeal
sage: CurvePointToIdeal(C, (0,0))
Ideal (x, y) of Multivariate Polynomial Ring in x, y over Rational Field
```

```
class sage.schemes.generic.divisor.Divisor_curve(v, parent=None, check=True, reduce=True)
Bases: Divisor_generic

For any curve C, use C.divisor(v) to construct a divisor on C. Here v can be either
• a rational point on $C$
• a list of rational points
• a list of 2-tuples $(c, P)$, where $c$ is an integer and $P$ is a rational point.

TODO: Divisors shouldn’t be restricted to rational points. The problem is that the divisor group is the formal sum of the group of points on the curve, and there’s no implemented notion of point on $E/K$ that has coordinates in $L$. This is what should be implemented, by adding an appropriate class to `schemes/generic/morphism.py`.

EXAMPLES:

```
sage: E = EllipticCurve([0, 0, 1, -1, 0])
sage: P = E(0,0)
sage: 10*P
(161/16 : -2065/64 : 1)
sage: D = E.divisor(P)
sage: D
(x, y)
sage: 10*D
10*(x, y)
sage: E.divisor([P, P])
2*(x, y)
sage: E.divisor([(3,P), (-4,5*P)])
3*(x, y) - 4*(x - 1/4*z, y + 5/8*z)
```

`coefficient(P)`

Return the coefficient of a given point $P$ in this divisor.

EXAMPLES:

```
sage: x,y = AffineSpace(2, GF(5), names='xy').gens()
sage: C = Curve(y^2 - x^9 - x)
sage: pts = C.rational_points(); pts
[(0, 0), (2, 2), (2, 3), (3, 1), (3, 4)]
sage: D = C.divisor(pts[0])
sage: D.coefficient(pts[0])
1
sage: D = C.divisor([(3, pts[0]), (-1, pts[1])]); D
3*(x, y) - (x - 2, y - 2)
sage: D.coefficient(pts[0])
3
sage: D.coefficient(pts[1])
-1
```

`support()`

Return the support of this divisor, which is the set of points that occur in this divisor with nonzero coefficients.

EXAMPLES:

```
sage: x,y = AffineSpace(2, GF(5), names='xy').gens()
sage: C = Curve(y^2 - x^9 - x)
sage: pts = C.rational_points(); pts
[(0, 0), (2, 2), (2, 3), (3, 1), (3, 4)]
sage: D = C.divisor_group()([(3, pts[0]), (-1, pts[1])]); D
```

(continues on next page)
\[ 3^*(x, y) - (x - 2, y - 2) \]

\texttt{sage: D.support()}
\[[0, 0], (2, 2)]

\textbf{class} \quad \texttt{sage.schemes.generic.divisor.Divisor\_generic}(v, parent, check=True, reduce=True)

\textbf{Bases:} \quad \texttt{FormalSum}

A Divisor.

\textbf{scheme()}

Return the scheme that this divisor is on.

\textbf{EXAMPLES:}

\begin{verbatim}
\texttt{sage: A.<x, y> = AffineSpace(2, GF(5))
\texttt{sage: C = Curve(y^2 - x^9 - x)
\texttt{sage: pts = C.rational_points(); pts
\[(0, 0), (2, 2), (2, 3), (3, 1), (3, 4)]
\texttt{sage: D = C.divisor(pts[0])*3 - C.divisor(pts[1]); D
3^*(x, y) - (x - 2, y - 2)
\texttt{sage: D.scheme()}
Affine Plane Curve over Finite Field of size 5 defined by -x^9 + y^2 - x
\end{verbatim}

\texttt{sage.schemes.generic.divisor.is\_Divisor(x)}

Test whether \(x\) is an instance of \texttt{Divisor\_generic}

\textbf{INPUT:}

- \(x\) – anything.

\textbf{OUTPUT:}

True or False.

\textbf{EXAMPLES:}

\begin{verbatim}
\texttt{sage: from sage.schemes.generic.divisor import is\_Divisor
\texttt{sage: x,y = AffineSpace(2, GF(5), names='xy').gens()
\texttt{sage: C = Curve(y^2 - x^9 - x)
\texttt{sage: is\_Divisor(C.divisor([]))}
True
\texttt{sage: is\_Divisor("Ceci n'est pas un diviseur")}
False
\end{verbatim}
DIVISOR GROUPS

AUTHORS:

- David Kohel (2006): Initial version

sage.schemes.generic.divisor_group.DivisorGroup(scheme, base_ring=None)

Return the group of divisors on the scheme.

INPUT:

- scheme – a scheme.
- base_ring – usually either \( \mathbb{Z} \) (default) or \( \mathbb{Q} \). The coefficient ring of the divisors. Not to be confused with the base ring of the scheme!

OUTPUT:

An instance of DivisorGroup_generic.

EXAMPLES:

```
sage: from sage.schemes.generic.divisor_group import DivisorGroup
sage: DivisorGroup(Spec(ZZ))
Group of ZZ-Divisors on Spectrum of Integer Ring
sage: DivisorGroup(Spec(ZZ), base_ring=QQ)
Group of QQ-Divisors on Spectrum of Integer Ring
```

class sage.schemes.generic.divisor_group.DivisorGroup_curve(scheme, base_ring)

Special case of the group of divisors on a curve.

class sage.schemes.generic.divisor_group.DivisorGroup_generic(scheme, base_ring)

Bases: FormalSums

The divisor group on a variety.

\textbf{base_extend}(R)

EXAMPLES:

```
sage: from sage.schemes.generic.divisor_group import DivisorGroup
sage: DivisorGroup(Spec(ZZ), ZZ).base_extend(QQ)
Group of QQ-Divisors on Spectrum of Integer Ring
sage: DivisorGroup(Spec(ZZ), ZZ).base_extend(GF(7))
Group of (Finite Field of size 7)-Divisors on Spectrum of Integer Ring
```
Divisor groups are unique:

```
sage: A.<x, y> = AffineSpace(2, CC)  # needs sage.rings.real_mpfr
sage: C = Curve(y^2 - x^9 - x)  # needs sage.rings.real_mpfr sage.schemes
sage: DivisorGroup(C, ZZ).base_extend(QQ) is DivisorGroup(C, QQ)  # needs sage.rings.real_mpfr sage.schemes
True
```

`scheme()`

Return the scheme supporting the divisors.

EXAMPLES:

```
sage: from sage.schemes.generic.divisor_group import DivisorGroup
sage: Div = DivisorGroup(Spec(ZZ))  # indirect test
sage: Div.scheme()
Spectrum of Integer Ring
```

`sage.schemes.generic.divisor_group.is_DivisorGroup(x)`

Return whether `x` is a `DivisorGroup_generic`.

INPUT:

- `x` – anything.

OUTPUT:

True or False.

EXAMPLES:

```
sage: from sage.schemes.generic.divisor_group import is_DivisorGroup, DivisorGroup
sage: Div = DivisorGroup(Spec(ZZ), base_ring=QQ)
sage: is_DivisorGroup(Div)
True
sage: is_DivisorGroup('not a divisor')
False
```
13.1 Affine $n$ space over a ring

```
sage.schemes.affine.affine_space.AffineSpace($n$, $R$=None, names=None,
ambient_projective_space=None,
default_embedding_index=None)
```

Return affine space of dimension $n$ over the ring $R$.

**EXAMPLES:**

The dimension and ring can be given in either order:

```sage
sage: AffineSpace(3, QQ, 'x')
Affine Space of dimension 3 over Rational Field
sage: AffineSpace(5, QQ, 'x')
Affine Space of dimension 5 over Rational Field
sage: A = AffineSpace(2, QQ, names='XY'); A
Affine Space of dimension 2 over Rational Field
sage: A.coordinate_ring()
Multivariate Polynomial Ring in X, Y over Rational Field
```

Use the divide operator for base extension:

```sage
sage: AffineSpace(5, names='x')/GF(17)
Affine Space of dimension 5 over Finite Field of size 17
```

The default base ring is $\mathbb{Z}$:

```sage
sage: AffineSpace(5, names='x')
Affine Space of dimension 5 over Integer Ring
```

There is also an affine space associated to each polynomial ring:

```sage
sage: R = GF(7)[['x', 'y', 'z']]
sage: A = AffineSpace(R); A
Affine Space of dimension 3 over Finite Field of size 7
sage: A.coordinate_ring() is R
True
```

**class** `sage.schemes.affine.affine_space.AffineSpace_field($n$, $R$, names, ambient_projective_space,
default_embedding_index)`

**Bases:** `AffineSpace_generic`
curve($F$)

Return a curve defined by $F$ in this affine space.

INPUT:

- $F$ – a polynomial, or a list or tuple of polynomials in the coordinate ring of this affine space

EXAMPLES:

```sage
sage: A.<x,y,z> = AffineSpace(QQ, 3)
sage: A.curve([y - x^4, z - y^5])
Affine Curve over Rational Field defined by -x^4 + y, -y^5 + z
```

line_through($p, q$)

Return the line through $p$ and $q$.

INPUT:

- $p, q$ – distinct rational points of the affine space

EXAMPLES:

```sage
sage: # needs sage.libs.singular sage.schemes
sage: A3.<x,y,z> = AffineSpace(3, QQ)
sage: p1 = A3(1, 2, 3)
sage: p2 = A3(4, 5, 6)
sage: L = A3.line_through(p1, p2); L
Affine Curve over Rational Field defined by -1/6*x + 1/6*y - 1/6, -1/6*x + 1/6*z - 1/3, -1/6*y + 1/6*z - 1/6, -1/6*x + 1/3*y - 1/6*z
sage: L(p1)
(1, 2, 3)
sage: L(p2)
(4, 5, 6)
sage: A3.line_through(p1, p1)
Traceback (most recent call last):
...  
ValueError: not distinct points
```

points_of_bounded_height(**kwds)

Return an iterator of the points in this affine space of absolute height of at most the given bound.

Bound check is strict for the rational field. Requires this space to be affine space over a number field. Uses the Doyle-Krumm algorithm 4 (algorithm 5 for imaginary quadratic) for computing algebraic numbers up to a given height [DK2013].

The algorithm requires floating point arithmetic, so the user is allowed to specify the precision for such calculations. Additionally, due to floating point issues, points slightly larger than the bound may be returned. This can be controlled by lowering the tolerance.

INPUT:

kwds:

- bound - a real number
- tolerance - a rational number in $(0, 1]$ used in doyle-krumm algorithm-4
- precision - the precision to use for computing the elements of bounded height of number fields

OUTPUT:
• an iterator of points in self

EXAMPLES:

```python
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: list(A.points_of_bounded_height(bound=3))
[(0, 0), (1, 0), (-1, 0), (1/2, 0), (2, 0), (-2, 0), (0, 1),
 (-1, 1), (-1, 1), (1/2, 1), (-1/2, 1), (2, 1), (-2, 1), (0, -1), (1, -1),
 (-1, -1), (1/2, -1), (-1/2, -1), (2, -1), (-2, -1), (0, 1/2), (1, 1/2),
 (-1, 1/2), (1/2, 1/2), (-1/2, 1/2), (2, 1/2), (-2, 1/2), (0, -1/2), (1, -1/2),
 (-1, -1/2), (1/2, -1/2), (-1/2, -1/2), (2, -1/2), (-2, -1/2), (0, 2), (1, 2),
 (-1, 2), (1/2, 2), (-1/2, 2), (2, 2), (-2, 2), (0, -2), (1, -2), (-1, -2),
 (1/2, -2), (-1/2, -2), (2, -2), (-2, -2)]
```

```python
sage: A.<x,y,z> = AffineSpace(QQ, 3)
sage: p = A(1,2,3)
sage: q = A(4,5,6)
sage: A.translation(p, q)
```

```
Scheme endomorphism of Affine Space of dimension 3 over Rational Field
Defn: Defined on coordinates by sending (x, y, z) to
(x + 3, y + 3, z + 3)
```

```python
sage: A.translation(p, q) == A.translation(A.origin(), q)
```

True

weil_restriction()

Compute the Weil restriction of this affine space over some extension field.

If the field is a finite field, then this computes the Weil restriction to the prime subfield.

OUTPUT: Affine space of dimension \( d \times \text{self.dimension_relative()} \) over the base field of self.base_ring().
EXAMPLES:

```python
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: K.<w> = NumberField(x^5 - 2)
sage: AK.<x,y> = AffineSpace(K, 2)
sage: AK.weil_restriction()
Affine Space of dimension 10 over Rational Field
sage: R.<x> = K[]
sage: L.<v> = K.extension(x^2 + 1)
sage: AL.<x,y> = AffineSpace(L, 2)
sage: AL.weil_restriction()
Affine Space of dimension 4 over Number Field in w
with defining polynomial x^5 - 2
```

```python
class sage.schemes.affine.affine_space.AffineSpace_finite_field(n, R, names, ambient_projective_space, default_embedding_index)
Bases: AffineSpace_field

```python
class sage.schemes.affine.affine_space.AffineSpace_generic(n, R, names, ambient_projective_space, default_embedding_index)
Bases: AmbientSpace, AffineScheme
```

Affine space of dimension \( n \) over the ring \( R \).

EXAMPLES:

```python
sage: X.<x,y,z> = AffineSpace(3, QQ)
sage: X.base_scheme()
Spectrum of Rational Field
sage: X.base_ring()
Rational Field
sage: X.category()
Category of schemes over Rational Field
sage: X.structure_morphism()
Scheme morphism:
  From: Affine Space of dimension 3 over Rational Field
  To:   Spectrum of Rational Field
  Defn: Structure map
```

Loading and saving:

```python
sage: loads(X.dumps()) == X
True
```

We create several other examples of affine spaces:

```python
sage: AffineSpace(5, PolynomialRing(QQ, 'z'), 'Z')
Affine Space of dimension 5 over Univariate Polynomial Ring in z over Rational Field
sage: AffineSpace(RealField(), 3, 'Z')
  # needs sage.rings.real_mpfr
Affine Space of dimension 3 over Real Field with 53 bits of precision
```

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(continued from previous page)

```
sage: AffineSpace(Qp(7), 2, 'x')
˓→ needs sage.rings.padics
Affine Space of dimension 2 over 7-adic Field with capped relative precision 20
```

Even 0-dimensional affine spaces are supported:

```
sage: AffineSpace(0)
Affine Space of dimension 0 over Integer Ring
```

**change_ring(R)**
Return an affine space over ring \( R \) and otherwise the same as this space.

INPUT:

- \( R \) – commutative ring or morphism.

OUTPUT: An affine space over \( R \).

**Note:** There is no need to have any relation between \( R \) and the base ring of this space, if you want to have such a relation, use `self.base_extend(R)` instead.

**EXAMPLES:**

```
sage: A.<x,y,z> = AffineSpace(3, ZZ)
sage: AQ = A.change_ring(QQ); AQ
Affine Space of dimension 3 over Rational Field
sage: AQ.change_ring(GF(5))
Affine Space of dimension 3 over Finite Field of size 5
```

```
sage: K.<w> = QuadraticField(5)
˓→ needs sage.rings.number_field
sage: A = AffineSpace(K, 2, 't')
˓→ needs sage.rings.number_field
sage: A.change_ring(K.embeddings(CC)[1])
˓→ needs sage.rings.number_field
Affine Space of dimension 2 over Complex Field with 53 bits of precision
```

**chebyshev_polynomial** \((n, \text{kind}='\text{first}', \text{monic}=False)\)
Generate an endomorphism of this affine line by a Chebyshev polynomial.

Chebyshev polynomials are a sequence of recursively defined orthogonal polynomials. Chebyshev of the first kind are defined as \( T_0(x) = 1, T_1(x) = x, \) and \( T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \). Chebyshev of the second kind are defined as \( U_0(x) = 1, U_1(x) = 2x, \) and \( U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \).

**INPUT:**

- \( n \) – a non-negative integer.
- `kind` – `first` or `second` specifying which kind of chebyshev the user would like to generate. Defaults to `first`.
- `monic` – `True` or `False` specifying if the polynomial defining the system should be monic or not. Defaults to `False`.

**OUTPUT:** `DynamicalSystem_affine`

**EXAMPLES:**

13.1. Affine \( n \) space over a ring

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sage: A.<x> = AffineSpace(QQ, 1)
sage: A.chebyshev_polynomial(5, 'first')  # needs sage.schemes
Dynamical System of Affine Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x) to (16*x^5 - 20*x^3 + 5*x)

sage: A.<x> = AffineSpace(QQ, 1)
sage: A.chebyshev_polynomial(3, 'second')  # needs sage.schemes
Dynamical System of Affine Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x) to (8*x^3 - 4*x)

sage: A.<x> = AffineSpace(QQ, 1)
sage: A.chebyshev_polynomial(3, 2)  # needs sage.schemes
Traceback (most recent call last):
  ... ValueError: keyword 'kind' must have a value of either 'first' or 'second'

sage: A.<x> = AffineSpace(QQ, 1)
# A.chebyshev_polynomial(-4, 'second')
Traceback (most recent call last):
  ... ValueError: first parameter 'n' must be a non-negative integer

sage: A = AffineSpace(QQ, 2, 'x')
sage: A.chebyshev_polynomial(2)
Traceback (most recent call last):
  ... TypeError: affine space must be of dimension 1

sage: A.<x> = AffineSpace(QQ, 1)
sage: A.chebyshev_polynomial(7, monic=True)  # needs sage.schemes
Dynamical System of Affine Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x) to (x^7 - 7*x^5 + 14*x^3 - 7*x)

sage: F.<t> = FunctionField(QQ)
sage: A.<x> = AffineSpace(F, 1)
sage: A.chebyshev_polynomial(4, monic=True)  # needs sage.schemes
Dynamical System of Affine Space of dimension 1 over Rational function field in t over Rational Field
  Defn: Defined on coordinates by sending (x) to (x^4 + (-4)*x^2 + 2)

coordinate_ring()

Return the coordinate ring of this scheme, if defined.

EXAMPLES:

sage: R = AffineSpace(2, GF(9,'alpha'), 'z').coordinate_ring(); R  # needs sage.rings.finite_rings

(continues on next page)
ngens()

Return the number of generators of self, i.e. the number of variables in the coordinate ring of self.

EXAMPLES:

```sage
sage: AffineSpace(3, QQ).ngens()
3
sage: AffineSpace(7, ZZ).ngens()
7
```

origin()

Return the rational point at the origin of this affine space.

EXAMPLES:

```sage
sage: A.<x,y,z> = AffineSpace(QQ, 3)
sage: A.origin()
(0, 0, 0)
sage: _ == A(0,0,0)
True
```

projective_embedding(i=None, PP=None)

Return a morphism from this space into an ambient projective space of the same dimension.

INPUT:

- `i` – integer (default: dimension of self = last coordinate) determines which projective embedding to compute. The embedding is that which has a 1 in the i-th coordinate, numbered from 0.
- `PP` – (default: None) ambient projective space, i.e., codomain of morphism; this is constructed if it is not given.

EXAMPLES:

```sage
sage: AA = AffineSpace(2, QQ, 'x')
sage: pi = AA.projective_embedding(0); pi
Scheme morphism:
  From: Affine Space of dimension 2 over Rational Field
  To:  Projective Space of dimension 2 over Rational Field
  Defn: Defined on coordinates by sending (x0, x1) to (1 : x0 : x1)
sage: z = AA(3, 4)
sage: pi(z)
(1/4 : 3/4 : 1)
sage: pi(AA(0,2))
(1/2 : 0 : 1)
sage: pi = AA.projective_embedding(1); pi
Scheme morphism:
  From: Affine Space of dimension 2 over Rational Field
  To:  Projective Space of dimension 2 over Rational Field
```

(continues on next page)
Defn: Defined on coordinates by sending \((x_0, x_1)\) to \((x_0 : 1 : x_1)\)

```sage
sage: pi(z)
(3/4 : 1/4 : 1)
sage: pi = AA.projective_embedding(2)
sage: pi(z)
(3 : 4 : 1)
sage: A.<x,y> = AffineSpace(ZZ, 2)
sage: A.projective_embedding(2).codomain().affine_patch(2) == A
True
```

**rational_points** *(F=None)*

Return the list of \(F\)-rational points on the affine space self, where \(F\) is a given finite field, or the base ring of self.

**EXAMPLES:**

```sage
sage: A = AffineSpace(1, GF(3))
sage: A.rational_points()
[(0), (1), (2)]
sage: A.rational_points(GF(3^2, 'b'))  # needs sage.rings.finite_rings
[(0), (b), (b + 1), (2*b + 1), (2), (2*b), (2*b + 2), (b + 2), (1)]
sage: AffineSpace(2, ZZ).rational_points(GF(2))
[(0, 0), (0, 1), (1, 0), (1, 1)]
```

**subscheme** *(X, **kwds)*

Return the closed subscheme defined by \(X\).

**INPUT:**

* \(X\) - a list or tuple of equations.

**EXAMPLES:**

```sage
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: X = A.subscheme([x, y^2, x*y^2]); X
Closed subscheme of Affine Space of dimension 2 over Rational Field defined by:
  x,
  y^2,
  x*y^2
sage: # needs sage.libs.singular
sage: X.defining_polynomials ()
(x, y^2, x*y^2)
sage: I = X.defining_ideal(); I
Ideal (x, y^2, x*y^2) of Multivariate Polynomial Ring in x, y over Rational Field
sage: I.groebner_basis()
[y^2, x]
sage: X.dimension()
0
```

(continues on next page)
sage: X.base_ring()
Rational Field
sage: X.base_scheme()
Spectrum of Rational Field
sage: X.structure_morphism()
Scheme morphism:
  From: Closed subscheme of Affine Space of dimension 2 over Rational Field
defined by: x, y^2, x*y^2
  To:   Spectrum of Rational Field
  Defn: Structure map
sage: X.dimension()
0

13.2 Morphisms on affine schemes

This module implements morphisms from affine schemes. A morphism from an affine scheme to an affine scheme is determined by rational functions that define what the morphism does on points in the ambient affine space. A morphism from an affine scheme to a projective scheme is determined by homogeneous polynomials.

EXAMPLES:

sage: A2.<x,y> = AffineSpace(QQ, 2)
sage: P2.<x0,x1,x2> = ProjectiveSpace(QQ, 2)
sage: A2.hom([x, x + y], A2)
Scheme endomorphism of Affine Space of dimension 2 over Rational Field
  Defn: Defined on coordinates by sending (x, y) to (x, x + y)
sage: A2.hom([1, x, x + y], P2)
Scheme morphism:
  From: Affine Space of dimension 2 over Rational Field
  To:   Projective Space of dimension 2 over Rational Field
  Defn: Defined on coordinates by sending (x, y) to (1 : x : x + y)

AUTHORS:

• David Kohel, William Stein: initial version
• Volker Braun (2011-08-08): renamed classes, more documentation, misc cleanups
• Ben Hutz (2013-03): iteration functionality and new directory structure for affine/projective
• Kwankyu Lee (2020-02): added indeterminacy_locus() and image()

class sage.schemes.affine.affine_morphism.SchemeMorphism_polynomial_affine_space(parent, polys, check=True)

Bases: SchemeMorphism_polynomial

A morphism of schemes determined by rational functions.

EXAMPLES:

```
sage: RA.<x,y> = QQ[]
sage: A2 = AffineSpace(RA)
sage: RP.<u,v,w> = QQ[]
sage: P2 = ProjectiveSpace(RP)
sage: H = A2.Hom(P2)
sage: f = H([x, y, 1])
sage: f
Scheme morphism:
From: Affine Space of dimension 2 over Rational Field
To: Projective Space of dimension 2 over Rational Field
Defn: Defined on coordinates by sending (x, y) to (x : y : 1)
```

as_dynamical_system()

Return this endomorphism as a DynamicalSystem_affine.

OUTPUT:

• DynamicalSystem_affine

EXAMPLES:

```
sage: A.<x,y,z> = AffineSpace(ZZ, 3)
sage: H = End(A)
sage: f = H([x^2, y^2, z^2])
sage: type(f.as_dynamical_system())
˓→ needs sage.schemes
<class 'sage.dynamics.arithmetic_dynamics.affine_ds.DynamicalSystem_affine'>
```

```
sage: A.<x,y> = AffineSpace(ZZ, 2)
sage: H = End(A)
sage: f = H([x^2 - y^2, y^2])
sage: type(f.as_dynamical_system())
˓→ needs sage.schemes
<class 'sage.dynamics.arithmetic_dynamics.affine_ds.DynamicalSystem_affine'>
```

```
sage: A.<x> = AffineSpace(GF(5), 1)
sage: H = End(A)
sage: f = H([x^2])
sage: type(f.as_dynamical_system())
˓→ needs sage.schemes
<class 'sage.dynamics.arithmetic_dynamics.affine_ds.DynamicalSystem_affine_finite_field'>
```
sage: P.<x,y> = AffineSpace(RR, 2)
sage: f = DynamicalSystem([x^2 + y^2, y^2], P)  # needs sage.schemes
sage: g = f.as_dynamical_system()  # needs sage.schemes
sage: g is f  # needs sage.schemes
True

default
Return the degree of the affine morphism.

EXAMPLES:

sage: R.<x> = AffineSpace(QQ, 1)
sage: H = Hom(R, R)
sage: f = H([x^7])
sage: f.degree()
7

sage: R.<x,y,z> = AffineSpace(QQ, 3)
sage: H = Hom(R, R)
sage: f = H([x^3, y^2 + 5, z^4 + y])
sage: f.degree()
4

global_height(prec=None)
Take the height of the homogenization, and return the global height of the coefficients as a projective point.

INPUT:

- prec – desired floating point precision (default: default RealField precision).

OUTPUT: A real number.

EXAMPLES:

sage: A.<x> = AffineSpace(QQ, 1)
sage: H = Hom(A, A)
sage: f = H([1/1331*x^2 + 4000])
sage: f.global_height()  # needs sage.symbolic
15.4877354584971

sage: A.<x> = AffineSpace(QQ, 1)
sage: H = Hom(A, A)
sage: f = H([13*w*x^2 + 4*y, 1/w*y^2])
sage: f.global_height(prec=2)
4.0

13.2. Morphisms on affine schemes
sage: A.<x> = AffineSpace(ZZ, 1)
sage: H = Hom(A, A)
sage: f = H([7*x^2 + 1513])
sage: f.global_height()
# needs sage.symbolic
7.32184971378836

sage: A.<x> = AffineSpace(QQ, 1)
sage: B.<y,z> = AffineSpace(QQ, 2)
sage: H = Hom(A, B)
sage: f = H([1/3*x^2 + 10, 7*x^3])
sage: f.global_height()
# needs sage.symbolic
3.40119738166216

sage: P.<x,y> = AffineSpace(QQ, 2)
sage: A.<z> = AffineSpace(QQ, 1)
sage: H = Hom(P, A)
sage: f = H([1/1331*x^2 + 4000*y])
sage: f.global_height()
# needs sage.symbolic
15.4877354584971

**homogenize**(n)

Return the homogenization of this map.

If it’s domain is a subscheme, the domain of the homogenized map is the projective embedding of the
domain. The domain and codomain can be homogenized at different coordinates: n[0] for the domain and
n[1] for the codomain.

**INPUT:**

- n – a tuple of nonnegative integers. If n is an integer, then the two values of the tuple are assumed to be the same

**OUTPUT:** a morphism from the projective embedding of the domain of this map

**EXAMPLES:**

sage: A.<x,y> = AffineSpace(ZZ, 2)
sage: H = Hom(A, A)
sage: f = H([(x^2-2)/x^5, y^2])
sage: f.homogenize(2)
Scheme endomorphism of Projective Space of dimension 2 over Integer Ring
  Defn: Defined on coordinates by sending (x0 : x1 : x2) to
    (x0^2*x2^5 - 2*x2^7 : x0^5*x1^2 : x0^5*x2^2)

sage: # needs sage.rings.real_mpfr
sage: A.<x,y> = AffineSpace(CC, 2)
sage: H = Hom(A, A)
sage: f = H([(x^2-2)/(x*y), y^2 - x])
sage: f.homogenize((2, 0))
Scheme endomorphism of Projective Space of dimension 2
  over Complex Field with 53 bits of precision

(continues on next page)
Defn: Defined on coordinates by sending \((x_0 : x_1 : x_2)\) to
\[
\frac{x_0 \cdot x_1 \cdot x_2^2}{x_0^2 \cdot x_2^2 + (-2.00000000000000) \cdot x_2^4 \cdot x_0^3 - x_0^2 \cdot x_1^2 \cdot x_2}
\]

```
sage: A.<x,y> = AffineSpace(ZZ, 2)  
sage: X = A.subscheme([x - y^2])  
sage: H = Hom(X, X)  
sage: f = H([9*y^2, 3*y])  
sage: f.homogenize(2)  
```

needs sage.libs.singular

Scheme endomorphism of Closed subscheme of Projective Space of dimension 2 over Integer Ring defined by: \(x_1^2 - x_0 \cdot x_2^2\)

Defn: Defined on coordinates by sending \((x_0 : x_1 : x_2)\) to
\[
(9 \cdot x_1^2 : 3 \cdot x_1 \cdot x_2 : x_2^2)
\]

```
sage: R.<t> = PolynomialRing(ZZ)  
sage: A.<x,y> = AffineSpace(R, 2)  
sage: H = Hom(A, A)  
sage: f = H([(x^2-2)/y, y^2 - x])  
sage: f.homogenize((2, 0))  
```

Scheme endomorphism of Projective Space of dimension 2 over Univariate Polynomial Ring in \(t\) over Integer Ring

Defn: Defined on coordinates by sending \((x_0 : x_1 : x_2)\) to
\[
(x_1 \cdot x_2^2 : x_0^2 \cdot x_2 + (-2) \cdot x_2^3 : x_1^3 - x_0 \cdot x_1 \cdot x_2)
\]

```
sage: A.<x> = AffineSpace(QQ, 1)  
sage: H = End(A)  
sage: f = H([x^2 - 1])  
sage: f.homogenize((1, 0))  
```

Scheme endomorphism of Projective Space of dimension 1 over Rational Field

Defn: Defined on coordinates by sending \((x_0 : x_1)\) to
\[
(x_1^2 : x_0^2 - x_1^2)
\]

```
sage: # needs sage.rings.number_field  
sage: R.<a> = PolynomialRing(QQbar)  
sage: A.<x,y> = AffineSpace(R, 2)  
sage: H = End(A)  
sage: f = H([QQbar(sqrt(2))*x*y, a*x^2])  
```

needs sage.symbolic

Scheme endomorphism of Projective Space of dimension 2 over Univariate Polynomial Ring in \(a\) over Algebraic Field

Defn: Defined on coordinates by sending \((x_0 : x_1 : x_2)\) to
\[
(1.414213562373095? \cdot x_0 \cdot x_1 : a \cdot x_0^2 : x_2^2)
\]

```
sage: P.<x,y,z> = AffineSpace(QQ, 3)  
sage: H = End(P)  
sage: f = H([(x^2 - 2)*x*y + z*x, z^2 - y^2 , 5*z*y])  
sage: f.homogenize(2).dehomogenize(2) == f  
```

True
sage: K.<c> = FunctionField(QQ)
sage: A.<x> = AffineSpace(K, 1)
sage: f = Hom(A, A)([x^2 + c])
sage: f.homogenize(1)
Scheme endomorphism of Projective Space of dimension 1
over Rational function field in c over Rational Field
Defn: Defined on coordinates by sending (x0 : x1) to
(x0^2 + c*x1^2 : x1^2)

sage: # needs sage.rings.number_field
sage: A.<z> = AffineSpace(QQbar, 1)
sage: H = End(A)
sage: f = H([2*z / (z^2 + 2*z + 3)])
sage: f.homogenize(1)
Scheme endomorphism of Projective Space of dimension 1
over Algebraic Field
Defn: Defined on coordinates by sending (x0 : x1) to
(x0*x1 : 1/2*x0^2 + x0*x1 + 3/2*x1^2)

sage: # needs sage.rings.number_field
sage: R.<c,d> = QQbar[]
sage: A.<x> = AffineSpace(R, 1)
sage: H = Hom(A, A)
sage: F = H([d*x^2 + c])
sage: F.homogenize(1)
Scheme endomorphism of Projective Space of dimension 1
over Multivariate Polynomial Ring in c, d over Algebraic Field
Defn: Defined on coordinates by sending (x0 : x1) to
(d*x0^2 + c*x1^2 : x1^2)

\textbf{jacobian()}\hspace{1cm}

Return the Jacobian matrix of partial derivative of this map.

The \((i, j)\) entry of the Jacobian matrix is the partial derivative \(\text{diff(functions}[i], \text{variables}[j])\).

\textbf{OUTPUT:}

\begin{itemize}
  \item matrix with coordinates in the coordinate ring of the map.
\end{itemize}

\textbf{EXAMPLES:}

sage: A.<z> = AffineSpace(QQ, 1)
sage: H = End(A)
sage: f = H([z^2 - 3/4])
sage: f.jacobian()  # needs sage.modules
\begin{bmatrix}
2^z
\end{bmatrix}

sage: A.<x,y> = AffineSpace(QQ, 2)
sage: H = End(A)
sage: f = H([x^3 - 25*x + 12*y, 5*y^2*x - 53*y + 24])
sage: f.jacobian()  # needs sage.modules

(continues on next page)
\[
\begin{bmatrix}
3x^2 - 25 & 12 \\
5y^2 & 10xy - 53
\end{bmatrix}
\]

```sage
A.<x,y> = AffineSpace(ZZ, 2)
sage: H = End(A)
sage: f = H([x^2 - xy)/(1+y), (5+y)/(2+x)])
sage: f.jacobian()
\[
\begin{bmatrix}
(2x - y)/(y + 1) & (-x^2 - x)/(y^2 + 2*y + 1)
\end{bmatrix}
\]
```

`local_height(v, prec=None)`

Return the maximum of the local heights of the coefficients in any of the coordinate functions of this map.

**INPUT:**

- `v` – a prime or prime ideal of the base ring.
- `prec` – desired floating point precision (default: default RealField precision).

**OUTPUT:**

- a real number.

**EXAMPLES:**

```sage
P.<x,y> = AffineSpace(QQ, 2)
sage: H = Hom(P, P)
sage: f = H([1/1331*x^2 + 1/4000*y^2, 210*x*y])
sage: f.local_height(1331)
```

```sage
7.19368581839511
```

```sage
P.<x,y,z> = AffineSpace(QQ, 3)
sage: H = Hom(P, P)
sage: f = H([4*x^2 + 3/100*y^2, 8/210*x*y, 1/10000*z^2])
sage: f.local_height(2)
```

```sage
2.77258872223978
```

```sage
P.<x,y,z> = AffineSpace(QQ, 3)
sage: H = Hom(P, P)
sage: f = H([4*x^2 + 3/100*y^2, 8/210*x*y, 1/10000*z^2])
sage: f.local_height(2, prec=2)
```

```sage
3.0
```

```sage
# needs sage.rings.number_field
R.<z> = PolynomialRing(QQ)
sage: K.<w> = NumberField(z^2 - 2)
sage: P.<x,y> = AffineSpace(K, 2)
sage: H = Hom(P, P)
sage: f = H([2*x^2 + w/3*y^2, 1/w*y^2])
```

(continues on next page)
local_height_arch(i, prec=None)

Return the maximum of the local height at the \(i\)-th infinite place of the coefficients in any of the coordinate functions of this map.

INPUT:

- \(i\) – an integer.
- \(prec\) – desired floating point precision (default: default RealField precision).

OUTPUT:

- a real number.

EXAMPLES:

```
sage: P.<x,y> = AffineSpace(QQ, 2)
sage: H = Hom(P, P)
sage: f = H([1/1331*x^2 + 1/4000*y^2, 210*x*y]);
sage: f.local_height_arch(0)  # needs sage.rings.real_mpfr
5.34710753071747
```

```
sage: P.<x,y> = AffineSpace(QQ, 2)
sage: H = Hom(P, P)
sage: f = H([1/1331*x^2 + 1/4000*y^2, 210*x*y]);
sage: f.local_height_arch(0, prec=5)  # needs sage.rings.real_mpfr
5.2
```

```
sage: # needs sage.rings.number_field
sage: R.<z> = PolynomialRing(QQ)
sage: K.<w> = NumberField(z^2 - 2)
sage: P.<x,y> = AffineSpace(K, 2)
sage: H = Hom(P, P)
sage: f = H([2*x^2 + w/3*y^2, 1/w*y^2])
sage: f.local_height_arch(1)
0.6931471805599453094172321214582
```

class sage.schemes.affine.affine_morphism.SchemeMorphism_polynomial_affine_space_field(parent, polys, check=True)

Bases: SchemeMorphism_polynomial_affine_space

image()

Return the scheme-theoretic image of the morphism.

OUTPUT: a subscheme of the ambient space of the codomain

EXAMPLES:
```python
sage: # needs sage.libs.singular
sage: A1.<w> = AffineSpace(QQ, 1)
sage: A2.<x,y> = AffineSpace(QQ, 2)
sage: f = A2.hom([x + y], A1)
sage: f.image()
Closed subscheme of Affine Space of dimension 1 over Rational Field defined by:
(no polynomials)
sage: f = A2.hom([x, x], A2)
sage: f.image()
Closed subscheme of Affine Space of dimension 2 over Rational Field defined by:
x - y
sage: f = A2.hom([x^2, x^3], A2)
sage: f.image()
Closed subscheme of Affine Space of dimension 2 over Rational Field defined by:
x^3 - y^2
sage: P2.<x0,x1,x2> = ProjectiveSpace(QQ, 2)
sage: f = A2.hom([x, x^2, x^3], P2)
sage: f.image()
Closed subscheme of Projective Space of dimension 2 over Rational Field defined by:
x1^2 - x0*x2
```

**indeterminacy_locus()**

Return the indeterminacy locus of this map as a rational map on the domain.

The indeterminacy locus is the intersection of all the base indeterminacy locuses of maps that define the same rational map as by this map.

OUTPUT: a subscheme of the domain of the map

EXAMPLES:

```python
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: H = End(A)
sage: f = H([x - y, x^2 - y^2])
sage: f.indeterminacy_locus()  # needs sage.libs.singular
Closed subscheme of Affine Space of dimension 2 over Rational Field defined by: 1
```

```python
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: f = A.hom([x, x/y], A)
sage: f.indeterminacy_locus()  # needs sage.libs.singular
Closed subscheme of Affine Space of dimension 2 over Rational Field defined by: y
```

**indeterminacy_points(F=None)**

Return the points in the indeterminacy locus of this map.

If the dimension of the indeterminacy locus is not zero, an error is raised.

INPUT:

- F – a field; if not given, the base ring of the domain is assumed

13.2. Morphisms on affine schemes
OUTPUT: indeterminacy points of the map defined over F

EXAMPLES:

```python
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: H = End(A)
sage: f = H([x - y, x^2 - y^2])
sage: f.indeterminacy_points()  # needs sage.libs.singular
[]
```

```python
sage: A2.<x,y> = AffineSpace(QQ, 2)
```

```python
sage: P2.<x0,x1,x2> = ProjectiveSpace(QQ, 2)
```

```python
sage: f = A2.hom([x*y, y, x], P2)
sage: f.indeterminacy_points()  # needs sage.libs.singular

```

```python
[(0, 0)]
```

`reduce_base_field()`

Return this map defined over the field of definition of the coefficients.

The base field of the map could be strictly larger than the field where all of the coefficients are defined. This function reduces the base field to the minimal possible. This can be done when the base ring is a number field, QQbar, a finite field, or algebraic closure of a finite field.

OUTPUT: a scheme morphism

EXAMPLES:

```python
sage: # needs sage.rings.finite_rings
sage: K.<t> = GF(5^4)
sage: A.<x> = AffineSpace(K, 1)
sage: A2.<a,b> = AffineSpace(K, 2)
sage: H = End(A)
sage: H2 = Hom(A, A2)
sage: H3 = Hom(A2, A)
sage: f = H([x^2 + 2*(t^3 + t^2 + t + 3)])
sage: f.reduce_base_field()
Scheme endomorphism of Affine Space of dimension 1 over Finite Field in t2 of size 5^2
Defn: Defined on coordinates by sending (x) to (x^2 + (2*t2))
sage: f2 = H2([x^2 + 4, 2*x])
sage: f2.reduce_base_field()
Scheme morphism:
From: Affine Space of dimension 1 over Finite Field in t of size 5
To: Affine Space of dimension 2 over Finite Field of size 5
Defn: Defined on coordinates by sending (x) to (x^2 - 1, 2*x)
sage: f3 = H3([a^2 + t*b])
sage: f3.reduce_base_field()
Scheme morphism:
From: Affine Space of dimension 2 over Finite Field in t of size 5^4
To: Affine Space of dimension 1 over Finite Field in t of size 5^4
Defn: Defined on coordinates by sending (a, b) to (a^2 + t*b)
```
sage: # needs sage.rings.number_field
sage: K.<v> = CyclotomicField(4)
sage: A.<x> = AffineSpace(K, 1)
sage: H = End(A)
sage: f = H([x^2 + v])

Scheme endomorphism of Affine Space of dimension 1 over Cyclotomic Field of order 4 and degree 2
Defn: Defined on coordinates by sending (x) to (x^2 + v)
sage: g = f.reduce_base_field(); g
Scheme endomorphism of Affine Space of dimension 1 over Cyclotomic Field of order 4 and degree 2
Defn: Defined on coordinates by sending (x) to (x^2 + v)
sage: g.base_ring() is K
True

sage: # needs sage.rings.number_field
sage: A.<x> = AffineSpace(QQbar, 1)
sage: H = End(A)
sage: f = H([(QQbar(sqrt(2))*x^2 + 1/QQbar(sqrt(3))) / (5*x)])

Scheme endomorphism of Affine Space of dimension 1 over Number Field in a with defining polynomial y^4 - 4*y^2 + 1 with a = ...?
Defn: Defined on coordinates by sending (x) to (((a^3 - 3*a)*x^2 + (-1/3*a^2 + 2/3))/(5*x))
To: Affine Space of dimension 2 over Number Field in a with defining polynomial $x^3 - x + 1$ with $a = -1.324717957244746$?
Defn: Defined on coordinates by sending $(x)$ to $(x^2 + a*x + 3, 5*x)$

```
sage: # needs sage.rings.number_field
sage: K.<v> = QuadraticField(2)
sage: A.<x> = AffineSpace(K, 1)
sage: H = End(A)
sage: f = H([3*x^2 + x + 1])
sage: f.reduce_base_field()
Scheme endomorphism of Affine Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x) to (3*x^2 + x + 1)
```

weil_restriction()
Compute the Weil restriction of this morphism over some extension field.

If the field is a finite field, then this computes the Weil restriction to the prime subfield.

A Weil restriction of scalars - denoted $\text{Res}_{L/k}$ - is a functor which, for any finite extension of fields $L/k$ and any algebraic variety $X$ over $L$, produces another corresponding variety $\text{Res}_{L/k}(X)$, defined over $k$. It is useful for reducing questions about varieties over large fields to questions about more complicated varieties over smaller fields. Since it is a functor it also applied to morphisms. In particular, the functor applied to a morphism gives the equivalent morphism from the Weil restriction of the domain to the Weil restriction of the codomain.

OUTPUT: Scheme morphism on the Weil restrictions of the domain and codomain of the map.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: K.<v> = QuadraticField(5)
sage: A.<x,y> = AffineSpace(K, 2)
sage: H = End(A)
sage: f = H([x^2 - y^2, y^2])
sage: f.weil_restriction()  # needs sage.libs.singular
Scheme endomorphism of Affine Space of dimension 4 over Rational Field
  Defn: Defined on coordinates by sending (z0, z1, z2, z3) to
    (z0^2 + 5*z1^2 - z2^2 - 5*z3^2, 2*z0*z1 - 2*z2*z3, z2^2 + 5*z3^2, 2*z2*z3)
```
```python
sage: # needs sage.rings.number_field
sage: K.<v> = QuadraticField(5)
sage: PS.<x,y> = AffineSpace(K, 2)
sage: H = Hom(PS, PS)
sage: f = H([x, y])
sage: F = f.weil_restriction()
sage: P = PS(2, 1)
sage: Q = P.weil_restriction()
sage: f(P).weil_restriction() == F(Q)
# needs sage.libs.singular
True
```

class sage.schemes.affine.affine_morphism.SchemeMorphism_polynomial_affine_space_finite_field(parent, polys, check=True):

Bases: SchemeMorphism_polynomial_affine_space_field

class sage.schemes.affine.affine_morphism.SchemeMorphism_polynomial_affine_subscheme_field(parent, polys, check=True):

Bases: SchemeMorphism_polynomial_affine_space_field

Morphisms from subschemes of affine spaces defined over fields.

image()

Return the scheme-theoretic image of the morphism.

OUTPUT: a subscheme of the ambient space of the codomain

EXAMPLES:

```python
sage: A1.<w> = AffineSpace(QQ, 1)
sage: A2.<x,y> = AffineSpace(QQ, 2)
sage: X = A2.subscheme([0])
sage: f = X.hom([x + y], A1)
sage: f.image()
# needs sage.libs.singular
Closed subscheme of Affine Space of dimension 1 over Rational Field defined by:
(no polynomials)
```

```python
sage: A2.<x,y> = AffineSpace(QQ, 2)
sage: X = A2.subscheme([x*y^2 - y^3 - 1])
sage: f = X.hom([y, y/x], A2)
sage: f.image()
# needs sage.libs.singular
Closed subscheme of Affine Space of dimension 2 over Rational Field defined by:
-x^3*y + x^3 - y
```

indeterminacy_locus()

Return the indeterminacy locus of this map.

The map defines a rational map on the domain. The output is the subscheme of the domain on which the rational map is not defined by any representative of the rational map. See representatives().

EXAMPLES:
```python
sage: A2.<x1,x2> = AffineSpace(QQ, 2)
sage: X = A2.subscheme(0)
sage: A1.<x> = AffineSpace(QQ, 1)
sage: f = X.hom([x1/x2], A1)
sage: f.indeterminacy_locus()  # needs sage.libs.singular
Closed subscheme of Affine Space of dimension 2 over Rational Field defined by:

sage: A2.<x1,x2> = AffineSpace(QQ, 2)
sage: X = A2.subscheme(0)
sage: P1.<a,b> = ProjectiveSpace(QQ, 1)
sage: f = X.hom([x1,x2], P1)
sage: L = f.indeterminacy_locus()  # needs sage.libs.singular
sage: L.rational_points()  # needs sage.libs.singular
[(0, 0)]
```

```python
sage: A2.<x,y> = AffineSpace(QQ, 2)
sage: X = A2.subscheme([x^2 - y^2 - y])
sage: A1.<a> = AffineSpace(QQ, 1)
sage: f = X.hom([x/y], A1)
sage: f.indeterminacy_locus()  # needs sage.libs.singular
Closed subscheme of Affine Space of dimension 2 over Rational Field defined by:

sage: A3.<x,y,z> = AffineSpace(QQ, 3)
sage: X = A3.subscheme(x^2 - y*z - x)
sage: A2.<a,b> = AffineSpace(QQ, 2)
sage: f = X.hom([y, y/x], A2)
sage: L = f.indeterminacy_locus(); L  # needs sage.libs.singular
Closed subscheme of Affine Space of dimension 3 over Rational Field defined by:

is_morphism()

Return True if the map is defined everywhere on the domain.

EXAMPLES:

```python
sage: P2.<x,y,z> = ProjectiveSpace(QQ,2)
sage: P1.<a,b> = ProjectiveSpace(QQ,1)
sage: X = P2.subscheme([x^2 - y^2 - y*z])
sage: f = X.hom([x,y], P1)
sage: f.is_morphism()  # needs sage.libs.singular
(continues on next page)
```
representatives()

Return all maps representing the same rational map as by this map.

EXAMPLES:

```sage
sage: A2.<x,y> = AffineSpace(QQ, 2)
sage: X = A2.subscheme(0)
sage: f = X.hom([x, x/y], A2)
sage: f.representatives()
[Scheme morphism:
  From: Closed subscheme of Affine Space of dimension 2 over Rational Field
defined by: 0
  To: Affine Space of dimension 2 over Rational Field
  Defn: Defined on coordinates by sending (x, y) to (x, x/y)]
```

```sage
sage: # needs sage.libs.singular
sage: A2.<x,y> = AffineSpace(QQ, 2)
sage: A1.<a> = AffineSpace(QQ, 1)
sage: X = A2.subscheme([x^2 - y^2 - y])
sage: f = X.hom([x/y], A1)
sage: f.representatives()
[Scheme morphism:
  From: Closed subscheme of Affine Space of dimension 2 over Rational Field
defined by: x^2 - y^2 - y
  To: Affine Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x, y) to (x/y),
Scheme morphism:
  From: Closed subscheme of Affine Space of dimension 2 over Rational Field
defined by: x^2 - y^2 - y
  To: Affine Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x, y) to ((y + 1)/x)]
```

```sage
g = _[1]
sage: g.representatives()
[Scheme morphism:
  From: Closed subscheme of Affine Space of dimension 2 over Rational Field
defined by: x^2 - y^2 - y
  To: Affine Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x, y) to (x/y),
Scheme morphism:
  From: Closed subscheme of Affine Space of dimension 2 over Rational Field
defined by: x^2 - y^2 - y
  To: Affine Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x, y) to ((y + 1)/x)]
```

```sage
sage: A2.<x,y> = AffineSpace(QQ, 2)
sage: P1.<a,b> = ProjectiveSpace(QQ, 1)
sage: X = A2.subscheme([x^2 - y^2 - y])
sage: f = X.hom([x, y], P1)
```
13.3 Points on affine varieties

Scheme morphism for points on affine varieties.

AUTHORS:

• David Kohel, William Stein
• Volker Braun (2011-08-08): Renamed classes, more documentation, misc cleanups.
• Ben Hutz (2013)

class sage.schemes.affine.affine_point.SchemeMorphism_point_affine(X, v, check=True)

A rational point on an affine scheme.

INPUT:

• X – a subscheme of an ambient affine space over a ring \( R \)
• v – a list/tuple/iterable of coordinates in \( R \)
• check – boolean (optional, default:True); whether to check the input for consistency

EXAMPLES:

```
sage: A = AffineSpace(2, QQ)
sage: A(1, 2)
(1, 2)
```

global_height(prec=None)

Returns the logarithmic height of the point.

INPUT:

• prec – desired floating point precision (default: default RealField precision).

OUTPUT:

• a real number.

EXAMPLES:
Todo: P-adic heights.

**homogenize**($n$)

Return the homogenization of the point at the $n$th coordinate.

**INPUT:**

- $n$ – integer between 0 and dimension of the map, inclusive.

**OUTPUT:**

- A point in the projectivization of the codomain of the map.

**EXAMPLES:**

```python
sage: A.<x,y> = AffineSpace(ZZ, 2)
sage: Q = A(2, 3)
sage: Q.homogenize(2).dehomogenize(2) == Q
True
::

sage: A.<x,y> = AffineSpace(QQ, 2)
sage: Q = A(2, 3)
sage: P = A(0, 1)
sage: Q.homogenize(2).codomain() == P.homogenize(2).codomain()
True
```

class sage.schemes.affine.affine_point.SchemeMorphism_point_affine_field($X$, $v$, check=True)

**Bases:** SchemeMorphism_point_affine

**intersection_multiplicity**($X$)

Return the intersection multiplicity of the codomain of this point and $X$ at this point.
This uses the intersection_multiplicity implementations for projective/affine subschemes. This point must be a point on an affine subscheme.

INPUT:

• \( X \) – a subscheme in the same ambient space as that of the codomain of this point.

OUTPUT: Integer.

EXAMPLES:

```python
sage: # needs sage.libs.singular
sage: A.<x,y> = AffineSpace(GF(17), 2)
sage: X = A.subscheme([y^2 - x^3 + 2*x^2 - x])
sage: Y = A.subscheme([y - 2*x + 2])
sage: Q1 = Y([1,0])
sage: Q1.intersection_multiplicity(X)
2
sage: Q2 = X([4,6])
sage: Q2.intersection_multiplicity(Y)
1
```

```python
sage: A.<x,y,z,w> = AffineSpace(QQ, 4)
sage: X = A.subscheme([x^2 - y*z^2, z - 2*w^2])
sage: Q = A([2,1,2,-1])
sage: Q.intersection_multiplicity(X)
Traceback (most recent call last):
  ...TypeError: this point must be a point on an affine subscheme
```

**multiplicity()**

Return the multiplicity of this point on its codomain.

Uses the subscheme multiplicity implementation. This point must be a point on an affine subscheme.

OUTPUT: an integer.

EXAMPLES:

```python
sage: A.<x,y,z> = AffineSpace(QQ, 3)
sage: X = A.subscheme([y^2 - x^7*z])
sage: Q1 = X([1,1,1])
sage: Q1.multiplicity()# needs sage.libs.singular
1
sage: Q2 = X([0,0,2])
sage: Q2.multiplicity()# needs sage.libs.singular
2
```

**weil_restriction()**

Compute the Weil restriction of this point over some extension field.

If the field is a finite field, then this computes the Weil restriction to the prime subfield.

A Weil restriction of scalars - denoted \( \text{Res}_{L/k} \) - is a functor which, for any finite extension of fields \( L/k \) and any algebraic variety \( X \) over \( L \), produces another corresponding variety \( \text{Res}_{L/k}(X) \), defined over \( k \). It is useful for reducing questions about varieties over large fields to questions about more complicated varieties

---

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over smaller fields. This functor applied to a point gives the equivalent point on the Weil restriction of its

codomain.

OUTPUT: Scheme point on the Weil restriction of the codomain of this point.

EXAMPLES:

```sage
sage: # needs sage.libs.singular sage.rings.finite_rings
sage: A.<x,y,z> = AffineSpace(GF(5^3, 't'), 3)
sage: X = A.subscheme([y^2 - x*z, z^2 + y])
sage: Y = X.weil_restriction()
sage: P = X([1, -1, 1])
sage: Q = P.weil_restriction(); Q
(1, 0, 0, 4, 0, 0, 1, 0, 0)
sage: Q.codomain() == Y
True
```

class sage.schemes.affine.affine_point.SchemeMorphism_point_affine_finite_field(X, v, check=True)

Bases: SchemeMorphism_point_affine_field

13.4 Subschemes of affine space

AUTHORS:

• David Kohel, William Stein (2005): initial version

• Ben Hutz (2013): affine subschemes

class sage.schemes.affine.affine_subscheme.AlgebraicScheme_subscheme_affine(A, polynomials, embedding_center=None, embedding_codomain=None, embedding_images=None)

Bases: AlgebraicScheme_subscheme

An algebraic subscheme of affine space.

INPUT:

• `A` – ambient affine space

• `polynomials` – single polynomial, ideal or iterable of defining polynomials

EXAMPLES:
sage: A3.<x, y, z> = AffineSpace(QQ, 3)
sage: A3.subscheme([x^2 - y*z])
Closed subscheme of Affine Space of dimension 3 over Rational Field defined by:
  x^2 - y*z

dimension()

Return the dimension of the affine algebraic subscheme.

EXAMPLES:

sage: # needs sage.libs.singular
sage: A.<x,y> = AffineSpace(2, QQ)
sage: A.subscheme([]).dimension()
2
sage: A.subscheme([x]).dimension()
1
sage: A.subscheme([x^5]).dimension()
1
sage: A.subscheme([x^2 + y^2 - 1]).dimension()
1
sage: A.subscheme([x*(x-1), y*(y-1)]).dimension()
0

Something less obvious:

sage: A.<x,y,z,w> = AffineSpace(4, QQ)
sage: X = A.subscheme([x^2, x^2*y^2 + z^2, z^2 - w^2, 10*x^2 + w^2 - z^2])
sage: X
Closed subscheme of Affine Space of dimension 4 over Rational Field defined by:
  x^2,
x^2*y^2 + z^2,
z^2 - w^2,
10*x^2 - z^2 + w^2
sage: X.dimension()              # needs sage.libs.singular
1

intersection_multiplicity(X, P)

Return the intersection multiplicity of this subscheme and the subscheme X at the point P.

The intersection of this subscheme with X must be proper, that is codim(self ∩ X) =
codim(self) + codim(X), and must also be finite. We use Serre’s Tor formula to compute
the intersection multiplicity. If I, J are the defining ideals of self, X, respectively, then this is
∑ᵢ₌₀⁻¹⁻¹length(Torᵢ^{O_A}(O_A/I, O_A/J)) where A is the affine ambient space of these subschemes.

INPUT:

• X – subscheme in the same ambient space as this subscheme.
• P – a point in the intersection of this subscheme with X.

OUTPUT: An integer.

EXAMPLES:
```
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = Curve([y^2 - x^3 - x^2], A)  # needs sage.libs.singular
sage: D = Curve([y^2 + x^3], A)  # needs sage.libs.singular
sage: Q = A([0,0])
sage: C.intersection_multiplicity(D, Q)  # needs sage.libs.singular
4

sage: # needs sage.rings.number_field
sage: R.<a> = QQ[]
sage: K.<b> = NumberField(a^6 - 3*a^5 + 5*a^4 - 5*a^3 + 5*a^2 - 3*a + 1)
sage: A.<x,y,z,w> = AffineSpace(K, 4)
sage: X = A.subscheme([x*y, y*z + 7, w^3 - x^3])
sage: Y = A.subscheme([x - z^3 + z + 1])
sage: Q = A([0,...
....: -7*b^5 + 21*b^4 - 28*b^3 + 21*b^2 - 21*b + 14,
....: -b^5 + 2*b^4 - 3*b^3 + 2*b^2 - 2*b,
....: 0])
sage: X.intersection_multiplicity(Y, Q)  # needs sage.intersection_multiplicity
3

sage: A.<x,y,z> = AffineSpace(QQ, 3)
sage: X = A.subscheme([z^2 - 1])
sage: Y = A.subscheme([z - 1, y - x^2])
sage: Q = A([1,1,1])
--- needs sage.intersection_multiplicity
Traceback (most recent call last):
...
TypeError: the intersection of this subscheme and (=Closed subscheme of AffineSpace of dimension 3 over Rational Field defined by: z - 1, -x^2 + y) must be proper and finite

sage: A.<x,y,z,w,t> = AffineSpace(QQ, 5)
sage: X = A.subscheme([x*y, t^2*w, w^3*z])
sage: Y = A.subscheme([y*w + z])
sage: Q = A([0,...
--- needs sage.intersection_multiplicity
Traceback (most recent call last):
...
TypeError: the intersection of this subscheme and (=Closed subscheme of AffineSpace of dimension 5 over Rational Field defined by: y*w + z) must be proper and finite
```

**is_smooth** *(point=None)*

Test whether the algebraic subscheme is smooth.

**INPUT:**
• point – A point or None (default). The point to test smoothness at.

OUTPUT:
Boolean. If no point was specified, returns whether the algebraic subscheme is smooth everywhere. Otherwise, smoothness at the specified point is tested.

EXAMPLES:

sage: A2.<x,y> = AffineSpace(2, QQ)
sage: cuspidal_curve = A2.subscheme([y^2 - x^3])
sage: cuspidal_curve
Closed subscheme of Affine Space of dimension 2 over Rational Field defined by:
-x^3 + y^2
sage: smooth_point = cuspidal_curve.point([1,1])
sage: smooth_point in cuspidal_curve
True
sage: singular_point = cuspidal_curve.point([0,0])
sage: singular_point in cuspidal_curve
True
sage: cuspidal_curve.is_smooth(smooth_point)
# needs sage.libs.singular
True
sage: cuspidal_curve.is_smooth(singular_point)
# needs sage.libs.singular
False
sage: cuspidal_curve.is_smooth()
# needs sage.libs.singular
False

multiplicity(P)
Return the multiplicity of P on this subscheme.

This is computed as the multiplicity of the local ring of this subscheme corresponding to P. This subscheme must be defined over a field. An error is raised if P is not a point on this subscheme.

INPUT:
• P – a point on this subscheme.

OUTPUT:
An integer.

EXAMPLES:

sage: A.<x,y,z,w> = AffineSpace(QQ, 4)
sage: X = A.subscheme([z*y - x^7, w - 2*z])
sage: Q1 = A([1,1/3,3,6])
sage: XMultiplicity(Q1)
# needs sage.libs.singular
1
sage: Q2 = A([0,0,0,0])
sage: XMultiplicity(Q2)
# needs sage.libs.singular
2
```python
sage: A.<x,y,z,w,v> = AffineSpace(GF(23), 5)
sage: C = A.curve([x^8 - y, y^7 - z, z^3 - 1, w^5 - v^3])
# needs sage.libs.singular sage.schemes
sage: Q = A([22,1,1,0,0])
sage: C.multiplicity(Q)
# needs sage.libs.singular sage.schemes
3
```

```python
sage: K.<a> = QuadraticField(-1)
sage: A.<x,y,z,w,t> = AffineSpace(K, 5)
sage: X = A.subscheme([y^7 - x^2*z^5 + z^3*t^8 - x^2*y^4*z - t^8])
sage: Q1 = A([1,1,0,1,-1])
sage: X.multiplicity(Q1)
# needs sage.libs.singular
1
sage: Q2 = A([0,0,0,-a,0])
sage: X.multiplicity(Q2)
# needs sage.libs.singular
7
```

Check that github issue #27479 is fixed:

```python
sage: A1.<x> = AffineSpace(QQ, 1)
sage: X = A1.subscheme([x^1789 + x])
sage: Q = X([0])
sage: X.multiplicity(Q)
# needs sage.libs.singular
1
```

**projective_closure**(i=None, PP=None)

Return the projective closure of this affine subscheme.

INPUT:

- i – (default: None) determines the embedding to use to compute the projective closure of this affine subscheme. The embedding used is the one which has a 1 in the i-th coordinate, numbered from 0.
- PP – (default: None) ambient projective space, i.e., ambient space of codomain of morphism; this is constructed if it is not given

OUTPUT: a projective subscheme

EXAMPLES:

```python
sage: A.<x,y,z,w> = AffineSpace(QQ, 4)
sage: X = A.subscheme([x^2 - y, x*y - z, y^2 - w,
....:                   x*z - w, y*z - x*w, z^2 - y*w])
sage: X.projective_closure()   # needs sage.libs.singular
Closed subscheme of Projective Space of dimension 4 over Rational Field
defined by:
    x0^2  -  x1*x4,  
x0*x1  -  x2*x4,
x1^2  -  x3*x4,
```

(continues on next page)
\[ x_0^2x_2 - x_3^2x_4, \]
\[ x_1^2x_2 - x_0^2x_3, \]
\[ x_2^2 - x_1x_3 \]

sage: \( \text{A.<x,y,z> = AffineSpace(QQ, 3)} \)
sage: \( \text{P.<a,b,c,d> = ProjectiveSpace(QQ, 3)} \)
sage: \( X = \text{A.subscheme([z - x^2 - y^2])} \)
sage: \( X.\text{projective_closure(1, P).ambient_space()} \) == P
   #→ needs sage.libs.singular
True

\textbf{projective\_embedding}(i=None, PP=None)

Return a morphism from this affine scheme into an ambient projective space of the same dimension.

The codomain of this morphism is the projective closure of this affine scheme in \( PP \), if given, or otherwise in a new projective space that is constructed.

INPUT:

- \( i \) – integer (default: dimension of self = last coordinate) determines which projective embedding to compute. The embedding is that which has a 1 in the \( i \)-th coordinate, numbered from 0.

- \( PP = \) (default: None) ambient projective space, i.e., ambient space of codomain of morphism; this is constructed if it is not given.

EXAMPLES:

\begin{verbatim}
sage: \( \text{A.<x, y, z> = AffineSpace(3, ZZ)} \)
sage: \( S = \text{A.subscheme([x*y - z])} \)
sage: \( S.\text{projective_embedding()} \)  #→ needs sage.libs.singular
Scheme morphism:
   From: Closed subscheme of Affine Space of dimension 3 over Integer Ring defined by: x*y - z
   To:  Closed subscheme of Projective Space of dimension 3 over Integer Ring defined by: x0*x1 - x2*x3
   Defn: Defined on coordinates by sending (x, y, z) to (x : y : z : 1)
\end{verbatim}

\begin{verbatim}
sage: \( \text{A.<x, y, z> = AffineSpace(3, ZZ)} \)
sage: \( P = \text{ProjectiveSpace(3, ZZ, 'u')} \)
sage: \( S = \text{A.subscheme([x^2 - y^2*z])} \)
sage: \( S.\text{projective_embedding(1, P)} \)  #→ needs sage.libs.singular
Scheme morphism:
   From: Closed subscheme of Affine Space of dimension 3 over Integer Ring defined by: x^2 - y^2*z
   To:  Closed subscheme of Projective Space of dimension 3 over Integer Ring defined by: u0^2 - u2*u3
   Defn: Defined on coordinates by sending (x, y, z) to (x : 1 : y : z)
\end{verbatim}

\begin{verbatim}
sage: \( \text{A.<x,y,z> = AffineSpace(QQ, 3)} \)
sage: \( X = \text{A.subscheme([y - x^2, z - x^3])} \)
sage: \( X.\text{projective_embedding()} \)  #→ needs sage.libs.singular
\end{verbatim}
Scheme morphism:
  From: Closed subscheme of Affine Space of dimension 3 over Rational Field
defined by: -x^2 + y, -x^3 + z
  To: Closed subscheme of Projective Space of dimension 3 over Rational Field
defined by: x0^2 - x1*x3, x0*x1 - x2*x3, x1^2 - x0*x2
  Defn: Defined on coordinates by sending (x, y, z) to (x : y : z : 1)

When taking a closed subscheme of an affine space with a projective embedding, the subscheme inherits
the embedding:

```python
sage: A.<u,v> = AffineSpace(2, QQ, default_embedding_index=1)
sage: X = A.subscheme(u - v)  # needs sage.libs.singular
sage: X.projective_embedding()  # needs sage.libs.singular
Scheme morphism:
  From: Closed subscheme of Affine Space of dimension 2 over Rational Field
defined by: u - v
  To: Closed subscheme of Projective Space of dimension 2 over Rational Field
defined by: x0 - x2
  Defn: Defined on coordinates by sending (u, v) to (u : 1 : v)
sage: phi = X.projective_embedding()  # needs sage.libs.singular
sage: psi = A.projective_embedding()
sage: phi(X(2, 2)) == psi(A(X(2, 2)))  # needs sage.libs.singular
True
```

### class `sage.schemes.affine.affine_subscheme.AlgebraicScheme_subscheme_affine_field`:

Bases: `AlgebraicScheme_subscheme_affine`

Algebraic subschemes of projective spaces defined over fields.

**tangent_space**(*p*)

Return the tangent space at the point *p*.

The points of the tangent space are the tangent vectors at *p*.

**INPUT:**

- *p* – a rational point

**EXAMPLES:**

```python
sage: A3.<x,y,z> = AffineSpace(3, QQ)
sage: X = A3.subscheme(z - x*y)
sage: X.tangent_space(A3.origin())  # needs sage.libs.singular
```
Closed subscheme of Affine Space of dimension 3 over Rational Field defined by:
\[z\]
\[\text{sage: X.tangent_space(X(1,1,1))} \quad \texttt{# needs sage.libs.singular}\]
Closed subscheme of Affine Space of dimension 3 over Rational Field defined by:
\[-x - y + z\]

Tangent space at a point may have higher dimension than the dimension of the point.

\[\text{sage: # needs sage.libs.singular}\]
\[\text{sage: C = Curve([x + y + z, x^2 - y^2*z^2 + z^3])}\]
\[\text{sage: C.singular_points()}\]
\[\{(0, 0, 0)\}\]
\[\text{sage: p = C(0,0,0)}\]
\[\text{sage: C.tangent_space(p)}\]
Closed subscheme of Affine Space of dimension 3 over Rational Field defined by:
\[x + y + z\]
\[\text{sage: _.dimension()}\]
\[2\]
\[\text{sage: q = C(1,0,-1)}\]
\[\text{sage: C.tangent_space(q)}\]
Closed subscheme of Affine Space of dimension 3 over Rational Field defined by:
\[x + y + z,\]
\[2x + 3z\]
\[\text{sage: _.dimension()}\]
\[1\]

13.5 Enumeration of rational points on affine schemes

Naive algorithms for enumerating rational points over \(\mathbb{Q}\) or finite fields over for general schemes.

**Warning:** Incorrect results and infinite loops may occur if using a wrong function.

(For instance using an affine function for a projective scheme or a finite field function for a scheme defined over an infinite field.)

**EXAMPLES:**

Affine, over \(\mathbb{Q}\):

\[\text{sage: from sage.schemes.affine.affine_rational_point import enum_affine_rational_field}\]
\[\text{sage: A.<x,y,z> = AffineSpace(3, QQ)}\]
\[\text{sage: S = A.subscheme([2^x - 3^y])}\]
\[\text{sage: enum_affine_rational_field(S, 2)}\]
\[\{(0, 0, -2), (0, 0, -1), (0, 0, -1/2), (0, 0, 0), (0, 0, 1/2), (0, 0, 1), (0, 0, 2)\} \]
Affine over a finite field:

```python
sage: from sage.schemes.affine.affine_rational_point import enum_affine_finite_field
sage: A.<w,x,y,z> = AffineSpace(4, GF(2))
sage: enum_affine_finite_field(A(GF(2)))
[(0, 0, 0, 0), (0, 0, 0, 1), (0, 0, 1, 0), (0, 0, 1, 1), (0, 1, 0, 0),
 (0, 1, 0, 1), (0, 1, 1, 0), (0, 1, 1, 1), (1, 0, 0, 0), (1, 0, 0, 1),
 (1, 0, 1, 0), (1, 0, 1, 1), (1, 1, 0, 0), (1, 1, 0, 1), (1, 1, 1, 0),
 (1, 1, 1, 1)]
```

AUTHORS:

- David R. Kohel <kohel@maths.usyd.edu.au>: original version.
- John Cremona and Charlie Turner <charlotteturner@gmail.com> (06-2010): improvements to clarity and documentation.

`sage.schemes.affine.affine_rational_point.enum_affine_finite_field(X)`

Enumerates affine points on scheme $X$ defined over a finite field.

**INPUT:**

- $X$ - a scheme defined over a finite field or a set of abstract rational points of such a scheme.

**OUTPUT:**

- a list containing the affine points of $X$ over the finite field, sorted.

**EXAMPLES:**

```python
sage: from sage.schemes.affine.affine_rational_point import enum_affine_finite_field
sage: F = GF(7)
sage: A.<w,x,y,z> = AffineSpace(4, F)
sage: C = A.subscheme([w^2 + x + 4, y*z*x - 6, z*y + w*x])
sage: enum_affine_finite_field(C(F))
[]
sage: C = A.subscheme([w^2 + x + 4, y*z*x - 6])
sage: enum_affine_finite_field(C(F))
[(0, 3, 1, 2), (0, 3, 2, 1), (0, 3, 3, 3), (0, 3, 4, 4), (0, 3, 5, 6),
 (0, 3, 6, 5), (1, 2, 1, 3), (1, 2, 2, 5), (1, 2, 3, 1), (1, 2, 4, 6),
 (1, 2, 5, 2), (1, 2, 6, 4), (2, 6, 1, 1), (2, 6, 2, 4), (2, 6, 3, 5),
 (2, 6, 4, 2), (2, 6, 5, 3), (2, 6, 6, 6), (3, 1, 1, 6), (3, 1, 2, 3),
 (3, 1, 3, 2), (3, 1, 4, 5), (3, 1, 5, 4), (3, 1, 6, 1), (4, 1, 1, 6),
 (4, 1, 2, 3), (4, 1, 3, 2), (4, 1, 4, 5), (4, 1, 5, 4), (4, 1, 6, 1),
 (5, 6, 1, 1), (5, 6, 2, 4), (5, 6, 3, 5), (5, 6, 4, 2), (5, 6, 5, 3),
 (5, 6, 6, 6), (6, 2, 1, 3), (6, 2, 2, 5), (6, 2, 3, 1), (6, 2, 4, 6),
 (6, 2, 5, 2), (6, 2, 6, 4)]
```

```python
sage: A.<x,y,z> = AffineSpace(3, GF(3))
sage: S = A.subscheme(x + y)
sage: enum_affine_finite_field(S)
[(0, 0, 0), (0, 0, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2),
 (2, 1, 0), (2, 1, 1), (2, 1, 2)]
```

**ALGORITHM:**

Checks all points in affine space to see if they lie on $X$. 

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**Warning:** If \( X \) is defined over an infinite field, this code will not finish!

**AUTHORS:**
- John Cremona and Charlie Turner (06-2010)

```python
code.schemes.affine.affine_rational_point.enum_affine_number_field(X, **kwds)
```

Enumerates affine points on scheme \( X \) defined over a number field. Simply checks all of the points of absolute height up to \( B \) and adds those that are on the scheme to the list.

This algorithm computes 2 lists: \( L \) containing elements \( x \) in \( K \) such that \( H_k(x) \leq B \), and a list \( L' \) containing elements \( x \) in \( K \) that, due to floating point issues, may be slightly larger than the bound. This can be controlled by lowering the tolerance.

**ALGORITHM:**
This is an implementation of the revised algorithm (Algorithm 4) in [DK2013]. Algorithm 5 is used for imaginary quadratic fields.

**INPUT:**
- `kwds`:
  - `bound` - a real number
  - `tolerance` - a rational number in \((0,1]\) used in doyle-krumm algorithm 4
  - `precision` - the precision to use for computing the elements of bounded height of number fields.

**OUTPUT:**
- a list containing the affine points of \( X \) of absolute height up to \( B \), sorted.

**EXAMPLES:**

```python
sage: # needs sage.rings.number_field
sage: from sage.schemes.affine.affine_rational_point import enum_affine_number_field
sage: K = NumberField(u^2 + 2, 'v')
```

```python
sage: A.<x,y,z> = AffineSpace(K, 3)
```

```python
sage: X = A.subscheme([y^2 - x])
```

```python
sage: enum_affine_number_field(X(K), bound=2**0.5)
```

```
[(0, 0, -1), (0, 0, -v), (0, 0, -1/2*v), (0, 0, 0), (0, 0, 1/2*v),
 (0, 0, v), (0, 0, 1), (1, -1, -1), (1, -1, -v), (1, -1, -1/2*v),
 (1, -1, 0), (1, -1, 1/2*v), (1, -1, v), (1, -1, 1), (1, 1, -1),
 (1, 1, -v), (1, 1, -1/2*v), (1, 1, 0), (1, 1, 1/2*v), (1, 1, v), (1, 1, 1)]
```

```python
sage: # needs sage.rings.number_field
sage: from sage.schemes.affine.affine_rational_point import enum_affine_number_field
```

```python
sage: K = NumberField(u^2 + 3, 'v')
```

```python
sage: A.<x,y> = AffineSpace(K, 2)
```

```python
sage: X = A.subscheme(x - y)
```

```python
sage: enum_affine_number_field(X, bound=3**0.25)
```

```
[(-1, -1), (-1/2*v - 1/2, -1/2*v - 1/2), (1/2*v - 1/2, 1/2*v - 1/2),
 (0, 0), (-1/2*v + 1/2, -1/2*v + 1/2), (1/2*v + 1/2, 1/2*v + 1/2), (1, 1)]
```
Enumerates affine rational points on scheme $X$ up to bound $B$.

**INPUT:**
- $X$ - a scheme or set of abstract rational points of a scheme.
- $B$ - a positive integer bound.

**OUTPUT:**
- a list containing the affine points of $X$ of height up to $B$, sorted.

**EXAMPLES:**

```python
sage: A.<x,y,z> = AffineSpace(3, QQ)
sage: from sage.schemes.affine.affine_rational_point import enum_affine_rational_field
sage: enum_affine_rational_field(A(QQ), 1)
[(-1, -1, -1), (-1, -1, 0), (-1, -1, 1), (-1, 0, -1), (-1, 0, 0), (-1, 0, 1),
 (-1, 1, -1), (-1, 1, 0), (-1, 1, 1), (0, -1, -1), (0, -1, 0), (0, -1, 1),
 (0, 0, -1), (0, 0, 0), (0, 0, 1), (0, 1, -1), (0, 1, 0), (0, 1, 1), (1, -1, -1),
 (1, -1, 0), (1, -1, 1), (1, 0, -1), (1, 0, 0), (1, 0, 1), (1, 1, -1), (1, 1, 0),
 (1, 1, 1)]
```

```python
sage: A.<w,x,y,z> = AffineSpace(4, QQ)
sage: S = A.subscheme([x^2 - y*z + 1, w^3 + z + y^2])
sage: enum_affine_rational_field(S(QQ), 1)
[(0, 0, -1, -1)]
sage: enum_affine_rational_field(S(QQ), 2)
[(0, 0, -1, -1), (1, -1, -1, -2), (1, 1, -1, -2)]
```

```python
sage: A.<x,y> = AffineSpace(2, QQ)
sage: C = Curve(x^2 + y - x) # needs sage.libs.singular
sage: enum_affine_rational_field(C, 10) # long time (3 s) # needs sage.libs.singular
[(-2, -6), (-1, -2), (-2/3, -10/9), (-1/2, -3/4), (-1/3, -4/9),
 (0, 0), (1/3, 2/9), (1/2, 1/4), (2/3, 2/9), (1, 0),
 (4/3, -4/9), (3/2, -3/4), (5/3, -10/9), (2, -2), (3, -6)]
```

**AUTHORS:**
- David R. Kohel <kohel@maths.usyd.edu.au>: original version.
- Charlie Turner (06-2010): small adjustments.
- Raman Raghukul 2018: updated.
13.6 Set of homomorphisms between two affine schemes

For schemes $X$ and $Y$, this module implements the set of morphisms $\text{Hom}(X,Y)$. This is done by $\text{SchemeHomset\_generic}$. As a special case, the Hom-sets can also represent the points of a scheme. Recall that the $K$-rational points of a scheme $X$ over $k$ can be identified with the set of morphisms $\text{Spec}(K) \to X$. In Sage the rational points are implemented by such scheme morphisms. This is done by $\text{SchemeHomset\_points}$ and its subclasses.

Note: You should not create the Hom-sets manually. Instead, use the $\text{Hom()}$ method that is inherited by all schemes.

AUTHORS:

- Ben Hutz (2018): add numerical point support

class sage.schemes.affine.affine_homset.SchemeHomset_points_affine($X$, $Y$, category=None, 
check=True, base=Integer Ring)

Bases: $\text{SchemeHomset\_points}$

Set of rational points of an affine variety.

INPUT:

See $\text{SchemeHomset\_generic}$.

EXAMPLES:

```
sage: from sage.schemes.affine.affine_homset import SchemeHomset_points_affine
sage: SchemeHomset_points_affine(Spec(QQ), AffineSpace(ZZ,2))
Set of rational points of Affine Space of dimension 2 over Rational Field
```

$\text{numerical\_points}(F=\text{None}, **kwds)$

Return some or all numerical approximations of rational points of an affine scheme. This is for dimension 0 subschemes only and the points are determined through a groebner calculation over the base ring and then numerically approximating the roots of the resulting polynomials. If the base ring is a number field, the embedding into $F$ must be known.

INPUT:

$F$ - numerical ring

$kwds$:

- $\text{zero\_tolerance}$ - positive real number (optional, default=10^(-10)). For numerically inexact fields, points are on the subscheme if they satisfy the equations to within tolerance.

OUTPUT: A list of points in the ambient space.

Warning: For numerically inexact fields the list of points returned may contain repeated or be missing points due to tolerance.

EXAMPLES:
sage: # needs sage.libs.singular sage.rings.number_field
sage: K.<v> = QuadraticField(3)
sage: A.<x,y> = AffineSpace(K, 2)
sage: X = A.subscheme([x^3 - v^2*y, y - v*x^2 + 3])
sage: L = X(K).numerical_points(F=RR); L  # abs tol 1e-14
[(−1.18738247880014, -0.558021142104134),
 (1.57693558184861, 1.30713548084184),
 (4.80659931965815, 37.0162574656220)]
sage: L[0].codomain()
Affine Space of dimension 2 over Real Field with 53 bits of precision

sage: A.<x,y> = AffineSpace(QQ, 2)
sage: X = A.subscheme([y^2 - x^2 - 3*x, x^2 - 10*y])
sage: len(X(QQ).numerical_points(F=ComplexField(100)))  # needs sage.libs.singular
4

sage: A.<x1, x2> = AffineSpace(QQ, 2)
sage: E = A.subscheme([30*x1^100 + 1000*x2^2 + 2000*x1*x2 + 1, x1 + x2])
sage: len(E(A.base_ring()).numerical_points(F=CDF, zero_tolerance=1e-9))  # needs sage.libs.singular
100

points(**kwds)
Return some or all rational points of an affine scheme.

For dimension 0 subschemes points are determined through a groebner basis calculation. For schemes or
subschemas with dimension greater than 1 points are determined through enumeration up to the specified
bound.

Over a finite field, all points are returned. Over an infinite field, all points satisfying the bound are returned.
For a zero-dimensional subscheme, all points are returned regardless of whether the field is infinite or not.

For number fields, this uses the Doyle-Krumm algorithm 4 (algorithm 5 for imaginary quadratic) for com-
puting algebraic numbers up to a given height [DK2013].

The algorithm requires floating point arithmetic, so the user is allowed to specify the precision for such
calculations. Additionally, due to floating point issues, points slightly larger than the bound may be returned.
This can be controlled by lowering the tolerance.

INPUT:

kwds:

• bound - real number (optional, default: 0). The bound for the height of the coordinates. Only used for
  subschemes with dimension at least 1.

• zero_tolerance - positive real number (optional, default=10^(-10)). For numerically inexact fields,
  points are on the subscheme if they satisfy the equations to within tolerance.

• tolerance - a rational number in (0,1] used in doyle-krumm algorithm-4 for enumeration over number
  fields.

• precision - the precision to use for computing the elements of bounded height of number fields.

OUTPUT:

• a list of rational points of a affine scheme

13.6. Set of homomorphisms between two affine schemes
Warning: For numerically inexact fields such as ComplexField or RealField the list of points returned is very likely to be incomplete. It may also contain repeated points due to tolerance.

EXAMPLES: The bug reported at #11526 is fixed:

```
sage: A2 = AffineSpace(ZZ, 2)
sage: F = GF(3)
sage: A2(F).points()
[[(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)]
```

```
sage: A.<x,y> = ZZ
sage: I = A.ideal(x^2 - y^2 - 1)
sage: V = AffineSpace(ZZ, 2)
sage: X = V.subscheme(I)
sage: N = X(ZZ)
sage: N.points(bound=1)
[(-1, 0), (1, 0)]
```

```
sage: u = QQ['u'].0
sage: K.<v> = NumberField(u^2 + 3)
# needs sage.rings.number_field
sage: A.<x,y> = AffineSpace(K, 2)
# needs sage.rings.number_field
sage: len(A(K).points(bound=2))
# needs sage.rings.number_field
1849
```

```
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: E = A.subscheme([x^2 + y^2 - 1, y^2 - x^3 + x^2 + x - 1])
sage: E(A.base_ring()).points()  # needs sage.libs.singular
[(-1, 0), (0, -1), (0, 1), (1, 0)]
```

```
sage: A.<x,y> = AffineSpace(CC, 2)
# needs sage.rings.complex_double
sage: E = A.subscheme([x1^2 + x2^2 + x1*x2, x1 + x2])
# needs sage.libs.singular sage.rings.real_mpfr
```

```
verbose 0 (...: affine_homset.py, points)
Warning: computations in the numerical fields are inexact; points may be computed partially or incorrectly.
[(-1.0000000000000000000000, 0.0000000000000000000000),
 (0.0000000000000000000000, 0.0000000000000000000000)]
```

```
sage: A.<x1,x2> = AffineSpace(CDF, 2)
# needs sage.rings.complex_double
```

```
verbose 0 (...: affine_homset.py, points)
```

(continues on next page)
Warning: computations in the numerical fields are inexact; points may be computed partially or incorrectly.

```
[(0.0, 0.0)]
```
14.1 Projective \(n\) space over a ring

EXAMPLES:

We construct projective space over various rings of various dimensions.

The simplest projective space:

```
sage: ProjectiveSpace(0)
Projective Space of dimension 0 over Integer Ring
```

A slightly bigger projective space over \(\mathbb{Q}\):

```
sage: X = ProjectiveSpace(1000, QQ); X
Projective Space of dimension 1000 over Rational Field
sage: X.dimension()
1000
```

We can use “over” notation to create projective spaces over various base rings.

```
sage: X = ProjectiveSpace(5)/QQ; X
Projective Space of dimension 5 over Rational Field
sage: X/CC
# needs sage.rings.real_mpfr
Projective Space of dimension 5 over Complex Field with 53 bits of precision
```

The third argument specifies the printing names of the generators of the homogeneous coordinate ring. Using the method `objgens()` you can obtain both the space and the generators as ready to use variables.

```
sage: P2, vars = ProjectiveSpace(10, QQ, 't').objgens()
sage: vars
(t0, t1, t2, t3, t4, t5, t6, t7, t8, t9, t10)
```

You can alternatively use the special syntax with `< and >`.

```
sage: P2.<x,y,z> = ProjectiveSpace(2, QQ)
sage: P2
Projective Space of dimension 2 over Rational Field
sage: P2.coordinate_ring()
Multivariate Polynomial Ring in x, y, z over Rational Field
```

The first of the three lines above is just equivalent to the two lines:
sage: P2 = ProjectiveSpace(2, QQ, 'xyz')
sage: x,y,z = P2.gens()

For example, we use \(x, y, z\) to define the intersection of two lines.

```
sage: V = P2.subscheme([x + y + z, x + y - z]); V
Closed subscheme of Projective Space of dimension 2 over Rational Field defined by:
  x + y + z,
  x + y - z
```

```
sage: V.dimension()                   # needs sage.libs.singular
0
```

AUTHORS:

- Ben Hutz: (June 2012): support for rings
- Ben Hutz (9/2014): added support for Cartesian products
- Rebecca Lauren Miller (March 2016): added point_transformation_matrix

sage.schemes.projective.projective_space.ProjectiveSpace(n, R=None, names=None)

Return projective space of dimension \(n\) over the ring \(R\).

EXAMPLES: The dimension and ring can be given in either order.

```
sage: ProjectiveSpace(3, QQ)
Projective Space of dimension 3 over Rational Field
sage: ProjectiveSpace(5, QQ)
Projective Space of dimension 5 over Rational Field
sage: P = ProjectiveSpace(2, QQ, names='XYZ'); P
Projective Space of dimension 2 over Rational Field
sage: P.coordinate_ring()
Multivariate Polynomial Ring in X, Y, Z over Rational Field
```

The divide operator does base extension.

```
sage: ProjectiveSpace(5)/GF(17)
Projective Space of dimension 5 over Finite Field of size 17
```

The default base ring is \(\mathbb{Z}\).

```
sage: ProjectiveSpace(5)
Projective Space of dimension 5 over Integer Ring
```

There is also an projective space associated each polynomial ring.

```
sage: R = GF(7)['x,y,z']
sage: P = ProjectiveSpace(R); P
Projective Space of dimension 2 over Finite Field of size 7
sage: P.coordinate_ring()
Multivariate Polynomial Ring in x, y, z over Finite Field of size 7
```

```
sage: P.coordinate_ring() is R
True
```
```python
sage: ProjectiveSpace(3, Zp(5), 'y')  # needs sage.rings.padics
Projective Space of dimension 3 over 5-adic Ring with capped relative precision 20
```

```python
sage: ProjectiveSpace(2, QQ, 'x,y,z')
Projective Space of dimension 2 over Rational Field
```

```python
sage: PS.<x,y> = ProjectiveSpace(1, CC); PS  # needs sage.rings.real_mpfr
Projective Space of dimension 1 over Complex Field with 53 bits of precision
```

```python
sage: R.<x,y,z> = QQ[]
sage: ProjectiveSpace(R).variable_names()
('x', 'y', 'z')
```

Projective spaces are not cached, i.e., there can be several with the same base ring and dimension (to facilitate gluing constructions).

```python
sage: R.<x> = QQ[]
sage: ProjectiveSpace(R)
Projective Space of dimension 0 over Rational Field
```

**class** `sage.schemes.projective.projective_space.ProjectiveSpace_field(n, R=Integer Ring, names=None)`

**Bases:** `ProjectiveSpace_ring`

**curve**(*F*)

Return a curve defined by F in this projective space.

**INPUT:**

* F – a polynomial, or a list or tuple of polynomials in the coordinate ring of this projective space

**EXAMPLES:**

```python
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: P.curve([y^2 - x*z])  # needs sage.schemes
Projective Plane Curve over Rational Field defined by y^2 - x*z
```

**line_through**(*p, q*)

Return the line through p and q.

**INPUT:**

* p, q – distinct rational points of the projective space

**EXAMPLES:**

```python
sage: P3.<x0,x1,x2,x3> = ProjectiveSpace(3, QQ)
sage: p1 = P3(1, 2, 3, 4)
sage: p2 = P3(4, 3, 2, 1)
sage: P3.line_through(p1, p2)  # needs sage.libs.singular sage.schemes
Projective Curve over Rational Field defined by
```

(continues on next page)
\[-5/4 \times x_0 + 5/2 \times x_1 - 5/4 \times x_2, \quad -5/2 \times x_0 + 15/4 \times x_1 - 5/4 \times x_3, \]
\[-5/4 \times x_0 + 15/4 \times x_2 - 5/2 \times x_3, \quad -5/4 \times x_1 + 5/2 \times x_2 - 5/4 \times x_3 \]

```
sage: p3 = P3(2,4,6,8)
sage: p3.line_through(p1, p3)
Traceback (most recent call last):
  ...
ValueError: not distinct points
```

**subscheme_from_Chow_form**(Ch, dim)

Returns the subscheme defined by the Chow equations associated to the Chow form Ch.

These equations define the subscheme set-theoretically, but only for smooth subschemes and hypersurfaces do they define the subscheme as a scheme.

**ALGORITHM:**

The Chow form is a polynomial in the Plucker coordinates. The Plucker coordinates are the bracket polynomials. We first re-write the Chow form in terms of the dual Plucker coordinates. Then we expand \(Ch(span(p, L))\) for a generic point \(p\) and a generic linear subspace \(L\). The coefficients as polynomials in the coordinates of \(p\) are the equations defining the subscheme. [DalbecSturmfels].

**INPUT:**

- Ch - a homogeneous polynomial.
- dim - the dimension of the associated scheme.

**OUTPUT:** a projective subscheme.

**EXAMPLES:**

```
sage: P = ProjectiveSpace(QQ, 4, 'z')
sage: R.<x0,x1,x2,x3,x4> = PolynomialRing(QQ)
sage: H = x1^2 + x2^2 + 5*x3*x4
sage: P.subscheme_from_Chow_form(H, 3)  
˓→ needs sage.modules
Closed subscheme of Projective Space of dimension 4 over Rational Field defined by:
-5*z0*z1 + z2^2 + z3^2
```

```
sage: P = ProjectiveSpace(QQ, 3, 'z')
sage: R.<x0,x1,x2,x3,x4,x5> = PolynomialRing(QQ)
sage: H = x1 - x2 - x3 + x5 + 2*x0
sage: P.subscheme_from_Chow_form(H, 1)  
˓→ needs sage.modules
Closed subscheme of Projective Space of dimension 3 over Rational Field defined by:
-z1 + z3,
z0 + z2 + z3,
-z1 - 2*z3,
z0 - z1 + 2*z2
```

```
sage: # needs sage.libs.singular
sage: P.<x0,x1,x2,x3> = ProjectiveSpace(GF(7), 3)
sage: X = P.subscheme([x3^2 + x1*x2, x2 - x0])
```

(continues on next page)
sage: Ch = X.Chow_form(); Ch
t0^2 - 2*t0*t3 + t3^2 - t2*t4 - t4*t5
sage: Y = P.subscheme_from_Chow_form(Ch, 1); Y
Closed subscheme of Projective Space of dimension 3 over Finite Field of size 7 defined by:
x1*x2 + x3^2,
-x0*x2 + x2^2,
-x0*x1 - x1*x2 - 2*x3^2,
x0^2 - x0*x2,
x0^2 + x3^2,
-2*x0*x3 + 2*x2*x3,
2*x0*x3 - 2*x2*x3,
x0^2 - 2*x0*x2 + x2^2
sage: I = Y.defining_ideal()
sage: I.saturation(I.ring().ideal(list(I.ring().gens())))[0]
Ideal (x0 - x2, x1*x2 + x3^2) of Multivariate Polynomial Ring in x0, x1, x2, x3 over Finite Field of size 7

class sage.schemes.projective.projective_space.ProjectiveSpace_finite_field(n, R=Integer Ring, names=None)

Bases: ProjectiveSpace_field

rational_points(F=None)

Return the list of F-rational points on this projective space, where F is a given finite field, or the base ring of this space.

EXAMPLES:

sage: P = ProjectiveSpace(1, GF(3))
sage: P.rational_points()
[(0 : 1), (1 : 0), (1 : 1), (2 : 1), (1 : 0)]
sage: sorted(P.rational_points(GF(3^2, 'b')), key=str)
(continues on next page)
class sage.schemes.projective.projective_space.ProjectiveSpace_rational_field(n, R=Integer Ring, names=None)

Bases: ProjectiveSpace_field

rational_points(bound=0)

Returns the projective points \( (x_0 : \cdots : x_n) \) over \( \mathbb{Q} \) with \( |x_i| \leq \text{bound} \).

ALGORITHM:

The very simple algorithm works as follows: every point \( (x_0 : \cdots : x_n) \) in projective space has a unique largest index \( i \) for which \( x_i \) is not zero. The algorithm then iterates downward on this index. We normalize by choosing \( x_i \) positive. Then, the points \( x_0, \ldots, x_{i-1} \) are the points of affine \( i \)-space that are relatively prime to \( x_i \). We access these by using the Tuples method.

INPUT:

- bound - integer.

EXAMPLES:

sage: PP = ProjectiveSpace(0, QQ)
sage: PP.rational_points(1)
[(1)]
sage: PP = ProjectiveSpace(1, QQ)
sage: PP.rational_points(2)
[(-2 : 1), (-1 : 1), (0 : 1), (1 : 1), (2 : 1), (-1/2 : 1), (1/2 : 1), (1 : 0)]
sage: PP = ProjectiveSpace(2, QQ)
sage: PP.rational_points(2)
[(-2 : -2 : 1), (-1 : -2 : 1), (0 : -2 : 1), (1 : -2 : 1), (2 : -2 : 1),
 (-2 : -1 : 1), (-1 : -1 : 1), (0 : -1 : 1), (1 : -1 : 1), (2 : -1 : 1),
 (-2 : 0 : 1), (-1 : 0 : 1), (0 : 0 : 1), (1 : 0 : 1), (2 : 0 : 1), (-2 : 1 :)
 \rightarrow 1),
 (-1 : 1 : 1), (0 : 1 : 1), (1 : 1 : 1), (2 : 1 : 1), (-2 : 2 : 1),
 (-1 : 2 : 1), (0 : 2 : 1), (1 : 2 : 1), (2 : 2 : 1), (-1/2 : -1 : 1),
 (1/2 : -1 : 1), (-1 : -1/2 : 1), (-1/2 : -1/2 : 1), (0 : -1/2 : 1),
 (1/2 : -1/2 : 1), (1 : -1/2 : 1), (-1/2 : 0 : 1), (1/2 : 0 : 1), (-1 : 1/2 :)
 \rightarrow 1),
 (-1/2 : 1/2 : 1), (0 : 1/2 : 1), (1/2 : 1/2 : 1), (1 : 1/2 : 1), (-1/2 : 1 :)
 \rightarrow 1),
 (1/2 : 1 : 1), (-2 : 1 : 0), (-1 : 1 : 0), (0 : 1 : 0), (1 : 1 : 0),
 (2 : 1 : 0), (-1/2 : 1 : 0), (1/2 : 1 : 0), (1 : 0 : 0)]

AUTHORS:

- Benjamin Antieau (2008-01-12)

class sage.schemes.projective.projective_space.ProjectiveSpace_ring(n, R=Integer Ring, names=None)

Bases: UniqueRepresentation, AmbientSpace

Projective space of dimension \( n \) over the ring \( R \).
EXAMPLES:

```python
sage: X.<x,y,z,w> = ProjectiveSpace(3, QQ)
sage: X.base_scheme()
Spectrum of Rational Field
sage: X.base_ring()
Rational Field
sage: X.structure_morphism()
Scheme morphism:
  From: Projective Space of dimension 3 over Rational Field
  To: Spectrum of Rational Field
  Defn: Structure map
sage: X.coordinate_ring()
Multivariate Polynomial Ring in x, y, z, w over Rational Field
```

Loading and saving:

```python
sage: loads(X.dumps()) == X
True
sage: P = ProjectiveSpace(ZZ, 1, 'x')
sage: loads(P.dumps()) is P
True
```

Equality and hashing:

```python
sage: ProjectiveSpace(QQ, 3, 'a') == ProjectiveSpace(ZZ, 3, 'a')
False
sage: ProjectiveSpace(ZZ, 1, 'a') == ProjectiveSpace(ZZ, 0, 'a')
False
sage: ProjectiveSpace(ZZ, 2, 'a') == AffineSpace(ZZ, 2, 'a')
False
sage: ProjectiveSpace(QQ, 3, 'a') != ProjectiveSpace(ZZ, 3, 'a')
True
sage: ProjectiveSpace(ZZ, 1, 'a') != ProjectiveSpace(ZZ, 0, 'a')
True
sage: ProjectiveSpace(ZZ, 2, 'a') != AffineSpace(ZZ, 2, 'a')
True
```

```
Lattes_map(E, m)

Given an elliptic curve E and an integer m return the Lattes map associated to multiplication by m.

In other words, the rational map on the quotient E/{±1} ≅ P¹ associated to [m] : E → E.

INPUT:

• E – an elliptic curve.
• m – an integer.
```

14.1. Projective space over a ring
OUTPUT: a dynamical system on this projective space.

EXAMPLES:

```python
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: E = EllipticCurve(QQ,[-1, 0])
# needs sage.schemes
sage: P.Lattes_map(E, 2)
# needs sage.schemes
Dynamical System of Projective Space of dimension 1 over Rational Field
Defn: Defined on coordinates by sending (x : y) to 
(1/4*x^4 + 1/2*x^2*y^2 + 1/4*y^4 :  x^3*y - x*y^3)
```

**affine_patch**(\(i, AA=None\))

Return the \(i^{th}\) affine patch of this projective space.

This is an ambient affine space \(A^n_R\), where \(R\) is the base ring of \(self\), whose “projective embedding" map is 1 in the \(i^{th}\) factor.

INPUT:

- \(i\) – integer between 0 and dimension of \(self\), inclusive.
- \(AA\) – (default: None) ambient affine space, this is constructed if it is not given.

OUTPUT:

- An ambient affine space with fixed projective_embedding map.

EXAMPLES:

```python
sage: PP = ProjectiveSpace(5) / QQ
sage: AA = PP.affine_patch(2)
sage: AA
Affine Space of dimension 5 over Rational Field
sage: AA.projective_embedding()
Scheme morphism:
From: Affine Space of dimension 5 over Rational Field
To: Projective Space of dimension 5 over Rational Field
Defn: Defined on coordinates by sending (x0, x1, x3, x4, x5) to 
(x0 : x1 : 1 : x3 : x4 : x5)
sage: AA.projective_embedding(0)
Scheme morphism:
From: Affine Space of dimension 5 over Rational Field
To: Projective Space of dimension 5 over Rational Field
Defn: Defined on coordinates by sending (x0, x1, x3, x4, x5) to 
(1 : x0 : x1 : x3 : x4 : x5)
```

**cartesian_product**(other)

Return the Cartesian product of this projective space and other.

INPUT:

- other - A projective space with the same base ring as this space.
OUTPUT:

• A Cartesian product of projective spaces.

EXAMPLES:

```python
sage: P1 = ProjectiveSpace(QQ, 1, 'x')
sage: P2 = ProjectiveSpace(QQ, 2, 'y')
sage: PP = P1.cartesian_product(P2); PP
Product of projective spaces P^1 x P^2 over Rational Field
sage: PP.gens()
(x0, x1, y0, y1, y2)
```

`change_ring(R)`

Return a projective space over ring `R`.

INPUT:

• `R` – commutative ring or morphism.

OUTPUT:

• projective space over `R`.

**Note:** There is no need to have any relation between `R` and the base ring of this space, if you want to have such a relation, use `self.base_extend(R)` instead.

EXAMPLES:

```python
sage: P.<x, y, z> = ProjectiveSpace(2, ZZ)
sage: PQ = P.change_ring(QQ); PQ
Projective Space of dimension 2 over Rational Field
sage: PQ.change_ring(GF(5))
Projective Space of dimension 2 over Finite Field of size 5
```

`chebyshev_polynomial(n, kind='first', monic=False)`

Generates an endomorphism of this projective line by a Chebyshev polynomial.

Chebyshev polynomials are a sequence of recursively defined orthogonal polynomials. Chebyshev of the first kind are defined as \( T_0(x) = 1, T_1(x) = x, \) and \( T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \) Chebyshev of the second kind are defined as \( U_0(x) = 1, U_1(x) = 2x, \) and \( U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x). \)

INPUT:

• `n` – a non-negative integer.

• `kind` – `first` or `second` specifying which kind of chebyshev the user would like to generate. Defaults to `first`.

14.1. Projective \( n \) space over a ring
• monic – True or False specifying if the polynomial defining the system should be monic or not. Defaults to False.

OUTPUT: DynamicalSystem_projective

EXAMPLES:

```
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: P.chebyshev_polynomial(5, 'first')  # needs sage.symbolic
Dynamical System of Projective Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x : y) to
  (16*x^5 - 20*x^3*y^2 + 5*x*y^4 : y^5)
```

```
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: P.chebyshev_polynomial(3, 'second')  # needs sage.symbolic
Dynamical System of Projective Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x : y) to
  (8*x^3 - 4*x*y^2 : y^3)
```

```
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: P.chebyshev_polynomial(3, 2)  # needs sage.symbolic
Traceback (most recent call last):
  ... ValueError: keyword 'kind' must have a value of either 'first' or 'second'
```

```
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: P.chebyshev_polynomial(-4, 'second')
Traceback (most recent call last):
  ... ValueError: first parameter 'n' must be a non-negative integer
```

```
sage: P = ProjectiveSpace(QQ, 2, 'x')
sage: P.chebyshev_polynomial(2)
Traceback (most recent call last):
  ... TypeError: projective space must be of dimension 1
```

```
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: P.chebyshev_polynomial(3, monic=True)  # needs sage.symbolic
Dynamical System of Projective Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x : y) to
  (x^3 - 3*x*y^2 : y^3)
```

```
sage: F.<t> = FunctionField(QQ)
sage: P.<y,z> = ProjectiveSpace(F, 1)
sage: P.chebyshev_polynomial(4, monic=True)  # needs sage.symbolic
Dynamical System of Projective Space of dimension 1
  over Rational function field in t over Rational Field
```

(continues on next page)
Defn: Defined on coordinates by sending \((y : z)\) to 
\((y^4 + (-4)y^2z^2 + 2z^4 : z^4)\)

**coordinate_ring()**

Return the coordinate ring of this scheme.

**EXAMPLES:**

```sage
sage: ProjectiveSpace(3, GF(19^2, 'alpha'), 'abcd').coordinate_ring()    # needs sage.rings.finite_rings
Multivariate Polynomial Ring in a, b, c, d over Finite Field in alpha of size 19^2
```

```sage
sage: ProjectiveSpace(3).coordinate_ring()
Multivariate Polynomial Ring in x0, x1, x2, x3 over Integer Ring
```

```sage
sage: ProjectiveSpace(2, QQ, ['alpha', 'beta', 'gamma']).coordinate_ring()
Multivariate Polynomial Ring in alpha, beta, gamma over Rational Field
```

**hyperplane_transformation_matrix**(plane_1, plane_2)

Return a PGL element sending plane_1 to plane_2.

plane_1 and plane_2 must be hyperplanes (subschemes of codimension 1, each defined by a single linear homogeneous equation).

**INPUT:**

- plane_1, plane_2 – hyperplanes of this projective space

**OUTPUT:** An element of PGL

**EXAMPLES:**

```sage
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: plane1 = P.subscheme(x)
sage: plane2 = P.subscheme(y)
sage: m = P.hyperplane_transformation_matrix(plane1, plane2); m    # needs sage.modules
[0 1]
[1 0]
sage: plane2(m*P((0,1)))    # needs sage.modules
(1 : 0)
```

```sage
sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: plane1 = P.subscheme(x + 2*y + z)
sage: plane2 = P.subscheme(2*x + y + z)
sage: m = P.hyperplane_transformation_matrix(plane1, plane2); m    # needs sage.modules
[1 0 0 0]
[0 4 0 0]
[0 0 2 0]
[0 0 0 1]
```
sage: P.<x,y> = ProjectiveSpace(ZZ, 1)
sage: plane1 = P.subscheme(x + y)
sage: plane2 = P.subscheme(y)
sage: P.hyperplane_transformation_matrix(plane1, plane2)  # needs sage.modules
[-1 0]
[ 1 1]

sage: # needs sage.rings.number_field
sage: K.<v> = CyclotomicField(3)
sage: P.<x,y,z> = ProjectiveSpace(K, 2)
sage: plane1 = P.subscheme(x - 2*v*y + z)
sage: plane2 = P.subscheme(x + v*y + v*z)
sage: m = P.hyperplane_transformation_matrix(plane1, plane2); m  # needs sage.modules
[ v 0 0]
[0 -2*v 0]
[0 0 1]

sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: K.<k> = NumberField(x^2 + 1)
sage: P.<x,y,z,w> = ProjectiveSpace(K, 3)
sage: plane1 = P.subscheme(k*x + 2*k*y + z)
sage: plane2 = P.subscheme(7*k*x + y + 9*z)
sage: m = P.hyperplane_transformation_matrix(plane1, plane2); m  # needs sage.modules
[ 1 0 0 0]
[0 14*k 0 0]
[0 0 7/9 0]
[0 0 0 1]

sage: # needs sage.rings.number_field
sage: K.<v> = CyclotomicField(3)
sage: R.<t> = K[]
sage: F.<w> = K.extension(t^5 + 2)
sage: G.<u> = F.absolute_field()
sage: P.<x,y,z> = ProjectiveSpace(G, 2)
sage: plane1 = P.subscheme(x - 2*u*y + z)
sage: plane2 = P.subscheme(x + u*y + z)
sage: P.hyperplane_transformation_matrix(plane1, plane2)  # needs sage.modules

sage: plane2(m*P((2*u, 1, 0)))  # needs sage.modules
(-u : 1 : 0)

sage: P.<x,y,z> = ProjectiveSpace(FiniteField(2), 2)
sage: plane1 = P.subscheme(x + y + z)
sage: plane2 = P.subscheme(z)
sage: P.hyperplane_transformation_matrix(plane1, plane2)  # needs sage.modules
[1 0 0]
(continues on next page)
Schemes, Release 10.2

[1 1 0]
[1 1 1]

is_linearly_independent(points, n=None)

Return whether the set of points is linearly independent. Alternatively, specify \( n \) to check if every subset of size \( n \) is linearly independent.

**INPUT:**

- `points` – a list of points in this projective space.
- `n` – (Optional) A positive integer less than or equal to the length of `points`. Specifies the size of the subsets to check for linear independence.

**OUTPUT:**

- True if `points` is linearly independent, False otherwise.

**EXAMPLES:**

```sage
R.<c> = QQ[]
P.<x,y,z> = ProjectiveSpace(R, 2)
points = [P((c, 0, 1)), P((0, c, 1)), P((1, 0, 1)), P((0, 0, 1))]
P.is_linearly_independent(points, 3) # needs sage.modules
False
```

14.1. Projective \( n \) space over a ring
needs sage.modules
True

sage: # needs sage.rings.number_field
sage: k.<k> = CyclotomicField(3)
sage: P.<x,y,z> = ProjectiveSpace(K, 2)
sage: points = [P((k, k^2, 1)), P((0, k, 1)), P((1, 0, 4)), P((0, 0, 1))]
sage: P.is_linearly_independent(points, 3)
needs sage.modules
True

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: points = [P((1, 0)), P((1, 1))]
sage: P.is_linearly_independent(points)
needs sage.modules
True

is_projective()
Return that this ambient space is projective \( n \)-space.

EXAMPLES:

sage: ProjectiveSpace(3, QQ).is_projective()
True

ngens()
Return the number of generators of this projective space.

This is the number of variables in the coordinate ring of self.

EXAMPLES:

sage: ProjectiveSpace(3, QQ).ngens()
4
sage: ProjectiveSpace(7, ZZ).ngens()
8

point(v, check=True)
Create a point on this projective space.

INPUT:

- \( v \) – anything that defines a point
- \( \text{check} \) – boolean (optional, default: True); whether to check the defining data for consistency

OUTPUT: A point of this projective space.

EXAMPLES:

sage: P2 = ProjectiveSpace(QQ, 2)
sage: P2.point([4,5])
(4 : 5 : 1)
sage: P = ProjectiveSpace(QQ, 1)
sage: P.point(infinity)
(1 : 0)

sage: P = ProjectiveSpace(QQ, 2)
sage: P.point(infinity)
Traceback (most recent call last):
...
ValueError: +Infinity not well defined in dimension > 1

sage: P = ProjectiveSpace(ZZ, 2)
sage: P.point([infinity])
Traceback (most recent call last):
...
ValueError: [+Infinity] not well defined in dimension > 1

point_transformation_matrix(points_source, points_target, normalize=True)

Returns a unique element of PGL that transforms one set of points to another.

Given a projective space of dimension n and a set of n+2 source points and a set of n+2 target points in the
same projective space, such that no n+1 points of each set are linearly dependent find the unique element
of PGL that translates the source points to the target points.

Warning: over non-exact rings such as the ComplexField, the returned matrix could be very far from
correct.

INPUT:
• points_source – points in source projective space.
• points_target – points in target projective space.
• normalize – (default: True) If the returned matrix should be normalized. Only works over exact
rings. If the base ring is a field, the matrix is normalized so that the last nonzero entry in the last row
is 1. If the base ring is a ring, then the matrix is normalized so that the entries are elements of the base
ring.

OUTPUT: Transformation matrix - element of PGL.

ALGORITHM:
See [Hutz2007], Proposition 2.16 for details.

EXAMPLES:

sage: P1.<a,b,c> = ProjectiveSpace(QQ, 2)
sage: points_source = [P1([1, 4, 1]), P1([1, 2, 2]), P1([3, 5, 1]), P1([1, -1, 1])]
sage: points_target = [P1([5, -2, 7]), P1([3, -2, 3]), P1([6, -5, 9]), P1([3, 6, 7])]
sage: m = P1.point_transformation_matrix(points_source, points_target); m
# needs sage.modules
[ -13/59  -128/59   -25/59]
[  538/177     8/59  26/177]
[  -45/59  -196/59    1]
sage: [m*points_source[i] == points_target[i] for i in range(4)]  # needs sage.modules
[True, True, True, True]

sage: P.<a,b> = ProjectiveSpace(GF(13), 1)
sage: points_source = [P([-6, 7]), P([1, 4]), P([3, 2])]
sage: points_target = [P([-1, 2]), P([0, 2]), P([-1, 6])]
sage: P.point_transformation_matrix(points_source, points_target)  # needs sage.modules
[10  4]
[10  1]

sage: P.<a,b> = ProjectiveSpace(QQ, 1)
sage: points_source = [P([-6, -4]), P([1, 4]), P([3, 2])]
sage: points_target = [P([-1, 2]), P([0, 2]), P([-7, -3])]
sage: P.point_transformation_matrix(points_source, points_target)  # needs sage.modules
Traceback (most recent call last):
...
ValueError: source points not independent

sage: R.<t> = FunctionField(QQ)
sage: P.<a,b> = ProjectiveSpace(R, 1)
sage: points_source = [P([-6*t, 7]), P([1, 4]), P([3, 2])]
sage: points_target = [P([-1, 2*t]), P([0, 2]), P([-1, 6])]
sage: P.point_transformation_matrix(points_source, points_target)  # needs sage.modules
[ (1/3*t + 7/12)/(t^2 - 53/24*t) (-1/12*t - 7/48)/(t^2 - 53/24*t)
  (-2/3*t^2 - 7/36*t - 35/12)/(t^2 - 53/24*t) 1]

sage: P1.<a,b,c> = ProjectiveSpace(RR, 2)
sage: points_source = [P1([1, 4, 1]), P1([1, 2, 2]), P1([3, 5, 1]), P1([1, -1, -1])]
sage: points_target = [P1([5, -2, 7]), P1([3, -2, 3]), P1([6, -5, 9]), P1([3, 6, 7])]
sage: P1.point_transformation_matrix(points_source, points_target)  # abs tol 1e-13 # needs sage.modules
....: points_target)
[-0.661947619047597 -0.609523809523810 -0.119047619047621]
[ 0.853968253968253 0.0380952380952380 0.0412698412698421]
[ -0.214285714285712 -0.933333333333333 0.280952380952379]

sage: P1.<a,b,c> = ProjectiveSpace(ZZ, 2)
sage: points_source = [P1([1, 4, 1]), P1([1, 2, 2]), P1([3, 5, 1]), P1([1, -1, -1])]
sage: points_target = [P1([5, -2, 7]), P1([3, -2, 3]), P1([6, -5, 9]), P1([3, 6, 7])]
sage: P1.point_transformation_matrix(points_source, points_target)  # needs sage.modules
(continues on next page)
points_of_bounded_height(**kwds)

Return an iterator of the points in self of absolute multiplicative height of at most the given bound.

ALGORITHM:
This is an implementation of Algorithm 6 in [Krumm2016].

INPUT:
kwds:
  • bound - a real number
  • precision - (default: 53) a positive integer

OUTPUT:
  • an iterator of points of bounded height

EXAMPLES:

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: sorted(list(P.points_of_bounded_height(bound=2)))
[(-2 : 1), (-1 : 1), (-1/2 : 1), (0 : 1),
 (1/2 : 1), (1 : 0), (1 : 1), (2 : 1)]

sage: u = QQ['u'].0
sage: P.<x,y,z> = ProjectiveSpace(NumberField(u^2 - 2, 'v'), 2)
  # needs sage.rings.number_field
sage: len(list(P.points_of_bounded_height(bound=2)))
sage: # needs sage.rings.number_field
sage: CF.<a> = CyclotomicField(3)

sage: R.<x> = CF[]

sage: L.<l> = CF.extension(x^3 + 2)

sage: Q.<x,y> = ProjectiveSpace(L, 1)

sage: sorted(list(Q.points_of_bounded_height(bound=1)))
[(0 : 1), (1 : 0), (a + 1 : 1), (a : 1), (-1 : 1), (-a - 1 : 1), (-a : 1), (1 : 1)]

sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]

sage: F.<a> = NumberField(x^4 - 8*x^2 + 3)

sage: P.<x,y,z> = ProjectiveSpace(F, 2)

sage: all(exp(p.global_height()) <= 1  
....: for p in P.points_of_bounded_height(bound=1))
True

sage: K.<a> = CyclotomicField(3)

sage: P.<x,y> = ProjectiveSpace(K, 1)

sage: len(list(P.points_of_bounded_height(bound=2)))
24

sage: R.<x> = QQ[]

sage: K.<v> = NumberField(x^5 + x^3 + 1)

sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]

sage: K.<v> = NumberField(x^5 + x^3 + 1)
.. code-block::

    sage: P.<x,y,z> = ProjectiveSpace(K, 2)
    sage: L = P.points_of_bounded_height(bound=1.2)
    sage: len(list(L))
    109

    sage: # needs sage.rings.number_field
    sage: K.<v> = QuadraticField(2)
    sage: P.<x,y> = ProjectiveSpace(K, 1)
    sage: sorted(list(P.points_of_bounded_height(bound=2)))
    [(-v - 2 : 1), (-v - 1 : 1), (-2 : 1), (-1/2*v - 1 : 1), (-v : 1), (-1 : 1),
     (-1/2*v : 1), (v - 2 : 1), (-1/2 : 1), (-v + 1 : 1), (1/2*v - 1 : 1), (0 : 1),
     (-1/2*v + 1 : 1), (v - 1 : 1), (1/2 : 1), (-v + 2 : 1), (1/2*v : 1), (1 : 0),
     (1 : 1), (v : 1), (1/2*v + 1 : 1), (2 : 1), (v + 1 : 1), (v + 2 : 1)]

    sage: # needs sage.rings.number_field
    sage: R.<x> = QQ[]
    sage: K.<a> = NumberField(3*x^2 + 1)
    sage: P.<z,w> = ProjectiveSpace(K, 1)
    sage: len(sorted(list(P.points_of_bounded_height(bound=1))))
    44

    sage: # needs sage.rings.number_field
    sage: R.<x> = QQ[]
    sage: K.<a> = NumberField(x^3 - 7)
    sage: O = K.maximal_order()
    sage: P.<z,w> = ProjectiveSpace(O, 1)
    sage: len(sorted(list(P.points_of_bounded_height(bound=2))))
    28

    sage: P.<w,z> = ProjectiveSpace(ZZ, 1)
    sage: sorted(list(P.points_of_bounded_height(bound=2)))
    [(-2 : -1), (-2 : 1), (-1 : -2), (-1 : -1),
     (-1 : 0), (-1 : 1), (-1 : 2), (0 : -1)]

    sage: R.<x> = QQ[]
    sage: P.<z,w> = ProjectiveSpace(R, 1)
    sage: P.points_of_bounded_height(bound=2)
    Traceback (most recent call last):
    ...
    NotImplementedError: self must be a projective space over
    a number field or a ring of integers

14.1. Projective $$n$$ space over a ring
sage: # needs sage.rings.number_field
sage: K.<i> = NumberField(x^2 + 1)
sage: PK.<t> = K[]

sage: L.<a> = K.extension(t^4 - i)
sage: P.<z,w> = ProjectiveSpace(L, 1)

sage: sorted(list(P.points_of_bounded_height(bound=1)))
[(0 : 1), (1 : 0), (a : 1), (a^2 : 1), (a^3 : 1), (i : 1),
 (i*a : 1), (i*a^2 : 1), (i*a^3 : 1), (-1 : 1), (-a : 1), (-a^2 : 1),
 (-a^3 : 1), (-i : 1), (-i*a : 1), (-i*a^2 : 1), (-i*a^3 : 1), (1 : 1)]

subscheme(X)

Return the closed subscheme defined by X.

INPUT:

• X - a list or tuple of equations.

EXAMPLES:

sage: A.<x,y,z> = ProjectiveSpace(2, QQ)
sage: X = A.subscheme([x*z^2, y^2*z, x*y^2]); X
Closed subscheme of Projective Space of dimension 2 over Rational Field defined by:
x*z^2,
y^2*z,
x*y^2

sage: X.defining_polynomials ()
(x*z^2, y^2*z, x*y^2)

sage: I = X.defining_ideal(); I
Ideal (x*z^2, y^2*z, x*y^2) of Multivariate Polynomial Ring in x, y, z
over Rational Field

sage: I.groebner_basis()  # needs sage.libs.singular
[x*y^2, y^2*z, x*z^2]

sage: X.dimension()  # needs sage.libs.singular
0

sage: X.base_ring()
Rational Field

sage: X.base_scheme()
Spectrum of Rational Field

sage: X.structure_morphism()
Scheme morphism:
From: Closed subscheme of Projective Space of dimension 2
over Rational Field defined by: x*z^2, y^2*z, x*y^2
To: Spectrum of Rational Field
Defn: Structure map

veronese_embedding(d, CS=None, order='lex')

Return the degree d Veronese embedding from this projective space.

INPUT:

• d – a positive integer.
• CS – a projective ambient space to embed into. If this projective space has dimension $N$, the dimension of CS must be $\binom{N+1}{d} - 1$. This is constructed if not specified. Default: None.

• order – a monomial order to use to arrange the monomials defining the embedding. The monomials will be arranged from greatest to least with respect to this order. Default: 'lex'.

OUTPUT:

• a scheme morphism from this projective space to CS.

EXAMPLES:

```python
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: vd = P.veronese_embedding(4, order='invlex')
```

Veronese surface:

```python
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: Q.<q,r,s,t,u,v> = ProjectiveSpace(QQ, 5)
sage: vd = P.veronese_embedding(2, Q)
```

Closed subscheme of Projective Space of dimension 5 over Rational Field defined by:

- $-u^2 + t*v$,
- $-s^2 + u*v$,
- $-s*t + r*u$,
- $-s^2 + q*v$,
- $-r*s + q*u$,
- $-r^2 + q*t$

```
sage.schemes.projective.projective_space.is_ProjectiveSpace(x)
```

Return True if $x$ is a projective space.

In other words, if $x$ is an ambient space $\mathbb{P}^n_R$, where $R$ is a ring and $n \geq 0$ is an integer.

EXAMPLES:

```python
sage: from sage.schemes.projective.projective_space import is_ProjectiveSpace
sage: is_ProjectiveSpace(ProjectiveSpace(5, names='x'))
True
```
14.2 Morphisms on projective schemes

This module defines morphisms from projective schemes. A morphism from a projective scheme to a projective scheme is defined by homogeneous polynomials of the same degree that define what the morphism does on points in the ambient projective space. A morphism from a projective scheme to an affine scheme is determined by rational function, that is, quotients of homogeneous polynomials of the same degree.

EXAMPLES:

```python
sage: P2.<x0,x1,x2> = ProjectiveSpace(QQ, 2)
sage: A2.<x,y> = AffineSpace(QQ, 2)
sage: P2.hom([x0, x1, x1 + x2], P2)
Scheme endomorphism of Projective Space of dimension 2 over Rational Field
Defn: Defined on coordinates by sending (x0 : x1 : x2) to (x0 : x1 : x1 + x2)
sage: P2.hom([x1/x0, (x1 + x2)/x0], A2)
Scheme morphism:
From: Projective Space of dimension 2 over Rational Field
To:  Affine Space of dimension 2 over Rational Field
Defn: Defined on coordinates by sending (x0 : x1 : x2) to (x1/x0, (x1 + x2)/x0)
```

AUTHORS:

- David Kohel, William Stein: initial version
- William Stein (2006-02-11): fixed bug where P(0,0,0) was allowed as a projective point
- Volker Braun (2011-08-08): renamed classes, more documentation, misc cleanups
- Ben Hutz (2013-03): iteration functionality and new directory structure for affine/projective, height functionality
- Brian Stout, Ben Hutz (2013-11): added minimal model functionality
- Dillon Rose (2014-01): speed enhancements
- Ben Hutz (2015-11): iteration of subschemes
- Kwankyu Lee (2020-02): added indeterminacy_locus() and image()
- Kwankyu Lee (2022-05): added graph(), projective_degrees(), and degree()

```python
class sage.schemes.projective.projective_morphism.SchemeMorphism_polynomial_projective_space(parent, polys, check=True):
    Bases: SchemeMorphism_polynomial
```

A morphism of schemes determined by rational functions that define what the morphism does on points in the ambient projective space.

EXAMPLES:
A more complicated example:

```
sage: P2.<x,y,z> = ProjectiveSpace(2, QQ)
sage: P1 = P2.subscheme(x - y)
sage: H12 = P1.Hom(P2)
sage: H12([x^2, x*z, z^2])
```

Scheme morphism:
\[
\text{From: Closed subscheme of Projective Space of dimension 2 over Rational Field defined by } x - y \\
\text{To: Projective Space of dimension 2 over Rational Field} \\
\text{Defn: Defined on coordinates by sending } (x : y : z) \text{ to } (x^2 : x*z : z^2)
\]

We illustrate some error checking:

```
sage: R.<x,y> = QQ[]
sage: P1 = ProjectiveSpace(R)
sage: H = P1.Hom(P1)
sage: f = H([x - y, x*y])
Traceback (most recent call last): 
  ... 
ValueError: polys (=[x - y, x*y]) must be of the same degree
```

```
sage: H([x - 1, x*y + x])
Traceback (most recent call last): 
  ... 
ValueError: polys (=[x - 1, x*y + x]) must be homogeneous
```
We can also compute the forward image of subschemes through elimination. In particular, let \( X = V(h_1, \ldots, h_t) \) and define the ideal \( I = (h_1, \ldots, h_t, y_0 - f_0(x), \ldots, y_n - f_n(x)) \). Then the elimination ideal \( I_{n+1} = I \cap K[y_0, \ldots, y_n] \) is a homogeneous ideal and \( f(X) = V(I_{n+1}) \):

```python
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: H = End(P)
sage: f = H([(x-2*y)^2, (x-2*z)^2, x^2])
sage: X = P.subscheme(y-z)
sage: f(f(f(X)))
```

Closed subscheme of Projective Space of dimension 2 over Rational Field defined by:
\( y - z \)

```python
sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: H = End(P)
sage: f = H([(x-2*y)^2, (x-2*z)^2, (x-2*w)^2, x^2])
sage: f(P.subscheme([x,y,z]))
```

Closed subscheme of Projective Space of dimension 3 over Rational Field defined by:
\( w, y, x \)

**as_dynamical_system()**

Return this endomorphism as a `DynamicalSystem_projective`.

**OUTPUT:**

- `DynamicalSystem_projective`

**EXAMPLES:**

```python
sage: P.<x,y,z> = ProjectiveSpace(ZZ, 2)
sage: H = End(P)
sage: f = H([x^2, y^2, z^2])
sage: type(f.as_dynamical_system())
```

`<class 'sage.dynamics.arithmetic_dynamics.projective_ds.DynamicalSystem_projective'>`

```python
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: H = End(P)
sage: f = H([x^2 - y^2, y^2])
sage: type(f.as_dynamical_system())
```

(continues on next page)
needs sage.schemes
<class 'sage.dynamics.arithmetic_dynamics.projective_ds.DynamicalSystem_projective_field'>

sage: P.<x,y> = ProjectiveSpace(GF(5), 1)
sage: H = End(P)
sage: f = H([x^2, y^2])
sage: type(f.as_dynamical_system()) # needs sage.schemes
class 'sage.dynamics.arithmetic_dynamics.projective_ds.DynamicalSystem_projective_finite_field'>

sage: P.<x,y> = ProjectiveSpace(RR, 1)
sage: f = DynamicalSystem([x^2 + y^2, y^2], P) # needs sage.schemes
sage: g = f.as_dynamical_system() # needs sage.schemes
sage: g is f # needs sage.schemes
True

degree()

Return the degree of this map.

The degree is defined as the degree of the homogeneous polynomials that are the coordinates of this map.

OUTPUT:

- A positive integer

EXAMPLES:

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: H = Hom(P, P)
sage: f = H([x^2 + y^2, y^2])
sage: f.degree()
2

sage: # needs sage.rings.real_mpfr
sage: P.<x,y,z> = ProjectiveSpace(CC, 2)
sage: H = Hom(P, P)
sage: f = H([x^3 + y^3, y^2*z, z*x*y])
sage: f.degree()
3

sage: R.<t> = PolynomialRing(QQ)
sage: P.<x,y,z> = ProjectiveSpace(R, 2)
sage: H = Hom(P, P)
sage: f = H([x^2 + t*y^2, (2-t)*y^2, z^2])
sage: f.degree()
2

14.2. Morphisms on projective schemes
sage: P.<x,y,z> = ProjectiveSpace(ZZ, 2)
sage: X = P.subscheme(x^2 - y^2)
sage: H = Hom(X, X)
sage: f = H([x^2, y^2, z^2])
sage: f.degree()
2

dehomogenize(n)

Return the standard dehomogenization at the n[0] coordinate for the domain and the n[1] coordinate for the codomain.

Note that the new function is defined over the fraction field of the base ring of this map.

INPUT:

• n – a tuple of nonnegative integers. If n is an integer, then the two values of the tuple are assumed to be the same.

OUTPUT:

• SchemeMorphism_polynomial_affine_space.

EXAMPLES:

sage: P.<x,y> = ProjectiveSpace(ZZ, 1)
sage: H = Hom(P, P)
sage: f = H([x^2 + y^2, y^2])
sage: f.dehomogenize(0)
Scheme endomorphism of Affine Space of dimension 1 over Integer Ring
Defn: Defined on coordinates by sending (y) to (y^2/(y^2 + 1))

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: H = Hom(P, P)
sage: f = H([x^2 - y^2, y^2])
sage: f.dehomogenize((0,1))
Scheme morphism:
    From: Affine Space of dimension 1 over Rational Field
    To:   Affine Space of dimension 1 over Rational Field
    Defn: Defined on coordinates by sending (y) to ((-y^2 + 1)/y^2)

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: H = Hom(P, P)
sage: f = H([x^2 + y^2, y^2 - z^2, 2*z^2])
sage: f.dehomogenize(2)
Scheme endomorphism of Affine Space of dimension 2 over Rational Field
Defn: Defined on coordinates by sending (x, y) to
(1/2*x^2 + 1/2*y^2, 1/2*y^2 - 1/2)

sage: R.<t> = PolynomialRing(QQ)
sage: P.<x,y,z> = ProjectiveSpace(FractionField(R),2)
sage: H = Hom(P,P)
sage: f = H([x^2 + t*y^2, t*y^2 - z^2, t*z^2])
sage: f.dehomogenize(2)
Scheme endomorphism of Affine Space of dimension 2 over Fraction Field of Univariate Polynomial Ring in t over Rational Field

(continues on next page)
Defn: Defined on coordinates by sending \((x, y)\) to 
\[
\left(\frac{1}{t}x^2 + y^2, y^2 - \frac{1}{t}\right)
\]

sage: \(P.<x,y,z> = \text{ProjectiveSpace}(\mathbb{Z}Z, 2)\)
sage: \(X = P.\text{subscheme}(x^2 - y^2)\)
sage: \(H = \text{Hom}(X, X)\)
sage: \(f = H([x^2, y^2, x*z])\)
sage: \(f.\text{dehomogenize}(2)\)

# needs sage.libs.singular

Scheme endomorphism of Closed subscheme of Affine Space of dimension 2 over Integer Ring defined by: \(x^2 - y^2\)
Defn: Defined on coordinates by sending \((x, y)\) to \((x, y^2/x)\)

sage: \(P.<x,y> = \text{ProjectiveSpace}(\mathbb{Q}Q, 1)\)
sage: \(H = \text{End}(P)\)
sage: \(f = H([x^2 - 2*x*y, y^2])\)
sage: \(f.\text{dehomogenize}(0).\text{homogenize}(0) == f\)
True

sage: # needs sage.rings.number_field
sage: \(K.<w> = \text{QuadraticField}(3)\)
sage: \(O = K.\text{ring_of_integers()}\)
sage: \(P.<x,y> = \text{ProjectiveSpace}(O, 1)\)
sage: \(H = \text{End}(P)\)
sage: \(f = H([x^2 - O(w)*y^2, y^2])\)
sage: \(f.\text{dehomogenize}(1)\)

Scheme endomorphism of Affine Space of dimension 1 over Maximal Order in Number Field in \(w\) with defining polynomial \(x^2 - 3\)
with \(w = 1.732050807568878?\)
Defn: Defined on coordinates by sending \((x)\) to \((x^2 - w)\)

sage: \(P1.<x,y> = \text{ProjectiveSpace}(\mathbb{Q}Q, 1)\)
sage: \(P2.<u,v,w> = \text{ProjectiveSpace}(\mathbb{Q}Q, 2)\)
sage: \(H = \text{Hom}(P2, P1)\)
sage: \(f = H([u*w, v^2 + w^2])\)
sage: \(f.\text{dehomogenize((2,1))}\)

Scheme morphism:
From: Affine Space of dimension 2 over Rational Field
To: Affine Space of dimension 1 over Rational Field
Defn: Defined on coordinates by sending \((u, v)\) to \((u/(v^2 + 1))\)

global_height\((\text{prec=None})\)

Return the global height of the coefficients as a projective point.

INPUT:

- \(\text{prec}\) – desired floating point precision (default: default \(\text{RealField}\) precision).

OUTPUT:

- a real number.

EXAMPLES:
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: H = Hom(P, P)
sage: f = H([1/1331*x^2 + 1/4000*y^2, 210*x*y]);
sage: f.global_height()
20.8348429892146

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: H = Hom(P, P)
sage: f = H([1/1331*x^2 + 1/4000*y^2, 210*x*y]);
sage: f.global_height(prec=11)
needs sage.symbolic
20.8

sage: P.<x,y,z> = ProjectiveSpace(ZZ, 2)
sage: H = Hom(P, P)
sage: f = H([4*x^2 + 100*y^2, 210*x*y, 10000*z^2]);
sage: f.global_height()
needs sage.symbolic
8.51719319141624

sage: # needs sage.rings.number_field
sage: R.<z> = PolynomialRing(QQ)
sage: K.<w> = NumberField(z^2 - 2)
sage: O = K.maximal_order()
sage: P.<x,y> = ProjectiveSpace(O, 1)
sage: H = Hom(P, P)
sage: f = H([2*x^2 + 3*O(w)*y^2, O(w)*y^2])
sage: f.global_height()
1.09861228866811

sage: # needs sage.rings.number_field sage.symbolic
sage: P.<x,y> = ProjectiveSpace(QQbar, 1)
sage: P2.<u,v,w> = ProjectiveSpace(QQbar, 2)
sage: H = Hom(P, P2)
sage: f = H([x^2 + QQbar(I)*x*y + 3*y^2, y^2, QQbar(sqrt(5))*x*y])
sage: f.global_height()
1.09861228866811

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: A.<z,w> = ProjectiveSpace(QQ, 1)
sage: H = Hom(P, A)
sage: f = H([1/1331*x^2 + 4000*y*z, y^2])
sage: f.global_height()
needs sage.symbolic
15.4877354584971

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem([[1/25*x^2 + 25/3*x*y + y^2, 1*y^2]])
sage: exp(f.global_height())
needs sage.symbolic

(continues on next page)
Scaling should not change the result:

```python
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem([1/25*x^2 + 25/3*x*y + y^2, 1*y^2])
sage: f.global_height()  # needs sage.symbolic
6.43775164973640
sage: c = 10000
sage: f.scale_by(c)
sage: f.global_height()  # needs sage.symbolic
6.43775164973640
```

**is_morphism()**

Return True if this map is a morphism.

The map is a morphism if and only if the ideal generated by the defining polynomials is the unit ideal (no common zeros of the defining polynomials).

**OUTPUT:**

- Boolean

**EXAMPLES:**

```python
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: H = Hom(P, P)
sage: f = H([x^2 + y^2, y^2])
sage: f.is_morphism()  # needs sage.libs.singular
True
```

```python
sage: P.<x,y,z> = ProjectiveSpace(RR, 2)
sage: H = Hom(P, P)
sage: f = H([x*z - y*z, x^2 - y^2, z^2])
sage: f.is_morphism()  # needs sage.libs.singular
False
```

```python
sage: R.<t> = PolynomialRing(GF(5))
sage: P.<x,y,z> = ProjectiveSpace(R, 2)
sage: H = Hom(P, P)
sage: f = H([x*z - t*y^2, x^2 - y^2, t*z^2])
sage: f.is_morphism()  # needs sage.libs.singular
True
```

Map that is not morphism on projective space, but is over a subscheme:

```python
sage: P.<x,y,z> = ProjectiveSpace(RR, 2)
sage: X = P.subscheme([x*y + y*z])
sage: H = Hom(X, X)
```
sage: f = H([x*z - y*z, x^2 - y^2, z^2])
sage: f.is_morphism()  
# Needs sage.libs.singular  
True

\texttt{local\_height}(v, prec=\texttt{None})

Return the maximum of the local height of the coefficients in any of the coordinate functions of this map.

INPUT:

- \( v \) – a prime or prime ideal of the base ring.
- \( \texttt{prec} \) – desired floating point precision (default: default RealField precision).

OUTPUT:

- a real number.

EXAMPLES:

```
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: H = Hom(P, P)
sage: f = H([1/1331*x^2 + 1/4000*y^2, 210*x*y])
sage: f.local_height(1331)  
# Needs sage.rings.real_mpfr  
7.19368581839511

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: H = Hom(P, P)
sage: f = H([1/1331*x^2 + 1/4000*y^2, 210*x*y])
sage: f.local_height(1331, prec=2)  
# Needs sage.rings.real_mpfr  
8.0
```

This function does not automatically normalize:

```
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: H = Hom(P, P)
sage: f = H([4*x^2 + 3/100*y^2, 8/210*x*y, 1/10000*z^2])
sage: f.local_height(2)  
# Needs sage.rings.real_mpfr  
2.77258872223978

sage: f.normalize_coordinates()  
# Needs sage.libs.singular

sage: f.local_height(2)  
# Needs sage.libs.singular

0.000000000000000
```

```
sage: # Needs sage.rings.number\_field
sage: R.<z> = PolynomialRing(QQ)
sage: K.<w> = NumberField(z^2 - 2)
sage: P.<x,y> = ProjectiveSpace(K, 1)
sage: H = Hom(P, P)
sage: f = H([2*x^2 + w/3*y^2, 1/w*y^2])
```

(continues on next page)
sage: f.local_height(K.ideal(3))
1.09861228866811

local_height_arch\(i, \text{prec=}\text{None}\)

Return the maximum of the local height at the \(i\)-th infinite place of the coefficients in any of the coordinate functions of this map.

INPUT:

• \(i\) – an integer.

• \(\text{prec}\) – desired floating point precision (default: default RealField precision).

OUTPUT:

• a real number.

EXAMPLES:

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: H = Hom(P, P)
sage: f = H([1/1331*x^2 + 1/4000*y^2, 210*x*y])
sage: f.local_height_arch(0) # needs sage.rings.real_mpfr
5.34710753071747

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: H = Hom(P, P)
sage: f = H([1/1331*x^2 + 1/4000*y^2, 210*x*y])
sage: f.local_height_arch(0, \text{prec}=5) # needs sage.rings.real_mpfr
5.2

sage: # needs sage.rings.number_field
sage: R.<z> = PolynomialRing(QQ)
sage: K.<w> = NumberField(z^2 - 2)
sage: P.<x,y> = ProjectiveSpace(K, 1)
sage: H = Hom(P, P)
sage: f = H([2*x^2 + w/3*y^2, 1/w*y^2])
sage: f.local_height_arch(1)
0.6931471805599453094172321214582

normalize_coordinates(**kwds)

Ensures that this morphism has integral coefficients. If the coordinate ring has a GCD, then it ensures that the coefficients have no common factor.

It also makes the leading coefficients of the first polynomial positive (if positive has meaning in the coordinate ring). This is done in place.

When \text{ideal} or \text{valuation} is specified, normalization occurs with respect to the absolute value defined by the \text{ideal} or \text{valuation}. That is, the coefficients are scaled such that one coefficient has absolute value 1 while the others have absolute value less than or equal to 1. Only supported when the base ring is a number field.

INPUT:

kwds:
• ideal – (optional) a prime ideal of the base ring of this morphism.

• valuation – (optional) a valuation of the base ring of this morphism.

OUTPUT:

• None.

EXAMPLES:

```python
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: H = Hom(P, P)
sage: f = H([5/4*x^3, 5*x*y^2])
sage: f.normalize_coordinates(); f
Scheme endomorphism of Projective Space of dimension 1 over Rational Field
   Defn: Defined on coordinates by sending (x : y) to (x^2 : 4*y^2)
```

```python
sage: P.<x,y,z> = ProjectiveSpace(GF(7), 2)
sage: X = P.subscheme(x^2 - y^2)
sage: H = Hom(X, X)
sage: f = H([x^3 + x*y^2, x*y^2, x*z^2])
# needs sage.libs.singular
sage: f.normalize_coordinates(); f
Scheme endomorphism of Closed subscheme of Projective Space of dimension 2
   over Finite Field of size 7 defined by: x^2 - y^2
   Defn: Defined on coordinates by sending (x : y : z) to (2*y^2 : y^2 : z^2)
```

```python
sage: R.<a,b> = QQ[]
sage: P.<x,y,z> = ProjectiveSpace(R, 2)
sage: H = End(P)
sage: f = H([a*(x*z + y^2)*x^2, a*b*(x*z + y^2)*y^2, a*(x*z + y^2)*z^2])
# needs sage.rings.number_field
sage: f.normalize_coordinates(); f
Scheme endomorphism of Projective Space of dimension 2 over
   Multivariate Polynomial Ring in a, b over Rational Field
   Defn: Defined on coordinates by sending (x : y : z) to (x^2 : b*y^2 : z^2)
```

```python
sage: K.<w> = QuadraticField(5)
sage: P.<x,y> = ProjectiveSpace(K, 1)
sage: f = DynamicalSystem([w*x^2 + (1/5*w)*y^2, w*y^2])
sage: f.normalize_coordinates(); f
Dynamical System of Projective Space of dimension 1 over Number Field in w
   with defining polynomial x^2 - 5 with w = 2.236067977499790?
   Defn: Defined on coordinates by sending (x : y) to (5*x^2 + y^2 : 5*y^2)
```

```
```
Defn: Defined on coordinates by sending \((x : y)\) to 
\((-100x^2 + (140b^2 + 140b + 140)x^y + (-77b^2 - 567b - 1057)y^2
: 100y^2)\)

We can used `ideal` to scale with respect to a norm defined by an ideal:

```
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem_projective([2*x^3, 2*x^2*y + 4*x*y^2])
sage: f.normalize_coordinates(ideal=2); f
Dynamical System of Projective Space of dimension 1 over Rational Field
Defn: Defined on coordinates by sending (x : y) to (x^3 : x^2*y + 2*x*y^2)
```

```
sage: # needs sage.rings.number_field
sage: R.<w> = QQ[]
sage: A.<a> = NumberField(w^2 + 1)
sage: P.<x,y,z> = ProjectiveSpace(A, 2)
sage: X = P.subscheme(x^2 - y^2)
sage: H = Hom(X, X)
sage: f = H([(a+1)*x^3 + 2*x*y^2, 4*x*y^2, 8*x*z^2])
sage: f.normalize_coordinates(ideal=A.prime_above(2)); f
Scheme endomorphism of Closed subscheme of Projective Space of dimension 2 over
Number Field in a with defining polynomial w^2 + 1 defined by:
(-a + 2)*x*y^2 : (-2*a + 2)*x*y^2 : (-4*a + 4)*x*z^2)
```

We can pass in a valuation to `valuation`:

```
sage: g = H([(a+1)*x^3 + 2*x*y^2, 4*x*y^2, 8*x*z^2])
#˓→ needs sage.rings.number_field
sage: g.normalize_coordinates(valuation=A.valuation(A.prime_above(2)))
#˓→ needs sage.rings.number_field
sage: g == f
#˓→ needs sage.rings.number_field
True
```

Check that #35797 is fixed:

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: K.<a> = NumberField(3*x^2 + 1)
sage: P.<z,w> = ProjectiveSpace(K, 1)
```

(continues on next page)
sage: f = DynamicalSystem_projective([a*(z^2 + w^2), z*w])
sage: f.normalize_coordinates(); f
Dynamical System of Projective Space of dimension 1 over
Number Field in a with defining polynomial 3*x^2 + 1
Defn: Defined on coordinates by sending (z : w) to
      ((-3/2*a + 1/2)*z^2 + (-3/2*a + 1/2)*w^2 : (-3/2*a - 3/2)*z*w)

sage: R.<a,b> = QQ[]
sage: P.<x,y,z> = ProjectiveSpace(FractionField(R), 2)
sage: H = End(P)
sage: f = H([a/b*(x*z + y^2)*x^2, a*b*(x*z + y^2)*y^2, a*(x*z + y^2)*z^2])
sage: f.normalize_coordinates(); f
Scheme endomorphism of Projective Space of dimension 2 over Fraction
Field of Multivariate Polynomial Ring in a, b over Rational Field
Defn: Defined on coordinates by sending (x : y : z) to
      (x^2 : (b^2)*y^2 : b*z^2)

scale_by(t)

Scales each coordinate by a factor of t.

A TypeError occurs if the point is not in the coordinate ring of the parent after scaling.

INPUT:

• t – a ring element.

OUTPUT:

• None.

EXAMPLES:

sage: A.<x,y> = ProjectiveSpace(QQ, 1)
sage: H = Hom(A, A)
sage: f = H([x^3 - 2*x*y^2, x^2*y])
sage: f.scale_by(1/x)
sage: f
Scheme endomorphism of Projective Space of dimension 1 over Rational Field
Defn: Defined on coordinates by sending (x : y) to (x^2 - 2*y^2 : x*y)

sage: R.<t> = PolynomialRing(QQ)
sage: P.<x,y> = ProjectiveSpace(R, 1)
sage: H = Hom(P,P)
sage: f = H([3/5*x^2, 6*y^2])
sage: f.scale_by(5/3*t); f
Scheme endomorphism of Projective Space of dimension 1 over
Univariate Polynomial Ring in t over Rational Field
Defn: Defined on coordinates by sending (x : y) to (t*x^2 : 10*t*y^2)

sage: P.<x,y,z> = ProjectiveSpace(GF(7), 2)
sage: X = P.subscheme(x^2 - y^2)
sage: H = Hom(X, X)
sage: f = H([x^2, y^2, z^2])
sage: f.scale_by(x - y); f

(continues on next page)
needs sage.libs.singular

Scheme endomorphism of Closed subscheme of Projective Space of dimension 2
over Finite Field of size 7 defined by: x^2 - y^2
Defn: Defined on coordinates by sending (x : y : z) to
(x^2 y^2 - y^3 : x^2 y^2 - y^3 : x^2 z^2 - y^2 z^2)

wronskian_ideal()

Return the ideal generated by the critical point locus.

This is the vanishing of the maximal minors of the Jacobian matrix. Not implemented for subvarieties.

OUTPUT: an ideal in the coordinate ring of the domain of this map.

EXAMPLES:

sage: # needs sage.rings.number_field
sage: R.<x> = PolynomialRing(QQ)
sage: K.<w> = NumberField(x^2 + 11)
sage: P.<x,y> = ProjectiveSpace(K, 1)
sage: H = End(P)
sage: f = H([x^2 - w*y^2, w*y^2])
sage: f.wronskian_ideal()
Ideal ((4*w)*x*y) of Multivariate Polynomial Ring in x, y
over Number Field in w with defining polynomial x^2 + 11

sage: # needs sage.rings.number_field
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: P2.<u,v,t> = ProjectiveSpace(K, 2)
sage: H = Hom(P, P2)
sage: f = H([x^2 - 2*y^2, y^2, x*y])
sage: f.wronskian_ideal()
Ideal (4*x*y, 2*x^2 + 4*y^2, -2*y^2) of
Multivariate Polynomial Ring in x, y over Rational Field

class sage.schemes.projective.projective_morphism.SchemeMorphism_polynomial_projective_space_field(parent, polys, check)

Bases: SchemeMorphism_polynomial_projective_space

base_indeterminacy_locus()

Return the base indeterminacy locus of this map.

The base indeterminacy locus is the set of points in projective space at which all of the defining polynomials
of the rational map simultaneously vanish.

OUTPUT: a subscheme of the domain of the map

EXAMPLES:

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: H = End(P)
sage: f = H([x*z - y*z, x^2 - y^2, z^2])
sage: f.base_indeterminacy_locus()
Closed subscheme of Projective Space of dimension 2 over Rational Field defined by:
\[ x^2 - y^2, \]
\[ z^2 \]

```
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: H = Hom(P)
sage: f = H([x^2, y^2, z^2])
sage: f.base_indeterminacy_locus()
Closed subscheme of Projective Space of dimension 2 over Rational Field
defined by:
\[ x^2, \]
\[ y^2, \]
\[ z^2 \]
```

```
sage: P1.<x,y,z> = ProjectiveSpace(RR, 2)
sage: P2.<t,u,v,w> = ProjectiveSpace(RR, 3)
sage: H = Hom(P1, P2)
sage: h = H([y^3*z^3, x^3*z^3, y^3*z^3, x^2*y^2*z^2])
sage: h.base_indeterminacy_locus()  # needs sage.rings.real_mpfr
Closed subscheme of Projective Space of dimension 2 over Real Field with 53 bits of precision defined by:
\[ y^3*z^3, \]
\[ x^3*z^3, \]
\[ y^3*z^3, \]
\[ x^2*y^2*z^2 \]
```

If defining polynomials are not normalized, output scheme will not be normalized:

```
sage: P.<x,y,z> = ProjectiveSpace(QQ,2)
sage: H = End(P)
sage: f = H([x*x^2,x*y^2,x*z^2])
sage: f.base_indeterminacy_locus()
Closed subscheme of Projective Space of dimension 2 over Rational Field
defined by:
\[ x^3, \]
\[ x*y^2, \]
\[ x*z^2 \]
```

\textbf{image()}

Return the scheme-theoretic image of the morphism.

\textbf{OUTPUT:} a subscheme of the ambient space of the codomain

\textbf{EXAMPLES:}

```
sage: P2.<x0,x1,x2> = ProjectiveSpace(QQ, 2)
sage: f = P2.hom([x0^3, x0^2*x1, x0*x1^2], P2)
sage: f.image()  # needs sage.libs.singular
Closed subscheme of Projective Space of dimension 2 over Rational Field defined by:
\[ x1^2 - x0*x2 \]
```

(continues on next page)
sage: f = P2.hom([x0 - x1, x0 - x2, x1 - x2], P2)
sage: f.image()  # needs sage.libs.singular
Closed subscheme of Projective Space of dimension 2 over Rational Field defined by:
x0 - x1 + x2

sage: P2.<x0,x1,x2> = ProjectiveSpace(QQ, 2)
sage: A2.<x,y> = AffineSpace(QQ, 2)
sage: f = P2.hom([1, x0/x1], A2)
sage: f.image()  # needs sage.libs.singular
Closed subscheme of Affine Space of dimension 2 over Rational Field defined by:
-x + 1

indeterminacy_locus()

Return the indeterminacy locus of this map as a rational map on the domain.

The indeterminacy locus is the intersection of all the base indeterminacy locuses of maps that define the same rational map as by this map.

OUTPUT: a subscheme of the domain of the map

EXAMPLES:

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: H = End(P)
sage: f = H([x^2, y^2, z^2])
sage: f.indeterminacy_locus()  # needs sage.libs.singular
... DeprecationWarning: The meaning of indeterminacy_locus() has changed. Read the docstring. See https://github.com/sagemath/sage/issues/29145 for details.
Closed subscheme of Projective Space of dimension 2 over Rational Field defined by:
z, y, x

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: H = End(P)
sage: f = H([x*z - y*z, x^2 - y^2, z^2])
sage: f.indeterminacy_locus()  # needs sage.libs.singular
Closed subscheme of Projective Space of dimension 2 over Rational Field defined by:
z, x^2 - y^2

There is related base_indeterminacy_locus() method. This computes the indeterminacy locus only from the defining polynomials of the map.
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: H = End(P)
sage: f = H([x*z - y*z, x^2 - y^2, z^2])
sage: f.base_indeterminacy_locus()
Closed subscheme of Projective Space of dimension 2 over Rational Field defined
by:
  x*z - y*z,
  x^2 - y^2,
  z^2

indeterminacy_points\((F=None, base=False)\)

Return the points in the indeterminacy locus of this map.

If the dimension of the indeterminacy locus is not zero, an error is raised.

INPUT:

- \(F\) – a field; if not given, the base ring of the domain is assumed
- \(base\) – if True, the base indeterminacy locus is used

OUTPUT: indeterminacy points of the map defined over \(F\)

EXAMPLES:

```
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: H = End(P)
sage: f = H([x*z - y*z, x^2 - y^2, z^2])
sage: f.indeterminacy_points()  # needs sage.libs.singular
... DeprecationWarning: The meaning of indeterminacy_locus() has changed. Read the docstring. See https://github.com/sagemath/sage/issues/29145 for details.
[(0 : 1 : -1), (0 : 1 : 1)]
```

```
sage: P1.<x,y,z> = ProjectiveSpace(RR, 2)
sage: P2.<t,u,v,w> = ProjectiveSpace(RR, 3)
sage: H = Hom(P1, P2)
sage: h = H([x + y, y, z + y, y])
sage: set_verbose(None)
sage: h.indeterminacy_points(base=True)  # needs sage.libs.singular
[]
sage: g = H([y^3*z^3, x^3*z^3, y^3*z^3, x^2*y^2*z^2])
sage: g.indeterminacy_points(base=True)  # needs sage.libs.singular
Traceback (most recent call last):
... ValueError: indeterminacy scheme is not dimension 0
```

(continues on next page)
rational_preimages(Q, k=1)

Determine all of the rational $k$-th preimages of $Q$ by this map.

Given a rational point $Q$ in the domain of this map, return all the rational points $P$ in the domain with $f^k(P) = Q$. In other words, the set of $k$-th preimages of $Q$. The map must be defined over a number field and be an endomorphism for $k > 1$. 

14.2. Morphisms on projective schemes
If \( Q \) is a subscheme, then return the subscheme that maps to \( Q \) by this map. In particular, 
\[
(f^{-k}(V(h_1, \ldots, h_t)) = V(h_1 \circ f^k, \ldots, h_t \circ f^k).
\]

**INPUT:**
- \( Q \) - a rational point or subscheme in the domain of this map.
- \( k \) - positive integer.

**OUTPUT:**
- a list of rational points or a subscheme in the domain of this map.

**EXAMPLES:**

```python
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: H = End(P)
sage: f = H([16*x^2 - 29*y^2, 16*y^2])
sage: f.rational_preimages(P(-1, 4)) # needs sage.libs.singular
[(-5/4 : 1), (5/4 : 1)]
```

A non-periodic example

```python
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: H = End(P)
sage: f = H([x^2 + y^2, 2*x*y])
sage: f.rational_preimages(P(17, 15)) # needs sage.libs.singular
[(3/5 : 1), (5/3 : 1)]
```

```python
sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: H = End(P)
sage: f = H([x^2 - 2*y^2 - 3*w^2, -2*x^2 + y^2 - 2*z^2 + 4*y*w + 3*w^2,
          x^2 - y^2 + 2*z^2 - 2*y*w - w^2, w^2])
sage: f.rational_preimages(P(0, -1, 0, 1)) # needs sage.libs.singular
[]
```

```python
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: H = End(P)
sage: f = H([x^2 + y^2, 2*x*y])
sage: f.rational_preimages(CC.0, 1) # needs sage.libs.singular
Traceback (most recent call last):
(continues on next page)
```
A number field example

```python
sage: # needs sage.rings.number_field
sage: z = QQ['z'].0
sage: K.<a> = NumberField(z^2 - 2)
sage: P.<x,y> = ProjectiveSpace(K, 1)
sage: H = End(P)
sage: f = H([x^2 + y^2, y^2])
sage: f.rational_preimages(P(3, 1)) # needs sage.libs.singular
[(-a : 1), (a : 1)]

sage: # needs sage.rings.number_field
sage: z = QQ['z'].0
sage: K.<a> = NumberField(z^2 - 2)
sage: P.<x,y,z> = ProjectiveSpace(K, 2)
sage: X = P.subscheme([x^2 - z^2])
sage: H = End(X)
sage: f = H([x^2 - z^2, a*y^2, z^2 - x^2])
sage: f.rational_preimages(X([1, 2, -1])) # needs sage.libs.singular
[]

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: X = P.subscheme([x^2 - z^2])
sage: H = End(X)
sage: f = H([x^2-z^2, y^2, z^2-x^2])
sage: f.rational_preimages(X([0, 1, 0])) # needs sage.libs.singular
Closed subscheme of Projective Space of dimension 2 over Rational Field defined by:
   x^2 - z^2,
   -x^2 + z^2,
   0,
   -x^2 + z^2

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: H = End(P)
sage: f = H([x^2 - y^2, y^2])
sage: f.rational_preimages(P.subscheme([x])) # needs sage.libs.singular
Closed subscheme of Projective Space of dimension 1 over Rational Field defined by: x^2 - y^2

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: H = End(P)
sage: f = H([x^2 - 29/16*y^2, y^2])
sage: f.rational_preimages(P(5/4, 1), k=4) # needs sage.libs.singular
```

14.2. Morphisms on projective schemes
reduce_base_field()

Return this map defined over the field of definition of the coefficients.

The base field of the map could be strictly larger than the field where all of the coefficients are defined. This function reduces the base field to the minimal possible. This can be done when the base ring is a number field, QQbar, a finite field, or algebraic closure of a finite field.

OUTPUT: A scheme morphism.

EXAMPLES:

sage: # needs sage.rings.finite_rings
sage: K.<t> = GF(3^4)
sage: P.<x,y> = ProjectiveSpace(K, 1)
sage: P2.<a,b,c> = ProjectiveSpace(K, 2)
sage: H = End(P)
sage: H2 = Hom(P, P2)
sage: H3 = Hom(P2, P)
sage: f = H([x^2 + (2*t^3 + 2*t^2 + 1)*y^2, y^2])
sage: f.reduce_base_field() # needs sage.libs.singular sage.modules
Scheme endomorphism of Projective Space of dimension 1 over Finite Field in t2 of size 3^2
  Defn: Defined on coordinates by sending (x : y) to (x^2 - y^2 : y^2 : -x*y)
sage: f2 = H2([x^2 + 5*y^2, y^2, 2*x*y])
sage: f2.reduce_base_field() # needs sage.libs.singular sage.modules
Scheme morphism:
  From: Projective Space of dimension 1 over Finite Field of size 3
  To: Projective Space of dimension 2 over Finite Field of size 3
  Defn: Defined on coordinates by sending (x : y) to (x^2 - y^2 : y^2 : -x*y)
sage: f3 = H3([a^2 + t*b^2, c^2])
sage: f3.reduce_base_field() # needs sage.libs.singular sage.modules
Scheme morphism:
  From: Projective Space of dimension 2 over Finite Field of size 3
  To: Projective Space of dimension 1 over Finite Field of size 3
  Defn: Defined on coordinates by sending (a : b : c) to (a^2 + t*b^2 : c^2)
sage: H = End(P)
sage: f = H([x^2 + 2*y^2, y^2])
sage: f.reduce_base_field()  
\texttt{\# needs sage.libs.singular}
Scheme endomorphism of Projective Space of dimension 1 over Rational Field 
  Defn: Defined on coordinates by sending (x : y) to (x^2 + 2*y^2 : y^2)

sage: # needs sage.rings.finite_rings
sage: K.<v> = GF(5)
sage: L = K.algebraic_closure()
sage: P.<x,y> = ProjectiveSpace(L, 1)
sage: H = End(P)
sage: f = H([(L.gen(2))*x^2 + L.gen(4)*y^2, x*y])
sage: f.reduce_base_field()  
\texttt{\# needs sage.libs.singular}
Scheme endomorphism of Projective Space of dimension 1 over Finite Field in z4 of size 5^4 
  Defn: Defined on coordinates by sending (x : y) to 
    ((z4^3 + z4^2 + z4 - 2)*x^2 + z4*y^2 : x*y)

sage: f = DynamicalSystem_projective([L.gen(3)*x^2 + L.gen(2)*y^2, x*y])  
\texttt{\# needs sage.schemes}

sage: f.reduce_base_field()  
\texttt{\# needs sage.libs.singular sage.schemes}
Dynamical System of Projective Space of dimension 1 over Finite Field in z6 of size 5^6 
  Defn: Defined on coordinates by sending (x : y) to 
    (-z6^5 + z6^4 - z6^3 - z6^2 - 2*z6 - 2)*x^2 
    + (z6^5 - 2*z6^4 + z6^2 - z6 + 1)*y^2 : x*y)

class sage.schemes.projective.projective_morphism.SchemeMorphism_polynomial_projective_space_finite_field

Bases: \texttt{SchemeMorphism_polynomial_projective_space_field}

class sage.schemes.projective.projective_morphism.SchemeMorphism_polynomial_projective_subscheme_field

Bases: \texttt{SchemeMorphism_polynomial_projective_space_field}

Morphisms from subschemes of projective spaces defined over fields.

\texttt{degree()}

Return the degree of this rational map.

\texttt{EXAMPLES:}

sage: # needs sage.schemes
sage: k = GF(11)
sage: E = EllipticCurve(k, [1,1])
sage: Q = E(6, 5)
sage: phi = E.scalar_multiplication(2)
sage: mor = phi.as_morphism()
graph()

Return the graph of this morphism.

The graph is a subscheme of the product of the ambient spaces of the domain and the codomain. If the ambient space of the codomain is an affine space, it is first embedded into a projective space.

EXAMPLES:

We get the standard quadratic curve as the graph of a quadratic function of an affine line.

```
sage: A1.<x> = AffineSpace(1, QQ)
sage: X = A1.subscheme(0)  # affine line
sage: phi = X.hom([x^2], A1)
sage: mor = phi.homogenize(0)  # needs sage/libs/singular
sage: G = mor.graph(); G
Closed subscheme of Product of projective spaces P^1 x P^1
over Rational Field defined by: x1^2*x2 - x0^2*x3
```

image()

Return the scheme-theoretic image of the morphism.

EXAMPLES:

```
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: P2.<x0,x1,x2> = ProjectiveSpace(QQ, 2)
sage: X = P2.subscheme([0])
sage: f = X.hom([x1,x0], P)
sage: f.image()  # needs sage/libs/singular
Closed subscheme of Projective Space of dimension 1 over Rational Field defined by:
(no polynomials)
```

```
sage: P2.<x,y,z> = ProjectiveSpace(QQ,2)
sage: X = P2.subscheme([x^3 - x*y^2 + y^3])
sage: f = X.hom([x^2*z, x*y, x^3 + y^3], P2)
sage: f.image()  # needs sage/libs/singular
Closed subscheme of Projective Space of dimension 2 over Rational Field defined by:
x^6 + 2*x^3*y^3 + x*y^5 + y^6 - x^3*y^2*z - y^5*z
```

indeterminacy_locus()

Return the indeterminacy locus of this map.
The map defines a rational map on the domain. The output is the subscheme of the domain on which the rational map is not defined by any representative of the rational map. See `representatives()`.

 EXAMPLES:

```
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: P2.<x0,x1,x2> = ProjectiveSpace(QQ, 2)
sage: X = P2.subscheme([0])
sage: f = X.hom([x1,x0], P)
sage: L = f.indeterminacy_locus()  # needs sage.libs.singular

sage: L.rational_points()  # needs sage.libs.singular
[(0 : 0 : 1)]
```

```
sage: P2.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: P1.<a,b> = ProjectiveSpace(QQ, 1)
sage: X = P2.subscheme([x^2 - y^2 - y*z])
sage: f = X.hom([x,y], P1)
sage: f.indeterminacy_locus()  # needs sage.libs.singular
Closed subscheme of Projective Space of dimension 2 over Rational Field defined by:
    z, y, x
```

```
sage: P3.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: P2.<a,b,c> = ProjectiveSpace(QQ, 2)
sage: X = P3.subscheme(x^2 - w*y - x*z)
sage: f = X.hom([x/z, y/x], A2)
sage: L = f.indeterminacy_locus()  # needs sage.libs.singular

sage: L.rational_points()  # needs sage.libs.singular
[(0 : 0 : 0 : 1), (0 : 1 : 0 : 0)]
```

```
sage: P3.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: A2.<a,b> = AffineSpace(QQ, 2)
sage: X = P3.subscheme(x^2 - w*y - x*z)
sage: f = X.hom([x/z, y/x], A2)
sage: L = f.indeterminacy_locus()  # needs sage.libs.singular

sage: L.rational_points()  # needs sage.libs.singular
[(0 : 0 : 0 : 1), (0 : 1 : 0 : 0)]
```

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sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: X = P.subscheme(x - y)
sage: H = End(X)
sage: f = H([x^2 - 4*y^2, y^2 - z^2, 4*z^2 - x^2])
sage: Z = f.indeterminacy_locus(); Z
Closed subscheme of Projective Space of dimension 2 over Rational Field defined by:
  z,
y,
x

\textbf{is\_morphism()}

Return True if the map is defined everywhere on the domain.

\textbf{EXAMPLES:}

sage: P2.<x,y,z> = ProjectiveSpace(QQ,2)
sage: P1.<a,b> = ProjectiveSpace(QQ, 1)
sage: X = P2.subscheme([x^2 - y^2 - y*z])
sage: f = X.hom([x,y], P1)
sage: f.is_morphism()  # needs sage.libs.singular
True

\textbf{projective\_degrees()}

Return the projective degrees of this rational map.

\textbf{EXAMPLES:}

sage: # needs sage.schemes
sage: k = GF(11)
sage: E = EllipticCurve(k, [1,1])
sage: Q = E(6, 5)
sage: phi = E.scalar_multiplication(2)
sage: mor = phi.as_morphism()
sage: mor.projective_degrees()
(12, 3)

\textbf{representatives()}

Return all maps representing the same rational map as by this map.

\textbf{EXAMPLES:}

sage: P2.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: X = P2.subscheme(0)
sage: f = X.hom([x^2*y, x^2*z, x*y*z], P2)
sage: f.representatives()  # needs sage.libs.singular
[Scheme morphism:
  From: Closed subscheme of Projective Space of dimension 2 over Rational Field defined by: 0
  To: Projective Space of dimension 2 over Rational Field
  Defn: Defined on coordinates by sending (x : y : z) to (x^2*y : x^2*z : y*z)
sage: P2.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: P1.<a,b> = ProjectiveSpace(QQ, 1)
sage: X = P2.subscheme([x^2 - y^2 - y*z])
sage: f = X.hom([x, y], P1)
sage: f.representatives()  # needs sage.libs.singular
[Scheme morphism:
  From: Closed subscheme of Projective Space of dimension 2
  over Rational Field defined by: x^2 - y^2 - y*z
  To: Projective Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x : y : z) to (y + z : x),
  Scheme morphism:
  From: Closed subscheme of Projective Space of dimension 2
  over Rational Field defined by: x^2 - y^2 - y*z
  To: Projective Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x : y : z) to (x : y)]
sage: g = _[0]  # needs sage.libs.singular
sage: g.representatives()  # needs sage.libs.singular
[Scheme morphism:
  From: Closed subscheme of Projective Space of dimension 2
  over Rational Field defined by: x^2 - y^2 - y*z
  To: Projective Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x : y : z) to (y + z : x),
  Scheme morphism:
  From: Closed subscheme of Projective Space of dimension 2
  over Rational Field defined by: x^2 - y^2 - y*z
  To: Projective Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x : y : z) to (x : y)]

sage: P2.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: X = P2.subscheme([x^2 - y^2 - y*z])
sage: A1.<a> = AffineSpace(QQ, 1)
sage: g = X.hom([y/x], A1)
sage: g.representatives()  # needs sage.libs.singular
[Scheme morphism:
  From: Closed subscheme of Projective Space of dimension 2
  over Rational Field defined by: x^2 - y^2 - y*z
  To: Affine Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x : y : z) to (y + z : x),
  Scheme morphism:
  From: Closed subscheme of Projective Space of dimension 2
  over Rational Field defined by: x^2 - y^2 - y*z
  To: Affine Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x : y : z) to (x : y)]
sage: g0, g1 = _  # needs sage.libs.singular
sage: emb = A1.projective_embedding(0)
sage: emb^g0  # needs sage.libs.singular
sage: emb^g1  # needs sage.libs.singular

(continues on next page)
Scheme morphism:
- From: Closed subscheme of Projective Space of dimension 2 over Rational Field defined by: \(x^2 - y^2 - yz\)
- To: Projective Space of dimension 1 over Rational Field
- Defn: Defined on coordinates by sending \((x : y : z)\) to \((y + z : x)\)

```
sage: emb^g1  
˓→ needs sage.libs.singular
```

Scheme morphism:
- From: Closed subscheme of Projective Space of dimension 2 over Rational Field defined by: \(x^2 - y^2 - yz\)
- To: Projective Space of dimension 1 over Rational Field
- Defn: Defined on coordinates by sending \((x : y : z)\) to \((x : y)\)

ALGORITHM:
- The algorithm is from Proposition 1.1 in [Sim2004].

### 14.3 Points on projective varieties

Scheme morphism for points on projective varieties

**AUTHORS:**
- David Kohel, William Stein
- William Stein (2006-02-11): fixed bug where \(P(0,0,0)\) was allowed as a projective point.
- Volker Braun (2011-08-08): Renamed classes, more documentation, misc cleanups.
- Ben Hutz (June 2012) added support for projective ring; (March 2013) iteration functionality and new directory structure for affine/projective, height functionality

**class** `sage.schemes.projective.projective_point.SchemeMorphism_point_abelian_variety_field(X, v, check=True)`

Based on `AdditiveGroupElement`, `SchemeMorphism_point_projective_field`

A rational point of an abelian variety over a field.

**EXAMPLES:**

```
sage: # needs sage.schemes
sage: E = EllipticCurve([0,0,1,-1,0])
sage: origin = E(0)
sage: origin.domain()  
Spectrum of Rational Field
sage: origin.codomain()  
Elliptic Curve defined by y^2 + y = x^3 - x over Rational Field
```

**class** `sage.schemes.projective.projective_point.SchemeMorphism_point_projective_field(X, v, check=True)`

Based on `SchemeMorphism_point_projective_ring`

A rational point of projective space over a field.

**INPUT:**
• **X** – a homset of a subscheme of an ambient projective space over a field \( K \).
• **v** – a list or tuple of coordinates in \( K \).
• **check** – boolean (optional, default: True). Whether to check the input for consistency.

### EXAMPLES:

```sage
sage: # needs sage.rings.real_mpfr
sage: P = ProjectiveSpace(3, RR)
sage: P(2, 3, 4, 5)
(0.400000000000000 : 0.600000000000000 : 0.800000000000000 : 1.000000000000000)
```

#### clear_denominators()

scales by the least common multiple of the denominators.

OUTPUT: None.

### EXAMPLES:

```sage
sage: R.<t> = PolynomialRing(QQ)
sage: P.<x,y,z> = ProjectiveSpace(FractionField(R), 2)
sage: Q = P([t, 3/t^2, 1])
sage: Q.clear_denominators(); Q
(t^3 : 3 : t^2)
```

```sage
sage: # needs sage.rings.number_field
sage: R.<x> = PolynomialRing(QQ)
sage: K.<w> = NumberField(x^2 - 3)
sage: P.<x,y,z> = ProjectiveSpace(K, 2)
sage: Q = P([1/w, 3, 0])
sage: Q.clear_denominators(); Q
(w : 9 : 0)
```

```sage
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: X = P.subscheme(x^2 - y^2)
sage: Q = X([1/2, 1/2, 1])
sage: Q.clear_denominators(); Q
(1 : 1 : 2)
```

```sage
sage: PS.<x,y> = ProjectiveSpace(QQ, 1)
sage: Q = PS.point([1, 2/3], False); Q
(1 : 2/3)
sage: Q.clear_denominators(); Q
(3 : 2)
```

#### intersection_multiplicity(X)

Return the intersection multiplicity of the codomain of this point and \( X \) at this point.

This uses the intersection_multiplicity implementations for projective/affine subschemes. This point must be a point of a projective subscheme.

INPUT:

• **X** – a subscheme in the same ambient space as that of the codomain of this point.

OUTPUT: Integer.
EXAMPLES:

```python
sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: X = P.subscheme([x*z - y^2])
sage: Y = P.subscheme([x^3 - y*w^2 + z*w^2, x*y - z*w])
sage: Q1 = X([1/2, 1/4, 1/8, 1])
sage: Q1.intersection_multiplicity(Y)  # needs sage.libs.singular
1
sage: Q2 = X([0,0,0,1])
sage: Q2.intersection_multiplicity(Y)  # needs sage.libs.singular
5
sage: Q3 = X([0,0,1,0])
sage: Q3.intersection_multiplicity(Y)  # needs sage.libs.singular
6
```

```
sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: X = P.subscheme([x^2 - y^2])
sage: Q = P([1,1,1,0]); Q
(1 : 1 : 1)
sage: Q.intersection_multiplicity(X)
Traceback (most recent call last):
  ...TypeError: this point must be a point on a projective subscheme
```

**multiplicity()**

Return the multiplicity of this point on its codomain.

Uses the subscheme multiplicity implementation. This point must be a point on a projective subscheme.

**OUTPUT:** an integer.

**EXAMPLES:**

```
sage: P.<x,y,z,w,t> = ProjectiveSpace(QQ, 4)
sage: X = P.subscheme([y^6 - x^3*w^2*t + t^5*w, x^2 - t^2])
sage: Q1 = X([1,0,2,1,1])
sage: Q1.multiplicity()  # needs sage.libs.singular
1
sage: Q2 = X([0,0,-2,1,0])
sage: Q2.multiplicity()  # needs sage.libs.singular
8
```

**normalize_coordinates()**

Normalizes the point so that the last non-zero coordinate is 1.

**OUTPUT:** None.

**EXAMPLES:**

```
sage: P.<x,y,z> = ProjectiveSpace(GF(5), 2)
sage: Q = P.point([GF(5)(1), GF(5)(3), GF(5)(0)], False); Q
(1 : 3 : 0)
```

(continues on next page)
sage: Q.normalize_coordinates(); Q
(2 : 1 : 0)

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: X = P.subscheme(x^2 - y^2);
sage: Q = X.point([23, 23, 46], False); Q
(23 : 23 : 46)
sage: Q.normalize_coordinates(); Q
(1/2 : 1/2 : 1)

class sage.schemes.projective.projective_point.SchemeMorphism_point_projective_finite_field(X, v, check=True)
Bases: SchemeMorphism_point_projective_field
class sage.schemes.projective.projective_point.SchemeMorphism_point_projective_ring(X, v, check=True)
Bases: SchemeMorphism_point
A rational point of projective space over a ring.

INPUT:

• X – a homset of a subscheme of an ambient projective space over a ring K.
• v – a list or tuple of coordinates in K.
• check – boolean (optional, default:True). Whether to check the input for consistency.

EXAMPLES:

sage: P = ProjectiveSpace(2, ZZ)
sage: P(2,3,4)
(2 : 3 : 4)

dehomogenize(n)
    Dehomogenizes at the nth coordinate.

INPUT:

• n – non-negative integer.

OUTPUT:

• SchemeMorphism_point_affine.

EXAMPLES:

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: X = P.subscheme(x^2 - y^2)
sage: Q = X(23, 23, 46)
sage: Q.dehomogenize(2) # needs sage.libs.singular
(1/2, 1/2)
sage: # needs sage.libs.pari
sage: R.<t> = PolynomialRing(QQ)
sage: S = R.quo(R.ideal(t^3))
sage: P.<x,y,z> = ProjectiveSpace(S, 2)
sage: Q = P(t, 1, 1)
sage: Q.dehomogenize(1)
(tbar, 1)

sage: P.<x,y,z> = ProjectiveSpace(GF(5), 2)
sage: Q = P(1, 3, 1)
sage: Q.dehomogenize(0)
(3, 1)

sage: P.<x,y,z> = ProjectiveSpace(GF(5), 2)
sage: Q = P(1, 3, 0)
sage: Q.dehomogenize(2)
Traceback (most recent call last):
... ValueError: can't dehomogenize at 0 coordinate

global_height(prec=None)
Return the absolute logarithmic height of the point.
INPUT:
• prec – desired floating point precision (default: default RealField precision).
OUTPUT:
• a real number.
EXAMPLES:

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: Q = P.point([4, 4, 1/30])
sage: Q.global_height()  # needs sage.symbolic
4.78749174278205

sage: P.<x,y,z> = ProjectiveSpace(ZZ, 2)
sage: Q = P([4, 1, 30])
sage: Q.global_height()  # needs sage.symbolic
3.40119738166216

sage: R.<x> = PolynomialRing(QQ)
sage: k.<w> = NumberField(x^2 + 5)  # needs sage.rings.number_field
sage: A = ProjectiveSpace(k, 2, 'z')  # needs sage.rings.number_field
sage: A([3, 5*w + 1, 1]).global_height(prec=100)  # needs sage.rings.number_field
2.4181409534757389986565376694
is_preperiodic$(f, \text{err}=0.1, \text{return_period}=\text{False})$

Determine if the point is preperiodic with respect to the map $f$.

This is implemented for both projective space and subschemes. There are two optional keyword arguments: $\text{error bound}$ sets the error bound used in the canonical height computation and $\text{return period}$ a boolean which controls if the period is returned if the point is preperiodic. If $\text{return period}$ is True and this point is not preperiodic, then $(0,0)$ is returned for the period.

**ALGORITHM:**

We know that a point is preperiodic if and only if it has canonical height zero. However, we can only compute the canonical height up to numerical precision. This function first computes the canonical height of the point to the given error bound. If it is larger than that error bound, then it must not be preperiodic. If it is less than the error bound, then we expect preperiodic. In this case we begin computing the orbit stopping if either we determine the orbit is finite, or the height of the point is large enough that it must be wandering. We can determine the height cutoff by computing the height difference constant, i.e., the bound between the height and the canonical height of a point (which depends only on the map and not the point itself). If the height of the point is larger than the difference bound, then the canonical height cannot be zero so the point cannot be preperiodic.

**INPUT:**

- $f$ – an endomorphism of this point’s codomain.

**kwds:**

- $\text{err}$ – a positive real number (optional - default: 0.1).
- $\text{return_period}$ – boolean (optional - default: False).

**OUTPUT:**

- boolean – True if preperiodic.
- if $\text{return_period}$ is True, then $(0,0)$ if wandering, and $(m,n)$ if preperiod $m$ and period $n$.

**EXAMPLES:**

```python
sage: P.<x,y> = ProjectiveSpace(QQ, 1)  # needs sage.schemes
sage: f = DynamicalSystem_projective([x^3 - 3*x*y^2, y^3], domain=P)  # needs sage.schemes
sage: Q = P(-1, 1)
sage: Q.is_preperiodic(f)  # (continues on next page)
```
needs sage.libs.singular sage.schemes
True

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: X = P.subscheme(z)
sage: f = DynamicalSystem([x^2 - y^2, y^2, z^2], domain=X) #
→needs sage.schemes
sage: p = X((-1, 1, 0))
sage: p.is_preperiodic(f, return_period=True) #
→needs sage.libs.singular sage.schemes
(0, 2)

sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2 - 29/16*y^2, y^2], domain=P) #
→needs sage.schemes
sage: Q = P(1, 4)
sage: Q.is_preperiodic(f, return_period=True) #
→needs sage.libs.singular sage.schemes
(1, 3)
sage: Q = P(1, 1)
sage: Q.is_preperiodic(f, return_period=True) #
→needs sage.libs.singular sage.schemes
(0, 0)

sage: # needs sage.rings.number_field
sage: R.<x> = PolynomialRing(QQ)
sage: K.<a> = NumberField(x^2 + 1)
sage: P.<x,y> = ProjectiveSpace(K, 1)
sage: f = DynamicalSystem_projective([x^5 + 5/4*x*y^4, y^5], domain=P) #
→needs sage.schemes
sage: Q = P([-1/2*a + 1/2, 1])
sage: Q.is_preperiodic(f) #
→needs sage.schemes
True
sage: Q = P([a, 1])
sage: Q.is_preperiodic(f) #
→needs sage.schemes
False

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: f = DynamicalSystem_projective([ 
....:      -38/45*x^2 + (2*y - 7/45*z)*x + (-1/2*y^2 - 1/2*y*z + z^2), 
....:      -67/90*x^2 + (2*y + z*157/90)*x - y*z, 
....:       z^2 
....: ], domain=P)
sage: Q = P([1, 3, 1])
sage: Q.is_preperiodic(f, return_period=True) #
→needs sage.libs.singular sage.schemes
(0, 9)

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sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)  

sage: f = DynamicalSystem_projective(
    (-y - w)*x + (-13/30*y^2 + 13/30*w*y + w^2),
    -1/2*x^2 + (-y + 3/2*w)*x + (-1/3*y^2 + 4/3*w*y),
    -3/2*z^2 + 5/2*z*w + w^2,
    w^2,
    domain=P)

sage: Q = P([3,0,4/3,1])

sage: Q.is_preperiodic(f, return_period=True)

sage: from sage.misc.verbose import set_verbose

sage: set_verbose(-1)

sage: P.<x,y,z> = ProjectiveSpace(QQbar, 2)

sage: f = DynamicalSystem_projective([x^2, QQbar(sqrt(-1))*y^2, z^2],
    domain=P)

sage: Q = P([1, 1, 1])

sage: Q.is_preperiodic(f)
True

sage: P.<x,y> = ProjectiveSpace(QQ, 1)

sage: f = DynamicalSystem_projective([16*x^2 - 29*y^2, 16*y^2], domain=P)

sage: Q = P(-1,4)

sage: Q.is_preperiodic(f)
True

sage: P.<x,y,z> = ProjectiveSpace(GF(3), 2)

sage: F = DynamicalSystem([x^2 - 2*y^2, y^2, z^2])

sage: Q = P(1, 1, 1)

sage: Q.is_preperiodic(F, return_period=True)

local_height(v, prec=None)

Returns the maximum of the local height of the coordinates of this point.

INPUT:

- v – a prime or prime ideal of the base ring.

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• **prec** – desired floating point precision (default: default RealField precision).

**OUTPUT:**

• a real number.

**EXAMPLES:**

```sage
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: Q = P.point([4, 4, 1/150], False)
sage: Q.local_height(5)  # needs sage.rings.real_mpfr
3.21887582486820
```

```sage
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: Q = P([4, 1, 30])
sage: Q.local_height(2)  # needs sage.rings.real_mpfr
0.693147180559945
```

**local_height_arch**(i, **prec=**None)

Returns the maximum of the local heights at the i-th infinite place of this point.

**INPUT:**

• i – an integer.

• prec – desired floating point precision (default: default RealField precision).

**OUTPUT:**

• a real number.

**EXAMPLES:**

```sage
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: Q = P.point([4, 4, 1/150], False)
sage: Q.local_height_arch(0)  # needs sage.rings.real_mpfr
1.38629436111989
```

```sage
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: Q = P([4, 1, 30])
sage: Q.local_height_arch(1)  # needs sage.rings.real_mpfr
3.401197381662155375413236691607
```

**multiplier**(f, n, **check=**True)

Returns the multiplier of this point of period n by the function f.

f must be an endomorphism of projective space.

**INPUT:**

• f - a endomorphism of this point’s codomain.

• n - a positive integer, the period of this point.

• check – check if P is periodic of period n, Default: True.

**OUTPUT:**
• a square matrix of size self.codomain().dimension_relative() in the base_ring of this point.

EXAMPLES:

```
sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: f = DynamicalSystem_projective([x^2, y^2, 4*w^2, 4*z^2], domain=P)  # needs sage.schemes
sage: Q = P.point([4, 4, 1, 1], False)
sage: Q.multiplier(f, 1)  # needs sage.schemes
[ 2 0 -8]
[ 0 2 -8]
[ 0 0 -2]
```

`normalize_coordinates()`

Removes the gcd from the coordinates of this point (including $-1$).

**Warning:** The gcd will depend on the base ring.

OUTPUT: None.

EXAMPLES:

```
sage: P = ProjectiveSpace(ZZ, 2, 'x')
sage: p = P([-5, -15, -20])
sage: p.normalize_coordinates(); p
(1 : 3 : 4)
sage: # needs sage.rings.padics
sage: P = ProjectiveSpace(Zp(7), 2, 'x')
sage: p = P([-5, -15, -2]).
sage: p.normalize_coordinates(); p
(5 + O(7^20) : 1 + 2*7 + O(7^20) : 2 + O(7^20))
sage: R.<t> = PolynomialRing(QQ)
sage: P = ProjectiveSpace(R, 1)
sage: p = P([3/5*t^3, 6*t, t])
sage: p.normalize_coordinates(); p
(3/5*t^2 : 6 : 1)
sage: R.<c> = PolynomialRing(QQ)
sage: P = ProjectiveSpace(R, 1)
sage: Q = P(2*c, 4*c)
sage: Q.normalize_coordinates(); Q
(2 : 4)
```

Since the base ring is a polynomial ring over a field, only the gcd $c$ is removed.
A polynomial ring over a ring gives the more intuitive result.

```
sage: R.<c> = PolynomialRing(ZZ)
sage: P = ProjectiveSpace(R, 1)
sage: Q = P(2*c, 4*c)
sage: Q.normalize_coordinates();Q
(1 : 2)
```

```
sage: # needs sage.libs.singular
sage: R.<t> = PolynomialRing(QQ, 1)
sage: S = R.quotient_ring(R.ideal(t^3))
sage: P.<x,y> = ProjectiveSpace(S, 1)
sage: Q = P(t + 1, t^2 + t)
sage: Q.normalize_coordinates()
sage: Q
(1 : tbar)
```

`scale_by(t)`

Scale the coordinates of the point by $t$.

A `TypeError` occurs if the point is not in the base_ring of the codomain after scaling.

**INPUT:**

- $t$ – a ring element.

**OUTPUT:** None.

**EXAMPLES:**

```
sage: R.<t> = PolynomialRing(QQ)
sage: P = ProjectiveSpace(R, 2, 'x')
sage: p = P([3/5*t^3, 6*t, t])
sage: p.scale_by(1/t); p
(3/5*t^2 : 6 : 1)
```

```
sage: # needs sage.libs.pari
sage: R.<t> = PolynomialRing(QQ)
sage: S = R.quotient_ring(R.ideal(t^3))
sage: P.<x,y,z> = ProjectiveSpace(S, 2)
sage: Q = P(t, 1, 1)
sage: Q.scale_by(t);Q
(tbar^2 : tbar : tbar)
```

```
sage: P.<x,y,z> = ProjectiveSpace(ZZ,2)
sage: Q = P(2, 2, 2)
sage: Q.scale_by(1/2);Q
(1 : 1 : 1)
```
14.4 Subschemes of projective space

AUTHORS:

- Volker Braun (2010-12-24): documentation of schemes and refactoring. Added coordinate neighborhoods and is_smooth()
- Ben Hutz (2013) refactoring

```python
class sage.schemes.projective.projective_subscheme.AlgebraicScheme_subscheme_projective(A, polynomials):

    Bases: AlgebraicScheme_subscheme

    Construct an algebraic subscheme of projective space.

    Warning: You should not create objects of this class directly. The preferred method to construct such subschemes is to use subscheme() method of projective space.
```

INPUT:

- A – ambient projective space.
- polynomials – single polynomial, ideal or iterable of defining homogeneous polynomials.

EXAMPLES:

```python
sage: P.<x, y, z> = ProjectiveSpace(2, QQ)
sage: P.subscheme([x^2 - y*z])
Closed subscheme of Projective Space of dimension 2 over Rational Field defined by:
        x^2 - y*z
```

```python
affine_patch(i, AA=None)

    Return the \( i \)th affine patch of this projective scheme.
    This is the intersection with this \( i \)th affine patch of its ambient space.

    INPUT:

    - \( i \) – integer between 0 and dimension of self, inclusive.
    - AA – (default: None) ambient affine space, this is constructed if it is not given.

    OUTPUT:

    An affine algebraic scheme with fixed embedding_morphism() equal to the default projective_embedding() map.
```

EXAMPLES:
sage: # needs sage.libs.singular
sage: PP = ProjectiveSpace(2, QQ, names='X,Y,Z')
sage: X,Y,Z = PP.gens()
sage: C = PP.subscheme(X^3*Y + Y^3*Z + Z^3*X)
sage: U = C.affine_patch(0); U
Closed subscheme of Affine Space of dimension 2 over Rational Field defined by:
Y^3*Z + Z^3 + Y
sage: U.embedding_morphism()
Scheme morphism:
From: Closed subscheme of Affine Space of dimension 2 over Rational Field
defined by: Y^3*Z + Z^3 + Y
to: Closed subscheme of Projective Space of dimension 2 over Rational Field
defined by: X^3*Y + Y^3*Z + X*Z^3
Defn: Defined on coordinates by sending (Y, Z) to (1 : Y : Z)
sage: U.projective_embedding() is U.embedding_morphism()
True

sage: A.<x,y,z> = AffineSpace(QQ, 3)
sage: X = A.subscheme([x - y*z])
sage: Y = X.projective_embedding(1).codomain()  # needs sage.libs.singular
sage: Y.affine_patch(1, A).ambient_space() == A  # needs sage.libs.singular
True

sage: P.<u,v,w> = ProjectiveSpace(2, ZZ)
sage: S = P.subscheme([u^2 - v*w])
sage: A.<x, y> = AffineSpace(2, ZZ)
sage: S.affine_patch(1, A)  # needs sage.libs.singular
Closed subscheme of Affine Space of dimension 2 over Integer Ring defined by:
x^2 - y
degree()
Return the degree of this projective subscheme.
If \( P(t) = a_m t^m + \ldots + a_0 \) is the Hilbert polynomial of this subscheme, then the degree is \( a_m m! \).
OUTPUT: Integer.
EXAMPLES:

sage: P.<x,y,z,w,t,u> = ProjectiveSpace(QQ, 5)
sage: X = P.subscheme([x^7 + x*y*z*t^4 - u^7])
sage: X.degree()  # needs sage.libs.singular
7
sage: P.<x,y,z,w> = ProjectiveSpace(GF(13), 3)
sage: X = P.subscheme([y^3 - w^3, x + 7*z])
sage: X.degree()  # needs sage.libs.singular
3

(continues on next page)
```
sage: # needs sage.libs.singular sage.schemes
sage: P.<x,y,z,w,u> = ProjectiveSpace(QQ, 4)
sage: C = P.curve([x^7 - y*z^3*w^2*u, w*x^2 - y*u^2, z^3 + y^3])
sage: C.degree()
63
```

dimension()

Return the dimension of the projective algebraic subscheme.

OUTPUT:

Integer.

EXAMPLES:

```
sage: # needs sage.libs.singular
sage: P2.<x,y,z> = ProjectiveSpace(2, QQ)
sage: P2.subscheme([x]).dimension()
1
sage: P2.subscheme([x^5]).dimension()
1
sage: P2.subscheme([x^2 + y^2 - z^2]).dimension()
1
sage: P2.subscheme([x*(x-z), y*(y-z)]).dimension()
0
```

Something less obvious:

```
sage: P3.<x,y,z,w,t> = ProjectiveSpace(4, QQ)
sage: X = P3.subscheme([x^2, x^2*y^2 + z^2*t^2, z^2 - w^2, 10*x^2 + w^2 - z^2])
sage: X
Closed subscheme of Projective Space of dimension 4 over Rational Field defined by:
  x^2,
  x^2*y^2 + z^2*t^2,
  z^2 - w^2,
  10*x^2 - z^2 + w^2
sage: X.dimension()  # needs sage.libs.singular
1
```

dual()

Return the projective dual of the given subscheme of projective space.

INPUT:

- X – A subscheme of projective space. At present, X is required to be an irreducible and reduced hypersurface defined over \( \mathbb{Q} \) or a finite field.

OUTPUT:

- The dual of X as a subscheme of the dual projective space.

EXAMPLES:
The dual of a smooth conic in the plane is also a smooth conic:

```
sage: R.<x, y, z> = QQ[]
sage: P.<x, y, z> = ProjectiveSpace(2, QQ)
sage: I = R.ideal(x^2 + y^2 + z^2)
sage: X = P.subscheme(I)
sage: X.dual()
# needs sage.libs.singular
Closed subscheme of Projective Space of dimension 2 over Rational Field
defined by:
y0^2 + y1^2 + y2^2
```

The dual of the twisted cubic curve in projective 3-space is a singular quartic surface. In the following example, we compute the dual of this surface, which by double duality is equal to the twisted cubic itself. The output is the twisted cubic as an intersection of three quadrics:

```
sage: R.<x, y, z, w> = QQ[]
sage: P.<x, y, z, w> = ProjectiveSpace(3, QQ)
sage: I = R.ideal(y^2*z^2 - 4*x*z^3 - 4*y^3*w + 18*x*y*z*w - 27*x^2*w^2)
sage: X = P.subscheme(I)
sage: X.dual()
# needs sage.libs.singular
Closed subscheme of Projective Space of dimension 3 over Rational Field defined by:
y2^2 - y1*y3, y1*y2 - y0*y3, y1^2 - y0*y2
```

The singular locus of the quartic surface in the last example is itself supported on a twisted cubic:

```
sage: X.Jacobian().radical()
# needs sage.libs.singular
Ideal (z^2 - 3*y*w, y*z - 9*x*w, y^2 - 3*x*z) of Multivariate Polynomial Ring in x, y, z, w over Rational Field
```

An example over a finite field:

```
sage: R = PolynomialRing(GF(61), 'a,b,c')
sage: P.<a, b, c> = ProjectiveSpace(2, R.base_ring())
sage: X = P.subscheme(R.ideal(a^5 + 2*a*b + 3*c*c))
sage: X.dual()
# needs sage.libs.singular sage.rings.finite_rings
Closed subscheme of Projective Space of dimension 2 over Finite Field of size 61 defined by:
y0^2 - 30*y1^2 - 20*y2^2
```

**intersection_multiplicity(X, P)**

Return the intersection multiplicity of this subscheme and the subscheme X at the point P.

This uses the intersection_multiplicity function for affine subschemes on affine patches of this subscheme and X that contain P.

**INPUT:**

- X – subscheme in the same ambient space as this subscheme.
- P – a point in the intersection of this subscheme with X.
OUTPUT: An integer.

EXAMPLES:

```python
sage: # needs sage.schemes
sage: P.<x,y,z> = ProjectiveSpace(GF(5), 2)
sage: C = Curve([x^4 - z^2*y^2], P)
sage: D = Curve([y^4*z - x^5 - x^3*z^2], P)
sage: Q1 = P([0,1,0])
sage: C.intersection_multiplicity(D, Q1)  # needs sage.libs.singular
4
sage: Q2 = P([0,0,1])
sage: C.intersection_multiplicity(D, Q2)  # needs sage.libs.singular
6
```

```python
sage: # needs sage.rings.number_field
sage: R.<a> = QQ[]
sage: K.<b> = NumberField(a^4 + 1)
sage: P.<x,y,z,w> = ProjectiveSpace(K, 3)
sage: X = P.subscheme([x^2 + y^2 - z*w])
sage: Y = P.subscheme([y*z - x*w, z - w])
sage: Q1 = P([b^2,1,0,0])
sage: X.intersection_multiplicity(Y, Q1)  # needs sage.libs.singular
1
sage: Q2 = P([1/2*b^3 - 1/2*b, 1/2*b^3 - 1/2*b, 1, 1])
```

```python
sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: X = P.subscheme([x^2 - z^2, y^3 - w*x^2])
sage: Y = P.subscheme([w^2 - 2*x*y + z^2, y^2 - w^2])
sage: Q = P([1,1,-1,1])
```

```python
sage: is_smooth(point=None)
```

Test whether the algebraic subscheme is smooth.

**INPUT:**

- **point** – A point or **None** (default). The point to test smoothness at.

**OUTPUT:**

Boolean. If no point was specified, returns whether the algebraic subscheme is smooth everywhere. Otherwise, smoothness at the specified point is tested.

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EXAMPLES:

```
sage: # needs sage.libs.singular
sage: P2.<x,y,z> = ProjectiveSpace(2, QQ)
sage: cuspidal_curve = P2.subscheme([y^2*z - x^3])
sage: cuspidal_curve
Closed subscheme of Projective Space of dimension 2 over Rational Field defined by:
-x^3 + y^2*z
sage: cuspidal_curve.is_smooth([1,1,1])
True
sage: cuspidal_curve.is_smooth([0,0,1])
False
sage: cuspidal_curve.is_smooth()
False
sage: P2.subscheme([y^2*z - x^3 + z^3 + 1/10*x*y*z]).is_smooth()
True
```

**multiplicity**($P$)

Return the multiplicity of $P$ on this subscheme.

This is computed as the multiplicity of the corresponding point on an affine patch of this subscheme that contains $P$. This subscheme must be defined over a field. An error is returned if $P$ not a point on this subscheme.

**INPUT:**

- $P$ – a point on this subscheme.

**OUTPUT:**

An integer.

**EXAMPLES:**

```
sage: P.<x,y,z,w,t> = ProjectiveSpace(QQ, 4)
sage: X = P.subscheme([y^2 - x*t, w^7 - t*w*x^5 - z^7])
sage: Q1 = P([0,0,1,1,1])
sage: X.multiplicity(Q1)  # needs sage.libs.singular
1
sage: Q2 = P([1,0,0,0,0])
sage: X.multiplicity(Q2)  # needs sage.libs.singular
3
sage: Q3 = P([0,0,0,0,1])
sage: X.multiplicity(Q3)  # needs sage.libs.singular
7
```

```
sage: # needs sage.rings.real_mpfr
sage: P.<x,y,z,w> = ProjectiveSpace(CC, 3)
sage: X = P.subscheme([z^5*x^2*w - y^8])
sage: Q = P([2,0,0,1])
sage: X.multiplicity(Q)  # needs sage.libs.singular
1
```

(continues on next page)
needs sage.libs.singular

```sage
sage: # needs sage.libs.singular sage.schemes
sage: P.<x,y,z,w> = ProjectiveSpace(GF(29), 3)
sage: C = Curve([y^17 - x^5*w^4*z^8, x*y - z^2], P)
sage: Q = P([3,0,0,1])
sage: C.multiplicity(Q)
8
```

**neighborhood**(point)

Return an affine algebraic subscheme isomorphic to a neighborhood of the point.

**INPUT:**
- point – a point of the projective subscheme.

**OUTPUT:**
An affine algebraic scheme (polynomial equations in affine space) result such that
- embedding_morphism is an isomorphism to a neighborhood of point
- embedding_center is mapped to point.

**EXAMPLES:**

```sage
sage: P.<x,y,z>= ProjectiveSpace(QQ, 2)
sage: S = P.subscheme(x + 2*y + 3*z)
sage: s = S.point([0,-3,2]); s
(0 : -3/2 : 1)
sage: patch = S.neighborhood(s); patch
Closed subscheme of Affine Space of dimension 2 over Rational Field defined by: x + 3*z
sage: patch.embedding_morphism()
Scheme morphism:
From: Closed subscheme of Affine Space of dimension 2 over Rational Field defined by: x + 3*z
To:  Closed subscheme of Projective Space of dimension 2 over Rational Field defined by: x + 2*y + 3*z
Defn: Defined on coordinates by sending (x, z) to (x : -3/2 : z + 1)
sage: patch.embedding_center()
(0, 0)
sage: patch.embedding_morphism()([0,0])
(0 : -3/2 : 1)
sage: patch.embedding_morphism()(patch.embedding_center())
(0 : -3/2 : 1)
```

**nth_iterate**(f, n)

The nth forward image of this scheme by the map f.

**INPUT:**
- f – a DynamicalSystem_projective with self in f.domain()
- n – a positive integer.

**OUTPUT:**
• A subscheme in \( f.\text{codomain}() \)

**EXAMPLES:**

```python
sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: f = DynamicalSystem_projective([y^2, z^2, x^2, w^2])          # needs sage.schemes
sage: f.nth_iterate(P.subscheme([x - w, y - z]), 3)                  # needs sage.libs.singular sage.schemes
Closed subscheme of Projective Space of dimension 3 over Rational Field
defined by:
  y - z,
  x - w
sage: PS.<x,y,z> = ProjectiveSpace(ZZ, 2)
sage: f = DynamicalSystem_projective([x^2, y^2, z^2])               # needs sage.schemes
sage: X = PS.subscheme([x - y])
sage: X.nth_iterate(f, -2)                                          # needs sage.libs.singular sage.schemes
Traceback (most recent call last):
... TypeError: must be a forward orbit
sage: PS.<x,y,z> = ProjectiveSpace(ZZ, 2)
sage: P2.<u,v,w> = ProjectiveSpace(QQ, 2)
sage: H = Hom(PS, P2)
sage: f = H([x^2, y^2, z^2])
sage: X = PS.subscheme([x - y])
sage: X.nth_iterate(f, 2)
Traceback (most recent call last):
... TypeError: map must be a dynamical system for iteration
sage: PS.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: f = DynamicalSystem_projective([x^2, y^2, z^2])               # needs sage.schemes
sage: X = PS.subscheme([x - y])
sage: X.nth_iterate(f, 2.5)                                         # needs sage.schemes
Traceback (most recent call last):
... TypeError: Attempt to coerce non-integral RealNumber to Integer
```

\[ \text{orbit}(f, N) \]

Return the orbit of this scheme by \( f \).

If \( N \) is an integer it returns \([self, f(self), \ldots, f^N(self)]\). If \( N \) is a list or tuple \( N = [m, k] \) it returns \([f^m(self), \ldots, f^k(self)]\).

**INPUT:**

• \( f \) – a \texttt{DynamicalSystem\_projective} with \texttt{self} in \( f.\text{domain}() \)

• \( N \) – a non-negative integer or list or tuple of two non-negative integers
OUTPUT:

• a list of projective subschemes

EXAMPLES:

```python
sage: # needs sage.libs.singular sage.schemes
sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: f = DynamicalSystem_projective([(x-2*y)^2, (x-2*z)^2,
.....: (x-2*w)^2, x^2])
sage: f.orbit(P.subscheme([x]), 5)
[Closed subscheme of Projective Space of dimension 3 over Rational Field
defined by: x,
Closed subscheme of Projective Space of dimension 3 over Rational Field
defined by: w,
Closed subscheme of Projective Space of dimension 3 over Rational Field
defined by: z - w,
Closed subscheme of Projective Space of dimension 3 over Rational Field
defined by: y - z,
Closed subscheme of Projective Space of dimension 3 over Rational Field
defined by: x - y,
Closed subscheme of Projective Space of dimension 3 over Rational Field
defined by: x - w]
```

```python
sage: PS.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: P1.<u,v> = ProjectiveSpace(QQ, 1)
sage: H = Hom(PS, P1)
sage: f = H([x^2, y^2])
sage: X = PS.subscheme([x - y])
sage: X.orbit(f, 2)
Traceback (most recent call last):
...TypeError: map must be a dynamical system for iteration
```

```python
sage: PS.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: f = DynamicalSystem_projective([x^2, y^2, z^2])
# needs sage.schemes
sage: X = PS.subscheme([x - y])
sage: X.orbit(f, [-1,2])
# needs sage.schemes
Traceback (most recent call last):
...TypeError: orbit bounds must be non-negative
```

point(v, check=True)

Create a point on this projective subscheme.

INPUT:

• v – anything that defines a point

• check – boolean (optional, default: True); whether to check the defining data for consistency

OUTPUT: A point of the subscheme.

EXAMPLES:
sage: P2.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: X = P2.subscheme([x - y, y - z])
sage: X.point([1,1,1])
(1 : 1 : 1)

sage: P2.<x,y> = ProjectiveSpace(QQ, 1)
sage: X = P2.subscheme([y])
sage: X.point(infinity)
(1 : 0)

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: X = P.subscheme(x^2 + 2*y^2)
sage: X.point(infinity)
Traceback (most recent call last):
  ...TypeError: Coordinates [1, 0] do not define a point on Closed subscheme of Projective Space of dimension 1 over Rational Field defined by:
  x^2 + 2*y^2

preimage(f, k=1, check=True)

The subscheme that maps to this scheme by the map \( f^k \).

In particular, \( f^{-k}(V(h_1, \ldots, h_t)) = V(h_1 \circ f^k, \ldots, h_t \circ f^k) \). Map must be a morphism and also must be an endomorphism for \( k > 1 \).

INPUT:

- \( f \) - a map whose codomain contains this scheme
- \( k \) - a positive integer
- check – Boolean, if False no input checking is done

OUTPUT:

a subscheme in the domain of \( f \)

EXAMPLES:

sage: PS.<x,y,z> = ProjectiveSpace(ZZ, 2)
sage: H = End(PS)
sage: f = H([y^2, x^2, z^2])
sage: X = PS.subscheme([x - y])
sage: X.preimage(f)

Closed subscheme of Projective Space of dimension 2 over Integer Ring defined by:
- x^2 + y^2

sage: P.<x,y,z,w,t> = ProjectiveSpace(QQ, 4)
sage: H = End(P)
sage: f = H([x^2 - y^2, y^2, z^2, w^2, t^2 + w^2])
sage: f.rational_preimages(P.subscheme([x - z, t^2, w - t]))

Closed subscheme of Projective Space of dimension 4 over Rational Field defined by:

(continues on next page)
x^2 - y^2 - z^2,
\[ w^4 + 2*w^2*t^2 + t^4, \]
-t^2

sage: P1.<x,y> = ProjectiveSpace(QQ, 1)
sage: P3.<u,v,w,t> = ProjectiveSpace(QQ, 3)
sage: H = Hom(P1, P3)
sage: X = P3.subscheme([u - v, 2*u - w, u + t])
sage: f = H([x^2, y^2, x^2 + y^2, x*y])
sage: X.preimage(f)  # needs sage.libs.singular
Closed subscheme of Projective Space of dimension 1 over Rational Field
defined by:
\[ x^2 - y^2, \]
\[ x^2 - y^2, \]
\[ x^2 + x*y \]

sage: P1.<x,y> = ProjectiveSpace(QQ, 1)
sage: P3.<u,v,w,t> = ProjectiveSpace(QQ, 3)
sage: H = Hom(P3, P1)
sage: X = P1.subscheme([x - y])
sage: f = H([u^2, v^2])
sage: X.preimage(f)  # needs sage.libs.singular
Traceback (most recent call last):
...
TypeError: map must be a morphism

sage: PS.<x,y,z> = ProjectiveSpace(ZZ, 2)
sage: H = End(PS)
sage: f = H([x^2, x^2, x^2])
sage: X = PS.subscheme([x - y])
sage: X.preimage(f)  # needs sage.libs.singular
Traceback (most recent call last):
...
TypeError: map must be a morphism

sage: PS.<x,y,z> = ProjectiveSpace(ZZ, 2)
sage: P1.<u,v> = ProjectiveSpace(ZZ, 1)
sage: Y = P1.subscheme([u^2 - v^2])
sage: H = End(PS)
sage: f = H([x^2, y^2, z^2])
sage: Y.preimage(f)  # needs sage.libs.singular
Traceback (most recent call last):
...
TypeError: subscheme must be in ambient space of codomain

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: Y = P.subscheme([x - y])
(continues on next page)
sage: H = End(P)
sage: f = H([x^2, y^2, z^2])
sage: Y.preimage(f, k=2)  # needs sage.libs.singular
Closed subscheme of Projective Space of dimension 2 over Rational Field
defined by:
x^4 - y^4

veronese_embedding(d, CS=None, order='lex')

Return the degree d Veronese embedding of this projective subscheme.

INPUT:

• d – a positive integer.

• CS – a projective ambient space to embed into. If the projective ambient space of this subscheme is of
dimension N, the dimension of CS must be (N+d) – 1. This is constructed if not specified. Default: None.

• order – a monomial order to use to arrange the monomials defining the embedding. The monomials
will be arranged from greatest to least with respect to this order. Default: 'lex'.

OUTPUT:

• a scheme morphism from this subscheme to its image by the degree d Veronese embedding.

EXAMPLES:

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: L = P.subscheme([y - x])
sage: v = L.veronese_embedding(2); v  # needs sage.libs.singular
Scheme morphism:
From: Closed subscheme of Projective Space of dimension 2
over Rational Field defined by: -x + y
To:  Closed subscheme of Projective Space of dimension 5
over Rational Field defined by:
- x4^2 + x3*x5,
x2 - x4,
x1 - x3,
x0 - x3
Defn: Defined on coordinates by sending (x : y : z) to
(x^2 : x*y : x*z : y^2 : y*z : z^2)
sage: v.codomain().degree()  # needs sage.libs.singular
2
sage: C = P.subscheme([y*z - x^2])
sage: C.veronese_embedding(2).codomain().degree()  # needs sage.libs.singular
4
twisted cubic:

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: Q.<u,v,s,t> = ProjectiveSpace(QQ, 3)
sage: P.subscheme([]).veronese_embedding(3, Q)  #
class sage.schemes.projective.projective_subscheme.AlgebraicScheme_subscheme_projective_field

Bases: AlgebraicScheme_subscheme_projective

Algebraic subschemes of projective spaces defined over fields.

Chow_form()

Return the Chow form associated to this subscheme.

For a $k$-dimensional subvariety of $\mathbb{P}^N$ of degree $D$. The $(N-k-1)$-dimensional projective linear subspaces of $\mathbb{P}^N$ meeting $X$ form a hypersurface in the Grassmannian $G(N-k-1, N)$. The homogeneous form of degree $D$ defining this hypersurface in Plucker coordinates is called the Chow form of $X$.

The base ring needs to be a number field, finite field, or $\mathbb{Q}$.

ALGORITHM:

For a $k$-dimensional subscheme $X$ consider the $k+1$ linear forms $l_i = u_{i0}x_0 + \cdots + u_{in}x_n$. Let $J$ be the ideal in the polynomial ring $K[x_i, u_{ij}]$ defined by the equations of $X$ and the $l_i$. Let $J'$ be the saturation of $J$ with respect to the irrelevant ideal of the ambient projective space of $X$. The elimination ideal $I = J' \cap K[u_{ij}]$ is a principal ideal, let $R$ be its generator. The Chow form is obtained by writing $R$ as a polynomial in Plucker coordinates (i.e. bracket polynomials). [DS1994].

OUTPUT: a homogeneous polynomial.

EXAMPLES:

```python
sage: P.<x0,x1,x2,x3> = ProjectiveSpace(GF(17), 3)
sage: X = P.subscheme([x3 + x1, x2 - x0, x2 - x3])
sage: X.Chow_form()  # needs sage.libs.singular
 t0 - t1 + t2 + t3
```

```python
sage: P.<x0,x1,x2,x3> = ProjectiveSpace(QQ, 3)
sage: X = P.subscheme([x3^2 - 101*x1^2 - 3*x2*x0])
sage: X.Chow_form()  # needs sage.libs.singular
 t0^2 - 101*t2^2 - 3*t1*t3
```
sage: # needs sage.libs.singular
sage: P.<x0,x1,x2,x3> = ProjectiveSpace(QQ, 3)
sage: X = P.subscheme([x0^2 - x1^2, x0*x3 - x1*x2, x1^2 - x2^2])
sage: Ch = X.Chow_form(); Ch
t2^3 + 2*t2^2*t3 + t2*t3^2 - 3*t1*t2*t4 - t1*t3*t4 + t0*t4^2 + t1^2*t5
sage: Y = P.subscheme_from_Chow_form(Ch, 1); Y
Closed subscheme of Projective Space of dimension 3 over Rational Field defined by:
    x2^2*x3 - x1*x3^2,
    x2^2*x3 + x1*x3^2,
    3*x1*x2*x2 - 3*x0*x2*x3,
    -3*x1*x2*x2 + 3*x0*x1*x3,
    x2^3 - x1*x2*x3,
    2*x0*x2*x2 - 2*x0*x1*x3,
    -x0*x1*x2 + x0*x^2*
    -x1^3 + x0*x1*x2,

sage: I = Y.defining_ideal()
sage: I.saturation(I.ring().ideal(list(I.ring().gens())))[0]
Ideal (x2^2 - x1*x3, x1*x2 - x0*x3, x1^2 - x0*x2)
of Multivariate Polynomial Ring in x0, x1, x2, x3 over Rational Field

\textbf{global\_height}(\texttt{prec=None})

Return the (projective) global height of the subscheme.

INPUT:

\begin{itemize}
\item \texttt{prec} – desired floating point precision (default: default \texttt{RealField} precision).
\end{itemize}

OUTPUT:

\begin{itemize}
\item a real number.
\end{itemize}

EXAMPLES:

sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: NF.<a> = NumberField(x^2 - 5)
sage: P.<x,y,z> = ProjectiveSpace(NF, 2)
sage: X = P.subscheme([x^2 + y*z, 2*y*z, 3*x*y])

\textbf{local\_height}(\texttt{v, prec=None})

Return the (projective) local height of the subscheme.

INPUT:

\begin{itemize}
\item \texttt{v} – a prime or prime ideal of the base ring.
\item \texttt{prec} – desired floating point precision (default: default \texttt{RealField} precision).
\end{itemize}
OUTPUT:

- a real number.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: NF.<a> = NumberField(x^2 - 5)
sage: I = NF.ideal(3)
sage: P.<x,y,z> = ProjectiveSpace(NF, 2)
sage: X = P.subscheme([3*x^2*y - 5*x^2*z, y^2])
sage: X.local_height(I)  # needs sage.libs.singular
0.000000000000000
```

```
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: X = P.subscheme([z^2 - 101*y^2 - 3*x^2*z])
sage: X.local_height(2)  # needs sage.libs.singular
0.000000000000000
```

`local_height_arch(i, prec=None)`

Return the local height at the i-th infinite place of the subscheme.

INPUT:

- i – an integer.
- prec – desired floating point precision (default: default `RealField` precision).

OUTPUT:

- a real number.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: NF.<a> = NumberField(x^2 - 5)
sage: P.<x,y,z> = ProjectiveSpace(NF, 2)
sage: X = P.subscheme([x^2 + y*z, 3*x*y])
sage: X.local_height_arch(1)  # needs sage.libs.singular
0.0000000000000000000000000000000
```

```
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: X = P.subscheme([z^2 - 101*y^2 - 3*x^2*z])
sage: X.local_height_arch(1)  # needs sage.libs.singular
4.61512051684126
```

14.4. Subschemes of projective space
14.5 Enumeration of rational points on projective schemes

Naive algorithms for enumerating rational points over $\mathbb{Q}$ or finite fields over for general schemes.

**Warning:** Incorrect results and infinite loops may occur if using a wrong function. (For instance using an affine function for a projective scheme or a finite field function for a scheme defined over an infinite field.)

**EXAMPLES:**

Projective, over $\mathbb{Q}$:

```sage
def enum_projective_rational_field(X, n):
    # Implementation
    return result
```

```sage:
from sage.schemes.projective.projective_rational_point import enum_projective_rational_field

P.<X,Y,Z> = ProjectiveSpace(2, QQ)
C = P.subscheme([X + Y - Z])
seq = enum_projective_rational_field(C, 3)
seq
```

```
[(-2 : 3 : 1), (-1 : 1 : 0), (-1 : 2 : 1), (-1/2 : 3/2 : 1),
 (0 : 1 : 1), (1/3 : 2/3 : 1), (1/2 : 1/2 : 1), (2/3 : 1/3 : 1),
 (1 : 0 : 1), (3/2 : -1/2 : 1), (2 : -1 : 1), (3 : -2 : 1)]
```

Projective over a finite field:

```sage
def enum_projective_finite_field(X):
    # Implementation
    return result
```

```sage:
from sage.schemes.projective.projective_rational_point import enum_projective_finite_field

E = EllipticCurve('72').change_ring(GF(19))
seq = enum_projective_finite_field(E)
seq
```

```
[(0 : 1 : 0), (1 : 0 : 1), (3 : 0 : 1), (4 : 9 : 1), (4 : 10 : 1),
 (9 : 15 : 1), (12 : 8 : 1), (12 : 11 : 1), (13 : 8 : 1), (13 : 11 : 1),
 (14 : 3 : 1), (14 : 16 : 1), (15 : 0 : 1), (16 : 9 : 1), (16 : 10 : 1),
 (17 : 7 : 1), (17 : 12 : 1), (18 : 9 : 1), (18 : 10 : 1)]
```

**AUTHORS:**

- David R. Kohel <kohel@maths.usyd.edu.au>: original version.
- John Cremona and Charlie Turner <charlotteturner@gmail.com> (06-2010): improvements to clarity and documentation.
- Raghukul Raman <raghukul.raman01@gmail.com> (2018): Added sieve algorithm.
sage: from sage.schemes.projective.projective_rational_point import enum_projective_finite_field
sage: F = GF(53)
sage: P.<X,Y,Z> = ProjectiveSpace(2, F)
sage: len(enum_projective_finite_field(P(F)))
2863
sage: 53^2 + 53 + 1
2863

sage: # needs sage.rings.finite_rings
sage: F = GF(9, 'a')
sage: P.<X,Y,Z> = ProjectiveSpace(2, F)
sage: C = Curve(X^3 - Y^3 + Z^2*Y)
# needs sage.schemes
sage: enum_projective_finite_field(C(F))
# needs sage.schemes
[(0 : 0 : 1), (0 : 1 : 1), (0 : 2 : 1), (1 : 1 : 0), (a + 1 : 2*a : 1),
 (a + 1 : 2*a + 1 : 1), (a + 1 : 2*a + 2 : 1), (2*a + 2 : a : 1),
 (2*a + 2 : a + 1 : 1), (2*a + 2 : a + 2 : 1)]

sage: F = GF(5)
sage: P2F.<X,Y,Z> = ProjectiveSpace(2, F)
sage: enum_projective_finite_field(P2F)
[(0 : 0 : 1), (0 : 1 : 0), (0 : 1 : 1), (0 : 2 : 1), (0 : 3 : 1), (0 : 4 : 1),
 (1 : 0 : 0), (1 : 1 : 0), (1 : 1 : 1), (1 : 2 : 1), (1 : 3 : 1),
 (1 : 4 : 1), (2 : 0 : 1), (2 : 1 : 0), (2 : 1 : 1), (2 : 2 : 1), (2 : 3 : 1),
 (2 : 4 : 1), (3 : 0 : 1), (3 : 1 : 0), (3 : 1 : 1), (3 : 2 : 1), (3 : 3 : 1),
 (3 : 4 : 1), (4 : 0 : 1), (4 : 1 : 0), (4 : 1 : 1), (4 : 2 : 1), (4 : 3 : 1),
 (4 : 4 : 1)]

ALGORITHM:
Checks all points in projective space to see if they lie on \(X\).

**Warning:** If \(X\) is defined over an infinite field, this code will not finish!

AUTHORS:
• John Cremona and Charlie Turner (06-2010).

sage.schemes.projective.projective_rational_point.enum_projective_number_field(X, **kwds)
Enumerates projective points on scheme \(X\) defined over a number field.
Simply checks all of the points of absolute height of at most \(B\) and adds those that are on the scheme to the list.
This algorithm computes 2 lists: \(L\) containing elements \(x\) in \(K\) such that \(H_k(x) \leq B\), and a list \(L'\) containing elements \(x\) in \(K\) that, due to floating point issues, may be slightly larger then the bound. This can be controlled by lowering the tolerance.

ALGORITHM:
This is an implementation of the revised algorithm (Algorithm 4) in [DK2013]. Algorithm 5 is used for imaginary quadratic fields.

INPUT:
knwds:
• bound - a real number
• tolerance - a rational number in (0,1] used in doyle-krumm algorithm-4
• precision - the precision to use for computing the elements of bounded height of number fields.

OUTPUT:
• a list containing the projective points of \(X\) of absolute height up to \(B\), sorted.

EXAMPLES:

```python
sage: # needs sage.rings.number_field
sage: from sage.schemes.projective.projective_rational_point import enum_projective_number_field
sage: u = QQ['u'].0
sage: K = NumberField(u^3 - 5, 'v')
sage: P.<x,y,z> = ProjectiveSpace(K, 2)
sage: X = P.subscheme([x - y])
sage: enum_projective_number_field(X(K), bound=RR(5^(1/3)), prec=2^10) # needs sage.symbolic
[(0 : 0 : 1), (1 : 1 : 0), (-1 : -1 : 1), (1 : 1 : 1)]
```

```python
sage: # needs sage.rings.number_field
sage: u = QQ['u'].0
sage: K = NumberField(u^2 + 3, 'v')
sage: A.<x,y> = ProjectiveSpace(K, 1)
sage: X = A.subscheme(x - y)
sage: enum_projective_number_field(X, bound=2)
[(1 : 1)]
```

`sage.schemes.projective.projective_rational_point.enum_projective_rational_field(X, B)`
Enumerates projective, rational points on scheme \(X\) of height up to bound \(B\).

INPUT:
• \(X\) - a scheme or set of abstract rational points of a scheme.
• \(B\) - a positive integer bound.

OUTPUT:
• a list containing the projective points of \(X\) of height up to \(B\), sorted.

EXAMPLES:

```python
sage: P.<X,Y,Z> = ProjectiveSpace(2, QQ)
sage: C = P.subscheme([X + Y - Z])
sage: from sage.schemes.projective.projective_rational_point import enum_projective_rational_field
sage: enum_projective_rational_field(C(QQ), 6)
```
sage: enum_projective_rational_field(C, 6) == enum_projective_rational_field(C(QQ), 6)
True

ALGORITHM:
We just check all possible projective points in correct dimension of projective space to see if they lie on \( X \).

AUTHORS:
• John Cremona and Charlie Turner (06-2010)

sage.schemes.projective.projective_rational_point.sieve(X, bound)

Returns the list of all projective, rational points on scheme \( X \) of height up to bound.

Height of a projective point \( X = (x_1, x_2, \ldots, x_n) \) is given by \( H_X = \max(y_1, y_2, \ldots, y_n) \), where the values \( y_i \) are the normalized coordinates such that all \( y_i \) are integers and \( \gcd(y_1, y_2, \ldots, y_n) = 1 \).

ALGORITHM:
Main idea behind the algorithm is to find points modulo primes and then reconstruct them using Chinese Remainder theorem. We find modulo primes parallelly and then lift them and apply LLL in parallel.

For the algorithm to work correctly, sufficient primes need to be present, these are calculated using the bound given in this([Hutz2015]) paper.

INPUT:
• \( X \) - a scheme with ambient space defined over projective space
• \( \text{bound} \) - a positive integer bound

OUTPUT:
• a list containing the projective rational points of \( X \) of height up to \( \text{bound} \), sorted

EXAMPLES:

sage: from sage.schemes.projective.projective_rational_point import sieve
sage: P.<x,y,z,q> = ProjectiveSpace(QQ, 3)
sage: Y = P.subscheme([x^2 - 3^2*y^2 + z*q, x + z + 4*q])
sage: sorted(sieve(Y, 12))  # long time
˓→ needs sage.libs.singular

(continues on next page)
14.6 Set of homomorphisms between two projective schemes

For schemes \( X \) and \( Y \), this module implements the set of morphisms \( \text{Hom}(X, Y) \). This is done by \texttt{SchemeHomset_generic}.

As a special case, the Hom-sets can also represent the points of a scheme. Recall that the \( K \)-rational points of a scheme \( X \) over \( k \) can be identified with the set of morphisms \( \text{Spec}(K) \to X \). In Sage the rational points are implemented by such scheme morphisms. This is done by \texttt{SchemeHomset_points} and its subclasses.

\textbf{Note:} You should not create the Hom-sets manually. Instead, use the \texttt{Hom()} method that is inherited by all schemes.

AUTHORS:

- Volker Braun (2011-08-11): significant improvement and refactoring.
- Ben Hutz (June 2012): added support for projective ring
- Ben Hutz (2018): add numerical point support

\texttt{class sage.schemes.projective.projective_homset.SchemeHomset_points_abelian_variety_field(\( X, Y \), category=None, check=True, base=Integer Ring)}

Bases: \texttt{SchemeHomset_points_projective_field}

Set of rational points of an Abelian variety.

INPUT:

See \texttt{SchemeHomset_generic}.

\texttt{base_extend(\( R \))}

Extend the base ring.

This is currently not implemented except for the trivial case \( R=\mathbb{Z} \).
INPUT:

- \( R \) – a ring.

EXAMPLES:

```python
sage: # needs sage.schemes
sage: E = EllipticCurve('37a')
sage: Hom = E.point_homset(); Hom
Abelian group of points on Elliptic Curve defined
by \( y^2 + y = x^3 - x \) over Rational Field
sage: Hom.base_ring()
Rational Field
sage: Hom.base_extend(QQ)
Traceback (most recent call last):
  ... NotImplementedError: Abelian variety point sets are not
implemented as modules over rings other than ZZ
```

### class sage.schemes.projective.projective_homset.SchemeHomset_points_projective_field(X, Y, category=None, check=True, base=Integer Ring)

Bases: `SchemeHomset_points`

Set of rational points of a projective variety over a field.

INPUT:

See `SchemeHomset_generic`.

EXAMPLES:

```python
sage: from sage.schemes.projective.projective_homset import SchemeHomset_points_projective_field
sage: SchemeHomset_points_projective_field(Spec(QQ), ProjectiveSpace(QQ,2))
Set of rational points of Projective Space of dimension 2 over Rational Field
```

#### numerical_points(F=None, **kwds)

Return some or all numerical approximations of rational points of a projective scheme.

This is for dimension 0 subschemes only and the points are determined through a groebner calculation over
the base ring and then numerically approximating the roots of the resulting polynomials. If the base ring is
a number field, the embedding into \( F \) must be known.

INPUT:

- \( F \) - numerical ring

kwds:

- `point_tolerance` - positive real number (optional, default: \( 10^{-10} \)). For numerically inexact fields, two points are considered the same if their coordinates are within tolerance.

- `zero_tolerance` - positive real number (optional, default: \( 10^{-10} \)). For numerically inexact fields, points are on the subscheme if they satisfy the equations to within tolerance.

OUTPUT: A list of points in the ambient space.
Warning: For numerically inexact fields the list of points returned may contain repeated or be missing points due to tolerance.

EXAMPLES:

```python
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: E = P.subscheme([y^3 - x^3 - x^2*z^2, x*y^2*z])
sage: L = E(QQ).numerical_points(F=RR); L
# needs sage.libs.singular
[(0.000000000000000 : 0.000000000000000 : 1.000000000000000),
 (1.000000000000000 : 1.000000000000000 : 0.0000000000000000)]
```

```python
sage: L[0].codomain()
# needs sage.libs.singular
Projective Space of dimension 2 over Real Field with 53 bits of precision
```

```python
sage: S.<a> = QQ[]
sage: K.<v> = NumberField(a^5 - 7, embedding=CC(7)**(1/5))
# needs sage.rings.number_field
sage: P.<x,y,z> = ProjectiveSpace(K, 2)
# needs sage.rings.number_field
sage: X = P.subscheme([x^2 - v^2*z^2, y - v*z])
# needs sage.rings.number_field
sage: len(X(K).numerical_points(F=CDF))
# needs sage.libs.singular sage.rings.number_field
2
```

```python
sage: P.<x1, x2, x3> = ProjectiveSpace(QQ, 2)
sage: E = P.subscheme([3000*x1^50 + 9875643*x2^2*x3^48 + 12334545*x2^50, x1 + x2])
# needs sage.libs.singular
sage: len(E(P.base_ring()).numerical_points(F=CDF, zero_tolerance=1e-6))
# needs sage.libs.singular
49
```

points(**kwds)

Return some or all rational points of a projective scheme.

For dimension 0 subschemes points are determined through a groebner basis calculation. For schemes or subschemes with dimension greater than 1 points are determined through enumeration up to the specified bound.

INPUT:

kwds:

- bound - real number (optional, default: 0). The bound for the coordinates for subschemes with dimension at least 1.
- precision - integer (optional, default: 53). The precision to use to compute the elements of bounded height for number fields.
- point_tolerance - positive real number (optional, default: $10^{-10}$). For numerically inexact fields, two points are considered the same if their coordinates are within tolerance.
- zero_tolerance - positive real number (optional, default: $10^{-10}$). For numerically inexact fields, points are on the subscheme if they satisfy the equations to within tolerance.
• tolerance - a rational number in \((0,1]\) used in doyle-krumm algorithm-4 for enumeration over number fields.

OUTPUT:

• a list of rational points of a projective scheme

**Warning:** For numerically inexact fields such as ComplexField or RealField the list of points returned is very likely to be incomplete. It may also contain repeated points due to tolerances.

**EXAMPLES:**

```python
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: P(QQ).points(bound=4)
[(-4 : 1), (-3 : 1), (-2 : 1), (-3/2 : 1), (-4/3 : 1), (-1 : 1),
 (-3/4 : 1), (-2/3 : 1), (-1/2 : 1), (-1/3 : 1), (-1/4 : 1), (0 : 1),
 (1/4 : 1), (1/3 : 1), (1/2 : 1), (2/3 : 1), (3/4 : 1), (1 : 0), (1 : 1),
 (4/3 : 1), (3/2 : 1), (2 : 1), (3 : 1), (4 : 1)]
```

```python
sage: u = QQ['u'].0
sage: K.<v> = NumberField(u^2 + 3)

```

```python
sage: P.<x,y,z> = ProjectiveSpace(K, 2)
```

```python
sage: len(P(K).points(bound=1.8))
```

```
sage: P1 = ProjectiveSpace(GF(2), 1)
sage: F.<a> = GF(4, 'a')

```

```python
sage: P1(F).points()
```

```
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: E = P.subscheme([y^3 - x^3 - x*z^2, x*y*z])

```

```python
sage: L = E(P.base_ring()).points(); sorted(L, key=str)
```

```
sage: # needs sage.rings.real_mpfr
sage: P.<x,y,z> = ProjectiveSpace(CC, 2)
sage: E = P.subscheme([y^3 - x^3 - x*z^2, x*y*z])

```

```python
sage: L = E(P.base_ring()).points(); sorted(L, key=str)
```

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\(\begin{align*}
(-0.500000000000000 - 0.866025403784439 \cdot i : 1.000000000000000 : 0.000000000000000), \\
(-1.00000000000000 - 0.000000000000000 : 1.000000000000000), \\
(0.000000000000000 : 0.000000000000000 : 1.000000000000000), \\
(1.00000000000000 : 1.000000000000000 : 0.000000000000000), \\
(1.00000000000000 \cdot i : 0.000000000000000 : 1.000000000000000) \\
\end{align*}\]

```
sage: L[0].codomain()  # needs sage.libs.singular
Projective Space of dimension 2 over Complex Field with 53 bits of precision
```

```
sage: # needs sage.rings.complex_double
sage: P.<x,y,z> = ProjectiveSpace(CDF, 2)
sage: E = P.subscheme([y^2 + x^2 + z^2, x*y*z])
sage: len(E(P.base_ring()).points())  # needs sage.libs.singular
verbose 0 (...: projective_homset.py, points) Warning: computations in the numerical fields are inexact; points may be computed partially or incorrectly.
6
```

class sage.schemes.projective.projective_homset.SchemeHomset_points_projective_ring(X, Y, category=None, check=True, base=Integer Ring)

Bases: SchemeHomset_points

Set of rational points of a projective variety over a commutative ring.

INPUT:

See SchemeHomset_generic.

EXAMPLES:

```
sage: from sage.schemes.projective.projective_homset import SchemeHomset_points_projective_ring
sage: SchemeHomset_points_projective_ring(Spec(ZZ), ProjectiveSpace(ZZ, 2))
Set of rational points of Projective Space of dimension 2 over Integer Ring
```

points\((B=0)\)

Return some or all rational points of a projective scheme.

INPUT:

• \(B\) – integer (optional, default=0). The bound for the coordinates.

EXAMPLES:

```
sage: from sage.schemes.projective.projective_homset import SchemeHomset_points_projective_ring
sage: H = SchemeHomset_points_projective_ring(Spec(ZZ), ProjectiveSpace(ZZ, 2))
sage: H.points(3)
[(0 : 0 : 1), (0 : 1 : -3), (0 : 1 : -2), (0 : 1 : -1), (0 : 1 : 0), (0 : 1 : 0)]
```

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Schemes, Release 10.2

class sage.schemes.projective.projective_homset.SchemeHomset_polynomial_projective_space(X, Y, category=None, check=True, base=None)

Bases: SchemeHomset_generic

Set of morphisms of a projective space.

EXAMPLES:

sage: P.<x,y,z> = ProjectiveSpace(2, QQ)
sage: Hom(P, P)
Set of morphisms
From: Projective Space of dimension 2 over Rational Field
To: Projective Space of dimension 2 over Rational Field

identity()
Return the identity morphism of this hom-set.

EXAMPLES:

```
sage: P.<x,y,z> = ProjectiveSpace(2, QQ)
sage: Hom(P, P)
Set of morphisms
  From: Projective Space of dimension 2 over Rational Field
  To:   Projective Space of dimension 2 over Rational Field
sage: _.identity()
Scheme endomorphism of Projective Space of dimension 2 over Rational Field
  Defn: Identity map
```
CHAPTER
FIFTEEN

PRODUCTS OF PROJECTIVE SPACES

15.1 Products of projective spaces

This class builds on the projective space class and its point and morphism classes.

Products of projective spaces of varying dimension are convenient ambient spaces for complete intersections.

Group actions on them, and the interplay with representation theory, provide many interesting examples of algebraic varieties.

EXAMPLES:

We construct products projective spaces of various dimensions over the same ring:

```sage
sage: P1 = ProjectiveSpace(ZZ, 1, 'x')
sage: P2 = ProjectiveSpace(ZZ, 2, 'y')
sage: ProductProjectiveSpaces([P1, P2])
Product of projective spaces P^1 x P^2 over Integer Ring
```

We can also construct the product by specifying the dimensions and the base ring:

```sage
sage: ProductProjectiveSpaces([1, 2, 3], QQ, 'z')
Product of projective spaces P^1 x P^2 x P^3 over Rational Field
sage: P2xP2 = ProductProjectiveSpaces([2, 2], QQ, names=['x', 'y'])
sage: P2xP2.coordinate_ring().inject_variables()
Defining x0, x1, x2, y0, y1, y2
```

```sage
sage.schemes.product_projective.space.ProductProjectiveSpaces(n, R=None, names='x')
```

Return the Cartesian product of projective spaces.

The input `n` is either a list of projective space over the same base ring or the list of dimensions. `R` the base ring, and `names` the variable names.

INPUT:

- `n` – a list of integers or a list of projective spaces
- `R` – a ring
- `names` – a string or list of strings

EXAMPLES:
```python
sage: P1 = ProjectiveSpace(QQ, 2, 'x')
sage: P2 = ProjectiveSpace(QQ, 3, 'y')
sage: ProductProjectiveSpaces([P1, P2])
Product of projective spaces P^2 x P^3 over Rational Field

sage: ProductProjectiveSpaces([2, 2], GF(7), 'y')
Product of projective spaces P^2 x P^2 over Finite Field of size 7

sage: P1 = ProjectiveSpace(ZZ, 2, 'x')
sage: P2 = ProjectiveSpace(QQ, 3, 'y')
sage: ProductProjectiveSpaces([P1, P2])
Traceback (most recent call last):
  ... AttributeError: components must be over the same base ring
```

```python
class sage.schemes.product_projective.space.ProductProjectiveSpaces_field(N, R=Rational Field, names=None):
    Bases: ProductProjectiveSpaces_ring

    points_of_bounded_height(**kwds)
    Returns an iterator of the points in this product of projective spaces with the absolute heights of the components of at most the given bound.

    Bound check is strict for the rational field. Requires the base field of this space to be a number field. Uses the Doyle-Krumm algorithm 4 (algorithm 5 for imaginary quadratic) for computing algebraic numbers up to a given height [DK2013].

    The algorithm requires floating point arithmetic, so the user is allowed to specify the precision for such calculations. Additionally, due to floating point issues, points slightly larger than the bound may be returned. This can be controlled by lowering the tolerance.

    INPUT:
    • bound – a real number
    • tolerance – a rational number in (0,1] used in doyle-krumm algorithm-4
    • precision – the precision to use for computing the elements of bounded height of number fields.

    OUTPUT: an iterator of points in this space

    EXAMPLES:
```
(1 : 0 , 0 : -1 : 1), (1 : 0 , 0 : 0 : 1), (1 : 0 , 0 : 1 : 0),
(1 : 0 , 0 : 1 : 1), (1 : 0 , 1 : -1 : 1), (1 : 0 , 1 : 0 : 0),
(1 : 0 , 1 : 0 : 1), (1 : 0 , 1 : 1 : 0), (1 : 0 , 1 : 1 : 1),
(1 : 1 , -1 : -1 : 1), (1 : 1 , -1 : 0 : 1), (1 : 1 , -1 : 1 : 0),
(1 : 1 , -1 : 1 : 1), (1 : 1 , 0 : -1 : 1), (1 : 1 , 0 : 0 : 1),
(1 : 1 , 0 : 1 : 0), (1 : 1 , 0 : 1 : 1), (1 : 1 , 1 : -1 : 1),
(1 : 1 , 1 : 0 : 0), (1 : 1 , 1 : 0 : 1), (1 : 1 , 1 : 1 : 0),
(1 : 1 , 1 : 1 : 1)

sage: u = QQ['u'].0
sage: P = ProductProjectiveSpaces([1, 1], NumberField(u^2 - 2, 'v'))
    # needs sage.rings.number_field
sage: sorted(list(P.points_of_bounded_height(bound=1.5)))
    # needs sage.rings.number_field
[(-v : 1 , -v : 1), (-v : 1 , -1 : 1), (-v : 1 , -1/2*v : 1), (-v : 1 , 0 : 1),
(-1 : 1 , 1/2*v : 1), (1/2*v : 1 , -v : 1), (1/2*v : 1 , -1/2*v : 1),
(1/2*v : 1 , 0 : 1), (1/2*v : 1 , 1/2*v : 1), (1/2*v : 1 , 1 : 0),
(-1/2*v : 1 , -v : 1), (-1/2*v : 1 , -1 : 1), (-1/2*v : 1 , -1/2*v : 1),
(-1/2*v : 1 , 0 : 1), (-1/2*v : 1 , 1/2*v : 1), (-1/2*v : 1 , 1 : 0),
(-1 : 1 , 1 : 1), (1 : 0 , 0 : 1), (1 : 0 , 1 : 1), (1 : 0 , v : 1),
(1 : 1 , -v : 1), (1 : 1 , -1 : 1), (1 : 1 , -1/2*v : 1), (1 : 1 , 0 : 1),
(1 : 1 , 1/2*v : 1), (1 : 1 , 1 : 0), (1 : 1 , v : 1), (1 : 1 , v : 1),
(1 : 2 : 1 , 2 : 1), (1 : 2 : 1 , 2 : 1), (1 : 2 : 1 , 2 : 1), (1 : 2 : 1 , 2 : 1)]

class sage.schemes.product_projective.space.ProductProjectiveSpaces_field(N,
    R=Rational
    Field,
    names=None)

Bases: ProductProjectiveSpaces_field

rational_points(F=None)
    Return the list of $F$-rational points on this product of projective spaces, where $F$ is a given finite field, or the base ring of this space.

EXAMPLES:

sage: P = ProductProjectiveSpaces([1, 1], GF(5))
sage: P.rational_points()
[(0 : 1 , 0 : 1), (1 : 1 , 0 : 1), (2 : 1 , 0 : 1), (3 : 1 , 0 : 1), (4 : 1 , 0 : 1),
(0 : 1 , 1 : 1), (1 : 1 , 1 : 1), (2 : 1 , 1 : 1), (3 : 1 , 1 : 1), (4 : 1 , 1 : 1),
(0 : 1 , 1 : 1), (1 : 1 , 1 : 1), (2 : 1 , 1 : 1), (3 : 1 , 1 : 1), (4 : 1 , 1 : 1),
(0 : 1 , 2 : 1), (1 : 1 , 2 : 1), (2 : 1 , 2 : 1), (3 : 1 , 2 : 1), (4 : 1 , 2 : 1),
(0 : 1 , 3 : 1), (1 : 1 , 3 : 1), (2 : 1 , 3 : 1), (3 : 1 , 3 : 1), (4 : 1 , 3 : 1),
(0 : 1 , 4 : 1), (1 : 1 , 4 : 1), (2 : 1 , 4 : 1), (3 : 1 , 4 : 1), (4 : 1 , 4 : 1),
(0 : 1 , 5 : 1), (1 : 1 , 5 : 1), (2 : 1 , 5 : 1), (3 : 1 , 5 : 1), (4 : 1 , 5 : 1)]


```python
sage: P = ProductProjectiveSpaces([1, 1], GF(2))
sage: sorted(P.rational_points(GF(2^2, 'a')), key=str)    # needs sage.rings.finite_rings
[(0 : 1 , 0 : 1), (0 : 1 , 1 : 0), (0 : 1 , 1 : 1), (0 : 1 , a + 1 : 1), (0 : 1 → , a : 1),
 (1 : 0 , 0 : 1), (1 : 0 , 1 : 0), (1 : 0 , 1 : 1), (1 : 0 , a + 1 : 1), (1 : 0 → , a : 1),
 (1 : 1 , 0 : 1), (1 : 1 , 1 : 0), (1 : 1 , 1 : 1), (1 : 1 , a + 1 : 1), (1 : 1 → , a : 1),
 (a + 1 : 1 , 0 : 1), (a + 1 : 1 , 1 : 0), (a + 1 : 1 , 1 : 1), (a + 1 : 1 , a → +1 : 1), (a + 1 : 1 , a : 1),
 (a : 1 , 0 : 1), (a : 1 , 1 : 0), (a : 1 , 1 : 1), (a : 1 , a + 1 : 1), (a : 1 → , a : 1)]
```

class sage.schemes.product_projective.space.ProductProjectiveSpaces_ring(N, R=Rational Field, names=None)

Bases: AmbientSpace

Cartesian product of projective spaces \( \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \).

EXAMPLES:

```python
sage: P.<x0,x1,x2,x3,x4> = ProductProjectiveSpaces([1, 2], QQ); P
Product of projective spaces P^1 x P^2 over Rational Field
sage: P.coordinate_ring()
Multivariate Polynomial Ring in x0, x1, x2, x3, x4 over Rational Field
sage: P[0]
Projective Space of dimension 1 over Rational Field
sage: P[1]
Projective Space of dimension 2 over Rational Field
sage: Q = P(6, 3, 2, 2, 2); Q
(2 : 1 , 1 : 1 : 1)
sage: Q[0]
(2 : 1)
sage: H = Hom(P,P)
sage: f = H([x0^2*x3, x2^2*x1^2, x2^2, 2^x3^x2, x4^2])
sage: f(Q)
(4 : 1 , 1 : 2 : 1)
```

affine_patch(I, return_embedding=False)

Return the \( I^{th} \) affine patch of this projective space product where \( I \) is a multi-index.

INPUT:

- \( I \) – a list or tuple of positive integers.
- \( \text{return_embedding} \) – Boolean, if true the projective embedding is also returned.
Schemes, Release 10.2

OUTPUT:
• An affine space.
• An embedding into a product of projective spaces (optional).

EXAMPLES:

```python
sage: PP = ProductProjectiveSpaces([2, 2, 2], ZZ, ‘x’)
sage: phi = PP.affine_patch([0, 1, 2], True)
sage: phi.domain()
Affine Space of dimension 6 over Integer Ring
sage: phi
Scheme morphism:
From: Affine Space of dimension 6 over Integer Ring
To: Product of projective spaces P^2 x P^2 x P^2 over Integer Ring
Defn: Defined on coordinates by sending (x0, x1, x2, x3, x4, x5) to
(1 : x0 : x1 , x2 : 1 : x3 , x4 : x5 : 1)
```

`change_ring(R)`

Return a product of projective spaces over a ring R and otherwise the same as this projective space.

INPUT:
• R – commutative ring or morphism

OUTPUT:
• Product of projective spaces over R.

**Note:** There is no need to have any relation between R and the base ring of this space, if you want to have such a relation, use `self.base_extend(R)` instead.

EXAMPLES:

```python
sage: T.<x,y,z,u,v,w> = ProductProjectiveSpaces([2, 2], QQ)
sage: T.change_ring(GF(17))
Product of projective spaces P^2 x P^2 over Finite Field of size 17
```

`components()`

Return the components of this product of projective spaces.

OUTPUT: a list of projective spaces

EXAMPLES:

```python
sage: P.<x,y,z,u,v> = ProductProjectiveSpaces(QQ, [2, 1])
sage: P.components()
[Projective Space of dimension 2 over Rational Field,
Projective Space of dimension 1 over Rational Field]
```

`dimension()`

Return the absolute dimension of the product of projective spaces.

OUTPUT: a positive integer

EXAMPLES:
sage: T.<x,y,z,u,v,w> = ProductProjectiveSpaces([2, 2], GF(17))
sage: T.dimension_absolute()  
4
sage: T.dimension()  
4

dimension_absolute()  
Return the absolute dimension of the product of projective spaces.  
OUTPUT: a positive integer

EXAMPLES:

sage: T.<x,y,z,u,v,w> = ProductProjectiveSpaces([2, 2], GF(17))
sage: T.dimension_absolute()  
4
sage: T.dimension()  
4

dimension_absolute_components()  
Return the absolute dimension of the product of projective spaces.  
OUTPUT: a list of positive integers

EXAMPLES:

sage: T.<x,y,z,u,v,w> = ProductProjectiveSpaces([2, 2], GF(17))
sage: T.dimension_absolute_components()  
[2, 2]
sage: T.dimension_components()  
[2, 2]

dimension_components()  
Return the absolute dimension of the product of projective spaces.  
OUTPUT: a list of positive integers

EXAMPLES:

sage: T.<x,y,z,u,v,w> = ProductProjectiveSpaces([2, 2], GF(17))
sage: T.dimension_absolute_components()  
[2, 2]
sage: T.dimension_components()  
[2, 2]

dimension_relative()  
Return the relative dimension of the product of projective spaces.  
OUTPUT: a positive integer

EXAMPLES:

sage: T.<a,x,y,z,u,v,w> = ProductProjectiveSpaces([3, 2], QQ)
sage: T.dimension_relative()  
5
dimension_relative_components()
Return the relative dimension of the product of projective spaces.
OUTPUT: a list of positive integers
EXAMPLES:

```
sage: T.<a,x,y,z,u,v,w> = ProductProjectiveSpaces([3, 2], QQ)
sage: T.dimension_relative_components()
[3, 2]
```

ngens()
Return the number of generators of this space.
This is the number of variables in the coordinate ring of the projective space.
OUTPUT: an integer
EXAMPLES:

```
sage: T = ProductProjectiveSpaces([1, 1, 1], GF(5), 'x')
sage: T.ngens()
6
```

num_components()
Returns the number of components of this space.
OUTPUT: an integer
EXAMPLES:

```
sage: T = ProductProjectiveSpaces([1, 1, 1], GF(5), 'x')
sage: T.num_components()
3
```

segre_embedding(PP=None, var='u')
Return the Segre embedding of this space into the appropriate projective space.
INPUT:
• PP – (default: None) ambient image projective space;
  this is constructed if it is not given.
• var – string, variable name of the image projective space, default 𝑢 (optional).
OUTPUT:
Hom – from this space to the appropriate subscheme of projective space.

Todo: Cartesian products with more than two components.

EXAMPLES:

```
sage: X.<y0,y1,y2,y3,y4,y5> = ProductProjectiveSpaces(ZZ, [2, 2])
sage: phi = X.segre_embedding(); phi
Scheme morphism:
From: Product of projective spaces P^2 x P^2 over Integer Ring
To: Scheme over Integer Ring defined by 'y0' = 'y0'
'... (continues on next page)
To: Closed subscheme of Projective Space of dimension 8 over Integer Ring defined by:
- \(u^5u^7 + u^4u^8\), \(-u^5u^6 + u^3u^8\), \(-u^4u^6 + u^3u^7\),
- \(-u^2u^7 + u^1u^8\), \(-u^2u^4 + u^1u^5\), \(-u^2u^6 + u^0u^8\),
- \(-u^1u^6 + u^0u^7\), \(-u^2u^3 + u^0u^5\), \(-u^1u^3 + u^0u^4\)

Defn: Defined by sending \((y_0 : y_1 : y_2, y_3 : y_4 : y_5)\) to
\((y_0y_3 : y_0y_4 : y_0y_5 : y_1y_3 : y_1y_4 : y_1y_5 : y_2y_3 : y_2y_4 : y_2y_5)\).

\[\text{sage: } T = \text{ProductProjectiveSpaces([1, 2], CC, 'z')}\]  # \text{\textless needs sage.rings.real_mpfr}
\[\text{sage: } T.\text{segre_embedding}()\]  # \text{\textless needs sage.libs.singular sage.rings.real_mpfr}

Scheme morphism:
From: Product of projective spaces \(P^1 \times P^2\) over Complex Field with 53 bits of precision
To: Closed subscheme of Projective Space of dimension 5 over Complex Field with 53 bits of precision defined by:
- \(-u^2u^4 + u^1u^5\), \(-u^2u^3 + u^0u^5\), \(-u^1u^3 + u^0u^4\)

Defn: Defined by sending \((z_0 : z_1, z_2 : z_3 : z_4)\) to
\((z_0z_2 : z_0z_3 : z_0z_4 : z_1z_2 : z_1z_3 : z_1z_4)\).

\[\text{sage: } T = \text{ProductProjectiveSpaces([1, 2, 1], QQ, 'z')}\]
\[\text{sage: } T.\text{segre_embedding}()\]  # \text{\textless needs sage.libs.singular}

Scheme morphism:
From: Product of projective spaces \(P^1 \times P^2 \times P^1\) over Rational Field
To: Closed subscheme of Projective Space of dimension 11 over Rational Field defined by:
- \(-u^9u^10 + u^8u^11\), \(-u^7u^10 + u^6u^11\), \(-u^7u^8 + u^6u^9\),
- \(-u^5u^10 + u^4u^11\), \(-u^5u^8 + u^4u^9\), \(-u^5u^6 + u^4u^7\),
- \(-u^5u^9 + u^3u^10\), \(-u^5u^8 + u^3u^10\), \(-u^5u^8 + u^2u^11\),
- \(-u^4u^8 + u^2u^10\), \(-u^3u^8 + u^2u^9\), \(-u^3u^6 + u^2u^7\),
- \(-u^3u^4 + u^2u^5\), \(-u^5u^7 + u^1u^11\), \(-u^5u^6 + u^1u^10\),
- \(-u^3u^7 + u^1u^9\), \(-u^3u^6 + u^1u^8\), \(-u^5u^6 + u^0u^11\),
- \(-u^4u^6 + u^0u^10\), \(-u^3u^6 + u^0u^9\), \(-u^2u^6 + u^0u^8\),
- \(-u^1u^6 + u^0u^7\), \(-u^1u^4 + u^0u^5\), \(-u^1u^2 + u^0u^3\)

Defn: Defined by sending \((z_0 : z_1, z_2 : z_3 : z_4, z_5 : z_6)\) to
\((z_0z_2z_5 : z_0z_2z_6 : z_0z_3z_5 : z_0z_3z_6 : z_0z_4z_5 : z_0z_4z_6:
: z_1z_2z_5 : z_1z_2z_6 : z_1z_3z_5 : z_1z_3z_6 : z_1z_4z_5 : z_1z_4z_6)\).

\text{subscheme}(X)

Return the closed subscheme defined by \(X\).

INPUT:
- \(X\) – a list or tuple of equations

OUTPUT:
AlgebraicScheme_subscheme_projective_cartesian_product.
EXAMPLES:

```python
sage: P.<x,y,z,w> = ProductProjectiveSpaces([1, 1], GF(5))
sage: X = P.subscheme([x - y, z - w]); X
Closed subscheme of Product of projective spaces P^1 x P^1
over Finite Field of size 5 defined by:
   x - y,
   z - w
sage: X.defining_polynomials()
[x - y, z - w]
sage: I = X.defining_ideal(); I
Ideal (x - y, z - w) of Multivariate Polynomial Ring in x, y, z, w
over Finite Field of size 5
sage: X.dimension()
0
sage: X.base_ring()
Finite Field of size 5
sage: X.base_scheme()
Spectrum of Finite Field of size 5
sage: X.structure_morphism()
Scheme morphism:
   From: Closed subscheme of Product of projective spaces P^1 x P^1
          over Finite Field of size 5 defined by: x - y, z - w
   To:   Spectrum of Finite Field of size 5
   Defn: Structure map
```


```python
sage.schemes.product_projective.space.is_ProductProjectiveSpaces(x)
```

Return True if \( x \) is a product of projective spaces.

This is an ambient space defined by \( \mathbb{P}^n_R \times \cdots \times \mathbb{P}^m_R \), where \( R \) is a ring and \( n, \ldots, m \geq 0 \) are integers.

OUTPUT: Boolean.

EXAMPLES:

```python
sage: is_ProductProjectiveSpaces(ProjectiveSpace(5, names='x'))
False
sage: is_ProductProjectiveSpaces(ProductProjectiveSpaces([1, 2, 3], ZZ, 'x'))
True
```

## 15.2 Set of homomorphisms

AUTHORS:

- Volker Braun and Ben Hutz (2014): initial version
- Raghukul Raman (2018): code cleanup and added support for rational field

### 15.2. Set of homomorphisms
class sage.schemes.product_projective.homset.SchemeHomset_points_product_projective_spaces_field(X, Y, category=None, check=True, base=IntegerRing)

Bases: SchemeHomset_points_product_projective_spaces_ring

points(**kwds)
Return some or all rational points of a projective scheme.

Over a finite field, all points are returned. Over an infinite field, all points satisfying the bound are returned. For a zero-dimensional subscheme, all points are returned regardless of whether the base ring is a field or not.

For number fields, this uses the Doyle-Krumm algorithm 4 (algorithm 5 for imaginary quadratic) for computing algebraic numbers up to a given height [DK2013] or uses the Chinese remainder theorem and points modulo primes for larger bounds.

The algorithm requires floating point arithmetic, so the user is allowed to specify the precision for such calculations. Additionally, due to floating point issues, points slightly larger than the bound may be returned. This can be controlled by lowering the tolerance.

INPUT:
- bound - a real number
- tolerance - a rational number in (0,1] used in Doyle-Krumm algorithm 4
- precision - the precision to use for computing the elements of bounded height of number fields.
- algorithm - either 'sieve' or 'enumerate' algorithms can be used over \(\mathbb{Q}\). If not specified, 'enumerate' is used only for small height bounds.

OUTPUT: A list of rational points of the projective scheme.

EXAMPLES:

```
sage: P.<x,y,z,w> = ProductProjectiveSpaces([1, 1], QQ)
sage: X = P.subscheme([x - y, z^2 - 2*w^2])
sage: X(P.base_ring()).points()  # needs sage.libs.singular
[]
```

```
sage: u = QQ['u'].0
sage: K = NumberField(u^2 - 2, 'v')  # needs sage.rings.number_field
sage: P.<x,y,z,w> = ProductProjectiveSpaces([1, 1], K)  # needs sage.sage.sage.rings.number_field
sage: X = P.subscheme([x^2 - y^2, z^2 - 2*w^2])
sage: sorted(X(P.base_ring()).points())  # needs sage.libs.singular sage.rings.number_field
[(-1 : 1 , -v : 1), (-1 : 1 , v : 1), (1 : 1 , -v : 1), (1 : 1 , v : 1)]
```

```
sage: u = QQ['u'].0
sage: K = NumberField(u^2 + 1, 'v')  # needs sage.libs.singular
(continues on next page)
needs sage.rings.number_field

sage: P.<x,y,z,w> = ProductProjectiveSpaces([1, 1], K)    # needs sage.rings.number_field

sage: P(K).points(bound=1)  # needs sage.libs.singular sage.rings.number_field
[(-1 : 1 , -1 : 1), (-1 : 1 , -v : 1), (-1 : 1 , 0 : 1), (-1 : 1 , v : 1),
 (-1 : 1 , 1 : 0), (-1 : 1 , 1 : 1), (-v : 1 , -1 : 1), (-v : 1 , -v : 1),
 (-v : 1 , 0 : 1), (-v : 1 , v : 1), (-v : 1 , 1 : 0), (-v : 1 , 1 : 1),
 (0 : 1 , -1 : 1), (0 : 1 , 1 : 1), (v : 1 , -1 : 1), (v : 1 , -v : 1),
 (v : 1 , 0 : 1), (v : 1 , v : 1), (v : 1 , 1 : 0), (v : 1 , 1 : 1),
 (1 : 0 , -1 : 1), (1 : 0 , -v : 1), (1 : 0 , 0 : 1), (1 : 0 , v : 1),
 (1 : 0 , 1 : 0), (1 : 0 , 1 : 1), (1 : 1 , -1 : 1), (1 : 1 , -v : 1),
 (1 : 1 , 0 : 1), (1 : 1 , v : 1), (1 : 1 , 1 : 0), (1 : 1 , 1 : 1)]

sage: P.<x,y,z,u,v> = ProductProjectiveSpaces([2, 1], GF(3))
sage: P(P.base_ring()).points()  # needs sage.libs.singular
[(0 : 0 : 1 , 0 : 1), (0 : 0 : 1 , 1 : 0), (0 : 0 : 1 , 1 : 1), (0 : 0 : 1 , 2 : 1),
 (0 : 1 : 0 , 0 : 1), (0 : 1 : 0 , 1 : 0), (0 : 1 : 0 , 1 : 1), (0 : 1 : 0 , 2 : 1),
 (0 : 1 : 1 , 0 : 1), (0 : 1 : 1 , 1 : 0), (0 : 1 : 1 , 1 : 1), (0 : 1 : 1 , 2 : 1),
 (0 : 2 : 1 , 0 : 1), (0 : 2 : 1 , 1 : 0), (0 : 2 : 1 , 1 : 1), (0 : 2 : 1 , 2 : 1),
 (1 : 0 : 0 , 0 : 1), (1 : 0 : 0 , 1 : 0), (1 : 0 : 0 , 1 : 1), (1 : 0 : 0 , 2 : 1),
 (1 : 0 : 1 , 0 : 1), (1 : 0 : 1 , 1 : 0), (1 : 0 : 1 , 1 : 1), (1 : 0 : 1 , 2 : 1),
 (1 : 1 : 0 , 0 : 1), (1 : 1 : 0 , 1 : 0), (1 : 1 : 0 , 1 : 1), (1 : 1 : 0 , 2 : 1),
 (1 : 1 : 1 , 0 : 1), (1 : 1 : 1 , 1 : 0), (1 : 1 : 1 , 1 : 1), (1 : 1 : 1 , 2 : 1),
 (1 : 2 : 1 , 0 : 1), (1 : 2 : 1 , 1 : 0), (1 : 2 : 1 , 1 : 1), (1 : 2 : 1 , 2 : 1),
 (2 : 0 : 1 , 0 : 1), (2 : 0 : 1 , 1 : 0), (2 : 0 : 1 , 1 : 1), (2 : 0 : 1 , 2 : 1),
 (2 : 1 : 0 , 0 : 1), (2 : 1 : 0 , 1 : 0), (2 : 1 : 0 , 1 : 1), (2 : 1 : 0 , 2 : 1),
 (2 : 1 : 1 , 0 : 1), (2 : 1 : 1 , 1 : 0), (2 : 1 : 1 , 1 : 1), (2 : 1 : 1 , 2 : 1),
 (2 : 2 : 1 , 0 : 1), (2 : 2 : 1 , 1 : 0), (2 : 2 : 1 , 1 : 1), (2 : 2 : 1 , 2 : 1)]

sage: PP.<x,y,z,u,v> = ProductProjectiveSpaces([2, 1], QQ)
sage: X = PP.subscheme([x + y, u^2 - v^2])
sage: X.rational_points(bound=2)  # needs sage.libs.singular
(... continue on next page...)
(-1 : 1 : 0 , 1 : 1),
(-1 : 1 : 1 , 0 : 1),
(-1 : 1 : 1 , 1 : 1),
(-1/2 : 1/2 : 1 , 0 : 1),
(-1/2 : 1/2 : 1 , 1 : 1),
(0 : 0 : 1 , 0 : 1),
(0 : 0 : 1 , 1 : 1),
(1/2 : -1/2 : 1 , 0 : 1),
(1/2 : -1/2 : 1 , 1 : 1),
(1 : -1 : 1 , 0 : 1),
(1 : -1 : 1 , 1 : 1),
(2 : -2 : 1 , 0 : 1),
(2 : -2 : 1 , 1 : 1)]

better to enumerate with low codimension:

```
sage: PP.<x,y,z,u,v,a,b,c> = ProductProjectiveSpaces([2, 1, 2], QQ)
sage: X = PP.subscheme([x*u^2*a, b*z*u*v, z*v^2*c])
sage: len(X.rational_points(bound=1, algorithm='enumerate'))  # needs sage.libs.singular
232
```

class sage.schemes.product_projective.homset.SchemeHomset_points_product_projective_spaces_ring(X, Y, category=None, check=True, base=Integer Ring)

Bases: SchemeHomset_points

Set of rational points of a product of projective spaces.

INPUT: See SchemeHomset_generic.

EXAMPLES:

```
sage: from sage.schemes.product_projective.homset import SchemeHomset_points_product_projective_spaces_ring
sage: SchemeHomset_points_product_projective_spaces_ring(....: Spec(QQ), ProductProjectiveSpaces([1, 1], QQ, 'z'))
Set of rational points of Product of projective spaces P^1 x P^1 over Rational Field
15.3 Polynomial morphisms for products of projective spaces

This class builds on the projective space class and its point and morphism classes.

EXAMPLES:

```
sage: P1xP1.<x,y,u,v> = ProductProjectiveSpaces(QQ, [1, 1])
sage: H = End(P1xP1)
sage: H([x^2*u, y^2*v, x*v^2, y*u^2])
Scheme endomorphism of Product of projective spaces P^1 x P^1 over Rational Field
Defn: Defined by sending (x : y , u : v) to (x^2*u : y^2*v , x*v^2 : y*u^2).
```

```python
class sage.schemes.product_projective.morphism.ProductProjectiveSpaces_morphism_ring(
    parent, polys, check=True)
```

Bases: SchemeMorphism_polynomial

The class of morphisms on products of projective spaces.

The components are projective space morphisms.

EXAMPLES:

```
sage: T.<x,y,z,w,u> = ProductProjectiveSpaces([2, 1], QQ)
sage: H = T.Hom(T)
sage: H([x^2, y^2, z^2, w^2, u^2])
Scheme endomorphism of Product of projective spaces P^2 x P^1 over Rational Field
Defn: Defined by sending (x : y : z , w : u) to (x^2 : y^2 : z^2 , w^2 : u^2).
```

```python
as_dynamical_system()
```

Return this endomorphism as a DynamicalSystem_product_projective.

OUTPUT:

- DynamicalSystem_product_projective

EXAMPLES:

```
sage: Z.<a,b,x,y,z> = ProductProjectiveSpaces([1, 2], ZZ)
sage: H = End(Z)
sage: f = H([a^3, b^3, x^2, y^2, z^2])
sage: type(f.as_dynamical_system())
#<class 'sage.dynamics.arithmetic_dynamics.product_projective_ds.DynamicalSystem_product_projective'>
```

```python
global_height(prec=None)
```

Return the maximum of the absolute logarithmic heights of the coefficients in any of the coordinate functions of this map.

INPUT:

- prec – desired floating point precision (default: default RealField precision).

OUTPUT:

- a real number.
Todo: Add functionality for \( \mathbb{Q} \), implement function to convert the map defined over \( \mathbb{Q} \) to map over a number field.

EXAMPLES:

```
sage: P1xP1.<x,y,u,v> = ProductProjectiveSpaces([1, 1], ZZ)
sage: H = End(P1xP1)
sage: f = H([x^2*u, 3*y^2*v, 5*x*v^2, y*u^2])
sage: f.global_height()  # needs sage.rings.real_mpfr
1.60943791243410
```

```
sage: # needs sage.rings.number_field
sage: u = QQ['u'].0
sage: R = NumberField(u^2 - 2, 'v')
sage: PP.<x,y,a,b> = ProductProjectiveSpaces([1, 1], R)
sage: H = End(PP)
sage: O = R.maximal_order()
sage: g = H([3*O(u)*x^2, 13*x*y, 7*a*y, 5*b*x + O(u)*a*y])
sage: g.global_height()  # needs sage.rings.real_mpfr
2.56494935746154
```

**is_morphism()**

Return True if this mapping is a morphism of products of projective spaces.

For each component space of the codomain of this mapping we consider the subscheme of the domain of this map generated by the corresponding coordinates of the map. This map is a morphism if and only if each of these subschemes has no points.

OUTPUT: Boolean.

EXAMPLES:

```
sage: Z.<a,b,x,y,z> = ProductProjectiveSpaces([1, 2], ZZ)
sage: H = End(Z)
sage: f = H([a^2, b^2, x*z - y*z, x^2 - y^2, z^2])
sage: f.is_morphism()  # needs sage.libs.singular
False
```

```
sage: P.<x,y,z,u,v,w> = ProductProjectiveSpaces([2, 2], QQ)
sage: H = End(P)
sage: f = H([u, v, w, u^2, v^2, w^2])
sage: f.is_morphism()  # needs sage.libs.singular
True
```

```
sage: P.<x,y,z,w,u> = ProductProjectiveSpaces([2, 1], QQ)
sage: Q.<a,b,c,d,e> = ProductProjectiveSpaces([1, 2], QQ)
sage: H = Hom(P, Q)
sage: f = H([x^2, y^2, u^3, w^3, u^3])
```

(continues on next page)
local_height(v, prec=None)

Return the maximum of the local height of the coefficients in any of the coordinate functions of this map.

INPUT:

• v – a prime or prime ideal of the base ring.
• prec – desired floating point precision (default: default RealField precision).

OUTPUT: A real number.

EXAMPLES:

```python
sage: T.<x,y,z,w,u> = ProductProjectiveSpaces([2, 1], QQ)
sage: H = T.Hom(T)
sage: f = H([4*x^2 + 3/100*y^2, 8/210*x*y, 1/10000*z^2, 20*w^2, 1/384*u*w])
sage: f.local_height(2)  # needs sage.libs.singular
4.85203026391962
```

```python
sage: # needs sage.rings.real_mpfr
sage: # needs sage.rings.number_field
sage: R.<z> = PolynomialRing(QQ)
sage: K.<w> = NumberField(z^2 - 5)
sage: P.<x,y,a,b> = ProductProjectiveSpaces([1, 1], K)
sage: H = Hom(P, P)
sage: f = H([2*x^2 + w/3*y^2, 1/w*y^2, a^2, 6*b^2 + 1/9*a*b])
sage: f.local_height(K.ideal(3))  # needs sage.rings.real_mpfr
2.19722457733622
```

### 15.4 Points for products of projective spaces

This class builds on the projective space class and its point and morphism classes.

**EXAMPLES:**

We construct products projective spaces of various dimensions over the same ring:

```python
sage: P1xP1.<x,y, u,v> = ProductProjectiveSpaces(QQ, [1, 1])
sage: P1xP1([2, 1, 3, 1])
(2 : 1 , 3 : 1)
```

```python
class sage.schemes.product_projective.point.ProductProjectiveSpaces_point_field(parent, polys, check=True)
```

**Bases:** ProductProjectiveSpaces_point_ring

**intersection_multiplicity(X)**

Return the intersection multiplicity of the codomain of this point and subscheme X at this point.
This uses the subscheme implementation of intersection\_multiplicity. This point must be a point on a subscheme of a product of projective spaces.

**INPUT:**

- **X** – a subscheme in the same ambient space as the codomain of this point.

**OUTPUT:** An integer.

**EXAMPLES:**

```sage
sage: PP.<x,y,z,u,v> = ProductProjectiveSpaces(QQ, [2, 1])
sage: X = PP.subscheme([y^2*z^3*u - x^5*v])
sage: Y = PP.subscheme([u^3 - v^3, x - y])
sage: Q = X([0,0,1,1,1])
sage: Q.intersection_multiplicity(Y) # needs sage.libs.singular
2
```

**multiplicity()**

Return the multiplicity of this point on its codomain.

This uses the subscheme implementation of multiplicity. This point must be a point on a subscheme of a product of projective spaces.

**OUTPUT:** an integer.

**EXAMPLES:**

```sage
sage: PP.<x,y,z,w,u,v,t> = ProductProjectiveSpaces(QQ, [3, 2])
sage: X = PP.subscheme([x^8*t - y^8*t + z^5*w^3*v])
sage: Q1 = X([1,1,0,0,-1,-1,1])
sage: Q1.multiplicity() # needs sage.libs.singular
1
sage: Q2 = X([0,0,0,1,0,1,1])
sage: Q2.multiplicity() # needs sage.libs.singular
5
sage: Q3 = X([0,0,0,1,1,0,0])
sage: Q3.multiplicity() # needs sage.libs.singular
6
```

### sage.schemes.product_projective.point.ProductProjectiveSpaces_point_finite_field

**class** `sage.schemes.product_projective.point.ProductProjectiveSpaces_point_finite_field`

**Bases:** `ProductProjectiveSpaces_point_field`

The class of points on products of projective spaces.

The components are projective space points.

**EXAMPLES:**

```python
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```
```python
sage: T.<x,y,z,w,u> = ProductProjectiveSpaces([2, 1], QQ)
sage: T.point([1, 2, 3, 4, 5])
(1/3 : 2/3 : 1 : 4/5 : 1)
```

**change_ring**(*R*, **kwds**)

Return a new `ProductProjectiveSpaces_point` which is this point coerced to *R*.

If the keyword check is True, then the initialization checks are performed. The user may specify the embedding into *R* with a keyword.

**INPUT:**

- *R* – ring.

**kwds:**

- check – Boolean.
- embedding – field embedding from the base ring of this point to *R*.

**OUTPUT:** `ProductProjectiveSpaces_point`.

**EXAMPLES:**

```python
sage: T.<x,y,z,u,v,w> = ProductProjectiveSpaces([1, 1, 1], ZZ)
sage: P = T.point([5, 3, 15, 4, 2, 6])
sage: P.change_ring(GF(3))
(1 : 0 : 1 : 0)
```

**dehomogenize**(*L*)

Dehomogenize *k*\(^{th}\) point at \(L[k]\)\(^{th}\) coordinate.

This function computes the appropriate affine patch using *L* and then returns the dehomogenized point on of this affine space.

**INPUT:**

- *L* - a list of non-negative integers

**OUTPUT:**

- `SchemeMorphism_point_affine`.

**EXAMPLES:**

```python
sage: PP = ProductProjectiveSpaces([2, 2, 2], QQ, 'x')
sage: A = PP([2, 4, 6, 23, 46, 23, 9, 3, 1])
sage: A.dehomogenize([0, 1, 2])
(2, 3, 1/2, 1/2, 9, 3)
```

```python
sage: # needs sage.rings.real_mpfr sage.symbolic
sage: PP.<a,b,x,y,z> = ProductProjectiveSpaces([1, 2], CC)
sage: X = PP.subscheme([a^2 + b^2])
sage: P = X([2, 2*i, -3, 6*i, 3 - 6*i])
sage: P.dehomogenize([1,0])
(-1.00000000000000*I, -2.00000000000000*I, -1.00000000000000 + 2.00000000000000*I)
```

### 15.4. Points for products of projective spaces
global_height\((prec=None)\)

Return the absolute logarithmic height of the point.

This function computes the maximum of global height of each component point in the product. Global height of component point is computed using function for projective point.

INPUT:

\* prec – desired floating point precision (default: default RealField precision).

OUTPUT: A real number.

EXAMPLES:

```
sage: PP = ProductProjectiveSpaces(QQ, [2, 2], 'x')
sage: Q = PP([1, 7, 5, 18, 2, 3])
sage: Q.global_height()  # needs sage.symbolic
2.89037175789616
```

```
sage: PP = ProductProjectiveSpaces(QQ, [2, 2], 'x')
sage: Q = PP([1, 7, 5, 18, 2, 3])
sage: Q.global_height()  # needs sage.symbolic
2.89037175789616
```

```
sage: PP = ProductProjectiveSpaces(QQ, [2, 2], 'x')
sage: Q = PP([1, 7, 5, 18, 2, 3])
sage: Q.global_height()  # needs sage.symbolic
2.89037175789616
```

local_height\((v, prec=None)\)

Return the maximum of the local height of the coordinates of this point.

This function computes the maximum of local height of each component point in the product. Local height of component point is computed using function for projective point.

INPUT:
• \( v \) – a prime or prime ideal of the base ring.

• \( \text{prec} \) – desired floating point precision (default: default RealField precision).

**OUTPUT:** A real number.

**EXAMPLES:**

```python
sage: PP = ProductProjectiveSpaces(QQ, [1, 1], 'x')
sage: A = PP([11, 5, 10, 2])
sage: A.local_height(5)  # needs sage.rings.real_mpfr
1.60943791243410
```

```python
sage: P = ProductProjectiveSpaces(QQ, [1, 2], 'x')
sage: Q = P([1, 4, 1/2, 2, 32])
sage: Q.local_height(2)  # needs sage.rings.real_mpfr
4.15888308335967
```

**normalize_coordinates()**

Remove common factors (componentwise) from the coordinates of this point (including \(-1\)).

**OUTPUT:** None.

**EXAMPLES:**

```python
sage: T.<x,y,z,u,v,w> = ProductProjectiveSpaces([2, 2], ZZ)
sage: P = T.point([5, 10, 15, 4, 2, 6]);
sage: P.normalize_coordinates()
sage: P
(1 : 2 : 3 , 2 : 1 : 3)
```

**scale_by(t)**

Scale the coordinates of the point by \( t \), done componentwise.

A **TypeError** occurs if the point is not in the base ring of the codomain after scaling.

**INPUT:**

• \( t \) – a ring element

**EXAMPLES:**

```python
sage: T.<x, y, z, u, v, w> = ProductProjectiveSpaces([1, 1, 1], ZZ)
sage: P = T.point([5, 10, 15, 4, 2, 6]);
sage: P.scale_by([2, 1, 1])
sage: P
(10 : 20 , 15 : 4 , 2 : 6)
```
15.5 Subschemes of products of projective spaces

AUTHORS:

- Ben Hutz (2014): subschemes of Cartesian products of projective space

```python
class sage.schemes.product_projective.subscheme.AlgebraicScheme_subscheme_product_projective(A, polynomials):
    # Class definition
```

Bases: `AlgebraicScheme_subscheme_projective`

Construct an algebraic subscheme of a product of projective spaces.

**Warning:** You should not create objects of this class directly. The preferred method to construct such subschemes is to use `subscheme()` method of `Product of Projective Spaces`.

**INPUT:**

- `A` – ambient `Product of Projective Spaces`.
- `polynomials` – single polynomial, ideal or iterable of defining multi-homogeneous polynomials.

**EXAMPLES:**

```python
sage: P.<x, y, u, v> = ProductProjectiveSpaces([1, 1], QQ)
sage: P.subscheme([u*x^2 - v*y*x])
Closed subscheme of Product of projective spaces P^1 x P^1 over Rational Field defined by:
x^2*u - x*y*v
```

```
affine_patch(I, return_embedding=False)
```

Return the \( I \)th affine patch of this projective scheme where \( I \) is a multi-index.

**INPUT:**

- `I` – a list or tuple of positive integers
- `return_embedding` – Boolean, if true the projective embedding is also returned

**OUTPUT:**

- An affine algebraic scheme
- An embedding into a product of projective space (optional)

**EXAMPLES:**

```python
sage: PP.<x,y,z,w,u,v> = ProductProjectiveSpaces([3, 1], QQ)
sage: W = PP.subscheme([y^2*z - x^3, z^2 - w^2, u^3 - v^3])
sage: W.affine_patch([0, 1], True)
(Closed subscheme of Affine Space of dimension 4 over Rational Field defined by:
x0^2*x1 - 1,
x1^2 - x2^2,
x3^3 - 1,
Scheme morphism:
```

(continues on next page)
From: Closed subscheme of Affine Space of dimension 4 over Rational Field defined by: \(x^2 \cdot x_1 - 1, x_1 \cdot x_2 - x_2 \cdot x_2, x_3 \cdot x^3 - 1\)
To: Closed subscheme of Product of projective spaces \(P^3 \times P^1\) over Rational Field defined by: \(-x^3 + y^2 \cdot z, z^2 - w^2, u^3 - v^3\)
Defn: Defined on coordinates by sending \((x_0, x_1, x_2, x_3)\) to 
\((1 : x_0 : x_1 : x_2, x_3 : 1))

**dimension()**

Return the dimension of the algebraic subscheme.

**OUTPUT:** An integer.

**EXAMPLES:**

```
sage: X.<x,y,z,w,u,v> = ProductProjectiveSpaces([2, 2], QQ)
sage: L = (-w - v) \cdot x +( -w \cdot y - u \cdot z)
sage: Q = ((-u \cdot w - v \cdot z) \cdot x^2 + ((-w \cdot z - u \cdot w) \cdot y + (-w^2 + w \cdot v) \cdot z) \cdot x
....: + ((-w \cdot z - u \cdot w) \cdot x^2 + (-w^2 - v \cdot z) \cdot x + (w^2 + v \cdot u) \cdot w) \cdot z^2))
sage: W = X.subscheme([L, Q])
sage: W.dimension() # needs sage.libs.singular
2
```

```
sage: PP.<x,y,u,v,s,t> = ProductProjectiveSpaces([2, 1, 1], QQ)
sage: X = PP.subscheme([x^3, x^5 + y^5, z^6, x \cdot u - v \cdot y, s^2 - t^2])
sage: X.dimension() # needs sage.libs.singular
-1
```

```
sage: PP = ProductProjectiveSpaces([2, 1, 3], CC, 't') # needs sage.rings.real_mpfr
sage: PP.subscheme([]).dimension() # needs sage.libs.singular sage.rings.real_mpfr
6
```

```
sage: PP = ProductProjectiveSpaces([1, 3, 1], ZZ, 't')
sage: PP.subscheme([]).dimension() # needs sage.libs.singular
5
```

```
sage: PP.<x,y,u,v,s,t> = ProductProjectiveSpaces([1,1,1], CC) # needs sage.rings.real_mpfr
sage: X = PP.subscheme([x^2 - y^2, u - v, s^2 - t^2]) # needs sage.libs.singular sage.rings.real_mpfr
sage: X.dimension() # needs sage.libs.singular sage.rings.real_mpfr
0
```

**intersection_multiplicity(X, P)**

Return the intersection multiplicity of this subscheme and the subscheme \(X\) at the point \(P\).
This uses the intersection_multiplicity function for affine subschemes on affine patches of this subscheme and \( X \) that contain \( P \).

**INPUT:**

- \( X \) – subscheme in the same ambient space as this subscheme.
- \( P \) – a point in the intersection of this subscheme with \( X \).

**OUTPUT:** An integer.

**EXAMPLES:**

Multiplicity of a fixed point of the map \( z^2 + \frac{1}{4} \):

```sage
sage: PP.<x,y,u,v> = ProductProjectiveSpaces(QQ, [1, 1])
sage: G = PP.subscheme([[x^2 + 1/4*y^2]*v - y*u])
sage: D = PP.subscheme([x*v - y*u])
sage: sorted(G.intersection(D).rational_points()) # needs sage.libs.singular
[(1/2 : 1 , 1/2 : 1), (1 : 0 , 1 : 0)]
sage: Q = PP([1/2,1,1/2,1])
sage: G.intersection_multiplicity(D, Q) # needs sage.libs.singular
2
```

```sage
sage: # needs sage.rings.finite_rings
sage: F.<a> = GF(4)
sage: PP.<x,y,z,u,v,w> = ProductProjectiveSpaces(F, [2, 2])
sage: X = PP.subscheme([[z^5 + 3*x*y^4 + 8*y^5, u^2 - v^2]])
sage: Y = PP.subscheme([x^6 + z^6, w*z - v*y])
sage: Q = PP([a,a+1,a,a,1])
sage: X.intersection_multiplicity(Y, Q) # needs sage.libs.singular
16
```

```sage
sage: PP.<x,y,z,u,v,w> = ProductProjectiveSpaces(QQ, [2, 2])
sage: X = PP.subscheme([[x^2*u^3 + y*z*u*v^2, x - y]])
sage: Y = PP.subscheme([u^3 - w^3, x*v - y*w, z^3*w^2 - y^3*u*v])
sage: Q = PP([0,0,1,0,1,0])
sage: X.intersection_multiplicity(Y, Q) # needs sage.libs.singular
Traceback (most recent call last):
... TypeError: the intersection of this subscheme and (=Closed subscheme of AffineSpace of dimension 4 over Rational Field defined by: x2^3 - x3^3, -x1*x3 + x0, -x1^3*x2 + x3^2) must be proper and finite
```

**is_smooth(point=None)**

Test whether the algebraic subscheme is smooth.

**EXAMPLES:**

```sage
sage: X.<x,y,z,w,u,v> = ProductProjectiveSpaces([2, 2],QQ)
sage: L = (-w - v)*x + (-w*y - u*z)
```
sage: Q = ((-u*w - v^2)*x^2 + ((-w^2 - u*w + (-u*v - u^2))*y + (-w^2 - u*v)*z)*x + ((-w^2 - u*w - u^2)*y^2 + (-u*w - v^2)*z*y + (-w^2 + (-v - u)*w)*z^2))

sage: W = X.subscheme([L, Q])

sage: W.is_smooth()
Traceback (most recent call last):
...
NotImplementedError: Not Implemented

**multiplicity**(*P*)

Return the multiplicity of *P* on this subscheme.

This is computed as the multiplicity of the corresponding point on an affine patch of this subscheme that contains *P*. This subscheme must be defined over a field. An error is returned if *P* not a point on this subscheme.

**INPUT:**

- *P* – a point on this subscheme.

**OUTPUT:** An integer.

**EXAMPLES:**

```sage
sage: PP.<x,y,z,w> = ProductProjectiveSpaces(QQ, [1, 1])
sage: X = PP.subscheme([x^4*z^3 - y^4*w^3])
sage: Q1 = PP([1,1,1,1])
sage: X.multiplicity(Q1)  # needs sage.libs.singular
1
```

```sage
sage: Q2 = PP([0,1,1,0])
sage: X.multiplicity(Q2)  # needs sage.libs.singular
3
```

```sage
sage: PP.<x,y,z,w,u> = ProductProjectiveSpaces(GF(11), [1,2])
sage: X = PP.subscheme([x^7*u - y^7*z, u^6*x^2 - w^3*z^3*x*y - w^6*y^2])
sage: Q1 = PP([1,0,10,1,0])
sage: X.multiplicity(Q1)  # needs sage.libs.singular sage.rings.finite_rings
1
```

```sage
sage: Q2 = PP([1,0,1,0,0])
sage: X.multiplicity(Q2)  # needs sage.libs.singular sage.rings.finite_rings
4
```

**segre_embedding**(*PP=None*)

Return the Segre embedding of this subscheme into the appropriate projective space.

**INPUT:**

- *PP* – (default: None) ambient image projective space; this is constructed if it is not given.

**OUTPUT:**

Hom from this subscheme to the appropriate subscheme of projective space.
EXAMPLES:

```sage
sage: X.<x,y,z,w,u,v> = ProductProjectiveSpaces([2, 2], QQ)
sage: P = ProjectiveSpace(QQ, 8, 't')
sage: L = (-w - v)*x + (-w*y - u*z)
sage: Q = ((-u*w - v^2)*x^2 + ((-w^2 - u*w + (-u*v - u^2))*y + (-w^2 - u*v)*z)*x
    + ((-w^2 - u*w - u^2)*y^2 + (-u*w - v^2)*z*y + (-w^2 + (-v - u)*w)*z^2))
sage: W = X.subscheme([L,Q])
sage: phi = W.segre_embedding(P)
```

```
Scheme morphism:
From: Closed subscheme of Product of projective spaces P^2 x P^1 x P^1 over Complex Field with 53 bits of precision defined by:
(no polynomials)
To: Closed subscheme of Projective Space of dimension 7 over Complex Field with 53 bits of precision defined by:
-u5*u6 + u4*u7, -u3*u6 + u2*u7, -u3*u4 + u2*u5, -u3*u5 + u1*u7, -u3*u4 + u1*u6, -u3*u4 + u0*u7, -u2*u4 + u0*u6, -u1*u4 + u0*u5, -u1*u2 + u0*u3
Defn: Defined by sending (x : y , u : v , s : t) to
(x*u*s : x*u*t : x*v*s : x*v*t : y*u*s : y*u*t : y*v*s : y*v*t).
```

```
sage: PP.<x,y,u,v,s,t> = ProductProjectiveSpaces([2, 1, 1], ZZ)
sage: PP.subscheme([x^3, u - v, s^2 - t^2]).segre_embedding()
```

```
Scheme morphism:
From: Closed subscheme of Product of projective spaces P^2 x P^1 x P^1 over Integer Ring defined by:
x^3, u - v, s^2 - t^2
To: Closed subscheme of Projective Space of dimension 11 over Integer Ring defined by:
u10^2 - u11^2, u9 - u11, u8 - u10, -u7*u10 + u6*u11, u6*u10 - u7*u11, u6^2 - u7^2, u5 - u7, u4 - u6, u3^3, -u3*u10 + u2*u11, u2*u10 - u3*u11, -u3*u6 + u2*u7, u2*u6 - u3*u7, u2*u3^2, u2*u2 - u3^2, u1 - u3, u0 - u2
Defn: Defined by sending (x : y : z , u : v , s : t) to
(x*u*s : x*u*t : x*v*s : x*v*t : y*u*s : y*u*t : y*v*s : y*v*t : z*u*s : z*u*t : z*v*s : z*v*t).
```
15.6 Enumeration of rational points on product projective schemes

Naive algorithms for enumerating rational points over \( \mathbb{Q} \), number fields or finite fields over general schemes.

**Warning:** Incorrect results and infinite loops may occur if using a wrong function. (For instance using an affine function for a product projective scheme or a finite field function for a scheme defined over an infinite field.)

**EXAMPLES:**

Product Projective, over \( \mathbb{Q} \):

```sage
sage: PP.<x,y,z> = ProductProjectiveSpaces([1,0], QQ)
sage: from sage.schemes.product_projective.rational_point import enum_product_projective_rational_field
sage: enum_product_projective_rational_field(PP, 3)
[(-3 : 1 , 1), (-2 : 1 , 1), (-3/2 : 1 , 1),
 (-1 : 1 , 1), (-2/3 : 1 , 1), (-1/2 : 1 , 1),
 (-1/3 : 1 , 1), (0 : 1 , 1), (1/3 : 1 , 1),
 (1/2 : 1 , 1), (2/3 : 1 , 1), (1 : 0 , 1),
 (1/2 : 1 , 1), (2/3 : 1 , 1), (1 : 1 , 1),
 (3 : 1 , 1)]
```

Product projective over finite field:

```sage
sage: P1.<x,y,a,b> = ProductProjectiveSpaces([1, 1], GF(7))
sage: X = P1.subscheme([2*x + 3*y])
sage: from sage.schemes.product_projective.rational_point import enum_product_projective_finite_field
sage: enum_product_projective_finite_field(X)
[(2 : 1 , 0 : 1), (2 : 1 , 1 : 0), (2 : 1 , 1 : 1),
 (2 : 1 , 2 : 1), (2 : 1 , 3 : 1), (2 : 1 , 4 : 1),
 (2 : 1 , 5 : 1), (2 : 1 , 6 : 1)]
```

**AUTHORS:**

- Volker Braun and Ben Hutz (2014): initial version
- Raghukul Raman (2018): code cleanup and added support for rational fields

`sage.schemes.product_projective.rational_point.enum_product_projective_finite_field(X)`

Enumerates projective points on scheme \( X \) defined over a finite field.

**INPUT:**

- \( X \) - a scheme defined over a finite field or a set of abstract rational points of such a scheme.

**OUTPUT:**

- a list containing the projective points of \( X \) over the finite field, sorted.

**EXAMPLES:**

```sage
sage: PP.<x,y,z,w> = ProductProjectiveSpaces([1, 1], GF(3))
sage: from sage.schemes.product_projective.rational_point import enum_product_projective_finite_field
sage: enum_product_projective_finite_field(PP)
(continues on next page)
```
sage: PP.<x0,x1,x2,x3> = ProductProjectiveSpaces([1, 1], GF(17))

sage: X = PP.subscheme([x0^2 + 2*x1^2])

sage: from sage.schemes.product_projective.rational_point import 
    enum_product_projective_finite_field

sage: len(enum_product_projective_finite_field(X))
36

sage.schemes.product_projective.rational_point. 
enum_product_projective_number_field(X, 
    **kwds)

Enumerates product projective points on scheme X defined over a number field.
Simply checks all of the points of absolute height of at most B and adds those that are on the scheme to the list.
This algorithm computes 2 lists: L containing elements x in K such that H_k(x) <= B, and a list L' containing 
elements x in K that, due to floating point issues, may be slightly larger then the bound. This can be controlled 
by lowering the tolerance.

ALGORITHM:
This is an implementation of the revised algorithm (Algorithm 4) in [DK2013]. Algorithm 5 is used for imaginary 
quadratic fields.

INPUT:
kwds:
    • bound - a real number
    • tolerance - a rational number in (0,1] used in doyle-krumm algorithm-4
    • precision - the precision to use for computing the elements of bounded height of number fields.

OUTPUT:
    • a list containing the product projective points of X of absolute height up to B, sorted.

EXAMPLES:

sage: # needs sage.rings.number_field
sage: u = QQ['u'].0
sage: K = NumberField(u^2 + 2, 'v')

sage: PP.<x,y,z,w> = ProductProjectiveSpaces([1, 1], K)

sage: X = PP.subscheme([x^2 + 2*y^2])

sage: from sage.schemes.product_projective.rational_point import 
    enum_product_projective_number_field

sage: enum_product_projective_number_field(X, bound=1.5)
[(v : 1 , -v : 1), (-v : 1 , v : 1), (-v : 1 , -1/2**v : 1),
 (-v : 1 , 0 : 1), (-v : 1 , 1/2**v : 1), (-v : 1 , v : 1),
 (-v : 1 , 1 : 0), (-v : 1 , 1 : 1), (v : 1 , -1 : 1),
 (v : 1 , -v : 1), (v : 1 , -1/2**v : 1), (v : 1 , 0 : 1),
]
Enumerate projective, rational points on scheme $X$ of height up to bound $B$.

**INPUT:**
- $X$ – a scheme or set of abstract rational points of a scheme
- $B$ – a positive integer bound

**OUTPUT:**
- a list containing the product projective points of $X$ of height up to $B$, sorted.

**EXAMPLES:**

```python
sage: PP.<x0,x1,x2,x3,x4> = ProductProjectiveSpaces([1, 2], QQ)
sage: from sage.schemes.product_projective.rational_point import enum_product_projective_rational_field
sage: enum_product_projective_rational_field(PP, 1)
[(v : 1 , 1/2*v : 1), (v : 1 , v : 1), (v : 1 , 1 : 0), (v : 1 , 1 : 1)]
```

```python
sage: PP.<x,y,z,u,v> = ProductProjectiveSpaces([2, 1], QQ)
sage: X = PP.subscheme([x^2 + x*y + y*z, u*u - v*u])
sage: from sage.schemes.product_projective.rational_point import enum_product_projective_rational_field
sage: enum_product_projective_rational_field(X, 4)
[(-2 : -4 : 1 , 0 : 1), (-2 : 4 : 1 , 1 : 1), (-1 : 1 : 0 , 0 : 1), (-1 : 1 : 0 , 1 : 1), (-1 : 1 : 1 , 1 : 1), (-2/3 : -4/3 : 1 , 1 : 1), (-1/2 : -1/2 : 1 , 0 : 1), (-1/2 : 1 : 1 , 0 : 1), (0 : 0 : 1 , 0 : 1), (0 : 0 : 1 , 1 : 1), (0 : 1 : 0 , 1 : 1), (1 : -1/2 : 1 , 1 : 1), (1 : 1 , 1 : 1)]
```

sage.schemes.product_projective.rational_point.sieve($X$, $bound$)

Returns the list of all rational points on scheme $X$ of height up to bound.

15.6. Enumeration of rational points on product projective schemes
ALGORITHM:

Main idea behind the algorithm is to find points modulo primes and then reconstruct them using Chinese remainder theorem. We compute the points modulo primes parallelly and then lift them via Chinese remainder theorem in parallel. The LLL reduction algorithm is applied to each component of the points, and finally the result is merged and converted to a point on the subscheme.

For the algorithm to work correctly, sufficient primes need to be chosen, these are determined using the bounds dependent on the bound given in [Hutz2015].

INPUT:

• X - a scheme with ambient space defined over a product of projective spaces
• bound - a positive integer bound

OUTPUT:

• a list containing the rational points of X of height up to bound, sorted

EXAMPLES:

```
sage: from sage.schemes.product_projective.rational_point import sieve
sage: PP.<x,y,z,u,v> = ProductProjectiveSpaces([2, 1], QQ)
sage: X = PP.subscheme([x^2 + y^2 - x*z, u*u - v*u])
sage: sieve(X, 2)

[(0 : 0 : 1 , 0 : 1), (0 : 0 : 1 , 1 : 1), (1/2 : -1/2 : 1 , 0 : 1),
 (1/2 : -1/2 : 1 , 1 : 1), (1/2 : 1/2 : 1 , 0 : 1), (1/2 : 1/2 : 1 , 1 : 1),
 (1 : 0 : 1 , 0 : 1), (1 : 0 : 1 , 1 : 1)]
```
16.1 Toric varieties

This module provides support for (normal) toric varieties, corresponding to rational polyhedral fans. See also fano_variety for a more restrictive class of (weak) Fano toric varieties.

An excellent reference on toric varieties is the book “Toric Varieties” by David A. Cox, John B. Little, and Hal Schenck [CLS2011].

The interface to this module is provided through functions AffineToricVariety() and ToricVariety(), although you may also be interested in normalize_names().

Note: We do NOT build “general toric varieties” from affine toric varieties. Instead, we are using the quotient representation of toric varieties with the homogeneous coordinate ring (a.k.a. Cox’s ring or the total coordinate ring). This description works best for simplicial fans of the full dimension.

AUTHORS:
• Andrey Novoseltsev (2010-05-17): initial version.
• Volker Braun (2010-07-24): Cohomology and characteristic classes added.

EXAMPLES:
We start with constructing the affine plane as an affine toric variety. First, we need to have a corresponding cone:

\[
\text{Sage: } \text{quadrant} = \text{Cone([[1,0], (0,1)])}
\]

If you don’t care about variable names and the base field, that’s all we need for now:

\[
\text{Sage: } \text{A2} = \text{AffineToricVariety(\text{quadrant})}
\text{Sage: } \text{A2}
2-d affine toric variety
\text{Sage: } \text{origin} = \text{A2(0,0)}
\text{Sage: } \text{origin}
[0 : 0]
\]

Only affine toric varieties have points whose (homogeneous) coordinates are all zero.

\[
\text{Sage: } \text{parent(origin)}
\text{Set of rational points of 2-d affine toric variety}
\]

As you can see, by default toric varieties live over the field of rational numbers:
While usually toric varieties are considered over the field of complex numbers, for computational purposes it is more convenient to work with fields that have exact representation on computers. You can also always do

```python
sage: C2 = AffineToricVariety(quadrant, base_field=CC)
sage: C2.base_ring()
Complex Field with 53 bits of precision
```

or even

```python
sage: F = CC["a, b"].fraction_field()
sage: F.inject_variables()
Defining a, b
sage: A2 = AffineToricVariety(quadrant, base_field=F)
sage: A2(a,b)
[a : b]
```

OK, if you need to work only with affine spaces, `AffineSpace()` may be a better way to construct them. Our next example is the product of two projective lines realized as the toric variety associated to the face fan of the “diamond”:

```python
sage: diamond = lattice_polytope.cross_polytope(2)
sage: diamond.vertices()
M( 1, 0), M( 0, 1),
M(-1, 0), M( 0, -1)
in 2-d lattice M
sage: fan = FaceFan(diamond)
sage: P1xP1 = ToricVariety(fan)
sage: P1xP1
2-d toric variety covered by 4 affine patches
sage: P1xP1.fan().rays()
M( 1, 0), M( 0, 1),
M(-1, 0), M( 0, -1)
in 2-d lattice M
sage: P1xP1.gens()
(z0, z1, z2, z3)
```

We got four coordinates - two for each of the projective lines, but their names are perhaps not very well chosen. Let’s make \((x, y)\) to be coordinates on the first line and \((s, t)\) on the second one:

```python
sage: P1xP1 = ToricVariety(fan, coordinate_names="x s y t")
sage: P1xP1.gens()
(x, s, y, t)
```

Now, if we want to define subschemes of this variety, the defining polynomials must be homogeneous in each of these pairs:

```python
sage: P1xP1.inject_variables()
Defining x, s, y, t
sage: P1xP1.subscheme(x)
```
Closed subscheme of 2-d toric variety covered by 4 affine patches defined by:
\[ x \]
\[ \text{sage: P1xP1.subscheme}(x^2 + y^2) \]
Closed subscheme of 2-d toric variety covered by 4 affine patches defined by:
\[ x^2 + y^2 \]
\[ \text{sage: P1xP1.subscheme}(x^2 + s^2) \]
Traceback (most recent call last):
...
ValueError: \( x^2 + s^2 \) is not homogeneous
on 2-d toric variety covered by 4 affine patches
\[ \text{sage: P1xP1.subscheme}([x^2*s^2 + x*y*t^2 + y^2*t^2, s^3 + t^3]) \]
Closed subscheme of 2-d toric variety covered by 4 affine patches defined by:
\[ x^2*s^2 + x*y*t^2 + y^2*t^2, \]
\[ s^3 + t^3 \]

While we don’t build toric varieties from affine toric varieties, we still can access the “building pieces”:

\[ \text{sage: patch = P1xP1.affine_patch(2)} \]
\[ \text{sage: patch} \]
2-d affine toric variety
\[ \text{sage: patch.fan().rays()} \]
\[ M(1, 0), \]
\[ M(0, 1) \]
in 2-d lattice M
\[ \text{sage: patch.embedding_morphism()} \]
Scheme morphism:
\[ \text{From: 2-d affine toric variety} \]
\[ \text{To: 2-d toric variety covered by 4 affine patches} \]
\[ \text{Defn: Defined on coordinates by sending [x : s] to [x : s : 1 : 1]} \]

The patch above was specifically chosen to coincide with our representation of the affine plane before, but you can get the other three patches as well. (While any cone of a fan will correspond to an affine toric variety, the main interest is usually in the generating fans as “the biggest” affine subvarieties, and these are precisely the patches that you can get from \texttt{affine_patch()}.)

All two-dimensional toric varieties are “quite nice” because any two-dimensional cone is generated by exactly two rays. From the point of view of the corresponding toric varieties, this means that they have at worst quotient singularities:

\[ \text{sage: P1xP1.is_orbifold()} \]
True
\[ \text{sage: P1xP1.is_smooth()} \]
True
\[ \text{sage: TV = ToricVariety(NormalFan(diamond))} \]
\[ \text{sage: TV.fan().rays()} \]
\[ N(1, 1), N(1, -1), \]
\[ N(-1, -1), N(-1, 1) \]
in 2-d lattice M
\[ \text{sage: TV.is_orbifold()} \]
True
\[ \text{sage: TV.is_smooth()} \]
False

In higher dimensions worse things can happen:
sage: TV3 = ToricVariety(NormalFan(lattice_polytope.cross_polytope(3)))
sage: TV3.fan().rays()
N( 1, -1, -1), N( 1, 1, -1), N( 1, 1, 1), N( 1, -1, 1),
N(-1, -1, 1), N(-1, -1, -1), N(-1, 1, -1), N(-1, 1, 1)
in 3-d lattice N
sage: TV3.is_orbifold()
False

Fortunately, we can perform a (partial) resolution:

sage: TV3_res = TV3.resolve_to_orbifold()
sage: TV3_res.is_orbifold()
True
sage: TV3_res.fan().ngenerating_cones()
12
sage: TV3.fan().ngenerating_cones()
6

In this example we had to double the number of affine patches. The result is still singular:

sage: TV3_res.is_smooth()
False

You can resolve it further using resolve() method, but (at least for now) you will have to specify which rays should be inserted into the fan. See also CPRFanoToricVariety(), which can construct some other "nice partial resolutions."

The intersection theory on toric varieties is very well understood, and there are explicit algorithms to compute many quantities of interest. The most important tools are the cohomology ring and the Chow group. For $d$-dimensional compact toric varieties with at most orbifold singularities, the rational cohomology ring $H^*(X, \mathbb{Q})$ and the rational Chow ring $A^*(X, \mathbb{Q}) = A_{d-*}(X) \otimes \mathbb{Q}$ are isomorphic except for a doubling in degree. More precisely, the Chow group has the same rank

$$A_{d-k}(X) \otimes \mathbb{Q} \simeq H^{2k}(X, \mathbb{Q})$$

and the intersection in of Chow cycles matches the cup product in cohomology.

In this case, you should work with the cohomology ring description because it is much faster. For example, here is a weighted projective space with a curve of $\mathbb{Z}_3$-orbifold singularities:

sage: P4_11133 = toric_varieties.P4_11133()
sage: P4_11133.is_smooth(), P4_11133.is_orbifold()
(False, True)
sage: cone = P4_11133.fan(3)[8]
sage: cone.is_smooth(), cone.is_simplicial()
(False, True)
sage: HH = P4_11133.cohomology_ring(); HH
Rational cohomology ring of a 4-d CPR-Fano toric variety covered by 5 affine patches
sage: P4_11133.cohomology_basis()  # needs sage.libs.singular
([[1],], ([z4],), ([z4^2],), ([z4^3],), ([z4^4],))

Every cone defines a torus orbit closure, and hence a (co)homology class:
We can compute intersection numbers by integrating top-dimensional cohomology classes:

```python
sage: D = P4_11133.divisor(0)
sage: HH(D)
[3*z4]
sage: P4_11133.integrate(HH(D)^4)
9
sage: P4_11133.integrate(HH(D) * HH(cone))
1
```

Although computationally less efficient, we can do the same computations with the rational Chow group:

```python
sage: AA = P4_11133.Chow_group(QQ)
sage: list(map(AA, P4_11133.fan(1)))
# long time (5s on sage.math, 2012)
[(0 | 0 | 0 | 3 | 0), (0 | 0 | 0 | 3 | 0), (0 | 0 | 0 | 1 | 0), (0 | 0 | 0 | 1 | 0), (0 | 0 | 0 | 1 | 0)]
sage: list(map(AA, P4_11133.fan(4)))
# long time (5s on sage.math, 2012)
[(1 | 0 | 0 | 0 | 0), (1 | 0 | 0 | 0 | 0), (1 | 0 | 0 | 0 | 0), (1 | 0 | 0 | 0 | 0), (1 | 0 | 0 | 0 | 0)]
sage: AA(cone).intersection_with_divisor(D)
# long time (4s on sage.math, 2013)
(1 | 0 | 0 | 0 | 0)
sage: AA(cone).intersection_with_divisor(D).count_points()
# long time
1
```

The real advantage of the Chow group is that

- it works just as well over \(\mathbb{Z}\), so torsion information is also easily available, and
- its combinatorial description also works over worse-than-orbifold singularities. By contrast, the cohomology groups can become very complicated to compute in this case, and one usually only has a spectral sequence but no toric algorithm.

Below you will find detailed descriptions of available functions. If you are familiar with toric geometry, you will likely see that many important objects and operations are unavailable. However, this module is under active development and hopefully will improve in future releases of Sage. If there are some particular features that you would like to see implemented ASAP, please consider reporting them to the Sage Development Team or even implementing them on your own as a patch for inclusion!

```python
sage.schemes.toric.variety.AffineToricVariety(cone, *args, **kwds)
```

Construct an affine toric variety.

**INPUT:**

- `cone` – strictly convex rational polyhedral cone.

This cone will be used to construct a rational polyhedral fan, which will be passed to `ToricVariety()` with the rest of positional and keyword arguments.

**OUTPUT:** A toric variety.
Note: The generating rays of the fan of this variety are guaranteed to be listed in the same order as the rays of the original cone.

EXAMPLES:

We will create the affine plane as an affine toric variety:

```python
sage: quadrant = Cone([(1,0), (0,1)])
sage: A2 = AffineToricVariety(quadrant)
sage: origin = A2(0,0)
sage: origin
[0 : 0]
sage: parent(origin)
Set of rational points of 2-d affine toric variety
```

Only affine toric varieties have points whose (homogeneous) coordinates are all zero.

**class** `sage.schemes.toric.variety.CohomologyClass(cohomology_ring, representative)`

Bases: `QuotientRingElement`

An element of the `CohomologyRing`.

**Warning:** You should not create instances of this class manually. The generators of the cohomology ring as well as the cohomology classes associated to cones of the fan can be obtained from `ToricVariety_field.cohomology_ring()`.

**EXAMPLES:**

```python
sage: # needs sage.libs.singular
sage: P2 = toric_varieties.P2()
sage: P2.cohomology_ring().gen(0)
[z]
sage: HH = P2.cohomology_ring()
sage: HH.gen(0)
[z]
sage: cone = P2.fan(1)[0]; HH(cone)
[z]
```

**deg()**

The degree of the cohomology class.

**OUTPUT:**

An integer $d$ such that the cohomology class is in degree $2d$. If the cohomology class is of mixed degree, the highest degree is returned.

**EXAMPLES:**

```python
sage: P2 = toric_varieties.P2()
sage: P2.cohomology_ring().gen(0).deg()  # needs sage.libs.singular
1
sage: P2.cohomology_ring().zero().deg()  # needs sage.libs.singular
0
```
exp()

Exponentiate self.

Note: The exponential \( \exp(x) \) of a rational number \( x \) is usually not rational. Therefore, the cohomology class must not have a constant (degree zero) part. The coefficients in the Taylor series of \( \exp \) are rational, so any cohomology class without constant term can be exponentiated.

OUTPUT:

The cohomology class \( \exp(\text{self}) \) if the constant part vanishes, otherwise a ValueError is raised.

EXAMPLES:

```python
sage: # needs sage.libs.singular
sage: P2 = toric_varieties.P2()
sage: H_class = P2.cohomology_ring().gen(0)
sage: H_class
[1]
sage: H_class.exp()
[1/2*z^2 + z + 1]
```

part_of_degree\((d)\)

Project the (mixed-degree) cohomology class to the given degree.

\[
pr_d : H^*(X_\Delta, \mathbb{Q}) \to H^{2d}(X_\Delta, \mathbb{Q})
\]

INPUT:

• An integer \( d \)

OUTPUT:

• The degree-2d part of the cohomology class.

EXAMPLES:

```python
sage: # needs sage.libs.singular
sage: P1xP1 = toric_varieties.P1xP1()
sage: t = P1xP1.cohomology_ring().gen(0)
sage: y = P1xP1.cohomology_ring().gen(2)
sage: 3*t + 4*t^2*y + y + t*y + t + 1
[t^2y + 4*t + y + 1]
sage: (3*t + 4*t^2*y + y + t*y + t + 1).part_of_degree(1)
[4*t + y]
```

class `sage.schemes.toric.variety.CohomologyRing`\((\text{variety})\)

Bases: `QuotientRing_generic`, `UniqueRepresentation`

The (even) cohomology ring of a toric variety.

Irregardless of the variety’s base ring, we always work with the variety over \( \mathbb{C} \) and its topology.

The cohomology is always the singular cohomology with \( \mathbb{Q} \)-coefficients. Note, however, that the cohomology of smooth toric varieties is torsion-free, so there is no loss of information in that case.
Currently, the toric variety must not be “too singular”. See `ToricVariety_field.cohomology_ring()` for a detailed description of which toric varieties are admissible. For such varieties the odd-dimensional cohomology groups vanish.

**Warning:** You should not create instances of this class manually. Use `ToricVariety_field.cohomology_ring()` to generate the cohomology ring.

**INPUT:**

- `variety` – a toric variety. Currently, the toric variety must be at least an orbifold. See `ToricVariety_field.cohomology_ring()` for a detailed description of which toric varieties are admissible.

**EXAMPLES:**

```sage
sage: P2 = toric_varieties.P2()
sage: P2.cohomology_ring()
Rational cohomology ring of a 2-d CPR-Fano toric variety covered by 3 affine patches
```

This is equivalent to:

```sage
sage: from sage.schemes.toric.variety import CohomologyRing/nsage: CohomologyRing(P2)
Rational cohomology ring of a 2-d CPR-Fano toric variety covered by 3 affine patches
```

**gen(i)**

Return the generators of the cohomology ring.

**INPUT:**

- `i` – integer.

**OUTPUT:**

The $i$-th generator of the cohomology ring. If we denote the toric variety by $X$, then this generator is associated to the ray $X.fan().ray(i)$, which spans the one-cone $X.fan(1)[i]$.

**EXAMPLES:**

```sage
sage: P2 = toric_varieties.P2()
sage: P2.cohomology_ring().gen(2) # needs sage.libs.singular
[z]
```

**gens()**

Return the generators of the cohomology ring.

**OUTPUT:**

A tuple of generators, one for each toric divisor of the toric variety $X$. The order is the same as the ordering of the rays of the fan $X.fan().rays()$, which is also the same as the ordering of the one-cones in $X.fan(1)$.

**EXAMPLES:**

```sage
sage: P2 = toric_varieties.P2()
sage: P2.cohomology_ring().gens() # needs sage.libs.singular
([z], [z], [z])
```
Construct a toric variety.

INPUT:

- **fan** – rational polyhedral fan;
- **coordinate_names** – names of variables for the coordinate ring, see `normalize_names()` for acceptable formats. If not given, indexed variable names will be created automatically;
- **names** – an alias of `coordinate_names` for internal use. You may specify either `names` or `coordinate_names`, but not both;
- **coordinate_indices** – list of integers, indices for indexed variables. If not given, the index of each variable will coincide with the index of the corresponding ray of the fan;
- **base_ring** – base ring of the toric variety (default: \( \mathbb{Q} \)). Must be a field.
- **base_field** – alias for `base_ring`. Takes precedence if both are specified.

OUTPUT: A toric variety.

EXAMPLES:

We will create the product of two projective lines:

```python
sage: fan = FaceFan(lattice_polytope.cross_polytope(2))
sage: fan.rays()
M( 1, 0), M( 0, 1),
M(-1, 0), M( 0, -1)
in 2-d lattice M
sage: P1xP1 = ToricVariety(fan)
sage: P1xP1.gens()
(z0, z1, z2, z3)
```

Let’s create some points:

```python
sage: P1xP1(1,1,1,1)
[1 : 1 : 1 : 1]
sage: P1xP1(0,1,1,1)
[0 : 1 : 1 : 1]
sage: P1xP1(0,1,0,1)
Traceback (most recent call last):
  ...  
TypeError: coordinates (0, 1, 0, 1) are in the exceptional set
```

We cannot set to zero both coordinates of the same projective line!

Let’s change the names of the variables. We have to re-create our toric variety:

```python
sage: P1xP1 = ToricVariety(fan, "x s y t")
sage: P1xP1.gens()
(x, s, y, t)
```

Now \((x, y)\) correspond to one line and \((s, t)\) to the other one.
Here is a shorthand for defining the toric variety and homogeneous coordinates in one go:

```
sage: P1xP1.<a,b,c,d> = ToricVariety(fan)
sage: (a^2+b^2) * (c+d)
```

```a^2*c + b^2*c + a^2*d + b^2*d```

class `sage.schemes.toric.variety.ToricVariety_field`

```
Bases: AmbientSpace

Construct a toric variety associated to a rational polyhedral fan.

**Warning:** This class does not perform any checks of correctness of input. Use `ToricVariety()` and `AffineToricVariety()` to construct toric varieties.

**INPUT:**
- `fan` – rational polyhedral fan;
- `coordinate_names` – names of variables, see `normalize_names()` for acceptable formats. If `None`, indexed variable names will be created automatically;
- `coordinate_indices` – list of integers, indices for indexed variables. If `None`, the index of each variable will coincide with the index of the corresponding ray of the fan;
- `base_field` – base field of the toric variety.

**OUTPUT:** A toric variety.

**Aut_dimension**()

Return the dimension of the automorphism group

There are three kinds of symmetries of toric varieties:

- Toric automorphisms (rescaling of homogeneous coordinates)
- Demazure roots. These are translations $x_i \rightarrow x_i + \epsilon x_m$ of a homogeneous coordinate $x_i$ by a monomial $x_m$ of the same homogeneous degree.
- Symmetries of the fan. These yield discrete subgroups.

**OUTPUT:**

An integer. The dimension of the automorphism group. Equals the dimension of the $M$-lattice plus the number of Demazure roots.

**EXAMPLES:**

```
sage: P2 = toric_varieties.P2()
sage: P2.Aut_dimension()
8
```
**Chern_character**(deg=None)

Return the Chern character (of the tangent bundle) of the toric variety.

**INPUT:**

- deg – integer (optional). The degree of the Chern character.

**OUTPUT:**

- If the degree is specified, the degree-deg part of the Chern character.
- If no degree is specified, the total Chern character.

**REFERENCES:**

- Wikipedia article Chern_character#The_Chern_character

**EXAMPLES:**

```sage
sage: # needs sage.libs.singular
sage: dP6 = toric_varieties.dP6()
sage: dP6.Chern_character()
[3*w^2 + y + 2*v + 2*z + w + 2]
sage: dP6.ch()
[3*w^2 + y + 2*v + 2*z + w + 2]
sage: dP6.ch(1) == dP6.c(1)
True
```

**Chern_class**(deg=None)

Return Chern classes of the (tangent bundle of the) toric variety.

**INPUT:**

- deg – integer (optional). The degree of the Chern class.

**OUTPUT:**

- If the degree is specified, the deg-th Chern class.
- If no degree is specified, the total Chern class.

**REFERENCES:**

- Wikipedia article Chern_class

**EXAMPLES:**

```sage
sage: # needs sage.libs.singular
sage: X = toric_varieties.dP6()
sage: X.Chern_class()
[-6*w^2 + y + 2*v + 2*z + w + 1]
sage: X.c()  
[-6*w^2 + y + 2*v + 2*z + w + 1]
sage: X.c(1)  
[y + 2*v + 2*z + w]
sage: X.c(2)  
[-6*w^2]
```

16.1. Toric varieties
Chow_group(base_ring=Integer Ring)

Return the toric Chow group.

INPUT:

• base_ring – either ZZ (default) or QQ. The coefficient ring of the Chow group.

OUTPUT: A sage.schemes.toric.chow_group.ChowGroup_class.

EXAMPLES:

sage: A = toric_varieties.P2().Chow_group(); A
Chow group of 2-d CPR-Fano toric variety covered by 3 affine patches
sage: A.gens()
(( 0 | 0 | 1 ), ( 0 | 1 | 0 ), ( 1 | 0 | 0 ))

Demazure_roots()

Return the Demazure roots.

OUTPUT:

The roots as points of the $M$-lattice.

REFERENCES:

• De1970
• Baz2011

EXAMPLES:

sage: P2 = toric_varieties.P2()
sage: P2.Demazure_roots()
(M(-1, 0), M(-1, 1), M(0, -1), M(0, 1), M(1, -1), M(1, 0))

Here are the remaining three examples listed in [Baz2011], Example 2.1 and 2.3:

sage: s = 3
sage: cones = [(0,1), (1,2), (2,3), (3,0)]
sage: Hs = ToricVariety(Fan(rays=[[1,0], (0,-1), (-1,s), (0,1)], cones=cones))
sage: Hs.Demazure_roots()
(M(-1, 0), M(1, 0), M(0, 1), M(1, 1), M(2, 1), M(3, 1))

sage: P11s = ToricVariety(Fan(rays=[[1,0], (0,-1), (-1,s)],
....: cones=[[0,1], (1,2), (2,0)]))
sage: P11s.Demazure_roots()
(M(-1, 0), M(1, 0), M(0, 1), M(1, 1), M(2, 1), M(3, 1))

sage: Bs = ToricVariety(Fan(rays=[[s,1], (s,-1), (-s,-1), (-s,1)], cones=cones))
sage: Bs.Demazure_roots()
()

Euler_number()

Return the topological Euler number of the toric variety.

Sometimes, this is also called the Euler characteristic. chi() is a synonym for Euler_number().

REFERENCES:
• Wikipedia article Euler_characteristic

EXAMPLES:

```
sage: # needs sage.libs.singular
sage: P1xP1 = toric_varieties.P1xP1()
sage: P1xP1.Euler_number()
4
sage: P1xP1.chi()
4
```

**K()**

Returns the canonical divisor of the toric variety.

EXAMPLES:

Let's test that the del Pezzo surface $dP_6$ has degree 6, as its name implies:

```
sage: dP6 = toric_varieties.dP6()
sage: HH = dP6.cohomology_ring()
sage: dP6.K()
-V(x) - V(u) - V(y) - V(v) - V(z) - V(w)
sage: dP6.integrate( HH(dP6.K())^2 )
#␣˓→needs sage.libs.singular
6
```

**Kaehler_cone()**

Return the closure of the Kähler cone of *self*.

OUTPUT: cone.

**Note:** This cone sits in the rational divisor class group of *self* and the choice of coordinates agrees with *rational_class_group()*.

EXAMPLES:

```
sage: P1xP1 = toric_varieties.P1xP1()
sage: Kc = P1xP1.Kaehler_cone()
sage: Kc
2-d cone in 2-d lattice
sage: Kc.rays()
Divisor class [0, 1],
Divisor class [1, 0]
in Basis lattice of The toric rational divisor class group of a 2-d CPR-Fano toric variety covered by 4 affine patches
sage: [ divisor_class.lift() for divisor_class in Kc.rays() ]
[V(y), V(t)]
sage: Kc.lattice()
Basis lattice of The toric rational divisor class group of a 2-d CPR-Fano toric variety covered by 4 affine patches
```

**Mori_cone()**

Returns the Mori cone of *self*.

OUTPUT: cone.
Note:

- The Mori cone is dual to the Kähler cone.
- We think of the Mori cone as living inside the row span of the Gale transform matrix (computed by `self.fan().Gale_transform()`).
- The points in the Mori cone are the effective curves in the variety.
- The $i$-th entry in each Mori vector is the intersection number of the curve corresponding to the generator of the $i$-th ray of the fan with the corresponding divisor class. The very last entry is associated to the origin of the fan lattice.
- The Mori vectors are also known as the gauged linear sigma model charge vectors.

EXAMPLES:

```python
sage: P4_11169 = toric_varieties.P4_11169_resolved()

sage: P4_11169.Mori_cone()
2-d cone in 7-d lattice

sage: P4_11169.Mori_cone().rays()
(3, 2, 0, 0, 0, 1, -6),
(0, 0, 1, 1, 1, -3, 0)
in Ambient free module of rank 7 over the principal ideal domain Integer Ring

Spec(cone=None, names=None)

Return the spectrum associated to the dual cone.

Let $\sigma \in \mathbb{N}_R$ be a cone and $\sigma^\vee \cap M$ the associated semigroup of lattice points in the dual cone. Then

$$S = \mathbb{C}[\sigma^\vee \cap M]$$

is a $\mathbb{C}$-algebra. It is spanned over $\mathbb{C}$ by the points of $\sigma \cap N$, addition is formal linear combination of lattice points, and multiplication of lattice points is the semigroup law (that is, addition of lattice points). The $\mathbb{C}$-algebra $S$ then defines a scheme $\text{Spec}(S)$.

For example, if $\sigma = \{(x, y)|x \geq 0, y \geq 0\}$ is the first quadrant then $S$ is the polynomial ring in two variables. The associated scheme is $\text{Spec}(S) = \mathbb{C}^2$.

The same construction works over any base field, this introduction only used $\mathbb{C}$ for simplicity.

INPUT:

- `cone` – a `Cone`. Can be omitted for an affine toric variety, in which case the (unique) generating cone is used.
- `names` – (optional). Names of variables for the semigroup ring, see `normalize_names()` for acceptable formats. If not given, indexed variable names will be created automatically.

OUTPUT:

The spectrum of the semigroup ring $\mathbb{C}[\sigma^\vee \cap M]$.

EXAMPLES:

```python
sage: quadrant = Cone([(1,0), (0,1)])

sage: AffineToricVariety(quadrant).Spec()
Spectrum of Multivariate Polynomial Ring in z0, z1 over Rational Field

A more interesting example:
Stanley_Reisner_ideal()

Return the Stanley-Reisner ideal.

OUTPUT:

- The Stanley-Reisner ideal in the polynomial ring over \( \mathbb{Q} \) generated by the homogeneous coordinates.

EXAMPLES:

```
sage: fan = Fan([[0,1,3], [3,4], [2,0], [1,2,4]], [[(-3, -2, 1), (0, 0, 1), (3, -2, 1), (-1, -1, 1), (1, -1, 1)]])
sage: X = ToricVariety(fan, coordinate_names='A B C D E', base_field=GF(5))
sage: SR = X.Stanley_Reisner_ideal(); SR
Ideal (A*E, C*D, A*B*C, B*D*E) of Multivariate Polynomial Ring in A, B, C, D, E over Rational Field
```

Td(deg=None)

Return the Todd class (of the tangent bundle) of the toric variety.

INPUT:

- \( \text{deg} \) – integer (optional). The desired degree part.

OUTPUT:

- If the degree is specified, the degree-\( \text{deg} \) part of the Todd class.
- If no degree is specified, the total Todd class.

REFERENCES:

- Wikipedia article Todd_class

EXAMPLES:

```
sage: # needs sage.libs.singular
dsage: dP6 = toric_varieties.dP6()
sage: dP6.Todd_class()
[-w^2 + 1/2*y + v + z + 1/2*w + 1]
sage: dP6.Td()
[-w^2 + 1/2*y + v + z + 1/2*w + 1]
sage: dP6.integrate( dP6.Td() )
1
```

Todd_class(deg=None)

Return the Todd class (of the tangent bundle) of the toric variety.

INPUT:

- \( \text{deg} \) – integer (optional). The desired degree part.

OUTPUT:

- If the degree is specified, the degree-\( \text{deg} \) part of the Todd class.
• If no degree is specified, the total Todd class.

REFERENCES:
• Wikipedia article Todd_class

EXAMPLES:

```sage
sage: # needs sage.libs.singular
sage: dP6 = toric_varieties.dP6()
sage: dP6.Todd_class()
[-w^2 + 1/2*y + v + z + 1/2*w + 1]
sage: dP6.Td()
[-w^2 + 1/2*y + v + z + 1/2*w + 1]
sage: dP6.integrate( dP6.Td() )
1
```

```
affine_algebraic_patch(cone=None, names=None)

Return the patch corresponding to cone as an affine algebraic subvariety.

INPUT:

• cone – a Cone σ of the fan. It can be omitted for an affine toric variety, in which case the single generating cone is used.

OUTPUT:

A affine algebraic subscheme corresponding to the patch Spec(σ∨ ∩ M) associated to the cone σ. See also affine_patch(), which expresses the patches as subvarieties of affine toric varieties instead.

EXAMPLES:

```sage
sage: cone = Cone([(0,1), (2,1)])
sage: A2Z2 = AffineToricVariety(cone)
sage: A2Z2.affine_algebraic_patch() # needs fpylll sage.libs.singular
Closed subscheme of Affine Space of dimension 3 over Rational Field defined by:
-z0*z1 + z2^2
sage: A2Z2.affine_algebraic_patch(Cone([(0,1)]), names='x, y, t') # needs fpylll sage.libs.singular
Closed subscheme of Affine Space of dimension 3 over Rational Field defined by:
1
```

```
affine_patch(i)

Return the i-th affine patch of self.

INPUT:

• i – integer, index of a generating cone of the fan of self.

OUTPUT:

• affine toric variety corresponding to the i-th generating cone of the fan of self. The result is cached, so the i-th patch is always the same object in memory.

See also affine_algebraic_patch(), which expresses the patches as subvarieties of affine space instead.

EXAMPLES:
```
```
sage: fan = FaceFan(lattice_polytope.cross_polytope(2))
sage: P1xP1 = ToricVariety(fan, "x s y t")
sage: patch0 = P1xP1.affine_patch(0)
sage: patch0
2-d affine toric variety
sage: patch0.embedding_morphism()
Scheme morphism:
  From: 2-d affine toric variety
  To: 2-d toric variety covered by 4 affine patches
  Defn: Defined on coordinates by sending [y : t] to [1 : 1 : y : t]
sage: patch1 = P1xP1.affine_patch(1)
sage: patch1.embedding_morphism()
Scheme morphism:
  From: 2-d affine toric variety
  To: 2-d toric variety covered by 4 affine patches
  Defn: Defined on coordinates by sending [s : y] to [1 : s : y : 1]
sage: patch1 is P1xP1.affine_patch(1)
True
```

\text{c(deg=\text{None})}

Return Chern classes of the (tangent bundle of the) toric variety.

INPUT:

- \text{deg} – integer (optional). The degree of the Chern class.

OUTPUT:

- If the degree is specified, the \text{deg}-th Chern class.
- If no degree is specified, the total Chern class.

REFERENCES:

- \text{Wikipedia article Chern_class}

EXAMPLES:

```
sage: # needs sage.libs.singular
sage: X = toric_varieties.dP6()
sage: X.Chern_class()
[-6*w^2 + y + 2*v + 2*z + w + 1]
sage: X.c()
[-6*w^2 + y + 2*v + 2*z + w + 1]
sage: X.c(1)
[y + 2*v + 2*z + w]
sage: X.c(2)
[-6*w^2]
sage: X.integrate( X.c(2) )
6
sage: X.integrate( X.c(2) ) == X.Euler_number()
True
```

cartesian_product \text{(other, coordinate_names=\text{None}, coordinate_indices=\text{None})}

Return the Cartesian product of \text{self} with \text{other}.

INPUT:
• other – a toric variety;

• coordinate_names – names of variables for the coordinate ring, see normalize_names() for acceptable formats. If not given, indexed variable names will be created automatically;

• coordinate_indices – list of integers, indices for indexed variables. If not given, the index of each variable will coincide with the index of the corresponding ray of the fan.

OUTPUT: A toric variety.

EXAMPLES:

```python
sage: P1 = ToricVariety(Fan([Cone([(1,)]), Cone([(-1,)])]))
sage: P1xP1 = P1.cartesian_product(P1);
P1xP1
2-d toric variety covered by 4 affine patches
sage: P1xP1.fan().rays()
N+N(-1, 0), N+N( 1, 0), N+N( 0, -1), N+N( 0, 1)
in 2-d lattice N+N
```

ch(deg=None)
Return the Chern character (of the tangent bundle) of the toric variety.

INPUT:

• deg – integer (optional). The degree of the Chern character.

OUTPUT:

• If the degree is specified, the degree-deg part of the Chern character.

• If no degree is specified, the total Chern character.

REFERENCES:

• Wikipedia article Chern_character#The_Chern_character

EXAMPLES:

```python
sage: # needs sage.libs.singular
sage: dP6 = toric_varieties.dP6()
sage: dP6.Chern_character()
[3*w^2 + y + 2*v + 2*z + w + 2]
sage: dP6.ch()
[3*w^2 + y + 2*v + 2*z + w + 2]
sage: dP6.ch(1) == dP6.c(1)
True
```

change_ring(F)
Return a toric variety over F and otherwise the same as self.

INPUT:

• F – field.

OUTPUT: toric variety over F.

Note: There is no need to have any relation between F and the base field of self. If you do want to have such a relation, use base_extend() instead.

EXAMPLES:
sage: P1xA1 = toric_varieties.P1xA1()
sage: P1xA1.base_ring()
Rational Field
sage: P1xA1_RR = P1xA1.change_ring(RR)
sage: P1xA1_RR.base_ring()
Real Field with 53 bits of precision
sage: P1xA1_QQ = P1xA1_RR.change_ring(QQ)
sage: P1xA1_QQ.base_ring()
Rational Field
sage: P1xA1_RR.base_extend(QQ)
Traceback (most recent call last):
...  
ValueError: no natural map from the base ring 
(=Real Field with 53 bits of precision) to R (=Rational Field)!
sage: R = PolynomialRing(QQ, 2, 'a')
sage: P1xA1.change_ring(R)
Traceback (most recent call last):
...  
TypeError: need a field to construct a toric variety; 
got Multivariate Polynomial Ring in a0, a1 over Rational Field

chi()

Return the topological Euler number of the toric variety.
Sometimes, this is also called the Euler characteristic. chi() is a synonym for Euler_number().

REFERENCES:

• Wikipedia article Euler_characteristic

EXAMPLES:

sage: # needs sage.libs.singular
sage: X = toric_varieties.dP8()
sage: X.cohomology_basis()
(continues on next page)

cohomology_basis(d=None)

Return a basis for the cohomology of the toric variety.

INPUT:

• d (optional) – integer.

OUTPUT:

• Without the optional argument, a list whose d-th entry is a basis for $H^{2d}(X, \mathbb{Q})$
• If the argument is an integer d, returns basis for $H^{2d}(X, \mathbb{Q})$

EXAMPLES:

sage: # needs sage.libs.singular
sage: X = toric_varieties.dP8()
sage: X.cohomology_basis()
cohomology_ring()

Return the cohomology ring of the toric variety.

OUTPUT:

• If the toric variety is over \( \mathbb{C} \) and has at most finite orbifold singularities: \( H^*(X, \mathbb{Q}) \) as a polynomial quotient ring.
• Other cases are not handled yet.

Note:

• Toric varieties over any field of characteristic 0 are treated as if they were varieties over \( \mathbb{C} \).
• The integral cohomology of smooth toric varieties is torsion-free, so in this case there is no loss of information when going to rational coefficients.
• \( \text{self.cohomology_ring().gen(i)} \) is the divisor class corresponding to the \( i \)-th ray of the fan.

EXAMPLES:

```python
sage: X = toric_varieties.dP6()
sage: X.cohomology_ring()
Rational cohomology ring of a 2-d CPR-Fano toric variety covered by 6 affine patches

sage: X.cohomology_ring().defining_ideal()
Ideal (-u - y + z + w, x - y - v + w, x*v, x*z, u*z, u*w, y*z, y*w, v*w) of Multivariate Polynomial Ring in x, u, y, v, z, w over Rational Field

sage: X.coordinate_ring()
Multivariate Polynomial Ring in s, t, x, y over Rational Field

sage: X.variable_names()
('x', 'u', 'y', 'v', 'z', 'w')
sage: X.cohomology_ring().gens()  # needs sage.libs.singular
([y + v - w], [-y + z + w], [y], [v], [z], [w])
```

coordinate_ring()

Return the coordinate ring of \( \text{self} \).

For toric varieties this is the homogeneous coordinate ring (a.k.a. Cox’s ring and total ring).

OUTPUT: A polynomial ring.

EXAMPLES:

```python
sage: P1xP1 = toric_varieties.P1xP1()
sage: P1xP1.coordinate_ring()
Multivariate Polynomial Ring in s, t, x, y over Rational Field
```
count_points()
Return the number of points of self.
This is an alias for point_set().cardinality(), see cardinality() for details.

EXAMPLES:

```python
sage: o = lattice_polytope.cross_polytope(3)
sage: V = ToricVariety(FaceFan(o))
sage: V2 = V.change_ring(GF(2))
sage: V2.point_set().cardinality()
27
sage: V2.count_points()
27
```

dimension_singularities()
Return the dimension of the singular set.

OUTPUT:

Integer. The dimension of the singular set of the toric variety. Often the singular set is a reducible subvariety, and this method will return the dimension of the largest-dimensional component.

Returns -1 if the toric variety is smooth.

EXAMPLES:

```python
sage: toric_varieties.P4_11169().dimension_singularities()
1
sage: toric_varieties.Conifold().dimension_singularities()
0
sage: toric_varieties.P2().dimension_singularities()
-1
```

divisor(arg, base_ring=None, check=True, reduce=True)
Return a divisor.

INPUT:

The arguments are the same as in sage.schemes.toric.divisor.ToricDivisor(), with the exception of defining a divisor with a single integer: this method considers it to be the index of a ray of the fan() of self.

OUTPUT:

• A sage.schemes.toric.divisor.ToricDivisor_generic

EXAMPLES:

```python
sage: dP6 = toric_varieties.dP6()
sage: dP6.coordinate_ring()
Multivariate Polynomial Ring in x, u, y, v, z, w over Rational Field
sage: dP6.divisor(list(range(6)))
V(u) + 2*V(y) + 3*V(v) + 4*V(z) + 5*V(w)
sage: dP6.inject_variables()
Defining x, u, y, v, z, w
sage: dP6.divisor(x*u^3)
V(x) + 3*V(u)
```
You can also construct divisors based on ray indices:

```python
sage: dP6.divisor(0)
V(x)
sage: for i in range(dP6.fan().nrays()):
    ....: print('{} : generated by ray {}'.format(dP6.divisor(i),
    ....:          dP6.fan().ray(i)))
V(x) : generated by ray N(0, 1)
V(u) : generated by ray N(-1, 0)
V(y) : generated by ray N(-1, -1)
V(v) : generated by ray N(0, -1)
V(z) : generated by ray N(1, 0)
V(w) : generated by ray N(1, 1)
```

**divisor_group** *(base_ring=Integer Ring)*

Return the group of Weil divisors.

INPUT:

- **base_ring** – the coefficient ring, usually ZZ (default) or QQ.

OUTPUT:

The (free abelian) group of Cartier divisors, that is, formal linear combinations of polynomial equations over the coefficient ring base_ring.

These need not be toric (=defined by monomials), but allow general polynomials. The output will be an instance of `sage.schemes.generic.divisor_group.DivisorGroup_generic`.

**Warning:** You almost certainly want the group of toric divisors, see `toric_divisor_group()`.

The toric divisor group is generated by the rays of the fan. The general divisor group has no toric functionality implemented.

**EXAMPLES:**

```python
sage: dP6 = toric_varieties.dP6()
sage: Div = dP6.divisor_group(); Div
Group of ZZ-Divisors on 2-d CPR-Fano toric variety covered by 6 affine patches
sage: Div(x)                          # needs sage.symbolic
V(x)
```

**embedding_morphism()**

Return the default embedding morphism of self.

Such a morphism is always defined for an affine patch of a toric variety (which is also a toric varieties itself).

**OUTPUT:**

- scheme morphism if the default embedding morphism was defined for self, otherwise a ValueError exception is raised.

**EXAMPLES:**
```python
sage: fan = FaceFan(lattice_polytope.cross_polytope(2))
sage: P1xP1 = ToricVariety(fan, "x s y t")
sage: P1xP1.embedding_morphism()
Traceback (most recent call last):
... ValueError: no default embedding was defined for this toric variety
sage: patch = P1xP1.affine_patch(0)
sage: patch
2-d affine toric variety
sage: patch.embedding_morphism()
Scheme morphism:
  From: 2-d affine toric variety
  To: 2-d toric variety covered by 4 affine patches
  Defn: Defined on coordinates by sending [y : t] to [1 : 1 : y : t]
```

### fan(dim=None, codim=None)

Return the underlying fan of self or its cones.

**INPUT:**

- dim – dimension of the requested cones;
- codim – codimension of the requested cones.

**OUTPUT:**

- rational polyhedral fan if no parameters were given. tuple of cones otherwise.

**EXAMPLES:**

```python
sage: fan = FaceFan(lattice_polytope.cross_polytope(2))
sage: P1xP1 = ToricVariety(fan)
sage: P1xP1.fan()
Rational polyhedral fan in 2-d lattice M
sage: P1xP1.fan() is fan
True
sage: P1xP1.fan(1)[0]
1-d cone of Rational polyhedral fan in 2-d lattice M
```

### inject_coefficients(scope=None, verbose=True)

Inject generators of the base field of self into scope.

This function is useful if the base field is the field of rational functions.

**INPUT:**

- scope – namespace (default: global, not just the scope from which this function was called);
- verbose – if True (default), names of injected generators will be printed.

**OUTPUT:** None.

**EXAMPLES:**

```python
sage: fan = FaceFan(lattice_polytope.cross_polytope(2))
sage: F = QQ["a, b"].fraction_field()
sage: P1xP1 = ToricVariety(fan, base_field=F)
sage: P1xP1.inject_coefficients()
Defining a, b
```
We check that we can use names \(a\) and \(b\), github issue #10498 is fixed:

```python
sage: a + b
a + b
sage: a + b in P1xP1.coordinate_ring()
True
```

**integrate**(cohomology_class)

Integrate a cohomology class over the toric variety.

**INPUT:**

- cohomology_class – A cohomology class given as a polynomial in self.cohomology_ring()

**OUTPUT:**

The integral of the cohomology class over the variety. The volume normalization is given by volume_class(), that is, self.integrate(self.volume_class()) is always one (if the volume class exists).

**EXAMPLES:**

```python
sage: # needs sage.libs.singular
sage: dP6 = toric_varieties.dP6()
sage: HH = dP6.cohomology_ring()
sage: D = [ HH(c) for c in dP6.fan(dim=1) ]
sage: matrix([[ dP6.integrate(D[i]*D[j]) for i in range(0,6) ] for j in range(0,6) ])
```

If the toric variety is an orbifold, the intersection numbers are usually fractional:

```python
sage: # needs sage.libs.singular
sage: P2_123 = toric_varieties.P2_123()
sage: HH = P2_123.cohomology_ring()
sage: D = [ HH(c) for c in P2_123.fan(dim=1) ]
sage: matrix([[ P2_123.integrate(D[i]*D[j]) for i in range(0,3) ] for j in range(0,3) ])
```

(continues on next page)
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......: .count_points() for i in range(0,3) ] for j in range(0,3) ]
[2/3 1 1/3]
[ 1 3/2 1/2]
[1/3 1/2 1/6]

**is_affine()**

Check if self is an affine toric variety.

An affine toric variety is a toric variety whose fan is the face lattice of a single cone. See also *AffineToricVariety()*.

OUTPUT: A boolean.

EXAMPLES:

```
sage: toric_varieties.A2().is_affine()
True
sage: toric_varieties.P1xA1().is_affine()
False
```

**is_complete()**

Check if self is complete.

OUTPUT:

• True if self is complete and False otherwise.

EXAMPLES:

```
sage: P1xP1 = toric_varieties.P1xP1()
sage: P1xP1.is_complete()
True
sage: P1xP1.affine_patch(0).is_complete()
False
```

**is_homogeneous**(polynomial)

Check if polynomial is homogeneous.

The coordinate ring of a toric variety is multigraded by relations between generating rays of the underlying fan.

INPUT:

• polynomial – polynomial in the coordinate ring of self or its quotient.

OUTPUT:

• True if polynomial is homogeneous and False otherwise.

EXAMPLES:

We will use the product of two projective lines with coordinates \((x, y)\) for one and \((s, t)\) for the other:

```
sage: P1xP1.<x,y,s,t> = toric_varieties.P1xP1()
sage: P1xP1.is_homogeneous(x - y)
True
sage: P1xP1.is_homogeneous(x*s + y*t)
True
```
Note that by homogeneous, we mean well-defined with respect to the homogeneous rescalings of self. So a polynomial that you would usually not call homogeneous can be homogeneous if there are no homogeneous rescalings, for example:

```python
sage: A1.<z> = toric_varieties.A1()
sage: A1.is_homogeneous(z^3 + z^7)
True
```

Finally, the degree group is really the Chow group $\text{A}_{d-1}(X)$ and can contain torsion. For example, take $\mathbb{C}^2/\mathbb{Z}_2$. Here, the Chow group is $\text{A}_{d-1}(\mathbb{C}^2/\mathbb{Z}_2) = \mathbb{Z}_2$ and distinguishes even-degree homogeneous polynomials from odd-degree homogeneous polynomials:

```python
sage: A2_Z2.<x,y> = toric_varieties.A2_Z2()
sage: A2_Z2.is_homogeneous(x + y + x^3 + y^5 + x^3*y^4)
True
sage: A2_Z2.is_homogeneous(x^2 + x*y + y^4 + (x*y)^5 + x^4*y^4)
True
sage: A2_Z2.is_homogeneous(x + y^2)
False
```

**is_isomorphic**(another)

Check if self is isomorphic to another.

**INPUT:**

• another - toric variety.

**OUTPUT:**

• True if self and another are isomorphic, False otherwise.

**EXAMPLES:**

```python
sage: TV1 = toric_varieties.P1xA1()
sage: TV2 = toric_varieties.P1xP1()

Only the most trivial case is implemented so far:

```python
sage: TV1.is_isomorphic(TV1)
True
sage: TV1.is_isomorphic(TV2)
Traceback (most recent call last):
...
NotImplementedError: isomorphism check is not yet implemented
```

**is_orbifold()**

Check if self has only quotient singularities.

A toric variety with at most orbifold singularities (in this sense) is often called a simplicial toric variety. In this package, we generally try to avoid this term since it mixes up differential geometry and cone terminology.
OUTPUT:
- True if \( \text{self} \) has at most quotient singularities by finite groups, False otherwise.

EXAMPLES:
```
sage: fan1 = FaceFan(lattice_polytope.cross_polytope(2))
sage: P1xP1 = ToricVariety(fan1)
sage: P1xP1.is_orbifold()
True
sage: fan2 = NormalFan(lattice_polytope.cross_polytope(3))
sage: TV = ToricVariety(fan2)
sage: TV.is_orbifold()
False
```

**is_smooth()**
Check if \( \text{self} \) is smooth.

OUTPUT:
- True if \( \text{self} \) is smooth and False otherwise.

EXAMPLES:
```
sage: fan1 = FaceFan(lattice_polytope.cross_polytope(2))
sage: P1xP1 = ToricVariety(fan1)
sage: P1xP1.is_smooth()
True
sage: fan2 = NormalFan(lattice_polytope.cross_polytope(2))
sage: TV = ToricVariety(fan2)
sage: TV.is_smooth()
False
```

**linear_equivalence_ideal()**
Return the ideal generated by linear relations

OUTPUT:
- The ideal generated by the linear relations of the rays in the polynomial ring over \( \mathbb{Q} \) generated by the homogeneous coordinates.

EXAMPLES:
```
sage: fan = Fan([[0,1,3], [3,4], [2,0], [1,2,4]],
.....: [(-3, -2, 1), (0, 0, 1), (3, -2, 1), (-1, -1, 1), (1, -1, 1)])
sage: X = ToricVariety(fan, coordinate_names='A B C D E', base_field=GF(5))
sage: lin = X.linear_equivalence_ideal(); lin
Ideal (-3*A + 3*C - D + E, -2*A - 2*C - D - E, A + B + C + D + E) of
Multivariate Polynomial Ring in A, B, C, D, E over Rational Field
```

**orbit_closure(cone)**
Return the orbit closure of \( \text{cone} \).

The cones \( \sigma \) of a fan \( \Sigma \) are in one-to-one correspondence with the torus orbits \( O(\sigma) \) of the corresponding toric variety \( X_\Sigma \). Each orbit is isomorphic to a lower dimensional torus (of dimension equal to the codimension of \( \sigma \)). Just like the toric variety \( X_\Sigma \) itself, these orbits are (partially) compactified by lower-dimensional orbits. In particular, one can define the closure \( V(\sigma) \) of the torus orbit \( O(\sigma) \) in the ambient toric variety \( X_\Sigma \), which is again a toric variety.
See Proposition 3.2.7 of [CLS2011] for more details.

INPUT:
• cone – a cone of the fan.

OUTPUT:
• a torus orbit closure associated to cone as a toric variety.

EXAMPLES:

```sage
sage: P1xP1 = toric_varieties.P1xP1()
sage: H = P1xP1.fan(1)[0]
sage: V = P1xP1.orbit_closure(H); V
1-d toric variety covered by 2 affine patches
sage: V.embedding_morphism()
Scheme morphism:
    From: 1-d toric variety covered by 2 affine patches
    To: 2-d CPR-Fano toric variety covered by 4 affine patches
    Defn: Defined by embedding the torus closure associated to the 1-d
cone of Rational polyhedral fan in 2-d lattice N.
sage: V.embedding_morphism().as_polynomial_map()
Scheme morphism:
    From: 1-d toric variety covered by 2 affine patches
    To: 2-d CPR-Fano toric variety covered by 4 affine patches
    Defn: Defined on coordinates by sending [z0 : z1] to [0 : 1 : z1 : z0]
```

`plot(**options)`
Plot self, i.e. the corresponding fan.

INPUT:
• any options for toric plots (see `toric_plotter.options`), none are mandatory.

OUTPUT: A plot.

**Note:** The difference between `X.plot()` and `X.fan().plot()` is that in the first case default ray labels correspond to variables of `X`.

**EXAMPLES:**

```sage
sage: X = toric_varieties.Cube_deformation(4)
sage: X.plot()  # needs sage.plot
Graphics3d Object
```

`rational_class_group()`
Return the rational divisor class group of self.

Let `X` be a toric variety.

The **Weil divisor class group** \(\text{Cl}(X)\) is a finitely generated abelian group and can contain torsion. Its rank equals the number of rays in the fan of \(X\) minus the dimension of \(X\).

The **rational divisor class group** is \(\text{Cl}(X) \otimes \mathbb{Q}\) and never includes torsion. If \(X\) is smooth, this equals the **Picard group** of \(X\), whose elements are the isomorphism classes of line bundles on \(X\). The group law
(which we write as addition) is the tensor product of the line bundles. The Picard group of a toric variety is always torsion-free.

**OUTPUT:** *rational divisor class group.*

---

**Note:**
- Coordinates correspond to the rows of `self.fan().gale_transform()`.
- `Kaehler_cone()` yields a cone in this group.

---

**EXAMPLES:**

```python
sage: P1xA1 = toric_varieties.P1xA1()
sage: P1xA1.rational_class_group()
```

The toric rational divisor class group of a 2-d toric variety covered by 2 affine patches

---

**resolve(**\*\**kwds)**

Construct a toric variety whose fan subdivides the fan of `self`.

The name of this function reflects the fact that usually such subdivisions are done for resolving singularities of the original variety.

**INPUT:**

This function accepts only keyword arguments, none of which are mandatory.

- `coordinate_names` – names for coordinates of the new variety. If not given, will be constructed from the coordinate names of `self` and necessary indexed ones. See `normalize_names()` for the description of acceptable formats;
- `coordinate_indices` – coordinate indices which should be used for indexed variables of the new variety;
- all other arguments will be passed to `subdivide()` method of the underlying rational polyhedral fan, see its documentation for the available options.

**OUTPUT:** A *toric variety*.

**EXAMPLES:**

First we will “manually” resolve a simple orbifold singularity:

```python
sage: cone = Cone((1, 1), (-1, 1))
sage: fan = Fan([cone])
sage: TV = ToricVariety(fan)
sage: TV.is_smooth()
False
sage: TV_res = TV.resolve(new_rays=[[0, 1]])
sage: TV_res.is_smooth()
True
sage: TV_res.fan().rays()
N( 1, 1),
N(-1, 1),
N( 0, 1)
in 2-d lattice N
sage: [cone.ambient_ray_indices() for cone in TV_res.fan()]
[(0, 2), (1, 2)]
```
Now let’s “automatically” partially resolve a more complicated fan:

```python
sage: fan = NormalFan(lattice_polytope.cross_polytope(3))
sage: TV = ToricVariety(fan)
sage: TV.is_smooth()
False
sage: TV.is_orbifold()
False
sage: TV.fan().nrays()
8
sage: TV.fan().ngenerating_cones()
6
sage: TV_res = TV.resolve(make_simplicial=True)
```

```python
sage: TV_res.is_smooth()
False
sage: TV_res.is_orbifold()
True
sage: TV_res.fan().nrays()
8
sage: TV_res.fan().ngenerating_cones()
12
sage: TV.gens()
(z0, z1, z2, z3, z4, z5, z6, z7)
sage: TV_res.gens()
(z0, z1, z2, z3, z4, z5, z6, z7)
sage: TV_res = TV.resolve(coordinate_names="x+",
..........: make_simplicial=True)
```

```python
sage: TV_res.gens()
(x0, x1, x2, x3, x4, x5, x6, x7)
```

**resolve_to_orbifold(**kwds**)**

Construct an orbifold whose fan subdivides the fan of self.

It is a synonym for `resolve()` with `make_simplicial=True` option.

**INPUT:**

- this function accepts only keyword arguments. See `resolve()` for documentation.

**OUTPUT:** A toric variety.

**EXAMPLES:**

```python
sage: fan = NormalFan(lattice_polytope.cross_polytope(3))
sage: TV = ToricVariety(fan)
sage: TV.is_orbifold()
False
sage: TV.fan().nrays()
8
sage: TV.fan().ngenerating_cones()
6
sage: TV_res = TV.resolve_to_orbifold()
sage: TV_res.is_orbifold()
True
sage: TV_res.fan().nrays()
8
```
### property sheaves

Return the factory object for sheaves on the toric variety.

See `sage.schemes.toric.sheaf.constructor.SheafLibrary` for details.

**EXAMPLES:**

```python
sage: dP6 = toric_varieties.dP6()
sage: dP6.sheaves
Sheaf constructor on 2-d CPR-Fano toric variety covered by 6 affine patches
sage: dP6.sheaves.trivial_bundle()
Rank 1 bundle on 2-d CPR-Fano toric variety covered by 6 affine patches.
```

### subscheme polynomials)

Return the subscheme of `self` defined by polynomials.

**INPUT:**

- `polynomials` – list of polynomials in the coordinate ring of `self`.

**OUTPUT:** A subscheme of a toric variety.

**EXAMPLES:**

We will construct a subscheme of the product of two projective lines with coordinates `(x, y)` for one and `(s, t)` for the other:

```python
sage: P1xP1.<x,y,s,t> = toric_varieties.P1xP1()
sage: X = P1xP1.subscheme([x*s + y*t, x^3 + y^3])
sage: X
Closed subscheme of 2-d CPR-Fano toric variety covered by 4 affine patches defined by:
    x*s + y*t,
    x^3 + y^3
sage: X.defining_polynomials()
(x*s + y*t, x^3 + y^3)
sage: X.defining_ideal()
Ideal (x*s + y*t, x^3 + y^3) of
Multivariate Polynomial Ring in x, y, s, t over Rational Field
sage: X.base_ring()
Rational Field
sage: X.base_scheme()
Spectrum of Rational Field
sage: X.structure_morphism()
Scheme morphism:
    From: Closed subscheme of 2-d CPR-Fano toric variety covered by 4 affine patches defined by: x*s + y*t, x^3 + y^3
    To:   Spectrum of Rational Field
    Defn: Structure map
```

### toric_divisor_group(base_ring=Integer Ring)

Return the group of toric (T-Weil) divisors.

**INPUT:**
• base_ring – the coefficient ring, usually ZZ (default) or QQ.

**OUTPUT:**

The free Abelian group of toric Weil divisors, that is, formal base_ring-linear combinations of codimension-one toric subvarieties. The output will be an instance of `sage.schemes.toric.divisor.ToricDivisorGroup`.

The \( i \)-th generator of the divisor group is the divisor where the \( i \)-th homogeneous coordinate vanishes, \( \{ z_i = 0 \} \).

**EXAMPLES:**

```python
sage: dP6 = toric_varieties.dP6()
sage: TDiv = dP6.toric_divisor_group(); TDiv
Group of toric ZZ-Weil divisors on 2-d CPR-Fano toric variety covered by 6 affine patches
sage: TDiv == dP6.toric_divisor_group()
True
sage: TDiv.gens()
(V(x), V(u), V(y), V(v), V(z), V(w))
sage: dP6.coordinate_ring()
Multivariate Polynomial Ring in x, u, y, v, z, w over Rational Field
```

**volume_class()**

Return the cohomology class of the volume form on the toric variety.

Note that we are using cohomology with compact supports. If the variety is non-compact this is dual to homology without any support condition. In particular, for non-compact varieties the volume form \( d\text{Vol} = \wedge_i (dx_i \wedge dy_i) \) does not define a (non-zero) cohomology class.

**OUTPUT:**

A `CohomologyClass`. If it exists, it is the class of the (properly normalized) volume form, that is, it is the Poincaré dual of a single point. If it does not exist, a `ValueError` is raised.

**EXAMPLES:**

```python
sage: P2 = toric_varieties.P2()
sage: P2.volume_class()
# needs sage.libs.singular
[z^2]
sage: A2_Z2 = toric_varieties.A2_Z2()
sage: A2_Z2.volume_class()
# needs sage.libs.singular
Traceback (most recent call last):
... ValueError: volume class does not exist
```

If none of the maximal cones is smooth things get more tricky. In this case no torus-fixed point is smooth. If we want to count an ordinary point as 1, then a \( G \)-orbifold point needs to count as \( \frac{1}{|G|} \). For example, take \( \mathbb{P}^1 \times \mathbb{P}^1 \) with inhomogeneous coordinates \((t, y)\). Take the quotient by the action \((t, y) \mapsto (-t, -y)\). The \( \mathbb{Z}_2 \)-invariant Weil divisors \( \{ t = 0 \} \) and \( \{ y = 0 \} \) intersect in a \( \mathbb{Z}_2 \)-fixed point, so they ought to have intersection number \( \frac{1}{2} \). This means that the cohomology class \([t] \cap [y]\) should be \( \frac{1}{2} \) times the volume class. Note that this is different from the volume normalization chosen in [KS]:
The fractional coefficients are also necessary to match the normalization in the rational Chow group for simplicial toric varieties:

```
sage: A = P1xP1_Z2.Chow_group(QQ)
sage: A(Dt).intersection_with_divisor(Dy).count_points()
1/2
```

\[ \text{sage.schemes.toric.variety.is\_CohomologyClass}(x) \]
Check whether \( x \) is a cohomology class of a toric variety.

**INPUT:**

- \( x \) – anything.

**OUTPUT:**

True or False depending on whether \( x \) is an instance of \textit{CohomologyClass}

**EXAMPLES:**

```
sage: P2 = toric_varieties.P2()
sage: HH = P2.cohomology_ring()
sage: from sage.schemes.toric.variety import is\_CohomologyClass
sage: is\_CohomologyClass( HH.one() )  # needs sage.libs.singular
True
sage: is\_CohomologyClass( HH(P2.fan(1)[0]) )  # needs sage.libs.singular
True
sage: is\_CohomologyClass('z')
False
```

\[ \text{sage.schemes.toric.variety.is\_ToricVariety}(x) \]
Check if \( x \) is a toric variety.

**INPUT:**

- \( x \) – anything.

**OUTPUT:**

- True if \( x \) is a \textit{toric variety} and False otherwise.
Note: While projective spaces are toric varieties mathematically, they are not toric varieties in Sage due to efficiency considerations, so this function will return False.

EXAMPLES:

```
sage: from sage.schemes.toric.variety import is_ToricVariety
sage: is_ToricVariety(1)
False
sage: fan = FaceFan(lattice_polytope.cross_polytope(2))
2-d toric variety covered by 4 affine patches
sage: P = ToricVariety(fan)
2-d toric variety covered by 4 affine patches
sage: is_ToricVariety(P)
True
sage: is_ToricVariety(ProjectiveSpace(2))
False
```

```
sage.schemes.toric.variety.normalize_names(names=None, ngens=None, prefix=None, indices=None, return_prefix=False)
```

Return a list of names in the standard form.

INPUT:

All input parameters are optional.

- `names` – names given either as a single string (with individual names separated by commas or spaces) or a list of strings with each string specifying a name. If the last name ends with the plus sign, “+”, this name will be used as `prefix` (even if `prefix` was given explicitly);

- `ngens` – number of names to be returned;

- `prefix` – prefix for the indexed names given as a string;

- `indices` – list of integers (default: `range(ngens)`) used as indices for names with `prefix`. If given, must be of length `ngens`;

- `return_prefix` – if True, the last element of the returned list will contain the prefix determined from `names` or given as the parameter `prefix`. This is useful if you may need more names in the future.

OUTPUT:

- list of names given as strings.

These names are constructed in the following way:

1. If necessary, split `names` into separate names.
2. If the last name ends with “+”, put it into `prefix`.
3. If `ngens` was given, add to the names obtained so far as many indexed names as necessary to get this number. If the k-th name of the `total` list of names is indexed, it is `prefix + str(indices[k])`. If there were already more names than `ngens`, discard “extra” ones.
4. Check if constructed names are valid. See `certify_names()` for details.
5. If the option `return_prefix=True` was given, add `prefix` to the end of the list.

EXAMPLES:

As promised, all parameters are optional:
One of the most common uses is probably this one:

```python
sage: normalize_names("x+", 4)
['x0', 'x1', 'x2', 'x3']
```

Now suppose that you want to enumerate your variables starting with one instead of zero:

```python
sage: normalize_names("x+", 4, indices=list(range(1,5)))
['x1', 'x2', 'x3', 'x4']
```

You may actually have an arbitrary enumeration scheme:

```python
sage: normalize_names("x+", 4, indices=[1, 10, 100, 1000])
['x1', 'x10', 'x100', 'x1000']
```

Now let's add some “explicit” names:

```python
sage: normalize_names("x y z t+", 4)
['x', 'y', 'z', 't3']
```

Note that the “automatic” name is \(t3\) instead of \(t0\). This may seem weird, but the reason for this behaviour is that the fourth name in this list will be the same no matter how many explicit names were given:

```python
sage: normalize_names("x y s+", 4)
['x', 'y', 't2', 't3']
```

This is especially useful if you get names from a user but want to specify all default names:

```python
sage: normalize_names("x, y", 4, prefix="t")
['x', 'y', 't2', 't3']
```

In this format, the user can easily override your choice for automatic names:

```python
sage: normalize_names("x y s+", 4, prefix="t")
['x', 'y', 's2', 's3']
```

Let's now use all parameters at once:

```python
sage: normalize_names("x, y, s+", 4, prefix="t",
.....: indices=list(range(1,5)), return_prefix=True)
['x', 'y', 's3', 's4', 's']
```

Note that you still need to give indices for all names, even if some of the first ones will be “wasted” because of the explicit names. The reason is the same as before - this ensures consistency of automatically generated names, no matter how many explicit names were given.

The prefix is discarded if \(\text{ngens}\) was not given:

```python
sage: normalize_names("alpha, beta, gamma, zeta+")
['alpha', 'beta', 'gamma']
```

Finally, let's take a look at some possible mistakes:
A more subtle one:

```
sage: normalize_names("x1", 4, prefix="x")
Traceback (most recent call last):
  ...  
ValueError: variable name 'x1' appears more than once
```

## 16.2 Fano toric varieties

This module provides support for (Crepant Partial Resolutions of) Fano toric varieties, corresponding to crepant sub-
divisions of face fans of reflexive lattice polytopes. The interface is provided via `CPRFanoToric Variety()`. A careful exposition of different flavours of Fano varieties can be found in the paper by Benjamin Nill [Nil2005]. The main goal of this module is to support work with Gorenstein weak Fano toric varieties. Such a variety corresponds to a coherent crepant refinement of the normal fan of a reflexive polytope \( \Delta \), where crepant means that primitive generators of the refining rays lie on the facets of the polar polytope \( \Delta^\circ \) and coherent (a.k.a. regular or projective) means that there exists a strictly upper convex piecewise linear function whose domains of linearity are precisely the maximal cones of the subdivision. These varieties are important for string theory in physics, as they serve as ambient spaces for mirror pairs of Calabi-Yau manifolds via constructions due to Victor V. Batyrev [Bat1994] and Lev A. Borisov [Bor1993].

From the combinatorial point of view “crepant” requirement is much more simple and natural to work with than “co-
herent.” For this reason, the code in this module will allow work with arbitrary crepant subdivisions without checking whether they are coherent or not. We refer to corresponding toric varieties as CPR-Fano toric varieties.

REFERENCES:
- [Bat1994]
- [Bor1993]
- [CD2007]
- [Nil2005]

AUTHORS:

EXAMPLES:
Most of the functions available for Fano toric varieties are the same as for general toric varieties, so here we will concentrate only on Calabi-Yau subvarieties, which were the primary goal for creating this module.

For our first example we realize the projective plane as a Fano toric variety:

```
sage: simplex = LatticePolytope([(1,0), (0,1), (-1,-1)])
sage: P2 = CPRFanoToricVariety(Delta_polar=simplex)
```

Its anticanonical “hypersurface” is a one-dimensional Calabi-Yau manifold:
In many cases it is sufficient to work with the “simplified polynomial moduli space” of anticanonical hypersurfaces:

```
sage: P2.anticanonical_hypersurface(monomial_points="simplified")
Closed subscheme of 2-d CPR-Fano toric variety covered by 3 affine patches defined by:
an0*z0^3 + an1*z1^3 + an6*z0*z1*z2 + an2*z2^3
```

The mirror family to these hypersurfaces lives inside the Fano toric variety obtained using `simplex` as `Delta` instead of `Delta_polar`:

```
sage: FTV = CPRFanoToricVariety(Delta=simplex, coordinate_points="all")
sage: FTV.anticanonical_hypersurface(monomial_points="simplified")
Closed subscheme of 2-d CPR-Fano toric variety covered by 9 affine patches defined by:
a2*z2^3*z3^2*z4*z5^2*z8 + a1*z1^3*z3*z4^2*z7^2*z9 + a3*z0*z2*z3*z4*z5*z7*z8*z9 + a0*z0^3*z5*z7*z8^2*z9^2
```

Here we have taken the resolved version of the ambient space for the mirror family, but in fact we don’t have to resolve singularities corresponding to the interior points of facets - they are singular points which do not lie on a generic anticanonical hypersurface:

```
sage: FTV = CPRFanoToricVariety(Delta=simplex, coordinate_points="all but facets")
sage: FTV.anticanonical_hypersurface(monomial_points="simplified")
Closed subscheme of 2-d CPR-Fano toric variety covered by 3 affine patches defined by:
an0*z0^3 + an1*z1^3 + an3*z0*z1*z2 + an2*z2^3
```

This looks very similar to our second version of the anticanonical hypersurface of the projective plane, as expected, since all one-dimensional Calabi-Yau manifolds are elliptic curves!

Now let’s take a look at a toric realization of $M$-polarized K3 surfaces studied by Adrian Clingher and Charles F. Doran in [CD2007]:

```
sage: p4318 = ReflexivePolytope(3, 4318)
sage: FTV = CPRFanoToricVariety(Delta_polar=p4318)
sage: FTV.anticanonical_hypersurface()
Closed subscheme of 3-d CPR-Fano toric variety covered by 4 affine patches defined by:
a0*z2^12 + a4*z2^6*z3^6 + a3*z3^12 + a8*z0*z1*z2*z3 + a2*z1^3 + a1*z0^4
```

Below you will find detailed descriptions of available functions. Current functionality of this module is very basic, but it is under active development and hopefully will improve in future releases of Sage. If there are some particular features that you would like to see implemented ASAP, please consider reporting them to the Sage Development Team or even implementing them on your own as a patch for inclusion!
• \( P_{\Delta} \) – *CPR-Fano toric variety* associated to a reflexive polytope \( \Delta \);

• see \( \text{CPRFanoToricVariety_field.anticanonical_hypersurface()} \) for documentation on all other acceptable parameters.

**OUTPUT:**

• *anticanonical hypersurface* of \( P_{\Delta} \) (with the extended base field, if necessary).

**EXAMPLES:**

```python
sage: P1xP1 = toric_varieties.P1xP1()
sage: import sage.schemes.toric.fano_variety as ftv
sage: ftv.AnticanonicalHypersurface(P1xP1)
Closed subscheme of 2-d CPR-Fano toric variety covered by 4 affine patches defined by:
\[ a_0 s^2 x^2 + a_3 t^2 x^2 + a_6 s t x y + a_1 s^2 y^2 + a_2 t^2 y^2 + a_2 s^2 t^2 \]
```

See \( \text{anticanonical_hypersurface()} \) for a more elaborate example.

`sage.schemes.toric.fano_variety.CPRFanoToricVariety(Delta=None, Delta_polar=None, coordinate_points=None, charts=None, coordinate_names=None, names=None, coordinate_name_indices=None, make_simplicial=False, base_ring=None, base_field=None, check=True)`

Construct a CPR-Fano toric variety.

**Note:** See documentation of the module *fano_variety* for the used definitions and supported varieties.

Due to the large number of available options, it is recommended to always use keyword parameters.

**INPUT:**

• \( \text{Delta} \) – reflexive *lattice polytope*. The fan of the constructed CPR-Fano toric variety will be a crepant subdivision of the *normal fan* of \( \text{Delta} \). Either \( \text{Delta} \) or \( \text{Delta_polar} \) must be given, but not both at the same time, since one is completely determined by another via *polar* method;

• \( \text{Delta_polar} \) – reflexive *lattice polytope*. The fan of the constructed CPR-Fano toric variety will be a crepant subdivision of the *face fan* of \( \text{Delta_polar} \). Either \( \text{Delta} \) or \( \text{Delta_polar} \) must be given, but not both at the same time, since one is completely determined by another via *polar* method;

• \( \text{coordinate_points} \) – list of integers or string. A list will be interpreted as indices of (boundary) points of \( \text{Delta_polar} \) which should be used as rays of the underlying fan. It must include all vertices of \( \text{Delta_polar} \) and no repetitions are allowed. A string must be one of the following descriptions of points of \( \text{Delta_polar} \):
  – “vertices” (default),
  – “all” (will not include the origin),
  – “all but facets” (will not include points in the relative interior of facets);

• \( \text{charts} \) – list of lists of elements from \( \text{coordinate_points} \). Each of these lists must define a generating cone of a fan subdividing the normal fan of \( \text{Delta} \). Default \( \text{charts} \) correspond to the normal fan of \( \text{Delta} \) without subdivision. The fan specified by \( \text{charts} \) will be subdivided to include all of the requested \( \text{coordinate_points} \);

• \( \text{coordinate_names} \) – names of variables for the coordinate ring, see \( \text{normalize_names()} \) for acceptable formats. If not given, indexed variable names will be created automatically;
• names – an alias of coordinate_names for internal use. You may specify either names or coordinate_names, but not both;
• coordinate_name_indices – list of integers, indices for indexed variables. If not given, the index of each variable will coincide with the index of the corresponding point of Delta_polar;
• make_simplicial – if True, the underlying fan will be made simplicial (default: False);
• base_ring – base field of the CPR-Fano toric variety (default: Q);
• base_field – alias for base_ring. Takes precedence if both are specified.
• check – by default the input data will be checked for correctness (e.g. that charts do form a subdivision of the normal fan of Delta). If you know for sure that the input is valid, you may significantly decrease construction time using check=False option.

OUTPUT: CPR-Fano toric variety.

EXAMPLES:

We start with the product of two projective lines:

```
sage: diamond = lattice_polytope.cross_polytope(2)
sage: diamond.vertices()
M( 1, 0), M( 0, 1),
M(-1, 0), M( 0, -1)
in 2-d lattice M
sage: P1xP1 = CPRFanoToricVariety(Delta_polar=diamond)
sage: P1xP1
2-d CPR-Fano toric variety covered by 4 affine patches
sage: P1xP1.fan()
Rational polyhedral fan in 2-d lattice M
sage: P1xP1.fan().rays()
M( 1, 0), M( 0, 1),
M(-1, 0), M( 0, -1)
in 2-d lattice M
```

“Unfortunately,” this variety is smooth to start with and we cannot perform any subdivisions of the underlying fan without leaving the category of CPR-Fano toric varieties. Our next example starts with a square:

```
sage: square = diamond.polar()
sage: square.vertices()
N( 1, 1), N( 1, -1),
N(-1, -1), N(-1, 1)
in 2-d lattice N
sage: square.points()
N( 1, 1), N( 1, -1), N(-1, -1), N(-1, 1), N( 0, -1), N( 0, 1), N( 1, 0)
in 2-d lattice N
sage: FTV = CPRFanoToricVariety(Delta_polar=square)
sage: FTV.fan().rays()
N( 1, 1), N( 1, -1), N(-1, -1), N(-1, 1), N( 0, -1), N( 0, 1), N( 1, 0)
in 2-d lattice N
```

We will construct several varieties associated to it:

```
sage: FTV = CPRFanoToricVariety(Delta_polar=square)
sage: FTV.fan().rays()
N( 1, 1), N( 1, -1),
N(-1, -1), N(-1, 1)
in 2-d lattice N
```
sage: FTV.gens()
(z0, z1, z2, z3)

sage: FTV = CPRFanoToricVariety(Delta_polar=square,
       coordinate_points=[0,1,2,3,8])

sage: FTV.fan().rays()
N( 1, 1), N( 1, -1), N(-1, -1), N(-1, 1), N( 1, 0)
in 2-d lattice N

sage: FTV.gens()
(z0, z1, z2, z3, z8)

sage: FTV = CPRFanoToricVariety(Delta_polar=square,
       coordinate_points=[8,0,2,1,3],
       coordinate_names="x+")

sage: FTV.fan().rays()
N( 1, 0), N( 1, 1), N(-1, -1), N( 1, -1), N(-1, 1)
in 2-d lattice N

sage: FTV.gens()
(x8, x0, x2, x1, x3)

sage: FTV = CPRFanoToricVariety(Delta_polar=square,
       coordinate_points="all",
       coordinate_names="x y Z+",
       coordinate_name_indices=list(range(8)))

sage: FTV.gens()
(x, y, Z2, Z3, Z4, Z5, Z7, Z8)

Note that $Z6$ is “missing”. This is due to the fact that the 6-th point of $square$ is the origin, and all automatically created names have the same indices as corresponding points of $Delta_polar()$. This is usually very convenient, especially if you have to work with several partial resolutions of the same Fano toric variety. However, you can change it, if you want:

sage: FTV = CPRFanoToricVariety(Delta_polar=square,
       coordinate_points="all",
       coordinate_names="x y Z+",
       coordinate_name_indices=list(range(8)))

sage: FTV.gens()
(x, y, Z2, Z3, Z4, Z5, Z7, Z8)

Note that you have to provide indices for all variables, including those that have “completely custom” names. Again, this is usually convenient, because you can add or remove “custom” variables without disturbing too much “automatic” ones:

sage: FTV = CPRFanoToricVariety(Delta_polar=square,
       coordinate_points="all",
       coordinate_names="x Z+",
       coordinate_name_indices=list(range(8)))

(continues on next page)
If you prefer to always start from zero, you will have to shift indices accordingly:

```python
sage: FTV = CPRFanoToricVariety(Delta_polar=square,
....: coordinate_points="all",
....: coordinate_names="x Z+",
....: coordinate_name_indices=[0] + list(range(7)))
```

```python
sage: FTV.gens()
(x, Z0, Z1, Z2, Z3, Z4, Z5, Z6)
```

```python
sage: FTV = CPRFanoToricVariety(Delta_polar=square,
....: coordinate_points="all",
....: coordinate_names="x y Z+",
....: coordinate_name_indices=[0]*2 + list(range(6)))
```

```python
sage: FTV.gens()
(x, y, Z0, Z1, Z2, Z3, Z4, Z5)
```

So you always can get any names you want, somewhat complicated default behaviour was designed with the hope that in most cases you will have no desire to provide different names.

Now we will use the possibility to specify initial charts:

```python
sage: charts = [(0,1), (1,2), (2,3), (3,0)]
```

(these charts actually form exactly the face fan of our square)

```python
sage: FTV = CPRFanoToricVariety(Delta_polar=square,
....: coordinate_points=[0,1,2,3,4],
....: charts=charts)
sage: FTV.fan().rays()
N( 1, 1), N( 1, -1), N(-1, -1), N(-1, 1), N(-1, 0)
in 2-d lattice N
sage: [cone.ambient_ray_indices() for cone in FTV.fan()]
[(0, 1), (1, 2), (2, 4), (3, 4), (0, 3)]
```

If charts are wrong, it should be detected:

```python
sage: bad_charts = charts + [(3,0)]
```

```python
sage: FTV = CPRFanoToricVariety(Delta_polar=square,
....: coordinate_points=[0,1,2,3,4],
....: charts=bad_charts)
```

Traceback (most recent call last):
...
ValueError: you have provided 5 cones, but only 4 of them are maximal!
Use discard_faces=True if you indeed need to construct a fan from these cones.

These charts are technically correct, they just happened to list one of them twice, but it is assumed that such a situation will not happen. It is especially important when you try to speed up your code:

```python
sage: FTV = CPRFanoToricVariety(Delta_polar=square,
....: coordinate_points=[0,1,2,3,4],
```
In this case you still get an error message, but it is harder to figure out what is going on. It may also happen that “everything will still work” in the sense of not crashing, but work with such an invalid variety may lead to mathematically wrong results, so use `check=False` carefully!

Here are some other possible mistakes:

```
sage: bad_charts = charts + [(0,2)]
sage: FTV = CPRFanoToricVariety(Delta_polar=square,
   ...: coordinate_points=[0,1,2,3,4],
   ...: charts=bad_charts)
Traceback (most recent call last):
...
ValueError: (0, 2) does not form a chart of a subdivision of the face fan of 2-d reflexive polytope #14 in 2-d lattice N!
sage: bad_charts = charts[:-1]
sage: FTV = CPRFanoToricVariety(Delta_polar=square,
   ...: coordinate_points=[0,1,2,3,4],
   ...: charts=bad_charts)
Traceback (most recent call last):
...
ValueError: given charts do not form a complete fan!
sage: FTV = CPRFanoToricVariety(Delta_polar=square,
   ...: coordinate_points=[1,2,3,4])
Traceback (most recent call last):
...
ValueError: all 4 vertices of Delta_polar must be used for coordinates!
Got: [1, 2, 3, 4]
sage: FTV = CPRFanoToricVariety(Delta_polar=square,
   ...: coordinate_points=[0,0,1,2,3,4])
Traceback (most recent call last):
...
ValueError: no repetitions are allowed for coordinate points!
Got: [0, 0, 1, 2, 3, 4]
sage: FTV = CPRFanoToricVariety(Delta_polar=square,
   ...: coordinate_points=[0,1,2,3,6])
Traceback (most recent call last):
...
ValueError: the origin (point #6) cannot be used for a coordinate!
Got: [0, 1, 2, 3, 6]
```

Here is a shorthand for defining the toric variety and homogeneous coordinates in one go:
```python
sage: P1xP1.<a,b,c,d> = CPRFanoToricVariety(Delta_polar=diamond)
sage: (a^2+b^2) * (c+d)
a^2*c + b^2*c + a^2*d + b^2*d
```

**class** `sage.schemes.toric.fano_variety.CPRFanoToricVariety_field(Delta_polar, fan, coordinate_points, point_to_ray, coordinate_names, coordinate_name_indices, base_field)`

**Bases:** `ToricVariety_field`

Construct a CPR-Fano toric variety associated to a reflexive polytope.

**Warning:** This class does not perform any checks of correctness of input and it does assume that the internal structure of the given parameters is coordinated in a certain way. Use `CPRFanoToricVariety()` to construct CPR-Fano toric varieties.

**Note:** See documentation of the module `fano_variety` for the used definitions and supported varieties.

**INPUT:**
- `Delta_polar` – reflexive polytope;
- `fan` – rational polyhedral fan subdividing the face fan of `Delta_polar`;
- `coordinate_points` – list of indices of points of `Delta_polar` used for rays of `fan`;
- `point_to_ray` – dictionary mapping the index of a coordinate point to the index of the corresponding ray;
- `coordinate_names` – names of the variables of the coordinate ring in the format accepted by `normalize_names()`;
- `coordinate_name_indices` – indices for indexed variables, if `None`, will be equal to `coordinate_points`;
- `base_field` – base field of the CPR-Fano toric variety.

**OUTPUT:** *CPR-Fano toric variety.*

**Delta()**

Return the reflexive polytope associated to `self`.

**OUTPUT:**
- reflexive `lattice polytope`. The underlying fan of `self` is a coherent subdivision of the `normal fan` of this polytope.

**EXAMPLES:**
```
sage: diamond = lattice_polytope.cross_polytope(2)
sage: P1xP1 = CPRFanoToricVariety(Delta_polar=diamond)
sage: P1xP1.Delta()
2-d reflexive polytope #14 in 2-d lattice N
sage: P1xP1.Delta() is diamond.polar()
True
```
**Delta.polar()**

Return polar of \textit{Delta()}.  

**OUTPUT:**

- reflexive lattice polytope. The underlying fan of \texttt{self} is a coherent subdivision of the face fan of this polytope.

**EXAMPLES:**

```sage
diamond = lattice_polytope.cross_polytope(2)
P1xP1 = CPRFanoToricVariety(Delta_polar=diamond)
P1xP1.Delta_polar()
```

2-d reflexive polytope #3 in 2-d lattice \(\mathbb{M}\)

```sage
P1xP1.Delta_polar() is diamond
```

True

```sage
P1xP1.Delta_polar() is P1xP1.Delta().polar()
```

True

**anticanonical_hypersurface(**kwds**)**

Return an anticanonical hypersurface of \texttt{self}.

**Note:** The returned hypersurface may be actually a subscheme of another CPR-Fano toric variety: if the base field of \texttt{self} does not include all of the required names for generic monomial coefficients, it will be automatically extended.

Below \(\Delta\) is the reflexive polytope corresponding to \texttt{self}, i.e. the fan of \texttt{self} is a refinement of the normal fan of \(\Delta\). This function accepts only keyword parameters.

**INPUT:**

- **monomial points** – a list of integers or a string. A list will be interpreted as indices of points of \(\Delta\) which should be used for monomials of this hypersurface. A string must be one of the following descriptions of points of \(\Delta\):
  - “vertices”,
  - “vertices+origin”,
  - “all”,
  - “simplified” (default) – all points of \(\Delta\) except for the interior points of facets, this choice corresponds to working with the “simplified polynomial moduli space” of anticanonical hypersurfaces;
- **coefficient_names** – names for the monomial coefficients, see \texttt{normalize_names()} for acceptable formats. If not given, indexed coefficient names will be created automatically;
- **coefficient_name_indices** – a list of integers, indices for indexed coefficients. If not given, the index of each coefficient will coincide with the index of the corresponding point of \(\Delta\);
- **coefficients** – as an alternative to specifying coefficient names and/or indices, you can give the coefficients themselves as arbitrary expressions and/or strings. Using strings allows you to easily add “parameters”: the base field of \texttt{self} will be extended to include all necessary names.

**OUTPUT:**

- an anticanonical hypersurface of \texttt{self} (with the extended base field, if necessary).

**EXAMPLES:**

We realize the projective plane as a Fano toric variety:
Its anticanonical “hypersurface” is a one-dimensional Calabi-Yau manifold:

```
sage: P2.anticanonical_hypersurface(monomial_points="all")
Closed subscheme of 2-d CPR-Fano toric variety
covered by 3 affine patches defined by:
a0*z0^3 + a9*z0^2*z1 + a7*z0*z1^2 + a1*z1^3 + a8*z0^2*z2 + a6*z0*z1*z2
+ a4*z1^2*z2 + a5*z0*z2^2 + a3*z1*z2^2 + a2*z2^3
```

In many cases it is sufficient to work with the “simplified polynomial moduli space” of anticanonical hypersurfaces:

```
sage: P2.anticanonical_hypersurface(monomial_points="simplified")
Closed subscheme of 2-d CPR-Fano toric variety
covered by 3 affine patches defined by:
a0*z0^3 + a1*z1^3 + a6*z0*z1*z2 + a2*z2^3
```

The mirror family to these hypersurfaces lives inside the Fano toric variety obtained using `simplex` as `Delta` instead of `Delta_polar`:

```
sage: FTV = CPRFanoToricVariety(Delta=simplex, coordinate_points="all")
sage: FTV.anticanonical_hypersurface(monomial_points="simplified")
Closed subscheme of 2-d CPR-Fano toric variety
covered by 9 affine patches defined by:
a2*z2^3*z3^2*z4*z5^2*z8 + a1*z1^3*z3^2*z4^2*z7^2*z9
+ a3*z0*z1*z2^3*z4*z5^2*z7^2*z9 + a0*z0^3*z5*z7^2*z8^2*z9^2
```

Here we have taken the resolved version of the ambient space for the mirror family, but in fact we don’t have to resolve singularities corresponding to the interior points of facets - they are singular points which do not lie on a generic anticanonical hypersurface:

```
sage: FTV = CPRFanoToricVariety(Delta=simplex, coordinate_points="all but facets")
sage: FTV.anticanonical_hypersurface(monomial_points="simplified")
Closed subscheme of 2-d CPR-Fano toric variety
covered by 3 affine patches defined by:
a0*z0^3 + a1*z1^3 + a3*z0*z1*z2 + a2*z2^3
```

This looks very similar to our second anticanonical hypersurface of the projective plane, as expected, since all one-dimensional Calabi-Yau manifolds are elliptic curves!

All anticanonical hypersurfaces constructed above were generic with automatically generated coefficients. If you want, you can specify your own names

```
sage: FTV.anticanonical_hypersurface(coefficient_names="a b c d")
Closed subscheme of 2-d CPR-Fano toric variety
covered by 3 affine patches defined by:
a^a*z0^3 + b*z1^3 + d*z0*z1*z2 + c*z2^3
```

or give concrete coefficients.
sage: FTV.anticanonical_hypersurface(coefficients=[1, 2, 3, 4])
Closed subscheme of 2-d CPR-Fano toric variety
covered by 3 affine patches defined by:
$z_0^3 + 2z_1^3 + 4z_0z_1z_2 + 3z_2^3$

or even mix numerical coefficients with some expressions

sage: H = FTV.anticanonical_hypersurface(
    ....: coefficients=[0, "t", "1/t", "psi/(psi^2 + phi)"
    )

sage: H
Closed subscheme of 2-d CPR-Fano toric variety
covered by 3 affine patches defined by:
$tz_1^3 + psi/(phi + psi^2)z_0z_1z_2 + 1/tz_2^3$

sage: R = H.ambient_space().base_ring()

sage: R
Fraction Field of Multivariate Polynomial Ring in phi, psi, t over Rational Field

cartesian_product(other, coordinate_names=None, coordinate_indices=None)

Return the Cartesian product of self with other.

INPUT:

• other – a (possibly CPR-Fano) toric variety;

• coordinate_names – names of variables for the coordinate ring, see normalize_names() for acceptable formats. If not given, indexed variable names will be created automatically;

• coordinate_indices – list of integers, indices for indexed variables. If not given, the index of each variable will coincide with the index of the corresponding ray of the fan.

OUTPUT:

• a toric variety, which is CPR-Fano if other was.

EXAMPLES:

sage: P1 = toric_varieties.P1()
sage: P2 = toric_varieties.P2()
sage: P1xP2 = P1.cartesian_product(P2); P1xP2
3-d CPR-Fano toric variety covered by 6 affine patches

sage: P1xP2.fan().rays()
N+N( 1, 0, 0), N+N(-1, 0, 0), N+N( 0, 1, 0),
N+N( 0, 0, 1), N+N( 0, -1, -1)
in 3-d lattice N+N

sage: P1xP2.Delta_polar()
3-d reflexive polytope in 3-d lattice N+N

change_ring(F)

Return a CPR-Fano toric variety over field F, otherwise the same as self.

INPUT:

• F – field.

OUTPUT: CPR-Fano toric variety over F.
Note: There is no need to have any relation between \( F \) and the base field of \( \textit{self} \). If you do want to have such a relation, use \texttt{base_extend()} instead.

EXAMPLES:

```python
sage: P1xP1 = toric_varieties.P1xP1()
sage: P1xP1.base_ring()
Rational Field
sage: P1xP1_RR = P1xP1.change_ring(RR)
sage: P1xP1_RR.base_ring()
Real Field with 53 bits of precision
sage: P1xP1_QQ = P1xP1_RR.change_ring(QQ)
sage: P1xP1_QQ.base_ring()
Rational Field
sage: P1xP1_RR.base_extend(QQ)
Traceback (most recent call last):
...:
ValueError: no natural map from the base ring (=Real Field with 53 bits of precision) to R (=Rational Field)!
```

```python
sage: R = PolynomialRing(QQ, 2, 'a')
sage: P1xP1.change_ring(R)
Traceback (most recent call last):
...:
TypeError: need a field to construct a Fano toric variety!
```

\texttt{coordinate_point_to_coordinate(point)}

Return the variable of the coordinate ring corresponding to \( \textit{point} \).

\textbf{INPUT:}

- \( \textit{point} \) – integer from the list of \texttt{coordinate_points()}.

\textbf{OUTPUT:}

- the corresponding generator of the coordinate ring of \( \textit{self} \).

\textbf{EXAMPLES:}

```python
sage: diamond = lattice_polytope.cross_polytope(2)
sage: FTV = CPRFanoToricVariety(diamond, coordinate_points=[0,1,2,3,8])
sage: FTV.coordinate_points()
(0, 1, 2, 3, 8)
sage: FTV.gens()
(0, 1, z1, z2, z3, z8)
sage: FTV.coordinate_point_to_coordinate(8)
z8
```

\texttt{coordinate_points()}

Return indices of points of \texttt{Delta_polar()} used for coordinates.

\textbf{OUTPUT:} tuple of integers.

\textbf{EXAMPLES:}
Sage: diamond = lattice_polytope.cross_polytope(2)
Sage: square = diamond.polar()
Sage: FTV = CPRFanoToricVariety(Delta_polar=square, coordinate_points=[0,1,2,3,8])
Sage: FTV.coordinate_points()
(0, 1, 2, 3, 8)
Sage: FTV.gens()
(z0, z1, z2, z3, z8)
Sage: FTV = CPRFanoToricVariety(Delta_polar=square, coordinate_points="all")
Sage: FTV.coordinate_points()
(0, 1, 2, 3, 4, 5, 7, 8)
Sage: FTV.gens()
(z0, z1, z2, z3, z4, z5, z7, z8)

Note that one point is missing, namely

Sage: square.origin()
6

nef_complete_intersection(nef_partition, **kwds)
Return a nef complete intersection in self.

Note: The returned complete intersection may be actually a subscheme of another CPR-Fano toric variety: if the base field of self does not include all of the required names for monomial coefficients, it will be automatically extended.

Below $\Delta$ is the reflexive polytope corresponding to self, i.e. the fan of self is a refinement of the normal fan of $\Delta$. Other polytopes are described in the documentation of nef-partitions of reflexive polytopes.

Except for the first argument, nef_partition, this method accepts only keyword parameters.

INPUT:

- nef_partition – a $k$-part nef-partition of $\Delta^\circ$, all other parameters (if given) must be lists of length $k$;

- monomial_points – the $i$-th element of this list is either a list of integers or a string. A list will be interpreted as indices of points of $\Delta_i$ which should be used for monomials of the $i$-th polynomial of this complete intersection. A string must be one of the following descriptions of points of $\Delta_i$:
  - “vertices”,
  - “vertices+origin”,
  - “all” (default),
when using this description, it is also OK to pass a single string as monomial_points instead of repeating it $k$ times;

- coefficient_names – the $i$-th element of this list specifies names for the monomial coefficients of the $i$-th polynomial, see normalize_names() for acceptable formats. If not given, indexed coefficient names will be created automatically;
• **coefficient_name_indices** – the \( i \)-th element of this list specifies indices for indexed coefficients of the \( i \)-th polynomial. If not given, the index of each coefficient will coincide with the index of the corresponding point of \( \Delta_i \);

• **coefficients** – as an alternative to specifying coefficient names and/or indices, you can give the coefficients themselves as arbitrary expressions and/or strings. Using strings allows you to easily add “parameters”: the base field of \( \text{self} \) will be extended to include all necessary names.

**OUTPUT:**

• a \textit{nef complete intersection} of \( \text{self} \) (with the extended base field, if necessary).

**EXAMPLES:**

We construct several complete intersections associated to the same nef-partition of the 3-dimensional reflexive polytope \#2254:

```python
sage: p = ReflexivePolytope(3, 2254)
sage: np = p.nef_partitions()[1]; np
Nef-partition {2, 3, 4, 7, 8} ⊔ {0, 1, 5, 6}
sage: X = CPRFanoToricVariety(Delta_polar=p)
sage: X.nef_complete_intersection(np)
Closed subscheme of 3-d CPR-Fano toric variety
covered by 10 affine patches defined by:
\[
a_0 z_1 z_4^2 z_5^2 z_7^3 + a_2 z_2 z_4 z_5 z_6 z_7^2 z_8^2 \\
+ a_3 z_2^2 z_3 z_4 z_7 z_8 + a_1 z_0 z_2, \\
b_3 z_1 z_4 z_5 z_6 z_7^2 z_8^2 + b_0 z_2 z_5 z_6^3 z_7 z_8^4 \\
+ b_2 z_2 z_3 z_6^2 z_8^3 + b_1 z_1 z_3^2 z_4 + b_4 z_0 z_1 z_5 z_6
\]
```

Now we include only monomials associated to vertices of \( \Delta_i \):

```python
sage: X.nef_complete_intersection(np, monomial_points="vertices")
Closed subscheme of 3-d CPR-Fano toric variety
covered by 10 affine patches defined by:
\[
a_0 z_1 z_4^2 z_5^2 z_7^3 + a_2 z_2 z_4 z_5 z_6 z_7^2 z_8^2 \\
+ a_3 z_2^2 z_3 z_4 z_7 z_8 + a_1 z_0 z_2, \\
b_3 z_1 z_4 z_5 z_6 z_7^2 z_8^2 + b_0 z_2 z_5 z_6^3 z_7 z_8^4 \\
+ b_2 z_2 z_3 z_6^2 z_8^3 + b_1 z_1 z_3^2 z_4 + b_4 z_0 z_1 z_5 z_6
\]
```

(effectively, we set \( b_5 = 0 \)). Next we provide coefficients explicitly instead of using default generic names:

```python
sage: X.nef_complete_intersection(np, 
.....: monomial_points="vertices", 
.....: coefficients=[("a", "a^2", "a/e", "c_i"), list(range(1,6))])
Closed subscheme of 3-d CPR-Fano toric variety
covered by 10 affine patches defined by:
\[
a z_1 z_4^2 z_5^2 z_7^3 + a/e z_2 z_4 z_5 z_6 z_7^2 z_8^2 \\
+ (c_i) z_2^2 z_3 z_4 z_7 z_8 + (a^2) z_0 z_2, \\
4 z_1 z_4 z_5 z_6 z_7^2 z_8^2 + z_2 z_5 z_6^3 z_7 z_8^4 \\
+ 3 z_2 z_3 z_6^2 z_8^3 + 2 z_1 z_3^2 z_4 + 5 z_0 z_1 z_5 z_6
\]
```

Finally, we take a look at the generic representative of these complete intersections in a completely resolved ambient toric variety:

```python
sage: X = CPRFanoToricVariety(Delta_polar=p, 
.....: coordinate_points="all")
```

(continues on next page)
Closed subscheme of 3-d CPR-Fano toric variety
covered by 22 affine patches defined by:
\[\begin{align*}
& a_2 z_2 z_4 z_5 z_6 z_7^2 z_8^2 z_9 + a_1 z_0 z_2,
& b_0 z_2 z_5 z_6 z_7 z_8^4 z_9 z_{10}^2 z_{11}^2 z_{12}^2 z_{13}^2,
& b_3 z_1 z_4 z_5 z_6 z_7^2 z_8^2 z_9 z_{10}^2 z_{11} z_{12} z_{13}^2,
& b_2 z_2 z_3 z_6 z_8^3 z_9^2 z_{10}^2 z_{11} z_{12} z_{13},
& b_1 z_1 z_3 z_4 z_5 z_6 z_7 z_8 z_9 z_{10} z_{11} z_{12} z_{13},
\end{align*}\]

This function accepts only keyword arguments, none of which are mandatory.

INPUT:

- `new_points` – list of integers, indices of boundary points of `Delta_polar()`, which should be added as rays to the subdividing fan;
- all other arguments will be passed to `resolve()` method of (general) toric varieties, see its documentation for details.

OUTPUT:

- CPR-Fano toric variety if there was no new_rays argument and toric variety otherwise.

EXAMPLES:

```python
sage: diamond = lattice_polytope.cross_polytope(2)
sage: FTV = CPRFanoToric Variety(Delta=diamond)
sage: FTV.coordinate_points()
(0, 1, 2, 3)
sage: FTV.gens()
(z_0, z_1, z_2, z_3)
sage: FTV_res = FTV.resolve(new_points=[6, 8])
Traceback (most recent call last):
  ... ValueError: the origin (point #6) cannot be used for subdivision!
sage: FTV_res = FTV.resolve(new_points=[8, 5]); FTV_res
2-d CPR-Fano toric variety covered by 6 affine patches
sage: FTV_res.coordinate_points()
(0, 1, 2, 3, 8, 5)
sage: FTV_res.gens()
(z_0, z_1, z_2, z_3, z_8, z_5)
sage: TV_res = FTV.resolve(new_rays=[[1, 2]]); TV_res
2-d toric variety covered by 5 affine patches
sage: TV_res.gens()
(z_0, z_1, z_2, z_3, z_4)
```
class sage.schemes.toric.fano_variety.NefCompleteIntersection(P_Delta, nef_partition, monomial_points='all', coefficient_names=None, coefficient_name_indices=None, coefficients=None)

Bases: AlgebraicScheme_subscheme_toric

Construct a nef complete intersection in a CPR-Fano toric variety.

INPUT:

- `P_Delta` – a CPR-Fano toric variety associated to a reflexive polytope $\Delta$;
- see CPRFanoToricVariety_field.nef_complete_intersection() for documentation on all other acceptable parameters.

OUTPUT:

- a nef complete intersection of `P_Delta` (with the extended base field, if necessary).

EXAMPLES:

```python
sage: o = lattice_polytope.cross_polytope(3)
sage: np = o.nef_partitions()[0]; np
Nef-partition {0, 1, 3} ▫ {2, 4, 5}
sage: X = CPRFanoToricVariety(Delta_polar=o)
sage: X.nef_complete_intersection(np)
Closed subscheme of 3-d CPR-Fano toric variety
covered by 8 affine patches defined by:
a2*z0^2*z1 + a5*z0*z1*z3 + a1*z1*z3^2 + a3*z0^2*z4 + a4*z0*z3*z4 + a0*z3^2*z4,
b1*z1*z2^2 + b2*z2^2*z4 + b5*z1*z2*z5 + b4*z2*z4*z5 + b3*z1*z5^2 + b0*z4*z5^2
```

See CPRFanoToricVariety_field.nef_complete_intersection() for a more elaborate example.

cohomology_class()

Return the class of self in the ambient space cohomology ring.

OUTPUT: A cohomology class.

EXAMPLES:

```python
sage: o = lattice_polytope.cross_polytope(3)
sage: np = o.nef_partitions()[0]; np
Nef-partition {0, 1, 3} ▫ {2, 4, 5}
sage: X = CPRFanoToricVariety(Delta_polar=o)
sage: CI = X.nef_complete_intersection(np); CI
Closed subscheme of 3-d CPR-Fano toric variety
covered by 8 affine patches defined by:
a2*z0^2*z1 + a5*z0*z1*z3 + a1*z1*z3^2 + a3*z0^2*z4 + a4*z0*z3*z4 + a0*z3^2*z4,
b1*z1*z2^2 + b2*z2^2*z4 + b5*z1*z2*z5 + b4*z2*z4*z5 + b3*z1*z5^2 + b0*z4*z5^2
sage: CI.cohomology_class()  # needs sage.libs.singular
[2*z3*z4 + 4*z3*z5 + 2*z4*z5]
```

nef_partition()

Return the nef-partition associated to self.

OUTPUT: A nef-partition.

EXAMPLES:
```python
sage: o = lattice_polytope.cross_polytope(3)
sage: np = o.nef_partitions()[0]; np
Nef-partition {0, 1, 3} △ {2, 4, 5}
sage: X = CPRFanoToricVariety(Delta_polar=o)
sage: CI = X.nef_complete_intersection(np); CI
Closed subscheme of 3-d CPR-Fano toric variety covered by 8 affine patches defined by:
\[ a2^*z0^2*z1 + a5^*z0*z1*z3 + a1^*z1*z3^2 + a3^*z0^2*z4 + a4^*z0^3*z4 + a0^*z3^2*z4, \]
\[ b1^*z1^2*z2 + b2^*z2^2*z4 + b5^*z1*z2*z5 + b4^*z2^4*z5 + b3^*z1*z5^2 + b0^*z4*z5^2 \]
sage: CI.nef_partition()
Nef-partition {0, 1, 3} △ {2, 4, 5}
sage: CI.nef_partition() is np
True
```

`sage.schemes.toric.fano_variety.add_variables(field, variables)`

Extend field to include all variables.

**INPUT:**
- *field* - a field;
- *variables* - a list of strings.

**OUTPUT:**
- a fraction field extending the original field, which has all variables among its generators.

**EXAMPLES:**

We start with the rational field and slowly add more variables:

```python
sage: from sage.schemes.toric.fano_variety import *
sage: F = add_variables(QQ, []); F           # No extension
Rational Field
sage: F = add_variables(QQ, ["a"]); F
Fraction Field of Univariate Polynomial Ring in a over Rational Field
sage: F = add_variables(F, ["a"]); F
Fraction Field of Univariate Polynomial Ring in a over Rational Field
sage: F = add_variables(F, ["b", "c"]); F
Fraction Field of Multivariate Polynomial Ring in a, b, c over Rational Field
sage: F = add_variables(F, ["c", "d", "b", "c", "d"]); F
Fraction Field of Multivariate Polynomial Ring in a, b, c, d over Rational Field
```

`sage.schemes.toric.fano_variety.is_CPRFanoToricVariety(x)`

Check if *x* is a CPR-Fano toric variety.

**INPUT:**
- *x* – anything.

**OUTPUT:**
- True if *x* is a CPR-Fano toric variety and False otherwise.

**Note:** While projective spaces are Fano toric varieties mathematically, they are not toric varieties in Sage due to efficiency considerations, so this function will return False.

**EXAMPLES:**
### 16.3 Library of toric varieties

This module provides a simple way to construct often-used toric varieties. Please see the help for the individual methods of `toric_varieties` for a more detailed description of which varieties can be constructed.

**AUTHORS:**

- Volker Braun (2010-07-02): initial version

**EXAMPLES:**

```python
sage: toric_varieties.dP6()
2-d CPR-Fano toric variety covered by 6 affine patches
```

You can assign the homogeneous coordinates to Sage variables either with `inject_variables()` or immediately during assignment like this:

```python
sage: P2.<x,y,z> = toric_varieties.P2()
sage: x^2 + y^2 + z^2
x^2 + y^2 + z^2
sage: P2.coordinate_ring()
Multivariate Polynomial Ring in x, y, z over Rational Field
```

**class** `sage.schemes.toric.library.ToricVarietyFactory`

Bases: `SageObject`

The methods of this class construct toric varieties.

**Warning:** You need not create instances of this class. Use the already-provided object `toric_varieties` instead.

`A(n, names='z+', base_ring=Rational Field)`

Construct the n-dimensional affine space.

**INPUT:**

- `n` – positive integer. The dimension of the affine space.
- `names` – string. Names for the homogeneous coordinates. See `normalize_names()` for acceptable formats.
- `base_ring` – a ring (default: `Q`). The base ring for the toric variety.
OUTPUT: A toric variety.

EXAMPLES:

```
sage: A3 = toric_varieties.A(3); A3
3-d affine toric variety
sage: A3.fan().rays()
N(1, 0, 0),
N(0, 1, 0),
N(0, 0, 1)
in 3-d lattice N
sage: A3.gens()
(z0, z1, z2)
```

\textbf{A1}(\textit{names}='z', \textit{base\_ring}=\textit{Rational Field})

Construct the affine line $\mathbb{A}^1$ as a toric variety.

INPUT:

- \textit{names} – string. Names for the homogeneous coordinates. See \textit{normalize\_names()} for acceptable formats.
- \textit{base\_ring} – a ring (default: \textit{Q}). The base ring for the toric variety.

OUTPUT: A toric variety.

EXAMPLES:

```
sage: A1 = toric_varieties.A1(); A1
1-d affine toric variety
sage: A1.fan().rays()
N(1)
in 1-d lattice N
sage: A1.gens()
(z,)
```

\textbf{A2}(\textit{names}='x y', \textit{base\_ring}=\textit{Rational Field})

Construct the affine plane $\mathbb{A}^2$ as a toric variety.

INPUT:

- \textit{names} – string. Names for the homogeneous coordinates. See \textit{normalize\_names()} for acceptable formats.
- \textit{base\_ring} – a ring (default: \textit{Q}). The base ring for the toric variety.

OUTPUT: A toric variety.

EXAMPLES:

```
sage: A2 = toric_varieties.A2(); A2
2-d affine toric variety
sage: A2.fan().rays()
N(1, 0),
N(0, 1)
in 2-d lattice N
sage: A2.gens()
(x, y)
```
**A2_Z2**\((\text{names}=\text{'x y', base\_ring}=\text{Rational Field})\)

Construct the orbifold \(\mathbb{A}^2/\mathbb{Z}_2\) as a toric variety.

**INPUT:**

- **names** – string. Names for the homogeneous coordinates. See \texttt{normalize\_names()} for acceptable formats.

- **base\_ring** – a ring (default: \(\mathbb{Q}\)). The base ring for the toric variety.

**OUTPUT:** A **toric variety**.

**EXAMPLES:**

```sage
sage: A2_Z2 = toric\_varieties.A2\_Z2(); A2_Z2
2-d affine toric variety
gens()
(x, y)
```

**BCdI0G**\((\text{names}=\text{'v1 v2 c1 c2 v4 v5 b e1 e2 e3 f g v6', base\_ring}=\text{Rational Field})\)

Construct the 5-dimensional toric variety studied in [BCdlOG2000], [HLY2002]

**INPUT:**

- **names** – string. Names for the homogeneous coordinates. See \texttt{normalize\_names()} for acceptable formats.

- **base\_ring** – a ring (default: \(\mathbb{Q}\)). The base ring for the toric variety.

**OUTPUT:** A **CPR-Fano toric variety**.

**EXAMPLES:**

```sage
sage: X = toric\_varieties.BCdlOG(); X
5-d CPR-Fano toric variety covered by 54 affine patches
gens()
(v1, v2, c1, c2, v4, v5, b, e1, e2, e3, f, g, v6)
```

**BCdI0G\_base**\((\text{names}=\text{'d4 d3 r2 r1 d2 u d1', base\_ring}=\text{Rational Field})\)

Construct the base of the \(\mathbb{P}^2(1,2,3)\) fibration \texttt{BCdI0G()}.

**INPUT:**

- **names** – string. Names for the homogeneous coordinates. See \texttt{normalize\_names()} for acceptable formats.

- **base\_ring** – a ring (default: \(\mathbb{Q}\)). The base ring for the toric variety.

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OUTPUT: A toric variety.

EXAMPLES:

```
sage: base = toric_varieties.BCd1OG_base(); base
3-d toric variety covered by 10 affine patches
sage: base.fan().rays()
N(-1, 0, 0), N(0, -1, 0), N(0, 0, -1), N(0, 0, 1),
N(0, 1, 2), N(0, 1, 3), N(1, 0, 4)
in 3-d lattice N
sage: base.gens()
(d4, d3, r2, r1, d2, u, d1)
```

Conifold(names='u x y v', base_ring=Rational Field)

Construct the conifold as a toric variety.

INPUT:

- names – string. Names for the homogeneous coordinates. See `normalize_names()` for acceptable formats.
- base_ring – a ring (default: Q). The base ring for the toric variety.

OUTPUT: A toric variety.

EXAMPLES:

```
sage: Conifold = toric_varieties.Conifold(); Conifold
3-d affine toric variety
sage: Conifold.fan().rays()
N(0, 0, 1), N(0, 1, 1),
N(1, 0, 1), N(1, 1, 1)
in 3-d lattice N
sage: Conifold.gens()
(u, x, y, v)
```

Cube_deformation(k, names=None, base_ring=Rational Field)

Construct, for each \( k \in \mathbb{Z}_{\geq 0} \), a toric variety with \( \mathbb{Z}_k \)-torsion in the Chow group. The fans of this sequence of toric varieties all equal the face fan of a unit cube topologically, but the \((1, 1, 1)\)-vertex is moved to \((1, 1, 2k+1)\). This example was studied in [FS1994].

INPUT:

- \( k \) – integer. The case \( k=0 \) is the same as `Cube_face_fan()`.
- names – string. Names for the homogeneous coordinates. See `normalize_names()` for acceptable formats.
- base_ring – a ring (default: Q). The base ring for the toric variety.

OUTPUT:

A toric variety \( X_k \). Its Chow group is \( A_1(X_k) = \mathbb{Z}_k \).

EXAMPLES:

```
sage: X_2 = toric_varieties.Cube_deformation(2); X_2
3-d toric variety covered by 6 affine patches
sage: X_2.fan().rays()
(continues on next page)```
N(1, 1, 1), N(1, -1, 1), N(-1, 1, 1), N(-1, 1, -1), N(-1, -1, 1), N(-1, -1, -1), N(-1, 1, -1), N(1, -1, -1), N(1, 1, -1)

in 3-d lattice N

sage: X_2.gens()
(z0, z1, z2, z3, z4, z5, z6, z7)

Cube_face_fan(names='z+', base_ring=Rational Field)
Construct the toric variety given by the face fan of the 3-dimensional unit lattice cube.

This variety has 6 conifold singularities but the fan is still polyhedral.

INPUT:
• names – string. Names for the homogeneous coordinates. See normalize_names() for acceptable formats.
• base_ring – a ring (default: Q). The base ring for the toric variety.

OUTPUT: A CPR-Fano toric variety.

EXAMPLES:

sage: Cube_face_fan = toric_varieties.Cube_face_fan(); Cube_face_fan
3-d CPR-Fano toric variety covered by 6 affine patches

sage: Cube_face_fan.fan().rays()
N(1, 1, 1), N(1, -1, 1), N(-1, 1, 1), N(-1, -1, 1),
N(-1, -1, -1), N(-1, 1, -1), N(1, -1, -1), N(1, 1, -1)
in 3-d lattice N

sage: Cube_face_fan.gens()
(z0, z1, z2, z3, z4, z5, z6, z7)

Cube_nonpolyhedral(names='z+', base_ring=Rational Field)
Construct the toric variety defined by a fan that is not the face fan of a polyhedron.

This toric variety is defined by a fan that is topologically like the face fan of a 3-dimensional cube, but with a different N-lattice structure.

INPUT:
• names – string. Names for the homogeneous coordinates. See normalize_names() for acceptable formats.
• base_ring – a ring (default: Q). The base ring for the toric variety.

OUTPUT: A toric variety.

Note:
• This is an example of a non-polyhedral fan.
• Its Chow group has torsion: $A_2(X) = \mathbb{Z}^5 \oplus \mathbb{Z}_2$

EXAMPLES:

sage: Cube_nonpolyhedral = toric_varieties.Cube_nonpolyhedral()  
sage: Cube_nonpolyhedral
3-d toric variety covered by 6 affine patches
sage: Cube_nonpolyhedral.fan().rays()
N( 1, 2, 3), N( 1, -1, 1), N(-1, 1, 1), N(-1, -1, 1),
N(-1, -1, -1), N(-1, 1, -1), N( 1, -1, -1), N( 1, 1, -1)
in 3-d lattice N
sage: Cube_nonpolyhedral.gens()
(z0, z1, z2, z3, z4, z5, z6, z7)

Cube_sublattice(names='z+', base_ring=Rational Field)

Construct the toric variety defined by a face fan over a 3-dimensional cube, but not the unit cube in the
N-lattice. See p. 65 of [Ful1993].

Its Chow group is $A_2(X) = \mathbb{Z}^5$, which distinguishes it from the face fan of the unit cube.

INPUT:

• names – string. Names for the homogeneous coordinates. See normalize_names() for acceptable
  formats.

• base_ring – a ring (default: $\mathbb{Q}$). The base ring for the toric variety.

OUTPUT: A CPR-Fano toric variety.

EXAMPLES:
sage: Cube_sublattice = toric_varieties.Cube_sublattice(); Cube_sublattice
3-d CPR-Fano toric variety covered by 6 affine patches
sage: Cube_sublattice.fan().rays()
N( 1, 0, 0), N( 0, 1, 0), N( 0, 0, 1), N(-1, 1, 1),
N(-1, 0, 0), N( 0, -1, 0), N( 0, 0, -1), N( 1, -1, -1)
in 3-d lattice N
sage: Cube_sublattice.gens()
(z0, z1, z2, z3, z4, z5, z6, z7)

P(n, names='z+', base_ring=Rational Field)

Construct the n-dimensional projective space $\mathbb{P}^n$.

INPUT:

• n – positive integer. The dimension of the projective space.

• names – string. Names for the homogeneous coordinates. See normalize_names() for acceptable
  formats.

• base_ring – a ring (default: $\mathbb{Q}$). The base ring for the toric variety.

OUTPUT: A CPR-Fano toric variety.

EXAMPLES:
sage: P3 = toric_varieties.P(3); P3
3-d CPR-Fano toric variety covered by 4 affine patches
sage: P3.fan().rays()
N( 1, 0, 0),
N( 0, 1, 0),
N( 0, 0, 1),
N(-1, -1, -1)
in 3-d lattice N
Construct the projective line $P^1$ as a toric variety.

**INPUT:**

- `names` – string. Names for the homogeneous coordinates. See `normalize_names()` for acceptable formats.
- `base_ring` – a ring (default: $\mathbb{Q}$). The base ring for the toric variety.

**OUTPUT:** A **CPR-Fano toric variety**.

**EXAMPLES:**

```sage
sage: P1 = toric_varieties.P1(); P1
1-d CPR-Fano toric variety covered by 2 affine patches
sage: P1.fan().rays()
N( 1),
N(-1)
in 1-d lattice N
sage: P1.gens()
(s, t)
```

Construct the Cartesian product $P^1 \times \mathbb{A}^1$ as a toric variety.

**INPUT:**

- `names` – string. Names for the homogeneous coordinates. See `normalize_names()` for acceptable formats.
- `base_ring` – a ring (default: $\mathbb{Q}$). The base ring for the toric variety.

**OUTPUT:** A **toric variety**.

**EXAMPLES:**

```sage
sage: P1xA1 = toric_varieties.P1xA1(); P1xA1
2-d toric variety covered by 2 affine patches
sage: P1xA1.fan().rays()
N( 1, 0),
N(-1, 0),
N( 0, 1)
in 2-d lattice N
sage: P1xA1.gens()
(s, t, z)
```

Construct the del Pezzo surface $P^1 \times P^1$ as a toric variety.

**INPUT:**

- `names` – string. Names for the homogeneous coordinates. See `normalize_names()` for acceptable formats.
- `base_ring` – a ring (default: $\mathbb{Q}$). The base ring for the toric variety.
OUTPUT: A **CPR-Fano toric variety**.

**EXAMPLES:**

```sage
P1xP1 = toric_varieties.P1xP1(); P1xP1
2-d CPR-Fano toric variety covered by 4 affine patches
sage: P1xP1.fan().rays()
N( 1, 0), N(-1, 0),
N( 0, 1), N( 0, -1)
in 2-d lattice N
sage: P1xP1.gens()
(s, t, x, y)
```

**P1xP1** \(_\text{Z}_2\) (**names**='s t x y', **base**=_Rational Field_)

Construct the toric \(\mathbb{Z}_2\)-orbifold of the del Pezzo surface \(\mathbb{P}^1 \times \mathbb{P}^1\) as a toric variety.

**INPUT:**

- **names** – string. Names for the homogeneous coordinates. See `normalize_names()` for acceptable formats.
- **base** – a ring (default: \(\mathbb{Q}\)). The base ring for the toric variety.

**OUTPUT: A **CPR-Fano toric variety**.**

**EXAMPLES:**

```sage
P1xP1_Z2 = toric_varieties.P1xP1_Z2(); P1xP1_Z2
2-d CPR-Fano toric variety covered by 4 affine patches
sage: P1xP1_Z2.fan().rays()
N( 1, 1), N(-1, -1),
N(-1, 1), N( 1, -1)
in 2-d lattice N
sage: P1xP1_Z2.gens()
(s, t, x, y)
```

**P2** (**names**='x y z', **base**=_Rational Field_)

Construct the projective plane \(\mathbb{P}^2\) as a toric variety.

**INPUT:**

- **names** – string. Names for the homogeneous coordinates. See `normalize_names()` for acceptable formats.
- **base** – a ring (default: \(\mathbb{Q}\)). The base ring for the toric variety.

**OUTPUT: A **CPR-Fano toric variety**.**

**EXAMPLES:**

```sage
P2 = toric_varieties.P2(); P2
2-d CPR-Fano toric variety covered by 3 affine patches
sage: P2.fan().rays()
N( 1, 0),
N( 0, 1),
N(-1, -1)
in 2-d lattice N
```

(continues on next page)
P2_112 \( (\text{name}='z+4', \text{base}_{\text{ring}}=\text{Rational Field}) \)

Construct the weighted projective space \( \mathbb{P}^2(1, 1, 2) \).

INPUT:

- \text{name} – string. Names for the homogeneous coordinates. See \text{normalize\_names()} for acceptable formats.
- \text{base}_{\text{ring}} – a ring (default: \text{Q}). The base ring for the toric variety.

OUTPUT: A \text{CPR-Fano toric variety}.

EXAMPLES:

```python
sage: P2_112 = toric_varieties.P2_112(); P2_112
2-d CPR-Fano toric variety covered by 3 affine patches
sage: P2_112.fan().rays()
N( 1, 0),
N( 0, 1),
N(-1, -2)
in 2-d lattice \text{N}
sage: P2_112.gens()
(z0, z1, z2)
```

P2_123 \( (\text{name}='z+4', \text{base}_{\text{ring}}=\text{Rational Field}) \)

Construct the weighted projective space \( \mathbb{P}^2(1, 2, 3) \).

INPUT:

- \text{name} – string. Names for the homogeneous coordinates. See \text{normalize\_names()} for acceptable formats.
- \text{base}_{\text{ring}} – a ring (default: \text{Q}). The base ring for the toric variety.

OUTPUT: A \text{CPR-Fano toric variety}.

EXAMPLES:

```python
sage: P2_123 = toric_varieties.P2_123(); P2_123
2-d CPR-Fano toric variety covered by 3 affine patches
sage: P2_123.fan().rays()
N( 1, 0),
N( 0, 1),
N(-2, -3)
in 2-d lattice \text{N}
sage: P2_123.gens()
(z0, z1, z2)
```

P4_11133 \( (\text{name}='z+4', \text{base}_{\text{ring}}=\text{Rational Field}) \)

Construct the weighted projective space \( \mathbb{P}^4(1, 1, 1, 3, 3) \).

INPUT:

- \text{name} – string. Names for the homogeneous coordinates. See \text{normalize\_names()} for acceptable formats.
- **base_ring** – a ring (default: $\mathbb{Q}$). The base ring for the toric variety.

**OUTPUT**: A **CPR-Fano toric variety**.

**EXAMPLES**:

```python
sage: P4_11133 = toric_varieties.P4_11133(); P4_11133
4-d CPR-Fano toric variety covered by 5 affine patches
sage: P4_11133.fan().rays()
N( 1, 0, 0, 0), N( 0, 1, 0, 0), N( 0, 0, 1, 0),
N( 0, 0, 0, 1), N(-3, -3, -1, -1)
in 4-d lattice N
sage: P4_11133.gens()
(z0, z1, z2, z3, z4)
```

**P4_11133_resolved** *(names='z+', base_ring=Rational Field)*

Construct the weighted projective space $\mathbb{P}^4(1,1,1,3,3)$.

**INPUT**:

- **names** – string. Names for the homogeneous coordinates. See **normalize_names()** for acceptable formats.
- **base_ring** – a ring (default: $\mathbb{Q}$). The base ring for the toric variety.

**OUTPUT**: A **CPR-Fano toric variety**.

**EXAMPLES**:

```python
sage: P4_11133_resolved = toric_varieties.P4_11133_resolved()
4-d CPR-Fano toric variety covered by 9 affine patches
sage: P4_11133_resolved.fan().rays()
N( 1, 0, 0, 0), N( 0, 1, 0, 0), N( 0, 0, 1, 0),
N( 0, 0, 0, 1), N(-9, -6, -1, -1)
in 4-d lattice N
sage: P4_11133_resolved.gens()
(z0, z1, z2, z3, z4, z5)
```

**P4_11169** *(names='z+', base_ring=Rational Field)*

Construct the weighted projective space $\mathbb{P}^4(1,1,1,6,9)$.

**INPUT**:

- **names** – string. Names for the homogeneous coordinates. See **normalize_names()** for acceptable formats.
- **base_ring** – a ring (default: $\mathbb{Q}$). The base ring for the toric variety.

**OUTPUT**: A **CPR-Fano toric variety**.

**EXAMPLES**:

```python
sage: P4_11169 = toric_varieties.P4_11169(); P4_11169
4-d CPR-Fano toric variety covered by 5 affine patches
sage: P4_11169.fan().rays()
N( 1, 0, 0, 0), N( 0, 1, 0, 0), N( 0, 0, 1, 0),
N( 0, 0, 0, 1), N(-9, -6, -1, -1)
in 4-d lattice N
```

(continues on next page)
sage: P4_11169.gens()
(z0, z1, z2, z3, z4)

\textbf{P4\_11169\_resolved}(\textit{names=\textquoteleft\texttt{z+}; base\_ring=\texttt{Rational Field})}

Construct the blow-up of the weighted projective space $\mathbb{P}^4(1,1,1,6,9)$ at its curve of $\mathbb{Z}_3$ orbifold fixed points.

INPUT:
\begin{itemize}
  \item \textit{names} – string. Names for the homogeneous coordinates. See \texttt{normalize\_names()} for acceptable formats.
  \item \textit{base\_ring} – a ring (default: \texttt{Q}). The base ring for the toric variety.
\end{itemize}

OUTPUT: A \textit{CPR-Fano toric variety}.

EXAMPLES:

\begin{verbatim}
sage: P4_11169_resolved = toric_varieties.P4_11169_resolved()
sage: P4_11169_resolved
4-d CPR-Fano toric variety covered by 9 affine patches
sage: P4_11169_resolved.fan().rays()
N( 1, 0, 0, 0), N( 0, 1, 0, 0), N( 0, 0, 1, 0),
N( 0, 0, 0, 1), N(-9, -6, -1, -1), N(-3, -2, 0, 0)
in 4-d lattice N
sage: P4_11169_resolved.gens()
(z0, z1, z2, z3, z4, z5)
\end{verbatim}

\textbf{WP}(\texttt{\*q, \**kw})

Construct weighted projective $n$-space over a field.

INPUT:
\begin{itemize}
  \item \textit{q} – a sequence of positive integers relatively prime to one another. The weights \(q\) can be given either as a list or tuple, or as positional arguments.
\end{itemize}

Two keyword arguments:
\begin{itemize}
  \item base\_ring – a field (default: \texttt{Q}).
  \item names – string or list (tuple) of strings (default ‘\texttt{z+}’). See \texttt{normalize\_names()} for acceptable formats.
\end{itemize}

OUTPUT:
\begin{itemize}
  \item A \textit{toric variety}. If \(q = (q_0, \ldots, q_n)\), then the output is the weighted projective space $\mathbb{P}(q_0, \ldots, q_n)$ over base\_ring. names are the names of the generators of the homogeneous coordinate ring.
\end{itemize}

EXAMPLES:

A hyperelliptic curve $C$ of genus 2 as a subscheme of the weighted projective plane $\mathbb{P}(1,3,1)$:

\begin{verbatim}
sage: X = toric_varieties.WP([1,3,1], names='x y z')
sage: X.inject_variables()
Defining x, y, z
sage: g = y^2 - (x^6-z^6)
sage: C = X.subscheme([g]); C
Closed subscheme of 2-d toric variety covered by 3 affine patches defined by:
-x^6 + z^6 + y^2
\end{verbatim}
dP6\((\text{names=}'x\ u\ v\ z\ w', \text{base\_ring=}'\text{Rational Field}')\)

Construct the del Pezzo surface of degree 6 (\(\mathbb{P}^2\) blown up at 3 points) as a toric variety.

**INPUT:**
- **names** – string. Names for the homogeneous coordinates. See \texttt{normalize\_names()} for acceptable formats.
- **base\_ring** – a ring (default: \(\mathbb{Q}\)). The base ring for the toric variety.

**OUTPUT:** A \texttt{CPR-Fano toric variety}.

**EXAMPLES:**

```sage
sage: dP6 = toric_varieties.dP6(); dP6
2-d CPR-Fano toric variety covered by 6 affine patches
sage: dP6.fan().rays()
N( 0, 1), N(-1, 0), N(-1, -1),
N( 0, -1), N( 1, 0), N( 1, 1)
in 2-d lattice N
sage: dP6.gens()
(x, u, v, z, w)
```

\[\text{dP6xdP6\((\text{names=}'x0\ x1\ x2\ x3\ x4\ y0\ y1\ y2\ y3\ y4\ y5', \text{base\_ring=}'\text{Rational Field}')\)}\]

Construct the product of two del Pezzo surfaces of degree 6 (\(\mathbb{P}^2\) blown up at 3 points) as a toric variety.

**INPUT:**
- **names** – string. Names for the homogeneous coordinates. See \texttt{normalize\_names()} for acceptable formats.
- **base\_ring** – a ring (default: \(\mathbb{Q}\)). The base ring for the toric variety.

**OUTPUT:** A \texttt{CPR-Fano toric variety}.

**EXAMPLES:**

```sage
sage: dP6xdP6 = toric_varieties.dP6xdP6(); dP6xdP6
4-d CPR-Fano toric variety covered by 36 affine patches
sage: dP6xdP6.fan().rays()
N( 0, 1, 0, 0), N(-1, 0, 0, 0), N(-1, -1, 0, 0),
N( 0, -1, 0, 0), N( 1, 0, 0, 0), N( 1, 1, 0, 0),
N( 0, 0, 0, 1), N( 0, 0, -1, 0), N( 0, 0, -1, -1),
N( 0, 0, 0, -1), N( 0, 0, 1, 0), N( 0, 0, 1, 1)
in 4-d lattice N
sage: dP6xdP6.gens()
(x0, x1, x2, x3, x4, x5, y0, y1, y2, y3, y4, y5)
```

dP7\((\text{names=}'x\ u\ v\ z', \text{base\_ring=}'\text{Rational Field}')\)

Construct the del Pezzo surface of degree 7 (\(\mathbb{P}^2\) blown up at 2 points) as a toric variety.

**INPUT:**
- **names** – string. Names for the homogeneous coordinates. See \texttt{normalize\_names()} for acceptable formats.
- **base\_ring** – a ring (default: \(\mathbb{Q}\)). The base ring for the toric variety.

**OUTPUT:** A \texttt{CPR-Fano toric variety}.

**EXAMPLES:**

```sage
sage: dP7 = toric_varieties.dP7(); dP7
2-d CPR-Fano toric variety covered by 2 affine patches
sage: dP7.fan().rays()
N( 0, 1), N(-1, 0),
N( 0, -1), N( 1, 0)
in 2-d lattice N
sage: dP7.gens()
(x, u, v, z)
```
```python
sage: dP7 = toric_varieties.dP7(); dP7
2-d CPR-Fano toric variety covered by 5 affine patches
sage: dP7.fan().rays()
N( 0, 1), N(-1, 0), N(-1, -1),
N( 0, -1), N( 1, 0)
in 2-d lattice N
sage: dP7.gens()
(x, u, y, v, z)
```

\textbf{dP8}(names='t x y z', base_ring=Rational Field)

Construct the del Pezzo surface of degree 8 (\( \mathbb{P}^2 \) blown up at 1 point) as a toric variety.

\textbf{INPUT:}

- **names** – string. Names for the homogeneous coordinates. See \textit{normalize_names()} for acceptable formats.
- **base_ring** – a ring (default: \( \mathbb{Q} \)). The base ring for the toric variety.

\textbf{OUTPUT:} A \textit{CPR-Fano toric variety}.

\textbf{EXAMPLES:}

```python
sage: dP8 = toric_varieties.dP8(); dP8
2-d CPR-Fano toric variety covered by 4 affine patches
sage: dP8.fan().rays()
N( 1, 1), N( 0, 1),
N(-1, -1), N( 1, 0)
in 2-d lattice N
sage: dP8.gens()
(t, x, y, z)
```

\textbf{torus}(n, names='z+', base_ring=Rational Field)

Construct the n-dimensional algebraic torus (\( \mathbb{F}^\times \))^n.

\textbf{INPUT:}

- **n** – non-negative integer. The dimension of the algebraic torus.
- **names** – string. Names for the homogeneous coordinates. See \textit{normalize_names()} for acceptable formats.
- **base_ring** – a ring (default: \( \mathbb{Q} \)). The base ring for the toric variety.

\textbf{OUTPUT:} A \textit{toric variety}.

\textbf{EXAMPLES:}

```python
sage: T3 = toric_varieties.torus(3); T3
3-d affine toric variety
sage: T3.fan().rays()
Empty collection
in 3-d lattice N
sage: T3.fan().virtual_rays()
N(1, 0, 0),
N(0, 1, 0),
N(0, 0, 1)
in 3-d lattice N
```

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16.4 Toric divisors and divisor classes

Let $X$ be a toric variety corresponding to a rational polyhedral fan $\Sigma$. A toric divisor $D$ is a $T$-Weil divisor over a given coefficient ring (usually $\mathbb{Z}$ or $\mathbb{Q}$), i.e. a formal linear combination of torus-invariant subvarieties of $X$ of codimension one. In homogeneous coordinates $[z_0 : \cdots : z_k]$, these are the subvarieties $\{z_i = 0\}$. Note that there is a finite number of such subvarieties, one for each ray of $\Sigma$. We generally identify

- Toric divisor $D$,
- Sheaf $\mathcal{O}(D)$ (if $D$ is Cartier, it is a line bundle),
- Support function $\phi_D$ (if $D$ is $\mathbb{Q}$-Cartier, it is a function linear on each cone of $\Sigma$).

**EXAMPLES:**

We start with an illustration of basic divisor arithmetic:

```python
sage: dP6 = toric_varieties.dP6()
sage: Dx, Du, Dy, Dv, Dz, Dw = dP6.toric_divisor_group().gens()
sage: Dx
V(x)
sage: -Dx
-V(x)
sage: 2*Dx
2*V(x)
sage: Dx^2
2*V(x)
sage: (1/2)*Dx + Dy/3 - Dz
1/2*V(x) + 1/3*V(y) - V(z)
sage: Dx.parent()
Group of toric ZZ-Weil divisors on 2-d CPR-Fano toric variety covered by 6 affine patches
sage: (Dx/2).parent()
Group of toric QQ-Weil divisors on 2-d CPR-Fano toric variety covered by 6 affine patches
```

Now we create a more complicated variety to demonstrate divisors of different types:

```python
sage: F = Fan(cones=[(0,1,2,3), (0,1,4)],
          rays=[(1,1,1), (1,-1,1), (1,-1,-1), (1,1,-1), (0,0,1)])
sage: X = ToricVariety(F)
sage: QQ_Cartier = X.divisor([2,2,1,1,1])
sage: Cartier = 2 * QQ_Cartier
sage: Weil = X.divisor([1,1,1,0,0])
sage: QQ_Weil = 1/2 * Weil
sage: [QQ_Weil.is_QQ_Weil(),
    QQ_Weil.is_Weil(),
    QQ_Weil.is_Wiel()]
```

Now create a more complicated variety to demonstrate divisors of different types:
The toric (Q-Weil) divisors on a toric variety $X$ modulo linear equivalence generate the divisor class group $\text{Cl}(X)$, implemented by \texttt{ToricRationalDivisorClassGroup}. If $X$ is smooth, this equals the Picard group $\text{Pic}(X)$. We continue using del Pezzo surface of degree 6 introduced above:

```
sage: Cl = dP6.rational_class_group(); Cl
The toric rational divisor class group of a 2-d CPR-Fano toric variety covered by 6 affine patches
sage: Cl.ngens()
4
sage: c0,c1,c2,c3 = Cl.gens()
sage: c = c0 + 2*c1 - c3; c
Divisor class [1, 2, 0, -1]
```

Divisors are mapped to their classes and lifted via:

```
sage: Dx.divisor_class()
Divisor class [1, 0, 0, 0]
sage: Dx.divisor_class() in Cl
True
sage: (-Dw + Dv + Dy).divisor_class()
Divisor class [1, 0, 0, 0]
sage: c0
Divisor class [1, 0, 0, 0]
sage: c0.lift()
V(x)
```

The (rational) divisor class group is where the Kaehler cone lives:

```
sage: Kc = dP6.Kaehler_cone(); Kc
4-d cone in 4-d lattice
sage: Kc.rays()
Divisor class [0, 1, 1, 0],
Divisor class [0, 0, 1, 1],
Divisor class [1, 1, 0, 0],
```

(continues on next page)
Divisor class \([1, 1, 1, 0]\),
Divisor class \([0, 1, 1, 1]\)
in Basis lattice of The toric rational divisor class group
of a 2-d CPR-Fano toric variety covered by 6 affine patches
\begin{verbatim}
sage: Kc.ray(1).lift()
V(x) + V(w)
\end{verbatim}

Given a divisor \(D\), we have an associated line bundle (or a reflexive sheaf, if \(D\) is not Cartier) \(\mathcal{O}(D)\). Its sections are:

\begin{verbatim}
sage: P2 = toric_varieties.P2()
sage: H = P2.divisor(0); H
V(x)
sage: H.sections()
(M(-1, 0), M(-1, 1), M(0, 0))
sage: H.sections_monomials()
(z, y, x)
\end{verbatim}

Note that the space of sections is always spanned by monomials. Therefore, we can grade the sections (as homogeneous monomials) by their weight under rescaling individual coordinates. This weight data amounts to a point of the dual lattice.

In the same way, we can grade cohomology groups by their cohomological degree and a weight:

\begin{verbatim}
sage: M = P2.fan().lattice().dual()
sage: H.cohomology(deg=0, weight=M(-1,0))
Vector space of dimension 1 over Rational Field
\end{verbatim}

Here is a more complicated example with \(h^1(dP_6, \mathcal{O}(D)) = 4\)

\begin{verbatim}
sage: D = dP6.divisor([0, 0, -1, 0, 2, -1])
sage: D.cohomology()
\{0: Vector space of dimension 0 over Rational Field,
1: Vector space of dimension 4 over Rational Field,
2: Vector space of dimension 0 over Rational Field\}
sage: D.cohomology(dim=True)
(0, 4, 0)
\end{verbatim}

AUTHORS:


\begin{verbatim}
sage.schemes.toric.divisor.ToricDivisor(toric_variety, arg=None, ring=None, check=True, reduce=True)
\end{verbatim}

Construct a divisor of \(toric\_variety\).

INPUT:

- \texttt{toric\_variety} – a \texttt{toric\_variety};
- \texttt{arg} – one of the following description of the toric divisor to be constructed:
  - \texttt{None} or \texttt{0} (the trivial divisor);
  - monomial in the homogeneous coordinates;
  - one-dimensional cone of the fan of \texttt{toric\_variety} or a lattice point generating such a cone;
- sequence of rational numbers, specifying multiplicities for each of the toric divisors.

- **ring** – usually either \( \mathbb{Z} \) or \( \mathbb{Q} \). The base ring of the divisor group. If **ring** is not specified, a coefficient ring suitable for \( arg \) is derived.

- **check** – bool (default: True). Whether to coerce coefficients into base ring. Setting it to False can speed up construction.

- **reduce** – reduce (default: True). Whether to combine common terms. Setting it to False can speed up construction.

**Warning:** The coefficients of the divisor must be in the base ring and the terms must be reduced. If you set **check=False** and/or **reduce=False** it is your responsibility to pass valid input data *arg*.

**OUTPUT:** A `sage.schemes.toric.divisor.ToricDivisor_generic`.

**EXAMPLES:**

```sage
sage: from sage.schemes.toric.divisor import ToricDivisor
sage: dP6 = toric_varieties.dP6()
sage: ToricDivisor(dP6, [(1,dP6.gen(2)), (1,dP6.gen(1))])
V(u) + V(y)
sage: ToricDivisor(dP6, (0,1,1,0,0,0), ring=QQ)
V(u) + V(y)
sage: dP6.inject_variables()
Defining x, u, y, v, z, w
sage: ToricDivisor(dP6, u + y)
Traceback (most recent call last):
  ...  
ValueError: u + y is not a monomial
sage: ToricDivisor(dP6, u*y)
V(u) + V(y)
```

We attempt to guess the correct base ring:

```sage
sage: ToricDivisor(dP6, [(1/2,u)])
1/2*V(u)
sage: _.parent()
Group of toric QQ-Weil divisors on
2-d CPR-Fano toric variety covered by 6 affine patches
sage: ToricDivisor(dP6, [(1/2,u), (1/2,u)])
V(u)
sage: _.parent()
Group of toric ZZ-Weil divisors on
2-d CPR-Fano toric variety covered by 6 affine patches
sage: ToricDivisor(dP6, [(u,u)])
```
class sage.schemes.toric.divisor.ToricDivisorGroup(toric_variety, base_ring)

Bases: DivisorGroup_generic

The group of (Q-T-Weil) divisors on a toric variety.

EXAMPLES:

```
sage: P2 = toric_varieties.P2()
sage: P2.toric_divisor_group()
Group of toric ZZ-Weil divisors on 2-d CPR-Fano toric variety covered by 3 affine patches
```

Element
alias of ToricDivisor_generic

base_extend(R)

Extend the scalars of self to R.

INPUT:

- R – ring.

OUTPUT:

- toric divisor group.

EXAMPLES:

```
sage: P2 = toric_varieties.P2()
sage: DivZZ = P2.toric_divisor_group()
sage: DivQQ = P2.toric_divisor_group(base_ring=QQ)
sage: DivZZ.base_extend(QQ) is DivQQ
True
```

gen(i)

Return the i-th generator of the divisor group.

INPUT:

- i – integer.

OUTPUT:

The divisor \( z_i = 0 \), where \( z_i \) is the i-th homogeneous coordinate.

EXAMPLES:

```
sage: P2 = toric_varieties.P2()
sage: TDiv = P2.toric_divisor_group()
sage: TDiv.gen(2)
V(z)
```
Schemes, Release 10.2

**gens()**

Return the generators of the divisor group.

**EXAMPLES:**

```python
sage: P2 = toric_varieties.P2()
sage: TDiv = P2.toric_divisor_group()
sage: TDiv.gens()
(V(x), V(y), V(z))
```

**ngens()**

Return the number of generators.

**OUTPUT:**

The number of generators of self, which equals the number of rays in the fan of the toric variety.

**EXAMPLES:**

```python
sage: P2 = toric_varieties.P2()
sage: TDiv = P2.toric_divisor_group()
sage: TDiv.ngens()
3
```

class sage.schemes.toric.divisor.ToricDivisor_generic(v, parent, check=True, reduce=True)

**Bases:** `Divisor_generic`

Construct a (toric Weil) divisor on the given toric variety.

**INPUT:**

- `v` – a list of tuples (multiplicity, coordinate).
- `check` – boolean. Type-check the entries of `v`, see `DivisorGroup_generic`.
- `reduce` – boolean. Combine coefficients in `v`, see `DivisorGroup_generic`.

**Warning:** Do not construct `ToricDivisor_generic` objects manually. Instead, use either the function `ToricDivisor()` or the method `divisor()` of toric varieties.

**EXAMPLES:**

```python
sage: dP6 = toric_varieties.dP6()
sage: ray = dP6.fan().ray(0)
sage: ray
N(0, 1)
sage: D = dP6.divisor(ray); D
V(x)
sage: D.parent()
Group of toric ZZ-Weil divisors on 2-d CPR-Fano toric variety covered by 6 affine patches
```

**Chern_character()**

Return the Chern character of the sheaf $\mathcal{O}(D)$ defined by the divisor $D$.

You can also use a shortcut `ch()`.
EXAMPLES:

```python
sage: dP6 = toric_varieties.dP6()
sage: N = dP6.fan().lattice()
sage: D3 = dP6.divisor(dP6.fan().cone_containing( N(0,1) ))
sage: D5 = dP6.divisor(dP6.fan().cone_containing( N(-1,-1) ))
sage: D6 = dP6.divisor(dP6.fan().cone_containing( N(0,-1) ))
sage: D = -D3 + 2*D5 - D6
sage: D.Chern_character() # needs sage.libs.singular
[5*w^2 + y - 2*v + w + 1]
sage: dP6.integrate(D.ch() * dP6.Td()) # needs sage.libs.singular
-4
```

**Chow_cycle**(ring=Integer Ring)

Returns the Chow homology class of the divisor.

**INPUT:**

- ring – Either ZZ (default) or QQ. The base ring of the Chow group.

**OUTPUT:**

The ChowCycle represented by the divisor.

**EXAMPLES:**

```python
sage: dP6 = toric_varieties.dP6()
sage: cone = dP6.fan(1)[5]
sage: D = dP6.divisor(cone); D
V(w)
sage: D.Chow_cycle() # needs sage.libs.singular
( 0 | -1, 0, 1, 1 | 0 )
sage: dP6.Chow_group()(cone)
( 0 | -1, 0, 1, 1 | 0 )
```

**Kodaira_map**(names=z)

Return the Kodaira map.

The Kodaira map is the rational map \( \Sigma \to \mathbb{P}^{n-1} \), where \( n \) equals the number of sections. It is defined by the monomial sections of the line bundle.

If the divisor is ample and the toric variety smooth or of dimension 2, then this is an embedding.

**INPUT:**

- names – string (optional; default 'z'). The variable names for the destination projective space.

**EXAMPLES:**

```python
sage: P1.<u,v> = toric_varieties.P1()
sage: D = -P1.K()
sage: D.Kodaira_map() # needs fpylll sage.libs.singular
Scheme morphism:
  From: 1-d CPR-Fano toric variety covered by 2 affine patches
  To:   Closed subscheme of Projective Space of dimension 2
```

(continues on next page)
over Rational Field defined by: \(-z1^2 + z0*z2\)
Defn: Defined on coordinates by sending \([u : v]\) to \((v^2 : u*v : u^2)\)

```
sage: dP6 = toric_varieties.dP6()
sage: D = -dP6.K()
sage: D.Kodaira_map(names='x')
# needs fpylll sage.libs.singular
```

Scheme morphism:
From: 2-d CPR-Fano toric variety covered by 6 affine patches
To: Closed subscheme of Projective Space of dimension 6
over Rational Field defined by:
\[-x1*x5 + x0*x6, -x2*x3 + x0*x5, -x1*x3 + x0*x4,
 x4*x5 - x3*x6, -x1*x2 + x0*x3, x3*x5 - x2*x6,
 x3*x4 - x1*x6, x3^2 - x1*x5, x2*x4 - x1*x5,
-x1*x5^2 + x2*x3*x6, -x1*x5^3 + x2^2*x6^2\]
Defn: Defined on coordinates by sending \([x : u : y : v : z : w]\) to
\((x*u^2*y^2*v : x^2*u^2*y*w : u*y^2*v^2*z : x*u*y*v*z*w :
 x^2*u*z*w^2 : y*v^2*z^2*w : x*v*z^2*w^2)\)

**ch()**

Return the Chern character of the sheaf \(O(D)\) defined by the divisor \(D\).

You can also use a shortcut \(ch()\).

**EXAMPLES:**

```
sage: dP6 = toric_varieties.dP6()
sage: N = dP6.fan().lattice()
sage: D3 = dP6.divisor(dP6.fan().cone_containing( N(0,1) ))
sage: D5 = dP6.divisor(dP6.fan().cone_containing( N(-1,-1) ))
sage: D6 = dP6.divisor(dP6.fan().cone_containing( N(0,-1) ))
sage: D = -D3 + 2*D5 - D6
sage: D.Chern_character()  # needs sage.libs.singular
[5*w^2 + y - 2*v + w + 1]
sage: dP6.integrate(D.ch() * dP6.Td())  # needs sage.libs.singular
-4
```

**coefficient(x)**

Return the coefficient of \(x\).

**INPUT:**

- \(x\) – one of the homogeneous coordinates, either given by the variable or its index.

**OUTPUT:**

The coefficient of \(x\).

**EXAMPLES:**

```
sage: P2 = toric_varieties.P2()
sage: D = P2.divisor((11,12,13)); D
11*V(x) + 12*V(y) + 13*V(z)
```

(continues on next page)
cohomology(weight=\texttt{None}, \texttt{deg}=\texttt{None}, \texttt{dim}=\texttt{False})

Return the cohomology of the line bundle associated to the Cartier divisor or reflexive sheaf associated to
the Weil divisor.

\textbf{Note:} The cohomology of a toric line bundle/reflexive sheaf is graded by the usual degree as well as by
the $M$-lattice.

\begin{itemize}
\item weight -- (optional) a point of the $M$-lattice.
\item deg -- (optional) the degree of the cohomology group.
\item dim -- boolean. If False (default), the cohomology groups are returned as vector spaces. If True, only the dimension of the vector space(s) is returned.
\end{itemize}

\textbf{OUTPUT:}

The vector space $H^{\text{deg}}(X, O(D))$ (if deg is specified) or a dictionary \{\text{degree}:\text{cohomology(degree)}\} of
all degrees between 0 and the dimension of the variety.

If weight is specified, return only the subspace $H^{\text{deg}}(X, O(D))_{\text{weight}}$ of the cohomology of the given weight.

If dim==True, the dimension of the cohomology vector space is returned instead of actual vector space.
Moreover, if deg was not specified, a vector whose entries are the dimensions is returned instead of a
dictionary.

\textbf{ALGORITHM:}

Roughly, Cech cohomology is used to compute the cohomology. For toric divisors, the local sections can be
chosen to be monomials (instead of general homogeneous polynomials), this is the reason for the extra grading by $m \in M$. General references would be [Ful1993], [CLS2011]. Here are some salient features of our implementation:

\begin{itemize}
\item First, a finite set of $M$-lattice points is identified that supports the cohomology. The toric divisor
determines a (polyhedral) chamber decomposition of $M_R$, see Section 9.1 and Figure 4 of [CLS2011]. The
cohomology vanishes on the non-compact chambers. Hence, the convex hull of the vertices of the
chamber decomposition contains all non-vanishing cohomology groups. This is returned by the
private method \_sheaf_cohomology_support().

It would be more efficient, but more difficult to implement, to keep track of all of the individual chambers.
We leave this for future work.

\item For each point $m \in M$, the weight-$m$ part of the cohomology can be rewritten as the cohomology of
a simplicial complex, see Exercise 9.1.10 of [CLS2011], [Per2007]. This is returned by the private
method \_sheaf_complex().

The simplicial complex is the same for all points in a chamber, but we currently do not make use of
this and compute each point $m \in M$ separately.
\end{itemize}
Finally, the cohomology (over $\mathbb{Q}$) of this simplicial complex is computed in the private method `_sheaf_cohomology()`. Summing over the supporting points $m \in M$ yields the cohomology of the sheaf $\mathcal{F}$.

**EXAMPLES:**

Example 9.1.7 of Cox, Little, Schenck: “Toric Varieties” [CLS2011]:

```python
sage: F = Fan(cones=[[0,1], [1,2], [2,3], [3,4], [4,5], [5,0]],
             rays=[[1,0], [1,1], [0,1], [-1,0], [-1,-1], [0,-1]])
sage: dP6 = ToricVariety(F)
sage: D3 = dP6.divisor(2)
sage: D5 = dP6.divisor(4)
sage: D6 = dP6.divisor(5)
sage: D = -D3 + 2*D5 - D6
sage: D.cohomology()
{0: Vector space of dimension 0 over Rational Field,
  1: Vector space of dimension 4 over Rational Field,
  2: Vector space of dimension 0 over Rational Field}
sage: D.cohomology(deg=1)
Vector space of dimension 4 over Rational Field
sage: M = F.dual_lattice()
sage: D.cohomology(weight=M(0,0))
{0: Vector space of dimension 0 over Rational Field,
  1: Vector space of dimension 1 over Rational Field,
  2: Vector space of dimension 0 over Rational Field}
sage: D.cohomology(weight=M(0,0), deg=1)
Vector space of dimension 1 over Rational Field
```

Note the different output options:

```python
sage: D.cohomology()
{0: Vector space of dimension 0 over Rational Field,
  1: Vector space of dimension 4 over Rational Field,
  2: Vector space of dimension 0 over Rational Field}
sage: D.cohomology(dim=True)
(0, 4, 0)
sage: D.cohomology(weight=M(0,0))
{0: Vector space of dimension 0 over Rational Field,
  1: Vector space of dimension 1 over Rational Field,
  2: Vector space of dimension 0 over Rational Field}
sage: D.cohomology(weight=M(0,0), dim=True)
(0, 1, 0)
sage: D.cohomology(deg=1)
Vector space of dimension 4 over Rational Field
sage: D.cohomology(deg=1, dim=True)
4
sage: D.cohomology(weight=M(0,0), deg=1)
Vector space of dimension 1 over Rational Field
sage: D.cohomology(weight=M(0,0), deg=1, dim=True)
1
```

---

16.4. Toric divisors and divisor classes 307
Here is a Weil (non-Cartier) divisor example:

```
sage: K = toric_varieties.Cube_nonpolyhedral().K()
sage: K.is_Weil()
True
sage: K.is_QQ_Cartier()
False
sage: K.cohomology(dim=True)
(0, 0, 0, 1)
```

**cohomology_class()**

Return the degree-2 cohomology class associated to the divisor.

**OUTPUT:**

The corresponding cohomology class as an instance of `CohomologyClass`. The cohomology class is the first Chern class of the associated line bundle \(\mathcal{O}(D)\).

**EXAMPLES:**

```
sage: dP6 = toric_varieties.dP6()
sage: D = dP6.divisor(dP6.fan().ray(0))
sage: D.cohomology_class()  # needs sage.libs.singular
[\text{y + v - w}]
```

**cohomology_support()**

Return the weights for which the cohomology groups do not vanish.

**OUTPUT:**

A tuple of dual lattice points. `self.cohomology(weight=m)` does not vanish if and only if \(m\) is in the output.

**Note:** This method is provided for educational purposes and it is not an efficient way of computing the cohomology groups.

**EXAMPLES:**

```
sage: F = Fan(cones=[(0,1), (1,2), (2,3), (3,4), (4,5), (5,0)],
        ....: rays=[(1,0), (1,1), (0,1), (-1,0), (-1,-1), (0,-1)])
sage: dP6 = ToricVariety(F)
sage: D3 = dP6.divisor(2)
sage: D5 = dP6.divisor(4)
sage: D6 = dP6.divisor(5)
sage: D = -D3 + 2*D5 - D6
sage: D.cohomology_support()
(M(0, 0), M(1, 0), M(2, 0), M(1, 1))
```

**divisor_class()**

Return the linear equivalence class of the divisor.

**OUTPUT:**

The class of the divisor in \(\text{Cl}(X) \otimes \mathbb{Q}\) as an instance of `ToricRationalDivisorClassGroup`.

**EXAMPLES:**

---
sage: dP6 = toric_varieties.dP6()
sage: D = dP6.divisor(0)
sage: D.divisor_class()
Divisor class [1, 0, 0, 0]

function_value(point)

Return the value of the support function at point.

Let $X$ be the ambient toric variety of self, $\Sigma$ the fan associated to $X$, and $N$ the ambient lattice of $\Sigma$.

INPUT:

- point – either an integer, interpreted as the index of a ray of $\Sigma$, or a point of the lattice $N$.

OUTPUT: An integer or a rational number.

EXAMPLES:

sage: P2 = toric_varieties.P2()
sage: D = P2.divisor([11,22,44])  # total degree 77
sage: D.function_value(0)
11
sage: N = P2.fan().lattice()
sage: D.function_value( N(1,1) )
33
sage: D.function_value( P2.fan().ray(0) )
11

is_Cartier()

Return whether the divisor is a Cartier-divisor.

Note: The sheaf $\mathcal{O}(D)$ associated to the given divisor $D$ is a line bundle if and only if the divisor is Cartier.

EXAMPLES:

sage: X = toric_varieties.P4_11169()
sage: D = X.divisor(3)
sage: D.is_Cartier()
False
sage: D.is_QQ_Cartier()
True

is_QQ_Cartier()

Return whether the divisor is a $\mathcal{Q}$-Cartier divisor.

A $\mathcal{Q}$-Cartier divisor is a divisor such that some multiple of it is Cartier.

EXAMPLES:

sage: X = toric_varieties.P4_11169()
sage: D = X.divisor(3)
sage: D.is_QQ_Cartier()
True
sage: X = toric_varieties.Cube_face_fan()

(continues on next page)
sage: D = X.divisor(3)
sage: D.is_QQ_Cartier()
False

is_QQ_Weil()
Return whether the divisor is a $\mathbb{Q}$-Weil-divisor.

Note: This function returns always True since ToricDivisor can only describe $\mathbb{Q}$-Weil divisors.

EXAMPLES:

sage: P2 = toric_varieties.P2()
sage: D = P2.divisor([1,2,3])
sage: D.is_QQ_Weil()
True
sage: (D/2).is_QQ_Weil()
True

is_Weil()
Return whether the divisor is a Weil-divisor.

EXAMPLES:

sage: P2 = toric_varieties.P2()
sage: D = P2.divisor([1,2,3])
sage: D.is_Weil()
True
sage: (D/2).is_Weil()
False

is_ample()
Return whether a $\mathbb{Q}$-Cartier divisor is ample.

OUTPUT:

• True if the divisor is in the ample cone, False otherwise.

Note:

• For a $\mathbb{Q}$-Cartier divisor, some positive integral multiple is Cartier. We return whether this associated divisor is ample, i.e. corresponds to an ample line bundle.

• In the orbifold case, the ample cone is an open and full-dimensional cone in the rational divisor class group ToricRationalDivisorClassGroup.

• If the variety has worse than orbifold singularities, the ample cone is a full-dimensional cone within the (not full-dimensional) subspace spanned by the Cartier divisors inside the rational (Weil) divisor class group, that is, ToricRationalDivisorClassGroup. The ample cone is then relative open (open in this subspace).

• See also is_nef().

• A toric divisor is ample if and only if its support function is strictly convex.

EXAMPLES:
sage: P2 = toric_varieties.P2()
sage: K = P2.K()
sage: (+K).is_ample()  
False
sage: (0*K).is_ample()  
False
sage: (-K).is_ample()  
True

Example 6.1.3, 6.1.11, 6.1.17 of [CLS2011]:

```python
sage: from itertools import product
sage: fan = Fan(cones=[(0,1), (1,2), (2,3), (3,0)],
.....: rays=[(-1,2), (0,1), (1,0), (0,-1)])
sage: F2 = ToricVariety(fan, 'u1, u2, u3, u4')
sage: D(a,b): return a*F2.divisor(2) + b*F2.divisor(3)
sage: [ (a,b) for a,b in product(range(-3,3), repeat=2)
.....: if D(a,b).is_ample() ]
[(1, 1), (1, 2), (2, 1), (2, 2)]
sage: [ (a,b) for a,b in product(range(-3,3), repeat=2)
.....: if D(a,b).is_nef() ]
[(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)]
```

A (worse than orbifold) singular Fano threefold:

```python
sage: points = [(1,0,0),(0,1,0),(0,0,1),(-2,0,-1),(-2,-1,0),(-3,-1,-1),(1,1,1)]
sage: facets = [[0,1,3],[0,1,6],[0,2,4],[0,2,6],[0,3,5],[0,4,5],[1,2,3,4,5,6]]
sage: X = ToricVariety(Fan(cones=facets, rays=points))
sage: X.rational_class_group().dimension()
4
sage: X.Kaehler_cone().rays()
Divisor class [1, 0, 0, 0]
in Basis lattice of The toric rational divisor class group
of a 3-d toric variety covered by 7 affine patches
sage: antiK = -X.K()
sage: antiK.divisor_class()
Divisor class [2, 0, 0, 0]
sage: antiK.is_ample()
True
```

**is_integral()**

Return whether the coefficients of the divisor are all integral.

**EXAMPLES:**

```python
sage: P2 = toric_varieties.P2()
sage: DZZ = P2.toric_divisor_group(base_ring=ZZ).gen(0); DZZ
V(x)
sage: DQQ = P2.toric_divisor_group(base_ring=QQ).gen(0); DQQ
V(x)
sage: DZZ.is_integral()
True
sage: DQQ.is_integral()
True
```
is_nef()
Return whether a Q-Cartier divisor is nef.

OUTPUT:
• True if the divisor is in the closure of the ample cone, False otherwise.

Note:
• For a Q-Cartier divisor, some positive integral multiple is Cartier. We return whether this associated divisor is nef.
• The nef cone is the closure of the ample cone.
• See also is_ample().
• A toric divisor is nef if and only if its support function is convex (but not necessarily strictly convex).
• A toric Cartier divisor is nef if and only if its linear system is basepoint free.

EXAMPLES:
```python
sage: P2 = toric_varieties.P2()
sage: K = P2.K()
sage: (+K).is_nef()
False
sage: (0*K).is_nef()
True
sage: (-K).is_nef()
True
```

Example 6.1.3, 6.1.11, 6.1.17 of [CLS2011]:
```python
sage: from itertools import product
sage: fan = Fan(cones=[((0,1), (1,2), (2,3), (3,0)],
.....: rays=[(-1,2), (0,1), (1,0), (0,-1)])
sage: F2 = ToricVariety(fan, 'u1, u2, u3, u4')
sage: def D(a,b):
.....: return a*F2.divisor(2) + b*F2.divisor(3)
sage: [ (a,b) for a,b in product(range(-3,3), repeat=2)
.....: if D(a,b).is_ample() ]
[(1, 1), (1, 2), (2, 1), (2, 2)]
sage: [ (a,b) for a,b in product(range(-3,3), repeat=2)
.....: if D(a,b).is_nef() ]
[(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)]
```

m(cone)
Return $m_\sigma$ representing $\phi_D$ on cone.

Let $X$ be the ambient toric variety of this divisor $D$ associated to the fan $\Sigma$ in lattice $N$. Let $M$ be the lattice dual to $N$. Given the cone $\sigma = \langle v_1, \ldots, v_k \rangle$ in $\Sigma$, this method searches for a vector $m_\sigma \in MQ$ such that $\phi_D(v_i) = \langle m_\sigma, v_i \rangle$ for all $i = 1, \ldots, k$, where $\phi_D$ is the support function of $D$.

INPUT:
• cone – A cone in the fan of the toric variety.

OUTPUT:
• If possible, a point of lattice $M$. 

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• If the dual vector cannot be chosen integral, a rational vector is returned.
• If there is no such vector (i.e. \texttt{self} is not even a \texttt{Q-Cartier divisor}), a \texttt{ValueError} is raised.

\textbf{EXAMPLES:}

```python
sage: F = Fan(cones=[(0,1,2,3), (0,1,4)],
....:    rays=[(1,1,1), (1,-1,1), (1,-1,-1), (1,1,-1), (0,0,1)])
sage: X = ToricVariety(F)
sage: square_cone = X.fan().cone_containing(0,1,2,3)
sage: triangle_cone = X.fan().cone_containing(0,1,4)
sage: ray = X.fan().cone_containing(0)
sage: QQ_Cartier = X.divisor([2,2,1,1,1])
sage: QQ_Cartier.m(ray)
\texttt{M(0, 2, 0)}
sage: QQ_Cartier.m(square_cone)
\texttt{(3/2, 0, 1/2)}
sage: QQ_Cartier.m(triangle_cone)
\texttt{M(1, 0, 1)}
sage: QQ_Cartier.m(Cone(triangle_cone))
\texttt{M(1, 0, 1)}
sage: Weil = X.divisor([1,1,1,0,0])
sage: Weil.m(square_cone)
Traceback (most recent call last):
...
\texttt{ValueError: V(z0) + V(z1) + V(z2) is not QQ-Cartier, cannot choose a dual vector on 3-d cone}
```

\texttt{monomial(point)}

Return the monomial in the homogeneous coordinate ring associated to the \texttt{point} in the dual lattice.

\textbf{INPUT:}

• \texttt{point} – a point in \texttt{self.variety().fan().dual_lattice()}

\textbf{OUTPUT:}

For a fixed divisor \(D\), the sections are generated by monomials in \texttt{ToricVariety.coordinate_ring}.
Alternatively, the monomials can be described as \(M\)-lattice points in the polyhedron \(D.polyhedron()\).
This method converts the points \(m \in M\) into homogeneous polynomials.

\textbf{EXAMPLES:}

```python
sage: P2 = toric_varieties.P2()
sage: O3_P2 = -P2.K()
sage: M = P2.fan().dual_lattice()
sage: O3_P2.monomial( M(0,0) )
x*y*z
```

\texttt{move_away_from(cone)}

Move the divisor away from the orbit closure of \texttt{cone}.

\textbf{INPUT:}

• A cone of the fan of the toric variety.
OUTPUT:

A (rationally equivalent) divisor that is moved off the orbit closure of the given cone.

Note: A divisor that is Weil but not Cartier might be impossible to move away. In this case, a ValueError is raised.

EXAMPLES:

```
sage: F = Fan(cones=[[(0,1,2,3), (0,1,4)],
       ....:         rays=[(1,1,1), (1,-1,1), (1,-1,-1), (1,1,-1), (0,0,1)]
       sage: X = ToricVariety(F)
       sage: square_cone = X.fan().cone_containing(0,1,2,3)
       sage: triangle_cone = X.fan().cone_containing(0,1,4)
       sage: line_cone = square_cone.intersection(triangle_cone)
       sage: Cartier = X.divisor([2,2,1,1,1])
       sage: Cartier
       2*V(z0) + 2*V(z1) + V(z2) + V(z3) + V(z4)
       sage: Cartier.move_away_from(line_cone)
       3*V(z2) + 3*V(z3) - V(z4)
       sage: QQ_Weil = X.divisor([1,0,1,1,0])
       sage: QQ_Weil.move_away_from(line_cone)
       2*V(z2) + V(z3) - 1/2*V(z4)
```
sage: fan = Fan(cones=[(0,1), (1,2), (2,3), (3,0)],
    ....: rays=[(-1,2), (0,1), (1,0), (0,-1)])
sage: F2 = ToricVariety(fan, 'u1, u2, u3, u4')
sage: D = F2.divisor(3)
sage: D.polyhedron().Vrepresentation()
(A vertex at (0, 0), A vertex at (2, 1), A vertex at (0, 1))
sage: Dprime = F2.divisor(1) + D
sage: Dprime.polyhedron().Vrepresentation()
(A vertex at (2, 1), A vertex at (0, 1), A vertex at (0, 0))
sage: D.is_ample()
False
sage: D.is_nef()
True
sage: Dprime.is_nef()
False

A more complicated example where $P_D$ is not a lattice polytope:

sage: X = toric_varieties.BCdlOG_base()
sage: antiK = -X.K()
sage: P_D = antiK.polyhedron()
sage: P_D
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 8 vertices
sage: P_D.Vrepresentation()
(A vertex at (1, -1, 0), A vertex at (1, -3, 1),
 A vertex at (1, 1, 1), A vertex at (-5, 1, 1),
 A vertex at (1, 1, -1/2), A vertex at (1, 1/2, -1/2),
 A vertex at (-1, -1, 0), A vertex at (-5, -3, 1))
sage: P_D.Hrepresentation()
(An inequality (-1, 0, 0) x + 1 >= 0, An inequality (0, -1, 0) x + 1 >= 0,
 An inequality (0, 0, -1) x + 1 >= 0, An inequality (1, 0, 4) x + 1 >= 0,
 An inequality (0, 1, 3) x + 1 >= 0, An inequality (0, 1, 2) x + 1 >= 0)
sage: P_D.integral_points()
((-1, -1, 0), (0, -1, 0), (1, -1, 0), (-1, 0, 0), (0, 0, 0),
 (1, 0, 0), (-1, 1, 0), (0, 1, 0), (1, 1, 0), (-5, -3, 1),
 (-4, -3, 1), (-3, -3, 1), (-2, -3, 1), (-1, -3, 1), (0, -3, 1),
 (1, -3, 1), (-5, -2, 1), (-4, -2, 1), (-3, -2, 1), (-2, -2, 1),
 (-1, -2, 1), (0, -2, 1), (1, -2, 1), (-5, -1, 1), (-4, -1, 1),
 (-3, -1, 1), (-2, -1, 1), (-1, -1, 1), (0, -1, 1), (1, -1, 1),
 (-5, 0, 1), (-4, 0, 1), (-3, 0, 1), (-2, 0, 1), (-1, 0, 1),
 (0, 0, 1), (1, 0, 1), (-5, 1, 1), (-4, 1, 1), (-3, 1, 1),
 (-2, 1, 1), (-1, 1, 1), (0, 1, 1), (1, 1, 1))

sections()

Return the global sections (as points of the $M$-lattice) of the line bundle (or reflexive sheaf) associated to the divisor.

OUTPUT: A tuple of points of lattice $M$.

EXAMPLES:

sage: P2 = toric_varieties.P2()
sage: P2.fan().nrays()
3

(continues on next page)
The divisor can be non-nef yet still have sections:

\[
sage: \text{rays} = [(1,0,0),(0,1,0),(0,0,1),(-2,0,-1),(-2,-1,0),(-3,-1,-1),(1,1,1),(-1,0,0)]
\]
\[
sage: \text{cones} = [[0,1,3],[0,1,6],[0,2,4],[0,2,6],[0,3,5],[0,4,5],[1,3,7],[1,6,7],
\]
\[
-2,4,7],[2,6,7],[3,5,7],[4,5,7]]
\]
\[
sage: X = ToricVariety(Fan(rays=\text{rays}, cones=\text{cones}))
\]
\[
sage: D = X.divisor(2); D
\]
\[
V(z_2)
\]
\[
sage: D.is_nef()
\]
False
\]
\[
sage: D.sections()
\]
(M(0, 0, 0),)
\]
\[
sage: D.cohomology(dim=True)
\]
(1, 0, 0, 0)

sections_monomials()

Return the global sections of the line bundle associated to the Cartier divisor.

The sections are described as monomials in the generalized homogeneous coordinates.

OUTPUT:

• tuple of monomials in the coordinate ring of \texttt{self}.

EXAMPLES:

\[
sage: P2 = toric_varieties.P2()
\]
\[
sage: P2.fan().nrays()
\]
3
\]
\[
sage: P2.divisor(0).sections_monomials()
\]
(z, y, x)
\]
\[
sage: P2.divisor(1).sections_monomials()
\]
(z, y, x)
\]
\[
sage: P2.divisor(2).sections_monomials()
\]
(z, y, x)

From [Cox] page 38:

\[
sage: lp = LatticePolytope([(1,0), (1,1), (0,1), (-1,0), (0,-1)])
\]
\[
sage: lp
\]
2-d reflexive polytope #5 in 2-d lattice \texttt{M}
\]
\[
sage: dP7 = ToricVariety(FaceFan(lp), \text{'x1, x2, x3, x4, x5'})
\]
\[
sage: AK = -dP7.K()
\]
\[
sage: AK.sections()
\]
(N(-1, 0), N(-1, 1), N(0, -1), N(0, 0),
N(0, 1), N(1, -1), N(1, 0), N(1, 1))
class sage.schemes.toric.divisor.ToricRationalDivisorClassGroup(toric_variety)

Bases: FreeModule_ambient_field, UniqueRepresentation

The rational divisor class group of a toric variety.

The **T-Weil divisor class group** $\mathcal{C}_l(X)$ of a toric variety $X$ is a finitely generated abelian group and can contain torsion. Its rank equals the number of rays in the fan of $X$ minus the dimension of $X$.

The **rational divisor class group** is $\mathcal{C}_l(X) \otimes \mathbb{Q}$ and never includes torsion. If $X$ is smooth, this equals the **Picard group** $\text{Pic}(X)$, whose elements are the isomorphism classes of line bundles on $X$. The group law (which we write as addition) is the tensor product of the line bundles. The Picard group of a toric variety is always torsion-free.

**Warning:** Do not instantiate this class yourself. Use `rational_class_group()` method of toric varieties if you need the divisor class group. Or you can obtain it as the parent of any divisor class constructed, for example, via `ToricDivisor_generic.divisor_class()`.

**INPUT:**

- `toric_variety` – toric variety `<sage.schemes.toric.variety.ToricVariety_field`

**OUTPUT:**

- rational divisor class group of a toric variety.

**EXAMPLES:**

```
sage: P2 = toric_varieties.P2()
sage: P2.rational_class_group()
The toric rational divisor class group of a 2-d CPR-Fano toric variety covered by 3 affine patches
sage: D = P2.divisor(0); D V(x)
sage: Dclass = D.divisor_class(); Dclass Divisor class [1]
sage: Dclass.lift()
V(y)
sage: Dclass.parent()
The toric rational divisor class group of a 2-d CPR-Fano toric variety covered by 3 affine patches
```

**Element**

alias of `ToricRationalDivisorClass`

class sage.schemes.toric.divisor.ToricRationalDivisorClassGroup_basis_lattice(group)

Bases: FreeModule_ambient_pid

Construct the basis lattice of the group.

**INPUT:**

- `group` – toric rational divisor class group.
OUTPUT:

- the basis lattice of group.

EXAMPLES:

```python
sage: P1xP1 = toric_varieties.P1xP1()
sage: L = P1xP1.Kaehler_cone().lattice()
sage: L
Basis lattice of The toric rational divisor class group of a 2-d CPR-Fano toric variety covered by 4 affine patches
sage: L.basis()
[Divisor class [1, 0],
 Divisor class [0, 1]]
```

**Element**

alias of `ToricRationalDivisorClass`

```python
sage.schemes.toric.divisor.is_ToricDivisor(x)
```

Test whether `x` is a toric divisor.

INPUT:

- `x` – anything.

OUTPUT:

- `True` if `x` is an instance of `ToricDivisor_generic` and `False` otherwise.

EXAMPLES:

```python
sage: from sage.schemes.toric.divisor import is_ToricDivisor
sage: is_ToricDivisor(1)
False
sage: P2 = toric_varieties.P2()
sage: D = P2.divisor(0); D
V(x)
sage: is_ToricDivisor(D)
True
```

16.5 Toric rational divisor classes

This module is a part of the framework for **toric varieties**.

AUTHORS:

- Volker Braun and Andrey Novoseltsev (2010-09-05): initial version.

```python
class sage.schemes.toric.divisor_class.ToricRationalDivisorClass
```

Create a toric rational divisor class.

**Warning:** You probably should not construct divisor classes explicitly.
INPUT:
• same as for Vector_rational_dense.

OUTPUT:
• toric rational divisor class.

lift()
Return a divisor representing this divisor class.

OUTPUT:
An instance of ToricDivisor representing self.

EXAMPLES:

```
sage: X = toric_varieties.Cube_nonpolyhedral()
sage: D = X.divisor([0,1,2,3,4,5,6,7]); D
V(z1) + 2*V(z2) + 3*V(z3) + 4*V(z4) + 5*V(z5) + 6*V(z6) + 7*V(z7)
sage: D.divisor_class()
Divisor class [29, 6, 8, 10, 0]
sage: Dequiv = D.divisor_class().lift(); Dequiv
15*V(z1) - 11*V(z2) - 9*V(z5) + 19*V(z6) + 10*V(z7)
sage: Dequiv == D
False
sage: Dequiv.divisor_class() == D.divisor_class()
True
```

sage.schemes.toric.divisor_class.is_ToricRationalDivisorClass(x)
Check if x is a toric rational divisor class.

INPUT:
• x – anything.

OUTPUT:
• True if x is a toric rational divisor class, False otherwise.

EXAMPLES:

```
sage: from sage.schemes.toric.divisor_class import is_ToricRationalDivisorClass
sage: is_ToricRationalDivisorClass(1)
False
sage: dP6 = toric_varieties.dP6()
sage: D = dP6.rational_class_group().gen(0); D
Divisor class [1, 0, 0, 0]
sage: is_ToricRationalDivisorClass(D)
True
```
16.6 The Chow group of a toric variety

In general, the Chow group is an algebraic version of a homology theory. That is, the objects are formal linear combinations of submanifolds modulo relations. In particular, the objects of the Chow group are formal linear combinations of algebraic subvarieties and the equivalence relation is rational equivalence. There is no relative version of the Chow group, so it is not a generalized homology theory.

The Chow groups of smooth or mildly singular toric varieties are almost the same as the homology groups:

- For smooth toric varieties, \( A_k(X) = H_{2k}(X, \mathbb{Z}) \). While they are the same, using the cohomology ring instead of the Chow group will be much faster! The cohomology ring does not try to keep track of torsion and uses Groebner bases to encode the cup product.
- For simplicial toric varieties, \( A_k(X) \otimes \mathbb{Q} = H_{2k}(X, \mathbb{Q}) \).

Note that in these cases the odd-dimensional (co)homology groups vanish. But for sufficiently singular toric varieties the Chow group differs from the homology groups (and the odd-dimensional homology groups no longer vanish). For singular varieties the Chow group is much easier to compute than the (co)homology groups.

The toric Chow group of a toric variety is the Chow group generated by the toric subvarieties, that is, closures of orbits under the torus action. These are in one-to-one correspondence with the cones of the fan and, therefore, the toric Chow group is a quotient of the free Abelian group generated by the cones. In particular, the toric Chow group has finite rank. One can show [FMSS1995] that the toric Chow groups equal the “full” Chow group of a toric variety, so there is no need to distinguish these in the following.

AUTHORS:

- Volker Braun (2010-08-09): Initial version

REFERENCES:

- Wikipedia article Chow_ring
- [FMSS1995]
- [Ful1993], Chapter 5.1, “Chow Groups”

EXAMPLES:

```python
sage: X = toric_varieties.Cube_deformation(7)
sage: X.is_smooth()
False
sage: X.is_orbifold()
False
sage: A = X.Chow_group()
sage: A.degree()
(Z, C_7, C_2 \times C_2 \times Z^5, Z)
sage: A.degree(2).ngens()
7
sage: a = sum( A.gen(i) * (i+1) for i in range(A.ngens()) )  # an element of A
sage: a  # long time (2s on sage.math, 2011)
( 9 | 1 mod 7 | 1 mod 2, 0 mod 2, 4, 5, 6, 7, 8 | 3 )
```

The Chow group elements are printed as \( ( a_0 \mid a_1 \mod 7 \mid a_2 \mod 2, a_3 \mod 2, a_4, a_5, a_6, a_7, a_8 \mid a_9 ) \), which denotes the element of the Chow group in the same basis as \( A.\text{degree}() \). The \( | \) separates individual degrees, so the example means:

- The degree-0 part is \( 3 \in \mathbb{Z} \).
- The degree-1 part is \( 1 \in \mathbb{Z}_7 \).
The torsion of the degree-2 Chow group is \((0, 1) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2\).

The free part of the degree-2 Chow group is \((4, 5, 6, 7, 8) \in \mathbb{Z}^5\).

The degree-3 part is \(9 \in \mathbb{Z}\).

Note that the generators \(A.gens()\) are not sorted in any way. In fact, they may be of mixed degree. Use \(A.gens\text{(degree=d)}\) to obtain the generators in a fixed degree \(d\). See \texttt{ChowGroup\_class.gens()} for more details.

Cones of toric varieties can determine their own Chow cycle:

```
sage: A = X.Chow_group(); A
Chow group of 3-d toric variety covered by 6 affine patches
sage: cone = X.fan(dim=2)[3]; cone
2-d cone of Rational polyhedral fan in 3-d lattice \(\mathbb{N}\)
sage: A_cone = A(cone); A_cone
( 0 | 6 mod 7 | 0 mod 2, 0 mod 2, 0, 0, 0, 0, 0 | 0 )
sage: A_cone.degree()
1
sage: 2 * A_cone
( 0 | 5 mod 7 | 0 mod 2, 0 mod 2, 0, 0, 0, 0, 0 | 0 )
sage: A_cone + A.gens(0)
( 0 | 6 mod 7 | 1 mod 2, 0 mod 2, 0, 0, 0, 0, 0 | 0 )
```

Chow cycles can be of mixed degrees:

```
sage: mixed = sum(A.gens()); mixed
( 1 | 4 mod 7 | 1 mod 2, 1 mod 2, 1, 1, 1, 1, 1 | 1 )
sage: mixed.project_to_degree(1)
( 0 | 4 mod 7 | 0 mod 2, 0 mod 2, 0, 0, 0, 0 | 0 )
sage: sum( mixed.project_to_degree(i) for i in range(X.dimension()+1) ) == mixed
True
```

class \texttt{sage.schemes.toric.chow_group.ChowCycle}(parent, v, check=True)

Bases: \texttt{FGP\_Element}

The elements of the Chow group.

\textbf{Warning:} Do not construct \texttt{ChowCycle} objects manually. Instead, use the parent \texttt{ChowGroup} to obtain generators or Chow cycles corresponding to cones of the fan.

**EXAMPLES:**

```
sage: P2 = toric_varieties.P2()
sage: A = P2.Chow_group()
sage: A.gens()
(( 0 | 0 | 1 ), ( 0 | 1 | 0 ), ( 1 | 0 | 0 ))
sage: cone = P2.fan(1)[0]
sage: A(cone)
( 0 | 1 | 0 )
sage: A( Cone([[1,0]]) )
( 0 | 1 | 0 )
```

\texttt{cohomology\_class()} 
Return the (Poincaré-dual) cohomology class.
Consider a simplicial cone of the fan, that is, a $d$-dimensional cone spanned by $d$ rays. Take the product of the corresponding $d$ homogeneous coordinates. This monomial represents a cohomology classes of the toric variety $X$, see \texttt{cohomology\_ring()}. Its cohomological degree is $2d$, which is the same degree as the Poincaré-dual of the (real) $\dim(X) - 2d$-dimensional torus orbit associated to the simplicial cone. By linearity, we can associate a cohomology class to each Chow cycle of a simplicial toric variety.

If the toric variety is compact and smooth, the associated cohomology class actually is the Poincaré dual (over the integers) of the Chow cycle. In particular, integrals of dual cohomology classes perform intersection computations.

If the toric variety is compact and has at most orbifold singularities, the torsion parts in cohomology and the Chow group can differ. But they are still isomorphic as rings over the rationals. Moreover, the normalization of integration (\texttt{volume\_class}) and \texttt{count\_points()} are chosen to agree.

**OUTPUT:**

The \texttt{CohomologyClass} which is associated to the Chow cycle.

If the toric variety is not simplicial, that is, has worse than orbifold singularities, there is no way to associate a cohomology class of the correct degree. In this case, \texttt{cohomology\_class()} raises a \texttt{ValueError}.

**EXAMPLES:**

```python
sage: # needs sage.libs.singular
sage: dP6 = toric_varieties.dP6()
sage: cone = dP6.fan().cone_containing(2,3)
sage: HH = dP6.cohomology_ring()
sage: A = dP6.Chow_group()
sage: HH(cone)
[-w^2]
sage: A(cone)
( 1 | 0, 0, 0, 0 | 0 )
sage: A(cone).cohomology_class()
[-w^2]
```

Here is an example of a toric variety with orbifold singularities, where we can also use the isomorphism with the rational cohomology ring:

```python
sage: # needs sage.libs.singular
sage: WP4 = toric_varieties.P4_11169()
sage: A = WP4.Chow_group()
sage: HH = WP4.cohomology_ring()
sage: cone3d = Cone([(0,0,1,0), (0,0,0,1), (-9,-6,-1,-1)])
sage: A(cone3d)
( 0 | -1 | 0 | 0 | 0 )
sage: HH(cone3d)
[3*z4^3]
sage: D = -WP4.K() # the anticanonical divisor
sage: A(D)
( 0 | 0 | 0 | -18 | 0 )
sage: HH(D)
[18*z4]
sage: WP4.integrate( A(cone3d).cohomology_class() * D.cohomology_class() )
1
sage: WP4.integrate( HH(cone3d) * D.cohomology_class() )
1
```

(continues on next page)
The degree of the Chow cycle.

OUTPUT:

Integer. The complex dimension of the subvariety representing the Chow cycle. Raises a `ValueError` if the Chow cycle is a sum of mixed degree cycles.

EXAMPLES:
intersection_with_divisor(divisor)

Intersect the Chow cycle with divisor.


INPUT:

- divisor – a ToricDivisor that can be moved away from the Chow cycle. For example, any Cartier divisor. See also ToricDivisor.move_away_from.

OUTPUT:

A new ChowCycle. If the divisor is not Cartier then this method potentially raises a ValueError, indicating that the divisor cannot be made transversal to the Chow cycle.

EXAMPLES:

```python
sage: dP6 = toric_varieties.dP6()
sage: cone = dP6.fan().cone_containing(2); cone
1-d cone of Rational polyhedral fan in 2-d lattice N
sage: D = dP6.divisor(cone); D
V(y)
sage: A = dP6.Chow_group()
sage: A(cone)
( 0 | 0, 0, 1, 0 | 0 )
sage: intersection = A(cone).intersection_with_divisor(D); intersection
( -1 | 0, 0, 0, 0 | 0 )
sage: intersection.count_points()
-1
```

You can do the same computation over the rational Chow group since there is no torsion in this case:

```python
sage: A_QQ = dP6.Chow_group(base_ring=QQ)
sage: A_QQ(cone)
( 0 | 0, 0, 1 | 0 )
sage: intersection_QQ = A_QQ(cone).intersection_with_divisor(D); intersection
( -1 | 0, 0, 0 | 0 )
sage: intersection_QQ.count_points()
-1
```

project_to_degree(degree)

Project a (mixed-degree) Chow cycle to the given degree.

INPUT:

- degree – integer. The degree to project to.

OUTPUT:
The projection of the Chow class to the given degree as a new \emph{ChowCycle} of the same Chow group.

**EXAMPLES:**

```python
sage: A = toric_varieties.P2().Chow_group()
sage: cycle = 10*A.gen(0) + 11*A.gen(1) + 12*A.gen(2); cycle
( 12 | 11 | 10 )
sage: cycle.project_to_degree(2)
( 0 | 0 | 10 )
```

```
class sage.schemes.toric.chow_group.ChowGroupFactory

Bases: UniqueFactory

Factory for \emph{ChowGroup_class}.

\textbf{create_key_and_extra_args}(\textit{toric\_variety, base\_ring=Integer Ring, check=True})

Create a key that uniquely determines the \emph{ChowGroup_class}.

INPUT:

- \textit{toric\_variety} – a toric variety.
- \textit{base\_ring} – either \texttt{Z} (default) or \texttt{Q}. The coefficient ring of the Chow group.
- \textit{check} – boolean (default: \texttt{True}).

**EXAMPLES:**

```python
sage: from sage.schemes.toric.chow_group import *
sage: P2 = toric_varieties.P2()
sage: ChowGroup(P2, ZZ, check=True) == ChowGroup(P2, ZZ, check=False)  # indirect doctest
True
```

\textbf{create_object}(\textit{version, key, **extra\_args})

Create a \emph{ChowGroup_class}.

INPUT:

- \textit{version} – object version. Currently not used.
- \textit{key} – a key created by \textbf{create_key_and_extra_args}.
- \textit{**extra\_args} – Currently not used.

**EXAMPLES:**

```python
sage: from sage.schemes.toric.chow_group import *
sage: P2 = toric_varieties.P2()
sage: ChowGroup(P2)  # indirect doctest
Chow group of 2-d CPR-Fano toric variety covered by 3 affine patches
```

```
class sage.schemes.toric.chow_group.ChowGroup_class(\textit{toric\_variety, base\_ring, check})

Bases: FGP_Module_class, WithEqualityById

The Chow group of a toric variety.

**EXAMPLES:**

16.6. The Chow group of a toric variety
sage: P2 = toric_varieties.P2()
sage: from sage.schemes.toric.chow_group import ChowGroup_class
sage: A = ChowGroup_class(P2, ZZ, True); A
Chow group of 2-d CPR-Fano toric variety covered by 3 affine patches
sage: A.an_element()
(0 | 0 | 1)

Element
alias of ChowCycle

coordinate_vector(chow_cycle, degree=None, reduce=True)

Return the coordinate vector of the chow_cycle.

INPUT:

• chow_cycle – a ChowCycle.
• degree – None (default) or an integer.
• reduce – boolean (default: True). Whether to reduce modulo the invariants.

OUTPUT:

• If degree is None (default), the coordinate vector relative to the basis self.gens() is returned.
• If some integer degree=d is specified, the chow_cycle is projected to the given degree and the coordinate vector relative to the basis self.gens(degree=d) is returned.

EXAMPLES:

sage: A = toric_varieties.P2().Chow_group()
sage: a = A.gen(0) + 2*A.gen(1) + 3*A.gen(2)
sage: A.coordinate_vector(a)
(1, 2, 3)
sage: A.coordinate_vector(a, degree=1)
(2)

degree(k=None)

Return the degree-k Chow group.

INPUT:

• k – an integer or None (default). The degree of the Chow group.

OUTPUT:

• if k was specified, the Chow group $A_k$ as an Abelian group.
• if k was not specified, a tuple containing the Chow groups in all degrees.

Note:

• For a smooth toric variety, this is the same as the Poincaré-dual cohomology group $H^{d-2k}(X, \mathbb{Z})$.
• For a simplicial toric variety ("orbifold"), $A_k(X) \otimes \mathbb{Q} = H^{d-2k}(X, \mathbb{Q})$.

EXAMPLES:

Four exercises from page 65 of [Ful1993]. First, an example with $A_1(X) = \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.
Sage:

```python
sage: X = ToricVariety(Fan(cones=[[0,1], [1,2], [2,0]],
                          rays=[[2,-1], [-1,2], [-1,-1]]))
sage: A = X.Chow_group()
sage: A.degree(1)
C3 x Z
```

Second, an example with $A_2(X) = \mathbb{Z}^2$:

```python
sage: points = [[1,0,0], [0,1,0], [0,0,1], [1,-1,1], [-1,0,-1]]
sage: l = LatticePolytope(points)
sage: l.show3d() # needs sage.plot
sage: X = ToricVariety(FaceFan(l))
sage: A = X.Chow_group()
sage: A.degree(2)
\mathbb{Z}^2
```

Third, an example with $A_2(X) = \mathbb{Z}^5$:

```python
sage: cube = [[ 1,0,0], [ 0, 1,0], [ 0,0, 1], [-1, 1,1], [-1,-1, 1],
           [-1,0,0], [ 0,-1,0], [ 0,0,-1], [ 1,-1,-1]]
sage: lat_cube = LatticePolytope(cube)
sage: X = ToricVariety(FaceFan((LatticePolytope(lat_cube))))
sage: X.Chow_group().degree(2)
\mathbb{Z}^5
```

Fourth, a fan that is not the fan over a polytope. Combinatorially, the fan is the same in the third example, only the coordinates of the first point are different. But the resulting fan is not the face fan of a cube, so the variety is “more singular”. Its Chow group has torsion, $A_2(X) = \mathbb{Z}^5 \oplus \mathbb{Z}/2$:

```python
sage: rays = [[ 1, 2, 3], [ 1,-1, 1], [-1, 1, 1], [-1,-1, 1],
           [-1, 1,-1], [ 1,-1, 1], [ 1, 1,-1]]
sage: cones = [[0,1,2,3], [4,5,6,7], [0,1,7,6],
           [4,5,3,2], [0,2,5,7], [4,6,1,3]]
sage: X = ToricVariety(Fan(cones, rays))
sage: X.Chow_group().degree(2) # long time (2s on sage.math, 2011)
C2 x \mathbb{Z}^5
```

Finally, Example 1.3 of [FS1994]:

```python
sage: def points_mod(k):
    ...
    return matrix([[ 1, 1, 2*k+1], [ 1,-1, 1],
                   [-1, 1, 1], [-1,-1, 1],
                   [-1, 1,-1], [ 1,-1, 1], [ 1, 1,-1]])

sage: def rays(k):
    ...
    return matrix([[ 1, 1, 1],
                   [ 1,-1, 1],
                   [-1, 1,-1], [ 1,-1,-1], [ 1, 1,-1]]).solve_left(points_mod(k)).rows()

sage: cones = [[0,1,2,3], [4,5,6,7], [0,1,7,6], [4,5,3,2], [0,2,5,7], [4,6,1,3]]

sage: X_Delta = lambda k: ToricVariety(Fan(cones=cones, rays=rays(k)))

sage: X_Delta(0).Chow_group().degree() # long time (3s on sage.math, 2011)
(Z, Z, Z^5, Z)

sage: X_Delta(1).Chow_group().degree() # long time (3s on sage.math, 2011)
```

(continues on next page)
sage: X_Delta(2).Chow_group().degree() # long time (3s on sage.math, 2011)
(Z, C2, Z^5, Z)
sage: X_Delta(2).Chow_group(base_ring=QQ).degree() # long time (4s on sage.math, 2011)
(Q, 0, Q^5, Q)

\[ (Z, 0, Z^5, Z) \]

\[ \textbf{sage: } X_{\Delta}(2).\text{Chow\_group}.\text{degree() } \text{# long time (3s on sage.math, 2011)} \]
\[ (Z, C2, Z^5, Z) \]

\[ \textbf{sage: } X_{\Delta}(2).\text{Chow\_group(base\_ring=QQ)}.\text{degree()} \text{# long time (4s on sage.math, 2011)} \]
\[ (Q, 0, Q^5, Q) \]

\section*{gens\text{\texttt{(degree=None)}}}

\textbf{Return the generators of the Chow group.}

\textbf{INPUT:}

- \textit{degree} – integer (optional). The degree of the Chow group.

\textbf{OUTPUT:}

- if no degree is specified, the generators of the whole Chow group. The chosen generators may be of mixed degree.

- if \texttt{degree=k} was specified, the generators of the degree-k part \(A_k\) of the Chow group.

\textbf{EXAMPLES:}

\[ \textbf{sage: } A = \text{toric\_varieties.P2().Chow\_group()} \]
\[ \textbf{sage: } A.gens() \]
\[ ((0 | 0 | 1), (0 | 1 | 0), (1 | 0 | 0)) \]

\[ \textbf{sage: } A.gens\text{\texttt{(degree=1)}} \]
\[ ((0 | 1 | 0),) \]

\section*{relation\_gens()}

\textbf{Return the Chow cycles equivalent to zero.}

For each \(d-k-1\)-dimensional cone \(\rho \in \Sigma^{(d-k-1)}\), the relations in \(A_k(X)\), that is the cycles equivalent to zero, are generated by

\[ 0 \overset{!}{=} \text{div}(u) = \sum_{\rho<\sigma \in \Sigma^{(n-\rho)}} \langle u, n_{\rho,\sigma} \rangle V(\sigma), \quad u \in M(\rho) \]

where \(n_{\rho,\sigma}\) is a (randomly chosen) lift of the generator of \(N_{\sigma}/N_{\rho} \cong \mathbb{Z}\). See also Exercise 12.5.7 of [CLS2011].

See also \texttt{relations()} to obtain the relations as submodule of the free module generated by the cones. Or use \texttt{self.relations().gens()} to list the relations in the free module.

\textbf{OUTPUT:}

A tuple of Chow cycles, each rationally equivalent to zero, that generates the rational equivalence.

\textbf{EXAMPLES:}

\[ \textbf{sage: } P2 = \text{toric\_varieties.P2()} \]
\[ \textbf{sage: } A = P2.\text{Chow\_group()} \]
\[ \textbf{sage: } \text{first} = A.\text{relation\_gens()}[0]; \text{first} \]
\[ (0 | 0 | 0) \]
\[ \textbf{sage: } \text{first.is\_zero()} \]
\[ \text{True} \]
\[ \textbf{sage: } \text{first.lift()} \]
\[ (0, 1, 0, -1, 0, 0, 0) \]
scheme()

Return the underlying toric variety.

OUTPUT: A ToricVariety.

EXAMPLES:

```sage
P2 = toric_varieties.P2()
sage: A = P2.Chow_group()
sage: A.scheme()
2-d CPR-Fano toric variety covered by 3 affine patches
sage: A.scheme() is P2
True
```

class sage.schemes.toric.chow_group.ChowGroup_degree_class(A, d)

Bases: SageObject

A fixed-degree subgroup of the Chow group of a toric variety.

**Warning:** Use `degree()` to construct `ChowGroup_degree_class` instances.

EXAMPLES:

```sage
P2 = toric_varieties.P2()
sage: A = P2.Chow_group(); A
Chow group of 2-d CPR-Fano toric variety covered by 3 affine patches
sage: A.degree()
(Z, Z, Z)
sage: A.degree(2)
Z
sage: type(_)
<class 'sage.schemes.toric.chow_group.ChowGroup_degree_class'>
```

gen(i)

Return the i-th generator of the Chow group of fixed degree.

INPUT:

- i – integer. The index of the generator to be returned.

OUTPUT: A Chow cycle.

EXAMPLES:

```sage
projective_plane = toric_varieties.P2()
sage: A2 = projective_plane.Chow_group().degree(2)
sage: A2.gen(0)
( 0 | 0 | 1 )
```

gens()

Return the generators of the Chow group of fixed degree.

OUTPUT: A tuple of Chow cycles of fixed degree generating `module()`.

EXAMPLES:
```python
sage: projective_plane = toric_varieties.P2()
sage: A2 = projective_plane.Chow_group().degree(2)
sage: A2.gens()
((0 | 0 | 1),)
```

**module()**

Return the submodule of the toric Chow group generated.

OUTPUT: A `sage.modules.fg_pid.fgp_module.FGP_Module_class`.

**EXAMPLES:**

```python
sage: projective_plane = toric_varieties.P2()
sage: A2 = projective_plane.Chow_group().degree(2)
sage: A2.module()
Finitely generated module V/W over Integer Ring with invariants (0)
```

**ngens()**

Return the number of generators.

OUTPUT: An integer.

**EXAMPLES:**

```python
sage: projective_plane = toric_varieties.P2()
sage: A2 = projective_plane.Chow_group().degree(2)
sage: A2.ngens()
1
```

`sage.schemes.toric.chow_group.is_ChowCycle(x)`

Return whether `x` is a `ChowCycle`.

**INPUT:**

- `x` – anything.

**OUTPUT:** True or False.

**EXAMPLES:**

```python
sage: P2 = toric_varieties.P2()
sage: A = P2.Chow_group()
sage: from sage.schemes.toric.chow_group import *
sage: is_ChowCycle(A)
False
sage: is_ChowCycle(A.an_element())
True
sage: is_ChowCycle('Victoria')
False
```

`sage.schemes.toric.chow_group.is_ChowGroup(x)`

Return whether `x` is a `ChowGroup_class`.

**INPUT:**

- `x` – anything.
OUTPUT: True or False.

EXAMPLES:

```
sage: P2 = toric_varieties.P2()
sage: A = P2.Chow_group()
sage: from sage.schemes.toric.chow_group import is_ChowGroup
dsage: is_ChowGroup(A)
True
dsage: is_ChowGroup('Victoria')
False
```

### 16.7 Toric ideals

A toric ideal (associated to an integer matrix $\mathbf{A}$) is an ideal of the form

$$ I_{\mathbf{A}} = \langle x^u - x^v : u, v \in \mathbb{Z}^n_\geq, u - v \in \ker(\mathbf{A}) \rangle $$

In other words, it is an ideal generated by irreducible “binomials”, that is, differences of monomials without a common factor. Since the Buchberger algorithm preserves this property, any Groebner basis is then also generated by binomials.

EXAMPLES:

```
sage: A = matrix([[1,1,1], [0,1,2]])
sage: IA = ToricIdeal(A)
sage: IA.ker()
Free module of degree 3 and rank 1 over Integer Ring
User basis matrix:
[-1 2 -1]
sage: IA
Ideal (-z1^2 + z0*z2) of Multivariate Polynomial
Ring in z0, z1, z2 over Rational Field
```

Here, the “naive” ideal generated by $z_0z_2 - z_1^2$ does already equal the toric ideal. But that is not true in general! For example, this toric ideal ([Stu1997], Example 1.2) is the twisted cubic and cannot be generated by $2 = \dim \ker(\mathbf{A})$ polynomials:

```
sage: A = matrix([[3,2,1,0], [0,1,2,3]])
sage: IA = ToricIdeal(A)
sage: IA.ker()
Free module of degree 4 and rank 2 over Integer Ring
User basis matrix:
[[-1 1 1 -1]
 [1 0 -1 0]
sage: IA
Ideal (-z1*z2 + z0^2*z3, -z1^2 + z0*z2, z2^2 - z1*z3) of Multivariate Polynomial
Ring in z0, z1, z2, z3 over Rational Field
```

The following family of toric ideals is from Example 4.4 of [Stu1997]. One can show that $I_d$ is generated by one quadric and $d$ binomials of degree $d$:

```
sage: def I(d):
....:     return ToricIdeal(matrix([[1,1,1,1,1], [0,1,1,0,0], [0,0,1,1,d]]))
```

(continues on next page)
Finally, the example in [SH1995b]

\begin{verbatim}
sage: A = matrix(ZZ, [ [15, 4, 14, 19, 2, 1, 10, 17],
    ....: [18, 11, 13, 5, 16, 16, 8, 19],
    ....: [11, 7, 8, 19, 15, 18, 14, 6],
    ....: [17, 10, 13, 17, 16, 14, 15, 18] ])
sage: IA = ToricIdeal(A)    # long time
sage: IA.ngens()            # long time
213
\end{verbatim}

AUTHORS:

- Volker Braun (2011-01-03): Initial version

class sage.schemes.toric.ideal.ToricIdeal(A, names='z', base_ring=Rational Field, polynomial_ring=None, algorithm='HostenSturmfels')

Bases: MPolynomialIdeal

This class represents a toric ideal defined by an integral matrix.

INPUT:

- A – integer matrix. The defining matrix of the toric ideal.
- names – string (optional). Names for the variables. By default, this is 'z' and the variables will be named z0, z1, ...
- polynomial_ring – a polynomial ring (optional). The polynomial ring to construct the ideal in.
  You may specify the ambient polynomial ring via the polynomial_ring parameter or via the names and base_ring parameter. A ValueError is raised if you specify both.
- algorithm – string (optional). The algorithm to use. For now, must be 'HostenSturmfels' which is the algorithm proposed by Hosten and Sturmfels in [SH1995b].

EXAMPLES:
sage: A = matrix([[1, 1, 1], [0, 1, 2]])
sage: ToricIdeal(A)
Ideal (-z1^2 + z0*z2) of Multivariate Polynomial Ring in z0, z1, z2 over Rational Field

First way of specifying the polynomial ring:

sage: ToricIdeal(A, names='x,y,z', base_ring=ZZ)
Ideal (-y^2 + x*z) of Multivariate Polynomial Ring in x, y, z over Integer Ring

Second way of specifying the polynomial ring:

sage: R.<x,y,z> = ZZ[]
sage: ToricIdeal(A, polynomial_ring=R)
Ideal (-y^2 + x*z) of Multivariate Polynomial Ring in x, y, z over Integer Ring

It is an error to specify both:

sage: ToricIdeal(A, names='x,y,z', polynomial_ring=R)
Traceback (most recent call last):
  ... ValueError: you must not specify both variable names and a polynomial ring

A()
Return the defining matrix.

OUTPUT: An integer matrix.

EXAMPLES:

sage: A = matrix([[1, 1, 1], [0, 1, 2]])
sage: IA = ToricIdeal(A)
sage: IA.A()
[1 1 1]
[0 1 2]

ker()
Return the kernel of the defining matrix.

OUTPUT: The kernel of self.A().

EXAMPLES:

sage: A = matrix([[1, 1, 1], [0, 1, 2]])
sage: IA = ToricIdeal(A)
sage: IA.ker()
Free module of degree 3 and rank 1 over Integer Ring
User basis matrix: [-1 2 -1]

nvariables()
Return the number of variables of the ambient polynomial ring.

OUTPUT: An integer. The number of columns of the defining matrix A().

EXAMPLES:

16.7. Toric ideals
There are three “obvious” ways to map toric varieties to toric varieties:

1. Polynomial maps in local coordinates, the usual morphisms in algebraic geometry.
2. Polynomial maps in the (global) homogeneous coordinates.
3. Toric morphisms, that is, algebraic morphisms equivariant with respect to the torus action on the toric variety.

Both 2 and 3 are special cases of 1, which is just to say that we always remain within the realm of algebraic geometry. But apart from that, none is included in one of the other cases. In the examples below, we will explore some algebraic maps that can or cannot be written as a toric morphism. Often a toric morphism can be written with polynomial maps in homogeneous coordinates, but sometimes it cannot.

The toric morphisms are perhaps the most mysterious at the beginning. Let us quickly review their definition (See Definition 3.3.3 of [CLS2011]). Let $\Sigma_1$ be a fan in $N_1, R$ and $\Sigma_2$ be a fan in $N_2, R$. A morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ of the associated toric varieties is toric if $\phi$ maps the maximal torus $T_{N_1} \subseteq X_{\Sigma_1}$ into $T_{N_2} \subseteq X_{\Sigma_2}$ and $\phi|_{T_N}$ is a group homomorphism.

The data defining a toric morphism is precisely what defines a fan morphism (see fan_morphism), extending the more familiar dictionary between toric varieties and fans. Toric geometry is a functor from the category of fans and fan morphisms to the category of toric varieties and toric morphisms.

Note: Do not create the toric morphisms (or any morphism of schemes) directly from the SchemeMorphism... classes. Instead, use the hom() method common to all algebraic schemes to create new homomorphisms.

### EXAMPLES:

First, consider the following embedding of $\mathbb{P}^1$ into $\mathbb{P}^2$

```
sage: P2.<x,y,z> = toric_varieties.P2()
sage: P1.<u,v> = toric_varieties.P1()
sage: P1.hom([0, u^2 + v^2, u*v], P2)
Scheme morphism:
  From: 1-d CPR-Fano toric variety covered by 2 affine patches
  To:  2-d CPR-Fano toric variety covered by 3 affine patches
  Defn: Defined on coordinates by sending [u : v] to [0 : u^2 + v^2 : u*v]
```

This is a well-defined morphism of algebraic varieties because homogeneously rescaled coordinates of a point of $\mathbb{P}^1$ map to the same point in $\mathbb{P}^2$ up to its homogeneous rescalings. It is not equivariant with respect to the torus actions

$C^\times \times \mathbb{P}^1, (\mu, [u : v]) \mapsto [u : \mu v]$  \text{ and }  $(C^\times)^2 \times \mathbb{P}^2, ((\alpha, \beta), [x : y : z]) \mapsto [x : \alpha y : \beta z],$

hence it is not a toric morphism. Clearly, the problem is that the map in homogeneous coordinates contains summands that transform differently under the torus action. However, this is not the only difficulty. For example, consider
sage: phi = P1.hom([0,u,v], P2); phi
Scheme morphism:
   From: 1-d CPR-Fano toric variety covered by 2 affine patches
   To: 2-d CPR-Fano toric variety covered by 3 affine patches
   Defn: Defined on coordinates by sending [u : v] to [0 : u : v]

This map is actually the embedding of the \texttt{orbit_closure()} associated to one of the rays of the fan of \(\mathbb{P}^2\). Now the morphism is equivariant with respect to some map \(\mathbb{C}^\times \to (\mathbb{C}^\times)^2\) of the maximal tori of \(\mathbb{P}^1\) and \(\mathbb{P}^2\). But this map of the maximal tori cannot be the same as \(\phi\) defined above. Indeed, the image of \(\phi\) completely misses the maximal torus \(\mathbb{T}_{\mathbb{P}^2} = \{[x : y : z] | x \neq 0, y \neq 0, z \neq 0\}\) of \(\mathbb{P}^2\).

Consider instead the following morphism of fans:

sage: fm = FanMorphism(matrix(ZZ, [[1,0]]), P1.fan(), P2.fan()); fm
Fan morphism defined by the matrix
[[1 0]]
Domain fan: Rational polyhedral fan in 1-d lattice \(\mathbb{N}\)
Codomain fan: Rational polyhedral fan in 2-d lattice \(\mathbb{N}\)

which also defines a morphism of toric varieties:

sage: P1.hom(fm, P2)
Scheme morphism:
   From: 1-d CPR-Fano toric variety covered by 2 affine patches
   To: 2-d CPR-Fano toric variety covered by 3 affine patches
   Defn: Defined by sending Rational polyhedral fan in 1-d lattice \(\mathbb{N}\)
to Rational polyhedral fan in 2-d lattice \(\mathbb{N}\).

The fan morphism map is equivalent to the following polynomial map:

sage: _.as_polynomial_map()
Scheme morphism:
   From: 1-d CPR-Fano toric variety covered by 2 affine patches
   To: 2-d CPR-Fano toric variety covered by 3 affine patches
   Defn: Defined on coordinates by sending [u : v] to [u : v : v]

Finally, here is an example of a fan morphism that cannot be written using homogeneous polynomials. Consider the blowup \(O_{\mathbb{P}^1}(2) \to \mathbb{C}^2/\mathbb{Z}_2\). In terms of toric data, this blowup is:

sage: A2_Z2 = toric_varieties.A2_Z2()
sage: A2_Z2.fan().rays()
N(1, 0),
N(1, 2)
in 2-d lattice \(\mathbb{N}\)
sage: O2_P1 = A2_Z2.resolve(new_rays=[[1,1]])
sage: blowup = O2_P1.hom(identity_matrix(2), A2_Z2)
sage: blowup.as_polynomial_map()
Traceback (most recent call last):
...
TypeError: the fan morphism cannot be written in homogeneous polynomials

If we denote the homogeneous coordinates of \(O_{\mathbb{P}^1}(2)\) by \(x, t, y\) corresponding to the rays \((1, 2), (1, 1),\) and \((1, 0)\) then the blow-up map is \([BB2013]\):

\[
f : O_{\mathbb{P}^1}(2) \to \mathbb{C}^2/\mathbb{Z}_2, \quad (x, t, y) \mapsto (x\sqrt{t}, y\sqrt{t})
\]
which requires square roots.

### 16.8.1 Fibrations

If a toric morphism is *dominant*, then all fibers over a fixed torus orbit in the base are isomorphic. Hence, studying the fibers is again a combinatorial question and Sage implements additional methods to study such fibrations that are not available otherwise (however, note that you can always `factor()` to pick out the part that is dominant over the image or its closure).

For example, consider the blow-up restricted to one of the two coordinate charts of $\mathbb{O}_P^1(2)$

```python
sage: O2_P1_chart = ToricVariety(Fan([O2_P1.fan().generating_cones()[0]]))
sage: single_chart = O2_P1_chart.hom(identity_matrix(2), A2_Z2)
sage: single_chart.is_dominant()
True
sage: single_chart.is_surjective()
False
```

```python
sage: fiber = single_chart.fiber_generic(); fiber
(0-d affine toric variety, 1)
sage: fiber[0].embedding_morphism().as_polynomial_map()
Scheme morphism:
    From: 0-d affine toric variety
    To: 2-d affine toric variety
    Defined on coordinates by sending [] to [1 : 1]
```

The fibers are labeled by torus orbits in the base, that is, cones of the codomain fan. In this case, the fibers over lower-dimensional torus orbits are:

```python
sage: A2_Z2_cones = flatten(A2_Z2.fan().cones())
sage: table([[cone.ambient_ray_indices(), single_chart.fiber_dimension(cone)]
for cone in A2_Z2_cones], header_row=True)
+--------+-----+
| cone   | dim |
| ()     | 0   |
| (0,)   | 0   |
| (1,)   | -1  |
| (0, 1) | 1   |
```

Let's look closer at the one-dimensional fiber. Although not the case in this example, connected components of fibers over higher-dimensional cones (corresponding to lower-dimensional torus orbits) of the base are often not irreducible. The irreducible components are labeled by the `primitive_preimage_cones()`, which are certain cones of the domain fan that map to the cone in the base that defines the torus orbit:

```python
sage: table([[cone.ambient_ray_indices(), single_chart.fan_morphism().primitive_preimage_cones(cone)]
for cone in A2_Z2_cones], header_row=True)
+---------------------------+
<table>
<thead>
<tr>
<th>base cone</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-d cone of Rational polyhedral fan in 2-d lattice N,</td>
</tr>
<tr>
<td>1-d cone of Rational polyhedral fan in 2-d lattice N,</td>
</tr>
</tbody>
</table>
```
(continues on next page)
(1, ) ()
(0, 1) (1-d cone of Rational polyhedral fan in 2-d lattice N,)

The fiber over the trivial cone is the generic fiber that we have already encountered. The interesting fiber is the one over the 2-dimensional cone, which represents the exceptional set of the blow-up in this single coordinate chart. Let's investigate further:

```sage
sage: fm = single_chart.fan_morphism()
sage: exceptional_cones = fm.primitive_preimage_cones(A2_Z2.fan(2)[0])
sage: exceptional_set = single_chart.fiber_component(exceptional_cones[0])
sage: exceptional_set
1-d affine toric variety
sage: exceptional_set.embedding_morphism().as_polynomial_map()
Scheme morphism:
  From: 1-d affine toric variety
  To: 2-d affine toric variety
  Defn: Defined on coordinates by sending [z0] to [z0 : 0]
```

So we see that the fiber over this point is an affine line. Together with another affine line in the other coordinate patch, this covers the exceptional $\mathbb{P}^1$ of the blowup $O_{\mathbb{P}^1}(2) \to \mathbb{C}^2/\mathbb{Z}_2$.

Here is an example with higher dimensional varieties involved:

```sage
sage: A3 = toric_varieties.A(3)
sage: P3 = toric_varieties.P(3)
sage: m = matrix([(2,0,0), (1,1,0), (3,1,0)])
sage: phi = A3.hom(m, P3)
sage: phi.as_polynomial_map()
Scheme morphism:
  From: 3-d affine toric variety
  To: 3-d CPR-Fano toric variety covered by 4 affine patches
  Defn: Defined on coordinates by sending [z0 : z1 : z2] to
        [z0^2*z1*z2^3 : z1*z2 : 1 : 1]
sage: phi.fiber_generic()
Traceback (most recent call last):
  ... AttributeError: 'SchemeMorphism_fan_toric_variety' object has no attribute 'fiber_generic'
```

Let's use factorization mentioned above:

```sage
sage: phi_i, phi_b, phi_s = phi.factor()
```

It is possible to study fibers of the last two morphisms or their composition:

```sage
sage: phi_d = phi_b * phi_s
sage: phi_d
Scheme morphism:
  From: 3-d affine toric variety
  To: 2-d toric variety covered by 3 affine patches
  Defn: Defined by sending Rational polyhedral fan in 3-d lattice N to
        Rational polyhedral fan in Sublattice <N(1, 0, 0), N(0, 1, 0)>.
sage: phi_d.as_polynomial_map()
```

(continues on next page)
Scheme morphism:
  From: 3-d affine toric variety
  To: 2-d toric variety covered by 3 affine patches
  Defn: Defined on coordinates by sending \([z0 : z1 : z2]\) to
  \([1 : z1*z2 : z0^2*z1*z2^3]\)

\[
\text{sage: } \phi_d.\text{codomain().fan().rays()}
\]
\[
N(-1, -1, 0),
N( 0, 1, 0),
N( 1, 0, 0)
\]
in Sublattice \(<N(1, 0, 0), N(0, 1, 0)\>

\[
\text{sage: for } c \text{ in } \phi_d.\text{codomain().fan():}
\]
  \[
  \text{c.ambient_ray_indices()}
  \]
(0, 1)
(0, 2)
(1, 2)

We see that codomain fan of this morphism is a projective plane, which can be verified by

\[
\text{sage: } \phi_d.\text{codomain().fan().is_isomorphic(toric_varieties.P2().fan())} \# \text{ known bug}
\]

True

(Unfortunately it cannot be verified correctly until github issue #16012 is fixed.)

We now have access to fiber methods:

\[
\text{sage: fiber = } \phi_d.\text{fiber_generic()}
\]

\text{fiber}

(1-d affine toric variety, 2)

\[
\text{sage: fiber[0].embedding_morphism()}
\]

Scheme morphism:
  From: 1-d affine toric variety
  To: 3-d affine toric variety
  Defn: Defined by sending
    Rational polyhedral fan in Sublattice \(<N(1, 1, -1)\>) to
    Rational polyhedral fan in 3-d lattice \(N\).

\[
\text{sage: fiber[0].embedding_morphism().as_polynomial_map()}
\]

Traceback (most recent call last):
...

NotImplementedError: polynomial representations for
fans with virtual rays are not implemented yet

\[
\text{sage: fiber[0].fan().rays()}
\]

Empty collection

in Sublattice \(<N(1, 1, -1)\>)

We see that generic fibers of this morphism consist of 2 one-dimensional tori each. To see what happens over boundary points we can look at fiber components corresponding to the cones of the domain fan:

\[
\text{sage: } \text{fm = } \phi_d.\text{fan_morphism()}
\]

\[
\text{sage: for } c \text{ in flatten(\phi_d.\text{domain().fan().cones()})�}
\]
  \[
  \text{fc, m = } \phi_d.\text{fiber_component(c, multiplicity=True)}
  \]
  \[
  \text{print("{} |-> {} ({} rays, multiplicity {}) over {}".format(}
  \]
  \[
  \text{c.ambient_ray_indices(), fc, fc.fan().nrays(),}
  \]
  \[
  \text{m, fm.image_cone(c).ambient_ray_indices())}
  \]

(continues on next page)
Schemes, Release 10.2

Now we see that over one of the coordinate lines of the projective plane we also have one-dimensional tori (but only one in each fiber), while over one of the points fixed by torus action we have two affine planes intersecting along an affine line. An alternative perspective is provided by cones of the codomain fan:

```python
sage: for c in flatten(phi_d.codomain().fan().cones()):
    ....: print("{} connected components over {}, each with {} irreducible components.".
    ....:       format(fm.index(c), c.ambient_ray_indices(),
    ....:              len(fm.primitive_preimage_cones(c)))))

2 connected components over (), each with 1 irreducible components.
None connected components over (0,), each with 0 irreducible components.
None connected components over (1,), each with 0 irreducible components.
1 connected components over (2,), each with 1 irreducible components.
None connected components over (0, 1), each with 0 irreducible components.
None connected components over (0, 2), each with 0 irreducible components.
1 connected components over (1, 2), each with 2 irreducible components.
```

```python
class sage.schemes.toric.morphism.SchemeMorphism_fan_fiber_component_toric_variety(toric_morphism, defining_cone)

Bases: SchemeMorphism

The embedding of a fiber component of a toric morphism.

Note that the embedding map of a fiber component of a toric morphism is itself not a toric morphism!

INPUT:

- **toric_morphism** – a toric morphism. The toric morphism whose fiber component we are describing.

- **defining_cone** – a cone of the fan of the domain of **toric_morphism**. See **fiber_component()** for details.

EXAMPLES:

```python
sage: polytope = Polyhedron(
    ....:     [(−3,0,−1,−1), (−1,2,−1,−1), (0,−1,0,0), (0,0,0,1), (0,0,1,0),
    ....:      (0,1,0,0), (0,2,−1,−1), (1,0,0,0), (2,0,−1,−1)])
sage: coarse_fan = FaceFan(polytope, lattice=ToricLattice(4))
```

```python
sage: P2 = toric_varieties.P2()
```

```python
sage: proj24 = matrix([[0,0],[1,0],[0,0],[0,1]])
```

```python
sage: fm = FanMorphism(proj24, coarse_fan, P2.fan(), subdivision=True)
```

```python
sage: ffm = fibration = ToricVariety(fm.domain_fan()).hom(fm, P2)
```

```python
sage: primitive_cones = ffm.primitive_preimage_cones(P2.fan()[0])
```

(continues on next page)
sage: fiber_component = fibration.fiber_component(primitive_cone)
sage: fiber_component
2-d toric variety covered by 4 affine patches
sage: fiber_component.embedding_morphism()
Scheme morphism:
  From: 2-d toric variety covered by 4 affine patches
  To: 4-d toric variety covered by 23 affine patches
  Defn: Defined by embedding a fiber component corresponding to
        1-d cone of Rational polyhedral fan in 4-d lattice N.

sage: fiber_component.embedding_morphism().as_polynomial_map()
Scheme morphism:
  From: 2-d toric variety covered by 4 affine patches
  To: 4-d toric variety covered by 23 affine patches
  Defn: Defined on coordinates by sending \([z_0 : z_1 : z_2 : z_3]\) to

sage: type(fiber_component.embedding_morphism())
<class 'sage.schemes.toric.morphism.SchemeMorphism_fan_fiber_component_toric_variety'
˓→>

as_polynomial_map()

Express the embedding morphism via homogeneous polynomials.

OUTPUT:

A \texttt{SchemeMorphism\_polynomial\_toric\_variety}. Raises a \texttt{ValueError} if the morphism cannot be
written in terms of homogeneous polynomials.

EXAMPLES:

sage: polytope = Polyhedron(
      ....: [(-3,0,-1,-1),(-1,2,-1,-1),(0,-1,0,0),(0,0,0,1),(0,0,1,0),
      ....: (0,1,0,0),(0,2,-1,-1),(1,0,0,0),(2,0,-1,-1)])
sage: coarse_fan = FaceFan(polytope, lattice=ToricLattice(4))
sage: P2 = toric_varieties.P2()
sage: proj24 = matrix([[0],[1],[0],[0],[0],[1]])
sage: fm = FanMorphism(proj24, coarse_fan, P2.fan(), subdivide=True)
sage: fibration = ToricVariety(fm.domain_fan()).hom(fm, P2)

sage: primitive_cone = Cone(((0, 1, 0, 0)))
sage: f = fibration.fiber_component(primitive_cone).embedding_morphism()
sage: f.as_polynomial_map()
Scheme morphism:
  From: 2-d toric variety covered by 4 affine patches
  To: 4-d toric variety covered by 23 affine patches
  Defn: Defined on coordinates by sending \([z_0 : z_1 : z_2 : z_3]\) to

sage: primitive_cone = Cone((-1, 2, -1, 0))
sage: f = fibration.fiber_component(primitive_cone).embedding_morphism()
sage: f.as_polynomial_map()
Traceback (most recent call last):
  ... ValueError: the morphism cannot be written using homogeneous polynomials
base_cone()

Return the base cone $\sigma$.

The fiber is constant over the base orbit closure $V(\sigma)$.

OUTPUT:

A cone of the base of the toric fibration.

EXAMPLES:

```python
sage: P1xP1 = toric_varieties.P1xP1()
sage: P1 = toric_varieties.P1()
sage: fc = P1xP1.hom(matrix([[1], [0]]), P1).fiber_component(Cone([[1, 0]]))
sage: f = fc.embedding_morphism()
sage: f.defining_cone().rays()
N(1, 0)
in 2-d lattice N
sage: f.base_cone().rays()
N(1)
in 1-d lattice N
```

defining_cone()

Return the cone corresponding to the fiber torus orbit.

OUTPUT:

A cone of the fan of the total space of the toric fibration.

EXAMPLES:

```python
sage: P1xP1 = toric_varieties.P1xP1()
sage: P1 = toric_varieties.P1()
sage: fc = P1xP1.hom(matrix([[1], [0]]), P1).fiber_component(Cone([[1, 0]]))
sage: f = fc.embedding_morphism()
sage: f.defining_cone().rays()
N(1, 0)
in 2-d lattice N
sage: f.base_cone().rays()
N(1)
in 1-d lattice N
```

pullback_divisor(divisor)

Pull back a toric divisor.

INPUT:

- divisor – a torus-invariant $\mathbb{Q}$-Cartier divisor on the codomain of the embedding map.

OUTPUT:

A divisor on the domain of the embedding map (irreducible component of a fiber of a toric morphism) that is isomorphic to the pull-back divisor $f^*(D)$ but with possibly different linearization.

EXAMPLES:

```python
sage: A1 = toric_varieties.A1()
sage: fan = Fan([[0, 1, 2]], [(1, 1, 0), (1, 0, 1), (1, -1, -1)]).subdivide(new_rays=[[1,
```

(continues on next page)
class sage.schemes.toric.morphism.SchemeMorphism_fan_toric_variety(parent, fan_morphism, check=True)

Bases: SchemeMorphism, Morphism

Construct a morphism determined by a fan morphism

**Warning:** You should not create objects of this class directly. Use the hom() method of toric varieties instead.

**INPUT:**

- **parent** – Hom-set whose domain and codomain are toric varieties.
- **fan_morphism** – A morphism of fans whose domain and codomain fans equal the fans of the domain and codomain in the parent Hom-set.
- **check** – boolean (optional, default: True). Whether to check the input for consistency.

**Warning:** A fibration is a dominant morphism; if you are interested in these then you have to make sure that your fan morphism is dominant. For example, this can be achieved by factoring the morphism. See SchemeMorphism_fan_toric_variety_dominant for additional functionality for fibrations.

**OUTPUT:** A SchemeMorphism_fan_toric_variety.

**EXAMPLES:**

```python
sage: P1xP1 = toric_varieties.P1xP1()
sage: P1 = toric_varieties.P1()
sage: f = P1.hom(matrix([[1,0]]), P1xP1); f
Scheme morphism:
  From: 1-d CPR-Fano toric variety covered by 2 affine patches
  To: 2-d CPR-Fano toric variety covered by 4 affine patches
  Defined by sending Rational polyhedral fan in 1-d lattice N
to Rational polyhedral fan in 2-d lattice N.
sage: type(f)
<class 'sage.schemes.toric.morphism.SchemeMorphism_fan_toric_variety'>
```

Slightly more explicit construction:

```python
sage: P1xP1 = toric_varieties.P1xP1()
sage: P1 = toric_varieties.P1()
sage: hom_set = P1xP1.Hom(P1)
```

(continues on next page)
sage: fm = FanMorphism(matrix(ZZ, [[1],[0]]), P1xP1.fan(), P1.fan())
sage: hom_set(fm)
Scheme morphism:
  From: 2-d CPR-Fano toric variety covered by 4 affine patches
  To: 1-d CPR-Fano toric variety covered by 2 affine patches
  Defn: Defined by sending Rational polyhedral fan in 2-d lattice N
to Rational polyhedral fan in 1-d lattice N.
sage: P1xP1.hom(fm, P1)
Scheme morphism:
  From: 2-d CPR-Fano toric variety covered by 4 affine patches
  To: 1-d CPR-Fano toric variety covered by 2 affine patches
  Defn: Defined by sending Rational polyhedral fan in 2-d lattice N
to Rational polyhedral fan in 1-d lattice N.

\textbf{as\_polynomial\_map()}\footnote{Express the morphism via homogeneous polynomials.}

OUTPUT: A \texttt{SchemeMorphism\_polynomial\_toric\_variety}.

 Raises a \texttt{TypeError} if the morphism cannot be written in terms of homogeneous polynomials.

\textbf{EXAMPLES:}

sage: A1 = toric\_varieties.A1()
sage: square = A1.hom(matrix([[2]]), A1)
sage: square.as\_polynomial\_map()
Scheme endomorphism of 1-d affine toric variety
  Defn: Defined on coordinates by sending \([z]\) to \([z^2]\)
sage: P1 = toric\_varieties.P1()
sage: patch = A1.hom(matrix([[1]]), P1)
sage: patch.as\_polynomial\_map()
Scheme morphism:
  From: 1-d affine toric variety
  To: 1-d CPR-Fano toric variety covered by 2 affine patches
  Defn: Defined on coordinates by sending \([z]\) to \([z : 1]\)

\textbf{factor()}\footnote{Factor \texttt{self} into injective * birational * surjective morphisms.}

OUTPUT:

\begin{itemize}
  \item a triple of toric morphisms \((\phi_1, \phi_b, \phi_s)\), such that \(\phi_s\) is surjective, \(\phi_b\) is birational, \(\phi_i\) is injective, and \texttt{self} is equal to \(\phi_i \circ \phi_b \circ \phi_s\).
\end{itemize}

The intermediate varieties are universal in the following sense. Let \texttt{self} map \(X\) to \(X'\) and let \(X_s, X_i\) sit in between, that is,

\[
X \rightarrow X_s \rightarrow X_i \rightarrow X'.
\]

Then any toric morphism from \(X\) coinciding with \texttt{self} on the maximal torus factors through \(X_s\) and any toric morphism into \(X'\) coinciding with \texttt{self} on the maximal torus factors through \(X_i\). In particular, \(X_i\) is the closure of the image of \texttt{self} in \(X'\).

See \texttt{factor()} for a description of the toric algorithm.
EXAMPLES:

We map an affine plane into a projective 3-space in such a way, that it becomes "a double cover of a chart of the blow up of one of the coordinate planes."

```python
sage: A2 = toric_varieties.A2()
sage: P3 = toric_varieties.P(3)
sage: m = matrix([[2,0,0], [1,1,0]])
sage: phi = A2.hom(m, P3)
sage: phi.as_polynomial_map()
Scheme morphism:
  From: 2-d affine toric variety
  To: 3-d CPR-Fano toric variety covered by 4 affine patches
  Defn: Defined on coordinates by sending [x : y] to
         [x^2*y : y : 1 : 1]

sage: phi.is_surjective(), phi.is_birational(), phi.is_injective()
(False, False, False)

sage: phi_i, phi_b, phi_s = phi.factor()
sage: phi_i.is_surjective(), phi_b.is_birational(), phi_i.is_injective()
(True, True, True)

sage: prod(phi.factor()) == phi
True

Double cover (surjective):

```python
sage: phi_s.as_polynomial_map()
Scheme morphism:
  From: 2-d affine toric variety
  To: 2-d affine toric variety
  Defn: Defined on coordinates by sending [x : y] to [x^2 : y]

Blowup chart (birational):

```python
sage: phi_b.as_polynomial_map()
Scheme morphism:
  From: 2-d affine toric variety
  To: 2-d toric variety covered by 3 affine patches
  Defn: Defined on coordinates by sending [z0 : z1] to [1 : z1 : z0*z1]

Coordinate plane inclusion (injective):

```python
sage: phi_i.as_polynomial_map()
Scheme morphism:
  From: 2-d toric variety covered by 3 affine patches
  To: 3-d CPR-Fano toric variety covered by 4 affine patches
  Defn: Defined on coordinates by sending [z0 : z1 : z2] to [z2 : z1 : z0 : z0]

fan_morphism()

Return the defining fan morphism.

OUTPUT: A FanMorphism.

EXAMPLES:
```python
sage: P1xP1 = toric_varieties.P1xP1()
sage: P1 = toric_varieties.P1()
sage: f = P1xP1.hom(matrix([[1],[0]]), P1)
```

```python
sage: f.fan_morphism()
Fan morphism defined by the matrix
[1]
[0]
Domain fan: Rational polyhedral fan in 2-d lattice \(\mathbb{N}\)
Codomain fan: Rational polyhedral fan in 1-d lattice \(\mathbb{N}\)
```

**is_birational()**

Check if \(self\) is birational.

See **is_birational()** for fan morphisms for a description of the toric algorithm.

**OUTPUT:**

Boolean. Whether \(self\) is birational.

**EXAMPLES:**

```python
sage: dP8 = toric_varieties.dP8()
sage: P2 = toric_varieties.P2()
sage: dP8.hom(identity_matrix(2), P2).is_birational()
True
```

```python
sage: X = toric_varieties.A(2)
sage: Y = ToricVariety(Fan([Cone([[(1,0), (1,1)]])]))
sage: m = identity_matrix(2)
sage: f = Y.hom(m, X)
sage: f.is_birational()
True
```

**is_bundle()**

Check if \(self\) is a bundle.

See **is_bundle()** for fan morphisms for details.

**OUTPUT:**

- True if \(self\) is a bundle, False otherwise.

**EXAMPLES:**

```python
sage: P1xP1 = toric_varieties.P1xP1()
sage: P1 = toric_varieties.P1()
sage: P1xP1.hom(matrix([[1],[0]]), P1).is_bundle()
True
```

**is_dominant()**

Return whether \(self\) is dominant.

See **is_dominant()** for fan morphisms for a description of the toric algorithm.

**OUTPUT:**

Boolean. Whether \(self\) is a dominant scheme morphism.

**EXAMPLES:**
sage: P1 = toric_varieties.P1()
sage: A1 = toric_varieties.A1()
sage: phi = A1.hom(identity_matrix(1), P1); phi
Scheme morphism:
   From: 1-d affine toric variety
   To: 1-d CPR-Fano toric variety covered by 2 affine patches
   Defn: Defined by sending Rational polyhedral fan in 1-d lattice N
to Rational polyhedral fan in 1-d lattice N.
sage: phi.is_dominant()
True
sage: phi.is_surjective()
False

is_fibration()
Check if self is a fibration.
See is_fibration() for fan morphisms for details.
OUTPUT:
• True if self is a fibration, False otherwise.
EXAMPLES:

sage: P1xP1 = toric_varieties.P1xP1()
sage: P1 = toric_varieties.P1()
sage: P1xP1.hom(matrix([[1], [0]]), P1).is_fibration()
True

is_injective()
Check if self is injective.
See is_injective() for fan morphisms for a description of the toric algorithm.
OUTPUT:
Boolean. Whether self is injective.
EXAMPLES:

sage: P1xP1 = toric_varieties.P1xP1()
sage: P1 = toric_varieties.P1()
sage: P1xP1.hom(matrix([[1], [0]]), P1).is_injective()
False

sage: X = toric_varieties.A(2)
sage: m = identity_matrix(2)
sage: f = X.hom(m, X)
sage: f.is_injective()
True

sage: Y = ToricVariety(Fan([Cone([1, 0), (1, 1)])])
sage: f = Y.hom(m, X)
sage: f.is_injective()
False
is_surjective()

Check if self is surjective.

See is_surjective() for fan morphisms for a description of the toric algorithm.

OUTPUT:

Boolean. Whether self is surjective.

EXAMPLES:

```python
sage: P1xP1 = toric_varieties.P1xP1()
sage: P1 = toric_varieties.P1()
sage: P1xP1.hom(matrix([[1],[0]]), P1).is_surjective()
True

sage: X = toric_varieties.A(2)
sage: m = identity_matrix(2)
sage: f = X.hom(m, X)
sage: f.is_surjective()
True

sage: Y = ToricVariety(Fan([Cone([(1,0), (1,1)])]))
sage: f = Y.hom(m, X)
sage: f.is_surjective()
False
```

pullback_divisor(divisor)

Pull back a toric divisor.

INPUT:

- divisor – a torus-invariant $\mathbb{Q}$-Cartier divisor on the codomain of self.

OUTPUT:

The pull-back divisor $f^*(D)$.

EXAMPLES:

```python
sage: A2_Z2 = toric_varieties.A2_Z2()
sage: A2 = toric_varieties.A2()
sage: f = A2.hom(matrix([[1],[0]], [[1,2]]), A2_Z2)
sage: f.pullback_divisor(A2_Z2.divisor([0]))
V(x)

sage: A1 = toric_varieties.A1()
sage: square = A1.hom(matrix([[2]]), A1)
sage: D = A1.divisor([0]; D
V(z)
sage: square.pullback_divisor(D)
2*V(z)
```

class sage.schemes.toric.morphism.SchemeMorphism_fan_toric_variety_dominant(parent, fan_morphism, check=True)

Bases: SchemeMorphism_fan_toric_variety

Construct a morphism determined by a dominant fan morphism.
A dominant morphism is one that is surjective onto a dense subset. In the context of toric morphisms, this means that it is onto the big torus orbit.

**Warning:** You should not create objects of this class directly. Use the hom() method of toric varieties instead.

**INPUT:**

See SchemeMorphism_fan_toric_variety. The given fan morphism must be dominant.

**OUTPUT:** A SchemeMorphism_fan_toric_variety_dominant.

**EXAMPLES:**

```python
sage: P2 = toric_varieties.P2()
sage: dP8 = toric_varieties.dP8()
sage: f = dP8.hom(identity_matrix(2), P2); f
Scheme morphism:
  From: 2-d CPR-Fano toric variety covered by 4 affine patches
  To: 2-d CPR-Fano toric variety covered by 3 affine patches
  Defn: Defined by sending Rational polyhedral fan in 2-d lattice N
to Rational polyhedral fan in 2-d lattice N.
sage: type(f)
<class 'sage.schemes.toric.morphism.SchemeMorphism_fan_toric_variety_dominant'>

fiber_component(domain_cone, multiplicity=False)

Return a fiber component corresponding to domain_cone.

**INPUT:**

- domain_cone – a cone of the domain fan of self.
- multiplicity (default: False) – whether to return the number of fiber components corresponding to domain_cone as well.

**OUTPUT:**

- either \( X \) or a tuple \((X, n)\), where \( X \) is a toric variety with the embedding morphism into domain of self and \( n \) is an integer.

Let \( \phi : \Sigma \to \Sigma' \) be the fan morphism corresponding to self. Let \( \sigma \in \Sigma \) and \( \sigma' \in \Sigma' \) be the image_cone() of \( \sigma \). The fiber over any point of the torus orbit corresponding to \( \sigma' \) consists of \( n \) isomorphic connected components with each component being a union of toric varieties intersecting along their torus invariant subvarieties. The latter correspond to preimage_cones() of \( \sigma' \) and \( X \) is one of the \( n \) components corresponding to \( \sigma \). The irreducible components correspond to primitive_preimage_cones().

**EXAMPLES:**

```python
sage: polytope = LatticePolytope(....: [(-3,0,-1,-1),(-1,2,-1,-1),(0,-1,0,0),(0,0,0,1),(0,0,1,0),....: (0,1,0,0),(0,2,-1,-1),(1,0,0,0),(2,0,-1,-1)])
sage: coarse_fan = FaceFan(polytope)
sage: P2 = toric_varieties.P2()
sage: proj24 = matrix([[0,0], [1,0], [0,0], [0,1]])
sage: fm = FanMorphism(proj24, coarse_fan, P2.fan(), subdivide=True)
sage: fibration = Toric Variety(fm.domain_fan()).hom(fm, P2)
sage: ffm = fibration.fan_morphism()
```
sage: primitive_cones = ffm.primitive_preimage_cones(P2.fan(1)[0])
sage: primitive_cone = primitive_cones[0]
sage: fibration.fiber_component(primitive_cone)
2-d toric variety covered by 4 affine patches
sage: fibration.fiber_component(primitive_cone, True)
(2-d toric variety covered by 4 affine patches, 1)

sage: for primitive_cone in primitive_cones:
    ....: print(fibration.fiber_component(primitive_cone))
2-d toric variety covered by 4 affine patches
2-d toric variety covered by 3 affine patches
2-d toric variety covered by 3 affine patches

fiber_dimension(codomain_cone)

Return the dimension of the fiber over a particular torus orbit in the base.

INPUT:

• codomain_cone – a cone σ of the codomain, specifying a torus orbit O(σ).

OUTPUT:

An integer. The dimension of the fiber over the torus orbit corresponding to codomain_cone. If the fiber is the empty set, -1 is returned. Note that all fibers over this torus orbit are isomorphic, and therefore have the same dimension.

EXAMPLES:

sage: P1xP1 = toric_varieties.P1xP1()
sage: P1 = toric_varieties.P1()
sage: f = P1xP1.hom(matrix([[1], [0]]), P1)
sage: f.fiber_dimension(P1.fan(0)[0])
1
sage: f.fiber_dimension(P1.fan(1)[0])
1
sage: f.fiber_dimension(P1.fan(1)[1])
1

Here is a more complicated example that is not a flat fibration:

sage: A2_Z2 = toric_varieties.A2_Z2()
sage: o2_p1 = A2_Z2.resolve(new_rays=[[1, 1]])
sage: blowup = o2_p1.hom(identity_matrix(2), A2_Z2)
sage: blowup.fiber_dimension(A2_Z2.fan(0)[0])
0
sage: blowup.fiber_dimension(A2_Z2.fan(1)[0])
0
sage: blowup.fiber_dimension(A2_Z2.fan(2)[0])
1

This corresponds to the three different fibers:

sage: blowup.fiber_generic()
(0-d affine toric variety, 1)
sage: blowup.fiber_component(Cone(([1, 0])))
0-d affine toric variety

`sage: blowup.fiber_component(Cone([(1,1)])`

1-d toric variety covered by 2 affine patches

`fiber_generic()`

Return the generic fiber.

**OUTPUT:**

- a tuple \((X, n)\), where \(X\) is a **toric variety** with the embedding morphism into domain of `self` and \(n\) is an integer.

The fiber over the base point with homogeneous coordinates \([1 : 1 : \cdots : 1]\) consists of \(n\) disjoint toric varieties isomorphic to \(X\). Note that fibers of a dominant toric morphism are isomorphic over all points of a fixed torus orbit of its codomain, in particular over all points of the maximal torus, so it makes sense to talk about "the generic" fiber.

The embedding of \(X\) is a toric morphism with the `domain_fan()` being the `kernel_fan()` of the defining fan morphism. By contrast, embeddings of fiber components over lower-dimensional torus orbits of the image are not toric morphisms. Use `fiber_component()` for the latter (non-generic) fibers.

**EXAMPLES:**

```python
sage: P1xP1 = toric_varieties.P1xP1()
sage: P1 = toric_varieties.P1()
sage: fiber = P1xP1.hom(matrix([[1],[0]]), P1).fiber_generic()
sage: fiber
(1-d toric variety covered by 2 affine patches, 1)
sage: f = fiber[0].embedding_morphism(); f
Scheme morphism:
  From: 1-d toric variety covered by 2 affine patches
  To: 2-d CPR-Fano toric variety covered by 4 affine patches
  Defn: Defined by sending Rational polyhedral fan in Sublattice <N(0, 1)> to Rational polyhedral fan in 2-d lattice N.
sage: f.as_polynomial_map()
Scheme morphism:
  From: 1-d toric variety covered by 2 affine patches
  To: 2-d CPR-Fano toric variety covered by 4 affine patches
  Defn: Defined on coordinates by sending [z0 : z1] to [1 : 1 : z0 : z1]
sage: A1 = toric_varieties.A1()
sage: fan = Fan([[0,1,2]], [(1,1,0), (1,0,1), (1,-1,-1)])
sage: fan = fan.subdivide(new_rays=[(1,0,0)])
sage: f = ToricVariety(fan).hom(matrix([[1],[0],[0]]), A1)
sage: f.fiber_generic()
(2-d affine toric variety, 1)
sage: _[0].fan().generating_cones()
(0-d cone of Rational polyhedral fan in Sublattice <N(0, 1, 0), N(0, 0, 1)>,)
```

`fiber_graph(codomain_cone)`

Return the fiber over a given torus orbit in the codomain.

**INPUT:**

- `codomain_cone` – a cone \(\sigma\) of the codomain, specifying a torus orbit \(O(\sigma)\).
OUTPUT:

A graph whose nodes are the irreducible components of a connected component of the fiber over a point of \(O(\sigma)\). If two irreducible components intersect, the corresponding nodes of the graph are joined by an edge. Note that irreducible components do not have to be of the same dimension.

See also:

\texttt{fiber\_component()}.

EXAMPLES:

```python
sage: polytope = Polyhedron(....: [(-3,0,-1,-1),(-1,2,-1,-1),(0,-1,0,0),(0,0,0,1),(0,0,1,0),....: (0,1,0,0),(0,2,-1,-1),(1,0,0,0),(2,0,-1,-1)])
sage: coarse_fan = FaceFan(polytope, lattice=ToricLattice(4))
sage: P2 = toric_varieties.P2()
sage: proj34 = block_matrix(2, 1, [zero_matrix(2,2),....: identity_matrix(2)])
sage: fm = FanMorphism(proj34, coarse_fan, P2.fan(), subdivide=True)
sage: fibration = Toric Variety(fm.domain_fan()).hom(fm, P2)

sage: fibration.fiber_graph(P2.fan(0)[0])
Graph on 1 vertex
sage: for c1 in P2.fan(1):
....:    fibration.fiber_graph(c1)
Graph on 1 vertex
Graph on 1 vertex
Graph on 4 vertices

sage: fibration.fiber_graph(P2.fan(1)[2]).get_vertices()
{0: 2-d toric variety covered by 4 affine patches,
  1: 2-d toric variety covered by 3 affine patches,
  2: 2-d toric variety covered by 3 affine patches,
  3: 2-d toric variety covered by 4 affine patches}

sage: fibration
Scheme morphism:
From: 4-d toric variety covered by 18 affine patches
To: 2-d CPR-Fano toric variety covered by 3 affine patches
Defn: Defined by sending Rational polyhedral fan in 4-d lattice \(\mathbb{N}\) to Rational polyhedral fan in 2-d lattice \(\mathbb{N}\).
```

class \texttt{sage.schemes.toric.morphism.SchemeMorphism\_orbit\_closure\_toric\_variety}(\textit{parent},
\textit{defining\_cone},
\textit{ray\_map})

Bases: \texttt{SchemeMorphism}, \texttt{Morphism}

The embedding of an orbit closure.

INPUT:

- \texttt{parent} – the parent homset.
- \texttt{defining\_cone} – the defining cone.
ray_map – a dictionary {ambient ray generator: orbit ray generator}. Note that the image of the ambient ray generator is not necessarily primitive.

**Warning:** You should not create objects of this class directly. Use the orbit_closure() method of toric varieties instead.

**EXAMPLES:**

```python
sage: P1xP1 = toric_varieties.P1xP1()
sage: H = P1xP1.fan(1)[0]
sage: V = P1xP1.orbit_closure(H)
sage: V.embedding_morphism()
Scheme morphism:
 From: 1-d toric variety covered by 2 affine patches
 To: 2-d CPR-Fano toric variety covered by 4 affine patches
Defn: Defined by embedding the torus closure associated to the 1-d cone of Rational polyhedral fan in 2-d lattice N.
```

**as_polynomial_map()**

Express the morphism via homogeneous polynomials.

**OUTPUT:**

A `SchemeMorphism_polynomial_toric_variety`. Raises a TypeError if the morphism cannot be written in terms of homogeneous polynomials.

The defining polynomials are not necessarily unique. There are choices if multiple ambient space ray generators project to the same orbit ray generator, and one such choice is made implicitly. The orbit embedding can be written as a polynomial map if and only if each primitive orbit ray generator is the image of at least one primitive ray generator of the ambient toric variety.

**EXAMPLES:**

```python
sage: P2 = toric_varieties.P2()
sage: V = P2.orbit_closure(P2.fan(1)[0]); V
1-d toric variety covered by 2 affine patches
sage: V.embedding_morphism().as_polynomial_map()
Scheme morphism:
 From: 1-d toric variety covered by 2 affine patches
 To: 2-d CPR-Fano toric variety covered by 3 affine patches
Defn: Defined on coordinates by sending [z0 : z1] to [0 : z1 : z0]
```

If the toric variety is singular, then some orbit closure embeddings cannot be written with homogeneous polynomials:

```python
sage: P2_112 = toric_varieties.P2_112()
sage: P1 = P2_112.orbit_closure(Cone([(1,0)]))
sage: P1.embedding_morphism().as_polynomial_map()
Traceback (most recent call last):
 ...
TypeError: the embedding cannot be written with homogeneous polynomials
```

**defining_cone()**

Return the cone corresponding to the torus orbit.

**OUTPUT:** A cone of the fan of the ambient toric variety.
EXAMPLES:
```python
sage: P2 = toric_varieties.P2()
sage: cone = P2.fan(1)[0]
sage: P1 = P2.orbit_closure(cone)
sage: P1.embedding_morphism().defining_cone()
1-d cone of Rational polyhedral fan in 2-d lattice N
sage: _ is cone
True
```

**pullback_divisor(divisor)**

Pull back a toric divisor.

**INPUT:**
- `divisor` – a torus-invariant $\mathbb{Q}$-Cartier divisor on the codomain of the embedding map.

**OUTPUT:**
A divisor on the domain of the embedding map (the orbit closure) that is isomorphic to the pull-back divisor $f^*(D)$ but with possibly different linearization.

**EXAMPLES:**
```python
sage: P2 = toric_varieties.P2()
sage: P1 = P2.orbit_closure(P2.fan(1)[0])
sage: f = P1.embedding_morphism()
sage: D = P2.divisor([1,2,3]); D
V(x) + 2*V(y) + 3*V(z)
sage: f.pullback_divisor(D)
4*V(z0) + 2*V(z1)
```

**class** `sage.schemes.toric.morphism.SchemeMorphism_point_toric_field(X, coordinates, check=True)`

Bases: `SchemeMorphism_point`, `Morphism`

A point of a toric variety determined by homogeneous coordinates in a field.

**Warning:** You should not create objects of this class directly. Use the `hom()` method of `toric varieties` instead.

**INPUT:**
- `X` – toric variety or subscheme of a toric variety.
- `coordinates` – list of coordinates in the base field of `X`.
- `check` – if True (default), the input will be checked for correctness.

**OUTPUT:** A `SchemeMorphism_point_toric_field`.

**class** `sage.schemes.toric.morphism.SchemeMorphism_polynomial_toric_variety(parent, polynomials, check=True)`

Bases: `SchemeMorphism_polynomial`, `Morphism`

A morphism determined by homogeneous polynomials.
INPUT:
Same as for SchemeMorphism_polynomial.

OUTPUT: A SchemeMorphism_polynomial_toric_variety.

as_fan_morphism()
Express the morphism as a map defined by a fan morphism.

OUTPUT: A SchemeMorphism_polynomial_toric_variety.

Raises a TypeError if the morphism cannot be written in such a way.

EXAMPLES:

```
sage: A1.<z> = toric_varieties.A1()
sage: P1 = toric_varieties.P1()
sage: patch = A1.hom([1,z], P1)
sage: patch.as_fan_morphism()
Traceback (most recent call last):
...
NotImplementedError: expressing toric morphisms as fan morphisms is not implemented yet
```

16.9 Subschemes of toric space

AUTHORS:


class sage.schemes.toric.toric_subscheme.AlgebraicScheme_subscheme_affine_toric(toric_variety, polynomials)

Bases: AlgebraicScheme_subscheme_toric

Construct an algebraic subscheme of an affine toric variety.

Warning: You should not create objects of this class directly. The preferred method to construct such subschemes is to use subscheme() method of toric varieties.

INPUT:

- toric_variety – ambient affine toric variety.
- polynomials – single polynomial, list, or ideal of defining polynomials in the coordinate ring of toric_variety.

OUTPUT:

A algebraic subscheme of an affine toric variety.
**dimension()**

Return the dimension of self.

OUTPUT: An integer.

EXAMPLES:

```
sage: # needs sage.libs.singular
sage: P1xP1.<s0,s1,t0,t1> = toric_varieties.P1xP1()
sage: P1 = P1xP1.subscheme(s0 - s1)
sage: P1.dimension()
1
```

A more complicated example where the ambient toric variety is not smooth:

```
sage: # needs sage.libs.singular
sage: X.<x,y> = toric_varieties.A2_Z2()
sage: X.is_smooth()  # False
sage: Y = X.subscheme([x*y, x^2]); Y
Closed subscheme of 2-d affine toric variety defined by:
   x*y,
   x^2
sage: Y.dimension()
1
```

**is_smooth(point=None)**

Test whether the algebraic subscheme is smooth.

INPUT:

* point – A point or None (default). The point to test smoothness at.

OUTPUT:

Boolean. If no point was specified, returns whether the algebraic subscheme is smooth everywhere. Otherwise, smoothness at the specified point is tested.

EXAMPLES:

```
sage: # needs sage.libs.singular
sage: A2.<x,y> = toric_varieties.A2()
sage: cuspidal_curve = A2.subscheme([-x^3 + y^2])
sage: cuspidal_curve.is_smooth([1,1])
True
sage: cuspidal_curve.is_smooth([0,0])
False
sage: circle = A2.subscheme(x^2 + y^2 - 1)
sage: circle.is_smooth([1,0])
True
sage: circle.is_smooth()
False
```

16.9. Subschemes of toric space
A more complicated example where the ambient toric variety is not smooth:

```python
sage: # needs sage.libs.singular
sage: X.<x,y> = toric_varieties.A2_Z2()  # 2-d affine space mod Z/2
sage: X.is_smooth()
False
sage: Y = X.subscheme([x*y, x^2])  # (twice the x=0 curve) mod Z/2
sage: Y
Closed subscheme of 2-d affine toric variety defined by:
x*y,
x^2
sage: Y.dimension()  # Y is a Weil divisor but not Cartier
1
sage: Y.is_smooth()
True
sage: Y.is_smooth([0,0])
True
```

```python
class sage.schemes.toric.toric_subscheme.AlgebraicScheme_subscheme_toric(toric_variety, polynomials)

Bases: AlgebraicScheme_subscheme

Construct an algebraic subscheme of a toric variety.

Warning: You should not create objects of this class directly. The preferred method to construct such subschemes is to use subscheme() method of toric varieties.

INPUT:

- toric_variety – ambient toric variety.
- polynomials – single polynomial, list, or ideal of defining polynomials in the coordinate ring of toric_variety.

OUTPUT: An algebraic subscheme of a toric variety.

affine_algebraic_patch(cone=None, names=None)

Return the affine patch corresponding to cone as an affine algebraic scheme.

INPUT:

- cone – a Cone σ of the fan. It can be omitted for an affine toric variety, in which case the single generating cone is used.

OUTPUT:

An affine algebraic subscheme corresponding to the patch \( Spec(\sigma^\vee \cap M) \) associated to the cone σ. See also affine_patch(), which expresses the patches as subvarieties of affine toric varieties instead.

REFERENCES:


EXAMPLES:
Here we got two defining equations. The first one describes the singularity of the ambient space and the second is the pull-back of $x^2 + y^2 - 1$

sage: $lp = LatticePolytope([(1,0,0), (1,1,0), (1,1,1), (1,0,1), (-2,-1,-1)],
....: lattice=ToricLattice(3))$
sage: $X.<x,y,u,v,t> = CPRFanoToricVariety(Delta_polar=lp)$
sage: $Y = X.subscheme(x^v + y^u + t)$
sage: $cone = Cone([(1,0,0), (1,1,0), (1,1,1), (1,0,1)])$
sage: $Y.affine_algebraic_patch(cone)$
Closed subscheme of Affine Space of dimension 4 over Rational Field defined by:
$\ z_0^*z_2 - z_1^*z_3,$
$z_1 + z_3 + 1$

affine_patch(i)

Return the i-th affine patch of self as an affine toric algebraic scheme.

INPUT:

• i – integer, index of a generating cone of the fan of the ambient space of self.

OUTPUT:

• subscheme of an affine toric variety corresponding to the pull-back of self by the embedding morphism of the i-th affine patch of the ambient space of self.

The result is cached, so the i-th patch is always the same object in memory.

EXAMPLES:

sage: $P1xP1 = toric_varieties.P1xP1()$
sage: $patch1 = P1xP1.affine_patch(1)$

(continues on next page)
sage: patch1.embedding_morphism()
Scheme morphism:
  From: 2-d affine toric variety
  To: 2-d CPR-Fano toric variety covered by 4 affine patches
  Defn: Defined on coordinates by sending \([t : x]\) to \([1 : t : x : 1]\)

sage: P1.inject_variables()
Defining s, t, x, y

sage: P1 = P1xP1.subscheme(x - y)
sage: subpatch = P1.affine_patch(1)
sage: subpatch
Closed subscheme of 2-d affine toric variety defined by:
  x - 1

\textbf{dimension()}

Return the dimension of self.

\textbf{OUTPUT:} An integer. If self is empty, \(-1\) is returned.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: # needs sage.libs.singular
sage: P1xP1 = toric_varieties.P1xP1()
sage: P1xP1.inject_variables()
Defining s, t, x, y
sage: P1 = P1xP1.subscheme(s - t)
sage: P1.dimension()
1
sage: P1xP1.subscheme([s - t, (s-t)^2]).dimension()
1
sage: P1xP1.subscheme([s, t]).dimension()
-1
\end{verbatim}

\textbf{fan()}

Return the fan of the ambient space.

\textbf{OUTPUT:} A fan.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: P2.<x,y,z> = toric_varieties.P(2)
sage: E = P2.subscheme([x^2 + y^2 + z^2])
sage: E.fan()
Rational polyhedral fan in 2-d lattice N
\end{verbatim}

\textbf{is_nondegenerate()}

Check if self is nondegenerate.

\textbf{OUTPUT:}

Whether the variety is nondegenerate, that is, the intersection with every open torus orbit is smooth and transversal.

\textbf{EXAMPLES:}
A K3 surface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$:

```python
sage: diamond = lattice_polytope.cross_polytope(3)
sage: fan = FaceFan(diamond)
sage: P1xP1xP1 = ToricVariety(fan)
sage: z0, z1, z2, z3, z4, z5 = P1xP1xP1.gens()
sage: t = 5
sage: F = z0^2*z1^2*z2^2 + z1^2*z2^2*z3^2 + z0^2*z2^2*z4^2 + z1^2*z3^2*z5^2 + z0^2*z1^2*z5^2 + z3^2*z4^2*z5^2
sage: X = P1xP1xP1.subscheme([F])
sage: X.is_smooth()  # True
sage: X.is_nondegenerate()  # False
```

Taking a random change of variables breaks the symmetry, but makes the surface nondegenerate:

```python
sage: F1 = F.subs(z0=1*z0 + 1*z3, z3=1*z0 + 2*z3,
               z1=-2*z1 + -1*z4, z4=1*z1 + 2*z4,
               z2=-3*z2 + -1*z5, z5=-3*z2 + 2*z5)
sage: Y = P1xP1xP1.subscheme([F1])
sage: Y.is_smooth()  # True
sage: Y.is_nondegenerate()  # True
```

This example is from Hamm, arXiv 1106.1826v1. It addresses an issue raised at github issue #15239:

```python
sage: X = toric_varieties.WP([1,4,2,3], names='z0 z1 z2 z3')
sage: X.inject_variables()
Defining z0, z1, z2, z3
sage: g0 = z1^3 + z2^6 + z3^4
sage: g = g0 - 2*z3^2*z0^6 + z2*z0^10 + z0^12
sage: Y = X.subscheme([g])
sage: Y.is_nondegenerate()  # False
```

It handles nonzero characteristic:

```python
sage: P2.<x,y,z> = toric_varieties.P2()
sage: f = x^5 + 2*x^y^4 + y^5 - 2*y^3*z^2 + x^5*z^4 - 2*z^5
```
sage: P2.change_ring(GF(5)).subscheme([f]).is_nondegenerate()
    True
sage: P2.change_ring(GF(7)).subscheme([f]).is_nondegenerate()
    False

\textbf{is\_schon()}

Check if \texttt{self} is schon (nondegenerate).

See \texttt{is\_nondegenerate()} for further documentation.

\textbf{EXAMPLES:}

sage: # needs sage.libs.singular
sage: P2.<x,y,z> = toric_varieties.P2()
sage: X = P2.subscheme([(x-y)^2*(x+y) + x*y*z + z^3])
sage: X.is_smooth()
    True
sage: X.is_schon()
    False

\textbf{is\_smooth}(\texttt{point}=\texttt{None})

Test whether the algebraic subscheme is smooth.

\textbf{INPUT:}

- \texttt{point} – A point or \texttt{None} (default). The point to test smoothness at.

\textbf{OUTPUT:}

Boolean. If no point was specified, returns whether the algebraic subscheme is smooth everywhere. Otherwise, smoothness at the specified point is tested.

\textbf{EXAMPLES:}

sage: # needs sage.libs.singular
sage: P2.<x,y,z> = toric_varieties.P2()
sage: cuspidal_curve = P2.subscheme([y^2*z - x^3])
sage: cuspidal_curve
    Closed subscheme of 2-d CPR-Fano toric variety covered by 3 affine patches defined by:
    -x^3 + y^2*z
sage: cuspidal_curve.is_smooth([1,1,1])
    True
sage: cuspidal_curve.is_smooth([0,0,1])
    False
sage: cuspidal_curve.is_smooth()
    False

Any sufficiently generic cubic hypersurface is smooth:

sage: P2.subscheme([y^2*z - x^3 + z^3 + 1/10*x*y*z]).is_smooth()
    # needs sage.libs.singular
    True

A more complicated example:
Sage: # needs sage.libs.singular
sage: dP6.<x0,x1,x2,x3,x4,x5> = toric_varieties.dP6()
sage: disjointP1s = dP6.subscheme(x0*x3)
sage: disjointP1s.is_smooth()
True
sage: intersectingP1s = dP6.subscheme(x0*x1)
sage: intersectingP1s.is_smooth()
False

A smooth hypersurface in a compact singular toric variety:

Sage: # needs sage.libs.singular
sage: lp = LatticePolytope([(1,0,0), (1,1,0), (1,1,1), (1,0,1), (-2,-1,-1)],
                          lattice=ToricLattice(3))
sage: X.<x,y,u,v,t> = CPRFanoToricVariety(Delta_polar=lp)
sage: Y = X.subscheme(x*v + y*u + t)
sage: cone = Cone([(1,0,0), (1,1,0), (1,1,1), (1,0,1)])
sage: Y.is_smooth()
True

neighborhood(point)

Return an toric algebraic scheme isomorphic to neighborhood of the point.

INPUT:

• point – a point of the toric algebraic scheme.

OUTPUT:

An affine toric algebraic scheme (polynomial equations in an affine toric variety) with fixed embedding_morphism() and embedding_center().

EXAMPLES:

Sage: # needs sage.libs.singular
sage: P.<x,y,z> = toric_varieties.P2()
sage: S = P.subscheme(x + 2*y + 3*z)
sage: s = S.point([0,-3,2]); s
[0 : -3 : 2]
sage: patch = S.neighborhood(s); patch
Closed subscheme of 2-d affine toric variety defined by:
  x + 2*y + 6
sage: patch.embedding_morphism()
Scheme morphism:
  From: Closed subscheme of 2-d affine toric variety defined by: x + 2*y + 6
  To:  Closed subscheme of 2-d CPR-Fano toric variety covered by 3 affine patches defined by: x + 2*y + 3*z
  Defn: Defined on coordinates by sending [x : y] to [-2*y - 6 : y : 2]
sage: patch.embedding_center()
[0 : -3]
sage: patch.embedding_morphism()(patch.embedding_center())
[0 : -3 : 2]

A more complicated example:

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16.10 Weierstrass form of a toric elliptic curve

There are 16 reflexive polygons in the plane, see ReflexivePolytopes(). Each of them defines a toric Fano variety. And each of them has a unique crepant resolution to a smooth toric surface (Section 10.4 in [CLS2011]) by subdividing the face fan. An anticanonical hypersurface defines an elliptic curve in this ambient space, which we call a toric elliptic curve. The purpose of this module is to write an anticanonical hypersurface equation in the short Weierstrass form \( y^2 = x^3 + fx + g \). This works over any base ring as long as its characteristic \( \neq 2, 3 \).

For an analogous treatment of elliptic curves defined as complete intersection in higher dimensional toric varieties, see the module weierstrass_higher.

Technically, this module computes the Weierstrass form of the Jacobian of the elliptic curve. This is why you will never have to specify the origin (or zero section) in the following.

It turns out [Bra2011] that the anticanonical hypersurface equation of any one of the above 16 toric surfaces is a specialization (that is, set one or more of the coefficients to zero) of the following three cases. In inhomogeneous coordinates, they are

- Cubic in \( \mathbb{P}^2 \):
  \[
  p(x, y) = a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}
  \]

- Biquadric in \( \mathbb{P}^1 \times \mathbb{P}^1 \):
  \[
  p(x, y) = a_{22}x^2y^2 + a_{21}x^2y + a_{20}x^2 + a_{12}xy^2 + a_{11}xy + xa_{10} + y^2a_{02} + ya_{01} + a_{00}
  \]

- Anticanonical hypersurface in weighted projective space \( \mathbb{P}^2[1, 1, 2] \):
  \[
  p(x, y) = a_{40}x^4 + a_{30}x^3 + a_{21}x^2y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}
  \]

EXAMPLES:

The main functionality is provided by WeierstrassForm(), which brings each of the above hypersurface equations into Weierstrass form:
```python
sage: R.<x,y> = QQ[]
sage: cubic = x^3 + y^3 + 1
sage: WeierstrassForm(cubic)
(0, -27/4)
sage: WeierstrassForm(x^4 + y^2 + 1)
(-4, 0)
sage: WeierstrassForm(x^2*y^2 + x^2 + y^2 + 1)
(-16/3, 128/27)
```

Only the affine span of the Newton polytope of the polynomial matters. For example:

```python
sage: R.<x,y,z> = QQ[]
sage: WeierstrassForm(x^3 + y^3 + z^3)
(0, -27/4)
sage: WeierstrassForm(x * cubic)
(0, -27/4)
```

This allows you to work with either homogeneous or inhomogeneous variables. For example, here is the del Pezzo surface of degree 8:

```python
sage: dP8 = toric_varieties.dP8()
sage: dP8.inject_variables()
Defining t, x, y, z
sage: WeierstrassForm(x*y^2 + y^2*z + t^2*x^3 + t^2*z^3)
(-3, -2)
sage: WeierstrassForm(x*y^2 + y^2 + x^3 + 1)
(-3, -2)
```

By specifying only certain variables we can compute the Weierstrass form over the polynomial ring generated by the remaining variables. For example, here is a cubic over $\mathbb{Q}[a]$

```python
sage: R.<a, x, y, z> = QQ[]
sage: cubic = x^3 + a*y^3 + a^2*z^3
sage: WeierstrassForm(cubic, variables=[x,y,z])
(0, -27/4*a^6)
```

REFERENCES:
- [Bra2011]
- [Du2010]
- [ARVT2005]
- [CLS2011]

```
sage.schemes.toric.weierstrass.Discriminant(polynomial, variables=None)
The discriminant of the elliptic curve.
```

INPUT:
See `WeierstrassForm()` for how to specify the input polynomial(s) and variables.

OUTPUT: The discriminant of the elliptic curve.

EXAMPLES:
sage: from sage.schemes.toric.weierstrass import Discriminant
sage: R.<x, y, z> = QQ[]
sage: Discriminant(x^3 + y^3 + z^3)
19683/16
sage: Discriminant(x*y*z)
0
sage: R.<w,x,y,z> = QQ[]
sage: quadratic1 = w^2 + x^2 + y^2
sage: quadratic2 = z^2 + w*x
sage: Discriminant([quadratic1, quadratic2])
-1/16

sage.schemes.toric.weierstrass.Newton_polygon_embedded(polynomial, variables)

Embed the Newton polytope of the polynomial in one of the three maximal reflexive polygons.

This function is a helper for WeierstrassForm()

INPUT:

Same as WeierstrassForm() with only a single polynomial passed.

OUTPUT:

A tuple \((\Delta, P(x, y))\) where

- \(\Delta\) is the Newton polytope of polynomial.
- \(P(x, y)\) equals the input polynomial but with redefined variables such that its Newton polytope is \(\Delta\).

EXAMPLES:

sage: from sage.schemes.toric.weierstrass import Newton_polygon_embedded
sage: R.<x,y,z> = QQ[]
sage: cubic = x^3 + y^3 + z^3
sage: Newton_polygon_embedded(cubic, [x,y,z])
(A 2-dimensional lattice polytope in ZZ^3 with 3 vertices, x^3 + y^3 + 1, (x, y))

sage: R.<a, x,y,z> = QQ[]
sage: cubic = x^3 + a*y^3 + a^2*z^3
sage: Newton_polygon_embedded(cubic, variables=[x,y,z])
(A 2-dimensional lattice polytope in ZZ^3 with 3 vertices, a^2*x^3 + y^3 + a, (x, y))

sage: R.<s,t,x,y> = QQ[]
sage: biquadric = (s+t)^2 * (x+y)^2
sage: Newton_polygon_embedded(biquadric, [s,t,x,y])
(A 2-dimensional lattice polytope in ZZ^4 with 4 vertices, s^2*t^2 + 2*s^2*t + 2*s*t^2 + s^2 + 4*s*t + t^2 + 2*s + 2*t + 1, (s, t))

sage.schemes.toric.weierstrass.Newton_polytope_vars_coeffs(polynomial, variables)

Return the Newton polytope in the given variables.

INPUT:
See `WeierstrassForm()` for how to specify the input polynomial and variables.

**OUTPUT:**

A dictionary with keys the integral values of the Newton polytope and values the corresponding coefficient of polynomial.

**EXAMPLES:**

```python
sage: from sage.schemes.toric.weierstrass import Newton_polytope_vars_coeffs
sage: R.<x,y,z,a30,a21,a12,a03,a20,a11,a02,a10,a01,a00> = QQ[]
sage: p = (a30*x^3 + a21*x^2*y + a12*x*y^2 + a03*y^3 + a20*x^2*z +
.....: a11*x*y*z + a02*y^2*z + a10*x*z^2 + a01*y*z^2 + a00*z^3)
sage: p_data = Newton_polytope_vars_coeffs(p, [x,y,z]); p_data
{(0, 0, 3): a00,
 (0, 1, 2): a01,
 (0, 2, 1): a02,
 (0, 3, 0): a03,
 (1, 0, 2): a10,
 (1, 1, 1): a11,
 (1, 2, 0): a12,
 (2, 0, 1): a20,
 (2, 1, 0): a21,
 (3, 0, 0): a30}
```

```python
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import LatticePolytope_PPL
sage: polytope = LatticePolytope_PPL(list(p_data)); polytope
A 2-dimensional lattice polytope in ZZ^3 with 3 vertices
sage: polytope.vertices()
((0, 0, 3), (3, 0, 0), (0, 3, 0))
sage: polytope.embed_in_reflexive_polytope()
The map A^*x+b with A=
[[-1  -1]
 [ 0   1]
 [ 1   0]
 b =
(3, 0, 0)
```

**sage.schemes.toric.weierstrass.WeierstrassForm**(polynomial, variables=\text{None}, transformation=\text{False})

Return the Weierstrass form of an elliptic curve inside either inside a toric surface or \(\mathbb{P}^3\).

**INPUT:**

- **polynomial** – either a polynomial or a list of polynomials defining the elliptic curve. A single polynomial can be either a cubic, a biquadric, or the hypersurface in \(\mathbb{P}^2[1, 1, 2]\). In this case the equation need not be in any standard form, only its Newton polyhedron is used. If two polynomials are passed, they must both be quadrics in \(\mathbb{P}^3\).

- **variables** – a list of variables of the parent polynomial ring or \text{None} (default). In the latter case, all variables are taken to be polynomial ring variables. If a subset of polynomial ring variables are given, the Weierstrass form is determined over the function field generated by the remaining variables.

- **transformation** – boolean (default: False). Whether to return the new variables that bring polynomial into Weierstrass form.

**OUTPUT:**

The pair of coefficients \((f, g)\) of the Weierstrass form \(y^2 = x^3 + fx + g\) of the hypersurface equation.
If `transformation=True`, a triple \((X, Y, Z)\) of polynomials defining a rational map of the toric hypersurface or complete intersection in \(\mathbb{P}^3\) to its Weierstrass form in \(\mathbb{P}^2[2, 3, 1]\) is returned. That is, the triple satisfies

\[ Y^2 = X^3 + fXZ^4 + gZ^6 \]

when restricted to the toric hypersurface or complete intersection.

**EXAMPLES:**

```python
sage: R.<x,y,z> = QQ[]
sage: cubic = x^3 + y^3 + z^3
sage: f, g = WeierstrassForm(cubic); (f, g)
(0, -27/4)
```

Same in inhomogeneous coordinates:

```python
sage: R.<x,y> = QQ[]
sage: cubic = x^3 + y^3 + 1
sage: f, g = WeierstrassForm(cubic); (f, g)
(0, -27/4)
```

```python
sage: X,Y,Z = WeierstrassForm(cubic, transformation=True); (X,Y,Z)
(-x^3*y^3 - x^3 - y^3,
  1/2*x^6*y^3 - 1/2*x^3*y^6 - 1/2*x^6 + 1/2*y^6 + 1/2*x^3 - 1/2*y^3,
  x*y)
```

Note that plugging in \([X : Y : Z]\) to the Weierstrass equation is a complicated polynomial, but contains the hypersurface equation as a factor:

```python
sage: -Y^2 + X^3 + f*X*Z^4 + g*Z^6
-1/4*x^12*y^6 - 1/2*x^9*y^9 - 1/4*x^6*y^12 + 1/2*x^12*y^3
- 7/2*x^9*y^9 + 7/2*x^6*y^12 - 7/2*x^12*y^3
- 45/4*x^6*y^6 - 7/2*x^3*y^9 - 7/2*x^9*y^3
- 7/2*x^3*y^6 - 1/2*y^9 - 1/4*x^6 + 1/2*x^3*y^3 - 1/4*y^6
sage: cubic.divides(-Y^2 + X^3 + f*X*Z^4 + g*Z^6)
True
```

Only the affine span of the Newton polytope of the polynomial matters. For example:

```python
sage: R.<x,y,z> = QQ[]
sage: cubic = x^3 + y^3 + z^3
sage: WeierstrassForm(cubic.subs(z=1))
(0, -27/4)
sage: WeierstrassForm(x * cubic)
(0, -27/4)
```

This allows you to work with either homogeneous or inhomogeneous variables. For example, here is the del Pezzo surface of degree 8:

```python
sage: dP8 = toric_varieties.dP8()
sage: dP8.inject_variables()
Defining t, x, y, z
sage: WeierstrassForm(x*y^2 + y^2*z + t^2*x^3 + t^2*z^3)
(-3, -2)
sage: WeierstrassForm(x*y^2 + y^2 + x^3 + 1)
(-3, -2)
```
By specifying only certain variables we can compute the Weierstrass form over the function field generated by the remaining variables. For example, here is a cubic over $\mathbb{Q}[a]$

```
sage: R.<a, x,y,z> = QQ[]
sage: cubic = x^3 + a*y^3 + a^2*z^3
sage: WeierstrassForm(cubic, variables=[x,y,z])
(0, -27/4*a^6)
```

```
sage.schemes.toric.weierstrass.WeierstrassForm_P1xP1(biquadric, variables=None)
```
Bring a biquadric into Weierstrass form

Input/output is the same as `WeierstrassForm()`, except that the input polynomial must be a standard biquadric in $\mathbb{P}^2$,

$$p(x, y) = a_{40}x^4 + a_{30}x^3 + a_{21}x^2y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}$$

EXAMPLES:

```
sage: from sage.schemes.toric.weierstrass import WeierstrassForm_P1xP1
sage: R.<x0,x1,y0,y1> = QQ[]
sage: biquadric = (x0^2*y0^2 + x0*x1*y0^2*2 + x1^2*y0^2*3
..... + x0^2*y0*y1*4 + x0*x1*y0*y1*5 + x1^2*y0*y1*6
..... + x0^2*y1^2*7 + x0*x1*y1^2*8)
sage: WeierstrassForm_P1xP1(biquadric, [x0, x1, y0, y1])
(1581/16, -3529/32)
```

Since there is no $x_1^2y_1^2$ term in biquadric, we can dehomogenize it and get a cubic:

```
sage: from sage.schemes.toric.weierstrass import WeierstrassForm_P2
sage: WeierstrassForm_P2(biquadric(x0=1,y0=1))
(1581/16, -3529/32)
```

```
sage.schemes.toric.weierstrass.WeierstrassForm_P2(cubic, variables=None)
```
Bring a cubic into Weierstrass form.

Input/output is the same as `WeierstrassForm()`, except that the input polynomial must be a standard cubic in $\mathbb{P}^2$,

$$p(x, y) = a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}$$

EXAMPLES:

```
sage: from sage.schemes.toric.weierstrass import WeierstrassForm_P2
sage: R.<x,y,z> = QQ[]
sage: WeierstrassForm_P2(x^3 + y^3 + z^3)
(0, -27/4)
sage: R.<x,y,z, a,b> = QQ[]
sage: WeierstrassForm_P2(-y^2*z + x^3 + a*x*z^2 + b*z^3, [x,y,z])
(a, b)
```

```
sage.schemes.toric.weierstrass.WeierstrassForm_P2_112(polynomial, variables=None)
```
Bring an anticanonical hypersurface in $\mathbb{P}^2[1, 1, 2]$ into Weierstrass form.

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Input/output is the same as \texttt{WeierstrassForm()}, except that the input polynomial must be a standard anticanonical hypersurface in weighted projective space \( \mathbb{P}^2[1,1,2] \):

\[
p(x,y) = a_{40}x^4 + a_{30}x^3 + a_{21}x^2y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}
\]

EXAMPLES:

```python
sage: from sage.schemes.toric.weierstrass import WeierstrassForm_P2_112
sage: fan = Fan(rays=[(1,0),(0,1),(-1,-2),(0,-1)],
.....: cones=[[0,1],[1,2],[2,3],[3,0]])
.....: P112.<x,y,z,t> = ToricVariety(fan)
sage: (-P112.K()).sections_monomials()
(z^4*t^2, x*z^3*t^2, x^2*z^2*t^2, x^3*z*t^2,
 x^4*t^2, y*z^2*t, x*y*z*t, x^2*y*t, y^2)
```

```python
sage: WeierstrassForm_P2_112(sum(_), [x,y,z,t])
(-97/48, 17/864)
```

\texttt{sage.schemes.toric.weierstrass.j_invariant} \( (\text{polynomial}, \text{variables=None}) \)

Return the \( j \)-invariant of the elliptic curve.

**INPUT:**

See \texttt{WeierstrassForm()} for how to specify the input polynomial(s) and variables.

**OUTPUT:**

The \( j \)-invariant of the (irreducible) cubic. Notable special values:

- The Fermat cubic: \( j(x^3 + y^3 + z^3) = 0 \)
- A nodal cubic: \( j(-y^2 + x^2 + x^3) = \infty \)
- A cuspidal cubic \( y^2 = x^3 \) has undefined \( j \)-invariant. In this case, a \texttt{ValueError} is raised.

**EXAMPLES:**

```python
sage: from sage.schemes.toric.weierstrass import j_invariant
sage: R.<x,y,z> = QQ[]
sage: j_invariant(x^3 + y^3 + z^3)
0
sage: j_invariant(-y^2 + x^2 + x^3)
+Infinity
sage: R.<x,y,z, a,b> = QQ[]
sage: j_invariant(-y^2*z + x^3 + a*x*z^2, [x,y,z])
1728
```

### 16.11 Map to the Weierstrass form of a toric elliptic curve

There are 16 reflexive polygons in 2-d. Each defines a toric Fano variety, which (since it is 2-d) has a unique crepant resolution to a smooth toric surface. An anticanonical hypersurface defines a genus one curve \( C \) in this ambient space, with Jacobian elliptic curve \( J(C) \) which can be defined by the Weierstrass model \( y^2 = x^3 + f(x) + g(x) \). The coefficients \( f \) and \( g \) can be computed with the \texttt{weierstrass} module. The purpose of this model is to give an explicit rational map \( C \to J(C) \). This is an \( n^2 \)-cover, where \( n \) is the minimal multi-section of \( C \).

Since it is technically often easier to deal with polynomials than with fractions, we return the rational map in terms of homogeneous coordinates. That is, the ambient space for the Weierstrass model is the weighted projective space.
With homogeneous coordinates \([X:Y:Z] = [\lambda^2X, \lambda^3Y, \lambda Z]\). The homogenized Weierstrass equation is
\[
Y^2 = X^3 + fXZ^4 + gZ^6
\]

**EXAMPLES:**

```
sage: R.<x,y> = QQ[]
sage: cubic = x^3 + y^3 + 1
sage: f, g = WeierstrassForm(cubic); (f,g)
(0, -27/4)
```

That is, this hypersurface \(C \in P^2\) has a Weierstrass equation \(Y^2 = X^3 + 0 \cdot XZ^4 - \frac{27}{4} Z^6\) where \([X:Y:Z]\) are projective coordinates on \(P^2[2,3,1]\). The form of the map \(C \to J(C)\) is:

```
sage: X,Y,Z = WeierstrassForm(cubic, transformation=True); (X,Y,Z)
(-x^3*y^3 - x^3 - y^3,
 1/2*x^6*y^3 - 1/2*x^3*y^6 - 1/2*x^6 + 1/2*y^6 + 1/2*x^3 - 1/2*y^3,
x*y)
```

Note that plugging in \([X:Y:Z]\) to the Weierstrass equation is a complicated polynomial, but contains the hypersurface equation as a factor:

```
sage: -Y^2 + X^3 + f*X*Z^4 + g*Z^6
-Y^2 + X^3 + f*X*Z^4 + g*Z^6
```

If you prefer you can also use homogeneous coordinates for \(C \in P^2\)

```
sage: R.<x,y,z> = QQ[]
sage: cubic = x^3 + y^3 + z^3
sage: f, g = WeierstrassForm(cubic); (f,g)
(0, -27/4)
sage: X,Y,Z = WeierstrassForm(cubic, transformation=True)
sage: cubic.divides(-Y^2 + X^3 + f*X*Z^4 + g*Z^6)
True
```

The 16 toric surfaces corresponding to the 16 reflexive polygons can all be blown down to \(P^2, P^1 \times P^1,\) or \(P^2[1,1,2]\). Their (and hence in all 16 cases) anticanonical hypersurface can equally be brought into Weierstrass form. For example, here is an anticanonical hypersurface in \(P^2[1,1,2]\)

```
sage: P2_112 = toric_varieties.P2_112()
sage: C = P2_112.anticanonical_hypersurface(coefficients=[1]*4); C
Closed subscheme of 2-d CPR-Fano toric variety covered by 3 affine patches defined by:
  z0^4 + z2^4 + z0*z1*z2 + z1^2
sage: eq = C.defining_polynomials()[0]
sage: f, g = WeierstrassForm(eq)
sage: X,Y,Z = WeierstrassForm(eq, transformation=True)
sage: (-Y^2 + X^3 + f*X*Z^4 + g*Z^6).reduce(C.defining_ideal())
0
```

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Finally, you sometimes have to manually specify the variables to use. This is either because the equation is degenerate or because it contains additional variables that you want to treat as coefficients:

```
sage: R.<a, x,y,z> = QQ[]
sage: cubic = x^3 + y^3 + z^3 + a*x*y*z
sage: f, g = WeierstrassForm(cubic, variables=[x,y,z])
sage: X,Y,Z = WeierstrassForm(cubic, variables=[x,y,z], transformation=True)
sage: cubic.divides(-Y^2 + X^3 + f*X*Z^4 + g*Z^6)
True
```

REFERENCES:

• [AKMMMP2002]

`sage.schemes.toric.weierstrass_covering.WeierstrassMap(polynomial, variables=None)`

Return the Weierstrass form of an anticanonical hypersurface.

You should use `sage.schemes.toric.weierstrass.WeierstrassForm()` with `transformation=True` to get the transformation. This function is only for internal use.

INPUT:

• `polynomial` – a polynomial. The toric hypersurface equation. Can be either a cubic, a biquadratic, or the hypersurface in $\mathbb{P}^2[1,1,2]$. The equation need not be in any standard form, only its Newton polyhedron is used.

• `variables` – a list of variables of the parent polynomial ring or `None` (default). In the latter case, all variables are taken to be polynomial ring variables. If a subset of polynomial ring variables are given, the Weierstrass form is determined over the function field generated by the remaining variables.

OUTPUT:

A triple $(X, Y, Z)$ of polynomials defining a rational map of the toric hypersurface to its Weierstrass form in $\mathbb{P}^2[2,3,1]$. That is, the triple satisfies

$$Y^2 = X^3 + fXZ^4 + gZ^6$$

when restricted to the toric hypersurface.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: cubic = x^3 + y^3 + z^3
sage: X,Y,Z = WeierstrassForm(cubic, transformation=True); (X,Y,Z)
(-x^3*y^3 - x^3*z^3 - y^3*z^3,
 1/2*x^6*y^3 - 1/2*x^3*y^6 - 1/2*x^6*z^3 + 1/2*y^6*z^3 + 1/2*x^3*z^6 - 1/2*y^3*z^6,
x^3*z
)sage: f, g = WeierstrassForm(cubic); (f,g)
(0, -27/4)
sage: cubic.divides(-Y^2 + X^3 + f*X*Z^4 + g*Z^6)
True
```

Only the affine span of the Newton polytope of the polynomial matters. For example:

```
sage: WeierstrassForm(cubic.subs(z=1), transformation=True)
(-x^3*y^3 - x^3 - y^3,
 1/2*x^6*y^3 - 1/2*x^3*y^6 - 1/2*x^6)
```

(continues on next page)
+ 1/2*y^6 + 1/2*x^3 - 1/2*y^3,
  x*y)
sage: WeierstrassForm(x * cubic, transformation=True)
(-x^3*y^3 - x^3*z^3 - y^3*z^3,
  1/2*x^6*y^3 - 1/2*x^3*y^6 - 1/2*x^6*z^3 + 1/2*y^6*z^3
  + 1/2*x^3*z^6 - 1/2*y^3*z^6,
  x*y*z)

This allows you to work with either homogeneous or inhomogeneous variables. For example, here is the del Pezzo surface of degree 8:

sage: dP8 = toric_varieties.dP8()
sage: dP8.inject_variables()
Defining t, x, y, z
sage: WeierstrassForm(x*y^2 + y^2*z + t^2*x^3 + t^2*z^3, transformation=True)
(-1/27*t^4*x^6 - 2/27*t^4*x^5*z - 5/27*t^4*x^4*z^2
 - 8/27*t^4*x^3*z^3 - 5/27*t^4*x^2*z^4 - 2/27*t^4*x*z^5
 - 1/27*t^4*z^6 - 4/81*t^2*x^2*y^2 - 4/81*t^2*x*y^2*z
 - 4/81*t^2*y^2*z^3 - 4/81*t^2*y^2*z^4 - 2/81*x^2*y^4
 - 4/81*x*y^4*z - 2/81*x*y^4*z^2,
  0,
  1/3*t^2*x^2*z + 1/3*t^2*x*z^2 - 1/9*x*y^2 - 1/9*y^2*z)
sage: WeierstrassForm(x*y^2 + y^2 + x^3 + 1, transformation=True)
(-1/27*x^6 - 4/81*x^4*y^2 - 2/81*x^2*y^4 - 2/27*x^5
 - 4/81*x^3*y^2 - 4/81*x*y^4 - 5/27*x^4 - 2/27*y^4 - 8/27*x^3
 - 4/81*x^2*y^2 - 5/27*x^2 - 4/81*y^2 - 2/27*x - 1/27,
  0,
 -1/9*x*y^2 + 1/3*x^2 - 1/9*y^2 + 1/3*x)

By specifying only certain variables we can compute the Weierstrass form over the function field generated by the remaining variables. For example, here is a cubic over $\mathbb{Q}[a]$

sage: R.<a, x,y,z> = QQ[]
sage: cubic = x^3 + a*y^3 + a^2*z^3
sage: WeierstrassForm(cubic, variables=[x,y,z], transformation=True)
(-a^9*y^3*z^3 - a^8*x^3*z^3 - a^7*x^3*y^3,
 -1/2*a^14*y^3*z^6 + 1/2*a^13*y^6*z^3 + 1/2*a^13*x^3*z^6
 - 1/2*a^11*x^3*y^6 - 1/2*a^11*x^6*z^3 + 1/2*a^10*x^6*y^3,
  a^3*x*y^z)

sage.schemes.toric.weierstrass_covering.WeierstrassMap_P1xP1(polynomial, variables=None)

Map an anticanonical hypersurface in $\mathbb{P}^1 \times \mathbb{P}^1$ into Weierstrass form.

Input/output is the same as WeierstrassMap(), except that the input polynomial must be a standard anticanonical hypersurface in the toric surface $\mathbb{P}^1 \times \mathbb{P}^1$:

EXAMPLES:

sage: from sage.schemes.toric.weierstrass_covering import WeierstrassMap_P1xP1
sage: from sage.schemes.toric import WeierstrassForm_P1xP1
sage: R.<x0,x1,y0,y1,a> = QQ[]
sage: biquadric = (x0^2*y0^2 + x1^2*y1^2 + x0^2*y1 + x1^2*y0 +
  + ....: a * x0^2*x1*y0^2*y1^3)

(continues on next page)
sage: f, g = WeierstrassForm_P1xP1(biquadric, [x0, x1, y0, y1]); (f,g)
(-625/48*a^4 + 25/3*a^2 - 16/3, 15625/864*a^6 - 625/36*a^4 - 100/9*a^2 + 128/27)
sage: X, Y, Z = WeierstrassMap_P1xP1(biquadric, [x0, x1, y0, y1])
sage: (-Y^2 + X^3 + f*X*Z^4 + g*Z^6).reduce(R.ideal(biquadric))
needs sage.libs.singular

sage: R = PolynomialRing(QQ, 'x,y,s,t', order='lex')
sage: R.inject_variables()
Defining x, y, s, t
sage: equation = (s^2*(x^2+2*x*y+3*y^2) + s*t*(4*x^2+5*x*y+6*y^2) + t^2*(7*x^2+8*x*y+9*y^2))
sage: X, Y, Z = WeierstrassMap_P1xP1(equation, [x,y,s,t])
sage: f, g = WeierstrassForm_P1xP1(equation, variables=[x,y,s,t])
sage: (-Y^2 + X^3 + f*X*Z^4 + g*Z^6).reduce(R.ideal(equation))
needs sage.libs.singular

sage.schemes.toric.weierstrass_covering.WeierstrassMap_P2(polynomial, variables=None)

Map a cubic to its Weierstrass form.

Input/output is the same as WeierstrassMap(), except that the input polynomial must be a cubic in \( \mathbb{P}^2 \),
\[
p(x, y) = a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}
\]

EXAMPLES:

```python
sage: from sage.schemes.toric.weierstrass import WeierstrassForm_P2
sage: from sage.schemes.toric.weierstrass_covering import WeierstrassMap_P2
sage: R.<x,y,z> = QQ[]
sage: equation = x^3 + y^3 + z^3 + x*y*z
sage: f, g = WeierstrassForm_P2(equation)
sage: X,Y,Z = WeierstrassMap_P2(equation)
sage: equation.divides(-Y^2 + X^3 + f*X*Z^4 + g*Z^6)
True
```

```python
sage: from sage.schemes.toric.weierstrass import WeierstrassForm_P2
sage: from sage.schemes.toric.weierstrass_covering import WeierstrassMap_P2
sage: R.<x,y> = QQ[]
sage: equation = x^3 + y^3 + 1
sage: f, g = WeierstrassForm_P2(equation)
sage: X,Y,Z = WeierstrassMap_P2(equation)
```
sage: equation.divides(-Y^2 + X^3 + f*X*Z^4 + g*Z^6)
True

sage.schemes.toric.weierstrass_covering.WeierstrassMap_P2_112(polynomial, variables=None)
Map an anticanonical hypersurface in P^2[1, 1, 2] into Weierstrass form.

Input/output is the same as WeierstrassMap(), except that the input polynomial must be a standard anticanonical hypersurface in weighted projective space P^2[1, 1, 2]:

\[ p(x, y) = a_{40}x^4 + a_{30}x^3 + a_{21}x^2y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00} \]

EXAMPLES:

sage: from sage.schemes.toric.weierstrass_covering import WeierstrassMap_P2_112
sage: from sage.schemes.toric.weierstrass import WeierstrassForm_P2_112

sage: R = PolynomialRing(QQ, 'x,y,a0,a1,a2,a3,a4', order='lex')
sage: R.inject_variables()
Defining x, y, a0, a1, a2, a3, a4
sage: equation = y^2 + a0*x^4 + 4*a1*x^3 + 6*a2*x^2 + 4*a3*x + a4
sage: X, Y, Z = WeierstrassMap_P2_112(equation, [x,y])
sage: f, g = WeierstrassForm_P2_112(equation, variables=[x,y])
sage: (-Y^2 + X^3 + f*X*Z^4 + g*Z^6).reduce(R.ideal(equation))
\$\$
\[ \text{needs sage.libs.singular} \]

Another example, this time in homogeneous coordinates:

sage: fan = Fan(rays=[(1,0),(0,1),(-1,-2),(0,-1)],
...: cones=[[0,1],[1,2],[2,3],[3,0]])
sage: P112.<x,y,z,t> = ToricVariety(fan)
sage: (-P112.K()).sections_monomials()
(z^4*t^2, x*z^3*t^2, x^2*z^2*t^2, x^3*z*t^2,
 x^4*t^2, y*z^2*t, x*y*z*t, x^2*y*t, y^2)
sage: C_eqn = sum(_)
sage: C = P112.subscheme(C_eqn)
sage: WeierstrassForm_P2_112(C_eqn, [x,y,z,t])
(-97/48, 17/864)
sage: X, Y, Z = WeierstrassMap_P2_112(C_eqn, [x,y,z,t])
sage: (-Y^2 + X^3 - 97/48*X*Z^4 + 17/864*Z^6).reduce(C.defining_ideal())
\$\$
\[ \text{needs sage.libs.singular} \]

16.12 Weierstrass for elliptic curves in higher codimension

The weierstrass module lets you transform a genus-one curve, given as a hypersurface in a toric surface, into Weierstrass form. The purpose of this module is to extend this to higher codimension subschemes of toric varieties. In general, this is an unsolved problem. However, for certain special cases this is known.

The simplest codimension-two case is the complete intersection of two quadratic equations in \( \mathbb{P}^3 \)
sage: R.<w,x,y,z> = QQ[]
sage: quadratic1 = w^2 + x^2 + y^2
sage: quadratic2 = z^2 + w*x
sage: WeierstrassForm([quadratic1, quadratic2])
(-1/4, 0)

Hence, the Weierstrass form of this complete intersection is $Y^2 = X^3 - \frac{1}{4} X Z^4$.

Helper function for `WeierstrassForm()`

Currently, only the case of the complete intersection of two quadratic equations in $\mathbb{P}^3$ is supported.

INPUT / OUTPUT:

See `WeierstrassForm()`

`sage.schemes.toric.weierstrass_higher.WeierstrassForm_P3(quadratic1, quadratic2, variables=None)`

Bring a complete intersection of two quadratics into Weierstrass form.

Input/output is the same as `sage.schemes.toric.weierstrass.WeierstrassForm()`, except that the two input polynomials must be quadratic polynomials in $\mathbb{P}^3$.

EXAMPLES:

```python
sage: from sage.schemes.toric.weierstrass_higher import WeierstrassForm_P3
sage: R.<w,x,y,z> = QQ[]
sage: quadratic1 = w^2 + x^2 + y^2
sage: quadratic2 = z^2 + w*x
sage: WeierstrassForm_P3(quadratic1, quadratic2)
(-1/4, 0)
```

`sage.schemes.toric.weierstrass_higher.WeierstrassMap_P3(quadratic1, quadratic2, variables=None)`

Bring a complete intersection of two quadratics into Weierstrass form.

Input/output is the same as `sage.schemes.toric.weierstrass.WeierstrassForm()`, except that the two input polynomials must be quadratic polynomials in $\mathbb{P}^3$.

EXAMPLES:

```python
sage: from sage.schemes.toric.weierstrass_higher import WeierstrassForm_P3
sage: from sage.schemes.toric.weierstrass_higher import WeierstrassMap_P3
sage: R.<w,x,y,z> = QQ[]
sage: quadratic1 = w^2 + x^2 + y^2
sage: quadratic2 = z^2 + w*x
sage: X, Y, Z = WeierstrassMap_P3(quadratic1, quadratic2)
sage: X
1/1024*w^8 + 3/256*w^6*x^2 + 19/512*w^4*x^4 + 3/256*w^2*x^6 + 1/1024*x^8
sage: Y
1/32768*w^12 - 7/16384*w^10*x^2 - 145/32768*w^8*x^4 - 49/8192*w^6*x^6
- 145/32768*w^4*x^8 - 7/16384*w^2*x^10 + 1/32768*x^12
sage: Z
-1/8*w^2*y*z + 1/8*x^2*y*z
sage: a, b = WeierstrassForm_P3(quadratic1, quadratic2); a, b
(-1/4, 0)
```
16.13 Set of homomorphisms between two toric varieties

For schemes $X$ and $Y$, this module implements the set of morphisms $\text{Hom}(X, Y)$. This is done by `SchemeHomset_generic`.

As a special case, the Hom-sets can also represent the points of a scheme. Recall that the $K$-rational points of a scheme $X$ over $k$ can be identified with the set of morphisms $\text{Spec}(K) \to X$. In Sage, the rational points are implemented by such scheme morphisms. This is done by `SchemeHomset_points` and its subclasses.

**Note:** You should not create the Hom-sets manually. Instead, use the Hom() method that is inherited by all schemes.

**AUTHORS:**

- Volker Braun (2012-02-18): Initial version

**EXAMPLES:**

Here is a simple example, the projection of $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$

```sage
sage: P1xP1 = toric_varieties.P1xP1()
sage: P1 = toric_varieties.P1()
sage: hom_set = P1xP1.Hom(P1); hom_set
Set of morphisms
From: 2-d CPR-Fano toric variety covered by 4 affine patches
To: 1-d CPR-Fano toric variety covered by 2 affine patches
```

In terms of the fan, we can define this morphism by the projection onto the first coordinate. The Hom-set can construct the morphism from the projection matrix alone:

```sage
sage: hom_set(matrix([[1],[0]]))
Scheme morphism:
  From: 2-d CPR-Fano toric variety covered by 4 affine patches
  To: 1-d CPR-Fano toric variety covered by 2 affine patches
  Defn: Defined by sending Rational polyhedral fan in 2-d lattice $N$
         to Rational polyhedral fan in 1-d lattice $N$.
sage: _(as_polynomial_map())
Scheme morphism:
  From: 2-d CPR-Fano toric variety covered by 4 affine patches
  To: 1-d CPR-Fano toric variety covered by 2 affine patches
  Defn: Defined on coordinates by sending \([s : t : x : y]\) to \([s : t]\)
```

In the case of toric algebraic schemes (defined by polynomials in toric varieties), this module defines the underlying morphism of the ambient toric varieties:
sage: P1xP1.inject_variables()
Defining s, t, x, y
sage: S = P1xP1.subscheme([s*t - t^2])
sage: type(S.Hom(S))
<class 'sage.schemes.toric.homset.SchemeHomset_toric_variety_with_category'>

Finally, you can have morphisms defined through homogeneous coordinates where the codomain is not implemented as a toric variety:

```
sage: P2_toric.<x,y,z> = toric_varieties.P2()
sage: P2_native.<u,v,w> = ProjectiveSpace(QQ, 2)
sage: toric_to_native = P2_toric.Hom(P2_native); toric_to_native
Set of morphisms
    From: 2-d CPR-Fano toric variety covered by 3 affine patches
    To: Projective Space of dimension 2 over Rational Field
sage: type(toric_to_native)
<class 'sage.schemes.toric.homset.SchemeHomset_toric_variety_with_category'>
sage: toric_to_native([x^2, y^2, z^2])
Scheme morphism:
    From: 2-d CPR-Fano toric variety covered by 3 affine patches
    To: Projective Space of dimension 2 over Rational Field
    Defn: Defined on coordinates by sending [x : y : z] to (x^2 : y^2 : z^2)
```

```
sage: native_to_toric = P2_native.Hom(P2_toric); native_to_toric
Set of morphisms
    From: Projective Space of dimension 2 over Rational Field
    To: 2-d CPR-Fano toric variety covered by 3 affine patches
sage: type(native_to_toric)
<class 'sage.schemes.projective.projective_homset.SchemeHomset_polynomial_projective_space_with_category'>
sage: native_to_toric([u^2, v^2, w^2])
Scheme morphism:
    From: Projective Space of dimension 2 over Rational Field
    To: 2-d CPR-Fano toric variety covered by 3 affine patches
    Defn: Defined on coordinates by sending (u : v : w) to [u^2 : v^2 : w^2]
```

class sage.schemes.toric.homset.SchemeHomset_points_subscheme_toric_field(X, Y, category=None, check=True, base=Integer Ring)

Bases: SchemeHomset_points_toric_base
cardinality()

Return the number of points of the toric variety.

OUTPUT:

An integer or infinity. The cardinality of the set of points.

EXAMPLES:

```
sage: # needs sage.libs.singular
sage: P2.<x,y,z> = toric_varieties.P2(base_ring=GF(5))
sage: cubic = P2.subscheme([x^3 + y^3 + z^3])
```
class sage.schemes.toric.homset.SchemeHomset_points_toric_base(X, Y, category=None, check=True, base=Integer Ring)

Bases: SchemeHomset_points

Base class for homsets with toric ambient spaces.

INPUT:

• same as for SchemeHomset_points.

OUTPUT: A scheme morphism of type SchemeHomset_points_toric_base.

EXAMPLES:

sage: P1xP1 = toric_varieties.P1xP1()
sage: P1xP1(QQ)
Set of rational points of 2-d CPR-Fano toric variety covered by 4 affine patches

is_finite()

Return whether there are finitely many points.

OUTPUT: A boolean.

EXAMPLES:

sage: P2 = toric_varieties.P2()
sage: P2.point_set().is_finite()
False
sage: P2.change_ring(GF(7)).point_set().is_finite()
True

class sage.schemes.toric.homset.SchemeHomset_points_toric_field(X, Y, category=None, check=True, base=Integer Ring)

Bases: SchemeHomset_points_toric_base

Set of rational points of a toric variety.

You should not use this class directly. Instead, use the point_set() method to construct the point set of a toric variety.

INPUT:

• same as for SchemeHomset_points.

OUTPUT: A scheme morphism of type SchemeHomset_points_toric_field.

EXAMPLES:

sage: P1xP1 = toric_varieties.P1xP1()
sage: P1xP1.point_set()
Set of rational points of 2-d CPR-Fano toric variety covered by 4 affine patches
The quotient $\mathbb{P}^2/\mathbb{Z}_3$ over $GF(7)$ by the diagonal action. This is tricky because the base field has a 3-rd root of unity:

```sage
sage: fan = NormalFan(ReflexivePolytope(2, 0))

sage: X = ToricVariety(fan, base_field=GF(7))

sage: point_set = X.point_set()

sage: point_set.cardinality()
21
```

As for a non-compact example, the blow-up of the plane is the line bundle $O_{\mathbb{P}^1}(-1)$. Its point set is the Cartesian product of the points on the base $\mathbb{P}^1$ with the points on the fiber:

```sage
sage: fan = Fan([Cone([1,0]), Cone([1,1])])

sage: blowup_plane = ToricVariety(fan, base_ring=GF(3))

sage: point_set = blowup_plane.point_set()

sage: sorted(point_set.list())

[[0 : 1 : 0], [0 : 1 : 1], [0 : 1 : 2],
 [1 : 0 : 0], [1 : 0 : 1], [1 : 0 : 2],
 [1 : 1 : 0], [1 : 1 : 1], [1 : 1 : 2],
 [1 : 2 : 0], [1 : 2 : 1], [1 : 2 : 2]]
```

Toric varieties with torus factors (that is, where the fan is not full-dimensional) also work:

```sage
sage: F_times_Fstar = ToricVariety(Fan([Cone([1,0])]), base_field=GF(3))

sage: sorted(F_times_Fstar.point_set().list())

[[0 : 1], [0 : 2], [1 : 1], [1 : 2], [2 : 1], [2 : 2]]
```

cardinality()

Return the number of points of the toric variety.

OUTPUT:

An integer or infinity. The cardinality of the set of points.

EXAMPLES:

```sage
sage: o = lattice_polytope.cross_polytope(3)

sage: V = ToricVariety(FaceFan(o))

sage: V.change_ring(GF(2)).point_set().cardinality()

27

sage: V.change_ring(GF(8, "a")).point_set().cardinality()

# needs sage.rings.finite_rings

729
```
For non-smooth varieties over finite fields, the homogeneous rescalings are solved. This is somewhat slower:

```
sage: fan = NormalFan(ReflexivePolytope(2, 0))
sage: X = ToricVariety(fan, base_field=GF(7))
sage: X.point_set().cardinality()
21
```

Fulton’s formula does not apply since the variety is not smooth. And, indeed, naive application gives a different result:

```
sage: q = X.base_ring().order()
sage: n = X.dimension()
sage: d = map(len, fan().cones())
sage: sum(dk * (q-1)**(n-k) for k, dk in enumerate(d))
57
```

Over infinite fields the number of points is not very tricky:

```
sage: V.count_points()
+Infinity
```

**ALGORITHM:**

Uses the formula in Fulton [Ful1993], section 4.5.

**AUTHORS:**

- Beth Malmskog (2013-07-14)
- Adriana Salerno (2013-07-14)
- Yiwei She (2013-07-14)
- Christelle Vincent (2013-07-14)
- Ursula Whitcher (2013-07-14)
sage: hom_set(matrix([[1],[0]]))
Scheme morphism:
    From: 2-d CPR-Fano toric variety covered by 4 affine patches
    To: 1-d CPR-Fano toric variety covered by 2 affine patches
    Defn: Defined by sending Rational polyhedral fan in 2-d lattice N
to Rational polyhedral fan in 1-d lattice N.

16.14 Enumerate points of a toric variety

The classes here are not meant to be instantiated manually. Instead, you should always use the methods of the point set of the variety.

In this module, points are always represented by tuples instead of Sage’s class for points of the toric variety. All Sage library code must then convert it to proper point objects before returning it to the user.

EXAMPLES:

sage: P2 = toric_varieties.P2(base_ring=GF(3))
sage: point_set = P2.point_set()
sage: point_set.cardinality()
13
sage: next(iter(point_set))
[0 : 0 : 1]
sage: list(point_set)[0:5]
[[0 : 0 : 1], [1 : 0 : 0], [0 : 1 : 0], [0 : 1 : 1], [0 : 1 : 2]]

class sage.schemes.toric.points.FiniteFieldPointEnumerator(fan, ring)

    Bases: NaiveFinitePointEnumerator

    cardinality()

        Return the cardinality of the point set.

        OUTPUT: An integer. The number of points.

        EXAMPLES:

sage: fan = NormalFan(ReflexivePolytope(2, 0))
sage: X = ToricVariety(fan, base_ring=GF(7))
sage: point_set = X.point_set()
sage: ffe = point_set._finite_field_enumerator()
sage: ffe.cardinality()
21

    cone_points_iter()

        Iterate over the open torus orbits and yield distinct points.

        OUTPUT:

        For each open torus orbit (cone): A triple consisting of the cone, the nonzero homogeneous coordinates in that orbit (list of integers), and the nonzero log coordinates of distinct points as a cokernel.

        EXAMPLES:
sage: fan = NormalFan(ReflexivePolytope(2, 0))
sage: X = ToricVariety(fan, base_ring=GF(7))
sage: point_set = X.point_set()
sage: ffe = point_set._finite_field_enumerator()
sage: cpi = ffe.cone_points_iter()
sage: cone, nonzero_points, cokernel = list(cpi)[5]
sage: cone
1-d cone of Rational polyhedral fan in 2-d lattice N
sage: cone.ambient_ray_indices()
(2,)
sage: nonzero_points
[0, 1]
sage: cokernel
Finitely generated module V/W over Integer Ring with invariants (2)
sage: list(cokernel)
[(0), (1)]
sage: [p.lift() for p in cokernel]
[(0, 0), (0, 1)]

exp(powers)
Return the component-wise exp of z

INPUT:

- powers – a list/tuple/iterable of integers.

OUTPUT:

Tuple of finite field elements. The powers of the multiplicative_generator().

EXAMPLES:

sage: # needs sage.rings.finite_rings
sage: F.<a> = GF(5^2)
sage: point_set = toric_varieties.P2_123(base_ring=F).point_set()
sage: ffe = point_set._finite_field_enumerator()
sage: powers = list(range(24))
sage: ffe.exp(powers)
(1, a, a + 3, 4*a + 3, 2*a + 2, 4*a + 1, 2, 2*a, 2*a + 1, 3*a + 1, 4*a + 4, 3*a + 2, 4, 4*a, 4*a + 2, a + 2, 3*a + 3, a + 4, 3, 3*a, 3*a + 4, 2*a + 4, a + 1, 2*a + 3)
sage: ffe.log(ffe.exp(powers)) == tuple(powers)
True

log(z)
Return the component-wise log of z

INPUT:

- z – a list/tuple/iterable of non-zero finite field elements.

OUTPUT:

Tuple of integers. The logarithm with base the multiplicative_generator().

EXAMPLES:
sage: # needs sage.rings.finite_rings
sage: F.<a> = GF(5^2)
sage: point_set = toric_varieties.P2_123(base_ring=F).point_set()
sage: ffe = point_set._finite_field_enumerator()
sage: z = tuple(a^i for i in range(25)); z
(1, a, a + 3, 4*a + 3, 2*a + 2, 4*a + 1, 2, 2*a, 2*a + 1, 3*a + 1,
  4*a + 4, 3*a + 2, 4, 4*a, 4*a + 2, a + 2, 3*a + 3, a + 4, 3, 3*a,
  3*a + 4, 2*a + 4, a + 1, 2*a + 3, 1)
sage: ffe.log(z)
(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16,
  17, 18, 19, 20, 21, 22, 23, 0)
sage: ffe.exp(ffe.log(z)) == z
True
sage: ffe.log(ffe.exp(range(24))) == tuple(range(24))
True

multiplicative_generator()

Return the multiplicative generator of the finite field.

OUTPUT: A finite field element.

EXAMPLES:

sage: point_set = toric_varieties.P2(base_ring=GF(5^2, 'a')).point_set()  # needs sage.rings.finite_rings
sage: ffe = point_set._finite_field_enumerator()  # needs sage.rings.finite_rings
sage: ffe.multiplicative_generator()  # needs sage.rings.finite_rings

multiplicative_group_order()

EXAMPLES:

sage: class Foo:
    ....:     def __init__(self, x):
    ....:         self._x = x
    ....:     @cached_method
    ....:     def f(self):
    ....:         return self._x^2
sage: a = Foo(2)
sage: print(a.f.cache)
None
sage: a.f()
4
sage: a.f.cache
4

rescaling_log_generators()

Return the log generators of rescalings().

OUTPUT:

A tuple containing the logarithms (see log()) of the generators of the multiplicative group of rescalings().
EXAMPLES:

```python
sage: point_set = toric_varieties.P2_123(base_ring=GF(5)).point_set()
sage: ffe = point_set._finite_field Enumerator()
sage: ffe.rescalings()
((1, 1, 1), (1, 4, 4), (4, 2, 3), (4, 3, 2))
sage: list(map(ffe.log, ffe.rescalings()))
[0, 2, 2, 2, 3]
sage: ffe.root_generator(2)
4
sage: ffe.root_generator(3)
1
sage: ffe.root_generator(4)
2
```

**root_generator**(n)

Return a generator for `roots()`.

INPUT:

• n integer.

OUTPUT: A multiplicative generator for `roots()`.

EXAMPLES:

```python
sage: point_set = toric_varieties.P2(base_ring=GF(5)).point_set()
sage: ffe = point_set._finite_field Enumerator()
sage: ffe.root_generator(2)
4
sage: ffe.root_generator(3)
1
sage: ffe.root_generator(4)
2
```

class sage.schemes.toric.points.FiniteFieldSubschemePointCreator

```
Bases: NaiveSubschemePointCreator
```

cardinality()

Return the cardinality of the point set.

OUTPUT: An integer. The number of points.

EXAMPLES:

```python
sage: fan = NormalFan(ReflexivePolytope(2, 0))
sage: X.<u,v,w> = ToricVariety(fan, base_ring=GF(7))
sage: Y = X.subscheme(u^3 + v^3 + w^3 + u*v*w)
sage: point_set = Y.point_set()

# needs fpylll
sage: list(point_set)
[[0 : 1 : 3],
 [1 : 0 : 3],
 [1 : 3 : 0],
 [1 : 1 : 6],
 [1 : 1 : 4],
 [1 : 3 : 2],
 [1 : 3 : 5]]

sage: ffe = point_set._enumerator()

# needs fpylll
sage: ffe.cardinality()
```
homogeneous_coordinates(log_t, nonzero_coordinates, cokernel)

Convert the log of inhomogeneous coordinates back to homogeneous coordinates.

INPUT:

- `log_t` – log of inhomogeneous coordinates of a point.
- `nonzero_coordinates` – the nonzero homogeneous coordinates in the patch.
- `cokernel` – the logs of the nonzero coordinates of all distinct points as a cokernel. See `FiniteFieldPointEnumerator.cone_points_iter()`.

OUTPUT:
The same point, but as a tuple of homogeneous coordinates.

EXAMPLES:

```python
sage: P2.<x,y,z> = toric_varieties.P2(base_ring=GF(7))
sage: X = P2.subscheme([x^3 + 2*y^3 + 3*z^3, x*y*z + x*y^2])
sage: point_set = X.point_set()
sage: ffe = point_set._enumerator()
sage: cone, nonzero_coordinates, cokernel = list(ffe.ambient.cone_points_iter())[5]
sage: ffe.homogeneous_coordinates([0], nonzero_coordinates, cokernel)
(1, 1, 0)
sage: ffe.homogeneous_coordinates([1], nonzero_coordinates, cokernel)
(1, 3, 0)
sage: ffe.homogeneous_coordinates([2], nonzero_coordinates, cokernel)
(1, 2, 0)
```

inhomogeneous_equations(ring, nonzero_coordinates, cokernel)

Inhomogenize the defining polynomials.

INPUT:

- `ring` – the polynomial ring for inhomogeneous coordinates.
- `nonzero_coordinates` – list of integers. The indices of the non-zero homogeneous coordinates in the patch.
- `cokernel` – the logs of the nonzero coordinates of all distinct points as a cokernel. See `FiniteFieldPointEnumerator.cone_points_iter()`.

EXAMPLES:

```python
sage: R.<s> = QQ[]
sage: P2.<x,y,z> = toric_varieties.P2(base_ring=GF(7))
sage: X = P2.subscheme([x^3 + 2*y^3 + 3*z^3, x*y*z + x*y^2])
sage: point_set = X.point_set()
sage: ffe = point_set._enumerator()
sage: cone, nonzero_coordinates, cokernel = list(ffe.ambient.cone_points_iter())[5]
```


**sage**

```python
cone.ambient_ray_indices(), nonzero_coordinates
((2,), [0, 1])
ffe.inhomogeneous_equations(R, nonzero_coordinates, cokernel)
[2*s^3 + 1, s^2]
```

**solutions** *(inhomogeneous_equations, log_range)*

Parallel version of `solutions_serial()`

**INPUT/OUTPUT:**

Same as `solutions_serial()`, except that the output points are in random order. Order depends on the number of processors and relative speed of separate processes.

**EXAMPLES:**

```python
R.<s> = GF(7)[]
P2.<x,y,z> = toric_varieties.P2(base_ring=GF(7))
X = P2.subscheme(1)
point_set = X.point_set()
ffe = point_set._enumerator()
ffe.solutions([s^2 - 1, s^6 - s^2], [range(6)])
sage: sorted(_)
[(0,), (3,)]
```

**solutions_serial** *(inhomogeneous_equations, log_range)*

Iterate over solutions in a range.

**INPUT:**

- `inhomogeneous_equations` – list/tuple/iterable of inhomogeneous equations (i.e. output from `inhomogeneous_equations()`).
- `log_range` – list/tuple/iterable of integer ranges. One for each inhomogeneous coordinate. The logarithms of the homogeneous coordinates.

**OUTPUT:**

All solutions (as tuple of log inhomogeneous coordinates) in the Cartesian product of the ranges.

**EXAMPLES:**

```python
R.<s> = GF(7)[]
P2.<x,y,z> = toric_varieties.P2(base_ring=GF(7))
X = P2.subscheme(1)
point_set = X.point_set()
ffe = point_set._enumerator()
ffe.solutions_serial([s^2 - 1, s^6 - s^2], [range(6)])
sage: list(_)
[(0,), (3,)]
```

**class** `sage.schemes.toric.points.InfinitePointEnumerator(fan, ring)`

**Bases:** `object`

Point enumerator for infinite fields.

**INPUT:**

16.14. Enumerate points of a toric variety 385
• fan – fan of the toric variety.
• ring – infinite base ring over which to enumerate points.

class sage.schemes.toric.points.NaiveFinitePointEnumerator(fan, ring)

Bases: object

The naive point enumerator.

This is very slow.

INPUT:

• fan – fan of the toric variety.
• ring – finite base ring over which to enumerate points.

EXAMPLES:

```
sage: from sage.schemes.toric.points import NaiveFinitePointEnumerator
sage: fan = toric_varieties.P2().fan()
sage: n = NaiveFinitePointEnumerator(fan, GF(3))
sage: next(iter(n))
(0, 0, 1)
```

cone_iter()

Iterate over all cones of the fan

OUTPUT:

Iterator over the cones, starting with the high-dimensional ones.

EXAMPLES:

```
sage: dP6 = toric_varieties.dP6(base_ring=GF(11))
sage: ne = dP6.point_set()._naive_enumerator()
sage: for cone in ne.cone_iter():
    print(cone.ambient_ray_indices())
(0, 1)
(1, 2)
(2, 3)
(3, 4)
(4, 5)
(5, )
```

coordinate_iter()

Iterate over all distinct homogeneous coordinates.

This method does NOT identify homogeneous coordinates that are equivalent by a homogeneous rescaling.

OUTPUT: An iterator over the points.

EXAMPLES:
```
sage: P2 = toric_varieties.P2(base_ring=GF(2))
sage: ni = P2.point_set()._naive_enumerator()
sage: list(ni.coordinate_iter())
[(0, 0, 1), (1, 0, 0), (0, 1, 0), (0, 1, 1),
 (1, 0, 1), (1, 1, 0), (1, 1, 1)]
```

```
sage: P1xP1 = toric_varieties.P1xP1(base_ring=GF(2))
sage: ni = P1xP1.point_set()._naive_enumerator()
sage: list(ni.coordinate_iter())
[(0, 1, 0, 1), (1, 0, 0, 1), (1, 0, 1, 0),
 (0, 1, 1, 0), (0, 1, 1, 1), (1, 0, 1, 1),
 (1, 1, 0, 1), (1, 1, 1, 0), (1, 1, 1, 1)]
```

### orbit(point)

Return the orbit of homogeneous coordinates under rescalings.

**OUTPUT:**

The set of all homogeneous coordinates that are equivalent to `point`.

**EXAMPLES:**

```
sage: P2_123 = toric_varieties.P2_123(base_ring=GF(7))
sage: ne = P2_123.point_set()._naive_enumerator()
sage: sorted(ne.orbit([1, 0, 0]))
[(1, 0, 0), (2, 0, 0), (4, 0, 0)]
```

```
sage: sorted(ne.orbit([0, 1, 0]))
[(0, 1, 0), (0, 6, 0)]
```

```
sage: sorted(ne.orbit([0, 0, 1]))
[(0, 0, 1), (0, 0, 2), (0, 0, 3), (0, 0, 4), (0, 0, 5), (0, 0, 6)]
```

```
sage: sorted(ne.orbit([1, 1, 0]))
[(1, 1, 0), (1, 6, 0), (2, 1, 0), (2, 6, 0), (4, 1, 0), (4, 6, 0)]
```

### rays()

Return all rays (real and virtual).

**OUTPUT:** Tuple of rays of the fan.

**EXAMPLES:**

```
sage: from sage.schemes.toric.points import NaiveFinitePointEnumerator
sage: fan = toric_varieties.torus(2).fan()
sage: fan.rays()
Empty collection in 2-d lattice N
```

```
sage: n = NaiveFinitePointEnumerator(fan, GF(3))
sage: n.rays()
N(1, 0), N(0, 1)
in 2-d lattice N
```

### rescalings()

Return the rescalings of homogeneous coordinates.

**OUTPUT:**

---

**16.14. Enumerate points of a toric variety**

---
A tuple containing all points that are equivalent to \([1 : 1 : \cdots : 1]\), the distinguished point of the big torus orbit.

EXAMPLES:

```sage
sage: P2_123 = toric_varieties.P2_123(base_ring=GF(5))
sage: ni = P2_123.point_set()._naive_enumerator()
sage: ni.rescalings()
((1, 1, 1), (1, 4, 4), (4, 2, 3), (4, 3, 2))

sage: dP8 = toric_varieties.dP8(base_ring=GF(3))
sage: ni = dP8.point_set()._naive_enumerator()
sage: ni.rescalings()
((1, 1, 1, 1), (1, 2, 2, 2), (2, 1, 2, 1), (2, 2, 1, 2))

sage: P1xP1 = toric_varieties.P1xP1(base_ring=GF(3))
sage: ni = P1xP1.point_set()._naive_enumerator()
sage: ni.rescalings()
((1, 1, 1, 1), (1, 1, 2, 2), (2, 2, 1, 1), (2, 2, 2, 2))
```

**roots**

Return the \(n\)-th roots in the base field.

**INPUT:**

- \(n\) integer.

**OUTPUT:**

Tuple containing all \(n\)-th roots (not only the primitive ones). In particular, 1 is included.

**EXAMPLES:**

```sage
sage: P2 = toric_varieties.P2(base_ring=GF(5))
sage: ne = P2.point_set()._naive_enumerator()
sage: ne.roots(2)
(1, 4)
sage: ne.roots(3)
(1,)
sage: ne.roots(4)
(1, 2, 3, 4)
```

**units**

Return the units in the base field.

**EXAMPLES:**

```sage
sage: P2 = toric_varieties.P2(base_ring=GF(5))
sage: ne = P2.point_set()._naive_enumerator()
sage: ne.units()
(1, 2, 3, 4)
```

**class** `sage.schemes.toric.points.NaiveSubschemePointEnumerator` *(polynomials, ambient)*

**Bases:** object

Point enumerator for algebraic subschemes of toric varieties.

**INPUT:**
• polynomials – list/tuple/iterable of polynomials. The defining polynomials.
• ambient – enumerator for ambient space points.

16.15 Construct sheaves on toric varieties

A toric vector bundle (on a toric variety) is a vector bundle that is equivariant with respect to the algebraic torus action.

sage.schemes.toric.sheaf.constructor.CotangentBundle(X)
Construct the cotangent bundle of a toric variety.

INPUT:
• X – a toric variety. The base space of the bundle.

OUTPUT:
The cotangent bundle as a Klyachko bundle.

EXAMPLES:
```
sage: dP7 = toric_varieties.dP7()
sage: from sage.schemes.toric.sheaf.constructor import CotangentBundle
sage: CotangentBundle(dP7)
Rank 2 bundle on 2-d CPR-Fano toric variety covered by 5 affine patches.
```

sage.schemes.toric.sheaf.constructor.LineBundle(X, D)
Construct the rank-1 bundle $O(D)$.

INPUT:
• X – a toric variety. The base space of the bundle.
• D – a toric divisor.

OUTPUT:
The line bundle $O(D)$ as a Klyachko bundle of rank 1.

EXAMPLES:
```
sage: X = toric_varieties.dP8()
sage: D = X.divisor(0)
sage: from sage.schemes.toric.sheaf.constructor import LineBundle
sage: O_D = LineBundle(X, D)
sage: O_D.cohomology(dim=True, weight=(0,0))
(1, 0, 0)
```

class sage.schemes.toric.sheaf.constructor.SheafLibrary(toric_variety)
Utility object to construct sheaves on toric varieties.

**Warning:** You should never construct instances manually. Can be accessed from a toric variety via the `sage.schemes.toric.variety.ToricVariety_field.sheaves` attribute.

EXAMPLES:
Klyachko(`multi_filtration`)
Construct a Klyachko bundle (sheaf) from filtration data.

**INPUT:**
- `multi_filtration` – a multi-filtered vectors space with multiple filtrations being indexed by the rays of the fan. Either an instance of `MultiFilteredVectorSpace()` or something (like a dictionary of ordinary filtered vector spaces).

**OUTPUT:**
The Klyachko bundle defined by the filtrations, one for each ray, of a vector space.

**EXAMPLES:**

```python
sage: P1 = toric_varieties.P1()
sage: v1, v2, v3 = [(1,0,0), (0,1,0), (0,0,1)]
sage: F1 = FilteredVectorSpace({1: [v1, v2, v3], 3: [v1]})
sage: F2 = FilteredVectorSpace({0: [v1, v2, v3], 2: [v2, v3]})
sage: P1 = toric_varieties.P1()
sage: r1, r2 = P1.fan().rays()
sage: F = MultiFilteredVectorSpace({r1:F1, r2:F2}); F
Filtrations
  N(-1): QQ^3 >= QQ^2 >= QQ^2 >= 0 >= 0
  N(1): QQ^3 >= QQ^3 >= QQ^1 >= QQ^1 >= 0
 sage: P1.sheaves.Klyachko(F)
Rank 3 bundle on 1-d CPR-Fano toric variety covered by 2 affine patches.
```

cotangent_bundle()
Return the cotangent bundle of the toric variety.

**OUTPUT:**
The cotangent bundle as a Klyachko bundle.

**EXAMPLES:**

```python
sage: dP6 = toric_varieties.dP6()
sage: TX = dP6.sheaves.tangent_bundle()
sage: TXdual = dP6.sheaves.cotangent_bundle()
sage: TXdual == TX.dual()
True
```

divisor(*args, **kwds)
Return a toric divisor.

**INPUT:**
This is just an alias for `sage.schemes.toricvariety.ToricVariety_field.divisor()`, see there for details.

By abuse of notation, you can usually use the divisor $D$ interchangeably with the line bundle $O(D)$.

**OUTPUT:**
A toric divisor.
EXAMPLES:

```python
sage: dP6 = toric_varieties.dP6()
sage: dP6.inject_variables()
Defining x, u, y, v, z, w
sage: D = dP6.sheaves.divisor(x*u^3); D
V(x) + 3*V(u)
sage: D == dP6.divisor(x*u^3)
True
```

```python
line_bundle(divisor)

Construct the rank-1 bundle $O(D)$.

INPUT:

- divisor – a toric divisor.

OUTPUT:

The line bundle $O(D)$ for the given divisor as a Klyachko bundle of rank 1.

EXAMPLES:

```python
sage: X = toric_varieties.dP8()
sage: D = X.divisor(0)
sage: O_D = X.sheaves.line_bundle(D)
sage: O_D.cohomology(dim=True, weight=(0,0))
(1, 0, 0)
```

```python
tangent_bundle()

Return the tangent bundle of the toric variety.

OUTPUT:

The tangent bundle as a Klyachko bundle.

EXAMPLES:

```python
sage: toric_varieties.dP6().sheaves.tangent_bundle()
Rank 2 bundle on 2-d CPR-Fano toric variety covered by 6 affine patches.
```

```python
trivial_bundle(rank=1)

Return the trivial bundle of rank $r$.

INPUT:

- rank – integer (optional; default: 1). The rank of the bundle.

OUTPUT:

The trivial bundle as a Klyachko bundle.

EXAMPLES:

```python
sage: P2 = toric_varieties.P2()
sage: I3 = P2.sheaves.trivial_bundle(3); I3
Rank 3 bundle on 2-d CPR-Fano toric variety covered by 3 affine patches.
sage: I3.cohomology(weight=(0,0), dim=True)
(3, 0, 0)
```
sage.schemes.toric.sheaf.constructor.TangentBundle(X)

Construct the tangent bundle of a toric variety.

INPUT:
  • X – a toric variety. The base space of the bundle.

OUTPUT:
The tangent bundle as a Klyachko bundle.

EXAMPLES:

```
sage: dP7 = toric_varieties.dP7()
sage: from sage.schemes.toric.sheaf.constructor import TangentBundle
sage: TangentBundle(dP7)
Rank 2 bundle on 2-d CPR-Fano toric variety covered by 5 affine patches.
```

sage.schemes.toric.sheaf.constructor.TrivialBundle(X, rank=1)

Return the trivial bundle of rank $r$.

INPUT:
  • X – a toric variety. The base space of the bundle.
  • rank – the rank of the bundle.

OUTPUT:
The trivial bundle as a Klyachko bundle.

EXAMPLES:

```
sage: P2 = toric_varieties.P2()
sage: from sage.schemes.toric.sheaf.constructor import TrivialBundle
sage: I3 = TrivialBundle(P2, 3); I3
Rank 3 bundle on 2-d CPR-Fano toric variety covered by 3 affine patches.
sage: I3.cohomology(weight=(0,0), dim=True)
(3, 0, 0)
```

## 16.16 Klyachko bundles and sheaves

Klyachko bundles are torus-equivariant bundles on toric varieties. That is, the action of the maximal torus on the toric variety lifts to an action on the bundle. There is an equivalence of categories between Klyachko bundles [Kly1990] and multiple filtrations (one for each ray of the fan) of a vector space. The multi-filtrations are implemented in `sage.modules.multi_filtered_vector_space`.

EXAMPLES:

```
sage: X = toric_varieties.dP6xdP6()
sage: TX = X.sheaves.tangent_bundle()
sage: Alt2TX = TX.exterior_power(2); Alt2TX
Rank 6 bundle on 4-d CPR-Fano toric variety covered by 36 affine patches.
sage: K = X.sheaves.line_bundle(X.K())
sage: antiK = X.sheaves.line_bundle(-X.K())
sage: (Alt2TX * K).cohomology(dim=True, weight=(0,0,0,0)) # long time
```

(continues on next page)
sage: G_sum = TX + X.sheaves.trivial_bundle(2)
sage: V_sum = G_sum.wedge(2) * K  # long time
sage: V_sum.cohomology(dim=True, weight=(0,0,0,0))  # long time
(0, 0, 18, 16, 1)
sage: Gtilde = G_sum.random_deformation()
sage: V = Gtilde.wedge(2) * K  # long time
sage: V.cohomology(dim=True, weight=(0,0,0,0))  # long time  # random failure (see #32773)
(0, 0, 3, 0, 0)

REFERENCES:

• [Kly1990]
• [BIP]

\texttt{sage.schemes.toric.sheaf.klyachko.Bundle(toric\_variety, multi\_filtration, check=True)}

Construct a Klyacho bundle

INPUT:

• \texttt{toric\_variety} – a toric variety. The base space of the bundle.

• \texttt{multi\_filtration} – a multi-filtered vectors space with multiple filtrations being indexed by the one-dimensional cones of the fan. Either an instance of \texttt{MultiFilteredVectorSpace()} or something (like a dictionary of ordinary filtered vector spaces).

EXAMPLES:

\begin{verbatim}
sage: P1 = toric_varieties.P1()
sage: v1, v2, v3 = [(1,0,0), (0,1,0), (0,0,1)]
sage: F1 = FilteredVectorSpace({1: [v1, v2, v3], 3: [v1]})
sage: F2 = FilteredVectorSpace({0: [v1, v2, v3], 2: [v2, v3]})
sage: P1 = toric_varieties.P1()
sage: r1, r2 = P1.fan().rays()
sage: F = MultiFilteredVectorSpace({r1: F1, r2: F2}); F
Filtrations
  N(-1): QQ^3 >= QQ^2 >= QQ^2 >= 0 >= 0
  N(1): QQ^3 >= QQ^3 >= QQ^1 >= QQ^1 >= 0
\end{verbatim}

You should use the \texttt{Klyachko()} method to construct instances:

\begin{verbatim}
sage: P1.sheaves.Klyachko(F)
Rank 3 bundle on 1-d CPR-Fano toric variety covered by 2 affine patches.
sage: P1.sheaves.Klyachko({r1: F1, r2: F2})  # alternative
Rank 3 bundle on 1-d CPR-Fano toric variety covered by 2 affine patches.
\end{verbatim}

The above is just a shorthand for:

\begin{verbatim}
sage: from sage.schemes.toric.sheaf.klyachko import Bundle
sage: Bundle(P1, F)
Rank 3 bundle on 1-d CPR-Fano toric variety covered by 2 affine patches.
\end{verbatim}
class sage.schemes.toric.sheaf.klyachko.KlyachkoBundle_class(toric_variety, multi_filtration, check=True)

Bases: SageObject

A toric bundle using Klyachko’s representation.

Warning: You should always use the Bundle() factory function to construct instances.

INPUT:

• toric_variety – a toric variety. The base space of the bundle.
• multi_filtration – a MultiFilteredVectorSpace() with index set the rays of the fan.
• check – boolean (default: True). Whether to perform consistency checks.

EXAMPLES:

sage: P1 = toric_varieties.P1()
sage: r1, r2 = P1.fan().rays()
sage: F = MultiFilteredVectorSpace({
....:     r1: FilteredVectorSpace(3,1),
....:     r2: FilteredVectorSpace(3,0)}); F
Filtrations
N(-1): QQ^3 >= 0 >= 0
N(1): QQ^3 >= QQ^3 >= 0
sage: from sage.schemes.toric.sheaf.klyachko import Bundle
sage: Bundle(P1, F)
Rank 3 bundle on 1-d CPR-Fano toric variety covered by 2 affine patches.

E_degree(alpha, m)

Return the vector subspace $E^\alpha(m)$.

INPUT:

• alpha – a ray of the fan. Can be specified by its index (an integer), a one-dimensional cone, or a $N$-lattice point.
• m – tuple of integers or $M$-lattice point. A point in the dual lattice of the fan.

OUTPUT:

The subspace $E^\alpha(\alpha*m)$ of the filtration indexed by the ray $\alpha$ and at the filtration degree $\alpha * m$.

EXAMPLES:

sage: X = toric_varieties.P2()
sage: M = X.fan().dual_lattice()
sage: V = X.sheaves.tangent_bundle()
sage: V.E_degree(X.fan().ray(0), (1,0))
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix: [1 0]
sage: V.E_degree(X.fan().[0], (1,0))
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix: [1 0]
sage: V.E_degree([0], (1,0))
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix: [1 0]
E_intersection($\sigma$, $m$)
Return the vector subspace $E^\sigma(m)$.
See [Kly1990], equation 4.1.

INPUT:
• $\sigma$ – a cone of the fan of the base toric variety.
• $m$ – tuple of integers or $M$-lattice point. A point in the dual lattice of the fan. Must be immutable.

OUTPUT: The subspace $E^\sigma(m)$.

EXAMPLES:

```
sage: X = toric_varieties.P2()
sage: fan = X.fan()
sage: V = X.sheaves.tangent_bundle()
sage: V.E_intersection(fan(1)[0], (1,0))
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix: [1 0]
sage: V.E_intersection(fan(2)[0], (-1,1))
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix: [0 1]
```

For the empty cone, this is always the whole vector space:

```
sage: V.E_intersection(fan(0)[0], (1,0))
Vector space of dimension 2 over Rational Field
```

E_quotient($\sigma$, $m$)
Return the vector space quotient $E^\sigma(m)$.
See [Kly1990], equation 4.1.

INPUT:
• $\sigma$ – a cone of the fan of the base toric variety.
• $m$ – tuple of integers or $M$-lattice point. A point in the dual lattice of the fan. Must be immutable.

OUTPUT: The subspace $E^\sigma(m)$.

EXAMPLES:

```
sage: X = toric_varieties.P2()
sage: fan = X.fan()
sage: M = fan.dual_lattice()
sage: cone = fan(1)[0]
sage: V = X.sheaves.tangent_bundle()
sage: m = M(1, 0)
m.set_immutable()
sage: V.E_quotient(cone, m)
Vector space quotient V/W of dimension 1 over Rational Field where
V: Vector space of dimension 2 over Rational Field
W: Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix: [1 0]
sage: V.E_quotient(fan(2)[0], (-1,1))
Vector space quotient V/W of dimension 0 over Rational Field
```

(continues on next page)
E_quotient_projection(sigma, tau, m)

Return the projection map $E_\sigma(m) \to E_\tau(m)$ where $\sigma$ is a face of $\tau$.

INPUT:

- sigma – a cone of the fan of the base toric variety.
- tau – a cone of the fan containing sigma.
- m – tuple of integers or $M$-lattice point. A point in the dual lattice of the fan. Must be immutable.

OUTPUT:

The restriction map $E_\sigma(m) \to E_\tau(m)$

EXAMPLES:

```
sage: P3 = toric_varieties.P(3)
sage: rays = [(1,0,0), (0,1,0), (0,0,1)]
sage: F1 = FilteredVectorSpace(rays, {0: [0], 1: [2], 2: [1]})
sage: F2 = FilteredVectorSpace(3, 0)
sage: r = P3.fan().rays()
sage: V = P3.sheaves.Klyachko({r[0]: F1, r[1]: F2, r[2]: F2, r[3]: F2})
sage: tau = Cone([(1,0,0), (0,1,0)])
sage: sigma = Cone([(1,0,0)])
sage: M = P3.fan().dual_lattice()
sage: m = M(2,1,0)
sage: m.set_immutable()
sage: V.E_quotient(sigma, m)
Vector space quotient V/W of dimension 2 over Rational Field where
V: Vector space of dimension 3 over Rational Field
W: Vector space of degree 3 and dimension 1 over Rational Field
  Basis matrix: [0 1 0]
sage: V.E_quotient(tau, m)
Vector space quotient V/W of dimension 2 over Rational Field where
V: Vector space of dimension 3 over Rational Field
W: Vector space of degree 3 and dimension 1 over Rational Field
  Basis matrix: [0 1 0]
sage: V.E_quotient_projection(sigma, tau, m)
Vector space morphism represented by the matrix:
[1 0]
[0 1]
Domain: Vector space quotient V/W of dimension 2 over Rational Field where
V: Vector space of dimension 3 over Rational Field
W: Vector space of degree 3 and dimension 1 over Rational Field
  Basis matrix: [0 1 0]
Codomain: Vector space quotient V/W of dimension 2 over Rational Field where
```
(continues on next page)
V: Vector space of dimension 3 over Rational Field
W: Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix: [0 1 0]

base_ring()
Return the base field.

OUTPUT: A field.

EXAMPLES:

```
sage: T_P2 = toric_varieties.P2().sheaves.tangent_bundle()
sage: T_P2.base_ring()
Rational Field
```

cohomology(degree=None, weight=None, dim=False)
Return the bundle cohomology groups.

INPUT:

- degree – None (default) or an integer. The degree of the cohomology group.
- weight – None (default) or a tuple of integers or a $M$-lattice point. A point in the dual lattice of the fan defining a torus character. The weight of the cohomology group.
- dim – Boolean (default: False). Whether to return vector spaces or only their dimension.

OUTPUT:

The cohomology group of given cohomological degree and torus weight.

- If no weight is specified, the unweighted group (sum over all weights) is returned.
- If no degree is specified, a dictionary whose keys are integers and whose values are the cohomology groups is returned. If, in addition, dim=True, then an integral vector of the dimensions is returned.

EXAMPLES:

```
sage: V = toric_varieties.P2().sheaves.tangent_bundle()
sage: V.cohomology(degree=0, weight=(0,0))
Vector space of dimension 2 over Rational Field
sage: V.cohomology(weight=(0,0), dim=True)
(2, 0, 0)
sage: for i,j in cartesian_product((list(range(-2,3)), list(range(-2,3))):
    HH = V.cohomology(weight=(i,j), dim=True)
    if HH.is_zero(): continue
    print("H^*i(P^2, TP^2)_M({}, {}) = {}".format(i,j,HH))
H^*i(P^2, TP^2)_M(-1, 0) = (1, 0, 0)
H^*i(P^2, TP^2)_M(-1, 1) = (1, 0, 0)
H^*i(P^2, TP^2)_M(0, -1) = (1, 0, 0)
H^*i(P^2, TP^2)_M(0, 0) = (2, 0, 0)
H^*i(P^2, TP^2)_M(0, 1) = (1, 0, 0)
H^*i(P^2, TP^2)_M(1, -1) = (1, 0, 0)
H^*i(P^2, TP^2)_M(1, 0) = (1, 0, 0)
```

cohomology_complex(m)
Return the “cohomology complex” $C^*(m)$
Schemes, Release 10.2

See [Kly1990], equation 4.2.

INPUT:

• \( m \) – tuple of integers or \( M \)-lattice point. A point in the dual lattice of the fan. Must be immutable.

OUTPUT:

The “cohomology complex” as a chain complex over the \texttt{base_ring()}.

EXAMPLES:

```
sage: P3 = toric_varieties.P(3)
sage: rays = [(1,0,0), (0,1,0), (0,0,1)]
sage: F1 = FilteredVectorSpace(rays, {0: [0], 1: [2], 2: [1]})
sage: F2 = FilteredVectorSpace(rays, {0: [1,2], 1: [0]})
sage: r = P3.fan().rays()
sage: F = CyclotomicField(3)
sage: P3 = toric_varieties.P(3).change_ring(F)
sage: V = P3.sheaves.Klyachko({r[0]: F1, r[1]: F2, r[2]: F2, r[3]: F2})
sage: V.cohomology_complex(m)
sage: tau = Cone(([1,0,0], (0,1,0)))
sage: sigma = Cone(([1, 0, 0]))
sage: m = M(1, 1, 0); m.set_immutable()
sage: V.cohomology_complex(m)
```

\textbf{direct_sum}(\texttt{other})

Return the sum of two vector bundles.

INPUT:

• \texttt{other} – a Klyachko bundle over the same base.

OUTPUT: The direct sum as a new Klyachko bundle.

EXAMPLES:

```
sage: X = toric_varieties.P2()
sage: V1 = X.sheaves.trivial_bundle(1)
sage: V2 = X.sheaves.trivial_bundle(2)
sage: V2.direct_sum(V1)
```

\textbf{dual}()

Return the dual bundle.

OUTPUT: The dual bundle as a new Klyachko bundle.

EXAMPLES:
sage: P1 = toric_varieties.P1()
sage: H = P1.divisor(0)
sage: L = P1.sheaves.line_bundle(H)
sage: L.dual()
Rank 1 bundle on 1-d CPR-Fano toric variety covered by 2 affine patches.
sage: L.dual() == P1.sheaves.line_bundle(-H)
True

exterior_power\((n)\)
Return the \(n\)-th exterior power.

INPUT:

• \(n\) – integer.

OUTPUT:

The \(n\)-th exterior power \(\bigwedge^n V\) of the bundle \(V\) as a new Klyachko bundle.

EXAMPLES:

sage: X = toric_varieties.P2_123()
sage: TX = X.sheaves.tangent_bundle()
sage: antiK = X.sheaves.line_bundle(-X.K())
sage: TX.exterior_power(2) == antiK
True
sage: TX.wedge(2) == antiK  # alias
True

fiber()
Return the generic fiber of the vector bundle.

OUTPUT: A vector space over \(\text{base_ring()}\).

EXAMPLES:

sage: T_P2 = toric_varieties.P2().sheaves.tangent_bundle()
sage: T_P2.fiber()
Vector space of dimension 2 over Rational Field

filtration_intersection\((\sigma, i)\)
Return the intersection of the filtered subspaces.

INPUT:

• \(\sigma\) – a cone of the fan of the base toric variety.
• \(i\) – integer. The filtration degree.

OUTPUT:

Let the cone be spanned by the rays \(\sigma = \langle r_1, \ldots, r_k \rangle\). This method returns the intersection

\[
\bigcap_{r \in \{r_1, \ldots, r_k\}} E^r(i)
\]

EXAMPLES:
sage: X = toric_varieties.P2()
sage: fan = X.fan()
sage: V = X.sheaves.tangent_bundle()
sage: V.filtration_intersection(fan(1)[0], 1)
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix: [1 0]
sage: V.filtration_intersection(fan(2)[0], 1)
Vector space of degree 2 and dimension 0 over Rational Field
Basis matrix: []

get_degree\( \text{(ray, i)} \)
Return the vector subspace \( E^\alpha(i) \).

- ray – Integer, a \( \mathbb{N} \)-lattice point, a one-dimensional cone, or None (default). Specifies a ray of the fan of the toric variety, either via its index or its generator.
- i – integer. The filtration degree.

OUTPUT:
A subspace of the fiber() vector space. The defining data of a Klyachko bundle.

EXAMPLES:
sage: TX = toric_varieties.dP6().sheaves.tangent_bundle()
sage: TX.get_degree(0, 1)
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix: [0 1]

get_filtration\( \text{(ray=None)} \)
Return the filtration associated to the ray.

INPUT:

- ray – Integer, a \( \mathbb{N} \)-lattice point, a one-dimensional cone, or None (default). Specifies a ray of the fan of the toric variety, either via its index or its generator.

OUTPUT:
The filtered vector space associated to the given ray. If no ray is specified, all filtrations are returned.

EXAMPLES:
sage: TX = toric_varieties.dP6().sheaves.tangent_bundle()
sage: TX.get_filtration(0)
\mathbb{Q}^2 \geq \mathbb{Q}^1 \geq 0
sage: TX.get_filtration([-1, -1])
\mathbb{Q}^2 \geq \mathbb{Q}^1 \geq 0
sage: TX.get_filtration(TX.variety().fan(1)[0])
\mathbb{Q}^2 \geq \mathbb{Q}^1 \geq 0
sage: TX.get_filtration()
Filtrations
\begin{align*}
N(-1, -1): & \mathbb{Q}^2 \geq \mathbb{Q}^1 \geq 0 \\
N(-1, 0): & \mathbb{Q}^2 \geq \mathbb{Q}^1 \geq 0 \\
N(0, -1): & \mathbb{Q}^2 \geq \mathbb{Q}^1 \geq 0 \\
N(0, 1): & \mathbb{Q}^2 \geq \mathbb{Q}^1 \geq 0 \\
N(1, 0): & \mathbb{Q}^2 \geq \mathbb{Q}^1 \geq 0 \\
N(1, 1): & \mathbb{Q}^2 \geq \mathbb{Q}^1 \geq 0
\end{align*}
is_isomorphic(other)
Test whether two bundles are isomorphic.
INPUT:
  • other – anything.
OUTPUT: A boolean.
EXAMPLES:

```
sage: X = toric_varieties.P2()
sage: T_X = X.sheaves.tangent_bundle()
sage: O_X = X.sheaves.trivial_bundle(1)
sage: T_X + O_X == O_X + T_X
False
sage: (T_X + O_X).is_isomorphic(O_X + T_X)
Traceback (most recent call last):
  ... 
NotImplementedError
```

random_deformation(epsilon=None)
Return a generic torus-equivariant deformation of the bundle.
INPUT:
  • epsilon – an element of the base ring. Scales the random deformation.
OUTPUT:
A new Klyachko bundle with randomly perturbed moduli. In particular, the same Chern classes.
EXAMPLES:

```
sage: P1 = toric_varieties.P1()
sage: H = P1.divisor(0)
sage: V = P1.sheaves.line_bundle(H) + P1.sheaves.line_bundle(-H)
sage: V.cohomology(dim=True, weight=(0,))
(1, 0)
sage: Vtilde = V.random_deformation()
sage: Vtilde.cohomology(dim=True, weight=(0,))  # random failure (see #32773)
(1, 0)
```

rank()
Return the rank of the vector bundle.
OUTPUT: An integer.
EXAMPLES:

```
sage: T_P2 = toric_varieties.P2().sheaves.tangent_bundle()
sage: T_P2.rank()
2
```

symmetric_power(n)
Return the n-th symmetric power.
INPUT:
  • n – integer.
OUTPUT: The \(n\)-th symmetric power as a new Klyachko bundle.

EXAMPLES:

```
sage: P1 = toric_varieties.P1()
sage: H = P1.divisor(0)
sage: L = P1.sheaves.line_bundle(H)
sage: (L + L).symmetric_power(2)
Rank 3 bundle on 1-d CPR-Fano toric variety covered by 2 affine patches.
sage: (L + L).symmetric_power(2) == L* L + L* L + L* L
True
```

**tensor_product**(other)

Return the sum of two vector bundles.

INPUT:

- • other – a Klyachko bundle over the same base.

OUTPUT: The tensor product as a new Klyachko bundle.

EXAMPLES:

```
sage: X = toric_varieties.P2()
sage: OX = X.sheaves.trivial_bundle(1)
sage: X.sheaves.tangent_bundle().tensor_product(OX)
Rank 2 bundle on 2-d CPR-Fano toric variety covered by 3 affine patches.
sage: OX == OX * OX
True
```

**variety**( )

Return the base toric variety.

OUTPUT: A toric variety.

EXAMPLES:

```
sage: X = toric_varieties.P2()
sage: V = X.sheaves.tangent_bundle(); V
Rank 2 bundle on 2-d CPR-Fano toric variety covered by 3 affine patches.
sage: V.variety() is X
True
```

**wedge**(n)

Return the \(n\)-th exterior power.

INPUT:

- • \(n\) – integer.

OUTPUT:

The \(n\)-th exterior power \(\wedge^n V\) of the bundle \(V\) as a new Klyachko bundle.

EXAMPLES:

```
sage: X = toric_varieties.P2_123()
sage: TX = X.sheaves.tangent_bundle()
sage: antiK = X.sheaves.line_bundle(-X.K())
```

(continues on next page)
sage: TX.exterior_power(2) == antiK
True
sage: TX.wedge(2) == antiK # alias
True

```
sage.schemes.toric.sheaf.klyachko.is_KlyachkoBundle(X)
Test whether X is a Klyachko bundle

INPUT:
• X – anything.

OUTPUT: A boolean.

EXAMPLES:
```
sage: from sage.schemes.toric.sheaf.klyachko import is_KlyachkoBundle
sage: is_KlyachkoBundle('test')
False
17.1 Cyclic covers over a finite field

The most interesting feature is computation of Frobenius matrix on Monsky-Washnitzer cohomology and the Frobenius polynomial.

REFERENCES:
- [ABCMT2019]

EXAMPLES:

```python
sage: p = 13
sage: x = PolynomialRing(GF(p),"x").gen()
sage: C = CyclicCover(4, x^4 + 1)
sage: C.frobenius_polynomial()
x^6 - 6*x^5 + 3*x^4 + 60*x^3 + 39*x^2 - 1014*x + 2197
sage: R.<t> = PowerSeriesRing(Integers())
sage: C.projective_closure().zeta_series(2,t)
1 + 8*t + 102*t^2 + O(t^3)
sage: C.frobenius_polynomial().reverse()(t)/((1-t)*(1-p*t)) + O(t^5)
1 + 8*t + 102*t^2 + 1384*t^3 + 18089*t^4 + O(t^5)
sage: p = 49999
sage: x = PolynomialRing(GF(p),"x").gen()
sage: CyclicCover(5, x^5 + x).frobenius_polynomial()
x^12 + 299994*x^10 + 37498500015*x^8 + 2499850002999980*x^6 + 93742500224997000015*x^4
+ 1874812507499850001499994*x^2 + 15623125093747500037499700001
sage: CyclicCover(5, 2*x^5 + x).frobenius_polynomial()  # long time
x^12 + 299994*x^10 + 37498500015*x^8 + 2499850002999980*x^6 + 93742500224997000015*x^4
+ 1874812507499850001499994*x^2 + 15623125093747500037499700001
sage: p = 107
sage: x = PolynomialRing(GF(p),"x").gen()
sage: CyclicCover(2, x^5 + x).frobenius_matrix()
[ 0(107^2) 89*107 + 0(107^2) 0(107^2) ]
-0(107^2)
[ 89*107 + 0(107^2) 0(107^2) 0(107^2) ]
-0(107^2)
[ 0(107^2) 0(107^2) 105 + 5*107 ]
-0(107^2)
[ 0(107^2) 0(107^2) 89 + 53*107 + 0(107^2) ]
```

(continues on next page)
Continued from previous page.

```python
sage: CyclicCover(2, 3*x^5 + x).frobenius_matrix()
[ O(107^2) 89*107 + O(107^2) O(107^2) ␣
 ˓→ O(107^2)]
[ 89*107 + O(107^2) O(107^2) O(107^2) ␣
 ˓→ O(107^2)]
[ 0(107^2) 0(107^2) 61 + 58*107 + O(107^2) ␣
 ˓→ O(107^2)]
[ 0(107^2) 0(107^2) 69 + 53*107 + O(107^2) ␣
 ˓→ O(107^2)]
```

```python
sage: CyclicCover(3, x^3 + x).frobenius_matrix()
[ 0 0 O(107) O(107)]
[ 0 0 52 + O(107) O(107)]
[ O(107) 35 + O(107) 0 0]
[44 + O(107) 0(107) 0 0]
```

```python
sage: CyclicCover(3, 3*x^3 + x).frobenius_matrix()
[ 0 0 O(107) O(107)]
[ 0 0 79 + O(107) O(107)]
[ O(107) 42 + O(107) 0 0]
[30 + O(107) 0(107) 0 0]
```

```
class sage.schemes.cyclic_covers.cycliccover_finite_field.CyclicCover_finite_field
(AA, r, f,
 names=None,
 verbose=0)

Bases: CyclicCover_generic

EXAMPLES:
```
```
sage: p = 13
sage: x = PolynomialRing(GF(p), "x").gen()
sage: C = CyclicCover(4, x^4 + 1)
sage: C.frobenius_polynomial()
x^6 - 6*x^5 + 3*x^4 + 60*x^3 + 39*x^2 - 1014*x + 2197
sage: R.<t> = PowerSeriesRing(Integers())
sage: C.projective_closure().zeta_series(2,t)
1 + 8*t + 102*t^2 + O(t^3)
sage: C.frobenius_polynomial().reverse()(t)/(1-t)*(1-p*t)) + O(t^5)
1 + 8*t + 102*t^2 + 1384*t^3 + 18089*t^4 + O(t^5)
```

```
frobenius_matrix(N=None)

Compute p-adic Frobenius matrix to precision p^N.

If N not supplied, a default value is selected, which is the minimum needed to recover the charpoly unambiguously.

EXAMPLES:
```
```
sage: p = 107
sage: x = PolynomialRing(GF(p), "x").gen()
sage: CyclicCover(2, x^5 + x).frobenius_matrix()
[ 0(107^2) 89*107 + O(107^2) O(107^2) ␣
 ˓→ 0(107^2)]
```

(continues on next page)
sage: CyclicCover(2, 3*x^5 + x).frobenius_matrix()

\[
\begin{bmatrix}
89*10^7 + O(10^7^2) & 0(10^7^2) & 0(10^7^2) \\
0(10^7^2) & 0(10^7^2) & 0(10^7^2) \\
5*10^7 + O(10^7^2) & 0(10^7^2) & 0(10^7^2) \\
0(10^7^2) & 0(10^7^2) & 89 + 53*10^7 + O(10^7^2) \\
0(10^7^2) & 0(10^7^2) & 0(10^7^2)
\end{bmatrix}
\]

sage: CyclicCover(3, x^3 + x).frobenius_matrix()

\[
\begin{bmatrix}
0 & 0 & O(10^7) \\
0 & 0 & 52 + O(10^7) \\
0(10^7) & 35 & O(10^7) \\
44 + O(10^7) & 0(10^7) & 0(10^7)
\end{bmatrix}
\]

sage: CyclicCover(3, 3*x^3 + x).frobenius_matrix()

\[
\begin{bmatrix}
0 & 0 & 0(10^7) \\
0 & 0 & 79 + O(10^7) \\
0(10^7) & 42 & O(10^7) \\
30 + O(10^7) & 0(10^7) & 0(10^7)
\end{bmatrix}
\]

\[\text{frobenius\_polynomial}()\]

Return the characteristic polynomial of Frobenius.

EXAMPLES:

Hyperelliptic curves:

sage: p = 11
sage: x = PolynomialRing(GF(p), "x").gen()
sage: f = x^7 + 4*x^2 + 10*x + 4
sage: CyclicCover(2, f).frobenius_polynomial() ==\
    HyperellipticCurve(f).frobenius_polynomial()
True
sage: f = 2*x^5 + 4*x^3 + x^2 + 2*x + 1
sage: CyclicCover(2, f).frobenius_polynomial() ==\
    HyperellipticCurve(f).frobenius_polynomial()
True
sage: f = 2*x^6 + 4*x^4 + x^3 + 2*x^2 + x
sage: CyclicCover(2, f).frobenius_polynomial() ==\
    HyperellipticCurve(f).frobenius_polynomial()
True
sage: p = 1117
sage: x = PolynomialRing(GF(p), "x").gen()
sage: f = x^9 + 4*x^2 + 10*x + 4
sage: P1 = CyclicCover(2, f).frobenius_polynomial()
sage: P2 = HyperellipticCurve(f).frobenius_polynomial()
sage: P1 == P2  # long time
Superelliptic curves:

\begin{verbatim}
sage: p = 11
sage: x = PolynomialRing(GF(p), "x").gen()
sage: CyclicCover(3, x^4 + 4*x^3 + 9*x^2 + 3*x + 1).frobenius_polynomial()
x^6 + 21*x^4 + 231*x^2 + 1331

sage: CyclicCover(4, x^3 + x + 1).frobenius_polynomial()
2*x^6 + 2*x^5 + 11*x^4 + 121*x^2 + 242*x + 1331

sage: p = 4999
sage: x = PolynomialRing(GF(p),"x").gen()
sage: CyclicCover(4, x^3 - 1).frobenius_polynomial() == 
....: CyclicCover(3, x^4 + 1).frobenius_polynomial()
True

sage: CyclicCover(3, x^4 + 4*x^3 + 9*x^2 + 3*x + 1).frobenius_polynomial()
x^6 + 180*x^5 + 20988*x^4 + 1854349*x^3 + 104919012*x^2 + 4498200180*x + 
→124925014999

sage: CyclicCover(4, x^5 + x + 1).frobenius_polynomial()
x^12 - 64*x^11 + 5018*x^10 - 488640*x^9 + 28119583*x^8 - 641791616*x^7 
+ 124245485932*x^6 - 3208316288384*x^5 + 702708407289583*x^4 - 
→61843359329111360*x^3 
+ 3133741752599645018*x^2 - 1998000799840001599936*x + 15606259372500374970001
\end{verbatim}

A non-monic example checking that github issue #29015 is fixed:

\begin{verbatim}
sage: a = 3
sage: K.<s> = GF(83^3);
sage: R.<x> = PolynomialRing(K)
sage: h = s*x^4 + x*3 + 8
sage: C = CyclicCover(a, h)
sage: C.frobenius_polynomial()
x^6 + 1563486*x^4 + 893980969482*x^2 + 186940255267540403
\end{verbatim}

Non-superelliptic curves:
sage: p = 13
sage: x = PolynomialRing(GF(p), "x").gen()
sage: C = CyclicCover(4, x^4 + 1)
sage: C.frobenius_polynomial()
x^6 - 6*x^5 + 3*x^4 + 60*x^3 + 39*x^2 - 1014*x + 2197
sage: R.<t> = PowerSeriesRing(Integers())
sage: C.projective_closure().zeta_series(2, t)
1 + 8*t + 102*t^2 + 0(t^3)
sage: C.frobenius_polynomial().reverse()(t)/((1-t)*(1-p*t)) + O(t^5)
1 + 8*t + 102*t^2 + 1384*t^3 + 18089*t^4 + O(t^5)

sage: x = PolynomialRing(GF(11), "x").gen()
sage: CyclicCover(4, x^6 - 11*x^3 + 70*x^2 - x + 961).frobenius_polynomial() # long time
x^14 + 14*x^13 + 287*x^12 + 3025*x^11 + 33275*x^10 + 381997*x^9 + 2254714*x^8 + 19487171

sage: x = PolynomialRing(GF(4999), "x").gen()
sage: CyclicCover(4, x^6 - 11*x^3 + 70*x^2 - x + 961).frobenius_polynomial() # long time
x^14 - 4*x^13 - 2822*x^12 - 30032*x^11 + 37164411*x^10 - 152369520*x^9 + 54217349361*x^8 - 1021791160888*x^7 + 27103252945639*x^6 - 380771457169520*x^5 + 4642764601604000589*x^4 - 18754988504199390032*x^3 - 880993477679457047178*x^2 - 62425037490001499880004*x + 7801569060312937447503499

sage: p = 11
sage: x = PolynomialRing(GF(p), "x").gen()
sage: CyclicCover(3, 5*x^3 - 5*x + 13).frobenius_polynomial()
x^2 + 11
sage: CyclicCover(3, x^6 + x^4 - x^3 + 2*x^2 - x - 1).frobenius_polynomial()
x^8 + 32*x^6 + 462*x^4 + 3872*x^2 + 14641

sage: p = 4999
sage: x = PolynomialRing(GF(p), "x").gen()
sage: CyclicCover(3, 5*x^3 - 5*x + 13).frobenius_polynomial()
x^2 - 47*x + 4999
sage: CyclicCover(3, x^6 + x^4 - x^3 + 2*x^2 - x - 1).frobenius_polynomial()
x^8 + 122*x^7 + 4594*x^6 - 639110*x^5 - 82959649*x^4 - 3194910890*x^3 + 114804064594*x^2 + 1524085182987*x + 624500149980001

sage: p = 11
sage: x = PolynomialRing(GF(p), "x").gen()
sage: CyclicCover(5, x^5 + x).frobenius_polynomial() # long time
x^12 + 4*x^11 + 22*x^10 + 108*x^9 + 503*x^8 + 1848*x^7 + 5588*x^6 + 20328*x^5 + 60863*x^4 + 143748*x^3 + 322102*x^2 + 644204*x + 1771561
sage: CyclicCover(5, 2*x^5 + x).frobenius_polynomial() # long time
x^12 - 9*x^11 + 42*x^10 - 108*x^9 + 47*x^8 + 1782*x^7 - 8327*x^6 + 19602*x^5 - 5687*x^4 + 143748*x^3 + 614922*x^2 - 1449459*x + 1771561

sage: p = 4999
sage: x = PolynomialRing(GF(p), "x").gen()
sage: CyclicCover(5, x^5 + x).frobenius_polynomial() # long time
x^12 + 299994*x^10 + 37498500015*x^8 + 249985000299980*x^6 + 93742500224997000015*x^4 + 1874812507499850001499994*x^2
(continues on next page)
17.2 Cyclic covers curves over a general ring

EXAMPLES:

\begin{verbatim}
sage: ZZx.<x> = ZZ[]
sage: C = CyclicCover(5, x^5 + x + 1); C
Cyclic Cover of P^1 over Integer Ring defined by y^5 = x^5 + x + 1
sage: C.genus()
6
sage: D = C.projective_closure(); D
Projective Plane Curve over Integer Ring defined by x0^5 + x0^4*x1 + x1^5 - x2^5
sage: D.change_ring(QQ).genus()
6
sage: C.change_ring(GF(5))
Traceback (most recent call last):
... ValueError: As the characteristic divides the order of the cover, this model is not smooth.
sage: GF7x.<x> = GF(7)[]
sage: C = CyclicCover(3, x^9 + x + 1)
sage: C
Cyclic Cover of P^1 over Finite Field of size 7 defined by y^3 = x^9 + x + 1
sage: C.genus()
7
sage: C.projective_closure()
Traceback (most recent call last):
... NotImplementedError: Weighted Projective Space is not implemented
\end{verbatim}

\textbf{class} \texttt{sage.schemes.cyclic_covers.cycliccover_generic.CyclicCover\_generic(AA, r, f, names=None)}

\textbf{Bases:} \texttt{AffinePlaneCurve}

Cyclic covers over a general ring

\textbf{INPUT:}

- \texttt{A} - ambient affine space
- \texttt{r} - degree of the cover
- \texttt{f} - univariate polynomial
- \texttt{names} (default: ["x","y"]) - names for the coordinate functions
base_extend($R$)

Return this CyclicCover over a new base ring $R$.

EXAMPLES:

```sage
sage: ZZx.<x> = ZZ[]
sage: C = CyclicCover(5, x^5 + x + 1)
sage: C.change_ring(GF(5))
Traceback (most recent call last):
... ValueError: As the characteristic divides the order of the cover,
this model is not smooth.
sage: C.change_ring(GF(3))
Traceback (most recent call last):
... ValueError: Not a smooth Cyclic Cover of P^1: singularity in the
provided affine patch.
sage: C.change_ring(GF(17))
Cyclic Cover of P^1 over Finite Field of size 17 defined by y^5 = x^5 + x + 1
```

change_ring($R$)

Return this CyclicCover over a new base ring $R$.

EXAMPLES:

```sage
sage: ZZx.<x> = ZZ[]
sage: C = CyclicCover(5, x^5 + x + 1)
sage: C.change_ring(GF(5))
Traceback (most recent call last):
... ValueError: As the characteristic divides the order of the cover,
this model is not smooth.
sage: C.change_ring(GF(3))
Traceback (most recent call last):
... ValueError: Not a smooth Cyclic Cover of P^1: singularity in the
provided affine patch.
sage: C.change_ring(GF(17))
Cyclic Cover of P^1 over Finite Field of size 17 defined by y^5 = x^5 + x + 1
```

cover_polynomial($K=None$, var='x')

Return the polynomial defining the cyclic cover.

EXAMPLES:

```sage
sage: ZZx.<x> = ZZ[]; CyclicCover(5, x^5 + x + 1).cover_polynomial()
x^5 + x + 1
```

genus()

The geometric genus of the curve.

EXAMPLES:

```sage
sage: ZZx.<x> = ZZ[]
sage: CyclicCover(5, x^5 + x + 1).genus()
```
is_singular()

Return if this curve is singular or not.

This just checks that the characteristic of the ring does not divide the order of the cover and that the defining polynomial of the cover is square free.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: CyclicCover(3, x^5 + x + 1).is_singular()
False
sage: CyclicCover(3, (x^5 + x + 1)^2, check_smooth=False).is_singular()
True
```

is_smooth()

Return if this curve is smooth or not.

This just checks that the characteristic of the ring does not divide the order of the cover and that the defining polynomial of the cover is square free.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: CyclicCover(3, x^5 + x + 1).is_smooth()
True
sage: CyclicCover(3, (x^5 + x + 1)^2, check_smooth=False).is_smooth()
False
```

order()

The order of the cover.

EXAMPLES:

```
sage: ZZx.<x> = ZZ[]
sage: CyclicCover(5, x^5 + x + 1).order()
5
sage: CyclicCover(3, x^5 + x + 1).order()
3
```

projective_closure(**kwds)

Return the projective closure of this affine curve.

EXAMPLES:

```
sage: GF7x.<x> = GF(7)[]
sage: CyclicCover(3, x^9 + x + 1).projective_closure()
Traceback (most recent call last):
...
NotImplementedError: Weighted Projective Space is not implemented
```

(continues on next page)
17.3 Computation of the Frobenius polynomial using Newton’s identities

sage.schemes.cyclic_covers.charpoly_frobenius.charpoly_frobenius(frob_matrix, charpoly_prec, p, weight, a=1, known_factor=[1])

Return the characteristic polynomial of the given Frobenius matrix.

INPUT:

- `frob_matrix` – a matrix representing the Frobenius matrix up to some precision
- `charpoly_prec` – a vector, such that `frob_matrix.change_ring(ZZ).charpoly()[i]` will be correct mod `p^i`, this can be easily deduced from the Hodge numbers and knowing the q-adic precision of `frob_matrix`
- `p` – prime
- `weight` – weight of the motive
- `a` – `p = q^a`
- `known_factor` – the list of coefficients of the known factor

OUTPUT:

A list of integers corresponding to the characteristic polynomial of the Frobenius action.

EXAMPLES:

```
sage: from sage.schemes.cyclic_covers.charpoly_frobenius import charpoly_frobenius
sage: M = Matrix([[0(17), 8 + 0(17)], [0(17), 15 + 0(17)]])
sage: charpoly_frobenius(M, [2, 1, 1], 17, 1, 1)
[17, 2, 1]
sage: R = Zq(17**2, names=('a',))
sage: M = Matrix(R, [[8*17 + 16*17**2 + O(17**3), 8 + 11*17 + O(17**2)],
                  [7*17**2 + O(17**3), 15 + 8*17 + O(17**2)]])
sage: charpoly_frobenius(M*M, [3, 2, 2], 17, 1, 2)
[289, 30, 1]
sage: M = Matrix([[8*31 + 8*31**2 + O(31**3), 0(31**3), 0(31**3), 0(31**3)],
               [0(31**3), 23*31 + 22*31**2 + O(31**3), 0(31**3), 0(31**3)],
               [0(31**3), 0(31**3), 27 + 7*31 + O(31**3), 0(31**3)],
               [0(31**3), 0(31**3), 0(31**3), 4 + 23*31 + O(31**3)]])
sage: charpoly_frobenius(M, [4, 3, 2, 2, 2], 31, 1, 1)
[961, 0, 46, 0, 1]
sage: M = Matrix([[4*43^2 + O(43^3), 17*43 + 11*43^2 + O(43^3), 0(43^3), 0(43^3),
               17 + 37*43 + O(43**2), 0(43**2)],
               [30*43 + 23*43^2 + O(43**2), 5*43 + O(43**2), 0(43**2), 0(43**2), 3 + O(43**3)]])
sage: charpoly_frobenius(M, [4, 3, 2, 2, 2, 2], 43, 1, 1)
```

(continues on next page)
38*43 + O(43^3), 0(43^3)),

....: (0(43^3), 0(43^3), 9*43 + 32*43^2 + O(43^3), 13 + 25*43 + O(43^3),

→ 0(43^3), 17 + 18*43 + O(43^3)),

....: (0(43^3), 0(43^3), 22*43 + 25*43^2 + O(43^3), 11 + 24*43 + O(43^3),

→ 3), 0(43^3), 36 + 5*43 + O(43^3)),

....: (42*43 + 15*43^2 + O(43^3), 22*43 + 8*43^2 + O(43^3), 0(43^3),

→ 0(43^3), 29 + 4*43 + O(43^3), 0(43^3)),

....: (0(43^3), 0(43^3), 6*43 + 19*43^2 + O(43^3), 8 + 24*43 + O(43^3),

→ 0(43^3), 31 + 42*43 + O(43^3))]

sage: charpoly_frobenius(M, [5, 4, 3, 2, 2, 2, 2], 43, 1, 1)
[79507, 27735, 6579, 1258, 153, 15, 1]

sage: M = Matrix([(1 + O(4999), O(4999), 0, 0),

....: (O(4999), 4860 + O(4999), 0, 0),

....: (0, 0, O(4999), 0(4999)),

....: (0, 0, O(4999), 1 + O(4999))])

sage: charpoly_frobenius(M, [2, 1, 1], 4999, 1, 1, [1, -2, 1])
[4999, 139, 1]

17.4 Cyclic cover curve constructor

sage.schemes.cyclic_covers.constructor.CyclicCover(r, f, names=None, check_smooth=True)

Return the cyclic cover of the projective line given by \( y^r = f \), for a univariate polynomial \( f \).

INPUT:

• \( r \) - the order of the cover

• \( f \) - univariate polynomial if not given, then it defaults to 0.

• names (default: ["x", "y"] ) - names for the coordinate functions

• check_squarefree (default: True) - test if the input defines a unramified cover of the projective line.

WARNING: When setting check_smooth=False or using a base ring that is not a field, the output curves are not to be trusted. For example, the output of is_singular or is_smooth only tests smoothness over the field of fractions.

NOTE: The words “cyclic cover” are usually used for covers of degree greater than two. We usually refer to smooth double covers of the projective line as “hyperelliptic curves” or “elliptic curves” if the genus is one. We allow such cases in this implementation, but we highly recommend to use the more specific constructors/classes HyperellipticCurve and EllipticCurve for a wider range of tools.

EXAMPLES:

Basic examples:

sage: R.<x> = QQ[]
sage: CyclicCover(2, x^5 + x + 1)
Cyclic Cover of P^1 over Rational Field defined by y^2 = x^5 + x + 1
sage: CyclicCover(3, x^5 + x + 1)
Cyclic Cover of P^1 over Rational Field defined by y^3 = x^5 + x + 1
sage: CyclicCover(5, x^5 + x + 1)
Cyclic Cover of P^1 over Rational Field defined by y^5 = x^5 + x + 1
sage: CyclicCover(15, x^9 + x + 1)
Cyclic Cover of P^1 over Rational Field defined by y^15 = x^9 + x + 1

sage: k.<a> = GF(9); R.<x> = k[]
# needs sage.rings.finite_rings
sage: CyclicCover(5, x^9 + x + 1)
# needs sage.rings.finite_rings
Cyclic Cover of P^1 over Finite Field in a of size 3^2 defined by y^5 = x^9 + x + 1
sage: CyclicCover(15, x^9 + x + 1)
# needs sage.rings.finite_rings
Traceback (most recent call last):
...
ValueError: As the characteristic divides the order of the cover, this model is not smooth.

We can change the names of the characteristic divides the order of the cover, this model is not smooth.

Double roots:

sage: P.<x> = GF(7)[]
sage: CyclicCover(2, (x^3-x+2)^2*(x^6-1))
Traceback (most recent call last):
...
ValueError: Not a smooth Cyclic Cover of P^1: singularity in the provided affine patch.

Cyclic Cover of P^1 over Finite Field of size 7 defined by y^2 = x^12 - 2*x^10 - 3*x^9 + x^8 + 3*x^7 + 3*x^6 + 2*x^4 + 3*x^3 - x^2 - 3*x + 3

Input with integer coefficients creates objects with the integers as base ring, but only checks smoothness over Q, not over Spec(Z). In other words, it is checked that the discriminant is non-zero, but it is not checked whether the discriminant is a unit in Z^*:

sage: R.<x> = ZZ[]
sage: CyclicCover(5, (x^3-x+2)*(x^6-1))
Cyclic Cover of P^1 over Integer Ring defined by y^5 = x^9 - x^7 + 2*x^6 - x^3 + x - 2
18.1 Elements of Berkovich space.

*Berkovich_Element* is an abstract parent class for elements of any Berkovich space. *Berkovich_Element_Cp_Affine* and *Berkovich_Element_Cp_Projective* implement elements of Berkovich space over \( \mathbb{C}_p \) and \( \mathbb{P}^1(\mathbb{C}_p) \). Elements are determined by specific data and fall into one of the four following types:

- Type I points are represented by a center.
- Type II points are represented by a center and a rational power of \( p \).
- Type III points are represented by a center and a non-negative real radius.
- Type IV points are represented by a finite list of centers and a finite list of non-negative radii.

For an exposition of Berkovich space over \( \mathbb{C}_p \), see Chapter 6 of [Ben2019]. For a more involved exposition, see Chapter 1 and 2 of [BR2010].

AUTHORS:

- Alexander Galarraga (2020-06-22): initial implementation

```python
class sage.schemes.berkovich.berkovich_cp_element.Berkovich_Element
    Bases: Element
    The parent class for any element of a Berkovich space.

class sage.schemes.berkovich.berkovich_cp_element.Berkovich_Element_Cp
    (parent, center, radius=None, power=None, prec=20, space_type=None, error_check=True)
```

Bases: *Berkovich_Element*

The abstract parent class for any element of Berkovich space over \( \mathbb{C}_p \).

This class should never be instantiated, instead use *Berkovich_Element_Cp_Affine* or *Berkovich_Element_Cp_Projective*.

EXAMPLES:

```
sage: B = Berkovich_Cp_Affine(3)
sage: B(2)
Type I point centered at 2 + 0(3^20)
```
Hsia_kernel(other, basepoint)

The Hsia kernel of this point and other, with basepoint basepoint.

The Hsia kernel with arbitrary basepoint is a generalization of the Hsia kernel at infinity.

INPUT:

• other – A point of the same Berkovich space as this point.
• basepoint – A point of the same Berkovich space as this point.

OUTPUT: A finite or infinite real number.

EXAMPLES:

```
sage: B = Berkovich_Cp_Projective(3)
sage: Q1 = B(2, 9)
sage: Q2 = B(1/27, 1/27)
sage: Q3 = B(1, 1/3)
sage: Q1.Hsia_kernel(Q2, Q3)
0.111111111111111
```

```
sage: B = Berkovich_Cp_Projective(3)
sage: Q1 = B(2, 9)
sage: Q2 = B(1/2)
sage: Q3 = B(1/2)
sage: Q1.Hsia_kernel(Q2, Q3)
+infinity
```

Hsia_kernel_infinity(other)

Return the Hsia kernel at infinity of this point with other.

The Hsia kernel at infinity is the natural extension of the absolute value on \( \mathbb{C}_p \) to Berkovich space.

INPUT:

• other – A point of the same Berkovich space as this point.

OUTPUT: A real number.

EXAMPLES:

```
sage: B = Berkovich_Cp_Affine(Qp(3))
sage: Q1 = B(1/4, 4)
sage: Q2 = B(1/4, 6)
sage: Q1.Hsia_kernel_infinity(Q2)
6.00000000000000
```

```
sage: B = Berkovich_Cp_Affine(Qp(3))
sage: Q1 = B(1/4, 4)
sage: Q2 = B(1/4, 6)
sage: Q1.Hsia_kernel_infinity(Q2)
6.00000000000000
```

# needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: A.<a> = NumberField(x^3 + 20)
sage: ideal = A.ideal(-1/2*a^2 + a - 3)
sage: B = Berkovich_Cp_Projective(A, ideal)
sage: Q1 = B(4)
big\_metric\texttt{(other)}

Return the path distance metric distance between this point and \texttt{other}.

Also referred to as the hyperbolic metric, or the big metric.

On the set of type II, III and IV points, the path distance metric is a metric. Following Baker and Rumely, we extend the path distance metric to type I points $x, y$ by $\rho(x, x) = 0$ and $\rho(x, y) = \infty$. See [BR2010].

INPUT:

* other – A point of the same Berkovich space as this point.

OUTPUT: A finite or infinite real number.

EXAMPLES:

\begin{verbatim}
sage: B = Berkovich_Cp_Affine(3)
sage: Q1 = B(1/4, 4)
sage: Q2 = B(1/4, 6)
sage: Q1.path_distance_metric(Q2)
0.369070246428542
sage: Q3 = B(1)
sage: Q3.path_distance_metric(Q1)
+infinity
sage: Q3.path_distance_metric(Q3)
0
\end{verbatim}

center\texttt{()}

Return the center of the corresponding disk (or sequence of disks) in $C_p$.

OUTPUT: An element of the base of the parent Berkovich space.

EXAMPLES:

\begin{verbatim}
sage: B = Berkovich_Cp_Affine(3)
sage: B(3, 1).center()
3 + O(3^21)
sage: C = Berkovich_Cp_Projective(3)
sage: C(3, 1).center()
(3 + O(3^21) : 1 + O(3^20))
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: A.<a> = NumberField(x^3 + 20)
sage: ideal = A.ideal(-1/2*a^2 + a - 3)
sage: B = Berkovich_Cp_Projective(A, ideal)
sage: B(a^2 + 4).center()
(a^2 + 4 : 1)
\end{verbatim}

18.1. Elements of Berkovich space.
**center_function()**

Return the function defining the centers of disks in the approximation.

Not defined unless this point is a type IV point created by using a univariate function to compute centers.

**OUTPUT:** A univariate function.

**EXAMPLES:**

```sage
sage: B = Berkovich_Cp_Projective(5)
sage: L.<t> = PolynomialRing(Qp(5))
sage: T = FractionField(L)
sage: f = T(1/t)
sage: Y = FractionField(R)
sage: g = (40*pi)/x
# needs sage.symbolic
sage: Q1 = B(f, g)
# needs sage.symbolic
sage: Q1.center_function()
(1 + O(5^20))/((1 + O(5^20))*t)
```

**diameter(basepoint=+Infinity)**

Generalized diameter function on Berkovich space.

If the basepoint is infinity, the diameter is equal to the limit of the radii of the corresponding disks in \( \mathbb{C}_p \).

If the basepoint is not infinity, the diameter is the Hsia kernel of this point with itself at basepoint basepoint.

**INPUT:**

- `basepoint` – (default = Infinity) A point of the same Berkovich space as this point.

**OUTPUT:** A real number.

**EXAMPLES:**

```sage
sage: B = Berkovich_Cp_Affine(3)
sage: Q1 = B(3)
sage: Q1.diameter()
0
sage: Q2 = B(1/2, 9)
sage: Q2.diameter()
9.00000000000000
```

The diameter of a type IV point is the limit of the radii:

```sage
sage: R.<x> = PolynomialRing(Qp(3))
sage: f = R(2)
sage: S.<y> = PolynomialRing(RR)
sage: S = FractionField(S)
sage: g = (y+1)/y
sage: B(f,g).diameter()
1.0
```
sage: B = Berkovich_Cp_Affine(3)
sage: Q1 = B(1/81, 1)
sage: Q2 = B(1/3)
sage: Q1.diameter(Q2)
0.00137174211248285

sage: Q2.diameter(Q2)
+infinity

**hyperbolic_metric**(other)
Return the path distance metric distance between this point and other.
Also referred to as the hyperbolic metric, or the big metric.
On the set of type II, III and IV points, the path distance metric is a metric. Following Baker and Rumely, we extend the path distance metric to type I points \( x, y \) by \( \rho(x, x) = 0 \) and \( \rho(x, y) = \infty \). See [BR2010].

**INPUT:**
- other – A point of the same Berkovich space as this point.

**OUTPUT:** A finite or infinite real number.

**EXAMPLES:**

sage: B = Berkovich_Cp_Affine(3)
sage: Q1 = B(1/4, 4)
sage: Q2 = B(1/4, 6)
sage: Q1.path_distance_metric(Q2)
0.369070246428542

sage: Q3 = B(1)
sage: Q3.path_distance_metric(Q1)
+infinity

sage: Q3.path_distance_metric(Q3)
0

**ideal()**
The ideal which defines an embedding of the base_ring into \( \mathbb{C}_p \).
If this Berkovich space is backed by a p-adic field, then an embedding is already specified, and this returns None.

**EXAMPLES:**

sage: B = Berkovich_Cp_Projective(QQ, 3)
sage: B(0).ideal()
3
::
sage: B = Berkovich_Cp_Projective(3)
sage: B(0).ideal()
**path_distance_metric**(other)

Return the path distance metric distance between this point and other.

Also referred to as the hyperbolic metric, or the big metric.

On the set of type II, III and IV points, the path distance metric is a metric. Following Baker and Rumely, we extend the path distance metric to type I points $x, y$ by $\rho(x, x) = 0$ and $\rho(x, y) = \infty$. See [BR2010].

**INPUT:**

- other – A point of the same Berkovich space as this point.

**OUTPUT:** A finite or infinite real number.

**EXAMPLES:**

```sage
sage: B = Berkovich_Cp_Affine(3)
sage: Q1 = B(1/4, 4)
sage: Q2 = B(1/4, 6)
sage: Q1.path_distance_metric(Q2)
0.369070246428542

sage: Q3 = B(1)
sage: Q3.path_distance_metric(Q1)
+infinity

sage: Q3.path_distance_metric(Q3)
0
```

**potential_kernel**(other, basepoint)

The potential kernel of this point with other, with basepoint basepoint.

The potential kernel is the hyperbolic distance between basepoint and the join of this point with other relative to basepoint.

**INPUT:**

- other – A point of the same Berkovich space as this point.
- basepoint – A point of the same Berkovich space as this point.

**OUTPUT:** A finite or infinite real number.

**EXAMPLES:**

```sage
sage: B = Berkovich_Cp_Projective(3)
sage: Q1 = B(27, 1)
sage: Q2 = B(1/3, 2)
sage: Q3 = B(1/9, 1/2)
sage: Q3.potential_kernel(Q1, Q2)
0.369070246428543

sage: B = Berkovich_Cp_Affine(3)
sage: Q1 = B(27, 1)
sage: Q2 = B(1/3, 2)
sage: Q3 = B(1/9, 1/2)
sage: Q3.potential_kernel(Q1, Q2)
0.369070246428543
```
power()

The power of \( p \) such that \( p^{power} = \) radius.

For type II points, always in \( Q \). For type III points, a real number. Not defined for type I or IV points.

OUTPUT:

- A rational for type II points.
- A real number for type III points.

EXAMPLES:

```python
sage: B = Berkovich_Cp_Affine(3)
sage: Q1 = B(1, 9)
sage: Q1.power()
2

sage: Q2 = B(1, 4)
sage: Q2.power()
1.26185950714291
```

prec()

Return the precision of a type IV point.

This integer is the number of disks used in the approximation of the type IV point. Not defined for type I, II, or III points.

OUTPUT: An integer.

EXAMPLES:

```python
sage: B = Berkovich_Cp_Affine(Qp(3))
sage: d = B([2, 2, 2], [1.761, 1.123, 1.112])
sage: d.precision()
3
```

precision()

Return the precision of a type IV point.

This integer is the number of disks used in the approximation of the type IV point. Not defined for type I, II, or III points.

OUTPUT: An integer.

EXAMPLES:

```python
sage: B = Berkovich_Cp_Affine(Qp(3))
sage: d = B([2, 2, 2], [1.761, 1.123, 1.112])
sage: d.precision()
3
```

prime()

The residue characteristic of the parent.

OUTPUT: A prime integer.

EXAMPLES:
sage: B = Berkovich_Cp_Affine(3)
sage: B(1).prime()
3

radius()
Radius of the corresponding disk (or sequence of disks) in \( C_p \).

OUTPUT:
• A non-negative real number for type I, II, or III points.
• A list of non-negative real numbers for type IV points.

EXAMPLES:
sage: B = Berkovich_Cp_Affine(3)
sage: Q1 = B(1, 2/5)
sage: Q1.radius()
0.400000000000000

sage: d = B([2, 2, 2], [1.761, 1.123, 1.112])
sage: d.radius()
[1.76100000000000, 1.12300000000000, 1.11200000000000]

radius_function()
Return the function defining the radii of disks in the approximation.

Not defined unless this point is a type IV point created by using a univariate function to compute radii.

OUTPUT: A univariate function.

EXAMPLES:
sage: B = Berkovich_Cp_Projective(5)
sage: L.<t> = PolynomialRing(Qp(5))
sage: T = FractionField(L)
sage: f = T(1/t)
sage: R.<x> = RR[]
sage: Y = FractionField(R)
sage: g = (40*pi)/x
# needs sage.symbolic
sage: Q1 = B(f, g)
# needs sage.symbolic
sage: Q1.radius_function()
# needs sage.symbolic
40.0000000000000*pi/x

small_metric(other)
Return the small metric distance between this point and other.

The small metric is an extension of twice the spherical distance on \( P^1(C_p) \).

INPUT:
• other – A point of the same Berkovich space as this point.

OUTPUT: A real number.

EXAMPLES:
```python
sage: B = Berkovich_Cp_Affine(3)
sage: Q1 = B(1/4, 4)
sage: Q2 = B(1/4, 6)
sage: Q1.small_metric(Q2)
0.0833333333333333
```

```python
sage: B = Berkovich_Cp_Projective(QQ, 5)
sage: Q1 = B(0, 1)
sage: Q2 = B(99)
sage: Q1.small_metric(Q2)
1.00000000000000
```

```python
sage: Q3 = B(1/4, 4)
sage: Q3.small_metric(Q2)
1.75000000000000
```

```python
sage: Q2.small_metric(Q3)
1.75000000000000
```

**spherical_kernel**(other)

The spherical kernel of this point with other.

The spherical kernel is one possible extension of the spherical distance on $P^1(C_p)$ to the projective Berkovich line. See [BR2010] for details.

**INPUT:**

- other -- A point of the same Berkovich space as this point.

**OUTPUT:** A real number.

**EXAMPLES:**

```python
sage: B = Berkovich_Cp_Projective(3)
sage: Q1 = B(2, 2)
sage: Q2 = B(1/9, 1)
sage: Q1.spherical_kernel(Q2)
0.500000000000000
```

```python
sage: Q3 = B(2)
sage: Q3.spherical_kernel(Q3)
0
```

**type_of_point()**

Return the type of this point of Berkovich space over $C_p$.

**OUTPUT:** An integer between 1 and 4 inclusive.

**EXAMPLES:**

```python
sage: B = Berkovich_Cp_Affine(3)
sage: B(1).type_of_point()
1
```
Bases: `Berkovich_Element_Cp`

Element class of the Berkovich affine line over $\mathbb{C}_p$.

Elements are categorized into four types, represented by specific data:

- **Type I** points are represented by a center in the base of the parent Berkovich space, which is $\mathbb{Q}_p$, a finite extension of $\mathbb{Q}_p$, or a number field.
- **Type II** points are represented by a center in the base of the parent Berkovich space, and a rational power of $p$.
- **Type III** points are represented by a center in the base of the parent Berkovich space, and a radius, a real number in $[0, \infty)$.
- **Type IV** points are represented by a finite list of centers in the base of the parent Berkovich space and a finite list of radii in $[0, \infty)$. Type IV points can be created from univariate functions, allowing for arbitrary precision.

**INPUT:**

- **center** – For type I, II, and III points, the center of the corresponding disk in $\mathbb{C}_p$. If the parent Berkovich space was created using a number field $K$, then `center` must be an element of $K$. Otherwise, `center` must be an element of a $p$-adic field. For type IV points, can be a list of centers used to approximate the point or a univariate function that computes the centers (computation starts at 1).
- **radius** – (optional) For type I, II, and III points, the radius of the corresponding disk in $\mathbb{C}_p$. Must coerce into the real numbers. For type IV points, can be a list of radii used to approximate the point or a univariate function that computes the radii (computation starts at 1).
- **power** – (optional) Rational number. Used for constructing type II points; specifies the power of $p$ such that $p^{\text{power}} = \text{radius}$.
- **prec** – (default: 20) The number of disks to be used to approximate a type IV point.
- **error_check** – (default: True) If error checking should be run on input. If input is correctly formatted, can be set to `False` for better performance. **WARNING:** with error check set to `False`, any error in the input will lead to incorrect results.

**EXAMPLES:**

Type I points can be created by specifying the corresponding point of $\mathbb{C}_p$:

```python
sage: B = Berkovich_Cp_Affine(Qp(3))
sage: B(4)
Type I point centered at 1 + 3 + O(3^20)
```

The center of a point can be an element of a finite extension of $\mathbb{Q}_p$:

```python
sage: A.<t> = Qq(27)
sage: B(1 + t)
Type I point centered at (t + 1) + O(3^20)
```
Type II and III points can be created by specifying a center and a radius:

```sage
B(2, 3**(1/2))  # not tested
Type II point centered at 2 + O(3^20) of radius 3^1/2
```

```sage
B(2, 1.6)
Type III point centered at 2 + O(3^20) of radius 1.60000000000000
```

Some type II points may be mistaken for type III points:

```sage
B(3, 3**0.5)  # not tested
Type III point centered at 3 + O(3^21) of radius 1.73205080756888
```

To avoid these errors, specify the power instead of the radius:

```sage
B(3, power=RR(1/100000))
Type II point centered at 3 + O(3^21) of radius 3^1/100000
```

Type IV points can be constructed in a number of ways, the first being from a list of centers and radii used to approximate the point:

```sage
B([Qp(3)(2), Qp(3)(2), Qp(3)(2)], [1.761, 1.123, 1.112])  # not tested
Type IV point of precision 3, approximated by disks centered at [2 + O(3^20), 2 + O(3^20)] ... with radii [1.76100000000000, 1.12300000000000] ...
```

Type IV points can be constructed from univariate functions, with arbitrary precision:

```sage
A.<t> = Qq(27)
sage: R.<x> = PolynomialRing(A)
sage: f = (1 + t)^2*x
sage: S.<y> = PolynomialRing(RR)
sage: S = FractionField(S)
sage: g = (y + 1)/y
sage: d = B(f, g, prec=100); d
Type IV point of precision 100 with centers given by 
(((t^2 + 2*t + 1) + O(3^20))*x and radii given by (y + 1.00000000000000)/y
```

For increased performance, `error_check` can be set to `False`. WARNING: with error check set to `False`, any error in the input will lead to incorrect results:

```sage
B(f, g, prec=100, error_check=False)
Type IV point of precision 100 with centers given by 
(((t^2 + 2*t + 1) + O(3^20))*x and radii given by (y + 1.00000000000000)/y
```

When creating a Berkovich space backed by a number field, points can be created similarly:

```sage
# needs sage.rings.number_field
R.<x> = QQ[]
sage: A.<a> = NumberField(x^3 + 20)
sage: ideal = A.prime_above(3)
sage: B = Berkovich_Cp_Projective(A, ideal)
sage: Q1 = B(a); Q1
Type I point centered at (a : 1)
```
sage: B(a + 1, 3)
˓→ needs sage.rings.number_field
Type II point centered at (a + 1 : 1) of radius 3^1

as_projective_point()

Return the corresponding point of projective Berkovich space.

We identify affine Berkovich space with the subset \( P^1_{\text{Berk}}(C_p) - \{(1 : 0)\} \).

EXAMPLES:

sage: B = Berkovich_Cp_Affine(5)
sage: B(5).as_projective_point()
Type I point centered at (5 + O(5^21) : 1 + O(5^20))
sage: B(0, 1).as_projective_point()
Type II point centered at (0 : 1 + O(5^20)) of radius 5^0

L.<t> = PolynomialRing(Qp(5))
sage: f = T(1/t)
sage: Y = FractionField(R)
sage: g = (40*pi)/x
˓→ needs sage.symbolic
sage: Q2 = B(f, g)
˓→ needs sage.symbolic
sage: Q2.as_projective_point()
˓→ needs sage.symbolic
Type IV point of precision 20 with centers given by (1 + O(5^20))/((1 + O(5^20))^t)
and radii given by 40.000000000000*pi/x

contained_in_interval(start, end)

Check if this point is an element of the interval \([start, end]\).

INPUT:

• start – A point of the same Berkovich space as this point.

• end – A point of the same Berkovich space as this point.

OUTPUT:

• True if this point is an element of \([start, end]\).

• False otherwise.

EXAMPLES:

sage: B = Berkovich_Cp_Projective((3))
sage: Q1 = B(2, 1)
sage: Q2 = B(2, 4)
sage: Q3 = B(1/3)
sage: Q2.contained_in_interval(Q1, Q3.join(Q1))
False
Sage:
\[ \text{sage: } Q4 = B(1/81, 1) \]
\[ \text{sage: } Q2 \text{.contained\_in\_interval}(Q1, Q4 \text{.join}(Q1)) \]
True

\text{gt}(\text{other})

Return True if this point is strictly greater than other in the standard partial order.

Roughly, the partial order corresponds to containment of the corresponding disks in \( \mathbb{C}_p \).

For example, let \( x \) and \( y \) be points of type II or III. If \( x \) has center \( c_1 \) and radius \( r_1 \) and \( y \) has center \( c_2 \) and radius \( r_2 \), \( x < y \) if and only if \( D(c_1, r_1) \) is a subset of \( D(c_2, r_2) \) in \( \mathbb{C}_p \).

INPUT:
- \text{other} – A point of the same Berkovich space as this point.

OUTPUT:
- True – If this point is greater than other in the standard partial order.
- False – Otherwise.

EXAMPLES:

```
\text{sage: } B = \text{Berkovich\_Cp\_Affine}(\text{QQ}, 3)
\text{sage: } Q1 = B(5, 3)
\text{sage: } Q2 = B(5, 1)
\text{sage: } Q1 \text{.gt}(Q2)
True
```

```
\text{sage: } Q3 = B(1/27)
\text{sage: } Q1 \text{.gt}(Q3)
False
```

\text{involution\_map()}

Return the image of this point under the involution map.

The involution map is the extension of the map \( z \mapsto 1/z \) on \( \mathbb{C}_p \) to Berkovich space.

For affine Berkovich space, not defined for the type I point centered at 0.

If zero is contained in every disk approximating a type IV point, then the image under the involution map is not defined. To avoid this error, increase precision.

OUTPUT: A point of the same Berkovich space.

EXAMPLES:

The involution map is \( 1/z \) on type I points:

```
\text{sage: } B = \text{Berkovich\_Cp\_Affine}(3)
\text{sage: } Q1 = B(1/2)
\text{sage: } Q1 \text{.involution\_map()}
Type I point centered at 2 + O(3^20)
```

```
\text{sage: } Q2 = B(0, 1/3)
\text{sage: } Q2 \text{.involution\_map()}
Type II point centered at 0 of radius 3^1
```
Join \( (\text{other}, \text{basepoint}=\text{+Infinity}) \)

Compute the join of this point and \text{other} with respect to \text{basepoint}.

The join is first point that lies on the intersection of the path from this point to \text{basepoint} and the path from \text{other} to \text{basepoint}.

**INPUT:**

- \text{other} – A point of the same Berkovich space as this point.
- \text{basepoint} – (default: Infinity) A point of the same Berkovich space as this point or Infinity.

**OUTPUT:** A point of the same Berkovich space.

**EXAMPLES:**

```plaintext
sage: B = Berkovich_Cp_Affine(3)
sage: Q1 = B(2, 1)
sage: Q2 = B(2, 2)
sage: Q1.join(Q2)
```

Type III point centered at 2 + 0(3^20) of radius 2.00000000000000

```plaintext
sage: Q3 = B(5)
sage: Q3.join(Q1)
```

Type II point centered at 2 + 0(3^20) of radius 3^0

```plaintext
sage: Q3.join(Q1, basepoint=Q2)
```

Type II point centered at 2 + 0(3^20) of radius 3^0

\lt \text{(other)}

Return True if this point is strictly less than \text{other} in the standard partial order.

Roughly, the partial order corresponds to containment of the corresponding disks in \(\mathbb{C}_p\).

For example, let \(x\) and \(y\) be points of type II or III. If \(x\) has center \(c_1\) and radius \(r_1\) and \(y\) has center \(c_2\) and radius \(r_2\), \(x < y\) if and only if \(D(c_1, r_1)\) is a subset of \(D(c_2, r_2)\) in \(\mathbb{C}_p\).

**INPUT:**

- \text{other} – A point of the same Berkovich space as this point.

**OUTPUT:**

- True – If this point is less than \text{other} in the standard partial order.
- False – Otherwise.

**EXAMPLES:**

```plaintext
sage: B = Berkovich_Cp_Projective(3)
sage: Q1 = B(5, 0.5)
sage: Q2 = B(5, 1)
sage: Q1.lt(Q2)
```

True
sage: Q3 = B(1)
sage: Q1.lt(Q3)
False

class sage.schemes.berkovich.berkovich_cp_element.Berkovich_Element_Cp_Projective(parent, center, radius=None, power=None, prec=20, error_check=True)

Bases: Berkovich_Element_Cp

Element class of the Berkovich projective line over $\mathbb{C}_p$.

Elements are categorized into four types, represented by specific data:

- Type I points are represented by a center in the base of the parent Berkovich space, which is projective space of dimension 1 over either $\mathbb{Q}_p$, a finite extension of $\mathbb{Q}_p$, or a number field.
- Type II points are represented by a center in the base of the parent Berkovich space, and a rational power of $p$.
- Type III points are represented by a center in the base of the parent Berkovich space, and by a radius, a real number, in $[0, \infty)$.
- Type IV points are represented by a finite list of centers in the base of the parent Berkovich space and a finite list of radii in $[0, \infty)$.

The projective Berkovich line is viewed as the one-point compactification of the affine Berkovich line. The projective Berkovich line therefore contains every point of the affine Berkovich line, along with a type I point centered at infinity.

INPUT:

- **center** – For type I, II, and III points, the center of the corresponding disk in $\mathbb{P}^1(\mathbb{C}_p)$. If the parent Berkovich space was created using a number field $K$, then center can be an element of $\mathbb{P}^1(K)$. Otherwise, center must be an element of a projective space of dimension 1 over a padic field. For type IV points, can be a list of centers used to approximate the point or a univariate function that computes the centers (computation starts at 1).
- **radius** – (optional) For type I, II, and III points, the radius of the corresponding disk in $\mathbb{C}_p$. Must coerce into the real numbers. For type IV points, can be a list of radii used to approximate the point or a univariate function that computes the radii (computation starts at 1).
- **power** – (optional) Rational number. Used for constructing type II points; specifies the power of $p$ such that $p^{\text{power}} = \text{radius}$
- **prec** – (default: 20) The number of disks to be used to approximate a type IV point
- **error_check** – (default: True) If error checking should be run on input. If input is correctly formatted, can be set to False for better performance. WARNING: with error check set to False, any error in the input will lead to incorrect results.

EXAMPLES:

Type I points can be created by specifying the corresponding point of $\mathbb{P}^1(\mathbb{C}_p)$:
```python
sage: S = ProjectiveSpace(Qp(5), 1)
sage: P = Berkovich_Cp_Projective(S); P
Projective Berkovich line over Cp(5) of precision 20

sage: a = S(0, 1)
sage: Q1 = P(a); Q1
Type I point centered at (0 : 1 + O(5^20))

sage: Q2 = P((1,0)); Q2
Type I point centered at (1 + O(5^20) : 0)

Type II and III points can be created by specifying a center and a radius:

```
as_affine_point()

Return the corresponding affine point after dehomogenizing at infinity.

OUTPUT: A point of affine Berkovich space.

EXAMPLES:

```
sage: B = Berkovich_Cp_Projective(5)
sage: B(5).as_affine_point()
Type I point centered at 5 + 0(5^21)
```

```
sage: Q = B(0, 1).as_affine_point(); Q
Type II point centered at 0 of radius 5^0
sage: Q.parent()
Affine Berkovich line over Cp(5) of precision 20
```

```
sage: L.<t> = PolynomialRing(Qp(5))
sage: R.<x> = RR[]
sage: g = (40*pi)/x
```

needs sage.symbolic

```
sage: Q2 = B(f, g)
sage: Q2.as_affine_point()
Type IV point of precision 20 with centers given by
and radii given by 40.0000000000000*pi/x
```

contained_in_interval(start, end)

Check if this point is an element of the interval [start, end].

INPUT:

- start – A point of the same Berkovich space as this point.
- end – A point of the same Berkovich space as this point.

OUTPUT:

- True if this point is an element of [start, end].
- False otherwise.

EXAMPLES:

```
sage: B = Berkovich_Cp_Projective(3)
sage: Q1 = B(2, 1)
sage: Q2 = B(2, 4)
sage: Q3 = B(1/3)
sage: Q2.contained_in_interval(Q1, Q3.join(Q1))
False
```
gt(other)

Return True if this point is strictly greater than other in the standard partial order.

Roughly, the partial order corresponds to containment of the corresponding disks in $C_p$.

For example, let $x$ and $y$ be points of type II or III. If $x$ has center $c_1$ and radius $r_1$ and $y$ has center $c_2$ and radius $r_2$, $x < y$ if and only if $D(c_1, r_1)$ is a subset of $D(c_2, r_2)$ in $C_p$.

INPUT:

- other – A point of the same Berkovich space as this point.

OUTPUT:

- True – If this point is greater than other in the standard partial order.
- False – Otherwise.

EXAMPLES:

```python
sage: B = Berkovich_Cp_Projective(QQ, 3)
sage: Q1 = B(5, 3)
sage: Q2 = B(5, 1)
sage: Q1.gt(Q2)
True

sage: Q3 = B(1/27)
sage: Q1.gt(Q3)
False
```

involution_map()

Return the image of this point under the involution map.

The involution map is the extension of the map $z \rightarrow 1/z$ on $P^1(C_p)$ to Berkovich space.

If zero is contained in every disk approximating a type IV point, then the image under the involution map is not defined. To avoid this error, increase precision.

OUTPUT: A point of the same Berkovich space.

EXAMPLES:

The involution map is $1/z$ on type I points:

```python
sage: B = Berkovich_Cp_Projective(3)
sage: Q1 = B(1/2)
sage: Q1.involution_map()
Type I point centered at $(2 + O(3^{20}) : 1 + O(3^{20}))$

sage: Q2 = B(0, 1/3)
sage: Q2.involution_map()
Type II point centered at $(0 : 1 + O(3^{20}))$ of radius $3^{1/3}$
```
sage: Q3 = B(1/3, 1/3)
sage: Q3.involution_map()
Type II point centered at \((3 + 0(3^{21}) : 1 + 0(3^{20}))\) of radius \(3^{-3}\)

join\((other, basepoint=+Infinity)\)
Compute the join of this point and \(other\), with respect to \(basepoint\).

The join is first point that lies on the intersection of the path from this point to \(basepoint\) and the path from \(other\) to \(basepoint\).

INPUT:
- \(other\) – A point of the same Berkovich space as this point.
- \(basepoint\) – (default: Infinity) A point of the same Berkovich space as this point, or infinity.

OUTPUT: A point of the same Berkovich space.

EXAMPLES:

```
sage: B = Berkovich_Cp_Projective(3)
sage: Q1 = B(2, 1)
sage: Q2 = B(2, 2)
sage: Q1.join(Q2)
Type III point centered at \((2 + 0(3^{20}) : 1 + 0(3^{20}))\) of radius 2.

sage: Q3 = B(5)
sage: Q3.join(Q1)
Type II point centered at \((2 + 3 + 0(3^{20}) : 1 + 0(3^{20}))\) of radius 3^0

sage: Q3.join(Q1, basepoint=Q2)
Type II point centered at \((2 + 0(3^{20}) : 1 + 0(3^{20}))\) of radius 3^0
```

lt\((other)\)
Return True if this point is strictly less than \(other\) in the standard partial order.

Roughly, the partial order corresponds to containment of the corresponding disks in \(C_p\).

For example, let \(x\) and \(y\) be points of type II or III. If \(x\) has center \(c_1\) and radius \(r_1\) and \(y\) has center \(c_2\) and radius \(r_2\), \(x < y\) if and only if \(D(c_1, r_1)\) is a subset of \(D(c_2, r_2)\) in \(C_p\).

INPUT:
- \(other\) – A point of the same Berkovich space as this point.

OUTPUT:
- True – If this point is less than \(other\) in the standard partial order.
- False – Otherwise.

EXAMPLES:

```
sage: B = Berkovich_Cp_Projective(3)
sage: Q1 = B(5, 0.5)
sage: Q2 = B(5, 1)
sage: Q1.lt(Q2)
True
```
18.2 Berkovich Space over $C_p$

The Berkovich affine line is the set of seminorms on $C_p[x]$, with the weakest topology that makes the map $|\cdot| \to |f|$ continuous for all $f \in C_p[x]$. The Berkovich projective line is the one-point compactification of the Berkovich affine line.

The two main classes are `Berkovich_Cp_Affine` and `Berkovich_Cp_Projective`, which implement the affine and projective lines, respectively.

`Berkovich_Cp_Affine` and `Berkovich_Cp_Projective` take as input one of the following: the prime $p$, a finite extension of $Q_p$, or a number field and a place.

For an exposition of Berkovich space over $C_p$, see Chapter 6 of [Ben2019]. For a more involved exposition, see Chapter 1 and 2 of [BR2010].

AUTHORS:
- Alexander Galarraga (2020-06-22): initial implementation

```python
sage: B = Berkovich_Cp_Projective(A, ideal)
sage: B.ideal() Fractional ideal (-a - 2)
```
sage: B = Berkovich_Cp_Projective(QQ, 3)
sage: B.ideal()
3

sage: B = Berkovich_Cp_Projective(Qp(3))
sage: B.ideal() is None
True

**is_number_field_base**()

Return True if this Berkovich space is backed by a number field.

OUTPUT:

- True if this Berkovich space was created with a number field.
- False otherwise.

EXAMPLES:

sage: B = Berkovich_Cp_Affine(Qp(3))
sage: B.is_number_field_base()
False

sage: B = Berkovich_Cp_Affine(QQ, 3)
sage: B.is_number_field_base()
True

**is_padic_base**()

Return True if this Berkovich space is backed by a p-adic field.

OUTPUT:

- True if this Berkovich space was created with a p-adic field.
- False otherwise.

EXAMPLES:

sage: B = Berkovich_Cp_Affine(Qp(3))
sage: B.is_padic_base()
True

sage: B = Berkovich_Cp_Affine(QQ, 3)
sage: B.is_padic_base()
False

**prime**()

The residue characteristic of the base.

EXAMPLES:

sage: B = Berkovich_Cp_Projective(3)
sage: B.prime()
3
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: A.<a> = NumberField(x^3 + 20)

sage: ideal = A.ideal(-1/2*a^2 + a - 3)
sage: B = Berkovich_Cp_Affine(A, ideal)
sage: B.residue_characteristic()
7

residue_characteristic()
The residue characteristic of the base.

EXAMPLES:

sage: B = Berkovich_Cp_Projective(3)
sage: B.prime()
3

sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: A.<a> = NumberField(x^3 + 20)

sage: ideal = A.ideal(-1/2*a^2 + a - 3)
sage: B = Berkovich_Cp_Affine(A, ideal)
sage: B.residue_characteristic()
7

class sage.schemes.berkovich.berkovich_space.Berkovich_Cp_Affine(base, ideal=None)

Bases: Berkovich_Cp

The Berkovich affine line over $\mathbb{C}_p$.

The Berkovich affine line is the set of seminorms on $\mathbb{C}_p[x]$, with the weakest topology such that the map $|\cdot| \to |f|$ is continuous for all $f \in \mathbb{C}_p[x]$.

We can represent the Berkovich affine line in two separate ways: either using a $p$-adic field to represent elements or using a number field to represent elements while storing an ideal of the ring of integers of the number field, which specifies an embedding of the number field into $\mathbb{C}_p$. See the examples.

INPUT:

- base – Three cases:
  - a prime number $p$. Centers of elements are then represented as points of $\mathbb{Q}_p$.
  - $\mathbb{Q}_p$ or a finite extension of $\mathbb{Q}_p$. Centers of elements are then represented as points of base.
  - A number field $K$. Centers of elements are then represented as points of $K$.

- ideal – (optional) a prime ideal of base. Must be specified if a number field is passed to base, otherwise it is ignored.

EXAMPLES:

sage: B = Berkovich_Cp_Affine(3); B
Affine Berkovich line over Cp(3) of precision 20

We can create elements:
For details on element creation, see the documentation of `Berkovich_Element_Cp_Affine`. Initializing by passing in $Q_p$ looks the same:

```python
sage: B = Berkovich_Cp_Affine(Qp(3)); B
Affine Berkovich line over Cp(3) of precision 20
```

However, this method allows for more control over behind-the-scenes conversion:

```python
sage: B = Berkovich_Cp_Affine(Qp(3, 1)); B
Affine Berkovich line over Cp(3) of precision 1
sage: B(1/2)
Type I point centered at 2 + 0(3)
```

Note that this point has very low precision, as $B$ was initialized with a p-adic field of capped-relative precision one. For high precision, pass in a high precision p-adic field:

```python
sage: B = Berkovich_Cp_Affine(Qp(3, 1000)); B
Affine Berkovich line over Cp(3) of precision 1000
```

Points of Berkovich space can be created from points of extensions of $Q_p$:

```python
sage: B = Berkovich_Cp_Affine(3)
sage: A.<a> = Qp(3).extension(x^3 - 3)
sage: B(a)
Type I point centered at a + O(a^61)
```

For exact computation, a number field can be used:

```python
sage: R.<x> = QQ[]  # needs sage.rings.number_field
sage: A.<a> = NumberField(x^3 + 20)  # needs sage.rings.number_field
sage: ideal = A.prime_above(3)  # needs sage.rings.number_field
sage: B = Berkovich_Cp_Affine(A, ideal); B  # needs sage.rings.number_field
Affine Berkovich line over Cp(3), with base Number Field in a with defining polynomial x^3 + 20
```

Number fields have a major advantage of exact computation.

Number fields also have added functionality. Arbitrary extensions of $Q$ are supported, while there is currently limited functionality for extensions of $Q_p$. As seen above, constructing a Berkovich space backed by a number field requires specifying an ideal of the ring of integers of the number field. Specifying the ideal uniquely specifies an embedding of the number field into $C_p$. 

18.2. Berkovich Space over $C_p$ 439
Unlike in the case where Berkovich space is backed by a p-adic field, any point of a Berkovich space backed by a number field must be centered at a point of that number field:

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: A.<a> = NumberField(x^3 + 20)
sage: ideal = A.prime_above(3)
sage: B = Berkovich_Cp_Affine(A, ideal)
sage: C.<c> = NumberField(x^2 + 1)
sage: B(c)
Traceback (most recent call last):
  ... ValueError: could not convert c to Number Field in a with defining polynomial x^3 + 20
```

**Element**

alias of *Berkovich_Element_Cp_Affine*

```python
class sage.schemes.berkovich.berkovich_space.Berkovich_Cp_Projective(base, ideal=None)
Bases: Berkovich_Cp
```

The Berkovich projective line over $\mathbb{C}_p$.

The Berkovich projective line is the one-point compactification of the Berkovich affine line.

We can represent the Berkovich projective line in two separate ways: either using a p-adic field to represent elements or using a number field to represent elements while storing an ideal of the ring of integers of the number field, which specifies an embedding of the number field into $\mathbb{C}_p$. See the examples.

**INPUT:**

- base – Three cases:
  - a prime number $p$. Centers of elements are then represented as points of projective space of dimension 1 over $\mathbb{Q}_p$.
  - $\mathbb{Q}_p$ or a finite extension of $\mathbb{Q}_p$. Centers of elements are then represented as points of projective space of dimension 1 over base.
  - A number field $K$. Centers of elements are then represented as points of projective space of dimension 1 over base.

- ideal – (optional) a prime ideal of base. Must be specified if a number field is passed to base, otherwise it is ignored.

**EXAMPLES:**

```
sage: B = Berkovich_Cp_Projective(3); B
Projective Berkovich line over Cp(3) of precision 20
```

Elements can be constructed:

```
sage: B(1/2)
Type I point centered at (2 + 3 + 3^2 + 3^3 + 3^4 + 3^5 + 3^6 + 3^7 + 3^8 + 3^9 + 3^10 + 3^11 + 3^12 + 3^13 + 3^14 + 3^15 + 3^16 + 3^17 + 3^18 + 3^19 + O(3^20) : 1 + O(3^20))
```

```
sage: B(2, 1)
Type II point centered at (2 + O(3^20) : 1 + O(3^20)) of radius 3^0
```
For details about element construction, see the documentation of \texttt{Berkovich\_Element\_Cp\_Projective}. Initializing a Berkovich projective line by passing in a p-adic space looks the same:

```python
sage: B = Berkovich\_Cp\_Projective(Qp(3)); B
Projective Berkovich line over Cp(3) of precision 20
```

However, this method allows for more control over behind-the-scenes conversion:

```python
sage: S = Qp(3, 1)
sage: B = Berkovich\_Cp\_Projective(S); B
Projective Berkovich line over Cp(3) of precision 1
sage: Q1 = B(1/2); Q1
Type I point centered at (2 + O(3) : 1 + O(3))
```

Note that this point has very low precision, as \(S\) has low precision cap. Berkovich space can also be created over a number field, as long as an ideal is specified:

```python
sage: R.<x> = QQ[]
sage: A.<a> = NumberField(x^2 + 1) # needs sage\_rings\_number\_field
sage: ideal = A.prime\_above(2) # needs sage\_rings\_number\_field
sage: B = Berkovich\_Cp\_Projective(A, ideal); B
Projective Berkovich line over Cp(2), with base Number Field in a with defining polynomial x^2 + 1
```

Number fields have the benefit that computation is exact, but lack support for all of \(\mathbb{C}_p\).

Number fields also have the advantage of added functionality, as arbitrary extensions of \(\mathbb{Q}\) can be constructed while there is currently limited functionality for extensions of \(\mathbb{Q}_p\). As seen above, constructing a Berkovich space backed by a number field requires specifying an ideal of the ring of integers of the number field. Specifying the ideal uniquely specifies an embedding of the number field into \(\mathbb{C}_p\).

Unlike in the case where Berkovich space is backed by a p-adic field, any point of a Berkovich space backed by a number field must be centered at a point of that number field:

```python
sage: # needs sage\_rings\_number\_field
sage: R.<x> = QQ[]
sage: A.<a> = NumberField(x^3 + 20)
sage: ideal = A.prime\_above(3)
sage: B = Berkovich\_Cp\_Projective(A, ideal)
sage: C.<c> = NumberField(x^2 + 1)
sage: B(c)
Traceback (most recent call last):
  ... TypeError: could not convert \(c\) to Projective Space of dimension 1 over Number Field in \(a\) with defining polynomial \(x^3 + 20\)
```

\textbf{Element}

\textit{alias of Berkovich\_Element\_Cp\_Projective}

\texttt{base\_ring()}

The base ring of this Berkovich Space.
OUTPUT: A field.

EXAMPLES:

```
sage: B = Berkovich_Cp_Projective(3)
sage: B.base_ring()
3-adic Field with capped relative precision 20

sage: C = Berkovich_Cp_Projective(ProjectiveSpace(Qp(3, 1), 1))
sage: C.base_ring()
3-adic Field with capped relative precision 1
```

```
sage: # needs sage.rings.number_field
sage: R, x = QQ[]
sage: A.<a> = NumberField(x^3 + 20)
sage: ideal = A.prime_above(3)
sage: D = Berkovich_Cp_Projective(A, ideal)
sage: D.base_ring()
Number Field in a with defining polynomial x^3 + 20
```

sage.schemes.berkovich.berkovich_space.is_Berkovich(space)
Check if space is a Berkovich space.

OUTPUT:

• True if space is a Berkovich space.
• False otherwise.

EXAMPLES:

```
sage: B = Berkovich_Cp_Projective(3)
sage: from sage.schemes.berkovich.berkovich_space import is_Berkovich
sage: is_Berkovich(B)
True
```

sage.schemes.berkovich.berkovich_space.is_Berkovich_Cp(space)
Check if space is a Berkovich space over Cp.

OUTPUT:

• True if space is a Berkovich space over Cp.
• False otherwise.

EXAMPLES:

```
sage: B = Berkovich_Cp_Projective(3)
sage: from sage.schemes.berkovich.berkovich_space import is_Berkovich
sage: is_Berkovich(B)
True
```
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