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CHAPTER ONE

SET CONSTRUCTIONS

1.1 Cartesian products

AUTHORS:

• Nicolas Thiery (2010-03): initial version

class sage.sets.cartesian_product.CartesianProduct

Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

A class implementing a raw data structure for Cartesian products of sets (and elements thereof). See cartesian_product for how to construct full fledged Cartesian products.

EXAMPLES:

sage: G = cartesian_product([GF(5), Permutations(10)])
sage: G.cartesian_factors()
(Finite Field of size 5, Standard permutations of 10)
sage: G.cardinality()
18144000
sage: G.random_element()   # random
(1, [4, 7, 6, 5, 10, 1, 3, 2, 8, 9])
sage: G.category()
Join of Category of finite monoids
and Category of Cartesian products of monoids
and Category of Cartesian products of finite enumerated sets

_cartesian_product_of_elements(elements)

Return the Cartesian product of the given elements.

This implements Sets.CartesianProducts.ParentMethods._cartesian_product_of_elements(). INPUT:

• elements – an iterable (e.g. tuple, list) with one element of each Cartesian factor of self

Warning: This is meant as a fast low-level method. In particular, no coercion is attempted. When coercion or sanity checks are desirable, please use instead self(elements) or self._element_constructor_(elements).

EXAMPLES:
sage: S1 = Sets().example()
sage: S2 = InfiniteEnumeratedSets().example()
sage: C = cartesian_product([S2, S1, S2])
sage: C._cartesian_product_of_elements([S2.an_element(), S1.an_element(), S2.an_element()])
(42, 47, 42)

class Element
   Bases: sage.structure.element_wrapper.ElementWrapperCheckWrappedClass

cartesian_factors()
   Return the tuple of elements that compose this element.

   EXAMPLES:

   sage: A = cartesian_product([ZZ, RR])
   sage: A((1, 1.23)).cartesian_factors()
   (1, 1.23000000000000)
   sage: type(_)
   <... 'tuple'>

cartesian_projection(i)
   Return the projection of self on the \(i\)-th factor of the Cartesian product, as per Sets.CartesianProducts.ElementMethods.cartesian_projection().

   INPUT:
   - \(i\) – the index of a factor of the Cartesian product

   EXAMPLES:

   sage: C = Sets().CartesianProducts().example(); C
   The Cartesian product of (Set of prime numbers (basic implementation), An example of an infinite enumerated set: the non negative integers, An example of a finite enumerated set: \(\{1,2,3\}\))
   sage: x = C.an_element(); x
   (47, 42, 1)
   sage: x.cartesian_projection(1)
   42

wrapped_class
   alias of builtins.tuple

an_element()
   EXAMPLES:

   sage: C = Sets().CartesianProducts().example(); C
   The Cartesian product of (Set of prime numbers (basic implementation), An example of an infinite enumerated set: the non negative integers, An example of a finite enumerated set: \(\{1,2,3\}\))
   sage: C.an_element()
   (47, 42, 1)

cartesian_factors()
   Return the Cartesian factors of self.

   See also:
   Sets.CartesianProducts.ParentMethods.cartesian_factors().

   EXAMPLES:
cartesian_projection\( (i) \)

Return the natural projection onto the \( i \)-th Cartesian factor of \( \text{self} \) as per \text{Sets}. CartesianProducts.ParentMethods.cartesian_projection().

INPUT:

- \( i \) – the index of a Cartesian factor of \( \text{self} \)

EXAMPLES:

```python
sage: C = Sets().CartesianProducts().example(); C
The Cartesian product of (Set of prime numbers (basic implementation), An example of an infinite enumerated set: the non negative integers, An example of a finite enumerated set: \{1,2,3\})
sage: x = C.an_element(); x
(47, 42, 1)
sage: pi = C.cartesian_projection(1)
sage: pi(x)
42
```

construction()

Return the construction functor and its arguments for this Cartesian product.

OUTPUT:

A pair whose first entry is a Cartesian product functor and its second entry is a list of the Cartesian factors.

EXAMPLES:

```python
sage: cartesian_product([ZZ, QQ]).construction()
(The cartesian_product functorial construction, (Integer Ring, Rational Field))
```

## 1.2 Families

A Family is an associative container which models a family \( (f_i)_{i \in I} \). Then, \( f[i] \) returns the element of the family indexed by \( i \). Whenever available, set and combinatorial class operations (counting, iteration, listing) on the family are induced from those of the index set. Families should be created through the \text{Family()} \ function.

AUTHORS:

- Nicolas Thiery (2008-02): initial release

class sage.sets.family.AbstractFamily

Bases: sage.structure.parent.Parent

The abstract class for family

Any family belongs to a class which inherits from \text{AbstractFamily}.  

1.2. Families
hidden_keys()
Returns the hidden keys of the family, if any.

EXAMPLES:

```python
sage: f = Family({3: 'a', 4: 'b', 7: 'd'})
sage: f.hidden_keys()
[]
```

inverse_family()
Returns the inverse family, with keys and values exchanged. This presumes that there are no duplicate values in self.

This default implementation is not lazy and therefore will only work with not too big finite families. It is also cached for the same reason:

```python
sage: Family({3: 'a', 4: 'b', 7: 'd'}).inverse_family()
Finite family {'a': 3, 'b': 4, 'd': 7}
sage: Family({3,4,7}).inverse_family()
Finite family {3: 0, 4: 1, 7: 2}
```

map(f, name=None)
Returns the family \((f(self[i]))_{i \in I}\), where \(I\) is the index set of self.

Todo: good name?

EXAMPLES:

```python
sage: f = Family({3: 'a', 4: 'b', 7: 'd'})
sage: g = f.map(lambda x: x+'1')
sage: list(g)
['a1', 'b1', 'd1']
```

zip(f, other, name=None)
Given two families with same index set \(I\) (and same hidden keys if relevant), returns the family \((f(self[i]), other[i])_{i \in I}\)

Todo: generalize to any number of families and merge with map?

EXAMPLES:

```python
sage: f = Family({3: 'a', 4: 'b', 7: 'd'})
sage: g = Family({3: '1', 4: '2', 7: '3'})
sage: h = f.zip(lambda x,y: x+y, g)
sage: list(h)
['a1', 'b2', 'd3']
```

class sage.sets.family.EnumeratedFamily(enumset)
Bases: sage.sets.family.LazyFamily

EnumeratedFamily turns an enumerated set \(c\) into a family indexed by the set \(\{0, \ldots, |c| - 1\}\).

Instances should be created via the Family() factory. See its documentation for examples and tests.

cardinality()
Return the number of elements in self.
EXAMPLES:

```python
sage: from sage.sets.family import EnumeratedFamily
sage: f = EnumeratedFamily(Permutations(3))
sage: f.cardinality()
6
sage: f = Family(NonNegativeIntegers())
sage: f.cardinality()
+Infinity
```

`sage.sets.family.Family(indices, function=None, hidden_keys=[], hidden_function=None, lazy=False, name=None)`

A Family is an associative container which models a family $(f_i)_{i \in I}$. Then, $f[i]$ returns the element of the family indexed by $i$. Whenever available, set and combinatorial class operations (counting, iteration, listing) on the family are induced from those of the index set.

There are several available implementations (classes) for different usages; Family serves as a factory, and will create instances of the appropriate classes depending on its arguments.

INPUT:

- indices – the indices for the family
- function – (optional) the function $f$ applied to all visible indices; the default is the identity function
- hidden_keys – (optional) a list of hidden indices that can be accessed through `my_family[i]`
- hidden_function – (optional) a function for the hidden indices
- lazy – boolean (default: False); whether the family is lazily created or not; if the indices are infinite, then this is automatically made True
- name – (optional) the name of the function; only used when the family is lazily created via a function

EXAMPLES:

In its simplest form, a list $l = [l_0, l_1, \ldots, l_\ell]$ or a tuple by itself is considered as the family $(l_i)_{i \in I}$ where $I$ is the set $\{0, \ldots, \ell\}$ where $\ell$ is `len(l) - 1`. So `Family(l)` returns the corresponding family:

```python
sage: f = Family([1,2,3])
sage: f
Family (1, 2, 3)
sage: f = Family((1,2,3))
sage: f
Family (1, 2, 3)
```

Instead of a list you can as well pass any iterable object:

```python
sage: f = Family(2*i+1 for i in [1,2,3])
sage: f
Family (3, 5, 7)
```

A family can also be constructed from a dictionary $t$. The resulting family is very close to $t$, except that the elements of the family are the values of $t$. Here, we define the family $(f_i)_{i \in \{3,4,7\}}$ with $f_3 = a$, $f_4 = b$, and $f_7 = d$:

```python
sage: f = Family({3: 'a', 4: 'b', 7: 'd'})
sage: f
Finite family {3: 'a', 4: 'b', 7: 'd'}
sage: f[7]
```

(continues on next page)
A family can also be constructed by its index set $\mathcal{I}$ and a function $f$, as in $(f(i))_{i \in \mathcal{I}}$:

```
sage: f = Family([3,4,7], lambda i: 2*i)
sage: f
Finite family {3: 6, 4: 8, 7: 14}
sage: f[7]
14
sage: list(f)
[6, 8, 14]
sage: [x for x in f]
[6, 8, 14]
sage: len(f)
3
```

By default, all images are computed right away, and stored in an internal dictionary:

```
sage: f = Family((3,4,7), lambda i: 2*i)
sage: f
Finite family {3: 6, 4: 8, 7: 14}
```

Note that this requires all the elements of the list to be hashable. One can ask instead for the images $f(i)$ to be computed lazily, when needed:

```
sage: f = Family([3,4,7], lambda i: 2*i, lazy=True)
sage: f
Lazy family (<lambda>(i))_{i in [3, 4, 7]}
sage: f[7]
14
sage: list(f)
[6, 8, 14]
sage: [x for x in f]
[6, 8, 14]
```

This allows in particular for modeling infinite families:

```
sage: f = Family(ZZ, lambda i: 2*i, lazy=True)
sage: f
Lazy family (<lambda>(i))_{i in Integer Ring}
sage: f.keys()
Integer Ring
```
sage: f[1]
2
sage: f[-5]
-10
sage: i = iter(f)
sage: next(i), next(i), next(i), next(i), next(i)
(0, 2, -2, 4, -4)

Note that the lazy keyword parameter is only needed to force laziness. Usually it is automatically set to a correct default value (ie: False for finite data structures and True for enumerated sets:

sage: f == Family(ZZ, lambda i: 2*i)
True

Beware that for those kind of families len(f) is not supposed to work. As a replacement, use the .cardinality() method:

sage: f = Family(Permutations(3), attrcall("to_lehmer_code")); f
Lazy family (i.to_lehmer_code())_{i in Standard permutations of 3}
sage: f == loads(dumps(f))
True

Caveat: Only certain families with lazy behavior can be pickled. In particular, only functions that work with Sage's pickle_function and unpickle_function (in sage.misc.fpickle) will correctly unpickle. The following two work:

sage: f = Family(Permutations(3), lambda p: p.to_lehmer_code()); f
Lazy family (<lambda>(i))_{i in Standard permutations of 3}
sage: f == loads(dumps(f))
True
sage: f = Family(Permutations(3), attrcall("to_lehmer_code")); f
Lazy family (i.to_lehmer_code())_{i in Standard permutations of 3}
sage: f == loads(dumps(f))
True

But this one does not:

sage: def plus_n(n):
...     return lambda x: x+n
sage: f = Family([1,2,3], plus_n(3), lazy=True); f
Lazy family (<lambda>(i))_{i in [1, 2, 3]}
sage: f == loads(dumps(f))
Traceback (most recent call last):
...   ValueError: Cannot pickle code objects from closures

Finally, it can occasionally be useful to add some hidden elements in a family, which are accessible as f[i], but do not appear in the keys or the container operations:

sage: f = Family([3,4,7], lambda i: 2*i, hidden_keys=[2])
sage: f
Finite family {3: 6, 4: 8, 7: 14}
sage: f.keys()
[3, 4, 7]
sage: f.hidden_keys()
[2]
The following example illustrates when the function is actually called:

```
 sage: def compute_value(i):
 ...:     print('computing 2*'+str(i))
 ...:     return 2*i
 sage: f = Family([3,4,7], compute_value, hidden_keys=[2])
```
```
computing 2*3
computing 2*4
computing 2*7
```
```
sage: f
Finite family {3: 6, 4: 8, 7: 14}
sage: f.keys()
[3, 4, 7]
sage: f.hidden_keys()
[2]
sage: f[7]
14
sage: f[2]
4
 computing 2*2
sage: f[2]
4
sage: list(f)
[6, 8, 14]
sage: [x for x in f]
[6, 8, 14]
sage: len(f)
3
```
Here is a close variant where the function for the hidden keys is different from that for the other keys:

```
sage: f = Family([3,4,7], lambda i: 2*i, hidden_keys=[2], hidden_function = lambda i: 3*i)
```
```
sage: f
Finite family {3: 6, 4: 8, 7: 14}
sage: f.keys()
[3, 4, 7]
sage: f.hidden_keys()
[2]
sage: f[7]
14
sage: f[2]
6
sage: list(f)
[6, 8, 14]
```
Family accept finite and infinite EnumeratedSets as input:

```python
sage: f = Family(FiniteEnumeratedSet([1,2,3]))
sage: f
Family (1, 2, 3)
sage: f = Family(NonNegativeIntegers())
sage: f
Family (Non negative integers)
sage: f = Family(FiniteEnumeratedSet([3,4,7]), lambda i: 2*i)
sage: f
Finite family {3: 6, 4: 8, 7: 14}
sage: f.keys()
{3, 4, 7}
sage: f[7]
14
sage: list(f)
[6, 8, 14]
sage: [x for x in f]
[6, 8, 14]
sage: len(f)
3
```

```python
class sage.sets.family.FiniteFamily(dictionary, keys=None)
```

A `FiniteFamily` is an associative container which models a finite family \((f_i)_{i \in I}\). Its elements \(f_i\) are therefore its values. Instances should be created via the `Family()` factory. See its documentation for examples and tests.

**EXAMPLES:**

We define the family \((f_i)_{i \in \{3,4,7\}}\) with \(f_3 = a, f_4 = b,\) and \(f_7 = d\):

```python
sage: from sage.sets.family import FiniteFamily
sage: f = FiniteFamily({3: 'a', 4: 'b', 7: 'd'})
```

Individual elements are accessible as in a usual dictionary:

```python
sage: f[7]
'd'
```

And the other usual dictionary operations are also available:

```python
sage: len(f)
3
sage: f.keys()
[3, 4, 7]
```

However \(f\) behaves as a container for the \(f_i\)'s:
The order of the elements can be specified using the \texttt{keys} optional argument:

```
sage: f = FiniteFamily({"a": "aa", "b": "bb", "c": "cc"}, keys = ["c", "a", "b" \rightarrow "a"])
sage: list(f)
['cc', 'aa', 'bb']
```

\textbf{cardinality()}

Returns the number of elements in self.

\textbf{EXAMPLES:}

```
sage: from sage.sets.family import FiniteFamily
sage: f = FiniteFamily({3: 'a', 4: 'b', 7: 'd'})
sage: f.cardinality()
3
```

\textbf{has_key\,(k)}

Returns whether \(k\) is a key of \texttt{self}

\textbf{EXAMPLES:}

```
sage: Family({"a":1, "b":2, "c":3}).has_key("a")
True
sage: Family({"a":1, "b":2, "c":3}).has_key("d")
False
```

\textbf{keys()}

Returns the index set of this family

\textbf{EXAMPLES:}

```
sage: f = Family(["c", "a", "b"], lambda x: x+x)
sage: f.keys()
['c', 'a', 'b']
```

\textbf{values()}

Returns the elements of this family

\textbf{EXAMPLES:}

```
sage: f = Family(["c", "a", "b"], lambda x: x+x)
sage: f.values()
['cc', 'aa', 'bb']
```

\textbf{class \texttt{sage.sets.family.FiniteFamilyWithHiddenKeys}(dictionary, hidden_keys, hidden_function, keys=None)}

A close variant of \texttt{FiniteFamily} where the family contains some hidden keys whose corresponding values are computed lazily (and remembered). Instances should be created via the \texttt{Family()} factory. See its documentation for examples and tests.

Caveat: Only instances of this class whose functions are compatible with \texttt{sage.misc.fpickle} can be pickled.
hidden_keys()
Returns self’s hidden keys.

EXAMPLES:

```python
sage: f = Family([3,4,7], lambda i: 2*i, hidden_keys=[2])
sage: f.hidden_keys()
[2]
```

class sage.sets.family.LazyFamily(set, function, name=None)

Bases: sage.sets.family.AbstractFamily

A LazyFamily(I, f) is an associative container which models the (possibly infinite) family \((f(i))_{i \in I}\).

Instances should be created via the `Family()` factory. See its documentation for examples and tests.

cardinality()
Return the number of elements in self.

EXAMPLES:

```python
sage: from sage.sets.family import LazyFamily
sage: f = LazyFamily([3,4,7], lambda i: 2*i)
sage: f.cardinality()
3
sage: l = LazyFamily(NonNegativeIntegers(), lambda i: 2*i)
sage: l.cardinality()
+Infinity
```

dkeys()
Returns self’s keys.

EXAMPLES:

```python
sage: from sage.sets.family import LazyFamily
sage: f = LazyFamily([3,4,7], lambda i: 2*i)
sage: f.keys()
[3, 4, 7]
```

class sage.sets.family.TrivialFamily(enumeration)

Bases: sage.sets.family.AbstractFamily

TrivialFamily turns a list/tuple \(c\) into a family indexed by the set \(\{0, \ldots, |c| - 1\}\).

Instances should be created via the `Family()` factory. See its documentation for examples and tests.

cardinality()
Return the number of elements in self.

EXAMPLES:

```python
sage: from sage.sets.family import TrivialFamily
sage: f = TrivialFamily([3,4,7])
sage: f.cardinality()
3
```

dkeys()
Returns self’s keys.

EXAMPLES:
1.3 Sets

AUTHORS:

• William Stein (2005) - first version
• William Stein (2006-02-16) - large number of documentation and examples; improved code
• Mike Hansen (2007-3-25) - added differences and symmetric differences; fixed operators
• Florent Hivert (2010-06-17) - Adapted to categories
• Nicolas M. Thiery (2011-03-15) - Added subset and superset methods
• Julian Rueth (2013-04-09) - Collected common code in Set_object_binary, fixed __hash__.

sage.sets.set.Set (X=None)
Create the underlying set of X.

If X is a list, tuple, Python set, or X.is_finite() is True, this returns a wrapper around Python’s enumerated immutable frozenset type with extra functionality. Otherwise it returns a more formal wrapper.

If you need the functionality of mutable sets, use Python’s builtin set type.

EXAMPLES:

sage: X = Set(GF(9,'a'))
sage: X
{0, 1, 2, a, a + 1, a + 2, 2*a, 2*a + 1, 2*a + 2}
sage: type(X)
<class 'sage.sets.set.Set_object_enumerated_with_category'>
sage: Y = X.union(Set(QQ))
sage: Y
Set-theoretic union of {0, 1, 2, a, a + 1, a + 2, 2*a, 2*a + 1, 2*a + 2} and Set of elements of Rational Field
sage: type(Y)
<class 'sage.sets.set.Set_object_union_with_category'>

Usually sets can be used as dictionary keys.

sage: d=Set({(2*I,1+I)}:10)
sage: d
{(2*I, 1+I): 10}  # key is randomly ordered
sage: d[Set((1+I,2*I))]  
10
sage: d[Set((1+I,2*I))]  
10

The original object is often forgotten.

sage: v = [1,2,3]
sage: X = Set(v)
sage: X
(continues on next page)
Set also accepts iterators, but be careful to only give *finite* sets:

```
sage: sorted(Set(range(1,6)))
[1, 2, 3, 4, 5]
sage: sorted(Set(list(range(1,6))))
[1, 2, 3, 4, 5]
sage: sorted(Set(iter(range(1,6))))
[1, 2, 3, 4, 5]
```

We can also create sets from different types:

```
sage: sorted(Set([Sequence([3,1], immutable=True), 5, QQ, Partition([3,1,1])]), key=str)
[5, Rational Field, [3, 1, 1], [3, 1]]
```

Sets with unhashable objects work, but with less functionality:

```
sage: A = Set([QQ, (3,1), 5])  # hashable
sage: sorted(A.list(), key=repr)
[(3, 1), 5, Rational Field]
sage: type(A)
<class 'sage.sets.set.Set_object_enumerated_with_category'>
sage: B = Set([QQ, [3,1], 5])  # unhashable
sage: sorted(B.list(), key=repr)
Traceback (most recent call last):
  ... AttributeError: 'Set_object_with_category' object has no attribute 'list'
sage: type(B)
<class 'sage.sets.set.Set_object_with_category'>
```

```
class sage.sets.set.Set_object (X, category=None)

Bases: sage.structure.parent.Set_generic

A set attached to an almost arbitrary object.

EXAMPLES:
```
```
sage: K = GF(19)
sage: Set(K)
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18}
sage: S = Set(K)
sage: latex(S)
\left\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\right\}
sage: TestSuite(S).run()
sage: latex(Set(ZZ))
\Bold{Z}
```
```
an_element ()

Return the first element of self returned by __iter__ ()
```
If `self` is empty, the exception `EmptySetError` is raised instead.

This provides a generic implementation of the method `_an_element_()` for all parents in `EnumeratedSets`.

**EXAMPLES:**

```python
sage: C = FiniteEnumeratedSets().example(); C
An example of a finite enumerated set: {1,2,3}
sage: C.an_element() # indirect doctest
1
sage: S = Set([])
sage: S.an_element()
Traceback (most recent call last):
  ...
EmptySetError
```

cardinality()

Return the cardinality of this set, which is either an integer or `Infinity`.

**EXAMPLES:**

```python
sage: Set(ZZ).cardinality()
+Infinity
sage: Primes().cardinality()
+Infinity
sage: Set(GF(5)).cardinality()
5
sage: Set(GF(5^2,'a')).cardinality()
25
```

difference(X)

Return the set difference `self - X`.

**EXAMPLES:**

```python
sage: X = Set(ZZ).difference(Primes())
sage: 4 in X
True
sage: 3 in X
False
sage: 4/1 in X
True
sage: X = Set(GF(9,'b')).difference(Set(GF(27,'c')))
sage: X
{0, 1, 2, b, b + 1, b + 2, 2*b, 2*b + 1, 2*b + 2}
```

intersection(X)

Return the intersection of `self` and `X`.

**EXAMPLES:**

```python
sage: X = Set(GF(9,'b')).difference(Set(GF(27,'b')))
sage: X
{0, 1, 2, b, b + 1, b + 2, 2*b, 2*b + 1, 2*b + 2}
```
is_empty()  
Return boolean representing emptiness of the set.

OUTPUT:
True if the set is empty. False if otherwise.

EXAMPLES:

```python
sage: Set([]).is_empty()
True
sage: Set([0]).is_empty()
False
sage: Set([1..100]).is_empty()
False
sage: Set(SymmetricGroup(2).list()).is_empty()
False
sage: Set(ZZ).is_empty()
False
```
sage: X = Primes()
sage: X.object()
Set of all prime numbers: 2, 3, 5, 7, ...

subsets\((size=None)\)

Return the Subsets object representing the subsets of a set. If size is specified, return the subsets of that size.

EXAMPLES:

sage: X = Set([1,2,3])
sage: list(X.subsets())

\[
\{
\},
\{1\},
\{2\},
\{3\},
\{1, 2\},
\{1, 3\},
\{2, 3\},
\{1, 2, 3\}\
\]

sage: list(X.subsets(2))

\[
\{(1, 2),
\{1, 3\},
\{2, 3\}\
\]

subsets_lattice()

Return the lattice of subsets ordered by containment.

EXAMPLES:

sage: X = Set([1,2,3])
sage: X.subsets_lattice()

Finite lattice containing 8 elements

sage: Y = Set()
sage: Y.subsets_lattice()

Finite lattice containing 1 elements

symmetric_difference\((X)\)

Returns the symmetric difference of self and X.

EXAMPLES:

sage: X = Set([1,2,3]).symmetric_difference(Set([3,4]))
sage: X

\{1, 2, 4\}

union\((X)\)

Return the union of self and X.

EXAMPLES:

sage: Set(QQ).union(Set(ZZ))

Set-theoretic union of Set of elements of Rational Field and Set of elements of Integer Ring

sage: Set(QQ) + Set(ZZ)

Set-theoretic union of Set of elements of Rational Field and Set of elements of Integer Ring

sage: X = Set(QQ).union(Set(GF(3))); X

Set-theoretic union of Set of elements of Rational Field and \{0, 1, 2\}

sage: 2/3 in X

True

sage: GF(3)(2) in X

True

sage: GF(5)(2) in X

False

sage: sorted(Set(GF(7)) + Set(GF(3)), key=int)

\[0, 0, 1, 1, 2, 2, 3, 4, 5, 6\]
class sage.sets.set.Set_object_binary (X, Y, op, latex_op)

Bases: sage.sets.set.Set_object

An abstract common base class for sets defined by a binary operation (ex. Set_object_union, Set_object_intersection, Set_object_difference, and Set_object_symmetric_difference).

INPUT:

• X, Y – sets, the operands to op
• op – a string describing the binary operation
• latex_op – a string used for rendering this object in LaTeX

EXAMPLES:

```
sage: X = Set(QQ^2)
sage: Y = Set(ZZ)
sage: from sage.sets.set import Set_object_binary
sage: S = Set_object_binary(X, Y, "union", "\cup"); S
Set-theoretic union of Set of elements of Vector space of dimension 2 over Rational Field and Set of elements of Integer Ring
```

class sage.sets.set.Set_object_difference (X, Y)

Bases: sage.sets.set.Set_object_binary

Formal difference of two sets.

is_finite ()

Return whether this set is finite.

EXAMPLES:

```
sage: X = Set(range(10))
sage: Y = Set(range(-10,5))
sage: Z = Set(QQ)
sage: X.difference(Y).is_finite()
True
sage: X.difference(Z).is_finite()
True
sage: Z.difference(X).is_finite()
False
sage: Z.difference(Set(ZZ)).is_finite()
Traceback (most recent call last):
  ... NotImpliedByError
```

class sage.sets.set.Set_object_enumerated (X)

Bases: sage.sets.set.Set_object

A finite enumerated set.

cardinality ()

Return the cardinality of self.

EXAMPLES:

```
sage: Set([1,1]).cardinality()
1
```
**difference (other)**

Return the set difference self - other.

**EXAMPLES:**

```
sage: X = Set([1,2,3,4])
sage: Y = Set([1,2])
sage: X.difference(Y)
{3, 4}
sage: Z = Set(ZZ)
sage: W = Set([2.5, 4, 5, 6])
sage: W.difference(Z)
{2.50000000000000}
```

**frozenset ()**

Return the Python frozenset object associated to this set, which is an immutable set (hence hashable).

**EXAMPLES:**

```
sage: X = Set(GF(8,'c'))
sage: X
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
sage: s = X.set(); s
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
sage: hash(s)
Traceback (most recent call last):
...
TypeError: unhashable type: 'set'
sage: s = X.frozenset(); s
frozenset({0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1})
sage: hash(s) != hash(tuple(X.set()))
True
sage: type(s)
<... 'frozenset'>
```

**intersection (other)**

Return the intersection of self and other.

**EXAMPLES:**

```
sage: X = Set(GF(8,'c'))
sage: Y = Set([GF(8,'c').0, 1, 2, 3])
sage: X.intersection(Y)
{1, c}
```

**is_finite ()**

Return True as this is a finite set.

**EXAMPLES:**

```
sage: Set(GF(19)).is_finite()
True
```

**issubset (other)**

Return whether self is a subset of other.

**INPUT:**
• other – a finite Set

EXAMPLES:

```sage
sage: X = Set([1,3,5])
sage: Y = Set([0,1,2,3,5,7])
sage: X.issubset(Y)
True
sage: Y.issubset(X)
False
sage: X.issubset(X)
True
```

issuperset (other)

Return whether self is a superset of other.

INPUT:

• other – a finite Set

EXAMPLES:

```sage
sage: X = Set([1,3,5])
sage: Y = Set([0,1,2,3,5])
sage: X.issuperset(Y)
False
sage: Y.issuperset(X)
True
sage: X.issuperset(X)
True
```

list ()

Return the elements of self, as a list.

EXAMPLES:

```sage
sage: X = Set(GF(8,'c'))
sage: X
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
sage: X.list()
[0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1]
sage: type(X.list())
<... 'list'>
```

Todo: FIXME: What should be the order of the result? That of self.object()? Or the order given by set(self.object())? Note that __getitem__() is currently implemented in term of this list method, which is really inefficient ...

random_element ()

Return a random element in this set.

EXAMPLES:

```sage
sage: Set([1,2,3]).random_element() # random
2
```

set ()

Return the Python set object associated to this set.
Python has a notion of finite set, and often Sage sets have an associated Python set. This function returns that set.

EXAMPLES:

```python
sage: X = Set(GF(8, 'c'))
sage: X
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
sage: X.set()
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
sage: type(X.set())
<... 'set'>
sage: type(X)
<class 'sage.sets.set.Set_object_enumerated_with_category'>
```

**symmetric_difference**(other)

Return the symmetric difference of self and other.

EXAMPLES:

```python
sage: X = Set([1,2,3,4])
sage: Y = Set([1,2])
sage: X.symmetric_difference(Y)
{3, 4}
sage: Z = Set(ZZ)
sage: W = Set([2.5, 4, 5, 6])
sage: U = W.symmetric_difference(Z)
sage: 2.5 in U
True
sage: 4 in U
False
sage: V = 2.5.symmetric_difference(W)
sage: V == U
True
sage: 2.5 in V
True
sage: 6 in V
False
```

**union**(other)

Return the union of self and other.

EXAMPLES:

```python
sage: X = Set(GF(8, 'c'))
sage: Y = Set([GF(8, 'c').0, 1, 2, 3])
sage: X
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
sage: sorted(Y)
[1, 2, 3, c]
sage: sorted(X.union(Y), key=str)
[0, 1, 2, 3, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1]
```

class sage.sets.set.Set_object_intersection(X, Y)

Formal intersection of two sets.

**is_finite**()

Return whether this set is finite.
EXAMPLES:

```python
sage: X = Set(IntegerRange(100))
sage: Y = Set(ZZ)
sage: X.intersection(Y).is_finite()
True
sage: Y.intersection(X).is_finite()
True
sage: Y.intersection(Set(QQ)).is_finite()
Traceback (most recent call last):
 ... 
NotImplementedError
```

```python
class sage.sets.set.Set_object_symmetric_difference(X, Y)
Bases: sage.sets.set.Set_object_binary

Formal symmetric difference of two sets.

is_finite()
Return whether this set is finite.

EXAMPLES:

```python
sage: X = Set(range(10))
sage: Y = Set(range(-10,5))
sage: Z = Set(QQ)
sage: X.symmetric_difference(Y).is_finite()
True
sage: X.symmetric_difference(Z).is_finite()
False
sage: Z.symmetric_difference(X).is_finite()
False
sage: Z.symmetric_difference(Set(ZZ)).is_finite()
Traceback (most recent call last):
 ... 
NotImplementedError
```

```python
class sage.sets.set.Set_object_union(X, Y)
Bases: sage.sets.set.Set_object_binary

A formal union of two sets.

cardinality()
Return the cardinality of this set.

EXAMPLES:

```python
sage: X = Set(GF(3)).union(Set(GF(2)))
sage: X
{0, 1, 2, 0, 1}
sage: X.cardinality()
5
sage: X = Set(GF(3)).union(Set(ZZ))
sage: X.cardinality()
+Infinity
```

is_finite()
Return whether this set is finite.

EXAMPLES:
sage: X = Set(range(10))
sage: Y = Set(range(-10,0))
sage: Z = Set(Primes())
sage: X.union(Y).is_finite()
True
sage: X.union(Z).is_finite()
False

sage.sets.set.has_finite_length(obj)
Return True if obj is known to have finite length.

This is mainly meant for pure Python types, so we do not call any Sage-specific methods.

EXAMPLES:

sage: from sage.sets.set import has_finite_length
sage: has_finite_length(tuple(range(10)))
True
sage: has_finite_length(list(range(10)))
True
sage: has_finite_length(set(range(10)))
True
sage: has_finite_length(iter(range(10)))
False
sage: has_finite_length(GF(17^127))
True
sage: has_finite_length(ZZ)
False

1.4 Disjoint-set data structure

The main entry point is DisjointSet() which chooses the appropriate type to return. For more on the data structure, see DisjointSet().

This module defines a class for mutable partitioning of a set, which cannot be used as a key of a dictionary, vertex of a graph etc. For immutable partitioning see SetPartition.

AUTHORS:

• Sébastien Labbé (2008) - Initial version.
• Sébastien Labbé (2009-11-24) - Pickling support
• Sébastien Labbé (2010-01) - Inclusion into sage (trac ticket #6775).

EXAMPLES:

Disjoint set of integers from 0 to n - 1:

sage: s = DisjointSet(6)
sage: s
{(0), {1}, {2}, {3}, {4}, {5}}
sage: s.union(2, 4)
sage: s.union(1, 3)
sage: s.union(5, 1)
sage: s
{(0), {1, 3, 5}, {2, 4}}
sage: s.find(3)
Disjoint set of hashables objects:

```python
sage: d = DisjointSet('abcde')
sage: d
{{'a'}, {'b'}, {'c'}, {'d'}, {'e'}}
sage: d.union('a','b')
sage: d.union('b','c')
sage: d.union('c','d')
sage: d
{{'a', 'b', 'c', 'd'}, {'e'}}
sage: d.find('c')
'a'
```

`sage.sets.disjoint_set.DisjointSet`(*arg*)

Constructs a disjoint set where each element of `arg` is in its own set. If `arg` is an integer, then the disjoint set returned is made of the integers from 0 to `arg - 1`.

A disjoint-set data structure (sometimes called union-find data structure) is a data structure that keeps track of a partitioning of a set into a number of separate, nonoverlapping sets. It performs two operations:

- `find()` – Determine which set a particular element is in.
- `union()` – Combine or merge two sets into a single set.

**REFERENCES:**
- Wikipedia article Disjoint-set_data_structure

**INPUT:**
- `arg` – non negative integer or an iterable of hashable objects.

**EXAMPLES:**

From a non-negative integer:

```python
sage: DisjointSet(5)
{{0}, {1}, {2}, {3}, {4}}
```

From an iterable:

```python
sage: DisjointSet('abcde')
{{'a'}, {'b'}, {'c'}, {'d'}, {'e'}}
sage: DisjointSet(range(6))
{{0}, {1}, {2}, {3}, {4}, {5}}
sage: DisjointSet(['yi',45,'cheval'])
{{'cheval'}, {'yi'}, {45}}
```

class `sage.sets.disjoint_set.DisjointSet_class`

Bases: `sage.structure.sage_object.SageObject`


**cardinality()**

Return the number of elements in `self`, not the number of subsets.
EXAMPLES:

```python
sage: d = DisjointSet(5)
sage: d.cardinality()
5
sage: d.union(2, 4)
sage: d.cardinality()
5
sage: d = DisjointSet(range(5))
```

```python
sage: d.cardinality()
5
sage: d.union(2, 4)
```

```python
sage: d.cardinality()
5
```

```
number_of_subsets()
```

Return the number of subsets in self.

EXAMPLES:

```python
sage: d = DisjointSet(5)
sage: d.number_of_subsets()
5
sage: d.union(2, 4)
sage: d.number_of_subsets()
4
sage: d = DisjointSet(range(5))
sage: d.number_of_subsets()
5
sage: d.union(2, 4)
sage: d.number_of_subsets()
4
```

```
class sage.sets.disjoint_set.DisjointSet_of_hashables
```

Bases: `sage.sets.disjoint_set.DisjointSet_class`

Disjoint set of hashables.

EXAMPLES:

```python
sage: d = DisjointSet('abcde')
sage: d
{{'a'}, {'b'}, {'c'}, {'d'}, {'e'}}
sage: d.union('a', 'c')
sage: d
{{'a', 'c'}, {'b'}, {'d'}, {'e'}}
sage: d.find('a')
'a'
```

```
element_to_root_dict()
```

Return the dictionary where the keys are the elements of self and the values are their representative inside a list.

EXAMPLES:

```python
sage: d = DisjointSet(range(5))
sage: d.union(2,3)
sage: d.union(4,1)
sage: e = d.element_to_root_dict()
(continues on next page)
sage: sorted(e.items())
[(0, 0), (1, 4), (2, 2), (3, 2), (4, 4)]
sage: WordMorphism(e)
WordMorphism: 0->0, 1->4, 2->2, 3->2, 4->4

\textbf{find} (e)

Return the representative of the set that \( e \) currently belongs to.

\textbf{INPUT:}

\begin{itemize}
  \item \( e \) – element in \( self \)
\end{itemize}

\textbf{EXAMPLES:}

sage: e = DisjointSet(range(5))
sage: e.union(4,2)
sage: e
{{0}, {1}, {2, 4}, {3}}
sage: e.find(2)
4
sage: e.find(4)
4
sage: e.union(1,3)
sage: e
{{0}, {1, 3}, {2, 4}}
sage: e.find(1)
1
sage: e.find(3)
1
sage: e.union(3,2)
sage: e
{{0}, {1, 2, 3, 4}}
sage: [e.find(i) for i in range(5)]
[0, 1, 1, 1, 1]
sage: e.find(5)
Traceback (most recent call last):
  ...KeyError: 5

\textbf{root_to_elements_dict} ()

Return the dictionary where the keys are the roots of \( self \) and the values are the elements in the same set.

\textbf{EXAMPLES:}

sage: d = DisjointSet(range(5))
sage: d.union(2,3)
sage: e = d.root_to_elements_dict()
sage: sorted(e.items())
[(0, [0]), (2, [2, 3]), (4, [1, 4])]

\textbf{to_digraph} ()

Return the current digraph of \( self \) where \((a, b)\) is an oriented edge if \( b \) is the parent of \( a \).

\textbf{EXAMPLES:}

sage: d = DisjointSet(range(5))
sage: d.union(2,3)

(continues on next page)
The result depends on the ordering of the union:

```
sage: d = DisjointSet(range(5))
sage: d.union(1,2)
sage: d.union(1,3)
sage: d.union(1,4)
sage: d
{{0}, {1, 2, 3, 4}}
sage: d.to_digraph().edges()
[(0, 0, None), (1, 1, None), (2, 1, None), (3, 1, None), (4, 1, None)]
```

The `union` method combines the set of `e` and the set of `f` into one:

Combine the set of `e` and the set of `f` into one. All elements in those two sets will share the same representative that can be gotten using `find`.

**INPUT:**
- `e` - element in `self`
- `f` - element in `self`

**EXAMPLES:**

```
sage: e = DisjointSet('abcde')
sage: e
{{'a'}, {'b'}, {'c'}, {'d'}, {'e'}}
sage: e.union('a','b')
sage: e
{{'a', 'b'}, {'c'}, {'d'}, {'e'}}
sage: e.union('c','e')
sage: e
{{'a', 'b', 'c', 'e'}, {'d'}}
sage: e.union('b','e')
sage: e
{{'a', 'b', 'c', 'e'}, {'d'}}
```

### class sage.sets.disjoint_set.DisjointSet_of_integers

Bases: `sage.sets.disjoint_set.DisjointSet_class` Disjoint set of integers from 0 to `n-1`.

**EXAMPLES:**

```
sage: d = DisjointSet(5)
sage: d
{{0}, {1}, {2}, {3}, {4}}
sage: d.union(2,4)
sage: d.union(0,2)
sage: d
```

{{0, 2, 4}, {1}, {3}}
sage: d.find(2)
2

**element_to_root_dict()**
Return the dictionary where the keys are the elements of `self` and the values are their representative inside a list.

**EXAMPLES:**

```python
sage: d = DisjointSet(5)
sage: d.union(2,3)
sage: d.union(4,1)
sage: e = d.element_to_root_dict(); e
{0: 0, 1: 4, 2: 2, 3: 2, 4: 4}
sage: WordMorphism(e)
WordMorphism: 0->0, 1->4, 2->2, 3->2, 4->4
```

**find(i)**
Return the representative of the set that `i` currently belongs to.

**INPUT:**

- `i` – element in `self`

**EXAMPLES:**

```python
sage: e = DisjointSet(5)
sage: e.union(4,2)
sage: e
{{0}, {1}, {2, 4}, {3}}
sage: e.find(2)
4
sage: e.find(4)
4
sage: e.union(1,3)
sage: e
{{0}, {1, 3}, {2, 4}}
sage: e.find(1)
1
sage: e.find(3)
1
sage: e.union(3,2)
sage: e
{{0}, {1, 2, 3, 4}}
sage: [e.find(i) for i in range(5)]
[0, 1, 1, 1, 1]
sage: e.find(5)
Traceback (most recent call last):
...
ValueError: i(=5) must be between 0 and 4
```

**root_to_elements_dict()**
Return the dictionary where the keys are the roots of `self` and the values are the elements in the same set as the root.

**EXAMPLES:**
```python
sage: d = DisjointSet(5)
sage: sorted(d.root_to_elements_dict().items())
[(0, [0]), (1, [1]), (2, [2]), (3, [3]), (4, [4])]
sage: d.union(2,3)
sage: sorted(d.root_to_elements_dict().items())
[(0, [0]), (1, [1]), (2, [2, 3]), (4, [4])]
sage: d.union(3,0)
sage: sorted(d.root_to_elements_dict().items())
[(1, [1]), (2, [0, 2, 3]), (4, [4])]
sage: d
{{0, 2, 3}, {1}, {4}}
to_digraph()
Return the current digraph of self where \((a, b)\) is an oriented edge if \(b\) is the parent of \(a\).
EXAMPLES:
```
1.5 Disjoint union of enumerated sets

AUTHORS:

- Florent Hivert (2010-03): classcall related stuff.
- Florent Hivert (2010-12): fixed facade element construction.

```python
sage: d
{{0, 1}, {2, 4}, {3}}
sage: d.union(1,4)
sage: d
{{0, 1, 2, 4}, {3}}
sage: d.union(1,5)
Traceback (most recent call last):
...
ValueError: j(=5) must be between 0 and 4
```

class sage.sets.disjoint_union Enumerated Sets

```
INPUT:

- family – a list (or iterable or family) of enumerated sets
- keepkey – a boolean
- facade – a boolean

This models the enumerated set obtained by concatenating together the specified ordered sets. The latter are
supposed to be pairwise disjoint; otherwise, a multiset is created.

The argument family can be a list, a tuple, a dictionary, or a family. If it is not a family it is first converted
into a family (see sage.sets.family.Family()).

Experimental options:

By default, there is no way to tell from which set of the union an element is generated. The option
keepkey=True keeps track of those by returning pairs (key, el) where key is the index of the set to
which el belongs. When this option is specified, the enumerated sets need not be disjoint anymore.

With the option facade=False the elements are wrapped in an object whose parent is the disjoint union
itself. The wrapped object can then be recovered using the value attribute.

The two options can be combined.

The names of those options is imperfect, and subject to change in future versions. Feedback welcome.

1.5. Disjoint union of enumerated sets
EXAMPLES:

The input can be a list or a tuple of FiniteEnumeratedSets:

```
sage: U1 = DisjointUnionEnumeratedSets((
    ....:     FiniteEnumeratedSet([1,2,3]),
    ....:     FiniteEnumeratedSet([4,5,6])))
sage: U1
Disjoint union of Family ({1, 2, 3}, {4, 5, 6})
sage: U1.list()
[1, 2, 3, 4, 5, 6]
sage: U1.cardinality()
6
```

The input can also be a dictionary:

```
sage: U2 = DisjointUnionEnumeratedSets({1: FiniteEnumeratedSet([1,2,3]),
    ....:     2: FiniteEnumeratedSet([4,5,6])})
sage: U2
Disjoint union of Finite family {1: {1, 2, 3}, 2: {4, 5, 6}}
sage: U2.list()
[1, 2, 3, 4, 5, 6]
sage: U2.cardinality()
6
```

However in that case the enumeration order is not specified.

In general the input can be any family:

```
sage: U3 = DisjointUnionEnumeratedSets(
    ....:     Family([2,3,4], Permutations, lazy=True))
sage: U3
Disjoint union of Lazy family (<class 'sage.combinat.permutation.Permutations'>
    \[\mapsto (i)\] \{i in [2, 3, 4]\})
sage: U3.cardinality()
32
sage: it = iter(U3)
sage: [next(it), next(it), next(it), next(it), next(it), next(it)]
[[], [1], [1, 2], [2, 1], [1, 2, 3], [1, 3, 2]]
sage: U3.unrank(18)
[2, 4, 1, 3]
```

This allows for infinite unions:

```
sage: U4 = DisjointUnionEnumeratedSets(
    ....:     Family(NonNegativeIntegers(), Permutations))
sage: U4
Disjoint union of Lazy family (<class 'sage.combinat.permutation.Permutations'>
    \[\mapsto (i)\] \{i in Non negative integers\})
sage: U4.cardinality()
+Infinity
sage: it = iter(U4)
sage: [next(it), next(it), next(it), next(it), next(it), next(it)]
[[], [1], [1, 2], [2, 1], [1, 2, 3], [1, 3, 2]]
sage: U4.unrank(18)
[2, 3, 1, 4]
```
Warning: Beware that some of the operations assume in that case that infinitely many of the enumerated sets are non empty.

Experimental options

We demonstrate the keepkey option:

```python
sage: Ukeep = DisjointUnionEnumeratedSets(
.....: Family(list(range(4)), Permutations), keepkey=True)
sage: it = iter(Ukeep)
sage: [next(it) for i in range(6)]
[(0, []), (1, [1]), (2, [1, 2]), (2, [2, 1]), (3, [1, 2, 3]), (3, [1, 3, 2])]
sage: type(next(it)[1])
<class 'sage.combinat.permutation.StandardPermutations_n_with_category.element_class'>
```

We now demonstrate the facade option:

```python
sage: UNoFacade = DisjointUnionEnumeratedSets(
.....: Family(list(range(4)), Permutations), facade=False)
sage: it = iter(UNoFacade)
sage: [next(it) for i in range(6)]
[[], [1], [1, 2], [2, 1], [1, 2, 3], [1, 3, 2]]
sage: el = next(it); el
[2, 1, 3]
sage: type(el)
<... 'sage.structure.element_wrapper.ElementWrapper'>
sage: el.parent() == UNoFacade
True
sage: elv = el.value; elv
[2, 1, 3]
sage: type(elv)
<class 'sage.combinat.permutation.StandardPermutations_n_with_category.element_class'>
```

The elements `el` of the disjoint union are simple wrapped elements. So to access the methods, you need to do `el.value`:

```python
sage: el[0]  # py2
Traceback (most recent call last):
...
TypeError: 'sage.structure.element_wrapper.ElementWrapper' object has no attribute '__getitem__'
sage: el[0]  # py3
Traceback (most recent call last):
...
TypeError: 'sage.structure.element_wrapper.ElementWrapper' object is not subscriptable
sage: el.value[0]
2
```

Possible extensions: the current enumeration order is not suitable for unions of infinite enumerated sets (except possibly for the last one). One could add options to specify alternative enumeration orders (anti-diagonal, round robin,...) to handle this case.

1.5. Disjoint union of enumerated sets
Inheriting from `DisjointUnionEnumeratedSets`

There are two different use cases for inheriting from `DisjointUnionEnumeratedSets`: writing a parent which happens to be a disjoint union of some known parents, or writing generic disjoint unions for some particular classes of `sage.categories.enumerated_sets.EnumeratedSets`.

- In the first use case, the input of the `__init__` method is most likely different from that of `DisjointUnionEnumeratedSets`. Then, one simply writes the `__init__` method as usual:

```python
sage: class MyUnion(DisjointUnionEnumeratedSets):
....:     def __init__(self):
....:         DisjointUnionEnumeratedSets.__init__(self,
....:             Family([1,2], Permutations))

sage: pp = MyUnion()

sage: pp.list()
[[1], [1, 2], [2, 1]]
```

In case the `__init__()` method takes optional arguments, or does some normalization on them, a specific method `__classcall_private__` is required (see the documentation of `UniqueRepresentation`).

- In the second use case, the input of the `__init__` method is the same as that of `DisjointUnionEnumeratedSets`; one therefore wants to inherit the `__classcall_private__()` method as well, which can be achieved as follows:

```python
sage: class UnionOfSpecialSets(DisjointUnionEnumeratedSets):
....:     __classcall_private__ = staticmethod(DisjointUnionEnumeratedSets.__
....:     classcall_private__)

sage: psp = UnionOfSpecialSets(Family([1,2], Permutations))

sage: psp.list()
[[1], [1, 2], [2, 1]]
```

`Element()`

`an_element()`

Return an element of this disjoint union, as per `Sets.ParentMethods.an_element()`.

**EXAMPLES:**

```python
sage: U4 = DisjointUnionEnumeratedSets(Family([3, 5, 7], Permutations))

sage: U4.an_element()
[1, 2, 3]
```

`cardinality()`

Returns the cardinality of this disjoint union.

**EXAMPLES:**

For finite disjoint unions, the cardinality is computed by summing the cardinalities of the enumerated sets:

```python
sage: U = DisjointUnionEnumeratedSets(Family([0,1,2,3], Permutations))

sage: U.cardinality()
10
```

For infinite disjoint unions, this makes the assumption that the result is infinite:

```python
sage: U = DisjointUnionEnumeratedSets(Family(NonNegativeIntegers(), Permutations))

(continues on next page)"
Warning: As pointed out in the main documentation, it is possible to construct examples where this is incorrect:

```python
sage: U = DisjointUnionEnumeratedSets(
    ....:     Family(NonNegativeIntegers(), lambda x: []))
sage: U.cardinality()  # Should be 0!
+Infinity
```

### 1.6 Enumerated set from iterator

**EXAMPLES:**

We build a set from the iterator `graphs` that returns a canonical representative for each isomorphism class of graphs:

```python
sage: from sage.sets.set_from_iterator import EnumeratedSetFromIterator
sage: E = EnumeratedSetFromIterator(
    ....:     graphs,
    ....:     name = "Graphs",
    ....:     category = InfiniteEnumeratedSets(),
    ....:     cache = True)
sage: E
Graphs
sage: E.unrank(0)
Graph on 0 vertices
sage: E.unrank(4)
Graph on 3 vertices
sage: E.cardinality()
+Infinity
sage: E.category()
Category of facade infinite enumerated sets
```

The module also provides decorator for functions and methods:

```python
sage: from sage.sets.set_from_iterator import set_from_function
sage: @set_from_function
def f(n): return xsrange(n)
sage: f(3)
{0, 1, 2}
sage: f(5)
{0, 1, 2, 3, 4}
sage: f(100)
{0, 1, 2, 3, 4, ...}
```

```python
sage: from sage.sets.set_from_iterator import set_from_method
class A:
    ....: @set_from_method
def f(self,n):
    ....:     return xsrange(n)
sage: a = A()
sage: a.f(3)
```

(continues on next page)
class sage.sets.set_from_iterator.Decorator
    Bases: object

    Abstract class that manage documentation and sources of the wrapped object.
    The method needs to be stored in the attribute self.f

class sage.sets.set_from_iterator.DummyExampleForPicklingTest
    Bases: object

    Class example to test pickling with the decorator set_from_method.

    Warning: This class is intended to be used in doctest only.

EXAMPLES:

sage: from sage.sets.set_from_iterator import DummyExampleForPicklingTest
sage: DummyExampleForPicklingTest().f()
{(10, 11, 12, 13, 14, ...)

f()

Returns the set between self.start and self.stop.

EXAMPLES:

sage: from sage.sets.set_from_iterator import DummyExampleForPicklingTest
sage: d = DummyExampleForPicklingTest()
sage: d.f()
{(10, 11, 12, 13, 14, ...)
sage: d.start = 4
sage: d.stop = 200
sage: d.f()
{(4, 5, 6, 7, 8, ...)

class sage.sets.set_from_iterator.EnumeratedSetFromIterator(f, args=None, kwds=None, name=None, category=None, cache=False)

    Bases: sage.structure.parent.Parent

    A class for enumerated set built from an iterator.

    INPUT:
    
    • f – a function that returns an iterable from which the set is built from
    • args – tuple – arguments to be sent to the function f
    • kwds – dictionary – keywords to be sent to the function f
    • name – an optional name for the set
• **category** – (default: None) an optional category for that enumerated set. If you know that your iterator will stop after a finite number of steps you should set it as `FiniteEnumeratedSets`, conversely if you know that your iterator will run over and over you should set it as `InfiniteEnumeratedSets`.

• **cache** – boolean (default: False) – Whether or not use a cache mechanism for the iterator. If True, then the function \( f \) is called only once.

**EXAMPLES:**

```python
sage: from sage.sets.set_from_iterator import EnumeratedSetFromIterator
sage: E = EnumeratedSetFromIterator(graphs, args = (7,))
sage: E
{Graph on 7 vertices, Graph on 7 vertices, Graph on 7 vertices, Graph on 7 vertices, Graph on 7 vertices, Graph on 7 vertices, Graph on 7 vertices, ...}
sage: E.category()
Category of facade enumerated sets
```

The same example with a cache and a custom name:

```python
sage: E = EnumeratedSetFromIterator(graphs, args = (8,),
.....: category = FiniteEnumeratedSets(),
.....: name = "Graphs with 8 vertices",
.....: cache = True)
sage: E
Graphs with 8 vertices
sage: E.unrank(3)
Graph on 8 vertices
sage: E.category()
Category of facade finite enumerated sets
```

**Note:** In order to make the `TestSuite` works, the elements of the set should have parents.

**clear_cache()**

Clear the cache.

**EXAMPLES:**

```python
sage: from itertools import count
sage: from sage.sets.set_from_iterator import EnumeratedSetFromIterator
sage: E = EnumeratedSetFromIterator(count, args=(1,), cache=True)
sage: e1 = E._cache
sage: e1
lazy list [1, 2, 3, ...]
sage: E.clear_cache()
sage: E._cache
lazy list [1, 2, 3, ...]
sage: e1 is E._cache
False
```

**is_parent_of(x)**

Test whether \( x \) is in `self`.

If the set is infinite, only the answer `True` should be expected in finite time.

**EXAMPLES:**
```python
sage: from sage.sets.set_from_iterator import EnumeratedSetFromIterator
sage: P = Partitions(12, min_part=2, max_part=5)
sage: E = EnumeratedSetFromIterator(P.__iter__)
sage: P([5,5,2]) in E
True
```

**unrank(i)**

Returns the element at position $i$.

**EXAMPLES:**

```python
sage: from sage.sets.set_from_iterator import EnumeratedSetFromIterator
sage: E = EnumeratedSetFromIterator(graphs, args=(8,), cache=True)
sage: F = EnumeratedSetFromIterator(graphs, args=(8,), cache=False)
sage: E.unrank(2)
Graph on 8 vertices
sage: E.unrank(2) == F.unrank(2)
True
```

```python
class sage.sets.set_from_iterator.EnumeratedSetFromIterator_function_decorator (f=None, name=None, **options)
Bases: sage.sets.set_from_iterator.Decorator

Decorator for EnumeratedSetFromIterator.

Name could be string or a function (args, kwds) -> string.

**Warning:** If you are going to use this with the decorator cached_function, you must place the cached_function first. See the example below.

**EXAMPLES:**

```python
sage: from sage.sets.set_from_iterator import set_from_function
sage: @set_from_function
....: def f(n):
....:     for i in range(n):
....:         yield i**2 + i + 1
sage: f(3)
{1, 3, 7}
sage: f(100)
{1, 3, 7, 13, 21, ...}
```

To avoid ambiguity, it is always better to use it with a call which provides optional global initialization for the call to EnumeratedSetFromIterator:

```python
sage: @set_from_function(category=InfiniteEnumeratedSets())
....: def Fibonacci():
....:     a = 1; b = 2
....:     while True:
....:         yield a
....:     a, b = b, a + b
sage: F = Fibonacci()
sage: F
{1, 2, 3, 5, 8, ...}
```
```
A simple example with many options:

```python
sage: @set_from_function
     ....:     name = "From \%(m)d to \%(n)d",
     ....:     category = FiniteEnumeratedSets()
     ....: def f(m, n): return xsrange(m, n+1)
sage: E = f(3,10); E
From 3 to 10
sage: E.list()
[3, 4, 5, 6, 7, 8, 9, 10]
sage: E = f(1,100); E
From 1 to 100
sage: E.cardinality()
100
sage: f(n=100, m=1) == E
True
```

An example which mixes together `set_from_function` and `cached_method`:

```python
sage: @cached_function
     ....: @set_from_function
     ....:     name = "Graphs on \%(n)d vertices",
     ....:     category = FiniteEnumeratedSets(),
     ....:     cache = True)
     ....: def Graphs(n):
     return graphs(n)
sage: Graphs(10)
Graphs on 10 vertices
sage: Graphs(10).unrank(0)
Graph on 10 vertices
sage: Graphs(10) is Graphs(10)
True
```

The `cached_function` must go first:

```python
sage: @set_from_function
     ....:     name = "Graphs on \%(n)d vertices",
     ....:     category = FiniteEnumeratedSets(),
     ....:     cache = True)
     ....: @cached_function
     ....: def Graphs(n):
     return graphs(n)
sage: Graphs(10)
Graphs on 10 vertices
sage: Graphs(10).unrank(0)
Graph on 10 vertices
sage: Graphs(10) is Graphs(10)
False
```

```
class sage.sets.set_from_iterator.EnumeratedSetFromIterator_method_caller (inst, f, name=None, **options)
Bases: sage.sets.set_from_iterator.Decorator
```

Caller for decorated method in class.

1.6. Enumerated set from iterator 37
INPUT:

- \texttt{inst} – an instance of a class
- \texttt{f} – a method of a class of \texttt{inst} (and not of the instance itself)
- \texttt{name} – optional – either a string (which may contains substitution rules from argument or a function \texttt{args,kwds -> string}.
- \texttt{options} – any option accepted by \texttt{EnumeratedSetFromIterator}

\begin{verbatim}
class sage.sets.set_from_iterator.EnumeratedSetFromIterator_method_decorator (f=None, **options)

Bases: object

Decorator for enumerated set built from a method.

INPUT:

- \texttt{f} – Optional function from which are built the enumerated sets at each call
- \texttt{name} – Optional string (which may contains substitution rules from argument) or a function \texttt{(args, kwds) -> string}
- any option accepted by \texttt{EnumeratedSetFromIterator}.

EXAMPLES:

\begin{verbatim}
sage: from sage.sets.set_from_iterator import set_from_method
sage: class A():
....:     def n(self): return 12
....:     @set_from_method
....:     def f(self): return xsrange(self.n())
sage: a = A()
sage: print (a.f.__class__)
<class 'sage.sets.set_from_iterator.EnumeratedSetFromIterator_method_caller'>
sage: a.f()
{0, 1, 2, 3, 4, ...}
sage: A.f(a)
{0, 1, 2, 3, 4, ...}
\end{verbatim}

A more complicated example with a parametrized name:

\begin{verbatim}
sage: class B():
....:     @set_from_method(
....:         name = "Graphs(%(n)d)",
....:         category = FiniteEnumeratedSets())
....:     def graphs(self, n):
....:         return graphs(n)
sage: b = B()
sage: G3 = b.graphs(3)
sage: G3
Graphs(3)
sage: G3.cardinality()
4
sage: b.graphs(b,3)
Graphs(3)
\end{verbatim}

And a last example with a name parametrized by a function:

\begin{verbatim}
\end{verbatim}

Chapter 1. Set Constructions
.. code::

    sage: class D():
    ....:     def __init__(self, name): self.name = str(name)
    ....:     def __str__(self): return self.name
    ....:     @set_from_method
    ....:     name = lambda self,n: str(self)*n,
    ....:     category = FiniteEnumeratedSets())
    ....:     def subset(self, n):
    ....:         return xsrange(n)

sage: d = D('a')
sage: E = d.subset(3); E
aaa
sage: E.list()  
[0, 1, 2]
sage: F = d.subset(n=10); F
aaaaaaaaaa
sage: F.list()  
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]

Todo: It is not yet possible to use set_from_method in conjunction with cached_method.

sage.sets.set_from_iterator.set_from_function
alias of sage.sets.set_from_iterator.EnumeratedSetFromIterator_function_decorator
sage.sets.set_from_iterator.set_from_method
alias of sage.sets.set_from_iterator.EnumeratedSetFromIterator_method_decorator

1.7 Finite Enumerated Sets

class sage.sets.finite_enumerated_set.FiniteEnumeratedSet(elements)
Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

A class for finite enumerated set.

Returns the finite enumerated set with elements in elements where element is any (finite) iterable object.

The main purpose is to provide a variant of list or tuple, which is a parent with an interface consistent with EnumeratedSets and has unique representation. The list of the elements is expanded in memory.

EXAMPLES:

sage: S = FiniteEnumeratedSet([1, 2, 3])
sage: S  
{1, 2, 3}
sage: S.list()  
[1, 2, 3]
sage: S.cardinality()  
3
sage: S.random_element()  # random
1
sage: S.first()  
1
sage: S.category()  
Category of facade finite enumerated sets
sage: TestSuite(S).run()
Note that being an enumerated set, the result depends on the order:

```python
sage: S1 = FiniteEnumeratedSet((1, 2, 3))
sage: S1
{1, 2, 3}
sage: S1.list()
[1, 2, 3]
sage: S1 == S
True
sage: S2 = FiniteEnumeratedSet((2, 1, 3))
sage: S2 == S
False
```

As an abuse, repeated entries in elements are allowed to model multisets:

```python
sage: S1 = FiniteEnumeratedSet((1, 2, 1, 2, 2, 3))
sage: S1
{1, 2, 1, 2, 2, 3}
```

Finally, the elements are not aware of their parent:

```python
sage: S.first().parent()
Integer Ring
```

**an_element()**

**cardinality()**

**first()**

Return the first element of the enumeration or raise an EmptySetError if the set is empty.

**EXAMPLES:**

```python
sage: S = FiniteEnumeratedSet('abc')
sage: S.first()
'a'
```

**index(x)**

Returns the index of x in this finite enumerated set.

**EXAMPLES:**

```python
sage: S = FiniteEnumeratedSet(['a','b','c'])
sage: S.index('b')
1
```

**is_parent_of(x)**

**last()**

Returns the last element of the iteration or raise an EmptySetError if the set is empty.

**EXAMPLES:**

```python
sage: S = FiniteEnumeratedSet([0,'a',1.23, 'd'])
sage: S.last()
'd'
```

**list()**

**random_element()**

Return a random element.
EXAMPLES:

```
sage: S = FiniteEnumeratedSet('abc')
sage: S.random_element() # random
'b'
```

**rank** \((x)\)

Returns the index of \(x\) in this finite enumerated set.

EXAMPLES:

```
sage: S = FiniteEnumeratedSet(['a','b','c'])
sage: S.index('b')
1
```

**unrank** \((i)\)

Return the element at position \(i\).

EXAMPLES:

```
sage: S = FiniteEnumeratedSet([1,'a',-51])
sage: S[0], S[1], S[2]
(1, 'a', -51)
sage: S[3]
Traceback (most recent call last):
  ... IndexError: tuple index out of range
sage: S[-1], S[-2], S[-3]
(-51, 'a', 1)
sage: S[-4]
Traceback (most recent call last):
  ... IndexError: list index out of range
```

1.8 Recursively enumerated set

A set \(S\) is called recursively enumerable if there is an algorithm that enumerates the members of \(S\). We consider here the recursively enumerated sets that are described by some seeds and a successor function successors. The successor function may have some structure (symmetric, graded, forest) or not. The elements of a set having a symmetric, graded or forest structure can be enumerated uniquely without keeping all of them in memory. Many kinds of iterators are provided in this module: depth first search, breadth first search or elements of given depth.

See Wikipedia article Recursively_enumerable_set.

See documentation of \texttt{RecursivelyEnumeratedSet()} below for the description of the inputs.

AUTHORS:

- Sébastien Labbé, April 2014, at Sage Days 57, Cernay-la-ville

EXAMPLES:
1.8.1 No hypothesis on the structure

What we mean by “no hypothesis” is that the set is not known to be a forest, symmetric, or graded. However, it may have other structure, like not containing an oriented cycle, that does not help with the enumeration.

In this example, the seed is 0 and the successor function is either $+2$ or $+3$. This is the set of non negative linear combinations of 2 and 3:

```
sage: succ = lambda a: [a+2,a+3]
sage: C = RecursivelyEnumeratedSet([0], succ)
sage: C
A recursively enumerated set (breadth first search)
```

Breadth first search:

```
sage: it = C.breadth_first_search_iterator()
sage: [next(it) for _ in range(10)]
[0, 2, 3, 4, 5, 6, 7, 8, 9, 10]
```

Depth first search:

```
sage: it = C.depth_first_search_iterator()
sage: [next(it) for _ in range(10)]
[0, 3, 6, 9, 12, 15, 18, 21, 24, 27]
```

1.8.2 Symmetric structure

The origin $(0, 0)$ as seed and the upper, lower, left and right lattice point as successor function. This function is symmetric since $p$ is a successor of $q$ if and only if $q$ is a successor or $p$:

```
sage: succ = lambda a: [(a[0]-1,a[1]), (a[0],a[1]-1), (a[0]+1,a[1]), (a[0],a[1]+1)]
sage: seeds = [(0,0)]
sage: C = RecursivelyEnumeratedSet(seeds, succ, structure='symmetric', enumeration='depth')
sage: C
A recursively enumerated set with a symmetric structure (depth first search)
```

In this case, depth first search is the default enumeration for iteration:

```
sage: it_depth = iter(C)
sage: [(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8), (0, 9)]
```

Breadth first search:

```
sage: it_breadth = C.breadth_first_search_iterator()
sage: [next(it_breadth) for _ in range(13)]
[(0, 0), (-1, 0), (0, -1), (1, 0), (0, 1), (-2, 0), (-1, -1), (-1, 1), (0, -2), (1, -1), (2, 0), (1, 1), (0, 2)]
```

Levels (elements of given depth):

```
sage: sorted(C.graded_component(0))
[(0, 0)]
sage: sorted(C.graded_component(1))
(continues on next page)


1.8.3 Graded structure

Identity permutation as seed and permutohedron_succ as successor function:

\[
\text{sage: succ = attrcall("permutohedron_succ")}
\]
\[
\text{sage: seed = [Permutation([1..5])]}
\]
\[
\text{sage: R = RecursivelyEnumeratedSet(seed, succ, structure='graded')}
\]
\[
\text{sage: R}
\]
A recursively enumerated set with a graded structure (breadth first search)

Depth first search iterator:

```
\text{sage: it_depth = R.depth_first_search_iterator()}
\text{sage: [next(it_depth) for _ in range(5)]}
```
```
[[1, 2, 3, 4, 5],
 [1, 2, 3, 5, 4],
 [1, 2, 5, 3, 4],
 [1, 2, 5, 4, 3],
 [1, 5, 2, 4, 3]]
```

Breadth first search iterator:

```
\text{sage: it_breadth = R.breadth_first_search_iterator()}
\text{sage: [next(it_breadth) for _ in range(5)]}
```
```
[[1, 2, 3, 4, 5],
 [2, 1, 3, 4, 5],
 [1, 3, 2, 4, 5],
 [1, 2, 4, 3, 5],
 [1, 2, 3, 5, 4]]
```

Elements of given depth iterator:

```
\text{sage: sorted(R.elements_of_depth_iterator(9))}
```
```
[[4, 5, 3, 2, 1], [5, 3, 4, 2, 1], [5, 4, 2, 3, 1], [5, 4, 3, 1, 2]]
```

```
\text{sage: list(R.elements_of_depth_iterator(10))}
```
```
[[5, 4, 3, 2, 1]]
```

Graded components (set of elements of the same depth):

```
\text{sage: sorted(R.graded_component(9))}
```
```
[[1, 2, 3, 4, 5]]
```

```
\text{sage: sorted(R.graded_component(11))}
```
```
[[1, 2, 3, 4, 5], [1, 3, 2, 4, 5], [2, 1, 3, 4, 5]]
```

```
\text{sage: sorted(R.graded_component(9))}
```
```
[[4, 5, 3, 2, 1], [5, 3, 4, 2, 1], [5, 4, 2, 3, 1], [5, 4, 3, 1, 2]]
```

```
\text{sage: sorted(R.graded_component(10))}
```
```
[[5, 4, 3, 2, 1]]
```

1.8. Recursively enumerated set
1.8.4 Forest structure

The set of words over the alphabet \{a, b\} can be generated from the empty word by appending letter a or b as a successor function. This set has a forest structure:

```
sage: seeds = ['']
sage: succ = lambda w: [w+'a', w+'b']
sage: C = RecursivelyEnumeratedSet(seeds, succ, structure='forest')
sage: C
An enumerated set with a forest structure
```

Depth first search iterator:

```
sage: it = C.depth_first_search_iterator()
sage: [next(it) for _ in range(6)]
['', 'a', 'aa', 'aaa', 'aaaa', 'aaaaa']
```

Breadth first search iterator:

```
sage: it = C.breadth_first_search_iterator()
sage: [next(it) for _ in range(6)]
['', 'a', 'b', 'aa', 'ab', 'ba']
```

1.8.5 Example: Forest structure

This example was provided by Florent Hivert.

How to define a set using those classes?

Only two things are necessary to define a set using a `RecursivelyEnumeratedSet` object (the other classes being very similar):

For the previous example, the two necessary pieces of information are:

- the initial element ""
- the function:

```
lambda x: [x + letter for letter in ['a', 'b', 'c']]
```

This would actually describe an infinite set, as such rules describes “all words” on 3 letters. Hence, it is a good idea to replace the function by:

```
lambda x: [x + letter for letter in ['a', 'b', 'c']] if len(x) < 2 else []
```

or even:

```
sage: def children(x):
.
.
.
..:    if len(x) < 2:
..:        for letter in ['a', 'b', 'c']:
..:            yield x+letter
```

We can then create the `RecursivelyEnumeratedSet` object with either:
1.8.6 Example: Forest structure 2

This example was provided by Florent Hivert.

Here is a little more involved example. We want to iterate through all permutations of a given set \( S \). One solution is to take elements of \( S \) one by one and insert them at every position. So a node of the generating tree contains two pieces of information:

- the list \( \text{lst} \) of already inserted element;
- the set \( \text{st} \) of the yet to be inserted element.

We want to generate a permutation only if \( \text{st} \) is empty (leaves on the tree). Also suppose for the sake of the example, that instead of list we want to generate tuples. This selection of some nodes and final mapping of a function to the element is done by the \( \text{post_process} = f \) argument. The convention is that the generated elements are the \( s := f(n) \), except when \( s \) not None when no element is generated at all. Here is the code:

```python
def children(node):
    (lst, st) = node
    st = set(st)  # make a copy
    if st:
        el = st.pop()
        for i in range(0, len(lst)+1):
            yield (lst[0:i]+[el]+lst[i:], st)
def post_process(node):
    (l, s) = node
    return tuple(l) if not s else None
S = RecursivelyEnumeratedSet( [([], {1,3,6,8})],
    children, post_process=post_process,
    structure='forest', enumeration='depth',
    category=FiniteEnumeratedSets())
S.list()
[(6, 3, 1, 8), (3, 6, 1, 8), (3, 1, 6, 8), (3, 1, 8, 6), (6, 1, 3, 8),
  (1, 6, 3, 8), (1, 3, 6, 8), (1, 3, 8, 6), (6, 1, 8, 3), (1, 6, 8, 3),
  (1, 8, 6, 3), (1, 8, 3, 6), (6, 3, 8, 1), (3, 6, 8, 1), (3, 8, 6, 1),
  (3, 8, 1, 6), (6, 8, 3, 1), (8, 6, 3, 1), (8, 3, 6, 1), (8, 3, 1, 6),
  (6, 8, 1, 3), (8, 6, 1, 3), (8, 1, 6, 3), (8, 1, 3, 6)]
S.cardinality()
24
```
Return a recursively enumerated set.

A set \( S \) is called recursively enumerable if there is an algorithm that enumerates the members of \( S \). We consider here the recursively enumerated set that are described by some \textit{seeds} and a successor function \textit{successors}.

Let \( U \) be a set and \( \text{successors} : U \to 2^U \) be a successor function associating to each element of \( U \) a subset of \( U \). Let \( \text{seeds} \) be a subset of \( U \). Let \( S \subseteq U \) be the set of elements of \( U \) that can be reached from a seed by applying recursively the \textit{successors} function. This class provides different kinds of iterators (breadth first, depth first, elements of given depth, etc.) for the elements of \( S \).

See Wikipedia article Recursively_enumerable_set.

**INPUT:**
- \textit{seeds} – list (or iterable) of hashable objects
- \textit{successors} – function (or callable) returning a list (or iterable) of hashable objects
- \textit{structure} – string (optional, default: None), structure of the set, possible values are:
  - None – nothing is known about the structure of the set.
  - 'forest' – if the \textit{successors} function generates a forest, that is, each element can be reached uniquely from a seed.
  - 'graded' – if the \textit{successors} function is graded, that is, all paths from a seed to a given element have equal length.
  - 'symmetric' – if the relation is symmetric, that is, \( y \) in \textit{successors}(x) if and only if \( x \) in \textit{successors}(y)
- \textit{enumeration} – 'depth', 'breadth', 'naive' or None (optional, default: None). The default enumeration for the \textit{__iter__} function.
- \textit{max_depth} – integer (optional, default: float("inf")), limit the search to a certain depth, currently works only for breadth first search
- \textit{post_process} – (optional, default: None), for forest only
- \textit{facade} – (optional, default: None)
- \textit{category} – (optional, default: None)

**EXAMPLES:**
A recursive set with no other information:

```python
sage: f = lambda a: [a+3, a+5]
sage: C = RecursivelyEnumeratedSet([0], f)
sage: C
A recursively enumerated set (breadth first search)
sage: it = iter(C)
sage: [next(it) for _ in range(10)]
[0, 3, 5, 6, 8, 10, 9, 11, 13, 15]
```

A recursive set with a forest structure:
A recursive set given by a symmetric relation:

```python
sage: f = lambda a: [a-1, a+1]
sage: C = RecursivelyEnumeratedSet([10, 15], f, structure='symmetric')
sage: C
A recursively enumerated set with a symmetric structure (breadth first search)
sage: it = iter(C)
sage: [next(it) for _ in range(7)]
[10, 15, 9, 11, 14, 16, 8]
```

A recursive set given by a graded relation:

```python
sage: f = lambda a: [a+1, a+I]
sage: C = RecursivelyEnumeratedSet([0], f, structure='graded')
sage: C
A recursively enumerated set with a graded structure (breadth first search)
sage: it = iter(C)
sage: [next(it) for _ in range(7)]
[0, I, 2, I + 1, 2*I, 3]
```

**Warning:** If you do not set the good structure, you might obtain bad results, like elements generated twice:

```python
sage: f = lambda a: [a-1, a+1]
sage: C = RecursivelyEnumeratedSet([0], f, structure='graded')
sage: it = iter(C)
sage: [next(it) for _ in range(7)]
[0, -1, 1, -2, 0, 2, -3]
```

class sage.sets.recursively.enumerated_set.RecursivelyEnumeratedSet_forest (roots=None, children=None, post_process=None, algorithm='depth', facade=None, category=None)
See also sage.combinat.backtrack.GenericBacktracker, RecursivelyEnumeratedSet_graded, and RecursivelyEnumeratedSet_symmetric.

INPUT:

- roots – a list (or iterable)
- children – a function returning a list (or iterable, or iterator)
- post_process – a function defined over the nodes of the forest (default: no post processing)
- algorithm – 'depth' or 'breadth' (default: 'depth')
- category – a category (default: EnumeratedSets)

The option post_process allows for customizing the nodes that are actually produced. Furthermore, if \( f(x) \) returns None, then \( x \) won’t be output at all.

EXAMPLES:

We construct the set of all binary sequences of length at most three, and list them:

```python
sage: from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_forest
sage: S = RecursivelyEnumeratedSet_forest([],
    ....: lambda l: [l+[0], l+[1]] if len(l) < 3 else [],
    ....: category=FiniteEnumeratedSets())
sage: S.list()
[[], [0], [0, 0], [0, 0, 0], [0, 0, 1], [0, 1], [0, 1, 0], [0, 1, 1],
 [1], [1, 0], [1, 0, 0], [1, 0, 1], [1, 1], [1, 1, 0], [1, 1, 1]]
```

RecursivelyEnumeratedSet_forest needs to be explicitly told that the set is finite for the following to work:

```python
sage: S.category()
Category of finite enumerated sets
sage: S.cardinality()
15
```

We proceed with the set of all lists of letters in 0, 1, 2 without repetitions, ordered by increasing length (i.e. using a breadth first search through the tree):

```python
sage: from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_forest
sage: tb = RecursivelyEnumeratedSet_forest([],
    ....: lambda l: [l + [i] for i in range(3) if i not in l],
    ....: algorithm = 'breadth',
    ....: category=FiniteEnumeratedSets())
sage: tb[0]
[]
sage: tb.cardinality()
16
sage: list(tb)
[[], [0], [1], [2],
 [0, 1], [0, 2], [1, 0], [1, 2], [2, 0], [2, 1],
 [0, 1, 2], [0, 2, 1], [1, 0, 2], [1, 2, 0], [2, 0, 1], [2, 1, 0]]
```

For infinite sets, this option should be set carefully to ensure that all elements are actually generated. The following example builds the set of all ordered pairs \((i, j)\) of nonnegative integers such that \( j \leq 1 \):

```python
sage: from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_forest
sage: S = RecursivelyEnumeratedSet_forest([],
    ....: lambda l: [l + [i] for i in range(3) if i not in l],
    ....: algorithm = 'breadth',
    ....: category=FiniteEnumeratedSets())
sage: S[0]
[]
sage: S.cardinality()
16
sage: list(S)
[(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (3, 0), (3, 1), (3, 2), (4, 0), (4, 1), (4, 2), (5, 0), (5, 1)]
```

```python
sage: from sage.sets.recursively enumerated_set import RecursivelyEnumeratedSet_f
sage: I = RecursivelyEnumeratedSet_forest([(0, 0)],
        ...: lambda l: [(l[0]+1, l[1]), (l[0], 1)]
        ...: if l[1] == 0 else [(l[0], l[1]+1)])
```

With a depth first search, only the elements of the form \((i, 0)\) are generated:

```python
sage: depth_search = I.depth_first_search_iterator()
sage: [next(depth_search) for i in range(7)]
[(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0)]
```

Using instead breadth first search gives the usual anti-diagonal iterator:

```python
sage: breadth_search = I.breadth_first_search_iterator()
sage: [next(breadth_search) for i in range(15)]
[(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), (3, 0), (2, 1), (1, 2), (0, 3), (4, 0), (3, 1), (2, 2), (1, 3), (0, 4)]
```

### Deriving subclasses

The class of a parent \(A\) may derive from `RecursivelyEnumeratedSet_forest` so that \(A\) can benefit from enumeration tools. As a running example, we consider the problem of enumerating integers whose binary expansion have at most three nonzero digits. For example, \(3 = 2^1 + 2^0\) has two nonzero digits. \(15 = 2^3 + 2^2 + 2^1 + 2^0\) has four nonzero digits. In fact, \(15\) is the smallest integer which is not in the enumerated set.

To achieve this, we use `RecursivelyEnumeratedSet_forest` to enumerate binary tuples with at most three nonzero digits, apply a post processing to recover the corresponding integers, and discard tuples finishing by zero.

A first approach is to pass the `roots` and `children` functions as arguments to `RecursivelyEnumeratedSet_forest.__init__()`:}

```python
sage: from sage.sets.recursively enumerated set import RecursivelyEnumeratedSet_f
sage: class A(UniqueRepresentation, RecursivelyEnumeratedSet_forest):
    ...: def __init__(self):
        ...:         RecursivelyEnumeratedSet_forest.__init__(self, []
        ...:         lambda x : [x+(0,), x+(1,)] if sum(x) < 3 else [],
        ...:         lambda x : sum(x[i]*2^i for i in range(len(x))) if sum(x) != 0
        ...:             and x[-1] != 0 else None,
        ...:             algorithm = 'breadth',
        ...:             category=InfiniteEnumeratedSets())
sage: MyForest = A(); MyForest
An enumerated set with a forest structure
```

An alternative approach is to implement `roots` and `children` as methods of the subclass (in fact they could also be attributes of \(A\)). Namely, \(A.roots()\) must return an iterable containing the enumeration generators,
and \texttt{A.children(x)} must return an iterable over the children of \(x\). Optionally, \(A\) can have a method or attribute such that \(A.post_process(x)\) returns the desired output for the node \(x\) of the tree:

\begin{verbatim}
Sage: from sage.sets.recursivelyEnumeratedSet import RecursivelyEnumeratedSet_
  \rightarrow forest
Sage: class A(UniqueRepresentation, RecursivelyEnumeratedSet_forest):
    ...:     def __init__(self):
    ...:         RecursivelyEnumeratedSet_forest.__init__(self, algorithm = 'breadth
    ...:        irim',
    ...:             category=InfiniteEnumeratedSets())
    ...:
    ...:     def roots(self):
    ...:         return [()]
    ...:
    ...:     def children(self, x):
    ...:         if sum(x) < 3:
    ...:             return [x+(0,), x+(1,)]
    ...:         else:
    ...:             return []
    ...:
    ...:     def post_process(self, x):
    ...:         if sum(x) == 0 or x[-1] == 0:
    ...:             return None
    ...:         else:
    ...:             return sum(x[i]*2^i for i in range(len(x)))
Sage: MyForest = A(); MyForest
An enumerated set with a forest structure
Sage: MyForest.category()
Category of infinite enumerated sets
Sage: p = iter(MyForest)
Sage: [next(p) for i in range(30)]
[1, 2, 3, 4, 6, 5, 7, 8, 12, 10, 14, 9, 13, 11, 16, 24, 20, 28, 18, 26, 22, 17, 25, 21, 19, 32, 48, 40, 56, 36]
\end{verbatim}

**Warning:** A \texttt{RecursivelyEnumeratedSet_forest} instance is picklable if and only if the input functions are themselves picklable. This excludes anonymous or interactively defined functions:

\begin{verbatim}
Sage: def children(x):
    ...:     return [x+1]
Sage: S = RecursivelyEnumeratedSet_forest([1], children, category=InfiniteEnumeratedSets())
Sage: dumps(S)
Traceback (most recent call last):
... PicklingError: Can't pickle <...function...>: attribute lookup ... failed
\end{verbatim}

Let us now fake \texttt{children} being defined in a Python module:

\begin{verbatim}
Sage: import __main__
Sage: __main__.children = children
Sage: S = RecursivelyEnumeratedSet_forest([1], children, category=InfiniteEnumeratedSets())
Sage: loads(dumps(S))
An enumerated set with a forest structure
\end{verbatim}

\texttt{breadth_first_search_iterator()} 
Return a breadth first search iterator over the elements of \texttt{self}
EXAMPLES:

```python
sage: from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_forest
sage: f = RecursivelyEnumeratedSet_forest([[]],
           lambda l: [l+[0], l+[1]] if len(l) < 3 else [])
sage: list(f.breadth_first_search_iterator())
[[], [0], [1], [0, 0], [0, 1], [1, 0], [1, 1], [0, 0, 0], [0, 0, 1], [0, 1, 0], [1, 0, 0], [1, 0, 1], [1, 1, 0], [1, 1, 1]]
sage: S = RecursivelyEnumeratedSet_forest([(0,0)],
           lambda x : [(x[0]+1, x[1]), (x[0], 1)] if x[1] == 0 else [(x[0], x[1]+1)])
sage: [i for i in S.children((0,0))]
[(1, 0), (0, 1)]
sage: [i for i in S.children((1,0))]
[(2, 0), (1, 1)]
sage: [i for i in S.children((1,1))]
[(1, 2)]
sage: [i for i in S.children((4,1))]
[(4, 2)]
sage: [i for i in S.children((4,0))]
[(5, 0), (4, 1)]
```

**children(x)**

Return the children of the element `x`

The result can be a list, an iterable, an iterator, or even a generator.

EXAMPLES:

```python
sage: from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_forest
sage: I = RecursivelyEnumeratedSet_forest([(0,0)],
           lambda l: [(l[0]+1, l[1]), (l[0], 1)] if l[1] == 0 else [(l[0], l[1]+1)])
sage: [i for i in I.children((0,0))]
[(1, 0), (0, 1)]
sage: [i for i in I.children((1,0))]
[(2, 0), (1, 1)]
sage: [i for i in I.children((1,1))]
[(1, 2)]
sage: [i for i in I.children((4,1))]
[(4, 2)]
sage: [i for i in I.children((4,0))]
[(5, 0), (4, 1)]
```

**depth_first_search_iterator()**

Return a depth first search iterator over the elements of `self`

EXAMPLES:

```python
sage: from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_forest
sage: f = RecursivelyEnumeratedSet_forest([[]],
           lambda l: [l+[0], l+[1]] if len(l) < 3 else [])
sage: list(f.depth_first_search_iterator())
[[], [0], [0, 0], [0, 1], [1, 0], [1, 1], [0, 0, 0], [0, 0, 1], [0, 1, 0], [1, 0, 0], [1, 0, 1], [1, 1, 0], [1, 1, 1]]
```

**elements_of_depth_iterator**(depth=0)

Return an iterator over the elements of `self` of given depth. An element of depth `n` can be obtained applying `n` times the children function from a root.

EXAMPLES:
sage: from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_forest

sage: S = RecursivelyEnumeratedSet_forest([(0,0)],
....: lambda x: [(x[0], x[1]+1)] if x[1] != 0 else [(x[0]+1,0), (x[0], 1)],
....: post_process = lambda x: x if ((is_prime(x[0]) and is_prime(x[1]))
....: and (x[0] - x[1]) == 2) else None)

sage: p = S.elements_of_depth_iterator(8)
sage: next(p)
(5, 3)

sage: S = RecursivelyEnumeratedSet_forest(NN, lambda x: [],
....: lambda x: x^2 if x.is_prime() else None)

sage: p = S.elements_of_depth_iterator(0)

sage: [next(p), next(p), next(p), next(p), next(p)]
[4, 9, 25, 49, 121]

map_reduce (map_function=None, reduce_function=None, reduce_init=None)
Apply a Map/Reduce algorithm on self

INPUT:

- **map_function** – a function from the element of self to some set with a reduce operation (e.g.: a monoid). The default value is the constant function 1.
- **reduce_function** – the reduce function (e.g.: the addition of a monoid). The default value is +.
- **reduce_init** – the initialisation of the reduction (e.g.: the neutral element of the monoid). The default value is 0.

Note: the effect of the default values is to compute the cardinality of self.

EXAMPLES:

sage: seeds = [(i,i, i) for i in range(1,10)]
sage: def succ(t):
....:     list, sum, last = t
....:     return [(list + [i], sum + i, i) for i in range(1, last)]

sage: F = RecursivelyEnumeratedSet(seeds, succ,
....:     structure='forest', enumeration='depth')

sage: y = var('y')
sage: def map_function(t):
....:     li, sum, _ = t
....:     return y ^ sum

sage: reduce_function = lambda x,y: x + y

sage: F.map_reduce(map_function, reduce_function, 0)
y^45 + y^44 + y^43 + 2*y^42 + 2*y^41 + 3*y^40 + 4*y^39 + 5*y^38 + 6*y^37 +
19*y^36 + 9*y^35 + 10*y^34 + 12*y^33 + 13*y^32 + 15*y^31 + 17*y^30 + 18*y^29 +
19*y^28 + 21*y^27 + 21*y^26 + 22*y^25 + 23*y^24 + 23*y^23 + 23*y^22 +
23*y^21 + 22*y^20 + 21*y^19 + 21*y^18 + 19*y^17 + 18*y^16 + 17*y^15 + 15*y^14 +
13*y^13 + 12*y^12 + 10*y^11 + 9*y^10 + 8*y^9 + 6*y^8 + 5*y^7 + 4*y^6 +
3*y^5 + 2*y^4 + 2*y^3 + y^2 + y

Here is an example with the default values:
sage: F.map_reduce()
511

See also:
sage.parallel.map_reduce

roots()
Return an iterable over the roots of self.

EXAMPLES:

```python
sage: from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_forest
sage: I = RecursivelyEnumeratedSet_forest([(0,0)], lambda l: [(l[0]+1, l[1]), (l[0], 1)] if l[1] == 0 else [(l[0], l[1]+1)])
sage: [i for i in I.roots()]
[(0, 0)]
sage: I = RecursivelyEnumeratedSet_forest([(0,0),(1,1)], lambda l: [(l[0]+1, l[1]), (l[0], l[1]+1)] if l[1] == 0 else [(l[0], l[1]+1)])
sage: [i for i in I.roots()]
[(0, 0), (1, 1)]
```

class sage.sets.recursively_enumerated_set.RecursivelyEnumeratedSet_generic

Bases: sage.structure.parent.Parent

A generic recursively enumerated set.

For more information, see `RecursivelyEnumeratedSet()`.

EXAMPLES:

```python
sage: f = lambda a: [a+1]
Different structure for the sets:

```

```python
sage: RecursivelyEnumeratedSet([0], f, structure=None)
A recursively enumerated set (breadth first search)
sage: RecursivelyEnumeratedSet([0], f, structure='graded')
A recursively enumerated set with a graded structure (breadth first search)
sage: RecursivelyEnumeratedSet([0], f, structure='symmetric')
A recursively enumerated set with a symmetric structure (breadth first search)
sage: RecursivelyEnumeratedSet([0], f, structure='forest')
An enumerated set with a forest structure
```

Different default enumeration algorithms:

```python
sage: RecursivelyEnumeratedSet([0], f, enumeration='breadth')
A recursively enumerated set (breadth first search)
sage: RecursivelyEnumeratedSet([0], f, enumeration='naive')
A recursively enumerated set (naive search)
sage: RecursivelyEnumeratedSet([0], f, enumeration='depth')
A recursively enumerated set (depth first search)
```

`breadth_first_search_iterator(max_depth=None)`
Iterate on the elements of self (breadth first).

This code remembers every element generated.

The elements are guaranteed to be enumerated in the order in which they are first visited (left-to-right traversal).
INPUT:

- `max_depth` – (default: `self._max_depth`) specifies the maximal depth to which elements are computed

EXAMPLES:

```
sage: f = lambda a: [a+3, a+5]
sage: C = RecursivelyEnumeratedSet([0], f)
sage: it = C.breadth_first_search_iterator()
sage: [next(it) for _ in range(10)]
[0, 3, 5, 6, 8, 10, 9, 11, 13, 15]
```

**depth_first_search_iterator()**

Iterate on the elements of `self` (depth first).

This code remembers every elements generated.

The elements are traversed right-to-left, so the last element returned by the successor function is visited first.

See Wikipedia article Depth-first_search.

EXAMPLES:

```
sage: f = lambda a: [a+3, a+5]
sage: C = RecursivelyEnumeratedSet([0], f)
sage: it = C.depth_first_search_iterator()
sage: [next(it) for _ in range(10)]
[0, 5, 10, 15, 20, 25, 30, 35, 40, 45]
```

**elements_of_depth_iterator**(depth)

Iterate over the elements of `self` of given depth.

An element of depth `n` can be obtained applying `n` times the successor function to a seed.

INPUT:

- `depth` – integer

OUTPUT:

An iterator.

EXAMPLES:

```
sage: f = lambda a: [a-1, a+1]
sage: S = RecursivelyEnumeratedSet([5, 10], f, structure='symmetric')
sage: it = S.elements_of_depth_iterator(2)
sage: sorted(it)
[3, 7, 8, 12]
```

**graded_component**(depth)

Return the graded component of given depth.

This method caches each lower graded component.

A graded component is a set of elements of the same depth where the depth of an element is its minimal distance to a root.

It is currently implemented only for graded or symmetric structure.

INPUT:
• depth – integer

OUTPUT:
A set.

EXAMPLES:

```python
sage: f = lambda a: [a+3, a+5]
sage: C = RecursivelyEnumeratedSet([0], f)
sage: C.graded_component(0)
Traceback (most recent call last):
  ...  
NotImplementedError: graded_component_iterator method currently implemented only for graded or symmetric structure
```

graded_component_iterator()
Iterate over the graded components of self.
A graded component is a set of elements of the same depth.
It is currently implemented only for graded or symmetric structure.

OUTPUT:
An iterator of sets.

EXAMPLES:

```python
sage: f = lambda a: [a+3, a+5]
sage: C = RecursivelyEnumeratedSet([0], f)
sage: it = C.graded_component_iterator()  # todo: not implemented
```

naive_search_iterator()
Iterate on the elements of self (in no particular order).
This code remembers every elements generated.

seeds()
Return an iterable over the seeds of self.

EXAMPLES:

```python
sage: R = RecursivelyEnumeratedSet([1], lambda x: [x+1, x-1])
sage: R.seeds()
[1]
```

successors
to_digraph (max_depth=None, loops=True, multiedges=True)
Return the directed graph of the recursively enumerated set.

INPUT:
• max_depth – (default: self._max_depth) specifies the maximal depth for which outgoing edges of elements are computed
• loops – (default: True) option for the digraph
• multiedges – (default: True) option of the digraph

OUTPUT:
A directed graph
Warning: If the set is infinite, this will loop forever unless max_depth is finite.

EXAMPLES:

```python
sage: child = lambda i: [(i+3) % 10, (i+8) % 10]
sage: R = RecursivelyEnumeratedSet([0], child)
sage: R.to_digraph()
Looped multi-digraph on 10 vertices
```

Digraph of an recursively enumerated set with a symmetric structure of infinite cardinality using max_depth argument:

```python
sage: succ = lambda a: [(a[0]-1,a[1]), (a[0],a[1]-1), (a[0]+1,a[1]), (a[0], a[1]+1)]
sage: seeds = [(0,0)]
sage: C = RecursivelyEnumeratedSet(seeds, succ, structure='symmetric')
sage: C.to_digraph(max_depth=3)
Looped multi-digraph on 41 vertices
```

The max_depth argument can be given at the creation of the set:

```python
sage: C = RecursivelyEnumeratedSet(seeds, succ, structure='symmetric', max_depth=2)
sage: C.to_digraph()
Looped multi-digraph on 25 vertices
```

Digraph of an recursively enumerated set with a graded structure:

```python
sage: f = lambda a: [a[0]+1, a[1]]
sage: C = RecursivelyEnumeratedSet([0], f, structure='graded')
sage: C.to_digraph(max_depth=4)
Looped multi-digraph on 21 vertices
```

```python
class sage.sets.recursively.enumerated_set.RecursivelyEnumeratedSet_graded
Bases: sage.sets.recursively.enumerated_set.RecursivelyEnumeratedSet_generic

Generic tool for constructing ideals of a graded relation.

INPUT:

- seeds – list (or iterable) of hashable objects
- successors – function (or callable) returning a list (or iterable)
- enumeration – 'depth', 'breadth' or None (default: None)
- max_depth – integer (default: float("inf"))

EXAMPLES:

```python
sage: f = lambda a: [(a[0]+1,a[1]), (a[0],a[1]+1)]
sage: C = RecursivelyEnumeratedSet([0], f, structure='graded', max_depth=3)
sage: C
A recursively enumerated set with a graded structure (breadth first search) with max_depth=3
sage: list(C)
[(0, 0),
 (1, 0), (0, 1),
```
breadth_first_search_iterator (max_depth=None)

Iterate on the elements of self (breadth first).

This iterator makes use of the graded structure by remembering only the elements of the current depth.

The elements are guaranteed to be enumerated in the order in which they are first visited (left-to-right traversal).

INPUT:

- max_depth – (default: self._max_depth) specifies the maximal depth to which elements are computed

EXAMPLES:

```sage
def f(a):
    return [a+1, a+I]

c = RecursivelyEnumeratedSet([0], f, structure='graded')
for i in range(5):
    sorted(c.graded_component(i))
```

graded_component (depth)

Return the graded component of given depth.

This method caches each lower graded component. See graded_component_iterator() to generate each graded component without caching the previous ones.

A graded component is a set of elements of the same depth where the depth of an element is its minimal distance to a root.

INPUT:

- depth – integer

OUTPUT:

A set.

EXAMPLES:

```sage
def f(a):
    return [a+1, a+I]

c = RecursivelyEnumeratedSet([0], f, structure='graded')
for i in range(5):
    sorted(c.graded_component(i))
```

graded_component_iterator ()

Iterate over the graded components of self.

A graded component is a set of elements of the same depth.

The algorithm remembers only the current graded component generated since the structure is graded.

OUTPUT:
An iterator of sets.

EXAMPLES:

```python
sage: f = lambda a: [(a[0]+1,a[1]), (a[0],a[1]+1)]
```

```python
sage: C = RecursivelyEnumeratedSet([(0,0)], f, structure='graded', max_depth=3)
```

```python
sage: it = C.graded_component_iterator()
```

```python
sage: for _ in range(4): sorted(next(it))
[(0, 0)]
[(0, 1), (1, 0)]
[(0, 2), (1, 1), (2, 0)]
[(0, 3), (1, 2), (2, 1), (3, 0)]
```

class sage.sets.recursivelyEnumeratedSet.RecursivelyEnumeratedSet_symmetric

Bases: sage.sets.recursivelyEnumeratedSet.RecursivelyEnumeratedSet_generic

Generic tool for constructing ideals of a symmetric relation.

INPUT:

- `seeds` – list (or iterable) of hashable objects
- `successors` – function (or callable) returning a list (or iterable)
- `enumeration` – 'depth', 'breadth' or None (default: None)
- `max_depth` – integer (default: float("inf"))

EXAMPLES:

```python
sage: f = lambda a: [a-1,a+1]
```

```python
sage: C = RecursivelyEnumeratedSet([0], f, structure='symmetric')
```

```python
sage: C
A recursively enumerated set with a symmetric structure (breadth first search)
```

```python
sage: it = iter(C)
```

```python
sage: [next(it) for _ in range(7)]
[0, -1, 1, -2, 2, -3, 3]
```

`breadth_first_search_iterator(max_depth=None)`

Iterate on the elements of `self` (breadth first).

This iterator makes use of the graded structure by remembering only the last two graded components since the structure is symmetric.

The elements are guaranteed to be enumerated in the order in which they are first visited (left-to-right traversal).

INPUT:

- `max_depth` – (default: `self._max_depth`) specifies the maximal depth to which elements are computed

EXAMPLES:

```python
sage: f = lambda a: [(a[0]-1,a[1]), (a[0],a[1]-1), (a[0]+1,a[1]), (a[0]+1,a[1]+1)]
```

```python
sage: C = RecursivelyEnumeratedSet([(0,0)], f, structure='symmetric')
```

```python
sage: s = list(C.breadth_first_search_iterator(max_depth=2)); s
[(0, 0), (-1, 0), (0, -1), (1, 0), (0, 1), (-2, 0), (-1, -1), (-1, 1), (0, -2), (1, -1), (2, 0), (1, 1), (0, 2)]
```
This iterator is used by default for symmetric structure:

```python
sage: it = iter(C)
sage: s == [next(it) for _ in range(13)]
```

True

**graded_component** *(depth)*

Return the graded component of given depth.

This method caches each lower graded component. See `graded_component_iterator()` to generate each graded component without caching the previous ones.

A graded component is a set of elements of the same depth where the depth of an element is its minimal distance to a root.

**INPUT:**

- `depth` – integer

**OUTPUT:**

A set.

**EXAMPLES:**

```python
sage: f = lambda a: [a-1, a+1]
sage: C = RecursivelyEnumeratedSet([10, 15], f, structure='symmetric')
sage: for i in range(5): sorted(C.graded_component(i))
[10, 15]
[9, 11, 14, 16]
[8, 12, 13, 17]
[7, 18]
[6, 19]
```

**graded_component_iterator()**

Iterate over the graded components of `self`.

A graded component is a set of elements of the same depth.

The enumeration remembers only the last two graded components generated since the structure is symmetric.

**OUTPUT:**

An iterator of sets.

**EXAMPLES:**

```python
sage: f = lambda a: [a-1, a+1]
sage: S = RecursivelyEnumeratedSet([10], f, structure='symmetric')
sage: it = S.graded_component_iterator()
sage: [sorted(next(it)) for _ in range(5)]
[[10], [9, 11], [8, 12], [7, 13], [6, 14]]
```

Starting with two generators:

```python
sage: f = lambda a: [a-1, a+1]
sage: S = RecursivelyEnumeratedSet([5, 10], f, structure='symmetric')
sage: it = S.graded_component_iterator()
sage: [sorted(next(it)) for _ in range(5)]
[[5, 10], [4, 6, 9, 11], [3, 7, 8, 12], [2, 13], [1, 14]]
```

Gaussian integers:
```python
sage: f = lambda a: [a+1, a+I]
sage: S = RecursivelyEnumeratedSet([0], f, structure='symmetric')
sage: it = S.graded_component_iterator()
sage: [sorted(next(it)) for _ in range(7)]
[[0],
 [I, 1],
 [2*I, I + 1, 2],
 [3*I, 2*I + 1, I + 2, 3],
 [4*I, 3*I + 1, 2*I + 2, I + 3, 4],
 [5*I, 4*I + 1, 3*I + 2, 2*I + 3, I + 4, 5],
 [6*I, 5*I + 1, 4*I + 2, 3*I + 3, 2*I + 4, I + 5, 6]]
```

**sage.sets.recursivelyEnumeratedSet.search_forest_iterator**

Return an iterator on the nodes of the forest having the given roots, and where children(x) returns the children of the node x of the forest. Note that every node of the tree is returned, not simply the leaves.

**INPUT:**
- roots – a list (or iterable)
- children – a function returning a list (or iterable)
- algorithm – 'depth' or 'breadth' (default: 'depth')

**EXAMPLES:**

We construct the prefix tree of binary sequences of length at most three, and enumerate its nodes:

```python
sage: from sage.sets.recursivelyEnumeratedSet import search_forest_iterator
sage: list(search_forest_iterator([], lambda l: [l+[0], l+[1]],
                             algorithm='depth'))
[[], [0], [1], [0, 0], [0, 1], [1, 0], [1, 1],
 [0, 0, 0], [0, 0, 1], [0, 1, 0], [0, 1, 1],
 [1, 0, 0], [1, 0, 1], [1, 1, 0], [1, 1, 1]]
```

By default, the nodes are iterated through by depth first search. We can instead use a breadth first search (increasing depth):

```python
sage: list(search_forest_iterator([], lambda l: [l+[0], l+[1]],
                             algorithm='breadth'))
[[], [0], [1], [0, 0], [0, 1], [1, 0], [1, 1],
 [0, 0, 0], [0, 0, 1], [0, 1, 0], [0, 1, 1],
 [1, 0, 0], [1, 0, 1], [1, 1, 0], [1, 1, 1]]
```

This allows for iterating through trees of infinite depth:

```python
sage: it = search_forest_iterator([], lambda l: [l+[0], l+[1]], algorithm='breadth')
sage: [next(it) for i in range(16)]
[[], [0], [1], [0, 0], [0, 1], [1, 0], [1, 1],
 [0, 0, 0], [0, 0, 1], [0, 1, 0], [0, 1, 1],
 [1, 0, 0], [1, 0, 1], [1, 1, 0], [1, 1, 1]]
```

Here is an iterator through the prefix tree of sequences of letters in 0, 1, 2 without repetitions, sorted by length; the leaves are therefore permutations:
1.9 Maps between finite sets

This module implements parents modeling the set of all maps between two finite sets. At the user level, any such parent should be constructed using the factory class `FiniteSetMaps` which properly selects which of its subclasses to use.

AUTHORS:

• Florent Hivert

class sage.sets.finite_set_maps.FiniteSetEndoMaps_N(n, action, category=None)
    Bases: sage.sets.finite_set_maps.FiniteSetMaps_MN

The sets of all maps from \{1, 2, \ldots, n\} to itself

Users should use the factory class `FiniteSetMaps` to create instances of this class.

INPUT:

• n – an integer.
• category – the category in which the sets of maps is constructed. It must be a sub-category of `Monoids().Finite()` and `EnumeratedSets().Finite()` which is the default value.

Element
    alias of `sage.sets.finite_set_map_cy.FiniteSetEndoMap_N`

an_element()
    Returns a map in self

EXAMPLES:

    sage: M = FiniteSetMaps(4)
    sage: M.an_element()
    [3, 2, 1, 0]

one()
    EXAMPLES:

    sage: M = FiniteSetMaps(4)
    sage: M.one()
    [0, 1, 2, 3]

class sage.sets.finite_set_maps.FiniteSetEndoMaps_Set(domain, action, category=None)
    Bases: sage.sets.finite_set_maps.FiniteSetMaps_Set, sage.sets.finite_set_maps.FiniteSetEndoMaps_N

The sets of all maps from a set to itself

Users should use the factory class `FiniteSetMaps` to create instances of this class.
INPUT:

- **domain** – an object in the category `FiniteSets()`.
- **category** – the category in which the sets of maps is constructed. It must be a sub-category of `Monoids().Finite()` and `EnumeratedSets().Finite()` which is the default value.

**Element**

alias of `sage.sets.finite_set_map_cy.FiniteSetEndoMap_Set`

---

**class** `sage.sets.finite_set_maps.FiniteSetMaps`

**Bases:** `sage.structure.unique_representation.UniqueRepresentation`, `sage.structure.parent.Parent`

Maps between finite sets

Constructs the set of all maps between two sets. The sets can be given using any of the three following ways:

1. an object in the category `Sets()`.
2. a finite iterable. In this case, an object of the class `FiniteEnumeratedSet` is constructed from the iterable.
3. an integer \( n \) designating the set \( \{0, 1, \ldots, n - 1\} \). In this case an object of the class `IntegerRange` is constructed.

**INPUT:**

- **domain** – a set, finite iterable, or integer.
- **codomain** – a set, finite iterable, integer, or `None` (default). In this last case, the maps are endo-maps of the domain.
- **action** – "left" (default) or "right". The side where the maps act on the domain. This is used in particular to define the meaning of the product (composition) of two maps.
- **category** – the category in which the sets of maps is constructed. By default, this is `FiniteMonoids()` if the domain and codomain coincide, and `FiniteEnumeratedSets()` otherwise.

**OUTPUT:**

an instance of a subclass of `FiniteSetMaps` modeling the set of all maps between `domain` and `codomain`.

**EXAMPLES:**

We construct the set \( M \) of all maps from \( \{a, b\} \) to \( \{3, 4, 5\} \):

```
sage: M = FiniteSetMaps(\"a", \"b\", [3, 4, 5]); M
Maps from {'a', 'b'} to {3, 4, 5}
sage: M.cardinality()
9
sage: M.domain()
{'a', 'b'}
sage: M.codomain()
{3, 4, 5}
sage: for f in M: print(f)
map: a -> 3, b -> 3
map: a -> 3, b -> 4
map: a -> 3, b -> 5
map: a -> 4, b -> 3
map: a -> 4, b -> 4
map: a -> 4, b -> 5
```
Elements can be constructed from functions and dictionaries:

\begin{verbatim}
 sage: M(lambda c: ord(c)-94)
 map: a -> 3, b -> 4
 sage: M.from_dict({'a':3, 'b':5})
 map: a -> 3, b -> 5
\end{verbatim}

If the domain is equal to the codomain, then maps can be composed:

\begin{verbatim}
 sage: M = FiniteSetMaps([1, 2, 3])
 sage: f = M.from_dict({1:2, 2:1, 3:3}); f
 map: 1 -> 2, 2 -> 1, 3 -> 3
 sage: g = M.from_dict({1:2, 2:3, 3:1}); g
 map: 1 -> 2, 2 -> 3, 3 -> 1
 sage: f * g
 map: 1 -> 1, 2 -> 3, 3 -> 2
\end{verbatim}

This makes $M$ into a monoid:

\begin{verbatim}
 sage: M.category()
 Category of finite enumerated monoids
 sage: M.one()
 map: 1 -> 1, 2 -> 2, 3 -> 3
\end{verbatim}

By default, composition is from right to left, which corresponds to an action on the left. If one specifies action to right, then the composition is from left to right:

\begin{verbatim}
 sage: M = FiniteSetMaps([1, 2, 3], action = 'right')
 sage: f = M.from_dict({1:2, 2:1, 3:3})
 sage: g = M.from_dict({1:2, 2:3, 3:1})
 sage: f * g
 map: 1 -> 3, 2 -> 2, 3 -> 1
\end{verbatim}

If the domains and codomains are both of the form \{0, ..., 2\}, then one can use the shortcut:

\begin{verbatim}
 sage: M = FiniteSetMaps(2,3); M
 Maps from \{0, 1\} to \{0, 1, 2\}
 sage: M.cardinality()
 9
\end{verbatim}

For a compact notation, the elements are then printed as lists \{f(i), i = 0, ..., 2\}:

\begin{verbatim}
 sage: list(M)
 [[0, 0], [0, 1], [0, 2], [1, 0], [1, 1], [1, 2], [2, 0], [2, 1], [2, 2]]
\end{verbatim}

**cardinality()**

The cardinality of self

**EXAMPLES:**
class sage.sets.finite_set_maps.FiniteSetMaps_MN(m, n, category=None)

Bases: sage.sets.finite_set_maps.FiniteSetMaps

The set of all maps from \{1, 2, \ldots, m\} to \{1, 2, \ldots, n\}.

Users should use the factory class `FiniteSetMaps` to create instances of this class.

**INPUT:**

- \(m, n\) – integers
- \(\text{category}\) – the category in which the sets of maps is constructed. It must be a sub-category of `EnumeratedSets().Finite()` which is the default value.

**Element**

alias of `sage.sets.finite_set_map_cy.FiniteSetMap_MN`

**an_element()**

Returns a map in `self`

**EXAMPLES:**

```
sage: M = FiniteSetMaps(4, 2)
sage: M.an_element()
[0, 0, 0, 0]
sage: M = FiniteSetMaps(0, 0)
sage: M.an_element()
[]
```

An exception `EmptySetError` is raised if this set is empty, that is if the codomain is empty and the domain is not.

```
sage: M = FiniteSetMaps(4, 0)
sage: M.cardinality() 0
sage: M.an_element() Traceback (most recent call last):  
  ... EmptySetError
```

**codomain()**

The codomain of `self`

**EXAMPLES:**

```
sage: FiniteSetMaps(3,2).codomain()
{0, 1}
```

**domain()**

The domain of `self`

**EXAMPLES:**

```
sage: FiniteSetMaps(3,2).domain()
{0, 1, 2}
```

class sage.sets.finite_set_maps.FiniteSetMaps_Set(domain, codomain, category=None)

Bases: sage.sets.finite_set_maps.FiniteSetMaps_MN

The sets of all maps between two sets

Users should use the factory class `FiniteSetMaps` to create instances of this class.
INPUT:

- domain – an object in the category `FiniteSets()`.
- codomain – an object in the category `FiniteSets()`.
- category – the category in which the sets of maps is constructed. It must be a sub-category of `EnumeratedSets().Finite()` which is the default value.

**Element**

alias of `sage.sets.finite_set_map_cy.FiniteSetMap_Set`

codomain()

The codomain of self

EXAMPLES:

```python
sage: FiniteSetMaps(['a', 'b'], [3, 4, 5]).codomain()
{3, 4, 5}
```

domain()

The domain of self

EXAMPLES:

```python
sage: FiniteSetMaps(['a', 'b'], [3, 4, 5]).domain()
{'a', 'b'}
```

from_dict(d)

Create a map from a dictionary

EXAMPLES:

```python
sage: M = FiniteSetMaps(['a', 'b'], [3, 4, 5])
sage: M.from_dict({'a': 4, 'b': 3})
map: a → 4, b → 3
```

1.10 Data structures for maps between finite sets

This module implements several fast Cython data structures for maps between two finite set. Those classes are not intended to be used directly. Instead, such a map should be constructed via its parent, using the class `FiniteSetMaps`.

EXAMPLES:

To create a map between two sets, one first creates the set of such maps:

```python
sage: M = FiniteSetMaps(['a', 'b'], [3, 4, 5])
```

The map can then be constructed either from a function:

```python
sage: f1 = M(lambda c: ord(c)-94); f1
map: a → 3, b → 4
```

or from a dictionary:

```python
sage: f2 = M.from_dict({'a': 3, 'b': 4}); f2
map: a → 3, b → 4
```

The two created maps are equal:
Internally, maps are represented as the list of the ranks of the images $f(x)$ in the co-domain, in the order of the domain:

```
sage: list(f2)
[0, 1]
```

A third fast way to create a map it to use such a list. It should be kept for internal use:

```
sage: f3 = M._from_list_([0, 1]); f3
map: a -> 3, b -> 4
sage: f1 == f3
True
```

AUTHORS:

- Florent Hivert

```python
class sage.sets.finite_set_map_cy.FiniteSetEndoMap_N
    Bases: sage.sets.finite_set_map_cy.FiniteSetMap_MN

Maps from range(n) to itself.

See also:
FiniteSetMap_MN for assumptions on the parent
```

```python
class sage.sets.finite_set_map_cy.FiniteSetEndoMap_Set
    Bases: sage.sets.finite_set_map_cy.FiniteSetMap_Set

Maps from a set to itself.

See also:
FiniteSetMap_Set for assumptions on the parent
```

```python
class sage.sets.finite_set_map_cy.FiniteSetMap_MN
    Bases: sage.structure.list_clone.ClonableIntArray

Data structure for maps from range(m) to range(n).

We assume that the parent given as argument is such that:
- \( m \) is stored in \( \text{self}.\text{parent}() \).\_m
- \( n \) is stored in \( \text{self}.\text{parent}() \).\_n
- the domain is in \( \text{self}.\text{parent}() \).\domain()
- the codomain is in \( \text{self}.\text{parent}() \).\domain()

check()
    Performs checks on self

    Check that \( \text{self} \) is a proper function and then calls \( \text{parent}.\text{check_element}(\text{self}) \) where
    \( \text{parent} \) is the parent of \( \text{self} \).

codomain()
    Returns the codomain of \( \text{self} \)
```

EXAMPLES:
sage: FiniteSetMaps(4, 3)([1, 0, 2, 1]).codomain()
{0, 1, 2}

domain()
Returns the domain of self

EXAMPLES:

sage: FiniteSetMaps(4, 3)([1, 0, 2, 1]).domain()
{0, 1, 2, 3}

fibers()
Returns the fibers of self

OUTPUT:

a dictionary \( d \) such that \( d[y] \) is the set of all \( x \) in \( \text{domain} \) such that \( f(x) = y \)

EXAMPLES:

sage: FiniteSetMaps(4, 3)([1, 0, 2, 1]).fibers()
{0: {1}, 1: {0, 3}, 2: {2}}
sage: F = FiniteSetMaps(\{'a', 'b', 'c'\})
sage: F.from_dict({'a': 'b', 'b': 'a', 'c': 'b'}).fibers() == {'a': {\'b\'}, 'b →\': {\'a\', \'c\'}}
True

getimage(i)
Returns the image of \( i \) by \( self \)

INPUT:

- \( i \) – any object.

Note: if you need speed, please use instead \_getimage()

EXAMPLES:

sage: fs = FiniteSetMaps(4, 3)([1, 0, 2, 1])
sage: fs.getimage(0), fs.getimage(1), fs.getimage(2), fs.getimage(3)
(1, 0, 2, 1)

image_set()
Returns the image set of \( self \)

EXAMPLES:

sage: FiniteSetMaps(4, 3)([1, 0, 2, 1]).image_set()
{0, 1, 2}
sage: FiniteSetMaps(4, 3)([1, 0, 0, 1]).image_set()
{0, 1}

items()
The items of \( self \)

Return the list of the ordered pairs \( (x, \text{self}(x)) \)

EXAMPLES:
```python
sage: FiniteSetMaps(4, 3)([1, 0, 2, 1]).items()
[(0, 1), (1, 0), (2, 2), (3, 1)]
```

**setimage** *(i, j)*

Set the image of *i* as *j* in *self*

**Warning:** *self* must be mutable; otherwise an exception is raised.

**INPUT:**

- *i, j* – two object's

**OUTPUT:** None

**Note:** if you need speed, please use instead `_setimage()`

**EXAMPLES:**

```python
sage: fs = FiniteSetMaps(4, 3)([1, 0, 2, 1])
sage: fs2 = copy(fs)
sage: fs2.setimage(2, 1)
sage: fs2
[1, 0, 1, 1]
sage: with fs.clone() as fs3:
    ....:     fs3.setimage(0, 2)
    ....:     fs3.setimage(1, 2)
sage: fs3
[2, 2, 2, 1]
```

**class** `sage.sets.finite_set_map_cy.FiniteSetMap_Set`

**Bases:** `sage.sets.finite_set_map_cy.FiniteSetMap_MN`

Data structure for maps

We assume that the parent given as argument is such that:

- the domain is in `parent.domain()`
- the codomain is in `parent.codomain()`
- `parent._m` contains the cardinality of the domain
- `parent._n` contains the cardinality of the codomain
- `parent._unrank_domain` and `parent._rank_domain` is a pair of reciprocal rank and unrank functions between the domain and range (`parent._m`).
- `parent._unrank_codomain` and `parent._rank_codomain` is a pair of reciprocal rank and unrank functions between the codomain and range (`parent._n`).

**classmethod** `from_dict` *(t, parent, d)*

Creates a `FiniteSetMap` from a dictionary

**Warning:** no check is performed!
classmethod `from_list`(t, parent, lst)

Creates a FiniteSetMap from a list

**Warning:** no check is performed!

`getimage`(i)

Returns the image of `i` by `self`

**INPUT:**

- `i` – an int

**EXAMPLES:**

```python
sage: F = FiniteSetMaps(['a', 'b', 'c', 'd'], ['u', 'v', 'w'])
sage: fs = F._from_list_([1, 0, 2, 1])
sage: list(map(fs.getimage, ['a', 'b', 'c', 'd']))
['v', 'u', 'w', 'v']
```

`image_set`()

Returns the image set of `self`

**EXAMPLES:**

```python
sage: F = FiniteSetMaps(['a', 'b', 'c'])
sage: sorted(F.from_dict({'a': 'b', 'b': 'a', 'c': 'b'}).image_set())
['a', 'b']
sage: F = FiniteSetMaps(['a', 'b', 'c'])
sage: F(lambda x: 'c').image_set()
{'c'}
```

`items`()

The items of `self`

Return the list of the couple `(x, self(x))`

**EXAMPLES:**

```python
sage: F = FiniteSetMaps(['a', 'b', 'c'])
sage: F.from_dict({'a': 'b', 'b': 'a', 'c': 'b'}).items()
[('a', 'b'), ('b', 'a'), ('c', 'b')]
```

`setimage`(i, j)

Set the image of `i` as `j` in `self`

**Warning:** `self` must be mutable otherwise an exception is raised.

**INPUT:**

- `i, j` – two object's

**OUTPUT:** None

**EXAMPLES:**
sage: F = FiniteSetMaps(["a", "b", "c", "d"], ["u", "v", "w"])  
sage: fs = F(lambda x: "v")  
sage: fs2 = copy(fs)  
sage: fs2.setimage("a", "w")  
sage: fs2  
map: a -> w, b -> v, c -> v, d -> v

sage: with fs.clone() as fs3:  
.....: fs3.setimage("a", "u")  
.....: fs3.setimage("c", "w")  
sage: fs3  
map: a -> u, b -> v, c -> w, d -> v

sage.sets.finite_set_map_cy.FiniteSetMap_Set_from_dict (t, parent, d)  
Creates a FiniteSetMap from a dictionary

**Warning:** no check is performed!

sage.sets.finite_set_map_cy.FiniteSetMap_Set_from_list (t, parent, lst)  
Creates a FiniteSetMap from a list

**Warning:** no check is performed!

sage.sets.finite_set_map_cy.fibers (f, domain)  
Returns the fibers of the function f on the finite set domain

**INPUT:**

- f – a function or callable
- domain – a finite iterable

**OUTPUT:**

- a dictionary d such that d[y] is the set of all x in domain such that f(x) = y

**EXAMPLES:**

```python
sage: from sage.sets.finite_set_map_cy import fibers, fibers_args
sage: fibers(lambda x: 1, [])
[]
sage: fibers(lambda x: x^2, [-1, 2, -3, 1, 3, 4])

{1: [1, -1], 4: {2}, 9: {3, -3}, 16: {4}}
sage: fibers(lambda x: 1, [-1, 2, -3, 1, 3, 4])

{1: [1, 2, 3, 4, -3, -1]}
sage: fibers(lambda x: 1, [1,1,1])

{1: [1]}
```

**See also:**

*fibers_args()* if one needs to pass extra arguments to f.

sage.sets.finite_set_map_cy.fibers_args (f, domain, *args, **opts)  
Returns the fibers of the function f on the finite set domain

It is the same as *fibers()* except that one can pass extra argument for f (with a small overhead)

**EXAMPLES:**
1.11 Totally Ordered Finite Sets

AUTHORS:

- Stepan Starosta (2012): Initial version

```python
class sage.sets.totally_ordered_finite_set.TotallyOrderedFiniteSet(elements, facade=True):
    pass
```

Totally ordered finite set.

This is a finite enumerated set assuming that the elements are ordered based upon their rank (i.e. their position in the set).

INPUT:

- `elements` – A list of elements in the set
- `facade` – (default: True) if True, a facade is used; it should be set to False if the elements do not inherit from Element or if you want a funny order. See examples for more details.

See also:

`FiniteEnumeratedSet`

EXAMPLES:

```python
sage: S = TotallyOrderedFiniteSet([1,2,3])
sage: S
{1, 2, 3}
sage: S.cardinality()
3
```

By default, totally ordered finite set behaves as a facade:

```python
sage: S(1).parent()
Integer Ring
```

It makes comparison fails when it is not the standard order:

```python
sage: T1 = TotallyOrderedFiniteSet([3,2,5,1])
sage: T1(3) < T1(1)
False
sage: T2 = TotallyOrderedFiniteSet([3,var('x')])
sage: T2(3) < T2(var('x'))
3 < x
```

To make the above example work, you should set the argument facade to False in the constructor. In that case, the elements of the set have a dedicated class:
```python
sage: A = TotallyOrderedFiniteSet([3,2,0,'a',7,(0,0),1], facade=False)
sage: A
(3, 2, 0, 'a', 7, (0, 0), 1)
sage: x = A.an_element()
sage: x
3
sage: x.parent()
(3, 2, 0, 'a', 7, (0, 0), 1)
sage: A(3) < A(2)
True
sage: A('a') < A(7)
True
sage: A(3) > A(2)
False
sage: A(1) < A(3)
False
sage: A(3) == A(3)
True

But then, the equality comparison is always False with elements outside of the set:

```python
sage: A(1) == 1
False
sage: 1 == A(1)
False
sage: 'a' == A('a')
False
sage: A('a') == 'a'
False
```

Since trac ticket #16280, totally ordered sets support elements that do not inherit from `sage.structure.element.Element`, whether they are facade or not:

```python
sage: S = TotallyOrderedFiniteSet(['a','b'])
sage: S('a')
'a'
sage: S = TotallyOrderedFiniteSet(['a','b'], facade = False)
sage: S('a')
'a'
```

Multiple elements are automatically deleted:

```python
sage: TotallyOrderedFiniteSet([1,1,2,1,2,2,5,4])
{1, 2, 5, 4}
```

**Element**

alias of `TotallyOrderedFiniteSetElement`

```python
le (x, y)
Return True if \( x \leq y \) for the order of self.
```
class sage.sets.totally_ordered_finite_set.TotallyOrderedFiniteSetElement (parent, data)

Element of a finite totally ordered set.

EXAMPLES:

```python
tsage: S = TotallyOrderedFiniteSet([2,7], facade=False)
tsage: x = S(2)
tsage: print(x)
2
tsage: x.parent()
{2, 7}
```

1.12 Set of all objects of a given Python class

sage.sets.pythonclass.Set_PythonType(typ)

Return the (unique) Parent that represents the set of Python objects of a specified type.

EXAMPLES:

```python
tsage: from sage.sets.pythonclass import Set_PythonType
tsage: Set_PythonType(list)
Set of Python objects of class 'list'
tsage: Set_PythonType(list) is Set_PythonType(list)
True
tsage: S = Set_PythonType(tuple)
tsage: S([1,2,3])
(1, 2, 3)
```

S is a parent which models the set of all lists:

```python
tsage: S.category()
Category of sets
```

class sage.sets.pythonclass.Set_PythonType_class

Bases: sage.structure.parent.Set_generic

The set of Python objects of a given class.

The elements of this set are not instances of Element; they are instances of the given class.

INPUT:

- typ – a Python (new-style) class

EXAMPLES:

```python
tsage: from sage.sets.pythonclass import Set_PythonType
tsage: S = Set_PythonType(int); S
Set of Python objects of class 'int'
tsage: int('1') in S
True
```
sage: Integer('1') in S
False

sage: Set_PythonType(2)
Traceback (most recent call last):
  ...
TypeError: must be initialized with a class, not 2

cardinality()

EXAMPLES:

sage: from sage.sets.pythonclass import Set_PythonType
sage: S = Set_PythonType(bool)
sage: S.cardinality()
2
sage: S = Set_PythonType(int)
sage: S.cardinality()
+Infinity

object()

EXAMPLES:

sage: from sage.sets.pythonclass import Set_PythonType
sage: Set_PythonType(tuple).object()
<... 'tuple'>
2.1 Integer Range

AUTHORS:

- Florent Hivert (2010-03): Added a class factory + cardinality method.
- Vincent Delecroix (2012-02): add methods rank/unrank, make it compliant with Python int.

class sage.sets.integer_range.IntegerRange
Bases:  sage.structure.unique_representation.UniqueRepresentation,  sage.
structure.parent.Parent

The class of Integer ranges

Returns an enumerated set containing an arithmetic progression of integers.

INPUT:

- begin – an integer, Infinity or -Infinity
- end – an integer, Infinity or -Infinity
- step – a non zero integer (default to 1)
- middle_point – an integer inside the set (default to None)

OUTPUT:

A parent in the category FiniteEnumeratedSets() or InfiniteEnumeratedSets() depending on the arguments defining self.

IntegerRange(i, j) returns the set of {i, i + 1, i + 2, ..., j - 1}. start (!) defaults to 0. When step is given, it specifies the increment. The default increment is 1. IntegerRange allows begin and end to be infinite.

IntegerRange is designed to have similar interface Python range. However, whereas range accept and returns Python int, IntegerRange deals with Integer.

If middle_point is given, then the elements are generated starting from it, in a alternating way: \{m, m + 1, m - 2, m + 2, m - 2\}.

EXAMPLES:

```sage```
list(IntegerRange(5))
[0, 1, 2, 3, 4]
```
sage: list(IntegerRange(2, 5))
```
When \( \text{begin} \) and \( \text{end} \) are both finite, \( \text{IntegerRange}(\text{begin}, \text{end}, \text{step}) \) is the set whose list of elements is equivalent to the python construction \( \text{range}(\text{begin}, \text{end}, \text{step}) \):

\[
\text{sage: } \text{list}(\text{IntegerRange}(4,105,3)) == \text{list}(\text{range}(4,105,3))
\]

\[
\text{True}
\]

\[
\text{sage: } \text{list}(\text{IntegerRange}(-54,13,12)) == \text{list}(\text{range}(-54,13,12))
\]

\[
\text{True}
\]

Except for the type of the numbers:

\[
\text{sage: } \text{type}(\text{IntegerRange}(-54,13,12)[0]), \text{type}(\text{list}(\text{range}(-54,13,12))[0])
\]

\[
(\text{<... 'sage.rings.integer.Integer'>}, \text{<... 'int'>})
\]

When \( \text{begin} \) is finite and \( \text{end} \) is \(+\text{Infinity}\), \( \text{self} \) is the infinite arithmetic progression starting from the \( \text{begin} \) by \( \text{step} \):
It is also possible to use the argument `middle_point` for other cases, finite or infinite. The set will be the same as if you didn’t give this extra argument but the enumeration will begin with this `middle_point`:

```
sage: I = IntegerRange(123, -12, -14); I
{123, 109, ..., -3}
sage: list(I)
[123, 109, 95, 81, 67, 53, 39, 25, 11, -3]
sage: J = IntegerRange(123, -12, -14, 25); J
Integer progression containing 25 with increment -14 and bounded with 123 and -12
sage: list(J)
[25, 11, 39, -3, 53, 67, 81, 95, 109, 123]
```

Remember that, like for `range`, if you define a non-empty set, `begin` is supposed to be included and `end` is supposed to be excluded. In the same way, when you define a set with a `middle_point`, the `begin` bound will be supposed to be included and the `end` bound supposed to be excluded:

```
sage: I = IntegerRange(-100, 100, 10, 0)
sage: J = list(range(-100, 100, 10))
sage: 100 in I
False
sage: 100 in J
False
sage: -100 in I
True
sage: -100 in J
True
sage: list(I)
[0, 10, -10, 20, -20, 30, -30, 40, -40, 50, -50, 60, -60, 70, -70, 80, -80, 90, -90, -100]
```

**Note:** The input is normalized so that:

```
sage: IntegerRange(1, 6, 2) is IntegerRange(1, 7, 2)
True
sage: IntegerRange(1, 8, 3) is IntegerRange(1, 10, 3)
True
```

**element_class**

alias of `sage.rings.integer.Integer`

**class sage.sets.integer_range.IntegerRangeEmpty**(elements)

Bases: `sage.sets.integer_range.IntegerRange`, `sage.sets.finite_enumerated_set.FiniteEnumeratedSet`

A singleton class for empty integer ranges

See `IntegerRange` for more details.

**class sage.sets.integer_range.IntegerRangeFinite**(begin, end, step=1)

Bases: `sage.sets.integer_range.IntegerRange`

The class of finite enumerated sets of integers defined by finite arithmetic progressions

See `IntegerRange` for more details.

### 2.1. Integer Range

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cardinality()
Return the cardinality of self

EXAMPLES:

```
sage: IntegerRange(123,12,-4).cardinality()
28
sage: IntegerRange(-57,12,8).cardinality()
9
sage: IntegerRange(123,12,4).cardinality()
0
```

rank(x)
EXAMPLES:

```
sage: I = IntegerRange(-57,36,8)
sage: I.rank(23)
10
sage: I.unrank(10)
23
sage: I.rank(22)
Traceback (most recent call last):
...  
IndexError: 22 not in self
sage: I.rank(87)
Traceback (most recent call last):
...  
IndexError: 87 not in self
```

unrank(i)
Return the i-th element of this integer range.

EXAMPLES:

```
sage: I = IntegerRange(1,13,5)
sage: I[0], I[1], I[2]
(1, 6, 11)
sage: I[3]
Traceback (most recent call last):
...  
IndexError: out of range
sage: I[-1]
11
sage: I[-4]
Traceback (most recent call last):
...  
IndexError: out of range
```

class sage.sets.integer_range.IntegerRangeFromMiddle
```
Bases: sage.sets.integer_range.IntegerRange
```
The class of finite or infinite enumerated sets defined with an inside point, a progression and two limits.

See `IntegerRange` for more details.

**next** *(elt)*

Return the next element of `elt` in `self`.

**EXAMPLES:**

```python
sage: from sage.sets.integer_range import IntegerRangeFromMiddle
sage: I = IntegerRangeFromMiddle(-100,100,10,0)
(10, -10, 20, -20, None)
sage: I = IntegerRangeFromMiddle(-Infinity,Infinity,10,0)
(10, -10, 20, -20, 110)
sage: I.next(1)
Traceback (most recent call last):
  ... LookupError: 1 not in Integer progression containing 0 with increment 10 and
  \text{bounded with -Infinity and +Infinity}
```

class `sage.sets.integer_range.IntegerRangeInfinite` *(begin, step=1)*

Bases: `sage.sets.integer_range.IntegerRange`

The class of infinite enumerated sets of integers defined by infinite arithmetic progressions.

See `IntegerRange` for more details.

**rank** *(x)*

EXAMPLES:

```python
sage: I = IntegerRange(-57,Infinity,8)
10
sage: I.rank(23)
10
sage: I.unrank(10)
23
sage: I.rank(22)
Traceback (most recent call last):
  ... IndexError: 22 not in self
```

**unrank** *(i)*

Returns the `i`-th element of `self`.

**EXAMPLES:**

```python
sage: I = IntegerRange(-8,Infinity,3)
sage: I.unrank(1)
-5
```
2.2 Positive Integers

class sage.sets.positive_integers.PositiveIntegers
    Bases: sage.sets.integer_range.IntegerRangeInfinite

The enumerated set of positive integers. To fix the ideas, we mean \( \{1, 2, 3, 4, 5, \ldots \} \).
This class implements the set of positive integers, as an enumerated set (see InfiniteEnumeratedSets).
This set is an integer range set. The construction is therefore done by IntegerRange (see IntegerRange).

EXAMPLES:

```
sage: PP = PositiveIntegers()
sage: PP
Positive integers
sage: PP.cardinality()
+Infinity
sage: TestSuite(PP).run()
sage: PP.list()
Traceback (most recent call last):
  ...
NotImplementedError: cannot list an infinite set
sage: it = iter(PP)
sage: (next(it), next(it), next(it), next(it), next(it))
(1, 2, 3, 4, 5)
sage: PP.first()
1
```

an_element()
Returns an element of self.

EXAMPLES:

```
sage: PositiveIntegers().an_element()
42
```

2.3 Non Negative Integers

class sage.sets.non_negative_integers.NonNegativeIntegers(category=None)
    Bases: sage.structure.unique_representation.UniqueRepresentation, sage.
           structure.parent.Parent

The enumerated set of non negative integers.
This class implements the set of non negative integers, as an enumerated set (see InfiniteEnumeratedSets).

EXAMPLES:

```
sage: NN = NonNegativeIntegers()
sage: NN
Non negative integers
sage: NN.cardinality()
+Infinity
sage: TestSuite(NN).run()
sage: NN.list()
```
Currently, this is just a “facade” parent; namely its elements are plain Sage Integers with Integer Ring as parent:

```sage```
x = NN(15); type(x)  
<... 'sage.rings.integer.Integer'>
sage: x.parent()  
Integer Ring
sage: x+3  
18
```

In a later version, there will be an option to specify whether the elements should have Integer Ring or Non negative integers as parent:

```sage```
NN = NonNegativeIntegers(facade = False)  # todo: not implemented
x = NN(5)  # todo: not implemented
sage: x.parent()  # todo: not implemented
Non negative integers
```

This runs generic sanity checks on NN:

```sage```
TestSuite(NN).run()
```

TODO: do not use NN any more in the doctests for NonNegativeIntegers.

**Element**

alias of `sage.rings.integer.Integer`

**an_element()**

EXAMPLES:

```sage```
NonNegativeIntegers().an_element()
42
```

**from_integer**

alias of `sage.rings.integer.Integer`

**next(o)**

EXAMPLES:

```sage```
NN = NonNegativeIntegers()
sage: NN.next(3)
4
```

**some_elements()**

EXAMPLES:
```python
sage: NonNegativeIntegers().some_elements()
[0, 1, 3, 42]
```

```
unrank(rnk)
EXAMPLES:

sage: NN = NonNegativeIntegers()
sage: NN.unrank(100)
100
```

### 2.4 The set of prime numbers

**AUTHORS:**
- William Stein (2005): original version

```python
class sage.sets.primes.Primes(proof)
```

Bases: `sage.structure.parent.Set_generic`, `sage.structure.unique_representation.UniqueRepresentation`

The set of prime numbers.

**EXAMPLES:**

```python
sage: P = Primes(); P
Set of all prime numbers: 2, 3, 5, 7, ...
```

We show various operations on the set of prime numbers:

```python
sage: P.cardinality()
+Infinity
sage: R = Primes()
sage: P == R
True
sage: 5 in P
True
sage: 100 in P
False
sage: len(P)
Traceback (most recent call last):
  ...
NotImplementedError: infinite set
```

**first()**

Return the first prime number.

**EXAMPLES:**

```python
sage: P = Primes()
sage: P.first()
2
```

**next**(pr)

Return the next prime number.
EXAMPLES:

```
sage: P = Primes()
sage: P.next(5)
7
```

**unrank**\( (n) \)
Return the n-th prime number.

EXAMPLES:

```
sage: P = Primes()
sage: P.unrank(0)
2
sage: P.unrank(5)
13
sage: P.unrank(42)
191
```

### 2.5 Subsets of the Real Line

This module contains subsets of the real line that can be constructed as the union of a finite set of open and closed intervals.

EXAMPLES:

```
sage: RealSet(0,1)
(0, 1)
sage: RealSet((0,1), [2,3])
(0, 1) + [2, 3]
sage: RealSet(-oo, oo)
(-oo, +oo)
```

Brackets must be balanced in Python, so the naive notation for half-open intervals does not work:

```
sage: RealSet([0,1))
Traceback (most recent call last):
...
SyntaxError: ...
```

Instead, you can use the following construction functions:

```
sage: RealSet.open_closed(0,1)
(0, 1]
sage: RealSet.closed_open(0,1)
[0, 1)
sage: RealSet.point(1/2)
[1/2]
sage: RealSet.unbounded_below_open(0)
(-oo, 0)
sage: RealSet.unbounded_below_closed(0)
[-oo, 0]
sage: RealSet.unbounded_above_open(1)
(1, +oo)
sage: RealSet.unbounded_above_closed(1)
[1, +oo)
```
Relations containing symbols and numeric values or constants:

```
sage: RealSet(x != 0)
(-oo, 0) + (0, +oo)
sage: RealSet(x == pi)
{pi}
sage: RealSet(x < 1/2)
(-oo, 1/2)
sage: RealSet(1/2 < x)
(1/2, +oo)
sage: RealSet(1.5 <= x)
[1.50000000000000, +oo)
```

Note that multiple arguments are combined as union:

```
sage: RealSet(x >= 0, x < 1)
(-oo, +oo)
sage: RealSet(x >= 0, x > 1)
[0, +oo)
sage: RealSet(x >= 0, x > -1)
(-1, +oo)
```

AUTHORS:

- Laurent Claessens (2010-12-10): Interval and ContinuousSet, posted to sage-devel at http://www.mail-archive.com/sage-support@googlegroups.com/msg21326.html.
- Ares Ribo (2011-10-24): Extended the previous work defining the class RealSet.
- Jordi Saludes (2011-12-10): Documentation and file reorganization.
- Volker Braun (2013-06-22): Rewrite

```python
class sage.sets.real_set.InternalRealInterval(lower, lower_closed, upper, upper_closed, check=True)
Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent
```

A real interval.

You are not supposed to create `RealInterval` objects yourself. Always use `RealSet` instead.

**INPUT:**

- `lower` – real or minus infinity; the lower bound of the interval.
- `lower_closed` – boolean; whether the interval is closed at the lower bound
- `upper` – real or (plus) infinity; the upper bound of the interval
- `upper_closed` – boolean; whether the interval is closed at the upper bound
- `check` – boolean; whether to check the other arguments for validity

**closure()**

Return the closure

**OUTPUT:**

The closure as a new `RealInterval`
contains \( x \)

Return whether \( x \) is contained in the interval

INPUT:

- \( x \) – a real number.

OUTPUT:

Boolean.

EXAMPLES:

```python
sage: i = RealSet.open_closed(0, 2)[0]; i
(0, 2]
sage: i.contains(0)
False
sage: i.contains(1)
True
sage: i.contains(2)
True
```

convex_hull \( \text{other} \)

Return the convex hull of the two intervals

OUTPUT:

The convex hull as a new RealInterval.

EXAMPLES:

```python
sage: I1 = RealSet.open(0, 1)[0]; I1
(0, 1)
sage: I2 = RealSet.closed(1, 2)[0]; I2
[1, 2]
sage: I1.convex_hull(I2)
(0, 2]
sage: I2.convex_hull(I1)
(0, 2]
sage: I1.convex_hull(I2.interior())
(0, 2)
sage: I1.closure().convex_hull(I2.interior())
[0, 2]
sage: I1.closure().convex_hull(I2)
[0, 2]
sage: I3 = RealSet.closed(1/2, 3/2)[0]; I3
[1/2, 3/2]
sage: I1.convex_hull(I3)
(0, 3/2]
```

```python
```

**element_class**

alias of `sage.rings.real_lazy.LazyFieldElement`
interior()

Return the interior

OUTPUT:

The interior as a new RealInterval

EXAMPLES:

```
sage: RealSet.closed(0, 1)[0].interior()
(0, 1)
sage: RealSet.open_closed(-oo, 1)[0].interior()
(-oo, 1)
sage: RealSet.closed_open(0, oo)[0].interior()
(0, +oo)
```

intersection(other)

Return the intersection of the two intervals

INPUT:

- other — a RealInterval

OUTPUT:

The intersection as a new RealInterval

EXAMPLES:

```
sage: I1 = RealSet.open(0, 2)[0]; I1
(0, 2)
sage: I2 = RealSet.closed(1, 3)[0]; I2
[1, 3]
sage: I1.intersection(I2)
[1, 2)
sage: I2.intersection(I1)
[1, 2)
sage: I1.closure().intersection(I2.interior())
(1, 2]
sage: I2.interior().intersection(I1.closure())
(1, 2]
sage: I3 = RealSet.closed(10, 11)[0]; I3
[10, 11]
sage: I1.intersection(I3)
(0, 0)
sage: I3.intersection(I1)
(0, 0)
```

is_connected(other)

Test whether two intervals are connected

OUTPUT:

Boolean. Whether the set-theoretic union of the two intervals has a single connected component.

EXAMPLES:

```
sage: I1 = RealSet.open(0, 1)[0]; I1
(0, 1)
sage: I2 = RealSet.closed(1, 2)[0]; I2
[1, 2]
```
is_empty()

Return whether the interval is empty

The normalized form of RealSet has all intervals non-empty, so this method usually returns False.

OUTPUT:

Boolean.

EXAMPLES:

```sage
sage: I = RealSet(0, 1)[0]
sage: I.is_empty()
False
```

is_point()

Return whether the interval consists of a single point

OUTPUT:

Boolean.

EXAMPLES:

```sage
sage: I = RealSet(0, 1)[0]
sage: I.is_point()
False
```

lower()

Return the lower bound

OUTPUT:

The lower bound as it was originally specified.

EXAMPLES:

```sage
sage: I = RealSet(0, 1)[0]
sage: I.lower()
0
```
lower_closed()  
Return whether the interval is open at the lower bound  

OUTPUT:  
Boolean.  

EXAMPLES:  

```
sage: I = RealSet.open_closed(0, 1)[0]; I  
(0, 1]
sage: I.lower_closed()  
False  
sage: I.lower_open()  
True  
sage: I.upper_closed()  
True  
sage: I.upper_open()  
False  
```

lower_open()  
Return whether the interval is closed at the upper bound  

OUTPUT:  
Boolean.  

EXAMPLES:  

```
sage: I = RealSet.open_closed(0, 1)[0]; I  
(0, 1]
sage: I.lower_closed()  
False  
sage: I.lower_open()  
True  
sage: I.upper_closed()  
True  
sage: I.upper_open()  
False  
```

upper()  
Return the upper bound  

OUTPUT:  
The upper bound as it was originally specified.  

EXAMPLES:  

```
sage: I = RealSet(0, 1)[0]  
sage: I.lower()  
0  
sage: I.upper()  
1  
```

upper_closed()  
Return whether the interval is closed at the lower bound
OUTPUT:
Boolean.

EXAMPLES:

```
sage: I = RealSet.open_closed(0, 1)[0]; I
(0, 1]
sage: I.lower_closed()
False
sage: I.lower_open()
True
sage: I.upper_closed()
True
sage: I.upper_open()
False
```

**upper_open()**
Return whether the interval is closed at the upper bound

OUTPUT:
Boolean.

EXAMPLES:

```
sage: I = RealSet.open_closed(0, 1)[0]; I
(0, 1]
sage: I.lower_closed()
False
sage: I.lower_open()
True
sage: I.upper_closed()
True
sage: I.upper_open()
False
```

**class** `sage.sets.real_set.RealSet(*intervals)`

**Bases:** `sage.structure.unique_representation.UniqueRepresentation`, `sage.structure.parent.Parent`

A subset of the real line

**INPUT:**
Arguments defining a real set. Possibilities are either two real numbers to construct an open set or a list/tuple/iterable of intervals. The individual intervals can be specified by either a `RealInterval`, a tuple of two real numbers (constructing an open interval), or a list of two number (constructing a closed interval).

**EXAMPLES:**

```
sage: RealSet(0,1)  # open set from two numbers
(0, 1)
sage: i = RealSet(0,1)[0]
sage: RealSet(i)  # interval
(0, 1)
sage: RealSet(i, (3,4))  # tuple of two numbers = open set
(0, 1) + (3, 4)
sage: RealSet(i, [3,4])  # list of two numbers = closed set
(0, 1) + [3, 4]
```
\textbf{an\_element} ()
\begin{itemize}
\item Return a point of the set
\end{itemize}
\textbf{OUTPUT:}
\begin{itemize}
\item A real number. \texttt{ValueError} if the set is empty.
\end{itemize}
\textbf{EXAMPLES:}
\begin{verbatim}
sage: RealSet.open_closed(0, 1).an_element()
sage: RealSet(0, 1).an_element()
sage: RealSet(-oo, +oo).an_element()
sage: RealSet(-oo, 7).an_element()
sage: RealSet(7, +oo).an_element()
\end{verbatim}

\textbf{static are\_pairwise\_disjoint} (*real\_set\_collection)
\begin{itemize}
\item Test whether sets are pairwise disjoint
\end{itemize}
\textbf{INPUT:}
\begin{itemize}
\item *real\_set\_collection – a list/tuple/iterable of \texttt{RealSet}.
\end{itemize}
\textbf{OUTPUT:}
\begin{itemize}
\item Boolean.
\end{itemize}
\textbf{EXAMPLES:}
\begin{verbatim}
sage: s1 = RealSet((0, 1), (2, 3))
sage: s2 = RealSet((1, 2))
sage: s3 = RealSet.point(3)
sage: RealSet.are_pairwise_disjoint(s1, s2, s3)
sage: RealSet.are_pairwise_disjoint(s1, s2, s3, [10,10])
sage: RealSet.are_pairwise_disjoint(s1, s2, s3, [-1, 1/2])
\end{verbatim}

\textbf{cardinality} ()
\begin{itemize}
\item Return the cardinality of the subset of the real line.
\end{itemize}
\textbf{OUTPUT:}
\begin{itemize}
\item Integer or infinity. The size of a discrete set is the number of points; the size of a real interval is Infinity.
\end{itemize}
\textbf{EXAMPLES:}
\begin{verbatim}
sage: RealSet([0, 0], [1, 1], [3, 3]).cardinality()
sage: RealSet(0,3).cardinality()
\end{verbatim}

\textbf{static closed} (lower, upper)
\begin{itemize}
\item Construct a closed interval
\end{itemize}
\textbf{INPUT:}
\begin{itemize}
\item lower, upper – two real numbers or infinity. They will be sorted if necessary.
\end{itemize}
OUTPUT:
A new RealSet.

EXAMPLES:

```
sage: RealSet.closed(1, 0)
[0, 1]
```

**static closed_open**(lower, upper)
Construct an half-open interval

INPUT:

- lower, upper – two real numbers or infinity. They will be sorted if necessary.

OUTPUT:
A new RealSet that is closed at the lower bound and open at the upper bound.

EXAMPLES:

```
sage: RealSet.closed_open(1, 0)
[0, 1)
```

**complement**()
Return the complement

OUTPUT:
The set-theoretic complement as a new RealSet.

EXAMPLES:

```
sage: RealSet(0,1).complement()
(-oo, 0] + [1, +oo)
sage: s1 = RealSet(0,2) + RealSet.unbounded_above_closed(10); s1
(0, 2] + [10, +oo)
sage: s1.complement()
(-oo, 0] + [2, 10)
sage: s2 = RealSet(1,3) + RealSet.unbounded_below_closed(-10); s2
(-oo, -10] + (1, 3)
sage: s2.complement()
(-10, 1] + [3, +oo)
```

**contains**(x)
Return whether x is contained in the set

INPUT:

- x – a real number.

OUTPUT:
Boolean.

EXAMPLES:

```
sage: s = RealSet(0,2) + RealSet.unbounded_above_closed(10); s
(0, 2] + [10, +oo)
sage: s.contains(1)
```

(continues on next page)
difference(*other)

Return self with other subtracted

INPUT:

• other – a RealSet or data that defines one.

OUTPUT:

The set-theoretic difference of self with other removed as a new RealSet.

EXAMPLES:

```
sage: s1 = RealSet(0,2) + RealSet.unbounded_above_closed(10); s1
(0, 2) + [10, +oo)
sage: s2 = RealSet(1,3) + RealSet.unbounded_below_closed(-10); s2
(-oo, -10] + (1, 3)
sage: s1.difference(s2)
(0, 1] + [10, +oo)
sage: s1 - s2  # syntactic sugar
(0, 1] + [10, +oo)
sage: s2.difference(s1)
(-oo, -10] + [2, 3)
sage: s2 - s1  # syntactic sugar
(-oo, -10] + [2, 3)
sage: s1.difference(1,11)
(0, 1] + [11, +oo)
```

get_interval(i)

Return the i-th connected component.

Note that the intervals representing the real set are always normalized, see normalize().

INPUT:

• i – integer.

OUTPUT:

The i-th connected component as a RealInterval.

EXAMPLES:

```
sage: s = RealSet(RealSet.open_closed(0,1), RealSet.closed_open(2,3))
sage: s.get_interval(0)
(0, 1]
sage: s[0]  # shorthand
(0, 1]
sage: s.get_interval(1)
[2, 3)
sage: s[0] == s.get_interval(0)
True
```

inf()

Return the infimum
OUTPUT:
A real number or infinity.

EXAMPLES:

```python
sage: s1 = RealSet(0,2) + RealSet.unbounded_above_closed(10); s1
(0, 2) + [10, +oo)
sage: s1.inf()
0

sage: s2 = RealSet(1,3) + RealSet.unbounded_below_closed(-10); s2
(-oo, -10] + (1, 3)
sage: s2.inf()
-Infinity
```

**intersection** (*other*)

Return the intersection of the two sets

**INPUT:**

* other – a `RealSet` or data that defines one.

**OUTPUT:**

The set-theoretic intersection as a new `RealSet`.

**EXAMPLES:**

```python
sage: s1 = RealSet(0,2) + RealSet.unbounded_above_closed(10); s1
(0, 2) + [10, +oo)
sage: s2 = RealSet(1,3) + RealSet.unbounded_below_closed(-10); s2
(-oo, -10] + (1, 3)
sage: s1.intersection(s2)
(1, 2)
sage: s1 & s2  # syntactic sugar
(1, 2)

sage: s1 = RealSet((0, 1), (2, 3)); s1
(0, 1) + (2, 3)
sage: s2 = RealSet([0, 1], [2, 3]); s2
[0, 1] + [2, 3]
sage: s3 = RealSet([1, 2]); s3
[1, 2]
sage: s1.intersection(s2)
(0, 1) + (2, 3)
sage: s1.intersection(s3)
{}
sage: s2.intersection(s3)
{1} + {2}
```

**is_disjoint_from** (*other*)

Test whether the two sets are disjoint

**INPUT:**

* other – a `RealSet` or data defining one.

**OUTPUT:**

Boolean.

**EXAMPLES:**

2.5. Subsets of the Real Line
is_empty()

Return whether the set is empty

EXAMPLES:

```python
sage: RealSet(0, 1).is_empty()
False
sage: RealSet(0, 0).is_empty()
True
```

is_included_in(*other)

Tests interval inclusion

INPUT:

- *args – a RealSet or something that defines one.

OUTPUT:

Boolean.

EXAMPLES:

```python
sage: I = RealSet((1,2))
sage: J = RealSet((1,3))
sage: K = RealSet((2,3))
sage: I.is_included_in(J)
True
sage: J.is_included_in(K)
False
```

n_components()

Return the number of connected components

See also get_interval()

EXAMPLES:

```python
sage: s = RealSet(RealSet.open_closed(0,1), RealSet.closed_open(2,3))
sage: s.n_components()
2
```

static normalize(intervals)

Bring a collection of intervals into canonical form

INPUT:

- intervals – a list/tuple/iterable of intervals.

OUTPUT:

A tuple of intervals such that
• they are sorted in ascending order (by lower bound)
• there is a gap between each interval
• all intervals are non-empty

EXAMPLES:

```
sage: i1 = RealSet((0, 1))[0]
sage: i2 = RealSet([1, 2])[0]
sage: i3 = RealSet((2, 3))[0]
sage: RealSet.normalize([i1, i2, i3])
((0, 3),)
sage: RealSet((0, 1), [1, 2], (2, 3))
(0, 3)
sage: RealSet((0, 1), (1, 2), (2, 3))
(0, 1) + (1, 2) + (2, 3)
sage: RealSet([0, 1], [2, 3])
[0, 1] + [2, 3]
sage: RealSet((0, 2), (1, 3))
(0, 3)
sage: RealSet(0,0)
{
```

**static open** (*lower*, *upper*)

Construct an open interval

**INPUT:**

• *lower*, *upper* – two real numbers or infinity. They will be sorted if necessary.

**OUTPUT:**

A new *RealSet*.

**EXAMPLES:**

```
sage: RealSet.open(1, 0)
(0, 1)
```

**static open_closed** (*lower*, *upper*)

Construct a half-open interval

**INPUT:**

• *lower*, *upper* – two real numbers or infinity. They will be sorted if necessary.

**OUTPUT:**

A new *RealSet* that is open at the lower bound and closed at the upper bound.

**EXAMPLES:**

```
sage: RealSet.open_closed(1, 0)
(0, 1]
```

**static point** (*p*)

Construct an interval containing a single point

**INPUT:**

• *p* – a real number.
OUTPUT:
A new `RealSet`.

EXAMPLES:

```python
sage: RealSet.open(1, 0)
(0, 1)
```

`sage().sup()`
Return the supremum

OUTPUT:
A real number or infinity.

EXAMPLES:

```python
sage: s1 = RealSet(0,2) + RealSet.unbounded_above_closed(10); s1
(0, 2) + [10, +oo)
sage: s1.sup()
+Infinity
sage: s2 = RealSet(1,3) + RealSet.unbounded_below_closed(-10); s2
(-oo, -10] + (1, 3)
sage: s2.sup()
3
```

`static unbounded_above_closed(bound)`
Construct a semi-infinite interval

INPUT:
• bound – a real number.

OUTPUT:
A new `RealSet` from the bound (including) to plus infinity.

EXAMPLES:

```python
sage: RealSet.unbounded_above_closed(1)
[1, +oo)
```

`static unbounded_above_open(bound)`
Construct a semi-infinite interval

INPUT:
• bound – a real number.

OUTPUT:
A new `RealSet` from the bound (excluding) to plus infinity.

EXAMPLES:

```python
sage: RealSet.unbounded_above_open(1)
(1, +oo)
```

`static unbounded_below_closed(bound)`
Construct a semi-infinite interval

INPUT:
• bound – a real number.

OUTPUT:

A new RealSet from minus infinity to the bound (including).

EXAMPLES:

```sage
sage: RealSet.unbounded_below_closed(1)
(-oo, 1]
```

**static unbounded_below_open(bound)**

Construct a semi-infinite interval

INPUT:

• bound – a real number.

OUTPUT:

A new RealSet from minus infinity to the bound (excluding).

EXAMPLES:

```sage
sage: RealSet.unbounded_below_open(1)
(-oo, 1)
```

**union(*other)**

Return the union of the two sets

INPUT:

• other – a RealSet or data that defines one.

OUTPUT:

The set-theoretic union as a new RealSet.

EXAMPLES:

```sage
sage: s1 = RealSet(0,2)
sage: s2 = RealSet(1,3)
sage: s1.union(s2)
(0, 3)
sage: s1.union(1,3)
(0, 3)
sage: s1 | s2  # syntactic sugar
(0, 3)
sage: s1 + s2  # syntactic sugar
(0, 3)
```
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