Tensors on free modules of finite rank

Release 9.6

The Sage Development Team

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This work is part of the SageManifolds project but it does not depend upon other SageManifolds classes. In other words, it constitutes a self-consistent subset that can be used independently of SageManifolds.
FREE MODULES OF FINITE RANK

The class `FiniteRankFreeModule` implements free modules of finite rank over a commutative ring.

A free module of finite rank over a commutative ring $R$ is a module $M$ over $R$ that admits a finite basis, i.e. a finite family of linearly independent generators. Since $R$ is commutative, it has the invariant basis number property, so that the rank of the free module $M$ is defined uniquely, as the cardinality of any basis of $M$.

No distinguished basis of $M$ is assumed. On the contrary, many bases can be introduced on the free module along with change-of-basis rules (as module automorphisms). Each module element has then various representations over the various bases.

Note: The class `FiniteRankFreeModule` does not inherit from class `FreeModule_generic` nor from class `CombinatorialFreeModule`, since both classes deal with modules with a distinguished basis (see details below). Accordingly, the class `FiniteRankFreeModule` inherits directly from the generic class `Parent` with the category set to `Modules` (and not to `ModulesWithBasis`).

Todo:

• implement submodules
• create a FreeModules category (cf. the TODO statement in the documentation of `Modules: Implement a “FreeModules(R)” category, when so prompted by a concrete use case`)

AUTHORS:

• Eric Gourgoulhon, Michal Bejger (2014-2015): initial version
• Travis Scrimshaw (2016): category set to Modules(ring).FiniteDimensional() (trac ticket #20770)
• Michael Jung (2019): improve treatment of the zero element
• Eric Gourgoulhon (2021): unicode symbols for tensor and exterior products

REFERENCES:

• Chap. 10 of R. Godement: Algebra [God1968]
• Chap. 3 of S. Lang: Algebra [Lan2002]

EXAMPLES:

Let us define a free module of rank 2 over $\mathbb{Z}$:

```sage
M = FiniteRankFreeModule(ZZ, 2, name='M') ; M
Rank-2 free module M over the Integer Ring
```
Tensors on free modules of finite rank, Release 9.6

We introduce a first basis on $M$:

```python
sage: e = M.basis('e'); e
Basis (e_0, e_1) on the Rank-2 free module M over the Integer Ring
```

The elements of the basis are of course module elements:

```python
sage: e[0]
Element e_0 of the Rank-2 free module M over the Integer Ring
sage: e[1]
Element e_1 of the Rank-2 free module M over the Integer Ring
```

We define a module element by its components w.r.t. basis $e$:

```python
sage: u = M([2,-3], basis=e, name='u')
sage: u.display(e)
u = 2 e_0 - 3 e_1
```

Module elements can be also be created by arithmetic expressions:

```python
sage: v = -2*u + 4*e[0]; v
Element of the Rank-2 free module M over the Integer Ring
sage: v.display(e)
6 e_1
```

We define a second basis on $M$ from a family of linearly independent elements:

```python
sage: f = M.basis('f', from_family=(e[0]-e[1], -2*e[0]+3*e[1])); f
Basis (f_0, f_1) on the Rank-2 free module M over the Integer Ring
```

We may of course express the elements of basis $e$ in terms of basis $f$:

```python
sage: e[0].display(f)
e_0 = 3 f_0 + f_1
sage: e[1].display(f)
e_1 = 2 f_0 + f_1
```

as well as any module element:

```python
sage: u.display(f)
u = -f_1
sage: v.display(f)
12 f_0 + 6 f_1
```
The two bases are related by a module automorphism:

\[
\text{sage: } a = M.\text{change\_of\_basis}(e, f) \; ; \; a \\
\text{Automorphism of the Rank-2 free module } M \text{ over the Integer Ring} \\
\text{sage: } a.\text{parent()} \\
\text{General linear group of the Rank-2 free module } M \text{ over the Integer Ring} \\
\text{sage: } a.\text{matrix}(e) \\
\begin{bmatrix}
1 & -2 \\
-1 & 3
\end{bmatrix}
\]

Let us check that basis \( f \) is indeed the image of basis \( e \) by \( a \):

\[
\text{sage: } f[0] == a(e[0]) \\
\text{True} \\
\text{sage: } f[1] == a(e[1]) \\
\text{True}
\]

The reverse change of basis is of course the inverse automorphism:

\[
\text{sage: } M.\text{change\_of\_basis}(f, e) == a^{-1} \\
\text{True}
\]

We introduce a new module element via its components w.r.t. basis \( f \):

\[
\text{sage: } v = M([2, 4], \text{basis}=f, \text{name}='v') \\
\text{sage: } v.\text{display}(f) \\
v = 2 f_0 + 4 f_1
\]

The sum of the two module elements \( u \) and \( v \) can be performed even if they have been defined on different bases, thanks to the known relation between the two bases:

\[
\text{sage: } s = u + v \; ; \; s \\
\text{Element } u+v \text{ of the Rank-2 free module } M \text{ over the Integer Ring} \\
\text{sage: } s.\text{display}(e) \\
u+v = -4 e_0 + 7 e_1 \\
\text{sage: } s.\text{display}(f) \\
u+v = 2 f_0 + 3 f_1
\]

Tensor products of elements are implemented:

\[
\text{sage: } t = u*v \; ; \; t \\
\text{Type-(2,0) tensor } u \otimes v \text{ on the Rank-2 free module } M \text{ over the Integer Ring} \\
\text{sage: } t.\text{parent()} \\
\text{Free module of type-(2,0) tensors on the Rank-2 free module } M \text{ over the Integer Ring} \\
\text{sage: } t.\text{display}(e) \\
u \otimes v = -12 e_0 \otimes e_0 + 20 e_0 \otimes e_1 + 18 e_1 \otimes e_0 - 30 e_1 \otimes e_1 \\
\text{sage: } t.\text{display}(f) \\
u \otimes v = -2 f_1 \otimes f_0 - 4 f_1 \otimes f_1
\]

We can access to tensor components w.r.t. to a given basis via the square bracket operator:
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The parent of the automorphism \( a \) is the group \( \text{GL}(M) \), but \( a \) can also be considered as a tensor of type \((1, 1)\) on \( M \):

\[
\text{sage: } a.parent()
\]
General linear group of the Rank-2 free module \( M \) over the Integer Ring

\[
\text{sage: } a.tensor_type()
\]
\((1, 1)\)

\[
\text{sage: } a.display(e)
\]
e\_0 \otimes e^0 - 2 e\_0 \otimes e^1 - e\_1 \otimes e^0 + 3 e\_1 \otimes e^1

\[
\text{sage: } a.display(f)
\]
f\_0 \otimes f^0 - 2 f\_0 \otimes f^1 - f\_1 \otimes f^0 + 3 f\_1 \otimes f^1

As such, we can form its tensor product with \( t \), yielding a tensor of type \((3, 1)\):

\[
\text{sage: } t \ast a
\]
Type-(3,1) tensor on the Rank-2 free module \( M \) over the Integer Ring

\[
\text{sage: } (t \ast a).display(e)
\]
-12 e\_0 \otimes e\_0 \otimes e\_0 \otimes e^0 + 24 e\_0 \otimes e\_0 \otimes e\_0 \otimes e^1 + 12 e\_0 \otimes e\_0 \otimes e\_1 \otimes e^0
-36 e\_0 \otimes e\_0 \otimes e\_1 \otimes e^1 + 20 e\_0 \otimes e\_1 \otimes e\_0 \otimes e^0 - 40 e\_0 \otimes e\_1 \otimes e\_0 \otimes e^1
-20 e\_0 \otimes e\_1 \otimes e\_0 \otimes e^0 + 60 e\_0 \otimes e\_1 \otimes e\_1 \otimes e^1 + 18 e\_1 \otimes e\_0 \otimes e\_0 \otimes e^0
-36 e\_1 \otimes e\_0 \otimes e\_0 \otimes e^1 - 18 e\_1 \otimes e\_0 \otimes e\_1 \otimes e^0 + 54 e\_1 \otimes e\_0 \otimes e\_1 \otimes e^1
-30 e\_1 \otimes e\_1 \otimes e\_0 \otimes e^0 + 60 e\_1 \otimes e\_1 \otimes e\_0 \otimes e^1 + 30 e\_1 \otimes e\_1 \otimes e\_1 \otimes e^0
-90 e\_1 \otimes e\_1 \otimes e\_1 \otimes e^1

The parent of \( t \otimes a \) is itself a free module of finite rank over \( \mathbb{Z} \):

\[
\text{sage: } T = (t \ast a).parent() ; T
\]
Free module of type-(3,1) tensors on the Rank-2 free module \( M \) over the Integer Ring

\[
\text{sage: } T.base_ring()
\]
Integer Ring

\[
\text{sage: } T.rank()
\]
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Differences between \texttt{FiniteRankFreeModule} and \texttt{FreeModule} (or \texttt{VectorSpace})

To illustrate the differences, let us create two free modules of rank 3 over $\mathbb{Z}$, one with \texttt{FiniteRankFreeModule} and the other one with \texttt{FreeModule}:

\begin{verbatim}
sage: M = FiniteRankFreeModule(ZZ, 3, name='M') ; M
Rank-3 free module M over the Integer Ring
sage: N = FreeModule(ZZ, 3) ; N
Ambient free module of rank 3 over the principal ideal domain Integer Ring
\end{verbatim}

The main difference is that \texttt{FreeModule} returns a free module with a distinguished basis, while \texttt{FiniteRankFreeModule} does not:

\begin{verbatim}
sage: N.basis()
[(1, 0, 0), (0, 1, 0), (0, 0, 1)]
sage: M.bases()
[]
sage: M.print_bases()
No basis has been defined on the Rank-3 free module M over the Integer Ring
\end{verbatim}

This is also revealed by the category of each module:

\begin{verbatim}
sage: M.category()
Category of finite dimensional modules over Integer Ring
sage: N.category()
Category of finite dimensional modules with basis over (euclidean domains and infinite enumerated sets and metric spaces)
\end{verbatim}

In other words, the module created by \texttt{FreeModule} is actually $\mathbb{Z}^3$, while, in the absence of any distinguished basis, no canonical isomorphism relates the module created by \texttt{FiniteRankFreeModule} to $\mathbb{Z}^3$:

\begin{verbatim}
sage: N is ZZ^3
True
sage: M is ZZ^3
False
sage: M == ZZ^3
False
\end{verbatim}

Because it is $\mathbb{Z}^3$, $N$ is unique, while there may be various modules of the same rank over the same ring created by \texttt{FiniteRankFreeModule}: they are then distinguished by their names (actually by the complete sequence of arguments of \texttt{FiniteRankFreeModule}):

\begin{verbatim}
sage: N1 = FreeModule(ZZ, 3) ; N1
Ambient free module of rank 3 over the principal ideal domain Integer Ring
sage: N1 is N  # FreeModule(ZZ, 3) is unique
True
sage: M1 = FiniteRankFreeModule(ZZ, 3, name='M_1') ; M1
Rank-3 free module M_1 over the Integer Ring
sage: M1 is M  # M1 and M are different rank-3 modules over ZZ
False
\end{verbatim}

(continues on next page)
As illustrated above, various bases can be introduced on the module created by \texttt{FiniteRankFreeModule}:

\begin{verbatim}
sage: e = M.basis('e') ; e
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
sage: f = M.basis('f', from_family=(-e[0], e[1]-e[2], -2*e[1]+3*e[2])) ; f
Basis (f_0,f_1,f_2) on the Rank-3 free module M over the Integer Ring
sage: M.bases()
[Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring, Basis (f_0,f_1,f_2) on the Rank-3 free module M over the Integer Ring]
\end{verbatim}

Each element of a basis is accessible via its index:

\begin{verbatim}
sage: e[0]
Element e_0 of the Rank-3 free module M over the Integer Ring
sage: e[0].parent()
Rank-3 free module M over the Integer Ring
sage: f[1]
Element f_1 of the Rank-3 free module M over the Integer Ring
sage: f[1].parent()
Rank-3 free module M over the Integer Ring
\end{verbatim}

while on module \(N\), the element of the (unique) basis is accessible directly from the module symbol:

\begin{verbatim}
sage: N.0
(1, 0, 0)
sage: N.1
(0, 1, 0)
sage: N.0.parent()
Ambient free module of rank 3 over the principal ideal domain Integer Ring
\end{verbatim}

The arithmetic of elements is similar; the difference lies in the display: a basis has to be specified for elements of \(M\), while elements of \(N\) are displayed directly as elements of \(\mathbb{Z}^3\):

\begin{verbatim}
sage: u = 2*e[0] - 3*e[2] ; u
Element of the Rank-3 free module M over the Integer Ring
sage: u.display(e)
2 e_0 - 3 e_2
sage: u.display(f)
-2 f_0 - 6 f_1 - 3 f_2
sage: u[e,:]
[2, 0, -3]
sage: u[f,:]
[-2, -6, -3]
sage: v = 2*N.0 - 3*N.2 ; v
(2, 0, -3)
\end{verbatim}

For the case of \(M\), in order to avoid to specify the basis if the user is always working with the same basis (e.g. only one basis has been defined), the concept of default basis has been introduced:
sage: M.default_basis()
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
sage: M.print_bases()
Bases defined on the Rank-3 free module M over the Integer Ring:
  - (e_0,e_1,e_2) (default basis)
  - (f_0,f_1,f_2)

This is different from the distinguished basis of N: it simply means that the mention of the basis can be omitted in function arguments:

sage: u.display()  # equivalent to u.display(e)
2 e_0 - 3 e_2
sage: u[:]
# equivalent to u[e,:]
[2, 0, -3]

At any time, the default basis can be changed:

sage: M.set_default_basis(f)
sage: u.display()
-2 f_0 - 6 f_1 - 3 f_2

Another difference between FiniteRankFreeModule and FreeModule is that for the former the range of indices can be specified (by default, it starts from 0):

sage: M = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1) ; M
Rank-3 free module M over the Integer Ring
sage: e = M.basis('e') ; e  # compare with (e_0,e_1,e_2) above
Basis (e_1,e_2,e_3) on the Rank-3 free module M over the Integer Ring
sage: e[1], e[2], e[3]
(Element e_1 of the Rank-3 free module M over the Integer Ring,
 Element e_2 of the Rank-3 free module M over the Integer Ring,
 Element e_3 of the Rank-3 free module M over the Integer Ring)

All the above holds for VectorSpace instead of FreeModule: the object created by VectorSpace is actually a Cartesian power of the base field:

sage: V = VectorSpace(QQ,3) ; V
Vector space of dimension 3 over Rational Field
sage: V.category()
Category of finite dimensional vector spaces with basis
  over (number fields and quotient fields and metric spaces)
sage: V is QQ^3
True
sage: V.basis()
[ (1, 0, 0), (0, 1, 0), (0, 0, 1) ]

To create a vector space without any distinguished basis, one has to use FiniteRankFreeModule:

sage: V = FiniteRankFreeModule(QQ, 3, name='V') ; V
3-dimensional vector space V over the Rational Field
sage: V.category()
Category of finite dimensional vector spaces over Rational Field
sage: V.bases()
[]
sage: V.print_bases()
No basis has been defined on the 3-dimensional vector space V over the Rational Field

The class `FiniteRankFreeModule` has been created for the needs of the `SageManifolds` project, where free modules do not have any distinguished basis. Too kinds of free modules occur in the context of differentiable manifolds (see here for more details):

- the tangent vector space at any point of the manifold (cf. `TangentSpace`);
- the set of vector fields on a parallelizable open subset $U$ of the manifold, which is a free module over the algebra of scalar fields on $U$ (cf. `VectorFieldFreeModule`).

For instance, without any specific coordinate choice, no basis can be distinguished in a tangent space.

On the other side, the modules created by `FreeModule` have much more algebraic functionalities than those created by `FiniteRankFreeModule`. In particular, submodules have not been implemented yet in `FiniteRankFreeModule`. Moreover, modules resulting from `FreeModule` are tailored to the specific kind of their base ring:

- free module over a commutative ring that is not an integral domain ($\mathbb{Z}/6\mathbb{Z}$):

  ```python
  sage: R = IntegerModRing(6) ; R
  Ring of integers modulo 6
  sage: FreeModule(R, 3)
  Ambient free module of rank 3 over Ring of integers modulo 6
  sage: type(FreeModule(R, 3))
  <class 'sage.modules.free_module.FreeModule_ambient_with_category'>
  ```

- free module over an integral domain that is not principal ($\mathbb{Z}[X]$):

  ```python
  sage: R.<X> = ZZ[] ; R
  Univariate Polynomial Ring in X over Integer Ring
  sage: FreeModule(R, 3)
  Ambient free module of rank 3 over the integral domain Univariate Polynomial Ring in X over Integer Ring
  sage: type(FreeModule(R, 3))
  <class 'sage.modules.free_module.FreeModule_ambient_domain_with_category'>
  ```

- free module over a principal ideal domain ($\mathbb{Z}$):

  ```python
  sage: R = ZZ ; R
  Integer Ring
  sage: FreeModule(R,3)
  Ambient free module of rank 3 over the principal ideal domain Integer Ring
  sage: type(FreeModule(R, 3))
  <class 'sage.modules.free_module.FreeModule_ambient_pid_with_category'>
  ```

On the contrary, all objects constructed with `FiniteRankFreeModule` belong to the same class:

```python
sage: R = IntegerModRing(6)
sage: type(FiniteRankFreeModule(R, 3))
```
**Differences between `FiniteRankFreeModule` and `CombinatorialFreeModule`**

An alternative to construct free modules in Sage is `CombinatorialFreeModule`. However, as `FreeModule`, it leads to a module with a distinguished basis:

```
sage: N = CombinatorialFreeModule(ZZ, [1,2,3]) ; N
Free module generated by {1, 2, 3} over Integer Ring
sage: N.category()
Category of finite dimensional modules with basis over Integer Ring
```

The distinguished basis is returned by the method `basis()`:

```
sage: b = N.basis() ; b
Finite family {1: B[1], 2: B[2], 3: B[3]}
sage: b[1]
B[1]
sage: b[1].parent()
Free module generated by {1, 2, 3} over Integer Ring
```

For the free module $M$ created above with `FiniteRankFreeModule`, the method `basis` has at least one argument: the symbol string that specifies which basis is required:

```
sage: e = M.basis('e') ; e
Basis (e_1,e_2,e_3) on the Rank-3 free module M over the Integer Ring
sage: e[1]
Element e_1 of the Rank-3 free module M over the Integer Ring
sage: e[1].parent()
Rank-3 free module M over the Integer Ring
```

The arithmetic of elements is similar:

```
Element of the Rank-3 free module M over the Integer Ring
```

One notices that elements of $N$ are displayed directly in terms of their expansions on the distinguished basis. For elements of $M$, one has to use the method `display()` in order to specify the basis:

```
sage: u.display(e)
2 e_1 - 5 e_3
```

The components on the basis are returned by the square bracket operator for $M$ and by the method `coefficient` for $N$:  

```
sage: u[e]
2
```

```
sage: u.coefficient(e[1])
2
```

The arithmetic of elements is similar:
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sage: [u[e,i] for i in {1,2,3}]
[2, 0, -5]
sage: u[e,:]
# a shortcut for the above
[2, 0, -5]
sage: [v.coefficient(i) for i in {1,2,3}]
[2, 0, -5]

class sage.tensor.modules.finite_rank_free_module.FiniteRankFreeModule(ring, rank, name=None, latex_name=None, start_index=0, output_formatter=None, category=None):
    Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent
    Free module of finite rank over a commutative ring.

A free module of finite rank over a commutative ring $R$ is a module $M$ over $R$ that admits a finite basis, i.e. a finite family of linearly independent generators. Since $R$ is commutative, it has the invariant basis number property, so that the rank of the free module $M$ is defined uniquely, as the cardinality of any basis of $M$.

No distinguished basis of $M$ is assumed. On the contrary, many bases can be introduced on the free module along with change-of-basis rules (as module automorphisms). Each module element has then various representations over the various bases.

Note: The class FiniteRankFreeModule does not inherit from class FreeModule_generic nor from class CombinatorialFreeModule, since both classes deal with modules with a distinguished basis (see details above). Moreover, following the recommendation exposed in trac ticket #16427 the class FiniteRankFreeModule inherits directly from Parent (with the category set to Modules) and not from the Cython class Module.

The class FiniteRankFreeModule is a Sage parent class, the corresponding element class being FiniteRankFreeModuleElement.

INPUT:
- ring – commutative ring $R$ over which the free module is constructed
- rank – positive integer; rank of the free module
- name – (default: None) string; name given to the free module
- latex_name – (default: None) string; LaTeX symbol to denote the freemodule; if none is provided, it is set to name
- start_index – (default: 0) integer; lower bound of the range of indices in bases defined on the free module
- output_formatter – (default: None) function or unbound method called to format the output of the tensor components; output_formatter must take 1 or 2 arguments: the first argument must be an element of the ring $R$ and the second one, if any, some format specification

EXAMPLES:
Free module of rank 3 over $\mathbb{Z}$:

sage: FiniteRankFreeModule._clear_cache_() # for doctests only
sage: M = FiniteRankFreeModule(ZZ, 3) ; M
Rank-3 free module over the Integer Ring

(continues on next page)
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(continued from previous page)

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M') ; M  # declaration with a name
Rank-3 free module M over the Integer Ring
sage: M.category()
Category of finite dimensional modules over Integer Ring
sage: M.base_ring()
Integer Ring
sage: M.rank()
3
```

If the base ring is a field, the free module is in the category of vector spaces:

```python
sage: V = FiniteRankFreeModule(QQ, 3, name='V') ; V
3-dimensional vector space V over the Rational Field
sage: V.category()
Category of finite dimensional vector spaces over Rational Field
```

The \LaTeX{} output is adjusted via the parameter `latex_name`:

```python
sage: latex(M)  # the default is the symbol provided in the string `name`
M
sage: M = FiniteRankFreeModule(ZZ, 3, name='M', latex_name=r'\mathcal{M}')
sage: latex(M)
\mathcal{M}
```

The free module `M` has no distinguished basis:

```python
sage: M in ModulesWithBasis(ZZ)
False
sage: M in Modules(ZZ)
True
```

In particular, no basis is initialized at the module construction:

```python
sage: M.print_bases()
No basis has been defined on the Rank-3 free module M over the Integer Ring
sage: M.bases()
[]
```

Bases have to be introduced by means of the method `basis()`, the first defined basis being considered as the `default basis`, meaning it can be skipped in function arguments required a basis (this can be changed by means of the method `set_default_basis()`):

```python
sage: e = M.basis('e') ; e
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
sage: M.default_basis()
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
```

A second basis can be created from a family of linearly independent elements expressed in terms of basis `e`:

```python
sage: f = M.basis('f', from_family=(-e[0], e[1]+e[2], 2*e[1]+3*e[2]))
sage: f
Basis (f_0,f_1,f_2) on the Rank-3 free module M over the Integer Ring
sage: M.print_bases()
```

(continues on next page)
Tensors on free modules of finite rank, Release 9.6

Bases defined on the Rank-3 free module \( M \) over the Integer Ring:
- \((e_0, e_1, e_2)\) (default basis)
- \((f_0, f_1, f_2)\)

```
sage: M.bases()
```

[Basis \((e_0, e_1, e_2)\) on the Rank-3 free module \( M \) over the Integer Ring,
Basis \((f_0, f_1, f_2)\) on the Rank-3 free module \( M \) over the Integer Ring]

\( M \) is a *parent* object, whose elements are instances of *FiniteRankFreeModuleElement* (actually a dynamically
generated subclass of it):

```
sage: v = M.an_element() ; v
Element of the Rank-3 free module \( M \) over the Integer Ring
sage: from sage.tensor.modules.free_module_element import FiniteRankFreeModuleElement
sage: isinstance(v, FiniteRankFreeModuleElement)
True
sage: v in M
True
sage: M.is_parent_of(v)
True
sage: v.display()  # expansion w.r.t. the default basis (e)
e_0 + e_1 + e_2
sage: v.display(f)
-f_0 + f_1
```

The test suite of the category of modules is passed:

```
sage: TestSuite(M).run()
```

Constructing an element of \( M \) from (the integer) 0 yields the zero element of \( M \):

```
sage: M(0)
Element zero of the Rank-3 free module \( M \) over the Integer Ring
sage: M(0) is M.zero()
True
```

Non-zero elements are constructed by providing their components in a given basis:

```
sage: v = M([-1,0,3]) ; v  # components in the default basis (e)
Element of the Rank-3 free module \( M \) over the Integer Ring
sage: v.display()  # expansion w.r.t. the default basis (e)
e_0 + 3 e_2
sage: v.display(f)
f_0 + 6 f_1 + 3 f_2
```

```
sage: v = M([-1,0,3], basis=f) ; v  # components in a specific basis
Element of the Rank-3 free module \( M \) over the Integer Ring
sage: v.display(f)
-f_0 + 3 f_2
sage: v.display()
e_0 + 6 e_1 + 9 e_2
sage: v = M([-1,0,3], basis=f, name='v') ; v
Element \( v \) of the Rank-3 free module \( M \) over the Integer Ring
sage: v.display(f)
```

(continues on next page)
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(continued from previous page)

\[
v = -f_0 + 3 f_2
\]

\[
\text{sage: } v.\text{display()}
\]

\[
v = e_0 + 6 e_1 + 9 e_2
\]

An alternative is to construct the element from an empty list of components and to set the nonzero components afterwards:

\[
\text{sage: } v = M([], \text{name}=\text{'v'})
\]

\[
\text{sage: } v[e,0] = -1
\]

\[
\text{sage: } v[e,2] = 3
\]

\[
\text{sage: } v.\text{display(e)}
\]

\[
v = -e_0 + 3 e_2
\]

Indices on the free module, such as indices labelling the element of a basis, are provided by the generator method \text{irange()}. By default, they range from 0 to the module’s rank minus one:

\[
\text{sage: } \text{list}(M.\text{irange()})
\]

\[
[0, 1, 2]
\]

This can be changed via the parameter \text{start_index} in the module construction:

\[
\text{sage: } M1 = \text{FiniteRankFreeModule}(\text{ZZ}, 3, \text{name}=\text{'M'}, \text{start_index}=1)
\]

\[
\text{sage: } \text{list}(M1.\text{irange()})
\]

\[
[1, 2, 3]
\]

The parameter \text{output_formatter} in the constructor of the free module is used to set the output format of tensor components:

\[
\text{sage: } N = \text{FiniteRankFreeModule}(\text{QQ}, 3, \text{output_formatter}=\text{Rational.numerical_approx})
\]

\[
\text{sage: } e = N.\text{basis}(\text{\textquote{e}})
\]

\[
\text{sage: } v = N([1/3, 0, -2], \text{basis}=e)
\]

\[
\text{sage: } v[e,:]
\]

\[
[0.3333333333333333, 0.000000000000000, -2.00000000000000]
\]

\[
\text{sage: } v.\text{display(e)} \quad \# \text{default format (53 bits of precision)}
\]

\[
0.3333333333333333 \, e_0 - 2.00000000000000 \, e_2
\]

\[
\text{sage: } v.\text{display(e, format_spec=10)} \quad \# 10 \text{ bits of precision}
\]

\[
0.33 \, e_0 - 2.0 \, e_2
\]

\textbf{Element}

\textit{alias of \text{sage.tensor.modules.free_module_element.FiniteRankFreeModuleElement}}

\textbf{alternating_contravariant_tensor}(\text{degree, name=None, latex_name=None})

Construct an alternating contravariant tensor on the free module.

\textbf{INPUT:}

\begin{itemize}
  \item \text{degree} – degree of the alternating contravariant tensor (i.e. its tensor rank)
  \item \text{name} – (default: None) string; name given to the alternating contravariant tensor
  \item \text{latex_name} – (default: None) string; LaTeX symbol to denote the alternating contravariant tensor; if none is provided, the LaTeX symbol is set to \text{name}
\end{itemize}

\textbf{OUTPUT:}

\begin{itemize}
  \item instance of \text{AlternatingContrTensor}
\end{itemize}
EXAMPLES:

Alternating contravariant tensor on a rank-3 module:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: a = M.alternating_contravariant_tensor(2, 'a'); a
Alternating contravariant tensor a of degree 2 on the
Rank-3 free module M over the Integer Ring
```

The nonzero components in a given basis have to be set in a second step, thereby fully specifying the
alternating form:

```
sage: e = M.basis('e'); e
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
sage: a.set_comp(e)[0,1] = 2
sage: a.set_comp(e)[1,2] = -3
sage: a.display(e)
a = 2 e_0∧e_1 - 3 e_1∧e_2
```

An alternating contravariant tensor of degree 1 is simply an element of the module:

```
sage: a = M.alternating_contravariant_tensor(1, 'a'); a
Element a of the Rank-3 free module M over the Integer Ring
```

See `AlternatingContrTensor` for more documentation.

```
alternating_form(degree, name=None, latex_name=None)
```

Construct an alternating form on the free module.

**INPUT:**

- `degree` – the degree of the alternating form (i.e. its tensor rank)
- `name` – (default: `None`) string; name given to the alternating form
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the alternating form; if none is pro-

**OUTPUT:**

- instance of `FreeModuleAltForm`

**EXAMPLES:**

Alternating forms on a rank-3 module:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: a = M.alternating_form(2, 'a'); a
Alternating form a of degree 2 on the
Rank-3 free module M over the Integer Ring
```

The nonzero components in a given basis have to be set in a second step, thereby fully specifying the
alternating form:

```
sage: e = M.basis('e'); e
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
sage: a.set_comp(e)[0,1] = 2
sage: a.set_comp(e)[1,2] = -3
sage: a.display(e)
a = 2 e^0∧e^1 - 3 e^1∧e^2
```
An alternating form of degree 1 is a linear form:

\[
\texttt{sage: } a = M.\text{alternating}\_\text{form}(1, 'a') \; ; \; a\\
\text{Linear form } a \text{ on the Rank-3 free module } M \text{ over the Integer Ring}
\]

To construct such a form, it is preferable to call the method \texttt{linear\_form()} instead:

\[
\texttt{sage: } a = M.\text{linear}\_\text{form('a')} \; ; \; a\\
\text{Linear form } a \text{ on the Rank-3 free module } M \text{ over the Integer Ring}
\]

See \texttt{FreeModuleAltForm} for more documentation.

\texttt{automorphism(matrix=None, basis=None, name=None, latex_name=None)}

Construct a module automorphism of \texttt{self}.

Denoting \texttt{self} by \(M\), an automorphism of \texttt{self} is an element of the general linear group \(GL(M)\).

INPUT:

- \texttt{matrix} – (default: \texttt{None}) matrix of size \text{rank}(M)\times\text{rank}(M) representing the automorphism with respect to \texttt{basis}; this entry can actually be any material from which a matrix of elements of \texttt{self} base ring can be constructed; the \textit{columns} of \texttt{matrix} must be the components w.r.t. \texttt{basis} of the images of the elements of \texttt{basis}. If \texttt{matrix} is \texttt{None}, the automorphism has to be initialized afterwards by method \texttt{set\_comp()} or via the operator \([\text{}]\).
- \texttt{basis} – (default: \texttt{None}) basis of \texttt{self} defining the matrix representation; if \texttt{None} the default basis of \texttt{self} is assumed.
- \texttt{name} – (default: \texttt{None}) string; name given to the automorphism
- \texttt{latex\_name} – (default: \texttt{None}) string; \LaTeX{} symbol to denote the automorphism; if none is provided, the \LaTeX{} symbol is set to \texttt{name}

OUTPUT:

- instance of \texttt{FreeModuleAutomorphism}

EXAMPLES:

Automorphism of a rank-2 free \(\mathbb{Z}\)-module:

\[
\texttt{sage: } M = \text{FiniteRankFreeModule}(\mathbb{Z}\mathbb{Z}, 2, \text{name}='M')\\
\texttt{sage: } e = M.\text{basis}('e')\\
\texttt{sage: } a = M.\text{automorphism(matrix=[[1,2],[1,3]], basis=e, name='a')} \; ; \; a\\
\text{Automorphism } a \text{ of the Rank-2 free module } M \text{ over the Integer Ring}\\
\texttt{sage: } a.\text{parent()}\\
\text{General linear group of the Rank-2 free module } M \text{ over the Integer Ring}\\
\texttt{sage: } a.\text{matrix}(e)\\
\begin{bmatrix}
1 & 2 \\
1 & 3 \\
\end{bmatrix}
\]

An automorphism is a tensor of type (1,1):

\[
\texttt{sage: } a.\text{tensor\_type()}\\
(1, 1)\\
\texttt{sage: } a.\text{display}(e)\\
a = e_0 \otimes e^0 + 2 e_0 \otimes e^1 + e_1 \otimes e^0 + 3 e_1 \otimes e^1
\]

The automorphism components can be specified in a second step, as components of a type-(1,1) tensor.
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Component by component specification:

\begin{verbatim}
sage: a2 = M.automorphism(name='a')
sage: a2[0,0] = 1  # component set in the module's default basis (e)
sage: a2[0,1] = 2
sage: a2[1,0] = 1
sage: a2[1,1] = 3
sage: a2.matrix(e)
[1 2]
[1 3]
sage: a2 == a
True
\end{verbatim}

See FreeModuleAutomorphism for more documentation.

bases()

Return the list of bases that have been defined on the free module self.

Use the method print_bases() to get a formatted output with more information.

OUTPUT:

- list of instances of class FreeModuleBasis

EXAMPLES:

Bases on a rank-3 free module:

\begin{verbatim}
sage: M = FiniteRankFreeModule(ZZ, 3, name='M_3', start_index=1)
sage: M.bases()
[]
sage: e = M.basis('e')
sage: M.bases()
[Basis (e_1,e_2,e_3) on the Rank-3 free module M_3 over the Integer Ring]
sage: f = M.basis('f')
sage: M.bases()
[Basis (e_1,e_2,e_3) on the Rank-3 free module M_3 over the Integer Ring,
Basis (f_1,f_2,f_3) on the Rank-3 free module M_3 over the Integer Ring]
\end{verbatim}

basis(symbol, latex_symbol=None, from_family=None, indices=None, latex_indices=None,
symbol_dual=None, latex_symbol_dual=None)

Define or return a basis of the free module self.

Let \( M \) denote the free module self and \( n \) its rank.

The basis can be defined from a set of \( n \) linearly independent elements of \( M \) by means of the argument from_family. If from_family is not specified, the basis is created from scratch and, at this stage, is unrelated to bases that could have been defined previously on \( M \). It can be related afterwards by means of the method set_change_of_basis().
If the basis specified by the given symbol already exists, it is simply returned, whatever the value of the arguments `latex_symbol` or `from_family`.

Note that another way to construct a basis of `self` is to use the method `new_basis()` on an existing basis, with the automorphism relating the two bases as an argument.

**INPUT:**

- `symbol` – either a string, to be used as a common base for the symbols of the elements of the basis, or a list/tuple of strings, representing the individual symbols of the elements of the basis
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the elements of the basis, or a list/tuple of strings, representing the individual LaTeX symbols of the elements of the basis; if `None`, `symbol` is used in place of `latex_symbol`
- `from_family` – (default: `None`) tuple or list of \( n \) linearly independent elements of the free module `self` (\( n \) being the rank of `self`)
- `indices` – (default: `None`; used only if `symbol` is a single string) list/tuple of strings representing the indices labelling the elements of the basis; if `None`, the indices will be generated as integers within the range declared on `self`
- `latex_indices` – (default: `None`) list/tuple of strings representing the indices for the LaTeX symbols of the elements of the basis; if `None`, `indices` is used instead
- `symbol_dual` – (default: `None`) same as `symbol` but for the dual basis; if `None`, `symbol` must be a string and is used for the common base of the symbols of the elements of the dual basis
- `latex_symbol_dual` – (default: `None`) same as `latex_symbol` but for the dual basis

**OUTPUT:**

- instance of `FreeModuleBasis` representing a basis on `self`

**EXAMPLES:**

Bases on a rank-3 free module:

\[\begin{align*}
\text{sage: } & M = \text{FiniteRankFreeModule}(\mathbb{Z}, 3, \text{name}='M') \\
\text{sage: } & e = M.basis('e') ; e \\
& \text{Basis } (e_0,e_1,e_2) \text{ on the Rank-3 free module } M \text{ over the Integer Ring} \\
\text{sage: } & e[0] \\
& \text{Element } e_0 \text{ of the Rank-3 free module } M \text{ over the Integer Ring} \\
\text{sage: } & latex(e) \\
& \left(e_{0},e_{1},e_{2}\right)
\end{align*}\]

The LaTeX symbol can be set explicitly:

\[\begin{align*}
\text{sage: } & \text{eps} = M.basis('eps', \text{latex_symbol=r'\epsilon'}) ; \text{eps} \\
& \text{Basis } (\text{eps}_0,\text{eps}_1,\text{eps}_2) \text{ on the Rank-3 free module } M \text{ over the Integer Ring} \\
\text{sage: } & latex(\text{eps}) \\
& \left(\epsilon_{0},\epsilon_{1},\epsilon_{2}\right)
\end{align*}\]

The indices can be customized:

\[\begin{align*}
\text{sage: } & f = M.basis('f', \text{indices=('x', 'y', 'z')) ; f} \\
& \text{Basis } (f_x,f_y,f_z) \text{ on the Rank-3 free module } M \text{ over the Integer Ring} \\
\text{sage: } & latex(f[1]) \\
& f_{y}
\end{align*}\]
By providing a list or a tuple for the argument symbol, one can have a different symbol for each element of the basis; it is then mandatory to specify some symbols for the dual basis:

```
sage: g = M.basis(('a', 'b', 'c'), symbol_dual=('A', 'B', 'C')); g
Basis (a,b,c) on the Rank-3 free module M over the Integer Ring
sage: g.dual_basis()
Dual basis (A,B,C) on the Rank-3 free module M over the Integer Ring
```

If the provided symbol and indices are that of an already defined basis, the latter is returned (no new basis is created):

```
sage: M.basis('e') is e
True
sage: M.basis('eps') is eps
True
sage: M.basis('e', indices=['x', 'y', 'z']) is e
False
sage: M.basis('e', indices=['x', 'y', 'z']) is 
.....: M.basis('e', indices=['x', 'y', 'z'])
True
```

The individual elements of the basis are labelled according the parameter start_index provided at the free module construction:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1)
sage: e = M.basis('e') ; e
Basis (e_1,e_2,e_3) on the Rank-3 free module M over the Integer Ring
sage: e[1]
Element e_1 of the Rank-3 free module M over the Integer Ring
```

Construction of a basis from a spanning family of linearly independent module elements:

```
sage: f1 = -e[2]
sage: f2 = 4*e[1] + 3*e[3]
sage: f3 = 7*e[1] + 5*e[3]
sage: f = M.basis('f', from_family=(f1,f2,f3))
sage: f[1].display()
f_1 = -e_2
sage: f[2].display()
f_2 = 4 e_1 + 3 e_3
sage: f[3].display()
f_3 = 7 e_1 + 5 e_3
```

The change-of-basis automorphisms have been registered:

```
sage: M.change_of_basis(e,f).matrix(e)
[ 0 4 7]
[-1 0 0]
[ 0 3 5]
sage: M.change_of_basis(f,e).matrix(e)
[ 0 -1 0]
[-5 0 7]
[ 3 0 -4]
sage: M.change_of_basis(f,e) == M.change_of_basis(e,f).inverse()
True
```
Check of the change-of-basis $e \rightarrow f$:

```
sage: a = M.change_of_basis(e,f) ; a
Automorphism of the Rank-3 free module $M$ over the Integer Ring
sage: all( f[i] == a(e[i]) for i in M.irange() )
True
```

Providing a family of module elements that are not linearly independent raise an error:

```
sage: g = M.basis('g', from_family=(f1, f2, f1+f2))
Traceback (most recent call last):
  ...
ValueError: the provided module elements are not linearly independent
```

For more documentation on bases see `FreeModuleBasis`.

```
change_of_basis(basis1, basis2)
```

Return a module automorphism linking two bases defined on the free module $M$.

If the automorphism has not been recorded yet (in the internal dictionary `self._basis_changes`), it is computed by transitivity, i.e. by performing products of recorded changes of basis.

**INPUT:**

- `basis1` – a basis of $M$, denoted $(e_i)$ below
- `basis2` – a basis of $M$, denoted $(f_i)$ below

**OUTPUT:**

- instance of `FreeModuleAutomorphism` describing the automorphism $P$ that relates the basis $(e_i)$ to the basis $(f_i)$ according to $f_i = P(e_i)$

**EXAMPLES:**

Changes of basis on a rank-2 free module:

```
sage: FiniteRankFreeModule._clear_cache_() # for doctests only
sage: M = FiniteRankFreeModule(ZZ, 2, name='M', start_index=1)
sage: e = M.basis('e')
sage: f = M.basis('f', from_family=(e[1]+2*e[2], e[1]+3*e[2]))
sage: P = M.change_of_basis(e,f) ; P
Automorphism of the Rank-2 free module $M$ over the Integer Ring
sage: P.matrix(e)
[1 1]
[2 3]
```

Note that the columns of this matrix contain the components of the elements of basis $f$ w.r.t. to basis $e$:

```
sage: f[1].display(e)
f_1 = e_1 + 2 e_2
sage: f[2].display(e)
f_2 = e_1 + 3 e_2
```

The change of basis is cached:

```
sage: P is M.change_of_basis(e,f)
True
```
Check of the change-of-basis automorphism:

```python
sage: f[1] == P(e[1])
True
sage: f[2] == P(e[2])
True
```

Check of the reverse change of basis:

```python
sage: M.change_of_basis(f,e) == P^(-1)
True
```

We have of course:

```python
sage: M.change_of_basis(e,e)
Identity map of the Rank-2 free module M over the Integer Ring
```

```python
sage: M.change_of_basis(e,e) is M.identity_map()
True
```

Let us introduce a third basis on \( M \):

```python
sage: h = M.basis('h', from_family=(3*e[1]+4*e[2], 5*e[1]+7*e[2]))
```

The change of basis \( e \to h \) has been recorded directly from the definition of \( h \):

```python
sage: Q = M.change_of_basis(e,h) ; Q.matrix(e)
\[
\begin{bmatrix}
3 & 5 \\
4 & 7 \\
\end{bmatrix}
\]
```

The change of basis \( f \to h \) is computed by transitivity, i.e. from the changes of basis \( f \to e \) and \( e \to h \):

```python
sage: R = M.change_of_basis(f,h) ; R
Automorphism of the Rank-2 free module M over the Integer Ring
```

```python
sage: R.matrix(e)
\[
\begin{bmatrix}
-1 & 2 \\
-2 & 3 \\
\end{bmatrix}
\]
```

```python
sage: R.matrix(f)
\[
\begin{bmatrix}
5 & 8 \\
-2 & -3 \\
\end{bmatrix}
\]
```

Let us check that \( R \) is indeed the change of basis \( f \to h \):

```python
sage: h[1] == R(f[1])
True
True
```

A related check is:

```python
sage: R == Q*P^(-1)
True
```

**default_basis()**

Return the default basis of the free module `self`. 

---

**Chapter 1. Free modules of finite rank**
The default basis is simply a basis whose name can be skipped in methods requiring a basis as an argument. By default, it is the first basis introduced on the module. It can be changed by the method set_default_basis().

OUTPUT:

- instance of FreeModuleBasis

EXAMPLES:

At the module construction, no default basis is assumed:

```
sage: M = FiniteRankFreeModule(ZZ, 2, name='M', start_index=1)
sage: M.default_basis()
```

No default basis has been defined on the Rank-2 free module $M$ over the Integer Ring

The first defined basis becomes the default one:

```
sage: e = M.basis('e') ; e
Basis (e_1,e_2) on the Rank-2 free module $M$ over the Integer Ring
sage: M.default_basis()
Basis (e_1,e_2) on the Rank-2 free module $M$ over the Integer Ring
```

```
sage: f = M.basis('f') ; f
Basis (f_1,f_2) on the Rank-2 free module $M$ over the Integer Ring
sage: M.default_basis()
Basis (e_1,e_2) on the Rank-2 free module $M$ over the Integer Ring
```

dual()

Return the dual module of self.

EXAMPLES:

Dual of a free module over $\mathbb{Z}$:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: M.dual()
Dual of the Rank-3 free module $M$ over the Integer Ring
sage: latex(M.dual())
$M^*$
```

The dual is a free module of the same rank as $M$:

```
sage: isinstance(M.dual(), FiniteRankFreeModule)
True
sage: M.dual().rank()
3
```

It is formed by alternating forms of degree 1, i.e. linear forms:

```
sage: M.dual() is M.dual_exterior_power(1)
True
sage: M.dual().an_element()
Linear form on the Rank-3 free module $M$ over the Integer Ring
sage: a = M.linear_form()
sage: a in M.dual()
True
```
The elements of a dual basis belong of course to the dual module:

```sage
sage: e = M.basis('e')
sage: e.dual_basis()[0] in M.dual()
True
```

**dual_exterior_power**(*p*)

Return the *p*-th exterior power of the dual of self.

If *M* stands for the free module self, the *p*-th exterior power of the dual of *M* is the set \( \Lambda^p(M^*) \) of all alternating forms of degree *p* on *M*, i.e. of all multilinear maps

\[
\underbrace{M \times \cdots \times M}_{p \text{ times}} \rightarrow R
\]

that vanish whenever any of two of their arguments are equal. \( \Lambda^p(M^*) \) is a free module of rank \( \binom{n}{p} \) over the same ring as *M*, where *n* is the rank of *M*.

**INPUT:**

- *p* – non-negative integer

**OUTPUT:**

- for *p* = 0, the base ring \( R \)
- for *p* ≥ 1, instance of *ExtPowerDualFreeModule* representing the free module \( \Lambda^p(M^*) \)

**EXAMPLES:**

Exterior powers of the dual of a free \( \mathbb{Z} \)-module of rank 3:

```sage
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: M.dual_exterior_power(0) # return the base ring
Integer Ring
sage: M.dual_exterior_power(1) # return the dual module
Dual of the Rank-3 free module M over the Integer Ring
sage: M.dual_exterior_power(1) is M.dual()
True
sage: M.dual_exterior_power(2)
2nd exterior power of the dual of the Rank-3 free module M over the Integer Ring
sage: M.dual_exterior_power(2).an_element()
Alternating form of degree 2 on the Rank-3 free module M over the Integer Ring
sage: M.dual_exterior_power(2).an_element().display()
e^0\wedge e^1
sage: M.dual_exterior_power(3)
3rd exterior power of the dual of the Rank-3 free module M over the Integer Ring
sage: M.dual_exterior_power(3).an_element()
Alternating form of degree 3 on the Rank-3 free module M over the Integer Ring
sage: M.dual_exterior_power(3).an_element().display()
e^0\wedge e^1\wedge e^2
```

See *ExtPowerDualFreeModule* for more documentation.

**endomorphism**(*matrix_rep*, **basis**=None, **name**=None, **latex_name**=None)

Construct an endomorphism of the free module self.

The returned object is a module morphism \( \phi : M \rightarrow M \), where *M* is self.

**INPUT:**
• **matrix_rep** – matrix of size \(\text{rank}(M) \times \text{rank}(M)\) representing the endomorphism with respect to basis; this entry can actually be any material from which a matrix of elements of self base ring can be constructed; the columns of matrix_rep must be the components w.r.t. basis of the images of the elements of basis.

• **basis** – (default: None) basis of self defining the matrix representation; if None the default basis of self is assumed.

• **name** – (default: None) string; name given to the endomorphism

• **latex_name** – (default: None) string; LaTeX symbol to denote the endomorphism; if none is provided, name will be used.

**OUTPUT:**

• the endomorphism \(\phi : M \rightarrow M\) corresponding to the given specifications, as an instance of :class:`FiniteRankFreeModuleMorphism`

**EXAMPLES:**

Construction of an endomorphism with minimal data (module’s default basis and no name):

```python
sage: M = FiniteRankFreeModule(ZZ, 2, name='M')
sage: e = M.basis('e')
sage: phi = M.endomorphism([[1,-2], [-3,4]]) ; phi
Generic endomorphism of Rank-2 free module M over the Integer Ring
sage: phi.matrix()  # matrix w.r.t the default basis
[ 1 -2]
[-3 4]
```

Construction with full list of arguments (matrix given a basis different from the default one):

```python
sage: a = M.automorphism() ; a[0,1], a[1,0] = 1, -1
sage: ep = e.new_basis(a, 'ep', latex_symbol="e")
sage: phi = M.endomorphism([[1,-2], [-3,4]], basis=ep, name='phi',
                        latex_name=r'\phi')
sage: phi
Generic endomorphism of Rank-2 free module M over the Integer Ring
sage: phi.matrix(ep)  # the input matrix
[ 1 -2]
[-3 4]
sage: phi.matrix()  # matrix w.r.t the default basis
[4 3]
[2 1]
```

See :class:`FiniteRankFreeModuleMorphism` for more documentation.

**exterior_power**(\(p\))

Return the \(p\)-th exterior power of self.

If \(M\) stands for the free module self, the \(p\)-th exterior power of \(M\) is the set \(\Lambda^p(M)\) of all alternating contravariant tensors of rank \(p\), i.e. of all multilinear maps

\[
\underbrace{M^* \times \cdots \times M^*}_{\text{\(p\) times}} \rightarrow R
\]

that vanish whenever any of two of their arguments are equal. \(\Lambda^p(M)\) is a free module of rank \(\binom{n}{p}\) over the same ring as \(M\), where \(n\) is the rank of \(M\).

**INPUT:**
• $p$ – non-negative integer

OUTPUT:
• for $p = 0$, the base ring $R$
• for $p = 1$, the free module $M$, since $\Lambda^1(M) = M$
• for $p \geq 2$, instance of $ExtPowerFreeModule$ representing the free module $\Lambda^p(M)$

EXAMPLES:
Exterior powers of the dual of a free $\mathbb{Z}$-module of rank 3:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: M.exterior_power(0)  # return the base ring
Integer Ring
sage: M.exterior_power(1)  # return the module itself
Rank-3 free module $M$ over the Integer Ring
sage: M.exterior_power(1) is M
True
sage: M.exterior_power(2)
2nd exterior power of the Rank-3 free module $M$ over the Integer Ring
sage: M.exterior_power(2).an_element()  # Alternating contravariant tensor of degree 2 on the Rank-3 free module $M$ over the Integer Ring
sage: M.exterior_power(2).an_element().display()
e_0 \wedge e_1
sage: M.exterior_power(3)
3rd exterior power of the Rank-3 free module $M$ over the Integer Ring
sage: M.exterior_power(3).an_element()  # Alternating contravariant tensor of degree 3 on the Rank-3 free module $M$ over the Integer Ring
sage: M.exterior_power(3).an_element().display()
e_0 \wedge e_1 \wedge e_2
```

See $ExtPowerFreeModule$ for more documentation.

`general_linear_group()`

Return the general linear group of self.

If self is the free module $M$, the general linear group is the group $GL(M)$ of automorphisms of $M$.

OUTPUT:
• instance of class $FreeModuleLinearGroup$ representing $GL(M)$

EXAMPLES:
The general linear group of a rank-3 free module:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: GL = M.general_linear_group() ; GL
General linear group of the Rank-3 free module $M$ over the Integer Ring
sage: GL.category()
Category of groups
sage: type(GL)
<class 'sage.tensor.modules.free_module_linear_group.FreeModuleLinearGroup_with_category'>
```
There is a unique instance of the general linear group:

```
sage: M.general_linear_group() is GL
True
```

The group identity element:

```
sage: GL.one()
Identity map of the Rank-3 free module M over the Integer Ring
sage: GL.one().matrix(e)
[1 0 0]
[0 1 0]
[0 0 1]
```

An element:

```
sage: GL.an_element()
Automorphism of the Rank-3 free module M over the Integer Ring
sage: GL.an_element().matrix(e)
[ 1 0 0]
[ 0 -1 0]
[ 0 0 1]
```

See `FreeModuleLinearGroup` for more documentation.

```
hom(codomain, matrix_rep, bases=None, name=None, latex_name=None)
```

Homomorphism from `self` to a free module.

Define a module homomorphism

\[ \phi : M \rightarrow N, \]

where \( M \) is `self` and \( N \) is a free module of finite rank over the same ring \( R \) as `self`.

**Note:** This method is a redefinition of `sage.structure.parent.Parent.hom()` because the latter assumes that `self` has some privileged generators, while an instance of `FiniteRankFreeModule` has no privileged basis.

**INPUT:**

- `codomain` – the target module \( N \)
- `matrix_rep` – matrix of size \( \text{rank}(N) \times \text{rank}(M) \) representing the homomorphism with respect to the pair of bases defined by `bases`; this entry can actually be any material from which a matrix of elements of \( R \) can be constructed; the columns of `matrix_rep` must be the components w.r.t. `basis_N` of the images of the elements of `basis_M`.
- `bases` – (default: None) pair \( (\text{basis}_M, \text{basis}_N) \) defining the matrix representation, `basis_M` being a basis of `self` and `basis_N` a basis of module \( N \); if None the pair formed by the default bases of each module is assumed.
- `name` – (default: None) string; name given to the homomorphism
- `latex_name` – (default: None) string; LaTeX symbol to denote the homomorphism; if None, `name` will be used.
OUTPUT:
  • the homomorphism $\phi : M \to N$ corresponding to the given specifications, as an instance of
    \texttt{FiniteRankFreeModuleMorphism}

EXAMPLES:
Homomorphism between two free modules over $\mathbb{Z}$:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: N = FiniteRankFreeModule(ZZ, 2, name='N')
sage: e = M.basis('e')
sage: f = N.basis('f')
sage: phi = M.hom(N, [[-1,2,0], [5,1,2]]) ; phi
Generic morphism:
  From: Rank-3 free module $M$ over the Integer Ring
  To:  Rank-2 free module $N$ over the Integer Ring
```

Homomorphism defined by a matrix w.r.t. bases that are not the default ones:

```
sage: ep = M.basis('ep', latex_symbol=r'\text{\text Ascii}e')
sage: fp = N.basis('fp', latex_symbol=r'\text{\text Ascii}f')
sage: phi = M.hom(N, [[3,2,1], [1,2,3]], bases=(ep, fp)) ; phi
Generic morphism:
  From: Rank-3 free module $M$ over the Integer Ring
  To:  Rank-2 free module $N$ over the Integer Ring
```

Call with all arguments specified:

```
sage: phi = M.hom(N, [[3,2,1], [1,2,3]], bases=(ep, fp),
    ....:     name='\phi', latex_name=r'\text{\phi}')
```

The parent:

```
sage: phi.parent() is Hom(M,N)
True
```

See class \texttt{FiniteRankFreeModuleMorphism} for more documentation.

\texttt{identity_map(name='Id', latex_name=None)}

Return the identity map of the free module \texttt{self}.

INPUT:

  • \texttt{name} -- (string; default: ‘Id’) name given to the identity identity map
  • \texttt{latex_name} -- (string; default: None) \LaTeX{} symbol to denote the identity map; if none is provided,
    the \LaTeX{} symbol is set to ‘\texttt{\texttt{mathrm[Id]’}} if \texttt{name} is ‘Id’ and to \texttt{name} otherwise

OUTPUT:

  • the identity map of \texttt{self} as an instance of \texttt{FreeModuleAutomorphism}

EXAMPLES:
Identity map of a rank-3 $\mathbb{Z}$-module:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: Id = M.identity_map() ; Id
```

(continues on next page)
Identity map of the Rank-3 free module $M$ over the Integer Ring

\[ \text{sage: Id.parent()} \]
General linear group of the Rank-3 free module $M$ over the Integer Ring

\[ \text{sage: Id.matrix(e)} \]
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

The default LaTeX symbol:

\[ \text{sage: latex(Id)} \]
\[ \mathrm{Id} \]

It can be changed by means of the method \text{set_name()}:

\[ \text{sage: Id.set_name(latex_name=r'\mathrm{1}_M')} \]
\[ \text{sage: latex(Id)} \]
\[ \mathrm{1}_M \]

The identity map is actually the identity element of $\text{GL}(M)$:

\[ \text{sage: Id is M.general_linear_group().one()} \]
\[ \text{True} \]

It is also a tensor of type-(1,1) on $M$:

\[ \text{sage: Id.tensor_type()} \]
\[ (1, 1) \]
\[ \text{sage: Id.comp(e)} \]
Kronecker delta of size 3x3
\[ \text{sage: Id[:]} \]
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Example with a LaTeX symbol different from the default one and set at the creation of the object:

\[ \text{sage: N = FiniteRankFreeModule(ZZ, 3, name='N')} \]
\[ \text{sage: f = N.basis('f')} \]
\[ \text{sage: Id = N.identity_map(name='Id_N', latex_name=r'\mathrm{Id}_N')} \]
\[ \text{sage: Id} \]
Identity map of the Rank-3 free module $N$ over the Integer Ring
\[ \text{sage: latex(Id)} \]
\[ \mathrm{Id}_N \]

\text{irange(start=None)}

Single index generator, labelling the elements of a basis of \text{self}.

**INPUT:**

- \text{start} – (default: None) integer; initial value of the index; if none is provided, \text{self._sindex} is assumed

**OUTPUT:**

- an iterable index, starting from \text{start} and ending at \text{self._sindex + self.rank()} - 1
EXAMPLES:

Index range on a rank-3 module:

```
sage: M = FiniteRankFreeModule(ZZ, 3)
sage: list(M.irange())
[0, 1, 2]
sage: list(M.irange(start=1))
[1, 2]
```

The default starting value corresponds to the parameter `start_index` provided at the module construction (the default value being 0):

```
sage: M1 = FiniteRankFreeModule(ZZ, 3, start_index=1)
sage: list(M1.irange())
[1, 2, 3]
sage: M2 = FiniteRankFreeModule(ZZ, 3, start_index=-4)
sage: list(M2.irange())
[-4, -3, -2]
```

**isomorphism_with_fixed_basis**(basis, codomain=None)

Construct the canonical isomorphism from the free module `self` to a free module in which `basis` of `self` is mapped to the distinguished basis of `codomain`.

INPUT:

- `basis` – the basis of `self` which should be mapped to the distinguished basis on `codomain`
- `codomain` – (default: None) the codomain of the isomorphism represented by a free module within the category `ModulesWithBasis` with the same rank and base ring as `self`; if None a free module represented by `CombinatorialFreeModule` is constructed

OUTPUT:

- a module morphism represented by `ModuleMorphismFromFunction`

EXAMPLES:

```
sage: V = FiniteRankFreeModule(QQ, 3, start_index=1); V
3-dimensional vector space over the Rational Field
sage: basis = e = V.basis("e"); basis
Basis (e_1,e_2,e_3) on the 3-dimensional vector space over the Rational Field
sage: phi_e = V.isomorphism_with_fixed_basis(basis); phi_e
Generic morphism:
From: 3-dimensional vector space over the Rational Field
to: Free module generated by {1, 2, 3} over Rational Field
sage: phi_e.codomain().category()
Category of finite dimensional vector spaces with basis over Rational Field
sage: phi_e(e[1] + 2 * e[2])
```

```
sage: abc = V.basis(['a', 'b', 'c'], symbol_dual=['d', 'e', 'f']); abc
Basis (a,b,c) on the 3-dimensional vector space over the Rational Field
sage: phi_abc = V.isomorphism_with_fixed_basis(abc); phi_abc
Generic morphism:
```
linear_form(name=None, latex_name=None)

Construct a linear form on the free module self.

A linear form on a free module $M$ over a ring $R$ is a map $M \to R$ that is linear. It can be viewed as a tensor of type $(0,1)$ on $M$.

INPUT:

- name – (default: None) string; name given to the linear form
- latex_name – (default: None) string; LaTeX symbol to denote the linear form; if none is provided, the LaTeX symbol is set to name

OUTPUT:

- instance of FreeModuleAltForm

EXAMPLES:

Linear form on a rank-3 free module:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
```

```python
sage: e = M.basis('e')
```

```python
sage: a = M.linear_form('A') ; a
```

Linear form $A$ on the Rank-3 free module $M$ over the Integer Ring

```python
sage: a[:] = [2,-1,3]  # components w.r.t. the module's default basis (e)
```

```python
sage: a.display()
A = 2 e^0 - e^1 + 3 e^2
```

A linear form maps module elements to ring elements:

```python
sage: v = M([1,1,1])
```

```python
sage: a(v)
4
```

Test of linearity:

```python
sage: u = M([-5,-2,7])
```

```python
sage: a(3*u - 4*v) == 3*a(u) - 4*a(v)
True
```

See FreeModuleAltForm for more documentation.
print_bases()
Display the bases that have been defined on the free module self.
Use the method bases() to get the raw list of bases.

EXAMPLES:
Bases on a rank-4 free module:

```sage
M = FiniteRankFreeModule(ZZ, 4, name='M', start_index=1)
M.print_bases()
No basis has been defined on the
Rank-4 free module M over the Integer Ring
sage: e = M.basis('e')
M.print_bases()
Bases defined on the Rank-4 free module M over the Integer Ring:
- (e_1,e_2,e_3,e_4) (default basis)
sage: f = M.basis('f')
M.print_bases()
Bases defined on the Rank-4 free module M over the Integer Ring:
- (e_1,e_2,e_3,e_4) (default basis)
- (f_1,f_2,f_3,f_4)
sage: M.set_default_basis(f)
M.print_bases()
Bases defined on the Rank-4 free module M over the Integer Ring:
- (e_1,e_2,e_3,e_4)
- (f_1,f_2,f_3,f_4) (default basis)
```

rank()
Return the rank of the free module self.
Since the ring over which self is built is assumed to be commutative (and hence has the invariant basis number property), the rank is defined uniquely, as the cardinality of any basis of self.

EXAMPLES:
Rank of free modules over Z:

```sage
M = FiniteRankFreeModule(ZZ, 3)
M.rank()
3
sage: M.tensor_module(0,1).rank()
3
sage: M.tensor_module(0,2).rank()
9
sage: M.tensor_module(1,0).rank()
3
sage: M.tensor_module(1,1).rank()
9
sage: M.tensor_module(1,2).rank()
27
sage: M.tensor_module(2,2).rank()
81
```

set_change_of_basis(basis1, basis2, change_of_basis, compute_inverse=True)
Relates two bases by an automorphism of self.
This updates the internal dictionary self._basis_changes.
INPUT:

- `basis1` – basis 1, denoted \((e_i)\) below
- `basis2` – basis 2, denoted \((f_i)\) below
- `change_of_basis` – instance of class `FreeModuleAutomorphism` describing the automorphism \(P\) that relates the basis \((e_i)\) to the basis \((f_i)\) according to \(f_i = P(e_i)\)
- `compute_inverse` (default: True) – if set to True, the inverse automorphism is computed and the change from basis \((f_i)\) to \((e_i)\) is set to it in the internal dictionary `self._basis_changes`

EXAMPLES:

Defining a change of basis on a rank-2 free module:

```python
sage: M = FiniteRankFreeModule(QQ, 2, name='M')
sage: e = M.basis('e')
sage: f = M.basis('f')
sage: a = M.automorphism()
sage: a[:,:] = [[1, 2], [-1, 3]]
sage: M.set_change_of_basis(e, f, a)
```

The change of basis and its inverse have been recorded:

```python
sage: M.change_of_basis(e,f).matrix(e)
[ 1 2]
[-1 3]
sage: M.change_of_basis(f,e).matrix(e)
[ 3/5 -2/5]
[ 1/5 1/5]
```

and are effective:

```python
sage: f[0].display(e)
f_0 = e_0 - e_1
sage: e[0].display(f)
e_0 = 3/5 f_0 + 1/5 f_1
```

`set_default_basis(basis)`

Sets the default basis of `self`.

The `default basis` is simply a basis whose name can be skipped in methods requiring a basis as an argument. By default, it is the first basis introduced on the module.

INPUT:

- `basis` – instance of `FreeModuleBasis` representing a basis on `self`

EXAMPLES:

Changing the default basis on a rank-3 free module:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1)
sage: e = M.basis('e') ; e
Basis \((e_1,e_2,e_3)\) on the Rank-3 free module \(M\) over the Integer Ring
sage: f = M.basis('f') ; f
Basis \((f_1,f_2,f_3)\) on the Rank-3 free module \(M\) over the Integer Ring
```

(continues on next page)
Basis \((e_1,e_2,e_3)\) on the Rank-3 free module \(M\) over the Integer Ring

\[
\begin{align*}
sage: & \ M.set_default_basis(f) \\
sage: & \ M.default_basis() \\
Basis \((f_1,f_2,f_3)\) on the Rank-3 free module \(M\) over the Integer Ring
\end{align*}
\]

**sym_bilinear_form** (*name=None, latex_name=None*)

Construct a symmetric bilinear form on the free module \(self\).

**INPUT:**

- **name** – (default: None) string; name given to the symmetric bilinear form
- **latex_name** – (default: None) string; \(\LaTeX\) symbol to denote the symmetric bilinear form; if none is provided, the \(\LaTeX\) symbol is set to \(name\)

**OUTPUT:**

- instance of **FreeModuleTensor** of tensor type \((0, 2)\) and symmetric

**EXAMPLES:**

Symmetric bilinear form on a rank-3 free module:

\[
\begin{align*}
sage: & \ M = \text{FiniteRankFreeModule}(\ZZ, 3, \text{name}='M') \\
sage: & \ a = M.sym_bilinear_form('A') ; a \\
\end{align*}
\]

A symmetric bilinear form is a type-\((0,2)\) tensor that is symmetric:

\[
\begin{align*}
sage: & \ a.\text{parent}() \\
\end{align*}
\]

Components with respect to a given basis:

\[
\begin{align*}
sage: & \ e = M.basis('e') \\
sage: & \ a[0,0], a[0,1], a[0,2] = 1, 2, 3 \\
sage: & \ a[1,1], a[1,2] = 4, 5 \\
sage: & \ a[2,2] = 6 \\
\end{align*}
\]

Only independent components have been set; the other ones are deduced by symmetry:

\[
\begin{align*}
sage: & \ a[1,0], a[2,0], a[2,1] \\
& (2, 3, 5) \\
sage: & \ a[:]
\end{align*}
\]

A symmetric bilinear form acts on pairs of module elements:
The sum of two symmetric bilinear forms is another symmetric bilinear form:

```plaintext
sage: b = M.sym_bilinear_form('B')
sage: b[0,0], b[0,1], b[1,2] = -2, 1, -3
sage: s = a + b ; s
Symmetric bilinear form A+B on the
   Rank-3 free module M over the Integer Ring
sage: a[:], b[:], s[:]
([1 2 3], [-2 1 0], [-1 3 3]
[2 4 5], [ 1 0 -3], [ 3 4 2]
[3 5 6], [ 0 -3 0], [ 3 2 6])
```

Adding a symmetric bilinear form with a non-symmetric one results in a generic type-(0, 2) tensor:

```plaintext
sage: c = M.tensor((0,2), name='C')
sage: c[0,1] = 4
sage: s = a + c ; s
Type-(0,2) tensor A+C on the Rank-3 free module M over the Integer Ring
sage: s.symmetries()
no symmetry; no antisymmetry
sage: s[:]
[1 6 3]
[2 4 5]
[3 5 6]
```

See `FreeModuleTensor` for more documentation.
• instance of `FreeModuleTensor` representing the tensor defined on `self` with the provided characteristics

**EXAMPLES:**

Tensors on a rank-3 free module:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: t = M.tensor((1,0), name='t') ; t
Element t of the Rank-3 free module M over the Integer Ring
sage: t = M.tensor((0,1), name='t') ; t
Linear form t on the Rank-3 free module M over the Integer Ring
sage: t = M.tensor((1,1), name='t') ; t
Type-(1,1) tensor t on the Rank-3 free module M over the Integer Ring
sage: t = M.tensor((0,2), name='t', sym=(0,1)) ; t
Symmetric bilinear form t on the Rank-3 free module M over the Integer Ring
sage: t = M.tensor((0,2), name='t', antisym=(0,1)) ; t
Alternating form t of degree 2 on the Rank-3 free module M over the Integer Ring
sage: t = M.tensor((1,2), name='t') ; t
Type-(1,2) tensor t on the Rank-3 free module M over the Integer Ring
```

See `FreeModuleTensor` for more examples and documentation.

**tensor_from_comp**(tensor_type, comp, name=None, latex_name=None)

Construct a tensor on `self` from a set of components.

The tensor symmetries are deduced from those of the components.

**INPUT:**

- `tensor_type` – pair `(k, l)` with `k` being the contravariant rank and `l` the covariant rank
- `comp` – instance of `Components` representing the tensor components in a given basis
- `name` – (default: None) string; name given to the tensor
- `latex_name` – (default: None) string; LaTeX symbol to denote the tensor; if none is provided, the LaTeX symbol is set to `name`

**OUTPUT:**

- instance of `FreeModuleTensor` representing the tensor defined on `self` with the provided characteristics.

**EXAMPLES:**

Construction of a tensor of rank 1:

```python
sage: from sage.tensor.modules.comp import Components, CompWithSym, ...
          CompFullySym, CompFullyAntiSym
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e') ; e
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
sage: c = Components(ZZ, e, 1)
sage: c[:]
[0, 0, 0]
```
Tensors on free modules of finite rank, Release 9.6

... (continued from previous page) ...

```
sage: c[:] = [-1,4,2]
sage: t = M.tensor_from_comp((1,0), c)
sage: t
Element of the Rank-3 free module M over the Integer Ring

sage: t.display(e)
-e_0 + 4 e_1 + 2 e_2

sage: t = M.tensor_from_comp((0,1), c) ; t
Linear form on the Rank-3 free module M over the Integer Ring

sage: t.display(e)
-e^0 + 4 e^1 + 2 e^2
```

Construction of a tensor of rank 2:

```
sage: c = CompFullySym(ZZ, e, 2)
sage: c[0,0], c[1,2] = 4, 5
sage: t = M.tensor_from_comp((0,2), c) ; t
Symmetric bilinear form on the Rank-3 free module M over the Integer Ring

sage: t.symmetries()
symmetry: (0, 1); no antisymmetry

sage: t.display(e)
4 e^0⊗e^0 + 5 e^1⊗e^2 + 5 e^2⊗e^1
```

```
sage: c = CompFullyAntiSym(ZZ, e, 2)
sage: c[0,1], c[1,2] = 4, 5
sage: t = M.tensor_from_comp((0,2), c) ; t
Alternating form of degree 2 on the Rank-3 free module M over the Integer Ring

sage: t.display(e)
4 e^0∧e^1 + 5 e^1∧e^2
```

tensor_module\((k, l)\)

Return the free module of all tensors of type \((k, l)\) defined on self.

**INPUT:**

- \(k\) – non-negative integer; the contravariant rank, the tensor type being \((k, l)\)
- \(l\) – non-negative integer; the covariant rank, the tensor type being \((k, l)\)

**OUTPUT:**

- instance of `TensorFreeModule` representing the free module \(T^{(k,l)}(M)\) of type-\((k, l)\) tensors on the free module self

**EXAMPLES:**

Tensor modules over a free module over \(\mathbb{Z}\):

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: T = M.tensor_module(1,2) ; T
Free module of type-\((1,2)\) tensors on the Rank-3 free module M over the Integer Ring

sage: T.an_element()
Type-\((1,2)\) tensor on the Rank-3 free module M over the Integer Ring
```
sage: M.tensor_module(1,2) is T
True

The base module is itself the module of all type-(1,0) tensors:

sage: M.tensor_module(1,0) is M
True

See `TensorFreeModule` for more documentation.

**zero()**

Return the zero element of `self`.

**EXAMPLES:**

Zero elements of free modules over \(\mathbb{Z}\):

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: M.zero()
Element zero of the Rank-3 free module M over the Integer Ring
sage: M.zero().parent() is M
True
sage: M.zero() is M(0)
True
sage: T = M.tensor_module(1,1)
sage: T.zero()
Type-(1,1) tensor zero on the Rank-3 free module M over the Integer Ring
sage: T.zero().parent() is T
True
sage: T.zero() is T(0)
True
```

Components of the zero element with respect to some basis:

```python
sage: e = M.basis('e')
sage: M.zero()[e,:]
[0, 0, 0]
sage: all(M.zero()[e,i] == M.base_ring().zero() for i in M.irange())
True
sage: T.zero()[e,:]
[0 0 0]
[0 0 0]
[0 0 0]
sage: M.tensor_module(1,2).zero()[e,:]
[[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
 [[0, 0, 0], [0, 0, 0], [0, 0, 0]],
 [[0, 0, 0], [0, 0, 0], [0, 0, 0]]]
```
ELEMENTS OF FREE MODULES OF FINITE RANK

The class `FiniteRankFreeModuleElement` implements elements of free modules of finite rank over a commutative ring.

AUTHORS:

- Eric Gourgoulhon (2017): class `FiniteRankFreeModuleElement` inherits from `AlternatingContrTensor`

REFERENCES:

- Chap. 21 of R. Godement: *Algebra* [God1968]
- Chap. 12 of J. M. Lee: *Introduction to Smooth Manifolds* [Lee2013] (only when the free module is a vector space)
- Chap. 2 of B. O’Neill: *Semi-Riemannian Geometry* [ONe1983]

```python
class sage.tensor.modules.free_module_element.FiniteRankFreeModuleElement(fmodule, name=None, latex_name=None)
```

Bases: `sage.tensor.modules.alternating_contr_tensor.AlternatingContrTensor`

Element of a free module of finite rank over a commutative ring.

Input:

- `fmodule` – free module $M$ of finite rank over a commutative ring $R$, as an instance of `FiniteRankFreeModule`.
- `name` – (default: None) name given to the element.
- `latex_name` – (default: None) LaTeX symbol to denote the element; if none is provided, the LaTeX symbol is set to `name`.

Note that for free modules of finite rank, this map is actually an isomorphism, enabling the canonical identification: $M^{**} = M$. 
EXAMPLES:

Let us consider a rank-3 free module $M$ over $\mathbb{Z}$:

```sage
definitions:
M = FiniteRankFreeModule(ZZ, 3, name='M')
e = M.basis('e'); e
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
```

There are three ways to construct an element of the free module $M$: the first one (recommended) is using the free module:

```sage:
v = M([2,0,-1], basis=e, name='v')
Element v of the Rank-3 free module M over the Integer Ring
sage:
v.display()  # expansion on the default basis (e)
v = 2 e_0 - e_2
sage:
v.parent() is M
True
```

The second way is to construct a tensor of type $(1,0)$ on $M$ (cf. the canonical identification $M^{**} = M$ recalled above):

```sage:
v2 = M.tensor((1,0), name='v')
v2[0], v2[2] = 2, -1 ; v2
Element v of the Rank-3 free module M over the Integer Ring
sage:
v2.display()  # expansion on the default basis (e)
v = 2 e_0 - e_2
sage:
v2 == v
True
```

Finally, the third way is via some linear combination of the basis elements:

```sage:
v3 = 2*e[0] - e[2]
v3.set_name('v')  # in this case, the name has to be set separately
Element v of the Rank-3 free module M over the Integer Ring
sage:
v3.display()
v = 2 e_0 - e_2
sage:
v3 == v
True
```

The canonical identification $M^{**} = M$ is implemented by letting the module elements act on linear forms, providing the same result as the reverse operation (cf. the map $\Phi$ defined above):

```sage:
a = M.linear_form(name='a')
a[:] = (2, 1, -3); a
Linear form a on the Rank-3 free module M over the Integer Ring
sage:
a(v)
7
sage:
a(v) == v(a)
True
```
ARITHMETIC EXAMPLES

Addition:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e') ; e
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
sage: a = M([0,1,3], name='a') ; a
Element a of the Rank-3 free module M over the Integer Ring
sage: a.display()
a = e_1 + 3 e_2
sage: b = M([2,-2,1], name='b') ; b
Element b of the Rank-3 free module M over the Integer Ring
sage: b.display()
b = 2 e_0 - 2 e_1 + e_2
sage: s = a + b ; s
Element a+b of the Rank-3 free module M over the Integer Ring
sage: s.display()
a+b = 2 e_0 - e_1 + 4 e_2
sage: all(s[i] == a[i] + b[i] for i in M.irange())
True
```

Subtraction:

```python
sage: s = a - b ; s
Element a-b of the Rank-3 free module M over the Integer Ring
sage: s.display()
a-b = -2 e_0 + 3 e_1 + 2 e_2
sage: all(s[i] == a[i] - b[i] for i in M.irange())
True
```

Multiplication by a scalar:

```python
sage: s = 2*a ; s
Element of the Rank-3 free module M over the Integer Ring
sage: s.display()
2 e_1 + 6 e_2
sage: a.display()
a = e_1 + 3 e_2
```

Tensor product:

```python
sage: s = a*b ; s
Type-(2,0) tensor a⊗b on the Rank-3 free module M over the Integer Ring
sage: s.symmetries()
no symmetry; no antisymmetry
sage: s[:]
[[0 0 0], [0 0 0], [0 0 0]]
sage: s = a⊗a⊗b ; s
Type-(3,0) tensor a⊗a⊗b on the Rank-3 free module M over the Integer Ring
sage: s[:]
[[[0, 0, 0], [0, 0, 0], [0, 0, 0]],...]
```

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Exterior product:

```
sage: s = a.wedge(b) ; s
Alternating contravariant tensor a∧b of degree 2 on the Rank-3 free module M over the Integer Ring
sage: s.display()
a∧b = -2 e_0∧e_1 - 6 e_0∧e_2 + 7 e_1∧e_2
```
CHAPTER THREE

FREE MODULE BASES

The class `FreeModuleBasis` implements bases on a free module $M$ of finite rank over a commutative ring, while the class `FreeModuleCoBasis` implements the dual bases (i.e. bases of the dual module $M^*$).

AUTHORS:

- Travis Scrimshaw (2016): ABC Basis_abstract and list functionality for bases (trac ticket #20770)
- Eric Gourgoulhon (2018): some refactoring and more functionalities in the choice of symbols for basis elements (trac ticket #24792)

REFERENCES:

- Chap. 10 of R. Godement: *Algebra* [God1968]
- Chap. 3 of S. Lang: *Algebra* [Lan2002]

```python
class sage.tensor.modules.free_module_basis.Basis_abstract(fmodule, symbol, latex_symbol, indices, latex_indices):
    Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.sage_object.SageObject

```

Abstract base class for (dual) bases of free modules.

**free_module()**

Return the free module of `self`.

**EXAMPLES:**

```
sage: M = FiniteRankFreeModule(QQ, 2, name='M', start_index=1)
sage: e = M.basis('e')
sage: e.free_module() is M
True
```

**set_name(symbol, latex_symbol=None, indices=None, latex_indices=None, index_position='down')**

Set (or change) the text name and LaTeX name of `self`.

**INPUT:**

- `symbol` – either a string, to be used as a common base for the symbols of the elements of `self`, or a list of strings, representing the individual symbols of the elements of `self`
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the elements of `self`, or a list of strings, representing the individual LaTeX symbols of the elements of `self`; if `None`, `symbol` is used in place of `latex_symbol`
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- **indices** – (default: None; used only if `symbol` is a single string) tuple of strings representing the indices labelling the elements of `self`; if None, the indices will be generated as integers within the range declared on the free module on which `self` is defined

- **latex_indices** – (default: None) list of strings representing the indices for the LaTeX symbols of the elements of `self`; if None, `indices` is used instead

- **index_position** – (default: 'down') determines the position of the indices labelling the individual elements of `self`; can be either 'down' or 'up'

**EXAMPLES:**

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e'); e
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
sage: e.set_name('f'); e
Basis (f_0,f_1,f_2) on the Rank-3 free module M over the Integer Ring
sage: e.set_name(['a', 'b', 'c']); e
Basis (a,b,c) on the Rank-3 free module M over the Integer Ring
sage: e.set_name('e', indices=['x', 'y', 'z']); e
Basis (e_x,e_y,e_z) on the Rank-3 free module M over the Integer Ring
sage: e.set_name('e', index_position='up'); e
Basis (e^0,e^1,e^2) on the Rank-3 free module M over the Integer Ring
sage: latex(e)
\left(e^{0},e^{1},e^{2}\right)
```

**class** `sage.tensor.modules.free_module_basis.FreeModuleBasis`

```
Bases: sage.tensor.modules.free_module_basis.Basis_abstract
```

Basis of a free module over a commutative ring \( R \).

**INPUT:**

- **fmodule** – free module \( M \) (as an instance of `FiniteRankFreeModule`)

- **symbol** – either a string, to be used as a common base for the symbols of the elements of the basis, or a tuple of strings, representing the individual symbols of the elements of the basis

- **latex_symbol** – (default: None) either a string, to be used as a common base for the LaTeX symbols of the elements of the basis, or a tuple of strings, representing the individual LaTeX symbols of the elements of the basis; if None, `symbol` is used in place of `latex_symbol`

- **indices** – (default: None; used only if `symbol` is a single string) tuple of strings representing the indices labelling the elements of the basis; if None, the indices will be generated as integers within the range
declared on \texttt{fmodule}

- \texttt{latex_indices} – (default: \texttt{None}) tuple of strings representing the indices for the LaTeX symbols of the elements of the basis; if \texttt{None}, \texttt{indices} is used instead

- \texttt{symbol_dual} – (default: \texttt{None}) same as \texttt{symbol} but for the dual basis; if \texttt{None}, \texttt{symbol} must be a string and is used for the common base of the symbols of the elements of the dual basis

- \texttt{latex_symbol_dual} – (default: \texttt{None}) same as \texttt{latex_symbol} but for the dual basis

\textbf{EXAMPLES:}

A basis on a rank-3 free module over $\mathbb{Z}$:

```
sage: M0 = FiniteRankFreeModule(ZZ, 3, name='M_0')
sage: from sage.tensor.modules.free_module_basis import FreeModuleBasis
sage: e = FreeModuleBasis(M0, 'e') ; e
Basis (e_0,e_1,e_2) on the Rank-3 free module M_0 over the Integer Ring
```

Instead of importing \texttt{FreeModuleBasis} in the global name space, it is recommended to use the module’s method \texttt{basis()}:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e') ; e
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
```

The individual elements constituting the basis are accessed via the square bracket operator:

```
sage: e[0]
Element e_0 of the Rank-3 free module M over the Integer Ring
sage: e[0] in M
True
```

The slice operator 
 \texttt{:} can be used to access to more than one element:

```
sage: e[0:2]
(\text{Element } e_0 \text{ of the Rank-3 free module } M \text{ over the Integer Ring, } \text{Element } e_1 \text{ of the Rank-3 free module } M \text{ over the Integer Ring})
sage: e[:]
(\text{Element } e_0 \text{ of the Rank-3 free module } M \text{ over the Integer Ring, } \text{Element } e_1 \text{ of the Rank-3 free module } M \text{ over the Integer Ring, } \text{Element } e_2 \text{ of the Rank-3 free module } M \text{ over the Integer Ring})
```

The LaTeX symbol can be set explicitly:

```
sage: latex(e)
\left(e_{0},e_{1},e_{2}\right)
sage: eps = M.basis('eps', latex_symbol=r'\epsilon') ; eps
Basis (eps_0,eps_1,eps_2) on the Rank-3 free module M over the Integer Ring
sage: latex(eps)
\left(\epsilon_{0},\epsilon_{1},\epsilon_{2}\right)
```

The individual elements of the basis are labelled according the parameter \texttt{start_index} provided at the free module construction:
It is also possible to fully customize the labels, via the argument indices:

```python
sage: f = M.basis('f', indices=('x', 'y', 'z')); f
Basis (f_x,f_y,f_z) on the Rank-3 free module M over the Integer Ring
sage: f[1]
Element f_x of the Rank-3 free module M over the Integer Ring
```

The symbol of each element of the basis can also be freely chosen, by providing a tuple of symbols as the first argument of `basis`; it is then mandatory to specify some symbols for the dual basis as well:

```python
sage: g = M.basis(('a', 'b', 'c'), symbol_dual=('A', 'B', 'C')); g
Basis (a,b,c) on the Rank-3 free module M over the Integer Ring
sage: g[1]
Element a of the Rank-3 free module M over the Integer Ring
sage: g.dual_basis()[1]
Linear form A on the Rank-3 free module M over the Integer Ring
```

dual_basis()
Return the basis dual to self.

OUTPUT:

• instance of FreeModuleCoBasis representing the dual of self

EXAMPLES:

Dual basis on a rank-3 free module:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1)
sage: e = M.basis('e'); e
Basis (e_1,e_2,e_3) on the Rank-3 free module M over the Integer Ring
sage: f = e.dual_basis(); f
Dual basis (e^1,e^2,e^3) on the Rank-3 free module M over the Integer Ring
```

Let us check that the elements of f are elements of the dual of M:

```python
sage: f[1] in M.dual()
True
sage: f[1]
Linear form e^1 on the Rank-3 free module M over the Integer Ring
```

and that f is indeed the dual of e:

```python
sage: f[1](e[1]), f[1](e[2]), f[1](e[3])
(1, 0, 0)
sage: f[2](e[1]), f[2](e[2]), f[2](e[3])
(0, 1, 0)
sage: f[3](e[1]), f[3](e[2]), f[3](e[3])
(0, 0, 1)
```
module()

Return the free module on which the basis is defined.

OUTPUT:

• instance of :class:`FiniteRankFreeModule` representing the free module of which `self` is a basis

EXAMPLES:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: e.module()
Rank-3 free module M over the Integer Ring
sage: e.module() is M
True
```

new_basis(
``change_of_basis``, `symbol`, `latex_symbol`=`None`, `indices`=`None`, `latex_indices`=`None`,
``symbol_dual`=`None`, `latex_symbol_dual`=`None``)

Define a new module basis from `self`.

The new basis is defined by means of a module automorphism.

INPUT:

• `change_of_basis` – instance of :class:`FreeModuleAutomorphism` describing the automorphism \(P\) that relates the current basis \((e_i)\) (described by `self`) to the new basis \((n_i)\) according to \(n_i = P(e_i)\)

• `symbol` – either a string, to be used as a common base for the symbols of the elements of the basis, or a tuple of strings, representing the individual symbols of the elements of the basis

• `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the elements of the basis, or a tuple of strings, representing the individual LaTeX symbols of the elements of the basis; if `None`, `symbol` is used in place of `latex_symbol`

• `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the elements of the basis; if `None`, the indices will be generated as integers within the range declared on the free module on which `self` is defined

• `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the elements of the basis; if `None`, `indices` is used instead

• `symbol_dual` – (default: `None`) same as `symbol` but for the dual basis; if `None`, `symbol` must be a string and is used for the common base of the symbols of the elements of the dual basis

• `latex_symbol_dual` – (default: `None`) same as `latex_symbol` but for the dual basis

OUTPUT:

• the new basis \((n_i)\), as an instance of :class:`FreeModuleBasis`

EXAMPLES:

Change of basis on a vector space of dimension 2:

```
sage: M = FiniteRankFreeModule(QQ, 2, name='M', start_index=1)
sage: e = M.basis('e')
sage: a = M.automorphism()
sage: a[:] = [[1, 2], [-1, 3]]
sage: f = e.new_basis(a, 'f') ; f
Basis \((f_1,f_2)\) on the 2-dimensional vector space \(M\) over the Rational Field
```

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```
sage: f[1].display()
f_1 = e_1 - e_2
sage: f[2].display()
f_2 = 2 e_1 + 3 e_2
sage: e[1].display(f)
e_1 = 3/5 f_1 + 1/5 f_2
sage: e[2].display(f)
e_2 = -2/5 f_1 + 1/5 f_2
```

Use of some keyword arguments:

```
sage: b = e.new_basis(a, 'b', indices=('x', 'y'),
........: symbol_dual=('A', 'B'))
sage: b
Basis (b_x,b_y) on the 2-dimensional vector space M over the Rational Field
sage: b.dual_basis()
Dual basis (A,B) on the 2-dimensional vector space M over the Rational Field
```

```
class sage.tensor.modules.free_module_basis.FreeModuleCoBasis(basis, symbol, latex_symbol=None, indices=None, latex_indices=None):

    Bases: sage.tensor.modules.free_module_basis.Basis_abstract

    Dual basis of a free module over a commutative ring.

    INPUT:

    • basis – basis of a free module \( M \) of which self is the dual (must be an instance of FreeModuleBasis)
    • symbol – either a string, to be used as a common base for the symbols of the elements of the cobasis, or a
tuple of strings, representing the individual symbols of the elements of the cobasis
    • latex_symbol – (default: None) either a string, to be used as a common base for the \( \text{LaTeX} \) symbols of
the elements of the cobasis, or a tuple of strings, representing the individual \( \text{LaTeX} \) symbols of the elements
of the cobasis; if None, symbol is used in place of latex_symbol
    • indices – (default: None; used only if symbol is a single string) tuple of strings representing the indices
labelling the elements of the cobasis; if None, the indices will be generated as integers within the range
declared on the free module on which the cobasis is defined
    • latex_indices – (default: None) tuple of strings representing the indices for the \( \text{LaTeX} \) symbols of the
elements of the cobasis; if None, indices is used instead

    EXAMPLES:

    Dual basis on a rank-3 free module:

    sage: M = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1)
sage: e = M.basis('e') ; e
    Basis (e_1,e_2,e_3) on the Rank-3 free module M over the Integer Ring
sage: from sage.tensor.modules.free_module_basis import FreeModuleCoBasis
sage: f = FreeModuleCoBasis(e, 'f') ; f
    Dual basis (f^1,f^2,f^3) on the Rank-3 free module M over the Integer Ring

    Instead of importing FreeModuleCoBasis in the global name space, it is recommended to use the method
dual_basis() of the basis e:
```
Let us check that the elements of $f$ are in the dual of $M$:

```
sage: f[1]
Linear form $e^1$ on the Rank-3 free module $M$ over the Integer Ring
sage: f[1] in M.dual()
True
```

and that $f$ is indeed the dual of $e$:

```
sage: f[1](e[1]), f[1](e[2]), f[1](e[3])
(1, 0, 0)
sage: f[2](e[1]), f[2](e[2]), f[2](e[3])
(0, 1, 0)
sage: f[3](e[1]), f[3](e[2]), f[3](e[3])
(0, 0, 1)
```
4.1 Tensor products of free modules

The class `TensorFreeModule` implements tensor products of the type

\[ T^{(k,l)}(M) = M \otimes \cdots \otimes M \otimes M^* \otimes \cdots \otimes M^*, \]

where \( M \) is a free module of finite rank over a commutative ring \( R \) and \( M^* = \text{Hom}_R(M, R) \) is the dual of \( M \). Note that \( T^{(1,0)}(M) = M \) and \( T^{(0,1)}(M) = M^* \).

Thanks to the canonical isomorphism \( M^{**} \simeq M \) (which holds since \( M \) is a free module of finite rank), \( T^{(k,l)}(M) \) can be identified with the set of tensors of type \((k, l)\) defined as multilinear maps

\[ \underbrace{M^* \times \cdots \times M^*} \times \underbrace{M \times \cdots \times M} \rightarrow R \]

\( k \) times \hspace{1cm} \( t \) times

Accordingly, `TensorFreeModule` is a Sage parent class, whose element class is `FreeModuleTensor`. \( T^{(k,l)}(M) \) is itself a free module over \( R \), of rank \( n^{k+l} \), \( n \) being the rank of \( M \). Accordingly the class `TensorFreeModule` inherits from the class `FiniteRankFreeModule`.

**Todo:** implement more general tensor products, i.e. tensor product of the type \( M_1 \otimes \cdots \otimes M_n \), where the \( M_i \)'s are \( n \) free modules of finite rank over the same ring \( R \).

**REFERENCES:**

- K. Conrad: Tensor products [Con2015]
- Chap. 21 (Exer. 4) of R. Godement: Algebra [God1968]
- Chap. 16 of S. Lang: Algebra [Lan2002]
As recalled above, $T^{(k,l)}(M)$ can be canonically identified with the set of tensors of type $(k, l)$ on $M$.

This is a Sage parent class, whose element class is FreeModuleTensor.

**INPUT:**

- `fmodule` – free module $M$ of finite rank over a commutative ring $R$, as an instance of FiniteRankFreeModule
- `tensor_type` – pair $(k, l)$ with $k$ being the contravariant rank and $l$ the covariant rank
- `name` – (default: None) string; name given to the tensor module
- `latex_name` – (default: None) string; LaTeX symbol to denote the tensor module; if none is provided, it is set to name

**EXAMPLES:**

Set of tensors of type $(1, 2)$ on a free $\mathbb{Z}$-module of rank 3:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: from sage.tensor.modules.tensor_free_module import TensorFreeModule
sage: T = TensorFreeModule(M, (1,2)) ; T
Free module of type-(1,2) tensors on the Rank-3 free module M over the Integer Ring
```

Instead of importing TensorFreeModule in the global name space, it is recommended to use the module’s method `tensor_module()`:

```python
sage: T = M.tensor_module(1,2) ; T
Free module of type-(1,2) tensors on the Rank-3 free module M over the Integer Ring
sage: latex(T)
T^{(1, 2)}(M)
```

The module $M$ itself is considered as the set of tensors of type $(1, 0)$:

```python
sage: M is M.tensor_module(1,0)
True
```

$T$ is a module (actually a free module) over $\mathbb{Z}$:

```python
sage: T.category()
Category of finite dimensional modules over Integer Ring
sage: T in Modules(ZZ)
True
sage: T.rank()
27
sage: T.base_ring()
Integer Ring
sage: T.base_module()
Rank-3 free module M over the Integer Ring
```

$T$ is a parent object, whose elements are instances of FreeModuleTensor:

```python
sage: t = T.an_element() ; t
Type-(1,2) tensor on the Rank-3 free module M over the Integer Ring
```
Elements can be constructed from $T$. In particular, 0 yields the zero element of $T$:

```python
sage: T(0)
Type-(1,2) tensor zero on the Rank-3 free module $M$ over the Integer Ring
sage: T(0) is T.zero()
True
```

while non-zero elements are constructed by providing their components in a given basis:

```python
sage: e
Basis (e_0,e_1,e_2) on the Rank-3 free module $M$ over the Integer Ring
sage: comp = [[[i-j+k for k in range(3)] for j in range(3)] for i in range(3)]
```

```python
sage: t = T(comp, basis=e, name='t') ; t
Type-(1,2) tensor $t$ on the Rank-3 free module $M$ over the Integer Ring
sage: t.comp(e)[:]
[[[0, 1, 2], [-1, 0, 1], [-2, -1, 0]],
 [[1, 2, 3], [0, 1, 2], [-1, 0, 1]],
 [[2, 3, 4], [1, 2, 3], [0, 1, 2]]]
sage: t.display(e)
t = e_0⊗e^0⊗e^1 + 2 e_0⊗e^0⊗e^2 - e_0⊗e^1⊗e^0 + e_0⊗e^1⊗e^2
- 2 e_0⊗e^2⊗e^0 - e_0⊗e^2⊗e^1 + e_1⊗e^0⊗e^0 + 2 e_1⊗e^0⊗e^1
+ 3 e_1⊗e^0⊗e^2 + e_1⊗e^1⊗e^0 + 2 e_1⊗e^1⊗e^1 - e_1⊗e^2⊗e^0
+ e_2⊗e^0⊗e^2 + 2 e_2⊗e^0⊗e^0 + 3 e_2⊗e^0⊗e^1 + 4 e_2⊗e^0⊗e^2
+ e_2⊗e^1⊗e^0 + 2 e_2⊗e^1⊗e^1 + 3 e_2⊗e^1⊗e^2 + e_2⊗e^2⊗e^1
+ 2 e_2⊗e^2⊗e^2
```

An alternative is to construct the tensor from an empty list of components and to set the nonzero components afterwards:

```python
sage: t = T([], name='t')
sage: t.set_comp(e)[0,1,1] = -3
sage: t.set_comp(e)[2,0,1] = 4
sage: t.display(e)
# notice that $t^2_{10}$ has be set equal to $t^2_{01}$ by symmetry
t = -3 e_0⊗e^1⊗e^1 + 4 e_2⊗e^0⊗e^1
```

See the documentation of `FreeModuleTensor` for the full list of arguments that can be provided to the `__call__` operator. For instance, to construct a tensor symmetric with respect to the last two indices:

```python
sage: t = T([], name='t', sym=(1,2))
sage: t.set_comp(e)[0,1,1] = -3
sage: t.set_comp(e)[2,0,1] = 4
sage: t.display(e)  # notice that $t^2_{\{10\}}$ has be set equal to $t^2_{\{01\}}$ by symmetry
t = -3 e_0⊗e^1⊗e^1 + 4 e_2⊗e^0⊗e^1 + 4 e_2⊗e^1⊗e^0
```

The tensor modules over a given module $M$ are unique.
Tensors on free modules of finite rank, Release 9.6

```sage
T is M.tensor_module(1,2)
True
```

There is a coercion map from $\Lambda^p(M^*)$, the set of alternating forms of degree $p$, to $T^{(0,p)}(M)$:

```sage
L2 = M.dual_exterior_power(2) ; L2
2nd exterior power of the dual of the Rank-3 free module M over the
Integer Ring
T02 = M.tensor_module(0,2) ; T02
Free module of type-(0,2) tensors on the Rank-3 free module M over the
Integer Ring
T02.has_coerce_map_from(L2)
True
```

Of course, for $p \geq 2$, there is no coercion in the reverse direction, since not every tensor of type $(0,p)$ is alternating:

```sage
L2.has_coerce_map_from(T02)
False
```

The coercion map $\Lambda^2(M^*) \to T^{(0,2)}(M)$ in action:

```sage
a = M.alternating_form(2, name='a') ; a
Alternating form a of degree 2 on the Rank-3 free module M over the
Integer Ring
a[0,1], a[1,2] = 4, -3
a.display(e)
a = 4 e^0 \wedge e^1 - 3 e^1 \wedge e^2
a.parent() is L2
True
ta = T02(a) ; ta
Type-(0,2) tensor a on the Rank-3 free module M over the Integer Ring
ta.display(e)
a = 4 e^0 \otimes e^1 - 4 e^1 \otimes e^0 - 3 e^1 \otimes e^2 + 3 e^2 \otimes e^1
ta.symmetries() # the antisymmetry is of course preserved
no symmetry; antisymmetry: (0, 1)
```

For the degree $p = 1$, there is a coercion in both directions:

```sage
L1 = M.dual_exterior_power(1) ; L1
Dual of the Rank-3 free module M over the Integer Ring
T01 = M.tensor_module(0,1) ; T01
Free module of type-(0,1) tensors on the Rank-3 free module M over the
Integer Ring
T01.has_coerce_map_from(L1)
True
L1.has_coerce_map_from(T01)
True
```

The coercion map $\Lambda^1(M^*) \to T^{(0,1)}(M)$ in action:

```sage
a = M.linear_form('a')
a[:] = -2, 4, 1 ; a.display(e)
a = -2 e^0 + 4 e^1 + e^2
```

(continues on next page)
sage: a.parent() is L1
True
sage: ta = T01(a) ; ta
Type-(0,1) tensor a on the Rank-3 free module M over the Integer Ring
sage: ta.display(e)
a = -2 e^0 + 4 e^1 + e^2
The coercion map $T^{(0,1)}(M) \rightarrow \Lambda^1(M^*)$ in action:

sage: ta.parent() is T01
True
sage: lta = L1(ta) ; lta
Linear form a on the Rank-3 free module M over the Integer Ring
sage: lta.display(e)
a = -2 e^0 + 4 e^1 + e^2
sage: lta == a
True

There is a canonical identification between tensors of type $(1, 1)$ and endomorphisms of module $M$. Accordingly, coercion maps have been implemented between $T^{(1,1)}(M)$ and $\text{End}(M)$ (the module of all endomorphisms of $M$, see \texttt{FreeModuleHomset}):}

sage: T11 = M.tensor_module(1,1) ; T11
Free module of type-(1,1) tensors on the Rank-3 free module M over the Integer Ring
sage: End(M)
Set of Morphisms from Rank-3 free module M over the Integer Ring
to Rank-3 free module M over the Integer Ring
in Category of finite dimensional modules over Integer Ring
sage: T11.has_coerce_map_from(End(M))
True
sage: End(M).has_coerce_map_from(T11)
True
The coercion map $\text{End}(M) \rightarrow T^{(1,1)}(M)$ in action:

sage: phi = End(M).an_element() ; phi
Generic endomorphism of Rank-3 free module M over the Integer Ring
sage: phi.matrix(e)
[1 1 1]
[1 1 1]
[1 1 1]
sage: tphi = T11(phi) ; tphi # image of phi by the coercion map
Type-(1,1) tensor on the Rank-3 free module M over the Integer Ring
sage: tphi[:]
[1 1 1]
[1 1 1]
[1 1 1]
sage: t = M.tensor((1,1))
sage: t[0,0], t[1,1], t[2,2] = -1,-2,-3
sage: t[:]
[-1 0 0]
[0 -2 0]

4.1. Tensor products of free modules
Tensors on free modules of finite rank, Release 9.6

[ 0 0 -3]
sage: s = t + phi ; s  # phi is coerced to a type-(1,1) tensor prior to the addition
Type-(1,1) tensor on the Rank-3 free module \( M \) over the Integer Ring
sage: s[:,:]
[ 0 1 1]
[ 1 -1 1]
[ 1 1 -2]

The coercion map \( T^{(1,1)}(M) \rightarrow \text{End}(M) \) in action:

sage: phi1 = End(M)(tphi) ; phi1
Generic endomorphism of Rank-3 free module \( M \) over the Integer Ring
sage: phi1 == phi
True
sage: s = phi + t ; s  # t is coerced to an endomorphism prior to the addition
Generic endomorphism of Rank-3 free module \( M \) over the Integer Ring
sage: s.matrix(e)
[ 0 1 1]
[ 1 -1 1]
[ 1 1 -2]

There is a coercion \( \text{GL}(M) \rightarrow T^{(1,1)}(M) \), i.e. from automorphisms of \( M \) to type-(1,1) tensors on \( M \):

sage: GL = M.general_linear_group() ; GL
General linear group of the Rank-3 free module \( M \) over the Integer Ring
sage: T11.has_coerce_map_from(GL)
True

The coercion map \( \text{GL}(M) \rightarrow T^{(1,1)}(M) \) in action:

sage: a = GL.an_element() ; a
Automorphism of the Rank-3 free module \( M \) over the Integer Ring
sage: a.matrix(e)
[ 1 0 0]
[ 0 -1 0]
[ 0 0 1]
sage: ta = T11(a) ; ta
Type-(1,1) tensor on the Rank-3 free module \( M \) over the Integer Ring
sage: ta.display(e)
e_0\otimes e^0 - e_1\otimes e^1 + e_2\otimes e^2
sage: a.display(e)
e_0\otimes e^0 - e_1\otimes e^1 + e_2\otimes e^2

Of course, there is no coercion in the reverse direction, since not every type-(1,1) tensor is invertible:

sage: GL.has_coerce_map_from(T11)
False

Element

alias of \sage.tensor.modules.free_module_tensor.FreeModuleTensor

base_module()

Return the free module on which \texttt{self} is constructed.

OUTPUT:
• instance of `FiniteRankFreeModule` representing the free module on which the tensor module is defined.

**EXAMPLES:**

Base module of a type-(1, 2) tensor module:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: T = M.tensor_module(1,2)
sage: T.base_module()  # Rank-3 free module M over the Integer Ring
sage: T.base_module() is M
True
```

`tensor_type()`

Return the tensor type of `self`.

**OUTPUT:**

• pair \((k, l)\) such that `self` is the module tensor product \(T^{(k,l)}(M)\)

**EXAMPLES:**

```python
sage: M = FiniteRankFreeModule(ZZ, 3)
sage: T = M.tensor_module(1,2)
sage: T.tensor_type()
(1, 2)
```

`zero()`

Return the zero of `self`.

**EXAMPLES:**

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: T11 = M.tensor_module(1,1)
sage: T11.zero()
Type-(1,1) tensor zero on the Rank-3 free module M over the Integer Ring

The zero element is cached:

```python
sage: T11.zero() is T11(0)
True
```

## 4.2 Tensors on free modules

The class `FreeModuleTensor` implements tensors on a free module \(M\) of finite rank over a commutative ring. A tensor of type \((k, l)\) on \(M\) is a multilinear map:

\[
M^* \times \cdots \times M^* \times M \times \cdots \times M \rightarrow R
\]

where \(R\) is the commutative ring over which the free module \(M\) is defined and \(M^* = \text{Hom}_R(M, R)\) is the dual of \(M\). The integer \(k + l\) is called the **tensor rank**. The set \(T^{(k,l)}(M)\) of tensors of type \((k, l)\) on \(M\) is a free module of finite rank over \(R\), described by the class `TensorFreeModule`. 

### 4.2. Tensors on free modules
Various derived classes of \texttt{FreeModuleTensor} are devoted to specific tensors:

- \texttt{AlternatingContrTensor} for fully antisymmetric type-\((k,0)\) tensors (alternating contravariant tensors);
  - \texttt{FiniteRankFreeModuleElement} for elements of \(M\), considered as type-(1,0) tensors thanks to the canonical identification \(M^{**} = M\) (which holds since \(M\) is a free module of finite rank);

- \texttt{FreeModuleAltForm} for fully antisymmetric type-(0,1) tensors (alternating forms);

- \texttt{FreeModuleAutomorphism} for type-(1,1) tensors representing invertible endomorphisms.

Each of these classes is a Sage \textit{element} class, the corresponding \textit{parent} class being:

- \texttt{TensorFreeModule} for \texttt{FreeModuleTensor};
- \texttt{FiniteRankFreeModule} for \texttt{FiniteRankFreeModuleElement};
- \texttt{ExtPowerFreeModule} for \texttt{AlternatingContrTensor};
- \texttt{ExtPowerDualFreeModule} for \texttt{FreeModuleAltForm};
- \texttt{FreeModuleLinearGroup} for \texttt{FreeModuleAutomorphism}.

**AUTHORS:**

- Michael Jung (2019): improve treatment of the zero element; add method \texttt{copy_from}

**REFERENCES:**

- Chap. 21 of R. Godement: \textit{Algebra} [God1968]
- Chap. 12 of J. M. Lee: \textit{Introduction to Smooth Manifolds} [Lee2013] (only when the free module is a vector space)
- Chap. 2 of B. O’Neill: \textit{Semi-Riemannian Geometry} [ONe1983]

**EXAMPLES:**

A tensor of type \((1,1)\) on a rank-3 free module over \(\mathbb{Z}\):

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: t = M.tensor((1,1), name='t') ; t
Type-(1,1) tensor t on the Rank-3 free module M over the Integer Ring
sage: t.parent()
Free module of type-(1,1) tensors on the Rank-3 free module M over the Integer Ring
sage: t.parent() is M.tensor_module(1,1)
True
sage: t in M.tensor_module(1,1)
True
```

Setting some component of the tensor in a given basis:

```python
sage: e = M.basis('e') ; e
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
sage: t.set_comp(e)[0,0] = -3  # the component [0,0] w.r.t. basis e is set to -3
```

The unset components are assumed to be zero:
The commands `t.set_comp(e)` and `t.comp(e)` can be abridged by providing the basis as the first argument in the square brackets:

```
sage: t[e,0,0] = -3
sage: t[e,:] = [-3 0 0]
```

Actually, since `e` is `M`'s default basis, the mention of `e` can be omitted:

```
sage: t[0,0] = -3
sage: t[:] = [-3 0 0]
```

For tensors of rank 2, the matrix of components w.r.t. a given basis is obtained via the function `matrix`:

```
sage: matrix(t.comp(e))
[[-3 0 0]
 [ 0 0 0]
 [ 0 0 0]]
sage: matrix(t.comp(e)).parent()
Full MatrixSpace of 3 by 3 dense matrices over Integer Ring
```

Tensor components can be modified (reset) at any time:

```
sage: t[0,0] = 0
sage: t[:] = [0 0 0]
```

Checking that `t` is zero:

```
sage: t.is_zero()
True
sage: t == 0
True
sage: t == M.tensor_module(1,1).zero()  # the zero element of the module of all type-(1, -1) tensors on M
True
```

The components are managed by the class `Components`:
Tensors on free modules of finite rank, Release 9.6

```python
sage: type(t.comp(e))
<class 'sage.tensor.modules.comp.Components'>
```

Only non-zero components are actually stored, in the dictionary `_comp` of class `Components`, whose keys are the indices:

```python
sage: t.comp(e)._comp
{}
sage: t.set_comp(e)[0,0] = -3 ; t.set_comp(e)[1,2] = 2
sage: t.comp(e)._comp  # random output order (dictionary)
{(0, 0): -3, (1, 2): 2}
sage: t.display(e)
t = -3 e_0 \otimes e^0 + 2 e_1 \otimes e^2
```

Further tests of the comparison operator:

```python
sage: t.is_zero()
False
sage: t == 0
False
sage: t == M.tensor_module(1,1).zero()
False
sage: t1 = t.copy()
sage: t1 == t
True
sage: t1[2,0] = 4
sage: t1 == t
False
```

As a multilinear map $M^* \times M \to \mathbb{Z}$, the type-(1,1) tensor $t$ acts on pairs formed by a linear form and a module element:

```python
sage: a = M.linear_form(name='a') ; a[:] = (2, 1, -3) ; a
Linear form a on the Rank-3 free module M over the Integer Ring
sage: b = M([1,-6,2], name='b') ; b
Element b of the Rank-3 free module M over the Integer Ring
sage: t(a,b)
-2
```

```python
class sage.tensor.modules.free_module_tensor.FreeModuleTensor(fmodule, tensor_type, name=None, latex_name=None, sym=None, antisym=None, parent=None)
```

Bases: `sage.structure.element.ModuleElementWithMutability`

Tensor over a free module of finite rank over a commutative ring.

This is a Sage element class, the corresponding parent class being `TensorFreeModule`.

**INPUT:**

- `fmodule` – free module $M$ of finite rank over a commutative ring $R$, as an instance of `FiniteRankFreeModule`
- `tensor_type` – pair $(k, \ell)$ with $k$ being the contravariant rank and $\ell$ the covariant rank
- `name` – (default: None) name given to the tensor
\* latex_name – (default: None) \LaTeX symbol to denote the tensor; if none is provided, the \LaTeX symbol is set to name

\* sym – (default: None) a symmetry or a list of symmetries among the tensor arguments: each symmetry is described by a tuple containing the positions of the involved arguments, with the convention position=0 for the first argument. For instance:

- sym = (0,1) for a symmetry between the 1st and 2nd arguments;
- sym = [(0,2), (1,3,4)] for a symmetry between the 1st and 3rd arguments and a symmetry between the 2nd, 4th and 5th arguments.

\* antisym – (default: None) antisymmetry or list of antisymmetries among the arguments, with the same convention as for sym

\* parent – (default: None) some specific parent (e.g. exterior power for alternating forms); if None, fmodule.tensor_module(k,l) is used

EXAMPLES:

A tensor of type (1, 1) on a rank-3 free module over \( \mathbb{Z} \):

\begin{verbatim}
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: t = M.tensor((1,1), name='t') ; t
Type-(1,1) tensor t on the Rank-3 free module M over the Integer Ring
\end{verbatim}

Tensors are Element objects whose parents are tensor free modules:

\begin{verbatim}
sage: t.parent()
Free module of type-(1,1) tensors on the Rank-3 free module M over the Integer Ring
sage: t.parent() is M.tensor_module(1,1)
True
\end{verbatim}

\textbf{add_comp}(basis=None)

Return the components of self w.r.t. a given module basis for assignment, keeping the components w.r.t. other bases.

To delete the components w.r.t. other bases, use the method \texttt{set_comp()} instead.

INPUT:

\* basis – (default: None) basis in which the components are defined; if none is provided, the components are assumed to refer to the module’s default basis

\textbf{Warning}: If the tensor has already components in other bases, it is the user’s responsibility to make sure that the components to be added are consistent with them.

OUTPUT:

\* components in the given basis, as an instance of the class \texttt{Components}; if such components did not exist previously, they are created

EXAMPLES:

Setting components of a type-(1,1) tensor:

\begin{verbatim}
"4.2. Tensors on free modules"
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: t = M.tensor((1,1), name='t')
sage: t.add_comp()[0,1] = -3
sage: t.display()
t = -3 e_0⊗e^1
sage: t.add_comp()[1,2] = 2
sage: t.display()
t = -3 e_0⊗e^1 + 2 e_1⊗e^2
sage: t.add_comp(e)
2-indices components w.r.t. Basis (e_0,e_1,e_2) on the
Rank-3 free module M over the Integer Ring
sage: f = M.basis('f')
sage: t.add_comp(f)[0,1] = 4
The components w.r.t. basis e have been kept:
sage: sorted(t._components, key=repr)
[Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring,
Basis (f_0,f_1,f_2) on the Rank-3 free module M over the Integer Ring]
sage: t.display(f)
t = 4 f_0⊗f^1
sage: t.display(e)
t = -3 e_0⊗e^1 + 2 e_1⊗e^2
Since zero is an immutable element, its components cannot be changed:
sage: z = M.tensor_module(1, 1).zero()
sage: z.add_comp(e)[0,1] = 1
Traceback (most recent call last):
  ... ValueError: the components of an immutable element cannot be changed

antisymmetrize(*pos, **kwargs)
Antisymmetrization over some arguments.
INPUT:
- pos – list of argument positions involved in the antisymmetrization (with the convention position=0 for the first argument); if none, the antisymmetrization is performed over all the arguments
- basis – (default: None) module basis with respect to which the component computation is to be performed; if none, the module's default basis is used if the tensor field has already components in it; otherwise another basis w.r.t. which the tensor has components will be picked

OUTPUT:
- the antisymmetrized tensor (instance of FreeModuleTensor)

EXAMPLES:
Antisymmetrization of a tensor of type (2,0):
Tensors on free modules of finite rank, Release 9.6

 sage: M = FiniteRankFreeModule(QQ, 3, name='M')
sage: e = M.basis('e')
sage: t = M.tensor((2,0))
sage: t[:,:] = [[1,-2,3], [4,5,6], [7,8,-9]]
sage: s = t.antisymmetrize() ; s
Alternating contravariant tensor of degree 2 on the 3-dimensional vector space M over the Rational Field
sage: s.symmetries()
no symmetry; antisymmetry: (0, 1)
sage: t[:], s[:]
( [ 1 -2 3] [ 0 -3 -2]  
 [ 4 5 6] [ 3 0 -1]  
 [ 7 8 -9], [ 2 1 0]  
)
sage: all(s[i,j] == 1/2*(t[i,j]-t[j,i]) # Check:
....:     for i in range(3) for j in range(3))
True
sage: s.antisymmetrize() == s # another test
True
sage: t.antisymmetrize() == t.antisymmetrize(0,1)
True

Antisymmetrization of a tensor of type (0,3) over the first two arguments:

 sage: t = M.tensor((0,3))
sage: t[:,:] = [[[1,2,3], [-4,5,6], [7,8,-9]],
.....:     [[10,-11,12], [13,14,-15], [16,17,18]],
.....:     [[19,-20,-21], [-22,23,24], [25,26,-27]]]
sage: s = t.antisymmetrize(0,1) ; s # (0,1) = the first two arguments
Type-(0,3) tensor on the 3-dimensional vector space M over the Rational Field
sage: s.symmetries()
no symmetry; antisymmetry: (0, 1)
sage: s[:]
[[[0, 0, 0], [-7, 8, -3], [-6, 14, 6]],
 [[7, -8, 3], [0, 0, 0], [19, -3, -3]],
 [6, -14, -6], [-19, 3, 3], [0, 0, 0]]
sage: all(s[i,j,k] == 1/2*(t[i,j,k]-t[j,i,k]) # Check:
....:     for i in range(3) for j in range(3) for k in range(3))
True
sage: s.antisymmetrize(0,1) == s # another test
True
sage: s.symmetrize(0,1) == 0 # of course
True

Instead of invoking the method antisymmetrize(), one can use the index notation with square brackets denoting the antisymmetrization; it suffices to pass the indices as a string inside square brackets:

 sage: s1 = t["_ij"k] ; s1
Type-(0,3) tensor on the 3-dimensional vector space M over the Rational Field
sage: s1.symmetries()

(continues on next page)
no symmetry; antisymmetry: (0, 1)
sage: s1 == s
True

The \LaTeX{} notation is recognized:

\begin{verbatim}
sage: t[\{ij\}k] == s
True
\end{verbatim}

Note that in the index notation, the name of the indices is irrelevant; they can even be replaced by dots:

\begin{verbatim}
sage: t[\{..\}] == s
True
\end{verbatim}

Antisymmetrization of a tensor of type \((0, 3)\) over the first and last arguments:

\begin{verbatim}
sage: s = t.antisymmetrize(0,2) ; s  # (0,2) = first and last arguments
Type-(0,3) tensor on the 3-dimensional vector space M over the Rational Field
sage: s.symmetries()
no symmetry; antisymmetry: (0, 2)
sage: s[:]
[[[0, -4, -8], [0, -4, 14], [0, -4, -17]],
 [[4, 0, 16], [4, 0, -19], [4, 0, -4]],
 [[8, -16, 0], [-14, 19, 0], [17, 4, 0]]
sage: all(s[i,j,k] == 1/2*(t[i,j,k]-t[k,j,i])  # Check:
....:   for i in range(3) for j in range(3) for k in range(3))
True
sage: s.antisymmetrize(0,2) == s  # another test
True
sage: s.symmetrize(0,2) == 0  # of course
True
sage: s.symmetrize(0,1) == 0  # no reason for this to hold
False
\end{verbatim}

Antisymmetrization of a tensor of type \((0, 3)\) over the last two arguments:

\begin{verbatim}
sage: s = t.antisymmetrize(1,2) ; s  # (1,2) = the last two arguments
Type-(0,3) tensor on the 3-dimensional vector space M over the Rational Field
sage: s.symmetries()
no symmetry; antisymmetry: (1, 2)
sage: s[:]
[[[0, 3, -2], [-3, 0, -1], [2, 1, 0]],
 [[0, -12, -2], [12, 0, -16], [2, 16, 0]],
 [[0, 1, -23], [-1, 0, -1], [23, 1, 0]]
sage: all(s[i,j,k] == 1/2*(t[i,j,k]-t[i,k,j])  # Check:
....:   for i in range(3) for j in range(3) for k in range(3))
True
sage: s.antisymmetrize(1,2) == s  # another test
True
sage: s.symmetrize(1,2) == 0  # of course
True
\end{verbatim}
The index notation can be used instead of the explicit call to \texttt{antisymmetrize()}: 

\begin{verbatim}
sage: t[_i[jk]] == t.antisymmetrize(1,2) True
\end{verbatim}

Full antisymmetrization of a tensor of type \((0,3)\):

\begin{verbatim}
sage: s = t.antisymmetrize() ; s
Alternating form of degree 3 on the 3-dimensional vector space \(M\) over the Rational Field
sage: s.symmetries()
no symmetry; antisymmetry: (0, 1, 2)
sage: s[:]
[[[0, 0, 0], [0, 0, 2/3], [0, -2/3, 0]],
 [[0, 0, -2/3], [0, 0, 0], [2/3, 0, 0]],
 [[0, 2/3, 0], [-2/3, 0, 0], [0, 0, 0]]]
sage: all(s[i,j,k] == 1/6*(t[i,j,k]-t[i,k,j]+t[j,k,i]-t[j,i,k]
....: +t[k,i,j]-t[k,j,i])
....: for i in range(3) for j in range(3) for k in range(3))
True
sage: s.antisymmetrize() == s
# another test
True
sage: s.symmetrize(0,1) == 0
# of course
True
sage: s.symmetrize(0,2) == 0
# of course
True
sage: s.symmetrize(1,2) == 0
# of course
True
sage: t.antisymmetrize() == t.antisymmetrize(0,1,2)
True
\end{verbatim}

The index notation can be used instead of the explicit call to \texttt{antisymmetrize()}: 

\begin{verbatim}
sage: t['_-ijk'] == t.antisymmetrize() True
sage: t['_-abc'] == t.antisymmetrize() True
sage: t['_-[]'] == t.antisymmetrize() True
sage: t['_-{{ijk}}'] == t.antisymmetrize() # LaTeX notation
True
\end{verbatim}

Antisymmetrization can be performed only on arguments on the same type:

\begin{verbatim}
sage: t = M.tensor((1,2))
sage: t[:]
[[[1, 2, 3], [-4, 5, 6], [7, 8, -9]],
[[10, -11, -12], [13, 14, -15], [16, 17, 18]],
[[19, -20, -21], [-22, 23, 24], [25, 26, -27]]]
sage: s = t.antisymmetrize(0,1)
Traceback (most recent call last):
... 
TypeError: 0 is a contravariant position, while 1 is a covariant position;
antisymmetrization is meaningful only on tensor arguments of the same type
sage: s = t.antisymmetrize(1,2) # OK: both 1 and 2 are covariant positions
\end{verbatim}
The order of positions does not matter:

```
sage: t.antisymmetrize(2,1) == t.antisymmetrize(1,2)
True
```

Again, the index notation can be used:

```
sage: t['^i_[jk]'] == t.antisymmetrize(1,2)
True
sage: t['^i_{[jk]}'] == t.antisymmetrize(1,2)  # LaTeX notation
True
```

The character '^' can be skipped:

```
sage: t['i_[jk]'] == t.antisymmetrize(1,2)
True
```

\[ \text{base_module()} \]

Return the module on which \texttt{self} is defined.

**OUTPUT:**

- instance of \texttt{FiniteRankFreeModule} representing the free module on which the tensor is defined.

**EXAMPLES:**

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: M.an_element().base_module()
Rank-3 free module $\mathbb{Z}^3$ over the Integer Ring
sage: t = M.tensor((2,1))
sage: t.base_module()
Rank-3 free module $\mathbb{Z}^3$ over the Integer Ring
sage: t.base_module() is M
True
```

\[ \text{common_basis}(\text{other}) \]

Find a common basis for the components of \texttt{self} and \texttt{other}.

In case of multiple common bases, the free module's default basis is privileged. If the current components of \texttt{self} and \texttt{other} are all relative to different bases, a common basis is searched by performing a component transformation, via the transformations listed in \texttt{self._fmodule._basis_changes}, still privileging transformations to the free module's default basis.

**INPUT:**

- \texttt{other} – a tensor (instance of \texttt{FreeModuleTensor})

**OUTPUT:**

- instance of \texttt{FreeModuleBasis} representing the common basis; if no common basis is found, \texttt{None} is returned

**EXAMPLES:**

Common basis for the components of two module elements:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1)
sage: e = M.basis('e')
sage: u = M([2,1,-5])
```

(continues on next page)
sage: f = M.basis('f')
sage: M._basis_changes.clear() # to ensure that bases e and f are unrelated at this stage
sage: v = M([0,4,2], basis=f)
sage: u.common_basis(v)

The above result is None since u and v have been defined on different bases and no connection between these bases have been set:

sage: list(u._components)
[Basis (e_1,e_2,e_3) on the Rank-3 free module M over the Integer Ring]
sage: list(v._components)
[Basis (f_1,f_2,f_3) on the Rank-3 free module M over the Integer Ring]

Linking bases e and f changes the result:

sage: a = M.automorphism()
sage: a[:] = [[0,0,1], [1,0,0], [0,-1,0]]
sage: M.set_change_of_basis(e, f, a)
sage: u.common_basis(v)

Basis (e_1,e_2,e_3) on the Rank-3 free module M over the Integer Ring

Indeed, v is now known in basis e:

sage: sorted(v._components, key=repr)
[Basis (e_1,e_2,e_3) on the Rank-3 free module M over the Integer Ring, Basis (f_1,f_2,f_3) on the Rank-3 free module M over the Integer Ring]

comp (basis=None, from_basis=None)
Return the components of self w.r.t to a given module basis.

If the components are not known already, they are computed by the tensor change-of-basis formula from components in another basis.

INPUT:
• basis – (default: None) basis in which the components are required; if none is provided, the components are assumed to refer to the module’s default basis
• from_basis – (default: None) basis from which the required components are computed, via the tensor change-of-basis formula, if they are not known already in the basis basis; if none, a basis from which both the components and a change-of-basis to basis are known is selected.

OUTPUT:
• components in the basis basis, as an instance of the class Components

EXAMPLES:

Components of a tensor of type-(1,1):

sage: M = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1)
sage: e = M.basis('e')
sage: t = M.tensor((1,1), name='t')
sage: t[1,2] = -3 ; t[3,3] = 2
sage: t.components()
2-indices components w.r.t. Basis (e_1,e_2,e_3)
A shortcut is \texttt{t.comp()}: 

\begin{verbatim}
sage: t.comp() is t.components()
True
\end{verbatim}

A direct access to the components w.r.t. the module’s default basis is provided by the square brackets applied to the tensor itself:

\begin{verbatim}
sage: t[1,2] is t.comp(e)[1,2]
True
sage: t[:]
[ 0 -3  0]
[ 0  0  0]
[ 0  0  2]
\end{verbatim}

Components computed via a change-of-basis formula:

\begin{verbatim}
sage: a = M.automorphism()
sage: a[:] = [[0,0,1], [1,0,0], [0,-1,0]]
sage: f = e.new_basis(a, 'f')
sage: t.comp(f)
\end{verbatim}

\begin{verbatim}
2-indices components w.r.t. Basis (f_1,f_2,f_3) on the Rank-3 free module M over the Integer Ring
tsage: t.comp(f)[:]
[ 0  0  0]
[ 0  2  0]
[-3  0  0]
\end{verbatim}

\texttt{components(basis=None, from\_basis=None)}

Return the components of \texttt{self} w.r.t. to a given module basis.

If the components are not known already, they are computed by the tensor change-of-basis formula from components in another basis.

INPUT:

\begin{itemize}
  \item \texttt{basis} – (default: \texttt{None}) basis in which the components are required; if none is provided, the components are assumed to refer to the module’s default basis
  \item \texttt{from\_basis} – (default: \texttt{None}) basis from which the required components are computed, via the tensor change-of-basis formula, if they are not known already in the basis \texttt{basis}; if none, a basis from which both the components and a change-of-basis to \texttt{basis} are known is selected.
\end{itemize}

OUTPUT:

\begin{itemize}
  \item components in the basis \texttt{basis}, as an instance of the class \texttt{Components}
\end{itemize}

EXAMPLES:
Components of a tensor of type-(1,1):

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1)
sage: e = M.basis('e')
sage: t = M.tensor((1,1), name='t')
sage: t[1,2] = -3 ; t[3,3] = 2
sage: t.components()
2-indices components w.r.t. Basis (e_1,e_2,e_3)
on the Rank-3 free module M over the Integer Ring
sage: t.components() is t.components(e)  # since e is M's default basis
True
sage: t.components()[:]
[ 0 -3 0]
[ 0  0 0]
[ 0  0 2]
```

A shortcut is `t.comp()`:

```
sage: t.comp() is t.components()
True
```

A direct access to the components w.r.t. the module’s default basis is provided by the square brackets applied to the tensor itself:

```
sage: t[1,2] is t.comp(e)[1,2]
True
sage: t[:]
[ 0 -3 0]
[ 0  0 0]
[ 0  0 2]
```

Components computed via a change-of-basis formula:

```
sage: a = M.automorphism()
sage: a[,] = [[0, 0, 1], [1, 0, 0], [0, -1, 0]]
sage: f = e.new_basis(a, 'f')
sage: t.comp(f)
2-indices components w.r.t. Basis (f_1,f_2,f_3)
on the Rank-3 free module M over the Integer Ring
sage: t.comp(f)[:]
[ 0  0 0]
[ 0  2 0]
[-3  0 0]
```

`contract(*args)`

Contraction on one or more indices with another tensor.

**INPUT:**

- `pos1` – positions of the indices in `self` involved in the contraction; `pos1` must be a sequence of integers, with 0 standing for the first index position, 1 for the second one, etc; if `pos1` is not provided, a single contraction on the last index position of `self` is assumed
- `other` – the tensor to contract with
- `pos2` – positions of the indices in `other` involved in the contraction, with the same conventions as for `pos1`; if `pos2` is not provided, a single contraction on the first index position of `other` is assumed
OUTPUT:

- tensor resulting from the contraction at the positions pos1 and pos2 of self with other

EXAMPLES:

Contraction of a tensor of type \((0,1)\) with a tensor of type \((1,0)\):

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: a = M.linear_form()  # tensor of type \((0,1)\) is a linear form
sage: a[:] = [-3,2,1]
sage: b = M([2,5,-2])  # tensor of type \((1,0)\) is a module element
sage: s = a.contract(b) ; s
2
sage: s in M.base_ring()  # check of the computation
True
sage: s == a[0]*b[0] + a[1]*b[1] + a[2]*b[2]
True
```

The positions of the contraction indices can be set explicitly:

```python
sage: s == a.contract(0, b, 0)  # check of the computation
True
sage: s == a.contract(0, b)
True
sage: s == a.contract(b, 0)
True
```

Instead of the explicit call to the method `contract()`, the index notation can be used to specify the contraction, via Einstein convention (summation on repeated indices); it suffices to pass the indices as a string inside square brackets:

```python
sage: s1 = a['_i']^b['^i'] ; s1
2
sage: s1 == s
True
```

In the present case, performing the contraction is identical to applying the linear form to the module element:

```python
sage: a.contract(b) == a(b)
True
```

or to applying the module element, considered as a tensor of type \((1,0)\), to the linear form:

```python
sage: a.contract(b) == b(a)
True
```

We have also:

```python
sage: a.contract(b) == b.contract(a)
True
```

Contraction of a tensor of type \((1,1)\) with a tensor of type \((1,0)\):
Tensors on free modules of finite rank

```python
sage: a = M.tensor((1,1))
sage: a[:,:] = [[-1,2,3],[4,-5,6],[7,8,9]]
sage: s = a.contract(b); s
Element of the Rank-3 free module $\mathcal{M}$ over the Integer Ring
sage: s.display()
2 e_0 - 29 e_1 + 36 e_2
```

Since the index positions have not been specified, the contraction takes place on the last position of $a$ (i.e. no. 1) and the first position of $b$ (i.e. no. 0):

```python
sage: a.contract(b) == a.contract(1, b, 0)
True
sage: a.contract(b) == b.contract(0, a, 1)
True
sage: a.contract(b) == b.contract(a, 1)
True
```

Using the index notation with Einstein convention:

```python
sage: a['^i_j']*b['^j'] == a.contract(b)
True
```

The index $i$ can be replaced by a dot:

```python
sage: a['^_j']*b['^j'] == a.contract(b)
True
```

and the symbol $^\flat$ may be omitted, the distinction between contravariant and covariant indices being the position with respect to the symbol $_\flat$:

```python
sage: a['^_.j']*b['^j'] == a.contract(b)
True
```

Contraction is possible only between a contravariant index and a covariant one:

```python
sage: a.contract(0, b)
Traceback (most recent call last):
  ...
TypeError: contraction on two contravariant indices not permitted
```

Contraction of a tensor of type $(2,1)$ with a tensor of type $(0,2)$:

```python
sage: a = a*b ; a
Type-(2,1) tensor on the Rank-3 free module $\mathcal{M}$ over the Integer Ring
sage: b = M.tensor((0,2))
sage: b[:,:] = [[-2,3,1], [0,-2,3], [4,-7,6]]
sage: s = a.contract(1, b, 1) ; s
Type-(1,2) tensor on the Rank-3 free module $\mathcal{M}$ over the Integer Ring
sage: s[:]
[[[-9, 16, 39], [18, -32, -78], [27, -48, -117]],
[[36, -64, -156], [-45, 80, 195], [54, -96, -234]],
[[63, -112, -273], [72, -128, -312], [81, -144, -351]]
```

Check of the computation:

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```sage
sage: all(s[i,j,k] == a[i,0,j]*b[k,0]+a[i,1,j]*b[k,1]+a[i,2,j]*b[k,2]
....: for i in range(3) for j in range(3) for k in range(3))
True
```

Using index notation:

```sage
sage: a['il_j']*b['_kl'] == a.contract(1, b, 1)
True
```

LaTeX notation are allowed:

```sage
sage: a['^il_j']*b['_{kl}'] == a.contract(1, b, 1)
True
```

Indices not involved in the contraction may be replaced by dots:

```sage
sage: a['.l_.']*b['_l.'] == a.contract(1, b, 1)
True
```

The two tensors do not have to be defined on the same basis for the contraction to take place, reflecting the fact that the contraction is basis-independent:

```sage
sage: A = M.automorphism()
sage: A[:] = [[0,0,1], [1,0,0], [0,-1,0]]
sage: h = e.new_basis(A, 'h')
sage: b.comp(h)[:]
# forces the computation of b's components w.r.t. basis h
[-2 -3  0]
[ 7  6 -4]
[ 3 -1 -2]
sage: b.del_other_comp(h) # deletes components w.r.t. basis e
sage: list(b._components) # indeed:
[Basis (h_0,h_1,h_2) on the Rank-3 free module M over the Integer Ring]
sage: list(a._components) # while a is known only in basis e:
[Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring]
sage: s1 = a.contract(1, b, 1) ; s1 # yet the computation is possible
Type-(1,2) tensor on the Rank-3 free module M over the Integer Ring
sage: s1 == s # ... and yields the same result as previously:
True
```

The contraction can be performed on more than a single index; for instance a 2-indices contraction of a type-(2,1) tensor with a type-(1,2) one is:

```sage
sage: a # a is a tensor of type-(2,1)
Type-(2,1) tensor on the Rank-3 free module M over the Integer Ring
sage: b = M([1,-1,2])*b # a tensor of type (1,2)
Type-(1,2) tensor on the Rank-3 free module M over the Integer Ring
sage: s = a.contract(1,2,b,1,0) ; s # the double contraction
Type-(1,1) tensor on the Rank-3 free module M over the Integer Ring
sage: s[:]
[-36  30  15]
[-252 210 105]
[-204 170  85]
sage: s == a['^k.l']*b['^l_k.'] # the same thing in index notation
True
```
**copy** *(name=None, latex_name=None)*

Return an exact copy of `self`.

The name and the derived quantities are not copied.

**INPUT:**

- `name` – (default: `None`) name given to the copy
- `latex_name` – (default: `None`) LaTeX symbol to denote the copy; if none is provided, the LaTeX symbol is set to `name`

**EXAMPLES:**

Copy of a tensor of type `(1,1)`:  

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1)
sage: e = M.basis('e')
sage: t = M.tensor((1,1), name='t')
sage: t[1,2] = -3 ; t[3,3] = 2
sage: t1 = t.copy()
sage: t1[:]
[ 0 -3 0]
[ 0 0 0]
[ 0 0 2]
sage: t1 == t
True
```

If the original tensor is modified, the copy is not:

```python
sage: t[2,2] = 4
sage: t1[:]
[ 0 -3 0]
[ 0 0 0]
[ 0 0 2]
sage: t1 == t
False
```

**copy_from**(other)

Make `self` a copy from `other`.

**INPUT:**

- `other` – other tensor in the very same module from which `self` should be a copy of

**Warning:** All previous defined components will be deleted!

**EXAMPLES:**

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1)
sage: e = M.basis('e')
sage: t = M.tensor((1,1), name='t')
sage: t[1,2] = -3 ; t[3,3] = 2
sage: s = M.tensor((1,1), name='s')
sage: s.copy_from(t)
sage: s[:]
```

(continues on next page)
If the original tensor is modified, the copy is not:

```python
sage: t[2,2] = 4
sage: s[:]
[0 -3 0]
[0 0 0]
[0 0 2]
sage: s == t
False
```

```
\textbf{del\_other\_comp}(basis=\texttt{None})

Delete all the components but those corresponding to \texttt{basis}.

\textbf{INPUT}:

\begin{itemize}
  \item \texttt{basis} – (default: \texttt{None}) basis in which the components are kept; if none the module’s default basis is assumed
\end{itemize}

\textbf{EXAMPLES}:

Deleting components of a module element:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1)
sage: e = M.basis('e')
sage: u = M([2,1,-5])
sage: f = M.basis('f')
sage: u.add_comp(f)[:] = [0,4,2]
sage: sorted(u._components, key=repr)
[Basis \{e_1,e_2,e_3\} on the Rank-3 free module M over the Integer Ring, Basis \{f_1,f_2,f_3\} on the Rank-3 free module M over the Integer Ring]
sage: u.del_other_comp(f)
sage: list(u._components)
[Basis \{f_1,f_2,f_3\} on the Rank-3 free module M over the Integer Ring]
sage: u.add_comp(e)[:] = [2,1,-5]
sage: sorted(u._components, key=repr)
[Basis \{e_1,e_2,e_3\} on the Rank-3 free module M over the Integer Ring, Basis \{f_1,f_2,f_3\} on the Rank-3 free module M over the Integer Ring]
sage: u.del_other_comp()  # default argument: basis = \texttt{e}
sage: list(u._components)
[Basis \{e_1,e_2,e_3\} on the Rank-3 free module M over the Integer Ring]
```

\textbf{disp}(basis=\texttt{None}, format_spec=\texttt{None})

Display \texttt{self} in terms of its expansion w.r.t. a given module basis.

The expansion is actually performed onto tensor products of elements of the given basis and of elements of its dual basis (see examples below). The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).
INPUT:

- **basis** – (default: None) basis of the free module with respect to which the tensor is expanded; if none is provided, the module’s default basis is assumed
- **format_spec** – (default: None) format specification passed to self._fmodule._output_formatter to format the output

EXAMPLES:

Display of a module element (type-(1, 0) tensor):

```python
sage: M = FiniteRankFreeModule(QQ, 2, name='M', start_index=1)
sage: e = M.basis('e') ; e
Basis (e_1,e_2) on the 2-dimensional vector space M over the Rational Field
sage: v = M([1/3,-2], name='v')
sage: v.display(e)
v = 1/3 e_1 - 2 e_2
```

A shortcut is `disp()`:

```python
sage: v.disp()
v = 1/3 e_1 - 2 e_2
```

Display of a linear form (type-(0, 1) tensor):

```python
sage: de = e.dual_basis() ; de
Dual basis (e^1,e^2) on the 2-dimensional vector space M over the Rational Field
Linear form on the 2-dimensional vector space M over the Rational Field
sage: w.set_name('w', latex_name='\omega')
sage: w.display()
w = -3/4 e^1 + e^2
```

Display of a type-(1, 1) tensor:

```python
sage: t = v*w ; t  # the type-(1,1) is formed as the tensor product of v by w
Type-(1,1) tensor v⊗w on the 2-dimensional vector space M over the Rational Field
sage: t.display()
v⊗w = -1/4 e_1⊗e^1 + 1/3 e_1⊗e^2 + 3/2 e_2⊗e^1 - 2 e_2⊗e^2
```

Display in a basis which is not the default one:
sage: a = M.automorphism(matrix=[[1,2],[3,4]], basis=e)
sage: f = e.new_basis(a, 'f')
sage: v.display(f)  # the components w.r.t basis f are first computed via the change-of-basis formula defined by a
v = -8/3 f_1 + 3/2 f_2
sage: w.display(f)
w = 9/4 f^1 + 5/2 f^2
sage: t.display(f)
v\otimes w = -6 f_1\otimes f^1 - 20/3 f_1\otimes f^2 + 27/8 f_2\otimes f^1 + 15/4 f_2\otimes f^2

Parallel computation:

sage: Parallelism().set('tensor', nproc=2)
sage: t2 = v*w
sage: t2.display(f)
v\otimes w = -6 f_1\otimes f^1 - 20/3 f_1\otimes f^2 + 27/8 f_2\otimes f^1 + 15/4 f_2\otimes f^2
sage: t2[f,:]==t[f,:]  # check of the parallel computation
True
sage: Parallelism().set('tensor', nproc=1)  # switch off parallelization

The output format can be set via the argument output_formatter passed at the module construction:

sage: N = FiniteRankFreeModule(QQ, 2, name='N', start_index=1, ....: output_formatter=Rational.numerical_approx)
sage: e = N.basis('e')
sage: v = N([1/3,-2], name='v')
sage: v.display()  # default format (53 bits of precision)
v = 0.333333333333333 e_1 -2.00000000000000 e_2
sage: latex(v.display())
v = 0.333333333333333 e_{1} -2.00000000000000 e_{2}

The output format is then controlled by the argument format_spec of the method display():

sage: v.display(format_spec=10)  # 10 bits of precision
v = 0.33 e_1 - 2.0 e_2

Check that the bug reported in trac ticket #22520 is fixed:

sage: M = FiniteRankFreeModule(SR, 3, name='M')  # optional - sage.symbolic
sage: e = M.basis('e')  # optional - sage.symbolic
sage: t = SR.var('t', domain='real')  # optional - sage.symbolic
sage: (t*e[0]).display()  # optional - sage.symbolic
t e_0

\textbf{display}(basis=None, format_spec=None)

Display self in terms of its expansion w.r.t. a given module basis.

The expansion is actually performed onto tensor products of elements of the given basis and of elements of its dual basis (see examples below). The output is either text-formatted (console mode) or \LaTeX-\textit{formatted} (notebook mode).

\textbf{INPUT:}

- \textbf{basis} – (default: None) basis of the free module with respect to which the tensor is expanded; if none is provided, the module’s default basis is assumed
**format_spec** – (default: None) format specification passed to self._fmodule._output_formatter to format the output

**EXAMPLES:**

Display of a module element (type-(1,0) tensor):

```
sage: M = FiniteRankFreeModule(QQ, 2, name='M', start_index=1)
sage: e = M.basis('e') ; e
Basis (e_1,e_2) on the 2-dimensional vector space M over the Rational Field
sage: v = M([1/3,-2], name='v')
sage: v.display(e)
v = 1/3 e_1 - 2 e_2
sage: latex(v.display())       # display in the notebook
v = \frac{1}{3} e_{1} -2 e_{2}
```

A shortcut is disp():

```
sage: v.disp()
v = 1/3 e_1 - 2 e_2
```

Display of a linear form (type-(0,1) tensor):

```
sage: de = e.dual_basis() ; de
Dual basis (e^1,e^2) on the 2-dimensional vector space M over the Rational Field
Linear form on the 2-dimensional vector space M over the Rational Field
sage: w.set_name('w', latex_name='\omega')
sage: w.display()
w = -3/4 e^1 + e^2
sage: latex(w.display())       # display in the notebook
\omega = -\frac{3}{4} e^{1} +e^{2}
```

Display of a type-(1,1) tensor:

```
sage: t = v*w ; t
# the type-(1,1) is formed as the tensor product of v by w
Type-(1,1) tensor v⊗w on the 2-dimensional vector space M over the Rational Field
sage: t.display()
v⊗w = -1/4 e_1⊗e^1 + 1/3 e_1⊗e^2 + 3/2 e_2⊗e^1 - 2 e_2⊗e^2
sage: latex(t.display())       # display in the notebook
v\otimes \omega = -\frac{1}{4} e_1\otimes e^1 + \frac{1}{3} e_1\otimes e^2 + \frac{3}{2} e_2\otimes e^1 - 2 e_2\otimes e^2
```

Display in a basis which is not the default one:

```
sage: a = M.automorphism(matrix=[[1,2],[3,4]], basis=e)
sage: f = e.new_basis(a, 'f')
sage: v.display(f)
# the components w.r.t basis f are first computed via the change-of-basis formula defined by a
```
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\[
v = -\frac{8}{3} f_1 + \frac{3}{2} f_2
\]

```
sage: v.display(f)
w = \frac{9}{4} f^1 + \frac{5}{2} f^2
tsage: t.display(f)
vw = -6 f_1 \otimes f^1 - \frac{20}{3} f_1 \otimes f^2 + \frac{27}{8} f_2 \otimes f^1 + \frac{15}{4} f_2 \otimes f^2
```

Parallel computation:

```
sage: Parallelism().set('tensor', nproc=2)
tsage: t2 = v*w
sage: t2.display(f)
t2[f,:]= t[f,:]
```

```
# check of the parallel computation
True
sage: Parallelism().set('tensor', nproc=1)
# switch off parallelization
```

The output format can be set via the argument `output_formatter` passed at the module construction:

```
sage: N = FiniteRankFreeModule(QQ, 2, name='N', start_index=1,
....:                            output_formatter=Rational.numerical_approx)
sage: e = N.basis('e')
sage: v = N([1/3,-2], name='v')
sage: v.display()
```

```
v = 0.333333333333333 e_1 - 2.00000000000000 e_2
sage: latex(v.display())
```

```
v = 0.333333333333333 e_{1} -2.00000000000000 e_{2}
```

The output format is then controlled by the argument `format_spec` of the method `display()`:

```
sage: v.display(format_spec=10)  # 10 bits of precision
v = 0.33 e_1 - 2.0 e_2
```

Check that the bug reported in trac ticket #22520 is fixed:

```
sage: M = FiniteRankFreeModule(SR, 3, name='M')  # optional - sage.symbolic
sage: e = M.basis('e')  # optional - sage.symbolic
sage: t = SR.var('t', domain='real')  # optional - sage.symbolic
sage: (t*e[0]).display()  # optional - sage.symbolic
```

```
t e_0
```

\texttt{display\_comp}(basis=None, format\_spec=None, symbol=None, latex\_symbol=None, index\_labels=None, index\_latex\_labels=None, only\_nonzero=True, only\_nonredundant=False)

Display the tensor components with respect to a given module basis, one per line.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- \texttt{basis} – (default: None) basis of the free module with respect to which the tensor components are defined; if None, the module’s default basis is assumed
- \texttt{format\_spec} – (default: None) format specification passed to \texttt{self._fmodule\_output\_formatter} to format the output
- \texttt{symbol} – (default: None) string (typically a single letter) specifying the symbol for the components; if None, the tensor name is used if it has been set, otherwise \texttt{X} is used

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• `latex_symbol` – (default: `None`) string specifying the \LaTeX{} symbol for the components; if `None`, the tensor \LaTeX{} name is used if it has been set, otherwise 'X' is used

• `index_labels` – (default: `None`) list of strings representing the labels of each of the individual indices; if `None`, integer labels are used

• `index_latex_labels` – (default: `None`) list of strings representing the \LaTeX{} labels of each of the individual indices; if `None`, integers labels are used

• `only_nonzero` – (default: `True`) boolean; if `True`, only nonzero components are displayed

• `only_nonredundant` – (default: `False`) boolean; if `True`, only nonredundant components are displayed in case of symmetries

**EXAMPLES:**

Display of the components of a type-(2, 1) tensor on a rank 2 vector space over \(\mathbb{Q}\):

```
sage: FiniteRankFreeModule._clear_cache_() # for doctests only	sage: M = FiniteRankFreeModule(QQ, 2, name='M', start_index=1)	sage: e = M.basis('e')	sage: t = M.tensor((2,1), name='T', sym=(0,1))
tsage: t[1,2,1], t[1,2,2], t[2,2,2] = 2/3, -1/4, 3
tsage: t.display()
T = 2/3 e_1 \otimes e_2 \otimes e^1 - 1/4 e_1 \otimes e_2 \otimes e^2 + 2/3 e_2 \otimes e_1 \otimes e^1
   - 1/4 e_2 \otimes e_1 \otimes e^2 + 3 e_2 \otimes e_2 \otimes e^2
sage: t.display_comp()
T^{12}_{\phantom{1}1} = 2/3
T^{12}_{\phantom{1}2} = -1/4
T^{21}_{\phantom{2}1} = 2/3
T^{21}_{\phantom{2}2} = -1/4
T^{22}_{\phantom{2}2} = 3
```

The \LaTeX{} output for the notebook:

```
sage: latex(t.display_comp())
\begin{array}{lcl}
  T_{\phantom{1}2,1} & = & \frac{2}{3} \\
  T_{\phantom{1}2,2} & = & -\frac{1}{4} \\
  T_{\phantom{2}1,1} & = & \frac{2}{3} \\
  T_{\phantom{2}1,2} & = & -\frac{1}{4} \\
  T_{\phantom{2}2,2} & = & 3
\end{array}
```

By default, only the non-vanishing components are displayed; to see all the components, the argument `only_nonzero` must be set to `False`:

```
sage: t.display_comp(only_nonzero=False)
T^{11}_{\phantom{1}1} = 0
T^{11}_{\phantom{1}2} = 0
T^{12}_{\phantom{1}1} = 2/3
T^{12}_{\phantom{1}2} = -1/4
T^{21}_{\phantom{2}1} = 2/3
T^{21}_{\phantom{2}2} = -1/4
T^{22}_{\phantom{2}1} = 0
T^{22}_{\phantom{2}2} = 3
```

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t being symmetric w.r.t. to its first two indices, one may ask to skip the components that can be deduced by symmetry:

```
sage: t.display_comp(only_nonredundant=True)
T^12_1 = 2/3
T^12_2 = -1/4
T^22_2 = 3
```

The index symbols can be customized:

```
sage: t.display_comp(index_labels=['x', 'y'])
T^xy_x = 2/3
T^xy_y = -1/4
T^yx_x = 2/3
T^yx_y = -1/4
T^yy_y = 3
```

Display of the components w.r.t. a basis different from the default one:

```
sage: t.display_comp(basis=f)
T^11_1 = 29/24
T^11_2 = 13/24
T^12_1 = 3/4
T^12_2 = 3/4
T^21_1 = 3/4
T^21_2 = 3/4
T^22_1 = 7/24
T^22_2 = 23/24
```

**pick_a_basis()**

Return a basis in which the tensor components are defined.

The free module’s default basis is privileged.

**OUTPUT:**

- instance of `FreeModuleBasis` representing the basis

**EXAMPLES:**

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: t = M.tensor((2,0), name='t')
sage: e = M.basis('e')
sage: t[0,1] = 4  # component set in the default basis (e)
sage: t.pick_a_basis()
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
sage: f = M.basis('f')
sage: t.add_comp(f)[2,1] = -4  # the components in basis e are not erased
sage: t.pick_a_basis()
```

**set_comp(basis=None)**

Return the components of `self` w.r.t. a given module basis for assignment.
The components with respect to other bases are deleted, in order to avoid any inconsistency. To keep them, use the method `add_comp()` instead.

**INPUT:**

- `basis` – (default: `None`) basis in which the components are defined; if none is provided, the components are assumed to refer to the module's default basis

**OUTPUT:**

- components in the given basis, as an instance of the class `Components`; if such components did not exist previously, they are created.

**EXAMPLES:**

Setting components of a type-(1,1) tensor:

```sage
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: t = M.tensor((1,1), name='t')
sage: t.set_comp()[0,1] = -3
sage: t.display()
t = -3 e_0⊗e^1
t = -3 e_0⊗e^1
sage: t.set_comp()[1,2] = 2
sage: t.display()
t = -3 e_0⊗e^1 + 2 e_1⊗e^2
sage: t.set_comp(e)
2-indices components w.r.t. Basis (e_0,e_1,e_2) on the
  Rank-3 free module M over the Integer Ring
```

Setting components in a new basis:

```sage
sage: f = M.basis('f')
sage: t.set_comp(f)[0,1] = 4
sage: list(t._components) # the components w.r.t. basis e have been deleted
[Basis (f_0,f_1,f_2) on the Rank-3 free module M over the Integer Ring]
sage: t.display(f)
t = 4 f_0⊗f^1
```

The components w.r.t. basis `e` can be deduced from those w.r.t. basis `f`, once a relation between the two bases has been set:

```sage
sage: a = M.automorphism()
sage: a[:,:] = [[0,0,1], [1,0,0], [0,1,0]]
sage: M.set_change_of_basis(e, f, a)
sage: t.display(e)
t = -4 e_1⊗e^2
sage: sorted(t._components, key=repr)
[Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring,
  Basis (f_0,f_1,f_2) on the Rank-3 free module M over the Integer Ring]
```

Since zero is an immutable element, its components cannot be changed:

```sage
sage: z = M.tensor_module(1, 1).zero()
sage: z.set_comp(e)[0,1] = 1
Traceback (most recent call last):
```

(continues on next page)
... Value Error: the components of an immutable element cannot be changed

**set_name**(name=None, latex_name=None)

Set (or change) the text name and \LaTeX{} name of `self`.

**INPUT:**

- **name** – (default: None) string; name given to the tensor
- **latex_name** – (default: None) string; \LaTeX{} symbol to denote the tensor; if None while `name` is provided, the \LaTeX{} symbol is set to `name`

**EXAMPLES:**

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: t = M.tensor((2,1)) ; t
Type-(2,1) tensor on the Rank-3 free module M over the Integer Ring
sage: t.set_name('t') ; t
Type-(2,1) tensor t on the Rank-3 free module M over the Integer Ring
sage: latex(t)
t
sage: t.set_name(latex_name=r'\tau') ; t
Type-(2,1) tensor t on the Rank-3 free module M over the Integer Ring
sage: latex(t)
\tau
```

**symmetries()**

Print the list of symmetries and antisymmetries of `self`.

**EXAMPLES:**

Various symmetries / antisymmetries for a rank-4 tensor:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: t = M.tensor((4,0), name='T') # no symmetry declared
sage: t.symmetries()
no symmetry; no antisymmetry
sage: t = M.tensor((4,0), name='T', sym=(0,1))
sage: t.symmetries()
symmetry: (0, 1); no antisymmetry
sage: t = M.tensor((4,0), name='T', sym=((0,1), (2,3)))
sage: t.symmetries()
symmetries: [(0, 1), (2, 3)]; no antisymmetry
sage: t = M.tensor((4,0), name='T', sym=((0,1), antisym=(2,3)))
sage: t.symmetries()
symmetry: (0, 1); antisymmetry: (2, 3)
```

**symmetrize**(pos=None, **kwargs)

Symmetrization over some arguments.

**INPUT:**

- **pos** – list of argument positions involved in the symmetrization (with the convention position=0 for the first argument); if none, the symmetrization is performed over all the arguments
• basis – (default: None) module basis with respect to which the component computation is to be performed; if none, the module’s default basis is used if the tensor field has already components in it; otherwise another basis w.r.t. which the tensor has components will be picked

OUTPUT:
• the symmetrized tensor (instance of FreeModuleTensor)

EXAMPLES:
Symmetrization of a tensor of type (2,0):

```
sage: M = FiniteRankFreeModule(QQ, 3, name='M')
sage: e = M.basis('e')
sage: t = M.tensor((2,0))
sage: t[:,:] = [[2,1,-3], [0,-4,5], [-1,4,2]]
sage: s = t.symmetrize() ; s
Type-(2,0) tensor on the 3-dimensional vector space M over the Rational Field
sage: t[:,:], s[:,:]
([ 2 1 -3]      [ 2 1/2 -2]
 [ 0 -4 5]      [1/2 -4 9/2]
[-1 4 2], [ -2 9/2 2])
sage: s.symmetries()
symmetry: (0, 1); no antisymmetry
sage: all(s[i,j] == 1/2*(t[i,j]+t[j,i])  # check:
....:   for i in range(3) for j in range(3))
True
```

Instead of invoking the method symmetrize(), one may use the index notation with parentheses to denote the symmetrization; it suffices to pass the indices as a string inside square brackets:

```
sage: t['(ij)']
Type-(2,0) tensor on the 3-dimensional vector space M over the Rational Field
sage: t['(ij)'].symmetries()
symmetry: (0, 1); no antisymmetry
sage: t['(ij)'] == t.symmetrize()
True
```

The indices names are not significant; they can even be replaced by dots:

```
sage: t['(..)'] == t.symmetrize()
True
```

The \LaTeX{} notation can be used as well:

```
sage: t['^{(ij)}'] == t.symmetrize()
True
```

Symmetrization of a tensor of type (0,3) on the first two arguments:

```
sage: t = M.tensor((0,3))
sage: t[:,:] = [[[1,2,3], [-4,5,6], [7,8,-9]],
(continues on next page)
```
....:  [[[10, -11, 12], [13, 14, -15], [16, 17, 18]],
....:  [[19, -20, -21], [-22, 23, 24], [25, 26, -27]]

sage: s = t.symmetrize(0,1) ; s  # (0,1) = the first two arguments
Type-(0,3) tensor on the 3-dimensional vector space M over the Rational Field
sage: s.symmetries()
symmetry: (0, 1); no antisymmetry
sage: s[:]
[[[1, 2, 3], [3, -3, 9], [13, -6, -15]],
[[3, -3, 9], [13, 14, -15], [-3, 20, 21]],
[[13, -6, -15], [-3, 20, 21], [25, 26, -27]]

sage: all(s[i,j,k] == 1/2*(t[i,j,k]+t[j,i,k])  # Check:
....:    for i in range(3) for j in range(3) for k in range(3))
True
sage: s.symmetrize(0,1) == s  # another test
True

Again the index notation can be used:

sage: t['(ij)k'] == t.symmetrize(0,1)
True
sage: t['(...)'] == t.symmetrize(0,1)  # no index name
True
sage: t['(ij)k'] == t.symmetrize(0,1)  # LaTeX notation
True
sage: t['(....)'] == t.symmetrize(0,1)  # this also allowed
True

Symmetrization of a tensor of type (0,3) on the first and last arguments:

sage: s = t.symmetrize(0,2) ; s  # (0,2) = first and last arguments
Type-(0,3) tensor on the 3-dimensional vector space M over the Rational Field
sage: s.symmetries()
symmetry: (0, 2); no antisymmetry
sage: s[:]
[[[1, 6, 11], [-4, 9, -8], [7, 12, 8]],
[[6, -11, -4], [9, 14, 4], [12, 17, 22]],
[[11, -6, -15], [-8, 4, 24], [8, 22, -27]]

sage: all(s[i,j,k] == 1/2*(t[i,j,k]+t[k,j,i])  # Check:
....:    for i in range(3) for j in range(3) for k in range(3))
True
sage: s.symmetrize(0,2) == s  # another test
True

Symmetrization of a tensor of type (0,3) on the last two arguments:

sage: s = t.symmetrize(1,2) ; s  # (1,2) = the last two arguments
Type-(0,3) tensor on the 3-dimensional vector space M over the Rational Field
sage: s.symmetries()
symmetry: (1, 2); no antisymmetry
sage: s[:]
(continues on next page)
Use of the index notation:

```
sage: t['_i(jk)'] == t.symmetrize(1,2)
True
sage: t['_(..)'] == t.symmetrize(1,2)
True
sage: t['_{i(jk)}'] == t.symmetrize(1,2) # LaTeX notation
True
```

Full symmetrization of a tensor of type (0,3):

```
sage: s = t.symmetrize() ; s
Type-(0,3) tensor on the 3-dimensional vector space M over the Rational Field

```

```
sage: s.symmetries()
symmetry: (0, 1, 2); no antisymmetry
```

```
sage: s[:]
[[[1, 8/3, 29/3], [8/3, 7/3, 0], [29/3, 0, -5/3]],
 [[[8/3, 7/3, 0], [7/3, 14, 25/3], [0, 25/3, 68/3]],
 [[[29/3, 0, -5/3], [0, 25/3, 68/3], [-5/3, 68/3, -27]]]
```

```
sage: all(s[i,j,k] == 1/6*(t[i,j,k]+t[i,k,j]+t[j,k,i]+t[j,i,k]+t[k,i,j]+t[k,j,i]) # Check:
.....: for i in range(3) for j in range(3) for k in range(3))
True
sage: s.symmetrize() == s # another test
True
```

Index notation for the full symmetrization:

```
sage: t['_(ijk)'] == t.symmetrize()
True
sage: t['_{(ijk)}'] == t.symmetrize() # LaTeX notation
True
```

Symmetrization can be performed only on arguments on the same type:

```
sage: t = M.tensor((1,2))
sage: t[:] = [[[1,2,3], [-4,5,6], [7,8,-9]],
[[10,-11,12], [13,14,-15], [16,17,18]],
[[19,-20,-21], [-22,23,24], [25,26,-27]]
sage: s = t.symmetrize(0,1)
Traceback (most recent call last):
...
TypeError: 0 is a contravariant position, while 1 is a covariant position;
```

(continues on next page)
symmetrization is meaningful only on tensor arguments of the same type

\begin{verbatim}
sage: s = t.symmetrize(1,2) # OK: both 1 and 2 are covariant positions
\end{verbatim}

The order of positions does not matter:

\begin{verbatim}
sage: t.symmetrize(2,1) == t.symmetrize(1,2)
True
\end{verbatim}

Use of the index notation:

\begin{verbatim}
sage: t[^i_(jk)] == t.symmetrize(1,2)
True
sage: t[^._(..)] == t.symmetrize(1,2)
True
\end{verbatim}

The character ^ can be skipped, the character _ being sufficient to separate contravariant indices from covariant ones:

\begin{verbatim}
sage: t[i_(jk)] == t.symmetrize(1,2)
True
\end{verbatim}

The LaTeX notation can be employed:

\begin{verbatim}
sage: t[^i{ij}_(jk)] == t.symmetrize(1,2)
True
\end{verbatim}

tensor_rank()

Return the tensor rank of self.

OUTPUT:

• integer \(k+1\), where \(k\) is the contravariant rank and \(l\) is the covariant rank

EXAMPLES:

\begin{verbatim}
sage: M = FiniteRankFreeModule(ZZ, 3)
sage: M.an_element().tensor_rank()
1
sage: t = M.tensor((2,1))
sage: t.tensor_rank()
3
\end{verbatim}

tensor_type()

Return the tensor type of self.

OUTPUT:

• pair \((k, l)\), where \(k\) is the contravariant rank and \(l\) is the covariant rank

EXAMPLES:

\begin{verbatim}
sage: M = FiniteRankFreeModule(ZZ, 3)
sage: M.an_element().tensor_type()
(1, 0)
sage: t = M.tensor((2,1))
sage: t.tensor_type()
(2, 1)
\end{verbatim}
trace \( (pos1=0, pos2=1) \)

Trace (contraction) on two slots of the tensor.

INPUT:

- \( pos1 \) – (default: 0) position of the first index for the contraction, with the convention \( pos1=0 \) for the first slot
- \( pos2 \) – (default: 1) position of the second index for the contraction, with the same convention as for \( pos1 \); the variance type of \( pos2 \) must be opposite to that of \( pos1 \)

OUTPUT:

- tensor or scalar resulting from the \( (pos1, pos2) \) contraction

EXAMPLES:

Trace of a type-(1,1) tensor:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e') ; e
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
sage: a = M.tensor((1,1), name='a') ; a
Type-(1,1) tensor a on the Rank-3 free module M over the Integer Ring
sage: a[:,:] = [[[1,2,3], [4,5,6], [7,8,9]]
sage: a.trace()
15
sage: a.trace(0,1)  # equivalent to above (contraction of slot 0 with slot 1)
15
sage: a.trace(1,0)  # the order of the slots does not matter
15
```

Instead of the explicit call to the method \texttt{trace()}, one may use the index notation with Einstein convention (summation over repeated indices); it suffices to pass the indices as a string inside square brackets:

```
sage: a['^i_i']
15
```

The letter ‘i’ to denote the repeated index can be replaced by any other letter:

```
sage: a['^s_s']
15
```

Moreover, the symbol ^ can be omitted:

```
sage: a['i_i']
15
```

The contraction on two slots having the same tensor type cannot occur:

```
sage: b = M.tensor((2,0), name='b') ; b
Type-(2,0) tensor b on the Rank-3 free module M over the Integer Ring
sage: b[:,:] = [[[1,2,3], [4,5,6], [7,8,9]]
sage: b.trace(0,1)
Traceback (most recent call last):
...  
IndexError: contraction on two contravariant indices is not allowed
```

The contraction either preserves or destroys the symmetries:
sage: b = M.alternating_form(2, 'b') ; b
Alternating form b of degree 2 on the Rank-3 free module M
over the Integer Ring
sage: b[0,1], b[0,2], b[1,2] = 3, 2, 1
sage: t = a\cdot b ; t
Type-(1,3) tensor a\otimes b on the Rank-3 free module M
over the Integer Ring

By construction, t is a tensor field antisymmetric w.r.t. its last two slots:

sage: t.symmetries()
no symmetry; antisymmetry: (2, 3)
sage: s = t.trace(0,1) ; s  # contraction on the first two slots
Alternating form of degree 2 on the
Rank-3 free module M over the Integer Ring
sage: s.symmetries()  # the antisymmetry is preserved
no symmetry; antisymmetry: (0, 1)
sage: s[:]
[ 0 45 30]
[-45 0 15]
[-30 -15 0]
sage: s == 15\cdot b  # check
True
sage: s = t.trace(0,2) ; s  # contraction on the first and third slots
Type-(0,2) tensor on the Rank-3 free module M over the Integer Ring
sage: s.symmetries()  # the antisymmetry has been destroyed by the above...
←contraction:
no symmetry; no antisymmetry
sage: s[:]  # indeed:
[-26 -4 6]
[-31 -2 9]
[-36 0 12]
sage: s[:] == matrix( [[sum(t[k,i,k,j] for k in M.irange()) for j in M.irange()] for i in M.irange()] )  # check
True

Use of index notation instead of trace():

sage: t[\textsuperscript{ikj}] == t.trace(0,1)
True
sage: t[\textsuperscript{ik_(kj)}] == t.trace(0,1)  # LaTeX notation
True
sage: t[\textsuperscript{ikj}] == t.trace(0,2)
True
sage: t[\textsuperscript{ikj}] == t.trace(0,3)
True

Index symbols not involved in the contraction may be replaced by dots:

sage: t[\textsuperscript{k..}] == t.trace(0,1)
True
sage: t[\textsuperscript{k..}] == t.trace(0,2)
True
4.3 Index notation for tensors

AUTHORS:
• Eric Gourgoulhon, Michal Bejger (2014-2015): initial version
• Léo Brunswic (2019): add multiple symmetries and multiple contractions

class sage.tensor.modules.tensor_with_indices.TensorWithIndices(tensor, indices)
    Bases: sage.structure.sage_object.SageObject

Index notation for tensors.
This is a technical class to allow one to write some tensor operations (contractions and symmetrizations) in index notation.

INPUT:
• tensor – a tensor (or a tensor field)
• indices – string containing the indices, as single letters; the contravariant indices must be stated first and separated from the covariant indices by the character _

EXAMPLES:

Index representation of tensors on a rank-3 free module:

```
sage: M = FiniteRankFreeModule(QQ, 3, name='M')
sage: e = M.basis('e')
sage: a = M.tensor((2,0), name='a')
a[:,:] = [[1,2,3], [4,5,6], [7,8,9]]
sage: b = M.tensor((0,2), name='b')
b[:,:] = [[-1,2,-3], [-4,5,6], [7,-8,9]]
sage: t = a*b ; t.set_name('t') ; t
```

Type-(2,2) tensor t on the 3-dimensional vector space M over the Rational Field

```
sage: from sage.tensor.modules.tensor_with_indices import TensorWithIndices
sage: T = TensorWithIndices(t, '^ij_kl') ; T
```

The TensorWithIndices object is returned by the square bracket operator acting on the tensor and fed with the string specifying the indices:

```
sage: a['^ij']
a^ij
sage: type(a['^ij'])
<class 'sage.tensor.modules.tensor_with_indices.TensorWithIndices'>
sage: b['_ef']
b_ef
sage: t['^ij_kl']
t^ij_kl
```

4.3. Index notation for tensors
The symbol ‘^’ may be omitted, since the distinction between covariant and contravariant indices is performed by the index position relative to the symbol ‘_’:

\[
\text{sage: } t['ij_kl']
\]
\[
t^\text{ij}_\text{kl}
\]

Also, LaTeX notation may be used:

\[
\text{sage: } t^{'ij'}_{kl}
\]
\[
t^\text{ij}_\text{kl}
\]

If some operation is asked in the index notation, the resulting tensor is returned, not a \texttt{TensorWithIndices} object; for instance, for a symmetrization:

\[
\text{sage: } s = t^{('ij')}_{kl} ; s
\]
\[
\# \text{ the symmetrization on } i,j \text{ is indicated by parentheses}
\]
\[
\text{Type-(2,2) tensor on the 3-dimensional vector space } \mathbb{M} \text{ over the Rational Field}
\]
\[
\text{sage: } s.\text{symmetries()}
\]
\[
\text{symmetry: } (0, 1); \text{ no antisymmetry}
\]
\[
\text{sage: } s == t.\text{symmetrize}(0,1)
\]
\[
\text{True}
\]

The letters denoting the indices can be chosen freely; since they carry no information, they can even be replaced by dots:

\[
\text{sage: } t^{('..')_{..}} == t.\text{symmetrize}(0,1)
\]
\[
\text{True}
\]

Similarly, for an antisymmetrization:

\[
\text{sage: } s = t^{('ij')[kl]} ; s
\]
\[
\# \text{ the symmetrization on } k,l \text{ is indicated by square brackets}
\]
\[
\text{Type-(2,2) tensor on the 3-dimensional vector space } \mathbb{M} \text{ over the Rational Field}
\]
\[
\text{sage: } s.\text{symmetries()}
\]
\[
\text{no symmetry; } \text{ antisymmetry: } (2, 3)
\]
\[
\text{sage: } s == t.\text{antisymmetrize}(2,3)
\]
\[
\text{True}
\]

One can also perform multiple symmetrization-antisymmetrizations:

\[
\text{sage: } aa = a^\times a
\]
\[
\text{sage: } aa^{('..')(..)} == aa.\text{symmetrize}(0,1).\text{symmetrize}(2,3)
\]
\[
\text{True}
\]
\[
\text{sage: } aa == aa^{('..')(..)} + aa^{('[..][..]')} + aa^{('[..][..]')} + aa^{('[..][..]')}
\]
\[
\text{True}
\]

Another example of an operation indicated by indices is a contraction:

\[
\text{sage: } s = t^{('ki)_kj'}
\]
\[
\# \text{ contraction on the repeated index } k
\]
\[
\text{Type-(1,1) tensor on the 3-dimensional vector space } \mathbb{M} \text{ over the Rational Field}
\]
\[
\text{sage: } s == t.\text{trace}(0,2)
\]
\[
\text{True}
\]

Indices not involved in the contraction may be replaced by dots:
The contraction of two tensors is indicated by repeated indices and the * operator:

```sage
s = a[^ik' ] * b[^kj'] ; s
Type-(1,1) tensor on the 3-dimensional vector space M over the Rational Field
s = a.contract(1, b, 0)
True
s = t[^.k_.. ] * b[^_.k'] ; s
Type-(1,3) tensor on the 3-dimensional vector space M over the Rational Field
s = t.contract(1, b, 1)
True
s = t[^{ik}_{jl}'] * b[^_{mk}'] == s # LaTeX notation
True
```

Contraction on two indices:

```sage
s = a[^kl'] * b[^kl'] ; s
s = a*b[^kl_kl']
True
s = a*b[^kl\'k\l']
True
s = a.contract(0,1, b, 0,1)
True
```

The square bracket operator acts in a similar way on `TensorWithIndices`:

```sage
b = +a[^ij' ] ; b._tensor.set_name("b") # create a copy of a[^ij' ]
b[^ij]
b[:]
[1 2 3]
[4 5 6]
[7 8 9]
b[0,0] == 1
True
b[^ji']
b[^ji]
b[^{(ij)}][:]
[1 3 5]
[3 5 7]
[5 7 9]
b[^{(ij)}] == b[^{(ij)}][^ij']
True
```

However, it keeps track of indices:

```sage
b[^ij'] = a[^ji']
b[: ] == a[: ]
```
False
\[
sage: b[:PI] == a[:].transpose()
\]
True

Arithmetics:
\[
\text{\texttt{sage: $2\cdot a[{}^i^j][]$}} \\
X^i_j \\
\text{\texttt{sage: $(2\cdot a[{}^i^j][])._tensor == 2\cdot a$}} \\
\text{\texttt{True}} \\
\text{\texttt{sage: $2\cdot t[{}^i^j_k^l][]$}} \\
X^i_j_k_l \\
\text{\texttt{sage: +a[{}^i^j][] + a^i_j}} \\
\text{\texttt{sage: +t[{}^i^j_k^l][] + t^i_j_k_l}} \\
\text{\texttt{sage: -a[{}^i^j][] - a^i_j}} \\
\text{\texttt{sage: -t[{}^i^j_k^l][] - t^i_j_k_l}} \\
\text{\texttt{sage: a["^i(...)","^i_j"] == 1/2*(a["^i_j"] + a["^j_i"])}} \\
\text{\texttt{True}}
\]

The output indices are the ones of the left term of the addition:
\[
\text{\texttt{sage: a["^i(...)","^j_i"] == 1/2*(a["^i_j"] + a["^j_i"])}} \\
\text{\texttt{False}} \\
\text{\texttt{sage: (a\cdot a)["^i...(ij)\"","^abij\"] == 1/2*((a\cdot a)["^abij\"] + (a\cdot a)["^jab\"])} \\
\text{\texttt{True}} \\
\text{\texttt{sage: c = 1/2*((a\cdot a)["^abij\"] + (a\cdot a)["^jab\"])} \\
\text{\texttt{from itertools import product}} \\
\text{\texttt{sage: all(c[i,j,k,l] == c[k,l,i,j] for i,j,k,l in product(range(3),repeat=4))}} \\
\text{\texttt{True}}
\]

Non-digit unicode identifier characters are allowed:
\[
\text{\texttt{sage: a["^\mu\chi\"]}} \\
\text{\texttt{a^\mu\chi}}
\]

Conventions are checked and non acceptable indices raise \texttt{ValueError}, for instance:
\[
\text{\texttt{sage: a['(..)']} # nested symmetries}} \\
\text{\texttt{Traceback (most recent call last):}} \\
\text{\texttt{...}} \\
\text{\texttt{ValueError: index conventions not satisfied}} \\
\text{\texttt{sage: a['..'] # unbalanced parenthesis}} \\
\text{\texttt{Traceback (most recent call last):}} \\
\text{\texttt{...}} \\
\text{\texttt{ValueError: index conventions not satisfied}} \\
\text{\texttt{sage: a['ii'] # repeated indices of the same type}} \\
\text{\texttt{Traceback (most recent call last):}} \\
\text{\texttt{...}} \\
\text{\texttt{ValueError: index conventions not satisfied: repeated indices of same type}}
\]
permute_indices(permutation)
    Return a tensor with indices with permuted indices.
    
    INPUT:
    
    • permutation – permutation that has to be applied to the indices the input should be a list containing the second line of the permutation in Cauchy notation.
    
    OUTPUT:
    
    • an instance of TensorWithIndices whose indices names and place are those of self but whose components have been permuted with permutation.
    
    EXAMPLES:

```
sage: M = FiniteRankFreeModule(QQ, 3, name='M')
sage: e = M.basis('e')
sage: a = M.tensor((2,0), name='a')
sage: a[:,:] = [[[1,2,3], [4,5,6], [7,8,9]]
sage: b = M.tensor((2,0), name='b')
sage: b[:,:] = [[-1,2,-3], [-4,5,6], [7,-8,9]]
sage: identity = [0,1]
sage: transposition = [1,0]
sage: a['ij'].permute_indices(identity) == a['ij']
    True
sage: a['ij'].permute_indices(transposition)[:,] == a[:,].transpose()
    True
sage: cycle = [1,2,3,0] # the cyclic permutation sending 0 to 1
sage: (a*b)[0,1,2,0] == (a*b)['ijkl'].permute_indices(cycle)[1,2,0,0]
    True
```

update()
    Return the tensor contains in self if it differs from that used for creating self, otherwise return self.

    EXAMPLES:

```
sage: from sage.tensor.modules.tensor_with_indices import TensorWithIndices
sage: M = FiniteRankFreeModule(QQ, 3, name='M')
sage: e = M.basis('e')
sage: a = M.tensor((1,1), name='a')
sage: a[:,:] = [[[1,-2,3], [-4,5,-6], [7,-8,9]]
sage: a_ind = TensorWithIndices(a, 'i_j') ; a_ind
    a^i_j
sage: a_ind.update()
    a^i_j
sage: a_ind.update() is a_ind
```

(continues on next page)
True

```
sage: a_ind = TensorWithIndices(a, 'k_k') ; a_ind
scalar
sage: a_ind.update()
```

15
5.1 Exterior powers of free modules

Given a free module $M$ of finite rank over a commutative ring $R$ and a positive integer $p$, the $p$-th exterior power of $M$ is the set $\Lambda^p(M)$ of all alternating contravariant tensors of degree $p$ on $M$, i.e. of all multilinear maps

$$\underbrace{M^* \times \cdots \times M^*}_{p \text{ times}} \rightarrow R$$

that vanish whenever any of two of their arguments are equal ($M^*$ stands for the dual of $M$). Note that $\Lambda^1(M) = M$. The exterior power $\Lambda^p(M)$ is a free module of rank $\left(\begin{array}{c} n \\ p \end{array}\right)$ over $R$, where $n$ is the rank of $M$.

Similarly, the $p$-th exterior power of the dual of $M$ is the set $\Lambda^p(M^*)$ of all alternating forms of degree $p$ on $M$, i.e. of all multilinear maps

$$\underbrace{M \times \cdots \times M}_{p \text{ times}} \rightarrow R$$

that vanish whenever any of two of their arguments are equal. Note that $\Lambda^1(M^*) = M^*$ (the dual of $M$). The exterior power $\Lambda^p(M^*)$ is a free module of rank $\left(\begin{array}{c} n \\ p \end{array}\right)$ over $R$, where $n$ is the rank of $M$.

The class `ExtPowerFreeModule` implements $\Lambda^p(M)$, while the class `ExtPowerDualFreeModule` implements $\Lambda^p(M^*)$.

AUTHORS:
- Eric Gourgoulhon: initial version, regarding $\Lambda^p(M^*)$ only (2015); add class for $\Lambda^p(M)$ (2017)

REFERENCES:
- K. Conrad: Exterior powers [Con2013]
- Chap. 19 of S. Lang: Algebra [Lan2002]
that vanish whenever any of two of their arguments are equal. Note that \( \Lambda^1(M^*) = M^* \) (the dual of \( M \)).

\( \Lambda^p(M^*) \) is a free module of rank \( \binom{n}{p} \) over \( R \), where \( n \) is the rank of \( M \). Accordingly, the class \texttt{ExtPowerDualFreeModule} inherits from the class \texttt{FiniteRankFreeModule}.

This is a Sage \textit{parent} class, whose \textit{element} class is \texttt{FreeModuleAltForm}.

**INPUT:**

- \textit{ffmodule} – free module \( M \) of finite rank, as an instance of \texttt{FiniteRankFreeModule}
- \textit{degree} – positive integer; the degree \( p \) of the alternating forms
- \textit{name} – (default: None) string; name given to \( \Lambda^p(M^*) \)
- \textit{latex_name} – (default: None) string; LaTeX symbol to denote \( \Lambda^p(M^*) \)

**EXAMPLES:**

2nd exterior power of the dual of a free \( \mathbb{Z} \)-module of rank 3:

```sage
M = FiniteRankFreeModule(ZZ, 3, name='M')
e = M.basis('e')
from sage.tensor.modules.ext_pow_free_module import ExtPowerDualFreeModule
A = ExtPowerDualFreeModule(M, 2) ; A
```

2nd exterior power of the dual of the Rank-3 free module \( M \) over the Integer Ring

Instead of importing \texttt{ExtPowerDualFreeModule} in the global name space, it is recommended to use the module’s method \texttt{dual_exterior_power()}:

```sage
A = M.dual_exterior_power(2) ; A
```

A is a module (actually a free module) over \( \mathbb{Z} \):

```sage
A.category()
Category of finite dimensional modules over Integer Ring
A in Modules(ZZ)
True
A.rank()
3
A.base_ring()
Integer Ring
A.base_module()
Rank-3 free module \( M \) over the Integer Ring
```

A is a \textit{parent} object, whose elements are alternating forms, represented by instances of the class \texttt{FreeModuleAltForm}:

```sage
a = A.an_element() ; a
```

Alternating form of degree 2 on the Rank-3 free module \( M \) over the Integer Ring

```sage
a.display()  # expansion with respect to \( M \)'s default basis (e)
e^0 \wedge e^1
```

(continues on next page)
Elements can be constructed from $A$. In particular, 0 yields the zero element of $A$:

\begin{Verbatim}
\textbf{sage: } A(0)  \\
Alternating form zero of degree 2 on the Rank-3 free module $M$ over the 
Integer Ring
\textbf{sage: } A(0) \textbf{ is } A.zero()
\textbf{True}
\end{Verbatim}

while non-zero elements are constructed by providing their components in a given basis:

\begin{Verbatim}
\textbf{sage: } e  \\
Basis $(e_0,e_1,e_2)$ on the Rank-3 free module $M$ over the Integer Ring
\textbf{sage: } comp = [[0,3,-1],[-3,0,4],[1,-4,0]]
\textbf{sage: } a = A(comp, basis=e, name='a') ; a  \\
Alternating form $a$ of degree 2 on the Rank-3 free module $M$ over the 
Integer Ring
\textbf{sage: } a.display(e)
\textbf{a} = 3 \text{ } e^0 \wedge e^1 - e^0 \wedge e^2 + 4 \text{ } e^1 \wedge e^2
\end{Verbatim}

An alternative is to construct the alternating form from an empty list of components and to set the nonzero components afterwards:

\begin{Verbatim}
\textbf{sage: } a = A([], name='a')
\textbf{sage: } a.set_comp(e)[0,1] = 3
\textbf{sage: } a.set_comp(e)[0,2] = -1
\textbf{sage: } a.set_comp(e)[1,2] = 4
\textbf{sage: } a.display(e)
\textbf{a} = 3 \text{ } e^0 \wedge e^1 - e^0 \wedge e^2 + 4 \text{ } e^1 \wedge e^2
\end{Verbatim}

The exterior powers are unique:

\begin{Verbatim}
\textbf{sage: } A \textbf{ is } M.dual_exterior_power(2)
\textbf{True}
\end{Verbatim}

The exterior power $\Lambda^1(M^*)$ is nothing but $M^*$:

\begin{Verbatim}
\textbf{sage: } M.dual_exterior_power(1) \textbf{ is } M.dual()
\textbf{True}
\textbf{sage: } M.dual()
Dual of the Rank-3 free module $M$ over the Integer Ring
\textbf{sage: } latex(M.dual())
$M^*$
\end{Verbatim}

Since any tensor of type $(0,1)$ is a linear form, there is a coercion map from the set $T^{(0,1)}(M)$ of such tensors to $M^*$.
Tensors on free modules of finite rank, Release 9.6

```python
sage: T01 = M.tensor_module(0,1) ; T01
Free module of type-(0,1) tensors on the Rank-3 free module M over the
Integer Ring
sage: M.dual().has_coerce_map_from(T01)
True

There is also a coercion map in the reverse direction:

```python
sage: T01.has_coerce_map_from(M.dual())
True
```

For a degree $p \geq 2$, the coercion holds only in the direction $\Lambda^p(M^*) \to T^{(0,p)}(M)$:

```python
sage: T02 = M.tensor_module(0,2) ; T02
Free module of type-(0,2) tensors on the Rank-3 free module M over the
Integer Ring
sage: T02.has_coerce_map_from(A)
True
sage: A.has_coerce_map_from(T02)
False
```

The coercion map $T^{(0,1)}(M) \to M^*$ in action:

```python
sage: b = T01([-2,1,4], basis=e, name='b') ; b
Type-(0,1) tensor b on the Rank-3 free module M over the Integer Ring
sage: b.display(e)
b = -2 e^0 + e^1 + 4 e^2
sage: lb = M.dual()(b) ; lb
Linear form b on the Rank-3 free module M over the Integer Ring
sage: lb.display(e)
b = -2 e^0 + e^1 + 4 e^2
```

The coercion map $M^* \to T^{(0,1)}(M)$ in action:

```python
sage: tlb = T01(lb) ; tlb
Type-(0,1) tensor b on the Rank-3 free module M over the Integer Ring
sage: tlb == b
True
```

The coercion map $\Lambda^2(M^*) \to T^{(0,2)}(M)$ in action:

```python
sage: ta = T02(a) ; ta
Type-(0,2) tensor a on the Rank-3 free module M over the Integer Ring
sage: ta.display(e)
a = 3 e^0 \wedge e^1 - e^0 e^1 - e^1 e^0 - 4 e^1 \wedge e^2 - 3 e^0 \wedge e^2 - 4 e^2 \wedge e^0 - 4 e^2 \wedge e^1 + 4 e^1 \wedge e^0 - 4 e^0 \wedge e^2
```

Element

alias of `sage.tensor.modules.free_module_alt_form.FreeModuleAltForm`

base_module()

Return the free module on which self is constructed.
OUTPUT:

• instance of `FiniteRankFreeModule` representing the free module on which the exterior power is defined.

EXAMPLES:

```
sage: M = FiniteRankFreeModule(ZZ, 5, name='M')
sage: A = M.dual_exterior_power(2)
sage: A.base_module()
Rank-5 free module M over the Integer Ring
sage: A.base_module() is M
True
```

degree()

Return the degree of self.

OUTPUT:

• integer \( p \) such that self is the exterior power \( \Lambda^p(M^*) \)

EXAMPLES:

```
sage: M = FiniteRankFreeModule(ZZ, 5, name='M')
sage: A = M.dual_exterior_power(2)
sage: A.degree()
2
sage: M.dual_exterior_power(4).degree()
4
```

zero()

Return the zero of self.

EXAMPLES:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: A = M.dual_exterior_power(2)
sage: A.zero()
Alternating form zero of degree 2 on the Rank-3 free module M over the Integer Ring
sage: A(0) is A.zero()
True
```

class `sage.tensor.modules.ext_pow_free_module.ExtPowerFreeModule`

Given a free module \( M \) of finite rank over a commutative ring \( R \) and a positive integer \( p \), the \( p \)-th exterior power of \( M \) is the set \( \Lambda^p(M) \) of all alternating contravariant tensors of degree \( p \) on \( M \), i.e. of all multilinear maps

\[
\underbrace{M^* \times \cdots \times M^*}_{p \text{ times}} \rightarrow R
\]

that vanish whenever any of two of their arguments are equal. Note that \( \Lambda^1(M) = M \).

\( \Lambda^p(M) \) is a free module of rank \( \binom{n}{p} \) over \( R \), where \( n \) is the rank of \( M \). Accordingly, the class `ExtPowerFreeModule` inherits from the class `FiniteRankFreeModule`.

5.1. Exterior powers of free modules
This is a Sage parent class, whose element class is AlternatingContrTensor

INPUT:

- `fmodule` – free module $M$ of finite rank, as an instance of FiniteRankFreeModule
- `degree` – positive integer; the degree $p$ of the alternating elements
- `name` – (default: None) string; name given to $\Lambda^p(M)$
- `latex_name` – (default: None) string; LaTeX symbol to denote $\Lambda^p(M)$

EXAMPLES:

2nd exterior power of the dual of a free $\mathbb{Z}$-module of rank 3:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: from sage.tensor.modules.ext_pow_free_module import ExtPowerFreeModule
sage: A = ExtPowerFreeModule(M, 2) ; A
2nd exterior power of the Rank-3 free module M over the Integer Ring
```

Instead of importing ExtPowerFreeModule in the global name space, it is recommended to use the module's method `exterior_power()`:

```python
sage: A = M.exterior_power(2) ; A
2nd exterior power of the Rank-3 free module M over the Integer Ring
sage: latex(A)
\Lambda^{2}\left(M\right)
```

$A$ is a module (actually a free module) over $\mathbb{Z}$:

```python
sage: A.category()
Category of finite dimensional modules over Integer Ring
sage: A in Modules(ZZ)
True
sage: A.rank()
3
sage: A.base_ring()
Integer Ring
sage: A.base_module()
Rank-3 free module M over the Integer Ring
```

$A$ is a parent object, whose elements are alternating contravariant tensors, represented by instances of the class AlternatingContrTensor:

```python
sage: a = A.an_element() ; a
Alternating contravariant tensor of degree 2 on the Rank-3 free module M over the Integer Ring
sage: a.display() # expansion with respect to M's default basis (e)
e_0 \wedge e_1
sage: from sage.tensor.modules.alternating_contr_tensor import AlternatingContrTensor
--AlternatingContrTensor
sage: isinstance(a, AlternatingContrTensor)
True
```

(continues on next page)
Elements can be constructed from $A$. In particular, $0$ yields the zero element of $A$:

\[
\text{sage: } A(0) \\
\text{Alternating contravariant tensor zero of degree 2 on the Rank-3 free module $M$ over the Integer Ring} \\
\text{sage: } A(0) \text{ is } A.\text{zero()} \\
\text{True}
\]

while non-zero elements are constructed by providing their components in a given basis:

\[
\text{sage: } e \\
\text{Basis } (e_0,e_1,e_2) \text{ on the Rank-3 free module $M$ over the Integer Ring} \\
\text{sage: } \text{comp} = \begin{bmatrix} 0,3,-1 \end{bmatrix}, \begin{bmatrix} -3,0,4 \end{bmatrix}, \begin{bmatrix} 1,-4,0 \end{bmatrix} \\
\text{sage: } a = A(\text{comp}, \text{basis}=e, \text{name}=\text{'}a\text{'}) ; a \\
\text{Alternating contravariant tensor } a \text{ of degree 2 on the Rank-3 free module } M \text{ over the Integer Ring} \\
\text{sage: } a.\text{display}(e) \\
a = 3 \ e_0 \wedge e_1 - e_0 \wedge e_2 + 4 \ e_1 \wedge e_2
\]

An alternative is to construct the alternating contravariant tensor from an empty list of components and to set the nonzero components afterwards:

\[
\text{sage: } a = A([], \text{name}=\text{'}a\text{'}) \\
\text{sage: } a.\text{set_comp}(e)[0,1] = 3 \\
\text{sage: } a.\text{set_comp}(e)[0,2] = -1 \\
\text{sage: } a.\text{set_comp}(e)[1,2] = 4 \\
\text{sage: } a.\text{display}(e) \\
a = 3 \ e_0 \wedge e_1 - e_0 \wedge e_2 + 4 \ e_1 \wedge e_2
\]

The exterior powers are unique:

\[
\text{sage: } A \text{ is } M.\text{exterior_power(2)} \\
\text{True}
\]

The exterior power $\Lambda^1(M)$ is nothing but $M$:

\[
\text{sage: } M.\text{exterior_power(1)} \text{ is } M \\
\text{True}
\]

For a degree $p \geq 2$, there is a coercion $\Lambda^p(M) \rightarrow T^{(p,0)}(M)$:

\[
\text{sage: } T20 = M.\text{tensor_module(2,0)} ; T20 \\
\text{Free module of type-(2,0) tensors on the Rank-3 free module } M \text{ over the Integer Ring} \\
\text{sage: } T20.\text{has_coerce_map_from}(A) \\
\text{True}
\]

Of course, there is no coercion in the reverse direction:

5.1. Exterior powers of free modules
The coercion map $\Lambda^2(M) \rightarrow T^{(2,0)}(M)$ in action:

```sage
sage: ta = T20(a) ; ta
Type-(2,0) tensor a on the Rank-3 free module M over the Integer Ring
sage: ta.display(e)
a = 3 e_0 \otimes e_1 - e_0 \otimes e_2 - 3 e_1 \otimes e_0 + 4 e_1 \otimes e_2 + e_2 \otimes e_0 - 4 e_2 \otimes e_1
sage: ta.display(e)
a = 3 e_0 \wedge e_1 - e_0 \wedge e_2 + 4 e_1 \wedge e_2
sage: ta.symmetries()  # the antisymmetry is of course preserved
no symmetry; antisymmetry: (0, 1)
sage: ta == a  # equality as type-(2,0) tensors
True
```

* Element

  alias of `sage.tensor.modules.alternating_contr_tensor.AlternatingContrTensor`

  * base_module()
    
    Return the free module on which `self` is constructed.

    **OUTPUT:**
    
    • instance of `FiniteRankFreeModule` representing the free module on which the exterior power is defined.

    **EXAMPLES:**

    ```sage
    sage: M = FiniteRankFreeModule(ZZ, 5, name='M')
sage: A = M.exterior_power(2)
sage: A.base_module()
    Rank-5 free module M over the Integer Ring
sage: A.base_module() is M
    True
    ```

  * degree()
    
    Return the degree of `self`.

    **OUTPUT:**
    
    • integer $p$ such that `self` is the exterior power $\Lambda^p(M)$

    **EXAMPLES:**

    ```sage
    sage: M = FiniteRankFreeModule(ZZ, 5, name='M')
sage: A = M.exterior_power(2)
sage: A.degree()
    2
sage: M.exterior_power(4).degree()
    4
    ```

  * zero()
    
    Return the zero of `self`.

    **EXAMPLES:**

    ```sage
    ```
5.2 Alternating contravariant tensors on free modules

Given a free module $M$ of finite rank over a commutative ring $R$ and a positive integer $p$, an alternating contravariant tensor of degree $p$ is a map

$$a : \underbrace{M^* \times \cdots \times M^*}_{p \text{ times}} \rightarrow R$$

that (i) is multilinear and (ii) vanishes whenever any of two of its arguments are equal ($M^*$ stands for the dual of $M$). $a$ is an element of the $p$-th exterior power of $M$, $\Lambda^p(M)$.

Alternating contravariant tensors are implemented via the class `AlternatingContrTensor`, which is a subclass of the generic tensor class `FreeModuleTensor`.

AUTHORS:
- Eric Gourgoulhon (2017): initial version

REFERENCES:
- Chap. 23 of R. Godement: Algebra [God1968]
- Chap. 15 of S. Lang: Algebra [Lan2002]

class `sage.tensor.modules.alternating_contr_tensor.AlternatingContrTensor`(`fmodule`, `degree`, `name=None`, `latex_name=None`)

Bases: `sage.tensor.modules.free_module_tensor.FreeModuleTensor`

Alternating contravariant tensor on a free module of finite rank over a commutative ring.

This is a Sage element class, the corresponding parent class being `ExtPowerFreeModule`.

INPUT:
- `fmodule` – free module $M$ of finite rank over a commutative ring $R$, as an instance of `FiniteRankFreeModule`
- `degree` – positive integer; the degree $p$ of the alternating contravariant tensor (i.e. the tensor rank)
- `name` – (default: None) string; name given to the alternating contravariant tensor
- `latex_name` – (default: None) string; LaTeX symbol to denote the alternating contravariant tensor; if none is provided, name is used

EXAMPLES:
Alternating contravariant tensor of degree 2 on a rank-3 free module:
Tensors on free modules of finite rank, Release 9.6

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1)
sage: e = M.basis('e')
sage: a = M.alternating_contravariant_tensor(2, name='a') ; a
Alternating contravariant tensor a of degree 2 on the Rank-3 free module M over the Integer Ring
sage: type(a)
<class 'sage.tensor.modules.ext_pow_free_module.ExtPowerFreeModule_with_category.element_class'>
sage: a.parent()
2nd exterior power of the Rank-3 free module M over the Integer Ring
sage: a[1,2], a[2,3] = 4, -3
sage: a.display(e)
a = 4 e_1 ∧ e_2 - 3 e_2 ∧ e_3
```

The alternating contravariant tensor acting on the dual basis elements:

```python
sage: f = e.dual_basis(); f
Dual basis (e^1,e^2,e^3) on the Rank-3 free module M over the Integer Ring
sage: a(f[1],f[2])
4
sage: a(f[1],f[3])
0
sage: a(f[2],f[3])
-3
sage: a(f[2],f[1])
-4
```

An alternating contravariant tensor of degree 1 is an element of the module $M$:

```python
sage: b = M.alternating_contravariant_tensor(1, name='b') ; b
Element b of the Rank-3 free module M over the Integer Ring
sage: b[:] = [2,-1,3]  # components w.r.t. the module's default basis (e)
sage: b.parent() is M
True
```

The standard tensor operations apply to alternating contravariant tensors, like the extraction of components with respect to a given basis:

```python
sage: a[e,1,2]
4
sage: a[1,2]  # since e is the module's default basis
4
sage: all( a[i,j] == - a[j,i] for i in {1,2,3} for j in {1,2,3} )
True
```

the tensor product:

```python
sage: c = b⊗b ; c
Type-(2,0) tensor b⊗b on the Rank-3 free module M over the Integer Ring
sage: c.symmetries()  
symmetry: (0, 1); no antisymmetry
```

(continues on next page)
sage: c.parent()
Free module of type-(2,0) tensors on the Rank-3 free module $M$
over the Integer Ring
sage: c.display(e)
b$\otimes$e = 4 e_1$\otimes$e_1 - 2 e_1$\otimes$e_2 + 6 e_1$\otimes$e_3 - 2 e_2$\otimes$e_1 + e_2$\otimes$e_2
- 3 e_2$\otimes$e_3 + 6 e_3$\otimes$e_1 - 3 e_3$\otimes$e_2 + 9 e_3$\otimes$e_3

the contractions:

sage: s = a.contract(w) ; s
Element of the Rank-3 free module $M$ over the Integer Ring
sage: s.display(e)
4 e_1 - 7 e_2 + 3 e_3

or tensor arithmetics:

sage: s = 3*a + c ; s
Type-(2,0) tensor on the Rank-3 free module $M$ over the Integer Ring
sage: s.parent()
Free module of type-(2,0) tensors on the Rank-3 free module $M$
over the Integer Ring
sage: s.display(e)
4 e_1$\otimes$e_1 + 10 e_1$\otimes$e_2 + 6 e_1$\otimes$e_3 - 14 e_2$\otimes$e_1 + e_2$\otimes$e_2
- 12 e_2$\otimes$e_3 + 6 e_3$\otimes$e_1 + 6 e_3$\otimes$e_2 + 9 e_3$\otimes$e_3

Note that tensor arithmetics preserves the alternating character if both operands are alternating:

sage: s = a - 2*a ; s
Alternating contravariant tensor of degree 2 on the Rank-3 free
module $M$ over the Integer Ring
sage: s.parent()  # note the difference with s = 3*a + c above
2nd exterior power of the Rank-3 free module $M$ over the Integer
Ring
sage: s == -a
True

An operation specific to alternating contravariant tensors is of course the exterior product:

sage: s = a.wedge(b) ; s
Alternating contravariant tensor $a$\wedge$b$ of degree 3 on the Rank-3 free
module $M$ over the Integer Ring
sage: s.parent()
3rd exterior power of the Rank-3 free module $M$ over the Integer Ring
sage: s.display(e)
a$\wedge$b = 6 e_1$\wedge$e_2$\wedge$e_3
True

The exterior product is nilpotent on module elements:

sage: s = b.wedge(b) ; s
Alternating contravariant tensor $b$\wedge$b$ of degree 2 on the Rank-3 free

module $M$ over the Integer Ring

```python
sage: s.display(e)
b∧b = 0
```

**degree()**

Return the degree of self.

**EXAMPLES:**

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: a = M.alternating_contravariant_tensor(2, name='a')
sage: a.degree()
2
```

**disp(basis=None, format_spec=None)**

Display the alternating contravariant tensor self in terms of its expansion w.r.t. a given module basis.

The expansion is actually performed onto exterior products of elements of basis (see examples below). The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

**INPUT:**

- `basis` – (default: `None`) basis of the free module with respect to which self is expanded; if none is provided, the module's default basis is assumed
- `format_spec` – (default: `None`) format specification passed to self._fmodule._output_formatter to format the output

**EXAMPLES:**

Display of an alternating contravariant tensor of degree 2 on a rank-3 free module:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: a = M.alternating_contravariant_tensor(2, 'a', latex_name=r'\alpha')
sage: a[0,1,2] = 3, 2, -1
sage: a.display()
a = 3 e_0\wedge e_1 + 2 e_0\wedge e_2 - e_1\wedge e_2
sage: latex(a.display())
\alpha = 3 e_{0}\wedge e_{1} + 2 e_{0}\wedge e_{2} -e_{1}\wedge e_{2}
```

Display of an alternating contravariant tensor of degree 3 on a rank-3 free module:

```python
sage: b = M.alternating_contravariant_tensor(3, 'b')
sage: b[0,1,2] = 4
sage: b.display()
b = 4 e_0\wedge e_1\wedge e_2
sage: latex(b.display())
b = 4 e_{0}\wedge e_{1}\wedge e_{2}
```

Display of a vanishing alternating contravariant tensor:

```python
sage: b[0,1,2] = 0  # the only independent component set to zero
sage: b.is_zero()
True
sage: b.display()
```
Display in a basis which is not the default one:

```
sage: aut = M.automorphism(matrix=[[0,1,0], [0,0,-1], [1,0,0]],
.....:                     basis=e)
sage: f = e.new_basis(aut, 'f')
sage: a.display(f)
a = -2 f_0\wedge f_1 - f_0\wedge f_2 - 3 f_1\wedge f_2
sage: a.display(f)    # shortcut notation
a = -2 f_0\wedge f_1 - f_0\wedge f_2 - 3 f_1\wedge f_2
```

The output format can be set via the argument `output_formatter` passed at the module construction:

```
sage: N = FiniteRankFreeModule(QQ, 3, name='N', start_index=1,
.....:                     output_formatter=Rational.numerical_approx)
sage: e = N.basis('e')
sage: a = N.alternating_contravariant_tensor(2, 'a')
sage: a[1,2], a[1,3], a[2,3] = 1/3, 5/2, 4
sage: a.display()    # default format (53 bits of precision)
a = 0.333333333333333 e_1\wedge e_2 + 2.50000000000000 e_1\wedge e_3
   + 4.00000000000000 e_2\wedge e_3
```

The output format is then controlled by the argument `format_spec` of the method `display()`:

```
sage: a.display(format_spec=10)    # 10 bits of precision
a = 0.33 e_1\wedge e_2 + 2.5 e_1\wedge e_3 + 4.0 e_2\wedge e_3
```

`display(basis=None, format_spec=None)`

Display the alternating contravariant tensor `self` in terms of its expansion w.r.t. a given module basis.

The expansion is actually performed onto exterior products of elements of `basis` (see examples below).

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

**INPUT:**

- `basis` – (default: None) basis of the free module with respect to which `self` is expanded; if none is provided, the module’s default basis is assumed
- `format_spec` – (default: None) format specification passed to `self._fmodule._output_formatter` to format the output

**EXAMPLES:**

Display of an alternating contravariant tensor of degree 2 on a rank-3 free module:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: a = M.alternating_contravariant_tensor(2, 'a', latex_name=r'\alpha')
sage: a[0,1], a[0,2], a[1,2] = 3, 2, -1
sage: a.display()
```

5.2. Alternating contravariant tensors on free modules
a = 3 e_0 ∧ e_1 + 2 e_0 ∧ e_2 - e_1 ∧ e_2
sage: latex(a.display())  # display in the notebook
\alpha = 3 e_{0}\wedge e_{1} + 2 e_{0}\wedge e_{2} -e_{1}\wedge e_{2}

Display of an alternating contravariant tensor of degree 3 on a rank-3 free module:

sage: b = M.alternating_contravariant_tensor(3, 'b')
sage: b[0,1,2] = 4
sage: b.display()
b = 4 e_0 ∧ e_1 ∧ e_2
sage: latex(b.display())
b = 4 e_{0}\wedge e_{1}\wedge e_{2}

Display of a vanishing alternating contravariant tensor:

sage: b[0,1,2] = 0  # the only independent component set to zero
sage: b.is_zero()
True
sage: b.display()
b = 0
sage: latex(b.display())
b = 0
sage: b[0,1,2] = 4  # value restored for what follows

Display in a basis which is not the default one:

sage: aut = M.automorphism(matrix=[[0,1,0], [0,0,-1], [1,0,0]],
.....: basis=e)
sage: f = e.new_basis(aut, 'f')
sage: a.display(f)
a = -2 f_0 ∧ f_1 - f_0 ∧ f_2 - 3 f_1 ∧ f_2
sage: a disp(f)  # shortcut notation
a = -2 f_0 ∧ f_1 - f_0 ∧ f_2 - 3 f_1 ∧ f_2
sage: b.display(f)
b = -4 f_0 ∧ f_1 ∧ f_2

The output format can be set via the argument output_formatter passed at the module construction:

sage: N = FiniteRankFreeModule(QQ, 3, name='N', start_index=1,
.....: output_formatter=Rational.numerical_approx)
sage: e = N.basis('e')
sage: a = N.alternating_contravariant_tensor(2, 'a')
sage: a[1,2], a[1,3], a[2,3] = 1/3, 5/2, 4
sage: a.display()  # default format (53 bits of precision)
a = 0.333333333333333 e_1 ∧ e_2 + 2.50000000000000 e_1 ∧ e_3
+ 4.00000000000000 e_2 ∧ e_3
sage: a.display(format_spec=10)  # 10 bits of precision
a = 0.33 e_1 ∧ e_2 + 2.5 e_1 ∧ e_3 + 4.0 e_2 ∧ e_3

interior_product(form)
Interior product with an alternating form.
If `self` is an alternating contravariant tensor $A$ of degree $p$ and $B$ is an alternating form of degree $q \geq p$ on the same free module, the interior product of $A$ by $B$ is the alternating form $\iota_A B$ of degree $q - p$ defined by

$$(\iota_A B)_{i_1 \ldots i_{q-p}} = A^{k_1 \ldots k_p} B_{k_1 \ldots k_p i_1 \ldots i_{q-p}}$$

**Note:** `A.interior_product(B)` yields the same result as `A.contract(0, \ldots, p-1, B, 0, \ldots, p-1)` (cf. `contract()`), but `interior_product` is more efficient, the alternating character of $A$ being not used to reduce the computation in `contract()`

**INPUT:**

- `form` – alternating form $B$ (instance of `FreeModuleAltForm`); the degree of $B$ must be at least equal to the degree of `self`

**OUTPUT:**

- element of the base ring (case $p = q$) or `FreeModuleAltForm` (case $p < q$) representing the interior product $\iota_A B$, where $A$ is `self`

See also:

`interior_product()` for the interior product of an alternating form by an alternating contravariant tensor

**EXAMPLES:**

Let us consider a rank-4 free module:

```python
sage: M = FiniteRankFreeModule(ZZ, 4, name='M', start_index=1)
sage: e = M.basis('e')
```

and various interior products on it, starting with a module element ($p=1$) and a linear form ($q=1$):

```python
sage: a = M([-2,1,2,3], basis=e, name='A')
sage: b = M.linear_form(name='B')
sage: b[:] = [2, 0, -3, 4]
sage: c = a.interior_product(b); c
2
sage: c == a.contract(b)
True
```

Case $p=1$ and $q=3$:

```python
sage: b = M.alternating_form(3, name='B')
sage: b[1,2,3], b[1,2,4], b[1,3,4], b[2,3,4] = 3, -1, 2, 5
sage: c = a.interior_product(b); c
Alternating form $i_A B$ of degree 2 on the Rank-4 free module $M$ over the Integer
˓→Ring
sage: c.display()
i_A B = 3 e^1 \wedge e^2 + 3 e^1 \wedge e^3 - 3 e^1 \wedge e^4 + 9 e^2 \wedge e^3 - 8 e^2 \wedge e^4 + e^3 \wedge e^4
sage: latex(c)
\iota_{A} B
sage: c == a.contract(b)
True
```

Case $p=2$ and $q=3$:

```python
```

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Case $p=2$ and $q=4$:

```python
sage: b = M.alternating_form(4, name='B')
sage: b[1,2,3,4] = 5
sage: c = a.interior_product(b); c
Alternating form $i_A B$ of degree 2 on the Rank-4 free module $M$ over the Integer Ring
display()
i_A B = 20 e^1 \wedge e^2 - 40 e^1 \wedge e^3 + 30 e^2 \wedge e^4 + 20 e^3 \wedge e^4
sage: c == a.contract(0, 1, b, 0, 1)
True
```

Case $p=2$ and $q=2$:

```python
sage: b = M.alternating_form(2)
sage: b[1,2], b[1,3], b[1,4] = 6, 0, -2
sage: b[2,3], b[2,4], b[3,4] = 2, 3, 4
sage: c = a.interior_product(b); c
48
sage: c == a.contract(0, 1, b, 0, 1)
True
```

Case $p=3$ and $q=3$:

```python
sage: a = M.alternating_contravariant_tensor(3, name='A')
sage: a[1,2,3], a[1,2,4], a[1,3,4], a[2,3,4] = -3, 2, 8, -5
sage: b = M.alternating_form(3, name='B')
sage: b[1,2,3], b[1,2,4], b[1,3,4], b[2,3,4] = 3, -1, 2, 5
sage: c = a.interior_product(b); c
-120
sage: c == a.contract(0, 1, 2, b, 0, 1, 2)
True
```

Case $p=3$ and $q=4$:

```python
sage: b = M.alternating_form(4, name='B')
sage: b[1,2,3,4] = 5
sage: c = a.interior_product(b); c
Linear form $i_A B$ on the Rank-4 free module $M$ over the Integer Ring
display()
i_A B = 150 e^1 + 240 e^2 - 60 e^3 - 90 e^4
sage: c == a.contract(0, 1, 2, b, 0, 1, 2)
True
```
Case $p=4$ and $q=4$:

```sage
sage: a = M.alternating_contravariant_tensor(4, name='A')
sage: a[1,2,3,4] = -2
sage: c = a.interior_product(b); c
-240
sage: c == a.contract(0, 1, 2, 3, b, 0, 1, 2, 3)
True
```

`wedge(other)`

Exterior product of `self` with the alternating contravariant tensor `other`.

**INPUT:**

- `other` – an alternating contravariant tensor

**OUTPUT:**

- instance of `AlternatingContrTensor` representing the exterior product `self ∧ other`

**EXAMPLES:**

Exterior product of two module elements:

```sage
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: a = M([1,-3,4], basis=e, name='A')
sage: b = M([2,-1,2], basis=e, name='B')
sage: c = a.wedge(b); c
Alternating contravariant tensor $A \wedge B$ of degree 2 on the Rank-3
free module $M$ over the Integer Ring
sage: c.display()
$A \wedge B = 5 \ e_0 \wedge e_1 - 6 \ e_0 \wedge e_2 - 2 \ e_1 \wedge e_2$
sage: latex(c)
$A\wedge B$
sage: latex(c.display())
$A\wedge B = 5 \ e_{0}\wedge e_{1} -6 \ e_{0}\wedge e_{2} -2 \ e_{1}\wedge e_{2}$
```

Test of the computation:

```sage
sage: a.wedge(b) == a*b - b*a
True
```

Exterior product of a module element and an alternating contravariant tensor of degree 2:

```sage
sage: d = M([-1,2,4], basis=e, name='D')
sage: s = d.wedge(c); s
Alternating contravariant tensor $D \wedge A \wedge B$ of degree 3 on the Rank-3
free module $M$ over the Integer Ring
sage: s.display()
$D \wedge A \wedge B = 34 \ e_0 \wedge e_1 \wedge e_2$
```

Test of the computation:

```sage
sage: s[0,1,2] == d[0]*c[1,2] + d[1]*c[2,0] + d[2]*c[0,1]
True
```
Let us check that the exterior product is associative:

```
sage: d.wedge(a.wedge(b)) == (d.wedge(a)).wedge(b)
True
```

and that it is graded anticommutative:

```
sage: a.wedge(b) == - b.wedge(a)
True
sage: d.wedge(c) == c.wedge(d)
True
```

### 5.3 Alternating forms on free modules

Given a free module \( M \) of finite rank over a commutative ring \( R \) and a positive integer \( p \), an *alternating form of degree* \( p \) on \( M \) is a map

\[
a : \underbrace{M \times \cdots \times M}_p \longrightarrow R
\]

that (i) is multilinear and (ii) vanishes whenever any of two of its arguments are equal. An alternating form of degree \( p \) is a tensor on \( M \) of type \((0, p)\).

Alternating forms are implemented via the class `FreeModuleAltForm`, which is a subclass of the generic tensor class `FreeModuleTensor`.

**AUTHORS:**

**REFERENCES:**
- Chap. 23 of R. Godement : *Algebra* [God1968]
- Chap. 15 of S. Lang : *Algebra* [Lan2002]

```python
class sage.tensor.modules.free_module_alt_form.FreeModuleAltForm(fmodule, degree, name=None, latex_name=None)
```

**INPUT:**
- `fmodule` – free module \( M \) of finite rank over a commutative ring \( R \), as an instance of `FiniteRankFreeModule`
- `degree` – positive integer; the degree \( p \) of the alternating form (i.e. its tensor rank)
- `name` – (default: `None`) string; name given to the alternating form
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the alternating form; if none is provided, name is used

**EXAMPLES:**
Alternating form of degree 2 on a rank-3 free module:
Tensors on free modules of finite rank, Release 9.6

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1)
sage: e = M.basis('e')
sage: a = M.alternating_form(2, name='a') ; a
Alternating form a of degree 2 on the
Rank-3 free module M over the Integer Ring
sage: type(a)
<class 'sage.tensor.modules.ext_pow_free_module.ExtPowerDualFreeModule_with_category.element_class'>
sage: a.parent()
2nd exterior power of the dual of the Rank-3 free module M over the Integer Ring
sage: a[1,2], a[2,3] = 4, -3
sage: a.display(e)
a = 4 e^1 \wedge e^2 - 3 e^2 \wedge e^3
```

The alternating form acting on the basis elements:

```python
sage: a(e[1],e[2])
4
sage: a(e[1],e[3])
0
sage: a(e[2],e[3])
-3
sage: a(e[2],e[1])
-4
```

An alternating form of degree 1 is a linear form:

```python
sage: b = M.linear_form('b') ; b
Linear form b on the Rank-3 free module M over the Integer Ring
sage: b[:] = [2,-1,3]  # components w.r.t. the module's default basis (e)
```

A linear form is a tensor of type (0,1):

```python
sage: b.tensor_type()
(0, 1)
```

It is an element of the dual module:

```python
sage: b.parent()
Dual of the Rank-3 free module M over the Integer Ring
sage: b.parent() is M.dual()
True
```

The members of a dual basis are linear forms:

```python
sage: e.dual_basis()[1]
Linear form e^1 on the Rank-3 free module M over the Integer Ring
sage: e.dual_basis()[2]
Linear form e^2 on the Rank-3 free module M over the Integer Ring
sage: e.dual_basis()[3]
Linear form e^3 on the Rank-3 free module M over the Integer Ring
```

Any linear form is expanded onto them:

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```
sage: b.display(e)
b = 2\ e^1 - \ e^2 + 3\ e^3
```

In the above example, an equivalent writing would have been `b.display()`, since the basis `e` is the module's default basis. A linear form maps module elements to ring elements:

```
sage: v = M([1,1,1])
sage: b(v)
4
```

Test of linearity:

```
sage: u = M([-5,-2,7])
sage: b(3*u - 4*v) == 3*b(u) - 4*b(v)
True
```

The standard tensor operations apply to alternating forms, like the extraction of components with respect to a given basis:

```
sage: a[e,1,2]
4
sage: a[1,2]  # since e is the module's default basis
4
sage: all( a[i,j] == - a[j,i] for i in {1,2,3} for j in {1,2,3} )
True
```

the tensor product:

```
sage: c = b⊗b ; c
Symmetric bilinear form b⊗b on the Rank-3 free module M over the Integer Ring
sage: c.parent()
Free module of type-(0,2) tensors on the Rank-3 free module M over the Integer Ring
sage: c.display(e)
b⊗b = 4\ e^1⊗e^1 - 2\ e^1⊗e^2 + 6\ e^1⊗e^3 - 2\ e^2⊗e^1 + e^2⊗e^2
- 3\ e^2⊗e^3 + 6\ e^3⊗e^1 - 3\ e^3⊗e^2 + 9\ e^3⊗e^3
```

the contractions:

```
sage: s = a.contract(v) ; s
Linear form on the Rank-3 free module M over the Integer Ring
sage: s.parent()
Dual of the Rank-3 free module M over the Integer Ring
sage: s.display(e)
4\ e^1 - 7\ e^2 + 3\ e^3
```

or tensor arithmetics:

```
sage: s = 3\ a + c ; s
Type-(0,2) tensor on the Rank-3 free module M over the Integer Ring
sage: s.parent()
```

(continues on next page)
Free module of type-\((0,2)\) tensors on the Rank-3 free module \(M\) over the Integer Ring

\[
\text{sage: } \text{s.display(e)}
\]
\[
4 \, e^1 \otimes e^1 + 10 \, e^1 \otimes e^2 + 6 \, e^1 \otimes e^3 - 14 \, e^2 \otimes e^1 + e^2 \otimes e^2
\]
\[
- 12 \, e^2 \otimes e^3 + 6 \, e^3 \otimes e^1 + 6 \, e^3 \otimes e^2 + 9 \, e^3 \otimes e^3
\]

Note that tensor arithmetics preserves the alternating character if both operands are alternating:

\[
\text{sage: } s = a - 2 \cdot a ; s
\]
Alternating form of degree 2 on the Rank-3 free module \(M\) over the Integer Ring

\[
\text{sage: } s.parent() \# \text{ note the difference with } s = 3 \cdot a + c \text{ above}
\]
2nd exterior power of the dual of the Rank-3 free module \(M\) over the Integer Ring

\[
\text{sage: } s == -a
\]
True

An operation specific to alternating forms is of course the exterior product:

\[
\text{sage: } s = a.\wedge(b) ; s
\]
Alternating form \(a \wedge b\) of degree 3 on the Rank-3 free module \(M\) over the Integer Ring

\[
\text{sage: } s.parent()
\]
3rd exterior power of the dual of the Rank-3 free module \(M\) over the Integer Ring

\[
\text{sage: } s.display(e)
\]
\(a \wedge b = 6 \, e^1 \wedge e^2 \wedge e^3\)

\[
\]
True

The exterior product is nilpotent on linear forms:

\[
\text{sage: } s = b.\wedge(b) ; s
\]
Alternating form zero of degree 2 on the Rank-3 free module \(M\) over the Integer Ring

\[
\text{sage: } s.display(e)
\]
\(\text{zero} = 0\)

\text{degree()}

Return the degree of \(\text{self}\).

\text{EXAMPLES:}

\[
\text{sage: } \text{M} = \text{FiniteRankFreeModule}(\text{ZZ}, 3, \text{name}='M')
\]
\[
\text{sage: } a = \text{M.alternating_form}(2, \text{name}='a')
\]
\[
\text{sage: } a.\text{degree()}
\]
\(2\)

\text{disp(basis=None, format_spec=None)}

Display the alternating form \(\text{self}\) in terms of its expansion w.r.t. a given module basis.

The expansion is actually performed onto exterior products of elements of the cobasis (dual basis) associated with \text{basis} (see examples below). The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

5.3. Alternating forms on free modules
INPUT:

- `basis` – (default: `None`) basis of the free module with respect to which the alternating form is expanded; if none is provided, the module’s default basis is assumed
- `format_spec` – (default: `None`) format specification passed to `self._fmodule._output_formatter` to format the output

EXAMPLES:

Display of an alternating form of degree 1 (linear form) on a rank-3 free module:

```sage
M = FiniteRankFreeModule(ZZ, 3, name='M')
e = M.basis('e')
e.dual_basis()
Dual basis (e^0,e^1,e^2) on the Rank-3 free module M over the Integer Ring
alpha = M.linear_form('a', latex_name=r'\alpha')
a[:] = [1,-3,4]
a.display(e)
a = e^0 - 3 e^1 + 4 e^2
alpha.display()
# display in the notebook
\alpha = e^{0} -3 e^{1} + 4 e^{2}
```

A shortcut is `disp()`:

```sage
alpha.disp()
a = e^0 - 3 e^1 + 4 e^2
```

Display of an alternating form of degree 2 on a rank-3 free module:

```sage
b = M.alternating_form(2, 'b', latex_name=r'\beta')
b[0,1], b[0,2], b[1,2] = 3, 2, -1
b.display()
b = 3 e^0 \wedge e^1 + 2 e^0 \wedge e^2 - e^1 \wedge e^2
latex(b.display())
# display in the notebook
\beta = 3 e^{0}\wedge e^{1} + 2 e^{0}\wedge e^{2} -e^{1}\wedge e^{2}
```

Display of an alternating form of degree 3 on a rank-3 free module:

```sage
c = M.alternating_form(3, 'c')
c[0,1,2] = 4
c.display()
c = 4 e^0 \wedge e^1 \wedge e^2
c.latex(c.display())
c = 4 e^{0}\wedge e^{1}\wedge e^{2}
```

Display of a vanishing alternating form:

```sage
c[0,1,2] = 0  # the only independent component set to zero
c.is_zero()
True
c.display()
c = 
latex(c.display())
```

Display in a basis which is not the default one:

```python
sage: aut = M.automorphism(matrix=[[0,1,0], [0,0,-1], [1,0,0]],
.....: basis=e)
sage: f = e.new_basis(aut, 'f')
sage: a.display(f)
a = 4 f^0 + f^1 + 3 f^2
sage: a.disp(f)  # shortcut notation
a = 4 f^0 + f^1 + 3 f^2
sage: b.display(f)
b = -2 f^0 \wedge f^1 - f^0 \wedge f^2 - 3 f^1 \wedge f^2
sage: c.display(f)
c = -4 f^0 \wedge f^1 \wedge f^2
```

The output format can be set via the argument `output_formatter` passed at the module construction:

```python
sage: N = FiniteRankFreeModule(QQ, 3, name='N', start_index=1,
.....: output_formatter=Rational.numerical_approx)
sage: e = N.basis('e')
sage: b = N.alternating_form(2, 'b')
sage: b[1,2], b[1,3], b[2,3] = 1/3, 5/2, 4
sage: b.display()  # default format (53 bits of precision)
b = 0.333333333333333 e^1 \wedge e^2 + 2.50000000000000 e^1 \wedge e^3
+ 4.00000000000000 e^2 \wedge e^3
```

The output format is then controlled by the argument `format_spec` of the method `display()`:

```python
sage: b.display(format_spec=10)  # 10 bits of precision
b = 0.33 e^1 \wedge e^2 + 2.5 e^1 \wedge e^3 + 4.0 e^2 \wedge e^3
```

Check that the bug reported in trac ticket #22520 is fixed:

```python
c = 0
sage: c[0,1,2] = 4  # value restored for what follows
```

5.3. Alternating forms on free modules

The expansion is actually performed onto exterior products of elements of the cobasis (dual basis) associated with basis (see examples below). The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

**INPUT:**

- `basis` – (default: `None`) basis of the free module with respect to which the alternating form is expanded; if none is provided, the module’s default basis is assumed
- `format_spec` – (default: `None`) format specification passed to `self._fmodule._output_formatter` to format the output.
EXAMPLES:

Display of an alternating form of degree 1 (linear form) on a rank-3 free module:

```sage
M = FiniteRankFreeModule(ZZ, 3, name='M')
e = M.basis('e')
e.dual_basis()
Dual basis (e^0,e^1,e^2) on the Rank-3 free module M over the Integer Ring
a = M.linear_form('a', latex_name=r'$\alpha$')
a[: ] = [1,-3,4]
a.display(e)
a = e^0 - 3 e^1 + 4 e^2
a.display()  # a shortcut since e is M's default basis
a = e^0 - 3 e^1 + 4 e^2
latex(a.display())  # display in the notebook
\alpha = e^{0} - 3 e^{1} + 4 e^{2}
```

A shortcut is disp():

```sage
a.disp()
a = e^0 - 3 e^1 + 4 e^2
```

Display of an alternating form of degree 2 on a rank-3 free module:

```sage
b = M.alternating_form(2, 'b', latex_name=r'$\beta$')
b[0,1], b[0,2], b[1,2] = 3, 2, -1
b.display()
b = 3 e^0 \wedge e^1 + 2 e^0 \wedge e^2 - e^1 \wedge e^2
latex(b.display())  # display in the notebook
\beta = 3 e^{0}\wedge e^{1} + 2 e^{0}\wedge e^{2} - e^{1}\wedge e^{2}
```

Display of an alternating form of degree 3 on a rank-3 free module:

```sage
c = M.alternating_form(3, 'c')
c[0,1,2] = 4
c.display()
c = 4 e^0 \wedge e^1 \wedge e^2
latex(c.display())
c = 4 e^{0}\wedge e^{1}\wedge e^{2}
```

Display of a vanishing alternating form:

```sage
c[0,1,2] = 0  # the only independent component set to zero
c.is_zero()
True
c.display()
c = 0
latex(c.display())
c = 0
c[0,1,2] = 4  # value restored for what follows
```

Display in a basis which is not the default one:

```sage
aut = M.automorphism(matrix=[[0,1,0], [0,0,-1], [1,0,0]], basis=e)
....:
```

(continues on next page)
sage: f = e.new_basis(aut, 'f')
sage: a.display(f)
a = 4 f^0 + f^1 + 3 f^2
sage: a.display(f)  # shortcut notation
a = 4 f^0 + f^1 + 3 f^2
sage: b.display(f)
b = -2 f^0∧f^1 - f^0∧f^2 - 3 f^1∧f^2
sage: c.display(f)
c = -4 f^0∧f^1∧f^2

The output format can be set via the argument `output_formatter` passed at the module construction:

```
sage: N = FiniteRankFreeModule(QQ, 3, name='N', start_index=1,
....:                        output_formatter=Rational.numerical_approx)
sage: e = N.basis('e')
sage: b.display()  # default format (53 bits of precision)
b = 0.333333333333333 e^1∧e^2 + 2.50000000000000 e^1∧e^3
+ 4.00000000000000 e^2∧e^3
```

The output format is then controlled by the argument `format_spec` of the method `display()`:

```
sage: b.display(format_spec=10)  # 10 bits of precision
b = 0.33 e^1∧e^2 + 2.5 e^1∧e^3 + 4.0 e^2∧e^3
```

Check that the bug reported in trac ticket #22520 is fixed:

```
sage: M = FiniteRankFreeModule(SR, 2, name='M')  # optional - sage.symbolic
sage: e = M.basis('e')  # optional - sage.symbolic
sage: a = M.alternating_form(2)  # optional - sage.symbolic
sage: a[0,1] = SR.var('t', domain='real')  # optional - sage.symbolic
sage: a.display()  # optional - sage.symbolic
```

```
interior_product(alt_tensor)
Interior product with an alternating contravariant tensor.

If `self` is an alternating form `A` of degree `p` and `B` is an alternating contravariant tensor of degree `q ≥ p` on the same free module, the interior product of `A` by `B` is the alternating contravariant tensor `ij...jq-p` defined by

\[(t_{AB})^{i_1...i_{q-p}} = A_{k_1...k_p} B^{k_1...k_p i_1...i_{q-p}}\]

Note: `A.interior_product(B)` yields the same result as `A.contract(0,..., p-1, B, 0,..., p-1)` (cf. `contract()`), but `interior_product` is more efficient, the alternating character of `A` being not used to reduce the computation in `contract()`

```
INPUT:

- `alt_tensor` – alternating contravariant tensor `B` (instance of `AlternatingContrTensor`); the degree of `B` must be at least equal to the degree of `self`

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OUTPUT:

- element of the base ring (case $p = q$) or `AlternatingContrTensor` (case $p < q$) representing the interior product $\iota_A B$, where $A$ is self

See also:

`interior_product()` for the interior product of an alternating contravariant tensor by an alternating form

EXAMPLES:

Let us consider a rank-3 free module:

```python
sage: M = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1)
sage: e = M.basis('e')
```

and various interior products on it, starting with a linear form ($p=1$) and a module element ($q=1$):

```python
sage: a = M.linear_form(name='A')
sage: a[:] = [-2, 4, 3]
sage: b = M([3, 1, 5], basis=e, name='B')
sage: c = a.interior_product(b); c
13
sage: c == a.contract(b)
True
```

Case $p=1$ and $q=2$:

```python
sage: b = M.alternating_contravariant_tensor(2, name='B')
sage: b[1,2], b[1,3], b[2,3] = 5, 2, 3
sage: c = a.interior_product(b); c
Element i_A B of the Rank-3 free module M over the Integer Ring
sage: c.display()
i_A B = -26 e_1 - 19 e_2 + 8 e_3
sage: latex(c)
\iota_{A} B
sage: c == a.contract(b)
True
```

Case $p=1$ and $q=3$:

```python
sage: b = M.alternating_contravariant_tensor(3, name='B')
sage: b[1,2,3] = 5
sage: c = a.interior_product(b); c
Alternating contravariant tensor i_A B of degree 2 on the Rank-3 free module M over the Integer Ring
sage: c.display()
i_A B = 15 e_1 \wedge e_2 - 20 e_1 \wedge e_3 - 10 e_2 \wedge e_3
sage: c == a.contract(b)
True
```

Case $p=2$ and $q=2$:

```python
sage: a = M.alternating_form(2, name='A')
sage: a[1,2], a[1,3], a[2,3] = 2, -3, 1
sage: b = M.alternating_contravariant_tensor(2, name='B')
sage: b[1,2], b[1,3], b[2,3] = 5, 2, 3
```

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(continued from previous page)

sage: c = a.interior_product(b); c
14
sage: c == a.contract(0, 1, b, 0, 1)  # contraction on all indices of a
True

Case $p=2$ and $q=3$:

sage: b = M.alternating_contravariant_tensor(3, name='B')
sage: b[1,2,3] = 5
sage: c = a.interior_product(b); c
Element $i_A B$ of the Rank-3 free module $M$ over the Integer Ring
sage: c.display()
i_A B = 10 e_1 + 30 e_2 + 20 e_3
sage: c == a.contract(0, 1, b, 0, 1)
True

Case $p=3$ and $q=3$:

sage: a = M.alternating_form(3, name='A')
sage: a[1,2,3] = -2
sage: c = a.interior_product(b); c
-60
sage: c == a.contract(0, 1, 2, b, 0, 1, 2)
True

\texttt{wedge}(\texttt{other})

Exterior product of self with the alternating form other.

INPUT:

- \texttt{other} – an alternating form

OUTPUT:

- instance of \texttt{FreeModuleAltForm} representing the exterior product self $\wedge$ other

EXAMPLES:

Exterior product of two linear forms:

sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: a = M.linear_form('A')
sage: a[:] = [1,-3,4]
sage: b = M.linear_form('B')
sage: b[:] = [2,-1,2]
sage: c = a.wedge(b) ; c
Alternating form $A \wedge B$ of degree 2 on the Rank-3 free module $M$ over the Integer Ring
sage: c.display()
$A \wedge B = 5 e^0 \wedge e^1 - 6 e^0 \wedge e^2 - 2 e^1 \wedge e^2$
sage: latex(c)
$A \wedge B$
sage: latex(c.display())
$A \wedge B = 5 e^0 \wedge e^1 - 6 e^0 \wedge e^2 - 2 e^1 \wedge e^2$

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Test of the computation:

```python
sage: a.wedge(b) == a*b - b*a
True
```

Exterior product of a linear form and an alternating form of degree 2:

```python
sage: d = M.linear_form('D')
sage: d[:] = [-1,2,4]
sage: s = d.wedge(c) ; s
Alternating form $D \wedge A \wedge B$ of degree 3 on the Rank-3 free module $M$
over the Integer Ring
sage: s.display()
$D \wedge A \wedge B = 34 \, e^0 \wedge e^1 \wedge e^2$
```

Test of the computation:

```python
sage: s[0,1,2] == d[0]*c[1,2] + d[1]*c[2,0] + d[2]*c[0,1]
True
```

Let us check that the exterior product is associative:

```python
sage: d.wedge(a.wedge(b)) == (d.wedge(a)).wedge(b)
True
```

and that it is graded anticommutative:

```python
sage: a.wedge(b) == - b.wedge(a)
True
sage: d.wedge(c) == c.wedge(d)
True
```
CHAPTER
SIX

MORPHISMS

6.1 Sets of morphisms between free modules

The class :class:`FreeModuleHomset` implements sets of homomorphisms between two free modules of finite rank over the same commutative ring.

AUTHORS:


REFERENCES:

- Chaps. 13, 14 of R. Godement : *Algebra* [God1968]
- Chap. 3 of S. Lang : *Algebra* [Lan2002]

.. class:: sage.tensor.modules.free_module_homset.FreeModuleHomset(fmodule1, fmodule2, name=None, latex_name=None)

Bases: :class:`sage.categories.homset.Homset`

Set of homomorphisms between free modules of finite rank over a commutative ring.

Given two free modules \( M \) and \( N \) of respective ranks \( m \) and \( n \) over a commutative ring \( R \), the class :class:`FreeModuleHomset` implements the set \( \text{Hom}(M, N) \) of homomorphisms \( M \to N \). The set \( \text{Hom}(M, N) \) is actually a free module of rank \( mn \) over \( R \), but this aspect is not taken into account here.

This is a Sage parent class, whose element class is :class:`FiniteRankFreeModuleMorphism`.

INPUT:

- \( fmodule1 \) – free module \( M \) (domain of the homomorphisms), as an instance of :class:`FiniteRankFreeModule`
- \( fmodule2 \) – free module \( N \) (codomain of the homomorphisms), as an instance of :class:`FiniteRankFreeModule`
- \( \text{name} \) – (default: None) string; name given to the hom-set; if none is provided, \( \text{Hom}(M, N) \) will be used
- \( \text{latex_name} \) – (default: None) string; LaTeX symbol to denote the hom-set; if none is provided, \( \text{Hom}(M, N) \) will be used

EXAMPLES:

Set of homomorphisms between two free modules over \( \mathbb{Z} \):

.. code::

    sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
    sage: N = FiniteRankFreeModule(ZZ, 2, name='N')
    sage: H = Hom(M, N)

Set of Morphisms from Rank-3 free module M over the Integer Ring
Hom-sets are cached:

```python
sage: H is Hom(M,N)
True
```

The \LaTeX{} formatting is:

```python
sage: latex(H)
\mathrm{Hom}\left(M,N\right)
```

As usual, the construction of an element is performed by the \_\_call\_ method; the argument can be the matrix representing the morphism in the default bases of the two modules:

```python
sage: e = M.basis('e')
sage: f = N.basis('f')
sage: phi = H([-1,2,0], [5,1,2]) ; phi
Generic morphism:
  From: Rank-3 free module M over the Integer Ring
  To: Rank-2 free module N over the Integer Ring
sage: phi.parent() is H
True
```

An example of construction from a matrix w.r.t. bases that are not the default ones:

```python
sage: ep = M.basis('ep', latex_symbol=r'\text{e}')
sage: fp = N.basis('fp', latex_symbol=r'\text{f}')
sage: phi2 = H([[3,2,1], [1,2,3]], bases=(ep,fp)) ; phi2
Generic morphism:
  From: Rank-3 free module M over the Integer Ring
  To: Rank-2 free module N over the Integer Ring
```

The zero element:

```python
sage: z = H.zero() ; z
Generic morphism:
  From: Rank-3 free module M over the Integer Ring
  To: Rank-2 free module N over the Integer Ring
sage: z.matrix(e,f)
[0 0 0]
[0 0 0]
```

The test suite for H is passed:

```python
sage: TestSuite(H).run()
```

The set of homomorphisms $M \to M$, i.e. endomorphisms, is obtained by the function `End`:
End(M) is actually identical to Hom(M, M):

```
sage: End(M) is Hom(M, M)
True
```

The unit of the endomorphism ring is the identity map:

```
sage: End(M).one()
Identity endomorphism of Rank-3 free module M over the Integer Ring
```

whose matrix in any basis is of course the identity matrix:

```
sage: End(M).one().matrix(e)
[1 0 0]
[0 1 0]
[0 0 1]
```

There is a canonical identification between endomorphisms of \( M \) and tensors of type \((1, 1)\) on \( M \). Accordingly, coercion maps have been implemented between \( \text{End}(M) \) and \( T^{(1,1)}(M) \) (the module of all type-(1, 1) tensors on \( M \), see `TensorFreeModule`):

```
sage: T11 = M.tensor_module(1,1) ; T11
Free module of type-(1,1) tensors on the Rank-3 free module M over the Integer Ring
sage: End(M).has_coerce_map_from(T11)
True
sage: T11.has_coerce_map_from(End(M))
True
```

See `TensorFreeModule` for examples of the above coercions.

There is a coercion \( \text{GL}(M) \to \text{End}(M) \), since every automorphism is an endomorphism:

```
sage: GL = M.general_linear_group() ; GL
General linear group of the Rank-3 free module M over the Integer Ring
sage: End(M).has_coerce_map_from(GL)
True
```

Of course, there is no coercion in the reverse direction, since only bijective endomorphisms are automorphisms:

```
sage: GL.has_coerce_map_from(End(M))
False
```

The coercion \( \text{GL}(M) \to \text{End}(M) \) in action:

```
sage: a = GL.an_element() ; a
Automorphism of the Rank-3 free module M over the Integer Ring
sage: a.matrix(e)
[1 0 0]
```

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\[
\begin{bmatrix}
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

sage: ea = End(M)(a) ; ea
Generic endomorphism of Rank-3 free module M over the Integer Ring
sage: ea.matrix(e)
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Element

alias of \(\text{sage.tensor.modules.free_module_morphism.FiniteRankFreeModuleMorphism}\)

\text{one()}

Return the identity element of \text{self} considered as a monoid (case of an endomorphism set).

This applies only when the codomain of \text{self} is equal to its domain, i.e. when \text{self} is of the type \(\text{Hom}(M, M)\).

OUTPUT:

- the identity element of \(\text{End}(M) = \text{Hom}(M, M)\), as an instance of \(\text{FiniteRankFreeModuleMorphism}\)

EXAMPLES:

Identity element of the set of endomorphisms of a free module over \(\mathbb{Z}\):

sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
  sage: e = M.basis('e')
  sage: H = End(M)
  sage: H.one()
Identity endomorphism of Rank-3 free module M over the Integer Ring
sage: H.one().matrix(e)
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
sage: H.one().is_identity()
True

NB: mathematically, \text{H.one()} coincides with the identity map of the free module \(M\). However the latter is considered here as an element of \(\text{GL}(M)\), the general linear group of \(M\). Accordingly, one has to use the coercion map \(\text{GL}(M) \to \text{End}(M)\) to recover \text{H.one()} from \text{M.identity_map()}:

sage: M.identity_map()
Identity map of the Rank-3 free module M over the Integer Ring
sage: M.identity_map().parent()
General linear group of the Rank-3 free module M over the Integer Ring
sage: H.one().parent()
Set of Morphisms from Rank-3 free module M over the Integer Ring to Rank-3 free module M over the Integer Ring in Category of finite dimensional modules over Integer Ring
sage: H.one() == H(M.identity_map())
True

Conversely, one can recover \text{M.identity_map()} from \text{H.one()} by means of a conversion \(\text{End}(M) \to \text{GL}(M)\):
6.2 Free module morphisms

The class `FiniteRankFreeModuleMorphism` implements homomorphisms between two free modules of finite rank over the same commutative ring.

AUTHORS:


REFERENCES:

- Chap. 3 of S. Lang : Algebra [Lan2002]

```python
sage: GL = M.general_linear_group()
sage: M.identity_map() == GL(M.one())
True
```

class `sage.tensor.modules.free_module_morphism.FiniteRankFreeModuleMorphism`(`parent`, `matrix_rep`, `bases=None`, `name=None`, `latex_name=None`, `is_identity=False`)

Bases: `sage.categories.morphism.Morphism`

Homomorphism between free modules of finite rank over a commutative ring.

An instance of this class is a homomorphism

$$\phi : M \rightarrow N,$$

where $M$ and $N$ are two free modules of finite rank over the same commutative ring $R$.

This is a Sage `element` class, the corresponding `parent` class being `FreeModuleHomset`.

INPUT:

- `parent` – hom-set Hom(M,N) to which the homomorphism belongs
- `matrix_rep` – matrix representation of the homomorphism with respect to the bases `bases`; this entry can actually be any material from which a matrix of size rank(N)*rank(M) of elements of $R$ can be constructed; the columns of the matrix give the images of the basis of $M$ (see the convention in the example below)
- `bases` – (default: `None`) pair (basis_M, basis_N) defining the matrix representation, basis_M being a basis of module M and basis_N a basis of module N; if `None` the pair formed by the default bases of each module is assumed.
- `name` – (default: `None`) string; name given to the homomorphism
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the homomorphism; if `None`, `name` will be used.
- `is_identity` – (default: `False`) determines whether the constructed object is the identity endomorphism; if set to `True`, then N must be M and the entry `matrix_rep` is not used.

EXAMPLES:

A homomorphism between two free modules over $\mathbb{Z}$ is constructed as an element of the corresponding hom-set, by means of the function `__call__`: 
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: N = FiniteRankFreeModule(ZZ, 2, name='N')
sage: e = M.basis('e'); f = N.basis('f')
sage: H = Hom(M,N); H
Set of Morphisms from Rank-3 free module M over the Integer Ring
to Rank-2 free module N over the Integer Ring
in Category of finite dimensional modules over Integer Ring
sage: phi = H([[2,-1,3], [1,0,-4]], name='phi', latex_name=r'$\phi$'); phi
Generic morphism:
  From: Rank-3 free module M over the Integer Ring
  To:  Rank-2 free module N over the Integer Ring

Since no bases have been specified in the argument list, the provided matrix is relative to the default bases of
modules M and N, so that the above is equivalent to:

sage: phi = H([[2,-1,3], [1,0,-4]], bases=(e,f), name='phi',
          latex_name=r'$\phi$'); phi
Generic morphism:
  From: Rank-3 free module M over the Integer Ring
  To:  Rank-2 free module N over the Integer Ring

An alternative way to construct a homomorphism is to call the method hom() on the domain:

sage: phi = M.hom(N, [[2,-1,3], [1,0,-4]], bases=(e,f), name='phi',
          latex_name=r'$\phi$'); phi
Generic morphism:
  From: Rank-3 free module M over the Integer Ring
  To:  Rank-2 free module N over the Integer Ring

The parent of a homomorphism is of course the corresponding hom-set:

sage: phi.parent() is H
True
sage: phi.parent() is Hom(M,N)
True

Due to Sage’s category scheme, the actual class of the homomorphism phi is a derived class of
FiniteRankFreeModuleMorphism:

sage: type(phi)
<class 'sage.tensor.modules.free_module_homset.FreeModuleHomset_with_category_with_˓
equality_by_id.element_class'>
sage: isinstance(phi, sage.tensor.modules.free_module_morphism.
              FiniteRankFreeModuleMorphism)
True

The domain and codomain of the homomorphism are returned respectively by the methods domain() and
codomain(), which are implemented as Sage’s constant functions:

sage: phi.domain()
Rank-3 free module M over the Integer Ring
sage: phi.codomain()
Rank-2 free module N over the Integer Ring
The matrix of the homomorphism with respect to a pair of bases is returned by the method `matrix()`:

```
sage: phi.matrix(e,f)
[ 2 -1  3]
[ 1  0 -4]
```

The convention is that the columns of this matrix give the components of the images of the elements of basis $e$ w.r.t basis $f$:

```
sage: phi(e[0]).display()
phi(e_0) = 2 f_0 + f_1
sage: phi(e[1]).display()
phi(e_1) = -f_0
sage: phi(e[2]).display()
phi(e_2) = 3 f_0 - 4 f_1
```

Test of the module homomorphism laws:

```
sage: phi(M.zero()) == N.zero()
True
sage: u = M([1,2,3], basis=e, name='u') ; u.display()
u = e_0 + 2 e_1 + 3 e_2
sage: v = M([-2,1,4], basis=e, name='v') ; v.display()
v = -2 e_0 + e_1 + 4 e_2
sage: phi(u).display()
phi(u) = 9 f_0 - 11 f_1
sage: phi(v).display()
phi(v) = 7 f_0 - 18 f_1
sage: phi(3*u + v).display()
34 f_0 - 51 f_1
sage: phi(3*u + v) == 3*phi(u) + phi(v)
True
```

The identity endomorphism:

```
sage: Id = End(M).one() ; Id
Identity endomorphism of Rank-3 free module $\mathbb{Z}$ over the Integer Ring
sage: Id.parent()
Set of Morphisms from Rank-3 free module $\mathbb{Z}$ over the Integer Ring
to Rank-3 free module $\mathbb{Z}$ over the Integer Ring
in Category of finite dimensional modules over Integer Ring
sage: Id.parent() is End(M)
True
```

The matrix of $\text{Id}$ with respect to the basis $e$ is of course the identity matrix:

```
sage: Id.matrix(e)
[1 0 0]
[0 1 0]
[0 0 1]
```

6.2. Free module morphisms
The identity acting on a module element:

```
sage: Id(v) is v
True
```

**is_identity()**

Check whether `self` is the identity morphism.

**EXAMPLES:**

```
sage: M = FiniteRankFreeModule(ZZ, 2, name='M')
sage: e = M.basis('e')
sage: phi = M.endomorphism([[1,0], [0,1]])
sage: phi.is_identity()
True
sage: (phi+phi).is_identity()
False
sage: End(M).zero().is_identity()
False
sage: a = M.automorphism() ; a[0,1], a[1,0] = 1, -1
sage: ep = e.new_basis(a, 'ep', latex_symbol='e''

sage: phi = M.endomorphism([[1,0], [0,1]], basis=ep)
sage: phi.is_identity()
True
```

Example illustrating that the identity can be constructed from a matrix that is not the identity one, provided that it is relative to different bases:

```
sage: phi = M.hom(M, [[0,1], [-1,0]], bases=(ep,e))
sage: phi.is_identity()
True
```

Of course, if we ask for the matrix in a single basis, it is the identity matrix:

```
sage: phi.matrix(e)
[1 0]
[0 1]
sage: phi.matrix(ep)
[1 0]
[0 1]
```

**is_injective()**

Determine whether `self` is injective.

**OUTPUT:**

- True if `self` is an injective homomorphism and False otherwise

**EXAMPLES:**

Homomorphisms between two $\mathbb{Z}$-modules:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: N = FiniteRankFreeModule(ZZ, 2, name='N')
sage: e = M.basis('e') ; f = N.basis('f')
sage: phi = M.hom(N, [[-1,2,0], [5,1,2]])
sage: phi.matrix(e,f)
```

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Indeed, phi has a non trivial kernel:

\[
\text{sage: } \phi(4e[0] + 2e[1] - 11e[2]).display()
\]
\[0\]

An injective homomorphism:

\[
\text{sage: } \psi = N.\text{hom}(M, [[1,-1], [0,3], [4,-5]])
\]
\[
\text{sage: } \psi.\text{matrix}(f,e)
\]
\[
\begin{bmatrix}
1 & -1 \\
0 & 3 \\
4 & -5
\end{bmatrix}
\]
\[
\text{sage: } \psi.\text{is_injective()}
\]
\[True\]

Of course, the identity endomorphism is injective:

\[
\text{sage: } \text{End}(M).\text{one().is_injective()}
\]
\[True\]
\[
\text{sage: } \text{End}(N).\text{one().is_injective()}
\]
\[True\]

is\_surjective()
Determine whether self is surjective.

OUTPUT:

• True if self is a surjective homomorphism and False otherwise

EXAMPLES:

This method has not been implemented yet:

\[
\text{sage: } M = \text{FiniteRankFreeModule}(\mathbb{Z}, 3, \text{name='M'})
\]
\[
\text{sage: } N = \text{FiniteRankFreeModule}(\mathbb{Z}, 2, \text{name='N'})
\]
\[
\text{sage: } e = M.\text{basis('e')}; f = N.\text{basis('f')}
\]
\[
\text{sage: } \phi = M.\text{hom}(N, [[-1,2,0], [5,1,2]])
\]
\[
\text{sage: } \phi.\text{is\_surjective()}
\]
Traceback (most recent call last):
...
NotImplementedError: FiniteRankFreeModuleMorphism.is\_surjective() has not been implemented yet

except for the identity endomorphism (!):

\[
\text{sage: } \text{End}(M).\text{one().is\_surjective()}
\]
\[True\]
\[
\text{sage: } \text{End}(N).\text{one().is\_surjective()}
\]
\[True\]
matrix(basis1=None, basis2=None)

Return the matrix of self w.r.t a pair of bases.

If the matrix is not known already, it is computed from the matrix in another pair of bases by means of the change-of-basis formula.

INPUT:

• basis1 – (default: None) basis of the domain of self; if none is provided, the domain’s default basis is assumed

• basis2 – (default: None) basis of the codomain of self; if none is provided, basis2 is set to basis1 if self is an endomorphism, otherwise, basis2 is set to the codomain’s default basis.

OUTPUT:

• the matrix representing the homomorphism self w.r.t to bases basis1 and basis2; more precisely, the columns of this matrix are formed by the components w.r.t. basis2 of the images of the elements of basis1.

EXAMPLES:

Matrix of a homomorphism between two \( \mathbb{Z} \)-modules:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: N = FiniteRankFreeModule(ZZ, 2, name='N')
sage: e = M.basis('e'); f = N.basis('f')
sage: phi = M.hom(N, [[-1,2,0], [5,1,2]])
sage: phi.matrix()  # default bases
[-1  2  0]
[ 5  1  2]
sage: phi.matrix(e, f)  # given bases
[-1  2  0]
[ 5  1  2]
sage: type(phi.matrix())
<class 'sage.matrix.matrix_integer_dense.Matrix_integer_dense'>
```

Matrix in bases different from those in which the homomorphism has been defined:

```
sage: a = M.automorphism(matrix=[[1,0,0],[0,1,2],[0,1,3]], basis=e)
sage: ep = e.new_basis(a, 'ep', latex_symbol='e')
sage: b = N.automorphism(matrix=[[3,5],[4,7]], basis=f)
sage: fp = f.new_basis(b, 'fp', latex_symbol='f')
sage: phi.matrix(ep, fp)
[ 32 -1 -12]
[-19  1   8]
```

Check of the change-of-basis formula:

```
sage: phi.matrix(ep, fp) == (b^(-1)).matrix(f) * phi.matrix(e,f) * a.matrix(e)
True
```

Single change of basis:

```
sage: phi.matrix(ep, f)
[ 1  2  4]
[-5  3  8]
sage: phi.matrix(ep,f) == phi.matrix(e,f) * a.matrix(e)
(continues on next page)
```
Matrix of an endomorphism:

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{bmatrix}
\]

6.3 General linear group of a free module

The set $GL(M)$ of automorphisms (i.e. invertible endomorphisms) of a free module of finite rank $M$ is a group under composition of automorphisms, named the general linear group of $M$. In other words, $GL(M)$ is the group of units (i.e. invertible elements) of $End(M)$, the endomorphism ring of $M$.

The group $GL(M)$ is implemented via the class `FreeModuleLinearGroup`.

AUTHORS:

- Eric Gourgoulhon (2015): initial version
- Michael Jung (2019): improve treatment of the identity element

REFERENCES:

- Chap. 15 of R. Godement: *Algebra* [God1968]

```python
class sage.tensor.modules.free_module_linear_group.FreeModuleLinearGroup(fmodule)
    Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

    General linear group of a free module of finite rank over a commutative ring.

    Given a free module of finite rank $M$ over a commutative ring $R$, the general linear group of $M$ is the group $GL(M)$ of automorphisms (i.e. invertible endomorphisms) of $M$. It is the group of units (i.e. invertible elements) of $End(M)$, the endomorphism ring of $M$.

    This is a Sage parent class, whose element class is `FreeModuleAutomorphism`.

INPUT:

- `fmodule` – free module $M$ of finite rank over a commutative ring $R$, as an instance of `FiniteRankFreeModule`
```
EXAMPLES:

General linear group of a free \( \mathbb{Z} \)-module of rank 3:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: from sage.tensor.modules.free_module_linear_group import FreeModuleLinearGroup
sage: GL = FreeModuleLinearGroup(M) ; GL
General linear group of the Rank-3 free module M over the Integer Ring
```

Instead of importing FreeModuleLinearGroup in the global name space, it is recommended to use the module’s method `general_linear_group()`:

```
sage: GL = M.general_linear_group() ; GL
General linear group of the Rank-3 free module M over the Integer Ring
sage: latex(GL)
\mathrm{GL}\left( M \right)
```

As most parents, the general linear group has a unique instance:

```
sage: GL is M.general_linear_group()
True
```

\( \text{GL}(M) \) is in the category of groups:

```
sage: GL.category()
Category of groups
sage: GL in Groups()
True
```

\( \text{GL} \) is a `parent` object, whose elements are automorphisms of \( M \), represented by instances of the class `FreeModuleAutomorphism`:

```
sage: GL.Element
<class 'sage.tensor.modules.free_module_automorphism.FreeModuleAutomorphism'>
sage: a = GL.an_element() ; a
Automorphism of the Rank-3 free module M over the Integer Ring
sage: a.matrix(e)
[ 1  0  0]
[ 0 -1  0]
[ 0  0  1]
sage: a in GL
True
sage: GL.is_parent_of(a)
True
```

As an endomorphism, \( a \) maps elements of \( M \) to elements of \( M \):

```
sage: v = M.an_element() ; v
Element of the Rank-3 free module M over the Integer Ring
sage: v.display()
e_0 + e_1 + e_2
sage: a(v)
Element of the Rank-3 free module M over the Integer Ring
```

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An automorphism can also be viewed as a tensor of type \((1, 1)\) on \(M\):

\[
\text{sage: a.tensor_type()}
\]

\((1, 1)\)

\[
\text{sage: a.display(e)}
\]

\[
e_0 \otimes e^0 - e_1 \otimes e^1 + e_2 \otimes e^2
\]

\[
\text{sage: type(a)}
\]

<class 'sage.tensor.modules.free_module_linear_group.FreeModuleLinearGroup_with_category.element_class'>

As for any group, the identity element is obtained by the method \(\text{one()}\):

\[
\text{sage: id = GL.one() ; id}
\]

Identity map of the Rank-3 free module \(M\) over the Integer Ring

\[
\text{sage: id*a == a}
\]

True

\[
\text{sage: a*id == a}
\]

True

\[
\text{sage: a*a^(-1) == id}
\]

True

\[
\text{sage: a^(-1)*a == id}
\]

True

The identity element is of course the identity map of the module \(M\):

\[
\text{sage: id(v) == v}
\]

True

\[
\text{sage: id.matrix(e)}
\]

\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

The module’s changes of basis are stored as elements of the general linear group:

\[
\text{sage: f = M.basis('f', from_family=(-e[1], 4*e[0]+3*e[2], 7*e[0]+5*e[2]))}
\]

\[
\text{sage: f}
\]

Basis \((f_0, f_1, f_2)\) on the Rank-3 free module \(M\) over the Integer Ring

\[
\text{sage: M.change_of_basis(e,f)}
\]

Automorphism of the Rank-3 free module \(M\) over the Integer Ring

\[
\text{sage: M.change_of_basis(e,f) in GL}
\]

True

\[
\text{sage: M.change_of_basis(e,f).parent()}
\]

General linear group of the Rank-3 free module \(M\) over the Integer Ring

\[
\text{sage: M.change_of_basis(e,f).matrix(e)}
\]

\[
\begin{bmatrix} 0 & 4 & 7 \\ -1 & 0 & 0 \\ 0 & 3 & 5 \end{bmatrix}
\]

\[
\text{sage: M.change_of_basis(e,f) == M.change_of_basis(f,e).inverse()}
\]

True
Since every automorphism is an endomorphism, there is a coercion $GL(M) \to \text{End}(M)$ (the endomorphism ring of module $M$):

```sage
sage: End(M).has_coerce_map_from(GL)
True
```

(see `FreeModuleHomset` for details), but not in the reverse direction, since only bijective endomorphisms are automorphisms:

```sage
sage: GL.has_coerce_map_from(End(M))
False
```

A bijective endomorphism can be converted to an element of $GL(M)$:

```sage
sage: h = M.endomorphism([[1,0,0], [0,-1,2], [0,1,-3]]) ; h
Generic endomorphism of Rank-3 free module M over the Integer Ring
sage: h.parent() is End(M)
True
sage: ah = GL(h) ; ah
Automorphism of the Rank-3 free module M over the Integer Ring
sage: ah.parent() is GL
True
```

As maps $M \to M$, $ah$ and $h$ are identical:

```sage
sage: v # recall
Element of the Rank-3 free module M over the Integer Ring
sage: ah(v) == h(v)
True
sage: ah.matrix(e) == h.matrix(e)
True
```

Of course, non-invertible endomorphisms cannot be converted to elements of $GL(M)$:

```sage
sage: GL(M.endomorphism([[0,0,0], [0,-1,2], [0,1,-3]]))
Traceback (most recent call last):
...
TypeError: the Generic endomorphism of Rank-3 free module M over the Integer Ring is not invertible
```

Similarly, there is a coercion $GL(M) \to T^{(1,1)}(M)$ (module of type-$(1,1)$ tensors):

```sage
sage: M.tensor_module(1,1).has_coerce_map_from(GL)
True
```

(see `TensorFreeModule` for details), but not in the reverse direction, since not every type-$(1,1)$ tensor can be considered as an automorphism:

```sage
sage: GL.has_coerce_map_from(M.tensor_module(1,1))
False
```

Invertible type-$(1,1)$ tensors can be converted to automorphisms:

```sage
sage: t = M.tensor((1,1), name='t')
sage: t[e,:,:] = [[-1,0,0], [0,1,2], [0,1,3]]
```

(continues on next page)
```python
sage: at = GL(t) ; at
Automorphism t of the Rank-3 free module M over the Integer Ring
sage: at.matrix(e)
[-1 0 0]
[ 0 1 2]
[ 0 1 3]
sage: at.matrix(e) == t[e,:]
True
```

Non-invertible ones cannot:

```python
sage: t0 = M.tensor((1,1), name='t_0')
sage: t0[e,0,0] = 1
sage: t0[e,:]
# the matrix is clearly not invertible
[1 0 0]
[0 0 0]
[0 0 0]
sage: GL(t0)
Traceback (most recent call last):
...
TypeError: the Type-(1,1) tensor t_0 on the Rank-3 free module M over
the Integer Ring is not invertible
sage: t0[e,1,1], t0[e,2,2] = 2, 3
sage: t0[e,:]
# the matrix is not invertible in Mat_3(ZZ)
[1 0 0]
[0 2 0]
[0 0 3]
sage: GL(t0)
Traceback (most recent call last):
...
TypeError: the Type-(1,1) tensor t_0 on the Rank-3 free module M over
the Integer Ring is not invertible
```

**Element**

alias of `sage.tensor.modules.free_module_automorphism.FreeModuleAutomorphism`

**base_module()**

Return the free module of which `self` is the general linear group.

OUTPUT:

- instance of `FiniteRankFreeModule` representing the free module of which `self` is the general linear group

EXAMPLES:

```python
sage: M = FiniteRankFreeModule(ZZ, 2, name='M')
sage: GL = M.general_linear_group()
sage: GL.base_module()
Rank-2 free module M over the Integer Ring
sage: GL.base_module() is M
True
```

**one()**

Return the group identity element of `self`.

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The group identity element is nothing but the module identity map.

**OUTPUT:**

- instance of `FreeModuleAutomorphism` representing the identity element.

**EXAMPLES:**

Identity element of the general linear group of a rank-2 free module:

```python
sage: M = FiniteRankFreeModule(ZZ, 2, name='M', start_index=1)
sage: GL = M.general_linear_group()
sage: GL.one()
Identity map of the Rank-2 free module M over the Integer Ring
```

The identity element is cached:

```python
sage: GL.one() is GL.one()
True
```

Check that the element returned is indeed the neutral element for the group law:

```python
sage: e = M.basis('e')
sage: a = GL([[3,4],[5,7]], basis=e) ; a
Automorphism of the Rank-2 free module M over the Integer Ring
sage: a.matrix(e)
[3 4]
[5 7]
sage: GL.one() * a == a
True
sage: a * GL.one() == a
True
sage: a * a^(-1) == GL.one()
True
sage: a^(-1) * a == GL.one()
True
```

The unit element of \( GL(M) \) is the identity map of \( M \):

```python
sage: GL.one()(e[1])
Element e_1 of the Rank-2 free module M over the Integer Ring
sage: GL.one()(e[2])
Element e_2 of the Rank-2 free module M over the Integer Ring
```

Its matrix is the identity matrix in any basis:

```python
sage: GL.one().matrix(e)
[1 0]
[0 1]
sage: f = M.basis('f', from_family=(e[1]+2*e[2], e[1]+3*e[2]))
sage: GL.one().matrix(f)
[1 0]
[0 1]
```
6.4 Free module automorphisms

Given a free module $M$ of finite rank over a commutative ring $R$, an *automorphism* of $M$ is a map

$$\phi : M \rightarrow M$$

that is linear (i.e. is a module homomorphism) and bijective.

Automorphisms of a free module of finite rank are implemented via the class `FreeModuleAutomorphism`.

**AUTHORS:**

- Eric Gourgoulhon (2015): initial version
- Michael Jung (2019): improve treatment of the identity element

**REFERENCES:**

- Chaps. 15, 24 of R. Godement: *Algebra* [God1968]

**class** `sage.tensor.modules.free_module_automorphism.FreeModuleAutomorphism`

```python

Automorphism of a free module of finite rank over a commutative ring.

This is a Sage *element* class, the corresponding *parent* class being `FreeModuleLinearGroup`.

This class inherits from the classes `FreeModuleTensor` and `MultiplicativeGroupElement`.

**INPUT:**

- `fmodule` – free module $M$ of finite rank over a commutative ring $R$, as an instance of `FiniteRankFreeModule`
- `name` – (default: None) name given to the automorphism
- `latex_name` – (default: None) LaTeX symbol to denote the automorphism; if none is provided, the LaTeX symbol is set to `name`
- `is_identity` – (default: False) determines whether the constructed object is the identity automorphism, i.e. the identity map of $M$ considered as an automorphism (the identity element of the general linear group)

**EXAMPLES:**

Automorphism of a rank-2 free module over $\mathbb{Z}$:

```python
sage: M = FiniteRankFreeModule(ZZ, 2, name='M', start_index=1)
sage: a = M.automorphism(name='a', latex_name=r'\alpha') ; a
Automorphism a of the Rank-2 free module M over the Integer Ring
sage: a.parent()
General linear group of the Rank-2 free module M over the Integer Ring
sage: a.parent() is M.general_linear_group()
True
sage: latex(a) \alpha
```

Setting the components of $a$ w.r.t. a basis of module $M$:
Tensors on free modules of finite rank, Release 9.6

```
sage: e = M.basis('e') ; e
Basis (e_1,e_2) on the Rank-2 free module M over the Integer Ring
sage: a[: ] = [[[1,2],[1,3]]
sage: a.matrix(e)
[1 2]  
[1 3]  
sage: a(e[1]).display()
a(e_1) = e_1 + e_2
sage: a(e[2]).display()
a(e_2) = 2 e_1 + 3 e_2
```

Actually, the components w.r.t. a given basis can be specified at the construction of the object:

```
sage: a = M.automorphism(matrix=[[1,2],[1,3]], basis=e, name='a',
        latex_name=r'$\alpha$') ; a
Automorphism a of the Rank-2 free module M over the Integer Ring
sage: a.matrix(e)
[1 2]  
[1 3]  
```

Since \(e\) is the module's default basis, it can be omitted in the argument list:

```
sage: a == M.automorphism(matrix=[[1,2],[1,3]], name='a',
        latex_name=r'$\alpha$')
True
```

The matrix of the automorphism can be obtained in any basis:

```
sage: f = M.basis('f', from_family=(3*e[1]+4*e[2], 5*e[1]+7*e[2])) ; f
Basis (f_1,f_2) on the Rank-2 free module M over the Integer Ring
sage: a.matrix(f)
[2 3]  
[1 2]  
```

Automorphisms are tensors of type \((1,1)\):

```
sage: a.tensor_type()
(1, 1)
sage: a.tensor_rank()
2
```

In particular, they can be displayed as such:

```
sage: a.display(e)
a = e_1\otimes e^1 + 2 e_1\otimes e^2 + e_2\otimes e^1 + 3 e_2\otimes e^2
sage: a.display(f)
a = 2 f_1\otimes f^1 + 3 f_1\otimes f^2 + f_2\otimes f^1 + 2 f_2\otimes f^2
```

The automorphism acting on a module element:

```
sage: v = M([-2,3], name='v') ; v
Element v of the Rank-2 free module M over the Integer Ring
sage: a(v)
Element a(v) of the Rank-2 free module M over the Integer Ring
```
A second automorphism of the module $M$:

```
sage: b = M.automorphism([[0,1],[-1,0]], name='b') ; b
Automorphism b of the Rank-2 free module M over the Integer Ring
sage: b.matrix(e)
[ 0 1]
[-1 0]
sage: b(e[1]).display()
b(e_1) = -e_2
sage: b(e[2]).display()
b(e_2) = e_1
```

The composition of automorphisms is performed via the multiplication operator:

```
sage: s = a*b ; s
Automorphism of the Rank-2 free module M over the Integer Ring
sage: s(v) == a(b(v))
True
sage: s.matrix(f)
[ 11 19]
[ -7 -12]
sage: s.matrix(f) == a.matrix(f) * b.matrix(f)
True
```

It is not commutative:

```
sage: a*b != b*a
True
```

In other words, the parent of $a$ and $b$, i.e. the group $GL(M)$, is not abelian:

```
sage: M.general_linear_group() in CommutativeAdditiveGroups()
False
```

The neutral element for the composition law is the module identity map:

```
sage: id = M.identity_map() ; id
Identity map of the Rank-2 free module M over the Integer Ring
sage: id.parent()
General linear group of the Rank-2 free module M over the Integer Ring
sage: id(v) == v
True
sage: id.matrix(f)
[1 0]
[0 1]
sage: id*a == a
True
sage: a*id == a
True
```

The inverse of an automorphism is obtained via the method `inverse()`, or the operator ~, or the exponent -1:
Tensors on free modules of finite rank, Release 9.6

```python
sage: a.inverse()
Automorphism a^(-1) of the Rank-2 free module M over the Integer Ring
sage: a.inverse() is -a
True
sage: a.inverse() is a^(-1)
True
sage: (a^(-1)).matrix(e)
[ 3 -2]
[-1 1]
sage: a*a^(-1) == id
True
sage: a^(-1)*a == id
True
sage: a^(-1)*s == b
True
sage: (a^(-1))(a(v)) == v
True

The module’s changes of basis are stored as automorphisms:

```python
sage: M.change_of_basis(e,f)
Automorphism of the Rank-2 free module M over the Integer Ring
sage: M.change_of_basis(e,f).parent()
General linear group of the Rank-2 free module M over the Integer Ring
sage: M.change_of_basis(e,f).matrix(e)
[3 5]
[4 7]
sage: M.change_of_basis(f,e) == M.change_of_basis(e,f).inverse()
True

The opposite of an automorphism is still an automorphism:

```python
sage: -a
Automorphism -a of the Rank-2 free module M over the Integer Ring
sage: (-a).parent()
General linear group of the Rank-2 free module M over the Integer Ring
sage: (-a).matrix(e) == - (a.matrix(e))
True

Adding two automorphisms results in a generic type-(1,1) tensor:

```python
sage: s = a + b ; s
Type-(1,1) tensor a+b on the Rank-2 free module M over the Integer Ring
sage: s.parent()
Free module of type-(1,1) tensors on the Rank-2 free module M over the Integer Ring
sage: a[:], b[:], s[:]
([1 2], [0 1], [1 3]
[1 3], [-1 0], [0 3])

To get the result as an endomorphism, one has to explicitly convert it via the parent of endomorphisms, End(M):
```
Tensors on free modules of finite rank, Release 9.6

```
sage: s = End(M)(a+b) ; s
generic endomorphism of Rank-2 free module M over the Integer Ring
sage: s(v) == a(v) + b(v)
True
sage: s.matrix(e) == a.matrix(e) + b.matrix(e)
True
sage: s.matrix(f) == a.matrix(f) + b.matrix(f)
True
```

```
add_comp(basis=None)

Return the components of self w.r.t. a given module basis for assignment, keeping the components w.r.t. other bases.

To delete the components w.r.t. other bases, use the method `set_comp()` instead.

INPUT:

• basis – (default: None) basis in which the components are defined; if none is provided, the components are assumed to refer to the module’s default basis

**Warning:** If the automorphism has already components in other bases, it is the user’s responsibility to make sure that the components to be added are consistent with them.

OUTPUT:

• components in the given basis, as an instance of the class `Components`; if such components did not exist previously, they are created

**EXAMPLES:**

Adding components to an automorphism of a rank-3 free Z-module:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: a = M.automorphism(name='a')
sage: a[e,:] = [[[1,0,0],[0,-1,2],[0,1,-3]]
sage: f = M.basis('f', from_family=(-e[0], 3*e[1]+4*e[2],
....: 5*e[1]+7*e[2])); f
basis (f_0,f_1,f_2) on the Rank-3 free module M over the Integer Ring
sage: a.add_comp(f)[:,:] = [[[1,0,0], [0, 80, 143], [0, -47, -84]]
```

The components in basis e have been kept:

```
sage: a._components
{Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring:
2-indices components w.r.t. Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring,
Basis (f_0,f_1,f_2) on the Rank-3 free module M over the Integer Ring:
2-indices components w.r.t. Basis (f_0,f_1,f_2) on the Rank-3 free module M over the Integer Ring}  
```

For the identity map, it is not permitted to invoke `add_comp()`:

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Indeed, the components are set automatically:

\begin{Verbatim}
\texttt{sage: id.comp(e)} \\
\texttt{Kronecker delta of size 3x3} \\
\texttt{sage: id.comp(f)} \\
\texttt{Kronecker delta of size 3x3}
\end{Verbatim}

\textbf{det()}

Return the determinant of \texttt{self}.

\textbf{OUTPUT:}

\begin{itemize}
  \item element of the base ring of the module on which \texttt{self} is defined, equal to the determinant of \texttt{self}.
\end{itemize}

\textbf{EXAMPLES:}

Determinant of an automorphism on a \(\mathbb{Z}\)-module of rank 2:

\begin{Verbatim}
\texttt{sage: M = FiniteRankFreeModule(ZZ, 2, name='M')} \\
\texttt{sage: e = M.basis('e')} \\
\texttt{sage: a = M.automorphism([[4,7],[3,5]], name='a')} \\
\texttt{sage: a.matrix(e)} \\
\begin{bmatrix} 4 & 7 \\ 3 & 5 \end{bmatrix} \\
\texttt{sage: a.det()} \\
-1 \\
\texttt{sage: det(a)} \\
-1 \\
\texttt{sage: -a.det()} \texttt{# determinant of the inverse automorphism} \\
-1 \\
\texttt{sage: id = M.identity_map()} \\
\texttt{sage: id.det()} \\
1
\end{Verbatim}

\textbf{inverse()}

Return the inverse automorphism.

\textbf{OUTPUT:}

\begin{itemize}
  \item instance of \texttt{FreeModuleAutomorphism} representing the automorphism that is the inverse of \texttt{self}.
\end{itemize}

\textbf{EXAMPLES:}

Inverse of an automorphism of a rank-3 free module:

\begin{Verbatim}
\texttt{sage: M = FiniteRankFreeModule(ZZ, 3, name='M')} \\
\texttt{sage: e = M.basis('e')} \\
\texttt{sage: a = M.automorphism(name='a')} \\
\texttt{sage: a[e,:] = [[1,0,0],[0,-1,2],[0,1,-3]]} \\
\texttt{sage: a.inverse()} \\
\texttt{Automorphism a^(-1) of the Rank-3 free module M over the Integer}
\end{Verbatim}

(continues on next page)
Ring
```
sage: a.inverse().parent()
General linear group of the Rank-3 free module M over the Integer Ring
```

Check that `a.inverse()` is indeed the inverse automorphism:
```
sage: a.inverse() * a
Identity map of the Rank-3 free module M over the Integer Ring
sage: a * a.inverse()
Identity map of the Rank-3 free module M over the Integer Ring
sage: a.inverse().inverse() == a
True
```

Another check is:
```
sage: a.inverse().matrix(e)

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & -3 & -2 \\
0 & -1 & -1 \\
\end{bmatrix}
\]
sage: a.inverse().matrix(e) == (a.matrix(e))^(-1)
True
```

The inverse is cached (as long as `a` is not modified):
```
sage: a.inverse() is a.inverse()
True
```

If `a` is modified, the inverse is automatically recomputed:
```
sage: a[0,0] = -1
sage: a.matrix(e)

\[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 2 \\
0 & 1 & -3 \\
\end{bmatrix}
\]
sage: a.inverse().matrix(e)  # compare with above

\[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & -3 & -2 \\
0 & -1 & -1 \\
\end{bmatrix}
\]
```

Shortcuts for `inverse()` are the operator ~ and the exponent -1:
```
sage: ~a is a.inverse()
True
sage: (a)^(-1) is a.inverse()
True
```

The inverse of the identity map is of course itself:
```
sage: id = M.identity_map()
sage: id.inverse() is id
True
```

and we have:

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If the name could cause some confusion, a bracket is added around the element before taking the inverse:

```
sage: c = M.automorphism(name='a^(-1)*b')
sage: c[e,: ] = [[1,0,0],[0,-1,1],[0,2,-1]]
sage: c.inverse()
Automorphism (a^(-1)*b)^(-1) of the Rank-3 free module M over the Integer Ring
```

```
matrix(basis1=None, basis2=None)
Return the matrix of self w.r.t a pair of bases.
If the matrix is not known already, it is computed from the matrix in another pair of bases by means of the change-of-basis formula.

INPUT:
• basis1 – (default: None) basis of the free module on which self is defined; if none is provided, the module's default basis is assumed
• basis2 – (default: None) basis of the free module on which self is defined; if none is provided, basis2 is set to basis1

OUTPUT:
• the matrix representing the automorphism self w.r.t to bases basis1 and basis2; more precisely, the columns of this matrix are formed by the components w.r.t. basis2 of the images of the elements of basis1.

EXAMPLES:
Matrices of an automorphism of a rank-3 free \( \mathbb{Z} \)-module:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1)
sage: e = M.basis('e')
sage: a = M.automorphism([[[-1,0,0],[0,1,2],[0,1,3]], name='a')
sage: a.matrix(e)
[-1 0 0]
[ 0 1 2]
[ 0 1 3]
sage: a.matrix()
[-1 0 0]
[ 0 1 2]
[ 0 1 3]
sage: f = M.basis('f', from_family=(-e[2], 4*e[1]+3*e[3], 7*e[1]+5*e[3])); f
Basis (f_1,f_2,f_3) on the Rank-3 free module M over the Integer Ring
sage: a.matrix(f)
[ 1 -6 -10]
[ -7 83 140]
[ 4 -48 -81]
```

Check of the above matrix:
```
sage: a(f[1]).display(f)
a(f_1) = f_1 - 7 f_2 + 4 f_3
sage: a(f[2]).display(f)
a(f_2) = -6 f_1 + 83 f_2 - 48 f_3
sage: a(f[3]).display(f)
a(f_3) = -10 f_1 + 140 f_2 - 81 f_3
```

Check of the change-of-basis formula:
```
sage: P = M.change_of_basis(e,f).matrix(e)
sage: a.matrix(f) == P^(-1) * a.matrix(e) * P
True
```

Check that the matrix of the product of two automorphisms is the product of their matrices:
```
sage: b = M.change_of_basis(e,f) ; b
Automorphism of the Rank-3 free module M over the Integer Ring
sage: b.matrix(e)
[ 0  4  7]
[-1  0  0]
[ 0  3  5]
sage: (a*b).matrix(e) == a.matrix(e) * b.matrix(e)
True
```

Check that the matrix of the inverse automorphism is the inverse of the automorphism’s matrix:
```
sage: (~a).matrix(e)
[-1  0  0]
[ 0  3 -2]
[ 0 -1  1]
sage: (~a).matrix(e) == ~(a.matrix(e))
True
```

Matrices of the identity map:
```
sage: id = M.identity_map()
sage: id.matrix(e)
[1 0 0]
[0 1 0]
[0 0 1]
sage: id.matrix(f)
[1 0 0]
[0 1 0]
[0 0 1]
```

```
set_comp(basis=None)

Return the components of self w.r.t. a given module basis for assignment.

The components with respect to other bases are deleted, in order to avoid any inconsistency. To keep them, use the method add_comp() instead.

INPUT:

- basis – (default: None) basis in which the components are defined; if none is provided, the components are assumed to refer to the module’s default basis

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```
OUTPUT:

• components in the given basis, as an instance of the class \texttt{Components}; if such components did not exist previously, they are created.

EXAMPLES:

Setting the components of an automorphism of a rank-3 free \( \mathbb{Z} \)-module:

\begin{verbatim}
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: a = M.automorphism(name='a')
sage: a.set_comp(e)
2-indices components w.r.t. Basis (e_0,e_1,e_2) on the Rank-3 free
module M over the Integer Ring
sage: a.set_comp(e)[:] = [[[1,0,0],[0,1,2],[0,1,3]]
sage: a.set_comp(e)
1 0 0
0 1 2
0 1 3

Since \( e \) is the module's default basis, one has:

\begin{verbatim}
sage: a.set_comp() is a.set_comp(e)
True
\end{verbatim}

The method \texttt{set_comp()} can be used to modify a single component:

\begin{verbatim}
sage: a.set_comp(e)[0,0] = -1
sage: a.matrix(e)
[-1 0 0]
[ 0 1 2]
[ 0 1 3]
\end{verbatim}

A short cut to \texttt{set_comp()} is the bracket operator, with the basis as first argument:

\begin{verbatim}
sage: a[e,:] = [[[1,0,0],[0,-1,2],[0,1,-3]]
sage: a.matrix(e)
1 0 0
0 -1 2
0 1 -3
sage: a[e,0,0] = -1
sage: a.matrix(e)
\end{verbatim}

The call to \texttt{set_comp()} erases the components previously defined in other bases; to keep them, use the method \texttt{add_comp()} instead:

\begin{verbatim}
sage: f = M.basis('f', from_family=(-e[0], 3*e[1]+4*e[2],
....: 5*e[1]+7*e[2])); f
Basis (f_0,f_1,f_2) on the Rank-3 free module M over the Integer
Ring
sage: a._components
{Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer
(continues on next page)}
Ring: 2-indices components w.r.t. Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring

\[
\text{sage: } a\text{.set_comp}(f)[:] = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1 
\end{bmatrix}
\]

The components w.r.t. basis e have been erased:

\[
\text{sage: } a\text{.comp}(e)
\]

Of course, they can be computed from those in basis f by means of a change-of-basis formula, via the method comp() or matrix():

\[
\text{sage: } a\text{.matrix}(e)
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 41 & -30 \\
0 & 56 & -41 
\end{bmatrix}
\]

For the identity map, it is not permitted to set components:

\[
\text{sage: } id = M\text{.identity_map()}
\text{sage: } id\text{.set_comp}(e)
\]

"..."

ValueError: the components of the identity map cannot be changed

Indeed, the components are set automatically:

\[
\text{sage: } id\text{.comp}(e)
\]

Kronecker delta of size 3x3

\[
\text{sage: } id\text{.comp}(f)
\]

Kronecker delta of size 3x3

\text{trace()}

Return the trace of \textit{self}.

\text{OUTPUT:}

\begin{itemize}
  \item element of the base ring of the module on which \textit{self} is defined, equal to the trace of \textit{self}.
\end{itemize}

\text{EXAMPLES:}

Trace of an automorphism on a \textit{Z}-module of rank 2:

\[
\text{sage: } M = \text{FiniteRankFreeModule(ZZ, 2, name='M')}
\text{sage: } e = M\text{.basis('e')}
\text{sage: } a = M\text{.automorphism([[4,7],[3,5]], name='a')}
\text{sage: } a\text{.matrix}(e)
\begin{bmatrix}
4 & 7 \\
3 & 5 
\end{bmatrix}
\text{sage: } a\text{.trace()}
9
\text{sage: } id = M\text{.identity_map()}
\]

(continues on next page)
sage: id.trace()
2
COMPONENTS AS INDEXED SETS OF RING ELEMENTS

The class `Components` is a technical class to take in charge the storage and manipulation of **indexed elements of a commutative ring** that represent the components of some “mathematical entity” with respect to some “frame”. Examples of entity/frame are vector/vector-space basis or vector field/vector frame on some manifold. More generally, the components can be those of a tensor on a free module or those of a tensor field on a manifold. They can also be non-tensorial quantities, like connection coefficients or structure coefficients of a vector frame.

The individual components are assumed to belong to a given commutative ring and are labelled by **indices**, which are **tuples of integers**. The following operations are implemented on components with respect to a given frame:

- arithmetics (addition, subtraction, multiplication by a ring element)
- handling of symmetries or antisymmetries on the indices
- symmetrization and antisymmetrization
- tensor product
- contraction

Various subclasses of class `Components` are

- `CompWithSym` for components with symmetries or antisymmetries w.r.t. index permutations
  - `CompFullySym` for fully symmetric components w.r.t. index permutations
  - `KroneckerDelta` for the Kronecker delta symbol
  - `CompFullyAntiSym` for fully antisymmetric components w.r.t. index permutations

**AUTHORS:**
- Joris Vankerschaver (2010): for the idea of storing only the non-zero components as dictionaries, whose keys are the component indices (implemented in the old class `DifferentialForm`; see trac ticket #24444)
- Marco Mancini (2015) : parallelization of some computations

**EXAMPLES:**

Set of components with 2 indices on a 3-dimensional vector space, the frame being some basis of the vector space:

```python
sage: from sage.tensor.modules.comp import Components
sage: V = VectorSpace(QQ,3)
sage: basis = V.basis() ; basis
[(1, 0, 0),
 (0, 1, 0),
 (0, 0, 1)]
```
Tensors on free modules of finite rank, Release 9.6

Actually, the frame can be any object that has some length, i.e. on which the function \texttt{len()} can be called:

\begin{verbatim}
\texttt{sage: basis1 = V.gens(); basis1}
\texttt{((1, 0, 0), (0, 1, 0), (0, 0, 1))}
\texttt{sage: c1 = Components(QQ, basis1, 2); c1}
\texttt{2-indices components w.r.t. ((1, 0, 0), (0, 1, 0), (0, 0, 1))}
\texttt{sage: basis2 = ['a', 'b', 'c']}
\texttt{sage: c2 = Components(QQ, basis2, 2); c2}
\texttt{2-indices components w.r.t. ['a', 'b', 'c']}
\end{verbatim}

A just created set of components is initialized to zero:

\begin{verbatim}
\texttt{sage: c.is_zero()}
\texttt{True}
\texttt{sage: c == 0}
\texttt{True}
\end{verbatim}

This can also be checked on the list of components, which is returned by the operator \texttt{[::]}:

\begin{verbatim}
\texttt{sage: c[:]
[0 0 0]
[0 0 0]
[0 0 0]}
\end{verbatim}

Individual components are accessed by providing their indices inside square brackets:

\begin{verbatim}
\texttt{sage: c[1,2] = -3}
\texttt{sage: c[:]
[ 0 0 0]
[ 0 0 -3]
[ 0 0 0]}
\texttt{sage: v = Components(QQ, basis, 1)}
\texttt{sage: v[:]
[0, 0, 0]}
\texttt{sage: v[0]}
\texttt{0}
\texttt{sage: v[:]=(-1,3,2)}
\texttt{sage: v[:]
[-1, 3, 2]}
\texttt{sage: v[0]}
\texttt{-1}
\end{verbatim}

Sets of components with 2 indices can be converted into a matrix:
sage: matrix(c)
[ 0  0  0]
[ 0  0 -3]
[ 0  0  0]
sage: matrix(c).parent()
Full MatrixSpace of 3 by 3 dense matrices over Rational Field

By default, the indices range from 0 to $n - 1$, where $n$ is the length of the frame. This can be changed via the argument start_index in the Components constructor:

sage: v1 = Components(QQ, basis, 1, start_index=1)
sage: v1[:]
[0, 0, 0]
sage: v1[0]
Traceback (most recent call last):
  ...  IndexError: index out of range: 0 not in [1, 3]
sage: v1[1]
0
sage: v1[:] = v[:]  # list copy of all components
sage: v1[:]
[-1, 3, 2]
sage: v1[1], v1[2], v1[3]
(-1, 3, 2)
sage: v[0], v[1], v[2]
(-1, 3, 2)

If some formatter function or unbound method is provided via the argument output_formatter in the Components constructor, it is used to change the output of the access operator [...]:

sage: a = Components(QQ, basis, 2, output_formatter=Rational.numerical_approx)
sage: a[1,2] = 1/3
sage: a[1,2]
0.33333333333333333333333333333
sage: a[1,2,10]  # here the format is 10, for 10 bits of precision
0.33
sage: a[1,2,100]
0.33333333333333333333333333333333

The format can be passed to the formatter as the last argument of the access operator [...]:

sage: a[1,2,10]  # here the format is 10, for 10 bits of precision
0.33
sage: a[1,2,100]
0.33333333333333333333333333333333

The raw (unformatted) components are then accessed by the double bracket operator:

sage: a[[1,2]]
1/3

For sets of components declared without any output formatter, there is no difference between [...] and [[...]]:

sage: c[1,2] = 1/3
sage: c[1,2], c[[1,2]]
(1/3, 1/3)

The formatter is also used for the complete list of components:
The complete list of components in raw form can be recovered by the double bracket operator, replacing \( a[[:]] \) by \( \text{slice}(\text{None}) \) (since \( a[[:]] \) generates a Python syntax error):

\[
\begin{align*}
sage: a[\text{slice}(\text{None})] \\
[ 0 \ 0 \ 0 ] \\
[ 0 \ 0 \ 1/3 ] \\
[ 0 \ 0 \ 0 ]
\end{align*}
\]

Another example of formatter: the Python built-in function \( \text{str()} \) to generate string outputs:

\[
\begin{align*}
sage: b = \text{Components}(\text{QQ}, \text{V.basis()}, 1, \text{output_formatter}=\text{str}) \\
sage: b[:]= (1, 0, -4) \\
sage: b[:]
[ '1', '0', '-4']
\end{align*}
\]

For such a formatter, 2-indices components are no longer displayed as a matrix:

\[
\begin{align*}
sage: b = \text{Components}(\text{QQ}, \text{basis}, 2, \text{output_formatter}=\text{str}) \\
sage: b[0,1]= 1/3 \\
sage: b[:]
[['0', '1/3', '0'], ['0', '0', '0'], ['0', '0', '0']]
\end{align*}
\]

But unformatted outputs still are:

\[
\begin{align*}
sage: b[\text{slice}(\text{None})] \\
[ 0 \ 1/3 \ 0 ] \\
[ 0 \ 0 \ 0 ] \\
[ 0 \ 0 \ 0 ]
\end{align*}
\]

Internally, the components are stored as a dictionary \( _\text{comp} \) whose keys are the indices; only the non-zero components are stored:

\[
\begin{align*}
sage: a[\cdot] \\
[0.000000000000000 0.000000000000000 0.000000000000000] \\
[0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000] \\
[0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000 0.333333333333333] \\
[0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000]
\end{align*}
\]

\[
\begin{align*}
sage: a._\text{comp} \\
\{(1, 2): 1/3\} \\
sage: v[:]= (-1, 0, 3) \\
sage: v._\text{comp} \\
\{(0,): -1, (2,): 3\}
\end{align*}
\]

In case of symmetries, only non-redundant components are stored:

\[
\begin{align*}
sage: a[\cdot] \\
[0.000000000000000 0.000000000000000 0.000000000000000] \\
[0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000 0.333333333333333] \\
[0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000]
\end{align*}
\]

\[
\begin{align*}
sage: a._\text{comp} \\
\{(1, 2): 1/3\} \\
sage: v[:]= (-1, 0, 3) \\
sage: v._\text{comp} \\
\{(0,): -1, (2,): 3\}
\end{align*}
\]
Tensors on free modules of finite rank, Release 9.6

```python
sage: from sage.tensor.modules.comp import CompFullyAntiSym
sage: c = CompFullyAntiSym(QQ, basis, 2)
sage: c[0,1] = 3
sage: c[:]
[ 0  3  0]
[-3  0  0]
[ 0  0  0]
```

```python
c._comp
{(0, 1): 3}
```

```python
class sage.tensor.modules.comp.CompFullyAntiSym(ring, frame, nb_indices, start_index=0, output_formatter=None):
    Bases: sage.tensor.modules.comp.CompWithSym

    Indexed set of ring elements forming some components with respect to a given “frame” that are fully antisymmetric with respect to any permutation of the indices.

    The “frame” can be a basis of some vector space or a vector frame on some manifold (i.e. a field of bases). The stored quantities can be tensor components or non-tensorial quantities.

    INPUT:

    • ring – commutative ring in which each component takes its value
    • frame – frame with respect to which the components are defined; whatever type frame is, it should have some method __len__() implemented, so that len(frame) returns the dimension, i.e. the size of a single index range
    • nb_indices – number of indices labeling the components
    • start_index – (default: 0) first value of a single index; accordingly a component index i must obey start_index <= i <= start_index + dim - 1, where dim = len(frame).
    • output_formatter – (default: None) function or unbound method called to format the output of the component access operator [...] (method__getitem__); output_formatter must take 1 or 2 arguments: the 1st argument must be an instance of ring and the second one, if any, some format specification.

    EXAMPLES:

    Antisymmetric components with 2 indices on a 3-dimensional space:

    ```python
    sage: from sage.tensor.modules.comp import CompWithSym, CompFullyAntiSym
    sage: V = VectorSpace(QQ, 3)
sage: c = CompFullyAntiSym(QQ, V.basis(), 2)
sage: c[0,1], c[0,2], c[1,2] = 3, 1/2, -1
sage: c[:]
# note that all components have been set according to the antisymmetry
[ 0  3 1/2]
[-3  0 -1]
[-1/2 1  0]
```

Internally, only non-redundant and non-zero components are stored:

```python
c._comp # random output order of the component dictionary
{(0, 1): 3, (0, 2): 1/2, (1, 2): -1}
```

Same thing, but with the starting index set to 1:
The values stored in \( c \) and \( c1 \) are equal:

\[
sage: c1[:] == c[:]
True
\]

but not \( c \) and \( c1 \), since their starting indices differ:

\[
sage: c1 == c
False
\]

Fully antisymmetric components with 3 indices on a 3-dimensional space:

\[
sage: a = CompFullyAntiSym(QQ, V.basis(), 3)
sage: a[0,1,2] = 3  # the only independent component in dimension 3
sage: a[:]
\[
\begin{bmatrix}
0 & 0 & 3 \\
0 & -3 & 0 \\
-3 & 0 & 0
\end{bmatrix}
\]

Setting a nonzero value incompatible with the antisymmetry results in an error:

\[
sage: a[0,1,0] = 4
Traceback (most recent call last):
...
ValueError: by antisymmetry, the component cannot have a nonzero value for the indices (0, 1, 0)
sage: a[0,1,0] = 0  # OK
sage: a[2,0,1] = 3  # OK
\]

The full antisymmetry is preserved by the arithmetics:

\[
sage: b = CompFullyAntiSym(QQ, V.basis(), 3)
sage: b[0,1,2] = -4
sage: s = a + 2*b ; s
Fully antisymmetric 3-indices components w.r.t. [ (0, 0, 0), (0, 1, 0), (0, 0, 1) ]
sage: a[:], b[:], s[:]
\[
\begin{bmatrix}
0 & 0 & 3 \\
0 & -3 & 0 \\
-3 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & -4 \\
0 & 4 & 0 \\
4 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
\]

(continues on next page)


It is lost if the added object is not fully antisymmetric:

```python
sage: b1 = CompWithSym(QQ, V.basis(), 3, antisym=(0,1))  # b1 has only antisymmetry on index positions (0,1)
sage: b1[0,1,2] = -4
sage: s = a + 2*b1 ; s  # the result has the same symmetry as b1:
3-indices components w.r.t. [ (1, 0, 0), (0, 1, 0), (0, 0, 1) ], with antisymmetry on the index positions (0, 1)
sage: a[:], b1[:], s[:]
([[[0, 0, 0], [0, 0, 3], [0, -3, 0]],
  [[0, 0, -3], [0, 0, 0], [3, 0, 0]],
  [[0, 3, 0], [-3, 0, 0], [0, 0, 0]]],
 [[0, 0, 0], [0, 0, 4], [0, 0, 0]],
 [[0, 0, 0], [0, 0, 0], [0, 0, 0]]),
 [[[0, 0, 0], [0, 0, -4], [0, 0, 0]],
  [[0, 0, 4], [0, 0, 0], [0, 0, 0]],
  [[0, 0, 0], [0, 0, 0], [0, 0, 0]]],
 [[[0, 0, 0], [0, 0, -3], [0, -3, 0]],
  [[0, 0, 5], [0, 0, 0], [3, 0, 0]],
  [[0, 3, 0], [-3, 0, 0], [0, 0, 0]]])
```

```python
sage: s = 2*b1 + a ; s  # the result has the same symmetry as b1:
3-indices components w.r.t. [ (1, 0, 0), (0, 1, 0), (0, 0, 1) ], with antisymmetry on the index positions (0, 1)
sage: 2*b1 + a == a + 2*b1
True
```

**interior_product**(other)

Interior product with another set of fully antisymmetric components.

The interior product amounts to a contraction over all the \( p \) indices of \self with the first \( p \) indices of \other, assuming that the number \( q \) of indices of \other obeys \( q \geq p \).

**Note:** \self.interior_product(other) yields the same result as \self.contract(0,..., p-1, other, 0,..., p-1) (cf. \contract()), but \interior_product is more efficient, the antisymmetry of \self being not used to reduce the computation in \contract().

**INPUT:**

- \other – fully antisymmetric components defined on the same frame as \self and with a number of indices at least equal to that of \self

**OUTPUT:**

- base ring element (case \( p = q \)) or set of components (case \( p < q \)) resulting from the contraction over all the \( p \) indices of \self with the first \( p \) indices of \other

**EXAMPLES:**

Interior product of a set of components $a$ with $p$ indices with a set of components $b$ with $q$ indices on a 4-dimensional vector space.

Case $p=2$ and $q=2$:

```python
sage: from sage.tensor.modules.comp import CompFullyAntiSym
sage: V = VectorSpace(QQ, 4)
sage: a = CompFullyAntiSym(QQ, V.basis(), 2)
sage: a[0,1], a[0,2], a[0,3] = -2, 4, 3
sage: a[1,2], a[1,3], a[2,3] = 5, -3, 1
sage: b = CompFullyAntiSym(QQ, V.basis(), 2)
sage: b[0,1], b[0,2], b[0,3] = 3, -4, 2
sage: b[1,2], b[1,3], b[2,3] = 2, 5, 1
sage: c = a.interior_product(b)
sage: c
-40
sage: c == a.contract(0, 1, b, 0, 1)
True
```

Case $p=2$ and $q=3$:

```python
sage: b = CompFullyAntiSym(QQ, V.basis(), 3)
sage: b[0,1,2], b[0,1,3], b[0,2,3], b[1,2,3] = 3, -4, 2, 5
sage: c = a.interior_product(b)
sage: c[:]
[58, 10, 6, 82]
sage: c == a.contract(0, 1, b, 0, 1)
True
```

Case $p=2$ and $q=4$:

```python
sage: b = CompFullyAntiSym(QQ, V.basis(), 4)
sage: b[0,1,2,3] = 5
sage: c = a.interior_product(b)
sage: c[:]
[[ 0  10  30  50]
 [ -10  0  30 -40]
 [ -30 -30  0 -20]
 [ -50  40  20  0]]
sage: c == a.contract(0, 1, b, 0, 1)
True
```

Case $p=3$ and $q=3$:

```python
sage: a = CompFullyAntiSym(QQ, V.basis(), 3)
sage: a[0,1,2], a[0,1,3], a[0,2,3], a[1,2,3] = 2, -1, 3, 5
sage: b = CompFullyAntiSym(QQ, V.basis(), 3)
sage: b[0,1,2], b[0,1,3], b[0,2,3], b[1,2,3] = -2, 1, 4, 2
sage: c = a.interior_product(b)
sage: c
102
sage: c == a.contract(0, 1, 2, b, 0, 1, 2)
True
```

Case $p=3$ and $q=4$:
Tensors on free modules of finite rank, Release 9.6

```python
sage: b = CompFullyAntiSym(QQ, V.basis(), 4)
sage: b[[0,1,2,3]] = 5
sage: c = a.interior_product(b)
sage: c[:]
[-150, 90, 30, 60]
sage: c == a.contract(0, 1, 2, b, 0, 1, 2)
True
```

Case p=4 and q=4:

```python
sage: a = CompFullyAntiSym(QQ, V.basis(), 4)
sage: a[[0,1,2,3]] = 3
sage: c = a.interior_product(b)
sage: c
360
sage: c == a.contract(0, 1, 2, 3, b, 0, 1, 2, 3)
True
```

class sage.tensor.modules.comp.CompFullySym(ring, frame, nb_indices, start_index=0, output_formatter=None)

Bases: sage.tensor.modules.comp.CompWithSym

Indexed set of ring elements forming some components with respect to a given “frame” that are fully symmetric with respect to any permutation of the indices.

The “frame” can be a basis of some vector space or a vector frame on some manifold (i.e. a field of bases). The stored quantities can be tensor components or non-tensorial quantities.

INPUT:

- `ring` – commutative ring in which each component takes its value
- `frame` – frame with respect to which the components are defined; whatever type `frame` is, it should have some method `__len__()` implemented, so that `len(frame)` returns the dimension, i.e. the size of a single index range
- `nb_indices` – number of indices labeling the components
- `start_index` – (default: 0) first value of a single index; accordingly a component index `i` must obey `start_index <= i <= start_index + dim - 1`, where `dim = len(frame)`. 
- `output_formatter` – (default: None) function or unbound method called to format the output of the component access operator `[...]` (method `__getitem__`); `output_formatter` must take 1 or 2 arguments: the 1st argument must be an instance of `ring` and the second one, if any, some format specification.

EXAMPLES:

Symmetric components with 2 indices on a 3-dimensional space:

```python
sage: from sage.tensor.modules.comp import CompFullySym, CompWithSym
sage: V = VectorSpace(QQ, 3)
sage: c = CompFullySym(QQ, V.basis(), 2)
sage: c[[0,0], [0,1], [1,2]] = 1, -2, 3
sage: c[:]
# note that c[1,0] and c[2,1] have been updated automatically (by symmetry)
[ 1 -2  0]
[-2  0  3]
[ 0  3  0]
```
Internally, only non-redundant and non-zero components are stored:

```
sage: c._comp  # random output order of the component dictionary
{(0, 0): 1, (0, 1): -2, (1, 2): 3}
```

Same thing, but with the starting index set to 1:

```
sage: c1 = CompFullySym(QQ, V.basis(), 2, start_index=1)
sage: c1[1,1], c1[1,2], c1[2,3] = 1, -2, 3
sage: c1[:]
[ 1 -2 0]
[-2 0 3]
[ 0 3 0]
```

The values stored in `c` and `c1` are equal:

```
sage: c1[:] == c[:]
True
```

but not `c` and `c1`, since their starting indices differ:

```
sage: c1 == c
False
```

Fully symmetric components with 3 indices on a 3-dimensional space:

```
sage: a = CompFullySym(QQ, V.basis(), 3)
sage: a[0,1,2] = 3
sage: a[:]
[[[0, 0, 0], [0, 0, 3], [0, 3, 0]],
 [[0, 0, 3], [0, 0, 0], [3, 0, 0]],
 [[0, 3, 0], [3, 0, 0], [0, 0, 0]]]
sage: a[0,1,0] = 4
sage: a[:]
[[[0, 4, 0], [4, 0, 3], [0, 3, 0]],
 [[4, 0, 3], [0, 0, 0], [3, 0, 0]],
 [[0, 3, 0], [3, 0, 0], [0, 0, 0]]]
```

The full symmetry is preserved by the arithmetics:

```
sage: b = CompFullySym(QQ, V.basis(), 3)
sage: b[0,0,0], b[0,1,0], b[1,0,2], b[1,2,2] = -2, 3, 1, -5
sage: s = a + 2*b ; s
Fully symmetric 3-indices components w.r.t. [ (1, 0, 0), (0, 1, 0), (0, 0, 1) ]
sage: a[:], b[:], s[:]
([[0, 4, 0], [4, 0, 3], [0, 3, 0]],
 [[4, 0, 3], [0, 0, 0], [3, 0, 0]],
 [[0, 3, 0], [3, 0, 0], [0, 0, 0]])
```

(continues on next page)
It is lost if the added object is not fully symmetric:

```python
sage: b1 = CompWithSym(QQ, V.basis(), 3, sym=(0,1))  # b1 has only symmetry on
        →index positions (0,1)
```

```python
sage: b1[0,0,0], b1[0,1,0], b1[1,0,2], b1[1,2,2] = -2, 3, 1, -5
```

```python
sage: s = a + 2*b1 ; s  # the result has the same symmetry as b1:
3-indices components w.r.t. [ (1, 0, 0),
(0, 1, 0),
(0, 0, 1) ], with symmetry on the index positions (0, 1)
```

```python
sage: a[:], b1[:], s[:]
([[0, 4, 0], [4, 0, 3], [0, 3, 0]],
[[0, 3, 0], [3, 0, 0], [0, 0, 0]],
[[[-2, 0, 0], [3, 0, 1], [0, 0, 0]],
[[3, 0, 1], [0, 0, 0], [0, 0, -5]],
[[0, 0, 0], [0, 0, -5], [0, 0, 0]]],
[[[-4, 4, 0], [10, 0, 5], [0, 3, 0]],
[[10, 0, 5], [0, 0, 0], [3, 0, -10]],
[[0, 3, 0], [3, 0, -10], [0, 0, 0]]])
```

```python
sage: s = 2*b1 + a ; s
3-indices components w.r.t. [ (1, 0, 0),
(0, 1, 0),
(0, 0, 1) ], with symmetry on the index positions (0, 1)
```

```python
sage: 2*b1 + a == a + 2*b1
True
```

```
```

class sage.tensor.modules.comp.CompWithSym
    Bases: sage.tensor.modules.comp.Components

Indexed set of ring elements forming some components with respect to a given “frame”, with symmetries or
antisymmetries regarding permutations of the indices.

The “frame” can be a basis of some vector space or a vector frame on some manifold (i.e. a field of bases).
The stored quantities can be tensor components or non-tensorial quantities, such as connection coefficients or
structure coefficients.

Subclasses of CompWithSym are

- CompFullySym for fully symmetric components.
- CompFullyAntiSym for fully antisymmetric components.

INPUT:

- ring – commutative ring in which each component takes its value
• frame – frame with respect to which the components are defined; whatever type frame is, it should have some method \_\_len\_\_(\_) implemented, so that len(frame) returns the dimension, i.e. the size of a single index range

• nb_indices – number of indices labeling the components

• start_index – (default: 0) first value of a single index; accordingly a component index i must obey start_index <= i <= start_index + dim - 1, where dim = len(frame).

• output_formatter – (default: None) function or unbound method called to format the output of the component access operator \[\ldots\] (method \_\_getitem\_\_); output_formatter must take 1 or 2 arguments: the 1st argument must be an instance of ring and the second one, if any, some format specification.

• sym – (default: None) a symmetry or a list of symmetries among the indices: each symmetry is described by a tuple containing the positions of the involved indices, with the convention position=0 for the first slot; for instance:
  – sym = (0, 1) for a symmetry between the 1st and 2nd indices
  – sym = [(0,2), (1,3,4)] for a symmetry between the 1st and 3rd indices and a symmetry between the 2nd, 4th and 5th indices.

• antisym – (default: None) antisymmetry or list of antisymmetries among the indices, with the same convention as for sym

EXAMPLES:

Symmetric components with 2 indices:

```
sage: from sage.tensor.modules.comp import Components, CompWithSym
sage: V = VectorSpace(QQ,3)
sage: c = CompWithSym(QQ, V.basis(), 2, sym=(0,1))  # for demonstration only: it is preferable to use CompFullySym in this case
sage: c[0,1] = 3
sage: c[:,:]  # note that c[1,0] has been set automatically
[0 3 0]
[3 0 0]
[0 0 0]
```

Antisymmetric components with 2 indices:

```
sage: c = CompWithSym(QQ, V.basis(), 2, antisym=(0,1))  # for demonstration only: it is preferable to use CompFullyAntiSym in this case
sage: c[0,1] = 3
sage: c[:,:]  # note that c[1,0] has been set automatically
[0 3 0]
[-3 0 0]
[0 0 0]
```

Internally, only non-redundant components are stored:

```
sage: c._comp
{(0, 1): 3}
```

Components with 6 indices, symmetric among 3 indices (at position (0, 1, 5)) and antisymmetric among 2 indices (at position (2, 4)):
Components with 4 indices, antisymmetric with respect to the first pair of indices as well as with the second pair of indices:

\[
\text{sage: } c = \text{CompWithSym}(\mathbb{Q}, \text{V.basis()}, 4, \text{antisym}=\{(0,1),(2,3)\})
\]

\[
\text{sage: } c[0,1,0,1] = 3
\]

\[
\text{sage: } c[0,1,1,0] \quad \# \text{antisymmetry on the first pair of indices}
-3
\]

\[
\text{sage: } c[1,0,1,0] \quad \# \text{antisymmetry on the second pair of indices}
-3
\]

\[
\text{sage: } c[1,0,1,0] \quad \# \text{consequence of the above}
3
\]

### ARITHMETIC EXAMPLES

Addition of a symmetric set of components with a non-symmetric one: the symmetry is lost:

\[
\text{sage: } V = \text{VectorSpace}(\mathbb{Q}, 3)
\]

\[
\text{sage: } a = \text{Components}(\mathbb{Q}, \text{V.basis()}, 2)
\]

\[
\text{sage: } a[:] = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \\ -7 & 8 & 9 \end{bmatrix}
\]

\[
\text{sage: } b = \text{CompWithSym}(\mathbb{Q}, \text{V.basis()}, 2, \text{sym}=(0,1)) \quad \# \text{for demonstration only: it is}\n\]

\[
\quad \rightarrow \text{preferable to declare } b = \text{CompFullySym}(\mathbb{Q}, \text{V.basis()}, 2)
\]

\[
\text{sage: } b[0,0], b[0,1], b[0,2] = 1, 2, 3
\]

\[
\text{sage: } b[1,1], b[1,2] = 5, 7
\]

\[
\text{sage: } b[2,2] = 11
\]

\[
\text{sage: } s = a + b \ ; \ s
\]

\[
\begin{align*}
\text{2-indices components w.r.t. } & [1, 0, 0], \\
& [0, 1, 0], \\
& [0, 0, 1]
\end{align*}
\]

\[
\text{sage: } a[:], b[:], s[:]
\]

\[
\begin{bmatrix}
1 & -2 & 3 \\
4 & 5 & -6 \\
-7 & 8 & 9
\end{bmatrix},
\begin{bmatrix}
1 & 2 & 3 \\
2 & 0 & 6 \\
6 & 10 & 1
\end{bmatrix},
\begin{bmatrix}
2 & 0 & 6 \\
6 & 10 & 1 \\
-4 & 15 & 20
\end{bmatrix}
\]

\[
\text{sage: } a + b == b + a
\]

True

Addition of two symmetric set of components: the symmetry is preserved:
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\[
\text{\texttt{sage: c = CompWithSym(QQ, V.basis(), 2, sym=(0,1)) \ # for demonstration only: it is prefer}\
\text{\texttt{able to declare c = CompFullySym(QQ, V.basis(), 2)}}
\]
\[
\text{\texttt{sage: c[0,0], c[0,1], c[0,2] = -4, 7, -8}}
\]
\[
\text{\texttt{sage: c[1,1], c[1,2] = 2, -4}}
\]
\[
\text{\texttt{sage: c[2,2] = 2}}
\]
\[
\text{\texttt{sage: s = b + c ; s}}
\]
\[
\text{\texttt{2-indices components w.r.t. [}}
\]
\[
\text{\texttt{(1, 0, 0),}}
\]
\[
\text{\texttt{(0, 1, 0),}}
\]
\[
\text{\texttt{(0, 0, 1)}}
\]
\[
\text{\texttt{], with symmetry on the index positions (0, 1)}}
\]
\[
\text{\texttt{sage: b[:], c[:], s[:]}}
\]
\[
\text{\texttt{([[ 1 2 3]}}
\]
\[
\text{\texttt{[-4 7 -8]}}
\]
\[
\text{\texttt{[-3 9 -5]}}
\]
\[
\text{\texttt{[ 2 5 7]}}
\]
\[
\text{\texttt{[ 7 2 -4]}}
\]
\[
\text{\texttt{[ 9 7 3]}}
\]
\[
\text{\texttt{[ 3 7 11], [-8 -4 2], [-5 3 13]}}
\]
\]
\[
\text{\texttt{sage: b + c == c + b}}
\]
\[
\text{\texttt{True}}
\]
\[
\text{\texttt{Check of the addition with counterparts not declared symmetric:}}
\]
\[
\text{\texttt{sage: bn = Components(QQ, V.basis(), 2)}}
\]
\[
\text{\texttt{sage: bn[:] = b[:]}}
\]
\[
\text{\texttt{sage: bn == b}}
\]
\[
\text{\texttt{True}}
\]
\[
\text{\texttt{sage: cn = Components(QQ, V.basis(), 2)}}
\]
\[
\text{\texttt{sage: cn[:] = c[:]}}
\]
\[
\text{\texttt{sage: cn == c}}
\]
\[
\text{\texttt{True}}
\]
\[
\text{\texttt{sage: bn + cn == b + c}}
\]
\[
\text{\texttt{True}}
\]
\[
\text{\texttt{Addition of an antisymmetric set of components with a non-symmetric one: the antisymmetry is lost:}}
\]
\[
\text{\texttt{sage: d = CompWithSym(QQ, V.basis(), 2, antisym=(0,1)) \ # for demonstration only: it is preferable to declare d = CompFullyAntiSym(QQ, V.basis(), 2)}}
\]
\[
\text{\texttt{sage: d[0,1], d[0,2], d[1,2] = 4, -1, 3}}
\]
\[
\text{\texttt{sage: s = a + d ; s}}
\]
\[
\text{\texttt{2-indices components w.r.t. [}}
\]
\[
\text{\texttt{(1, 0, 0),}}
\]
\[
\text{\texttt{(0, 1, 0),}}
\]
\[
\text{\texttt{(0, 0, 1)}}
\]
\[
\text{\texttt{]}}
\]
\[
\text{\texttt{sage: a[:], d[:], s[:]}}
\]
\[
\text{\texttt{([[ 1 -2 3]}}
\]
\[
\text{\texttt{[ 0 4 -1]}}
\]
\[
\text{\texttt{[ 1 2 2]}}
\]
\[
\text{\texttt{[ 4 5 -6]}}
\]
\[
\text{\texttt{[-4 0 3]}}
\]
\[
\text{\texttt{[ 0 5 -3]}}
\]
\[
\text{\texttt{[-7 8 9], [ 1 -3 0], [-6 5 9]}}
\]
\]
\[
\text{\texttt{sage: d + a == a + d}}
\]
\[
\text{\texttt{True}}
\]
\[
\text{\texttt{Addition of two antisymmetric set of components: the antisymmetry is preserved:}}
\]

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\[
\text{sage: } e = \text{CompWithSym}(\mathbb{Q}, V.\text{basis}(), 2, \text{antisym}=(0,1)) \quad \text{# for demonstration only: it is preferable to declare } e = \text{CompFullyAntiSym}(\mathbb{Q}, V.\text{basis}(), 2) \\
\text{sage: } e[0,1], e[0,2], e[1,2] = 2, 3, -1 \\
\text{sage: } s = d + e; s \\
2\text{-indices components w.r.t. } [(1, 0, 0), (0, 1, 0), (0, 0, 1)], \text{ with antisymmetry on the index positions (0, 1)} \\
\text{sage: } d[:,], e[:,], s[:]
\]

\[
\begin{bmatrix}
0 & 4 & -1 \\
-4 & 0 & 3 \\
1 & -3 & 0
\end{bmatrix}, 
\begin{bmatrix}
0 & 2 & 3 \\
-2 & 0 & -1 \\
-3 & 1 & 0
\end{bmatrix}, 
\begin{bmatrix}
0 & 6 & 2 \\
-6 & 0 & 2 \\
-2 & -2 & 0
\end{bmatrix}
\]

\[
\text{sage: } e + d == d + e \\
\text{True}
\]

\textbf{antisymmetrize(*pos)}

Antisymmetrization over the given index positions.

\textbf{INPUT:}

\begin{itemize}
\item pos – list of index positions involved in the antisymmetrization (with the convention position=0 for the first slot); if none, the antisymmetrization is performed over all the indices
\end{itemize}

\textbf{OUTPUT:}

\begin{itemize}
\item an instance of \texttt{CompWithSym} describing the antisymmetrized components
\end{itemize}

\textbf{EXAMPLES:}

Antisymmetrization of 3-indices components on a 3-dimensional space: 

\[
\text{sage: from sage.tensor.modules.comp import Components, CompWithSym, \\
\text{.. import} \text{CompFullySym, CompFullyAntiSym} \\
\text{sage: } V = \text{VectorSpace}(\mathbb{Q}, 3) \\
\text{sage: } a = \text{Components}(\mathbb{Q}, V.\text{basis}(), 1) \\
\text{sage: } a[:,] = (-2, 1, 3) \\
\text{sage: } b = \text{CompFullyAntiSym}(\mathbb{Q}, V.\text{basis}(), 2) \\
\text{sage: } b[0,1], b[0,2], b[1,2] = (4, 1, 2) \\
\text{sage: } c = a \cdot b; c \quad \text{# tensor product of a by b} \\
3\text{-indices components w.r.t. } [(1, 0, 0), (0, 1, 0), (0, 0, 1)], \text{ with antisymmetry on the index positions (1, 2)} \\
\text{sage: } s = c.\text{antisymmetrize}(); s \\
\text{Fully antisymmetric 3\text{-indices components w.r.t. } [(1, 0, 0), (0, 1, 0), (0, 0, 1)} \\
\text{sage: } c[:,], s[:]
\]

\[
\begin{bmatrix}
[[0, -8, -2], [8, 0, -4], [2, 4, 0]], \\
[[0, 4, 1], [-4, 0, 2], [-1, -2, 0]], \\
[[0, 0, 0], [-2, 4, -1], [-2, -2, 0]], \\
\end{bmatrix}
\]

(continues on next page)
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([[0, 12, 3], [-12, 0, 6], [-3, -6, 0]],
 [[0, 0, 0], [0, 0, 7/3], [0, -7/3, 0]],
 [[0, 0, -7/3], [0, 0, 0], [7/3, 0, 0]],
 [[0, 7/3, 0], [-7/3, 0, 0], [0, 0, 0]]
)

Check of the antisymmetrization:

```
sage: all(s[i,j,k] == (c[i,j,k]-c[i,k,j]+c[j,k,i]-c[j,i,k]+c[k,i,j]-c[k,j,i])/6
.....: for i in range(3) for j in range(3) for k in range(3))
True
```

Antisymmetrization over already antisymmetric indices does not change anything:

```
sage: s1 = s.antisymmetrize(1,2) ; s1
Fully antisymmetric 3-indices components w.r.t. 
(1, 0, 0),
(0, 1, 0),
(0, 0, 1)
]
```

```
sage: s1 == s
True
```

```
sage: c1 = c.antisymmetrize(1,2) ; c1
3-indices components w.r.t. 
(1, 0, 0),
(0, 1, 0),
(0, 0, 1)
], with antisymmetry on the index positions (1, 2)
```

```
sage: c1 == c
True
```

But in general, antisymmetrization may alter previous antisymmetries:

```
sage: c2 = c.antisymmetrize(0,1) ; c2  # the antisymmetry (2,3) is lost:
3-indices components w.r.t. 
(1, 0, 0),
(0, 1, 0),
(0, 0, 1)
], with antisymmetry on the index positions (0, 1)
```

```
sage: c2 == c
False
```

```
sage: c = s*a ; c
4-indices components w.r.t. 
(1, 0, 0),
(0, 1, 0),
(0, 0, 1)
], with antisymmetry on the index positions (0, 1, 2)
```

```
sage: s = c.antisymmetrize(1,3) ; s
4-indices components w.r.t. 
(1, 0, 0),
(0, 1, 0),
(0, 0, 1)
], with antisymmetry on the index positions (1, 3),
   with antisymmetry on the index positions (0, 2)
```
Partial antisymmetrization of 4-indices components with a symmetry on the first two indices:

```python
sage: a = CompFullySym(QQ, V.basis(), 2)
sage: a[:,] = [[-2,1,3], [1,0,-5], [3,-5,4]]
sage: b = Components(QQ, V.basis(), 2)
sage: b[:,] = [[1,2,3], [5,7,11], [13,17,19]]
sage: c = a*b ; c
```

4-indices components w.r.t. [(1, 0, 0), (0, 1, 0), (0, 0, 1)], with symmetry on the index positions (0, 1)

Some check of the antisymmetrization:

```python
sage: all(s[2,2,i,j] == (c[2,2,i,j] - c[2,2,j,i])/2 ....: for i in range(3) for j in range(i,3))
True
```

The full antisymmetrization results in zero because of the symmetry on the first two indices:

```python
sage: s = c.antisymmetrize() ; s
```

Fully antisymmetric 4-indices components w.r.t. [(1, 0, 0), (0, 1, 0), (0, 0, 1)]

Similarly, the partial antisymmetrization on the first two indices results in zero:

```python
sage: s = c.antisymmetrize(0,1) ; s
```

4-indices components w.r.t. [(1, 0, 0), (0, 1, 0), (0, 0, 1)], with antisymmetry on the index positions (0, 1)

The partial antisymmetrization on the positions (0, 2) destroys the symmetry on (0, 1):
\begin{verbatim}
sage: s = c.antisymmetrize(0,2) ; s
4-indices components w.r.t. [
(1, 0, 0),
(0, 1, 0),
(0, 0, 1)]
, with antisymmetry on the index positions (0, 2)
sage: s != 0
True
sage: s[0,1,2,1]
27/2
sage: s[1,0,2,1]  # the symmetry (0,1) is lost
-2
sage: s[2,1,0,1]  # the antisymmetry (0,2) holds
-27/2

non_redundant_index_generator()
Generator of indices, with only ordered indices in case of symmetries, so that only non-redundant indices
are generated.

OUTPUT:
• an iterable index

EXAMPLES:
Indices on a 2-dimensional space:
\begin{verbatim}
sage: from sage.tensor.modules.comp import Components, CompWithSym, \....: CompFullySym, CompFullyAntiSym
sage: V = VectorSpace(QQ, 2)
sage: c = CompFullySym(QQ, V.basis(), 2)
sage: list(c.non_redundant_index_generator())
[(0, 0), (0, 1), (1, 1)]
sage: c = CompFullySym(QQ, V.basis(), 2, start_index=1)
sage: list(c.non_redundant_index_generator())
[(1, 1), (1, 2), (2, 2)]
sage: c = CompFullyAntiSym(QQ, V.basis(), 2)
sage: list(c.non_redundant_index_generator())
[(0, 1)]
\end{verbatim}

Indices on a 3-dimensional space:
\begin{verbatim}
sage: V = VectorSpace(QQ, 3)
sage: c = CompFullySym(QQ, V.basis(), 2)
sage: list(c.non_redundant_index_generator())
[(0, 0), (0, 1), (0, 2), (1, 1), (1, 2), (2, 2)]
sage: c = CompFullySym(QQ, V.basis(), 2, start_index=1)
sage: list(c.non_redundant_index_generator())
[(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)]
sage: c = CompFullyAntiSym(QQ, V.basis(), 2)
sage: list(c.non_redundant_index_generator())
[(0, 1), (0, 2), (1, 2)]
sage: c = CompWithSym(QQ, V.basis(), 3, sym=(1,2))  # symmetry on the last two
indices
sage: list(c.non_redundant_index_generator())
\end{verbatim}
\end{verbatim}
\[(0, 0, 0), (1, 0, 0), (2, 0, 0), (0, 0, 1), (1, 0, 1),
(2, 0, 1), (0, 0, 2), (1, 0, 2), (2, 0, 2), (0, 1, 1),
(1, 1, 1), (2, 1, 1), (0, 1, 2), (1, 1, 2), (2, 1, 2),
(0, 2, 2), (1, 2, 2), (2, 2, 2)\]

\[\text{sage: } c = \text{CompWithSym}(\mathbb{Q}, V.\text{basis}(), 3, \text{antisym}=(1, 2)) \quad \text{# antisymmetry on the last two indices}\]

\[\text{sage: } \text{list}(c.\text{non_redundant_index_generator}())\]
\[(0, 0, 1), (1, 0, 1), (2, 0, 1), (0, 0, 2), (1, 0, 2),
(2, 0, 2), (0, 1, 2), (1, 1, 2), (2, 1, 2)\]

\[\text{sage: } c = \text{CompFullySym}(\mathbb{Q}, V.\text{basis}(), 3)\]

\[\text{sage: } \text{list}(c.\text{non_redundant_index_generator}())\]
\[(0, 0, 0), (0, 0, 1), (0, 0, 2), (0, 1, 1), (0, 1, 2), (0, 2, 2),
(1, 1, 1), (1, 1, 2), (1, 2, 2), (2, 2, 2)\]

\[\text{sage: } c = \text{CompFullyAntiSym}(\mathbb{Q}, V.\text{basis}(), 3)\]

\[\text{sage: } \text{list}(c.\text{non_redundant_index_generator}())\]
\[(0, 1, 2)\]

\[\text{sage: } \text{V = VectorSpace}((\mathbb{Q}, 4))\]

\[\text{sage: } c = \text{Components}(\mathbb{Q}, V.\text{basis}(), 1)\]

\[\text{sage: } \text{list}(c.\text{non_redundant_index_generator}())\]
\[(0,), (1,), (2,), (3,)]\]

\[\text{sage: } c = \text{CompFullyAntiSym}(\mathbb{Q}, V.\text{basis}(), 2)\]

\[\text{sage: } \text{list}(c.\text{non_redundant_index_generator}())\]
\[(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\]

\[\text{sage: } c = \text{CompFullyAntiSym}(\mathbb{Q}, V.\text{basis}(), 3)\]

\[\text{sage: } \text{list}(c.\text{non_redundant_index_generator}())\]
\[(0, 1, 2), (0, 1, 3), (0, 2, 3), (1, 2, 3)\]

\[\text{sage: } c = \text{CompFullyAntiSym}(\mathbb{Q}, V.\text{basis}(), 4)\]

\[\text{sage: } \text{list}(c.\text{non_redundant_index_generator}())\]
\[(0, 1, 2, 3)\]

\[\text{sage: } c = \text{CompFullyAntiSym}(\mathbb{Q}, V.\text{basis}(), 5)\]

\[\text{sage: } \text{list}(c.\text{non_redundant_index_generator}()) \quad \text{# nothing since } c \text{ is identically zero in this case (for 5 > 4)}\]

\[
\text{swap\_adjacent\_indices}(\text{pos1, pos2, pos3})
\]

Swap two adjacent sets of indices.

This method is essentially required to reorder the covariant and contravariant indices in the computation of a tensor product.

The symmetries are preserved and the corresponding indices are adjusted consequently.

**INPUT:**

- **pos1** – position of the first index of set 1 (with the convention position=0 for the first slot)
- **pos2** – position of the first index of set 2 = 1 + position of the last index of set 1 (since the two sets are adjacent)
- **pos3** – 1 + position of the last index of set 2

**OUTPUT:**

- Components with index set 1 permuted with index set 2.
EXAMPLES:

Swap of the index in position 0 with the pair of indices in position (1,2) in a set of components antisymmetric with respect to the indices in position (1,2):

```python
sage: from sage.tensor.modules.comp import CompWithSym
sage: V = VectorSpace(QQ, 3)
sage: c = CompWithSym(QQ, V.basis(), 3, antisym=(1,2))
sage: c[0,0,1], c[0,0,2], c[0,1,2] = (1,2,3)
sage: c[1,0,1], c[1,0,2], c[1,1,2] = (4,5,6)
sage: c[2,0,1], c[2,0,2], c[2,1,2] = (7,8,9)
sage: c[:]
[(0, 1, 2), (-1, 0, 3), (-2, -3, 0)],
[(0, 4, 5), (-4, 0, 6), (-5, -6, 0)],
[(0, 7, 8), (-7, 0, 9), (-8, -9, 0)]
sage: c1 = c.swap_adjacent_indices(0,1,3)
sage: c1._antisym  # c is antisymmetric with respect to the last pair of indices...
[(1, 2)]
sage: c1._antisym  #...while c1 is antisymmetric with respect to the first pair of indices ...
[(0, 1)]
sage: c[0,1,2] 3
sage: c1[1,2,0] 3
sage: c1[2,1,0] -3
```

\texttt{symmetrize(*pos)}

Symmetrization over the given index positions.

**INPUT:**

- pos – list of index positions involved in the symmetrization (with the convention position=0 for the first slot); if none, the symmetrization is performed over all the indices

**OUTPUT:**

- an instance of \texttt{CompWithSym} describing the symmetrized components

**EXAMPLES:**

Symmetrization of 3-indices components on a 3-dimensional space:

```python
sage: from sage.tensor.modules.comp import Components, CompWithSym, 
.....: CompFullySym, CompFullyAntiSym
sage: V = VectorSpace(QQ, 3)
sage: c = Components(QQ, V.basis(), 3)
sage: c[:] = [[[1,2,3], [4,5,6], [7,8,9]], [[10,11,12], [13,14,15], [16,17,18]],
...: [[19,20,21], [22,23,24], [25,26,27]]
sage: cs = c.symmetrize(0,1) ; cs
3-indices components w.r.t. [ (1, 0, 0),
(0, 1, 0),
(0, 0, 1) ], with symmetry on the index positions (0, 1)
```

(continues on next page)
sage: s = cs.symmetrize() ; s
Fully symmetric 3-indices components w.r.t. [ (1, 0, 0), (0, 1, 0), (0, 0, 1) ]
sage: cs[:], s[:]

([[[1, 2, 3], [7, 8, 9], [13, 14, 15]], [[7, 8, 9], [13, 14, 15], [19, 20, 21]], [[13, 14, 15], [19, 20, 21], [25, 26, 27]]],
[[[1, 16/3, 29/3], [16/3, 29/3, 14], [29/3, 14, 55/3]], [[16/3, 29/3, 14], [29/3, 14, 55/3], [14, 55/3, 68/3]], [[29/3, 14, 55/3], [14, 55/3, 68/3], [55/3, 68/3, 27]])
sage: s == c.symmetrize()  # should be true
True
sage: s1 = cs.symmetrize(0,1) ; s1  # should return a copy of cs
3-indices components w.r.t. [ (1, 0, 0), (0, 1, 0), (0, 0, 1) ], with symmetry on the index positions (0, 1)
sage: s1 == cs  # check that s1 is a copy of cs
True

Let us now start with a symmetry on the last two indices:

sage: cs1 = c.symmetrize(1,2) ; cs1
3-indices components w.r.t. [ (1, 0, 0), (0, 1, 0), (0, 0, 1) ], with symmetry on the index positions (1, 2)
sage: s2 = cs1.symmetrize() ; s2
Fully symmetric 3-indices components w.r.t. [ (1, 0, 0), (0, 1, 0), (0, 0, 1) ]
sage: s2 == c.symmetrize()  # should be true
True

Symmetrization alters pre-existing symmetries: let us symmetrize w.r.t. the index positions (1, 2) a set of components that is symmetric w.r.t. the index positions (0, 1):

sage: cs = c.symmetrize(0,1) ; cs
3-indices components w.r.t. [ (1, 0, 0), (0, 1, 0), (0, 0, 1) ], with symmetry on the index positions (0, 1)
sage: css = cs.symmetrize(1,2)
sage: css  # the symmetry (0,1) has been lost:

(continues on next page)
3-indices components w.r.t. [ (1, 0, 0),
(0, 1, 0),
(0, 0, 1) ]
], with symmetry on the index positions (1, 2)
sage: css[:]
[[[1, 9/2, 8], [9/2, 8, 23/2], [8, 23/2, 15]],
[[7, 21/2, 14], [21/2, 14, 35/2], [14, 35/2, 21]],
[[13, 33/2, 20], [33/2, 20, 47/2], [20, 47/2, 27]]
sage: cs[:]
[[[1, 2, 3], [7, 8, 9], [13, 14, 15]],
[[7, 8, 9], [13, 14, 15], [19, 20, 21]],
[[13, 14, 15], [19, 20, 21], [25, 26, 27]]
sage: css == c.symmetrize()  # css differs from the full symmetrized version
False
sage: css.symmetrize() == c.symmetrize()  # one has to symmetrize css over all indices to recover it
True

Another example of symmetry alteration: symmetrization over (0, 1) of a 4-indices set of components that is symmetric w.r.t. (1, 2, 3):
sage: v = Components(QQ, V.basis(), 1)
sage: v[:] = (-2,1,4)
sage: a = v*s ; a
4-indices components w.r.t. [ (1, 0, 0),
(0, 1, 0),
(0, 0, 1) ]
], with symmetry on the index positions (1, 2, 3)
sage: a1 = a.symmetrize(0,1) ; a1  # the symmetry (1,2,3) has been reduced to (2, 3):
4-indices components w.r.t. [ (1, 0, 0),
(0, 1, 0),
(0, 0, 1) ]
], with symmetry on the index positions (0, 1), with symmetry on the index positions (2, 3)
sage: a1._sym  # a1 has two distinct symmetries:
[(0, 1), (2, 3)]
sage: a[0,1,2,0] == a[0,0,2,1]  # a is symmetric w.r.t. positions 1 and 3
True
sage: a1[0,1,2,0] == a1[0,0,2,1]  # a1 is not
False
sage: a1[0,1,2,0] == a1[1,0,2,0]  # but it is symmetric w.r.t. position 0 and 1
True
sage: a[0,1,2,0] == a[1,0,2,0]  # while a is not
False

Partial symmetrization of 4-indices components with an antisymmetry on the last two indices:
sage: a = Components(QQ, V.basis(), 2)
(continues on next page)
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sage: a[:,:] = [[-1,2,3], [4,5,-6], [7,8,9]]
sage: b = CompFullyAntiSym(QQ, V.basis(), 2)
sage: b[0,1], b[0,2], b[1,2] = (2, 4, 8)
sage: c = a*b ; c

4-indices components w.r.t. [(1, 0, 0),
(0, 1, 0),
(0, 0, 1)], with antisymmetry on the index positions (2, 3)

sage: s = c.symmetrize(0,1) ; s # symmetrization on the first two indices

4-indices components w.r.t. [(1, 0, 0),
(0, 1, 0),
(0, 0, 1)], with symmetry on the index positions (0, 1), with antisymmetry on the index positions (2, 3)

sage: s[0,1,2,1] == (c[0,1,2,1] + c[1,0,2,1]) / 2 # check of the symmetrization
True

sage: s = c.symmetrize() ; s # symmetrization over all the indices

Fully symmetric 4-indices components w.r.t. [(1, 0, 0),
(0, 1, 0),
(0, 0, 1)]

sage: s == 0 # the full symmetrization results in zero due to the antisymmetry on the last two indices
True

sage: s = c.symmetrize(2,3) ; s

4-indices components w.r.t. [(1, 0, 0),
(0, 1, 0),
(0, 0, 1)], with symmetry on the index positions (2, 3)

sage: s == 0 # must be zero since the symmetrization has been performed on the antisymmetric indices
True

sage: s = c.symmetrize(0,2) ; s

4-indices components w.r.t. [(1, 0, 0),
(0, 1, 0),
(0, 0, 1)], with symmetry on the index positions (0, 2)

sage: s != 0 # s is not zero, but the antisymmetry on (2,3) is lost because the position 2 is involved in the new symmetry
True

Partial symmetrization of 4-indices components with an antisymmetry on the last three indices:

sage: a = Components(QQ, V.basis(), 1)
sage: a[:] = (1, -2, 3)
sage: b = CompFullyAntiSym(QQ, V.basis(), 3)
sage: b[0,1,2] = 4

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\[ \text{sage: } c = a^b ; c \]
4-indices components w.r.t.
\[ (1, 0, 0), \]
\[ (0, 1, 0), \]
\[ (0, 0, 1) \]
with antisymmetry on the index positions \((1, 2, 3)\)

\[ \text{sage: } s = c.symmetrize(0,1) ; s \]
4-indices components w.r.t.
\[ (1, 0, 0), \]
\[ (0, 1, 0), \]
\[ (0, 0, 1) \]
with symmetry on the index positions \((0, 1)\),
with antisymmetry on the index positions \((2, 3)\)

Note that the antisymmetry on \((1, 2, 3)\) has been reduced to \((2, 3)\) only:

\[ \text{sage: } s = c.symmetrize(1,2) ; s \]
4-indices components w.r.t.
\[ (1, 0, 0), \]
\[ (0, 1, 0), \]
\[ (0, 0, 1) \]
with symmetry on the index positions \((1, 2)\)

\[ \text{sage: } s == 0 \# \text{ because } (1,2) \text{ are involved in the original antisymmetry} \]
True

\textbf{trace}(pos1, pos2)
Index contraction, taking care of the symmetries.

\textbf{INPUT:}

\begin{itemize}
\item pos1 – position of the first index for the contraction (with the convention position=0 for the first slot)
\item pos2 – position of the second index for the contraction
\end{itemize}

\textbf{OUTPUT:}

\begin{itemize}
\item set of components resulting from the (pos1, pos2) contraction
\end{itemize}

\textbf{EXAMPLES:}

Self-contraction of symmetric 2-indices components:

\[ \text{sage: from sage.tensor.modules.comp import Components, CompWithSym, } \backslash 
\text{ ...: CompFullySym, CompFullyAntiSym} \]
\[ \text{sage: } V = \text{VectorSpace}(\text{QQ}, 3) \]
\[ \text{sage: } a = \text{CompFullySym}(\text{QQ}, V.\text{basis}(), 2) \]
\[ \text{sage: } a[:\:] = [[1,2,3],[2,4,5],[3,5,6]] \]
\[ \text{sage: } a.\text{trace}(0,1) \]
11
\[ \text{sage: } a[0,0] + a[1,1] + a[2,2] \]
11

Self-contraction of antisymmetric 2-indices components:

\[ \text{sage: } b = \text{CompFullyAntiSym}(\text{QQ}, V.\text{basis}(), 2) \]
\[ \text{sage: } b[0,1], b[0,2], b[1,2] = (3, -2, 1) \]
Self-contraction of 3-indices components with one symmetry:

\[
sage: v = Components(QQ, V.basis(), 1)
sage: v[:] = (-2, 4, -8)
sage: c = v*b ; c
3-indices components w.r.t. [(1, 0, 0), (0, 1, 0), (0, 0, 1)], with antisymmetry on the index positions (1, 2)
sage: s = c.trace(0,1) ; s
1-index components w.r.t. [(1, 0, 0), (0, 1, 0), (0, 0, 1)]
\[
\begin{bmatrix}
-28 \\
 2 \\
 8 \\
\end{bmatrix}
\]
\[
sage: \left[\sum v[k]^* b[k,i] \text{ for } k \text{ in range}(3) \right] \text{ for } i \text{ in range}(3) \] # check
\[
\begin{bmatrix}
-28 \\
 2 \\
 8 \\
\end{bmatrix}
\]
\[
sage: s = c.trace(1,2) ; s
1-index components w.r.t. [(1, 0, 0), (0, 1, 0), (0, 0, 1)]
\[
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]
\[
sage: s[:] \text{ is zero by antisymmetry}
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]
\[
sage: c = b*v ; c
3-indices components w.r.t. [(1, 0, 0), (0, 1, 0), (0, 0, 1)], with antisymmetry on the index positions (0, 1)
sage: s = c.trace(0,1)
sage: s[:] \text{ is zero by antisymmetry}
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]
\[
sage: s = c.trace(1,2) ; s[:]
\begin{bmatrix}
28 \\
-2 \\
-8 \\
\end{bmatrix}
\]
\[
sage: \left[\sum b[i,k]* v[k] \text{ for } k \text{ in range}(3) \right] \text{ for } i \text{ in range}(3) \] # check
\[
\begin{bmatrix}
28 \\
-2 \\
-8 \\
\end{bmatrix}
\]
\]

Self-contraction of 4-indices components with two symmetries:

\[
sage: c = a*b ; c
4-indices components w.r.t. [(1, 0, 0), (0, 1, 0), (0, 0, 1)]
\]

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], with symmetry on the index positions (0, 1), with antisymmetry on the index positions (2, 3)
sage: s = c.trace(0,1) ; s  # the symmetry on (0,1) is lost:
Fully antisymmetric 2-indices components w.r.t. [ (1, 0, 0),
(0, 1, 0),
(0, 0, 1) ]
sage: s[:]
[ 0 33 -22]
[-33 0 11]
[ 22 -11 0]
sage: [[sum(c[k,k,i] for k in range(3)) for j in range(3)] for i in range(3)]
# check
[[0, 33, -22], [-33, 0, 11], [22, -11, 0]]
sage: s = c.trace(1,2) ; s  # both symmetries are lost by this contraction
2-indices components w.r.t. [ (1, 0, 0),
(0, 1, 0),
(0, 0, 1) ]
sage: s[:]
[ 0 0 0]
[-2 1 0]
[-3 3 -1]
sage: [[sum(c[i,k,k] for k in range(3)) for j in range(3)] for i in range(3)]
# check
[[0, 0, 0], [-2, 1, 0], [-3, 3, -1]]

class sage.tensor.modules.comp.Components(ring, frame, nb_indices, start_index=0,
output_formatter=None)

Bases: sage.structure.sage_object.SageObject

Indexed set of ring elements forming some components with respect to a given “frame”.

The “frame” can be a basis of some vector space or a vector frame on some manifold (i.e. a field of bases).
The stored quantities can be tensor components or non-tensorial quantities, such as connection coefficients or
structure coefficients. The symmetries over some indices are dealt by subclasses of the class Components.

INPUT:

• ring – commutative ring in which each component takes its value
• frame – frame with respect to which the components are defined; whatever type frame is, it should have
a method __len__() implemented, so that len(frame) returns the dimension, i.e. the size of a single
index range
• nb_indices – number of integer indices labeling the components
• start_index – (default: 0) first value of a single index; accordingly a component index i must obey
start_index <= i <= start_index + dim - 1, where dim = len(frame).
• output_formatter – (default: None) function or unbound method called to format the output of the
component access operator [ ... ] (method __getitem__); output_formatter must take 1 or 2 arguments:
the 1st argument must be an element of ring and the second one, if any, some format specification.

EXAMPLES:
Set of components with 2 indices on a 3-dimensional vector space, the frame being some basis of the vector space:

```
sage: from sage.tensor.modules.comp import Components
sage: V = VectorSpace(QQ,3)
sage: basis = V.basis() ; basis
[(1, 0, 0),
 (0, 1, 0),
 (0, 0, 1)
]
sage: c = Components(QQ, basis, 2) ; c
2-indices components w.r.t. [(1, 0, 0),
 (0, 1, 0),
 (0, 0, 1)]
```

Actually, the frame can be any object that has some length, i.e. on which the function `len()` can be called:

```
sage: basis1 = V.gens() ; basis1
((1, 0, 0), (0, 1, 0), (0, 0, 1))
sage: c1 = Components(QQ, basis1, 2) ; c1
2-indices components w.r.t. ((1, 0, 0), (0, 1, 0), (0, 0, 1))
sage: basis2 = ['a', 'b', 'c']
sage: c2 = Components(QQ, basis2, 2) ; c2
2-indices components w.r.t. ['a', 'b', 'c']
```

By default, the indices range from 0 to \( n - 1 \), where \( n \) is the length of the frame. This can be changed via the argument `start_index`:

```
sage: c1 = Components(QQ, basis, 2, start_index=1)
sage: c1[0,1]
Traceback (most recent call last):
  ...
IndexError: index out of range: 0 not in [1, 3]
sage: c[0,1]  # for c, the index 0 is OK
0
sage: c[0,1] = -3
sage: c1[:][:] = c[:][:] # list copy of all components
sage: c1[1,2]  # (1,2) = (0,1) shifted by 1
-3
```

If some formatter function or unbound method is provided via the argument `output_formatter`, it is used to change the output of the access operator `[..., ]`:

```
sage: a = Components(QQ, basis, 2, output_formatter=Rational.numerical_approx)
sage: a[1,2] = 1/3
sage: a[1,2]
0.333333333333333
```

The format can be passed to the formatter as the last argument of the access operator `[..., ]`:
The raw (unformatted) components are then accessed by the double bracket operator:

```
sage: a[[1,2]]
1/3
```

For sets of components declared without any output formatter, there is no difference between [...] and [[...]]:

```
sage: c[1,2] = 1/3
sage: c[1,2], c[[1,2]]
(1/3, 1/3)
```

The formatter is also used for the complete list of components:

```
sage: a[:]
[0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000]  
[0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000]  
[0.000000000000000 0.000000000000000 0.000000000000000 0.333333333333333 0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000 0.000000000000000]
```

```
sage: a[10] # with a format different from the default one (53 bits)
[0.00 0.00 0.00]  
[0.00 0.00 0.33]  
[0.00 0.00 0.00]
```

The complete list of components in raw form can be recovered by the double bracket operator, replacing : by slice(None) (since a[:]) generates a Python syntax error):

```
sage: a[[slice(None)]]
[ 0 0 0]  
[ 0 0 1/3]  
[ 0 0 0]
```

Another example of formatter: the Python built-in function str() to generate string outputs:

```
sage: b = Components(QQ, V.basis(), 1, output_formatter=str)
sage: b[:] = (1, 0, -4)
sage: b[:]
['1', '0', '-4']
```

For such a formatter, 2-indices components are no longer displayed as a matrix:

```
sage: b = Components(QQ, V.basis(), 2, output_formatter=str)
sage: b[0,1] = 1/3
sage: b[:]
[['0', '1/3', '0'], ['0', '0', '0'], ['0', '0', '0']]
```

But unformatted outputs still are:

```
sage: b[[slice(None)]]
[ 0 1/3 0]
```

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Internally, the components are stored as a dictionary \(_\text{comp}\) whose keys are the indices; only the non-zero components are stored:

```python
sage: a[:]
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.333333333333333]
[0.000000000000000 0.000000000000000 0.000000000000000]
sage: a._comp
{(1, 2): 1/3}
sage: v = Components(QQ, basis, 1)
sage: v[:] = (-1, 0, 3)
sage: v._comp    # random output order of the component dictionary
{(0,): -1, (2,): 3}
```

**ARITHMETIC EXAMPLES:**

Unary plus operator:

```python
sage: a = Components(QQ, basis, 1)
sage: a[:] = (-1, 0, 3)
sage: s = +a ; s[:]
[-1, 0, 3]
sage: +a == a
True
```

Unary minus operator:

```python
sage: s = -a ; s[:]
[1, 0, -3]
```

Addition:

```python
sage: b = Components(QQ, basis, 1)
sage: b[:] = (2, 1, 4)
sage: s = a + b ; s[:]
[1, 1, 7]
sage: a + b == b + a
True
sage: a + (-a) == 0
True
```

Subtraction:

```python
sage: s = a - b ; s[:]
[-3, -1, -1]
sage: s + b == a
True
sage: a - b == -(b - a)
True
```
Multiplication by a scalar:

\[
\text{sage: } s = 2 \ast a \; ; \; s[:]
\]
\[-2, 0, 6\]

Division by a scalar:

\[
\text{sage: } s = a/2 \; ; \; s[:]
\]
\[-1/2, 0, 3/2\]
\[
\text{sage: } 2 \ast (a/2) == a
\]
True

Tensor product (by means of the operator \(*\)):

\[
\text{sage: } c = a \ast b \; ; \; c
\]
2-indices components w.r.t. [ (1, 0, 0), (0, 1, 0), (0, 0, 1) ]
\[
\text{sage: } a[:], b[:]
\]
((-1, 0, 3), (2, 1, 4))
\[
\text{sage: } c[:]
\]
[-2 -1 -4]
[ 0 0 0]
[ 6 3 12]
\[
\text{sage: } d = c \ast a \; ; \; d
\]
3-indices components w.r.t. [ (1, 0, 0), (0, 1, 0), (0, 0, 1) ]
\[
\text{sage: } d[:]
\]
[[[2, 0, -6], [1, 0, -3], [4, 0, -12]],
 [[0, 0, 0], [0, 0, 0], [0, 0, 0]],
 [[-6, 0, 18], [-3, 0, 9], [-12, 0, 36]]]
\[
\text{sage: } d[0,1,2] == a[0] \ast b[1] \ast a[2]
\]
True

antisymmetrize(*pos)

Antisymmetrization over the given index positions

INPUT:

- pos – list of index positions involved in the antisymmetrization (with the convention position=0 for the first slot); if none, the antisymmetrization is performed over all the indices

OUTPUT:

- an instance of CompWithSym describing the antisymmetrized components.

EXAMPLES:

Antisymmetrization of 2-indices components:

\[
\text{sage: from sage.tensor.modules.comp import Components}
\]
\[
\text{sage: V = VectorSpace(QQ, 3)}
\]
sage: c = Components(QQ, V.basis(), 2)
sage: c[:] = [[1,2,3], [4,5,6], [7,8,9]]
sage: s = c.antisymmetrize(); s

Fully antisymmetric 2-indices components w.r.t. [ (1, 0, 0), (0, 1, 0), (0, 0, 1) ]

sage: c[:], s[:]
([1 2 3] [ 0 -1 -2]
[4 5 6] [ 1 0 -1]
[7 8 9], [ 2 1 0])
sage: c.antisymmetrize() == c.antisymmetrize(0,1)
True

Full antisymmetrization of 3-indices components:

sage: c = Components(QQ, V.basis(), 3)
sage: c[:] = [[[-1,-2,3], [4,-5,4], [-7,8,9]], [[10,10,12], [13,-14,15], [-16, -17,19]], [[-19,20,21], [1,2,3], [-25,26,27]]]
sage: s = c.antisymmetrize(); s

Fully antisymmetric 3-indices components w.r.t. [ (1, 0, 0), (0, 1, 0), (0, 0, 1) ], with antisymmetry on the index positions (0, 1)

sage: c[:], s[:]
..., [[[-1, -2, 3], [4, -5, 4], [-7, 8, 9]], [[10, 10, 12], [13, -14, 15], [-16, 17, 19]], [[-19, 20, 21], [1, 2, 3], [-25, 26, 27]]]

sage: all(s[i,j,k] == (c[i,j,k]-c[i,k,j]+c[j,k,i]-c[j,i,k]+c[k,i,j]-c[k,j,i])/6.

# Check of the result:
....: for i in range(3) for j in range(3) for k in range(3))
True
sage: c.symmetrize() == c.symmetrize(0,1,2)
True

Partial antisymmetrization of 3-indices components:

sage: s = c.antisymmetrize(0,1); s  # antisymmetrization on the first two indices

3-indices components w.r.t. [ (1, 0, 0), (0, 1, 0), (0, 0, 1) ], with antisymmetry on the index positions (0, 1)

sage: c[:], s[:]

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The order of index positions in the argument does not matter:

\[
\text{sage: } \text{c.antisymmetrize}(1,0) == \text{c.antisymmetrize}(0,1) \\
\text{True}
\]

\[
\text{sage: } \text{c.antisymmetrize}(2,1) == \text{c.antisymmetrize}(1,2) \\
\text{True}
\]

\[
\text{sage: } \text{c.antisymmetrize}(2,0) == \text{c.antisymmetrize}(0,2) \\
\text{True}
\]
**contract(*args)**

Contraction on one or many indices with another instance of `Components`.

**INPUT:**

- `pos1` – positions of the indices in `self` involved in the contraction; `pos1` must be a sequence of integers, with 0 standing for the first index position, 1 for the second one, etc. If `pos1` is not provided, a single contraction on the last index position of `self` is assumed
- `other` – the set of components to contract with
- `pos2` – positions of the indices in `other` involved in the contraction, with the same conventions as for `pos1`. If `pos2` is not provided, a single contraction on the first index position of `other` is assumed

**OUTPUT:**

- set of components resulting from the contraction

**EXAMPLES:**

Contraction of a 1-index set of components with a 2-index one:

```python
sage: from sage.tensor.modules.comp import Components
gage: V = VectorSpace(QQ, 3)
sage: a = Components(QQ, V.basis(), 1)
sage: a[:] = (-1, 2, 3)
sage: b = Components(QQ, V.basis(), 2)
sage: b[::] = [[1,2,3], [4,5,6], [7,8,9]]
sage: s0 = a.contract(0, b, 0) ; s0
1-index components w.r.t. [(1, 0, 0), (0, 1, 0), (0, 0, 1)]
sage: s0[:]
[28, 32, 36]
sage: s0[:] == [sum(a[j]*b[j,i] for j in range(3)) for i in range(3)]
# check
True
sage: s1 = a.contract(0, b, 1) ; s1[:]
[12, 24, 36]
sage: s1[:] == [sum(a[j]*b[i,j] for j in range(3)) for i in range(3)]
# check
True
```

Parallel computations (see `Parallelism`):

```python
sage: Parallelism().set('tensor', nproc=2)
sage: Parallelism().get('tensor')
2
sage: s0_par = a.contract(0, b, 0) ; s0_par
1-index components w.r.t. [(1, 0, 0), (0, 1, 0), (0, 0, 1)]
sage: s0_par[:]
[28, 32, 36]
sage: s0_par == s0
True
```

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\begin{verbatim}
sage: s1_par = a.contract(0, b, 1) ; s1_par[:]
[12, 24, 36]
sage: s1_par == s1
True
sage: Parallelism().set('tensor', nproc = 1)  # switch off parallelization

Contraction on 2 indices:

sage: c = a*b ; c
3-indices components w.r.t. [ (1, 0, 0),
(0, 1, 0),
(0, 0, 1) ]
sage: s = c.contract(1,2, b, 0,1) ; s
1-index components w.r.t. [ (1, 0, 0),
(0, 1, 0),
(0, 0, 1) ]
sage: s[:]
[-285, 570, 855]
sage: [sum(sum(c[i,j,k]*b[j,k] for k in range(3))
....: for j in range(3)) for i in range(3)]
[-285, 570, 855]

Parallel computation:

sage: Parallelism().set('tensor', nproc=2)
sage: c_par = a*b ; c_par
3-indices components w.r.t. [ (1, 0, 0),
(0, 1, 0),
(0, 0, 1) ]
sage: c_par == c
True
sage: s_par = c_par.contract(1,2, b, 0,1) ; s_par
1-index components w.r.t. [ (1, 0, 0),
(0, 1, 0),
(0, 0, 1) ]
sage: s_par[:]
[-285, 570, 855]
sage: s_par == s
True
sage: Parallelism().set('tensor', nproc=1)  # switch off parallelization

Consistency check with \texttt{trace()}:

sage: b = a*a ; b  # the tensor product of a with itself

Fully symmetric 2-indices components w.r.t. [ (continues on next page)
\end{verbatim}
(continued from previous page)

```plaintext
(1, 0, 0),
(0, 1, 0),
(0, 0, 1)
]
sage: b[:]
[ 1 -2 -3
[-2 4 6
[-3 6 9
sage: b.trace(0,1)
14
sage: a.contract(0, a, 0) == b.trace(0,1)
True
```

**copy()**

Return an exact copy of `self`.

**EXAMPLES:**

Copy of a set of components with a single index:

```plaintext
sage: from sage.tensor.modules.comp import Components
sage: V = VectorSpace(QQ,3)
sage: a = Components(QQ, V.basis(), 1)
sage: a[:] = -2, 1, 5
sage: b = a.copy() ; b
1-index components w.r.t. [ (1, 0, 0),
(0, 1, 0),
(0, 0, 1)
]
sage: b[:]
[-2, 1, 5]
sage: b == a
True
sage: b is a  # b is a distinct object
False
```

**display**

Display all the components, one per line.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

**INPUT:**

- `symbol` – string (typically a single letter) specifying the symbol for the components
- `latex_symbol` – (default: `None`) string specifying the LaTeX symbol for the components; if `None`, `symbol` is used
- `index_positions` – (default: `None`) string of length the number of indices of the components and composed of characters ‘d’ (for “down”) or ‘u’ (for “up”) to specify the position of each index: ‘d’ corresponds to a subscript and ‘u’ to a superscript. If `index_positions` is `None`, all indices are printed as subscripts
- `index_labels` – (default: `None`) list of strings representing the labels of each of the individual indices within the index range defined at the construction of the object; if `None`, integer labels are used

```plaintext
display(symbol, latex_symbol=None, index_positions=None, index_labels=None, index_latex_labels=None, format_spec=None, only_nonzero=True, only_nonredundant=False)
```
• `index_latex_labels` – (default: None) list of strings representing the LaTeX labels of each of the individual indices within the index range defined at the construction of the object; if None, integers labels are used

• `format_spec` – (default: None) format specification passed to the output formatter declared at the construction of the object

• `only_nonzero` – (default: True) boolean; if True, only nonzero components are displayed

• `only_nonredundant` – (default: False) boolean; if True, only nonredundant components are displayed in case of symmetries

EXAMPLES:

Display of 3-indices components w.r.t. to the canonical basis of the free module \( \mathbb{Z}^2 \) over the integer ring:

```python
sage: from sage.tensor.modules.comp import Components
sage: c = Components(ZZ, (ZZ^2).basis(), 3)
sage: c[0,1,0], c[1,0,1], c[1,1,1] = -2, 5, 3
sage: c.display('c')
c_010 = -2
c_101 = 5
c_111 = 3
```

By default, only nonzero components are shown; to display all the components, it suffices to set the parameter `only_nonzero` to False:

```python
sage: c.display('c', only_nonzero=False)
c_000 = 0
c_001 = 0
c_010 = -2
c_011 = 0
c_100 = 0
c_101 = 5
c_110 = 0
c_111 = 3
```

By default, all indices are printed as subscripts, but any index position can be specified:

```python
sage: c.display('c', index_positions='udd')
c^0_10 = -2
c^1_01 = 5
c^1_11 = 3
sage: c.display('c', index_positions='udu')
c^0_1^0 = -2
c^1_0^1 = 5
c^1_1^1 = 3
sage: c.display('c', index_positions='ddu')
c_01^0 = -2
c_10^1 = 5
c_11^1 = 3
```

The LaTeX output is performed as an array, with the symbol adjustable if it differs from the text symbol:

```python
sage: latex(c.display('c', latex_symbol=r'\Gamma', index_positions='udd'))
\begin{array}{lcl}
```

(continues on next page)
The index labels can differ from integers:

```
sage: c.display('c', index_labels=['x', 'y'])
c_{xyx} = -2
c_{yxy} = 5
c_{yyy} = 3
```

If the index labels are longer than a single character, they are separated by a comma:

```
sage: c.display('c', index_labels=['r', 'th'])
c_{r,th,r} = -2
c_{th,r,th} = 5
c_{th,th,th} = 3
```

The LaTeX labels for the indices can be specified if they differ from the text ones:

```
sage: c.display('c', index_labels=['r', 'th'], index_latex_labels=['r', r'\theta'])
c_{r,\theta,r} = -2
c_{\theta,r,\theta} = 5
c_{\theta,\theta,\theta} = 3
```

The display of components with symmetries is governed by the parameter `only_nonredundant`:

```
sage: from sage.tensor.modules.comp import CompWithSym
sage: c = CompWithSym(ZZ, (ZZ^2).basis(), 3, sym=(1,2)) ; c
3-indices components w.r.t. [ (1, 0), (0, 1) ], with symmetry on the index positions (1, 2)
sage: c[0,0,1] = 2
sage: c.display('c')
c_001 = 2
c_010 = 2
sage: c.display('c', only_nonredundant=True)
c_001 = 2
```

If some nontrivial output formatter has been set, the format can be specified by means of the argument `format_spec`:

```
sage: c = Components(QQ, (QQ^3).basis(), 2, output_formatter=Rational.numerical_approx)
sage: c[0,1] = 1/3
sage: c[2,1] = 2/7
sage: c.display('C') # default format (53 bits of precision)
C_01 = 0.333333333333333
C_21 = 0.285714285714286
sage: c.display('C', format_spec=10) # 10 bits of precision
# (continues on next page)
```
Check that the bug reported in trac ticket #22520 is fixed:

```python
sage: c = Components(SR, [1, 2], 1)  # optional - sage.symbolic
sage: c[0] = SR.var('t', domain='real')  # optional - sage.symbolic
sage: c.display('c')  # optional - sage.symbolic
```

**index_generator()**

Generator of indices.

**OUTPUT:**

- an iterable index

**EXAMPLES:**

Indices on a 3-dimensional vector space:

```python
sage: from sage.tensor.modules.comp import Components
sage: V = VectorSpace(QQ,3)
sage: c = Components(QQ, V.basis(), 1)
sage: list(c.index_generator())
[(0,), (1,), (2,)]
sage: c = Components(QQ, V.basis(), 1, start_index=1)
sage: list(c.index_generator())
[(1,), (2,), (3,)]
sage: c = Components(QQ, V.basis(), 2)
sage: list(c.index_generator())
[(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)]
```

**is_zero()**

Return True if all the components are zero and False otherwise.

**EXAMPLES:**

A just-created set of components is initialized to zero:

```python
sage: from sage.tensor.modules.comp import Components
sage: V = VectorSpace(QQ,3)
sage: c = Components(QQ, V.basis(), 1)
sage: c.is_zero()
True
sage: c[:]
[0, 0, 0]
sage: c[0] = 1 ; c[:]
[1, 0, 0]
sage: c.is_zero()
False
sage: c[0] = 0 ; c[:]
[0, 0, 0]
sage: c.is_zero()
True
```

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It is equivalent to use the operator `==` to compare to zero:

```
sage: c == 0
True
sage: c != 0
False
```

Comparing to a nonzero number is meaningless:

```
sage: c == 1
Traceback (most recent call last):
...  
TypeError: cannot compare a set of components to a number
```

**non_redundant_index_generator()**

Generator of non redundant indices.

In the absence of declared symmetries, all possible indices are generated. So this method is equivalent to `index_generator()`. Only versions for derived classes with symmetries or antisymmetries are not trivial.

**OUTPUT:**

- an iterable index

**EXAMPLES:**

Indices on a 3-dimensional vector space:

```
sage: from sage.tensor.modules.comp import Components
sage: V = VectorSpace(QQ,3)
sage: c = Components(QQ, V.basis(), 2)
sage: list(c.non_redundant_index_generator())
[(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0),
 (2, 1), (2, 2)]
sage: c = Components(QQ, V.basis(), 2, start_index=1)
sage: list(c.non_redundant_index_generator())
[(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1),
 (3, 2), (3, 3)]
```

**swap_adjacent_indices(pos1, pos2, pos3)**

Swap two adjacent sets of indices.

This method is essentially required to reorder the covariant and contravariant indices in the computation of a tensor product.

**INPUT:**

- `pos1` – position of the first index of set 1 (with the convention `position=0` for the first slot)
- `pos2` – position of the first index of set 2 equals 1 plus the position of the last index of set 1 (since the two sets are adjacent)
- `pos3` – 1 plus position of the last index of set 2

**OUTPUT:**

- Components with index set 1 permuted with index set 2.

**EXAMPLES:**

Swap of the two indices of a 2-indices set of components:
Swap of two pairs of indices on a 4-indices set of components:

```python
sage: c = V.basis().create_tensor((3, 2))
sage: c[0,1,1,2] = 16
sage: c1 = c.swap_adjacent_indices(0,1,2)
sage: c[0,1,1,2], c1[0,1,1,2]
(16, 24)
```

**`symmetrize(*pos)`**

Symmetrization over the given index positions.

**INPUT:**

- `pos` – list of index positions involved in the symmetrization (with the convention position=0 for the first slot); if none, the symmetrization is performed over all the indices

**OUTPUT:**

- an instance of `CompWithSym` describing the symmetrized components

**EXAMPLES:**

Symmetrization of 2-indices components:

```python
sage: from sage.tensor.modules.comp import Components
sage: V = VectorSpace(QQ, 3)
sage: c = Components(QQ, V.basis(), 2)
sage: c[0,1] = [[1,2,3], [4,5,6], [7,8,9]]
sage: s = c.symmetrize() ; s
Fully symmetric 2-indices components w.r.t. [(1, 0, 0), (0, 1, 0), (0, 0, 1)]
```

(continues on next page)
sage: c[:], s[:]
([1 2 3] [1 3 5]
[4 5 6] [3 5 7]
[7 8 9], [5 7 9])
sage: c.symmetrize() == c.symmetrize(0,1)
True

Full symmetrization of 3-indices components:

sage: c = Components(QQ, V.basis(), 3)
sage: c[:] = [[[1, 2, 3], [4, 5, 6], [7, 8, 9]], [[10, 11, 12], [13, 14, 15], [16, 17, 18]],
           [[19, 20, 21], [22, 23, 24], [25, 26, 27]]]
sage: s = c.symmetrize() ; s
Fully symmetric 3-indices components w.r.t. 
[(1, 0, 0),
(0, 1, 0),
(0, 0, 1)]
sage: c[:], s[:]
([[[1, 2, 3], [4, 5, 6], [7, 8, 9]],
  [[10, 11, 12], [13, 14, 15], [16, 17, 18]],
  [[19, 20, 21], [22, 23, 24], [25, 26, 27]]),
([[[1, 2, 3], [7, 8, 9], [13, 14, 15]],
  [[7, 8, 9], [13, 14, 15], [19, 20, 21]],
  [[13, 14, 15], [19, 20, 21], [25, 26, 27]]])
sage: all(s[i,j,k] == (c[i,j,k]+c[i,k,j]+c[j,k,i]+c[j,i,k]+c[k,i,j]+c[k,j,i])/6
   # Check of the result:
   ....: for i in range(3) for j in range(3) for k in range(3))
True
sage: c.symmetrize() == c.symmetrize(0,1,2)
True

Partial symmetrization of 3-indices components:

sage: s = c.symmetrize(0,1) ; s  # symmetrization on the first two indices
3-indices components w.r.t. 
[(1, 0, 0),
(0, 1, 0),
(0, 0, 1)]
with symmetry on the index positions (0, 1)
sage: c[:], s[:]
([[[1, 2, 3], [4, 5, 6], [7, 8, 9]],
  [[10, 11, 12], [13, 14, 15], [16, 17, 18]],
  [[19, 20, 21], [22, 23, 24], [25, 26, 27]]),
([[[1, 2, 3], [7, 8, 9], [13, 14, 15]],
  [[7, 8, 9], [13, 14, 15], [19, 20, 21]],
  [[13, 14, 15], [19, 20, 21], [25, 26, 27]]))
sage: all(s[i,j,k] == (c[i,j,k]+c[i,k,j])/2  # Check of the result:
   ....: for i in range(3) for j in range(3) for k in range(3))
True
sage: s = c.symmetrize(1,2) ; s  # symmetrization on the last two indices
3-indices components w.r.t. 
(1, 0, 0),
(0, 1, 0),
(0, 0, 1)
], with symmetry on the index positions (1, 2)
sage: c[:], s[:]
([[1, 2, 3], [4, 5, 6], [7, 8, 9]],
 [[10, 11, 12], [13, 14, 15], [16, 17, 18]],
 [[19, 20, 21], [22, 23, 24], [25, 26, 27]]),
([[1, 3, 5], [3, 5, 7], [5, 7, 9]],
 [[10, 12, 14], [12, 14, 16], [14, 16, 18]],
 [[19, 21, 23], [21, 23, 25], [23, 25, 27]])

sage: all(s[i,j,k] == (c[i,j,k]+c[i,k,j])/2  # Check of the result:
....: for i in range(3) for j in range(3) for k in range(3))
True

trace(pos1, pos2)
Index contraction.

INPUT:
• pos1 – position of the first index for the contraction (with the convention position=0 for the first slot)
• pos2 – position of the second index for the contraction

OUTPUT:
• set of components resulting from the (pos1, pos2) contraction

EXAMPLES:
Self-contraction of a set of components with 2 indices:

sage: from sage.tensor.modules.comp import Components
sage: V = VectorSpace(QQ, 3)
sage: c = Components(QQ, V.basis(), 2)
sage: c[:] = [[1,2,3], [4,5,6], [7,8,9]]
sage: c.trace(0,1)
15
Three self-contractions of a set of components with 3 indices:

```python
sage: v = Components(QQ, V.basis(), 1)
sage: v[: ] = (-1,2,3)
sage: a = c*v ; a
3-indices components w.r.t. [(1, 0, 0),
(0, 1, 0),
(0, 0, 1)]
sage: s = a.trace(0,1) ; s  # contraction on the first two indices
1-index components w.r.t. [(1, 0, 0),
(0, 1, 0),
(0, 0, 1)]
sage: s[: ]
[-15, 30, 45]
sage: [sum(a[j,j,i] for j in range(3)) for i in range(3)]  # check
[-15, 30, 45]
sage: s = a.trace(0,2) ; s[: ]  # contraction on the first and last indices
[28, 32, 36]
sage: [sum(a[j,i,j] for j in range(3)) for i in range(3)]  # check
[28, 32, 36]
sage: s = a.trace(1,2) ; s[: ]  # contraction on the last two indices
[12, 24, 36]
sage: [sum(a[i,j,j] for j in range(3)) for i in range(3)]  # check
[12, 24, 36]
```

**class** `sage.tensor.modules.comp.KroneckerDelta(ring, frame, start_index=0, output_formatter=None)`

Bases: `sage.tensor.modules.comp.CompFullySym`

Kronecker delta $\delta_{ij}$.

**INPUT:**

- `ring` – commutative ring in which each component takes its value
- `frame` – frame with respect to which the components are defined; whatever type `frame` is, it should have some method `__len__()` implemented, so that `len(frame)` returns the dimension, i.e. the size of a single index range
- `start_index` – (default: 0) first value of a single index; accordingly a component index $i$ must obey $\text{start_index} \leq i \leq \text{start_index} + \text{dim} - 1$, where $\text{dim} = \text{len(frame)}$.
- `output_formatter` – (default: None) function or unbound method called to format the output of the component access operator `[...]` (method `__getitem__`); `output_formatter` must take 1 or 2 arguments: the first argument must be an instance of `ring` and the second one, if any, some format specification

**EXAMPLES:**

The Kronecker delta on a 3-dimensional space:
One can read, but not set, the components of a Kronecker delta:

```
sage: d[1,1]
1
sage: d[1,1] = 2
```

Traceback (most recent call last):
  ...
TypeError: the components of a Kronecker delta cannot be changed

Examples of use with output formatters:

```
sage: d = KroneckerDelta(QQ, V.basis(), output_formatter=Rational.numerical_approx)
sage: d[:]
# default format (53 bits of precision)
[ 1.000000000000000000000000000000000000000000000000000000000000000
  0.000000000000000000000000000000000000000000000000000000000000000
  0.000000000000000000000000000000000000000000000000000000000000000]
sage: d[:10]
# format = 10 bits of precision
[ 1.00 0.00 0.00]
[0.00 1.00 0.00]
[0.00 0.00 1.00]
sage: d = KroneckerDelta(QQ, V.basis(), output_formatter=str)
sage: d[:]
[['1', '0', '0'], ['0', '1', '0'], ['0', '0', '1']]
```
This module defines helper functions that are not class methods.

AUTHORS:

- Joris Vankerschaver (2010): for the function is_atomic()
- Michael Jung (2020): extended usage of is_atomic()

class sage.tensor.modules.format_utilities.FormattedExpansion(txt=\textit{None}, latex=\textit{None})

Bases: sage.structure.sage_object.SageObject

Helper class for displaying tensor expansions.

EXAMPLES:

```python
sage: from sage.tensor.modules.format_utilities import FormattedExpansion
sage: f = FormattedExpansion('v', r'\tilde v')
sage: f
v
sage: latex(f)
\textit{v}
```

```python
def format_mul_latex(name1, operator, name2)

Helper function for LaTeX names of results of multiplication or tensor product.

EXAMPLES:

```python
sage: from sage.tensor.modules.format_utilities import format_mul_latex
sage: format_mul_latex('a', '*', 'b')
'a*b'
```

sage.tensor.modules.format_utilities.format_mul_txt(name1, operator, name2)

Helper function for text-formatted names of results of multiplication or tensor product.

EXAMPLES:

```
sage: from sage.tensor.modules.format_utilities import format_mul_txt
sage: format_mul_txt('a', '*', 'b')
'a*b'
sage: format_mul_txt('a+b', '*', 'c')
'(a+b)*c'
sage: format_mul_txt('a', '*', 'b+c')
'a*(b+c)'
sage: format_mul_txt('a+b', '*', 'c+d')
'(a+b)*(c+d)'
sage: format_mul_txt(None, '*', 'b')
sage: format_mul_txt('a', '*', None)
```

sage.tensor.modules.format_utilities.format_unop_latex(operator, name)

Helper function for LaTeX names of results of unary operator.

EXAMPLES:

```
sage: from sage.tensor.modules.format_utilities import format_unop_latex
sage: format_unop_latex('-', 'a')
'-a'
sage: format_unop_latex('-', 'a+b')
'-\left(a+b\right)'
sage: format_unop_latex('-', '(a+b)')
'-(a+b)'
sage: format_unop_latex('-', None)
```

sage.tensor.modules.format_utilities.format_unop_txt(operator, name)

Helper function for text-formatted names of results of unary operator.

EXAMPLES:

```
sage: from sage.tensor.modules.format_utilities import format_unop_txt
sage: format_unop_txt('-', 'a')
'-a'
sage: format_unop_txt('-', 'a+b')
'-(a+b)'
sage: format_unop_txt('-', '(a+b)')
'-(a+b)'
sage: format_unop_txt('-', None)
```

sage.tensor.modules.format_utilities.is_atomic(expr, sep=['+', '-'])

Helper function to check whether some LaTeX expression is atomic.

Adapted from method _is_atomic() of class DifferentialFormFormatter written by Joris Vankerschaver (2010) and modified by Michael Jung (2020).

INPUT:

- expr – string representing the expression (e.g. LaTeX string)
- sep – (default: ['+', '-']) a list of strings representing the operations (e.g. LaTeX strings)

OUTPUT:
• True if the operations are enclosed in parentheses and False otherwise.

**EXAMPLES:**

```python
sage: from sage.tensor.modules.format_utilities import is_atomic
sage: is_atomic("2*x")
True
sage: is_atomic("2+x")
False
sage: is_atomic("(2+x)")
True
```

Moreover the separator can be changed:

```python
sage: is_atomic("a*b", sep=['*'])
False
sage: is_atomic("(a*b)", sep=['*'])
True
sage: is_atomic("a mod b", sep=['mod'])
False
sage: is_atomic("(a mod b)", sep=['mod'])
True
```

**sage.tensor.modules.format_utilities.is_atomic_wedge_latex(expression)**

Helper function to check whether LaTeX-formatted expression is atomic in terms of wedge products.

Adapted from method _is_atomic() of class DifferentialFormFormatter written by Joris Vankerschaver (2010) and modified by Michael Jung (2020).

**INPUT:**

• expression – string representing the LaTeX expression

**OUTPUT:**

• True if wedge products are enclosed in parentheses and False otherwise.

**EXAMPLES:**

```python
sage: from sage.tensor.modules.format_utilities import is_atomic_wedge_latex
sage: is_atomic_wedge_latex(r"a")
True
sage: is_atomic_wedge_latex(r"a\wedge b")
False
sage: is_atomic_wedge_latex(r"(a\wedge b)\wedge c")
False
sage: is_atomic_wedge_latex(r"((a\wedge b)\wedge c)")
True
sage: is_atomic_wedge_latex(r"(a\wedge b)\wedge \theta")
True
sage: is_atomic_wedge_latex(r"(\omega\wedge\theta)")
False
sage: is_atomic_wedge_latex(r"(\omega\wedge(\theta+a))")
False
```
sage.tensor.modules.format_utilities.is_atomic_wedge_txt(expression)
Helper function to check whether some text-formatted expression is atomic in terms of wedge products.

Adapted from method _is_atomic() of class DifferentialFormFormatter written by Joris Vankerschaver (2010) and modified by Michael Jung (2020).

INPUT:
• expression – string representing the text-formatted expression

OUTPUT:
• True if wedge products are enclosed in parentheses and False otherwise.

EXAMPLES:

```python
sage: from sage.tensor.modules.format_utilities import is_atomic_wedge_txt
sage: is_atomic_wedge_txt("a")
True
sage: is_atomic_wedge_txt(r"a∧b")
False
sage: is_atomic_wedge_txt(r"(a∧b)")
True
sage: is_atomic_wedge_txt(r"(a∧b)∧c")
False
sage: is_atomic_wedge_txt(r"(a∧b∧c)")
True
```
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